# MATH 25: PRECALCULUS



*Jennifer Moorhouse*
Hartnell College



# PreCalculus (Abramson OpenStax) for Hartnell Math 25

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# 1: FUNCTIONS


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# 1.0: Prelude to Functions

Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. The *Standard and Poor's Index* tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1100 by the beginning of 2000. That five-year period became known as the "dot-com bubble" because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

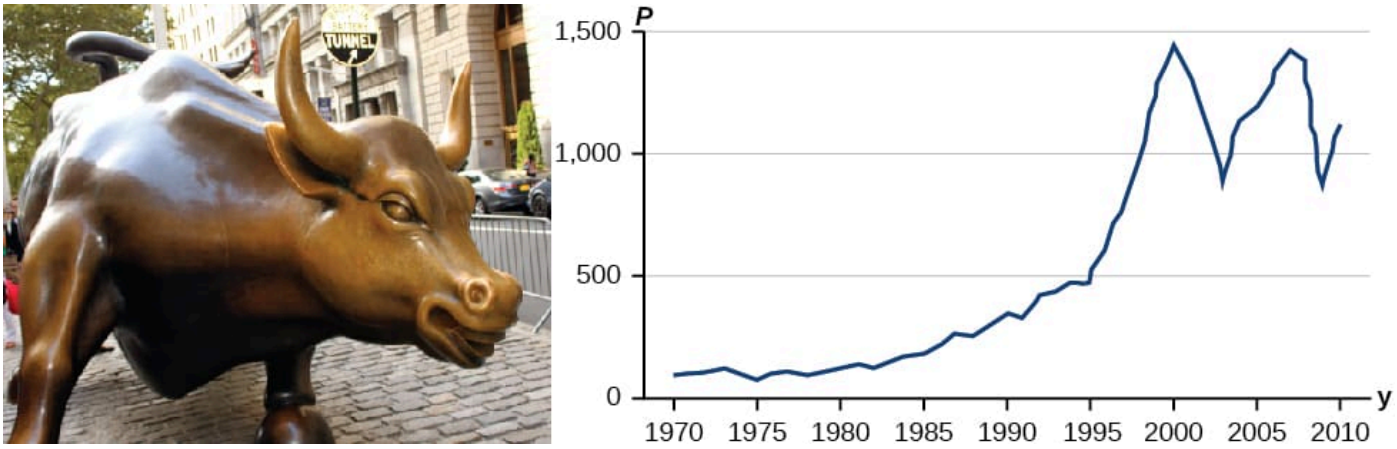



Figure 1.0.1: Figure of a Bull and a graph of market prices. Standard and Poor's Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

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# 1.1: Functions and Function Notation

**Learning Objectives**

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

## Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\} \tag{1.1.1}$$

The domain is $\{1,2,3,4,5\}$. The range is $\{2,4,6,8,10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter $x$. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter $y$.

A function $f$ is a relation that assigns a single value in the range to each value in the domain. In other words, no $x$-values are repeated. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$.

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\} \tag{1.1.2}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$. For example, the term "odd" corresponds to three values from the range, $\{1,3,5\}$, and the term "even" corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure $1.1.1$ compares relations that are functions and not functions.

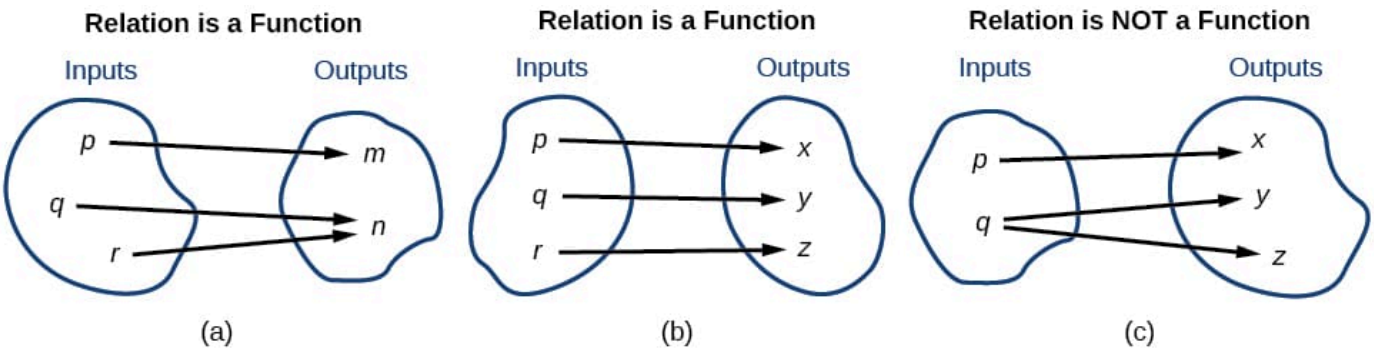



Figure 1.1.1: (a) This relationship is a function because each input is associated with a single output. Note that input $q$ and $r$ both give output $n$. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input $q$ is associated with two different outputs.

## Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.

## How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

## Example 1.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure 1.1.2 consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?




Figure 1.1.2: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

**Solution**

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure 1.1.3.




Figure 1.1.3: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure 1.1.4.

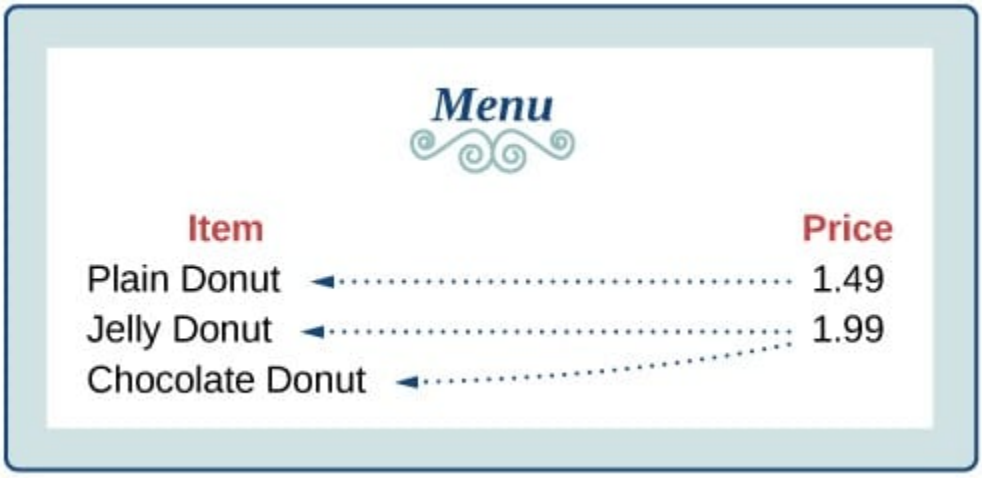



Figure 1.1.4: Association of the prices to the donuts.

Therefore, the item is a not a function of price.

## Example 1.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 1.1.1 shows a possible rule for assigning grade points.

*Table 1.1.1: Class grade points.*

| **Percent grade** | 0–56 | 57–61 | 62–66 | 67–71 | 72–77 | 78–86 | 87–91 | 92–100 |
|---|---|---|---|---|---|---|---|---|
| **Grade point average** | 0.0 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |

**Solution**

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

### Exercise 1.1.2

Table 1.1.2 lists the five greatest baseball players of all time in order of rank.

*Table 1.1.2: Five greatest baseball players.*

| Player | Rank |
|---|---|
| Babe Ruth | 1 |
| Willie Mays | 2 |
| Ty Cobb | 3 |
| Walter Johnson | 4 |
| Hank Aaron | 5 |

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

*Answer a available at the end of this section.*

## Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables $h$ for height and $a$ for age. The letters $f$, $g$,and $h$ are often used to represent functions just as we use $x$, $y$,and $z$ to represent numbers and $A$, $B$, and $C$ to represent sets.

$$\begin{aligned} &h \text{ is } f \text{ of } a && \text{We name the function } f\text{; height is a function of age.} \\ &h = f(a) && \text{We use parentheses to indicate the function input.} \\ &f(a) && \text{We name the function } f \text{ ; the expression is read as "} f \text{ of } a\text{."} \end{aligned} \tag{1.1.1}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that $h$ depends on $a$. The value $a$ must be put into the function $h$ to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add $a$ and $b$, and the result is the input for the function $f$." The operations must be performed in this order to obtain the correct result.

### Function Notation

The notation $y = f(x)$ defines a function named $f$. This is read as "$y$ is a function of $x$." The letter $x$ represents the input value, or independent variable. The letter $y$, or $f(x)$, represents the output value, or dependent variable.

## Example 1.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

**Solution**

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function $f$, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

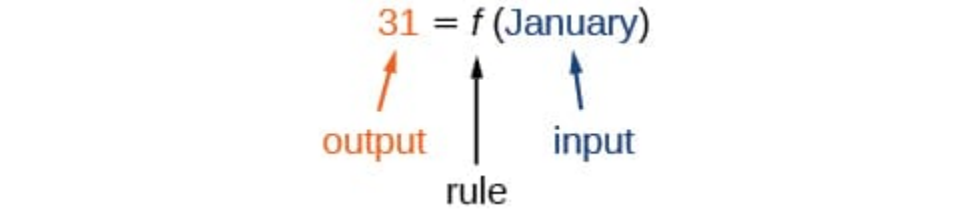



Figure 1.1.5: The function $31 = f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March}) = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, $d$ (the output), is dependent on the name of the month, $m$ (the input).

**Analysis**

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

## Example 1.1.3B: Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, $N$, in a town in year $y$. What does $f(2005) = 300$ represent?

**Solution**

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers $(N)$, is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

## Exercise 1.1.3

Use function notation to express the weight of a pig in pounds as a function of its age in days $d$.

*Answer available at the end of this section.*

## Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "$y$ is a function of $x$?"

*Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as $f$, which is a rule or procedure, and the output y we get by applying $f$ to a particular input $x$. This is why we usually use notation such as $y = f(x), P = W(d)$, and so on.*

## Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 1.1.3 lists the input number of each month ($\text{January} = 1$, $\text{February} = 2$, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function $f$ where $D = f(m)$ identifies months by an integer rather than by name.

*Table 1.1.3: Months and number of days per month.*

| **Month number, $m$ (input)** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Days in month, $D$ (output)** | 31 | 28 | 31 | 30 | 31 | 30 | 31 | 31 | 30 | 31 | 30 | 31 |

Table 1.1.4 defines a function $Q = g(n)$ Remember, this notation tells us that $g$ is the name of the function that takes the input $n$ and gives the output $Q$.

*Table 1.1.4: Function $Q = g(n)$*

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $Q$ | 8 | 6 | 7 | 6 | 8 |

Table 1.1.5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

*Table 1.1.5: Age of children and their corresponding heights.*

| **Age in years, $a$ (input)** | 5 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| **Height in inches, $h$ (output)** | 40 | 42 | 44 | 47 | 50 | 52 | 54 |

### How To: Given a table of input and output values, determine whether the table represents a function

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

### Example 1.1.5: Identifying Tables that Represent Functions

Which table, Table 1.1.6, Table 1.1.7, or Table 1.1.8, represents a function (if any)?

*Table 1.1.6*

| Input | Output |
|---|---|
| 2 | 1 |
| 5 | 3 |

| Input | Output |
|---|---|
| 8 | 6 |

*Table* 1.1.7

| Input | Output |
|---|---|
| -3 | 5 |
| 0 | 1 |
| 4 | 5 |

*Table* 1.1.8

| Input | Output |
|---|---|
| 1 | 0 |
| 5 | 2 |
| 5 | 4 |

**Solution**

Table 1.1.6 and Table 1.1.7 define functions. In both, each input value corresponds to exactly one output value. Table 1.1.8 does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 1.1.6 can be represented by writing

$$f(2) = 1,\ f(5) = 3, \text{ and } f(8) = 6$$

Similarly, the statements

$$g(-3) = 5,\ g(0) = 1, \text{ and } g(4) = 5$$

represent the function in Table 1.1.7.

Table 1.1.8 cannot be expressed in a similar way because it does not represent a function.

## Exercise 1.1.5

Does Table 1.1.9 represent a function?

*Table* 1.1.9

| Input | Output |
|---|---|
| 1 | 10 |
| 2 | 100 |
| 3 | 1000 |

*Answer available at the end of this section.*

# Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

## Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

### How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

### Example $1.1.6A$: Evaluating Functions at Specific Values

1. Evaluate $f(x) = x^2 + 3x - 4$ at

a. $2$
b. $a$
c. $a + h$
d. Evaluate $\frac{f(a+h)-f(a)}{h}$

**Solution**

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} \\ &= \frac{(2ah + h^2 + 3h)}{h} \\ &= \frac{h(2a + h + 3)}{h} && \text{Factor out h.} \\ &= 2a + h + 3 && \text{Simplify.} \end{aligned}$$

## Example 1.1.6$B$: Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

**Solution**

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p \\ h(4) &= (4)^2 + 2(4) \\ &= 16 + 8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

## Exercise 1.1.6

Given the function $g(m) = \sqrt{m - 4}$, evaluate $g(5)$.

*Answer available at the end of this section.*

## Example 1.1.7: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

**Solution**

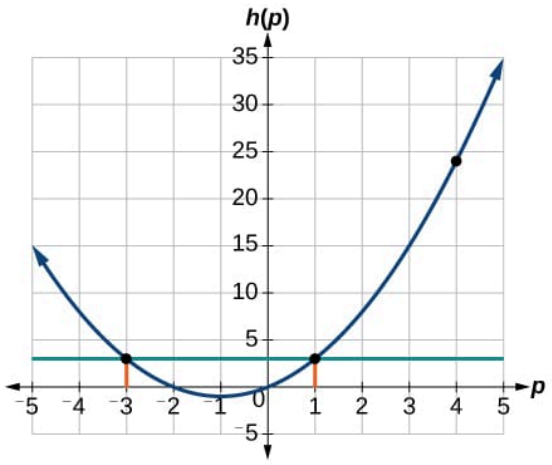
$$\begin{aligned} h(p) &= 3 \\ p^2 + 2p &= 3 && \text{Substitute the original function} \\ p^2 + 2p - 3 &= 0 && \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 && \text{Factor.} \end{aligned}$$

If $(p + 3)(p - 1) = 0$, either $(p + 3) = 0$ or $(p - 1) = 0$ (or both of them equal $0$). We will set each factor equal to $0$ and solve for $p$ in each case.

$$(p + 3) = 0,\ p = -3$$

$$(p - 1) = 0,\ p = 1$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as in Figure $1.1.6$. The graph verifies that $h(1) = h(-3) = 3$ and $h(4) = 24$.




| p | −3 | −2 | 0 | 1 | 4 |
|---|---|---|---|---|---|
| h(p) | 3 | 0 | 0 | 3 | 24 |

Figure 1.1.6: Graph of $h(p) = p^2 + 2p$

**Exercise 1.1.7**

Given the function $g(m) = \sqrt{m-4}$, solve $g(m) = 2$.

*Answer available at the end of this section.*

## Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in **equation** form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n + 6p = 12$ expresses a functional relationship between $n$ and $p$. We can rewrite it to decide if $p$ is a function of $n$.

**How to: Given a function in equation form, write its algebraic formula.**

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

**Example 1.1.8A: Finding an Equation of a Function**

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$, if possible.

**Solution**

To express the relationship in this form, we need to be able to write the relationship where $p$ is a function of $n$, which means writing it as $p = [\text{expression involving } n]$.

$$
\begin{aligned}
2n + 6p &= 12 \\
6p &= 12 - 2n && \text{Subtract 2n from both sides.} \\
p &= \frac{12 - 2n}{6} && \text{Divide both sides by 6 and simplify.} \\
p &= \frac{12}{6} - \frac{2n}{6} \\
p &= 2 - \frac{1}{3}n
\end{aligned}
$$

Therefore, $p$ as a function of $n$ is written as

$$p = f(n) = 2 - \frac{1}{3}n$$

**Analysis**

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

## Example 1.1.8$B$: Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with $x$ as input and $y$ as output? If so, express the relationship as a function $y = f(x)$.

**Solution**

First we subtract $x^2$ from both sides.

$$y^2 = 1 - x^2$$

We now try to solve for $y$ in this equation.

$$y = \pm\sqrt{1 - x^2}$$

$$\text{so, } y = \sqrt{1 - x^2} \text{ and } y = -\sqrt{1 - x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$ .

## Exercise 1.1.8

If $x - 8y^3 = 0$, express $y$ as a function of $x$.

*Answer available at the end of this section.*

## Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only $x$ that equals $y$. However, each $x$ does determine a unique value for $y$, and there are mathematical procedures by which $y$ can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for $y$ as a function of $x$, even though the formula cannot be written explicitly.

# Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table (Table 1.1.10).

*Table* 1.1.10

| Pet Memory | span in hours |
|---|---|
| Puppy | 0.008 |
| Adult Dog | 0.083 |
| Cat | 3 |
| Goldfish | 2160 |
| Beta Fish | 3600 |

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function $P$. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function $P$ at the input value of "goldfish." We would write $P(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

## How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

## Example 1.1.9: Evaluating and Solving a Tabular Function

Using Table 1.1.11,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

*Table* 1.1.11

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $g(n)$ | 8 | 6 | 7 | 6 | 8 |

**Solution**

a. Evaluating $g(3)$ means determining the output value of the function $g$ for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. Table 1.1.12 shows two solutions: 2 and 4.

*Table* 1.1.12

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $g(n)$ | 8 | 6 | 7 | 6 | 8 |

When we input 2 into the function $g$, our output is 6. When we input 4 into the function $g$, our output is also 6.

### Exercise 1.1.1

Using Table 1.1.12, evaluate $g(1)$.

*Answer available at the end of this section.*

## Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

### Example 1.1.10: Reading Function Values from a Graph

Given the graph in Figure 1.1.7,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

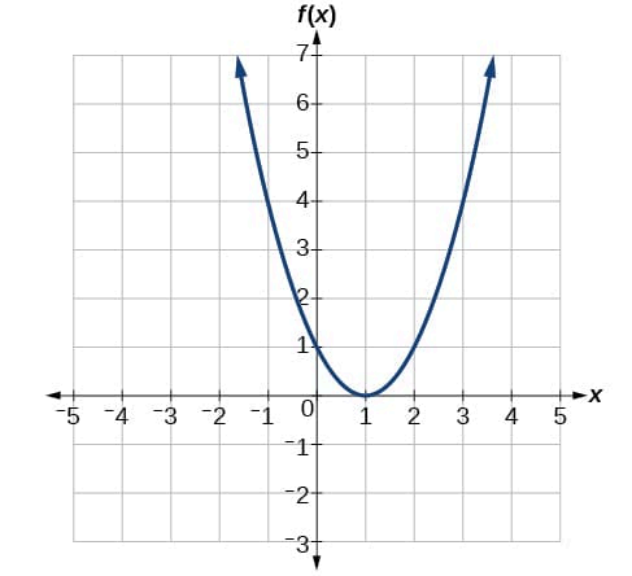
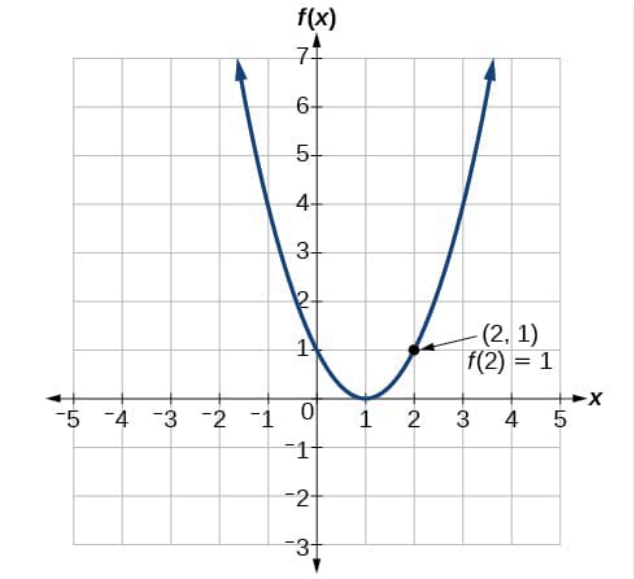



Figure 1.1.7: Graph of a positive parabola centered at $(1, 0)$.

**Solution**

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 1.1.8.




1.1.8: Graph of a positive parabola centered at $(1, 0)$ with the labeled point $(2, 1)$ where $f(2) = 1$.

To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: –1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is –1 or 3, the output is 4. See Figure 1.1.9.

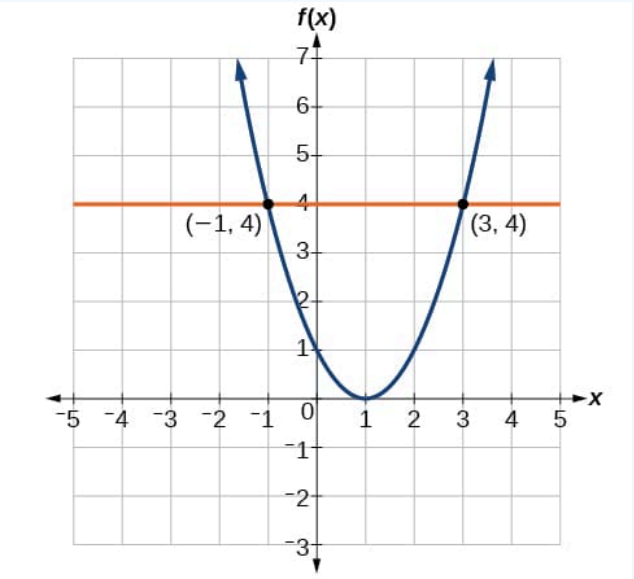



Figure 1.1.9: Graph of an upward-facing parabola with a vertex at $(0, 1)$ and labeled points at $(-1, 4)$ and $(3, 4)$. A line at $y = 4$ intersects the parabola at the labeled points.

**Exercise 1.1.10**

Given the graph in Figure 1.1.7, solve $f(x) = 1$.

*Answer available at the end of this section.*

## Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown

in the Figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 1.1.13.

*Table 1.1.13: Letter grades and decimal equivalents.*

| Letter Grade | Grade Point Average |
|---|---|
| A | 4.0 |
| B | 3.0 |
| C | 2.0 |
| D | 1.0 |

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figures 1.1.1*a* and 1.1.1*b*. The function in part (a) shows a relationship that is not a one-to-one function because inputs $q$ and $r$ both give output $n$. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

## One-to-One Functions

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

## Example 1.1.11: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

**Solution**

A circle of radius $r$ has a unique area measure given by $A = \pi r^2$, so for any input, $r$, there is only one output, $A$. The area is a function of radius$r$.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure $A$ is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

## Exercise 1.1.11*A*

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

*Answer available at the end of this section.*

## Exercise 1.1.11*B*

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

*Answer available at the end of this section.*

## Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value $x$ and the output $y$, and we say $y$ is a function of $x$, or $y = f(x)$ when the function is named $f$. The graph of the function is the set of all points $(x, y)$ in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure $1.1.10$ tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points $(x, y)$ satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

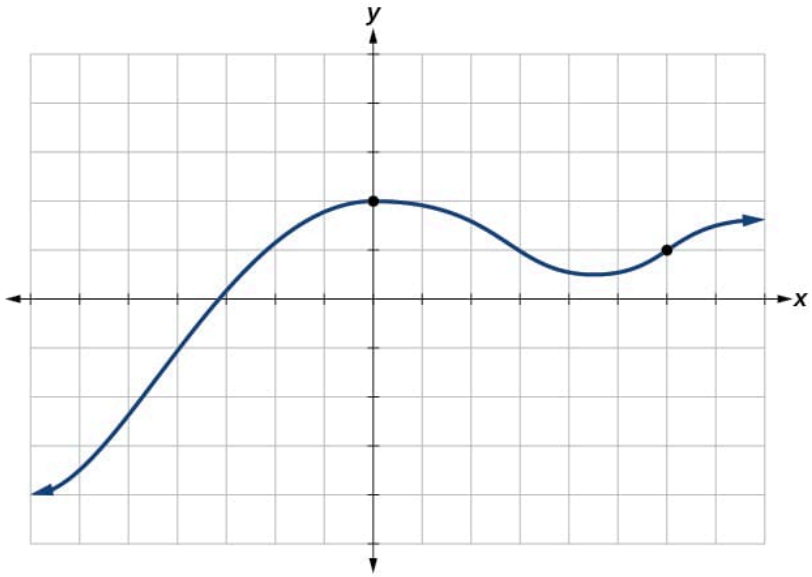



Figure $1.1.10$: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure $1.1.11$.

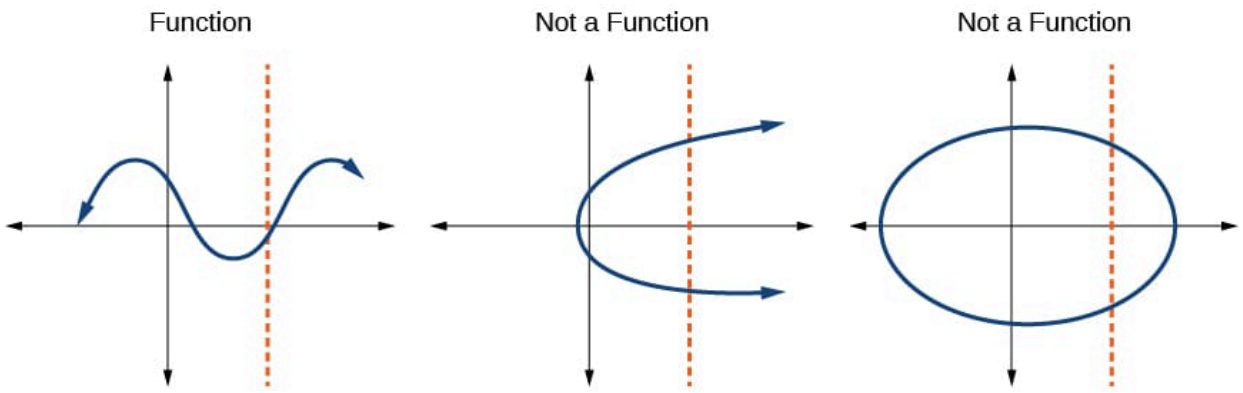



Figure 1.1.11: Three graphs visually showing what is and is not a function.

## Howto: Given a graph, use the vertical line test to determine if the graph represents a function

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

## Example 1.1.12: Applying the Vertical Line Test

Which of the graphs in Figure 1.1.12 represent(s) a function $y = f(x)$?

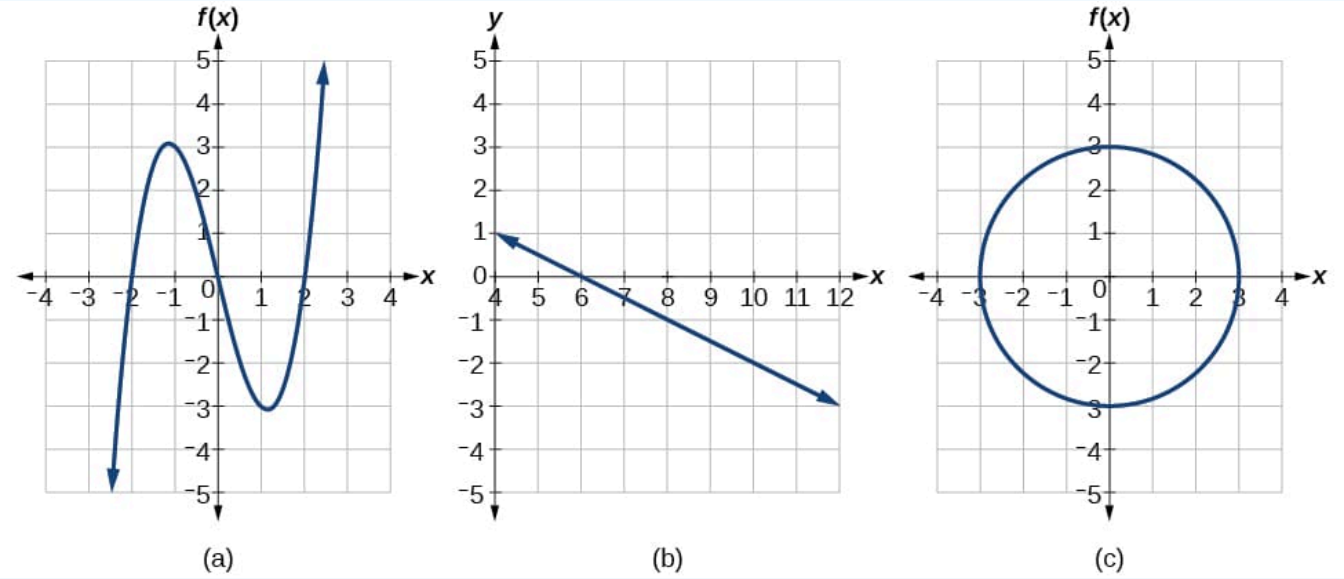



Figure 1.1.12: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

**Solution**

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 1.1.12. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure 1.1.13.

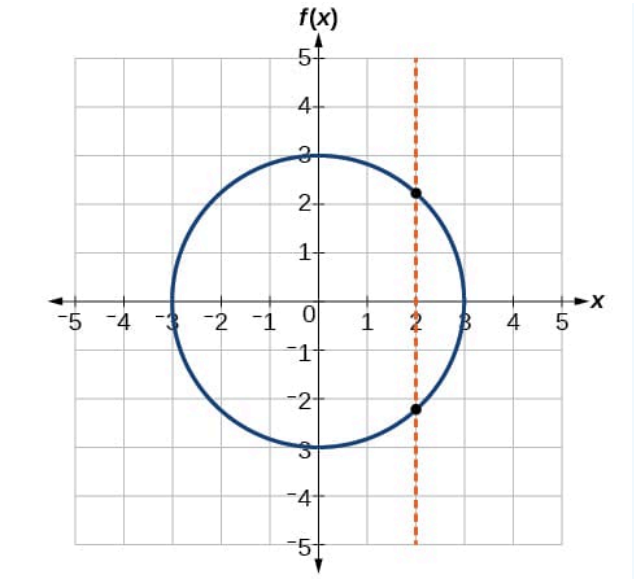



Figure 1.1.13: Graph of a circle.

## Exercise 1.1.12

Does the graph in Figure 1.1.14 represent a function?

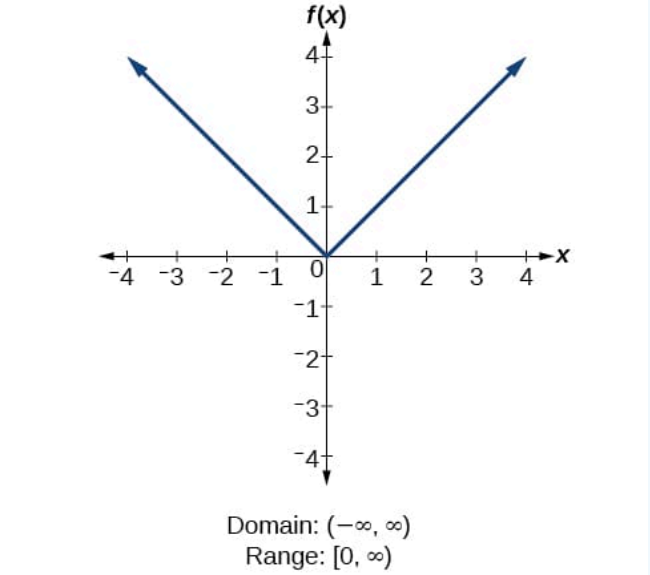



Figure 1.1.14: Graph of absolute value function.

*Answer available at the end of this section.*

# Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

**Howto: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function**

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

**Example 1.1.13: Applying the Horizontal Line Test**

Consider the functions shown in Figure $1.1.12a$ and Figure $1.1.12b$. Are either of the functions one-to-one?

**Solution**

The function in Figure $1.1.12a$ is not one-to-one. The horizontal line shown in Figure $1.1.15$ intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

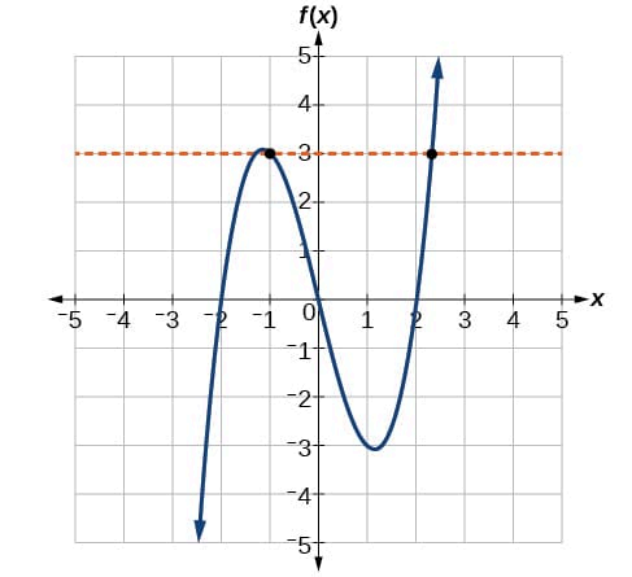



Figure $1.1.15$: Graph of a polynomial with a horizontal line crossing through 2 points

The function in Figure $1.1.12b$ is one-to-one. Any horizontal line will intersect a diagonal line at most once.

**Exercise 1.1.13**

Is the graph shown in Figure $1.1.13$ one-to-one?

*Answer available at the end of this section.*

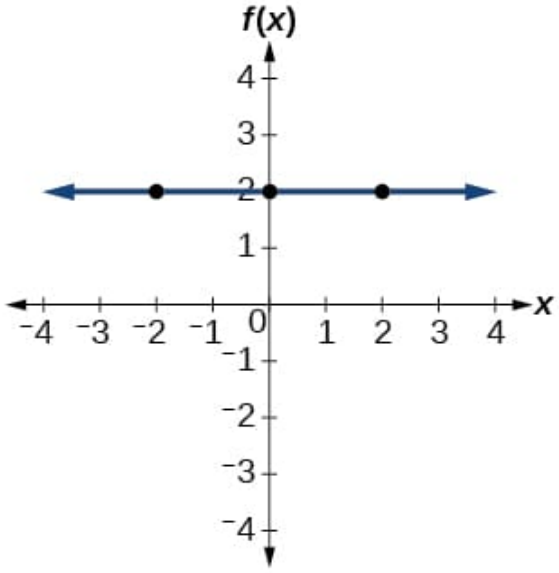
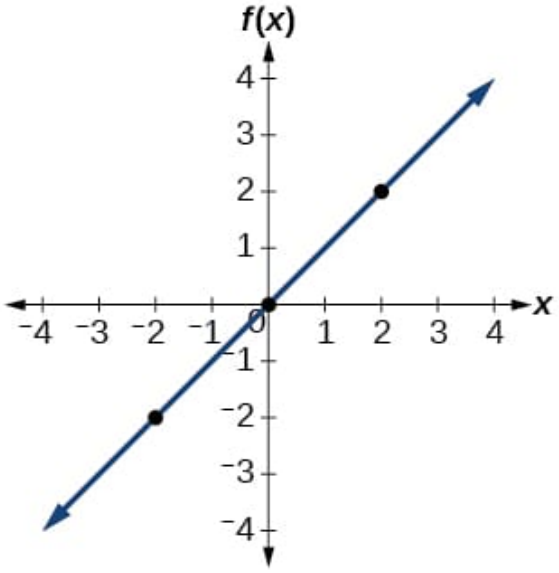
## Identifying Basic Toolkit Functions

In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout

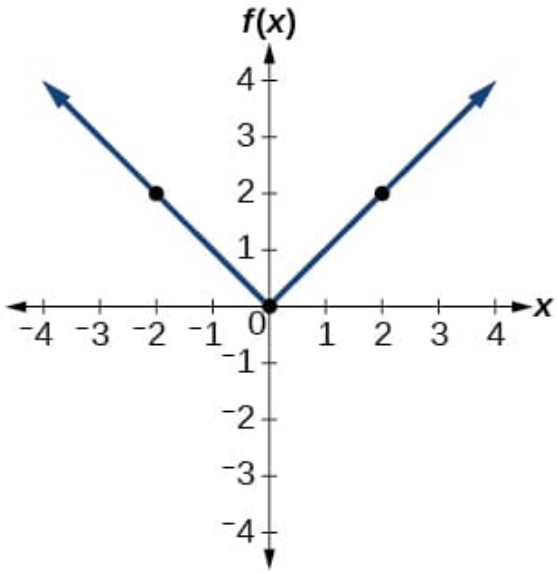
this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table 1.1.14.

*Table 1.1.14: Toolkit Functions*

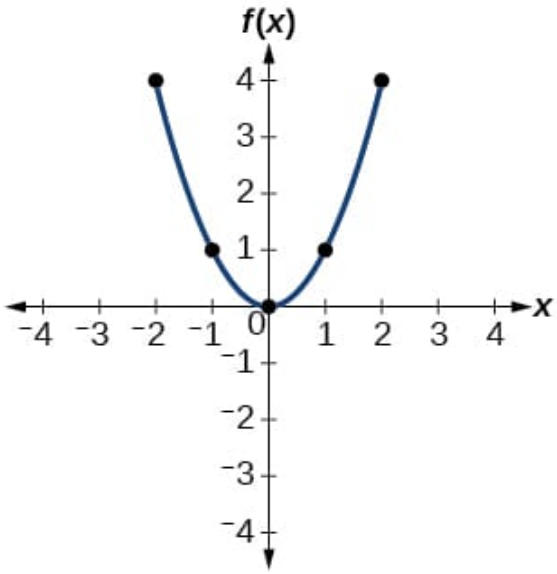
| Name | Function | Graph |
|---|---|---|
| Constant | $f(x) = c$ where $c$ is a constant |   |
| Identity | $f(x) = x$ |   |

| x | f(x) |
|---|---|
| −2 | 2 |
| 0 | 2 |
| 2 | 2 |

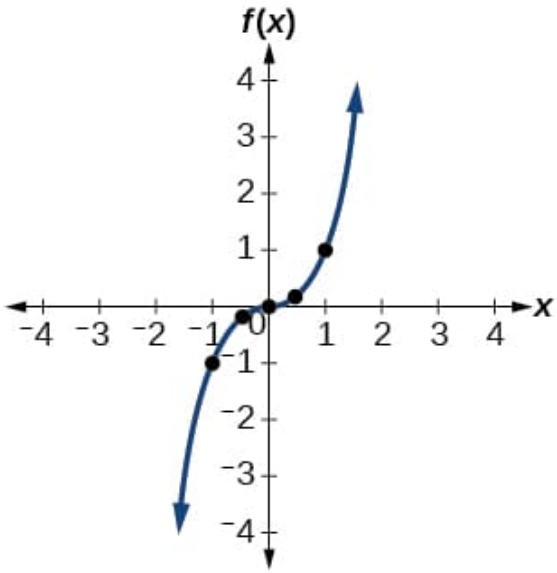
| x | f(x) |
|---|---|
| −2 | −2 |
| 0 | 0 |
| 2 | 2 |

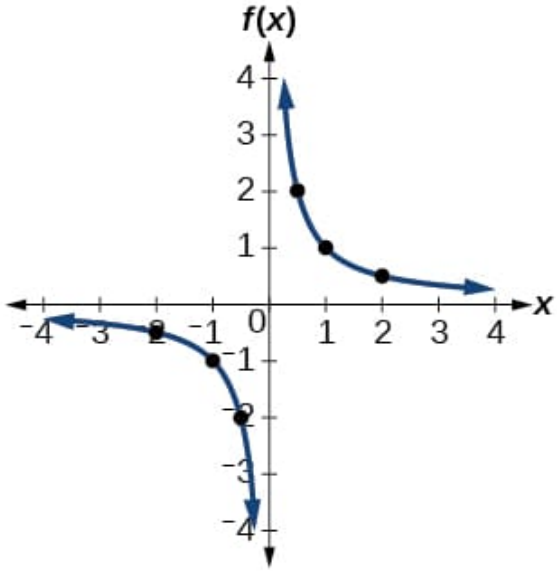
| Name | Function | Graph |
|---|---|---|
| Absolute Value | $f(x) = \|x\|$ |   |

| x | f(x) |
|---|---|
| −2 | 2 |
| 0 | 0 |
| 2 | 2 |

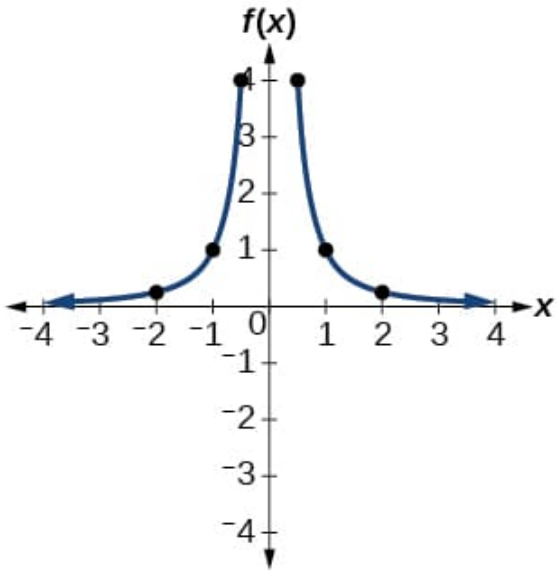
| Name | Function | Graph |
|---|---|---|
| Quadratic | $f(x) = x^2$ |   |

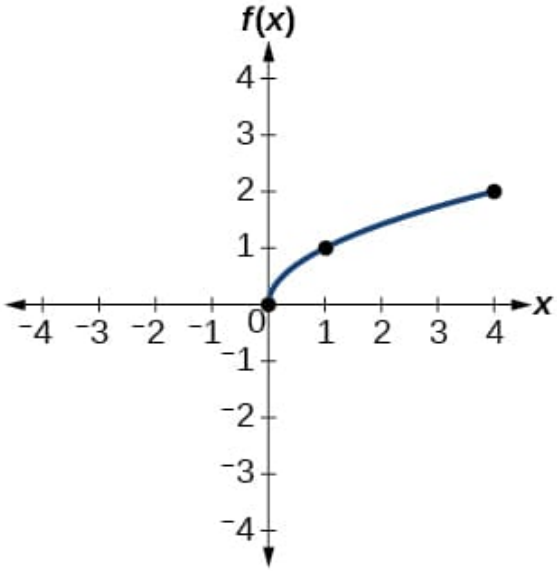
| x | f(x) |
|---|---|
| −2 | 4 |
| −1 | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Cubic</td><td>$f(x) = x^3$</td><td> 

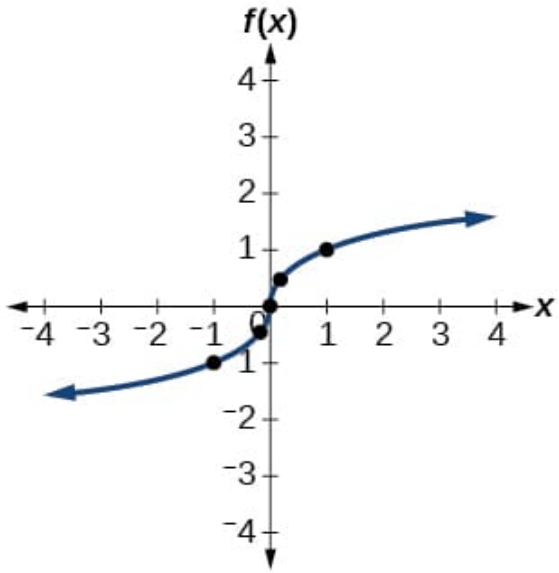
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−0.125</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.5</td><td>0.125</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
<tr><td>reciprocal</td><td>$f(x) = \frac{1}{x}$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−0.5</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−2</td></tr>
<tr><td>0.5</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.5</td></tr>
</table></td></tr>
</table>

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Reciprocal squared</td><td>$f(x) = \frac{1}{x^2}$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>0.25</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>−0.5</td><td>4</td></tr>
<tr><td>0.5</td><td>4</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.25</td></tr>
</table></td></tr>
<tr><td>Square root</td><td>$f(x) = \sqrt{x}$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>4</td><td>2</td></tr>
</table></td></tr>
</table>

| Name | Function | Graph |
|---|---|---|
| Cube root | $f(x) = \sqrt[3]{x}$ |  |



| x | f(x) |
|---|---|
| −1 | −1 |
| −0.125 | −0.5 |
| 0 | 0 |
| 0.125 | 0.5 |
| 1 | 1 |

## Key Equations

- Constant function $f(x) = c$, where $c$ is a constant
- Identity function $f(x) = x$
- Absolute value function $f(x) = |x|$
- Quadratic function $f(x) = x^2$
- Cubic function $f(x) = x^3$
- Reciprocal function $f(x) = \dfrac{1}{x}$
- Reciprocal squared function $f(x) = \frac{1}{x^2}$
- Square root function $f(x) = \sqrt{x}$
- Cube root function $f(x) = 3\sqrt{x}$

## Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$.
- In tabular form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- The graph of a one-to-one function passes the horizontal line test.

## Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.
2 www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

# Glossary

**dependent variable**
an output variable

**domain**
the set of all possible input values for a relation

**function**
a relation in which each input value yields a unique output value

**horizontal line test**
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

**independent variable**
an input variable

**input**
each object or value in a domain that relates to another object or value by a relationship known as a function

**one-to-one function**
a function for which each value of the output is associated with a unique input value

**output**
each object or value in the range that is produced when an input value is entered into a function

**range**
the set of output values that result from the input values in a relation

**relation**
a set of ordered pairs

**vertical line test**
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

---

## Answers

**Exercise** 1.1.2

Yes

yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

**Exercise** 1.1.3

$w = f(d)$

**Exercise** 1.1.5

yes

**Exercise** 1.1.6

$g(5) = 1$

**Exercise** 1.1.7

$m = 8$

**Exercise** 1.1.8

$y = f(x) = \frac{\sqrt[3]{x}}{2}$

**Exercise** 1.1.1

$g(1) = 8$

**Exercise** 1.1.10

$x = 0$ or $x = 2$

**Exercise** 1.1.11$A$

a. yes, because each bank account has a single balance at any given time;

b. no, because several bank account numbers may have the same balance;

c. no, because the same output may correspond to more than one input.

**Exercise** 1.1.11$B$

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

**Exercise** 1.1.12

yes

**Exercise** 1.1.13

No, because it does not pass the horizontal line test.

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# 1.2: Domain and Range

**Learning Objectives**

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—I am Legend, Hannibal, The Ring, The Grudge, and The Conjuring. Figure 1.2.1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

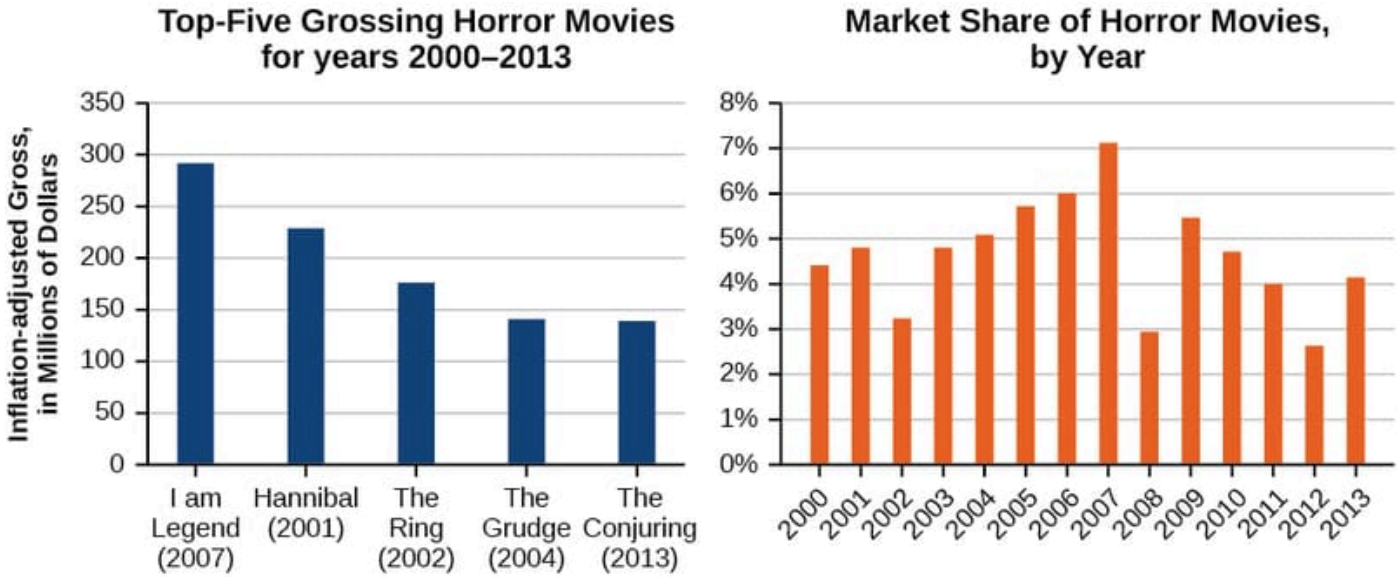



Figure 1.2.1: Graph of the Top-Five Grossing Horror Movies for years 2000-2003, and a Graph of the Market Share of Horror Movies by Year. Based on data compiled by www.the-numbers.com.

## Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (Figure 1.2.2).

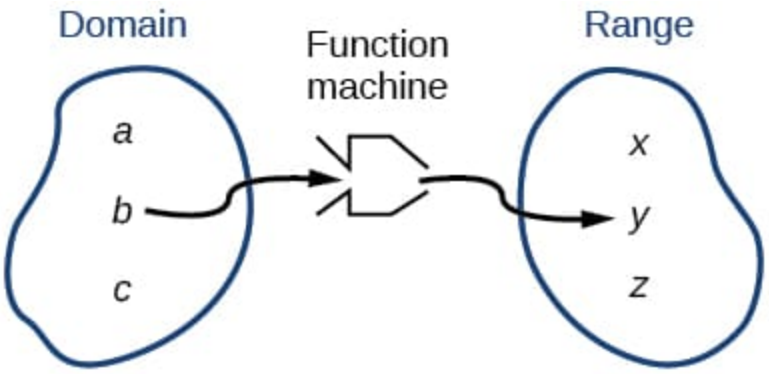



Figure 1.2.2: Diagram of how a function relates two relations

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [ when the set includes the endpoint and a parenthesis ( to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, ( or ), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [ or ], are used to indicate that an endpoint is included, called inclusive.

See Figure 1.2.3 for a summary of interval notation.

| Inequality | Interval Notation | Graph on Number Line | Description |
|---|---|---|---|
| $x > a$ | $(a, \infty)$ |  | $x$ is greater than $a$ |
| $x < a$ | $(-\infty, a)$ |  | $x$ is less than $a$ |
| $x \geq a$ | $[a, \infty)$ |  | $x$ is greater than or equal to $a$ |
| $x \leq a$ | $(-\infty, a]$ |  | $x$ is less than or equal to $a$ |
| $a < x < b$ | $(a, b)$ |  | $x$ is strictly between $a$ and $b$ |
| $a \leq x < b$ | $[a, b)$ |  | $x$ is between $a$ and $b$, to include $a$ |
| $a < x \leq b$ | $(a, b]$ |  | $x$ is between $a$ and $b$, to include $b$ |
| $a \leq x \leq b$ | $[a, b]$ |  | $x$ is between $a$ and $b$, to include $a$ and $b$ |

Figure 1.2.3: Summary of interval notation.

### Example 1.2.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2,10),(3,10),(4,20),(5,30),(6,40)\}$.

**Solution**

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2,3,4,5,6\}$$

### Exercse 1.2.1

Find the domain of the function:

$$\{(-5,4),(0,0),(5,-4),(10,-8),(15,-12)\}$$

*Answer available at the end of this section.*

### How To: Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

### Example 1.2.2: Finding the Domain of a Function

Find the domain of the function $f(x)=x^2-1$.

**Solution**

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty,\infty)$.

### Exercse 1.2.2

Find the domain of the function:

$$f(x)=5-x+x^3$$

*Answer available at the end of this section.*

### Howto: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.

3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

## Example 1.2.3: Finding the Domain of a Function Involving a Denominator

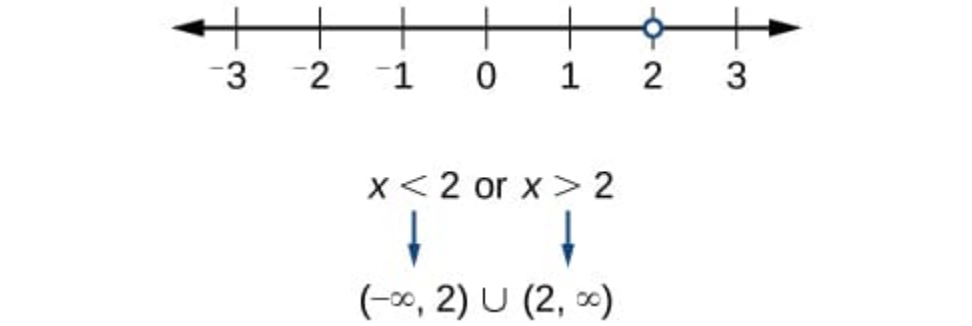
Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

**Solution**

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2 - x &= 0 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty, 2) \cup (2, \infty)$.




In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

## Exercse 1.2.3

Find the domain of the function:

$$f(x) = \frac{1+4x}{2x-1}$$

*Answer available at the end of this section.*

## How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

## Example 1.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x) = \sqrt{7 - x}.$$

**Solution**

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7 - x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

**Exercse 1.2.4**

Find the domain of the function

$$f(x) = \sqrt{5 + 2x}.$$

*Answer available at the end of this section.*

**Q&A: Can there be functions in which the domain and range do not intersect at all?**

*Yes. For example, the function $f(x) = -\dfrac{1}{\sqrt{x}}$ has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

## Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure 1.2.4 compares inequality notation, set-builder notation, and interval notation.

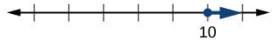
| | Inequality Notation | Set-builder Notation | Interval Notation |
|---|---|---|---|
|   | $5 < h \leq 10$ | $\{h \mid 5 < h \leq 10\}$ | $(5, 10]$ |
|   | $5 \leq h < 10$ | $\{h \mid 5 \leq h < 10\}$ | $[5, 10)$ |
|   | $5 < h < 10$ | $\{h \mid 5 < h < 10\}$ | $(5, 10)$ |
|   | $h < 10$ | $\{h \mid h < 10\}$ | $(-\infty, 10)$ |
|   | $h \geq 10$ | $\{h \mid h \geq 10\}$ | $[10, \infty)$ |
|   | All real numbers | $\mathbb{R}$ | $(-\infty, \infty)$ |

Figure 1.2.4: Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets$\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3\} = (-\infty, -3] \cup [3, \infty) \tag{1.2.1}$$

## Set-Builder Notation and Interval Notation

*Set-builder notation* is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|\text{ statement about x}\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \leq 12\}$$

*Interval notation* is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

**Given a line graph, describe the set of values using interval notation.**

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [ with each end value to be included in the set (solid dot) or ( for each excluded end value (open dot).
3. At the right end of each interval, use ] with each end value to be included in the set (filled dot) or ) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

## Example 1.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure 1.2.5 using inequality notation, set-builder notation, and interval notation.

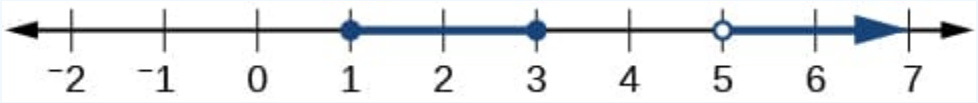



Figure 1.2.5: Line graph of $1 \leq x \leq 3$ or $x > 5$.

**Solution**

To describe the values, $x$, included in the intervals shown, we would say, "$x$ is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x | 1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

## Exercse 1.2.5

Given Figure 1.2.6, specify the graphed set in

a. words
b. set-builder notation
c. interval notation

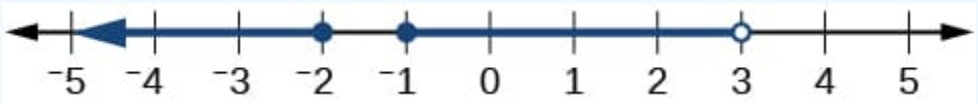



Figure 1.2.6: Line graph of $-2 \leq x$, $-1 \leq x < 3$.leq}x\), $-1 \leq x < 3$.

*Answer a available at the end of this section.*

## Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 1.2.7.

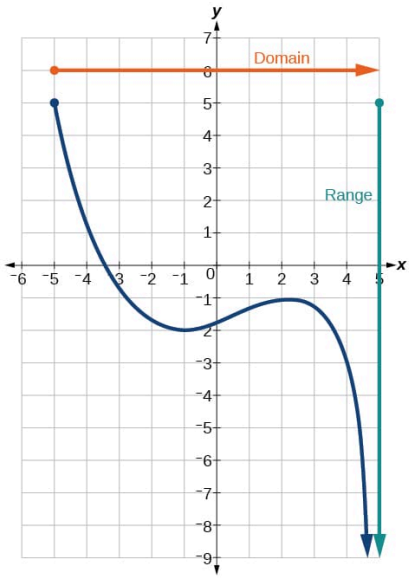



Figure 1.2.7: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from –5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

## Example 1.2.6$A$: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

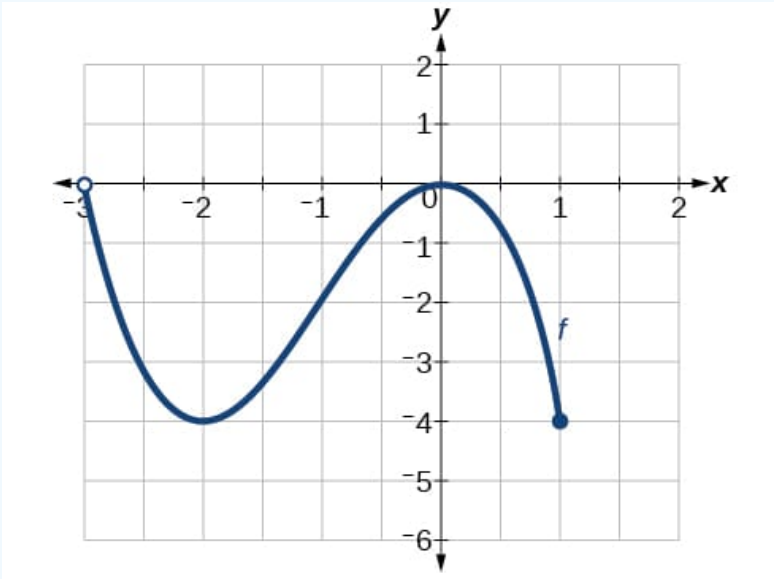



Figure 1.2.8: Graph of a function from (-3, 1].

**Solution**

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See Figure 1.2.9.

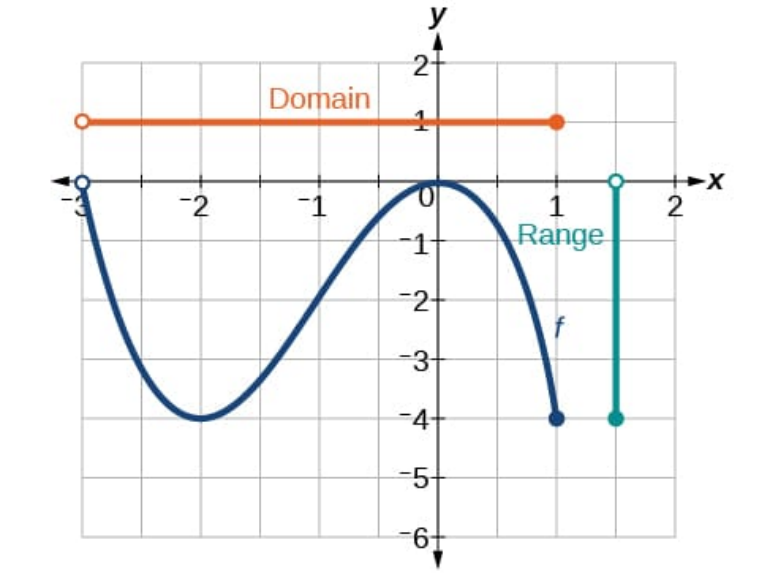



Figure 1.2.9: Graph of the previous function shows the domain and range

## Example 1.2.6B: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure 1.2.10.

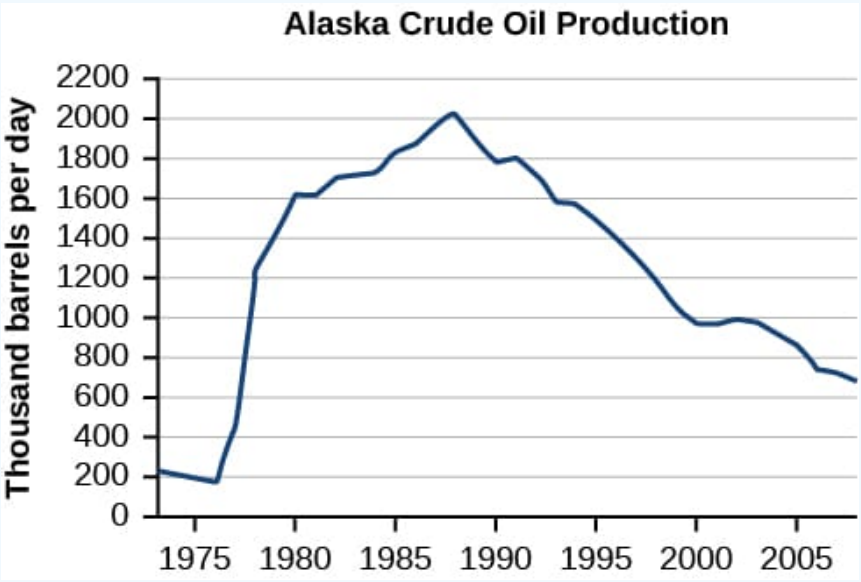



Figure 1.2.10: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

**Solution**

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is $[1973, 2008]$, and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

**Exercse 1.2.6**

Given Figure 1.2.11, identify the domain and range using interval notation.

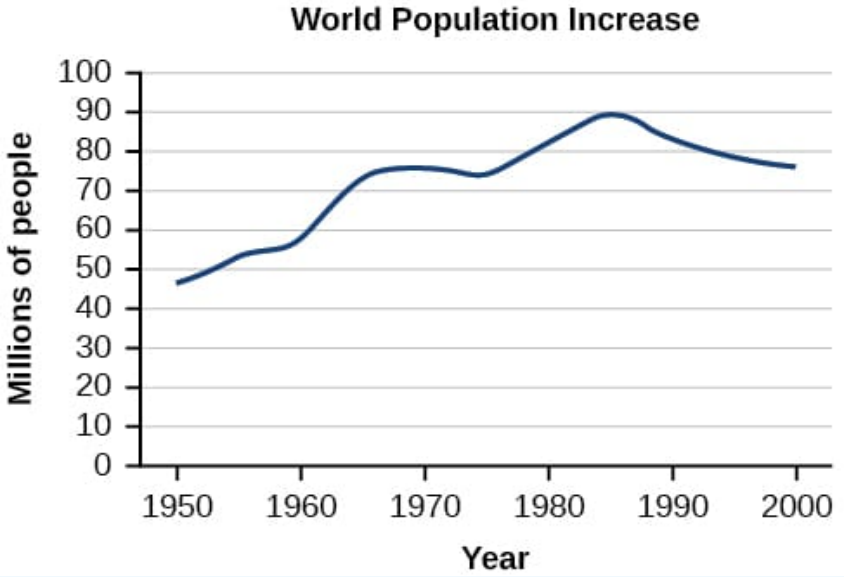



Figure 1.2.11: Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.

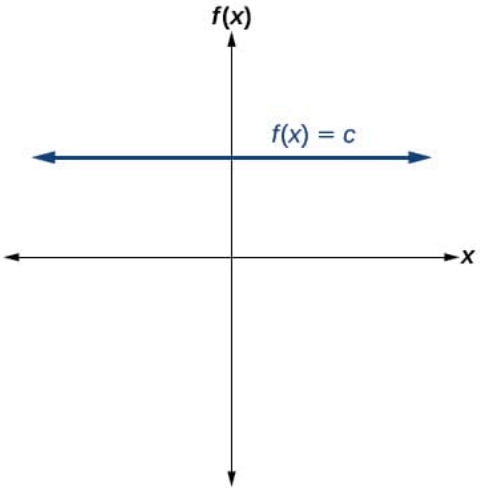
*Answer available at the end of this section.*

**Can a function's domain and range be the same?**

*Yes. For example, the domain and range of the cube root function are both the set of all real numbers.*

## Finding Domains and Ranges of the Toolkit Functions

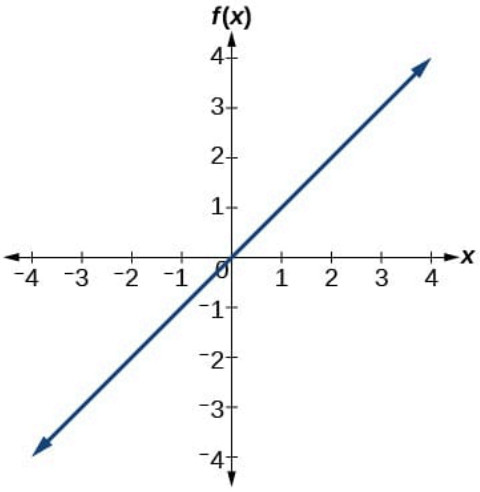
We will now return to our set of toolkit functions to determine the domain and range of each.




Domain: $(-\infty, \infty)$
Range: $[c,c]$

Figure 1.2.12: Constant function $f(x) = c$.

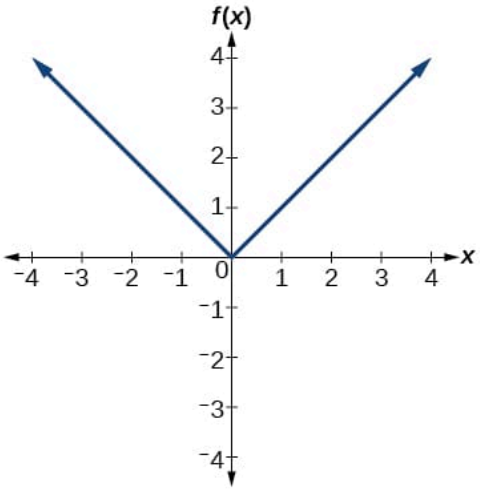
For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant $c$, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with $c$.




Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

***Figure*** 1.2.13***:*** *Identity function f(x)=x.*

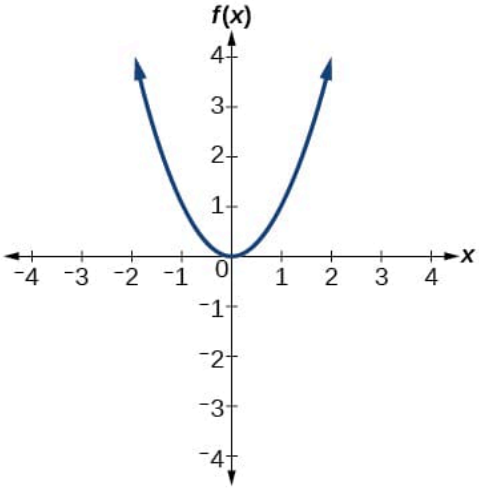
For the **identity function** $f(x) = x$, there is no restriction on $x$. Both the domain and range are the set of all real numbers.




Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 1.2.14: Absolute function $f(x) = |x|$.

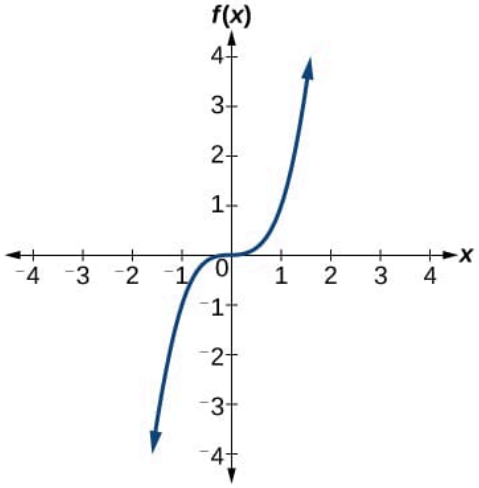
For the **absolute value function** $f(x) = |x|$, there is no restriction on $x$. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.




Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 1.2.15: Quadratic function $f(x) = x^2$.

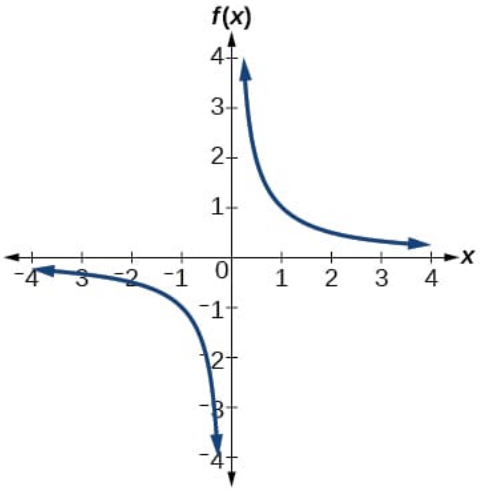
For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.




Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 1.2.16: Cubic function $f(x) - x^3$.

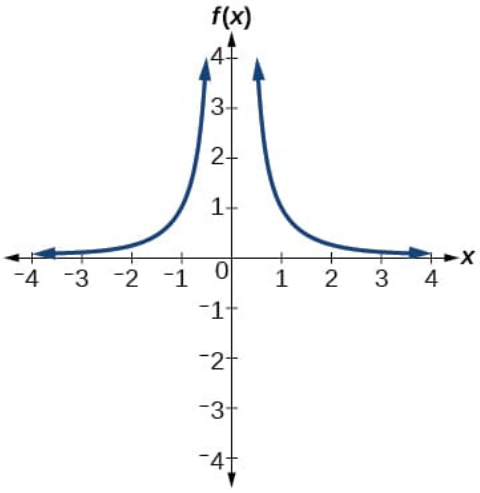
For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.




Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 1.2.17: Reciprocal function $f(x) = \frac{1}{x}$.

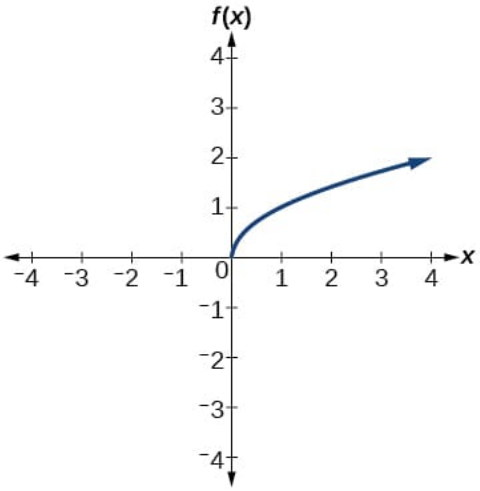
For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x|x \neq 0\}$,the set of all real numbers that are not zero.




Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 1.2.18: Reciprocal squared function $f(x) = \frac{1}{x^2}$

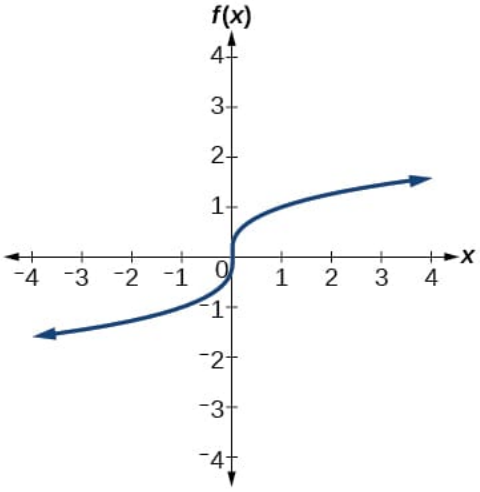
For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.




Domain: [0, ∞)
Range: [0, ∞)

***Figure*** $1.2.19$***:*** *Square root function* $f(x) = \sqrt{(x)}$.

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number $x$ is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us $x$.




Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure $1.2.20$: Cube root function $f(x) = \sqrt[3]{x}$.

For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

**How To: Given the formula for a function, determine the domain and range.**

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

## Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

**Solution**

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

## Example $1.2.7B$: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

**Solution**

We cannot evaluate the function at –1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

## Example $1.2.7C$: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

**Solution**

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as $x$ increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

**Analysis**

Figure $1.2.19$ represents the function $f$.

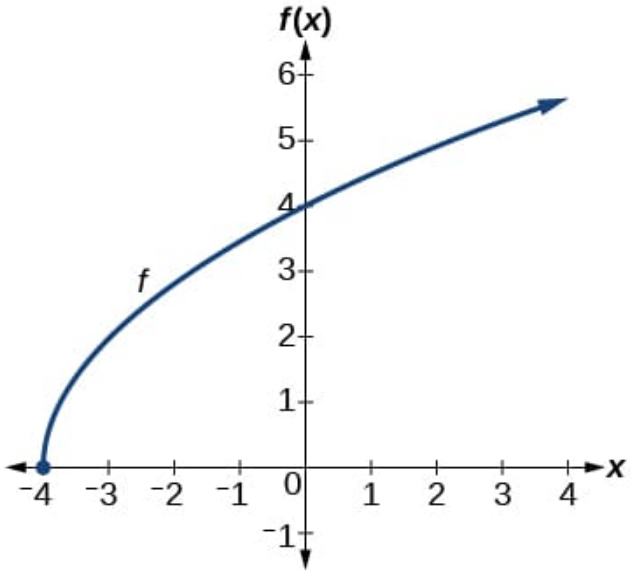



Figure 1.2.19: Graph of a square root function at $(-4, 0)$.

**Exercise 1.2.7**

Find the domain and range of

$f(x) = \sqrt{-2 - x}$.

*Answer available at the end of this section.*

## Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, **absolute value** can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

**Piecewise Function**

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain,

and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1} & \text{if x is in domain 1} \\ \text{formula 2} & \text{if x is in domain 2} \\ \text{formula 3} & \text{if x is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

**How To: Given a piecewise function, write the formula and identify the domain for each interval.**

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

## Example 1.2.8A: Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a **function** relating the number of people, $n$, to the cost, $C$.

**Solution**

Two different formulas will be needed. For $n$-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

**Analysis**

The function is represented in Figure 1.2.20. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

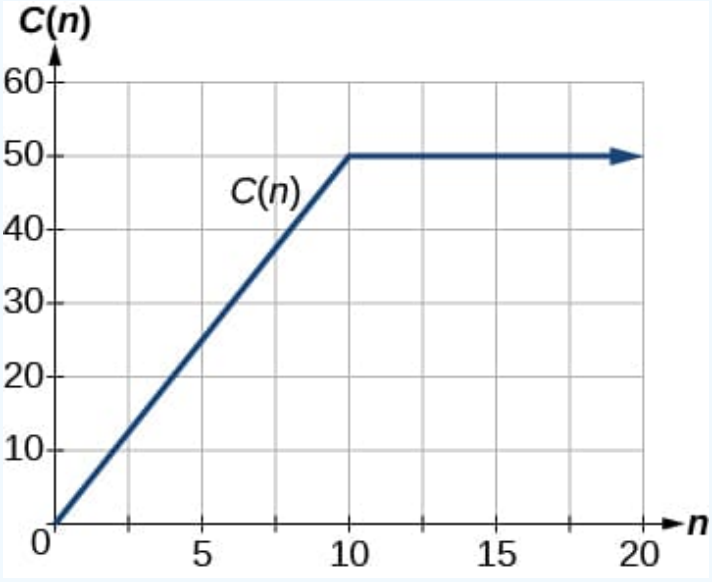



Figure 1.2.20

## Example $1.2.8B$: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g-2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

**Soltuion**

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, C(4), we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4-2) = \$45$$

**Analysis**

The function is represented in Figure $1.2.21$. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

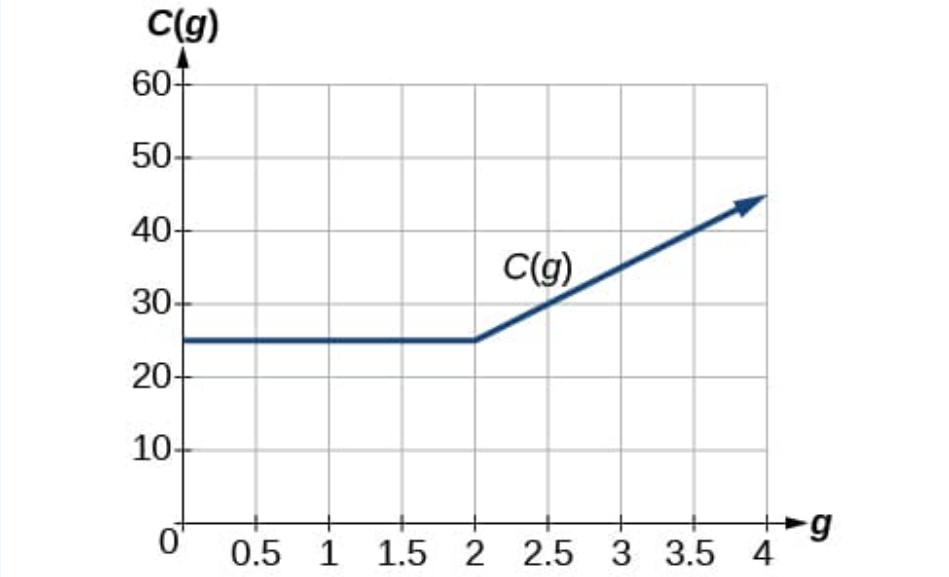



Figure $1.2.21$

**How To: Given a piecewise function, sketch a graph.**

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

## Example $1.2.8C$: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 & \text{if } 1 < x \leq 2 \\ x & \text{if } x > 2 \end{cases}$$

**Solution**

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 1.2.20 shows the three components of the piecewise function graphed on separate coordinate systems.

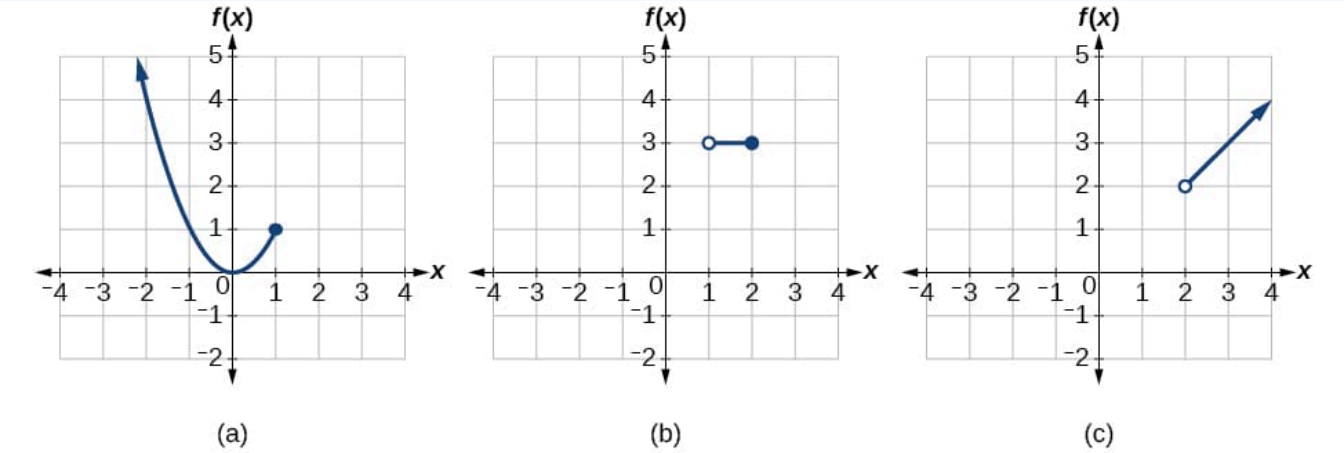



***Figure** 1.2.20**:** Graph of each part of the piece-wise function f(x)*

(a)$f(x) = x^2$ if $x \leq 1$; (b) $f(x) = 3$ if $1 < x \leq 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure 1.2.21.

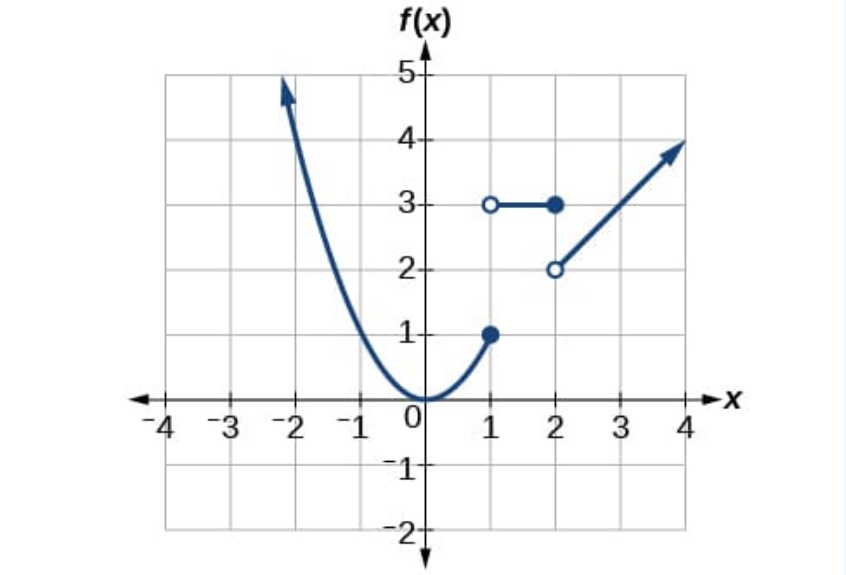



Figure 1.2.21: Graph of the entire function.

**Analysis**

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

**Exercise 1.2.8**

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

*Answer available at the end of this section.*

**Can more than one formula from a piecewise function be applied to a value in the domain?**

*No. Each value corresponds to one equation in a piecewise formula.*

## Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.
- A piecewise function is described by more than one formula.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain.

## Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014
2 www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

## Glossary

**interval notation**

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

**piecewise function**

a function in which more than one formula is used to define the output

**set-builder notation**

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

## Answers

**Exercse** 1.2.1

$\{-5, 0, 5, 10, 15\}$

**Exercse** 1.2.2

$(-\infty, \infty)$

**Exercse** 1.2.3

$$(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

**Exercse** 1.2.4

$$[-2.5, \infty)$$

**Exercse** 1.2.5

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

$\{x | x \leq -2 or -1 \leq x < 3\}$

$(-\infty, -2] \cup [-1, 3)$

**Exercse** 1.2.6

domain =$[1950, 2002]$

range = $[47,000,000, 89,000,000]$

**Exercise** 1.2.7

domain: $(-\infty, -2]$

range: $[0, \infty)$

**Exercise** 1.2.8

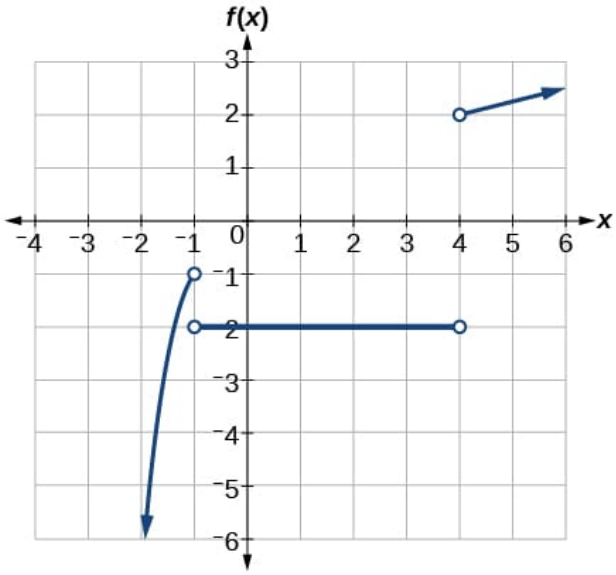



Figure 1.2.22

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# 1.3: Rates of Change and Behavior of Graphs

**Learning Objectives**

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

Gasoline costs have experienced some wild fluctuations over the last several decades. Table 1.3.1 lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

*Table* 1.3.1

| $y$ | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 |
|---|---|---|---|---|---|---|---|---|
| $C(y)$ | 2.31 | 2.62 | 2.84 | 3.30 | 2.41 | 2.84 | 3.58 | 3.68 |

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from $2.31 to $3.68, an increase of $1.37. While this is interesting, it might be more useful to look at how much the price changed per year. In this section, we will investigate changes such as these.

## Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 1.3.1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter $\Delta$ (delta) signifies the change in a quantity; we read the ratio as "delta-$y$ over delta-$x$" or "the change in $y$ divided by the change in $x$." Occasionally we write $\Delta f$ instead of $\Delta y$, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196 \text{ dollars per year.} \tag{1.3.1}$$

On average, the price of gas increased by about 19.6¢ each year. Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by $4,000 per quarter

## Definition: Rate of Change

A **rate of change** describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{1.3.2}$$

## How To...

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values $x_1$ and $x_2$.

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

## Example 1.3.1: Computing an Average Rate of Change

Using the data in Table 1.3.1, find the average rate of change of the price of gasoline between 2007 and 2009.

**Solution**

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} \\ &= \frac{-\$0.43}{2 \text{ years}} \\ &= -\$0.22 \text{ per year}\end{aligned}$$

**Analysis**

Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

## Exercise 1.3.1

Using the data in Table 1.3.1, find the average rate of change between 2005 and 2010.

*Solution available at the end of this section.*

## Example 1.3.2: Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in Figure 1.3.1, find the average rate of change on the interval $[-1, 2]$.

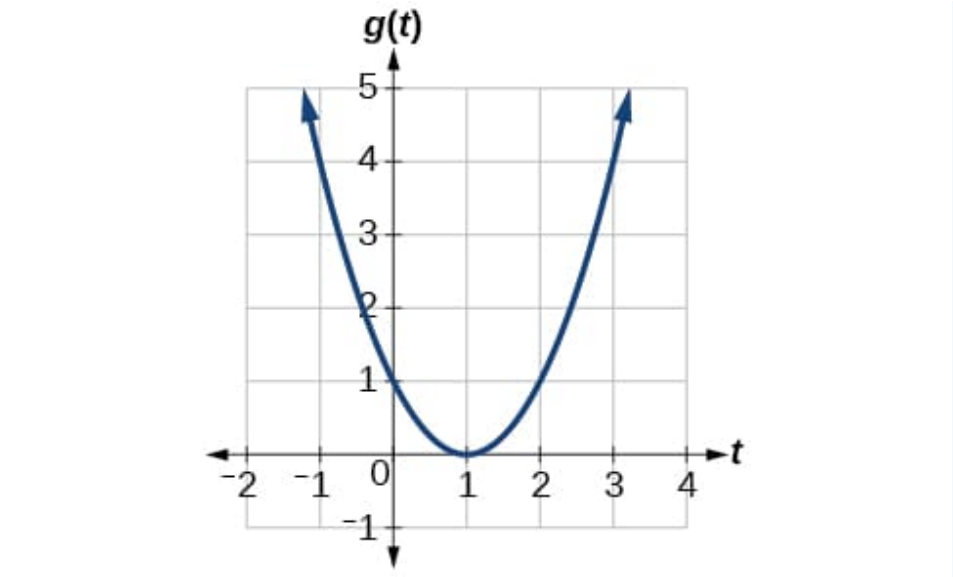



Figure 1.3.1: Graph of a parabola.

**Solution**

At $t = -1$, Figure 1.3.2 shows $g(-1) = 4$. At $t = 2$,the graph shows $g(2) = 1$.

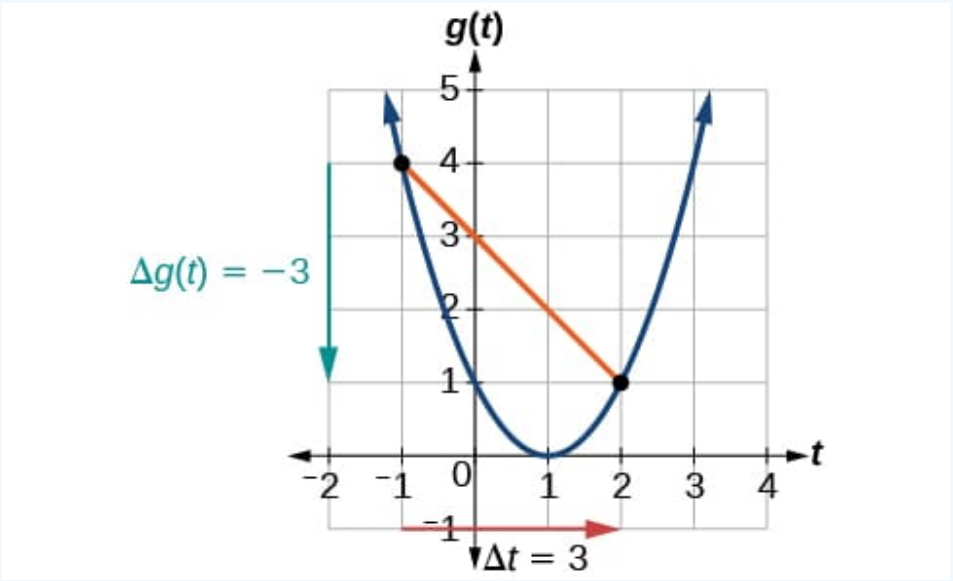



Figure 1.3.2: Graph of a parabola with a line from points (-1, 4) and (2, 1) to show the changes for g(t) and t.

The horizontal change $\Delta t = 3$ is shown by the red arrow, and the vertical change $\Delta g(t) = -3$ is shown by the turquoise arrow. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)} = \frac{-3}{3} = -1 \tag{1.3.3}$$

**Analysis**

Note that the order we choose is very important. If, for example, we use $\frac{y_2 - y_1}{x_1 - x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as $(x_1, y_1)$ and $(x_2, y_2)$.

## Example 1.3.3: Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table 1.3.2. Find her average speed over the first 6 hours.

*Table* 1.3.2

| $t$ **(hours)** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $D(t)$ **(miles)** | 10 | 55 | 90 | 153 | 214 | 240 | 292 | 300 |

**Solution**

Here, the average speed is the average rate of change. She traveled 292 miles in 6 hours, for an average speed of

$$\begin{aligned}\frac{292-10}{6-0} &= \frac{282}{6}\\ &= 47\end{aligned}$$

The average speed is about 47 miles per hour.

**Analysis**

Because the speed is not constant, the average speed depends on the interval chosen. For the interval $[2,3]$, the average speed is 63 miles per hour.

## Example 1.3.4: Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x)=x^2-\frac{1}{x}$ on the interval $[2,4]$.

**Solution**

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2-\frac{1}{2} & f(4) &= 4^2-\frac{1}{4}\\ &= 4-\frac{1}{2} & &= 16-\frac{1}{4}\\ &= 72 & &= \frac{63}{4}\end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned}\text{Average rate of change} &= \frac{f(4)-f(2)}{4-2}\\ &= \frac{\frac{63}{4}-\frac{7}{2}}{4-2}\\ &= \frac{\frac{49}{4}}{2}\\ &= \frac{49}{8}\end{aligned}$$

## Exercise 1.3.2

Find the average rate of change of $f(x)=x-2\sqrt{x}$ on the interval $[1,9]$.

*Solution available at the end of this section.*

## Example 1.3.5: Finding the Average Rate of Change of a Force

The electrostatic force $F$, measured in newtons, between two charged particles can be related to the distance between the particles $d$,in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

**Solution**

We are computing the average rate of change of $F(d) = \dfrac{2}{d^2}$ on the interval $[2, 6]$.

$$\begin{aligned}
\text{Average rate of change} &= \frac{F(6) - F(2)}{6 - 2} \\
&= \frac{\frac{2}{6^2} - \frac{2}{2^2}}{6 - 2} && \text{Simplify} \\
&= \frac{\frac{2}{36} - \frac{2}{4}}{4} \\
&= \frac{-\frac{16}{36}}{4} && \text{Combine numerator terms.} \\
&= -\frac{1}{9} && \text{Simplify}
\end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

## Example 1.3.6: Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving $a$.

**Solution**

We use the average rate of change formula.

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a) - g(0)}{a - 0} && \text{Evaluate.} \\
&= \frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a - 0} && \text{Simplify.} \\
&= \frac{a^2 + 3a + 1 - 1}{a} && \text{Simplify and factor.} \\
&= \frac{a(a + 3)}{a} && \text{Divide by the common factor a.} \\
&= a + 3
\end{aligned}$$

This result tells us the average rate of change in terms of a between $t = 0$ and any other point $t = a$. For example, on the interval $[0, 5]$, the average rate of change would be $5 + 3 = 8$.

## Exercise 1.3.3

Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

*Solution available at the end of this section.*

# Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. Figure $1.3.3$ shows examples of increasing and decreasing intervals on a function.

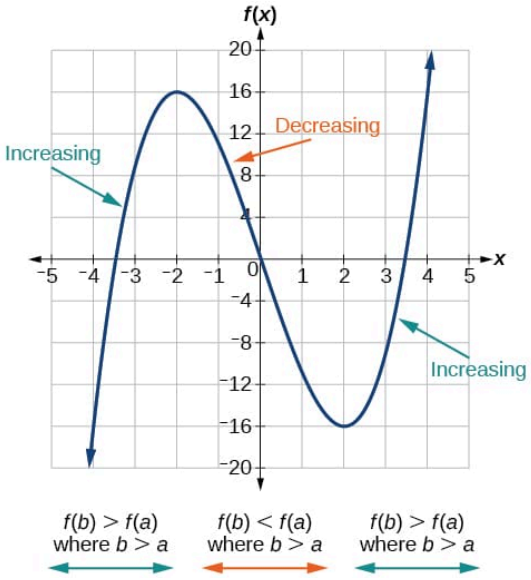



Figure $1.3.3$: The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is "local minima." Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is "extremum.") Often, the term local is replaced by the term relative. In this text, we will use the term local.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of local extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in Figure $1.3.4$, the local maximum is 16, and it occurs at $x = -2$. The local minimum is –16 and it occurs at $x = 2$.

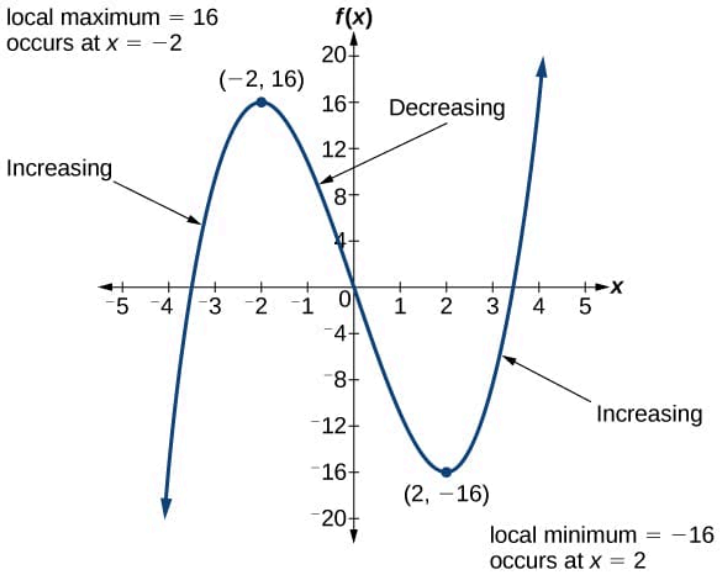



Figure 1.3.4: Graph of a polynomial that shows the increasing and decreasing intervals and local maximum.maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure 1.3.5 illustrates these ideas for a local maximum.

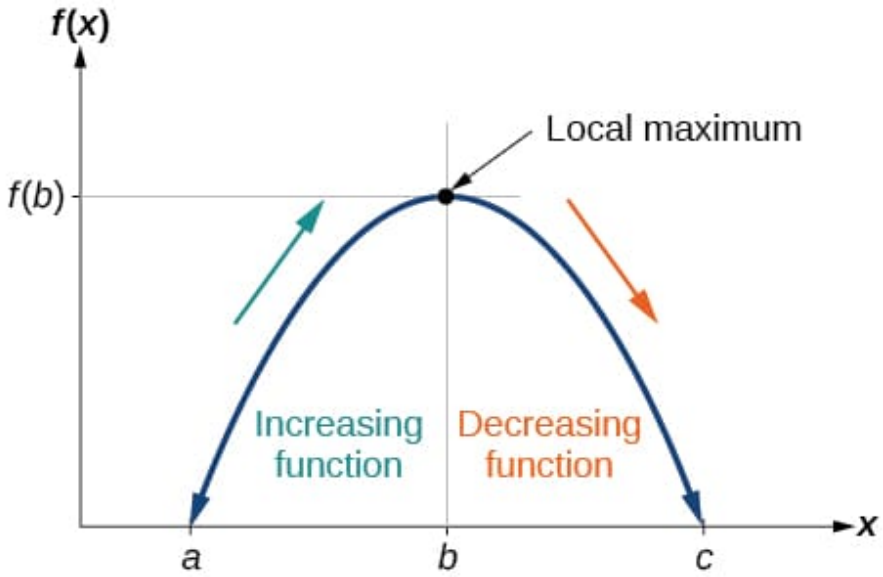



Figure 1.3.5: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

## Local Minima and Local Maxima

- A function $f$ is an **increasing function** on an open interval if $f(b) > f(a)$ for every $a$, $b$ interval where $b > a$.
- A function $f$ is a **decreasing function** on an open interval if $f(b) < f(a)$ for every $a$, $b$ interval where $b > a$.

A function $f$ has a local maximum at a point $b$ in an open interval $(a, c)$ if $f(b)$ is greater than or equal to $f(x)$ for every point $x$ ($x$ does not equal $b$) in the interval. Likewise, $f$ has a local minimum at a point $b$ in $(a, c)$ if $f(b)$ is less than or equal to $f(x)$ for every $x$ ($x$ does not equal $b$) in the interval.

## Example 1.3.7 Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in Figure 1.3.6, identify the intervals on which the function appears to be increasing.

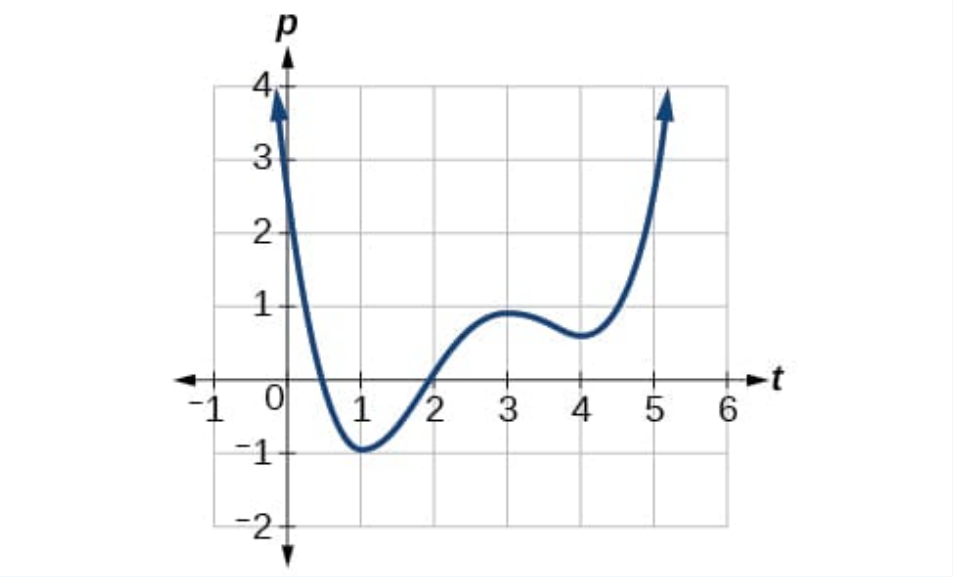



Figure 1.3.6: Graph of a polynomial.

**Solution**

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

**Analysis**

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

## Example 1.3.8: Finding Local Extrema from a Graph

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

**Solution**

Using technology, we find that the graph of the function looks like that in Figure 1.3.7. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$

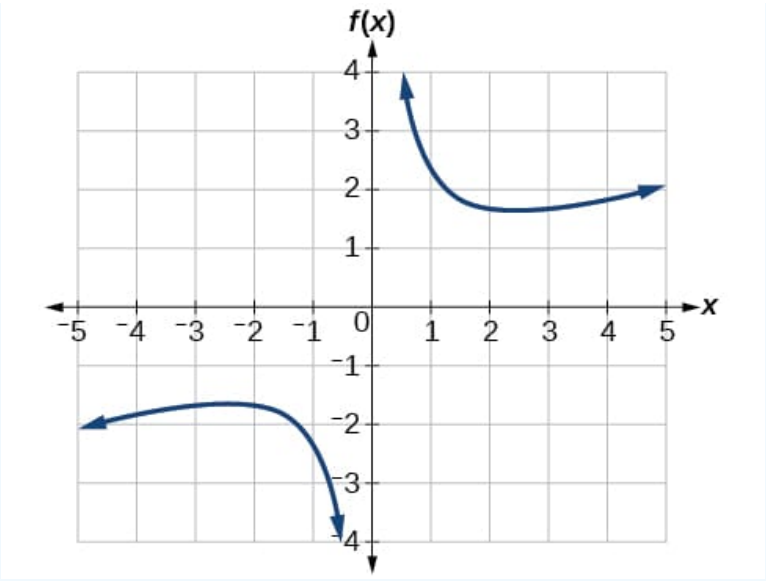



Figure 1.3.7: Graph of a reciprocal function.

**Analysis**

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Figure 1.3.8 provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

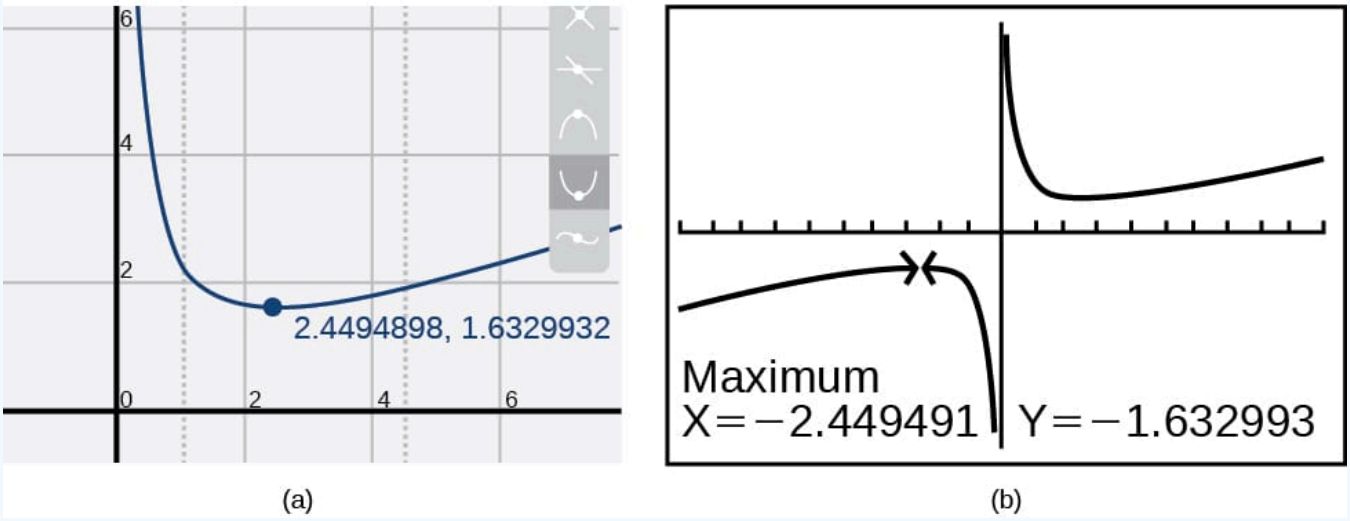



Figure 1.3.8: Graph of the reciprocal function on a graphing calculator.

Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

## Exercise 1.3.8

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

*Solution available at the end of this section.*

### Example 1.3.9: Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in Figure 1.3.9, find all local maxima and minima.

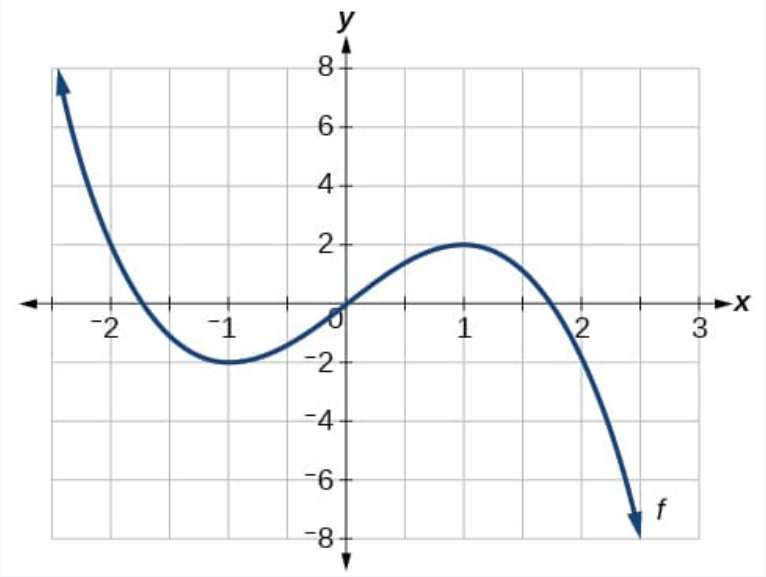



Figure 1.3.9: Graph of a polynomial.

**Solution**

Observe the graph of $f$. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is –2.

## Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in Figure 1.3.10, Figure 1.3.11, and Figure 1.3.12.

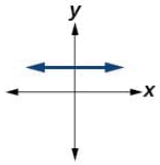
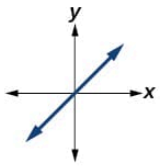
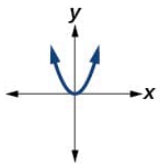
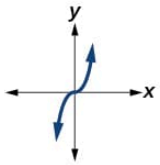
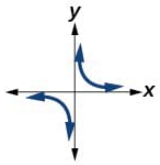
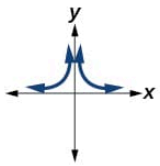
| Function | Increasing/Decreasing | Example |
|---|---|---|
| Constant Function<br>$f(x) = c$ | Neither increasing nor decreasing |  |
| Identity Function<br>$f(x) = x$ | Increasing |  |
| Quadratic Function<br>$f(x) = x^2$ | Increasing on $(0, \infty)$<br>Decreasing on $(-\infty, 0)$<br>Minimum at $x = 0$ |  |

Figure 1.3.10

| Function | Increasing/Decreasing | Example |
|---|---|---|
| Cubic Function<br>$f(x) = x^3$ | Increasing |  |
| Reciprocal<br>$f(x) = \frac{1}{x}$ | Decreasing $(-\infty, 0) \cup (0, \infty)$ |  |
| Reciprocal Squared<br>$f(x) = \frac{1}{x^2}$ | Increasing on $(-\infty, 0)$<br>Decreasing on $(0, \infty)$ |  |

.

Figure 1.3.11

| Function | Increasing/Decreasing | Example |
|---|---|---|
| Constant Function $f(x) = c$ | Neither increasing nor decreasing |  |
| Identity Function $f(x) = x$ | Increasing |  |
| Quadratic Function $f(x) = x^2$ | Increasing on $(0, \infty)$<br>Decreasing on $(-\infty, 0)$<br>Minimum at $x = 0$ |  |
| Cubic Function $f(x) = x^3$ | Increasing |  |
| Reciprocal $f(x) = \frac{1}{x}$ | Decreasing $(-\infty, 0) \cup (0, \infty)$ |  |
| Reciprocal Squared $f(x) = \frac{1}{x^2}$ | Increasing on $(-\infty, 0)$<br>Decreasing on $(0, \infty)$ |  |
| Cube Root $f(x) = \sqrt[3]{x}$ | Increasing |  |
| Square Root $f(x) = \sqrt{x}$ | Increasing on $(0, \infty)$ |  |
| Absolute Value $f(x) = \lvert x \rvert$ | Increasing on $(0, \infty)$<br>Decreasing on $(\infty, 0)$ |  |

***Figure*** 1.3.12

## Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively. To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function (Figure 1.3.13).

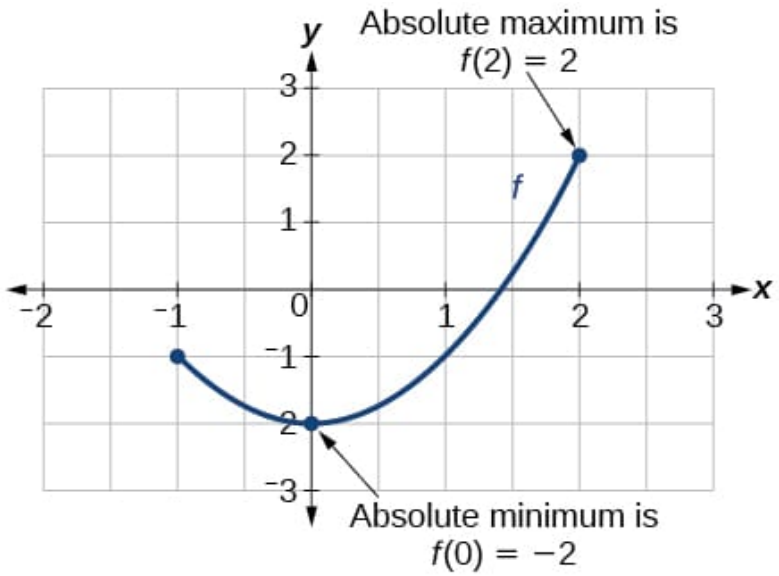



Figure 1.3.13: Graph of a segment of a parabola with an absolute minimum at (0, -2) and absolute maximum at (2, 2).

Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

### Absolute Maxima and Minima

- The **absolute maximum** of $f$ at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all $x$ in the domain of $f$.
- The **absolute minimum** of $f$ at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all $x$ in the domain of $f$.

### Example 1.3.10: Finding Absolute Maxima and Minima from a Graph

For the function f shown in Figure 1.3.14, find all absolute maxima and minima.

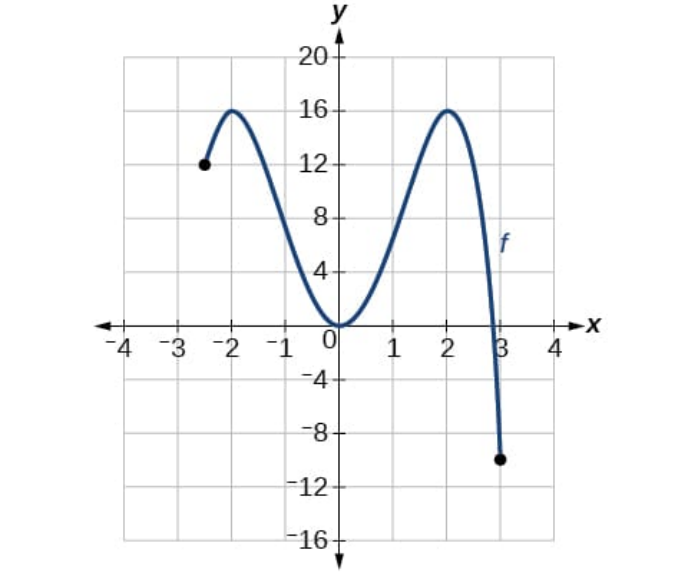



Figure 1.3.14: Graph of a polynomial.

**Solution**

Observe the graph of $f$. The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at x=3, because it is the lowest point on the domain of the function's graph. The absolute minimum is the y-coordinate at x=3,which is–10.

## Key Equations

- Average rate of change: $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

## Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See Example.
- Identifying points that mark the interval on a graph can be used to find the average rate of change. See Example.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See Example.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See Example and Example.
- The average rate of change can sometimes be determined as an expression. See Example.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See Example.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See Example and Example.
- The highest and lowest points on a graph indicate the maxima and minima. See Example.

## Answers

**Exercise** 1.3.1

$\frac{\$2.84 - \$2.315}{5 \text{ years}} = \frac{\$0.535}{5 \text{ years}} = \$0.106 \text{per year.}$

**Exercise** 1.3.2

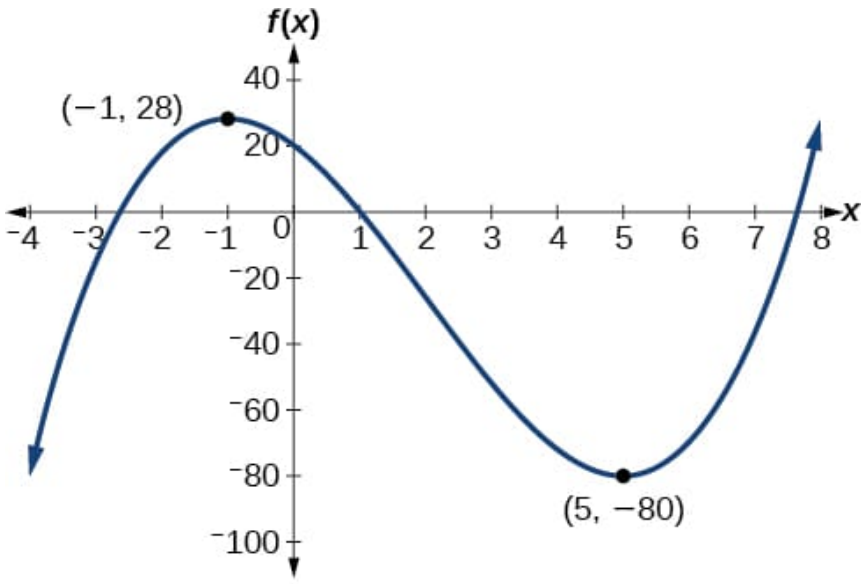
$\frac{1}{2}$

**Exercise** 1.3.3

$a + 7$

**Exercise** 1.3.8

The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.




*Graph of a polynomial with a local maximum at (-1, 28) and local minimum at (5, -80).*

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# 1.4: Composition of Functions

**Learning Objectives**

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost $C$ of heating a house for a given average daily temperature in $T$ degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $Cost = C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the **cost function** at that temperature. We would write $C(T(5))$.

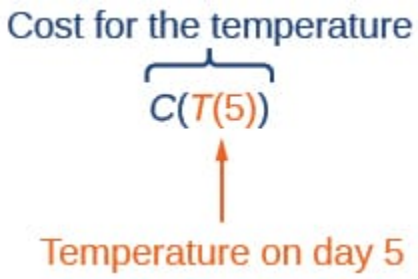



Figure 1.4.1: Explanation of $C(T(5))$, which is the cost for the temperature and $T(5)$ is the temperature on day 5.

By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

## Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year $y$, and we want $T$ to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have

the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, $fg$, and $\frac{f}{g}$ by the relations.

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f-g)(x) &= f(x)-g(x)\\ (fg)(x) &= f(x)g(x)\\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\end{aligned}$$

## Example 1.4.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x)=x-1$ and $g(x)=x^2-1$. Are they the same function?

**Solution**

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ &= x^2-x\\ &= x(x-1)\end{aligned}$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= g(x)f(x)\\ \left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ &= \frac{(x+1)(x-1)}{x-1}\\ &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x\neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

## Exercise 1.4.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

$$f(x)=x-1$$

and

$$g(x)=x^2-1$$

Are they the same function?

*Answer available at the end of this section.*

# Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f \circ g(x) = f(g(x)) \tag{1.4.1}$$

We read the left-hand side as"$f$ composed with $g$ at $x$," and the right-hand side as"$f$ of $g$ of $x$."The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function $g$ takes the input $x$ first and yields an output $g(x)$. Then the function $f$ takes $g(x)$ as an input and yields an output $f(g(x))$.

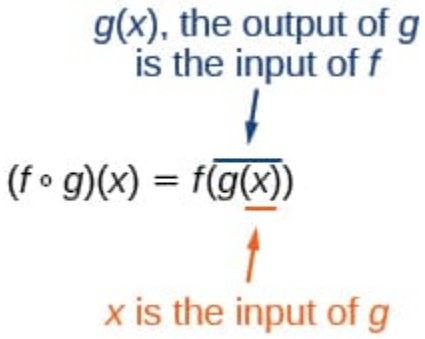



Figure 1.4.2: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all $x$. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

## Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input $x$ and functions $f$ and $g$, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \tag{1.4.2}$$

The domain of the composite function $f \circ g$ is all $x$ such that $x$ is in the domain of $g$ and $g(x)$ is in the domain of $f$.

It is important to realize that the product of functions $fg$ is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

## Example 1.4.2: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x) = 2x + 1 \quad g(x) = 3 - x$$

**Solution**

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3 - x) + 1 \\ &= 6 - 2x + 1 \\ &= 7 - 2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3 - (2x + 1) \\ &= 3 - 2x - 1 \\ &= 2 - 2x \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

## Example 1.4.3: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing $s$ sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in $t$ minutes. Interpret $c(s(3))$.

**Solution**

The inside expression in the composition is $s(3)$. Because the input to the $s$-function is time, $t = 3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

## Example 1.4.4: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in $x$ hours and $g(y)$ gives the gallons of gas used in driving $y$ miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

**Solution**

The function $y = f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, $g$, driving a certain number of miles, $f(x)$, in $x$ hours.

**Question/Answer**

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order

**Exercise 1.4.2**

The gravitational force on a planet a distance $r$ from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force $F$ is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

*Answer available at the end of this section.*

# Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

## Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

**Example 1.4.5: Using a Table to Evaluate a Composite Function**

Using Table 1.4.1, evaluate $f(g(3))$ and $g(f(3))$.

*Table* 1.4.1

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 1 | 6 | 3 |
| 2 | 8 | 5 |
| 3 | 3 | 2 |
| 4 | 1 | 7 |

**Solution**

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function $g$ : $g(3) = 2$. We can then use that result as the input to the function $f$, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function $f$, we find that $f(2) = 8$.

$$g(3) = 2$$

$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for $g$, we can evaluate

$$g(f(3)) = g(3) = 2$$

Table 1.4.2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

*Table* 1.4.2

| $x$ | $g(x)$ | $f(g(x))$ | $f(x)$ | $g(f(x))$ |
|---|---|---|---|---|
| 3 | 2 | 8 | 3 | 2 |

**Exercise 1.4.3**

Using Table 1.4.1, evaluate $f(g(1))$ and $g(f(4))$.

*Answer available at the end of this section.*

## Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

**How To ...**

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

## Example 1.4.6: Using a Graph to Evaluate a Composite Function

Using Figure 1.4.3, evaluate $f(g(1))$.

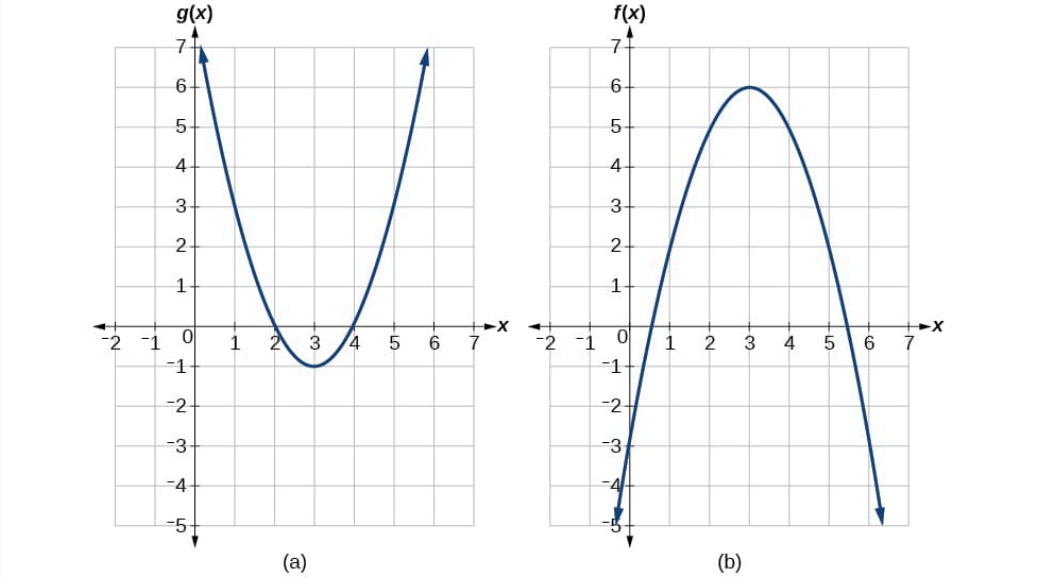



Figure 1.4.3: Two graphs of a positive and negative parabola.

**Solution**

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 1.4.4.

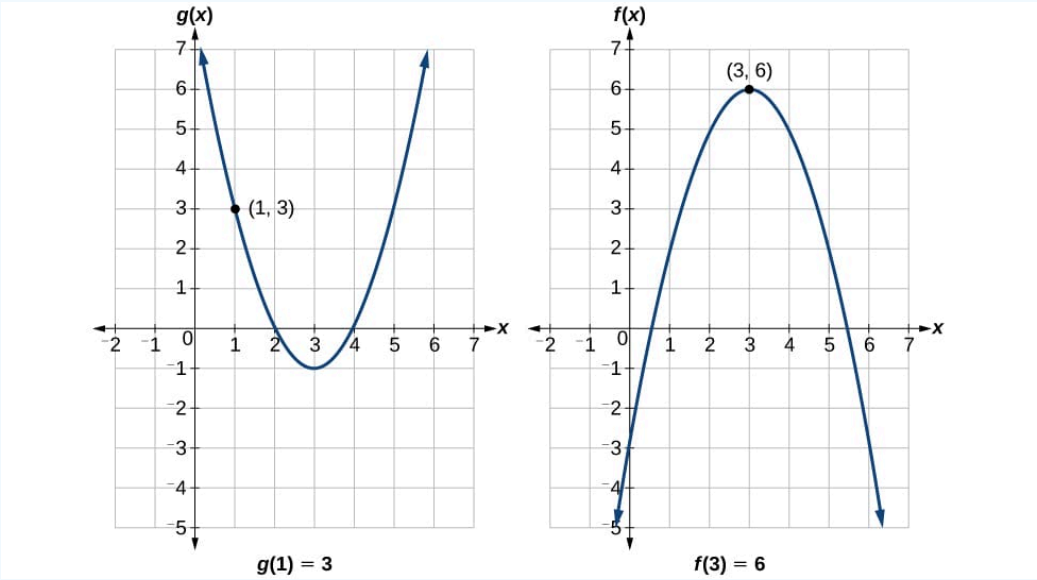



Figure 1.4.4: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function $f$.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

**Analysis**

Figure 1.4.5 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

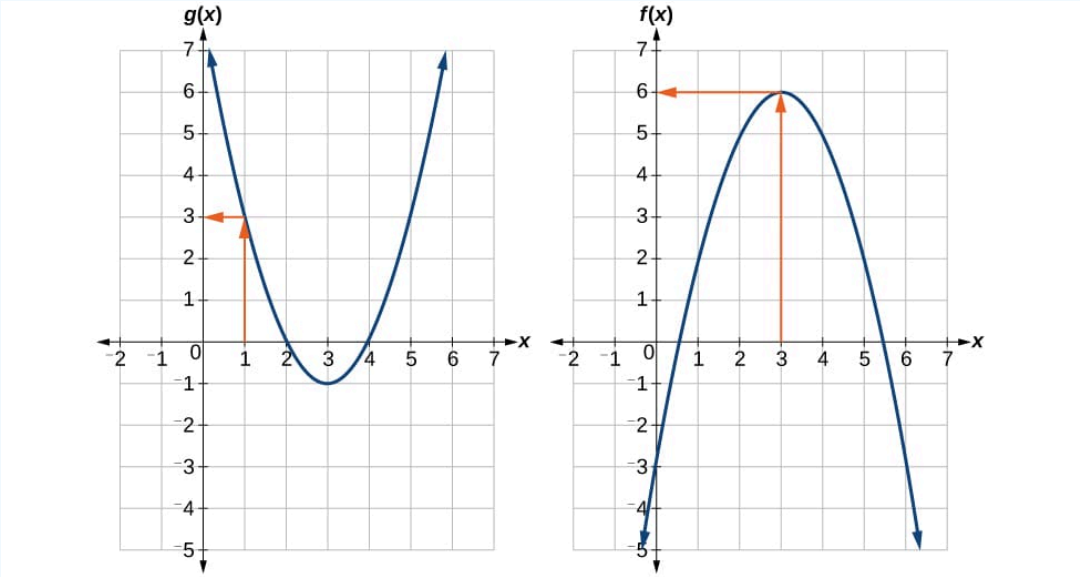



Figure 1.4.5: Two graphs of a positive and negative parabola.

**Exercise 1.4.4**

Using Figure 1.4.3, evaluate $g(f(2))$.

*Answer available at the end of this section.*

## Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

**How To...**

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

**Example 1.4.7: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input**

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

**Solution**

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$h(1) = 3(1) + 2$$
$$h(1) = 5$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$f(h(1)) = f(5)$$
$$f(h(1)) = 5^2 - 5$$
$$f(h(1)) = 20$$

**Analysis**

It makes no difference what the input variables $t$ and $x$ were called in this problem because we evaluated for specific numerical values.

**Exercise 1.4.5**

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

*Answer a available at the end of this section.*

## Finding the Domain of a Composite Function

As we discussed previously, the **domain of a composite function** such as $f \circ g$ is dependent on the domain of $g$ and the domain of $f$. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions $f$ and $g$ separately. If we write the composite function for an input $x$ as $f(g(x))$, we can see right away that $x$ must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of $f$, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of $g$ that produce outputs from $g$ belonging to the domain of $f$. Note that the domain of $f$ composed with $g$ is the set of all $x$ such that $x$ is in the domain of $g$ and g(x)\) is in the domain of $f$.

**Definition: Domain of a Composite Function**

The **domain of a composite function** $f(g(x))$ is the set of those inputs $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$.

**How To...**

Given a function composition $f(g(x))$, determine its domain.

1. Find the domain of $g$.
2. Find the domain of $f$.
3. Find those inputs $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$. That is, exclude those inputs $x$ from the domain of $g$ for which $g(x)$ is not in the domain of $f$. The resulting set is the domain of $f \circ g$.

## Example 1.4.8A: Finding the Domain of a Composite Function

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

**Solution**

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of $f$ consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of $x$ for which $g(x) = 1$.

$$\begin{aligned} \frac{4}{3x-2} &= 1 \\ 4 &= 3x-2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

## Example 1.4.8B: Finding the Domain of a Composite Function Involving Radicals

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \sqrt{x+2} \text{ and } g(x) = \sqrt{3-x}$$

**Solution**

Because we cannot take the square root of a negative number, the domain of $g$ is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$, $\sqrt{3-x}+2 \geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $\sqrt{3-x} \geq 0$,or, $3-x \geq 0$, which gives a domain of $(-\infty, 3]$.

**Analysis**

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of $f$, though they must be in the domain of $g$.

## Exercise 1.4.6

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \frac{1}{x-2} \text{ and } g(x) = \sqrt{x+4}$$

*Answer available at the end of this section.*

## Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to **decompose a composite function**, so we may choose the decomposition that appears to be most expedient.

### Example 1.4.9: Decomposing a Function

Write $f(x) = \sqrt{5-x^2}$ as the composition of two functions.

**Solution**

We are looking for two functions, $g$ and $h$, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5 - x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5 - x^2 \text{ and } g(x) = \sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g(5 - x^2) = \sqrt{5 - x^2}$$

### Exercise 1.4.7

Write $f(x) = \dfrac{4}{3 - \sqrt{4 + x^2}}$ as the composition of two functions.

*Answer available at the end of this section.*

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions (http://openstaxcollege.org/l/compfunction)
- Composite Function Notation Application (http://openstaxcollege.org/l/compfuncnot)
- Composite Functions Using Graphs (http://openstaxcollege.org/l/compfuncgraph)
- Decompose Functions (http://openstaxcollege.org/l/decompfunction)
- Composite Function Values (http://openstaxcollege.org/l/compfuncvalue)

## Key Equation

- Composite function $(f \circ g)(x) = f(g(x))$

## Key Concepts

- We can perform algebraic operations on functions. See Example.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example and Example.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the

outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example.
- A composite function can be evaluated from a graph. See Example.
- A composite function can be evaluated from a formula. See Example.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Example and Example.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See Example.

# Glossary

**composite function**

the new function formed by function composition, when the output of one function is used as the input of another

## Answers

**Exercise** 1.4.1

$(fg)(x) = f(x)g(x) = (x-1)(x2-1) = x^3 - x^2 - x + 1$ $(f-g)(x) = f(x) - g(x) = (x-1) - (x^2-1) = x - x^2$

No, the functions are not the same.

**Exercise** 1.4.2

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance $r$ from the Sun (due to gravity), but $G(a(F))$ does not make sense.

**Exercise** 1.4.3

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

**Exercise** 1.4.4

$g(f(2)) = g(5) = 3$

**Exercise** 1.4.5

8

20

**Exercise** 1.4.6

$[-4, 0) \cup (0, \infty)$

**Exercise** 1.4.7

Possible answers:

$g(x) = \sqrt{4 + x^2}$

$h(x) = \dfrac{4}{3 - x}$

$f = h \circ g$

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# 1.5: Transformation of Functions

**Learning Objectives**

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.



Figure 1.5.1: (credit: "Misko"/Flickr)

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

## Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically $k$ units. See Figure 1.5.2 for an example.

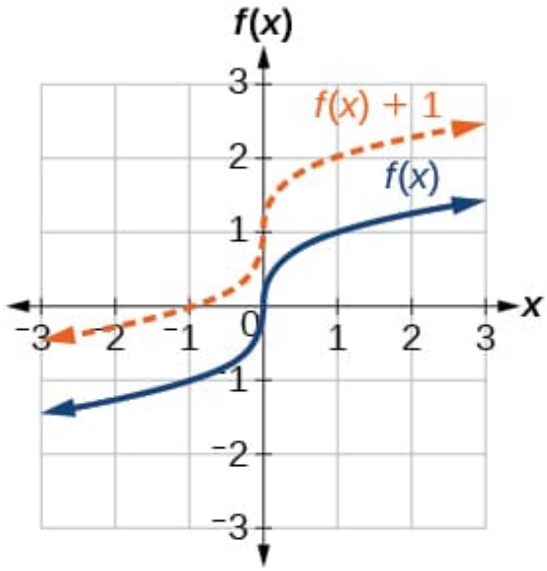



Figure 1.5.2: Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of $y$ is replaced by $y + k$, so the $y$-value increases or decreases depending on the value of $k$. The result is a shift upward or downward.

## Definition: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where $k$ is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by $k$ units. If $k$ is positive, the graph will shift up. If $k$ is negative, the graph will shift down.

## Example 1.5.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 1.5.3 shows the area of open vents $V$ (in square feet) throughout the day in hours after midnight, $t$. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

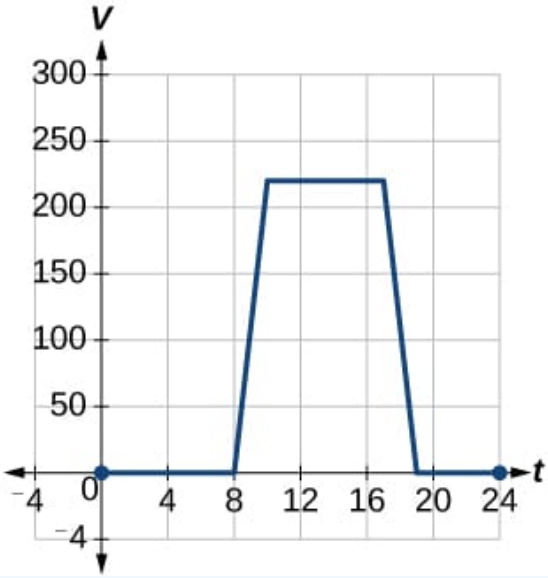



Figure 1.5.3

**Solution**

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 1.5.4.

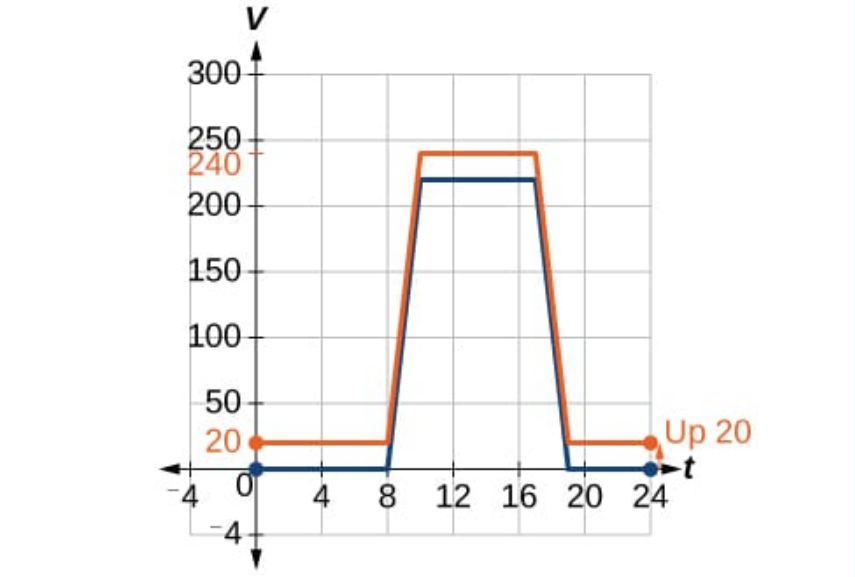



Figure 1.5.4

Notice that in Figure 1.5.4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20 \tag{1.5.1}$$

This notation tells us that, for any value of $t$,$S(t)$ can be found by evaluating the function $V$ at the same input and then adding 20 to the result. This defines $S$ as a transformation of the function $V$, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 1.5.1.

*Table* 1.5.1

| $t$ | 0 | 8 | 10 | 17 | 19 | 24 |
|---|---|---|---|---|---|---|
| $V(t)$ | 0 | 0 | 220 | 220 | 0 | 0 |
| $S(t)$ | 20 | 20 | 240 | 240 | 20 | 20 |

How To...

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

## Example 1.5.2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 1.5.2. Create a table for the function $g(x) = f(x) - 3$.

*Table* 1.5.2

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

**Solution**

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of $g$ by subtracting 3 from the output values of $f$. For example:

$$\begin{aligned} f(x) &= 1 && \text{Given} \\ g(x) &= f(x) - 3 && \text{Given Transformation} \\ g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 1.5.3.

*Table* 1.5.3

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |
| $g(x)$ | -2 | 0 | 4 | 8 |

**Analysis**

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

**Exercise 1.5.1**

The function $h(t) = -4.9t^2 + 30t$ gives the height $h$ of a ball (in meters) thrown upward from the ground after $t$ seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

*Answer available at the end of this section.*

## Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 1.5.4.

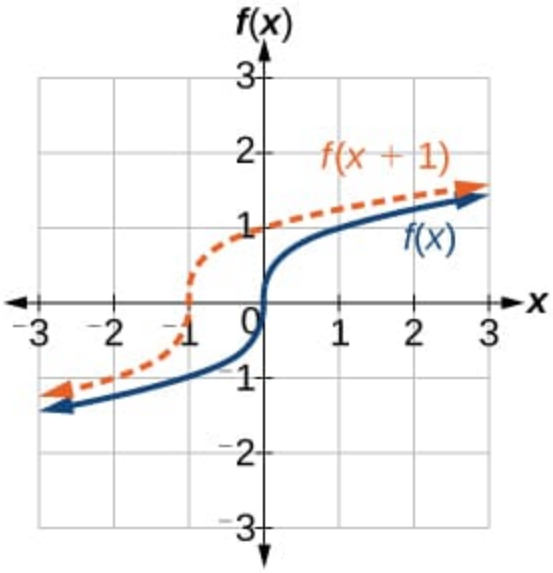



Figure 1.5.4: Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = -1$ shifts the graph to the left, that is, towards negative values of $x$.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in $f$.

### Definition: Horizontal Shift

Given a function $f$, a new function $g(x) = f(x-h)$, where $h$ is a constant, is a **horizontal shift** of the function $f$. If $h$ is positive, the graph will shift right. If $h$ is negative, the graph will shift left.

### Example 1.5.4: Adding a Constant to an Input

Returning to our building airflow example from Figure 1.5.2, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

**Solution**

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function $V$ was 2 hours later. For example, in the original function $V$, the airflow starts to change at 8 a.m., whereas for the function $F$, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 1.5.5. Notice also that the vents first opened to $220\text{ft}^2$ at 10 a.m. under the original plan, while under the new plan the vents reach $220\text{ft}^2$ at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

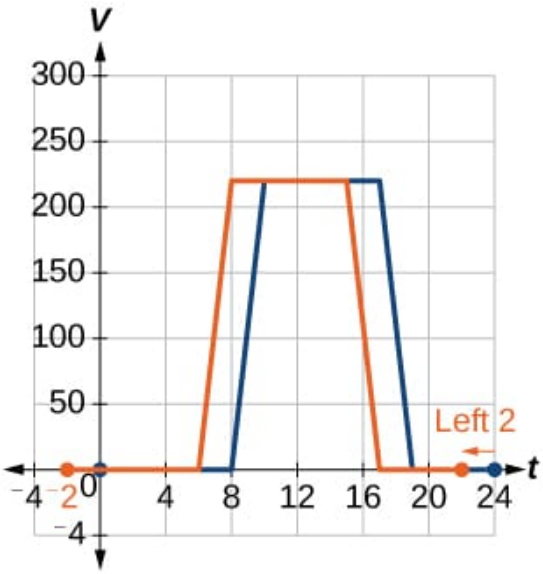



Figure 1.5.5

**Analysis**

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of $V$ to find the corresponding output for $F$ : $F(t) = V(t+2)$.

## How To...

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

## Example 1.5.5: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 1.5.4. Create a table for the function $g(x) = f(x-3)$.

*Table* 1.5.4

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

**Solution**

The formula $g(x) = f(x-3)$ tells us that the output values of $g$ are the same as the output value of $f$ when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function $g$, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function $f$.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 1.5.5.

*Table* 1.5.5

| $x$ | 5 | 7 | 9 | 11 |
|---|---|---|---|---|
| $x-3$ | 2 | 4 | 6 | 8 |
| $f(x)$ | 1 | 3 | 7 | 11 |
| $g(x)$ | 1 | 3 | 7 | 11 |

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, $x$, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

**Analysis**

Figure 1.5.6 represents both of the functions. We can see the horizontal shift in each point.

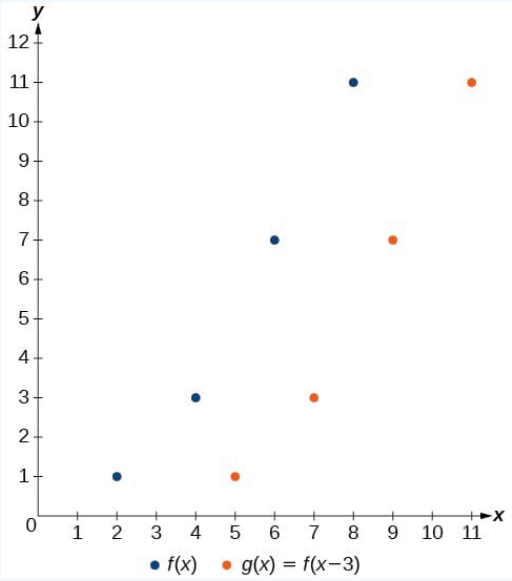



Figure 1.5.6: Graph of the points from the previous table for $f(x)$ and $g(x) = f(x-3)$

## Example 1.5.6: Identifying a Horizontal Shift of a Toolkit Function

Figure 1.5.7 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

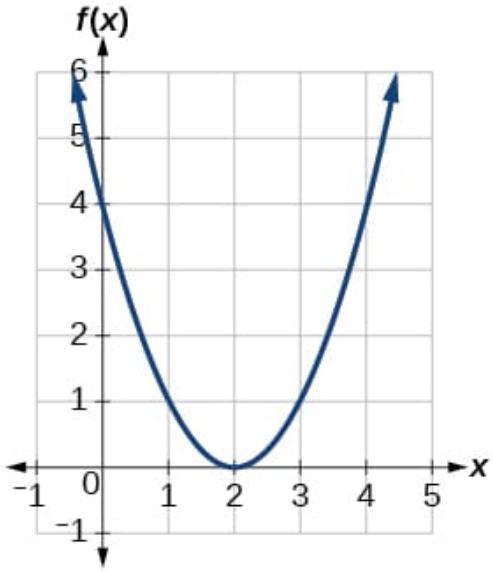



Figure 1.5.7: Graph of a parabola

**Solution**

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the $x$-values are shifted to the right 2 units. The vertex used to be at $(0, 0)$, but now the vertex is at $(2, 0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x - 2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the $x$-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x - 2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x - 2) \\ g(x) &= f(x - 2) = (x - 2)^2 \end{aligned}$$

**Analysis**

To determine whether the shift is $+2$ or $-2$, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function $f$, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x - 2) = f(0) = 0$. For this to work, we will need to subtract 2 units from our input values.

## Example 1.5.7: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive $m$ miles. Interpret $G(m) + 10$ and $G(m + 10)$

**Solution**

$G(m) + 10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive $m$ miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m + 10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than $m$ miles. The graph would indicate a horizontal shift.

**Exercise 1.5.7**

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

*Answer available at the end of this section.*

## Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output $(y-)$ axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input $(x-)$ axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

**How To...**

**Given a function and both a vertical and a horizontal shift, sketch the graph.**

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

**Example 1.5.8: Graphing Combined Vertical and Horizontal Shifts**

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x + 1) - 3$.

**Solution**

The function $f$ is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of $h$ has transformed $f$ in two ways: $f(x + 1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x + 1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 1.5.9.

Let us follow one point of the graph of $f(x) = |x|$.

- The point $(0, 0)$ is transformed first by shifting left 1 unit:$(0, 0) \to (-1, 0)$
- The point $(-1, 0)$ is transformed next by shifting down 3 units:$(-1, 0) \to (-1, -3)$

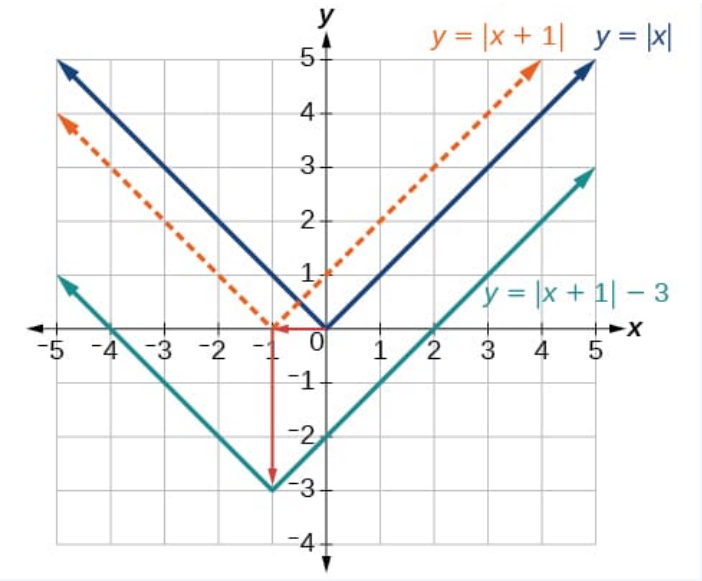



Figure 1.5.9: Graph of an absolute function, $y = |x|$, and how it was transformed to $y = |x + 1| - 3$

Figure 1.5.10 shows the graph of $h$.

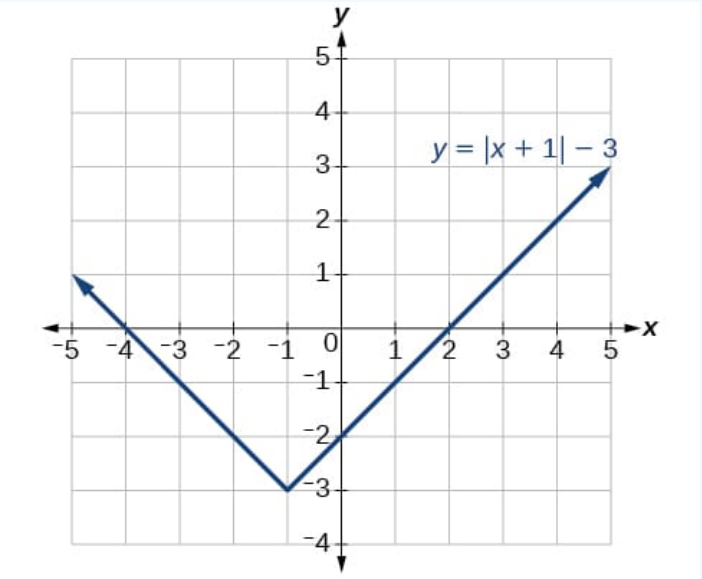



Figure 1.5.10: The final function $y = |x + 1| - 3$.

## Exercise 1.5.8

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

*Answer available at the end of this section.*

## Example 1.5.9: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 1.5.12, which is a transformation of the toolkit square root function.

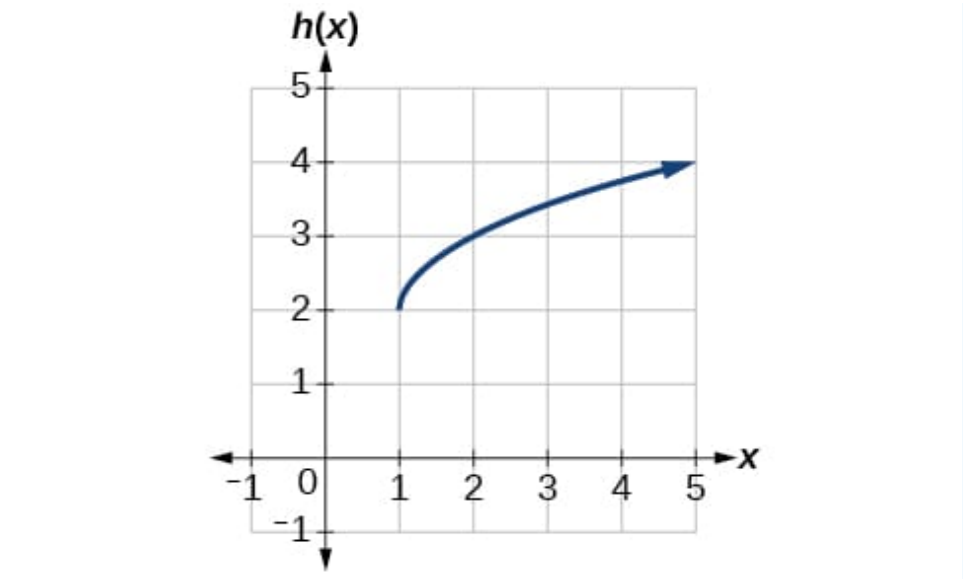



Figure 1.5.12: Graph of a square root function transposed right one unit and up 2.

**Solution**

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1) + 2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1} + 2$$

**Analysis**

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

**Exercise 1.5.9**

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

*Answer available at the end of this section.*

## Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 1.5.13**.**

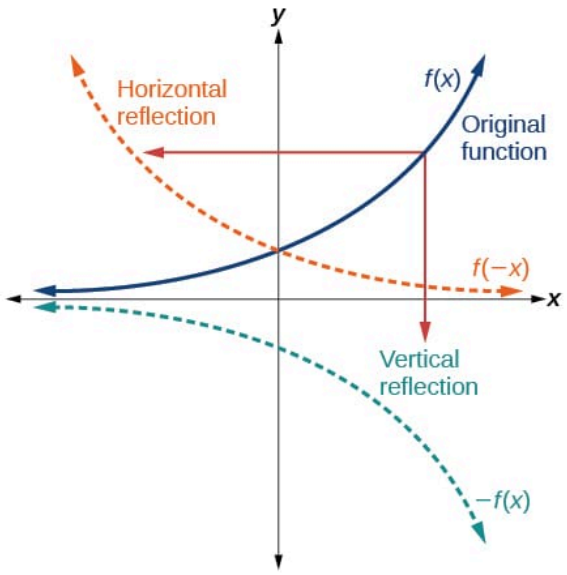



Figure 1.5.13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

## Definitions: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

## How To...

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

## Example 1.5.10: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

**Solution**

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 1.5.14.

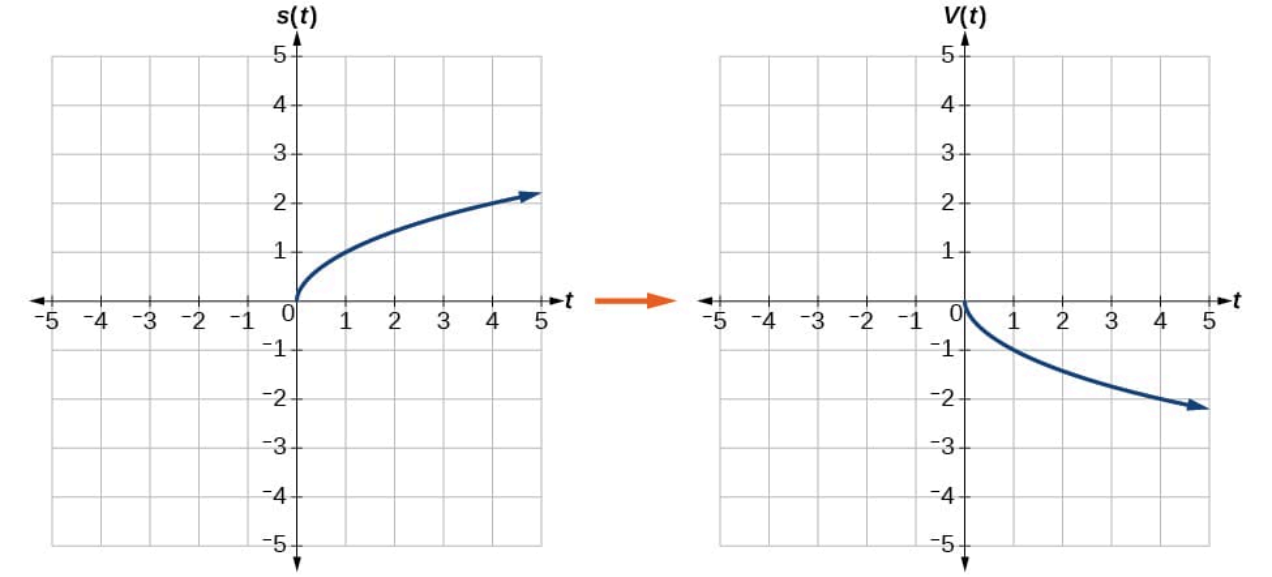



Figure 1.5.14: Graph of the vertical reflection of the square root function.

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 1.5.15.

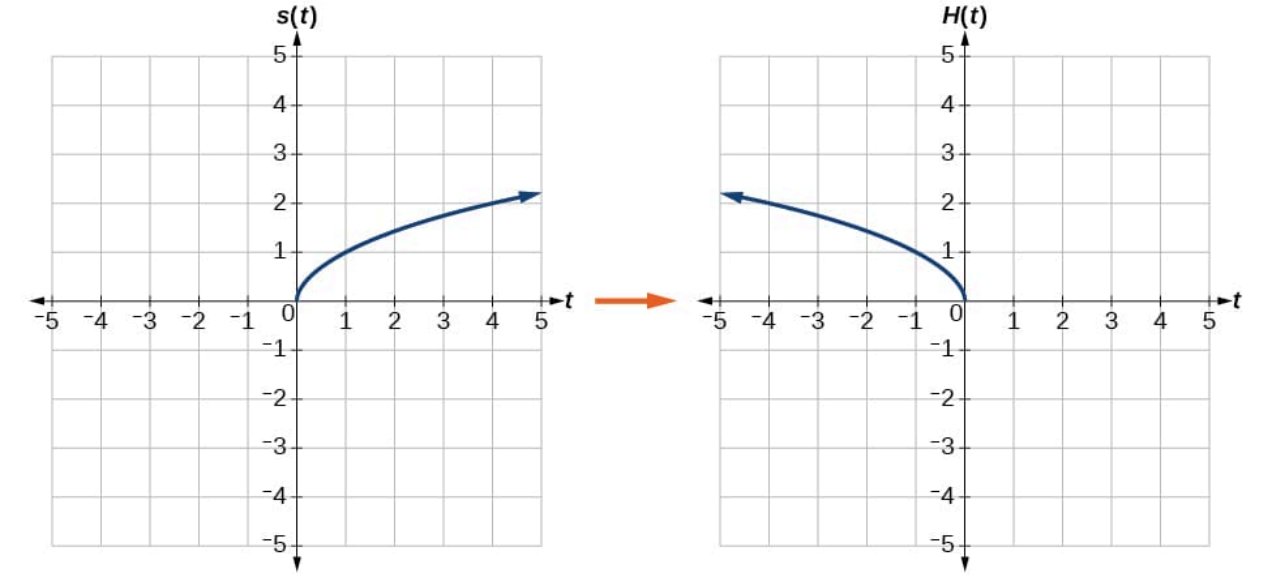



Figure 1.5.15: Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

## Exercise 1.5.5

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

*Answer available at the end of this section.*

## Example 1.5.11: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 1.5.6. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

*Table* 1.5.6

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 1.5.7.

*Table* 1.5.7

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x)$ | -1 | -3 | -7 | -11 |

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 1.5.8.

*Table* 1.5.8

| $x$ | -2 | -4 | -6 | -8 |
|---|---|---|---|---|
| $h(x)$ | 1 | 3 | 7 | 11 |

## Exercise 1.5.6

A function $f(x)$ is given as Table 1.5.9. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

*Table* 1.5.9

| $x$ | -2 | 0 | 2 | 4 |
|---|---|---|---|---|
| $f(x)$ | 5 | 10 | 15 | 20 |

*Answer available at the end of this section.*

## Example 1.5.12: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where $k$ is the percentage of mastery that can be achieved after $t$ practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 1.5.18. Sketch a graph of $k(t)$.

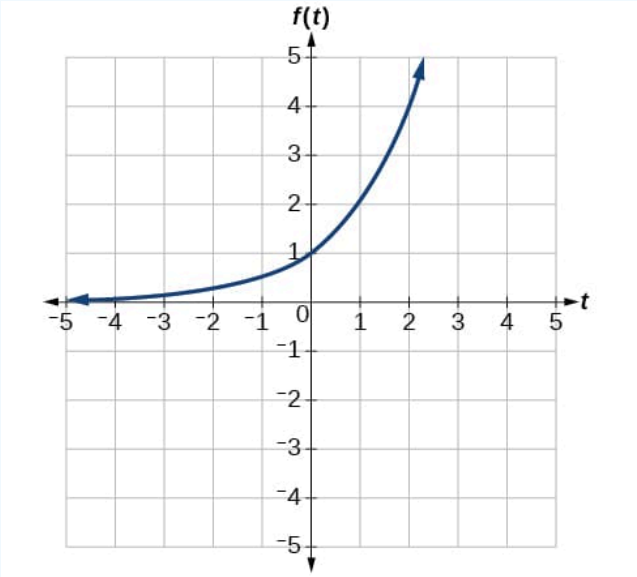



Figure 1.5.18: Graph of $k(t)$

**Solution**

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

- First, we apply a horizontal reflection: $(0, 1)$ $(-1, 2)$.
- Then, we apply a vertical reflection: $(0, -1)$ $(-1, -2)$.
- Finally, we apply a vertical shift: $(0, 0)$ $(-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 1.5.19, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

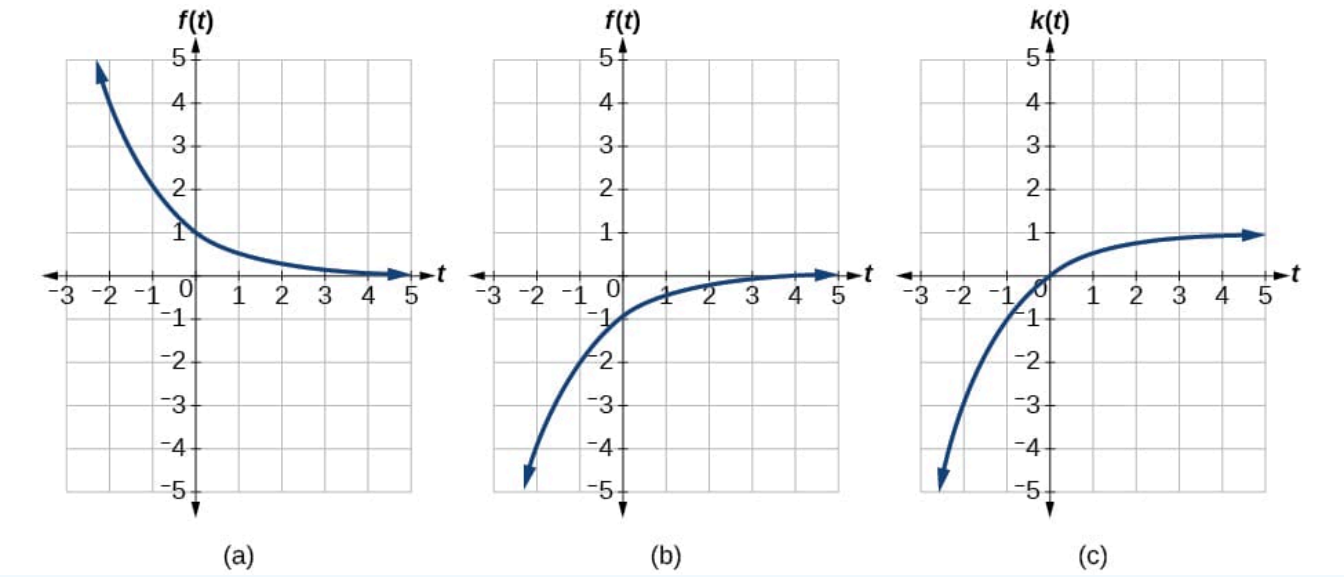



Figure 1.5.19: Graphs of all the transformations.

**Analysis**

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

## Exercise 1.5.7

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

*Answer available at the end of this section.*

# Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure 1.5.21.

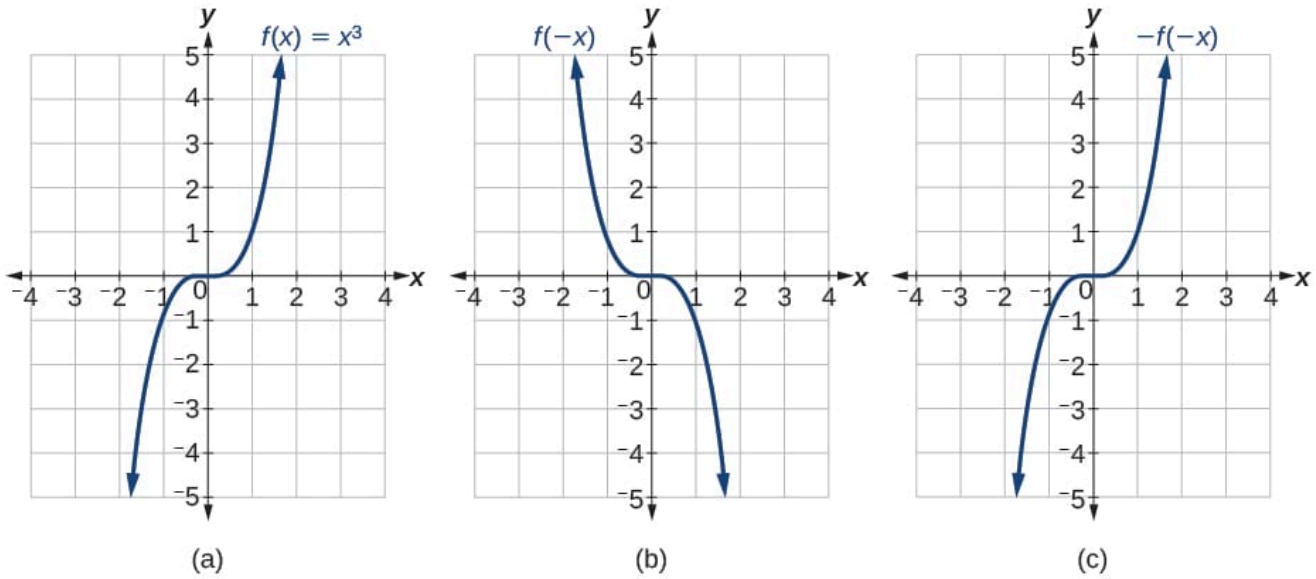



Figure 1.5.21: (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

### Definitions: Even and Odd Functions

A function is called an **even function** if for every input $x$

$f(x) = f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input $x$

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

### How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

### Example 1.5.13: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

**Solution**

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

**Analysis**

Consider the graph of $f$ in Figure $1.5.22$. Notice that the graph is symmetric about the origin. For every point $(x, y)$ on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of $f$, and the corresponding point $(-1, -3)$ is also on the graph.

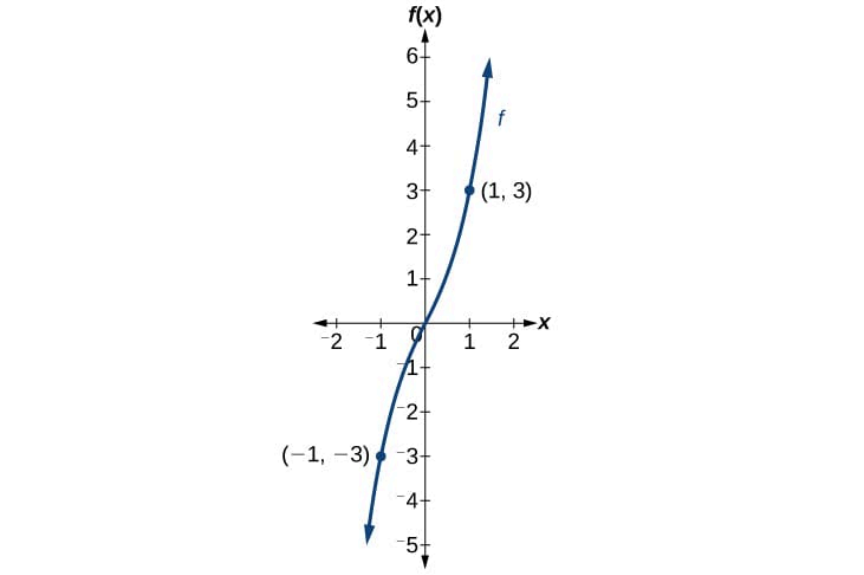



Figure $1.5.22$: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

**Exercise $1.5.8$**

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

*Answer available at the end of this section.*

# Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

## Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1,

we get a **vertical compression**. Figure $1.5.23$ shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

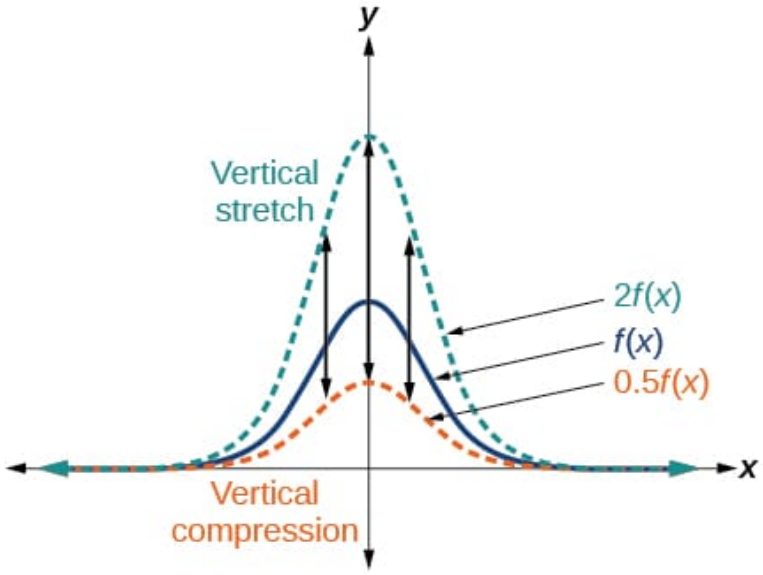



Figure $1.5.23$: Vertical stretch and compression

## Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where $a$ is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

## How To...

Given a function, graph its vertical stretch.

a. Identify the value of $a$.
b. Multiply all range values by $a$
c. If $a > 1$, the graph is stretched by a factor of $a$.
d. If $0 < a < 1$, the graph is compressed by a factor of $a$.
e. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

## Example 1.5.14: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure $1.5.24$.

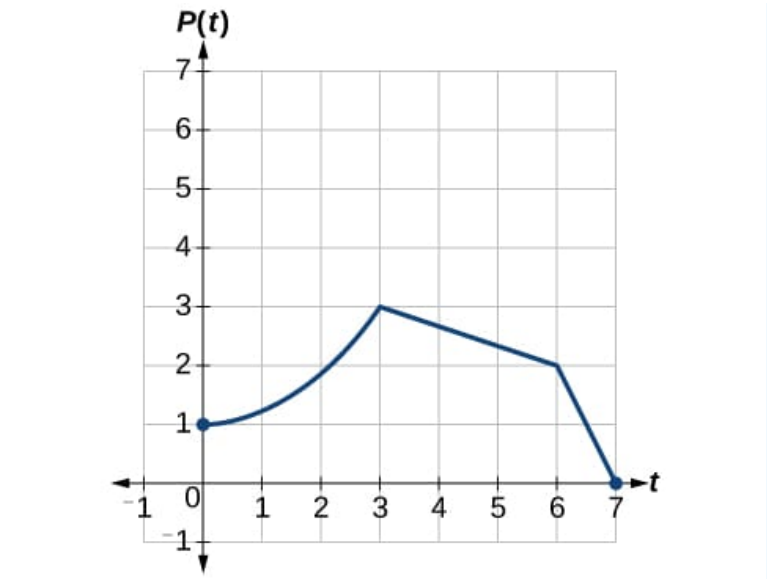



Figure 1.5.24: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, $Q$, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

**Solution**

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 1.5.25.

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2) \tag{1.5.2}$$

$$(3, 3) \to (3, 6) \tag{1.5.3}$$

$$(6, 2) \to (6, 4) \tag{1.5.4}$$

$$(7, 0) \to (7, 0) \tag{1.5.5}$$

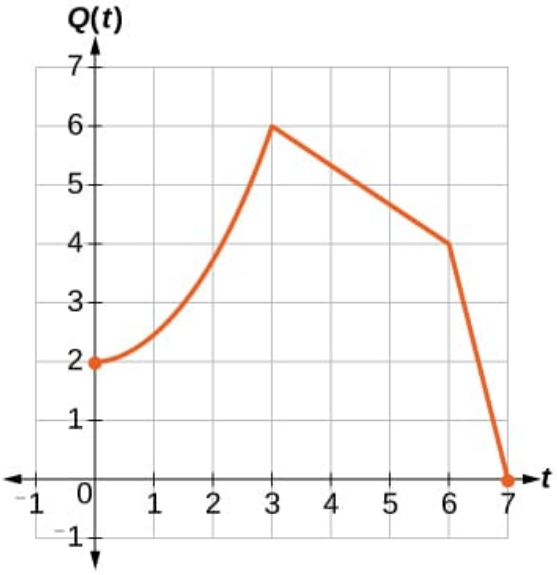



Figure 1.5.25: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input $t$, the value of the function $Q$ is twice the value of the function $P$. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, $t$, stay the same while the output values are twice as large as before.

**How To...**

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of $a$.
2. Multiply all of the output values by $a$.

**Example 1.5.15: Finding a Vertical Compression of a Tabular Function**

A function $f$ is given as Table 1.5.12. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

*Table* 1.5.12

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

**Solution**

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of $g$ are half of the output values of $f$ with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 1.5.13.

*Table* 1.5.13

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x)$ | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{7}{2}$ | $\frac{11}{2}$ |

**Analysis**

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

## Exercise 1.5.9

A function $f$ is given as Table 1.5.14. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

*Table* 1.5.14

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 12 | 16 | 20 | 0 |

*Answer available at the end of this section.*

## Example 1.5.16: Recognizing a Vertical Stretch

The graph in Figure 1.5.26 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

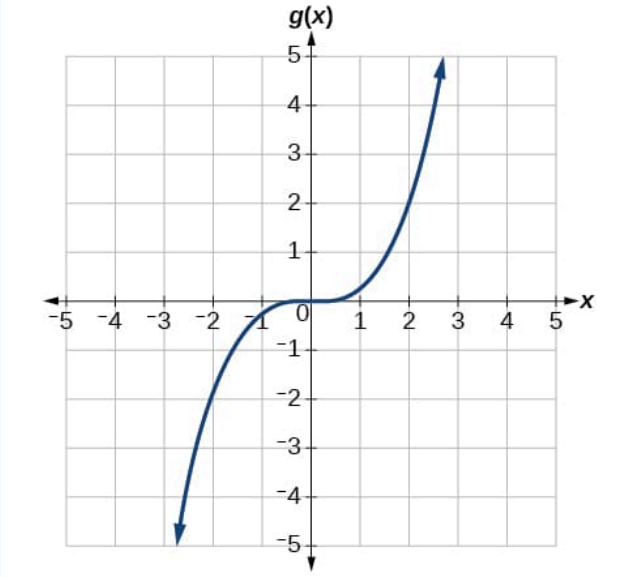



Figure 1.5.26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of $g$ are $\frac{1}{4}$ the outputs of the function $f$ because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for $g$ by using the definition of the function $f$.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3. \tag{1.5.6}$$

**Exercise 1.5.1**

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

*Answer available at the end of this section.*

## Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

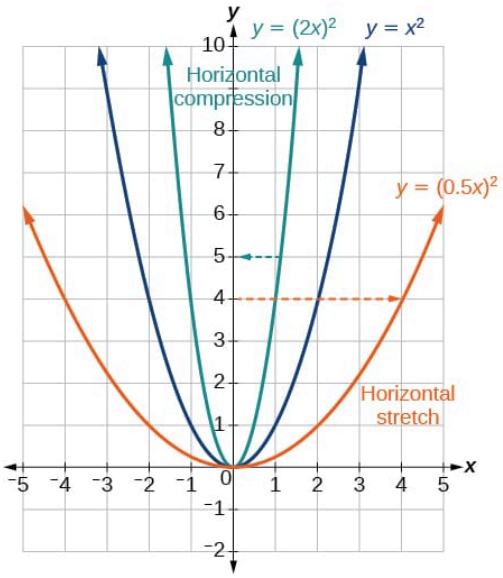



Figure 1.5.27: Graph of the vertical stretch and compression of $x^2$.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 1.5.27. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

**Definitions: Horizontal Stretches and Compressions**

Given a function $f(x)$, a new function $g(x) = f(bx)$, where $b$ is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

**How To...**

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

## Example 1.5.17: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, $R$, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

**Solution**

Symbolically, we could write

$$R(1) = P(2), \tag{1.5.7}$$
$$R(2) = P(4), \quad \text{and in general,} \tag{1.5.8}$$
$$R(t) = P(2t). \tag{1.5.9}$$

See Figure 1.5.28 for a graphical comparison of the original population and the compressed population.

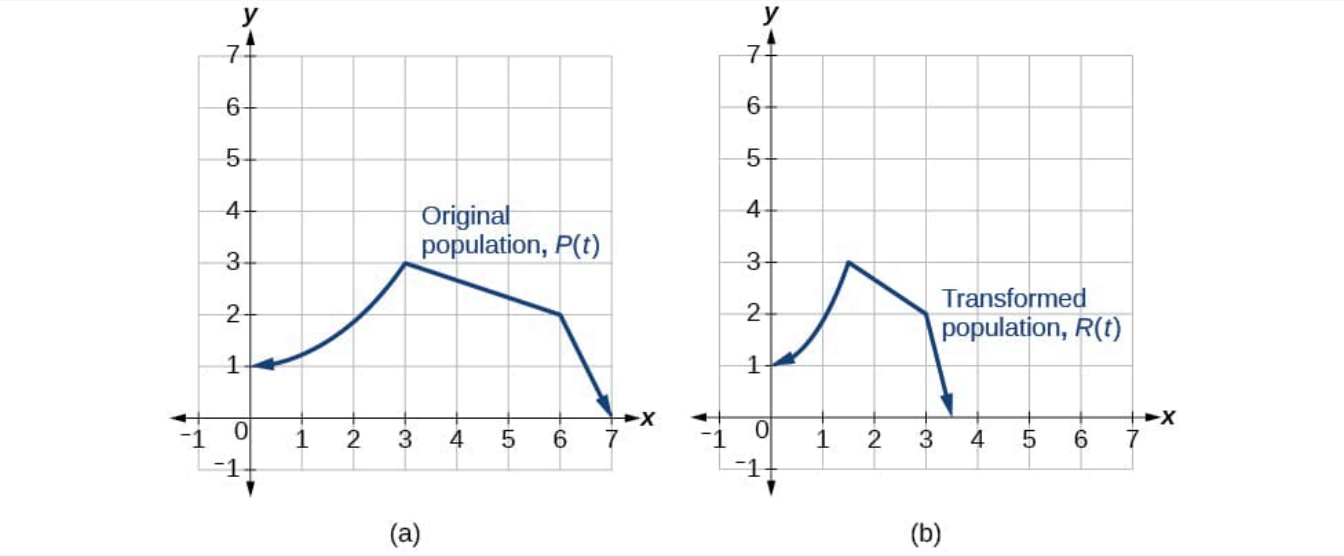



Figure 1.5.28: (a) Original population graph (b) Compressed population graph

**Analysis**

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

## Example 1.5.18: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 1.5.16. Create a table for the function $g(x) = f(\frac{1}{2}x)$.

*Table* 1.5.16

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for $g$ are the same as the output values for the function $f$ at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to $g$ will need to be twice as large to get inputs for $f$ that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1 \tag{1.5.10}$$

We do the same for the other values to produce Table 1.5.17.

*Table* 1.5.17

| $x$ | 4 | 8 | 12 | 16 |
|---|---|---|---|---|
| $g(x)$ | 1 | 3 | 7 | 11 |

Figure 1.5.29 shows the graphs of both of these sets of points.

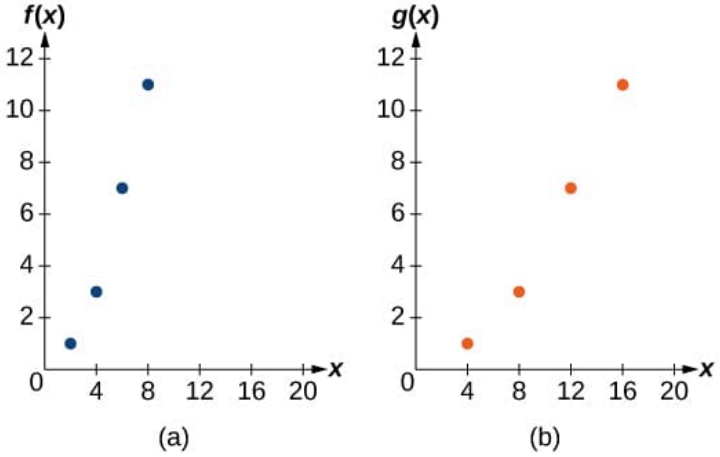



Figure 1.5.29: Graph of the previous table.

**Analysis**

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

## Example 1.5.19: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 1.5.30.

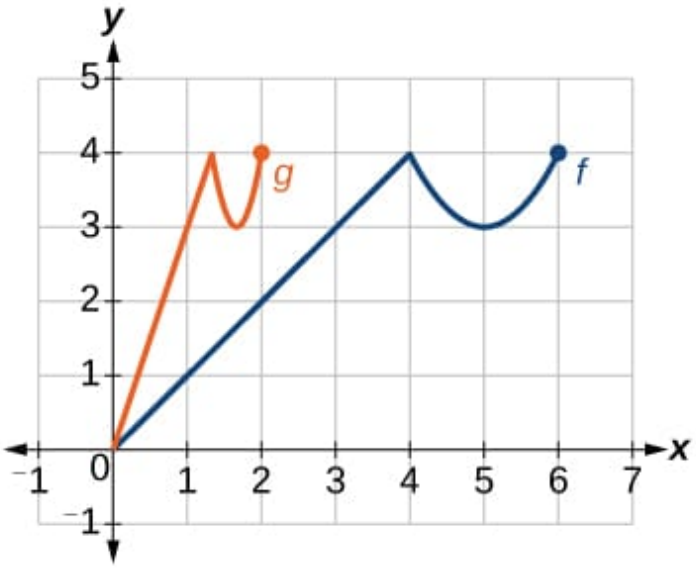



Figure 1.5.30: Graph of $f(x)$ being vertically compressed to $g(x)$.

**Solution**

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

**Analysis**

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

**Exercise 1.5.11**

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

*Answer available at the end of this section.*

## Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x) + 3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x + 3)$, for example, we have to think about how the inputs to the function $g$ relate to the inputs to the function $f$. Suppose we know $f(7) = 12$. What input to $g$ would produce that output? In other words, what value of $x$ will allow $g(x) = f(2x + 3) = 12$? We would need $2x + 3 = 7$. To solve for $x$, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f(b(x+\frac{p}{b}))$$

Let's work through an example.

$$f(x)=(2x+4)^2$$

We can factor out a 2.

$$f(x)=(2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

## Combining Transformations

- When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by $a$ and then vertically shift by $k$.
- When combining horizontal transformations written in the form $f(bx+h)$, first horizontally shift by $h$ and then horizontally stretch by $\frac{1}{b}$.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by $h$.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

## Example 1.5.20: Finding a Triple Transformation of a Tabular Function

Given Table 1.5.18 for the function $f(x)$, create a table of values for the function $g(x)=2f(3x)+1$.

*Table* 1.5.18

| $x$ | 6 | 12 | 18 | 24 |
|---|---|---|---|---|
| $f(x)$ | 10 | 14 | 15 | 17 |

**Solution**

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each $x$-value by $\frac{1}{3}$.See Table 1.5.19.

*Table* 1.5.19

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(3x)$ | 10 | 14 | 15 | 17 |

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 1.5.20.

*Table* 1.5.20

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $2f(3x)$ | 20 | 28 | 30 | 34 |

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 1.5.21.

*Table* 1.5.21

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x) = 2f(3x) + 1 + 1$ | 21 | 29 | 31 | 35 |

## Example 1.5.21: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 1.5.31 to sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$.

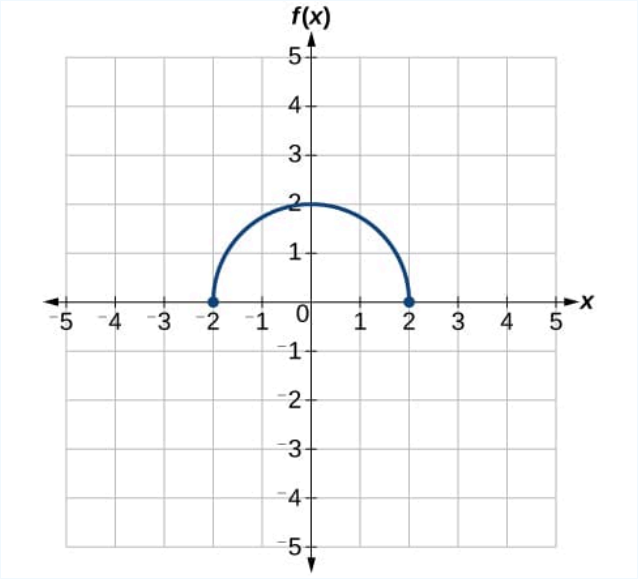



Figure 1.5.31: Graph of a half-circle.

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x + 1\right) - 3 = f\left(\frac{1}{2}(x + 2)\right) - 3 \tag{1.5.11}$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0, 2)$ remains at $(0, 2)$ while the point $(2, 0)$ will stretch to $(4, 0)$. See Figure 1.5.32 .

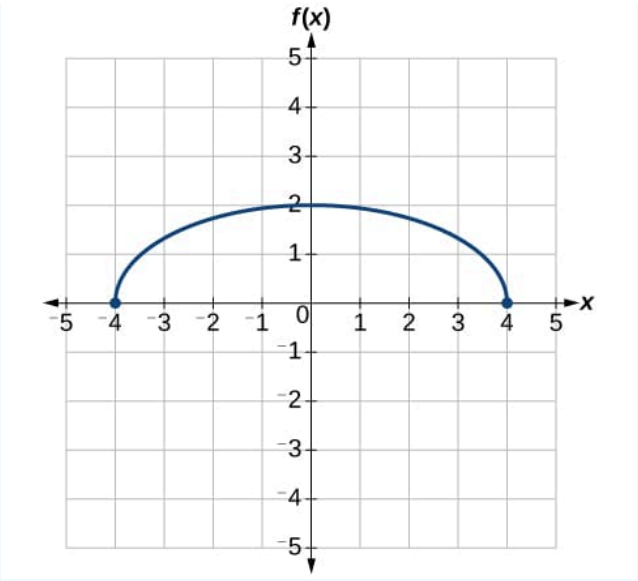



Figure 1.5.32: Graph of a vertically stretch half-circle.

Next, we horizontally shift left by 2 units, as indicated by $x + 2$. See Figure 1.5.33.

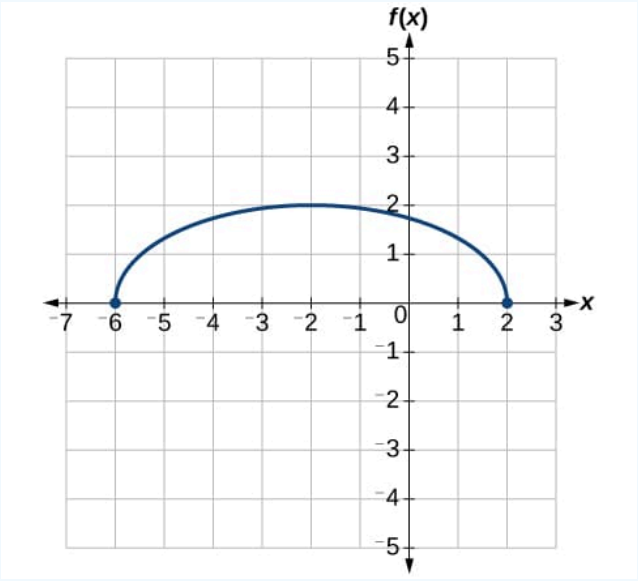



Figure 1.5.33: Graph of a vertically stretch and translated half-circle.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the –3 on the outside of the function. See Figure 1.5.34.

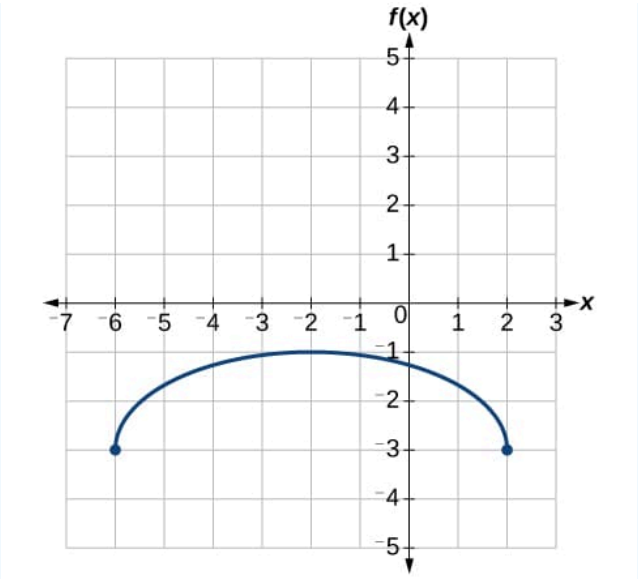



Figure 1.5.34: Graph of a vertically stretch and translated half-circle.

## Key Equations

- Vertical shift $g(x) = f(x) + k$ (up for $k > 0$)
- Horizontal shift $g(x) = f(x - h)$(right) for $h > 0$
- Vertical reflection $g(x) = -f(x)$
- Horizontal reflection $g(x) = f(-x)$
- Vertical stretch $g(x) = af(x)$ (a>0 )
- Vertical compression $g(x) = af(x)$ (0<a<1)
- Horizontal stretch $g(x) = f(bx)(0 < b < 1)$
- Horizontal compression $g(x) = f(bx)$ (b>1)

## Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

# Glossary

**even function**

a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

**horizontal compression**
a transformation that compresses a function's graph horizontally, by multiplying the input by a constant b>1

**horizontal reflection**
a transformation that reflects a function's graph across the y-axis by multiplying the input by –1

**horizontal shift**
a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

**horizontal stretch**
a transformation that stretches a function's graph horizontally by multiplying the input by a constant 0<b<1

**odd function**
a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

**vertical compression**
a function transformation that compresses the function's graph vertically by multiplying the output by a constant 0<a<1

**vertical reflection**
a transformation that reflects a function's graph across the x-axis by multiplying the output by –1

**vertical shift**
a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output

**vertical stretch**
a transformation that stretches a function's graph vertically by multiplying the output by a constant a>1

## Answers

**Exercise** 1.5.1

$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

**Exercise** 1.5.7

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

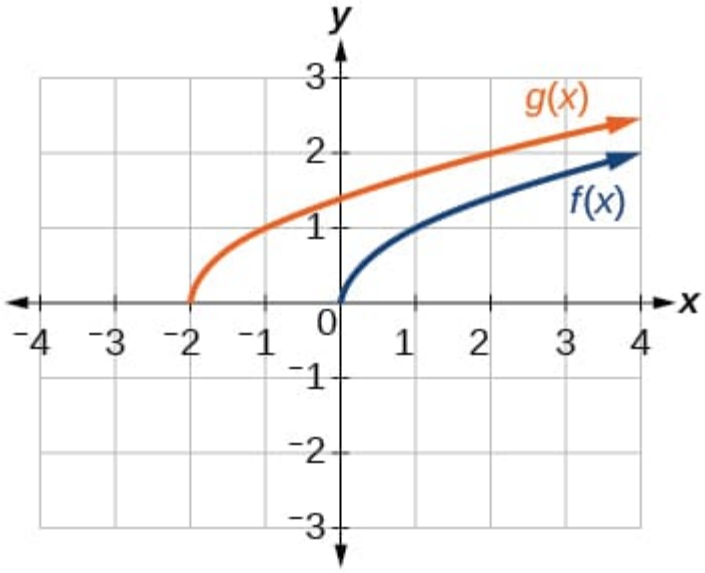



Figure 1.5.8

**Exercise** 1.5.8

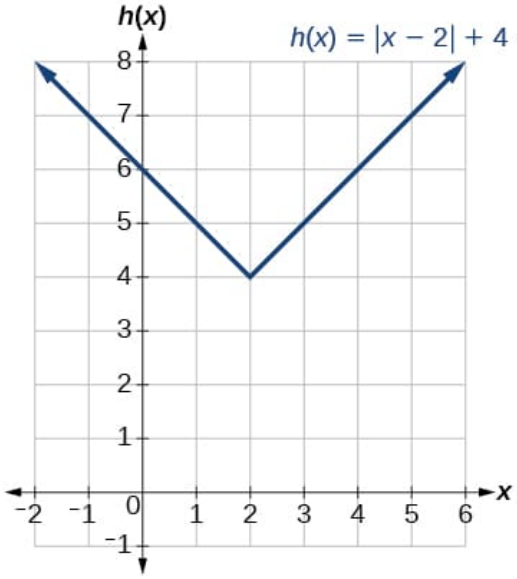



Figure 1.5.11

**Exercise** 1.5.9

$$g(x) = \frac{1}{x-1} + 1$$

**Exercise** 1.5.5

a.

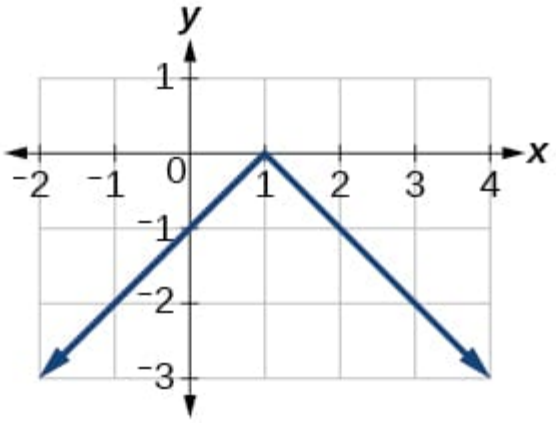



Figure 1.5.16: Graph of a vertically reflected absolute function.

b.

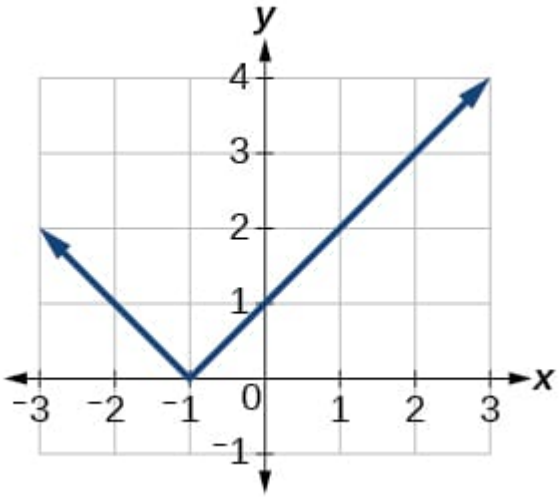



Figure 1.5.17: Graph of an absolute function translated one unit left.

**Exercise** 1.5.6

**a.** $g(x) = -f(x)$

*Table* 1.5.10

| $x$ | -2 | 0 | 2 | 4 |
|---|---|---|---|---|
| $g(x)$ | -5 | -10 | -15 | -20 |

**b.** $h(x) = f(-x)$

*Table* 1.5.11

| $x$ | -2 | 0 | 2 | -4 |
|---|---|---|---|---|
| $h(x)$ | 15 | 10 | 5 | 20 |

**Exercise** 1.5.7

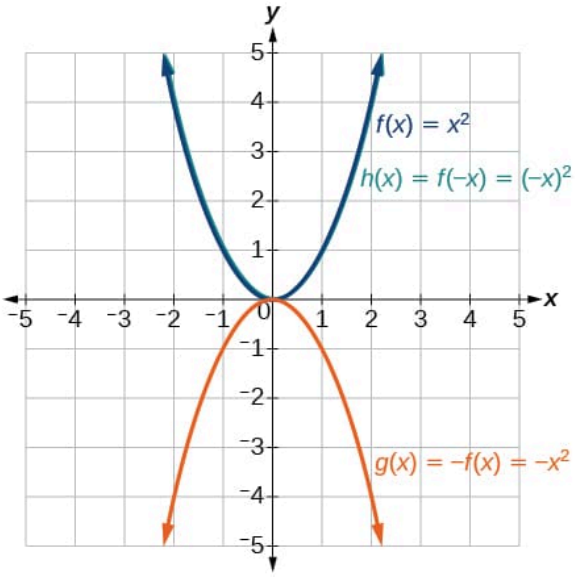



Figure 1.5.20: Graph of $x^2$ and its reflections.

Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

**Exercise** 1.5.8

even

**Exercise** 1.5.9

*Table* 1.5.15

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x)$ | 9 | 12 | 15 | 0 |

**Exercise** 1.5.1

$g(x) = 3x - 2$

**Exercise** 1.5.11

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

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# 1.6: Absolute Value Functions

**Learning Objectives**

- Graph an absolute value function.
- Solve an absolute value equation.

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate **absolute value functions**.

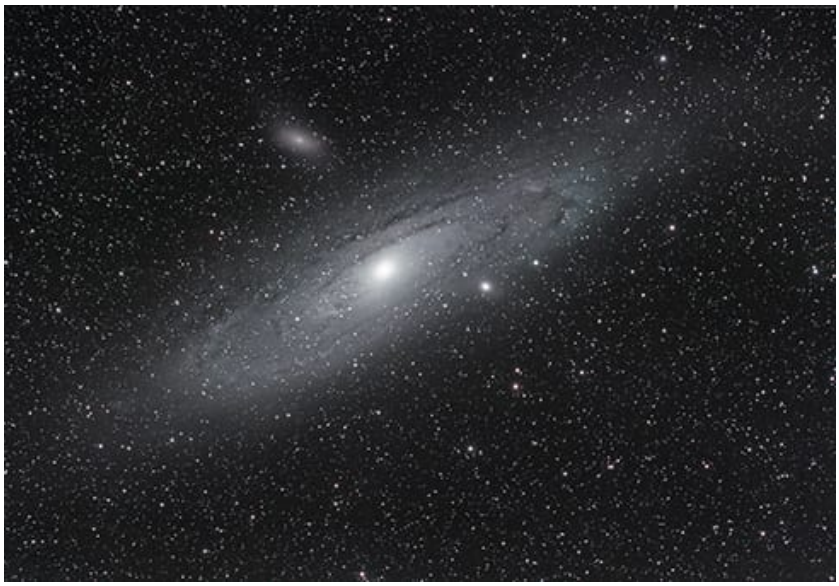


Figure 1.6.1: Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

## Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the **absolute value** function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

**Absolute Value Function**

The absolute value function can be defined as a piecewise function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{1.6.1}$$

**Example 1.6.1: Determine a Number within a Prescribed Distance**

Describe all values $x$ within or including a distance of 4 from the number 5.

**Solution**

We want the distance between $x$ and 5 to be less than or equal to 4. We can draw a number line, such as the one in , to represent the condition to be satisfied.

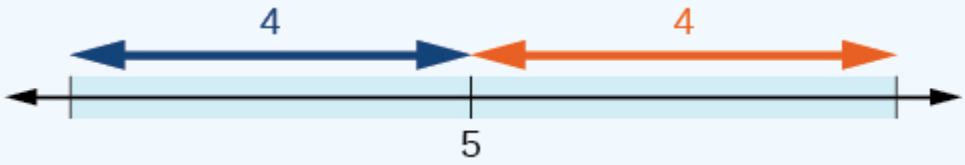



Figure 1.6.2: Number line describing the difference of the distance of 4 away from 5.

The distance from $x$ to 5 can be represented using the absolute value as $|x-5|$. We want the values of $x$ that satisfy the condition $|x-5| \leq 4$.

**Analysis**

Note that

$$\begin{aligned} -4 &\leq x-5 \qquad & x-5 &\leq 4 \\ 1 &\leq x & x &\leq 9 \end{aligned}$$

So $|x-5| \leq 4$ is equivalent to $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

## Exercise 1.6.1

Describe all values $x$ within a distance of 3 from the number 2.

*Answer available at the end of this section.*

## Example 1.6.2: Resistance of a Resistor

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often ±1%, ±5%, or ±10%.

Suppose we have a resistor rated at 680 ohms, ±5%. Use the absolute value function to express the range of possible values of the actual resistance.

**Solution**

5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance $R$ in ohms,

$$|R-680| \leq 34$$

## Exercise 1.6.2

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

*Answer available at the end of this section.*

# Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 1.6.3.

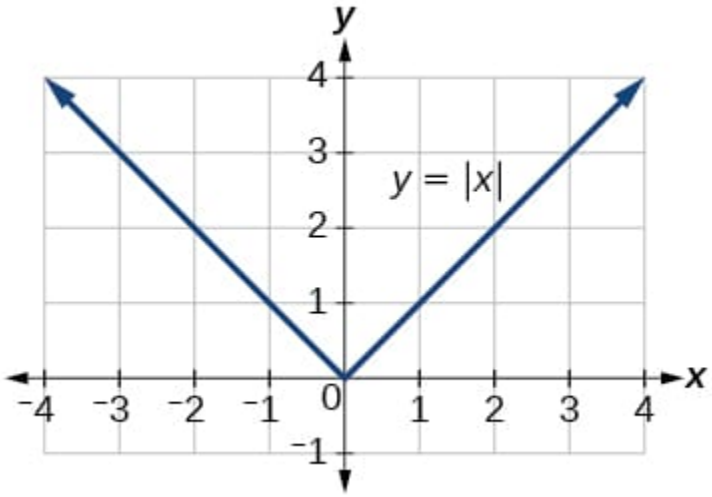



Figure 1.6.3: Graph of an absolute function.

Figure 1.6.3 shows the graph of $y = 2|x–3| + 4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

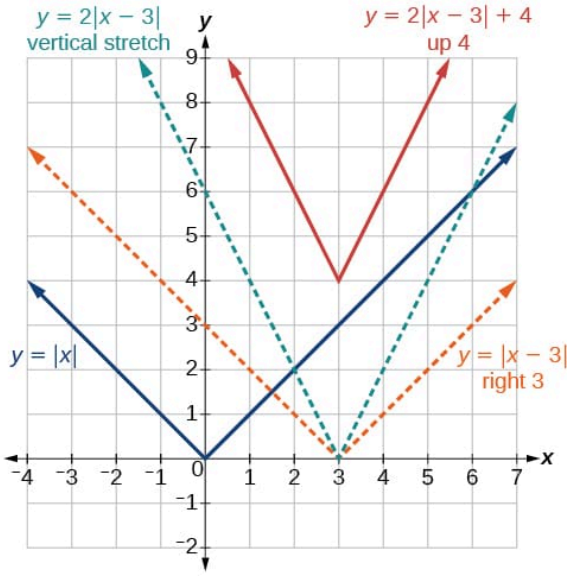



Figure 1.6.4: Graph of the different types of transformations for an absolute function.

## Example 1.6.3: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 1.6.5.

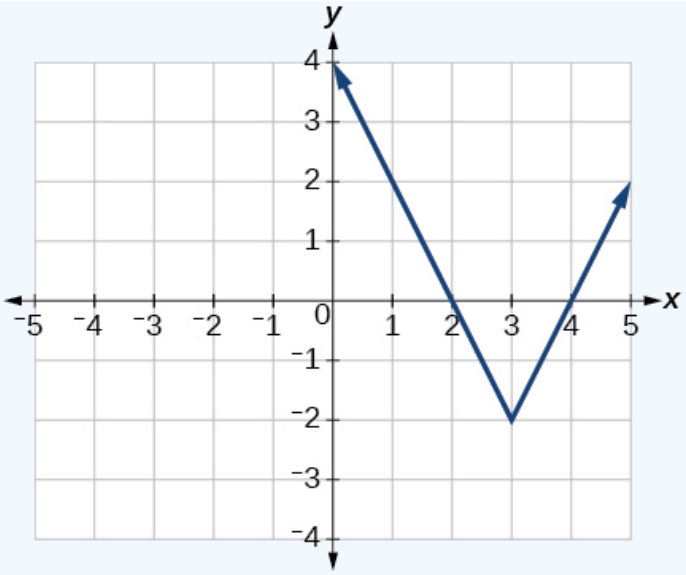



Figure 1.6.5: Graph of an absolute function.

**Solution**

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 1.6.6.

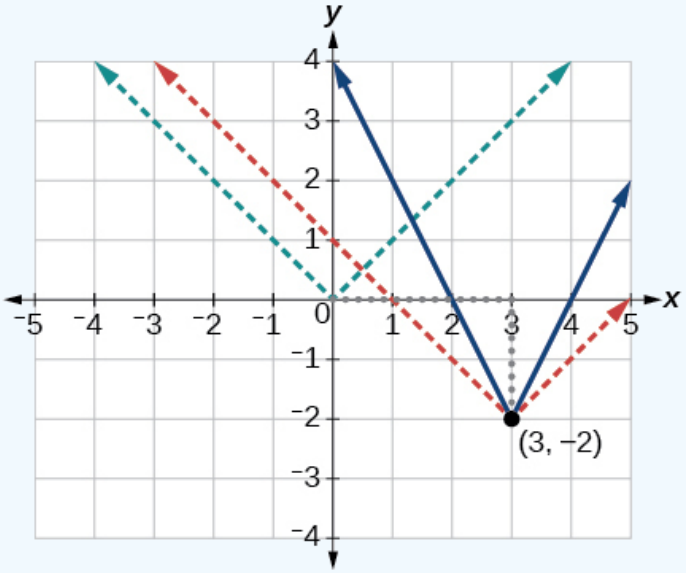



Figure 1.6.6: Graph of two transformations for an absolute function at $(3, -2)$.

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 1.6.7.

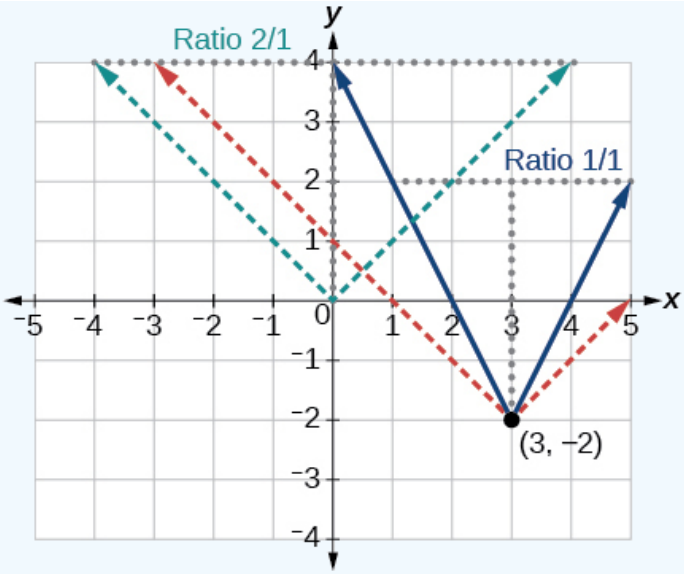



Figure 1.6.7: Graph of two transformations for an absolute function at $(3, -2)$ and the ratios between the two different transformations.

From this information we can write the equation

$$f(x) = 2|x-3| - 2, \quad \text{treating the stretch as a vertial stretch, or}$$
$$f(x) = |2(x-3)| - 2, \quad \text{treating the stretch as a horizontal compression.}$$

**Analysis**

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

**Q & A**

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

*Answer available at the end of this section.*

**Exercise 1.6.3**

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

*Answer available at the end of this section.*

**Q & A**

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

*Answer available at the end of this section.*

## Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation

such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$2x - 6 = 8 \quad \text{or} \quad 2x - 6 = -8$$

$$\begin{aligned} 2x &= 14 & 2x &= -2 \\ x &= 7 & x &= -1 \end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An **absolute value equation** is an equation in which the unknown variable appears in absolute value bars. For example,

$$\begin{aligned} |x| &= 4, \\ |2x - 1| &= 3, \\ |5x + 2| - 4 &= 9. \end{aligned}$$

## Solutions to Absolute Value Equations

For real numbers $A$ and $B$, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

## How To ...

Given the formula for an absolute value function, find the horizontal intercepts of its graph.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $-A = B$, assuming $B > 0$.
3. Solve for $x$.

## Example 1.6.4: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x + 1| - 7$, find the values of $x$ such that $f(x) = 0$.

**Solution**

$$\begin{aligned} 0 &= |4x + 1| - 7 && && \text{Substitute 0 for f(x).} \\ 7 &= |4x + 1| && && \text{Isolate the absolute value on one side of the equation.} \\ 7 &= 4x + 1 && \text{or } -7 = 4x + 1 && \text{Break into two separate equations and solve.} \\ 6 &= 4x && -8 = 4x \\ x &= \frac{6}{4} = 1.5 && x = \frac{-8}{4} = -2 \end{aligned}$$

The function outputs 0 when $x = 1.5$ or $x = -2$ (Figure 1.6.9).

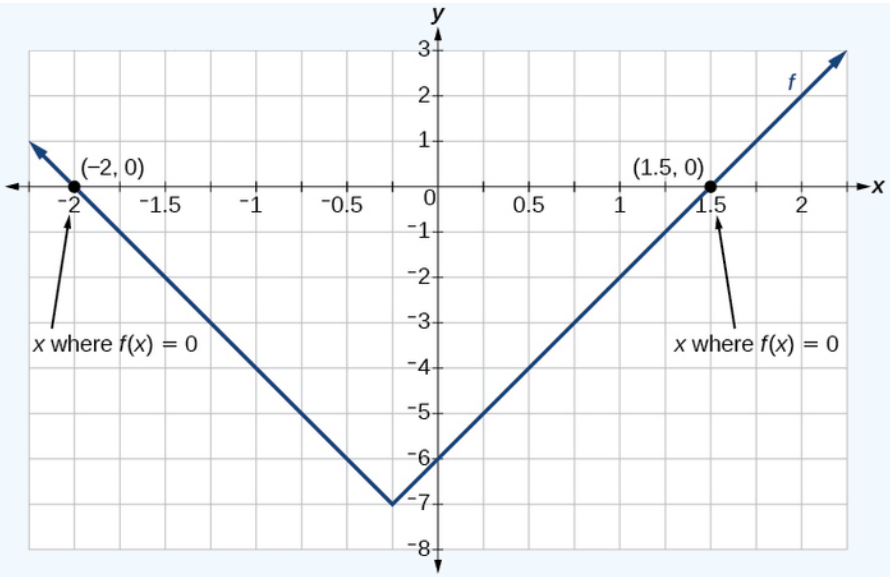



Figure 1.6.9: Graph of an absolute function with x-intercepts at -2 and 1.5.

## Exercise 1.6.4

For the function $f(x) = |2x - 1| - 3$, find the values of $x$ such that $f(x) = 0$.

**Solution**

$x = -1$ or $x = 2$

## Q & A

Should we always expect two answers when solving $|A| = B$?

**Answer**

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x - 5| = 1$.

## How To ...

Given an absolute value equation, solve it.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $A = -B$.
3. Solve for $x$.

## Example 1.6.5: Solving an Absolute Value Equation

Solve $1 = 4|x - 2| + 2$.

**Solution**

Isolating the absolute value on one side of the equation gives the following.

$$1 = 4|x-2| + 2$$
$$-1 = 4|x-2|$$
$$-\frac{1}{4} = |x-2|$$

The absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value. At this point, we notice that this equation has no solutions.

Q & A

In Example $1.6.5$, if $f(x) = 1$ and $g(x) = 4|x-2| + 2$ were graphed on the same set of axes, would the graphs intersect?

**Answer**

No. The graphs of $f$ and $g$ would not intersect, as shown in Figure $1.6.10$. This confirms, graphically, that the equation $1 = 4|x-2| + 2$ has no solution.

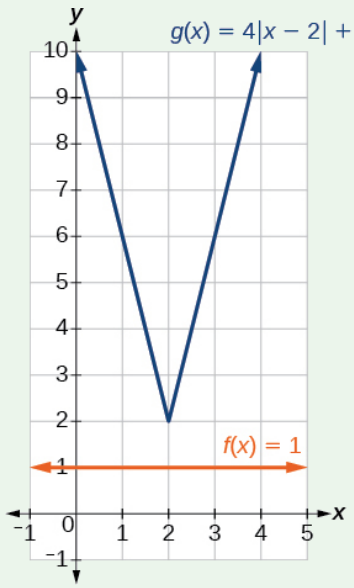



Figure $1.6.10$: Graph of $g(x) = 4|x-2| + 2$ and $f(x) = 1$.

Find where the graph of the function $f(x) = -|x+2| + 3$ intersects the horizontal and vertical axes.

$f(0) = 1$, so the graph intersects the vertical axis at $(0, 1)$. $f(x) = 0$ when $x = -5$ and $x = 1$ so the graph intersects the horizontal axis at $(-5, 0)$ and $(1, 0)$.

## Solving an Absolute Value Inequality

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an **absolute value inequality** to solve such an equation. An absolute value inequality is an equation of the form

$$|A| < B,\ |A| \leq B, |A| > B,$$

or

$$|A| \geq B,$$

where an expression $A$ (and possibly but not usually $B$) depends on a variable $x$. Solving the inequality means finding the set of all $x$ that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as

$$|x| < 200$$

or

$$-200 < x < 200$$

Suppose we want to know all possible returns on an investment if we could earn some amount of money within $200 of $600. We can solve algebraically for the set of values $x$ such that the distance between $x$ and 600 is less than 200. We represent the distance between $x$ and 600 as $|x - 600|$.

$$|x - 600| < 200 \tag{1.6.2}$$

or

$$-200 < x - 600 < 200 \tag{1.6.3}$$

$$-200 + 600 < x - 600 + 600 < 200 + 600$$

$$400 < x < 800$$

This means our returns would be between $400 and $800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

### How To ...

Given an absolute value inequality of the form $|x - A| \leq B$ for real numbers $a$ and $b$ where $b$ is positive, solve the absolute value inequality algebraically.

1. Find boundary points by solving $|x - A| = B$.
2. Test intervals created by the boundary points to determine where $|x - A| \leq B$.
3. Write the interval or union of intervals satisfying the inequality in interval, inequality, or set-builder notation.

### Example 1.6.6: Solving an Absolute Value Inequality

Solve $|x - 5| \leq 4$.

**Solution**

With both approaches, we will need to know first where the corresponding equality is true. In this case we first will find where $|x - 5| = 4$. We do this because the absolute value is a function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x - 5| = 4$.

$$\begin{aligned} x - 5 &= 4 \quad \text{or} \quad x = 9 \\ x - 5 &= -4 \quad\quad\;\; x = 1 \end{aligned}$$

After determining that the absolute value is equal to 4 at $x = 1$ and $x = 9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:

$$x < 1, \; 1 < x < 9, \text{ and } x > 9.$$

To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table 1.6.1.

*Table* 1.6.1

| Interval test $x$ | $f(x)$ | $<4$ **or** $>4$ | |
|---|---|---|---|
| $x<1$ | 0 | $\|0-5\|=5$ | Greater than |
| $1<x<9$ | 6 | $\|6-5\|=1$ | Less than |
| $x>9$ | 11 | $\|11-5\|=6$ | Greater than |

Because $1 \leq x \leq 9$ is the only interval in which the output at the test value is less than 4, we can conclude that the solution to $|x-5| \leq 4$ is $1 \leq x \leq 9$, or $[1,9]$.

To use a graph, we can sketch the function $f(x)=|x-5|$. To help us see where the outputs are 4, the line $g(x)=4$ could also be sketched as in Figure 1.6.11.

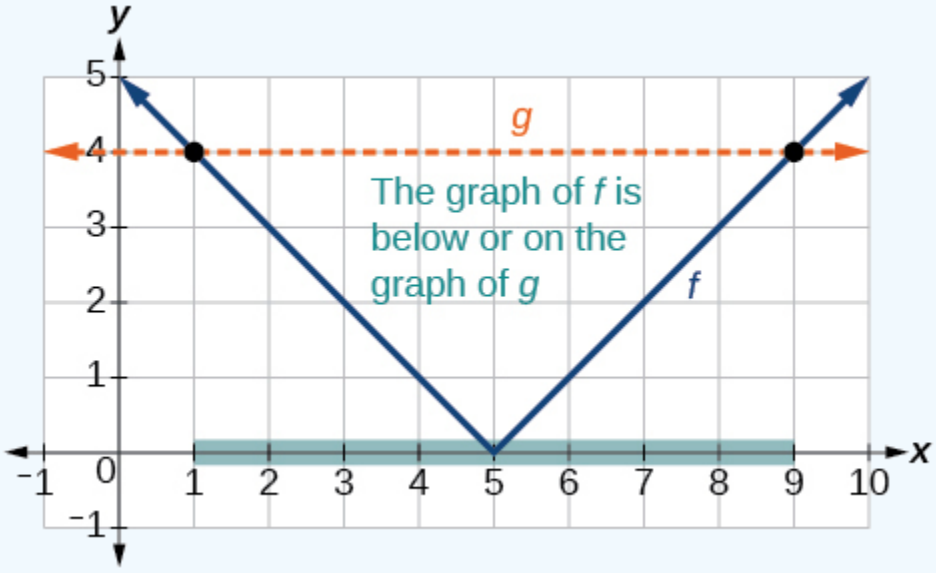



Figure 1.6.11: Graph to find the points satisfying an absolute value inequality.

We can see the following:

- The output values of the absolute value are equal to 4 at $x=1$ and $x=9$.
- The graph of $f$ is below the graph of $g$ on $1<x<9$. This means the output values of $f(x)$ are less than the output values of $g(x)$.
- The absolute value is less than or equal to 4 between these two points, when $1 \leq x \leq 9$. In interval notation, this would be the interval $[1,9]$.

**Analysis**

For absolute value inequalities,

$$\begin{array}{ll} |x-A|<C, & |x-A|>C, \\ -C<x-A<C, & x-A<-C \text{ or } x-A>C. \end{array}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for this example, we could use this alternative approach.

$$
\begin{aligned}
|x-5| &\leq 4 \\
-4 \leq x-5 &\leq 4 && \text{Rewrite by removing the absolute value bars.} \\
-4+5 \leq x-5+5 &\leq 4+5 && \text{Isolate the x.} \\
1 \leq x &\leq 9
\end{aligned}
$$

## Exercise 1.6.5

Solve $|x+2| \leq 6$.

*Answer available at the end of this section.*

## How To ...

Given an absolute value function, solve for the set of inputs where the output is positive (or negative).

1. Set the function equal to zero, and solve for the boundary points of the solution set.
2. Use test points or a graph to determine where the function's output is positive or negative.

## Example 1.6.7: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the function $f(x) = -\frac{1}{2}|4x-5|+3$, determine the $x$-values for which the function values are negative.

**Solution**

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x-5|+3 < 0$. We begin by isolating the absolute value.

$$
\begin{aligned}
-\frac{1}{2}|4x-5| &< -3 \quad \text{Multiply both sides by } -2, \text{ and reverse the inequality.} \\
|4x-5| &> 6
\end{aligned}
$$

Next we solve for the equality $|4x-5| = 6$.

$$
\begin{aligned}
4x-5 &= 6 \qquad 4x-5 = -6 \\
4x-6 &= 6
\end{aligned}
$$

or

$$
\begin{aligned}
4x &= -1 \\
x &= \frac{11}{4} \qquad x = -\frac{1}{4}
\end{aligned}
$$

Now, we can examine the graph of $f$ to observe where the output is negative. We will observe where the branches are below the $x$-axis. Notice that it is not even important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$ and that the graph has been reflected vertically. See Figure 1.6.12.

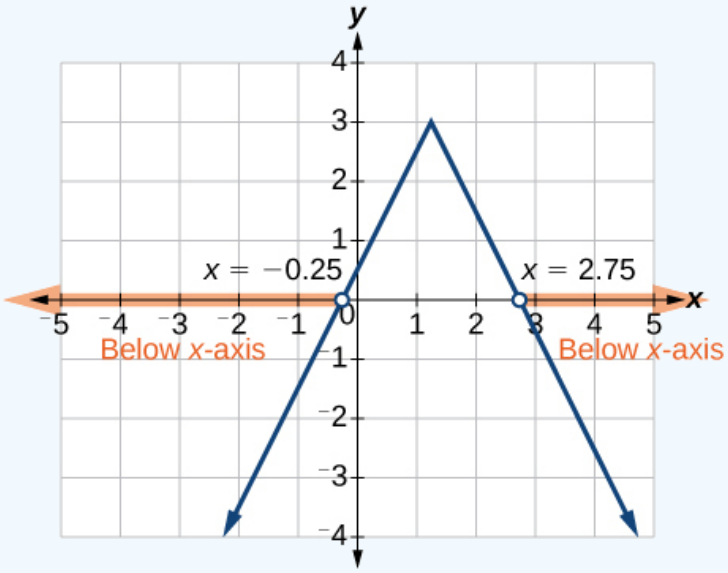



Figure 1.6.12: Graph of an absolute function with x-intercepts at -0.25 and 2.75.]

We observe that the graph of the function is below the $x$-axis left of $x = -\frac{1}{4}$ and right of $x = \frac{11}{4}$. This means the function values are negative to the left of the first horizontal intercept at $x = -\frac{1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality.

$$x < -\frac{1}{4} \text{ or } x > 1\frac{1}{4}$$

In interval notation, this would be $(-\infty, -0.25) \cup (2.75, \infty)$.

## Exercise 1.6.6

Solve $-2|k-4| \leq -6$.

*Answer available at the end of this section.*

# Key Concepts

- The absolute value function is commonly used to measure distances between points.
- Applied problems, such as ranges of possible values, can also be solved using the absolute value function.
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable.
- An absolute value equation may have one solution, two solutions, or no solutions.
- An absolute value inequality is similar to an absolute value equation but takes the form | A |<B, | A |≤B, | A |>B, or | A |≥B.It can be solved by determining the boundaries of the solution set and then testing which segments are in the set.
- Absolute value inequalities can also be solved graphically.

# Glossary

**absolute value equation**
an equation of the form $|A| = B$, with $B \geq 0$; it will have solutions when $A = B$ or $A = -B$

**absolute value inequality**
a relationship in the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$

## Answers

**Exercise** 1.6.1

$|x-2| \leq 3$

**Exercise** 1.6.2

Using the variable $p$ for passing, $|p-80| \leq 20$

**Q & A**

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for $x$ and $f(x)$.

$$f(x) = a|x-3| - 2$$

Now substituting in the point $(1, 2)$

$$\begin{aligned} 2 &= a|1-3| - 2 \\ 4 &= 2a \\ a &= 2 \end{aligned}$$

**Exercise** 1.6.3

$f(x) = -|x+2| + 3$

**Q & A**

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (Figure 1.6.8).

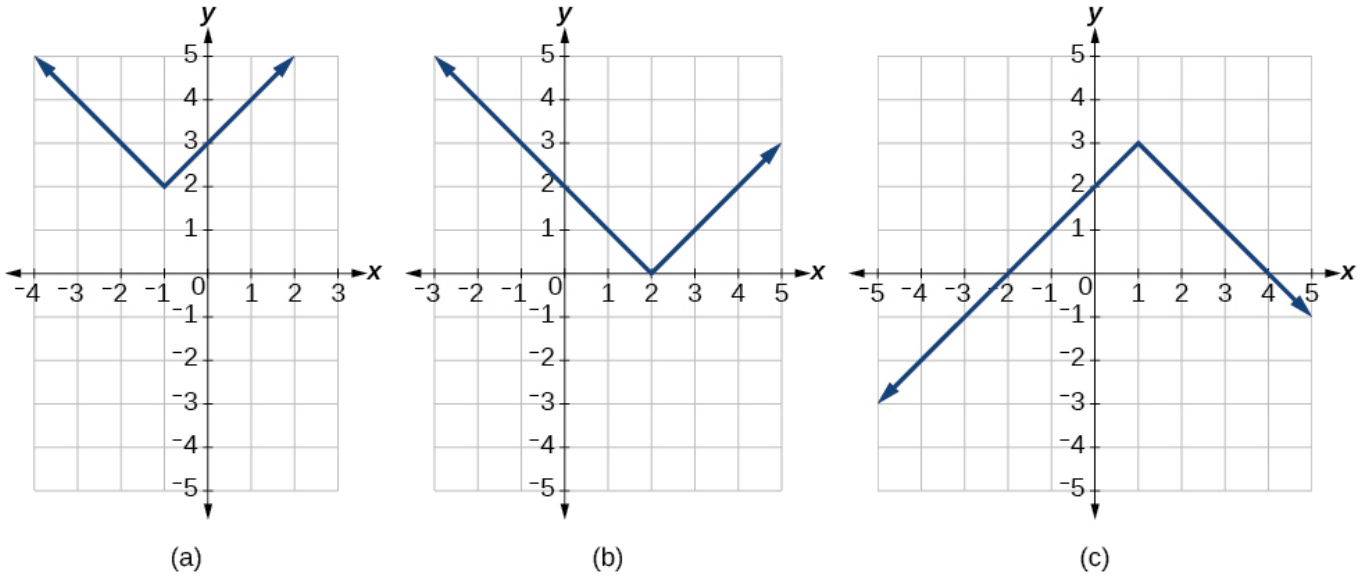



Figure 1.6.8: (a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

**Exercise** 1.6.5

$-8 \leq x \leq 4$

**Exercise** 1.6.6

$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$

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# 1.7: Inverse Functions

**Learning Objectives**

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure 1.7.1 provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

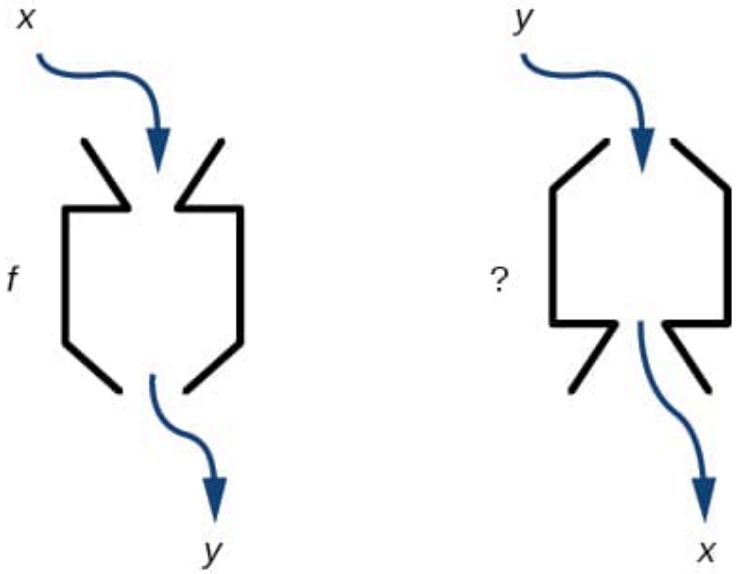



***Figure** 1.7.1**:** Can a function "machine" operate in reverse?*

## Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the **Celsius** scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees **Fahrenheit** to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32) \tag{1.7.1}$$

and substitutes 75 for $F$ to calculate

$$\frac{5}{9}(75 - 32) \approx 24^{\circ} \tag{1.7.2}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure 1.7.2 for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

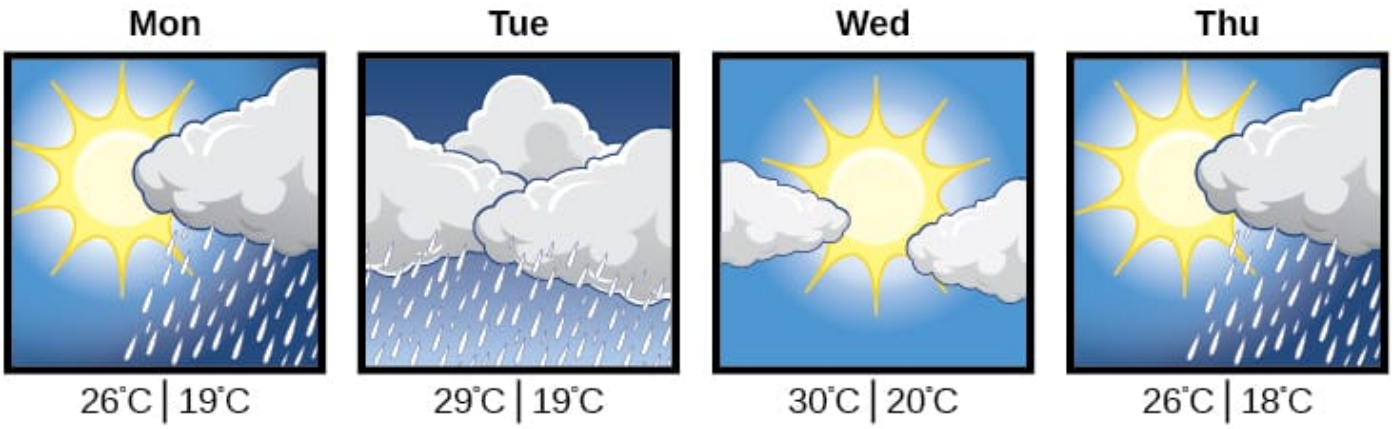



Figure 1.7.2: A forecast of Monday's through Thursday's weather.

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for $F$ after substituting a value for $C$. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F - 32) \tag{1.7.3}$$

$$26 \cdot \frac{9}{5} = F - 32 \tag{1.7.4}$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79 \tag{1.7.5}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "$f$ inverse of $x$." The raised –1 is part of the notation. It is not an exponent; it does not imply a power of –1 . In other words, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of $f$ and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number $a$, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x \tag{1.7.6}$$

This holds for all $x$ in the domain of $f$. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a **reciprocal**, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x \tag{1.7.7}$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x \tag{1.7.8}$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

## Definition: Inverse Function

For any **one-to-one function** $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of $f$ if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all $x$ in the domain of $f$. It also follows that $f(f^{-1}(x)) = x$ for all $x$ in the domain of $f^{-1}$ if $f^{-1}$ is the inverse of $f$.

The notation $f^{-1}$ is read "$f$ inverse." Like any other function, we can use any variable name as the input for $f^{-1}$, so we will often write $f^{-1}(x)$, which we read as "$f$ inverse of $x$." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)} \tag{1.7.9}$$

and not all functions have inverses.

## Example 1.7.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

**Solution**

The inverse function reverses the input and output quantities, so if

$$\begin{aligned} f(2) = 4, \text{ then } f^{-1}(4) = 2; \\ f(5) = 12, \text{ then } f^{-1}(12) = 5 \end{aligned} \tag{1.7.10}$$

.

Alternatively, if we want to name the inverse function $g$, then $g(4) = 2$ and $g(12) = 5$.

**Analysis**

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table 1.7.1.

*Table* 1.7.1

| $(x, f(x))$ | $(x, g(x))$ |
|---|---|
| $(2, 4)$ | $(4, 2)$ |
| $(5, 12)$ | $(12, 5)$ |

## Exercise 1.7.1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function $h$?

*Answer available at the end of this section.*

## How To: Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.

2. If both statements are true, then $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

## Example 1.7.2: Testing Inverse Relationships Algebraically

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

**Solution**

$$g(f(x)) = \frac{1}{(\frac{1}{x+2}) - 2} \tag{1.7.11}$$
$$= x + 2 - 2 \tag{1.7.12}$$
$$= x \tag{1.7.13}$$

so

$$g = f^{-1} \text{ and } f = g^{-1} \tag{1.7.14}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$f(g(x)) = \frac{1}{\frac{1}{x} - 2 + 2} \tag{1.7.15}$$
$$= \frac{1}{\frac{1}{x}} \tag{1.7.16}$$
$$= x \tag{1.7.17}$$

**Analysis**

Notice the inverse operations are in reverse order of the operations from the original function.

## Exercise 1.7.2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

*Answer available at the end of this section.*

## Example 1.7.3: Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

**Solution**

$$f(g(x)) = \frac{x^3}{27} \neq x \tag{1.7.18}$$

No, the functions are not inverses.

**Analysis**

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

## Exercise 1.7.3

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

*Answer available at the end of this section.*

## Finding Domain and Range of Inverse Functions

The outputs of the function $f$ are the inputs to $f^{-1}$, so the range of $f$ is also the domain of $f^{-1}$. Likewise, because the inputs to $f$ are the outputs of $f^{-1}$, the domain of $f$ is the range of $f^{-1}$. We can visualize the situation as in Figure $1.7.3$.

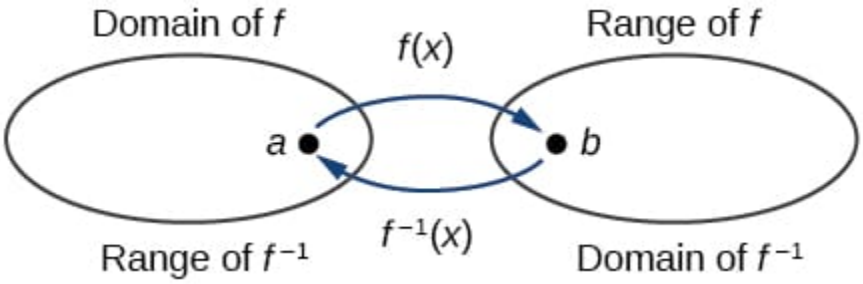



***Figure*** $1.7.3$***:*** *Domain and range of a function and its inverse.*

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its range limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x - 1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of $f$ = range of $f^{-1} = [1, \infty)$.
- The domain of $f^{-1}$ = range of $f = [0, \infty)$.

**Is it possible for a function to have more than one inverse?**

*No. If two supposedly different functions, say, $g$ and h, both meet the definition of being inverses of another function $f$, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.*

**Note: Domain and Range of Inverse Functions**

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

## How To: Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

## Example 1.7.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 1.7.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

*Table* 1.7.2

| Constant | Identity | Quadratic | Cubic | Reciprocal |
|---|---|---|---|---|
| $f(x) = c$ | $f(x) = x$ | $f(x) = x^2$ | $f(x) = x^3$ | $f(x) = \frac{1}{x}$ |
| Reciprocal squared | Cube Root | Square Root | Absolute Value | |
| $f(x) = \frac{1}{x^2}$ | $f(x) = \sqrt[3]{x}$ | $f(x) = \sqrt{x}$ | $f(x) = \|x\|$ | |

**Solution**

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

**Analysis**

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 1.7.4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

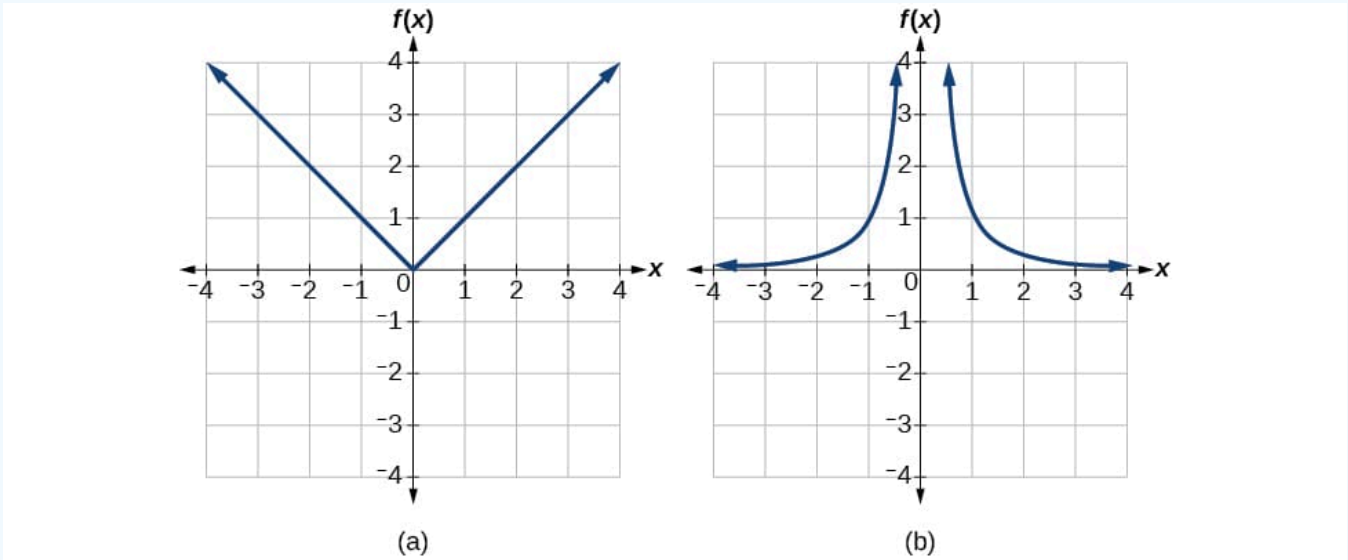



***Figure*** 1.7.4***:*** *(a) Absolute value (b) Reciprocal squared*

Try It 1.7.4: The domain of function $f$ is $(1, \infty)$ and the range of function $f$ is $(-\infty, -2)$. Find the domain and range of the inverse function.

**Solution**

The domain of function $f^{-1}$ is $(-\infty, -2)$ and the range of function $f^{-1}$ is $(1, \infty)$.

# Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

## Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

### Example 1.7.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 1.7.3, showing distance in miles that a car has traveled in $t$ minutes. Find and interpret $f^{-1}(70)$

*Table* 1.7.3

| $t$ **(minutes)** | 30 | 50 | 70 | 90 |
|---|---|---|---|---|
| $f(t)$ **(miles)** | 20 | 40 | 60 | 70 |

The inverse function takes an output of $f$ and returns an input for $f$. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function $f$, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value $a$ so that $f(a) = 70$. In this case, we are looking for a $t$ so that $f(t) = 70$, which is when $t = 90$.

### Exercise 1.7.5

Using Table 1.7.4, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

*Table* 1.7.4

| $t$ **(minutes)** | 30 | 50 | 60 | 70 | 90 |
|---|---|---|---|---|---|
| $f(t)$ **(miles)** | 20 | 40 | 50 | 60 | 70 |

*Answer available at the end of this section.*

## Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its

graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

**How To: Given the graph of a function, evaluate its inverse at specific points.**

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

## Example 1.7.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 1.7.5. Find $g(3)$ and $g^{-1}(3)$.

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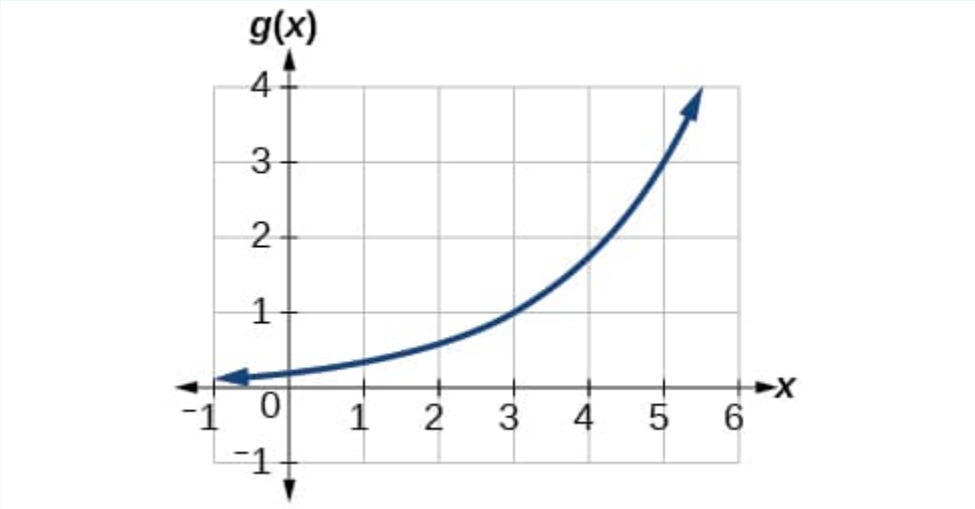



Figure 1.7.5: Graph of $g(x)$

**Solution**

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of $x$ for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 1.7.6.

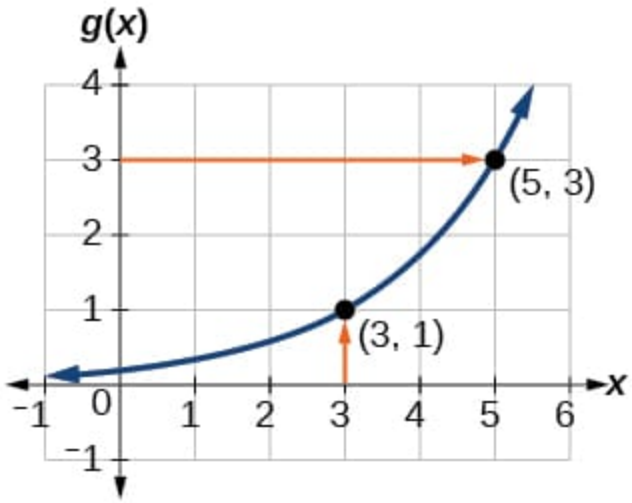



Figure 1.7.6: Graph of $g(x)$.

**Exercise 1.7.6**

Using the graph in Figure 1.7.6, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

*Answer a available at the end of this section.*

## Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, $y$ as a function of $x$— we can often find the inverse function by solving to obtain $x$ as a function of $y$.

**How To: Given a function represented by a formula, find the inverse.**

1. Make sure $f$ is a one-to-one function.
2. Solve for $x$
3. Interchange $x$ and $y$.

**Example 1.7.7: Inverting the Fahrenheit-to-Celsius Function**

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32) \tag{1.7.19}$$

**Solution**

$$C = \frac{5}{9}(F - 32) \tag{1.7.20}$$

$$C \cdot \frac{9}{5} = F - 32 \tag{1.7.21}$$

$$F = \frac{9}{5}C + 32 \tag{1.7.22}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32) \tag{1.7.23}$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32. \tag{1.7.24}$$

In this case, we introduced a function $h$ to represent the conversion because the input and output variables are descriptive, and writing $C^{-1}$ could get confusing.

## Exercise 1.7.7

Solve for $x$ in terms of $y$ given $y = \frac{1}{3}(x - 5)$

*Answer available at the end of this section.*

## Example 1.7.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

**Solution**

$$y = \frac{2}{x - 3 + 4} \quad \text{Set up an equation.} \tag{1.7.25}$$

$$y - 4 = \frac{2}{x - 3} \quad \text{Subtract 4 from both sides.} \tag{1.7.26}$$

$$x - 3 = \frac{2}{y - 4} \quad \text{Multiply both sides by x−3 and divide by y−4.} \tag{1.7.27}$$

$$x = \frac{2}{y - 4} + 3 \quad \text{Add 3 to both sides.} \tag{1.7.28}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

**Analysis**

The domain and range of $f$ exclude the values 3 and 4, respectively. $f$ and $f^{-1}$ are equal at two points but are not the same function, as we can see by creating Table 1.7.5.

*Table* 1.7.5

| $x$ | 1 | 2 | 5 | $f^{-1}(y)$ |
|---|---|---|---|---|
| $f(x)$ | 3 | 2 | 5 | $y$ |

## Example 1.7.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x - 4}$.

**Solution**

$$y = 2 + \sqrt{x - 4} \tag{1.7.29}$$

$$(y - 2)^2 = x - 4 \tag{1.7.30}$$

$$x = (y - 2)^2 + 4 \tag{1.7.31}$$

So $f^{-1}(x) = (x - 2)^2 + 4$.

The domain of $f$ is $[4, \infty)$. Notice that the range of $f$ is $[2, \infty)$, so this means that the domain of the inverse function $f^{-1}$ is also $[2, \infty)$

**Analysis**

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real $x$. However, $f^{-1}$ itself must have an inverse (namely, $f$ ) so we have to restrict the domain of $f^{-1}$ to $[2, \infty)$ in order to make $f^{-1}$ a one-to-one function. This domain of $f^{-1}$ is exactly the range of $f$.

**Exercise 1.7.8**

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

*Answer available at the end of this section.*

## Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure 1.7.7.

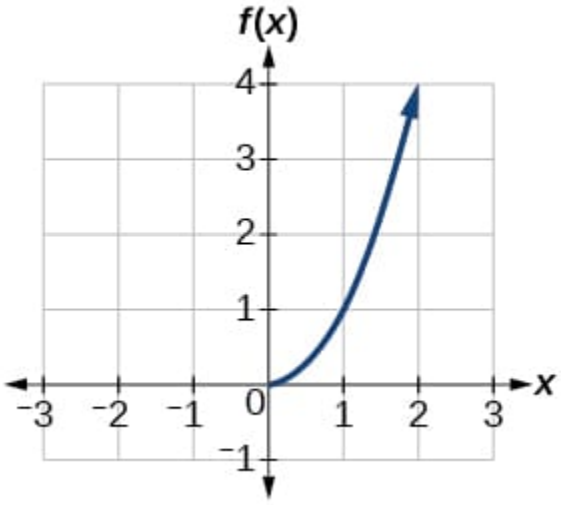



***Figure** 1.7.7**:** Quadratic function with domain restricted to $[0, \infty)$.*

**Restricting the domain** to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both $f$ and $f^{-1}$ on the same set of axes, using the x-axis for the input to both $f$ and $f^{-1}$?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 1.7.8**.**

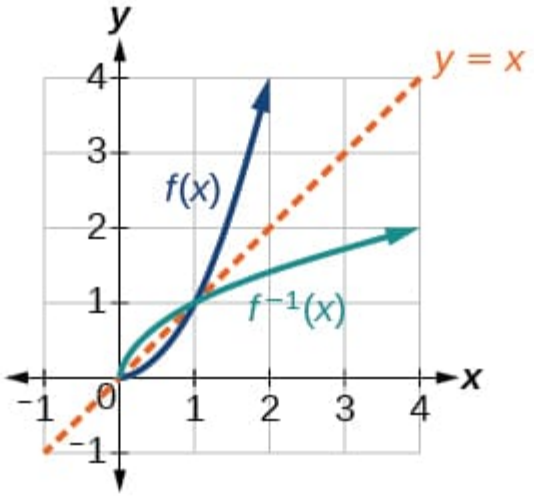



***Figure*** 1.7.8**:** *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

## Example 1.7.10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 1.7.9, sketch a graph of $f^{-1}(x)$.

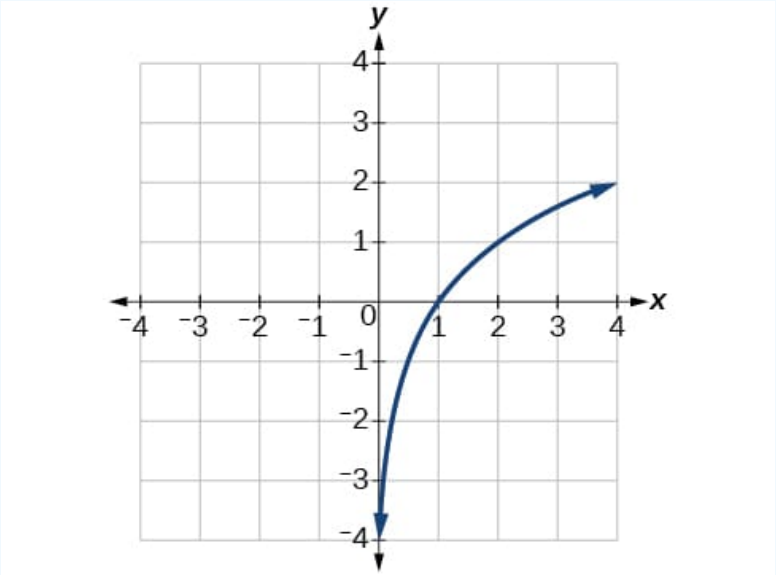



Figure 1.7.9: Graph of $f^(-1)(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 1.7.10.

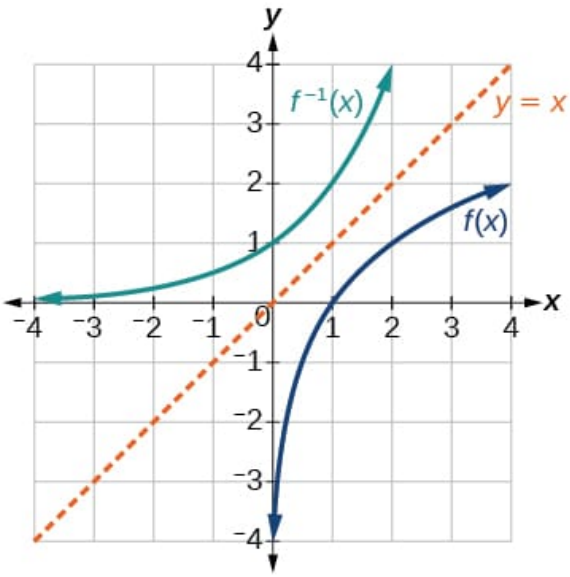



Figure 1.7.10: The function and its inverse, showing reflection about the identity line

**Exercise 1.7.1**

Draw graphs of the functions $f$ and $f^{-1}$ from Example 1.7.8.

*Answer available at the end of this section.*

**Is there any function that is equal to its own inverse?**

*Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because*

$$\frac{1}{\frac{1}{x}} = x \tag{1.7.32}$$

*Any function $f(x) = c - x$, where $c$ is a constant, is also equal to its own inverse.*

## Key Concepts

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$.
- Each of the toolkit functions has an inverse.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse.
- The inverse of a function can be determined at specific points on its graph.
- To find the inverse of a formula, solve the equation $y = f(x)$ for $x$ as a function of $y$. Then exchange the labels $x$ and $y$.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$.

### Answers

**Exercise** 1.7.1

$h(2) = 6$

**Exercise** 1.7.2

Yes

**Exercise** 1.7.3

Yes

**Exercise** 1.7.5

$f(60) = 50$. In 60 minutes, 50 miles are traveled.
$f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

**Exercise** 1.7.6

3

5.6

**Exercise** 1.7.7

$x = 3y + 5$

**Exercise** 1.7.8

$f^{-1}(x) = (2 - x)^2$; domain of $f$: $[0, \infty)$; domain of $f^{-1}$: $(-\infty, 2]$

**Exercise** 1.7.1

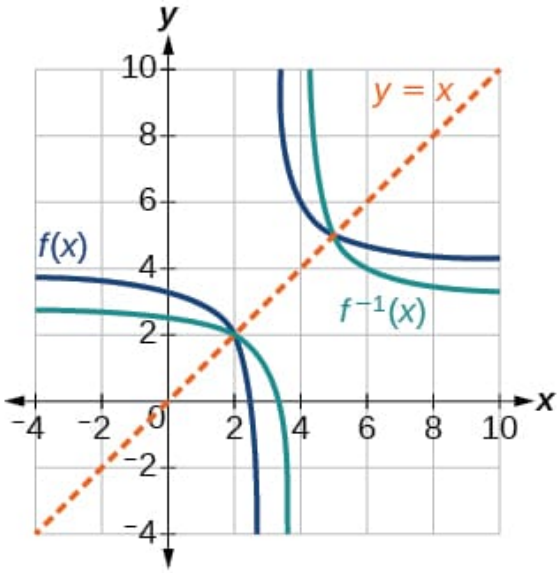



Figure 1.7.11: Graph of $f(x)$ and $f^{(}-1)(x)$.

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# 1.E: Functions (Exercises)

## 1.1: Functions and Function Notation

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

### Verbal

1) What is the difference between a relation and a function?

**Answer**

A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

2) What is the difference between the input and the output of a function?

3) Why does the vertical line test tell us whether the graph of a relation represents a function?

**Answer**

When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

4) How can you determine if a relation is a one-to-one function?

5) Why does the horizontal line test tell us whether the graph of a function is one-to-one?

**Answer**

When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

### Algebraic

**For the exercises 6-7, determine whether the relation represents a function.**

6) $\{(a,b),(c,d),(a,c)\}$

7) $\{(a,b),(b,c),(c,c)\}$

**Answer**

function

**For the exercises 8-26, determine whether the relation represents $y$ as a function of $x$.**

8) $5x+2y=10$

9) $y=x^2$

**Answer**

function

10) $x = y^2$

11) $3x^2 + y = 14$

**Answer**

function

12) $2x + y^2 = 6$

13) $y = -2x^2 + 40x$

**Answer**

function

14) $y = \dfrac{1}{x}$

15) $x = \dfrac{3y + 5}{7y - 1}$

**Answer**

function

16) $x = \sqrt{1 - y^2}$

17) $y = \dfrac{3x + 5}{7x - 1}$

**Answer**

function

18) $x^2 + y^2 = 9$

19) $2xy = 1$

**Answer**

function

20) $x = y^3$

21) $y = x^3$

**Answer**

function

22) $y = \sqrt{1 - x^2}$

23) $x = \pm\sqrt{1 - y}$

**Answer**

function

24) $y = \pm\sqrt{1-x}$

25) $y^2 = x^2$

**Answer**

not a function

26) $y^3 = x^2$

**For the exercises 27-39, evaluate the function $f$ at the indicated values $f(-3), f(2), f(-a), -f(a), f(a+h)$.**

27) $f(x) = 2x - 5$

**Answer**

$f(-3) = -11$; $f(2) = -1$; $f(-a) = -2a - 5$; $-f(a) = -2a + 5$; $f(a+h) = 2a + 2h - 5$

28) $f(x) = -5x^2 + 2x - 1$

29) $f(x) = \sqrt{2-x} + 5$

**Answer**

$f(-3) = \sqrt{5+5}$; $f(2) = 5$; $f(-a) = \sqrt{2+a+5}$; $-f(a) = -\sqrt{2-a} - 5$; $f(a+h) = \sqrt{2-a-h} + 5$

30) $f(x) = \dfrac{6x-1}{5x+2}$

31) $f(x) = |x-1| - |x+1|$

**Answer**

$f(-3) = 2$; $f(2) = 1 - 3 = -2$; $f(-a) = |-a-1| - |-a+1|$; $-f(a) = -|a-1| + |a+1|$; $f(a+h) = |a+h-1| - |a+h+1|$

32) Given the function $g(x) = 5 - x^2$, evaluate $g(x+h) - g(x)h$, $h \neq 0$.

33) Given the function $g(x) = x^2 + 2x$, evaluate $\dfrac{g(x) - g(a)}{x-a}$, $x \neq a$.

**Answer**

$\dfrac{g(x) - g(a)}{x-a} = x + a + 2$, $x \neq a$

34) Given the function $k(t) = 2t - 1$:

a. Evaluate $k(2)$.
b. Solve $k(t) = 7$.

35) Given the function $f(x) = 8 - 3x$:

a. Evaluate $f(-2)$.
b. Solve $f(x) = -1$.

**Answer**

a. $f(-2) = 14$; b. $x = 3$

36) Given the function $p(c) = c^2 + c$:

a. Evaluate $p(-3)$.
b. Solve $p(c) = 2$.

37) Given the function $f(x) = x^2 - 3x$:

a. Evaluate $f(5)$.
b. Solve $f(x) = 4$.

**Answer**

a. $f(5) = 10$; b. $x = -1$ or $x = 4$

38) Given the function $f(x) = \sqrt{x+2}$:

a. Evaluate $f(7)$.
b. Solve $f(x) = 4$.

39) Consider the relationship $3r + 2t = 18$.

a. Write the relationship as a function $r = f(t)$.
b. Evaluate $f(-3)$.
c. Solve $f(t) = 2$.

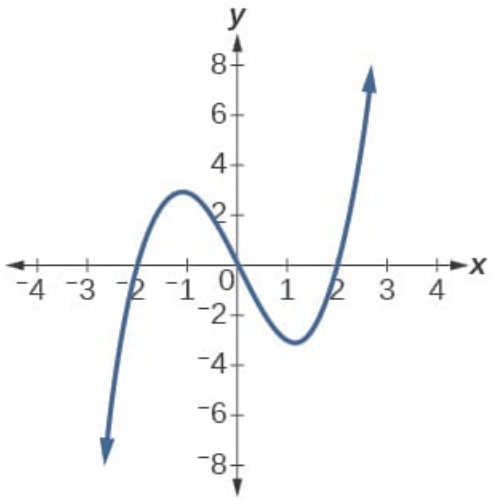
**Answer**

a. $f(t) = 6 - \frac{2}{3}t$; b. $f(-3) = 8$; c. $t = 6$

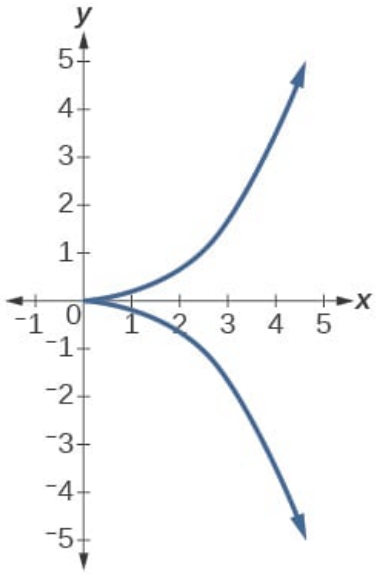
## Graphical

**For the exercises 40-54, use the vertical line test to determine which graphs show relations that are functions.**

40)



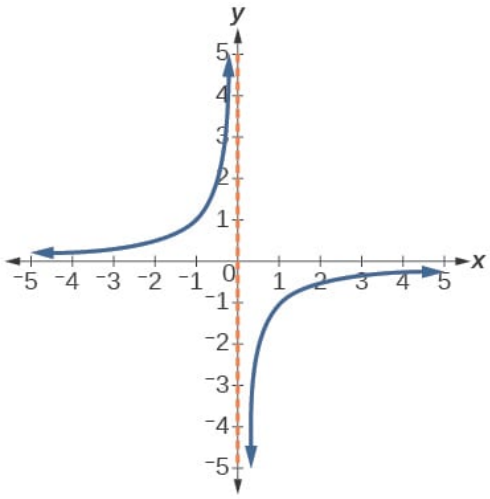

41)



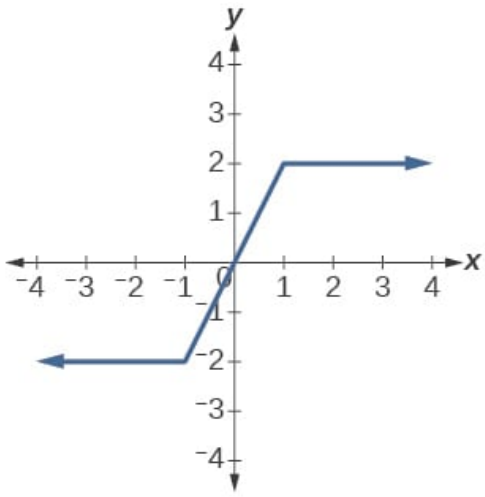

**Answer**

not a function

42)



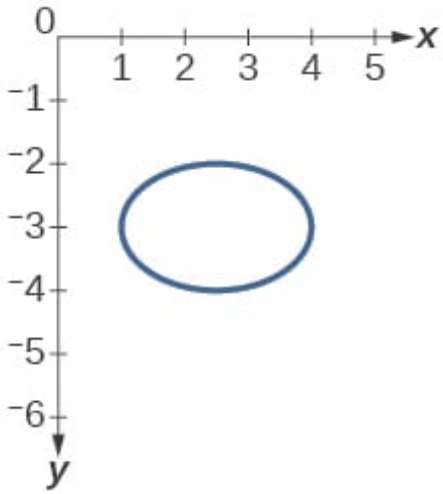

43)



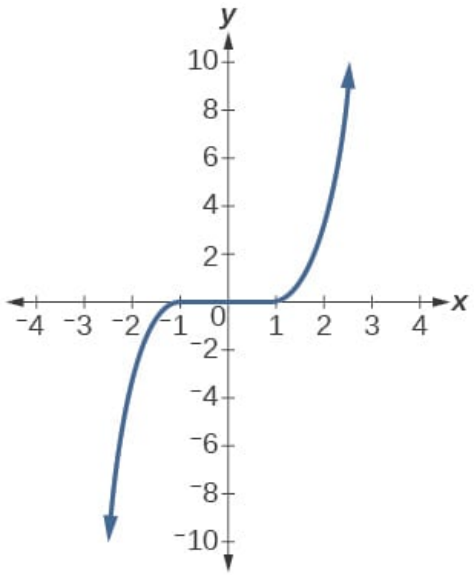

**Answer**

function

44)



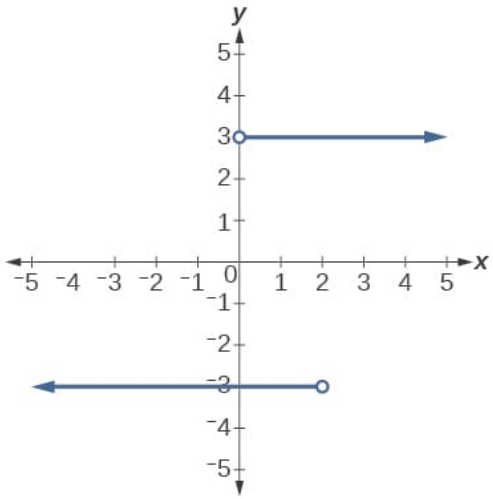

45)



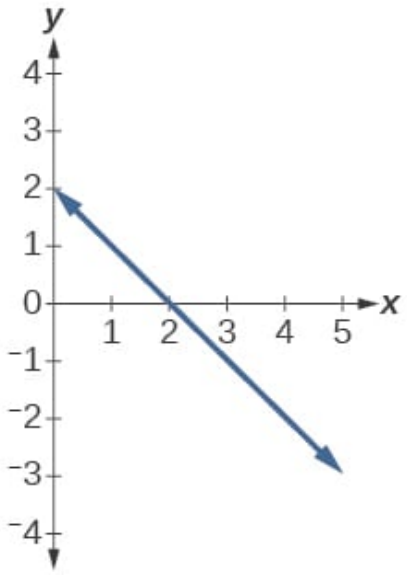

**Answer**

function

46)



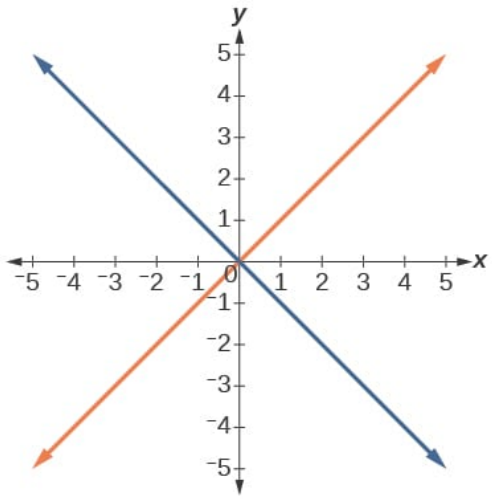

47)



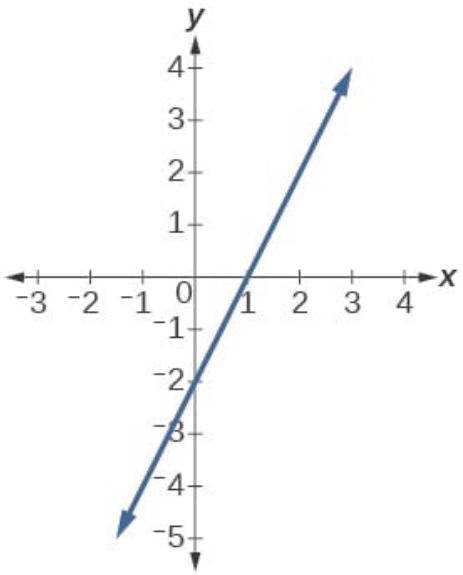

**Answer**

function

48)



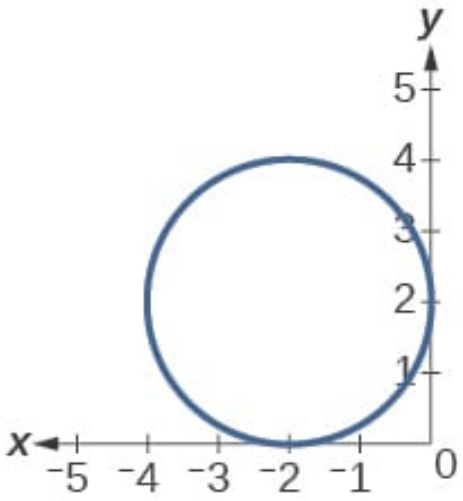

49)



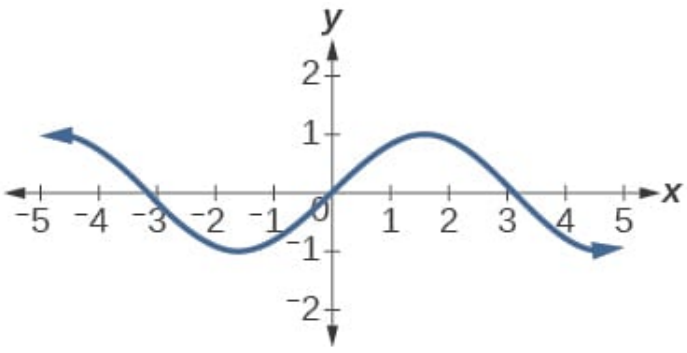

**Answer**

function

50)




51)

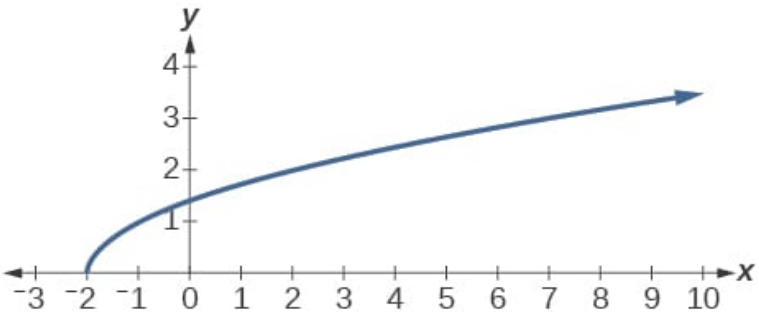


**Answer**

function

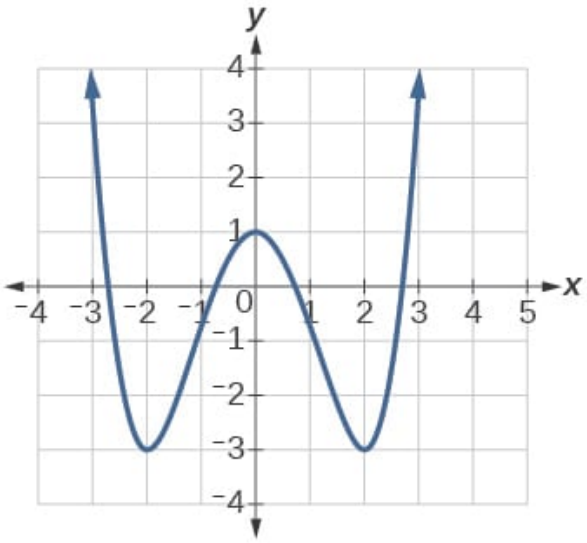
52) Given the following graph,

- Evaluate $f(-1)$.
- Solve for $f(x) = 3$.




53) Given the following graph,

- Evaluate $f(0)$.
- Solve for $f(x) = -3$.

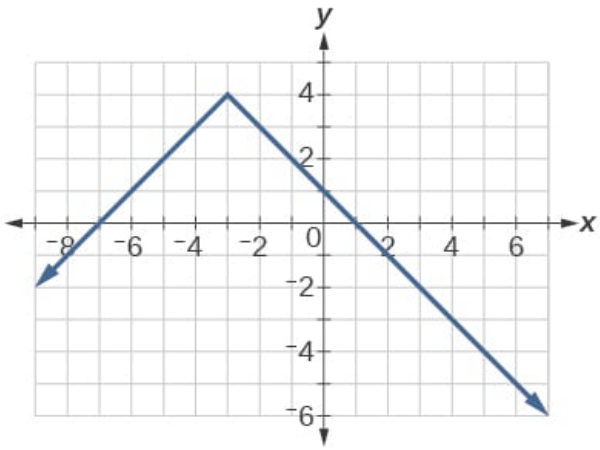



**Answer**

a.$f(0) = 1$; b. $f(x) = -3$, $x = -2$ or $x = 2$

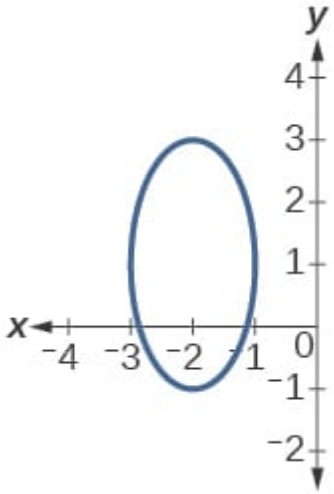
54) Given the following graph,

- Evaluate $f(4)$.
- Solve for $f(x) = 1$.




**For the exercises 55-59, determine if the given graph is a one-to-one function.**

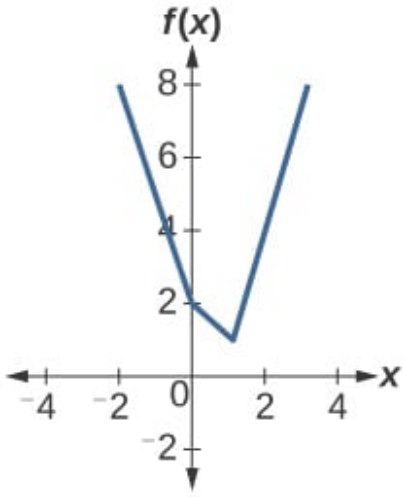
55)



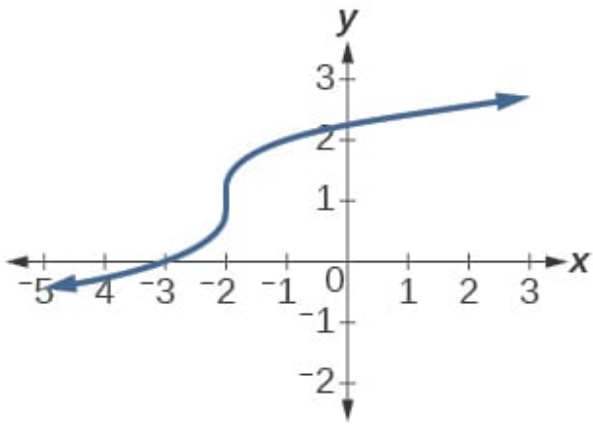

**Answer**

not a function so it is also not a one-to-one function

56)



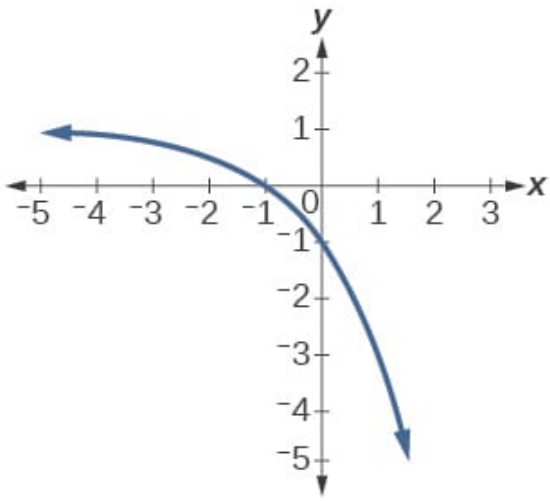

57)



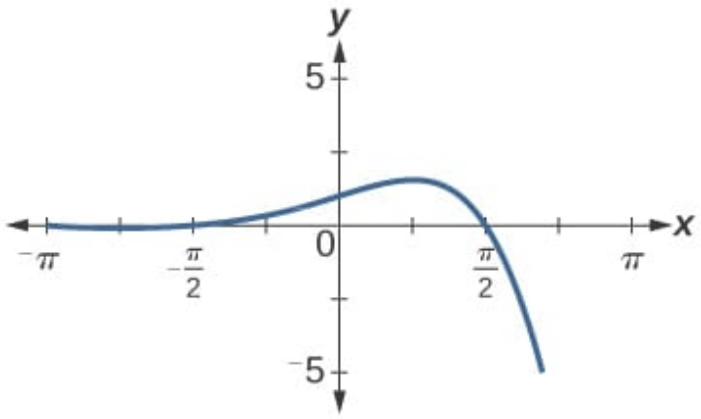

**Answer**

one-to-one function

58)




59)




**Answer**

function, but not one-to-one

## Numeric

**For the exercises 60-62, determine whether the relation represents a function.**

60) $\{(-1,-1),(-2,-2),(-3,-3)\}$

61) $\{(3,4),(4,5),(5,6)\}$

**Answer**

function

62) $(2,5),(7,11),(15,8),(7,9)\}$

**For the exercises 63-65, determine if the relation represented in table form represents $y$ as a function of $x$.**

63)

| $x$ | 5 | 10 | 15 |
|---|---|---|---|
| $y$ | 3 | 8 | 14 |

**Answer**

function

64)

| $x$ | 5 | 10 | 15 |
|---|---|---|---|
| $y$ | 3 | 8 | 8 |

65)

| $x$ | 5 | 10 | 10 |
|---|---|---|---|
| $y$ | 3 | 8 | 14 |

**Answer**

not a function

**For the exercises 66-67, use the function $f$ represented in the Table below**

| $x$ | $f(x)$ |
|---|---|
| 0 | 74 |
| 1 | 28 |
| 2 | 1 |
| 3 | 53 |
| 4 | 56 |
| 5 | 3 |
| 6 | 36 |
| 7 | 45 |
| 8 | 14 |
| 9 | 47 |

66) Evaluate $f(3)$.

67) Solve $f(x) = 1$.

**Answer**

$f(x) = 1, x = 2$

**For the exercises 68-73, evaluate the function $f$ at the values $f(-2)$, $f(-1)$, $f(0)$, $f(1)$,and $f(2)$.**

68) $f(x) = 4 - 2x$

69) $f(x) = 8 - 3x$

**Answer**

$f(-2) = 14; f(-1) = 11; f(0) = 8; f(1) = 5; f(2) = 2$

70) $f(x) = 8x^2 - 7x + 3$

71) $f(x) = 3 + \sqrt{x+3}$

**Answer**

$f(-2) = 4;\ f(-1) = 4.414;\ f(0) = 4.732;\ f(1) = 4.5;\ f(2) = 5.236$

72) $f(x) = \dfrac{x-2}{x+3}$

73) $f(x) = 3^x$

**Answer**

$f(-2) = \frac{1}{9};\ f(-1) = \frac{1}{3};\ f(0) = 1;\ f(1) = 3;\ f(2) = 9$

**For the exercises 74-75, evaluate the expressions, given functions $f$, $g$, and $h$:**

- $f(x) = 3x - 2$
- $g(x) = 5 - x^2$
- $h(x) = -2x^2 + 3x - 1$

74) $3f(1) - 4g(-2)$

75) $f\left(\frac{7}{3}\right) - h(-2)$

**Answer**

$20$

## Technology

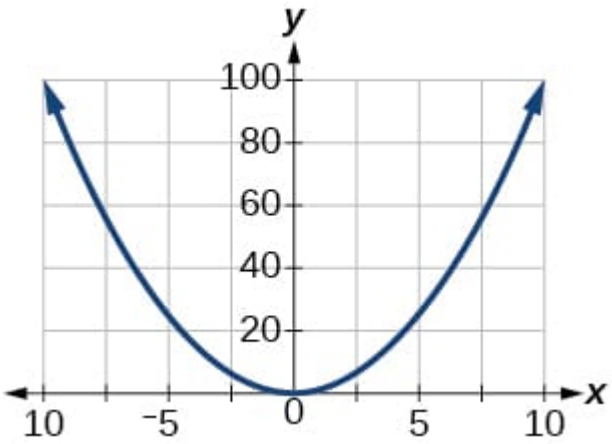
**For the exercises 76-78, graph $y = x^2$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.**

76) $[-0.1, 0.1]$

77) $[-10, 10]$

**Answer**

$[0, 100]$



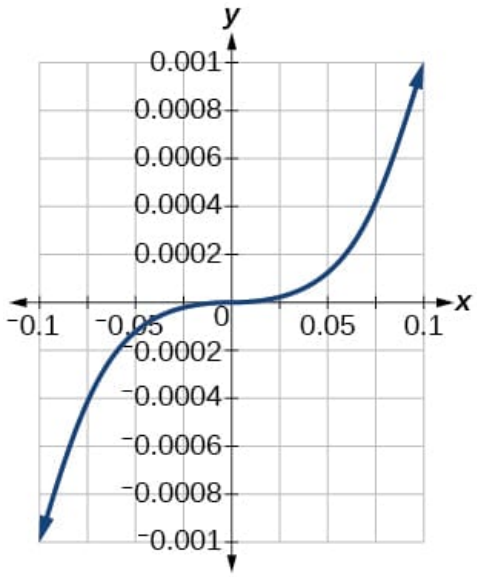

78) $[-100, 100]$

**For the exercises 79-81, graph $y = x^3$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.**

79) $[-0.1, 0.1]$

**Answer**

$[-0.001, 0.001]$

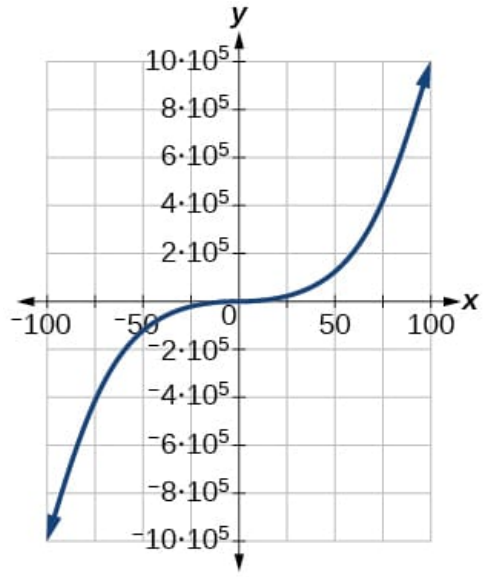



80) $[-10, 10]$

81) $[-100, 100]$

**Answer**

$[-1,000,000, 1,000,000]$



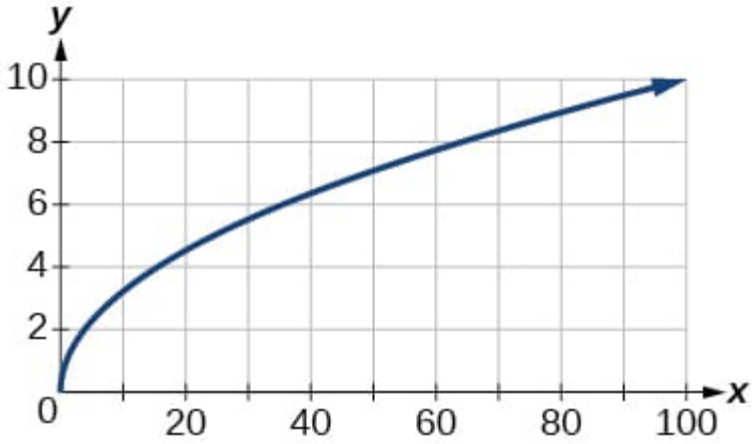

**For the exercises 82-84, graph $y = \sqrt{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.**

82) $[0, 0.01]$

83) $[0, 100]$

**Answer**

$[0, 10]$



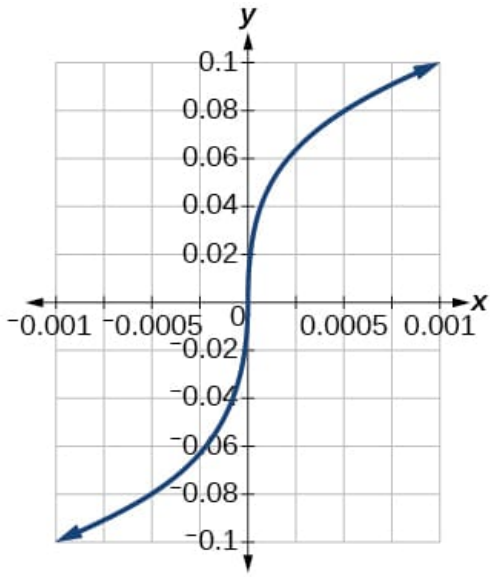

84) $[0, 10, 000]$

**For the exercises 85-87, graph $y = \sqrt[3]{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.**

85) $[-0.001, 0.001]$

**Answer**

$[-0.1, 0.1]$

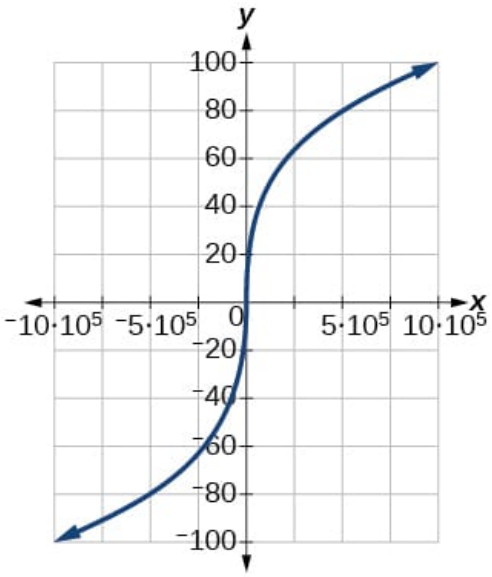



86) $[-1000, 1000]$

87) $[-1, 000, 000, 1, 000, 000]$

**Answer**

$[-100, 100]$




## Real-World Applications

88) The amount of garbage, $G$, produced by a city with population $p$ is given by $G = f(p)$. $G$ is measured in tons per week, and $p$ is measured in thousands of people.

The town of Tola has a population of $40,000$ and produces $13$ tons of garbage each week. Express this information in terms of the function $f$.Tola has a population of $40,000$ and produces $13$ tons of garbage each week. Express this information in terms of the function $f$.
Explain the meaning of the statement $f(5) = 2$.

89) The number of cubic yards of dirt, $D$, needed to cover a garden with area a square feet is given by $D = g(a)$.

a. A garden with area $5000\ ft^2$ requires $50\ yd^3$ of dirt. Express this information in terms of the function $g$.
b. Explain the meaning of the statement $g(100) = 1$.

**Answer**

a. $g(5000) = 50$; b. The number of cubic yards of dirt required for a garden of $100$ square feet is $1$.

90) Let $f(t)$ be the number of ducks in a lake $t$ years after 1990. Explain the meaning of each statement:

a. f(5)=30
b. f(10)=40

91) Let $h(t)$ be the height above ground, in feet, of a rocket $t$ seconds after launching. Explain the meaning of each statement:

a. $h(1) = 200$
b. $h(2) = 350$

**Answer**

a. The height of a rocket above ground after $1$ second is $200$ ft. b. the height of a rocket above ground after $2$ seconds is $350$ ft.

92) Show that the function $f(x) = 3(x-5)^2 + 7$ is not one-to-one.

# 1.2: Domain and Range

In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions.

## Verbal

1) Why does the domain differ for different functions?

**Answer**

The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

2) How do we determine the domain of a function defined by an equation?

3) Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.

**Answer**

There is no restriction on x for $f(x) = \sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x-values are restricted for $f(x) = \sqrt{x}$ to nonnegative numbers and the domain is $[0, \infty)$.

4) When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?

5) How do you graph a piecewise function?

**Answer**

Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints with an open circle. Use an arrow to indicate $-\infty$ or $\infty$. Combine the graphs to find the graph of the piecewise function.

## Algebraic

**For the exercises 6-25, find the domain of each function using interval notation.**

6) $f(x) = -2x(x-1)(x-2)$

7) $f(x) = 5 - 2x^2$

**Answer**

$(-\infty, \infty)$

8) $f(x) = 3\sqrt{x-2}$

9) $f(x) = 3 - \sqrt{6-2x}$

**Answer**

$(-\infty, 3]$

10) $f(x) = \sqrt{4-3x}$

11) $f(x) = \sqrt{x^2 + 4}$

**Answer**

$$(-\infty, \infty)$$

12) $f(x) = \sqrt[3]{1 - 2x}$

13) $f(x) = \sqrt[3]{x - 1}$

**Answer**

$$(-\infty, \infty)$$

14) $f(x) = \dfrac{9}{x - 6}$

15) $f(x) = \dfrac{3x + 1}{4x + 2}$

**Answer**

$$\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$$

16) $f(x) = \dfrac{\sqrt{x + 4}}{x - 4}$

17) $f(x) = \dfrac{x - 3}{x^2 + 9x - 22}$

**Answer**

$$(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$$

18) $f(x) = \dfrac{1}{x^2 - x - 6}$

19) $f(x) = \dfrac{2x^3 - 250}{x^2 - 2x - 15}$

**Answer**

$$(-\infty, -3) \cup (-3, 5) \cup (5, \infty)$$

20) $\dfrac{5}{\sqrt{x - 3}}$

21) $\dfrac{2x + 1}{\sqrt{5 - x}}$

**Answer**

$$(-\infty, 5)$$

22) $\dfrac{\sqrt{x - 4}}{\sqrt{x - 6}}$

23) $\dfrac{\sqrt{x - 6}}{\sqrt{x - 4}}$

**Answer**

$[6, \infty)$

24) $f(x) = \dfrac{x}{x}$

25) $f(x) = \dfrac{x^2 - 9x}{x^2 - 81}$

**Answer**

$(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$

26) Find the domain of the function $f(x) = \sqrt{2x^3 - 50x}$ by:

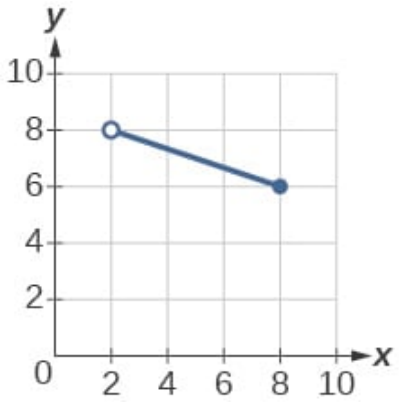
a. using algebra

b. graphing the function in the radicand and determining intervals on the x-axis for which the radicand is nonnegative.

## Graphical

**For the exercises 27-37, write the domain and range of each function using interval notation.**

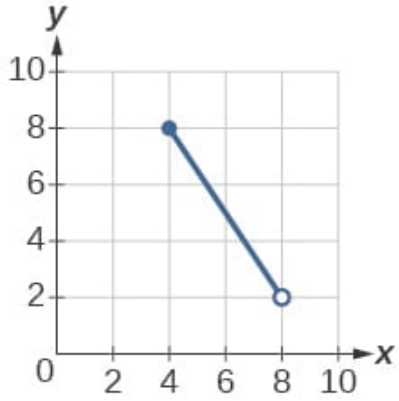
27)



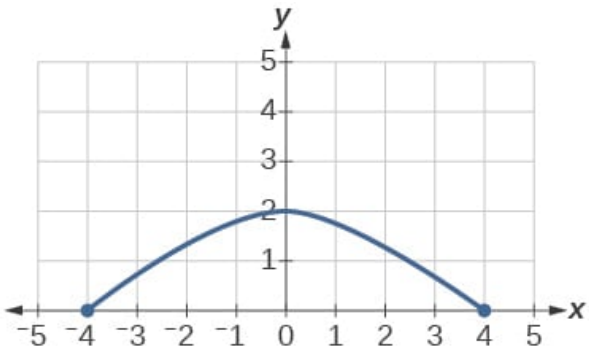

**Answer**

domain: $(2, 8]$, range $[6, 8)$

28)



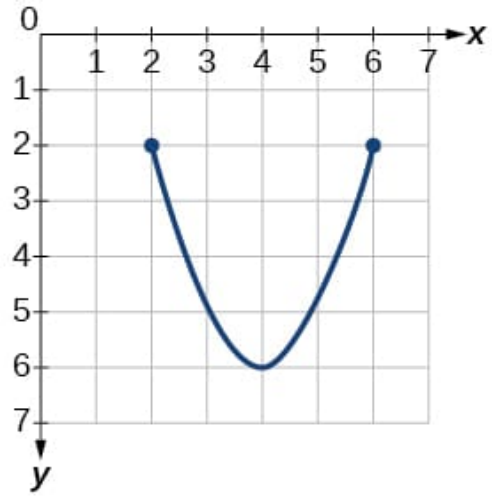
29)



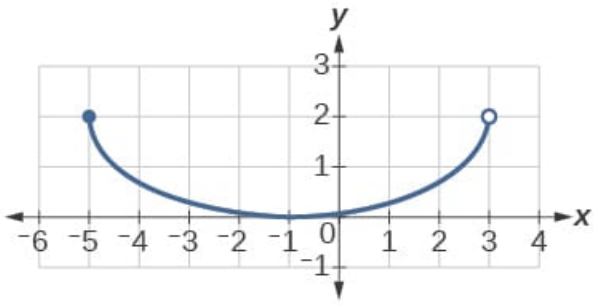

**Answer**

domain: $[-4, 4]$, range: $[0, 2]$

30)



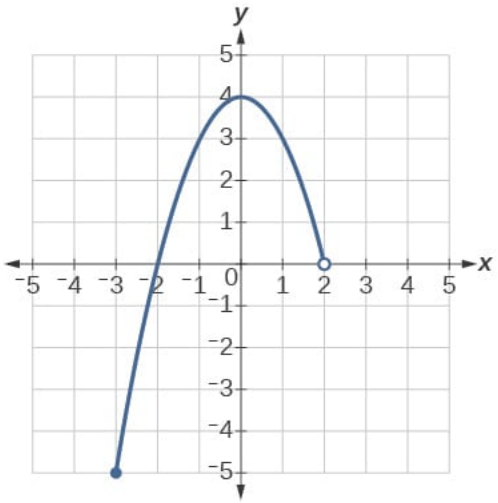

31)



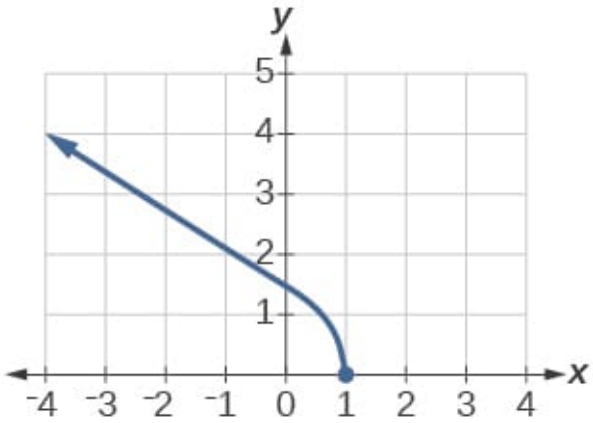

**Answer**

domain: $[-5, 3)$, range: $[0, 2]$

32)



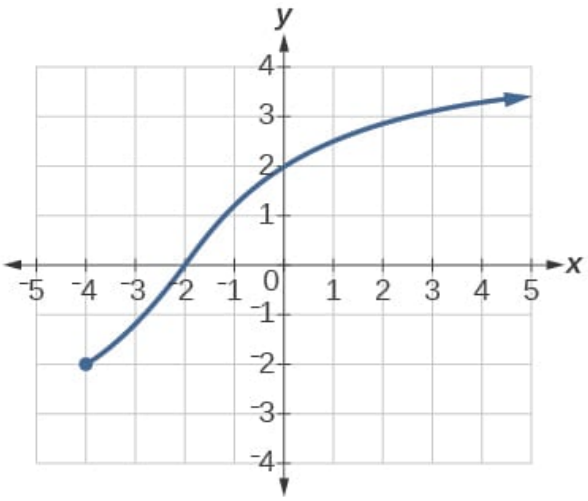

33)



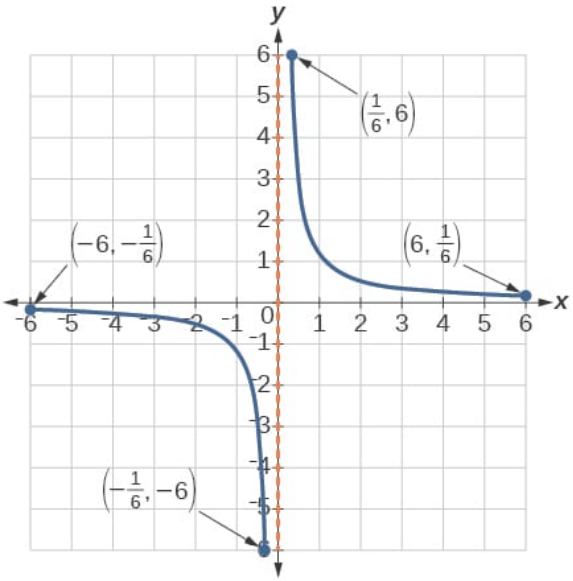

**Answer**

domain: $(-\infty, 1]$, range: $[0, \infty)$

34)



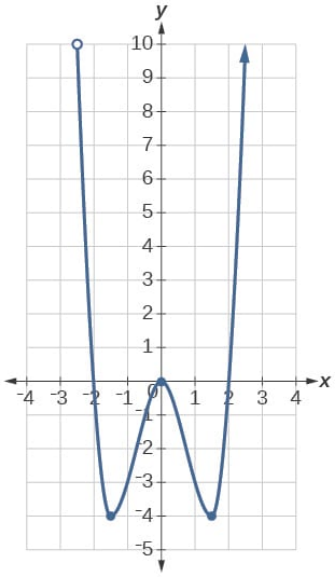

35)



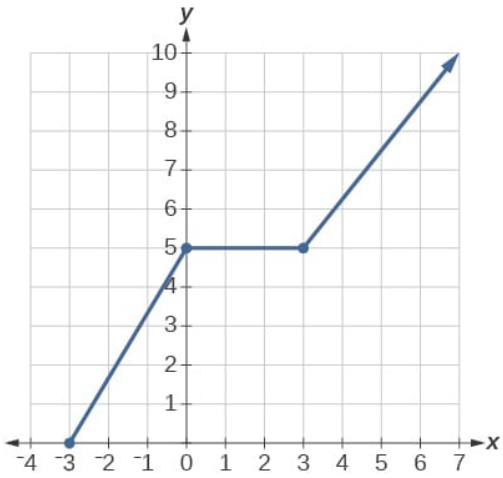

**Answer**

domain: $[-6, -16] \cup [16, 6]$; range: $[-6, -16] \cup [16, 6]$

36)




37)




**Answer**

domain: $[-3, \infty)$; range: $[0, \infty)$

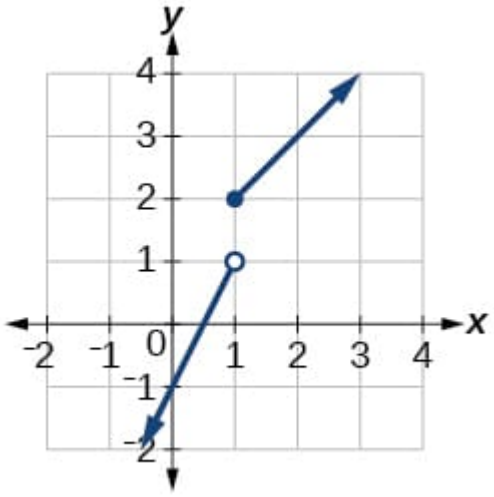
**For the exercises 38-45, sketch a graph of the piecewise function. Write the domain in interval notation.**

38) $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

39) $f(x) = \begin{cases} 2x-1 & \text{if } x < 1 \\ 1+x & \text{if } x \geq 1 \end{cases}$

**Answer**

domain: $(-\infty, \infty)$

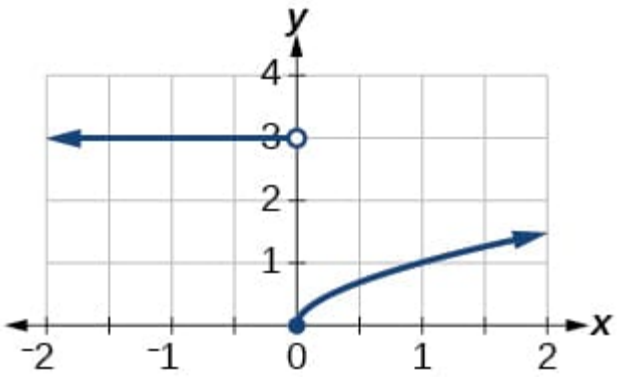



40) $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x-1 & \text{if } x > 0 \end{cases}$

41) $f(x) = \begin{cases} 3 & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$

**Answer**

domain: $(-\infty, \infty)$

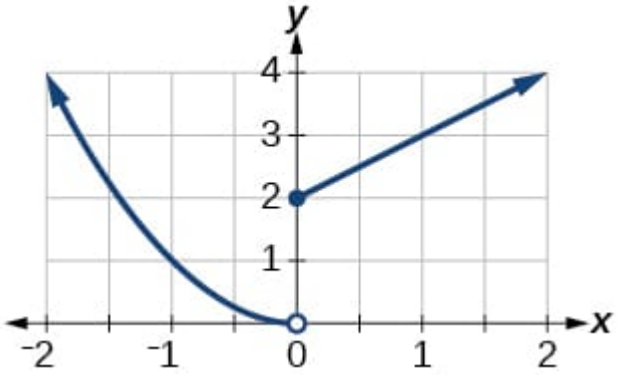



42) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 - x & \text{if } x > 0 \end{cases}$

43) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x + 2 & \text{if } x \geq 0 \end{cases}$

**Answer**

domain: $(-\infty, \infty)$

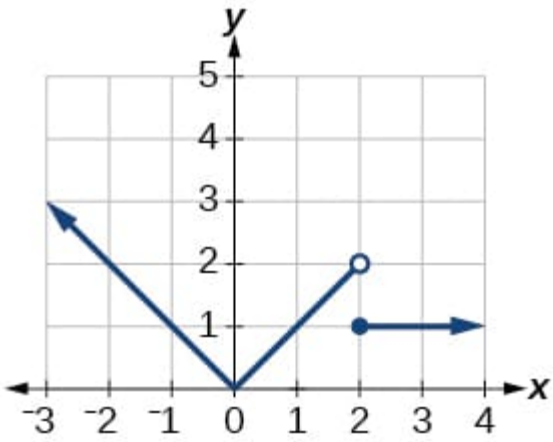



44) $f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ x^3 & \text{if } x \geq 1 \end{cases}$

45) $f(x) = \begin{cases} |x| & \text{if } x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$

**Answer**

domain: $(-\infty, \infty)$




## Numeric

**For the exercises 46-48, given each function $f$,evaluate $f(-3)$, $f(-2)$, $f(-1)$, and $f(0)$.**

46) $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

47) $f(x) = \begin{cases} 1 & \text{if } x \leq -3 \\ 0 & \text{if } x > -3 \end{cases}$

**Answer**

$f(-3) = 1;\ f(-2) = 0;\ f(-1) = 0;\ f(0) = 0$

48) $f(x) = \begin{cases} -2x^2+3 & \text{if } x \leq -1 \\ 5x-7 & \text{if } x > -1 \end{cases}$

**For the exercises 49-51, given each function $f$, evaluate $f(-1)$, $f(0)$, $f(2)$, and $f(4)$.**

49) $f(x) = \begin{cases} 7x+3 & \text{if } x < 0 \\ 7x+6 & \text{if } x \geq 0 \end{cases}$

**Answer**

$f(-1) = -4;\ f(0) = 6;\ f(2) = 20;\ f(4) = 34$

50) $f(x) = \begin{cases} x^2-2 & \text{if } x < 2 \\ 4+|x-5| & \text{if } x \geq 2 \end{cases}$

51) $f(x) = \begin{cases} 5x & \text{if } x < 0 \\ 3 & \text{if } 0 \geq x \leq 2 \\ x^2 & \text{if } x > 3 \end{cases}$

**Answer**

$f(-1) = -5;\ f(0) = 3;\ f(2) = 3;\ f(4) = 16$

**For the exercises 52-54, write the domain for the piecewise function in interval notation.**

52) $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

53) $f(x) = \begin{cases} x^2-2 & \text{if } x < 1 \\ -x^2+2 & \text{if } x > 1 \end{cases}$

**Answer**

domain: $(-\infty, 1) \cup (1, \infty)$

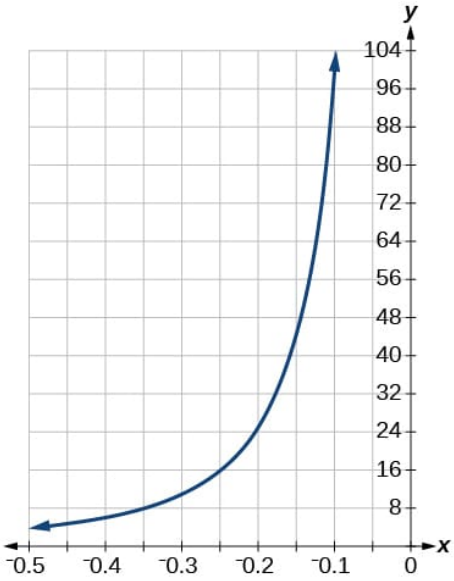
54) $f(x) = \begin{cases} x^2 - 3 & \text{if } x < 0 \\ -3x^2 & \text{if } x \geq 2 \end{cases}$

## Technology

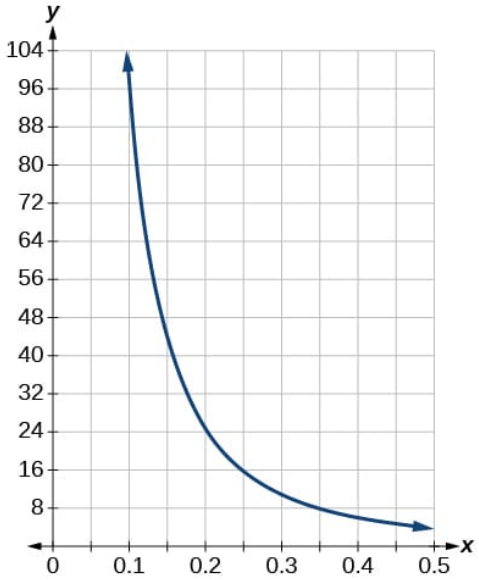
55) Graph $y = \frac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

**Answer**

window: $[-0.5, -0.1]$; range: $[4, 100]$




window: $[0.1, 0.5]$; range: $[4, 100]$



56) Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

## Extension

57) Suppose the range of a function $f$ is $[-5, 8]$. What is the range of $|f(x)|$?

**Answer**

$[0, 8]$

58) Create a function in which the range is all nonnegative real numbers.

59) Create a function in which the domain is $x > 2$.

**Answer**

Many answers. One function is $f(x) = \frac{1}{\sqrt{x-2}}$.

## Real-World Applications

60) The height $h$ of a projectile is a function of the time $t$ it is in the air. The height in feet for $t$ seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

**Answer**

The domain is $[0, 6]$; it takes 6 seconds for the projectile to leave the ground and return to the ground

61) The cost in dollars of making $x$ items is given by the function $C(x) = 10x + 500$.

a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.

b. What is the cost of making $25$ items?

c. Suppose the maximum cost allowed is $\$1500$. What are the domain and range of the cost function, $C(x)$?

# 1.3: Rates of Change and Behavior of Graphs

In this section, we will investigate changes in functions. For example, a rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. Identifying points that mark the interval on a graph can be used to find the average rate of change. Comparing pairs of input and output values in a table can also be used to find the average rate of change.

## Verbal

1) Can the average rate of change of a function be constant?

**Answer**

Yes, the average rate of change of all linear functions is constant.

2) If a function $f$ is increasing on $(a, b)$ and decreasing on $(b, c)$, then what can be said about the local extremum of $f$ on $(a, c)$?

3) How are the absolute maximum and minimum similar to and different from the local extrema?

**Answer**

The absolute maximum and minimum relate to the entire graph, whereas the local extrema relate only to a specific region around an open interval.

4) How does the graph of the absolute value function compare to the graph of the quadratic function, $y = x^2$, in terms of increasing and decreasing intervals?

## Algebraic

**For the exercises 5-15, find the average rate of change of each function on the interval specified for real numbers $b$ or $h$.**

5) $f(x) = 4x^2 - 7$ on $[1, b]$

**Answer**

$$4(b + 1)$$

6) $g(x) = 2x^2 - 9$ on $[4, b]$

7) $p(x) = 3x + 4$ on $[2, 2 + h]$

**Answer**

$$3$$

8) $k(x) = 4x - 2$ on $[3, 3 + h]$

9) $f(x) = 2x^2 + 1$ on $[x, x + h]$

**Answer**

$$4x + 2h$$

10) $g(x) = 3x^2 - 2$ on $[x, x + h]$

11) $a(t) = \dfrac{1}{t + 4}$ on $[9, 9 + h]$

**Answer**

$$\frac{-1}{13(13 + h)}$$

12) $b(x) = 1x + 3$ on $[1, 1 + h]$

13) $j(x) = 3x^3$ on $[1, 1 + h]$

**Answer**

$$3h^2 + 9h + 9$$

14) $r(t) = 4t^3$ on $[2, 2 + h]$

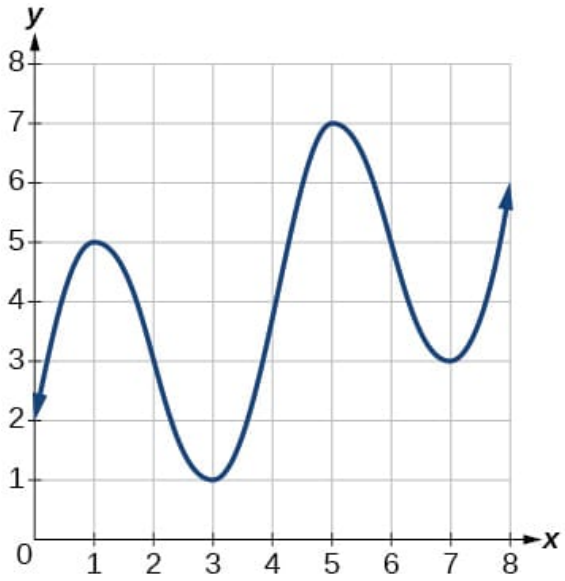
15) $\dfrac{f(x + h) - f(x)}{h}$ given $f(x) = 2x^2 - 3x$ on $[x, x + h]$

**Answer**

$$4x + 2h - 3$$

## Graphical

**For the exercises 16-17, consider the graph of $f$ shown in the Figure below**




*Graph of a polynomial.*

16) Estimate the average rate of change from $x = 1$ to $x = 4$.

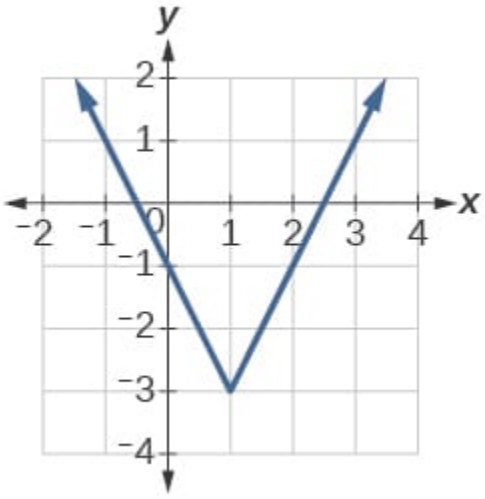
17) Estimate the average rate of change from $x = 2$ to $x = 5$.

**Answer**

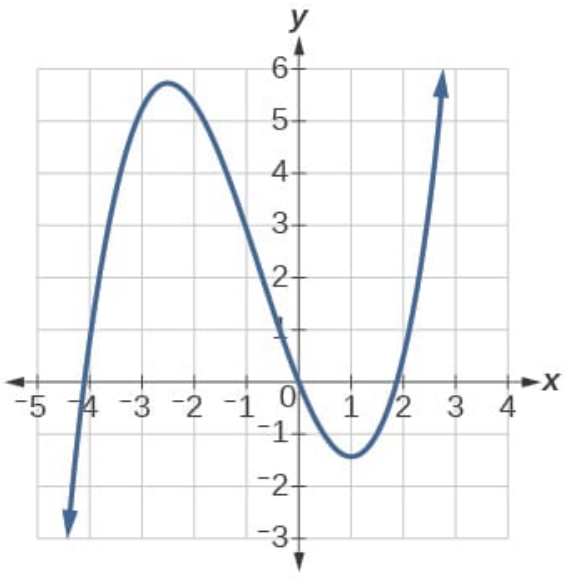
$$\frac{4}{3}$$

**For the following exercises, use the graph of each function to estimate the intervals on which the function is increasing or decreasing.**

18)



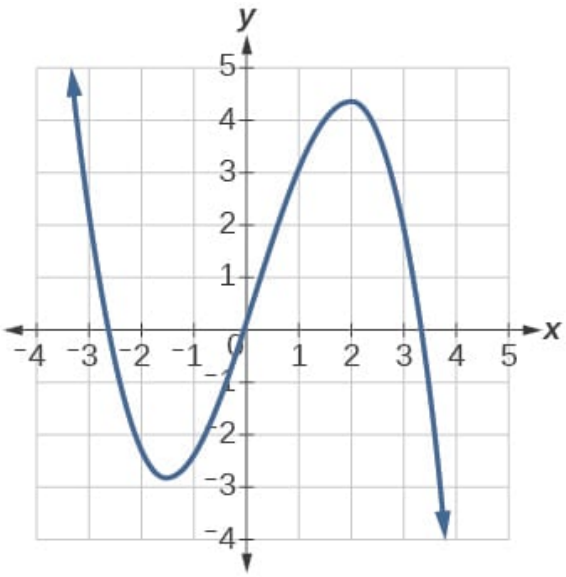

19)



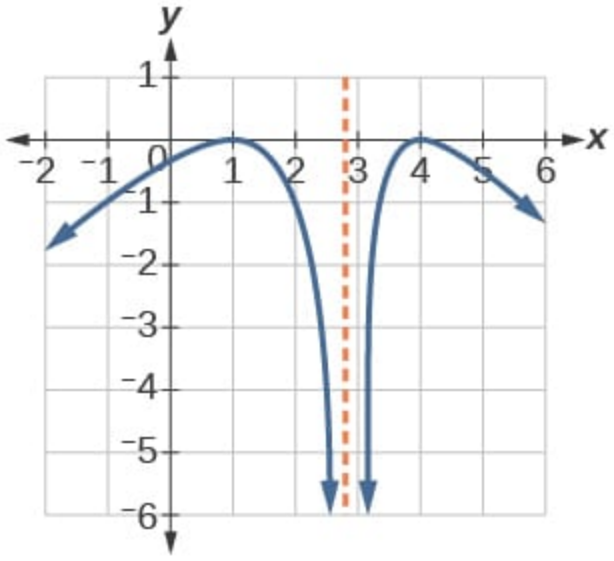

**Answer**

increasing on $(-\infty, -2.5) \cup (1, \infty)$, decreasing on $(-2.5, 1)$

20)



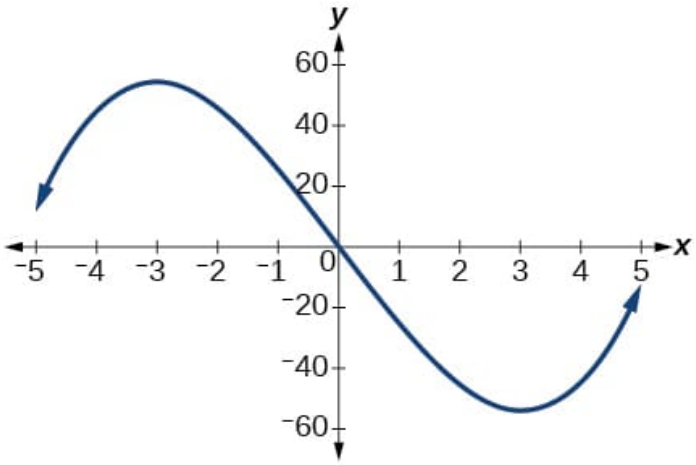

21)




**Answer**

increasing on $(-\infty, 1) \cup (3, 4)$, decreasing on $(1, 3) \cup (4, \infty)$

**For the exercises 22-23, consider the graph shown in the Figure below.**




*Graph of a cubic function.*

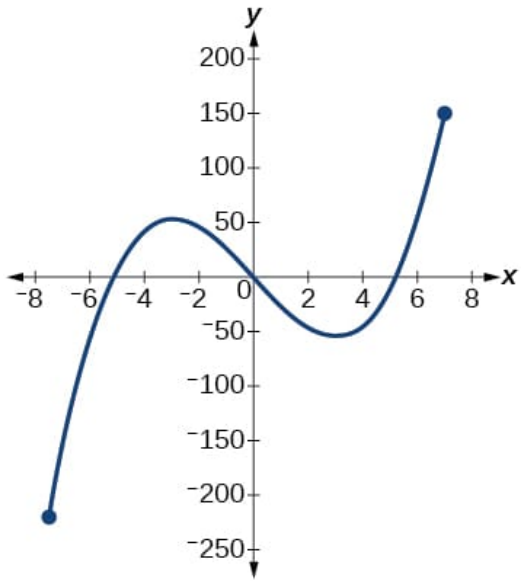
22) Estimate the intervals where the function is increasing or decreasing.

23) Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

**Answer**

local maximum: $(-3, 60)$, local minimum: $(3, -60)$

**For the exercises 24-25, consider the graph in the Figure below.**




*Graph of a cubic function.*

24) If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.

25) If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

**Answer**

absolute maximum at approximately $(7, 150)$, absolute minimum at approximately $(-7.5, -220)$

## Numeric

26) The Table below gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (a) between 2001 and 2002, and (b) between 2001 and 2004?

| Year | Sales (millions of dollars) |
|---|---|
| 1998 | 201 |
| 1999 | 219 |
| 2000 | 233 |
| 2001 | 243 |
| 2002 | 249 |
| 2003 | 251 |
| 2004 | 249 |
| 2005 | 243 |
| 2006 | 233 |

27) The Table below gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of population (a) between 2002 and 2004, and (b) between 2002 and 2006?

| Year | Population (thousands) |
|---|---|
| 2000 | 87 |

| Year | Population (thousands) |
|---|---|
| 2001 | 84 |
| 2002 | 83 |
| 2003 | 80 |
| 2004 | 77 |
| 2005 | 76 |
| 2006 | 78 |
| 2007 | 81 |
| 2008 | 85 |

**Answer**

a. $-3000$; b. $-1250$

**For the exercises 28-34, find the average rate of change of each function on the interval specified.**

28) $f(x) = x^2$ on $[1, 5]$

29) $h(x) = 5 - 2x^2$ on $[-2, 4]$

**Answer**

$-4$

30) $q(x) = x^3$ on $[-4, 2]$

31) $g(x) = 3x^3 - 1$ on $[-3, 3]$

**Answer**

$27$

32) $y = \dfrac{1}{x}$ on $[1, 3]$

33) $p(t) = \dfrac{(t^2 - 4)(t + 1)}{t^2 + 3}$ on $[-3, 1]$

**Answer**

$-0.167$

34) $k(t) = 6t^2 + \dfrac{4}{t^3}$ on $[-1, 3]$

## Technology

**For the exercises 35-40, use a graphing utility to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.**

35) $f(x) = x^4 - 4x^3 + 5$

**Answer**

Local minimum at $(3, -22)$, decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$

36) $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

37) $g(t) = t\sqrt{t+3}$

**Answer**

Local minimum at $(-2, -2)$, decreasing on $(-3, -2)$, increasing on $(-2, \infty)$

38) $k(t) = 3t^{\frac{2}{3}} - t$

39) $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

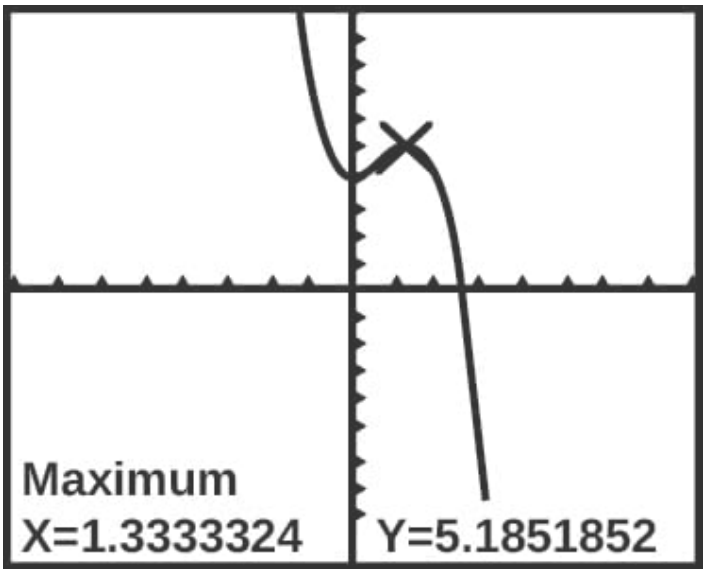
**Answer**

Local maximum at $(-0.5, 6)$, local minima at $(-3.25, -47)$ and $(2.1, -32)$, decreasing on $(-\infty, -3.25)$ and $(-0.5, 2.1)$, increasing on $(-3.25, -0.5)$ and $(2.1, \infty)$

40) $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

## Extension

41) The graph of the function $f$ is shown in the Figure below:




*Graph of f(x) on a graphing calculator.*

Based on the calculator screen shot, the point $(1.333, 5.185)$ is which of the following?

a. a relative (local) maximum of the function
b. the vertex of the function
c. the absolute maximum of the function
d. a zero of the function

**Answer**

a

42) Let $f(x) = \frac{1}{x}$. Find a number $c$ such that the average rate of change of the function f on the interval $(1, c)$ is $-\frac{1}{4}$.

43) Let $f(x) = \frac{1}{x}$. Find the number $b$ such that the average rate of change of f on the interval $(2, b)$ is $-\frac{1}{10}$.

**Answer**

$b = 5$

## Real-World Applications

44) At the start of a trip, the odometer on a car read $21,395$. At the end of the trip, $13.5$ hours later, the odometer read $22,125$. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?

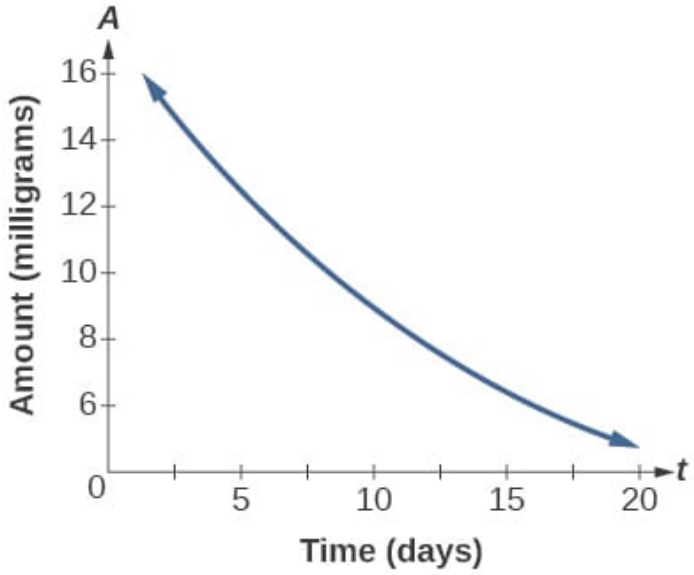
45) A driver of a car stopped at a gas station to fill up his gas tank. He looked at his watch, and the time read exactly 3:40 p.m. At this time, he started pumping gas into the tank. At exactly 3:44, the tank was full and he noticed that he had pumped $10.7$ gallons. What is the average rate of flow of the gasoline into the gas tank?

**Answer**

$2.7$ gallons per minute

46) Near the surface of the moon, the distance that an object falls is a function of time. It is given by $d(t) = 2.6667t^2$, where $t$ is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.

47) The graph in the Figure below illustrates the decay of a radioactive substance over $t$ days.




*Graph of an exponential function.*

Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

**Answer**

approximately $-0.6$ milligrams per day

# 1.4: Composition of Functions

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year.

The cost depends on the temperature, and the temperature depends on the day. By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

## Verbal

1) How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

**Answer**

Find the numbers that make the function in the denominator $g$ equal to zero, and check for any other domain restrictions on $f$ and $g$, such as an even-indexed root or zeros in the denominator.

2) What is the composition of two functions, $f \circ g$?

3) If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

**Answer**

Yes. Sample answer: Let $f(x) = x + 1$ and $g(x) = x - 1$. Then $f(g(x)) = f(x - 1) = (x - 1) + 1 = x$ and $g(f(x)) = g(x + 1) = (x + 1) - 1 = x$. So $f \circ g = g \circ f$.

4) How do you find the domain for the composition of two functions, $f \circ g$?

## Algebraic

5) Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$,find $f + g$, $f - g$, $fg$, and $\frac{f}{g}$.Determine the domain for each function in interval notation.

**Answer**

$(f + g)(x) = 2x + 6$, domain: $(-\infty, \infty)$

$(f - g)(x) = 2x^2 + 2x - 6$, domain: $(-\infty, \infty)$

$(fg)(x) = -x^4 - 2x^3 + 6x^2 + 12x$, domain: $(-\infty, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{x^2 + 2x}{6 - x^2}$, domain: $(-\infty, -\sqrt{6}) \cup (\sqrt{6}, \sqrt{6}) \cup (\sqrt{6}, \infty)$

6) Given $f(x) = -3x^2 + x$ and $g(x) = 5$, find $f + g$, $f - g$, $fg$, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

7) Given $f(x) = 2x^2 + 4x$ and $g(x) = \frac{1}{2x}$,find $f + g$, $f - g$, $fg$, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

**Answer**

$(f + g)(x) = \frac{4x^3 + 8x^2 + 1}{2x}$, domain: $(-\infty, 0) \cup (0, \infty)$

$(f - g)(x) = \frac{4x3 + 8x2 - 1}{2x}$, domain: $(-\infty, 0) \cup (0, \infty)$

$(fg)(x) = x + 2$, domain: $(-\infty, 0) \cup (0, \infty)$

$(fg)(x) = 4x^3 + 8x^2$, domain: $(-\infty, 0) \cup (0, \infty)$

8) Given $f(x) = \frac{1}{x - 4}$ and $g(x) = \frac{1}{6 - x}$, find$f + g$, $f - g$, $fg$, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

9) Given $f(x) = 3x^2$ and $g(x) = \sqrt{x-5}$, find $f+g$, $f-g$, $fg$, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

**Answer**

$(f+g)(x) = 3x^2 + \sqrt{x-5}$, domain: $[5, \infty)$

$(f-g)(x) = 3x^2 - \sqrt{x-5}$, domain: $[5, \infty)$

$(fg)(x) = 3x^2\sqrt{x-5}$, domain: $[5, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{3x^2}{\sqrt{x-5}}$, domain: $(5, \infty)$

10) Given $f(x) = \sqrt{x}$ and $g(x) = |x-3|$, find $\frac{g}{f}$. Determine the domain of the function in interval notation.

11) Given $f(x) = 2x^2 + 1$ and $g(x) = 3x - 5$, find the following:

a. $f(g(2))$
b. $f(g(x))$
c. $g(f(x))$
d. $(g \circ g)(x)$
e. $(f \circ f)(-2)$

**Answer**

a. $3$
b. $f(g(x)) = 2(3x-5)^2 + 1$
c. $f(g(x)) = 6x^2 - 2$
d. $(g \circ g)(x) = 3(3x-5) - 5 = 9x - 20$
e. $(f \circ f)(-2) = 163$

**For the exercises 12-17, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.**

12) $f(x) = x^2 + 1$, $g(x) = \sqrt{x+2}$

13) $f(x) = \sqrt{x} + 2$, $g(x) = x^2 + 3$

**Answer**

$f(g(x)) = \sqrt{x+3}$, $g(f(x)) = x + 4\sqrt{x} + 7$

14) $f(x) = |x|$, $g(x) = 5x + 1$

15) $f(x) = \sqrt[3]{x}$, $g(x) = \frac{x+1}{x^3}$

**Answer**

$f(g(x)) = \sqrt[3]{\frac{x+1}{x^3}} = \frac{\sqrt[3]{x+1}}{x}$, $g(f(x)) = \frac{\sqrt[3]{x}+1}{x}$

16) $f(x) = \frac{1}{x-6}$, $g(x) = \frac{7}{x} + 6$

17) $f(x) = \frac{1}{x-4}$, $g(x) = \frac{2}{x} + 4$

**Answer**

$(f \circ g)(x) = \frac{1}{\frac{2}{x} + 4 - 4} = \frac{x}{2}$, $(g \circ f)(x) = 2x - 4$

**For the exercises 18-25, use each set of functions to find f(g(h(x))). Simplify your answers.**

18) $f(x) = x^4 + 6$, $g(x) = x - 6$, and $h(x) = \sqrt{x}$

19) $f(x) = x^2 + 1$, $g(x) = \dfrac{1}{x}$, and $h(x) = x + 3$

**Answer**

$$f(g(h(x))) = \left(\frac{1}{x+3}\right)^2 + 1$$

20) Given $f(x) = \dfrac{1}{x}$ and $g(x) = x - 3$, find the following:

a. $(f \circ g)(x)$
b. the domain of $(f \circ g)(x)$ in interval notation
c. $(g \circ f)(x)$
d. the domain of $(g \circ f)(x)$
e. $\left(\dfrac{f}{g}\right) x$

21) Given $f(x) = \sqrt{2 - 4x}$ and $g(x) = -\dfrac{3}{x}$, find the following:

a. $(g \circ f)(x)$
b. the domain of $(g \circ f)(x)$ in interval notation

**Answer**

a. $(g \circ f)(x) = -\dfrac{3}{\sqrt{2-4x}}$
b. $\left(-\infty, \dfrac{1}{2}\right)$

22) Given the functions $f(x) = \dfrac{1-x}{x}$ and $g(x) = \dfrac{1}{1+x^2}$, find the following:

a. $(g \circ f)(x)$
b. $(g \circ f)(2)$

23) Given functions $p(x) = \dfrac{1}{\sqrt{x}}$ and $m(x) = x^2 - 4$, state the domain of each of the following functions using interval notation:

a. $\dfrac{p(x)}{m(x)}$
b. $p(m(x))$
c. $m(p(x))$

**Answer**

a. $(0, 2) \cup (2, \infty)$
b. $(-\infty, -2) \cup (2, \infty)$
c. $(0, \infty)$

24) Given functions $q(x) = \dfrac{1}{\sqrt{x}}$ and $h(x) = x^2 - 9$, state the domain of each of the following functions using interval notation.

a. $\dfrac{q(x)}{h(x)}$
b. $q(h(x))$
c. $h(q(x))$

25) For $f(x) = \dfrac{1}{x}$ and $g(x) = \sqrt{x-1}$, write the domain of $(f \circ g)(x)$ in interval notation.

**Answer**

$(1, \infty)$

**For the exercises 26-41, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x) = f(g(x))$.**

26) $h(x) = (x+2)^2$

27) $h(x) = (x-5)^3$

**Answer**

sample: $f(x) = x^3$

$g(x) = x - 5$

28) $h(x) = \dfrac{3}{x-5}$

29) $h(x) = \dfrac{4}{(x+2)^2}$

**Answer**

sample: $f(x) = \dfrac{4}{x}$

$g(x) = (x+2)^2$

30) $h(x) = 4 + \sqrt[3]{x}$

31) $h(x) = \sqrt[3]{\dfrac{1}{2x-3}}$

**Answer**

sample: $f(x) = \sqrt[3]{x}$

$g(x) = \dfrac{1}{2x-3}$

32) $h(x) = \dfrac{1}{(3x^2-4)^{-3}}$

33) $h(x) = \sqrt[4]{\dfrac{3x-2}{x+5}}$

**Answer**

sample: $f(x) = \sqrt[4]{x}$

$g(x) = \dfrac{3x-2}{x+5}$

34) $h(x) = (\dfrac{8+x^3}{8-x^3})^4$

35) $h(x) = \sqrt{2x+6}$

**Answer**

sample: $f(x) = \sqrt{x}$

$$g(x) = 2x + 6$$

36) $h(x) = (5x - 1)^3$

37) $h(x) = \sqrt[3]{x - 1}$

**Answer**

sample: $f(x) = \sqrt[3]{x}$
$g(x) = (x - 1)$

38) $h(x) = |x^2 + 7|$

39) $h(x) = \dfrac{1}{(x - 2)^3}$

**Answer**

sample: $f(x) = x^3$
$g(x) = \dfrac{1}{x - 2}$

40) $h(x) = \left(\dfrac{1}{2x - 3}\right)^2$

41) $h(x) = \sqrt{\dfrac{2x - 1}{3x + 4}}$

**Answer**

sample: $f(x) = \sqrt{x}$
$g(x) = \dfrac{2x - 1}{3x + 4}$

## Graphical

**For the exercises 42-49, use the graphs of $f$,shown in Figure 1, and $g$,shown in Figure 2, to evaluate the expressions.**

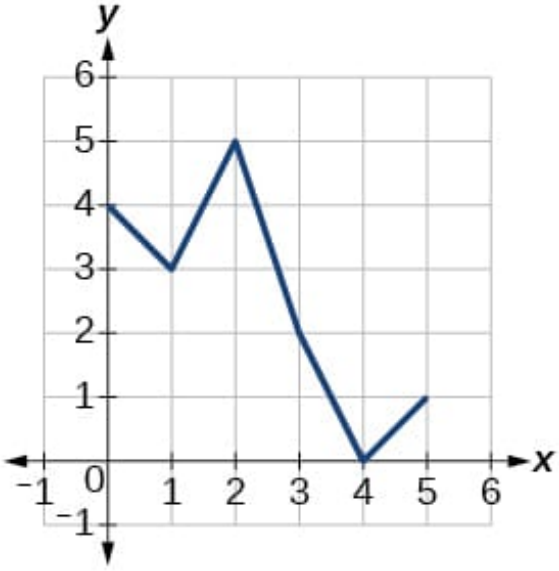



Figure 1: Graph of a function.

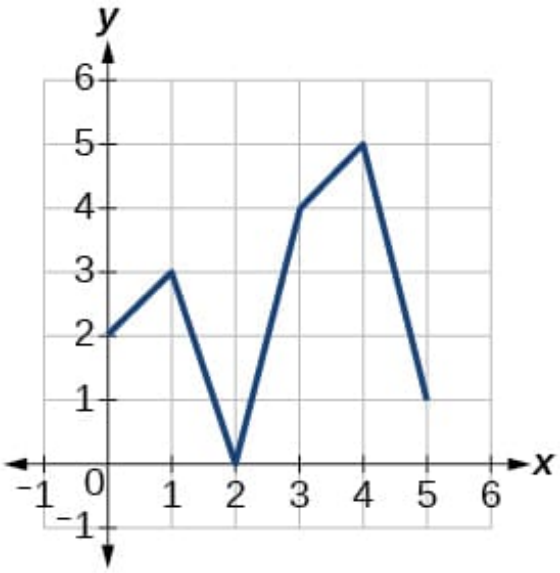



***Figure 2:*** *Graph of a function.*

42) $f(g(3))$

43) $f(g(1))$

**Answer**

$2$

44) $g(f(1))$

45) $g(f(0))$

**Answer**

$5$

46) $f(f(5))$

47) $f(f(4))$

**Answer**

$4$

48) $g(g(2))$

49) $g(g(0))$

**Answer**

$0$

**For the exercises 50-57, use graphs of $f(x)$,shown in Figure 3, $g(x)$,shown in Figure 4, and $h(x)$,shown in Figure 5, to evaluate the expressions.**

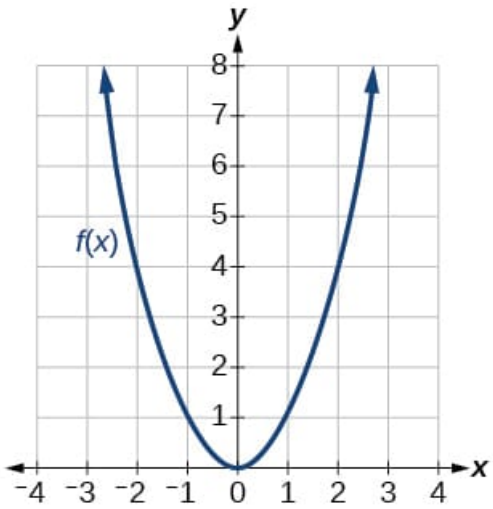



Figure 3: Graph of a parabola.

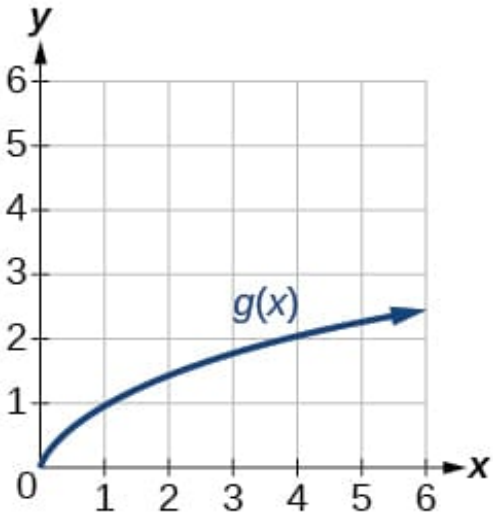



***Figure 4:*** *Graph of a square root function.*

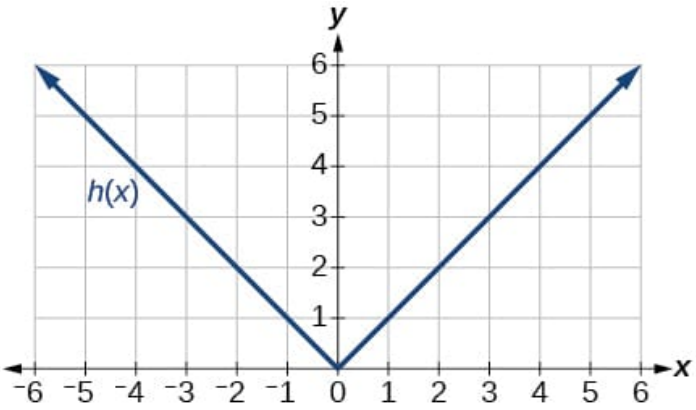



Figure 5: Graph of an absolute value function.

50) $g(f(1))$

51) $g(f(2))$

**Answer**

$2$

52) $f(g(4))$

53) $f(g(1))$

**Answer**

$1$

54) $f(h(2))$

55) $h(f(2))$

**Answer**

$4$

56) $f(g(h(4)))$

57) $f(g(f(-2)))$

**Answer**

$4$

## Numeric

**For the exercises 58-65, use the function values for $f$ and $g$ shown in the Table below to evaluate each expression.**

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 0 | 7 | 9 |
| 1 | 6 | 5 |
| 2 | 5 | 6 |
| 3 | 8 | 2 |
| 4 | 4 | 1 |
| 5 | 0 | 8 |
| 6 | 2 | 7 |
| 7 | 1 | 3 |
| 8 | 9 | 4 |
| 9 | 3 | 0 |

58) $f(g(8))$

59) $f(g(5))$

**Answer**

$9$

60) $g(f(5))$

61) $g(f(3))$

**Answer**

$4$

62) $f(f(4))$

63) $f(f(1))$

**Answer**

$2$

64) $g(g(2))$

65) $g(g(6))$

**Answer**

$3$

**For the exercises 66-71, use the function values for $f$ and $g$ shown in the Table below to evaluate the expressions.**

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| -3 | 11 | -8 |
| -2 | 9 | -3 |
| -1 | 7 | 0 |
| 0 | 5 | 1 |
| 1 | 3 | 0 |
| 2 | 1 | -3 |
| 3 | -1 | -8 |

66) $(f \circ g)(1)$

67) $(f \circ g)(2)$

**Answer**

$11$

68) $(g \circ f)(2)$

69) $(g \circ f)(3)$

**Answer**

$0$

70) $(g \circ g)(1)$

71) $(f \circ f)(3)$

**Answer**

$7$

**For the exercises 72-75, use each pair of functions to find $f(g(0))$ and $g(f(0))$.**

72) $f(x) = 4x + 8$, $g(x) = 7 - x^2$

73) $f(x) = 5x + 7$, $g(x) = 4 - 2x^2$

**Answer**

$f(g(0)) = 27$, $g(f(0)) = -94$

74) $f(x) = \sqrt{x+4}$, $g(x) = 12 - x^3$

75) $f(x) = \frac{1}{x+2}$, $g(x) = 4x + 3$

**Answer**

$f(g(0)) = \frac{1}{5}$, $g(f(0)) = 5$

**For the exercises 76-79, use the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$ to evaluate or find the composite function as indicated.**

76) $f(g(2))$

77) $f(g(x))$

**Answer**

$18x^2 + 60x + 51$

78) $g(f(-3))$

79) $(g \circ g)(x)$

**Answer**

$g \circ g(x) = 9x + 20$

## Extensions

**For the exercises 80-84, use $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$.**

80) Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Compare the two answers.

81) Find $(f \circ g)(2)$ and $(g \circ f)(2)$.

**Answer**

$2$

82) What is the domain of $(g \circ f)(x)$?

83) What is the domain of $(f \circ g)(x)$?

**Answer**

$(-\infty, \infty)$

84) Let $f(x) = \frac{1}{x}$.

a. Find $(f \circ f)(x)$.

b. Is $(f \circ f)(x)$ for any function $f$ the same result as the answer to part (a) for any function? Explain.

**For the exercises 85-87, let $F(x) = (x+1)^5$, $f(x) = x^5$, and $g(x) = x + 1$.**

85) True or False: $(g \circ f)(x) = F(x)$.

**Answer**

False

86) True or False: $(f \circ g)(x) = F(x)$.

**For the exercises 87-89, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x-2}$.**

87) $(f \circ g)(6)$; $(g \circ f)(6)$

**Answer**

$(f \circ g)(6) = 6$; $(g \circ f)(6) = 6$

88) $(g \circ f)(a)$; $(f \circ g)(a)$

89) $(f \circ g)(11)$; $(g \circ f)(11)$

**Answer**

$(f \circ g)(11) = 11$, $(g \circ f)(11) = 11$

## Real-World Applications

90) The function $D(p)$ gives the number of items that will be demanded when the price is $p$. The production cost $C(x)$ is the cost of producing $x$ items. To determine the cost of production when the price is $\$6$, you would do which of the following?

a. Evaluate $D(C(6))$
b. Evaluate $C(D(6))$
c. Solve $D(C(x)) = 6$
d. Solve $C(D(p)) = 6$

91) The function $A(d)$ gives the pain level on a scale of $0$ to $10$ experienced by a patient with $d$ milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after $t$ minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of $4$?

a. Evaluate $A(m(4))$
b. Evaluate $m(A(4))$
c. Solve $A(m(t)) = 4$
d. Solve $m(A(d)) = 4$

**Answer**

c

92) A store offers customers a $30\%$ discount on the price $x$ of selected items. Then, the store takes off an additional $15\%$ at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price $x$. (Hint: Use function composition to find your answer.)

93) A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.

**Answer**

$A(t) = \pi(25\sqrt{t+2})^2$ and $A(2) = \pi(25\sqrt{4})^2 = 2500\pi$ square inches

94) A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t) = 2t + 1$, express the area burned as a function of time, $t$ (minutes).

95) Use the function you found in the previous exercise to find the total area burned after $5$ minutes.

**Answer**

$A(5) = \pi(2(5) + 1)2 = 121\pi$ square units

96) The radius $r$, in inches, of a spherical balloon is related to the volume, $V$, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after $t$ seconds is given by $V(t) = 10 + 20t$.

a. Find the composite function $r(V(t))$.
b. Find the exact time when the radius reaches 10 inches.

97) The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where $T$ is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where $t$ is the time in hours.

a. Find the composite function $N(T(t))$.
b. Find the time (round to two decimal places) when the bacteria count reaches 6752.

**Answer**

a. $N(T(t)) = 23(5t + 1.5)^2 - 56(5t + 1.5) + 1$
b. $3.38$ hours

# 1.5: Transformation of Functions

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

## Verbal

1) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?

**Answer**

A horizontal shift results when a constant is added to or subtracted from the input. A vertical shifts results when a constant is added to or subtracted from the output.

2) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?

3) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?

**Answer**

A horizontal compression results when a constant greater than $1$ is multiplied by the input. A vertical compression results when a constant between $0$ and $1$ is multiplied by the output.

4) When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with

respect to the $x$-axis from a reflection with respect to the $y$-axis?

5) How can you determine whether a function is odd or even from the formula of the function?

**Answer**

For a function $f$, substitute (–x) for (x) in $f(x)$. Simplify. If the resulting function is the same as the original function, $f(-x) = f(x)$, then the function is even. If the resulting function is the opposite of the original function, $f(-x) = -f(x)$, then the original function is odd. If the function is not the same or the opposite, then the function is neither odd nor even.

## Algebraic

6) Write a formula for the function obtained when the graph of $f(x) = \sqrt{x}$ is shifted up $1$ unit and to the left $2$ units.

7) Write a formula for the function obtained when the graph of $f(x) = |x|$ is shifted down $3$ units and to the right $1$ unit.

**Answer**

$$g(x) = |x - 1| - 3$$

8) Write a formula for the function obtained when the graph of $f(x) = \dfrac{1}{x}$ is shifted down $4$ units and to the right $3$ units.

9) Write a formula for the function obtained when the graph of $f(x) = \dfrac{1}{x^2}$ is shifted up $2$ units and to the left $4$ units.

**Answer**

$$g(x) = \frac{1}{(x+4)^2} + 2$$

**For the exercises 10-19, describe how the graph of the function is a transformation of the graph of the original function $f$.**

10) $y = f(x - 49)$

11) $y = f(x + 43)$

**Answer**

The graph of $f(x + 43)$ is a horizontal shift to the left $43$ units of the graph of $f$.

12) $y = f(x + 3)$

13) $y = f(x - 4)$

**Answer**

The graph of $f(x - 4)$ is a horizontal shift to the right $4$ units of the graph of $f$.

14) $y = f(x) + 5$

15) $y = f(x) + 8$

**Answer**

The graph of $f(x) + 8$ is a vertical shift up $8$ units of the graph of $f$.

16) $y = f(x) - 2$

17) $y = f(x) - 7$

**Answer**

The graph of $f(x) - 7$ is a vertical shift down $7$ units of the graph of $f$.

18) $y = f(x - 2) + 3$

19) $y = f(x + 4) - 1$

**Answer**

The graph of $f(x + 4) - 1$ is a horizontal shift to the left $4$ units and a vertical shift down $1$ unit of the graph of $f$.

**For the exercises 20-23, determine the interval(s) on which the function is increasing and decreasing.**

20) $f(x) = 4(x + 1)^2 - 5$

21) $g(x) = 5(x + 3)^2 - 2$

**Answer**

decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$

22) $a(x) = \sqrt{-x + 4}$

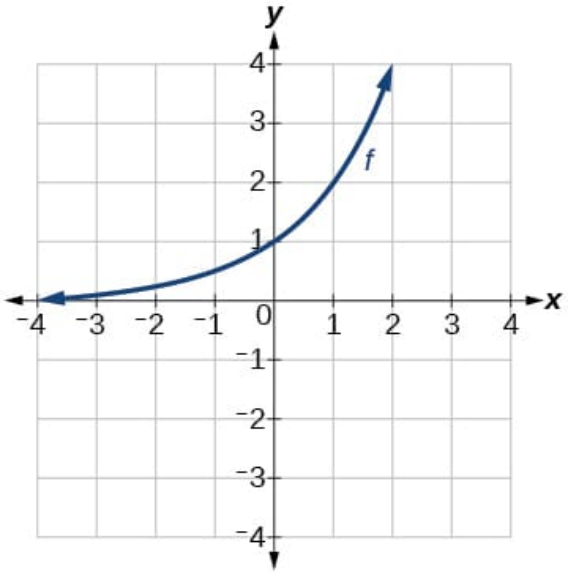
23) $k(x) = -3\sqrt{x} - 1$

**Answer**

decreasing on $(0, \infty)$

## Graphical

**For the exercises 24-26, use the graph of $f(x) = 2^x$ shown in the Figure below to sketch a graph of each transformation of $f(x)$.**

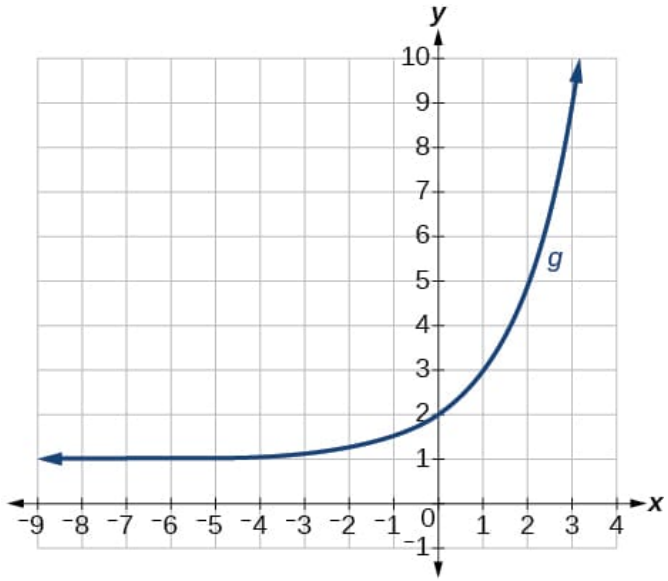



Graph of $f(x)$.

24) $h(x) = 2^x - 3$

25) $g(x) = 2^x + 1$

**Answer**



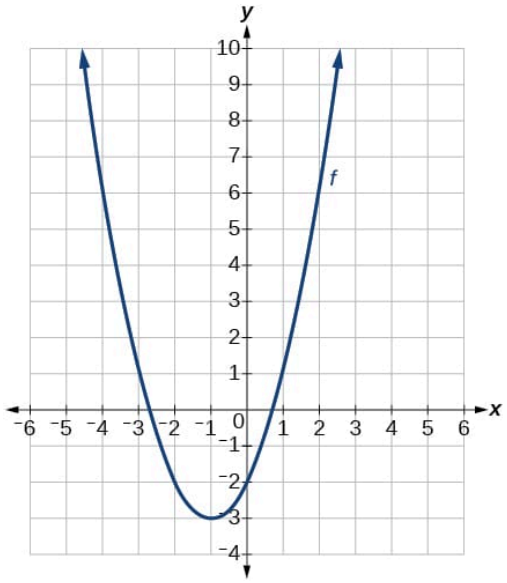

Graph of $g(x)$

26) $w(x) = 2^{x-1}$

**For the exercises 27-30, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.**

27) $f(t) = (t+1)^2 - 3$

**Answer**

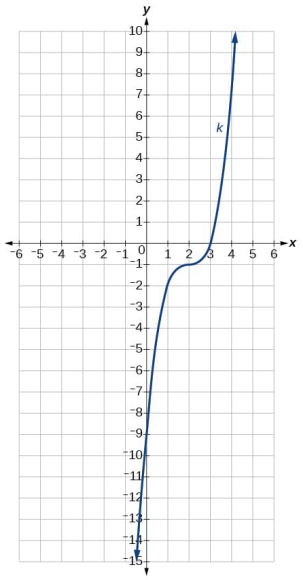



Graph of $f(t)$.

28) $h(x) = |x - 1| + 4$

29) $k(x) = (x-2)^3 - 1$

**Answer**




Graph of $k(x)$.

30) $m(t) = 3 + \sqrt{t+2}$

## Numeric

31) Tabular representations for the functions $f$, $g$, and $h$ are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | -2 | -1 | -3 | 1 | 2 |

| $x$ | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $g(x)$ | -2 | -1 | -3 | 1 | 2 |

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $h(x)$ | -1 | 0 | -2 | 2 | 3 |

**Answer**

$g(x) = f(x-1)$, $h(x) = f(x) + 1$

32) Tabular representations for the functions $f$, $g$, and $h$ are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

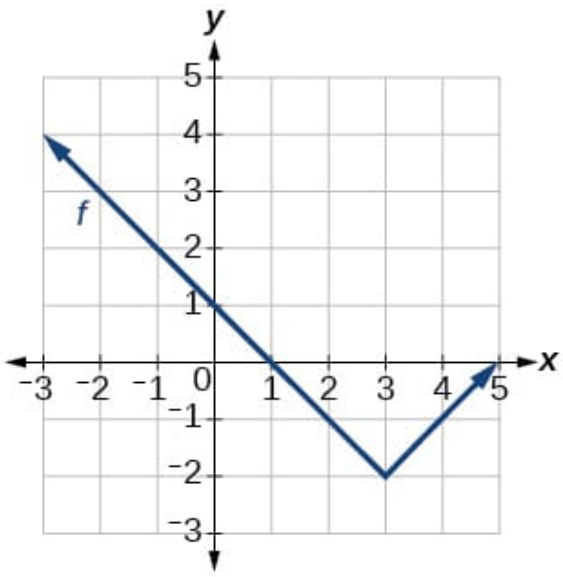
| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | -1 | -3 | 4 | 2 | 1 |

| $x$ | -3 | -2 | -1 | 0 | 1 |
|---|---|---|---|---|---|
| $g(x)$ | -1 | -3 | 4 | 2 | 1 |

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $h(x)$ | -2 | -4 | 3 | 1 | 0 |

**For the exercises 33-40, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.**

33)

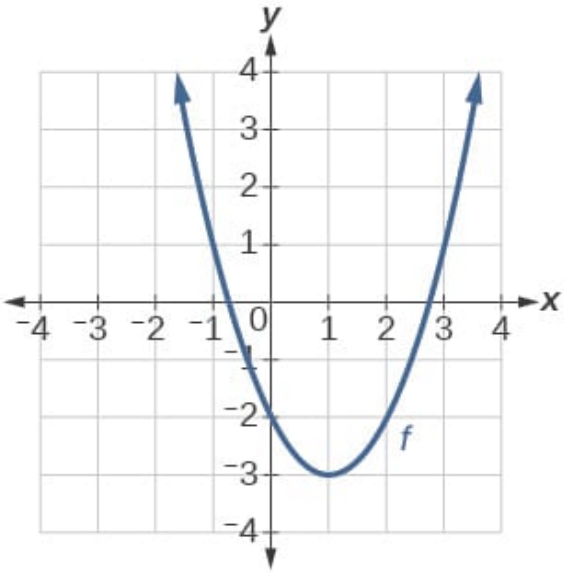



*Graph of an absolute function.*

**Answer**

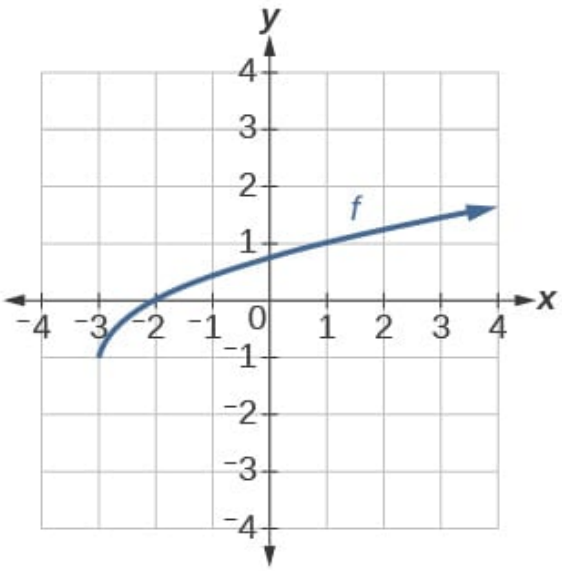
$$f(x) = |x - 3| - 2$$

34)




*Graph of a parabola.*

35)

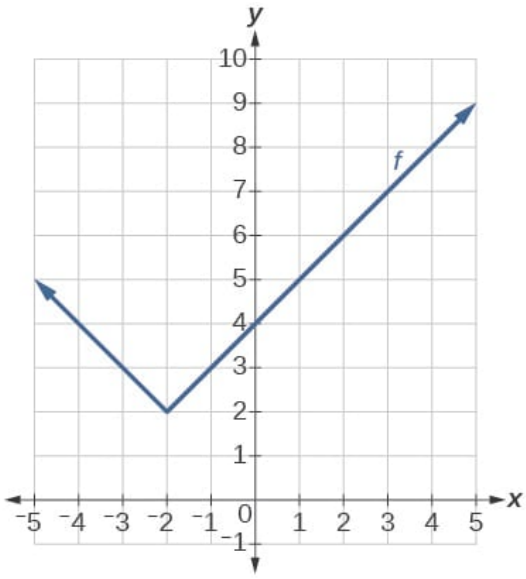



*Graph of a square root function.*

**Answer**

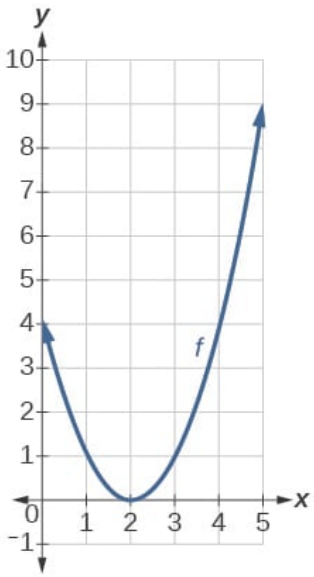
$f(x) = \sqrt{x+3} - 1$

36)




*Graph of an absolute function.*

37)

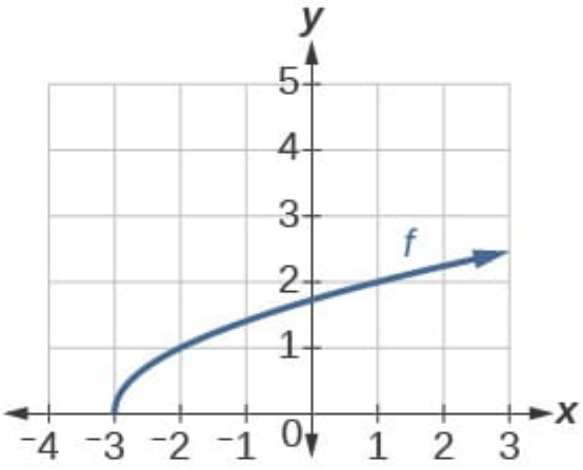



*Graph of a parabola*

**Answer**

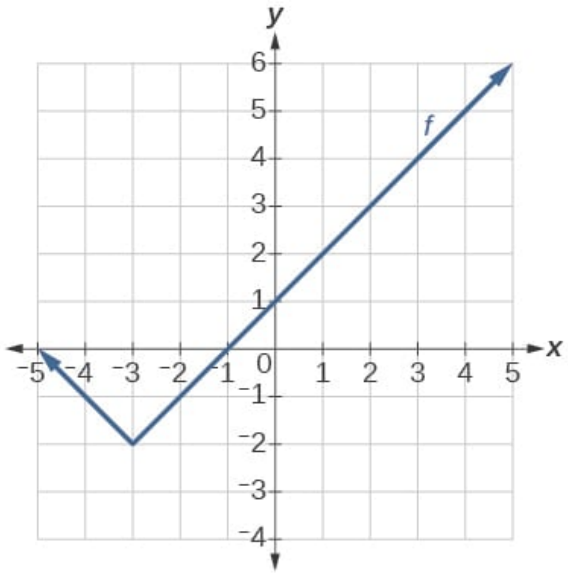
$$f(x) = (x-2)^2$$

38)




*Graph of a square root function.*

39)

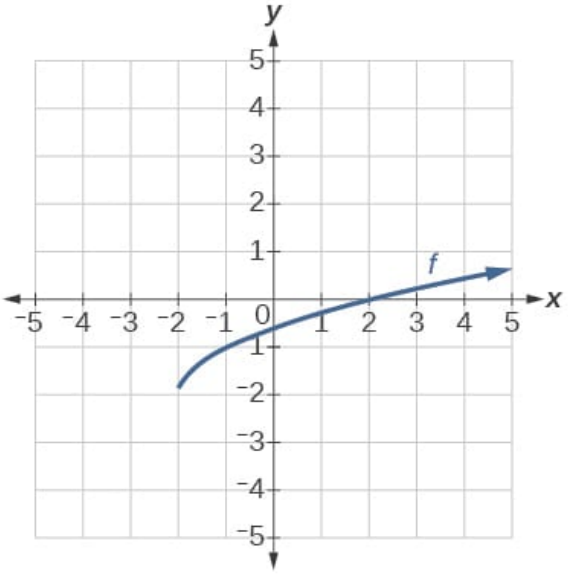



*Graph of an absolute function.*

**Answer**

$$f(x) = |x + 3| - 2$$

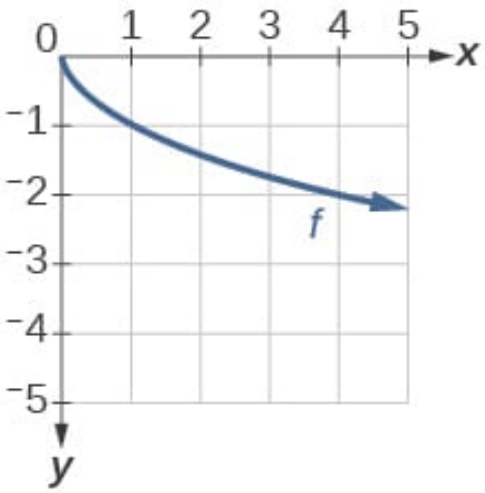
40)




*Graph of a square root function.*

**For the exercises 41-42, use the graphs of transformations of the square root function to find a formula for each of the functions.**

41)

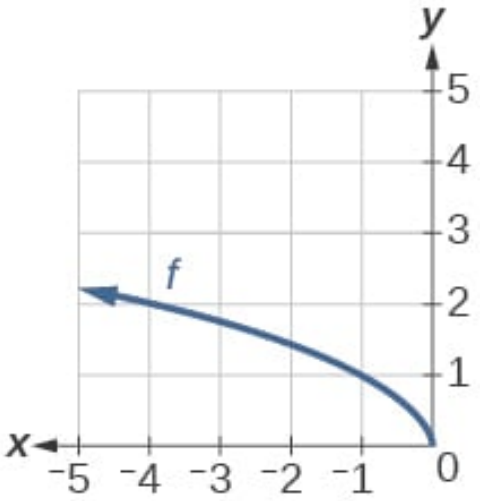



*Graph of a square root function.*

**Answer**

$f(x) = -\sqrt{x}$

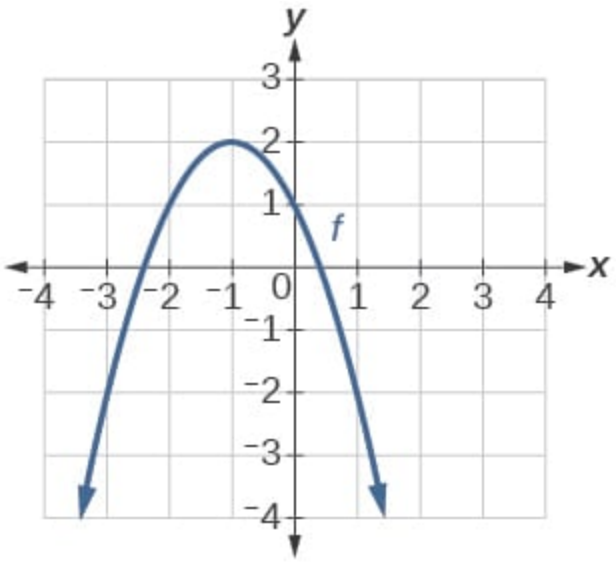
42)




*Graph of a square root function.*

**For the exercises 43-46, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.**

43)

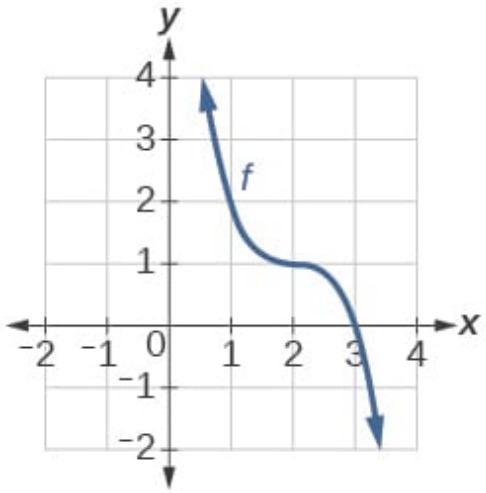



*Graph of a parabola.*

**Answer**

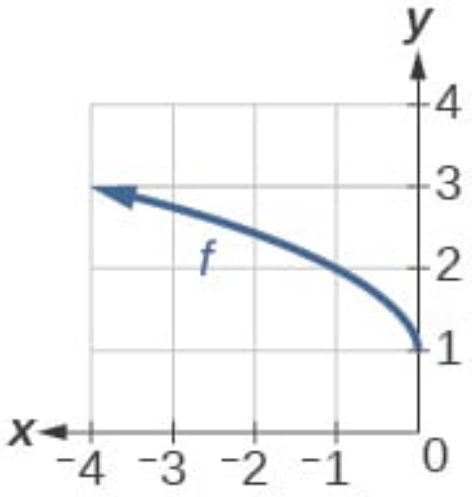
$$f(x) = -(x+1)^2 + 2$$

44)




*Graph of a cubic function.*

45)

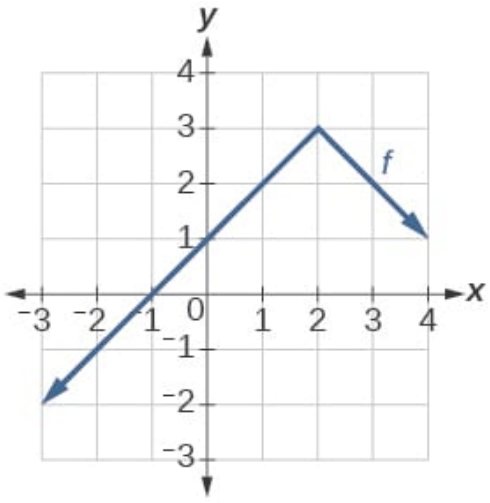



*Graph of a square root function.*

**Answer**

$f(x) = \sqrt{-x} + 1$

46)




*Graph of an absolute function.*

**For the exercises 47-52, determine whether the function is odd, even, or neither.**

47) $f(x) = 3x^4$

**Answer**

even

48) $g(x) = \sqrt{x}$

49) $h(x) = \frac{1}{x} + 3x$

**Answer**

odd

50) $f(x) = (x-2)^2$

51) $g(x) = 2x^4$

**Answer**

even

52) $h(x) = 2x - x^3$

**For the exercises 53-62, describe how the graph of each function is a transformation of the graph of the original function $f$.**

53) $g(x) = -f(x)$

**Answer**

The graph of $g$ is a vertical reflection (across the x-axis) of the graph of $f$.

54) $g(x) = f(-x)$

55) $g(x) = 4f(x)$

**Answer**

The graph of $g$ is a vertical stretch by a factor of 4 of the graph of $f$.

56) $g(x) = 6f(x)$

57) $g(x) = f(5x)$

**Answer**

The graph of $g$ is a horizontal compression by a factor of $15$ of the graph of $f$.

58) $g(x) = f(2x)$

59) $g(x) = f\left(\frac{1}{3}x\right)$

**Answer**

The graph of $g$ is a horizontal stretch by a factor of 3 of the graph of $f$.

60) $g(x) = f\left(\frac{1}{5}x\right)$

61) $g(x) = 3f(-x)$

**Answer**

The graph of $g$ is a horizontal reflection across the y-axis and a vertical stretch by a factor of 3 of the graph of $f$.

62) $g(x) = -f(3x)$

**For the exercises 63-68, write a formula for the function $g$ that results when the graph of a given toolkit function is transformed as described.**

63) The graph of $f(x) = |x|$ is reflected over the $y$-axis and horizontally compressed by a factor of $\frac{1}{4}$.

**Answer**

$$g(x) = |-4x|$$

64) The graph of $f(x) = \sqrt{x}$ is reflected over the $x$-axis and horizontally stretched by a factor of $2$.

65) The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left $2$ units and down $3$ units.

**Answer**

$$g(x) = \frac{1}{3(x+2)^2} - 3$$

66) The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of $8$, then shifted to the right $4$ units and up $2$ units.

67) The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right $5$ units and up $1$ unit.

**Answer**

$$g(x) = \frac{1}{2}(x-5)^2 + 1$$

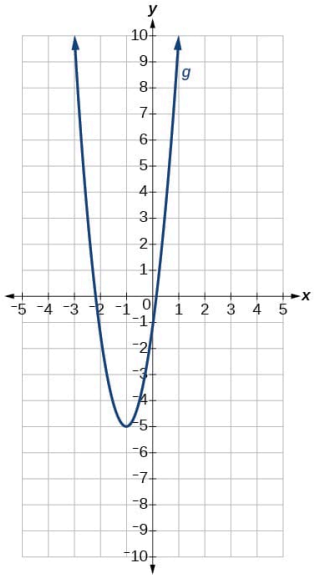
68) The graph of $f(x) = x^2$ is horizontally stretched by a factor of $3$, then shifted to the left $4$ units and down $3$ units.

**For the exercises 69-77, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.**

69) $g(x) = 4(x+1)^2 - 5$

**Answer**

The graph of the function $f(x) = x^2$ is shifted to the left $1$ unit, stretched vertically by a factor of $4$, and shifted down $5$ units.



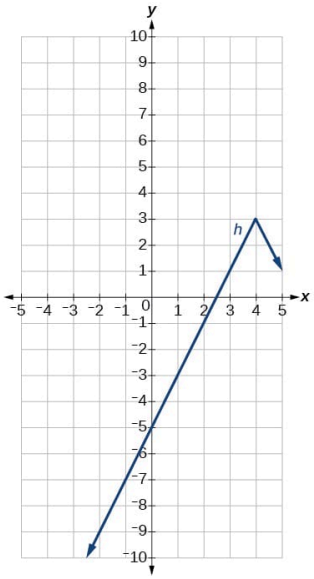

Graph of a parabola.

70) $g(x) = 5(x+3)^2 - 2$

71) $h(x) = -2|x-4| + 3$

**Answer**

The graph of $f(x) = |x|$ is stretched vertically by a factor of $2$, shifted horizontally $4$ units to the right, reflected across the horizontal axis, and then shifted vertically $3$ units up.



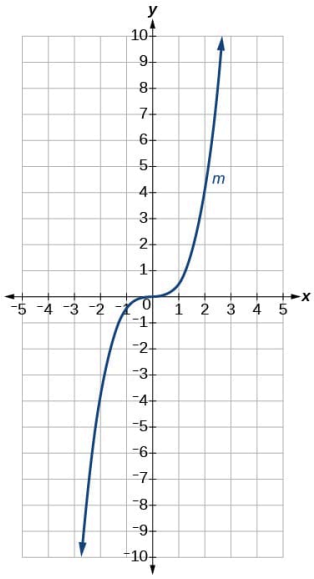

Graph of an absolute value function.

72) $k(x) = -3\sqrt{x} - 1$

73) $m(x) = \frac{1}{2}x^3$

**Answer**

The graph of the function $f(x) = x^3$ is compressed vertically by a factor of $\frac{1}{2}$.



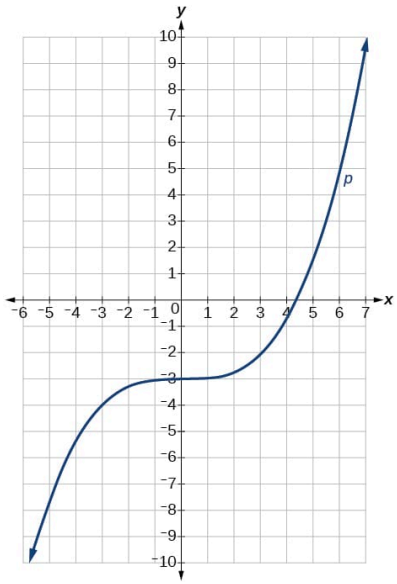
Graph of a cubic function.

74) $n(x) = \frac{1}{3}|x - 2|$

75) $p(x) = (\frac{1}{3}x)^3 - 3$

**Answer**

The graph of the function is stretched horizontally by a factor of $3$ and then shifted vertically downward by $3$ units.



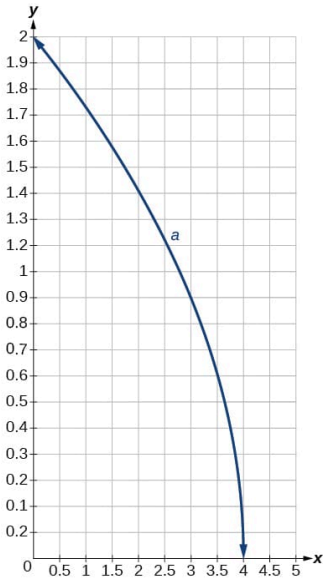

Graph of a cubic function.

76) $q(x) = (\frac{1}{4}x)^3 + 1$

77) $a(x) = \sqrt{-x + 4}$

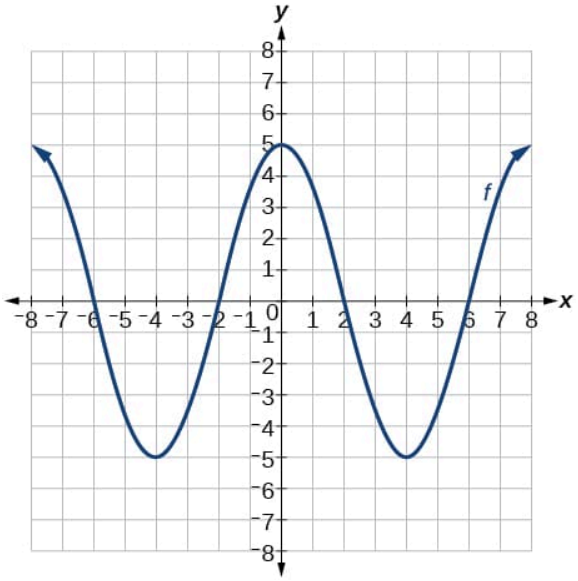
**Answer**

The graph of $f(x) = \sqrt{x}$ is shifted right $4$ units and then reflected across the vertical line $x = 4$.




Graph of a square root function.

**For the exercises 78-81, use the graph in the Figure below to sketch the given transformations.**

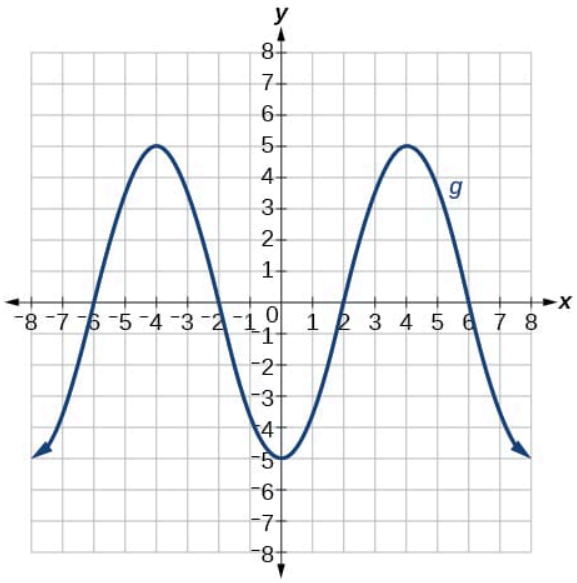



*Graph of a polynomial.*

78) $g(x) = f(x) - 2$

79) $g(x) = -f(x)$

**Answer**

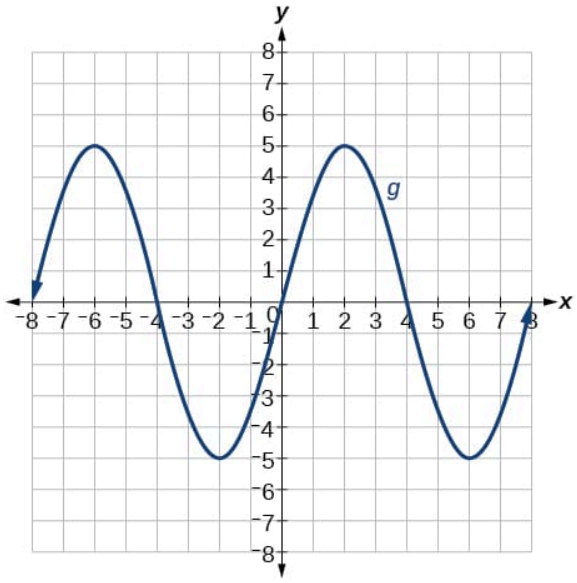



*Graph of a polynomial.*

80) $g(x) = f(x + 1)$

81) $g(x) = f(x - 2)$

**Answer**




Graph of a polynomial.

# 1.6: Absolute Value Functions

Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate absolute value functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

## Verbal

1) How do you solve an absolute value equation?

**Answer**

Isolate the absolute value term so that the equation is of the form $|A| = B$. Form one equation by setting the expression inside the absolute value symbol, $A$, equal to the expression on the other side of the equation, $B$. Form a second equation by setting $A$ equal to the opposite of the expression on the other side of the equation, $-B$. Solve each equation for the variable.

2) How can you tell whether an absolute value function has two $x$-intercepts without graphing the function?

3) When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?

**Answer**

The graph of the absolute value function does not cross the $x$-axis, so the graph is either completely above or completely below the $x$-axis.

4) How can you use the graph of an absolute value function to determine the $x$-values for which the function values are negative?

5) How do you solve an absolute value inequality algebraically?

**Answer**

First determine the boundary points by finding the solution(s) of the equation. Use the boundary points to form possible solution intervals. Choose a test value in each interval to determine which values satisfy the inequality.

## Algebraic

6) Describe all numbers $x$ that are at a distance of $4$ from the number $8$. Express this using absolute value notation.

7) Describe all numbers $x$ that are at a distance of $\frac{1}{2}$ from the number $-4$. Express this using absolute value notation.

**Answer**

$$|x + 4| = \frac{1}{2}$$

8) Describe the situation in which the distance that point $x$ is from $10$ is at least $15$ units. Express this using absolute value notation.

9) Find all function values $f(x)$ such that the distance from $f(x)$ to the value $8$ is less than $0.03$ units. Express this using absolute value notation.

**Answer**

$$|f(x) - 8| < 0.03$$

**For the exercises 10-24, solve the equations below and express the answer using set notation.**

10) $|x + 3| = 9$

11) $|6 - x| = 5$

**Answer**

$1, 11$

12) $|5x - 2| = 11$

13) $|4x - 2| = 11$

**Answer**

$$\left\{\frac{9}{4}, \frac{13}{4}\right\}$$

14) $2|4 - x| = 7$

15) $3|5 - x| = 5$

**Answer**

$$\left\{\frac{10}{3}, \frac{20}{3}\right\}$$

16) $3|x + 1| - 4 = 5$

17) $5|x - 4| - 7 = 2$

**Answer**

$$\left\{\frac{11}{5}, \frac{29}{5}\right\}$$

18) $0 = -|x - 3| + 2$

19) $2|x - 3| + 1 = 2$

**Answer**

$$\left\{\frac{5}{2}, \frac{7}{2}\right\}$$

20) $|3x - 2| = 7$

21) $|3x - 2| = -7$

**Answer**

No solution

22) $\left|\frac{1}{2}x - 5\right| = 11$

23) $\left|\frac{1}{3}x + 5\right| = 14$

**Answer**

$\{-57, 27\}$

24) $-\left|\frac{1}{3}x + 5\right| + 14 = 0$

**For the exercises 25-28, find the $x$- and $y$-intercepts of the graphs of each function.**

25) $f(x) = 2|x + 1| - 10$

**Answer**

$(0, -8)$; $(-6, 0)$, $(4, 0)$

26) $f(x) = 4|x - 3| + 4$

27) $f(x) = -3|x - 2| - 1$

**Answer**

$(0, -7)$; no $x$-intercepts

28) $f(x) = -2|x + 1| + 6$

**For the exercises 29-36, solve each inequality and write the solution in interval notation.**

29) $|x - 2| > 10$

**Answer**

$(-\infty, -8) \cup (12, \infty)$

30) $2|v - 7| - 4 \geq 42$

31) $|3x - 4| \geq 8$

**Answer**

$-\frac{4}{3} \leq x \leq 4$

32) $|x - 4| \geq 8$

33) $|3x - 5| \geq -13$

**Answer**

$\left(-\infty, -\frac{8}{3}\right] \cup [6, \infty)$

34) $|3x - 5| \geq -13$

35) $|\frac{3}{4}x - 5| \geq 7$

**Answer**

$\left(-\infty, -\frac{8}{3}\right] \cup [16, \infty)$

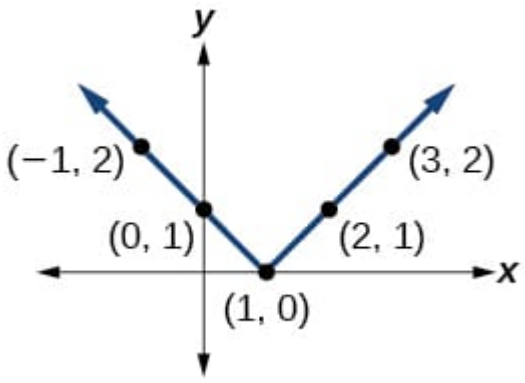
36) $\left|\frac{3}{4}x - 5\right| + 1 \leq 16$

## Graphical

**For the exercises 37-39, graph the absolute value function. Plot at least five points by hand for each graph.**

37) $y = |x - 1|$

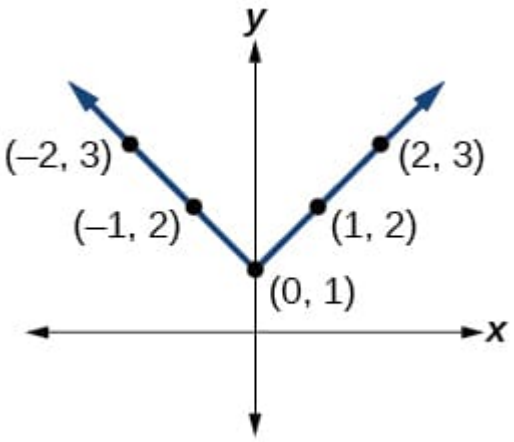
**Answer**




38) $y = |x + 1|$

39) $y = |x| + 1$

**Answer**

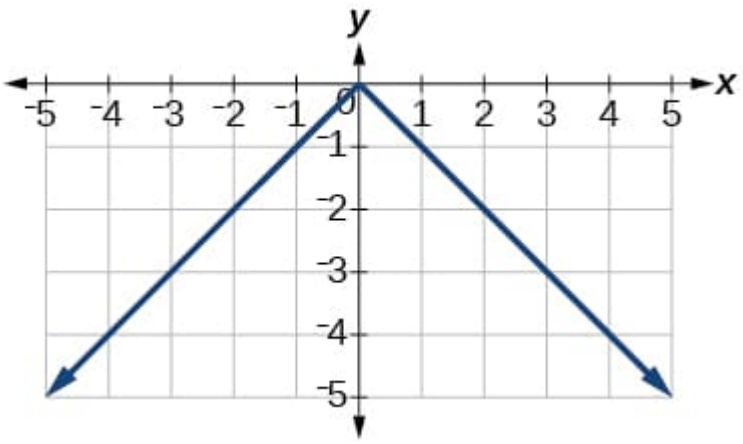



**For the exercises 40-52, graph the given functions by hand.**

40) $y = |x| - 2$

41) $y = -|x|$

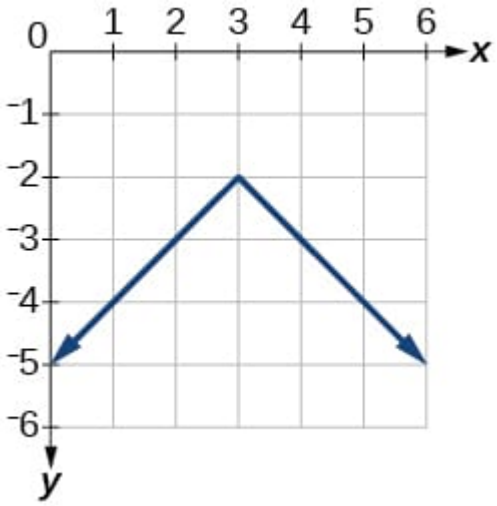
**Answer**




42) $y = -|x| - 2$

43) $y = -|x - 3| - 2$

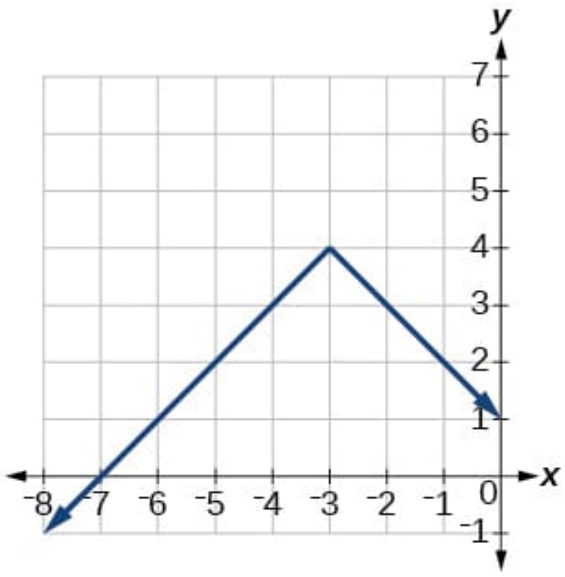
**Answer**




44) $f(x) = -|x - 1| - 2$

45) $f(x) = -|x + 3| + 4$

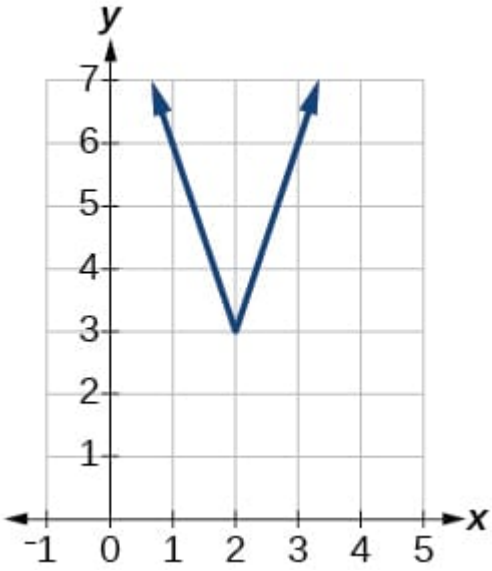
**Answer**




46) $f(x) = 2|x + 3| + 1$

47) $f(x) = 3|x - 2| + 3$

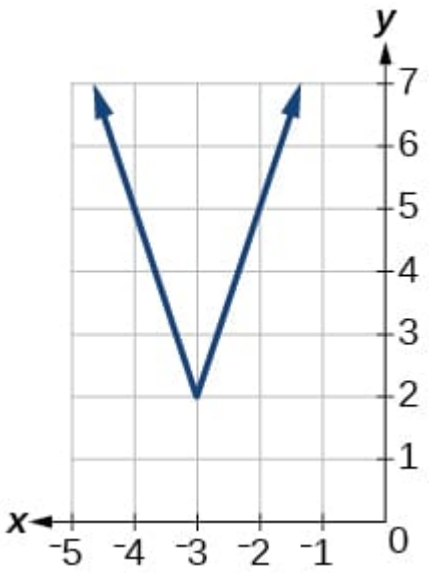
**Answer**




48) $f(x) = |2x - 4| - 3$

49) $f(x) = |3x + 9| + 2$

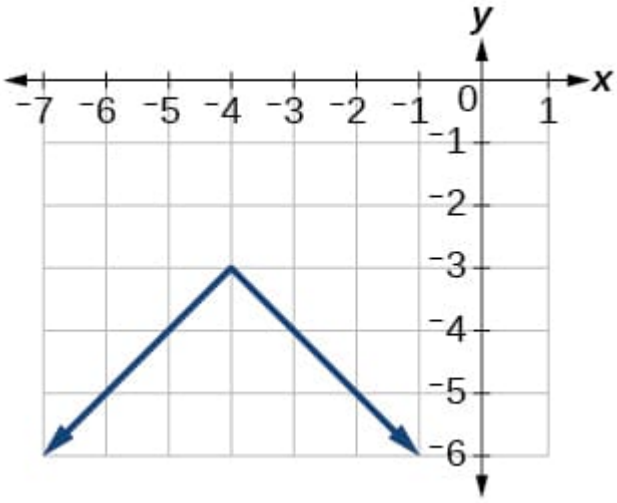
**Answer**




50) $f(x) = -|x - 1| - 3$

51) $f(x) = -|x + 4| - 3$

**Answer**



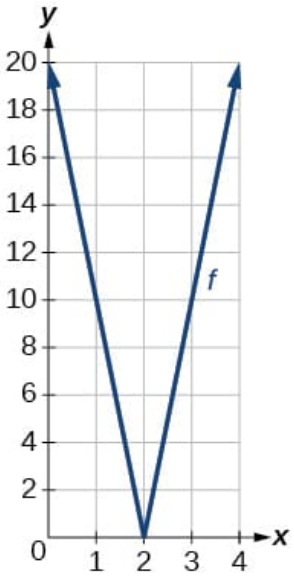

52) $f(x) = \frac{1}{2}|x + 4| - 3$

## Technology

53) Use a graphing utility to graph $f(x) = 10|x - 2|$ on the viewing window $[0, 4]$. Identify the corresponding range. Show the graph.

**Answer**

range: $[0, 20]$



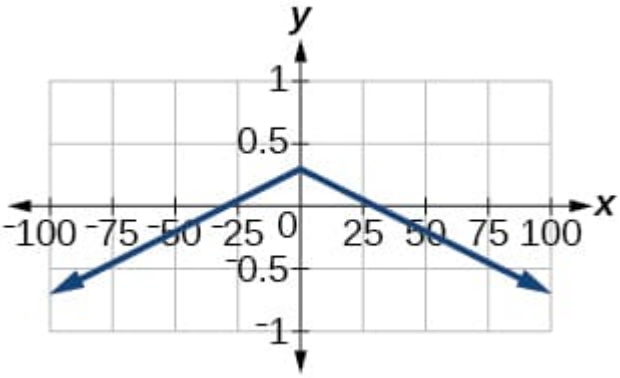

54) Use a graphing utility to graph $f(x) = -100|x| + 100$ on the viewing window $[-5, 5]$. Identify the corresponding range. Show the graph.

**For the exercises 55-56, graph each function using a graphing utility. Specify the viewing window.**

55) $f(x) = -0.1|0.1(0.2 - x)| + 0.3$

**Answer**

$x$-intercepts:




56) $f(x) = 4 \times 10^9|x - (5 \times 10^9)| + 2 \times 10^9$

## Extensions

**For the exercises 57-59, solve the inequality.**

57) $\left|-2x - \frac{2}{3}(x + 1)\right| + 3 > -1$

**Answer**

$(-\infty, \infty)$

58) If possible, find all values of a such that there are no $x$-intercepts for $f(x) = 2|x + 1| + a$.

59) If possible, find all values of a such that there are no $y$-intercepts for $f(x) = 2|x + 1| + a$.

**Answer**

There is no solution for a that will keep the function from having a $y$-intercept. The absolute value function always crosses the $y$-intercept when $x = 0$.

## Real-World Applications

60) Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least $100$ miles and $x$ represents the distance from city B to city A, express this using absolute value notation.

61) The true proportion $p$ of people who give a favorable rating to Congress is $8\%$ with a margin of error of $1.5\%$. Describe this statement using an absolute value equation.

**Answer**

$$|p - 0.08| \leq 0.015$$

62) Students who score within $18$ points of the number $82$ will pass a particular test. Write this statement using absolute value notation and use the variable $x$ for the score.

63) A machinist must produce a bearing that is within $0.01$ inches of the correct diameter of $5.0$ inches. Using $x$ as the diameter of the bearing, write this statement using absolute value notation.

**Answer**

$$|x - 5.0| \leq 0.01$$

64) The tolerance for a ball bearing is $0.01$. If the true diameter of the bearing is to be $2.0$ inches and the measured value of the diameter is $x$ inches, express the tolerance using absolute value notation.

# 1.7: Inverse Functions

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. In this section, we will consider the reverse nature of functions.

## Verbal

1) Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

**Answer**

Each output of a function must have exactly one output for the function to be one-to-one. If any horizontal line crosses the graph of a function more than once, that means that $y$-values repeat and the function is not one-to-one. If no horizontal line crosses the graph of the function more than once, then no $y$-values repeat and the function is one-to-one.

2) Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?

3) Can a function be its own inverse? Explain.

**Answer**

Yes. For example, $f(x) = \dfrac{1}{x}$ is its own inverse.

4) Are one-to-one functions either always increasing or always decreasing? Why or why not?

5) How do you find the inverse of a function algebraically?

**Answer**

Given a function $y = f(x)$, solve for $x$ in terms of $y$. Interchange the $x$ and $y$. Solve the new equation for $y$. The expression for $y$ is the inverse, $y = f^{-1}(x)$.

## Algebraic

6) Show that the function $f(x) = a - x$ is its own inverse for all real numbers $a$.

**For the exercises 7-12, find $f^{-1}(x)$ for each function.**

7) $f(x) = x + 3$

**Answer**

$$f^{-1}(x) = x - 3$$

8) $f(x) = x + 5$

9) $f(x) = 2 - x$

**Answer**

$$f^{-1}(x) = 2 - x$$

10) $f(x) = 3 - x$

11) $f(x) = \dfrac{x}{x+2}$

**Answer**

$$f^{-1}(x) = \frac{-2x}{x-1}$$

12) $f(x) = \dfrac{2x+3}{5x+4}$

**For the exercises 13-16, find a domain on which each function $f$ is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of $f$ restricted to that domain.**

13) $f(x) = (x+7)^2$

**Answer**

domain of $f(x)$: $[-7, \infty)$; $f^{-1}(x) = \sqrt{x} - 7$

14) $f(x) = (x-6)^2$

15) $f(x) = x^2 - 5$

**Answer**

domain of $f(x)$: $[0, \infty)$; $f^{-1}(x) = \sqrt{x+5}$

16) Given $f(x) = \dfrac{x}{2+x}$ and $g(x) = \dfrac{2x}{1-x}$ :

a. Find $f(g(x))$ and $g(f(x))$.
b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

**Answer**

a. $f(g(x)) = x$ and $g(f(x)) = x$

b. This tells us that $f$ and $g$ are inverse functions

**For the exercises 17-18, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.**

17) $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$

**Answer**

$f(g(x)) = x$, $g(f(x)) = x$

18) $f(x) = -3x + 5$ and $g(x) = \dfrac{x-5}{-3}$

## Graphical

**For the exercises 19-22, use a graphing utility to determine whether each function is one-to-one.**

19) $f(x) = \sqrt{x}$

**Answer**

one-to-one

20) $f(x) = \sqrt[3]{3x+1}$

21) $f(x) = -5x + 1$

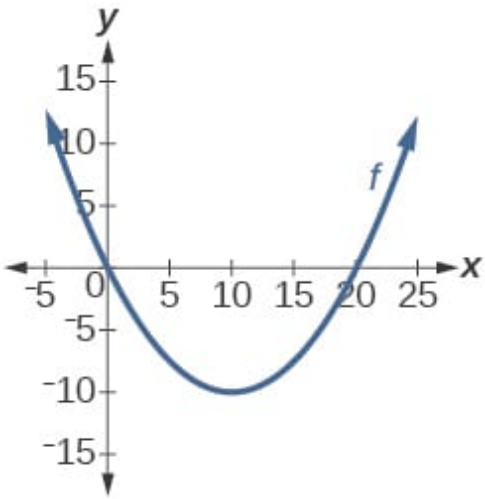
**Answer**

one-to-one

22) $f(x) = x^3 - 27$

**For the exercises 23-24, determine whether the graph represents a one-to-one function.**

23)



**Answer**

not one-to-one

24)

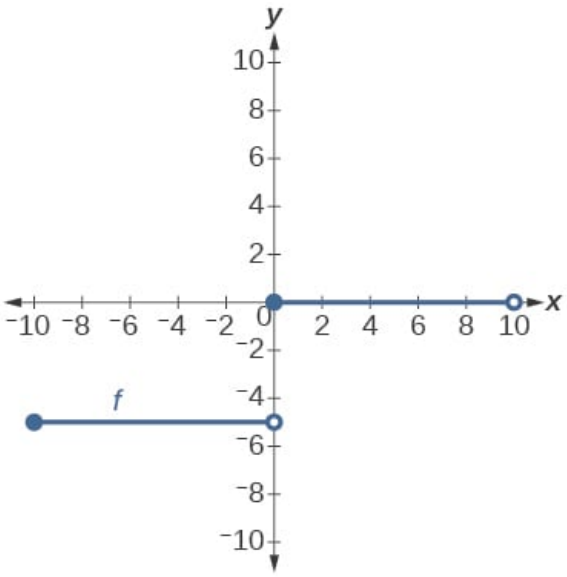
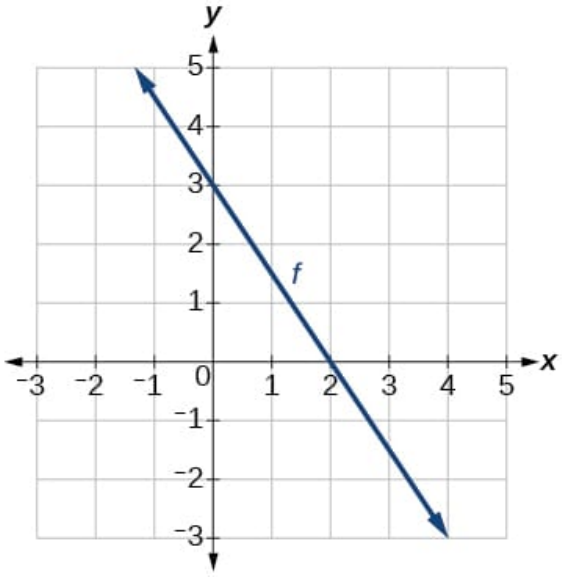



Figure below.




*Graph of a line*

25) Find $f(0)$.

**Answer**

$3$

26) Solve $f(x) = 0$.

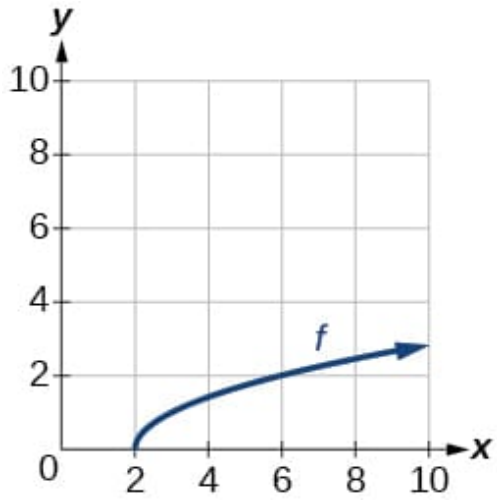
27) Find $f^{-1}(0)$.

**Answer**

$2$

28) Solve $f^{-1}(x) = 0$.

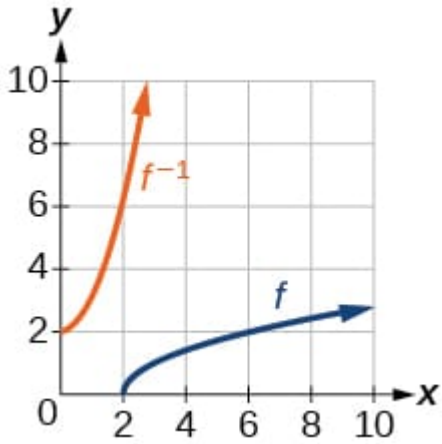
**For the exercises 29-32, use the graph of the one-to-one function shown in the Figure below**




*Graph of a square root function.*

29) Sketch the graph of $f^{-1}$.

**Answer**




30) Find $f(6)$ and $f^{-1}(2)$.

31) If the complete graph of $f$ is shown, find the domain of $f$.

**Answer**

$[2, 10]$

32) If the complete graph of $f$ is shown, find the range of $f$

## Numeric

**For the exercises 33-36, evaluate or solve, assuming that the function $f$ is one-to-one.**

33) If $f(6) = 7$, find $f^{-1}(7)$.

**Answer**

$6$

34) If $f(3) = 2$, find $f^{-1}(2)$.

35) If $f^{-1}(-4) = -8$, find $f(-8)$.

**Answer**

$-4$

36) If $f^{-1}(-2)$=–1, find $f(-1)$.

**For the exercises 37-40, use the values listed in the Table below to evaluate or solve.**

| $x$ | $f(x)$ |
|---|---|
| 0 | 8 |
| 1 | 0 |
| 2 | 7 |
| 3 | 4 |
| 4 | 2 |
| 5 | 6 |
| 6 | 5 |
| 7 | 3 |
| 8 | 9 |
| 9 | 1 |

37) Find $f(1)$.

**Answer**

$0$

38) Solve $f(x) = 3$.

39) Find $f^{-1}(0)$.

**Answer**

$1$

40) Solve $f^{-1}(x) = 7$.

41) Use the tabular representation of $f$ in the Table below to create a table for $f^{-1}(x)$.

| $x$ | 3 | 6 | 9 | 13 | 14 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 4 | 7 | 12 | 16 |

**Answer**

| $x$ | 1 | 4 | 7 | 12 | 16 |
|---|---|---|---|---|---|
| $f^{-1}(x)$ | 3 | 6 | 9 | 13 | 14 |

## Technology

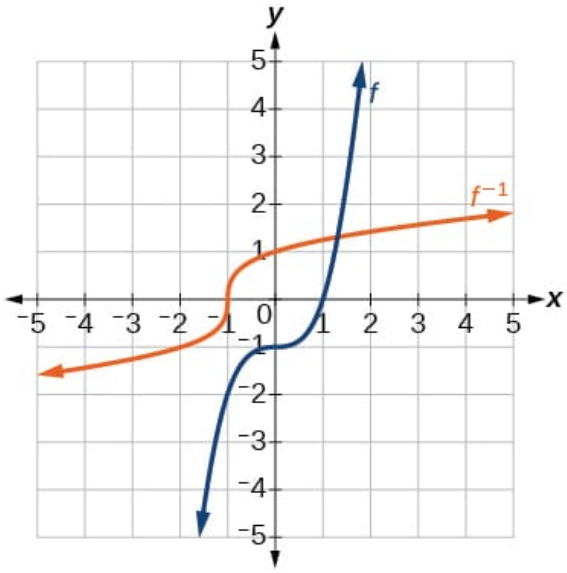
**For the exercises 42-44, find the inverse function. Then, graph the function and its inverse.**

42) $f(x) = \dfrac{3}{x-2}$

43) $f(x) = x^3 - 1$

**Answer**

$f^{-1}(x) = (1+x)^{1/3}$




44) Find the inverse function of $f(x) = \dfrac{1}{x-1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

## Real-World Applications

45) To convert from $x$ degrees Celsius to $y$ degrees Fahrenheit, we use the formula $f(x) = \dfrac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.

**Answer**

$f^{-1}(x) = \dfrac{5}{9}(x - 32)$. Given the Fahrenheit temperature, $x$, this formula allows you to calculate the Celsius temperature.

46) The circumference $C$ of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.

47) A car travels at a constant speed of $50$ miles per hour. The distance the car travels in miles is a function of time, $t$, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function

$t(d)$. Find $t(180)$ and interpret its meaning.

**Answer**

$t(d) = \frac{d}{50}$, $t(180) = \frac{180}{50}$. The time for the car to travel $180$ miles is $3.6$ hours.

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# 1.R: Functions (Review)

## 1.1: Functions and Function Notation

**For the exercises 1-4, determine whether the relation is a function.**

1) $\{(a,b),(c,d),(e,d)\}$

**Answer**

function

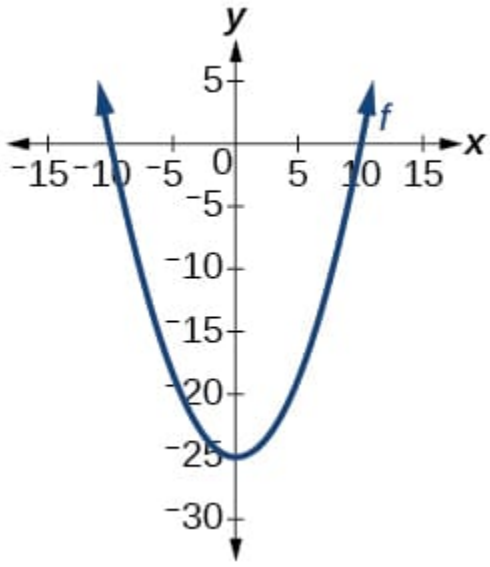
2) $\{(5,2),(6,1),(6,2),(4,8)\}$

3) $y^2+4=x$, for $x$ the independent variable and $y$ the dependent variable

**Answer**

not a function

4) Is the graph in the Figure below a function?




**For the exercises 5-6, evaluate the function at the indicated values:** $f(-3); f(2); f(-a); -f(a); f(a+h)$

5) $f(x)=-2x^2+3x$

**Answer**

$$f(-3)=-27; f(2)=-2; f(-a)=-2a^2-3a; -f(a)=2a^2-3a; f(a+h)=-2a^2+3a-4ah+3h-2h^2$$

6) $f(x)=2|3x-1|$

**For the exercises 7-8, determine whether the functions are one-to-one.**

7) $f(x)=-3x+5$

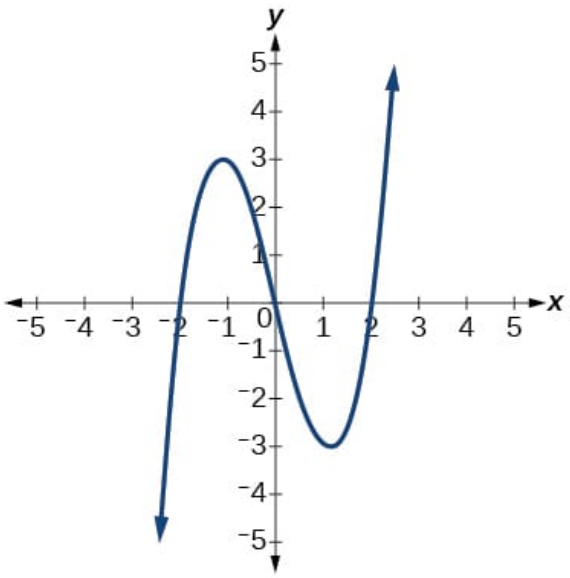
**Answer**

one-to-one

8) $f(x) = |x - 3|$

**For the exercises 9-11, use the vertical line test to determine if the relation whose graph is provided is a function.**

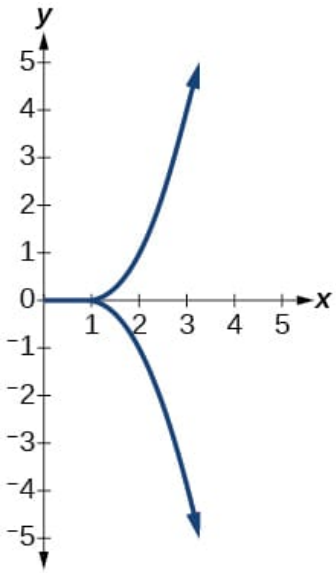
9)



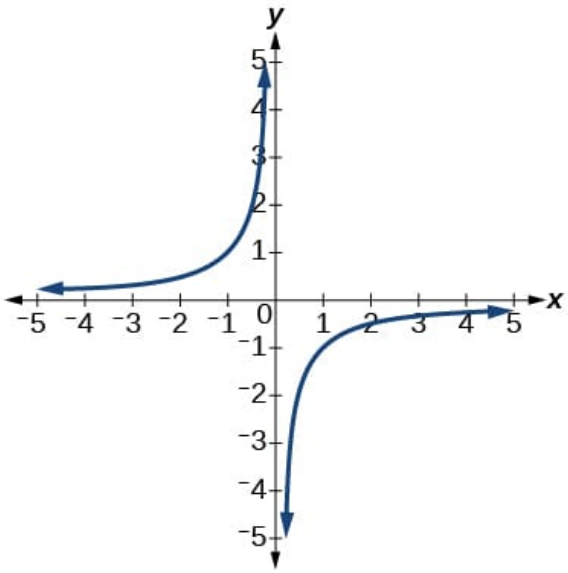

**Answer**

function

10)




11)




**Answer**

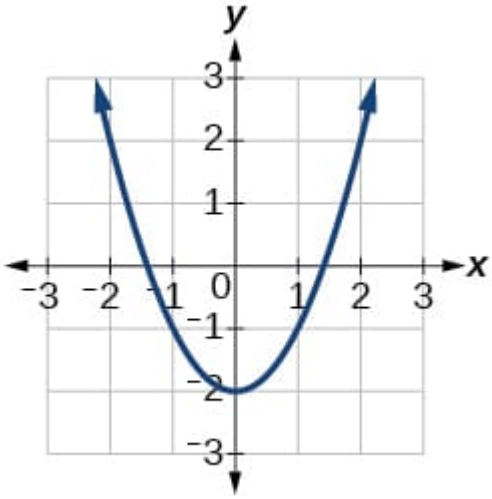
function

**For the exercises 12-13, graph the functions.**

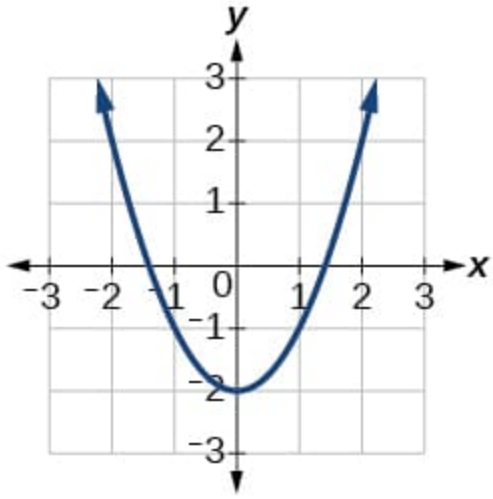
12) $f(x) = |x + 1|$

13) $f(x) = x^2 - 2$

**Answer**




**For the exercises 14-17, use the Figure below to approximate the values.**




14) $f(2)$

15) $f(-2)$

**Answer**

$2$

16) If $f(x) = -2$, then solve for $x$

17) If $f(x) = 1$, then solve for $x$

**Answer**

$x = -1.8$ or $x = 1.8$

**For the exercises 18-19, use the function $h(t) = -16t^2 + 80t$ to find the values.**

18) $\frac{h(2) - h(1)}{2 - 1}$

19) $\frac{h(a) - h(1)}{a - 1}$

**Answer**

$\frac{-64 + 80a - 16a^2}{-1 + a} = -16a + 64$

# 1.2: Domain and Range

**For the exercises 1-4, find the domain of each function, expressing answers using interval notation.**

1) $f(x) = \frac{2}{3x + 2}$

2) $f(x) = \frac{x-3}{x^2-4x-12}$

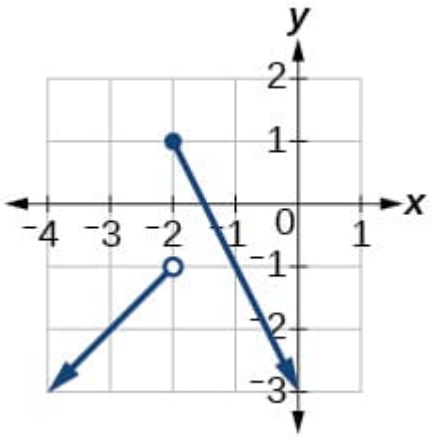
**Answer**

$(-\infty, -2) \cup (-2, 6) \cup (6, \infty)$

3)

4) Graph this piecewise function: $f(x) = \begin{cases} x+1 & x < -2 \\ -2x-3 & x \geq -2 \end{cases}$

**Answer**




## 1.3: Rates of Change and Behavior of Graphs

**For the exercises 1-3, find the average rate of change of the functions from $x = 1$ to $x = 2$**

1) $f(x) = 4x - 3$

2) $f(x) = 10x^2 + x$

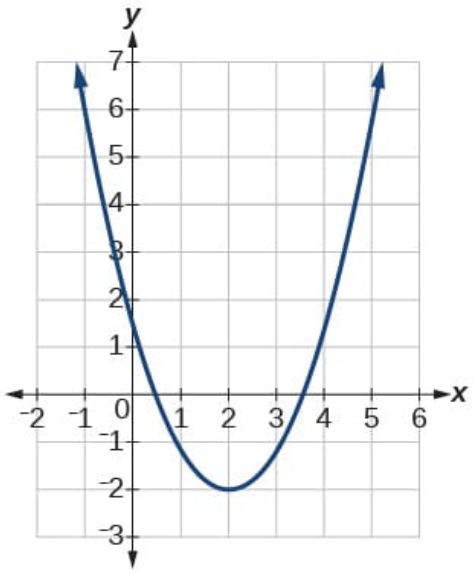
**Answer**

$31$

3) $f(x) = -\dfrac{2}{x^2}$

**For the exercises 4-6, use the graphs to determine the intervals on which the functions are increasing, decreasing, or constant.**

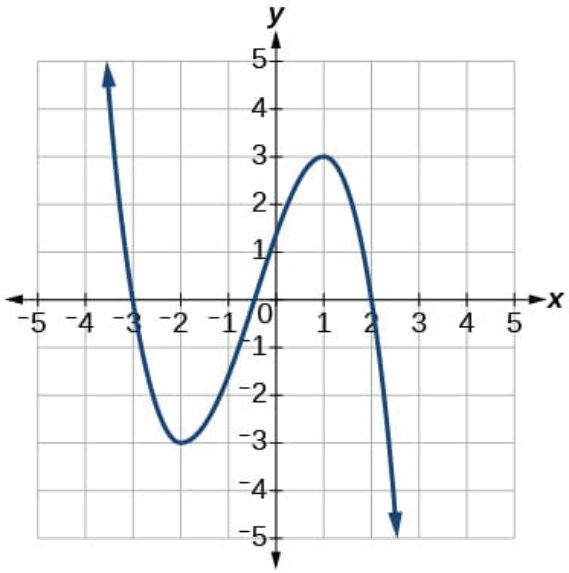
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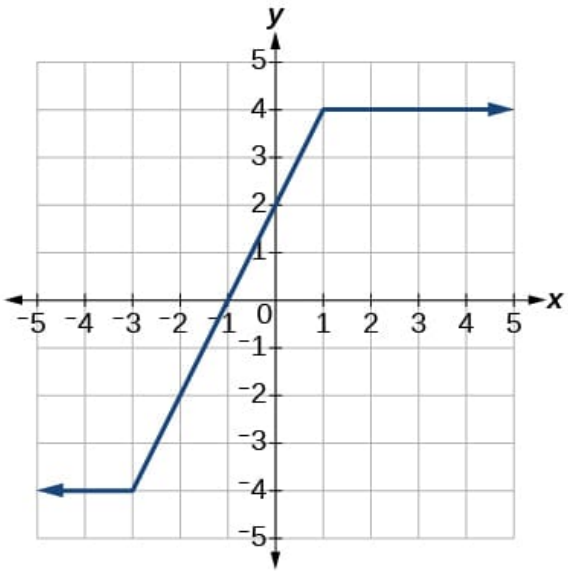

**Answer**

increasing $(2, \infty)$; decreasing $(-\infty, 2)$

5)




6)




**Answer**

increasing $(-3, 1)$; constant $(-\infty, -3) \cup (1, \infty)$

7) Find the local minimum of the function graphed in Exercise 4.

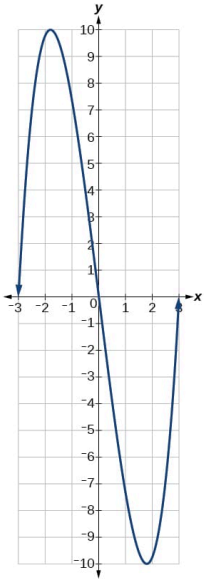
8) Find the local extrema for the function graphed in Exercise 5.

**Answer**

local minimum $(-2, -3)$; local maximum $(1, 3)$

9) For the graph in the Figure in Exercise 10, the domain of the function is $[-3, 3]$. The range is $[-10, 10]$. Find the absolute minimum of the function on this interval.

10) Find the absolute maximum of the function graphed in the Figure below.



**Answer**

$(-1.8, 10)$

# 1.4: Composition of Functions

**For the exercises 1-5, find $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions.**

1) $f(x) = 4 - x, g(x) = -4x$

2) $f(x) = 3x + 2, g(x) = 5 - 6x$

**Answer**

$(f \circ g)(x) = 17 - 18x; (g \circ f)(x) = -7 - 18x$

3) $f(x) = x^2 + 2x, g(x) = 5x + 1$

4) $f(x) = \sqrt{x+2}, g(x) = \dfrac{1}{x}$

**Answer**

$(f \circ g)(x) = \sqrt{\dfrac{1}{x} + 2}; (g \circ f)(x) = \dfrac{1}{\sqrt{x+2}}$

5) $f(x) = \dfrac{x+3}{2}, g(x) = \sqrt{1-x}$

**For the exercises 6-9, find $(f \circ g)$ and the domain for $(f \circ g)(x)$ for each pair of functions.**

6) $f(x) = \frac{x+1}{x+4}, g(x) = \frac{1}{x}$

**Answer**

$(f \circ g)(x) = \dfrac{1+x}{1+4x}, x \neq 0, x \neq -\dfrac{1}{4}$

7) $f(x) = \dfrac{1}{x+3}, g(x) = \dfrac{1}{x-9}$

8) $f(x) = \dfrac{1}{x}, g(x) = \sqrt{x}$

**Answer**

$(f \circ g)(x) = \frac{1}{\sqrt{x}}, x > 0$

9) $f(x) = \frac{1}{x^2-1}, g(x) = \sqrt{x+1}$

**For the exercises 10-11, express each function $H$ as a composition of two functions $f$ and $g$ where $H(x) = (f \circ g)(x)$**

10) $H(x) = \sqrt{\frac{2x-1}{3x+4}}$

**Answer**

sample: $g(x) = \dfrac{2x-1}{3x+4}; f(x) = \sqrt{x}$

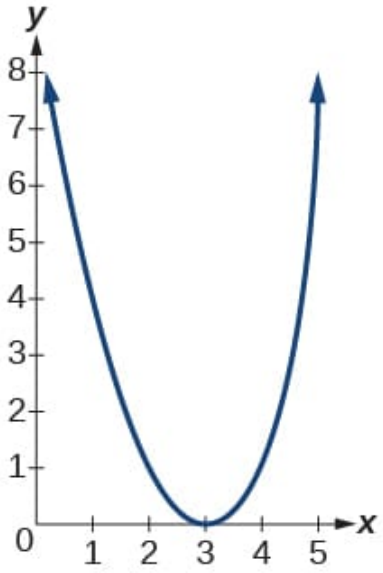
11) $H(x) = \dfrac{1}{(3x^2-4)^{-3}}$

# 1.5: Transformation of Functions

**For the exercises 1-8, sketch a graph of the given function.**

1) $f(x) = (x-3)^2$

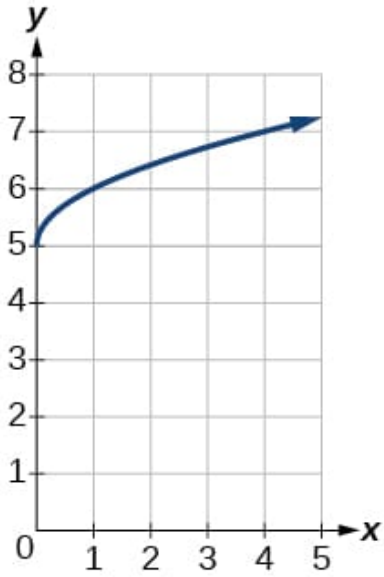
**Answer**




2) $f(x) = (x+4)^3$

3) $f(x) = \sqrt{x} + 5$

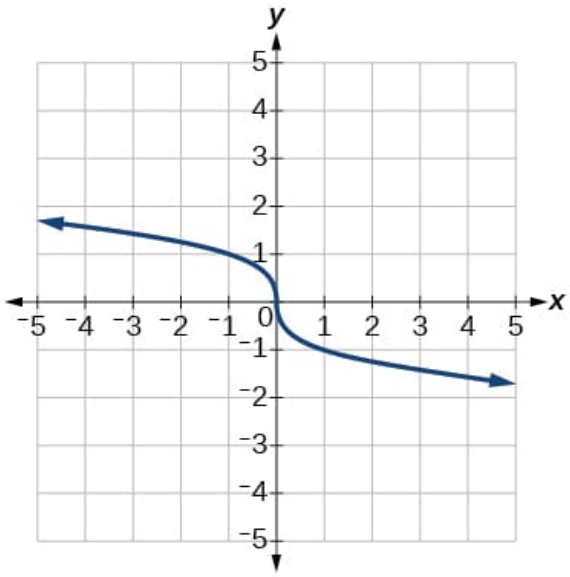
**Answer**




4) $f(x) = -x^3$

5) $f(x) = \sqrt[3]{-x}$

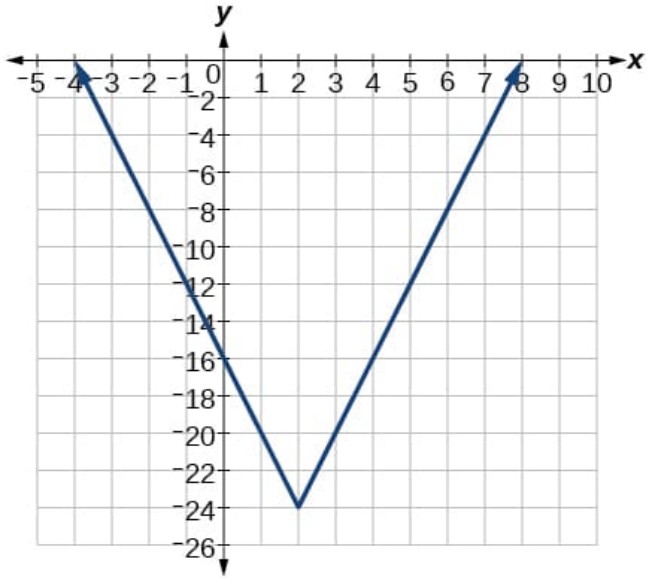
**Answer**




6) $f(x) = 5\sqrt{-x} - 4$

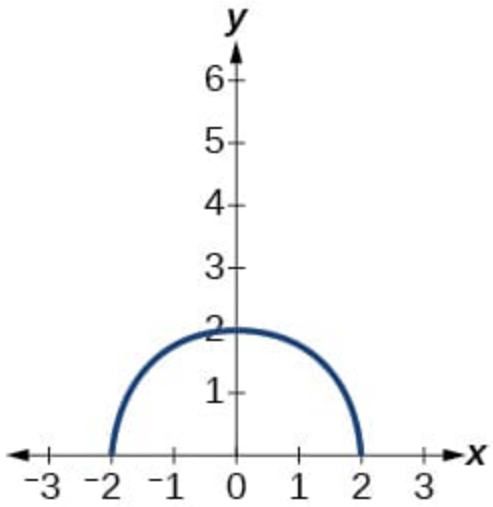
7) $f(x) = 4[|x - 2| - 6]$

**Answer**



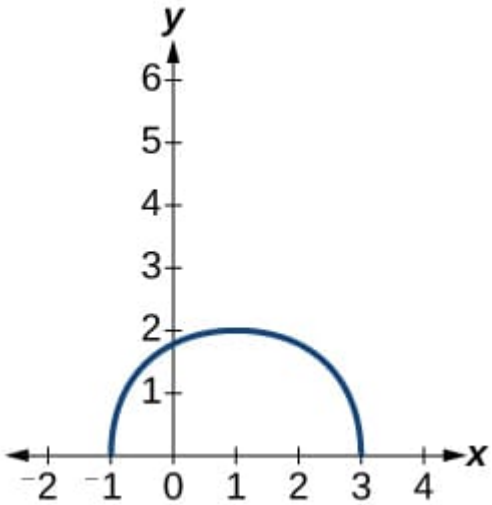

8) $f(x) = -(x + 2)^2 - 1$

**For the exercises 9-10, sketch the graph of the function $g$ if the graph of the function $f$ is shown in the Figure below.**




9) $g(x) = f(x - 1)$

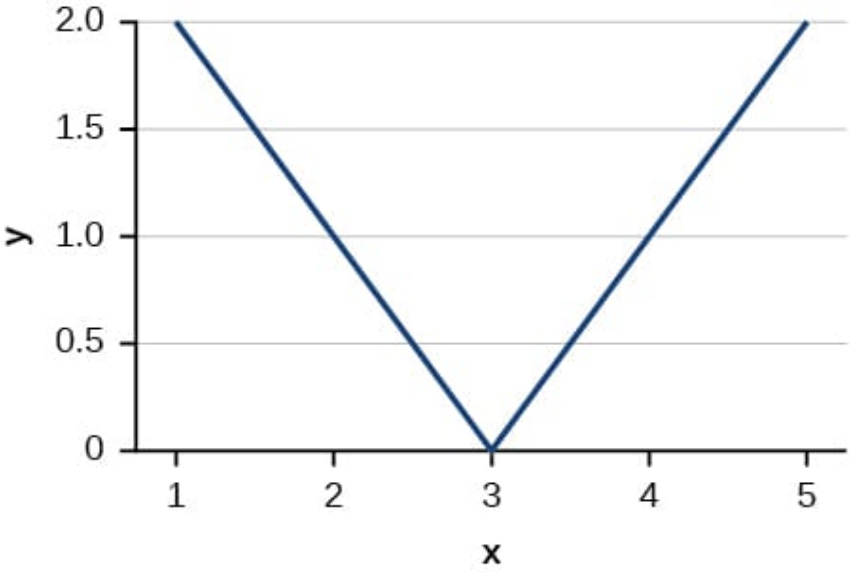
**Answer**




10) $g(x) = 3f(x)$

**For the exercises 11-12, write the equation for the standard function represented by each of the graphs below.**

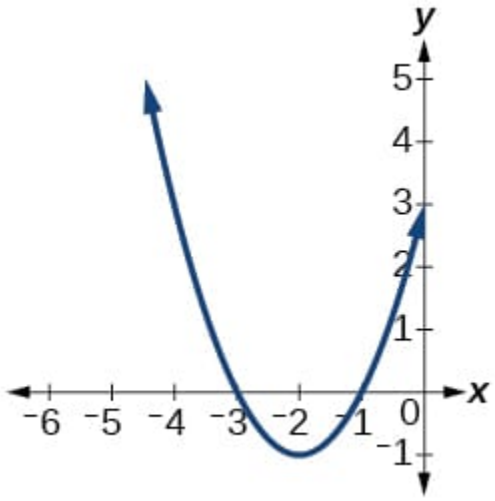
11)




**Answer**

$f(x) = |x - 3|$

12)




**For the exercises 13-15, determine whether each function below is even, odd, or neither.**

13) $f(x) = 3x^4$

**Answer**

even

14) $g(x) = \sqrt{x}$

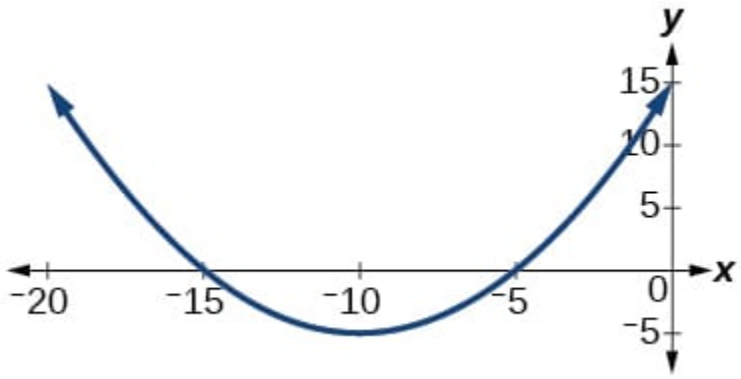
15) $h(x) = \frac{1}{x} + 3x$

**Answer**

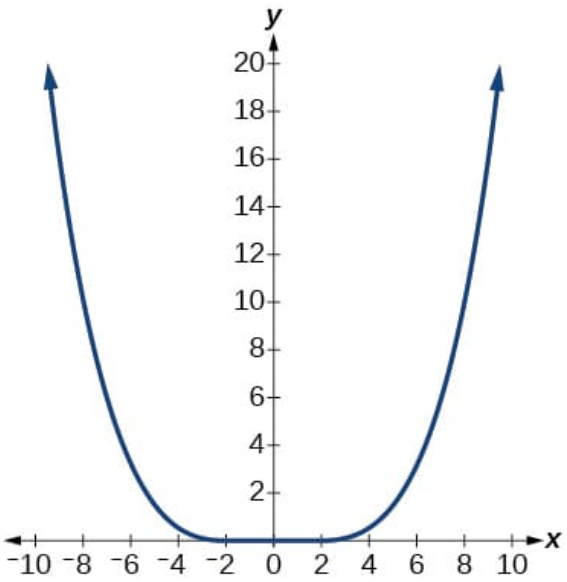
odd

**For the exercises 16-18, analyze the graph and determine whether the graphed function is even, odd, or neither.**

16)



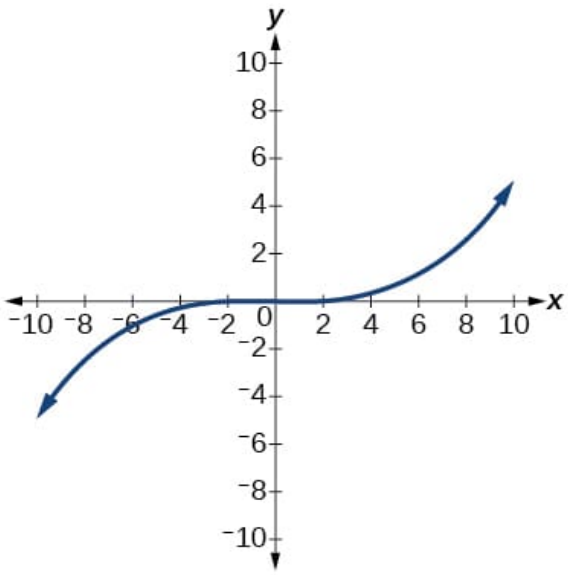

17)




**Answer**

even

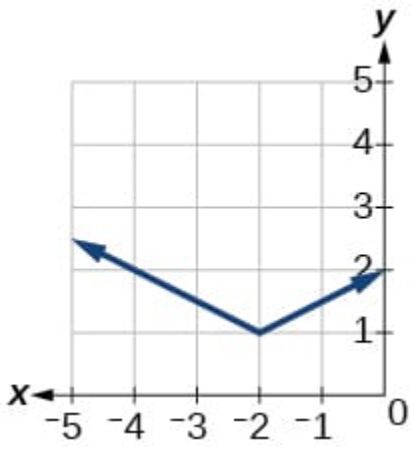
18)




# 1.6: Absolute Value Functions

**For the exercises 1-3, write an equation for the transformation of $f(x) = |x|$.**

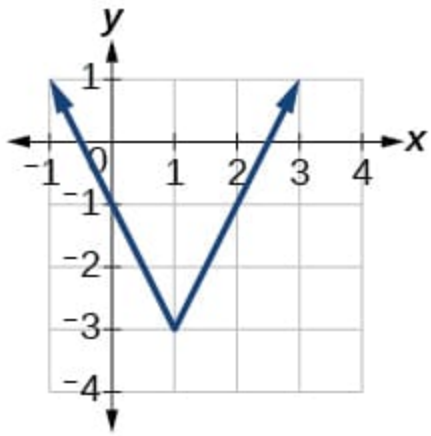
1)



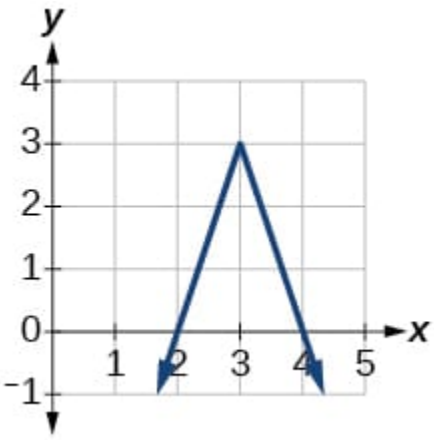

**Answer**

$$f(x) = \frac{1}{2}|x + 2| + 1$$

2)




3)




**Answer**

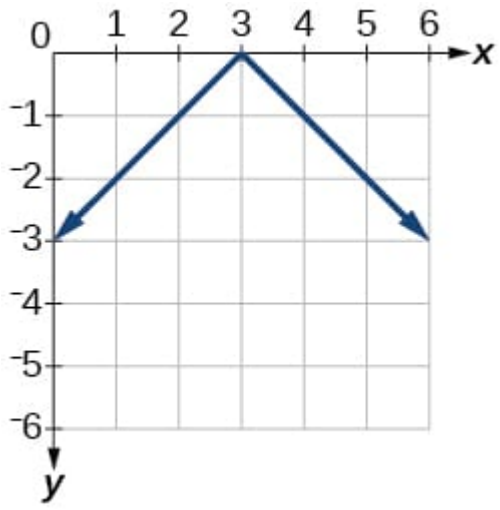
$$f(x) = -3|x - 3| + 3$$

**For the exercises 4-6, graph the absolute value function.**

4) $f(x) = |x - 5|$

5) $f(x) = -|x - 3|$

**Answer**




6) $f(x) = |2x - 4|$

**For the exercises 7-8, solve the absolute value equation.**

7) $|x + 4| = 18$

**Answer**

$$x = -22, x = 14$$

8) $\left|\frac{1}{3}x + 5\right| = \left|\frac{3}{4}x - 2\right|$

**For the exercises 9-10, solve the inequality and express the solution using interval notation.**

9) $|3x - 2| < 7$

**Answer**

$$\left(-\frac{5}{3}, 3\right)$$

10) $\left|\frac{1}{3}x - 2\right| \leq 7$

# 1.7: Inverse Functions

**For the exercises 1-2, find $f^{-1}(x)$ for each function.**

1) $f(x) = 9 + 10x$

2) $f(x) = \frac{x}{x + 2}$

**Answer**

$$f^{-1}(x) = \frac{-2x}{x - 1}$$

3) For the following exercise, find a domain on which the function $f$ is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of $f$ restricted to that domain.

$$f(x) = x^2 + 1 \tag{1.R.1}$$

4) Given $f(x) = x^3 - 5$ and $g(x) = \sqrt[3]{x+5}$:

a. Find $f(g(x))$ and $g(f(x))$.
b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

**Answer**

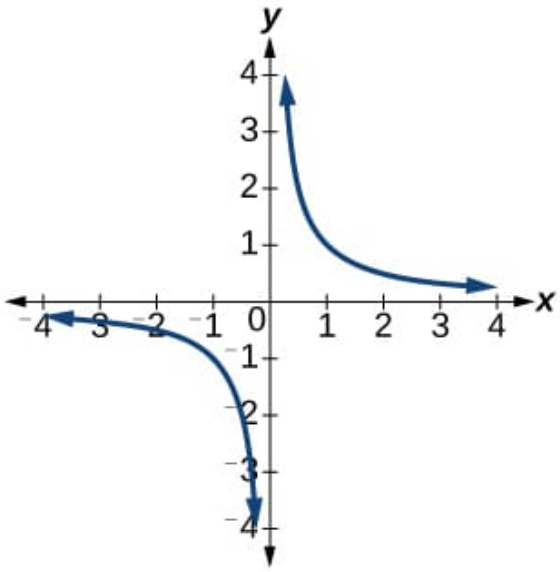
a. $f(g(x)) = x$ and $g(f(x)) = x$
b. This tells us that $f$ and $g$ are inverse functions

**For the exercises 5-8, use a graphing utility to determine whether each function is one-to-one.**

5) $f(x) = \dfrac{1}{x}$

**Answer**

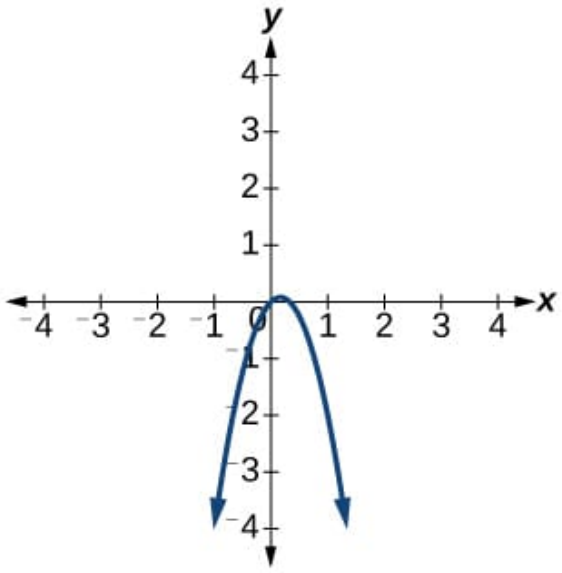
The function is one-to-one.




6) $f(x) = -3x^2 + x$

**Answer**

The function is not one-to-one.




7) If $f(5) = 2$, find $f^{-1}(2)$

**Answer**

$5$

8) If $f(1) = 4$, find $f^{-1}(4)$

## Practice Test

**For the exercises 1-2, determine whether each of the following relations is a function.**

1) $y = 2x + 8$

**Answer**

The relation is a function.

2) $\{(2, 1), (3, 2), (-1, 1), (0, -2)\}$

**For the exercises 3-4, evaluate the function $f(x) = -3x^2 + 2x$ at the given input.**

3) $f(-2)$

**Answer**

$-16$

4) $f(a)$

5) Show that the function $f(x) = -2(x - 1)^2 + 3$ is not one-to-one.

**Answer**

The graph is a parabola and the graph fails the horizontal line test.

6) Write the domain of the function $f(x) = \sqrt{3 - x}$ in interval notation.

7) Given $f(x) = 2x^2 - 5x$, find $f(a + 1) - f(1)$

**Answer**

$2a^2 - a$

8) Graph the function $f(x) = \begin{cases} x+1 & \text{if} \quad -2 < x < 3 \\ -x & \text{if} \quad x \geq 3 \end{cases}$

9) Find the average rate of change of the function $f(x) = 3 - 2x^2 + x$ by finding $\frac{f(b) - f(a)}{b - a}$

**Answer**

$-2(a+b) + 1$

**For the exercises 10-11, use the functions $f(x) = 3 - 2x^2 + x$ and $g(x) = \sqrt{x}$ to find the composite functions.**

10) $(g \circ f)(x)$

11) $(g \circ f)(1)$

**Answer**

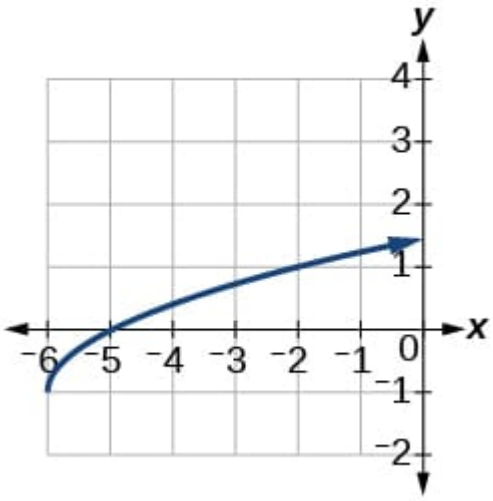
$\sqrt{2}$

12) Express $H(x) = \sqrt[3]{5x^2 - 3x}$ a composition of two functions, $f$ and $g$, where $(f \circ g)(x) = H(x)$

**For the exercises 13-14, graph the functions by translating, stretching, and/or compressing a toolkit function.**

13) $f(x) = \sqrt{x+6} - 1$

**Answer**




14) $f(x) = \frac{1}{x+2} - 1$

**For the exercises 15-17, determine whether the functions are even, odd, or neither.**

15) $f(x) = -\frac{5}{x^2} + 9x^6$

**Answer**

even

16) $f(x) = -\frac{5}{x^3} + 9x^5$

17) $f(x) = \frac{1}{x}$

**Answer**

odd

18) Graph the absolute value function $f(x) = -2|x - 1| + 3$.

19) Solve $|2x - 3| = 17$.

**Answer**

$x = -7$ and $x = 10$

20) Solve $-\left|\frac{1}{3}x - 3\right| \geq 17$. Express the solution in interval notation.

**For the exercises 21-22, find the inverse of the function.**

21) $f(x) = 3x - 5$

**Answer**

$f^{-1}(x) = \frac{x + 5}{3}$

22) $f(x) = \frac{4}{x + 7}$

**For the exercises 23-26, use the graph of $g$ shown in the Figure below.**

23) On what intervals is the function increasing?

**Answer**

$(-\infty, -1.1)$ and $(1.1, \infty)$

24) On what intervals is the function decreasing?

25) Approximate the local minimum of the function. Express the answer as an ordered pair.

**Answer**

$(1.1, -0.9)$

26) Approximate the local maximum of the function. Express the answer as an ordered pair.

**For the exercises 27-29, use the graph of the piecewise function shown in the Figure below.**

27) Find $f(2)$.

**Answer**

$f(2) = 2$

28) Find $f(-2)$.

29) Write an equation for the piecewise function.

**Answer**

$$f(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$$

**For the exercises 30-35, use the values listed in the Table below.**

| $x$ | $F(x)$ |
|---|---|
| 0 | 1 |
| 1 | 3 |
| 2 | 5 |
| 3 | 7 |
| 4 | 9 |
| 5 | 11 |
| 6 | 13 |
| 7 | 15 |
| 8 | 17 |

30) Find $F(6)$.

31) Solve the equation $F(x) = 5$

**Answer**

$x = 2$

32) Is the graph increasing or decreasing on its domain?

33) Is the function represented by the graph one-to-one?

**Answer**

yes

34) Find $F^{-1}(15)$.

35) Given $f(x) = -2x + 11$, find $f^{-1}(x)$.

**Answer**

$$f^{-1}(x) = -\frac{x - 11}{2}$$

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# 2: LINEAR FUNCTIONS

Recall that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.


2.0: Prelude to Linear Functions

2.1: Linear Functions

2.2: Graphs of Linear Functions

2.3: Modeling with Linear Functions

2.4: Fitting Linear Models to Data

2.E: Linear Functions(Exercises)

2.R: Linear Functions(Review)


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# 2.0: Prelude to Linear Functions

Imagine placing a plant in the ground one day and finding that it has doubled its height just a few days later. Although it may seem incredible, this can happen with certain types of bamboo species. These members of the grass family are the fastest-growing plants in the world. One species of bamboo has been observed to grow nearly 1.5 inches every hour.1 In a twenty-four hour period, this bamboo plant grows about 36 inches, or an incredible 3 feet! A constant rate of change, such as the growth cycle of this bamboo plant, is a linear function.

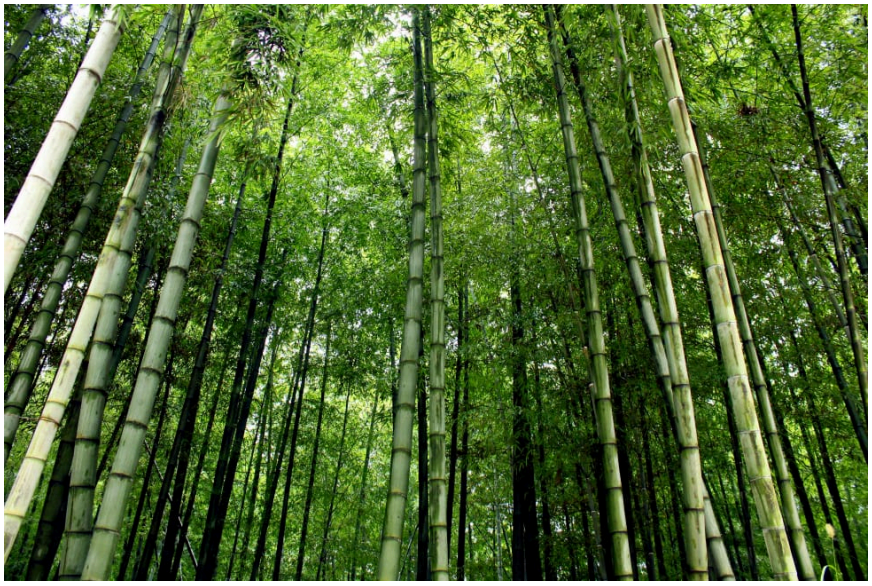


***Figure*** 2.0.1***:*** *A bamboo forest in China (credit: "JFXie"/Flickr)*

Recall from Functions and Function Notation that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

## Footnotes

1 www.guinnessworldrecords.com/...growing-plant/

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# 2.1: Linear Functions

Learning Objectives

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Interpret slope as a rate of change.
- Write and interpret an equation for a linear function.
- Graph linear functions.
- Determine whether lines are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train (maglev train in the world, the Shanghai MagLev Train (Figure 2.1.1). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes.




Figure 2.1.1: Shanghai MagLev Train (credit: “kanegen”/Flickr)

Suppose a maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.

## Representing Linear Functions

The function describing the train’s motion is a **linear function**, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train’s motion as a function using each method.

## Representing a Linear Function in Word Form

Let's begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train's distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

## Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as the slope-intercept form of a line, where $x$ is the input value, $m$ is the rate of change, and $b$ is the initial value of the dependent variable.

$$\begin{aligned} &\text{Equation form} && y = mx + b \\ &\text{Equation notation} && f(x) = mx + b \end{aligned}$$

In the example of the train, we might use the notation $D(t)$ in which the total distance $D$ is a function of the time $t$. The rate, $m$, is 83 meters per second. The initial value of the dependent variable $b$ is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$$D(t) = 83t + 250 \tag{2.1.1}$$

## Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 2.1.2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

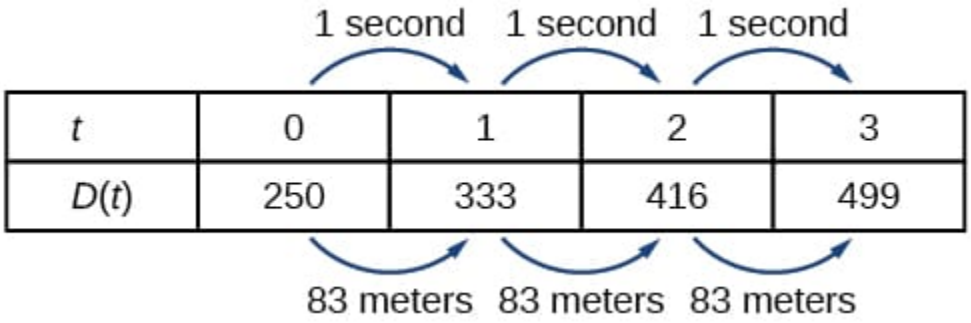



Figure 2.1.2: Tabular representation of the function $D$ showing selected input and output values

**Q/A? Can the input in the previous example be any real number?**

No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

## Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph, represented in Figure 2.1.3. Notice the graph is a line. When we plot a linear function, the graph is always

a line.

The rate of change, which is constant, determines the slant, or **slope** of the line. The point at which the input value is zero is the vertical intercept, or **y-intercept**, of the line. We can see from the graph in Figure $2.1.3$ that the y-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

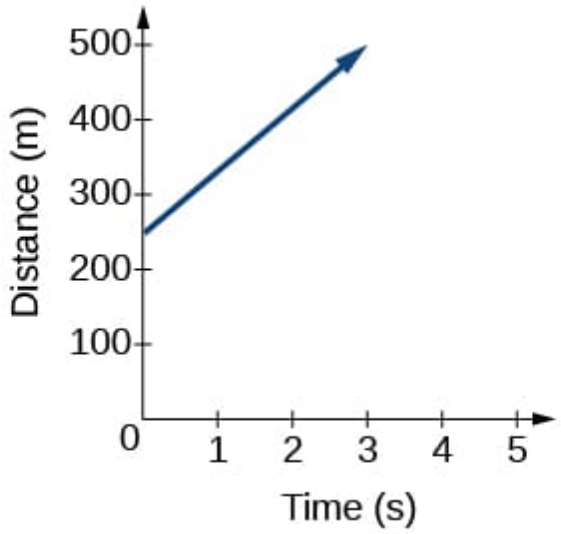



Figure $2.1.3$: The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1$. Ask yourself what numbers can be input to the function, that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

## Definition: Linear Function

A **linear function** is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b \tag{2.1.2}$$

where $b$ is the initial or starting value of the function (when input, $x = 0$), and $m$ is the constant rate of change, or **slope** of the function. The **y-intercept** is at $(0, b)$.

## Example $2.1.1$: Using a Linear Function to Find the Pressure on a Diver

The pressure, $P$, in pounds per square inch (PSI) on the diver in Figure $2.1.4$ depends upon her depth below the water surface, $d$, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.



Figure 2.1.4: (credit: Ilse Reijs and Jan-Noud Hutten)Hutten)

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

**Analysis**

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

## Determining whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an **increasing function** has a positive slope. A line with a positive slope slants upward from left to right as in Figure 2.1.5(a). For a **decreasing function**, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 2.1.5(b). If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 2.1.5(c).

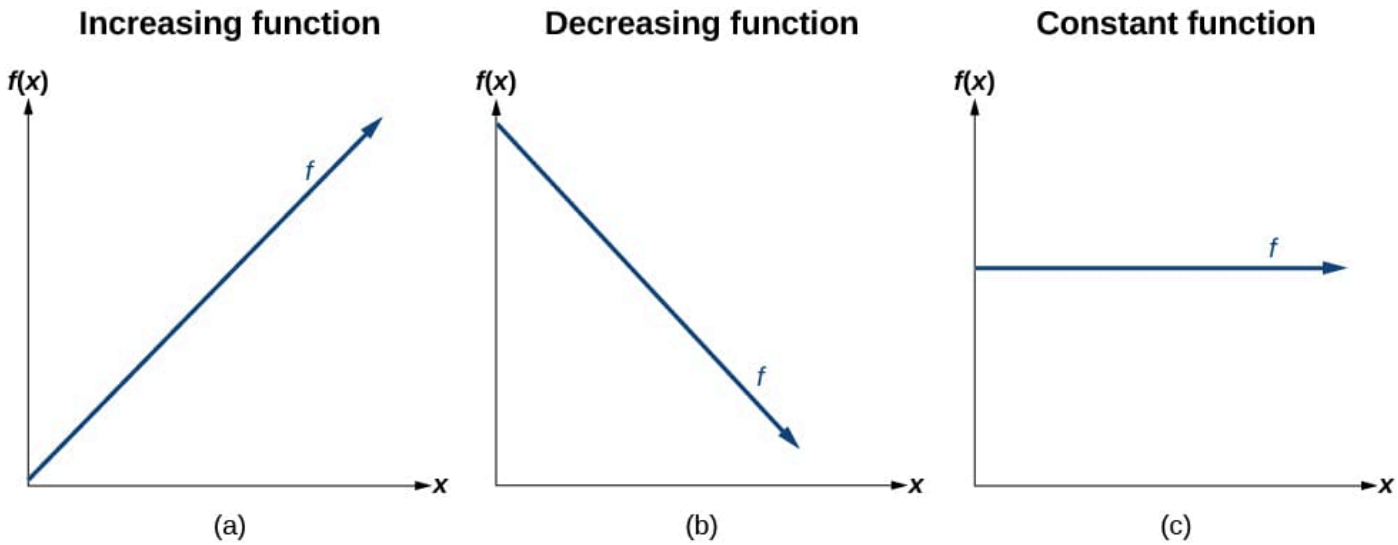



Figure 2.1.5: Three graphs depicting an increasing function, a decreasing function, and a constant function.]
Increasing and Decreasing Functions

## Increasing and Decreasing Functions

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is an decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

## Example 2.1.2: decide whether a Function is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

a. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
b. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
c. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

**Solution**

Analyze each function.

a. The function can be represented as $f(x) = 60x$ where $x$ is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
b. The function can be represented as $f(x) = 500 - 60x$ where $x$ is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after $x$ days.
c. The cost function can be represented as $f(x) = 50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

# Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, $x_1$ and $x_2$, and two corresponding values for the output, $y_1$ and $y_2$—which can be represented by a set of points, $(x_1, y_1)$ and $(x_2, y_2)$—we can calculate the slope $m$, as follows

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}}$$
$$= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where $\Delta y$ is the vertical displacement and $\Delta x$ is the horizontal displacement. Note in function notation two corresponding values for the output $y_1$ and $y_2$ for the function $f$, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Figure 2.1.6 indicates how the slope of the line between the points, $(x_1, y_1)$ and $(x_2, y_2)$, is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.

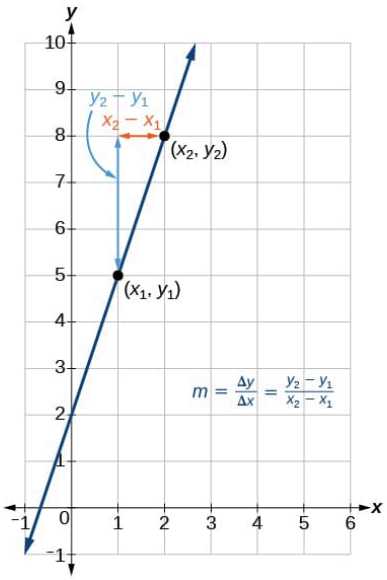



Figure 2.1.6: The slope of a function is calculated by the change in $y$ divided by the change in $x$. It does not matter which coordinate is used as the $(x_2, y_2)$ and which is the $(x_1, y_1)$,as long as each calculation is started with the elements from the same coordinate pair.

## Q & A

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

## Note: Calculate Slope

The slope or rate of change of a function, $m$, can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.1.3}$$

where $x_1$ and $x_2$ are input values, $y_1$ and $y_2$ are output values.

**How To: How to: Given two points from a linear function, calculate and interpret the slope.**

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

### Example 2.1.3: Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3,-2)$ and $(8,1)$ are points on the line, find the slope. Is this function increasing or decreasing?

**Solution**

The coordinate pairs are $(3,-2)$ and $(8,1)$. To find the rate of change, we divide the change in output by the change in input.

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{1-(-2)}{8-3} = \frac{3}{5} \tag{2.1.4}$$

We could also write the slope as $m = 0.6$. The function is increasing because $m > 0$.

**Analysis**

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used.

### Try it! 2.1.1

If $f(x)$ is a linear function, and $(2,3)$ and $(0,4)$ are points on the line, find the slope. Is this function increasing or decreasing?

*Answer available at the end of this section.*

### Example 2.1.4: Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

The rate of change relates the change in population to the change in time. The population increased by $27,800 - 23,400 = 4400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4,400 \text{ people}}{4 \text{ years}} = 1,100 \frac{\text{people}}{\text{year}} \tag{2.1.5}$$

So the population increased by 1,100 people per year.

**Analysis**

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

**Try it! 2.1.2**

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

*Answer available at the end of this section.*

## Writing the Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the **point-slope form**.

$$y - y_1 = m(x - x_1) \tag{2.1.6}$$

The point-slope form is derived from the slope formula.

$$\begin{aligned} m &= \frac{y - y_1}{x - x_1} && \text{assuming } x \neq x_1 \\ m(x - x_1) &= \frac{y - y_1}{x - x_1}(x - x_1) && \text{Multiply both sides by } (x - x_1). \\ m(x - x_1) &= y - y_1 && \text{Simplify} \\ y - y_1 &= m(x - x_1) && \text{Rearrange} \end{aligned}$$

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, $y - 4 = -\frac{1}{2}(x - 6)$. We can convert it to the slope-intercept form as shown.

$$\begin{aligned} y - 4 &= -\frac{1}{2}(x - 6) \\ y - 4 &= -\frac{1}{2}x + 3 && \text{Distribute the } -\frac{1}{2}. \\ y &= -\frac{1}{2}x + 7 && \text{Add 4 to each side.} \end{aligned}$$

Therefore, the same line can be described in slope-intercept form as $y = \frac{1}{2}x + 7$.

**Note: Point-Slope Form of a Linear Equation**

The **point-slope** form of a linear equation takes the form

$$y - y_1 = m(x - x_1) \tag{2.1.7}$$

where $m$ is the slope, $x_1$ and $y_1$ are the $x$ and $y$ coordinates of a specific point through which the line passes.

## Writing the Equation of a Line Using a Point and the Slope

The point-slope form is particularly useful if we know one point and the slope of a line. Suppose, for example, we are told that a line has a slope of 2 and passes through the point $(4, 1)$. We know that $m = 2$ and that $x_1 = 4$ and $y_1 = 1$. We can substitute these values into the general point-slope equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 4) \end{aligned}$$

If we wanted to then rewrite the equation in slope-intercept form, we apply algebraic techniques.

$$\begin{aligned} y-1 &= 2(x-4) \\ y-1 &= 2x-8 && \text{Distribute the 2.} \\ y &= 2x-7 && \text{Add 1 to each side.} \end{aligned}$$

Both equations, $y-1=2(x-4)$ and $y=2x$–7, describe the same line. See Figure 2.1.7.

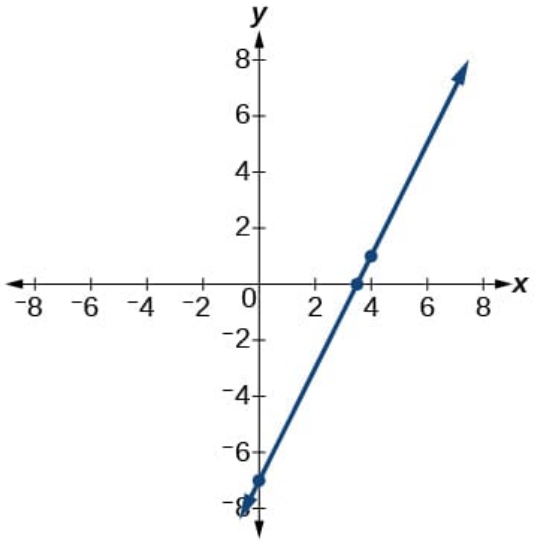



Figure 2.1.7

## Example 2.1.5: Writing Linear Equations Using a Point and the Slope

Write the point-slope form of an equation of a line with a slope of 3 that passes through the point $(6,-1)$. Then rewrite it in the slope-intercept form.

**Solution**

Let's figure out what we know from the given information. The slope is 3, so $m=3$. We also know one point, so we know $x_1=6$ and $y_1=-1$. Now we can substitute these values into the general point-slope equation.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-(-1) &= 3(x-6) && \text{Substitute known values.} \\ y+1 &= 3(x-6) && \text{Distribute } -1 \text{ to find point-slope form.} \end{aligned}$$

Then we use algebra to find the slope-intercept form.

$$\begin{aligned} y+1 &= 3(x-6) \\ y+1 &= 3x-18 && \text{Distribute 3.} \\ y &= 3x-19 && \text{Simplify to slope-intercept form.} \end{aligned}$$

## Try it! 2.1.3

Write the point-slope form of an equation of a line with a slope of –2 that passes through the point $(-2,2)$. Then rewrite it in the slope-intercept form.

*Answer available at the end of this section.*

## Writing the Equation of a Line Using Two Points

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points $(0, 1)$ and $(3, 2)$. We can use the coordinates of the two points to find the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 1}{3 - 0} \\ &= \frac{1}{3} \end{aligned}$$

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use $(0, 1)$ for our point.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{3}(x - 0) \end{aligned}$$

As before, we can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y - 1 &= \frac{1}{3}(x - 0) && \\ y - 1 &= \frac{1}{3}x && \text{Distribute the } \frac{1}{3}. \\ y &= \frac{1}{3}x + 1 && \text{Add 1 to each side.} \end{aligned}$$

Both equations describe the line shown in Figure 2.1.8.

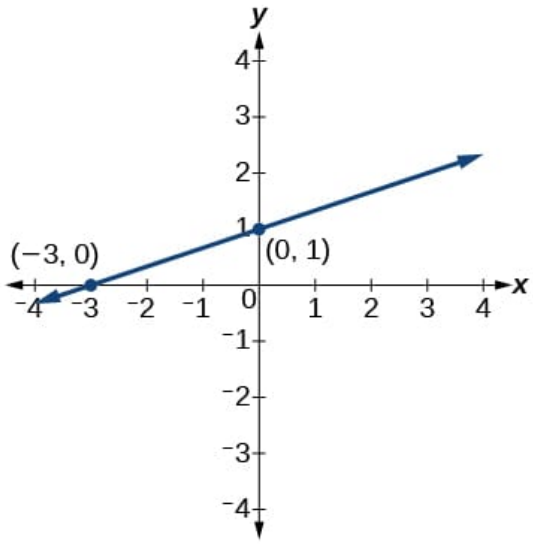



Figure 2.1.8

### Example 2.1.6: Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points $(5, 1)$ and $(8, 7)$. Then rewrite it in the slope-intercept form.

Let's begin by finding the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7-1}{8-5} \\ &= \frac{6}{3} \\ &= 2 \end{aligned}$$

So $m = 2$. Next, we substitute the slope and the coordinates for one of the points into the general point-slope equation. We can choose either point, but we will use $(5, 1)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 5) \end{aligned}$$

The point-slope equation of the line is $y_2–1 = 2(x_2–5)$. To rewrite the equation in slope-intercept form, we use algebra.

$$\begin{aligned} y - 1 &= 2(x - 5) \\ y - 1 &= 2x - 10 \\ y &= 2x - 9 \end{aligned}$$

The slope-intercept equation of the line is $y = 2x–9$.

**Try It! 2.1.4**

Write the point-slope form of an equation of a line that passes through the points $(–1, 3)$ and $(0, 0)$. Then rewrite it in the slope-intercept form.

*Answer available at the end of this section.*

## Writing and Interpreting an Equation for a Linear Function

Now that we have written equations for linear functions in both the slope-intercept form and the point-slope form, we can choose which method to use based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function $f$ in Figure 2.1.9.

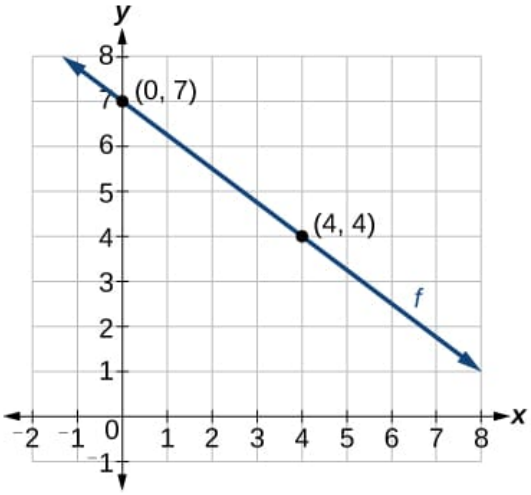



Figure 2.1.9: Graph depicting how to calculate the slope of a line.

We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let's choose $(0, 7)$ and $(4, 4)$. We can use these points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-7}{4-0} \\ &= -\frac{3}{4} \end{aligned}$$

Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -\frac{3}{4}(x - 4) \end{aligned}$$

If we want to rewrite the equation in the slope-intercept form, we would find

$$\begin{aligned} y - 4 &= -\frac{3}{4}(x - 4) \\ y - 4 &= -\frac{3}{4}x + 3 \\ y &= -\frac{3}{4}x + 7 \end{aligned}$$

If we wanted to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the y-axis when the output value is 7. Therefore, $b = 7$. We now have the initial value $b$ and the slope $m$ so we can substitute $m$ and $b$ into the slope-intercept form of a line.

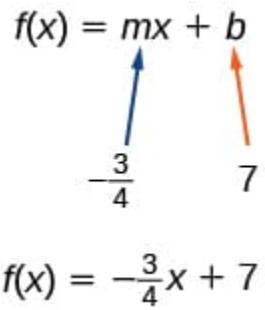



Figure 2.1.10

So the function is $f(x) = -\frac{3}{4}x + 7$, and the linear equation would be $y = -\frac{3}{4}x + 7$.

**How To: How to: Given the graph of a linear function, write an equation to represent the function.**

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the y-axis to identify the y-intercept by visual inspection.
4. Substitute the slope and y-intercept into the slope-intercept form of a line equation.

**Example 2.1.7: Writing an Equation for a Linear Function**

Write an equation for a linear function given a graph of $f$ shown in Figure 2.1.11.

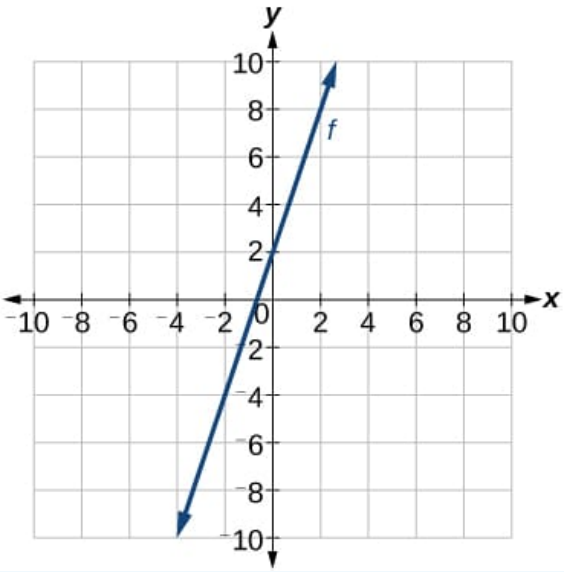



Figure 2.1.11

**Solution**

Identify two points on the line, such as $(0, 2)$ and $(-2, -4)$. Use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 2}{-2 - 0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= 3(x - (-2)) \\ y + 4 &= 3(x + 2) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y + 4 &= 3(x + 2) \\ y + 4 &= 3x + 6 \\ y &= 3x + 2 \end{aligned}$$

**Analysis**

This makes sense because we can see from Figure 2.1.12 that the line crosses the y-axis at the point $(0, 2)$, which is the y-intercept, so $b = 2$.

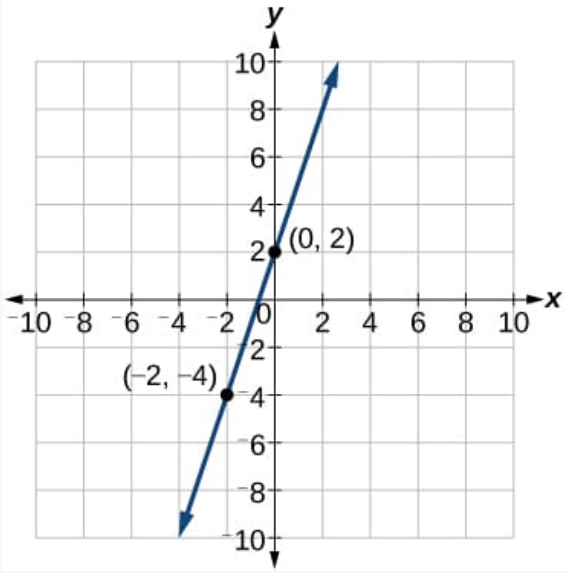



Figure 2.1.12: Graph of an increasing line with points at $(0, 2)$ and $(-2, -4)$.

## Example 2.1.8: Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function $C$ where $C(x)$ is the cost for $x$ items produced in a given month.

**Solution**

The fixed cost is present every month, $1,250. The costs that can vary include the cost to produce each item, which is $37.50 for Ben. The variable cost, called the marginal cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs, represented by $C(x) = 1250 + 37.5x$.

**Analysis**

If Ben produces 100 items in a month, his monthly cost is represented by

$$\begin{aligned} C(100) &= 1250 + 37.5(100) \\ &= 5000 \end{aligned}$$

So his monthly cost would be $5,000.

## Example 2.1.9: Writing an Equation for a Linear Function Given Two Points

If $f$ is a linear function, with $f(3) = -2$, and $f(8) = 1$, find an equation for the function in slope-intercept form.

**Solution**

We can write the given points using coordinates.

$$\begin{aligned} f(3) = -2 &\rightarrow (3, 2) \\ f(8) = 1 &\rightarrow (8, 1) \end{aligned}$$

We can then use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-2)}{8 - 3} \\ &= \frac{3}{5} \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= \frac{3}{5}(x - 3) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y + 2 &= \frac{3}{5}(x - 3) \\ y + 2 &= \frac{3}{5}x - \frac{9}{5} \\ y &= \frac{3}{5}x - \frac{19}{5} \end{aligned}$$

**Try It! 2.1.5**

If $f(x)$ is a linear function, with $f(2) = -11$, and $f(4) = -25$, find an equation for the function in slope-intercept form.

*Answer available at the end of this section.*

## Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

**How To: How to: Given a linear function $f$ and the initial value and rate of change, evaluate $f(c)$.**

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

**Example 2.1.10: Using a Linear Function to Determine the Number of Songs in a Music Collection**

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, $N$, in his collection as a function of time, $t$, the number of months. How many songs will he own in a year?

**Solution**

The initial value for this function is 200 because he currently owns 200 songs, so $N(0) = 200$, which means that $b = 200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m = 15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.

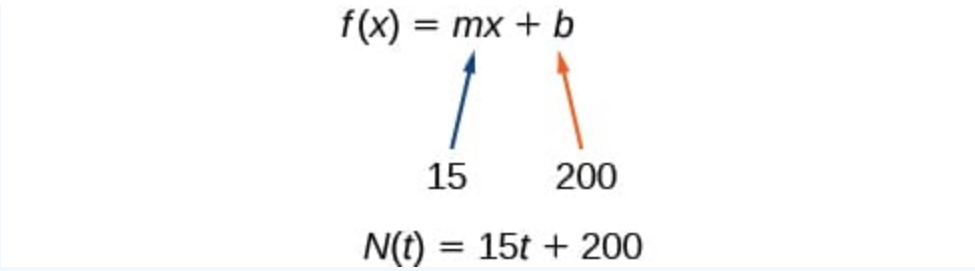



Figure 2.1.13

We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have in 1 year (12 months). In other words, we can evaluate the function at $t = 12$.

$$\begin{aligned} N(12) &= 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

**Analysis**

Notice that $N$ is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

## Example 2.1.11: Using a Linear Function to Calculate Salary Plus Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income, I,depends on the number of new policies, $n$, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

**Solution**

The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$\begin{aligned} m &= \frac{920 - 760}{5 - 3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy. Therefore, Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value.

$$\begin{aligned} I(n) &= 80n + b \\ 760 &= 80(3) + b \text{ when } n = 3, I(3) = 760 \\ 760 - 80(3) &= b \\ 520 &= b \end{aligned}$$

The value of $b$ is the starting value for the function and represents Ilya's income when $n = 0$, or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya's base salary is $520 per week and he earns an additional $80 commission for each policy sold.

## Example 2.1.12: Using Tabular Form to Write an Equation for a Linear Function

Table 2.1.1 relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

*Table* 2.1.1

| **w, number of weeks** | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| **P(w), number of rats** | 1000 | 1080 | 1160 | 1240 |

**Solution**

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for $m$, we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w) = 40w + 1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

$$\begin{aligned} m &= \frac{1240 - 1080}{6 - 2} \\ &= \frac{160}{4} \\ &= 40 \end{aligned}$$

### Is the initial value always provided in a table of values?

*No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for $b$.*

### Try It 2.1.6

A new plant food was introduced to a young tree to test its effect on the height of the tree. Table 2.1.2 shows the height of the tree, in feet, $x$ months since the measurements began. Write a linear function, $H(x)$, where $x$ is the number of months since the start of the experiment.

*Table* 2.1.2

| $x$ | 0 | 2 | 4 | 8 | 12 |
|---|---|---|---|---|---|
| $H(x)$ | 12.5 | 13.5 | 14.5 | 16.5 | 18.5 |

*Answer available at the end of this section.*

## Key Equations

- slope-intercept form of a line: $f(x) = mx + b$
- slope: $m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$
- point-slope form of a line: $y - y_1 = m(x - x_1)$

## Key Concepts

- The ordered pairs given by a linear function represent points on a line.
- Linear functions can be represented in words, function notation, tabular form, and graphical form.
- The rate of change of a linear function is also known as the slope.
- An equation in the slope-intercept form of a line includes the slope and the initial value of the function.
- The initial value, or y-intercept, is the output value when the input of a linear function is zero. It is the y-value of the point at which the line crosses the y-axis.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope.
- A decreasing linear function results in a graph that slants downward from left to right and has a negative slope.
- A constant linear function results in a graph that is a horizontal line.
- Analyzing the slope within the context of a problem indicates whether a linear function is increasing, decreasing, or constant.
- The slope of a linear function can be calculated by dividing the difference between y-values by the difference in corresponding x-values of any two points on the line.
- The slope and initial value can be determined given a graph or any two points on the line.
- One type of function notation is the slope-intercept form of an equation.
- The point-slope form is useful for finding a linear equation when given the slope of a line and one point.
- The point-slope form is also convenient for finding a linear equation when given two points through which a line passes.
- The equation for a linear function can be written if the slope $m$ and initial value $b$ are known.
- A linear function can be used to solve real-world problems.
- A linear function can be written from tabular form.

## Footnotes

1 www.chinahighlights.com/shang...glev-train.htm
2 www.cbsnews.com/8301-501465_1...ay-study-says/

## Glossary

**decreasing linear function**

a function with a negative slope: If $f(x) = mx + b$, then $m < 0$.

**increasing linear function**

a function with a positive slope: If $f(x) = mx + b$, then $m > 0$.

**linear function**

a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

**point-slope form**

the equation for a line that represents a linear function of the form \(y–y_1=m(x–x_1)

**slope**

the ratio of the change in output values to the change in input values; a measure of the steepness of a line

**slope-intercept form**

the equation for a line that represents a linear function in the form $f(x) = mx + b$

**y-intercept**

the value of a function when the input value is zero; also known as initial value

## Answers

**Try it!** 2.1.1

$m = \frac{4-3}{0-2} = \frac{1}{-2} = -\frac{1}{2}$; decreasing because $m < 0$.

**Try it!** 2.1.2

$m = \frac{1,868-1,442}{2,012-2,009} = \frac{426}{3} = 142$ people per year

**Try it!** 2.1.3

$y - 2 = -2(x + 2)$ ; $y = -2x - 2$

**Try It!** 2.1.4

$y - 0 = -3(x - 0)$; $y = -3x$.

**Try It!** 2.1.5

$y = -7x + 3$

**Try It** 2.1.6

$H(x) = 0.5x + 12.5$

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# 2.2: Graphs of Linear Functions

**Learning Objectives**

- Graph linear functions.
- Write the equation for a linear function from the graph of a line.
- Given the equations of two lines, determine whether their graphs are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.
- Solve a system of linear equations.

Two competing telephone companies offer different payment plans. The two plans charge the same rate per long distance minute, but charge a different monthly flat fee. A consumer wants to determine whether the two plans will ever cost the same amount for a given number of long distance minutes used. The total cost of each payment plan can be represented by a linear function. To solve the problem, we will need to compare the functions. In this section, we will consider methods of comparing functions using graphs.

## Graphing Linear Functions

Previously, we saw that that the graph of a linear function is a straight line. We were also able to see the points of the function as well as the initial value from a graph. By graphing two functions, then, we can more easily compare their characteristics. There are three basic methods of graphing linear functions:

1. Plot the points and then drawing a line through the points.
2. Use the y-intercept and slope.
3. Use transformations of the identity function $f(x) = x$.

### Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x) = 2x$, we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point $(1, 2)$. Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point $(2, 4)$. Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

**How To: Given a linear function, graph by plotting points.**

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

**Example 2.2.1: Graphing by Plotting Points**

Graph $f(x) = -\frac{2}{3}x + 5$ by plotting points.

**Solution**

Begin by choosing input values. This function includes a fraction with a denominator of 3, so let's choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value, and use the output value to identify coordinate pairs.

$$x = 0 \qquad f(0) = -\frac{2}{3}(0) + 5 = 5 \rightarrow (0, 5)$$
$$x = 3 \qquad f(3) = -\frac{2}{3}(3) + 5 = 3 \rightarrow (3, 3)$$
$$x = 6 \qquad f(6) = -\frac{2}{3}(6) + 5 = 1 \rightarrow (6, 1)$$

Plot the coordinate pairs and draw a line through the points. Figure 2.2.1 represents the graph of the function $f(x) = -\frac{2}{3}x + 5$ .

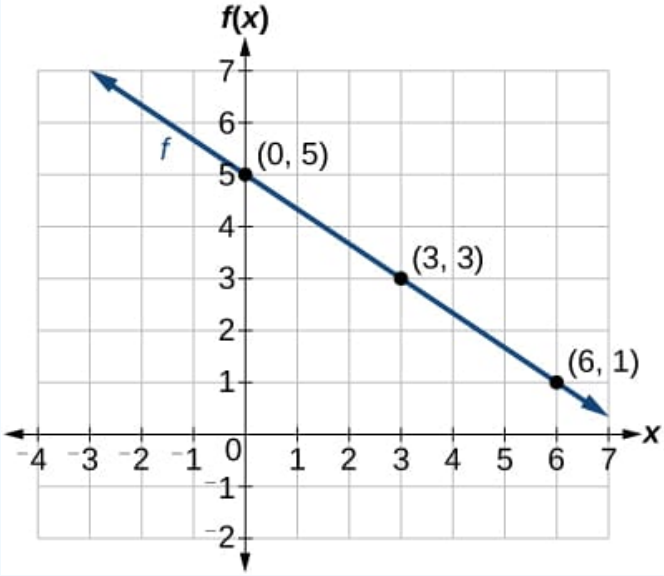



Figure 2.2.1: The graph of the linear function $f(x) = -\frac{2}{3}x + 5$.

**Analysis**

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative constant rate of change in the equation for the function.

**Exercise 2.2.1**

Graph $f(x) = -\frac{3}{4}x + 6$ by plotting points.

*Answer available at the end of this section.*

## Graphing a Function Using y-intercept and Slope

Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its **y-intercept**, which is the point at which the input value is zero. To find the y-intercept, we can set $x = 0$ in the equation.

The other characteristic of the linear function is its slope $m$, which is a measure of its steepness. Recall that the slope is the rate of change of the function. The slope of a function is equal to the ratio of the change in outputs to the change in inputs. Another way to think about the slope is by dividing the vertical difference, or rise, by the horizontal difference, or run. We encountered both the y-intercept and the slope in Linear Functions.

Let's consider the following function.

$$f(x) = \frac{1}{2}x + 1 \tag{2.2.1}$$

The slope is $\frac{1}{2}$. Because the slope is positive, we know the graph will slant upward from left to right. The y-intercept is the point on the graph when $x = 0$. The graph crosses the y-axis at $(0, 1)$. Now we know the slope and the y-intercept. We can begin graphing by plotting the point $(0, 1)$ We know that the slope is rise over run, $m = \frac{\text{rise}}{\text{run}}$. From our example, we have $m = \frac{1}{2}$, which means that the rise is 1 and the run is 2. So starting from our y-intercept $(0, 1)$, we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in Figure 2.2.3.

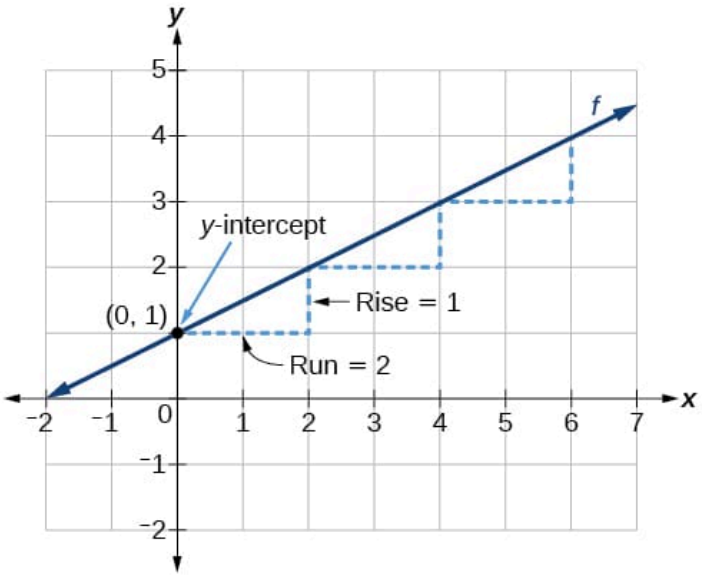



Figure 2.2.3: Graph of $f(x)$ showing slope and y-intercept

## Graphical Interpretation of a Linear Function

In the equation $f(x) = mx + b$

- $b$ is the y-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the y-axis.
- $m$ is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.2.2}$$

## Exercise 2.2.1

Do all linear functions have y-intercepts?

*Answer available at the end of this section.*

## How To: Given the equation for a linear function, graph the function using the y-intercept and slope.

1. Evaluate the function at an input value of zero to find the y-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the y-intercept.
4. Use $\frac{\text{rise}}{\text{run}}$ to determine at least two more points on the line.

5. Sketch the line that passes through the points.

### Example 2.2.2: Graphing by Using the y-intercept and Slope

Graph $f(x) = -\frac{2}{3}x + 5$ using the y-intercept and slope.

**Solution**

Evaluate the function at $x = 0$ to find the y-intercept. The output value when $x = 0$ is 5, so the graph will cross the y-axis at $(0, 5)$.

According to the equation for the function, the slope of the line is $-\frac{2}{3}$. This tells us that for each vertical decrease in the "rise" of –2 units, the "run" increases by 3 units in the horizontal direction. We can now graph the function by first plotting the y-intercept on the graph in Figure 2.2.4. From the initial value $(0, 5)$ we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then draw a line through the points.

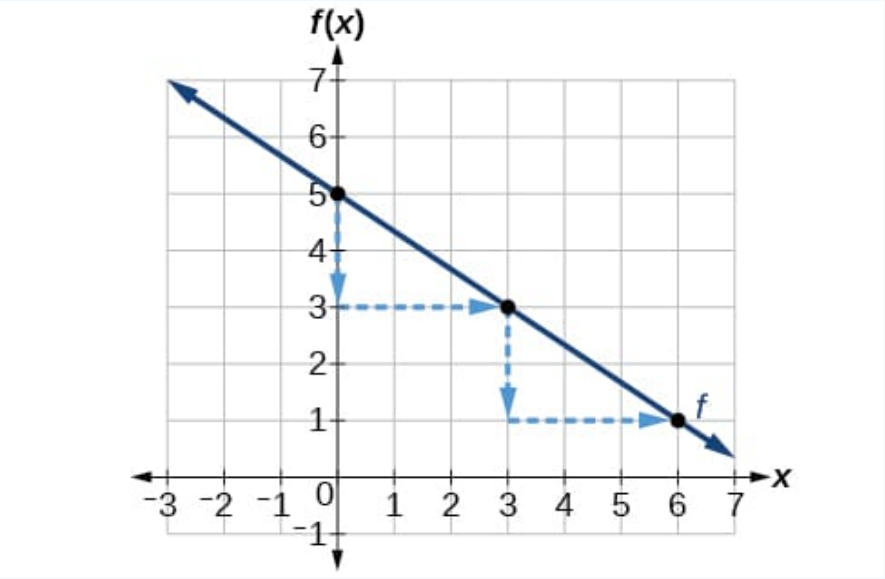



Figure 2.2.4: Graph of $f(x)$

**Analysis**

The graph slants downward from left to right, which means it has a negative slope as expected.

### Exercise 2.2.2

Find a point on the graph we drew in Example 2.2.2 that has a negative x-value.

*Answer available at the end of this section.*

## Graphing a Function Using Transformations

Another option for graphing is to use **transformations** of the identity function $f(x) = x$. A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

## Vertical Stretch or Compression

In the equation $f(x) = mx$, the $m$ is acting as the **vertical stretch** or **compression** of the identity function. When $m$ is negative, there is also a vertical reflection of the graph. Notice in Figure $2.2.5$ that multiplying the equation of $f(x) = x$ by $m$ stretches the graph of $f$ by a factor of $m$ units if $m > 1$ and compresses the graph of $f$ by a factor of $m$ units if $0 < m < 1$. This means the larger the absolute value of $m$, the steeper the slope.

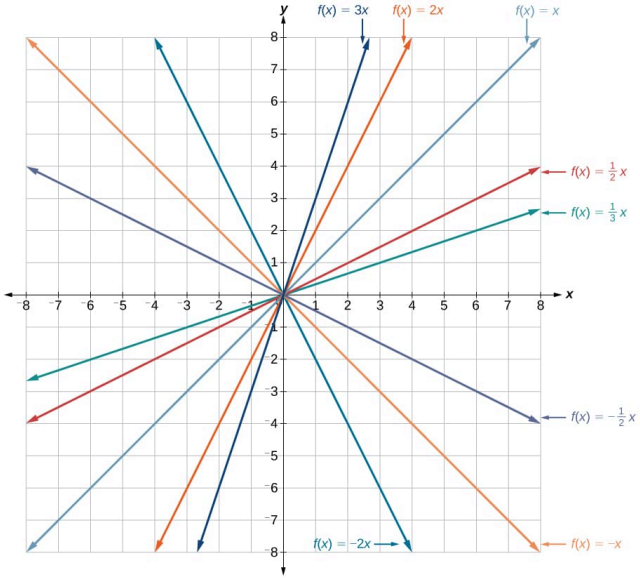



Figure $2.2.5$: Vertical stretches and compressions and reflections on the function $f(x) = x$.

## Vertical Shift

In $f(x) = mx + b$, the $b$ acts as the **vertical shift**, moving the graph up and down without affecting the slope of the line. Notice in Figure $2.2.6$ that adding a value of $b$ to the equation of $f(x) = x$ shifts the graph of $f$ a total of $b$ units up if $b$ is positive and $|b|$ units down if $b$ is negative.

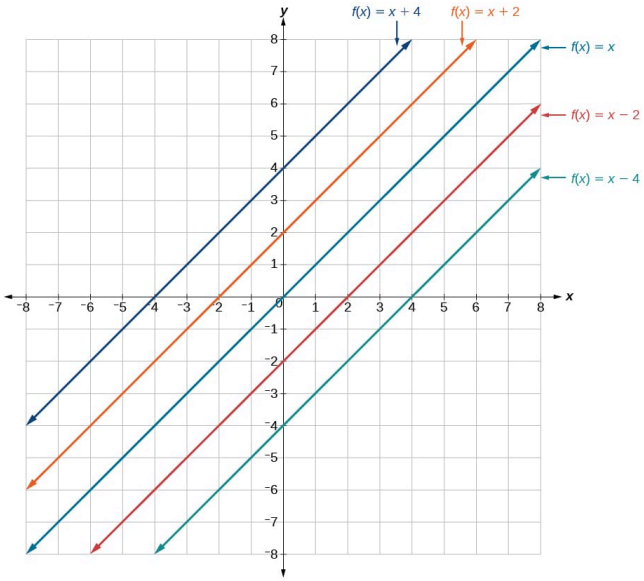



Figure $2.2.6$: This graph illustrates vertical shifts of the function $f(x) = x$.

Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

**How To: Given the equation of a linear function, use transformations to graph the linear function in the form $f(x) = mx + b$.**

1. Graph $f(x) = x$.
2. Vertically stretch or compress the graph by a factor $m$.
3. Shift the graph up or down $b$ units.

## Example 2.2.3: Graphing by Using Transformations

Graph $f(x) = \frac{1}{2}x - 3$ using transformations.

**Solution**

The equation for the function shows that $m = \frac{1}{2}$ so the identity function is vertically compressed by $\frac{1}{2}$. The equation for the function also shows that $b = -3$ so the identity function is vertically shifted down 3 units. First, graph the identity function, and show the vertical compression as in Figure 2.2.7.

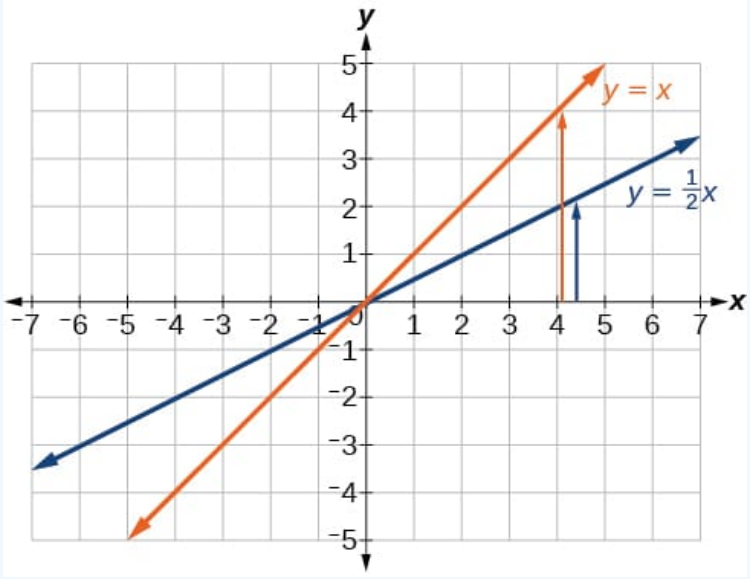



Figure 2.2.7: The function, $y = x$, compressed by a factor of $\frac{1}{2}$.

Then show the vertical shift as in Figure 2.2.8.

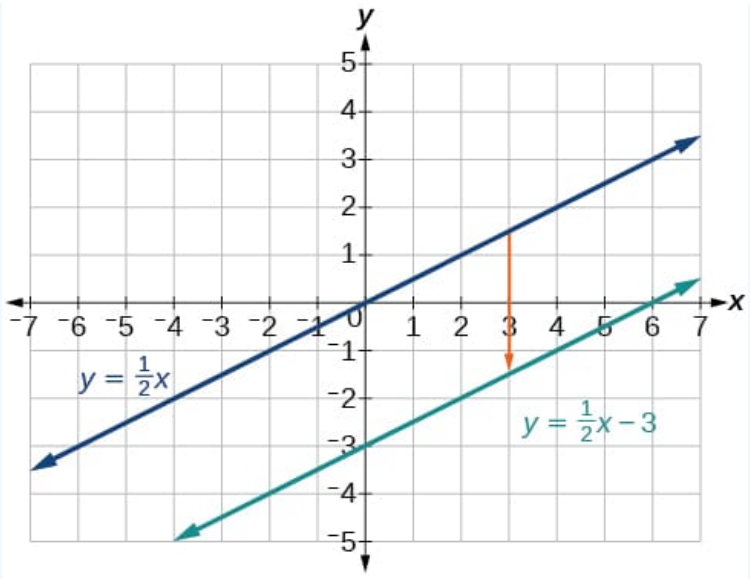



Figure 2.2.8: The function $y = \frac{1}{2}x$, shifted down 3 units.

**Exercise 2.2.3**

Graph $f(x) = 4 + 2x$, using transformations.

*Answer available at the end of this section.*

**In Example 2.2.3, could we have sketched the graph by reversing the order of the transformations?**

*No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following the order: Let the input be 2.*

$$f(2) = \frac{1}{2}(2) - 3 \tag{2.2.3}$$
$$= 1 - 3 \tag{2.2.4}$$
$$= -2 \tag{2.2.5}$$

## Writing the Equation for a Function from the Graph of a Line

Recall that in Linear Functions, we wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at Figure 2.2.10. We can see right away that the graph crosses the y-axis at the point $(0, 4)$ so this is the y-intercept.

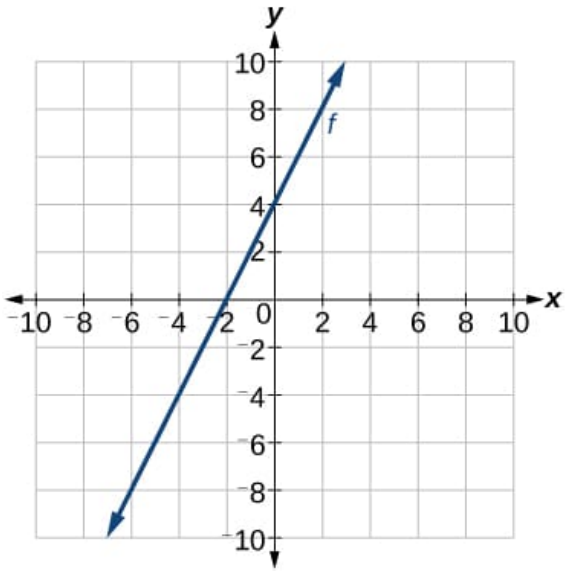



Figure 2.2.10: Graph of a linear function

Then we can calculate the slope by finding the rise and run. We can choose any two points, but let's look at the point (–2,0). To get from this point to the y-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be

$$m = \frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2 \tag{2.2.6}$$

Substituting the slope and y-intercept into the slope-intercept form of a line gives

$$y = 2x + 4 \tag{2.2.7}$$

**How To: Given a graph of linear function, find the equation to describe the function.**

1. Identify the y-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the y-intercept and slope into the slope-intercept form of a line.

## Example 2.2.4: Matching Linear Functions to Their Graphs

Match each equation of the linear functions with one of the lines in Figure 2.2.11.

a. $f(x) = 2x + 3$
b. $g(x) = 2x - 3$
c. $h(x) = -2x + 3$
d. $j(x) = \frac{1}{2}x + 3$

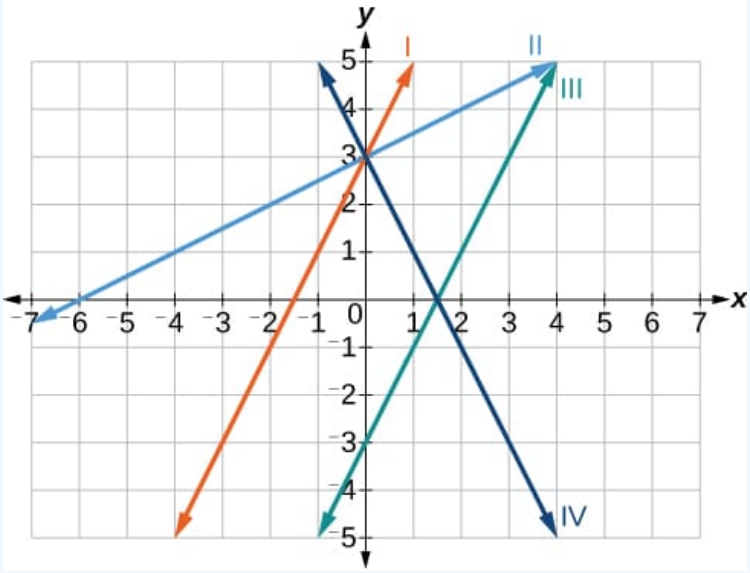



Figure 2.2.11: Graph of four linear functions

***Solution***

Analyze the information for each function.

a. This function has a slope of 2 and a y-intercept of 3. It must pass through the point $(0, 3)$ and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function $g$ has the same slope, but a different y-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through $(0, 3)$ so $f$ must be represented by line I.
b. This function also has a slope of 2, but a y-intercept of –3. It must pass through the point $(0, -3)$ and slant upward from left to right. It must be represented by line III.
c. This function has a slope of –2 and a y-intercept of 3. This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.
d. This function has a slope of $\frac{1}{2}$ and a y-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. Lines I and II pass through $(0, 3)$, but the slope of $j$ is less than the slope of $f$ so the line for $j$ must be flatter. This function is represented by Line II.

Now we can re-label the lines as in Figure 2.2.12.

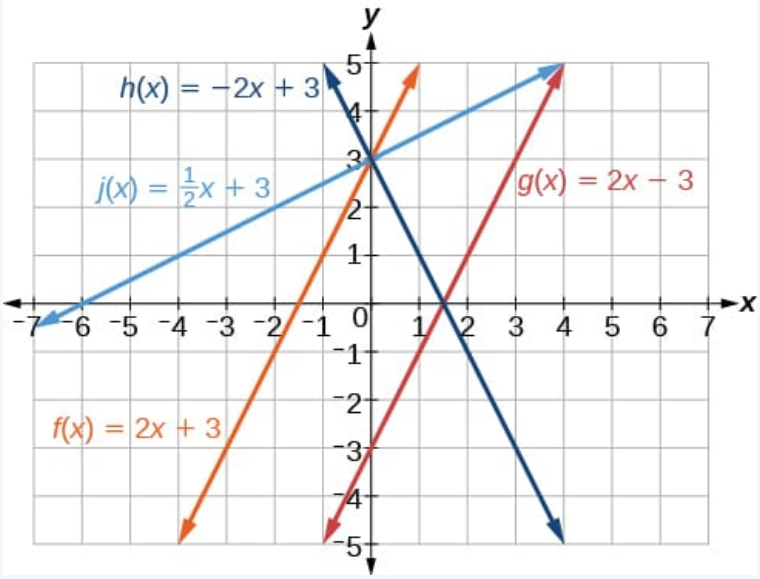



Figure 2.2.12

## Finding the x-intercept of a Line

So far, we have been finding the y-intercepts of a function: the point at which the graph of the function crosses the y-axis. A function may also have an x-intercept, which is the x-coordinate of the point where the graph of the function crosses the x-axis. In other words, it is the input value when the output value is zero.

To find the x-intercept, set a function $f(x)$ equal to zero and solve for the value of $x$. For example, consider the function shown.

$$f(x) = 3x - 6 \tag{2.2.8}$$

Set the function equal to 0 and solve for $x$.

$$0 = 3x - 6 \tag{2.2.9}$$
$$6 = 3x \tag{2.2.10}$$
$$2 = x \tag{2.2.11}$$
$$x = 2 \tag{2.2.12}$$

The graph of the function crosses the x-axis at the point $(2, 0)$.

**Do all linear functions have x-intercepts?**

Figure 2.2.13.

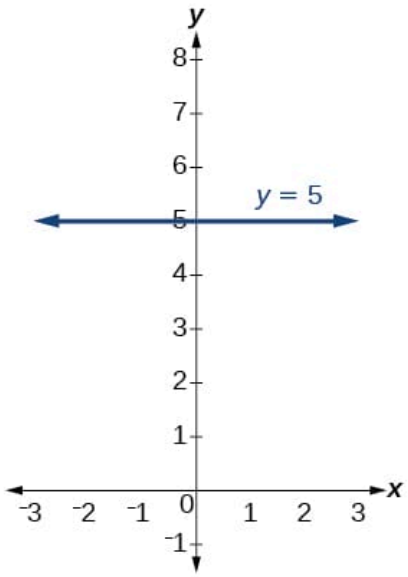



Figure 2.2.13: Graph of $y = 5$.

**Note: x-intercept**

The **x-intercept** of the function is value of $x$ when $f(x) = 0$. It can be solved by the equation $0 = mx + b$.

**Example 2.2.5: Finding an x-intercept**

Find the x-intercept of $f(x) = \frac{1}{2} - 3$.

**Solution**

Set the function equal to zero to solve for $x$.

$$0 = \frac{1}{2}x - 3$$
$$3 = \frac{1}{2}x$$
$$6 = x$$
$$x = 6$$

The graph crosses the x-axis at the point $(6, 0)$.

**Analysis**

A graph of the function is shown in Figure 2.2.14. We can see that the x-intercept is $(6, 0)$ as we expected.

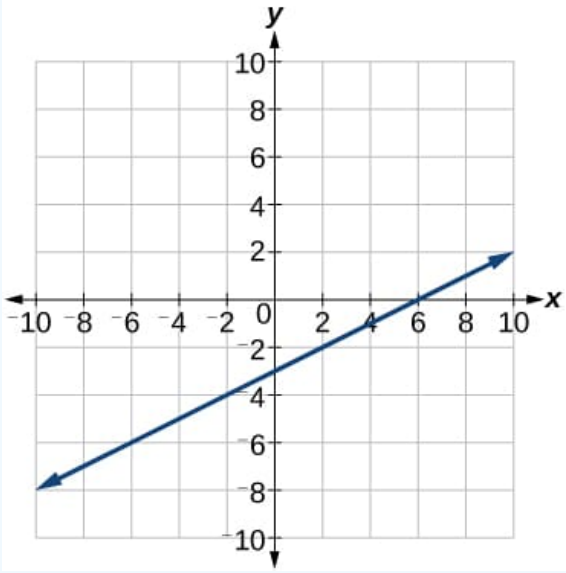



Figure 2.2.14: The graph of the linear function $f(x) = \frac{1}{2}x - 3$.

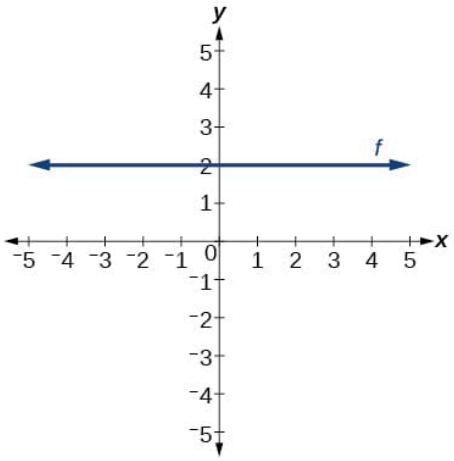
**Exercise 2.2.5**

Find the x-intercept of $f(x) = \frac{1}{4}x - 4$.

*Answer available at the end of this section.*

## Describing Horizontal and Vertical Lines

There are two special cases of lines on a graph—horizontal and vertical lines. A **horizontal line** indicates a constant output, or y-value. In Figure 2.2.15, we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m = 0$ in the equation $f(x) = mx + b$, the equation simplifies to $f(x) = b$. In other words, the value of the function is a constant. This graph represents the function $f(x) = 2$.




| $x$ | −4 | −2 | 0 | 2 | 4 |
|---|---|---|---|---|---|
| $y$ | 2 | 2 | 2 | 2 | 2 |

Figure 2.2.15: A horizontal line representing the function $f(x) = 2$.

A **vertical line** indicates a constant input, or x-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.

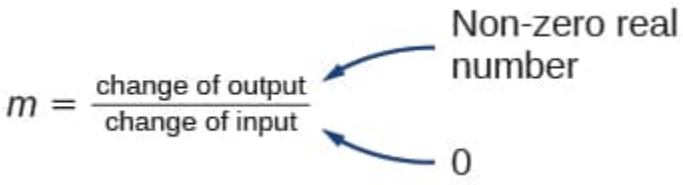
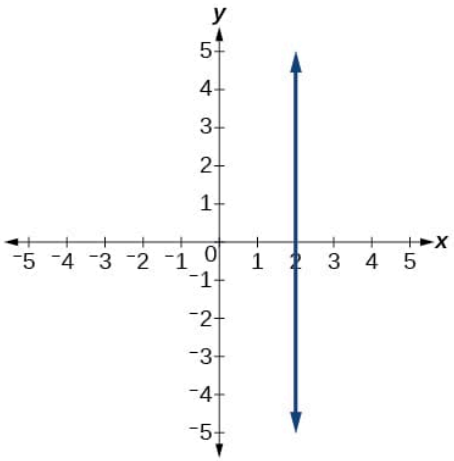



Figure 2.2.16

Notice that a vertical line, such as the one in Figure 2.2.17, has an x-intercept, but no y-intercept unless it's the line $x = 0$. This graph represents the line $x = 2$.




| $x$ | 2 | 2 | 2 | 2 | 2 |
|---|---|---|---|---|---|
| $y$ | −4 | −2 | 0 | 2 | 4 |

Figure 2.2.17: The vertical line, $x = 2$, which does not represent a function.

## Definitions: Horizontal and Vertical Lines

Lines can be horizontal or vertical.

- A **horizontal line** is a line defined by an equation in the form $f(x) = b$.
- A **vertical line** is a line defined by an equation in the form $x = a$.

## Example 2.2.6: Writing the Equation of a Horizontal Line

Write the equation of the line graphed in Figure 2.2.18.

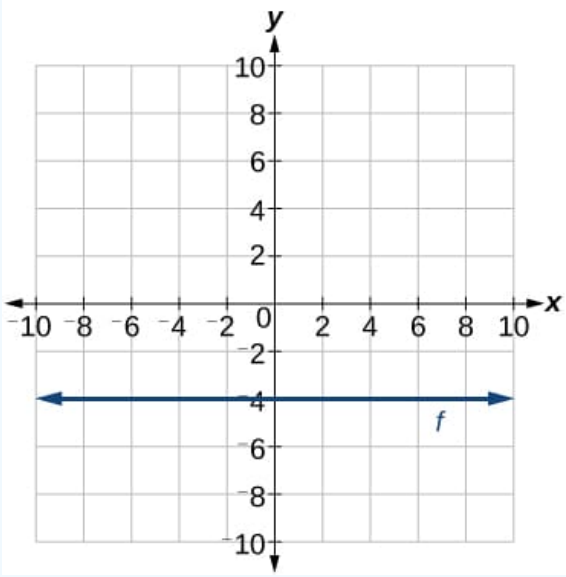



Figure 2.2.18: Graph of $x = 7$.

**Solution**

For any x-value, the y-value is –4, so the equation is $y = -4$.

## Example 2.2.7: Writing the Equation of a Vertical Line

Write the equation of the line graphed in Figure 2.2.19.

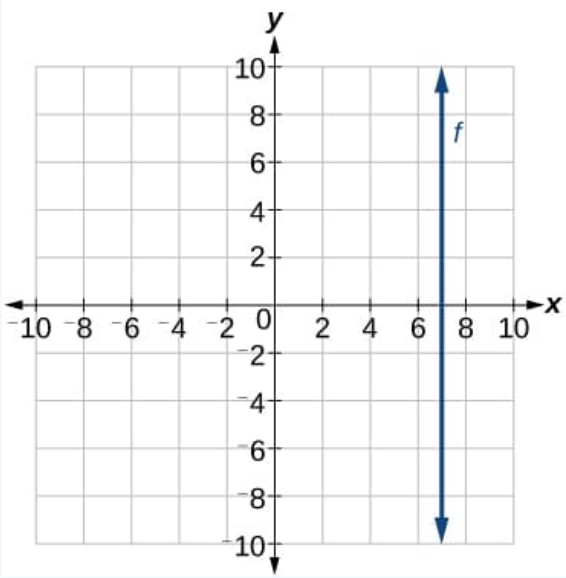



Figure 2.2.19: Graph of a vertical line

**Solution**

The constant x-value is 7, so the equation is $x = 7$.

# Determining Whether Lines are Parallel or Perpendicular

The two lines in Figure 2.2.20 are **parallel lines**: they will never intersect. Notice that they have exactly the same steepness, which means their slopes are identical. The only difference between the two lines is the y-intercept. If we shifted one line vertically toward the y-intercept of the other, they would become the same line.

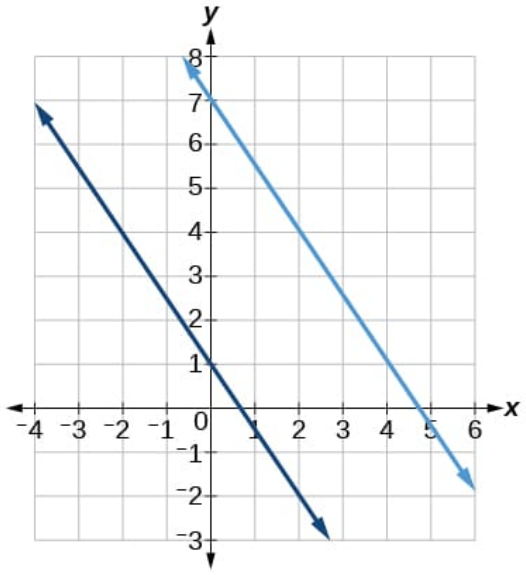



Figure 2.2.20: Parallel lines.

We can determine from their equations whether two lines are parallel by comparing their slopes. If the slopes are the same and the y-intercepts are different, the lines are parallel. If the slopes are different, the lines are not parallel.

$$\left.\begin{aligned} f(x) &= -2x+6 \\ f(x) &= -2x-4 \end{aligned}\right\} \text{ parallel} \qquad \left.\begin{aligned} f(x) &= 3x+2 \\ f(x) &= 2x+2 \end{aligned}\right\} \text{ not parallel}$$

Figure 2.2.21 are perpendicular.

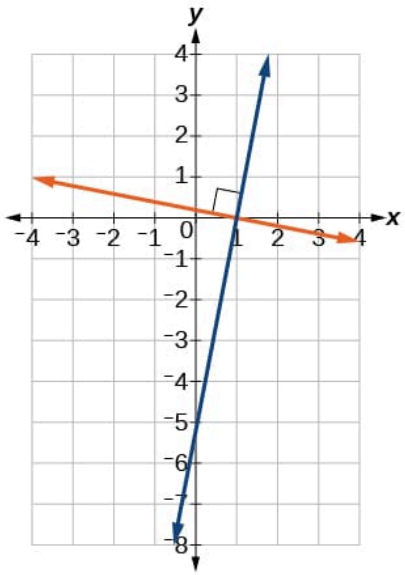



Figure 2.2.21: Perpendicular lines.

Perpendicular lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1.

So, if $m_1$ and $m_2$ are negative reciprocals of one another, they can be multiplied together to yield –1.

$$m_1 m_2 = -1 \tag{2.2.13}$$

To find the reciprocal of a number, divide 1 by the number. So the reciprocal of 8 is $\frac{1}{8}$, and the reciprocal of $\frac{1}{8}$ is 8. To find the negative reciprocal, first find the reciprocal and then change the sign.

As with parallel lines, we can determine whether two lines are perpendicular by comparing their slopes, assuming that the lines are neither horizontal nor perpendicular. The slope of each line below is the negative reciprocal of the other so the lines are perpendicular.

$$f(x) = \frac{1}{4}x + 2 \qquad \text{negative reciprocal of } \frac{1}{4} \text{ is -4} \tag{2.2.14}$$

$$f(x) = -4x + 3 \qquad \text{negative reciprocal of -4 is } \frac{1}{4} \tag{2.2.15}$$

The product of the slopes is –1.

$$-4\left(\frac{1}{4}\right) = -1 \tag{2.2.16}$$

## Definitions: Parallel and Perpendicular Lines

Two lines are **parallel lines** if they do not intersect. The slopes of the lines are the same.

$f(x) = m_1 x + b_1$ and $g(x) = m_2 x + b_2$ are parallel if $m_1 = m_2$.

If and only if $b_1 = b_2$ and $m_1 = m_2$, we say the lines coincide. Coincident lines are the same line.

Two lines are **perpendicular lines** if they intersect at right angles.

$f(x) = m_1 x + b_1$ and $g(x) = m_2 x + b_2$ are perpendicular if $m_1 m_2 = -1$, and so $m_2 = \frac{-1}{m_1}$.

## Example 2.2.8: Identifying Parallel and Perpendicular Lines

Given the functions below, identify the functions whose graphs are a pair of parallel lines and a pair of perpendicular lines.

$$f(x) = 2x + 3 \qquad h(x) = -2x + 2 \tag{2.2.17}$$

$$g(x) = \frac{1}{2}x - 4 \qquad f(x) = 2x - 6 \tag{2.2.18}$$

**Solution**

Parallel lines have the same slope. Because the functions $f(x) = 2x + 3$ and $j(x) = 2x - 6$ each have a slope of 2, they represent parallel lines. Perpendicular lines have negative reciprocal slopes. Because –2 and $\frac{1}{2}$ are negative reciprocals, the equations, $g(x) = \frac{1}{2}x - 4$ and $h(x) = -2x + 2$ represent perpendicular lines.

**Analysis**

A graph of the lines is shown in Figure 2.2.22.

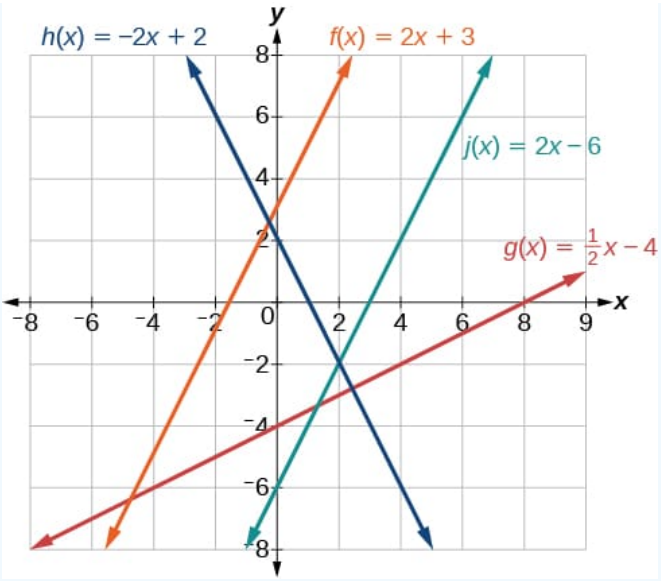



Figure 2.2.22: Graph of $h(x)$ (blue), $f(x)$ (orange), $j(x)$ (green), and $g(x)$ (red)

The graph shows that the lines $f(x) = 2x + 3$ and $j(x) = 2x–6$ are parallel, and the lines $g(x) = \frac{1}{2}x–4$ and $h(x) = −2x + 2$ are perpendicular.

# Writing the Equation of a Line Parallel or Perpendicular to a Given Line

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

## Writing Equations of Parallel Lines

Suppose for example, we are given the following equation.

$$f(x) = 3x + 1$$

We know that the slope of the line formed by the function is 3. We also know that the y-intercept is $(0, 1)$. Any other line with a slope of 3 will be parallel to $f(x)$. So the lines formed by all of the following functions will be parallel to $f(x)$.

$$\begin{aligned} g(x) &= 3x + 6 \\ h(x) &= 3x + 1 \\ p(x) &= 3x + \frac{2}{3} \end{aligned}$$

Suppose then we want to write the equation of a line that is parallel to $f$ and passes through the point $(1, 7)$. We already know that the slope is 3. We just need to determine which value for $b$ will give the correct line. We can begin with the point-slope form of an equation for a line, and then rewrite it in the slope-intercept form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 7 &= 3(x - 1) \\ y - 7 &= 3x - 3 \\ y &= 3x + 4 \end{aligned}$$

So $g(x) = 3x + 4$ is parallel to $f(x) = 3x + 1$ and passes through the point $(1, 7)$.

**How to ...**

Given the equation of a function and a point through which its graph passes, write the equation of a line parallel to the given line that passes through the given point.

1. Find the slope of the function.
2. Substitute the given values into either the general point-slope equation or the slope-intercept equation for a line.
3. Simplify.

**Example 2.2.9: Finding a Line Parallel to a Given Line**

Find a line parallel to the graph of $f(x) = 3x + 6$ that passes through the point $(3, 0)$.

**Solution**

The slope of the given line is 3. If we choose the slope-intercept form, we can substitute $m = 3$, $x = 3$, and $f(x) = 0$ into the slope-intercept form to find the y-intercept.

$$\begin{aligned} g(x) &= 3x + b \\ 0 &= 3(3) + b \\ b &= -9 \end{aligned}$$

The line parallel to $f(x)$ that passes through $(3, 0)$ is $g(x) = 3x - 9$.

**Analysis**

We can confirm that the two lines are parallel by graphing them. Figure 2.2.23 shows that the two lines will never intersect.

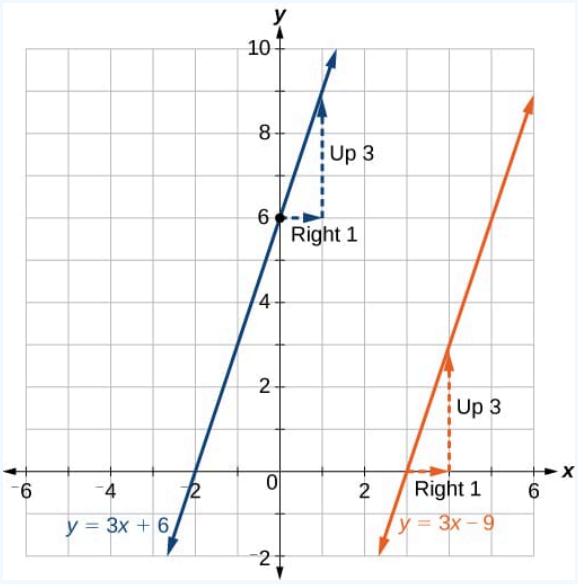



Figure 2.2.23: Graph of two functions where the blue line is $y = 3x + 6$, and the orange line is $y = 3x - 9$.

## Writing Equations of Perpendicular Lines

We can use a very similar process to write the equation for a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the following function:

$$f(x) = 2x + 4$$

The slope of the line is 2, and its negative reciprocal is $-\frac{1}{2}$. Any function with a slope of $-\frac{1}{2}$ will be perpendicular to $f(x)$. So the lines formed by all of the following functions will be perpendicular to $f(x)$.

$$g(x) = -\frac{1}{2}x + 4$$

$$h(x) = -\frac{1}{2}x + 2$$

$$p(x) = -\frac{1}{2}x - \frac{1}{2}$$

As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose then we want to write the equation of a line that is perpendicular to $f(x)$ and passes through the point $(4, 0)$. We already know that the slope is $-\frac{1}{2}$. Now we can use the point to find the y-intercept by substituting the given values into the slope-intercept form of a line and solving for $b$.

$$\begin{aligned} g(x) &= mx + b \\ 0 &= -\frac{1}{2}(4) + b \\ 0 &= -2 + b \\ 2 &= b \\ b &= 2 \end{aligned}$$

The equation for the function with a slope of $-\frac{1}{2}$ and a y-intercept of 2 is

$$g(x) = -\frac{1}{2}x + 2 \tag{2.2.19}$$

So $g(x) = -\frac{1}{2}x + 2$ is perpendicular to $f(x) = 2x + 4$ and passes through the point $(4, 0)$. Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

**A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not –1. Doesn't this fact contradict the definition of perpendicular lines?**

*No. For two perpendicular linear functions, the product of their slopes is –1. However, a vertical line is not a function so the definition is not contradicted.*

**How To...**

Given the equation of a function and a point through which its graph passes, write the equation of a line perpendicular to the given line.

1. Find the slope of the function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for $x$ and $y$ from the coordinate pair provided into $g(x) = mx + b$.
4. Solve for $b$.
5. Write the equation for the line.

**Example 2.2.10: Finding the Equation of a Perpendicular Line**

Find the equation of a line perpendicular to $f(x) = 3x + 3$ that passes through the point $(3, 0)$.

**Solution**

The original line has slope $m = 3$, so the slope of the perpendicular line will be its negative reciprocal, or $-\frac{1}{3}$. Using this slope and the given point, we can find the equation for the line.

$$g(x) = -\frac{1}{3}x + b$$
$$0 = -\frac{1}{3}(3) + b$$
$$1 = b$$
$$b = 1$$

The line perpendicular to $f(x)$ that passes through $(3, 0)$ is $g(x) = -\frac{1}{3}x + 1$.

**Analysis**

A graph of the two lines is shown in Figure 2.2.24 below.

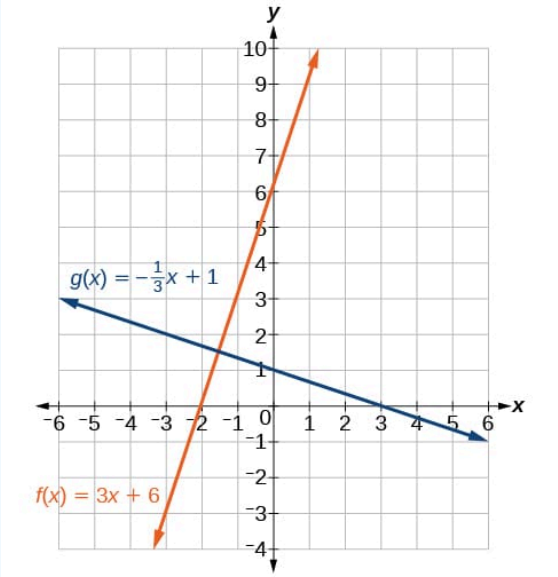



Figure 2.2.24: Graph of two functions where the blue line is $g(x) = -\frac{1}{3}x + 1$, and the orange line is $f(x) = 3x + 6$.

## Exercise 2.2.5

Given the function $h(x) = 2x - 4$, write an equation for the line passing through $(0, 0)$ that is

a. parallel to $h(x)$
b. perpendicular to $h(x)$

*Answer available at the end of this section.*

## How to...

Given two points on a line and a third point, write the equation of the perpendicular line that passes through the point.

1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.
3. Use the slope-intercept form or point-slope form to write the equation by substituting the known values.
4. Simplify.

### Example 2.2.11: Finding the Equation of a Line Perpendicular to a Given Line Passing through a Point

A line passes through the points $(-2, 6)$ and $(4, 5)$. Find the equation of a perpendicular line that passes through the point $(4, 5)$.

From the two points of the given line, we can calculate the slope of that line.

$$\begin{aligned} m_1 &= \frac{5-6}{4-(-2)} \\ &= \frac{-1}{6} \\ &= -\frac{1}{6} \end{aligned}$$

Find the negative reciprocal of the slope.

$$\begin{aligned} m_2 &= \frac{-1}{-\frac{1}{6}} \\ &= -1\left(-\frac{6}{1}\right) \\ &= 6 \end{aligned}$$

We can then solve for the y-intercept of the line passing through the point $(4, 5)$.

$$\begin{aligned} g(x) &= 6x + b \\ 5 &= 6(4) + b \\ 5 &= 24 + b \\ -19 &= b \\ b &= -19 \end{aligned}$$

The equation for the line that is perpendicular to the line passing through the two given points and also passes through point $(4, 5)$ is

$$y = 6x - 19$$

### Exercise 2.2.1

A line passes through the points, $(-2, -15)$ and $(2, -3)$. Find the equation of a perpendicular line that passes through the point, $(6, 4)$.

*Answer available at the end of this section.*

## Solving a System of Linear Equations Using a Graph

A system of linear equations includes two or more linear equations. The graphs of two lines will intersect at a single point if they are not parallel. Two parallel lines can also intersect if they are coincident, which means they are the same line and they intersect at every point. For two lines that are not parallel, the single point of intersection will satisfy both equations and therefore represent the solution to the system.

To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal to one another, and solve for the input that satisfies the equation.

## Example 2.2.12: Finding a Point of Intersection Algebraically

Find the point of intersection of the lines $h(t) = 3t - 4$ and $j(t) = 5 - t$.

**Solution**

Set $h(t) = j(t)$.

$$3t - 4 = 5 - t \tag{2.2.20}$$
$$4t = 9 \tag{2.2.21}$$
$$t = \frac{9}{4} \tag{2.2.22}$$

This tells us the lines intersect when the input is $\frac{9}{4}$.

We can then find the output value of the intersection point by evaluating either function at this input.

$$j\left(\frac{9}{4}\right) = 5 - \frac{9}{4} \tag{2.2.23}$$
$$= \frac{11}{4} \tag{2.2.24}$$

These lines intersect at the point $\left(\frac{9}{4}, \frac{11}{4}\right)$.

**Analysis**

Looking at Figure 2.2.25, this result seems reasonable.

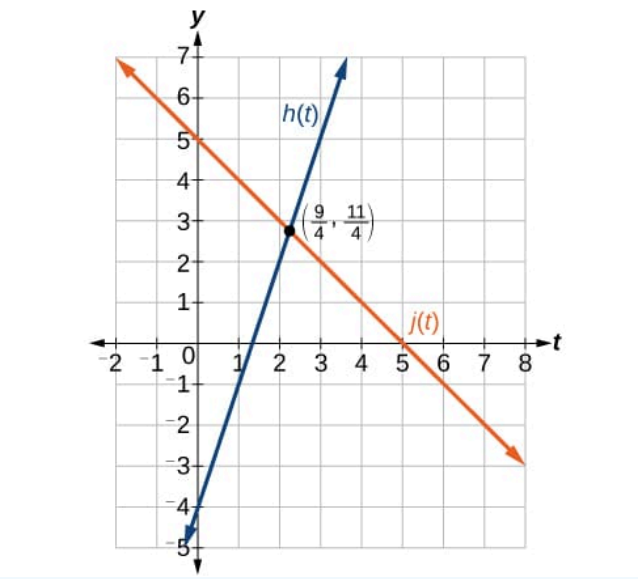



Figure 2.2.25: Graph of two functions $h(t) = 3t - 4$ and $j(t) = t + 5$ and their intersection at $(\frac{9}{4}, \frac{11}{4})$.

**If we were asked to find the point of intersection of two distinct parallel lines, should something in the solution process alert us to the fact that there are no solutions?**

*Yes. After setting the two equations equal to one another, the result would be the contradiction "*$0 = \text{non-zero real number}$*".*

## Exercise 2.2.7

Look at the graph in Figure 2.2.23 and identify the following for the function $j(t)$ :

a. y-intercept
b. x-intercept(s)
c. slope
d. Is $j(t)$ parallel or perpendicular to $h(t)$ (or neither)?
e. Is $j(t)$ an increasing or decreasing function (or neither)?
f. Write a transformation description for $j(t)$ from the identity toolkit function $f(x) = x$.

*Answer available at the end of this section.*

## Example 2.2.13: Finding a Break-Even Point

A company sells sports helmets. The company incurs a one-time fixed cost for $250,000. Each helmet costs $120 to produce, and sells for $140.

a. Find the cost function, $C$, to produce $x$ helmets, in dollars.
b. Find the revenue function, $R$, from the sales of $x$ helmets, in dollars.
c. Find the break-even point, the point of intersection of the two graphs $C$ and $R$.

**Solution**

a. The cost function in the sum of the fixed cost, $125,000, and the variable cost, $120 per helmet.

$$C(x) = 120x + 250,000$$

b. The revenue function is the total revenue from the sale of $x$ helmets, $R(x) = 140x$.

c. The break-even point is the point of intersection of the graph of the cost and revenue functions. To find the x-coordinate of the coordinate pair of the point of intersection, set the two equations equal, and solve for $x$.

$$\begin{aligned} C(x) &= R(x) \\ 250,000 + 120x &= 140x \\ 250,000 &= 20x \\ x &= 12,500 \end{aligned}$$

To find $y$, evaluate either the revenue or the cost function at 12,500.

$$\begin{aligned} R(20) &= 140(12,500) \\ &= \$1,750,000 \end{aligned}$$

The break-even point is $(12,500, 1,750,000)$.

**Analysis**

This means if the company sells 12,500 helmets, they break even; both the sales and cost incurred equaled 1.75 million dollars. See Figure 2.2.26.

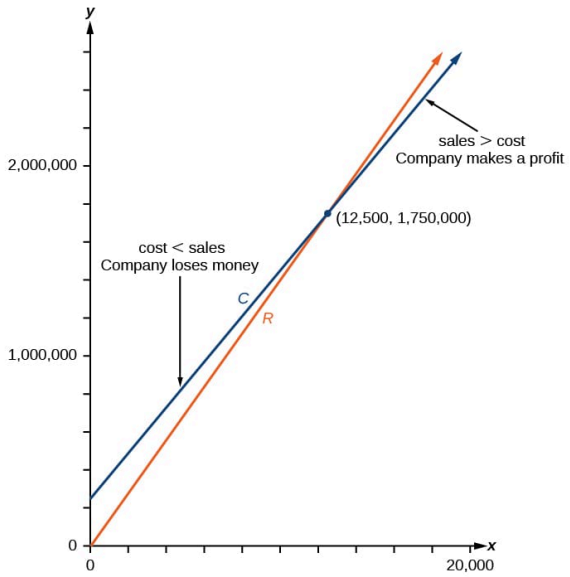



Figure 2.2.26: Graph of the two functions, $C(x)$ and $R(x)$ where it shows that below $(12500, 1750000)$ the company loses money and above that point the company makes a profit.

## Key Concepts

- Linear functions may be graphed by plotting points or by using the y-intercept and slope.
- Graphs of linear functions may be transformed by using shifts up, down, left, or right, as well as through stretches, compressions, and reflections.
- The y-intercept and slope of a line may be used to write the equation of a line.
- The x-intercept is the point at which the graph of a linear function crosses the x-axis.
- Horizontal lines are written in the form, $f(x) = b$.
- Vertical lines are written in the form, $x = b$.
- Parallel lines have the same slope.
- Perpendicular lines have negative reciprocal slopes, assuming neither is vertical.
- A line parallel to another line, passing through a given point, may be found by substituting the slope value of the line and the x- and y-values of the given point into the equation, $f(x) = mx + b$, and using the $b$ that results. Similarly, the point-slope form of an equation can also be used.
- A line perpendicular to another line, passing through a given point, may be found in the same manner, with the exception of using the negative reciprocal slope.
- A system of linear equations may be solved setting the two equations equal to one another and solving for $x$. The y-value may be found by evaluating either one of the original equations using this x-value.
- A system of linear equations may also be solved by finding the point of intersection on a graph.

## Glossary

horizontal line
a line defined by $f(x) = b$, where $b$ is a real number. The slope of a horizontal line is 0.

parallel lines
two or more lines with the same slope

perpendicular lines
two lines that intersect at right angles and have slopes that are negative reciprocals of each other

vertical line
a line defined by $x = a$, where a is a real number. The slope of a vertical line is undefined.

x-intercept

the point on the graph of a linear function when the output value is 0; the point at which the graph crosses the horizontal axis

## Answers

**Exercise** 2.2.1

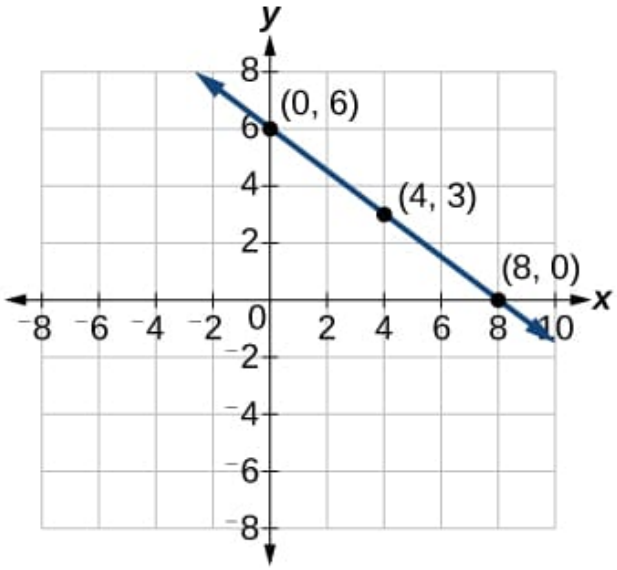



Figure 2.2.2

**Exercise** 2.2.1

Yes. All linear functions cross the y-axis and therefore have y-intercepts. (Note: A vertical line parallel to the y-axis does not have a y-intercept, but it is not a function.)

**Exercise** 2.2.2

Possible answers include $(-3, 7)$, $(-6, 9)$, or $(-9, 11)$.

**Exercise** 2.2.3

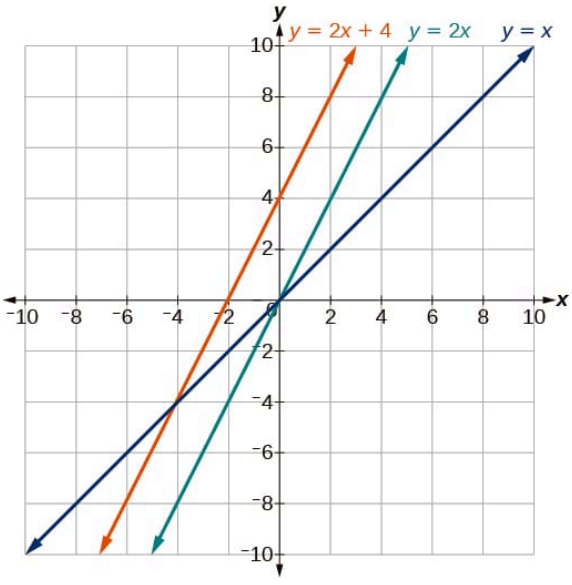



Figure 2.2.9

**Exercise** 2.2.5

$(16, 0)$

**Exercise** 2.2.5

$f(x) = 2x$ $g(x) = -\frac{1}{2}x$

**Exercise** 2.2.1

$y = -\frac{1}{3}x + 6$

**Exercise** 2.2.7

a. $(0, 5)$
b. (5, 0)
c. Slope -1
d. Neither parallel nor perpendicular
e. Decreasing function
f. Given the identity function, perform a vertical flip (over the t-axis) and shift up 5 units.

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# 2.3: Modeling with Linear Functions

**Learning Objectives**

- Build linear models from verbal descriptions.
- Model a set of data with a linear function.

Emily is a college student who plans to spend a summer in Seattle. She has saved $3,500 for her trip and anticipates spending $400 each week on rent, food, and activities. How can we write a linear model to represent her situation? What would be the x-intercept, and what can she learn from it? To answer these and related questions, we can create a model using a linear function. Models such as this one can be extremely useful for analyzing relationships and making predictions based on those relationships. In this section, we will explore examples of **linear function** models.



Figure 2.3.1: (credit: EEK Photography/Flickr)

## Identifying Steps to Model and Solve Problems

When **modeling** scenarios with linear functions and solving problems involving quantities with a **constant rate of change**, we typically follow the same problem strategies that we would use for any type of function. Let's briefly review them:

Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
When needed, write a formula for the function.
Solve or evaluate the function using the formula.
Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
Clearly convey your result using appropriate units, and answer in full sentences when necessary.

# Building Linear Models

Now let's take a look at the student in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.

- Output: $M$, money remaining, in dollars
- Input: $t$, time, in weeks

So, the amount of money remaining depends on the number of weeks: $M(t)$

We can also identify the initial value and the rate of change.

- Initial Value: She saved $3,500, so $3,500 is the initial value for M.
- Rate of Change: She anticipates spending $400 each week, so –$400 per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The **rate of change is constant**, so we can start with the linear model $M(t) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = mt + b$$

$$-400 \qquad 3500$$

$$M(t) = -400t + 3500$$

Figure 2.3.2

To find the x-intercept, we set the output to zero, and solve for the input.

$$\begin{aligned} 0 &= -400t + 3500 \\ t &= \frac{3500}{400} \\ &= 8.75 \end{aligned}$$

The x-intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn't make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved $3,500, but the scenario discussed poses the question once she saved $3,500 because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the x-intercept, unless Emily will use a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \leq t \leq 8.75$.

In the above example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

# Using a Given Intercept to Build a Model

Some real-world problems provide the y-intercept, which is the constant or initial value. Once the y-intercept is known, the x-intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan

balance is $1,000. She plans to pay $250 per month until her balance is $0. The y-intercept is the initial amount of her debt, or $1,000. The rate of change, or slope, is -$250 per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$\begin{aligned} f(x) &= mx + b \\ &= -250x + 1000 \end{aligned}$$

Now we can set the function equal to 0, and solve for $x$ to find the x-intercept.

$$\begin{aligned} 0 &= -250 + 1000 \\ 1000 &= 250x \\ 4 &= x \\ x &= 4 \end{aligned}$$

The x-intercept is the number of months it takes her to reach a balance of $0. The x-intercept is 4 months, so it will take Hannah four months to pay off her loan.

## Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

**How To: Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.**

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given x-value.
6. Use the model to identify an x-value that results in a given y-value.
7. Answer the question posed.

### Example 2.3.1: Using a Linear Model to Investigate a Town's Population

A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. Assume this trend continues.

a. Predict the population in 2013.
b. Identify the year in which the population will reach 15,000.

**Solution**

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the y-intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004:

- Input: $t$, years since 2004
- Output: $P(t)$, the town's population

To predict the population in 2013 ($t = 9$), we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.

$$m = \frac{\text{change in output}}{\text{change in input}} \tag{2.3.1}$$

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the y-intercept of the function. The year 2009 would correspond to $t = 5$, giving the point $(5, 8100)$.

The two coordinate pairs are $(0, 6200)$ and $(5, 8100)$. Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.

$$\begin{aligned} m &= \frac{8100 - 6200}{5 - 0} \\ &= \frac{1900}{5} \\ &= 380 \text{ people per year} \end{aligned}$$

We already know the y-intercept of the line, so we can immediately write the equation:

$$P(t) = 380t + 6200 \tag{2.3.2}$$

To predict the population in 2013, we evaluate our function at $t = 9$.

$$\begin{aligned} P(9) &= 380(9) + 6,200 \\ &= 9,620 \end{aligned}$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for $t$.

$$\begin{aligned} 15000 &= 380t + 6200 \\ 8800 &= 380t \\ t &\approx 23.158 \end{aligned}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

## Exercise 2.3.1A

A company sells doughnuts. They incur a fixed cost of $25,000 for rent, insurance, and other expenses. It costs $0.25 to produce each doughnut.

a. Write a linear model to represent the cost C of the company as a function of $x$, the number of doughnuts produced.
b. Find and interpret the y-intercept.

**Solution**

a. $C(x) = 0.25x + 25,000$ b. The y-intercept is $(0, 25,000)$. If the company does not produce a single doughnut, they still incur a cost of $25,000.

## Exercise 2.3.1B

A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

a. Predict the population in 2014.
b. Identify the year in which the population will reach 54,000.

**Solution**

a. 41,100 b. 2020

## Using a Diagram to Model a Problem

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometrical shapes or figures are drawn. Distances are often traced out. If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

### Example 2.3.2: Using a Diagram to Model Distance Walked

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

**Solution**

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question:

"How long will it take them to be 2 miles apart?"

In this problem, our changing quantities are time and position, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define our input and output variables.

- Input: $t$, time in hours.
- Output: $A(t)$, distance in miles, and $E(t)$, distance in miles

Because it is not obvious how to define our output variable, we'll start by drawing a picture such as Figure 2.3.3.

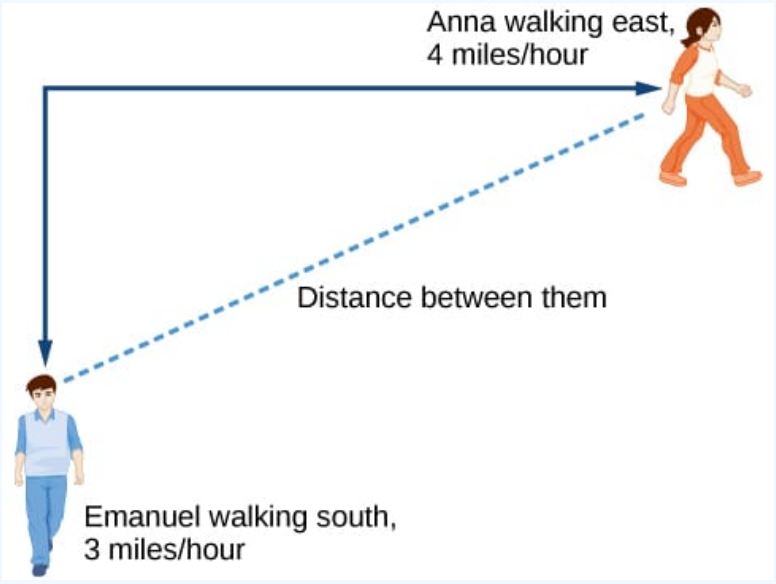



Figure 2.3.3

- Initial Value: They both start at the same intersection so when $t = 0$, the distance traveled by each person should also be 0. Thus the initial value for each is 0.

- Rate of Change: Anna is walking 4 miles per hour and Emanuel is walking 3 miles per hour, which are both rates of change. The slope for $A$ is 4 and the slope for $E$ is 3.

Using those values, we can write formulas for the distance each person has walked.

$$A(t) = 4t \tag{2.3.3}$$

$$E(t) = 3t \tag{2.3.4}$$

For this problem, the distances from the starting point are important. To notate these, we can define a coordinate system, identifying the "starting point" at the intersection where they both started. Then we can use the variable, $A$, which we introduced above, to represent Anna's position, and define it to be a measurement from the starting point in the eastward direction. Likewise, can use the variable, $E$, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system, we specified both the starting point of the measurement and the direction of measure.

We can then define a third variable, $D$, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the diagram is often helpful, as we can see from Figure 2.3.4.

Recall that we need to know how long it takes for $D$, the distance between them, to equal 2 miles. Notice that for any given input $t$, the outputs $A(t)$, $E(t)$, and $D(t)$ represent distances.

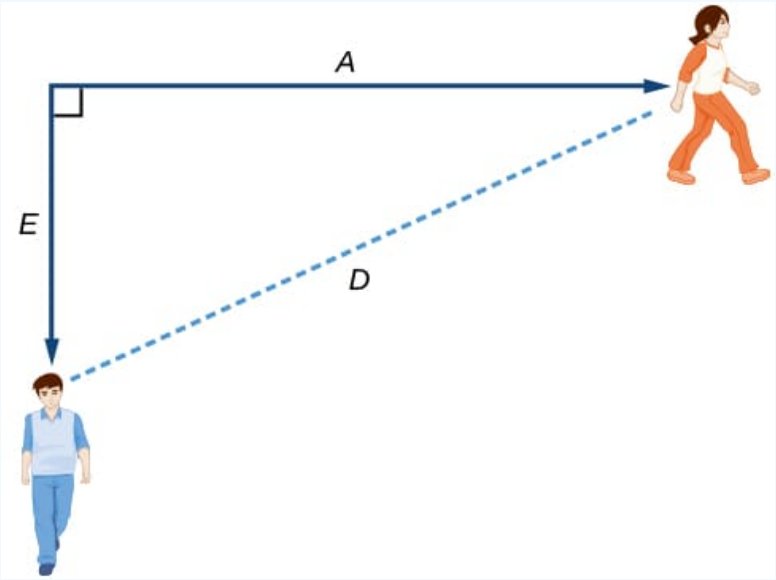



Figure 2.3.4: We can use the Pythagorean Theorem because we have drawn a right angle.

Using the Pythagorean Theorem, we get:

$$\begin{aligned} d(t)^2 &= A(t)^2 + E(t)^2 \\ &= (4t)^2 + (3t)^2 \\ &= 16t^2 + 9t^2 \\ &= 25t^2 \\ D(t) &= \pm\sqrt{25t^2} \qquad \text{Solve for } D(t) \text{ using the square root} \\ &= \pm 5|t| \end{aligned}$$

In this scenario we are considering only positive values of $t$, so our distance $D(t)$ will always be positive. We can simplify this answer to $D(t) = 5t$. This means that the distance between Anna and Emanuel is also a linear function. Because D is a linear function, we can now answer the question of when the distance between them will reach 2 miles. We will set the output $D(t) = 2$ and solve for $t$.

$$D(t) = 2$$
$$5t = 2$$
$$t = \frac{2}{5} = 0.4$$

They will fall out of radio contact in 0.4 hours, or 24 minutes.

**Should I draw diagrams when given information based on a geometric shape?**

*Yes. Sketch the figure and label the quantities and unknowns on the sketch.*

## Example 2.3.3: Using a Diagram to Model Distance between Cities

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

**Solution**

It might help here to draw a picture of the situation. See Figure 2.3.5. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts Agritown at coordinates $(30, 10)$, and Eastborough at $(20, 0)$.

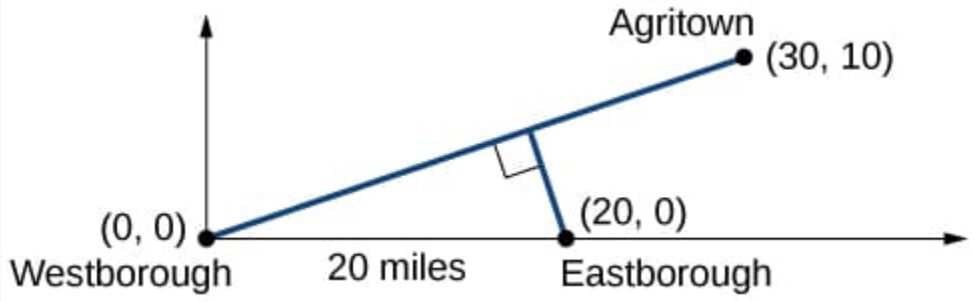



Figure 2.3.5

Using this point along with the origin, we can find the slope of the line from Westborough to Agritown:

$$m = \frac{10 - 0}{30 - 0} = \frac{1}{3} \tag{2.3.5}$$

The equation of the road from Westborough to Agritown would be

$$W(x) = \frac{1}{3}x \tag{2.3.6}$$

From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Because the town of Eastborough is at the point $(20, 0)$, we can find the equation:

$$\begin{aligned} E(x) &= -3x + b \\ 0 &= -3(20) + b \qquad \text{Substitute in } (20, 0) \\ b &= 60 \\ E(x) &= -3x + 60 \end{aligned}$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$$\begin{aligned}
\frac{1}{3}x &= -3x + 60 \\
\frac{10}{3}x &= 60 \\
10x &= 180 \\
x &= 18 \qquad \text{Substituting this back into } W(x) \\
y &= W(18) \\
&= \frac{1}{3}(18) \\
&= 6
\end{aligned}$$

The roads intersect at the point $(18, 6)$. Using the distance formula, we can now find the distance from Westborough to the junction.

$$\begin{aligned}
\text{distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(18 - 0)^2 + (6 - 0)^2} \\
&\approx 18.743 \text{miles}
\end{aligned}$$

**Analysis**

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

**Exercise 2.3.2**

There is a straight road leading from the town of Timpson to Ashburn 60 miles east and 12 miles north. Partway down the road, it junctions with a second road, perpendicular to the first, leading to the town of Garrison. If the town of Garrison is located 22 miles directly east of the town of Timpson, how far is the road junction from Timpson?

**Solution**

21.15 miles

## Building Systems of Linear Models

Real-world situations including two or more linear functions may be modeled with a **system of linear equations**. Remember, when solving a system of linear equations, we are looking for points the two lines have in common. Typically, there are three types of answers possible, as shown in Figure 2.3.6.

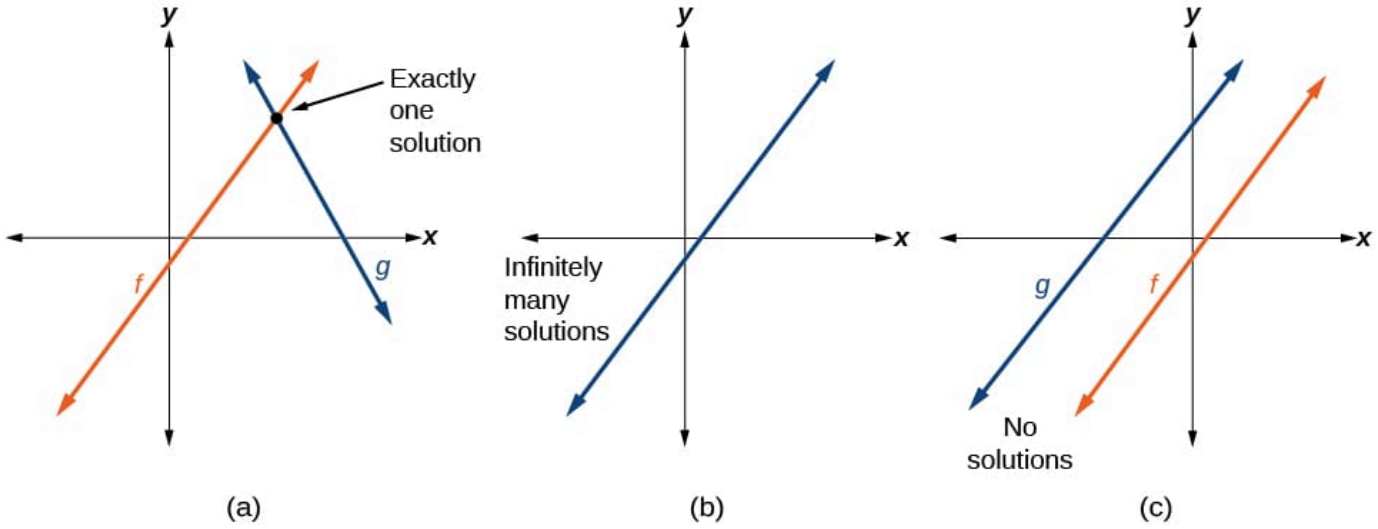



Figure 2.3.6

**How To: Given a situation that represents a system of linear equations, write the system of equations and identify the solution.**

a. Identify the input and output of each linear model.
b. Identify the slope and y-intercept of each linear model.
c. Find the solution by setting the two linear functions equal to another and solving for $x$,or find the point of intersection on a graph.

## Example 2.3.4: Building a System of Linear Models to Choose a Truck Rental Company

Jamal is choosing between two truck-rental companies. The first, Keep on Trucking, Inc., charges an up-front fee of $20, then 59 cents a mile[1]. The second, Move It Your Way, charges an up-front fee of $16, then 63 cents a mile. When will Keep on Trucking, Inc. be the better choice for Jamal?

**Solution**

The two important quantities in this problem are the cost and the number of miles driven. Because we have two companies to consider, we will define two functions.

- Input: $d$, distance driven in miles
- Outputs: $K(d)$ : cost, in dollars, for renting from Keep on Trucking

  $M(d)$ : cost, in dollars, for renting from Move It Your Way
- Initial Value: Up-front fee: $K(0) = 20$ and $M(0) = 16$
- Rate of Change: $K(d) = \frac{\$0.59}{\text{mile}}$ and $P(d) = \frac{\$0.63}{\text{mile}}$

A linear function is of the form $f(x) = mx + b$. Using the rates of change and initial charges, we can write the equations

$$K(d) = 0.59d + 20$$

$$M(d) = 0.63d + 16$$

Using these equations, we can determine when Keep on Trucking, Inc. will be the better choice. Because all we have to make that decision from is the costs, we are looking for when Keep on Trucking, Inc. will cost less, or when $K(d) < M(d)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, and then see where the $K(d)$ function is smaller.

These graphs are sketched in Figure 2.3.7, with $K(d)$ in blue.

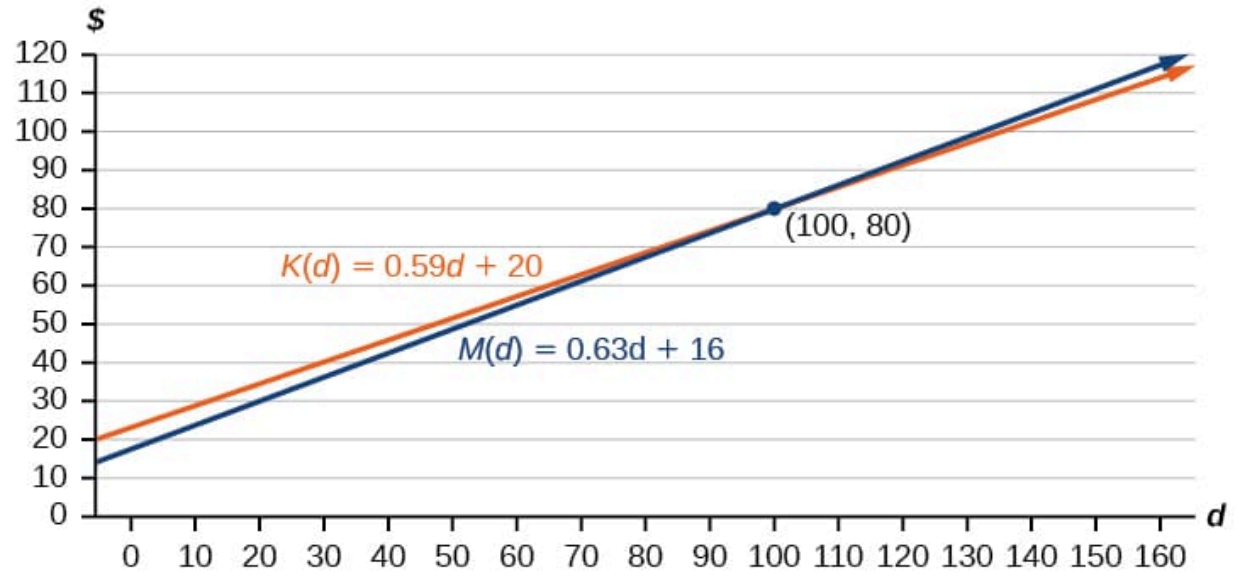



Figure 2.3.7

To find the intersection, we set the equations equal and solve:

$$\begin{aligned} K(d) &= M(d) \\ 0.59d + 20 &= 0.63d + 16 \\ 4 &= 0.04d \\ 100 &= d \\ d &= 100 \end{aligned}$$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that $K(d)$ is growing at a slower rate, we can conclude that Keep on Trucking, Inc. will be the cheaper price when more than 100 miles are driven, that is $d > 100$.

## Key Concepts

- We can use the same problem strategies that we would use for any type of function.
- When modeling and solving a problem, identify the variables and look for key values, including the slope and y-intercept.
- Draw a diagram, where appropriate.
- Check for reasonableness of the answer.
- Linear models may be built by identifying or calculating the slope and using the y-intercept.
- The x-intercept may be found by setting $y = 0$, which is setting the expression $mx + b$ equal to 0.
- The point of intersection of a system of linear equations is the point where the x- and y-values are the same.
- A graph of the system may be used to identify the points where one line falls below (or above) the other line.

## Footnotes

1 Rates retrieved Aug 2, 2010 from www.budgettruck.com and http://www.uhaul.com/

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# 2.4: Fitting Linear Models to Data

**Learning Objectives**

- Draw and interpret scatter diagrams.
- Use a graphing utility to find the line of best fit.
- Distinguish between linear and nonlinear relations.
- Fit a regression line to a set of data and use the linear model to make predictions.

A professor is attempting to identify trends among final exam scores. His class has a mixture of students, so he wonders if there is any relationship between age and final exam scores. One way for him to analyze the scores is by creating a diagram that relates the age of each student to the exam score received. In this section, we will examine one such diagram known as a scatter plot.

## Drawing and Interpreting Scatter Plots

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a **linear model**, or a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions. Figure 2.4.1 shows a sample scatter plot.




Figure 2.4.1: A scatter plot of age and final exam score variables

Notice this scatter plot does not indicate a **linear relationship**. The points do not appear to follow a trend. In other words, there does not appear to be a relationship between the age of the student and the score on the final exam.

**Example 2.4.1: Using a Scatter Plot to Investigate Cricket Chirps**

Table shows the number of cricket chirps in 15 seconds, for several different air temperatures, in degrees Fahrenheit[1]. Plot this data, and determine whether the data appears to be linearly related.

*Table* 2.4.1

| **Chirps** | 44 | 35 | 20.4 | 33 | 31 | 35 | 18.5 | 37 | 26 |
|---|---|---|---|---|---|---|---|---|---|
| **Temperature** | 80.5 | 70.5 | 57 | 66 | 68 | 72 | 52 | 73.5 | 53 |

**Solution**

Plotting this data, as depicted in Figure 2.4.2 suggests that there may be a trend. We can see from the trend in the data that the number of chirps increases as the temperature increases. The trend appears to be roughly linear, though certainly not perfectly so.

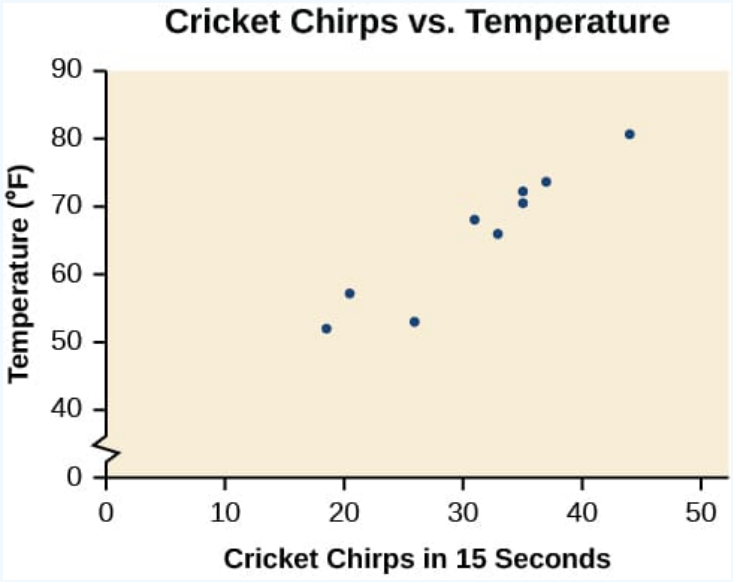



Figure 2.4.2: The line regression is generally positive.

## Finding the Line of Best Fit

Once we recognize a need for a linear function to model that data, the natural follow-up question is "what is that linear function?" One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the y-intercept. We can approximate the slope of the line by extending it until we can estimate the $\frac{\text{rise}}{\text{run}}$.

### Example 2.4.2: Finding a Line of Best Fit

Find a linear function that fits the data in Table 2.4.1 by "eyeballing" a line that seems to fit.

**Solution**

On a graph, we could try sketching a line.

Using the starting and ending points of our hand drawn line, points $(0, 30)$ and $(50, 90)$, this graph has a slope of

$$m = \frac{60}{50} = 1.2 \tag{2.4.1}$$

and a y-intercept at 30. This gives an equation of

$$T(c) = 1.2c + 30 \tag{2.4.2}$$

where $c$ is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit. The resulting equation is represented in Figure 2.4.3.

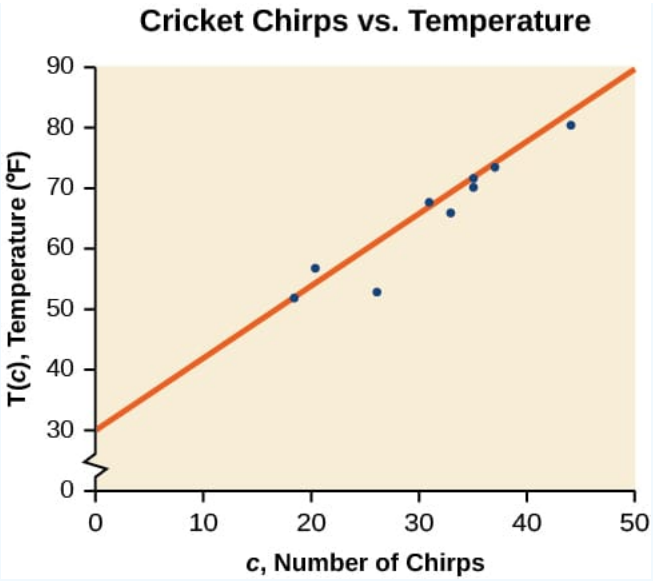



Figure 2.4.3: Scatter plot, showing the line of best fit.

**Analysis**

This linear equation can then be used to approximate answers to various questions we might ask about the trend.

## Recognizing Interpolation or Extrapolation

While the data for most examples does not fall perfectly on the line, the equation is our best guess as to how the relationship will behave outside of the values for which we have data. We use a process known as **interpolation** when we predict a value inside the domain and range of the data. The process of **extrapolation** is used when we predict a value outside the domain and range of the data.

Figure 2.4.4 compares the two processes for the cricket-chirp data addressed in Example 2.4.2. We can see that interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44. Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

There is a difference between making predictions inside the domain and range of values for which we have data and outside that domain and range. Predicting a value outside of the domain and range has its limitations. When our model no longer applies after a certain point, it is sometimes called **model breakdown**. For example, predicting a cost function for a period of two years may involve examining the data where the input is the time in years and the output is the cost. But if we try to extrapolate a cost when $x = 50$, that is in 50 years, the model would not apply because we could not account for factors fifty years in the future.

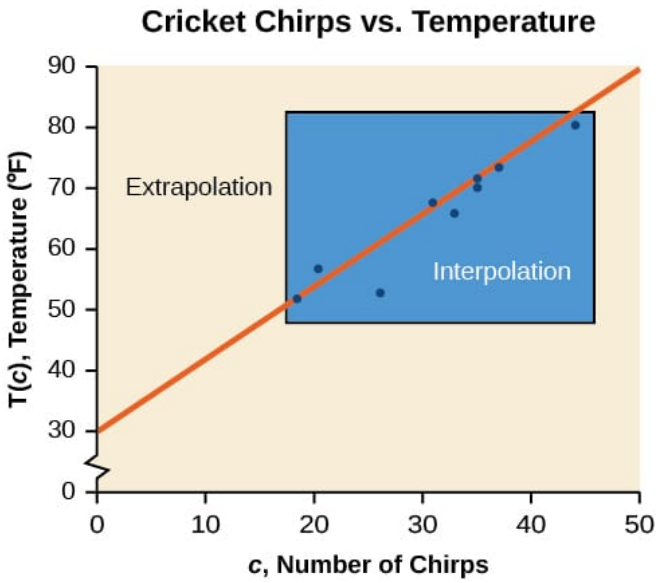



Figure 2.4.4: Interpolation occurs within the domain and range of the provided data whereas extrapolation occurs outside.

## Interpolation and Extrapolation

Different methods of making predictions are used to analyze data.

- The method of **extrapolation** involves predicting a value outside the domain and/or range of the data.
- **Model breakdown** occurs at the point when the model no longer applies.

## Example 2.4.3: Understanding Interpolation and Extrapolation

Use the cricket data from Table 2.4.1 to answer the following questions:

a. Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.
b. Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

**Solution**

a. The number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:

$$T(30) = 30 + 1.2(30) \tag{2.4.3}$$
$$= 66 \text{ degrees} \tag{2.4.4}$$

Based on the data we have, this value seems reasonable.
b. The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation because 40 is outside the range of our data. Using our model:

$$40 = 30 + 1.2c \tag{2.4.5}$$
$$10 = 1.2c \tag{2.4.6}$$
$$c \approx 8.33 \tag{2.4.7}$$

We can compare the regions of interpolation and extrapolation using Figure 2.4.5.

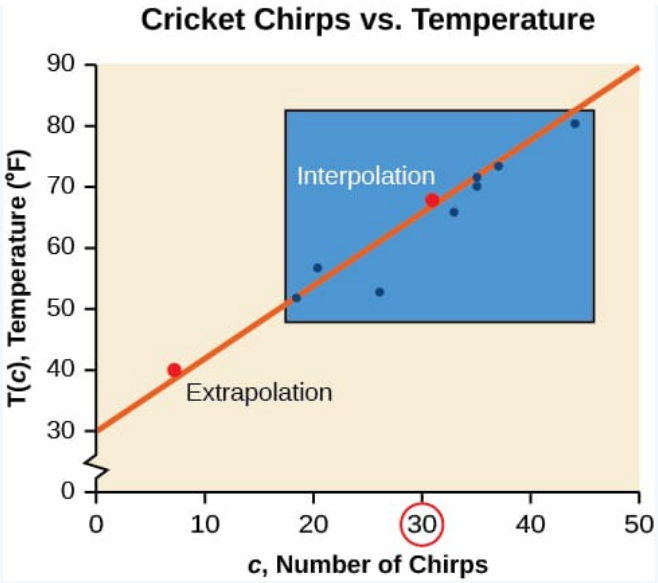



Figure 2.4.5: Scatter plot, showing the line of best fit and where interpolation and extrapolation occurs.

**Analysis**

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

### Exercise 2.4.1

According to the data from Table 2.4.1, what temperature can we predict it is if we counted 20 chirps in 15 seconds?

**Solution**

54°F

## Finding the Line of Best Fit Using a Graphing Utility

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values[2]. One such technique is called **least squares regression** and can be computed by many graphing calculators, spreadsheet software, statistical software, and many web-based calculators[3]. Least squares regression is one means to determine the line that best fits the data, and here we will refer to this method as linear regression.

**How To: Given data of input and corresponding outputs from a linear function, find the best fit line using linear regression.**

1. Enter the input in List 1 (L1).
2. Enter the output in List 2 (L2).
3. On a graphing utility, select Linear Regression (LinReg).

### Example 2.4.4: Finding a Least Squares Regression Line

Find the least squares regression line using the cricket-chirp data in Table 2.4.1.

**Solution**

Enter the input (chirps) in List 1 (L1).
Enter the output (temperature) in List 2 (L2). See Table 2.4.2.

*Table* 2.4.2

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **L1** | 44 | 35 | 20.4 | 33 | 31 | 35 | 18.5 | 37 | 26 |
| **L2** | 80.5 | 70.5 | 57 | 66 | 68 | 72 | 52 | 73.5 | 53 |

On a graphing utility, select Linear Regression (LinReg). Using the cricket chirp data from earlier, with technology we obtain the equation:

$$T(c) = 30.281 + 1.143c \tag{2.4.8}$$

**Analysis**

Notice that this line is quite similar to the equation we "eyeballed" but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

$$T(30) = 30.281 + 1.143(30) \tag{2.4.9}$$
$$= 64.571 \tag{2.4.10}$$
$$\approx 64.6 \text{ degrees} \tag{2.4.11}$$

The graph of the scatter plot with the least squares regression line is shown in Figure 2.4.6.

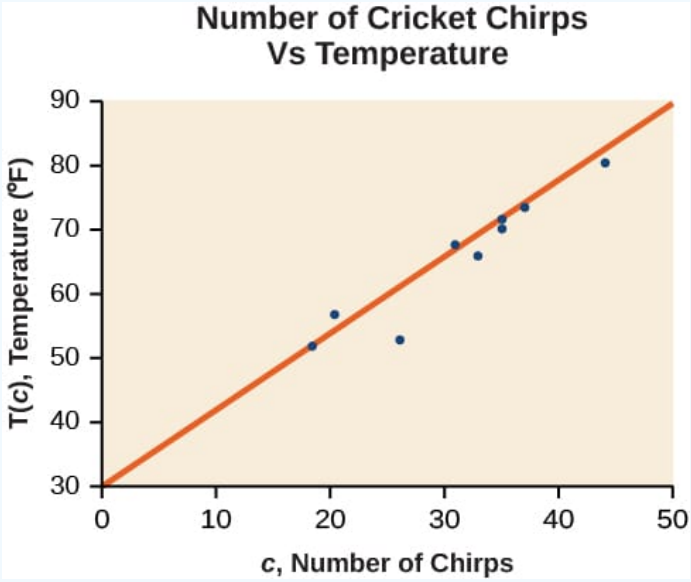



Figure 2.4.6: Scatter plot, showing the line of best fit.

**Will there ever be a case where two different lines will serve as the best fit for the data?**

*No. There is only one best fit line.*

## Distinguishing Between Linear and Non-Linear Models

As we saw above with the cricket-chirp model, some data exhibit strong linear trends, but other data, like the final exam scores plotted by age, are clearly nonlinear. Most calculators and computer software can also provide us with the **correlation coefficient**, which is a measure of how closely the line fits the data. Many graphing calculators require the user to turn a "diagnostic on" selection to find the correlation coefficient, which mathematicians label as $r$. The correlation coefficient provides an easy way to get an idea of how close to a line the data falls.

We should compute the correlation coefficient only for data that follows a linear pattern or to determine the degree to which a data set is linear. If the data exhibits a nonlinear pattern, the correlation coefficient for a linear regression is meaningless. To get a sense for the relationship between the value of $r$ and the graph of the data, Figure 2.4.7 shows some large data sets with their correlation coefficients. Remember, for all plots, the horizontal axis shows the input and the vertical axis shows the output.

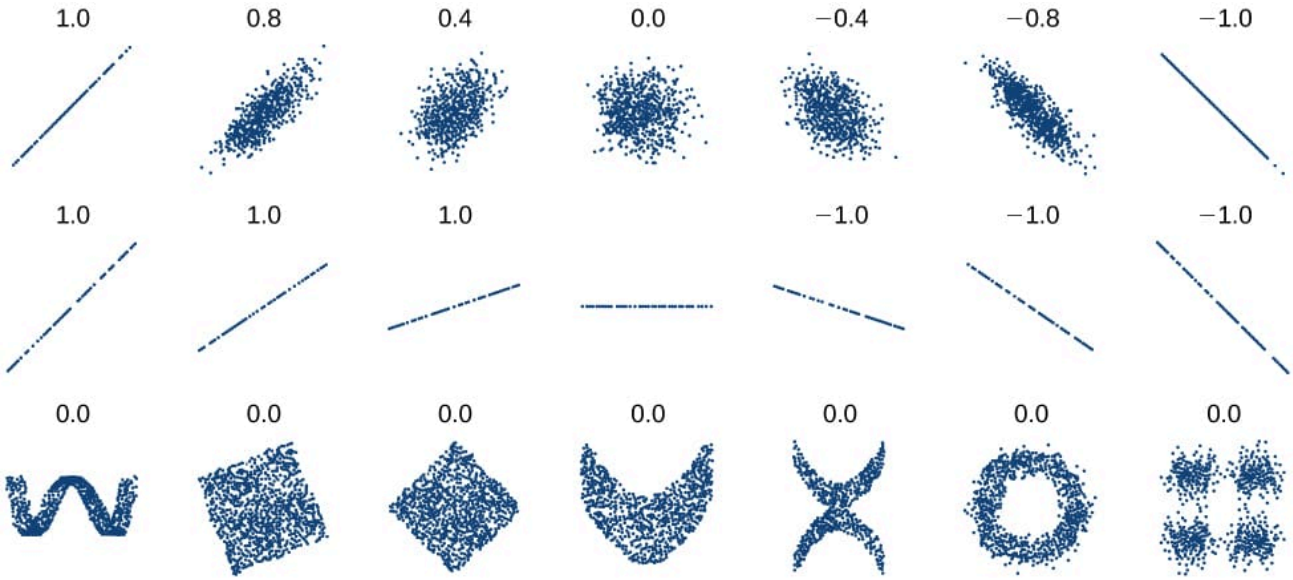



Figure 2.4.7: Plotted data and related correlation coefficients. (credit: "DenisBoigelot," Wikimedia Commons)

### Correlation Coefficient

The **correlation coefficient** is a value, $r$, between –1 and 1.

- $r > 0$ suggests a positive (increasing) relationship
- $r < 0$ suggests a negative (decreasing) relationship
- The closer the value is to 0, the more scattered the data.
- The closer the value is to 1 or –1, the less scattered the data is.

### Example 2.4.5: Finding a Correlation Coefficient

Calculate the correlation coefficient for cricket-chirp data in Table 2.4.1.

**Solution**

Because the data appear to follow a linear pattern, we can use technology to calculate $r$. Enter the inputs and corresponding outputs and select the Linear Regression. The calculator will also provide you with the correlation coefficient, $r = 0.9509$. This value is very close to 1, which suggests a strong increasing linear relationship.

Note: For some calculators, the Diagnostics must be turned "on" in order to get the correlation coefficient when linear regression is performed: [2nd]>[0]>[alpha][x–1], then scroll to DIAGNOSTICSON.

## Predicting with a Regression Line

Once we determine that a set of data is linear using the correlation coefficient, we can use the regression line to make predictions. As we learned above, a regression line is a line that is closest to the data in the scatter plot, which means that only one such line is a best fit for the data.

## Example 2.4.6: Using a Regression Line to Make Predictions

Gasoline consumption in the United States has been steadily increasing. Consumption data from 1994 to 2004 is shown in Table 2.4.3. Determine whether the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

*Table* 2.4.3

| **Year** | '94 | '95 | '96 | '97 | '98 | '99 | '00 | '01 | '02 | '03 | '04 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Consumption (billions of gallons)** | 113 | 116 | 118 | 119 | 123 | 125 | 126 | 128 | 131 | 133 | 136 |

The scatter plot of the data, including the least squares regression line, is shown in Figure 2.4.8.

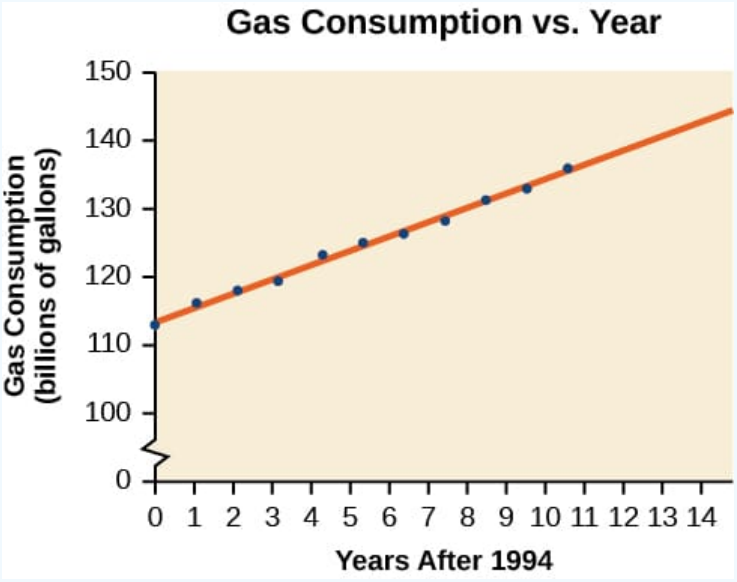



Figure 2.4.8: Scatter plot, showing the line of best fit.

We can introduce new input variable, $t$,representing years since 1994.

The least squares regression equation is:

$$C(t) = 113.318 + 2.209t \tag{2.4.12}$$

Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.

Using this to predict consumption in 2008 $(t = 14)$,

$$C(14) = 113.318 + 2.209(14) \tag{2.4.13}$$
$$= 144.244 \tag{2.4.14}$$

The model predicts 144.244 billion gallons of gasoline consumption in 2008.

## Exercise 2.4.1

Use the model we created using technology in Example 2.4.6 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

*Answer available at the end of this section.*

# Key Concepts

- Scatter plots show the relationship between two sets of data.
- Scatter plots may represent linear or non-linear models.
- The line of best fit may be estimated or calculated, using a calculator or statistical software.
- Interpolation can be used to predict values inside the domain and range of the data, whereas extrapolation can be used to predict values outside the domain and range of the data.
- The correlation coefficient, $r$, indicates the degree of linear relationship between data.
- A regression line best fits the data.
- The least squares regression line is found by minimizing the squares of the distances of points from a line passing through the data and may be used to make predictions regarding either of the variables.

## Answers

**Exercise** 2.4.1

150.871 billion gallons; extrapolation

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# 2.E: Linear Functions(Exercises)

## 2.1 Linear Functions

### Verbal

1) Terry is skiing down a steep hill. Terry's elevation, $E(t)$, in feet after $t$ seconds is given by $E(t) = 3000 - 70t$. Write a complete sentence describing Terry's starting elevation and how it is changing over time.

**Answer**

Terry starts at an elevation of $3000$ feet and descends $70$ feet per second.

2) Maria is climbing a mountain. Maria's elevation, $E(t)$, in feet after $t$ minutes is given by $E(t) = 1200 + 40t$. Write a complete sentence describing Maria's starting elevation and how it is changing over time.

3) Jessica is walking home from a friend's house. After $2$ minutes she is $1.4$ miles from home. Twelve minutes after leaving, she is $0.9$ miles from home. What is her rate in miles per hour?

**Answer**

$3$ miles per hour

4) Sonya is currently $10$ miles from home and is walking farther away at $2$ miles per hour. Write an equation for her distance from home $t$ hours from now.

5) A boat is $100$ miles away from the marina, sailing directly toward it at $10$ miles per hour. Write an equation for the distance of the boat from the marina after $t$ hours.

**Answer**

$d(t) = 100 - 10t$

6) Timmy goes to the fair with $\$40$. Each ride costs $\$2$. How much money will he have left after riding $n$ rides?

### Algebraic

**For the exercises 7-14, determine whether the equation of the curve can be written as a linear function.**

7) $y = \frac{1}{4}x + 6$

**Answer**

Yes

8) $y = 3x - 5$

9) $y = 3x^2 - 2$

**Answer**

No

10) $3x + 5y = 15$

11) $3x^2 + 5y = 15$

**Answer**

No

12) $3x + 5y^2 = 15$

13) $-2x^2 + 3y^2 = 6$

**Answer**

No

14) $-\frac{x-3}{5} = 2y$

**For the exercises 15-24, determine whether each function is increasing or decreasing.**

15) $f(x) = 4x + 3$

**Answer**

Increasing

16) $g(x) = 5x + 6$

17) $a(x) = 5 - 2x$

**Answer**

Decreasing

18) $b(x) = 8 - 3x$

19) $h(x) = -2x + 4$

**Answer**

Decreasing

20) $k(x) = -4x + 1$

21) $j(x) = \frac{1}{2}x - 3$

**Answer**

Increasing

22) $p(x) = \frac{1}{4}x - 5$

23) $n(x) = -\frac{1}{3}x - 2$

**Answer**

Decreasing

24) $m(x) = -\frac{3}{8}x + 3$

**For the exercises 25-29, find the slope of the line that passes through the two given points.**

25) $(2, 4)$ and $(4, 10)$

**Answer**

$3$

26) $(1, 5)$ and $(4, 11)$

27) $(-1, 4)$ and $(5, 2)$

**Answer**

$-\frac{1}{3}$

28) $(8, -2)$ and $(4, 6)$

29) (6, 11) and (–4, 3)

**Answer**

$\frac{4}{5}$

**For the exercises 30-37, given each set of information, find a linear equation satisfying the conditions, if possible.**

30) $f(-5) = -4$, and $f(5) = 2$

31) $f(-1) = 4$ and $f(5) = 1$

**Answer**

$f(x) = -\frac{1}{2}x + \frac{7}{2}$

32) $(2, 4)$ and $(4, 10)$

33) Passes through $(1, 5)$ and $(4, 11)$

**Answer**

$y = 2x + 3$

34) Passes through $(-1, 4)$ and $(5, 2)$

35) Passes through $(-2, 8)$ and $(4, 6)$

**Answer**

$y = -\frac{1}{3}x + \frac{22}{3}$

36) $x$ intercept at (–2, 0) and $y$ intercept at (0,–3)

37) $x$ intercept at (–5, 0) and $y$ intercept at (0, 4)

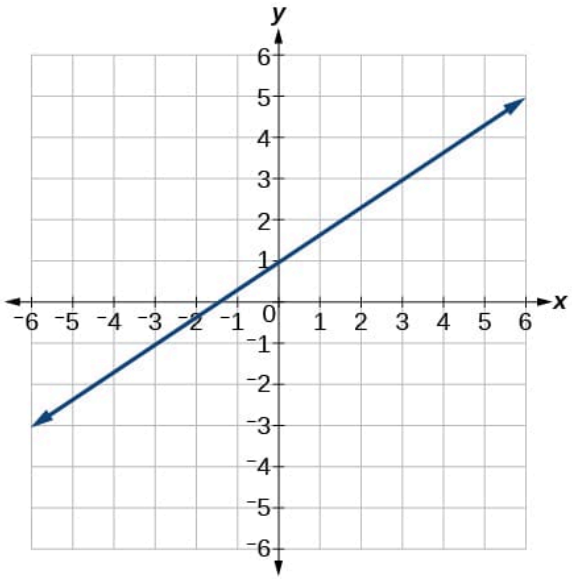
**Answer**

$y = \frac{4}{5}x + 4$

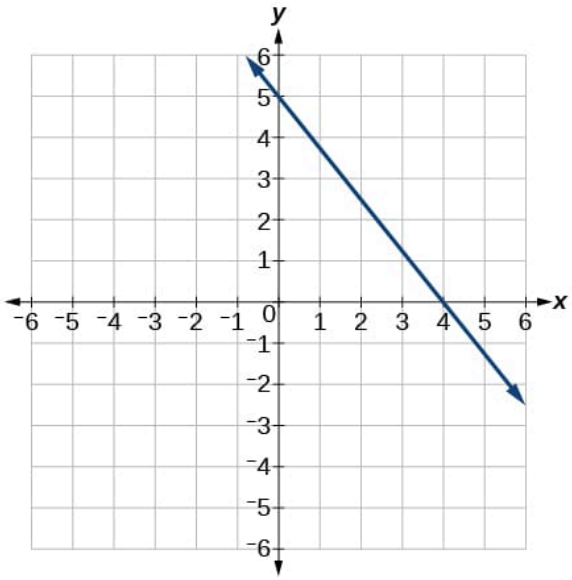
## Graphical

**For the exercises 38-,40 find the slope of the lines graphed.**

38)



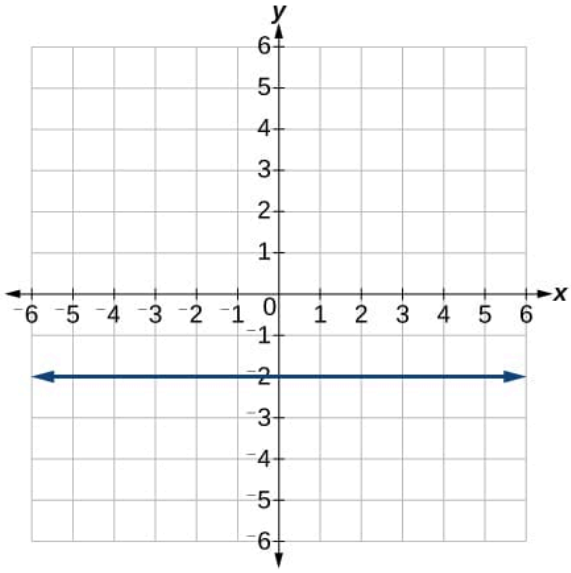

39)




**Answer**

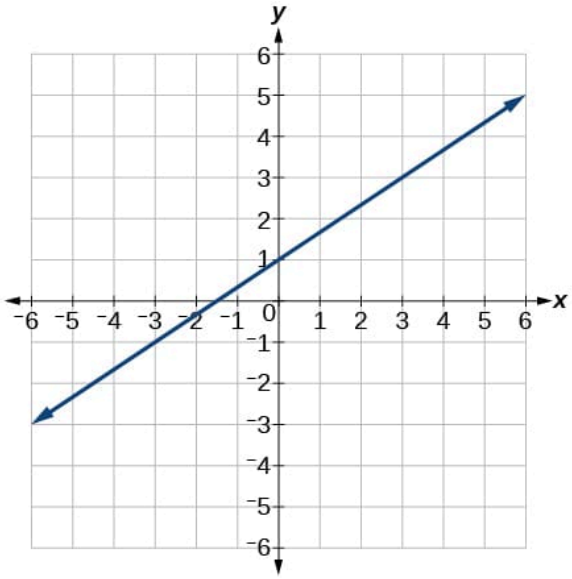
$$-\frac{5}{4}$$

40)




**For the exercises 41-46, write an equation for the lines graphed.**

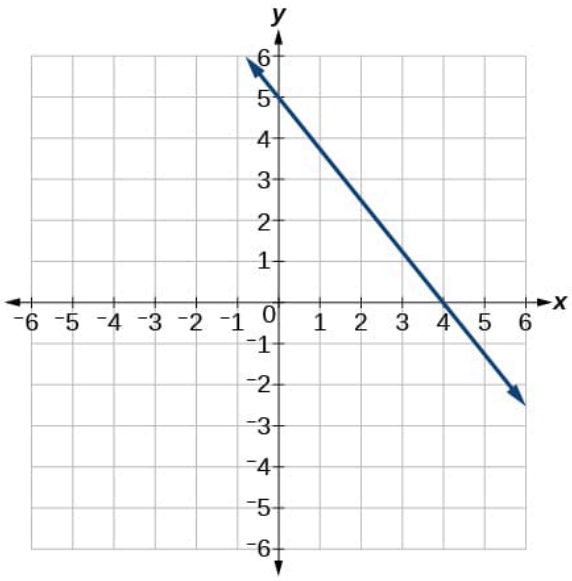
41)



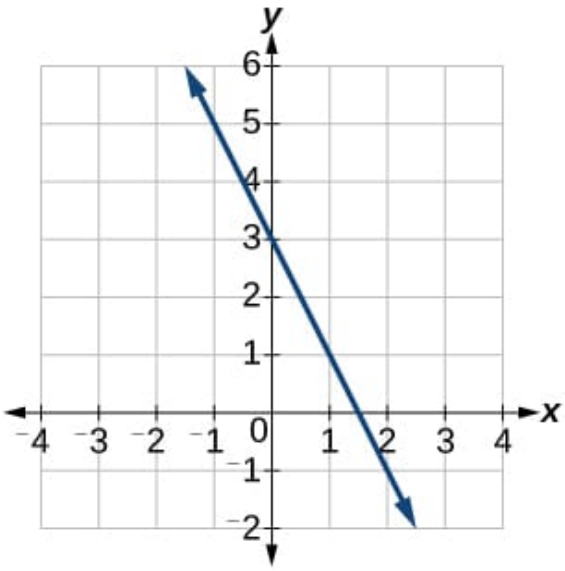

**Answer**

$$y = \frac{2}{3}x + 1$$

42)



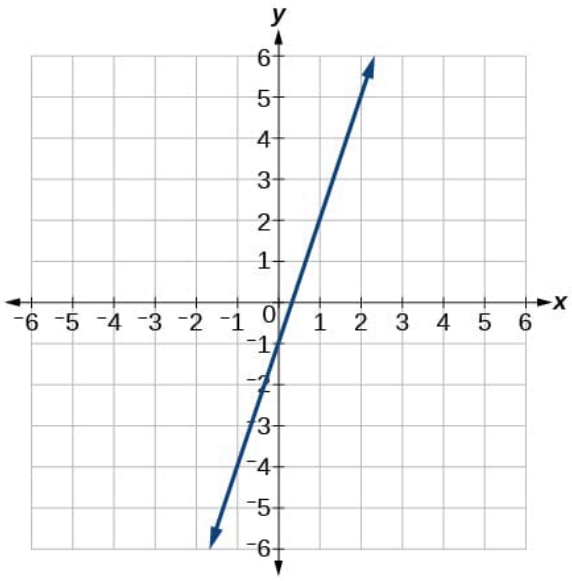

43)



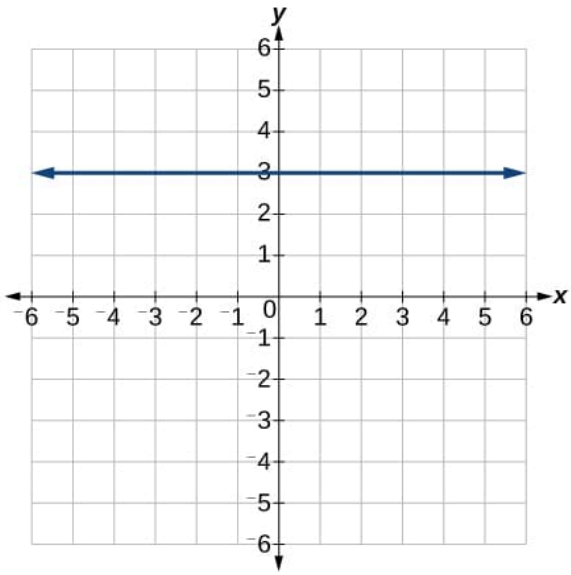

**Answer**

$y = -2x + 3$

44)



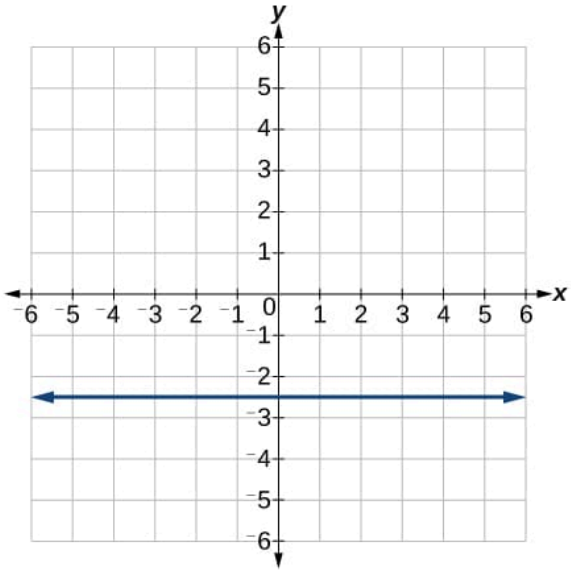

45)




**Answer**

$y = 3$

46)




## Numeric

**For the exercises 47-51, which of the tables could represent a linear function? For each that could be linear, find a linear equation that models the data.**

47)

| $x$ | 0 | 5 | 10 | 15 |
|---|---|---|---|---|
| $g(x)$ | 5 | −10 | −25 | −40 |

**Answer**

Linear, $g(x) = -3x + 5$

48)

| $x$ | 0 | 5 | 10 | 15 |
|---|---|---|---|---|
| $h(x)$ | 5 | 30 | 105 | 230 |

| $x$ | 0 | 5 | 10 | 15 |
|---|---|---|---|---|
| $f(x)$ | −5 | 20 | 45 | 70 |

**Answer**

Linear, $f(x) = 5x - 5$

49)

| $x$ | 5 | 10 | 20 | 25 |
|---|---|---|---|---|
| $k(x)$ | 28 | 13 | 58 | 73 |

| $x$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $g(x)$ | 6 | −19 | −44 | −69 |

**Answer**

Linear, $g(x) = -\frac{25}{2}x + 6$

50)

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | –4 | 16 | 36 | 56 |


| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | –4 | 16 | 36 | 56 |


**Answer**

Linear, $f(x) = 10x - 24$

51)

| $x$ | 0 | 2 | 6 | 8 |
|---|---|---|---|---|
| $k(x)$ | 6 | 31 | 106 | 231 |

## Technology

52) If $f$ is a linear function, $f(0.1) = 11.5$, and $f(0.4) = -5.9$, find an equation for the function.

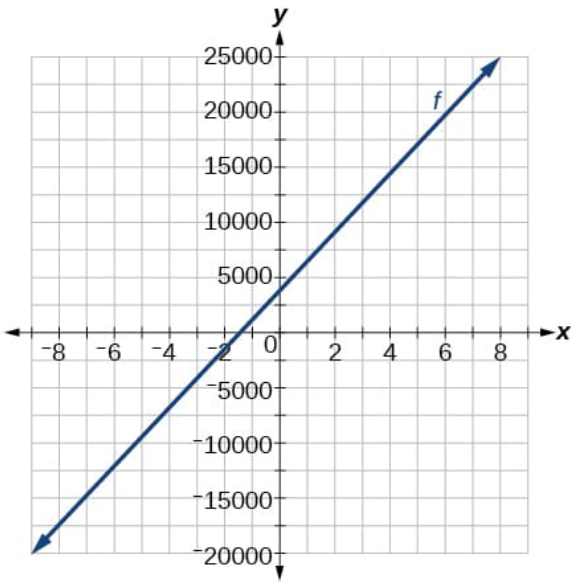
**Answer**

$f(x) = -58x + 17.3$

53) Graph the function $f$ on a domain of $[-10, 10] : f(x) = 0.02x - 0.01$. Enter the function in a graphing utility. For the viewing window, set the minimum value of $x$ to be –10 and the maximum value of $x$ to be 10.

54) Graph the function $f$ on a domain of $[-10, 10] : fx) = 2,500x + 4,000$

**Answer**



55) The Table below shows the input, $w$, and output, $k$, for a linear function $k$.

a. Fill in the missing values of the table.
b. Write the linear function $k$, round to $3$ decimal places.

| $w$ | -10 | 5.5 | 67.5 | b |
|---|---|---|---|---|
| $k$ | 30 | -26 | a | -44 |

56) The Table below shows the input, $p$, and output, $q$, for a linear function $q$.

a. Fill in the missing values of the table.
b. Write the linear function $k$.

| $p$ | 0.5 | 0.8 | 12 | b |
|---|---|---|---|---|
| $q$ | 400 | 700 | a | 1,000,000 |

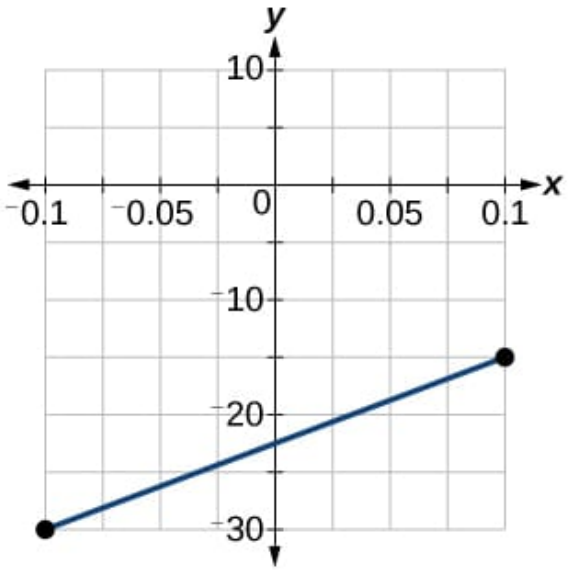
**Answer**

a. $a = 11{,}900$; $b = 1001.1$
b. $q(p) = 1000p - 100$

57) Graph the linear function $f$ on a domain of $[-10, 10]$ for the function whose slope is $\frac{1}{8}$ and y-intercept is $\frac{31}{16}$. Label the points for the input values of $-10$ and $10$.

58) Graph the linear function $f$ on a domain of $[-0.1, 0.1]$ for the function whose slope is $75$ and $y$-intercept is $-22.5$. Label the points for the input values of $-0.1$ and $0.1$.

**Answer**




59) Graph the linear function $f$ where $f(x) = ax + b$ on the same set of axes on a domain of $[-4, 4]$ for the following values of $a$ and $b$.

i. $a = 2; b = 3$
ii. $a = 2; b = 4$
iii. $a = 2; b = -4$
iv. $a = 2; b = -5$

## Extensions

60) Find the value of $x$ if a linear function goes through the following points and has the following slope: $(x, 2), (-4, 6), m = 3$

**Answer**

$$x = -\frac{16}{3}$$

61) Find the value of $y$ if a linear function goes through the following points and has the following slope: $(10, y), (25, 100), m = -5$

62) Find the equation of the line that passes through the following points: $(a, b)$ and $(a, b + 1)$

**Answer**

$$x = a$$

63) Find the equation of the line that passes through the following points: $(2a, b)$ and (a, b+1)

64) Find the equation of the line that passes through the following points: $(a, 0)$ and $(c, d)$

**Answer**

$$y = \frac{d}{c - a}x - \frac{ad}{c - a}$$

## Real-World Applications

65) At noon, a barista notices that she has $\$20$ in her tip jar. If she makes an average of $\$0.50$ from each customer, how much will she have in her tip jar if she serves $n$ more customers during her shift?

66) A gym membership with two personal training sessions costs $\$125$, while gym membership with five personal training sessions costs $\$260$. What is cost per session?

**Answer**

$\$45$ per training session.

67) A clothing business finds there is a linear relationship between the number of shirts, $n$, it can sell and the price, $p$, it can charge per shirt. In particular, historical data shows that $1,000$ shirts can be sold at a price of $\$30$, while $3,000$ shirts can be sold at a price of $\$22$. Find a linear equation in the form $p(n) = mn + b$ that gives the price $p$ they can charge for $n$ shirts.

68) A phone company charges for service according to the formula: $C(n) = 24 + 0.1$n, where $n$ is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

**Answer**

The rate of change is $0.1$. For every additional minute talked, the monthly charge increases by $\$0.1$ or $10$ cents. The initial value is $24$. When there are no minutes talked, initially the charge is $\$24$.

69) A farmer finds there is a linear relationship between the number of bean stalks, $n$, she plants and the yield, $y$, each plant produces. When she plants $30$ stalks, each plant yields $30$ oz of beans. When she plants $34$ stalks, each plant produces $28$ oz of beans. Find a linear relationships in the form $y = mn + b$ that gives the yield when $n$ stalks are planted.

70) A city's population in the year 1960 was $287,500$. In 1989 the population was $275,900$. Compute the rate of growth of the population and make a statement about the population rate of change in people per year.

**Answer**

The slope is $-400$. This means for every year between 1960 and 1989, the population dropped by $400$ per year in

the city.

71) A town's population has been growing linearly. In 2003, the population was $45,000$, and the population has been growing by $1,700$ people each year. Write an equation, $P(t)$, for the population $t$ years after 2003.

72) Suppose that average annual income (in dollars) for the years 1990 through 1999 is given by the linear function: $I(x) = 1054x + 23{,}286$, where $x$ is the number of years after 1990. Which of the following interprets the slope in the context of the problem?

a. As of 1990, average annual income was $\$23,286$.
b. In the ten-year period from 1990–1999, average annual income increased by a total of $\$1,054$.
c. Each year in the decade of the 1990s, average annual income increased by $1,054.
d. Average annual income rose to a level of $23,286 by the end of 1999.

**Answer**

c

73) When temperature is $0$ degrees Celsius, the Fahrenheit temperature is $32$. When the Celsius temperature is $100$, the corresponding Fahrenheit temperature is $212$. Express the Fahrenheit temperature as a linear function of $C$, the Celsius temperature, $F(C)$.

a. Find the rate of change of Fahrenheit temperature for each unit change temperature of Celsius.
b. Find and interpret $F(28)$.
c. Find and interpret $F(-40)$.

## 2.2 Graphs of Linear Functions

Verbal

1) If the graphs of two linear functions are parallel, describe the relationship between the slopes and the $y$-intercepts.

**Answer**

The slopes are equal; $y$-intercepts are not equal.

2) If the graphs of two linear functions are perpendicular, describe the relationship between the slopes and the $y$-intercepts.

3) If a horizontal line has the equation $f(x) = a$ and a vertical line has the equation $x = a$, what is the point of intersection? Explain why what you found is the point of intersection.

**Answer**

The point of intersection is $(a, a)$. This is because for the horizontal line, all of the $y$ coordinates are a and for the vertical line, all of the $x$ coordinates are a. The point of intersection will have these two characteristics.

4) Explain how to find a line parallel to a linear function that passes through a given point.

5) Explain how to find a line perpendicular to a linear function that passes through a given point.

**Answer**

First, find the slope of the linear function. Then take the negative reciprocal of the slope; this is the slope of the perpendicular line. Substitute the slope of the perpendicular line and the coordinate of the given point into the equation $y = mx + b$ and solve for $b$. Then write the equation of the line in the form $y = mx + b$ by substituting in $m$ and $b$.

## Algebraic

**For the exercises 6-11, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:**

6) $4x - 7y = 10$ $7x + 4y = 1$

7) $3y + x = 12$ $-y = 8x + 1$

**Answer**

neither parallel or perpendicular

8) $3y + 4x = 12$ $-6y = 8x + 1$

9) $6x - 9y = 10$ $3x + 2y = 1$

**Answer**

perpendicular

10) $y = \frac{2}{3}x + 1$ $3x + 2y = 1$

11) $y = \frac{3}{4}x + 1$ $-3x + 4y = 1$

**Answer**

parallel

**For the exercises 12-17, find the $x$- and $y$-intercepts of each equation**

12) $f(x) = -x + 2$

13) $g(x) = 2x + 4$

**Answer**

$(-2, 0); (0, 4)$

14) $h(x) = 3x - 5$

15) $k(x) = -5x + 1$

**Answer**

$\left(\frac{1}{5}, 0\right); (0, 1)$

16) $-2x + 5y = 20$

17) $7x + 2y = 56$

**Answer**

$(8, 0); (0, 28)$

**For the exercises 18-23, use the descriptions of each pair of lines given below to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?**

18)

Line 1: Passes through $(0, 6)$ and $(3, -24)$

Line 2: Passes through $(-1, 19)$ and $(8, -71)$

19)

Line 1: Passes through $(-8, -55)$ and $(10, 89)$

Line 2: Passes through $(9, -44)$ and $(4, -14)$

**Answer**

Line 1: $m = 8$ Line 2: $m = -6$ Neither

20)

Line 1: Passes through $(2, 3)$ and (4,–1)

Line 2: Passes through $(6, 3)$ and $(8, 5)$

21)

Line 1: Passes through $(1, 7)$ and $(5, 5)$

Line 2: Passes through $(-1, -3)$ and $(1, 1)$

**Answer**

Line 1: $m = -12$ Line 2: $m = 2$ Perpendicular

22)

Line 1: Passes through $(0, 5)$ and $(3, 3)$

Line 2: Passes through $(1, -5)$ and $(3, -2)$

23)

Line 1: Passes through $(2, 5)$ and $(5, -1)$

Line 2: Passes through $(-3, 7)$ and $(3, -5)$

**Answer**

Line 1: $m = -2$ Line 2: $m = -2$ Parallel

24) Write an equation for a line parallel to $f(x) = -5x - 3$ and passing through the point $(2, -12)$.

25) Write an equation for a line parallel to $g(x) = 3x - 1$ and passing through the point $(4, 9)$.

**Answer**

$g(x) = 3x - 3$

26) Write an equation for a line perpendicular to $h(t) = -2t + 4$ and passing through the point $(-4, -1)$.

27) Write an equation for a line perpendicular to $p(t) = 3t + 4$ and passing through the point $(3, 1)$.

**Answer**

$$p(t) = -\frac{1}{3}t + 2$$

28) Find the point at which the line $f(x) = -2x - 1$ intersects the line $g(x) = -x$.

29) Find the point at which the line $f(x) = 2x + 5$ intersects the line $g(x) = -3x - 5$.

**Answer**

$$(-2, 1)$$

30) Use algebra to find the point at which the line $f(x) = -\frac{4}{5}x + \frac{274}{25}$ intersects the line $h(x) = \frac{9}{4}x + \frac{73}{10}$.

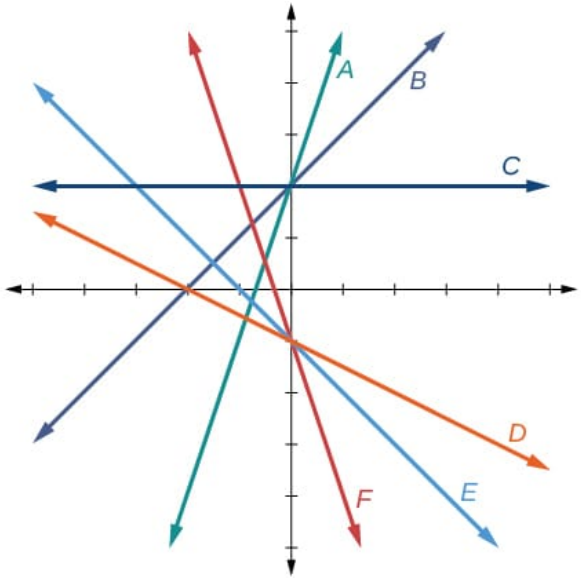
31) Use algebra to find the point at which the line $f(x) = \frac{7}{4}x + \frac{457}{60}$ intersects the line $g(x) = \frac{4}{3}x + \frac{31}{5}$.

**Answer**

$$\left(-\frac{17}{5}, \frac{5}{3}\right)$$

## Graphical

**For the exercises 32-37, match the given linear equation with its graph in the Figure below.**




32) $f(x) = -x - 1$

33) $f(x) = -2x - 1$

**Answer**

F

34) $f(x) = -\frac{1}{2}x - 1$

35) $f(x) = 2$

**Answer**

C

36) $f(x) = 2 + x$

37) $f(x) = 3x + 2$

**Answer**

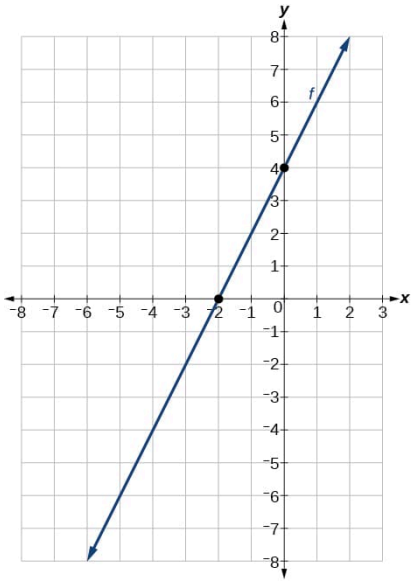
A

**For the exercises 38-43, sketch a line with the given features.**

38) An $x$-intercept of $(-4, 0)$ and $y$-intercept of $(0, -2)$

39) An $x$-intercept of $(-2, 0)$ and $y$-intercept of $(0, 4)$

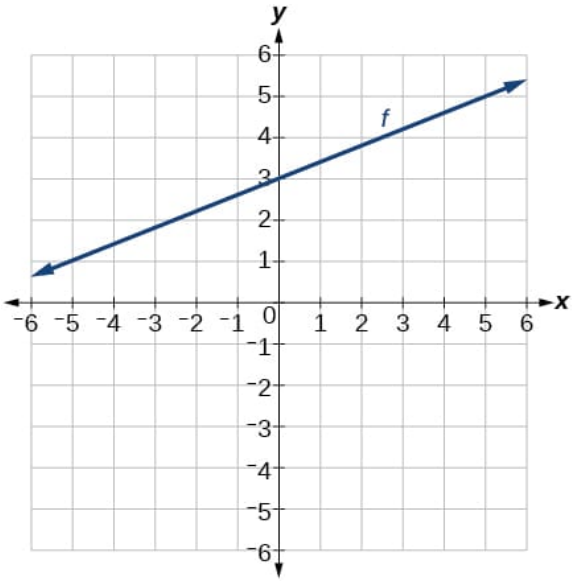
**Answer**




40) A $y$-intercept of $(0, 7)$ and slope $-\frac{3}{2}$

41) A $y$-intercept of $(0, 3)$ and slope $\frac{2}{5}$

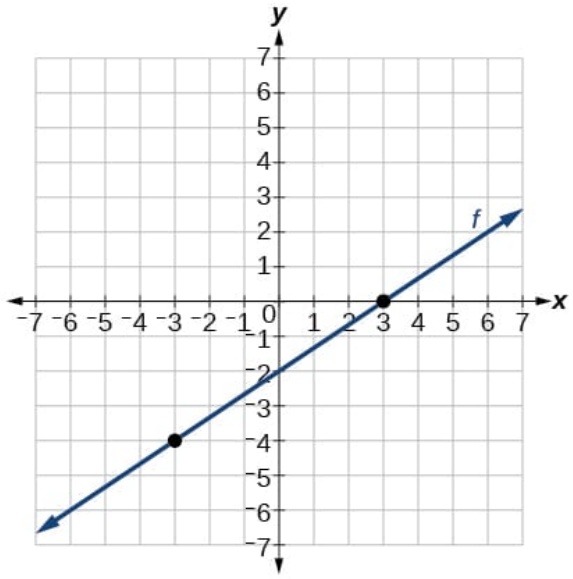
**Answer**




42) Passing through the points $(-6,-2)$ and $(6,-6)$

43) Passing through the points $(-3,-4)$ and $(3,0)$

**Answer**

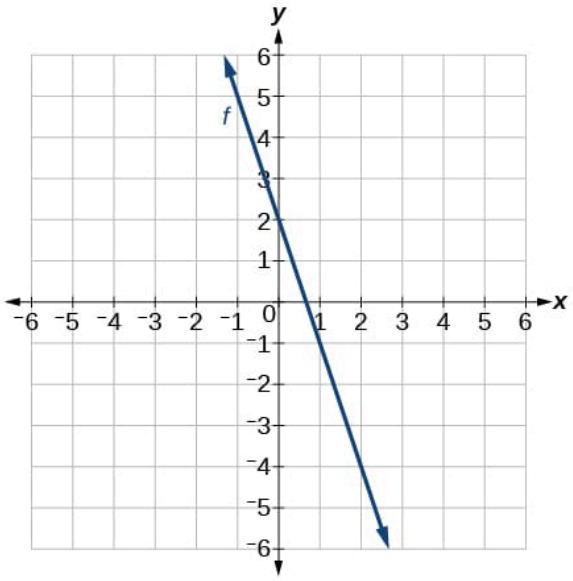



**For the exercises 44-58, sketch the graph of each equation.**

44) $f(x) = -2x - 1$

45) $g(x) = -3x + 2$

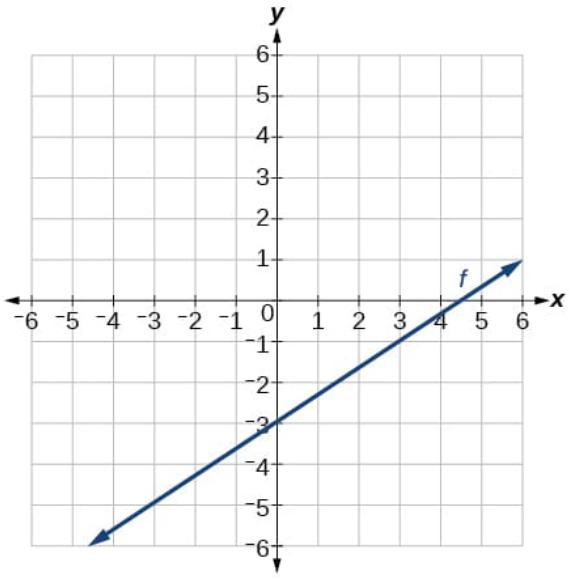
**Answer**




46) $h(x) = \frac{1}{3}x + 2$

47) $k(x) = \frac{2}{3}x - 3$

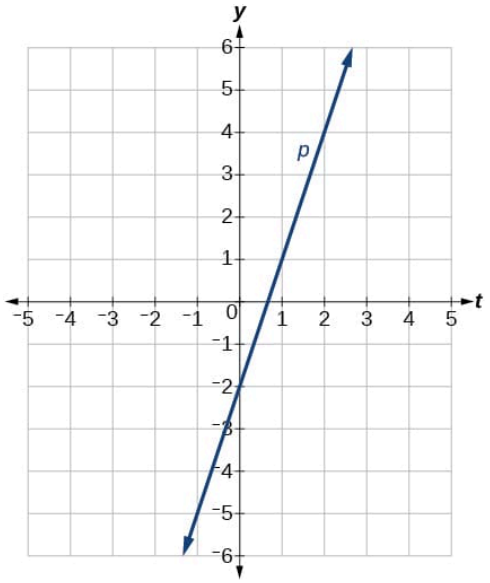
**Answer**




48) $f(t) = 3 + 2t$

49) $p(t) = -2 + 3t$

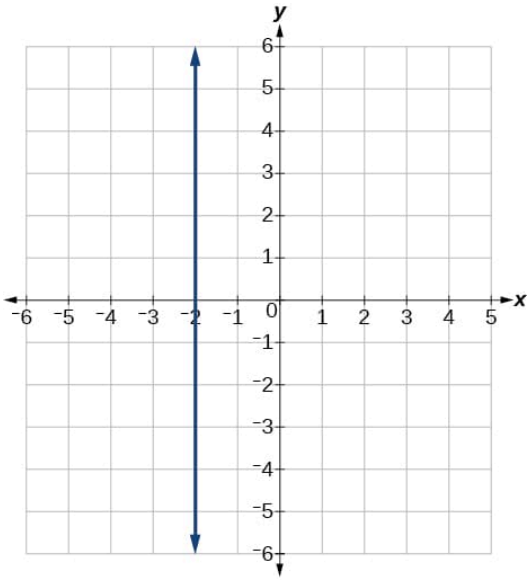
**Answer**




50) $x = 3$

51) $x = -2$

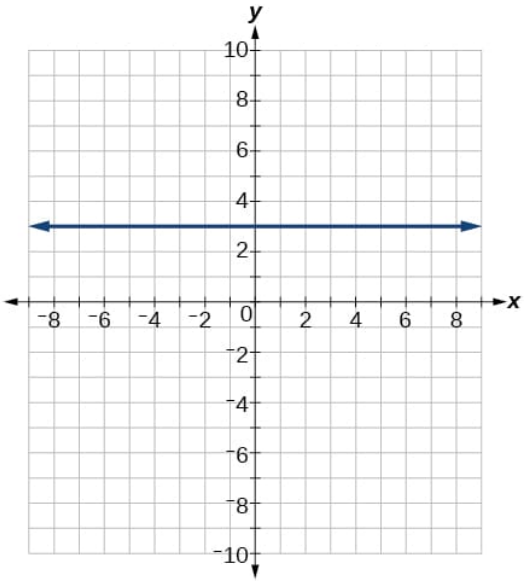
**Answer**




52) $r(x) = 4$

53) $q(x) = 3$

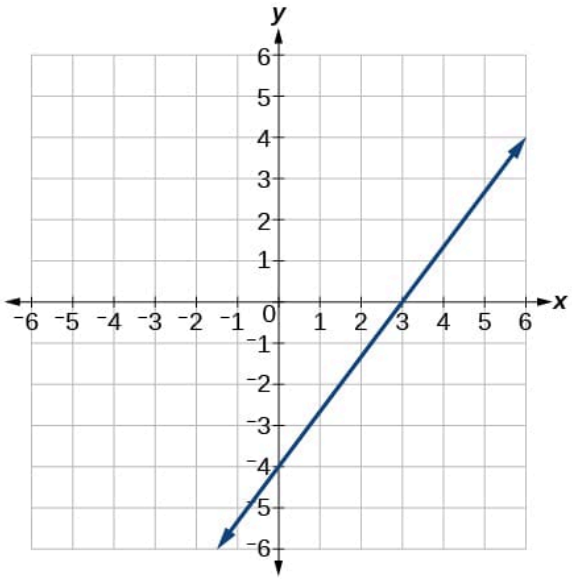
**Answer**




54) $4x = -9y + 36$

55) $\frac{x}{3} - \frac{y}{4} = 1$

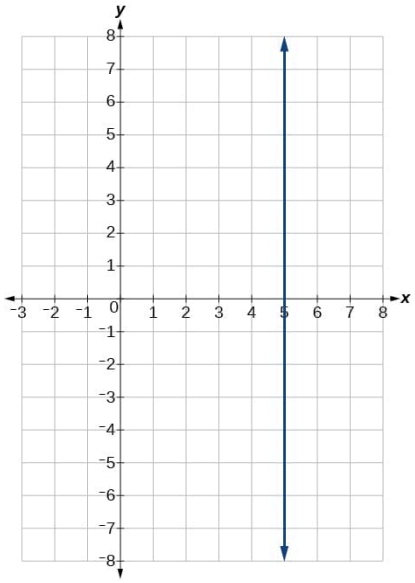
**Answer**




56) $3x - 5y = 15$

57) $3x = 15$

**Answer**




58) $3y = 12$

59) If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $\frac{3}{4}$, a shift right by $2$, and a shift down by $4$

a. Write an equation for $g(x)$.
b. What is the slope of this line?
c. Find the $y$-intercept of this line.

**Answer**

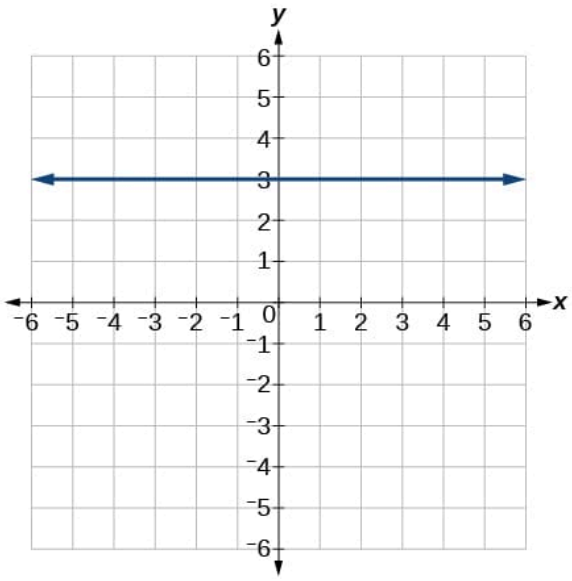
a. $g(x) = 0.75x - 5.5$
b. $0.75$
c. $0, -5.5)$

60) If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $\frac{1}{3}$, a shift left by $1$, and a shift up by $3$

a. Write an equation for $g(x)$.
b. What is the slope of this line?
c. Find the $y$-intercept of this line.

**For the exercises 61-64, write the equation of the line shown in the graph.**

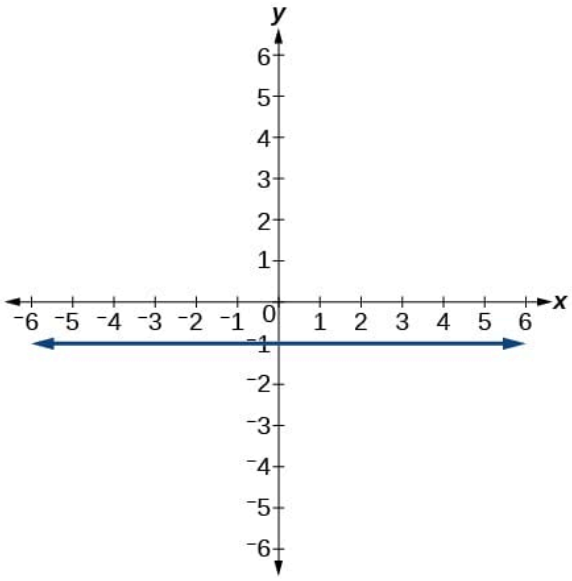
61)



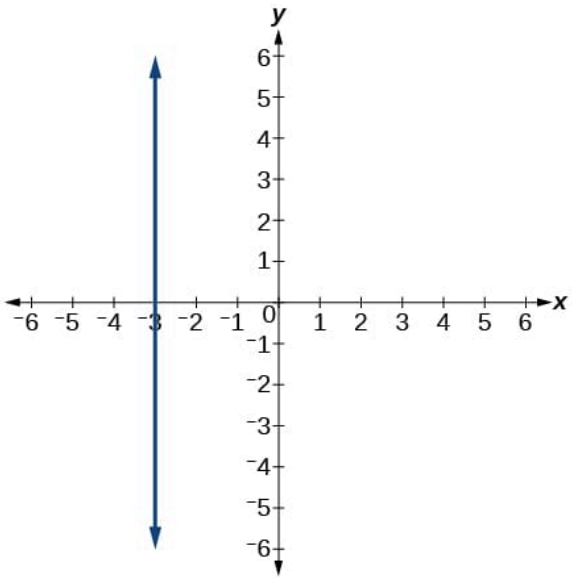

**Answer**

$y = 3$

62)



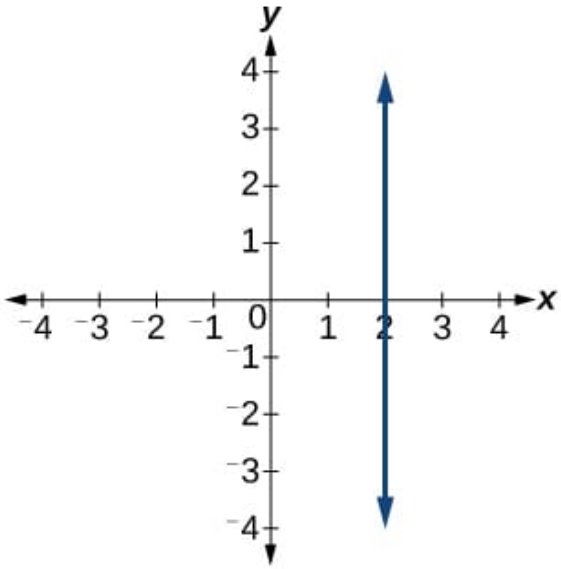

63)




**Answer**

$x = -3$

64)




**For the exercises 65-69, find the point of intersection of each pair of lines if it exists. If it does not exist, indicate that there is no point of intersection.**

65) $y = \frac{3}{4}x + 1 - 3x + 4y = 12$

**Answer**

no point of intersection

66) $2x - 3y = 12\ 5y + x = 30$

67) $2x = y - 3\ y + 4x = 15$

**Answer**

$(2, 7)$

68) $x - 2y + 2 = 3\ x - y = 3$

69) $5x + 3y = -65\ x - y = -5$

**Answer**

$(-10, -5)$

## Extensions

70) Find the equation of the line parallel to the line $g(x) = -0.01x + 2.01$ through the point $(1, 2)$.

71) Find the equation of the line perpendicular to the line $g(x) = -0.01x + 2.01$ through the point $(1, 2)$.

**Answer**

$y = 100x - 98$

**For the exercises 72-73, use the functions $f(x) = -0.1x + 200$ and $g(x) = 20x + 0.1$.**

72) Find the point of intersection of the lines $f$ and $g$.

73) Where is $f(x)$ greater than $g(x)$? Where is $g(x)$ greater than $f(x)$?

**Answer**

$x < \frac{1999}{201} x > \frac{1999}{201}$

## Real-World Applications

74) A car rental company offers two plans for renting a car.

Plan A: $\$30$ per day and $\$0.18$ per mile
Plan B: $\$50$ per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

75) A cell phone company offers two plans for minutes.

Plan A: $\$20$ per month and $\$1$ for every one hundred texts.
Plan B: $\$50$ per month with free unlimited texts.

How many texts would you need to send per month for plan B to save you money?

**Answer**

Less than $3000$ texts

76) A cell phone company offers two plans for minutes.

Plan A: $\$15$ per month and $\$2$ for every $300$ texts.
Plan B: $\$25$ per month and $\$0.50$ for every $100$ texts.

How many texts would you need to send per month for plan B to save you money?

## 2.3 Modeling with Linear Functions

# Verbal

1) Explain how to find the input variable in a word problem that uses a linear function.

**Answer**

Determine the independent variable. This is the variable upon which the output depends.

2) Explain how to find the output variable in a word problem that uses a linear function.

3) Explain how to interpret the initial value in a word problem that uses a linear function.

**Answer**

To determine the initial value, find the output when the input is equal to zero.

4) Explain how to determine the slope in a word problem that uses a linear function.

# Algebraic

5) Find the area of a parallelogram bounded by the $y$ axis, the line $x = 3$, the line $f(x) = 1 + 2x$,and the line parallel to $f(x)$ passing through (2, 7).

**Answer**

$6$ square units

6) Find the area of a triangle bounded by the $x$-axis, the line $f(x) = 12 - \frac{1}{3}x$, and the line perpendicular to $f(x)$ that passes through the origin.

7) Find the area of a triangle bounded by the $y$-axis, the line $f(x) = 9 - \frac{6}{7}x$, and the line perpendicular to $f(x)$ that passes through the origin.

**Answer**

$20.012$ square units

8) Find the area of a parallelogram bounded by the $x$-axis, the line $g(x) = 2$, the line $f(x) = 3x$, and the line parallel to $f(x)$ passing through (6,1).

**For the exercises 9-10, consider this scenario: A town's population has been decreasing at a constant rate. In 2010 the population was $5,900$. By 2012 the population had dropped $4,700$. Assume this trend continues.**

9) Predict the population in 2016.

**Answer**

$2,300$

10) Identify the year in which the population will reach $0$.

**For the exercises 11-12, consider this scenario: A town's population has been increased at a constant rate. In 2010 the population was $46,020$. By 2012 the population had increased to $52,070$. Assume this trend continues.**

11) Predict the population in 2016.

**Answer**

$64,170$

12) Identify the year in which the population will reach $75,000$.

**For the exercises 13-18, consider this scenario: A town has an initial population of $75,000$. It grows at a constant rate of $2,500$ per year for $5$ years.**

13) Find the linear function that models the town's population $P$ as a function of the year, $t$, where $t$ is the number of years since the model began.

**Answer**

$P(t) = 75,000 + 2500t$

14) Find a reasonable domain and range for the function $P$.

15) If the function $P$ is graphed, find and interpret the $x$- and $y$-intercepts.

**Answer**

$(-30, 0)$ Thirty years before the start of this model, the town had no. $(0, 75,000)$ Initially, the town had a population of $75,000$.

16) If the function $P$ is graphed, find and interpret the slope of the function.

17) When will the output reached $100,000$?

**Answer**

Ten years after the model began.

18) What is the output in the year $12$ years from the onset of the model?

**For the exercises 19-24, consider this scenario: The weight of a newborn is $7.5$ pounds. The baby gained one-half pound a month for its first year.**

19) Find the linear function that models the baby's weight $W$ as a function of the age of the baby, in months, $t$.

**Answer**

$W(t) = 7.5t + 0.5$

20) Find a reasonable domain and range for the function $W$.

21) If the function $W$ is graphed, find and interpret the $x$- and $y$-intercepts.

**Answer**

$(-15, 0)$: The $x$-intercept is not a plausible set of data for this model because it means the baby weighed $0$ pounds $15$ months prior to birth. $(0, 7.5)$: The baby weighed $7.5$ pounds at birth.

22) If the function $W$ is graphed, find and interpret the slope of the function.

23) When did the baby weight $10.4$ pounds?

**Answer**

At age $5.8$ months.

24) What is the output when the input is $6.2$? Interpret your answer.

**For the exercises 25-30, consider this scenario: The number of people afflicted with the common cold in the winter months steadily decreased by $205$ each year from 2005 until 2010. In 2005, $12,025$ people were afflicted.**

25) Find the linear function that models the number of people inflicted with the common cold $C$ as a function of the year, $t$.

**Answer**

$C(t) = 12,025 - 205t$

26) Find a reasonable domain and range for the function $C$.

27) If the function $C$ is graphed, find and interpret the $x$- and $y$-intercepts.

**Answer**

$(58.7, 0)$: In roughly $59$ years, the number of people inflicted with the common cold would be $0$. $(0, 12,025)$: Initially there were $12,025$ people afflicted by the common cold.

28) If the function $C$ is graphed, find and interpret the slope of the function.

29) When will the output reach $0$?

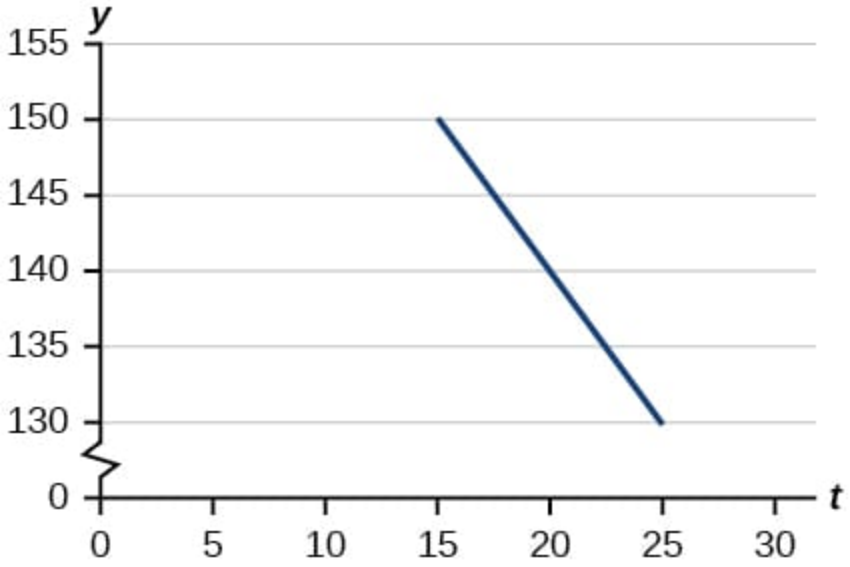
**Answer**

$2064$

30) In what year will the number of people be $9,700$?

# Graphical

**For the exercises 31-34, use the graph in the Figure below, which shows the profit, $y$, in thousands of dollars, of a company in a given year, $t$, where $t$ represents the number of years since 1980.**




31) Find the linear function $y$, where $y$ depends on $t$, the number of years since 1980.

**Answer**

$$y = -2t + 180$$

32) Find and interpret the $y$-intercept.

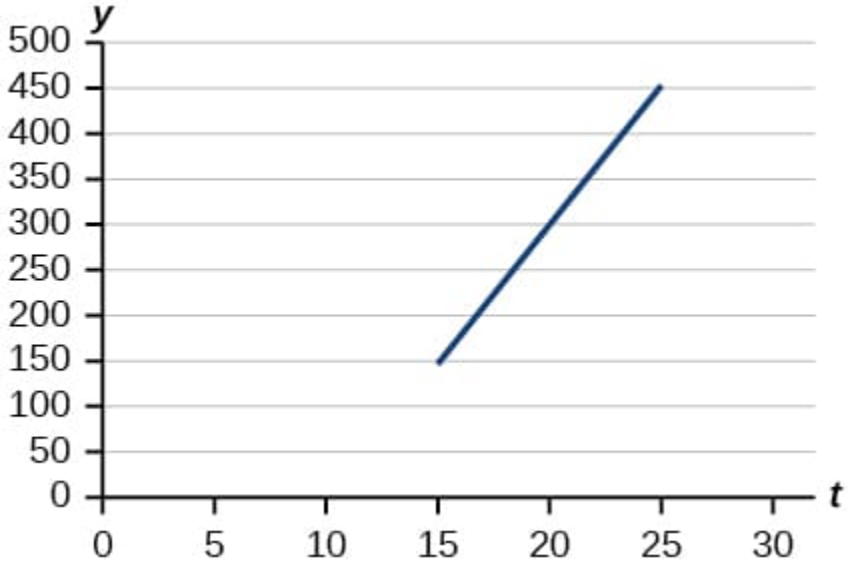
33) Find and interpret the $x$-intercept.

**Answer**

In 2070, the company's profit will be zero.

34) Find and interpret the slope.

**For the exercises 35-38, use the graph in the Figure below, which shows the profit, $y$, in thousands of dollars, of a company in a given year, $t$, where $t$ represents the number of years since 1980.**



35) Find the linear function $y$, where $y$ depends on $t$, the number of years since 1980.

**Answer**

$y = 30t - 300$

36) Find and interpret the $y$-intercept.

37) Find and interpret the $x$-intercept.

**Answer**

$(10, 0)$ In 1990, the profit earned zero profit.

38) Find and interpret the slope.

# Numeric

**For the exercises 39-41, use the median home values in Mississippi and Hawaii (adjusted for inflation) shown in the Table below. Assume that the house values are changing linearly.**

| Year | Mississippi | Hawaii |
|---|---|---|
| 1950 | $25,200 | $74,400 |
| 2000 | $71,400 | $272,700 |

39) In which state have home values increased at a higher rate?

**Answer**

Hawaii

40) If these trends were to continue, what would be the median home value in Mississippi in 2010?

41) If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

**Answer**

During the year 1933

**For the exercises 42-44, use the median home values in Indiana and Alabama (adjusted for inflation) shown in the Table below. Assume that the house values are changing linearly.**

| Year | Indiana | Alabama |
|---|---|---|
| 1950 | $37,700 | $27,100 |
| 2000 | $94,300 | $85,100 |

42) In which state have home values increased at a higher rate?

43) If these trends were to continue, what would be the median home value in Indiana in 2010?

**Answer**

$\$105,620$

44) If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

### Real-World Applications

45) In 2004, a school population was $1001$. By 2008 the population had grown to $1697$. Assume the population is changing linearly.

a. How much did the population grow between the year 2004 and 2008?
b. How long did it take the population to grow from $1001$ students to $1697$ students?
c. What is the average population growth per year?
d. What was the population in the year 2000?
e. Find an equation for the population, $P$, of the school $t$ years after 2000.
f. Using your equation, predict the population of the school in 2011.

**Answer**

a. $696$ people
b. $4$ years
c. $174$ people per year
d. $305$ people
e. $P(t) = 305 + 174t$
f. $2219$ people

46) In 2003, a town's population was $1431$. By 2007 the population had grown to $2134$. Assume the population is changing linearly.

a. How much did the population grow between the year 2003 and 2007?
b. How long did it take the population to grow from $1431$ people to $2134$ people?
c. What is the average population growth per year?
d. What was the population in the year 2000?
e. Find an equation for the population, $P$ of the town $t$ years after 2000.
f. Using your equation, predict the population of the town in 2014.

47) A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used on the phone. If a customer uses $410$ minutes, the monthly cost will be $\$71.50$. If the customer uses $720$ minutes, the monthly cost will be $\$118$.

a. Interpret the slope and $y$-intercept of the equation.

b. Use your equation to find the total monthly cost if $687$ minutes are used.

**Answer**

a. $C(x) = 0.15x + 10$
b. The flat monthly fee is $\$10$ and there is an additional $\$0.15$ fee for each additional minute used
c. $\$113.05$

48) A phone company has a monthly cellular data plan where a customer pays a flat monthly fee of $\$10$ and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses $20$ MB, the monthly cost will be $\$11.20$. If the customer uses $130$ MB, the monthly cost will be $\$17.80$.

a. Interpret the slope and $y$-intercept of the equation.

b. Use your equation to find the total monthly cost if $250$ MB are used.

49) In 1991, the moose population in a park was measured to be $4,360$. By 1999, the population was measured again to be $5,880$. Assume the population continues to change linearly.

a. What does your model predict the moose population to be in 2003?

**Answer**

a. $P(t) = 190t + 4360$
b. $6640$ moose

50) In 2003, the owl population in a park was measured to be $340$. By 2007, the population was measured again to be $285$. The population changes linearly. Let the input be years since 1990.

a. What does your model predict the owl population to be in 2012?

51) The Federal Helium Reserve held about $16$ billion cubic feet of helium in 2010 and is being depleted by about $2.1$ billion cubic feet each year.

a. In 2015, what will the helium reserves be?

b. If the rate of depletion doesn't change, in what year will the Federal Helium Reserve be depleted?

**Answer**

a. $R(t) = 16 - 2.1t$
b. $5.5$ billion cubic feet
c. During the year 2017

52) Suppose the world's oil reserves in 2014 are $1,820$ billion barrels. If, on average, the total reserves are decreasing by $25$ billion barrels of oil each year:

a. Seven years from now, what will the oil reserves be?

b. If the rate at which the reserves are decreasing is constant, when will the world's oil reserves be depleted?

53) You are choosing between two different prepaid cell phone plans. The first plan charges a rate of $26$ cents per minute. The second plan charges a monthly fee of $\$19.95$ plus $11$ cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

**Answer**

More than $133$ minutes

54) You are choosing between two different window washing companies. The first charges $\$5$ per window. The second charges a base fee of $\$40$ plus $\$3$ per window. How many windows would you need to have for the second company to be preferable?

55) When hired at a new job selling jewelry, you are given two pay options:

Option A: Base salary of $\$17,000$ a year with a commission of $12\%$ of your sales
Option B: Base salary of $\$20,000$ a year with a commission of $5\%$ of your sales

How much jewelry would you need to sell for option A to produce a larger income?

**Answer**

More than $\$42,857.14$ worth of jewelry

56) When hired at a new job selling electronics, you are given two pay options:

Option A: Base salary of $\$14,000$ a year with a commission of $10\%$ of your sales

Option B: Base salary of $\$19,000$ a year with a commission of $4\%$ of your sales

How much electronics would you need to sell for option A to produce a larger income?

57) When hired at a new job selling electronics, you are given two pay options:

Option A: Base salary of $\$20,000$ a year with a commission of $12\%$ of your sales

Option B: Base salary of $\$26,000$ a year with a commission of $3\%$ of your sales

How much electronics would you need to sell for option A to produce a larger income?

**Answer**

$\$66,666.67$

58) When hired at a new job selling electronics, you are given two pay options:

Option A: Base salary of $\$10,000$ a year with a commission of $9\%$ of your sales

Option B: Base salary of $\$20,000$ a year with a commission of $4\%$ of your sales

How much electronics would you need to sell for option A to produce a larger income?

# 2.4 Fitting Linear Models to Data

## Verbal

1) Describe what it means if there is a model breakdown when using a linear model.

**Answer**

When our model no longer applies, after some value in the domain, the model itself doesn't hold.

2) What is interpolation when using a linear model?

3) What is extrapolation when using a linear model?

**Answer**

We predict a value outside the domain and range of the data.

4) Explain the difference between a positive and a negative correlation coefficient.

5) Explain how to interpret the absolute value of a correlation coefficient.

**Answer**

The closer the number is to $1$, the less scattered the data, the closer the number is to $0$, the more scattered the data.

## Algebraic

6) A regression was run to determine whether there is a relationship between hours of TV watched per day (x) and number of sit-ups a person can do (y). The results of the regression are given below. Use this to predict the number of sit-ups a person who watches $11$ hours of TV can do.

$$\begin{aligned} y &= ax + b \\ a &= -1.341 \\ b &= 32.234 \\ r &= -0.896 \end{aligned}$$

7) A regression was run to determine whether there is a relationship between the diameter of a tree ($x$,in inches) and the tree's age ($y$,in years). The results of the regression are given below. Use this to predict the age of a tree with diameter $10$ inches.

$$\begin{aligned} y &= ax + b \\ a &= 6.301 \\ b &= -1.044 \\ r &= -0.970 \end{aligned}$$

**Answer**

$61.966$ years

**For the exercises 8-11, draw a scatter plot for the data provided. Does the data appear to be linearly related?**

8)

| 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| –22 | –19 | –15 | –11 | –6 | –2 |

9)

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 46 | 50 | 59 | 75 | 100 | 136 |

**Answer**

No

10)

| 100 | 250 | 300 | 450 | 600 | 750 |
|---|---|---|---|---|---|
| 12 | 12.6 | 13.1 | 14 | 14.5 | 15.2 |

11)

| 1 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|
| 1 | 9 | 28 | 65 | 125 | 216 |

**Answer**

No

12) For the following data, draw a scatter plot. If we wanted to know when the population would reach $15,000$, would the answer involve interpolation or extrapolation? Eyeball the line, and estimate the answer.

| Year | Population |
|---|---|
| 1990 | 11,500 |
| 1995 | 12,100 |
| 2000 | 12,700 |
| 2005 | 13,000 |
| 2010 | 13,750 |

13) For the following data, draw a scatter plot. If we wanted to know when the temperature would reach $28^{\circ}F$, would the answer involve interpolation or extrapolation? Eyeball the line and estimate the answer.

| Temperature, °F | 16 | 18 | 20 | 25 | 30 |
|---|---|---|---|---|---|
| Time, seconds | 46 | 50 | 54 | 55 | 62 |

**Answer**

Interpolation. About $60\ ^{\circ}F$.

## Graphical

**For the exercises 14-17, match each scatterplot with one of the four specified correlations in Figure ab and Figure cd.**

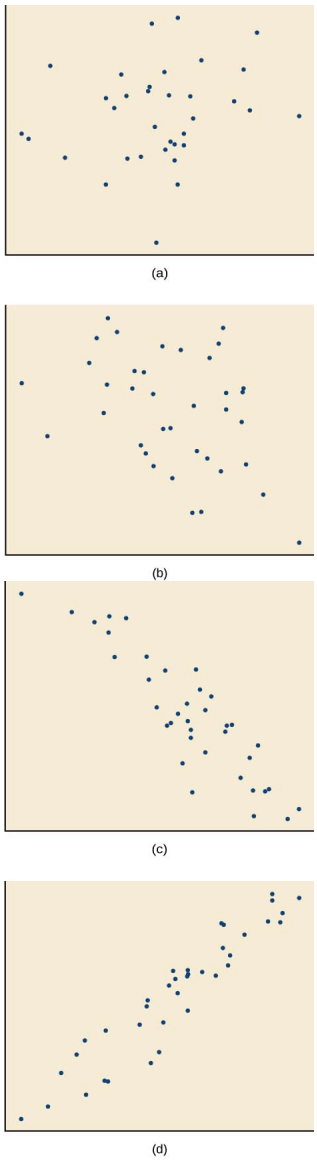


Figure ab Figure cd

14) $r = 0.95$

15) $r = -0.89$

**Answer**

C

16) $r = 0.26$

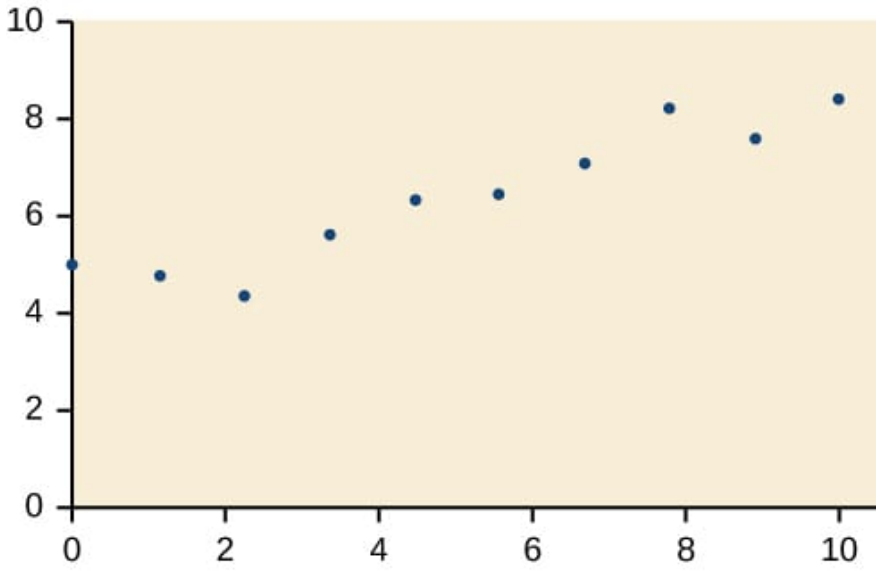
17) $r = -0.39$

**Answer**

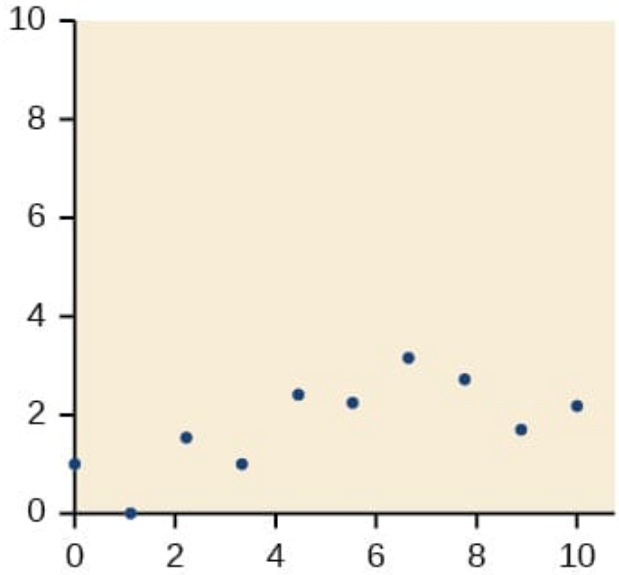
B

**For the exercises 18-21, draw a best-fit line for the plotted data.**

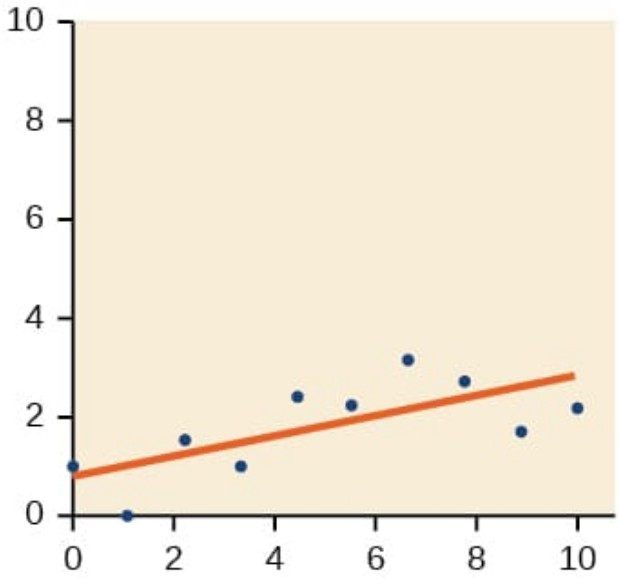
18)



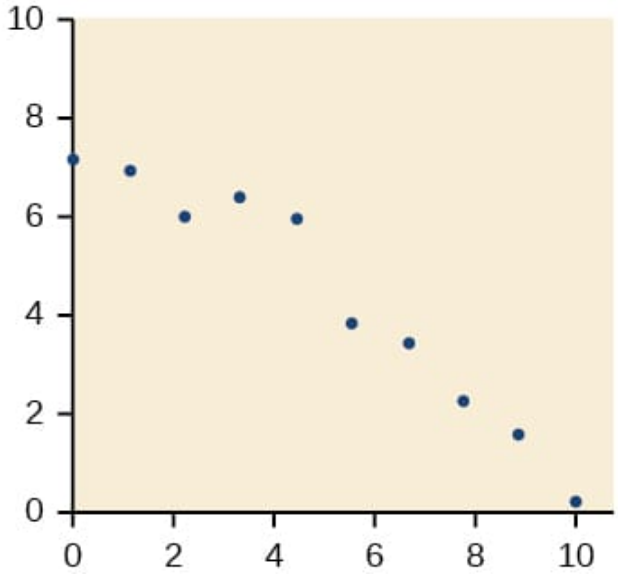

19)



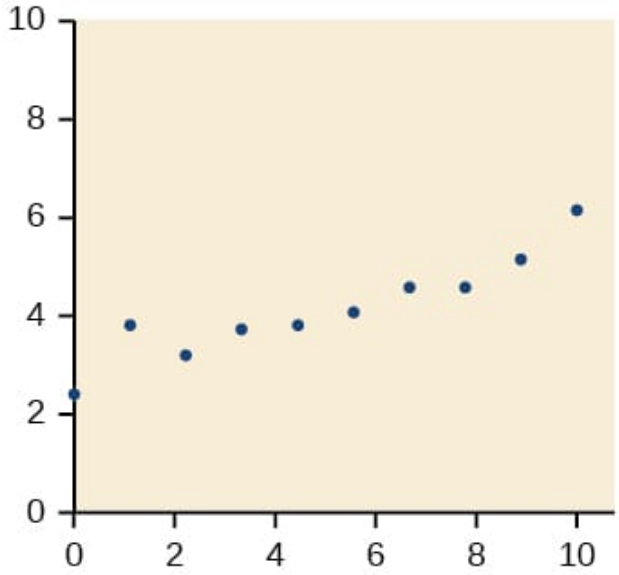
**Answer**



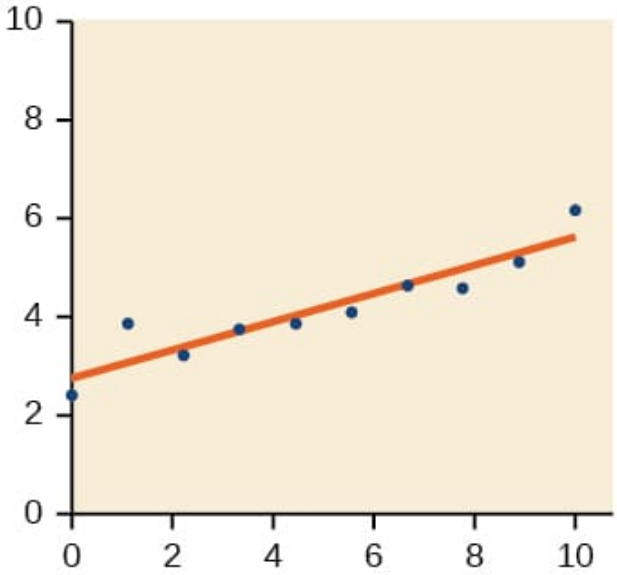

20)




21)




**Answer**




## Numeric

22) The U.S. Census tracks the percentage of persons $25$ years or older who are college graduates. That data for several years is given in the Table below. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the percentage exceed $35\%$?

| Year | Percent Graduates |
|---|---|
| 1990 | 21.3 |
| 1992 | 21.4 |
| 1994 | 22.2 |
| 1996 | 23.6 |
| 1998 | 24.4 |

| Year | Percent Graduates |
|---|---|
| 2000 | 25.6 |
| 2002 | 26.7 |
| 2004 | 27.7 |
| 2006 | 28 |
| 2008 | 29.4 |

23) The U.S. import of wine (in hectoliters) for several years is given in the Table. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will imports exceed $12,000$ hectoliters?

| Year | Imports |
|---|---|
| 1992 | 2665 |
| 1994 | 2688 |
| 1996 | 3565 |
| 1998 | 4129 |
| 2000 | 4584 |
| 2002 | 5655 |
| 2004 | 6549 |
| 2006 | 7950 |
| 2008 | 8487 |
| 2009 | 9462 |

**Answer**

Yes, trend appears linear because $r = 0.985$ and will exceed $12,000$ near midyear, 2016, $24.6$ years since 1992.

24) The Table shows the year and the number of people unemployed in a particular city for several years. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the number of unemployed reach $5$?

| Year | Number Unemployed |
|---|---|
| 1990 | 750 |
| 1992 | 670 |
| 1994 | 650 |
| 1996 | 605 |
| 1998 | 550 |
| 2000 | 510 |
| 2002 | 460 |
| 2004 | 420 |
| 2006 | 380 |
| 2008 | 320 |

## Technology

**For the exercises 25-31, use each set of data to calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to $3$ decimal places of accuracy.**

25)

| $x$ | 8 | 15 | 26 | 31 | 56 |
|---|---|---|---|---|---|
| $y$ | 23 | 41 | 53 | 72 | 103 |

**Answer**

$y = 1.640x + 13.800, r = 0.987$

26)

| $x$ | 5 | 7 | 10 | 12 | 15 |
|---|---|---|---|---|---|
| $y$ | 4 | 12 | 17 | 22 | 24 |

27)

| $x$ | $y$ | $x$ | $y$ |
|---|---|---|---|
| 3 | 21.9 | 11 | 15.76 |
| 4 | 22.22 | 12 | 13.68 |
| 5 | 22.74 | 13 | 14.1 |
| 6 | 22.26 | 14 | 14.02 |
| 7 | 20.78 | 15 | 11.94 |
| 8 | 17.6 | 16 | 12.76 |
| 9 | 16.52 | 17 | 11.28 |
| 10 | 18.54 | 18 | 9.1 |

**Answer**

$y = -0.962x + 26.86, r = -0.965$

28)

| $x$ | $y$ |
|---|---|
| 4 | 44.8 |
| 5 | 43.1 |
| 6 | 38.8 |
| 7 | 39 |
| 8 | 38 |
| 9 | 32.7 |
| 10 | 30.1 |
| 11 | 29.3 |
| 12 | 27 |

| $x$ | $y$ |
|---|---|
| 13 | 25.8 |

29)

| $x$ | $y$ |
|---|---|
| 21 | 17 |
| 25 | 11 |
| 30 | 2 |
| 31 | –1 |
| 40 | –18 |
| 50 | –40 |

**Answer**

$y = -1.981x + 60.197; r = -0.998$

30)

| $x$ | $y$ |
|---|---|
| 100 | 2000 |
| 80 | 1798 |
| 60 | 1589 |
| 55 | 1580 |
| 40 | 1390 |
| 20 | 1202 |

31)

| $x$ | $y$ |
|---|---|
| 900 | 70 |
| 988 | 80 |
| 1000 | 82 |
| 1010 | 84 |
| 1200 | 105 |
| 1205 | 108 |

**Answer**

$y = 0.121x - 38.841, r = 0.998$

## Extensions

32) Graph $f(x) = 0.5x + 10$. Pick a set of $5$ ordered pairs using inputs $x = -2, 1, 5, 6, 9$ and use linear regression to verify that the function is a good fit for the data.

33) Graph $f(x) = -2x - 10$. Pick a set of $5$ ordered pairs using inputs $x = -2, 1, 5, 6, 9$ and use linear regression to verify the function.

**Answer**

$(-2, -6), (1, -12), (5, -20), (6, -22), (9, -28); y = -2x - 10$

**For the exercises 34-36, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs shows dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span, (number of units sold, profit) for specific recorded years:**

$$(46, 1, 600), (48, 1, 550), (50, 1, 505), (52, 1, 540), (54, 1, 495)$$

34) Use linear regression to determine a function $P$ where the profit in thousands of dollars depends on the number of units sold in hundreds.

35) Find to the nearest tenth and interpret the $x$-intercept.

**Answer**

$(189.8, 0)$ If $18,980$ units are sold, the company will have a profit of zero dollars.

36) Find to the nearest tenth and interpret the $y$-intercept.

## Real-World Applications

**For the exercises 37-38, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs shows the population and the year over the ten-year span, (population, year) for specific recorded years:**

$$(2500, 2000), (2650, 2001), (3000, 2003), (3500, 2006), (4200, 2010)$$

37) Use linear regression to determine a function $y$, where the year depends on the population. Round to three decimal places of accuracy.

**Answer**

$y = 0.00587x + 1985.41$

38) Predict when the population will hit $8,000$.

**For the exercises 39-40, consider this scenario: The profit of a company increased steadily over a ten-year span. The following ordered pairs show the number of units sold in hundreds and the profit in thousands of over the ten year span, (number of units sold, profit) for specific recorded years:**

$$(46, 250), (48, 305), (50, 350), (52, 390), (54, 410)$$

39) Use linear regression to determine a function $y$, where the profit in thousands of dollars depends on the number of units sold in hundreds.

**Answer**

$y = 20.25x - 671.5$

40) Predict when the profit will exceed one million dollars.

**For the exercises 41-42, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs show dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span (number of units sold, profit) for specific recorded years:**

$$(46, 250), (48, 225), (50, 205), (52, 180), (54, 165)$$

41) Use linear regression to determine a function $y$, where the profit in thousands of dollars depends on the number of units sold in hundreds.

**Answer**

$$y = -10.75x + 742.50$$

42) Predict when the profit will dip below the $\$25,000$ threshold.

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# 2.R: Linear Functions(Review)

## 2.1 Linear Functions

1) Determine whether the algebraic equation is linear: $2x + 3y = 7$

**Answer**

Yes

2) Determine whether the algebraic equation is linear: $6x^2 - y = 5$

3) Determine whether the function is increasing or decreasing: $f(x) = 7x - 2$

**Answer**

Increasing

4) Determine whether the function is increasing or decreasing: $g(x) = -x + 2$

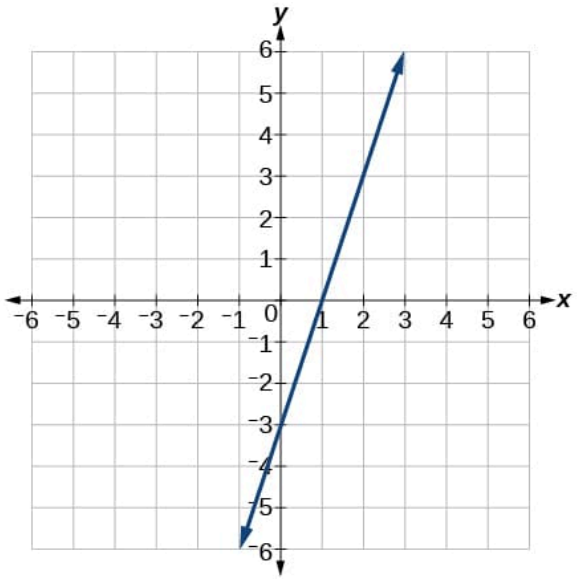
5) Given each set of information, find a linear equation that satisfies the given conditions, if possible. Passes through $(7, 5)$ and $(3, 17)$

**Answer**

$y = -3x + 26$

6) Given each set of information, find a linear equation that satisfies the given conditions, if possible. $x$ -intercept at $(6, 0)$ and $y$ -intercept at $(0, 10)$

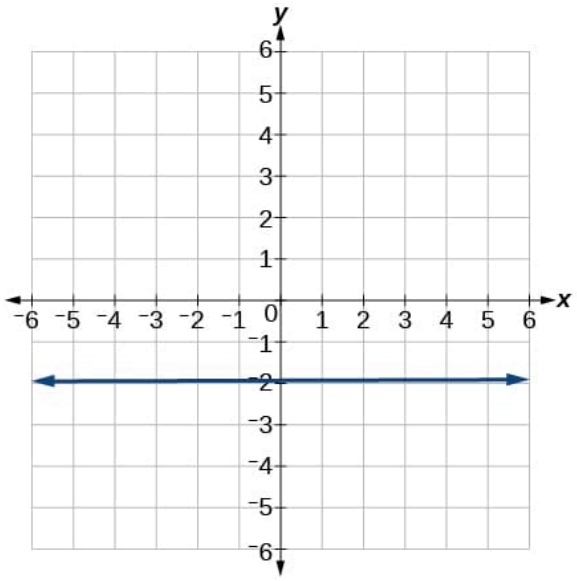
7) Find the slope of the line shown in the line graph.



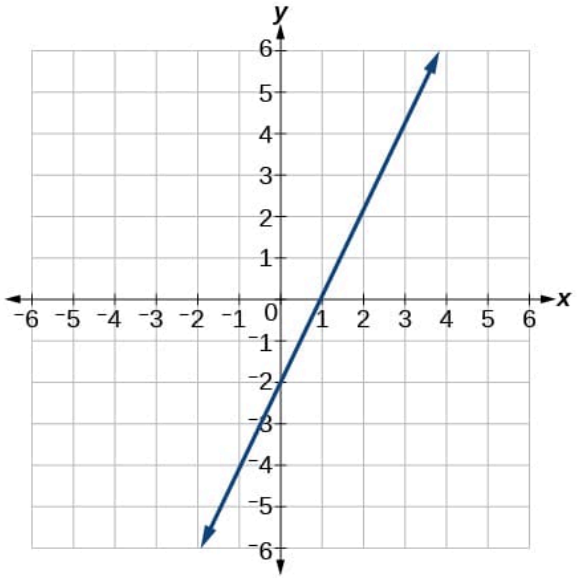

**Answer**

3

8) Find the slope of the line graphed.




9) Write an equation in slope-intercept form for the line shown.




**Answer**

$y = 2x - 2$

10) Does the following table represent a linear function? If so, find the linear equation that models the data.

| $x$ | -4 | 0 | 2 | 10 |
|---|---|---|---|---|
| $g(x)$ | 18 | -2 | -12 | -52 |

11) Does the following table represent a linear function? If so, find the linear equation that models the data.

| $x$ | 6 | 8 | 12 | 26 |
|---|---|---|---|---|
| $g(x)$ | -8 | -12 | -18 | -46 |

**Answer**

Not linear

12) On June 1st, a company has $\$4,000,000$ profit. If the company then loses $150,000$ dollars per day thereafter in the month of June, what is the company's profit n$^{th}$ day after June 1st?

## 2.2 Graphs of Linear Functions

**For the exercises 1-2, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:**

1) $\begin{array}{l} 2x-6y=12 \\ -x+3y=1 \end{array}$

**Answer**

parallel

2) $\begin{array}{l} y=\frac{1}{3}x-2 \\ 3x+y=-9 \end{array}$

**For the exercises 3-4, find the $x$- and $y$- intercepts of the given equation**

3) $7x+9y=-63$

**Answer**

$(-9,0);(0,-7)$

4) $f(x)=2x-1$

**For the exercises 5-11, use the descriptions of the pairs of lines to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?**

5)

Line 1: Passes through $(5,11)$ and $(10,1)$
Line 2 : Passes through $(-1,3)$ and $(-5,11)$

**Answer**

Line 1 : $m=-2$; Line 2 : $m=-2$; Parallel

6)

Line 1: Passes through $(8,-10)$ and $(0,-26)$
Line 2: Passes through $(2,5)$ and $(4,4)$

7) Write an equation for a line perpendicular to $f(x)=5x-1$ and passing through the point $(5,20)$.

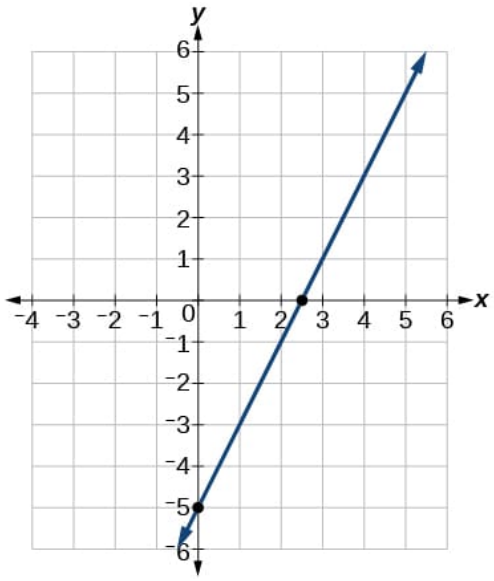
**Answer**

$y=-0.2x+21$

8) Find the equation of a line with a $y$ -intercept of $(0,2)$ and slope $-\frac{1}{2}$.

9) Sketch a graph of the linear function $f(t)=2t-5$.

**Answer**




10) Find the point of intersection for the $2$ linear functions: $\begin{aligned} x &= y + 6 \\ 2x - y &= 13 \end{aligned}$

11) A car rental company offers two plans for renting a car.

Plan A: $25$ dollars per day and $10$ cents per mile

Plan B: $50$ dollars per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

**Answer**

$250$

## 2.3 Modeling with Linear Functions

1) Find the area of a triangle bounded by the $y$ axis, the line $f(x) = 10 - 2x$, and the line perpendicular to $f$ that passes through the origin.

2) A town's population increases at a constant rate. In 2010 the population was $55,000$. By 2012 the population had increased to $76,000$. If this trend continues, predict the population in 2016.

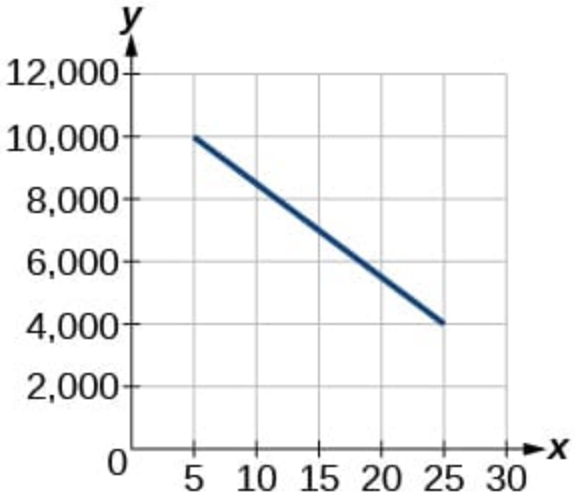
**Answer**

$118,000$

3) The number of people afflicted with the common cold in the winter months dropped steadily by $50$ each year since 2004 until 2010. In 2004, $875$ people were inflicted.

4) Find the linear function that models the number of people afflicted with the common cold $C$ as a function of the year, $t$. When will no one be afflicted?

**For the exercises 5-6, use the graph in the Figure below showing the profit, $y$, in thousands of dollars, of a company in a given year, $x$, where $x$ represents years since 1980.**




5) Find the linear function $y$, where $y$ depends on $x$, the number of years since 1980.

**Answer**

$$y = -300x + 11,500$$

6) Find and interpret the $y$-intercept.

7) For the following exercise, consider this scenario: In 2004, a school population was $1,700$. By 2012 the population had grown to $2,500$. Assume the population is changing linearly.

a. How much did the population grow between the year 2004 and 2012?
b. What is the average population growth per year?
c. Find an equation for the population, $P$, of the school $t$ years after 2004.

**Answer**

a. $800$
b. $100$ students per year
c. $P(t) = 100t + 1700$

**For the exercises 8-9, consider this scenario: In 2000, the moose population in a park was measured to be $6,500$. By 2010, the population was measured to be $12,500$. Assume the population continues to change linearly.**

8) Find a formula for the moose population, $P$.

9) What does your model predict the moose population to be in 2020?

**Answer**

$$18,500$$

**For the exercises 10-, consider this scenario: The median home values in subdivisions Pima Central and East Valley (adjusted for inflation) are shown in the Table below. Assume that the house values are changing linearly.**

| Year | Pima Central | East Valley |
|---|---|---|
| 1970 | 32,000 | 120,250 |
| 2010 | 85,000 | 150,000 |

10) In which subdivision have home values increased at a higher rate?

11) If these trends were to continue, what would be the median home value in Pima Central in 2015?

**Answer**

$\$91,625$

## 2.4 Fitting Linear Models to Data

1) Draw a scatter plot for the data in the Table below. Then determine whether the data appears to be linearly related.

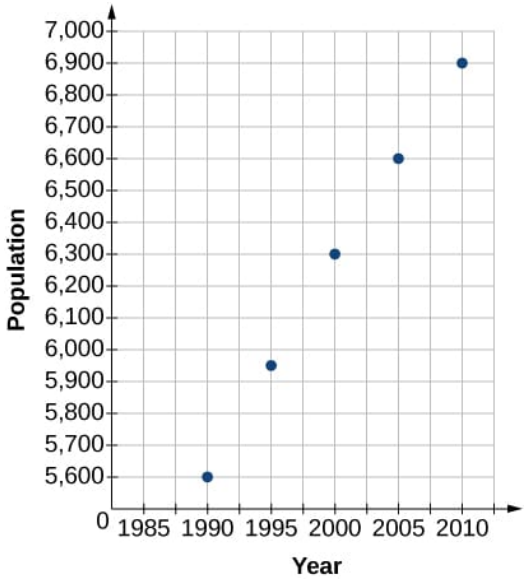
| 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| -105 | -50 | 1 | 55 | 105 | 160 |

2) Draw a scatter plot for the data in the Table. If we wanted to know when the population would reach $15,000$, would the answer involve interpolation or extrapolation?

| Year | Population |
|---|---|
| 1990 | 5,600 |
| 1995 | 5,950 |
| 2000 | 6,300 |
| 2005 | 6,600 |
| 2010 | 6,900 |

**Answer**

Extrapolation

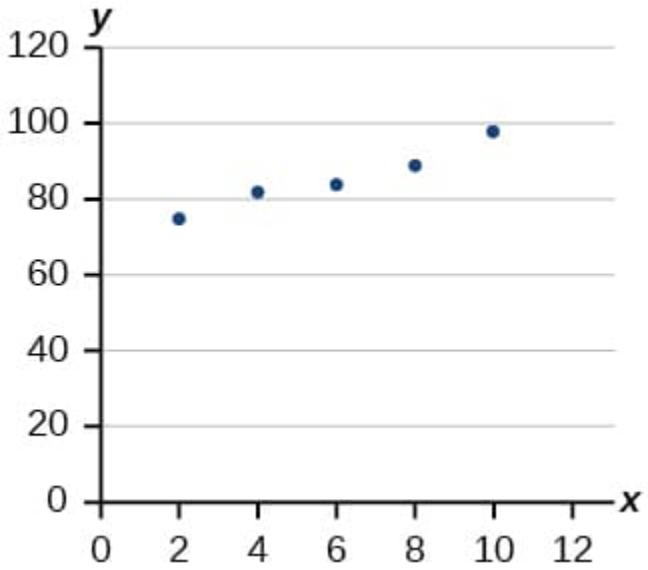



3) Eight students were asked to estimate their score on a $10$-point quiz. Their estimated and actual scores are given in the Table. Plot the points, then sketch a line that fits the data.

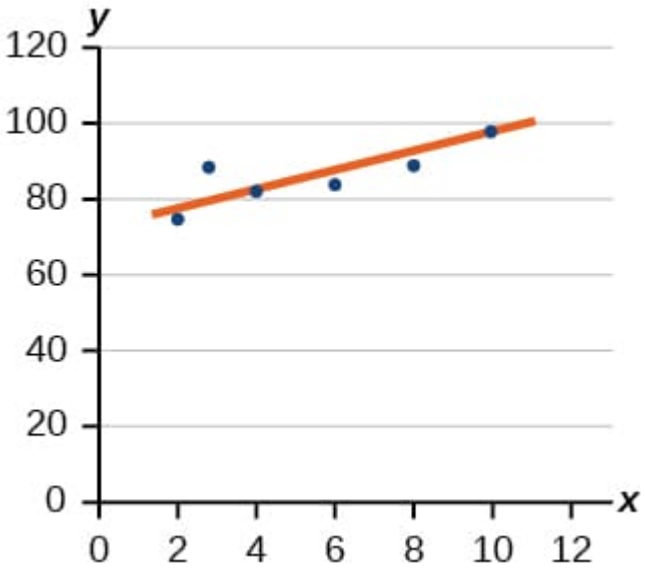
| Predicted | Actual |
|---|---|
| 6 | 6 |

| Predicted | Actual |
|---|---|
| 7 | 7 |
| 7 | 8 |
| 8 | 8 |
| 7 | 9 |
| 9 | 10 |
| 10 | 10 |
| 10 | 9 |

4) Draw a best-fit line for the plotted data.




**Answer**




**For the exercises 5-6, consider the data in the Table below, which shows the percent of unemployed in a city of people 25 years or older who are college graduates is given below, by year.**

| Year | Percent Graduates |
|---|---|
| 2000 | 6.5 |
| 2002 | 7.0 |
| 2005 | 7.4 |
| 2007 | 8.2 |
| 2010 | 9.0 |

5) Determine whether the trend appears to be linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

6) In what year will the percentage exceed $12\%$?

**Answer**

Midway through 2024

7) Based on the set of data given in the Table, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

| $x$ | 17 | 20 | 23 | 26 | 29 |
|---|---|---|---|---|---|
| $y$ | 15 | 25 | 31 | 37 | 40 |

8) Based on the set of data given in the Table, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

| $x$ | 10 | 12 | 15 | 18 | 20 |
|---|---|---|---|---|---|
| $y$ | 36 | 34 | 30 | 28 | 22 |

**Answer**

$y = -1.294x + 49.412; r = -0.974$

**For the exercises 9-12, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs show the population and the year over the ten-year span (population, year) for specific recorded years:** $(3,600, 2000); (4,000, 2001); (4,700, 2003); (6,000, 2006)$

9) Use linear regression to determine a function $y$, where the year depends on the population, to three decimal places of accuracy.

10) Predict when the population will hit $12,000$.

**Answer**

Early in 2022

11) What is the correlation coefficient for this model to three decimal places of accuracy?

12) According to the model, what is the population in 2014?

**Answer**

$7,660$

# Practice Test

1) Determine whether the following algebraic equation can be written as a linear function: $2x + 3y = 7$

**Answer**

Yes

2) Determine whether the following function is increasing or decreasing: $f(x) = -2x + 5$

3) Determine whether the following function is increasing or decreasing: $f(x) = 7x + 9$

**Answer**

Increasing

4) Given the following set of information, find a linear equation satisfying the conditions, if possible. Passes through $(5, 1)$ and $(3, -9)$

5) Given the following set of information, find a linear equation satisfying the conditions, if possible. $x$ intercept at $(-4, 0)$ and $y$ -intercept at $(0, -6)$

**Answer**

$y = -1.5x - 6$

6) Find the slope of the line in the Figure below.

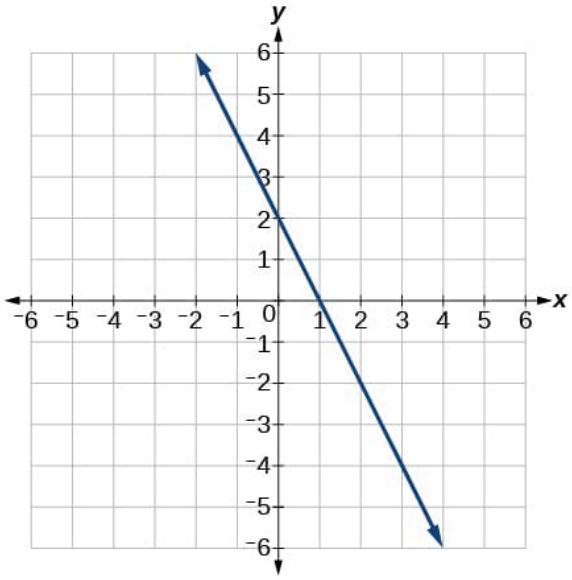
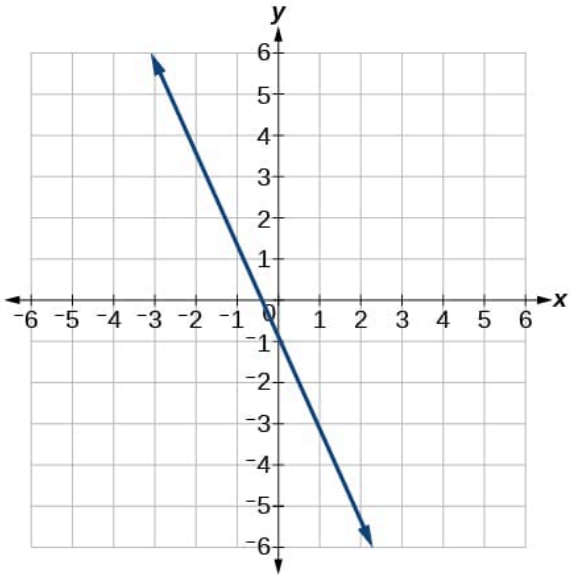



Figure below.




**Answer**

$y = -2x - 1$

8) Does the Table represent a linear function? If so, find a linear equation that models the data.

| $x$ | -6 | 0 | 2 | 4 |
|---|---|---|---|---|
| $g(x)$ | 14 | 32 | 38 | 44 |

9) Does the Table represent a linear function? If so, find a linear equation that models the data.

| $x$ | 1 | 3 | 7 | 11 |
|---|---|---|---|---|
| $g(x)$ | 4 | 9 | 19 | 12 |

**Answer**

No

10) At 6 AM, an online company has sold $120$ items that day. If the company sells an average of $30$ items per hour for the remainder of the day, write an expression to represent the number of items that were sold $n$ after 6 AM.

**For the exercises 11-22, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:**

11) $y = \frac{3}{4}x - 9$
$-4x - 3y = 8$

**Answer**

Perpendicular

12) $-2x + y = 3$
$3x + \frac{3}{2}y = 5$

13) Find the $x$ - and $y$ -intercepts of the equation $2x + 7y = -14$.

**Answer**

$(-7, 0); (0, -2)$

14) Given below are descriptions of two lines. Find the slopes of Line 1 and Line 2. Is the pair of lines parallel, perpendicular, or neither?

Line $1$ : Passes through $(-2, -6)$ and $(3, 14)$
Line $2$ : Passes through $(2, 6)$ and $(4, 14)$

15) Write an equation for a line perpendicular to $f(x) = 4x + 3$ and passing through the point $(8, 10)$.

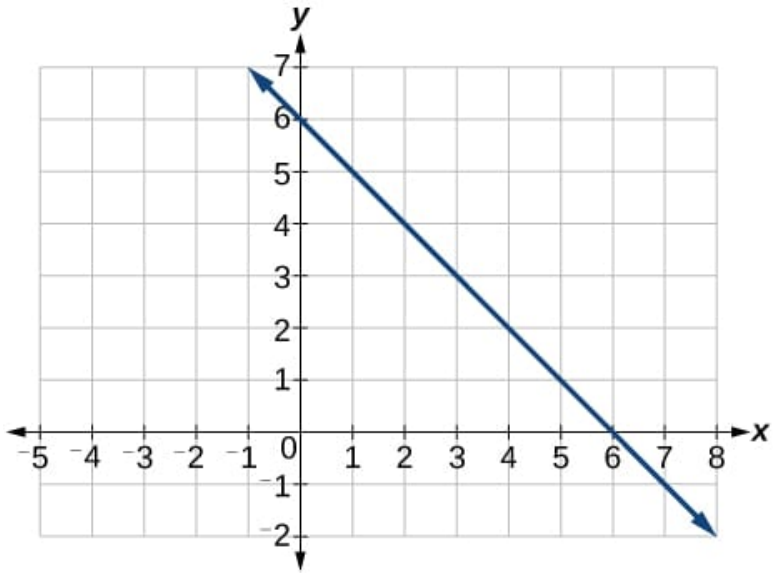
**Answer**

$y = -0.25x + 12$

16) Sketch a line with a $y$-intercept of $(0, 5)$ and slope $-\frac{5}{2}$.

17) Graph of the linear function $f(x) = -x + 6$.

**Answer**




18) For the two linear functions, find the point of intersection: $\begin{aligned} x &= y + 2 \\ 2x - 3y &= -1 \end{aligned}$

19) A car rental company offers two plans for renting a car.

Plan A: $\$25$ per day and $\$0.10$ per mile
Plan B: $\$40$ per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

**Answer**

$150$

20) Find the area of a triangle bounded by the $y$ axis, the line $f(x) = 12 - 4x$, and the line perpendicular to $f$ that passes through the origin.

21) A town's population increases at a constant rate. In 2010 the population was $65,000$. By 2012 the population had increased to

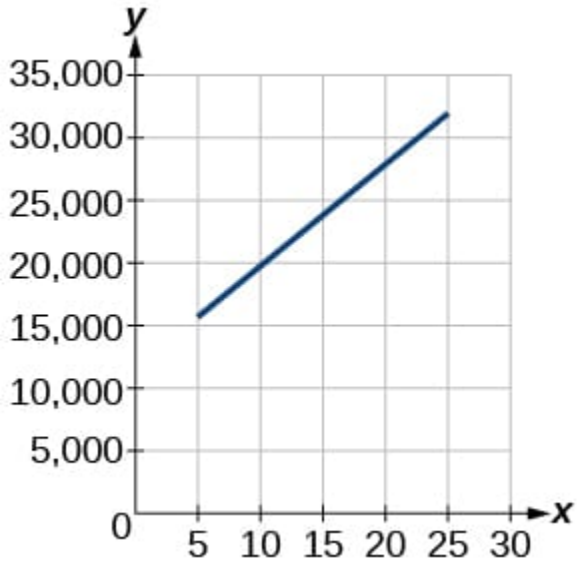
$90,000$. Assuming this trend continues, predict the population in 2018.

**Answer**

$165,000$

22) The number of people afflicted with the common cold in the winter months dropped steadily by $25$ each year since 2002 until 2012. In 2002, $8,040$ people were inflicted. Find the linear function that models the number of people afflicted with the common cold $C$ as a function of the year, $t$. When will less than $6,000$ people be afflicted?

**For the exercises 23-24, use the graph in Figure, showing the profit, $y$, in thousands of dollars, of a company in a given year, $x$, where $x$ represents years since 1980.**




23) Find the linear function $y$, where $y$ depends on $x$, the number of years since 1980.

**Answer**

$y = 875x + 10,675$

24) Find and interpret the $y$-intercept.

25) In 2004, a school population was $1250$. By 2012 the population had dropped to $875$. Assume the population is changing linearly.

a. How much did the population drop between the year 2004 and 2012?
b. What is the average population decline per year?
c. Find an equation for the population, $P$, of the school $t$ years after 2004.

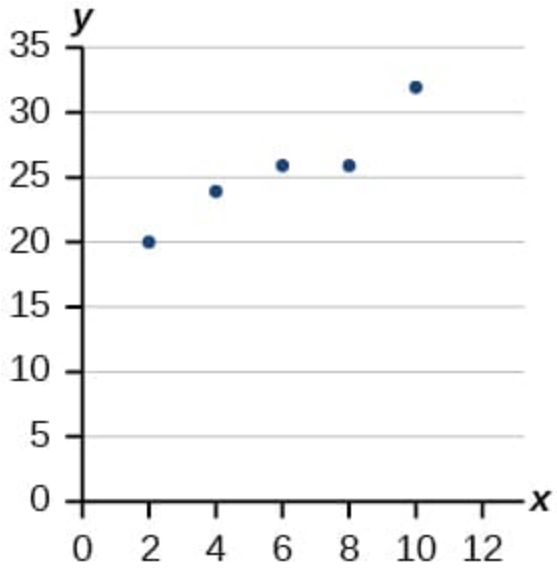
**Answer**

a. $375$
b. dropped an average of $46.875$, or about $47$ people per year
c. $y = -46.875t + 1250$

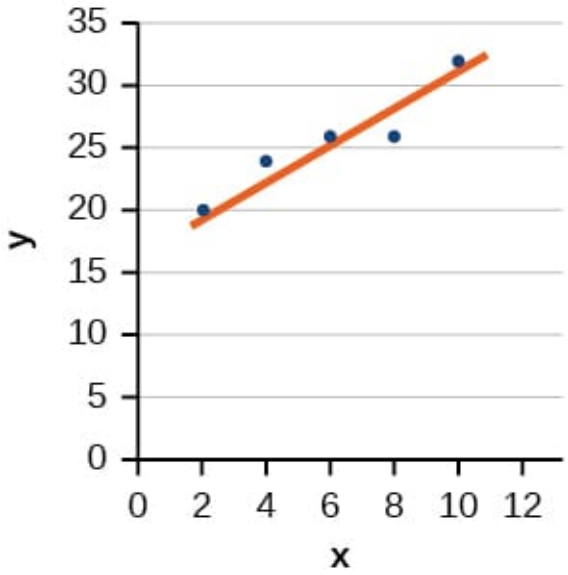
26) Draw a scatter plot for the data provided in the Table. Then determine whether the data appears to be linearly related.

| 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| -450 | -200 | 10 | 265 | 500 | 755 |

27) Draw a best-fit line for the plotted data.




**Answer**




**For the exercises 28-29, use the Table below, which shows the percent of unemployed persons $25$ years or older who are college graduates in a particular city, by year.**

| Year | Percent Graduates |
|---|---|
| 2000 | 8.5 |
| 2002 | 8.0 |
| 2005 | 7.2 |
| 2007 | 6.7 |
| 2010 | 6.4 |

28) Determine whether the trend appears linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

29) In what year will the percentage drop below $4\%$?

**Answer**

Early in 2018

30) Based on the set of data given in the Table below, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient. Round to three decimal places of accuracy.

| $x$ | 16 | 18 | 20 | 24 | 26 |
|---|---|---|---|---|---|
| $y$ | 106 | 110 | 115 | 120 | 125 |

**For the exercises 31-33, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs shows the population (in hundreds) and the year over the ten-year span, (population, year) for specific recorded years:** $(4, 500, 2000); (4, 700, 2001); (5, 200, 2003); (5, 800, 2006)$

31) Use linear regression to determine a function y, where the year depends on the population. Round to three decimal places of accuracy.

**Answer**

$y = 0.00455x + 1979.5$

32) Predict when the population will hit $20,000$.

33) What is the correlation coefficient for this model?

**Answer**

$r = 0.999$

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# 3: POLYNOMIAL AND RATIONAL FUNCTIONS

In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.


3.0: Prelude to Polynomial and Rational Functions

3.1: Complex Numbers

3.2: Quadratic Functions

3.3: Power Functions and Polynomial Functions

3.4: Graphs of Polynomial Functions

3.5: Dividing Polynomials

3.6: Zeros of Polynomial Functions

3.7: Rational Functions

3.8: Inverses and Radical Functions

3.9: Modeling Using Variation

3.E: Polynomial and Rational Functions(Exercises)

3.R: Polynomial and Rational Functions(Review)


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# 3.0: Prelude to Polynomial and Rational Functions

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.




***Figure*** 3.0.1***:*** *35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit "film": modification of work by Horia Varlan; credit "memory cards": modification of work by Paul Hudson)*

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# 3.1: Complex Numbers

**Learning Objectives**

- Express square roots of negative numbers as multiples of $i$.
- Plot complex numbers on the complex plane.
- Add and subtract complex numbers.
- Multiply and divide complex numbers.

The study of mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. For example, we still have no solution to equations such as

$$x^2 + 4 = 0 \tag{3.1.1}$$

Our best guesses might be +2 or –2. But if we test +2 in this equation, it does not work. If we test –2, it does not work. If we want to have a solution for this equation, we will have to go farther than we have so far. After all, to this point we have described the square root of a negative number as undefined. Fortunately, there is another system of numbers that provides solutions to problems such as these. In this section, we will explore this number system and how to work within it.

## Expressing Square Roots of Negative Numbers as Multiples of *i*

We know how to find the square root of any positive real number. In a similar way, we can find the square root of a negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number. The imaginary number i is defined as the square root of negative 1.

$$\sqrt{-1} = i \tag{3.1.2}$$

So, using properties of radicals,

$$i^2 = (\sqrt{-1})^2 = -1 \tag{3.1.3}$$

We can write the square root of any negative number as a multiple of i. Consider the square root of –25.

$$\sqrt{-25} = \sqrt{25 \cdot (-1)} \tag{3.1.4}$$
$$= \sqrt{25}\sqrt{-1} \tag{3.1.5}$$
$$= 5i \tag{3.1.6}$$

We use 5*i* and not –5*i* because the principal root of 25 is the positive root.

A **complex number** is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a + bi$ where $a$ is the real part and $bi$ is the imaginary part. For example, $5 + 2i$ is a complex number. So, too, is $3 + 4\sqrt{3}i$.

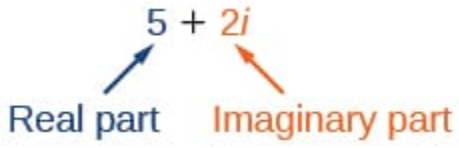



Figure 3.1.1

Imaginary numbers are distinguished from real numbers because a squared imaginary number produces a negative real number. Recall, when a positive real number is squared, the result is a positive real number and when a negative real number is squared,

again, the result is a positive real number. Complex numbers are a combination of real and imaginary numbers.

### Imaginary and Complex Numbers

A complex number is a number of the form $a + bi$ where

- $a$ is the real part of the complex number.
- $bi$ is the imaginary part of the complex number.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and $b$ is not equal to 0, the complex number is called an **imaginary number**. An imaginary number is an even root of a negative number.

### Standard Form

Given an imaginary number, express it in standard form.

1. Write $\sqrt{-a}$ as $\sqrt{a}\sqrt{-1}$.
2. Express $\sqrt{-1}$ as $i$.
3. Write $\sqrt{a} \cdot i$ in simplest form.

### Example 3.1.1: Expressing an Imaginary Number in Standard Form

Express $\sqrt{-9}$ in standard form.

**Solution**

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$$

In standard form, this is $0 + 3i$.

### Exercise 3.1.1

Express $\sqrt{-24}$ in standard form.

*Answer available at the end of this section.*

## Plotting a Complex Number on the Complex Plane

We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number we need to address the two components of the number. We use the complex plane, which is a coordinate system in which the horizontal axis represents the real component and the vertical axis represents the imaginary component. Complex numbers are the points on the plane, expressed as ordered pairs $(a, b)$, where $a$ represents the coordinate for the horizontal axis and $b$ represents the coordinate for the vertical axis.

Let's consider the number $-2 + 3i$. The real part of the complex number is–2 and the imaginary part is $3i$. We plot the ordered pair $(-2, 3)$ to represent the complex number $-2 + 3i$ as shown in Figure 3.1.2

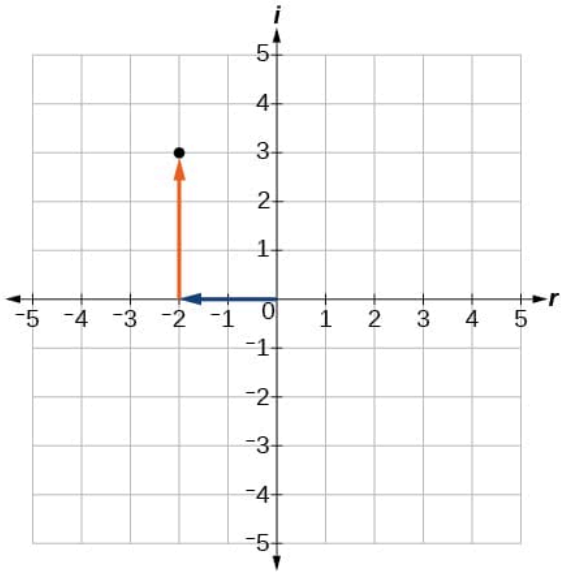



Figure 3.1.2: Plot of a complex number, $-2+3i$. Note that the real part $(-2)$ is plotted on the x-axis and the imaginary part $(3i)$ is plotted on the y-axis.

## Complex Plane

In the complex plane, the horizontal axis is the real axis, and the vertical axis is the imaginary axis as shown in Figure 3.1.3.

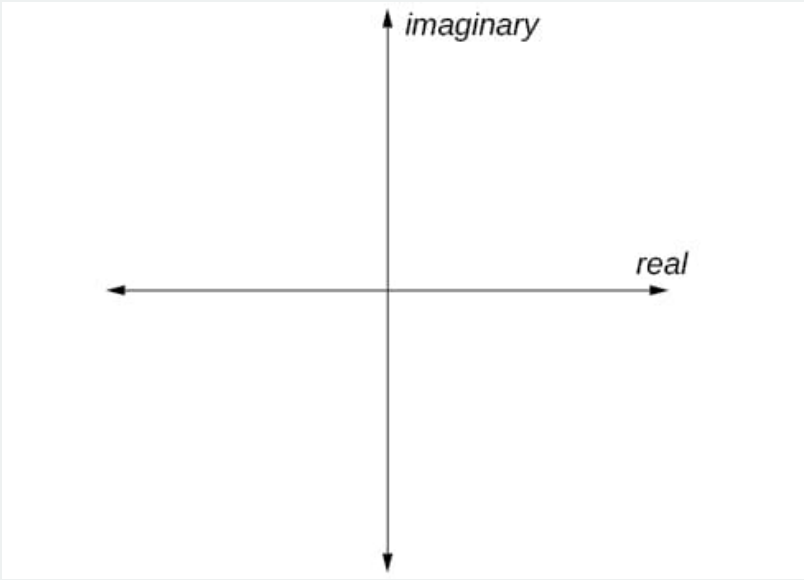



Figure 3.1.3: The complex plane showing that the horizontal axis (in the real plane, the x-axis) is known as the real axis and the vertical axis (in the real plane, the y-axis) is known as the imaginary axis.

## How To ...

Given a complex number, represent its components on the complex plane.

1. Determine the real part and the imaginary part of the complex number.
2. Move along the horizontal axis to show the real part of the number.
3. Move parallel to the vertical axis to show the imaginary part of the number.
4. Plot the point.

### Example 3.1.2: Plotting a Complex Number on the Complex Plane

Plot the complex number $3 - 4i$ on the complex plane.

**Solution**

The real part of the complex number is 3, and the imaginary part is $-4i$. We plot the ordered pair $(3, -4)$ as shown in Figure 3.1.4.

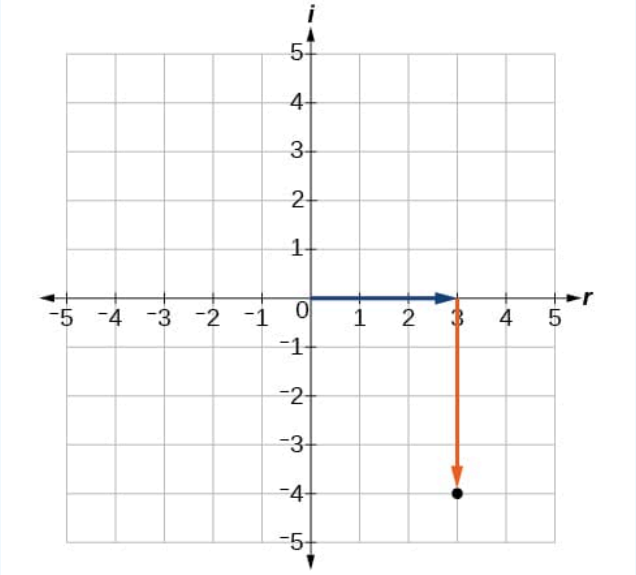



Figure 3.1.4: Plot of a complex number, $3 - 4i$. Note that the real part $(3)$ is plotted on the x-axis and the imaginary part $(-4i)$ is plotted on the y-axis.

### Exercise 3.1.1

Plot the complex number $-4 - i$ on the complex plane.

*Answer available at the end of this section.*

## Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and combine the imaginary parts.

### Complex Numbers: Addition and Subtraction

Adding complex numbers:

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{3.1.7}$$

Subtracting complex numbers:

$$(a + bi) - (c + di) = (a - c) + (b - d)i \tag{3.1.8}$$

**How To...**

Given two complex numbers, find the sum or difference.

1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

**Example 3.1.3: Adding Complex Numbers**

Add $3-4i$ and $2+5i$.

**Solution**

We add the real parts and add the imaginary parts.

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i \\ (3-4i)+(2+5i) &= (3+2)+(-4+5)i \\ &= 5+i\end{aligned}$$

**Exercise 3.1.3**

Subtract $2+5i$ from $3–4i$.

*Answer available at the end of this section.*

# Multiplying Complex Numbers

Multiplying complex numbers is much like multiplying binomials. The major difference is that we work with the real and imaginary parts separately.

## Multiplying a Complex Numbers by a Real Number

Let's begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. So, for example,

$$\begin{aligned}3(6+2i) &= (3\cdot 6)+(3\cdot 2i) && \text{Distribute.} \\ &= 18+6i && \text{Simplify.}\end{aligned}$$

Figure 3.1.6

**How To...**

Given a complex number and a real number, multiply to find the product.

1. Use the distributive property.
2. Simplify.

### Example 3.1.4: Multiplying a Complex Number by a Real Number

Find the product $4(2+5i)$.

**Solution**

Distribute the 4.

$$\begin{aligned} 4(2+5i) &= (4\cdot 2)+(4\cdot 5i) \\ &= 8+20i \end{aligned}$$

### Exercise 3.1.4

Find the product $-4(2+6i)$.

*Answer available at the end of this section.*

## Multiplying Complex Numbers Together

Now, let's multiply two complex numbers. We can use either the distributive property or the FOIL method. Recall that FOIL is an acronym for multiplying First, Outer, Inner, and Last terms together. Using either the distributive property or the FOIL method, we get

$$(a+bi)(c+di) = ac+adi+bci+bdi^2$$

Because $i^2=-1$, we have

$$(a+bi)(c+di) = ac+adi+bci-bd$$

To simplify, we combine the real parts, and we combine the imaginary parts.

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

### How To...

Given two complex numbers, multiply to find the product.

1. Use the distributive property or the FOIL method.
2. Simplify.

### Example 3.1.5: Multiplying a Complex Number by a Complex Number

Multiply $(4+3i)(2-5i)$.

**Solution**

Use $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$

$$\begin{aligned} (4+3i)(2-5i) &= (4\cdot 2-3\cdot(-5))+(4\cdot(-5)+3\cdot 2)i \\ &= (8+15)+(-20+6)i \\ &= 23-14i \end{aligned}$$

**Exercise 3.1.5**

Multiply $(3-4i)(2+3i)$.

*Answer available at the end of this section.*

## Dividing Complex Numbers

Division of two complex numbers is more complicated than addition, subtraction, and multiplication because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator. We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the **complex conjugate** of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of $a+bi$ is $a-bi$.

Note that complex conjugates have a reciprocal relationship: The complex conjugate of $a+bi$ is $a-bi$, and the complex conjugate of $a-bi$ is $a+bi$. Further, when a quadratic equation with real coefficients has complex solutions, the solutions are always complex conjugates of one another.

Suppose we want to divide $c+di$ by $a+bi$, where neither a nor $b$ equals zero. We first write the division as a fraction, then find the complex conjugate of the denominator, and multiply.

$$\frac{c+di}{a+bi} \text{ where } a\neq 0 \text{ and } b\neq 0$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(c+di)}{(a+bi)}\cdot\frac{(a-bi)}{(a-bi)}=\frac{(c+di)(a-bi)}{(a+bi)(a-bi)}$$

Apply the distributive property.

$$=\frac{ca-cbi+adi-bdi^2}{a^2-abi+abi-b^2i^2}$$

Simplify, remembering that $i^2=-1$.

$$=\frac{ca-cbi+adi-bd(-1)}{a^2-abi+abi-b^2(-1)}$$
$$=\frac{(ca+bd)+(ad-cb)i}{a^2+b^2}$$

**Definition: The Complex Conjugate**

The *complex conjugate* of a complex number $a+bi$ is $a-bi$. It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

**Example 3.1.6: Finding Complex Conjugates**

Find the complex conjugate of each number.

a. $2+i\sqrt{5}$
b. $-\frac{1}{2}i$

**Solution**

a. The number is already in the form $a+bi$. The complex conjugate is $a-bi$, or $2-i\sqrt{5}$.
b. We can rewrite this number in the form $a+bi$ as $0-\frac{1}{2}i$. The complex conjugate is $a-bi$, or $0+\frac{1}{2}i$. This can be written simply as $\frac{1}{2}i$.

**Analysis**

Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. To obtain a real number from an imaginary number, we can simply multiply by $i$.

How To...

Given two complex numbers, divide one by the other.

1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

## Example 3.1.7: Dividing Complex Numbers

Divide $(2+5i)$ by $(4-i)$.

**Solution**

We begin by writing the problem as a fraction.

$$\frac{(2+5i)}{(4-i)}$$

Then we multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(2+5i)}{(4-i)}\cdot\frac{(4+i)}{(4+i)}$$

To multiply two complex numbers, we expand the product as we would with polynomials (the process commonly called FOIL).

$$\frac{(2+5i)}{(4-i)}\cdot\frac{(4+i)}{(4+i)}=\frac{8+2i+20i+5i^2}{16+4i-4i-i^2} \tag{3.1.9}$$

$$=\frac{8+2i+20i+5(-1)}{16+4i-4i-(-1)} \qquad \text{because } i^2=-1 \tag{3.1.10}$$

$$=\frac{3+22i}{17} \tag{3.1.11}$$

$$=\frac{3}{17}+\frac{22}{17}i \qquad \text{Separate real and imaginary parts.} \tag{3.1.12}$$

Note that this expresses the quotient in standard form.

## Example 3.1.8: Substituting a Complex Number into a Polynomial Function

Let $f(x)=x^2-5x+2$. Evaluate $f(3+i)$.

**Solution**

Substitute $x = 3 + i$ into the function $f(x) = x^2 - 5x + 2$ and simplify.

$$\begin{aligned} f(3+i) &= (3+i)^2 - 5(3+i) + 2 && \text{Substitute 3+i for x} \\ &= (3 + 6i + i^2) - (15 + 5i) + 2 && \text{Multiply} \\ &= 9 + 6i + (-1) - 15 - 5i + 2 && \text{Subsitute -1 for } i^2 \\ &= -5 + i && \text{Combine like terms} \end{aligned}$$

**Analysis**

We write $f(3 + i) = -5 + i$. Notice that the input is $3 + i$ and the output is $-5 + i$.

## Exercise 3.1.8

Let $f(x) = 2x^2 - 3x$. Evaluate $f(8 - i)$.

*Answer available at the end of this section.*

## Example 3.1.9: Substituting an Imaginary Number in a Rational Function

Let $f(x) = \frac{2+x}{x+3}$. Evaluate $f(10i)$.

**Solution**

Substitute $x = 10i$ and simplify.

$$\begin{aligned} &\frac{2 + 10i}{10i + 3} && \text{Substitute } 10i \text{ for } x. \\ &\frac{2 + 10i}{3 + 10i} && \text{Rewrite the denominator in standard form.} \\ &\frac{2 + 10i}{3 + 10i} \cdot \frac{3 - 10i}{3 - 10i} && \text{Multiply the numerator and denominator by the complex conjugate of the denominator.} \\ &\frac{6 - 20i + 30i - 100i^2}{9 - 30i + 30i - 100i^2} && \text{Multiply using the distributive property or the FOIL method.} \\ &\frac{6 - 20i + 30i - 100(-1)}{9 - 30i + 30i - 100(-1)} && \text{Substitute } -1 \text{ for } i^2. \\ &\frac{106 + 10i}{109} && \text{Simplify.} \\ &\frac{106}{109} + \frac{10}{109} && \text{Separate the real and imaginary parts.} \end{aligned}$$

## Exercise 3.1.9

Let $f(x) = \frac{x+1}{x-4}$. Evaluate $f(-i)$.

*Answer available at the end of this section.*

# Simplifying Powers of $i$

The powers of $i$ are cyclic. Let's look at what happens when we raise $i$ to increasing powers.

$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$
$$i^4 = i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1$$
$$i^5 = i^4 \cdot i = 1 \cdot i = i$$

We can see that when we get to the fifth power of $i$, it is equal to the first power. As we continue to multiply $i$ by itself for increasing powers, we will see a cycle of 4. Let's examine the next 4 powers of $i$.

$$i^6 = i^5 \cdot i = i \cdot i = i^2 = -1$$
$$i^7 = i^6 \cdot i = i^2 \cdot i = i^3 = -i$$
$$i^8 = i^7 \cdot i = i^3 \cdot i = i^4 = 1$$
$$i^9 = i^8 \cdot i = i^4 \cdot i = i^5 = i$$

### Example 3.1.10: Simplifying Powers of $i$

Evaluate $i^{35}$.

**Solution**

Since $i^4 = 1$, we can simplify the problem by factoring out as many factors of i^4 as possible. To do so, first determine how many times 4 goes into 35: $35 = 4 \cdot 8 + 3$.

$$i^{35} = i^{4 \cdot 8 + 3} = i^{4 \cdot 8} \cdot i^3 = (i^4)^8 \cdot i^3 = 1^8 \cdot i^3 = i^3 = -i$$

### Q & A

Can we write $i^35$ in other helpful ways?

> As we saw in Example 3.1.10, we reduced $i^{35}$ to $i^3$ by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of $i^{35}$ may be more useful. Table 3.1.1 shows some other possible factorizations.

*Table 3.1.1: Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.*

| Factorization of $i^{35}$ | $i^{34} \cdot i$ | $i^{33} \cdot i^2$ | $i^{31} \cdot i^4$ | $i^{19} \cdot i^{16}$ |
|---|---|---|---|---|
| **Reduced form** | $\left(i^2\right)^{17} \cdot i$ | $i^{33} \cdot (-1)$ | $i^{31} \cdot 1$ | $i^{19} \cdot \left(i^4\right)^4$ |
| **Simplified form** | $(-1)^{17} \cdot i$ | $-i^{33}$ | $i^{31}$ | $i^{19}$ |

## Key Concepts

- The square root of any negative number can be written as a multiple of $i$.
- To plot a complex number, we use two number lines, crossed to form the complex plane. The horizontal axis is the real axis, and the vertical axis is the imaginary axis.
- Complex numbers can be added and subtracted by combining the real parts and combining the imaginary parts.
- Complex numbers can be multiplied and divided.
- To multiply complex numbers, distribute just as with polynomials.
- To divide complex numbers, multiply both the numerator and denominator by the complex conjugate of the denominator to eliminate the complex number from the denominator.
- The powers of $i$ are cyclic, repeating every fourth one.

# Glossary

**complex conjugate**
the complex number in which the sign of the imaginary part is changed and the real part of the number is left unchanged; when added to or multiplied by the original complex number, the result is a real number

**complex number**
the sum of a real number and an imaginary number, written in the standard form $a + bi$, where $a$ is the real part, and $bi$ is the imaginary part

**complex plane**
a coordinate system in which the horizontal axis is used to represent the real part of a complex number and the vertical axis is used to represent the imaginary part of a complex number

**imaginary number**
a number in the form bi where $i = \sqrt{-1}$

## Answers

**Exercise** 3.1.1
$\sqrt{-24} = 0 + 2i\sqrt{6}$

**Exercise** 3.1.1

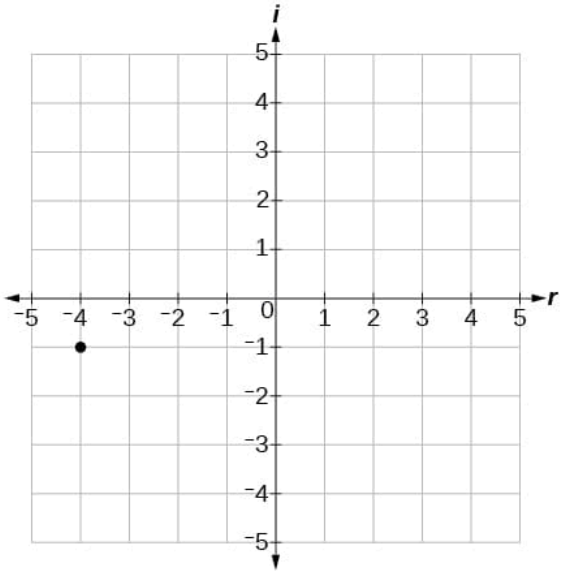



Figure 3.1.5

**Exercise** 3.1.3
$(3 - 4i) - (2 + 5i) = 1 - 9i$

**Exercise** 3.1.4
$-8 - 24i$

**Exercise** 3.1.5
$18 + i$

**Exercise** 3.1.8
$102 - 29i$

**Exercise** 3.1.9
$-\frac{3}{17} + \frac{5i}{17}$

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# 3.2: Quadratic Functions

**Learning Objectives**

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.

Curved antennas, such as the ones shown in Figure 3.2.1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

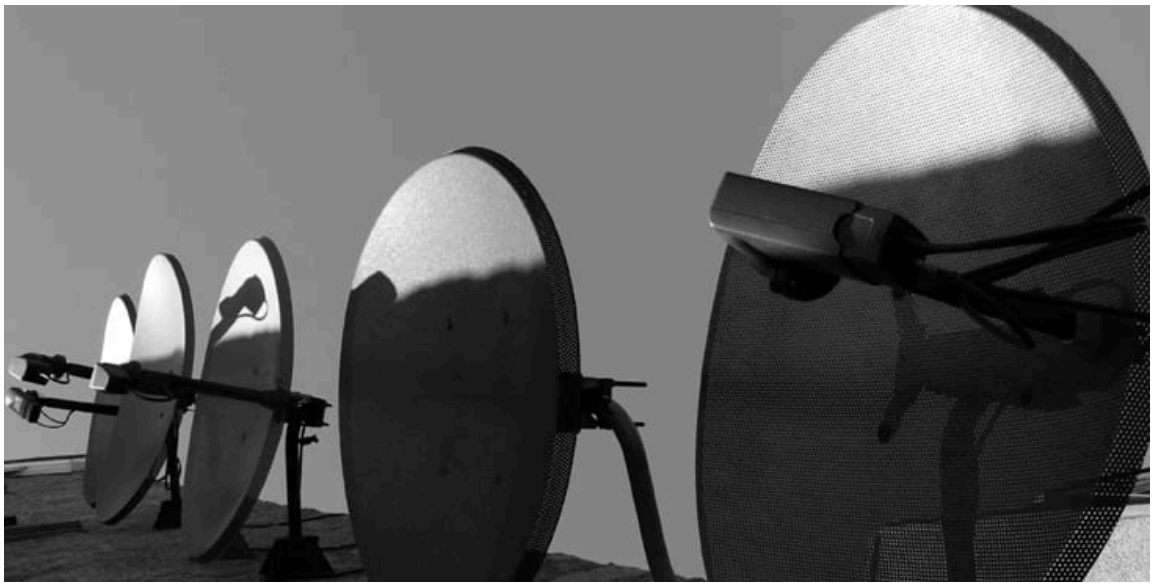


***Figure** 3.2.1**:** An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)*

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

## Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 3.2.2.

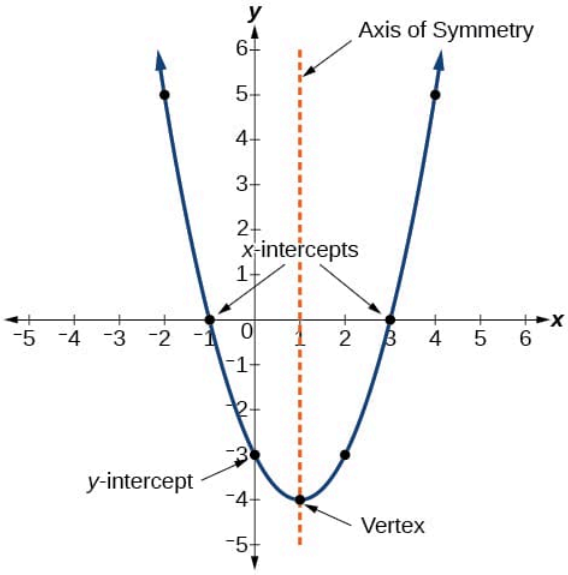



Figure 3.2.2: Graph of a parabola showing where the $x$ and $y$ intercepts, vertex, and axis of symmetry are.

The y-intercept is the point at which the parabola crosses the $y$-axis. The x-intercepts are the points at which the parabola crosses the $x$-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of $x$ at which $y = 0$.

## Example 3.2.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 3.2.3.

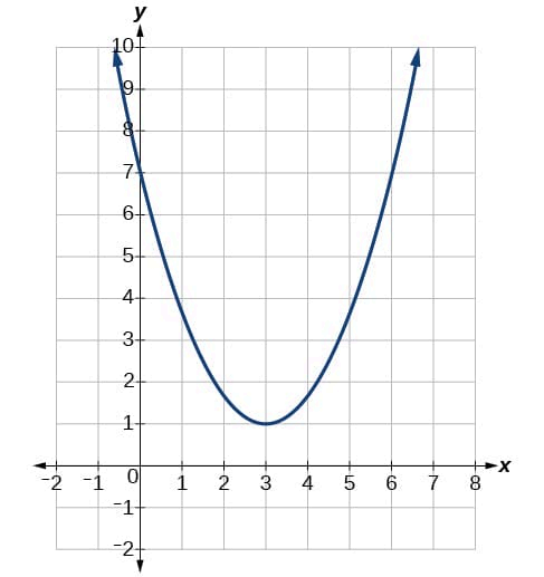



Figure 3.2.3.

**Solution**

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the $y$-axis at $(0, 7)$ so this is the y-intercept.

# Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c \tag{3.2.1}$$

where $a$, $b$, and $c$ are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of $x$ halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure $3.2.4$ represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

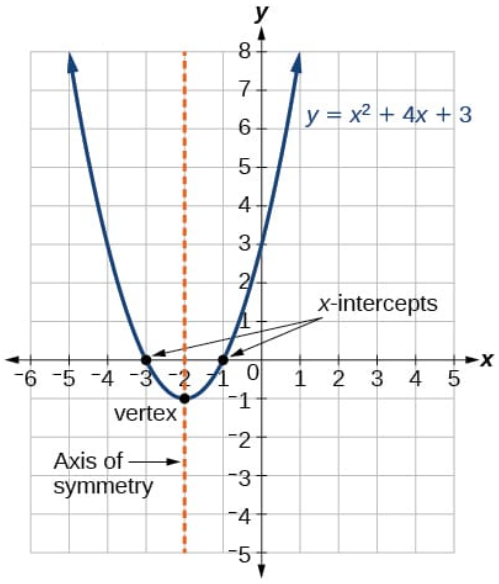



Figure $3.2.4$: Graph of a parabola showing where the $x$ and $y$ intercepts, vertex, and axis of symmetry are for the function $y = x^2 + 4x + 3$.

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k \tag{3.2.2}$$

where $(h, k)$ is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure $3.2.5$ represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x–h = x + 2$ in this example, $h = –2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

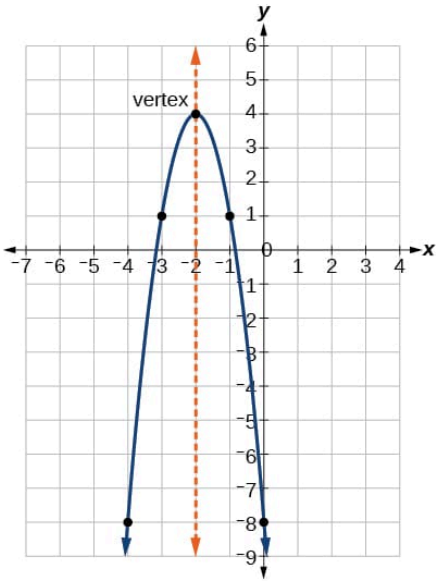



Figure 3.2.5: Graph of a parabola showing where the $x$ and $y$ intercepts, vertex, and axis of symmetry are for the function $y = -3(x + 2)^2 + 4$.

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure 3.2.6 is the graph of this basic function.

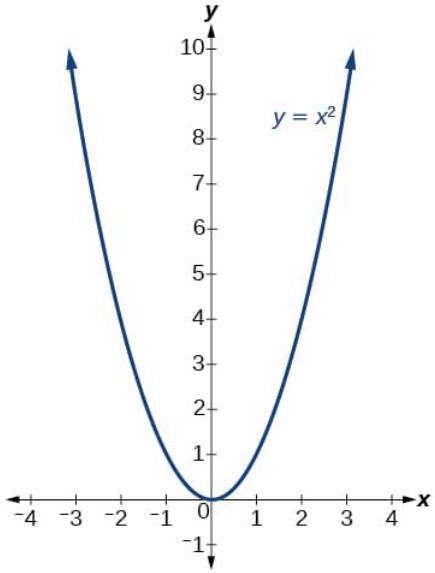



Figure 3.2.6: Graph of $y = x^2$.

If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 3.2.5, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 3.2.5, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of $a$ indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure 3.2.5, $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$a(x - h)^2 + k = ax^2 + bx + c$$
$$ax^2 - 2ahx + (ah^2 + k) = ax^2 + bx + c$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah = b, \text{ so } h = -\frac{b}{2a}.$$

This is the **axis of symmetry** we defined earlier. Setting the constant terms equal:

$$\begin{aligned} ah^2 + k &= c \\ k &= c - ah^2 \\ &= c - a \cdot \left(-\frac{b}{2a}\right)^2 \\ &= c - \frac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that $k$ is the output value of the function when the input is $h$, so $f(h) = k$.

## Definitions: Forms of Quadratic Functions

A quadratic function is a function of degree two. The graph of a *quadratic function* is a parabola.

- The *general form* of a quadratic function is $f(x) = ax^2 + bx + c$ where $a$, $b$, and $c$ are real numbers and $a \neq 0$.
- The standard form of a quadratic function is $f(x) = a(x - h)^2 + k$.
- The vertex $(h, k)$ is located at

$$h = -\frac{b}{2a}, \; k = f(h) = f(\frac{-b}{2a}). \tag{3.2.3}$$

## HOWTO: Write a quadratic function in a general form

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is $h$. Identify the vertical shift of the parabola; this value is $k$.
2. Substitute the values of the horizontal and vertical shift for $h$ and $k$. in the function $f(x) = a(x-h)^2 + k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for $x$ and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. If the parabola opens up, $a > 0$. If the parabola opens down, $a < 0$ since this means the graph was reflected about the x-axis.
6. Expand and simplify to write in general form.

## Example 3.2.2: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function $g$ in Figure 3.2.7 as a transformation of $f(x) = x^2$, and then expand the formula, and simplify terms to write the equation in general form.

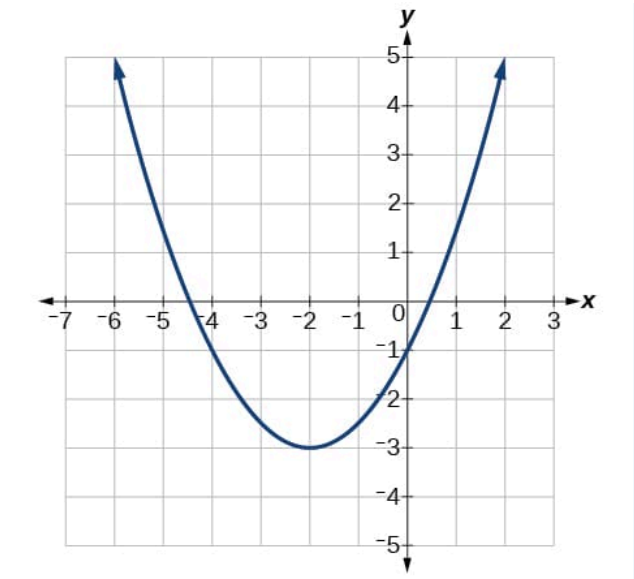



Figure 3.2.7: Graph of a parabola with its vertex at $(-2, -3)$.

**Solution**

We can see the graph of $g$ is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x + 2)^2$–3.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$-1 = a(0 + 2)^2 - 3 \tag{3.2.4}$$
$$2 = 4a \tag{3.2.5}$$
$$a = \frac{1}{2} \tag{3.2.6}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x + 2)^2$–3.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$g(x) = \frac{1}{2}(x + 2)^2 - 3 \tag{3.2.7}$$
$$= \frac{1}{2}(x + 2)(x + 2) - 3 \tag{3.2.8}$$
$$= \frac{1}{2}(x^2 + 4x + 4) - 3 \tag{3.2.9}$$
$$= \frac{1}{2}x^2 + 2x + 2 - 3 \tag{3.2.10}$$
$$= \frac{1}{2}x^2 + 2x - 1 \tag{3.2.11}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

**Analysis**

We can check our work using the table feature on a graphing utility. First enter $\mathrm{Y1} = \frac{1}{2}(\mathrm{x} + 2)^2 - 3$. Next, select $\mathrm{TBLSET}$, then use $\mathrm{TblStart} = -6$ and $\Delta\mathrm{Tbl} = 2$, and select $\mathrm{TABLE}$. See Table 3.2.1

*Table* 3.2.1

| $x$ | -6 | -4 | -2 | 0 | 2 |
|---|---|---|---|---|---|

| $y$ | -5 | -1 | -3 | -1 | 5 |
|---|---|---|---|---|---|

The ordered pairs in the table correspond to points on the graph.

## Exercise 3.2.2

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 3.2.8. Find an equation for the path of the ball. Does the shooter make the basket?

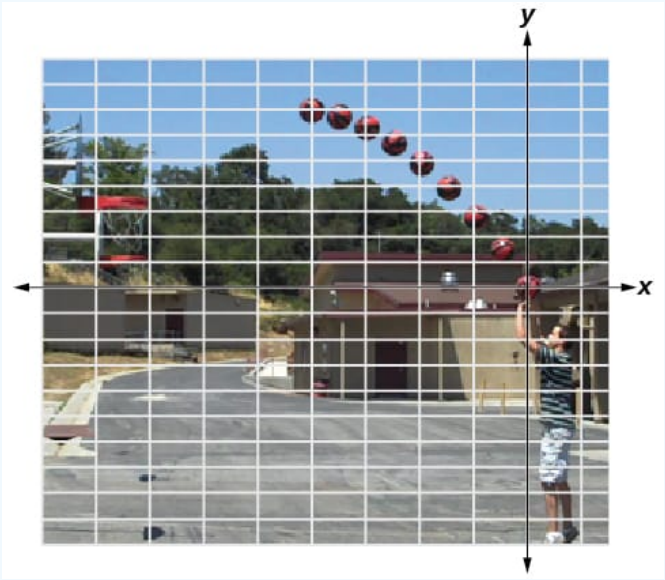



***Figure*** 3.2.8***:*** *Stop motioned picture of a boy throwing a basketball into a hoop to show the parabolic curve it makes. (credit: modification of work by Dan Meyer)*

*Answer available at the end of this section.*

**How To: Given a quadratic function in general form, find the vertex of the parabola.**

1. Identify $a$, $b$, and $c$.
2. Find $h$, the x-coordinate of the vertex, by substituting $a$ and $b$ into $h = -\frac{b}{2a}$.
3. Find $k$, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

## Example 3.2.3: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 – 6x + 7$. Rewrite the quadratic in standard form (vertex form).

**Solution**

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} \tag{3.2.12}$$
$$= -\frac{-6}{2(2)} \tag{3.2.13}$$
$$= \frac{6}{4} \tag{3.2.14}$$
$$= \frac{3}{2} \tag{3.2.15}$$

The vertical coordinate of the vertex will be at

$$k = f(h) \tag{3.2.16}$$
$$= f\left(\frac{3}{2}\right) \tag{3.2.17}$$
$$= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \tag{3.2.18}$$
$$= \frac{5}{2} \tag{3.2.19}$$

Rewriting into standard form, the stretch factor will be the same as the $a$ in the original quadratic.

$$f(x) = ax^2 + bx + c \tag{3.2.20}$$
$$f(x) = 2x^2 - 6x + 7$$

Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} \tag{3.2.21}$$

**Analysis**

One reason we may want to identify the vertex of the parabola is that this point will inform us what the maximum or minimum value of the function is, $(k)$,and where it occurs, $(h)$.

**Exercise 3.2.3**

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

*Answer available at the end of this section.*

## Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

**Definition: Domain and Range of a Quadratic Function**

The domain of any **quadratic function** is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive $a$ value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $[f(-\frac{b}{2a}), \infty)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f(-\frac{b}{2a})$, or $(-\infty, f(-\frac{b}{2a})]$.

The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive $a$ value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative $a$ value is $f(x) \leq k$.

**How To: Given a quadratic function, find the domain and range.**

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether $a$ is positive or negative. If $a$ is positive, the parabola has a minimum. If $a$ is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, $k$.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

### Example 3.2.4: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

**Solution**

As with any quadratic function, the domain is all real numbers.

Because $a$ is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} \tag{3.2.22}$$
$$= -\frac{9}{2(-5)} \tag{3.2.23}$$
$$= \frac{9}{10} \tag{3.2.24}$$

The maximum value is given by $f(h)$.

$$f(\frac{9}{10}) = 5(\frac{9}{10})^2 + 9(\frac{9}{10}) - 1 \tag{3.2.25}$$
$$= \frac{61}{20} \tag{3.2.26}$$

The range is $f(x) \leq \frac{61}{20}$, or $(-\infty, \frac{61}{20}]$.

### Exercise 3.2.4

Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

*Answer available at the end of this section.*

## Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the **parabola**. We can see the maximum and minimum values in Figure 3.2.9.

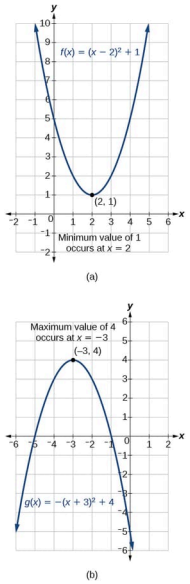



Figure 3.2.9: Minimum and maximum of two quadratic functions.

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

## Example 3.2.5: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length $L$.
b. What dimensions should she make her garden to maximize the enclosed area?

**Solution**

Let's use a diagram such as Figure 3.2.10 to record the given information. It is also helpful to introduce a temporary variable, $W$, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

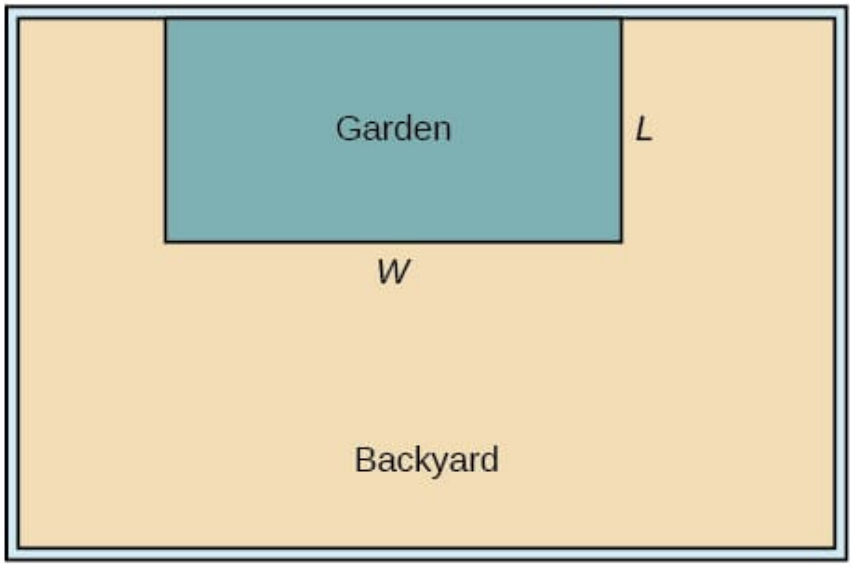



Figure 3.2.10: Diagram of the garden and the backyard.

a. We know we have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, $W$, in terms of $L$.

$$W = 80 - 2L \tag{3.2.27}$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80 - 2L) \tag{3.2.28}$$
$$A(L) = 80L - 2L^2 \tag{3.2.29}$$

This formula represents the area of the fence in terms of the variable length $L$. The function, written in general form, is

$$A(L) = -2L^2 + 80L \tag{3.2.30}$$

.

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since $a$ is the coefficient of the squared term, $a = -2$, $b = 80$, and $c = 0$.

To find the vertex:

$$h = -\frac{80}{2(-2)} \qquad k = A(20) \tag{3.2.31}$$
$$= 20 \qquad \text{and} \qquad = 80(20) - 2(20)^2 \tag{3.2.32}$$
$$= 800 \tag{3.2.33}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

**Analysis**

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 3.2.11.

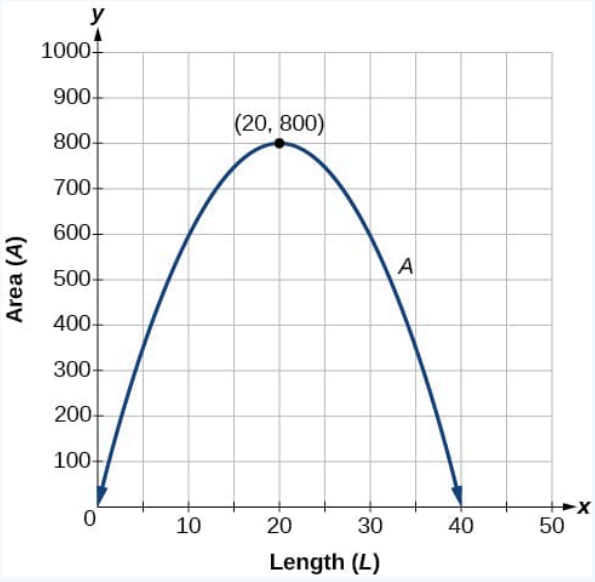



Figure 3.2.11: Graph of the parabolic function $A(L) = -2L^2 + 80L$

**How To: Given an application involving revenue, use a quadratic equation to find the maximum.**

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

## Example 3.2.6: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

**Solution**

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, $p$ for price per subscription and $Q$ for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} \tag{3.2.34}$$
$$= -\frac{5,000}{2} \tag{3.2.35}$$
$$= -2,500 \tag{3.2.36}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$Q = -2500p + b \qquad \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \tag{3.2.37}$$
$$84,000 = -2500(30) + b \qquad \text{Solve for } b \tag{3.2.38}$$
$$b = 159,000 \tag{3.2.39}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = pQ \tag{3.2.40}$$
$$\text{Revenue} = p(-2,500p + 159,000) \tag{3.2.41}$$
$$\text{Revenue} = -2,500p^2 + 159,000p \tag{3.2.42}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)} \tag{3.2.43}$$
$$= 31.8 \tag{3.2.44}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{maximum revenue} = -2,500(31.8)^2 + 159,000(31.8) \tag{3.2.45}$$
$$= 2,528,100 \tag{3.2.46}$$

**Analysis**

This could also be solved by graphing the quadratic as in Figure 3.2.12. We can see the maximum revenue on a graph of the quadratic function.

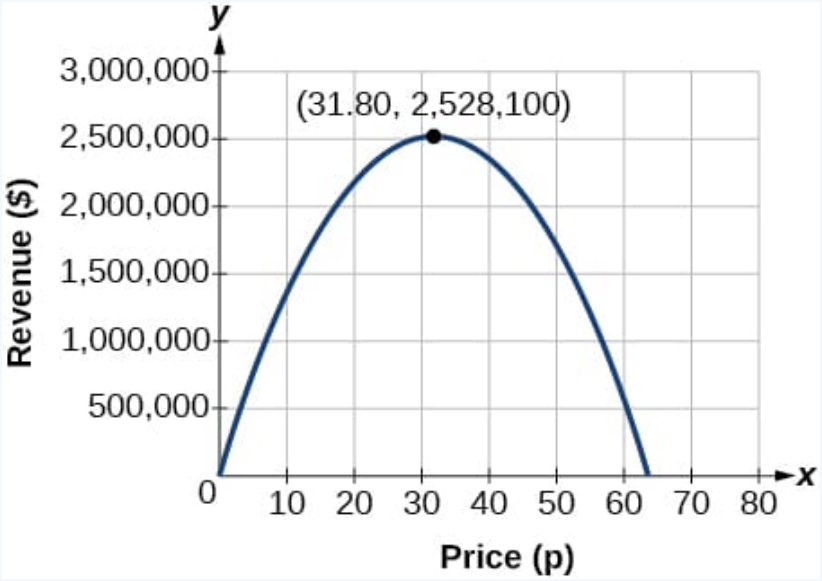



Figure 3.2.12: Graph of the parabolic function

## Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure 3.2.13 that the number of x-intercepts can vary depending upon the location of the graph.

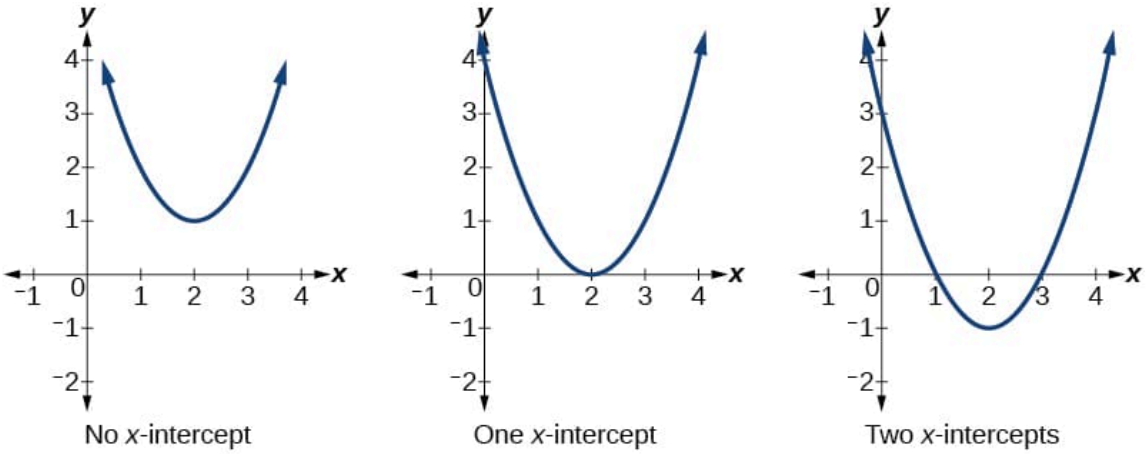



Figure 3.2.13: Number of x-intercepts of a parabola.

**Given a quadratic function $f(x)$, find the y- and x-intercepts.**

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x) = 0$ to find the x-intercepts.

### Example 3.2.7: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

**Solution**

We find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2 + 5(0) - 2 \tag{3.2.47}$$
$$= -2 \tag{3.2.48}$$

So the y-intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2 \tag{3.2.49}$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2) \tag{3.2.50}$$

$$0 = 3x - 1 \qquad 0 = x + 2 \tag{3.2.51}$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -2 \tag{3.2.52}$$

So the x-intercepts are at $(\frac{1}{3}, 0)$ and $(-2, 0)$.

**Analysis**

By graphing the function, we can confirm that the graph crosses the $y$-axis at $(0, -2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure 3.2.14.

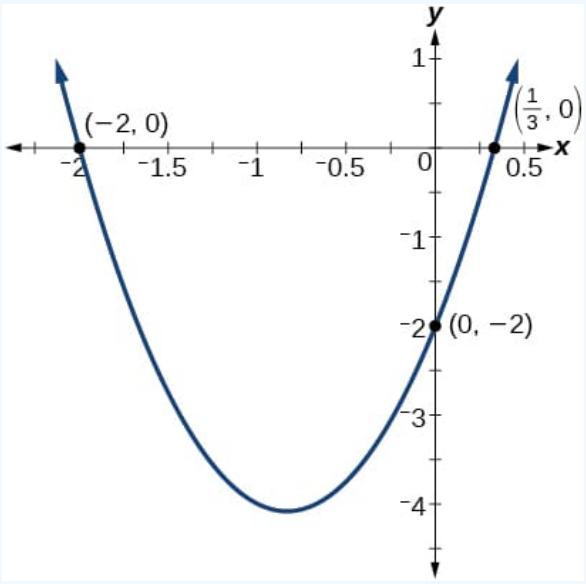



Figure 3.2.14: Graph of a parabola.

## Rewriting Quadratics in Standard Form

In Example 3.2.7, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

**How To: Given a quadratic function, find the x-intercepts by rewriting in standard form.**

1. Substitute a and $b$ into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find $k$.
3. Rewrite the quadratic in standard form using $h$ and $k$.
4. Solve for when the output of the function will be zero to find the x-intercepts.

## Example 3.2.8: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

**Solution**

We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x - h)^2 + k$$

We know that $a = 2$. Then we solve for $h$ and $k$.

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x + 1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x + 1)^2 - 6 \\ 6 &= 2(x + 1)^2 \\ 3 &= (x + 1)^2 \\ x + 1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$.

**Analysis**

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See Figure 3.2.15.

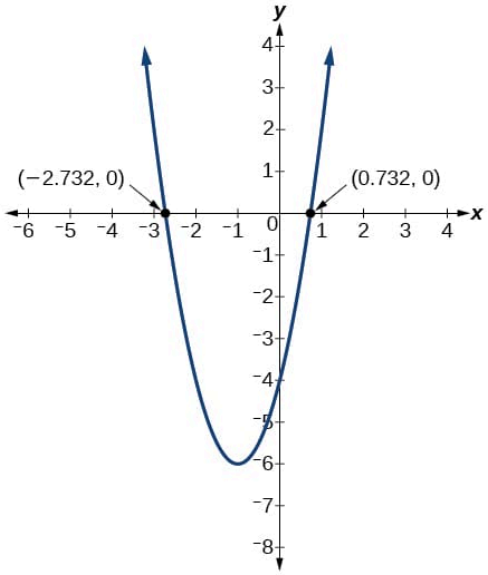



Figure 3.2.15: Graph of a parabola which has the following x-intercepts: $(-2.732, 0)$ and $(0.732, 0)$.

## Exercise 3.2.1

In Try It 3.2.1, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x-intercepts (if any).

*Answer available at the end of this section.*

## Example 3.2.9: Solving a Quadratic Equation with the Quadratic Formula

Solve $x^2 + x + 2 = 0$.

**Solution**

Let's begin by writing the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

When applying the quadratic formula, we identify the coefficients $a$, $b$ and $c$. For the equation $x^2 + x + 2 = 0$, we have $a = 1$, $b = 1$, and $c = 2$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (2)}}{2 \cdot 1} \\ &= \frac{-1 \pm \sqrt{1 - 8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $x = \frac{-1+i\sqrt{7}}{2}$ and $x = \frac{-1-i\sqrt{7}}{2}$ or $x = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $x = \frac{-1}{2} - \frac{i\sqrt{7}}{2}$.

## Example 3.2.10: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

When does the ball reach the maximum height?
What is the maximum height of the ball?
When does the ball hit the ground?

The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{80}{2(-16)} \tag{3.2.53}$$
$$= \frac{80}{32} \tag{3.2.54}$$
$$= \frac{5}{2} \tag{3.2.55}$$
$$= 2.5 \tag{3.2.56}$$

The ball reaches a maximum height after 2.5 seconds.

To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$k = H(-\frac{b}{2a}) \tag{3.2.57}$$
$$= H(2.5) \tag{3.2.58}$$
$$= -16(2.5)^2 + 80(2.5) + 40 \tag{3.2.59}$$
$$= 140 \tag{3.2.60}$$

The ball reaches a maximum height of 140 feet.

To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$.

We use the quadratic formula.

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \tag{3.2.61}$$
$$= \frac{-80 \pm \sqrt{8960}}{-32} \tag{3.2.62}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458 \tag{3.2.63}$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 3.2.16.

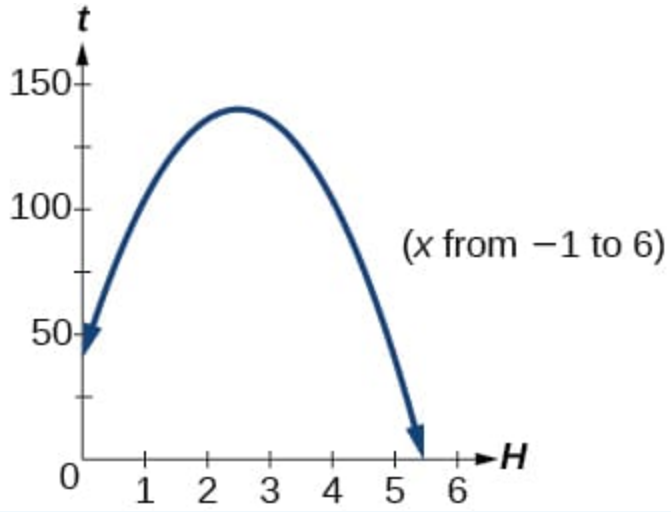



Figure 3.2.16

Try It 3.2.5: A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

**Solution**

a. 3 seconds b. 256 feet c. 7 seconds

## Key Equations

- general form of a quadratic function: $f(x) = ax^2 + bx + c$
- the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- standard form of a quadratic function: $f(x) = a(x - h)^2 + k$

## Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the $y$-axis.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph.
- The vertex can be found from an equation representing a quadratic function. .
- The domain of a quadratic function is all real numbers. The range varies with the function.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue.
- Some quadratic equations must be solved by using the quadratic formula.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems.

# Glossary

**axis of symmetry**
a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

**general form of a quadratic function**
the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where $a, b,$ and $c$ are real numbers and a≠0.

**standard form of a quadratic function**
the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where $(h, k)$ is the vertex.

**vertex**
the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

**vertex form of a quadratic function**
another name for the standard form of a quadratic function

**zeros**
in a given function, the values of $x$ at which $y = 0$, also called roots

## Answers

**Exercise** 3.2.2

The path passes through the origin and has vertex at $(-4, 7)$, so $h(x) = -\frac{7}{16}(x + 4)^2 + 7$. To make the shot, $h(-7.5)$ would need to be about 4 but $h(-7.5) \approx 1.64$; he doesn't make it.

**Exercise** 3.2.3

$g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x - 3)^2 + 4$ in standard form.

**Exercise** 3.2.4

The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $\left[\frac{8}{11}, \infty\right)$.

**Exercise** 3.2.1

y-intercept at $(0, 13)$, No x-intercepts

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# 3.3: Power Functions and Polynomial Functions

**Learning Objectives**

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table 3.3.1.

*Table* 3.3.1

| **Year** | 2009 | 2010 | 2011 | 2012 | 2013 |
|---|---|---|---|---|---|
| **Bird Population** | 800 | 897 | 992 | 1,083 | 1,169 |

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$, where $P(t)$ represents the bird population on the island $t$ years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.



Figure 3.3.1: (credit: Jason Bay, Flickr)

## Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius $r$ is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius $r$ is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, $\pi$ or $\frac{4}{3}\pi$, multiplied by a variable $r$ raised to a power.

### Definition: Power Function

A *power function* is a function that can be represented in the form

$$f(x) = kx^p \tag{3.3.1}$$

where $k$ and $p$ are real numbers, and $k$ is known as the *coefficient*.

### Q&A: Is $f(x) = 2^x$ a power function?

*No. A power function contains a variable base raised to a fixed power (Equation 3.3.1). This function has a constant base raised to a variable power. This is called an exponential function, not a power function. This function will be discussed later.*

### Example 3.3.1: Identifying Power Functions

Which of the following functions are power functions?

| | |
|---|---|
| $f(x) = 1$ | Constant function |
| $f(x) = x$ | Identify function |
| $f(x) = x^2$ | Quadratic function |
| $f(x) = x^3$ | Cubic function |
| $f(x) = \frac{1}{x}$ | Reciprocal function |
| $f(x) = \frac{1}{x^2}$ | Reciprocal squared function |
| $f(x) = \sqrt{x}$ | Square root function |
| $f(x) = \sqrt[3]{x}$ | Cube root function |

**Solution**

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

### Exercise 3.3.1

Which functions are power functions?

- $f(x) = 2x^2 \cdot 4x^3$
- $g(x) = -x^5 + 5x^3 - 4x$

- $h(x) = \frac{2x^5-1}{3x^2+4}$

*Answer available at the end of this section.*

## Identifying End Behavior of Power Functions

Figure 3.3.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

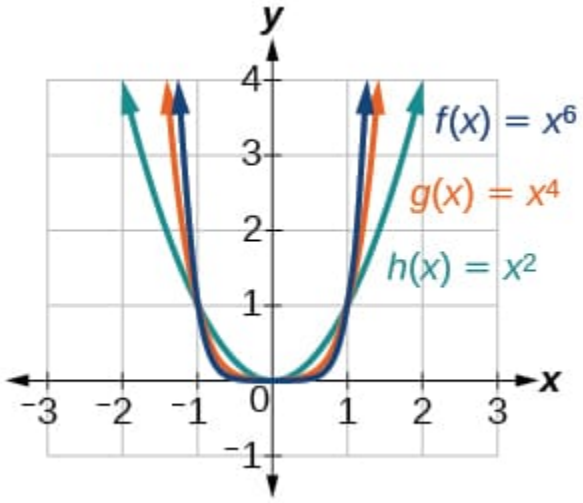



Figure 3.3.2: Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol $\infty$ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that $x$ is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as $x$ approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty,\ f(x) \to \infty$$

Figure 3.3.3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

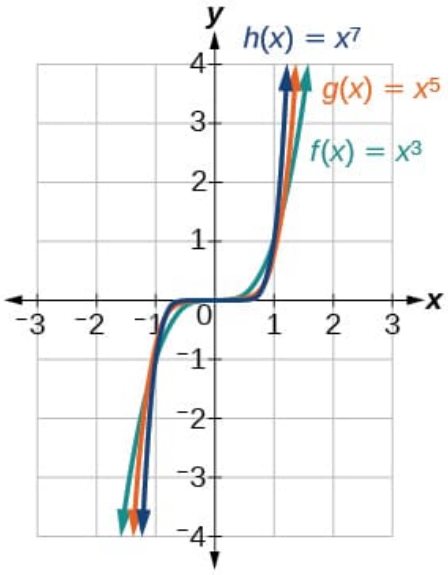



Figure 3.3.3: Odd-power function

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 3.3.2 we see that even functions of the form $f(x) = x^n$, $n$ even, are symmetric about the $y$-axis. In Figure 3.3.3 we see that odd functions of the form $f(x) = x^n$, $n$ odd, are symmetric about the origin.

For these odd power functions, as $x$ approaches negative infinity, $f(x)$ decreases without bound. As $x$ approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$
$$\text{as } x \to \infty,\ f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small $(x \to -\infty)$ and get very large $x \to \infty$ is referred to as the *end behavior* of the function. We can use words or symbols to describe end behavior.

Figure 3.3.4 shows the end behavior of power functions in the form $f(x) = kx^n$ where $n$ is a non-negative integer depending on the power and the constant.

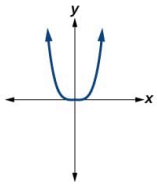
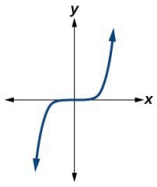
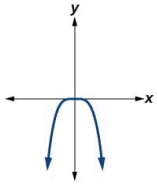
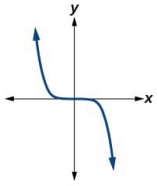
| | Even power | Odd power |
|---|---|---|
| Positive constant $k > 0$ |   $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to \infty$ |   $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to \infty$ |
| Negative constant $k < 0$ |   $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to -\infty$ |   $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to -\infty$ |

Figure 3.3.4

**How To: Given a power function $f(x) = kx^n$ where $n$ is a non-negative integer, identify the end behavior.**

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure 3.3.4 to identify the end behavior.

## Example 3.3.2: Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

**Solution**

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As $x$ approaches infinity, the output (value of $f(x)$ ) increases without bound. We write as $x \to \infty$, $f(x) \to \infty$. As $x$ approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty$, $f(x) \to \infty$. We can graphically represent the function as shown in Figure 3.3.5.

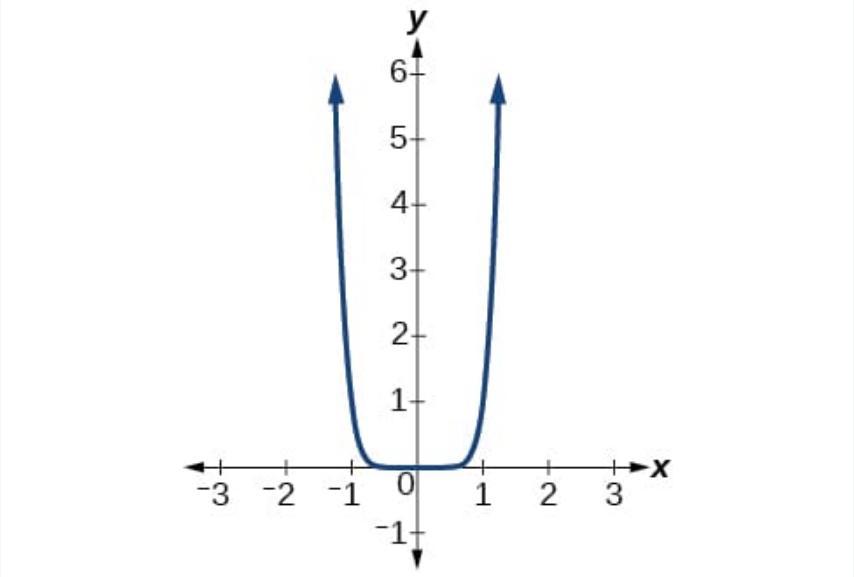



Figure 3.3.5: Graph of $f(x) = x^8$.

## Example 3.3.3: Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

**Solution**

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the $x$-axis of the graph of $f(x) = x^9$. Figure 3.3.6 shows that as $x$ approaches infinity, the output decreases without bound. As $x$ approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty$$
$$\text{as } x \to \infty,\ f(x) \to -\infty$$

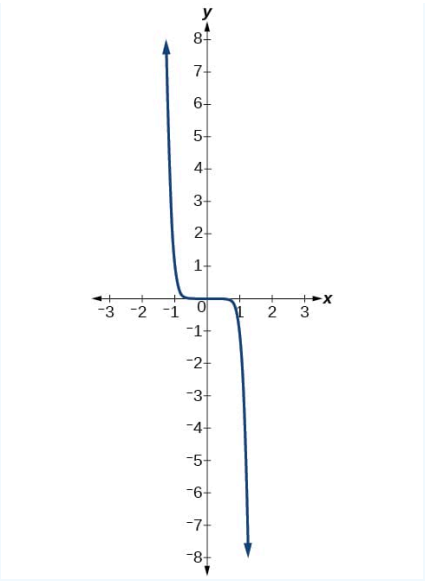



Figure 3.3.6: Graph of $f(x) = -x^9$.

**Analysis**

We can check our work by using the table feature on a graphing utility.

*Table* 3.3.2

| $x$ | $f(x)$ |
|---|---|
| -10 | 1,000,000,000 |
| -5 | 1,953,125 |
| 0 | 0 |
| 5 | -1,953,125 |
| 10 | -1,000,000,000 |

We can see from Table 3.3.2 that, when we substitute very small values for $x$, the output is very large, and when we substitute very large values for $x$, the output is very small (meaning that it is a very large negative value).

**Exercise 3.3.2**

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

*Answer available at the end of this section.*

# Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius $r$ of the spill depends on the number of weeks $w$ that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24+8w) \\ &= \pi(24+8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

### Definition: Polynomial Functions

Let $n$ be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{3.3.2}$$

This is called the general form of a polynomial function. Each $a_i$ is a coefficient and can be any real number. Each product $a_i x^i$ is a *term of a polynomial function*.

### Example 3.3.4: Identifying Polynomial Functions

Which of the following are polynomial functions?

- $f(x) = 2x^3 \cdot 3x + 4$
- $g(x) = -x(x^2 - 4)$
- $h(x) = 5\sqrt{x} + 2$

**Solution**

The first two functions are examples of polynomial functions because they can be written in the form of Equation 3.3.2, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

## Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

### Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.

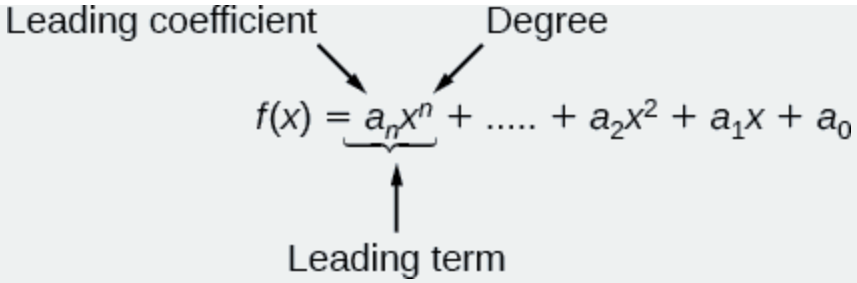



Figure 3.3.7

When a polynomial is written in this way, we say that it is in *general form*.

### How To: Given a polynomial function, identify the degree and leading coefficient

1. Find the highest power of $x$ to determine the degree function.
2. Identify the term containing the highest power of $x$ to find the leading term.
3. Identify the coefficient of the leading term.

### Example 3.3.5: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x) = 3 + 2x^2 - 4x^3$$

$$g(t) = 5t^5 - 2t^3 + 7t$$

$$h(p) = 6p - p^3 - 2$$

**Solution**

For the function $f(x)$, the highest power of $x$ is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, –4.

For the function $g(t)$, the highest power of $t$ is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of $p$ is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, –1.

### Exercise 3.3.3

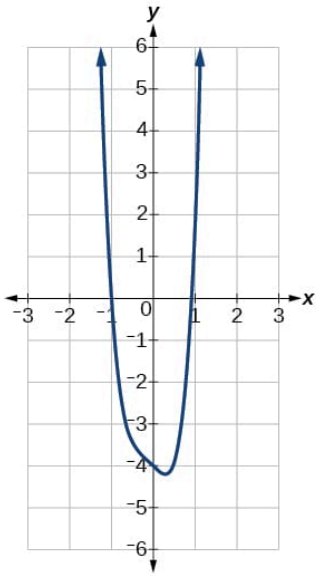
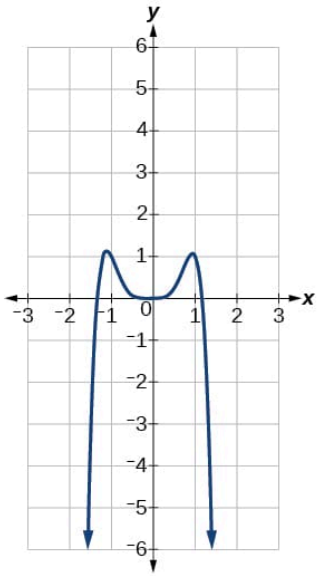
Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

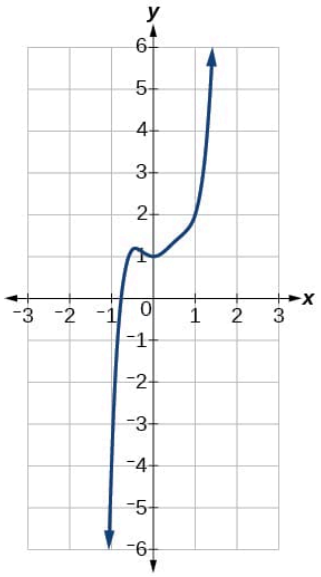
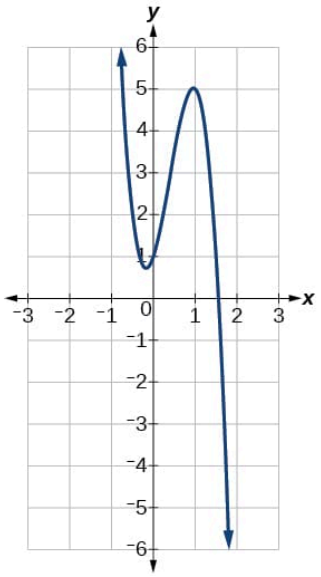
*Answer available at the end of this section.*

## Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as $x$ gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree (Table 3.3.3).

*Table* 3.3.3

| Polynomial Function | Leading Term | Graph of Polynomial Function |
|---|---|---|
| $f(x) = 5x^4 + 2x^3 - x - 4$ | $5x^4$ |  |
| $f(x) = -2x^6 - x^5 + 3x^4 + x^3$ | $-2x^6$ |  |

| Polynomial Function | Leading Term | Graph of Polynomial Function |
|---|---|---|
| $f(x) = 3x^5 - 4x^4 + 2x^2 + 1$ | $3x^5$ |  |
| $f(x) = -6x^3 + 7x^2 + 3x + 1$ | $-6x^3$ |  |

## Example 3.3.6: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 3.3.8.

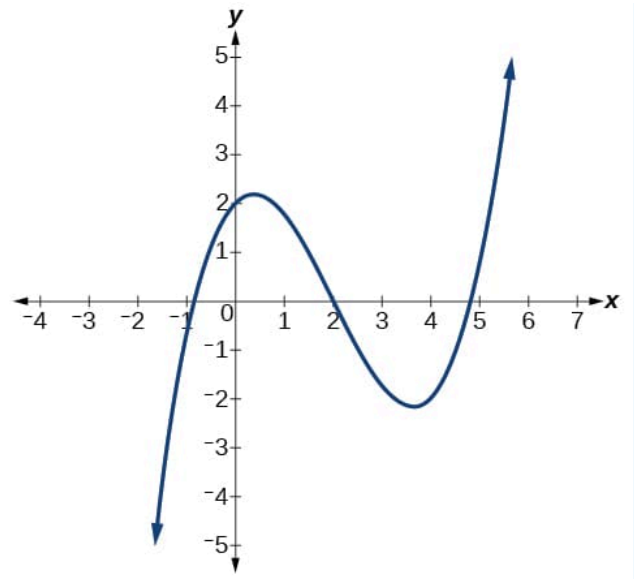



Figure 3.3.8.

**Solution**

As the input values $x$ get very large, the output values $f(x)$ increase without bound. As the input values $x$ get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to \infty,\ f(x) \to \infty$$

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

In words, we could say that as $x$ values approach infinity, the function values approach infinity, and as $x$ values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

## Exercise 3.3.1

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 3.3.9.

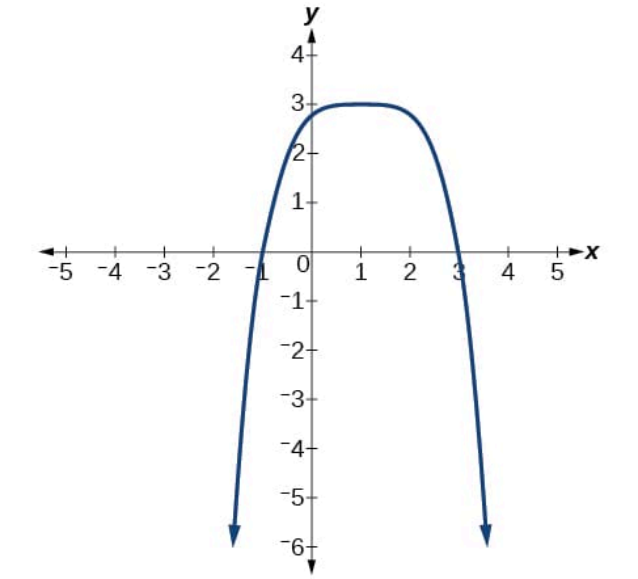



Figure 3.3.9

*Answer available at the end of this section.*

## Example 3.3.7: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

**Solution**

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4 - 9x^3 + 12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

$$\text{as } x \to \infty,\ f(x) \to -\infty$$

## Exercise 3.3.7

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

*Answer available at the end of this section.*

# Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the $y$-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one $y$-intercept $(0, a_0)$. The $x$-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one $x$-intercept. See Figure $3.3.10$.

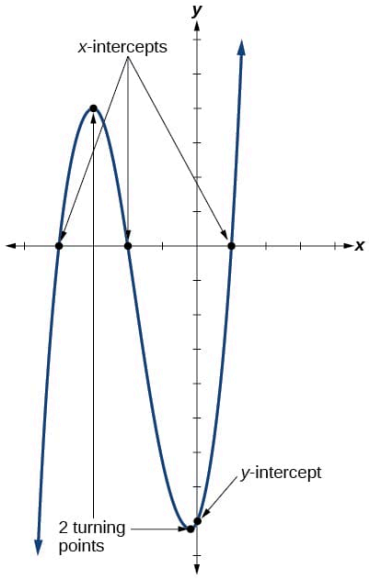



Figure $3.3.10$

## Defintion: Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The $y$-intercept is the point at which the function has an input value of zero. The $x$-intercepts are the points at which the output value is zero.

**How To: Given a polynomial function, determine the intercepts.**

1. Determine the $y$-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the $x$-intercepts by solving for the input values that yield an output value of zero.

## Example $3.3.8$: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the $y$- and $x$-intercepts.

**Solution**

The $y$-intercept occurs when the input is zero, so substitute $0$ for $x$.

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The $y$-intercept is $(0, 8)$.

The $x$-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{array}{rcrcr} x-2=0 & \text{or} & x+1=0 & \text{or} & x-4=0 \\ x=2 & \text{or} & x=-1 & \text{or} & x=4 \end{array}$$

The $x$-intercepts are $(2,0)$,$(-1,0)$, and $(4,0)$.

We can see these intercepts on the graph of the function shown in Figure 3.3.11.

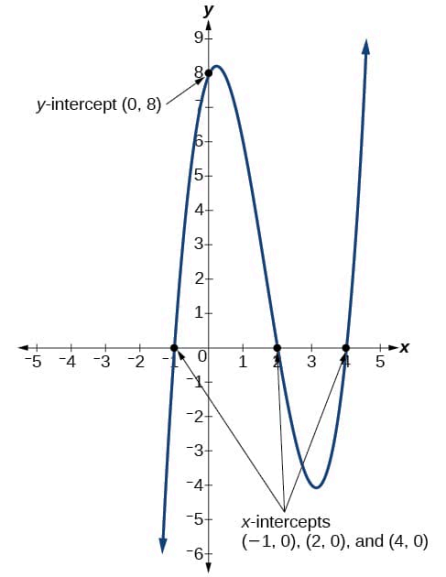



Figure 3.3.11: Graph of $f(x)=(x-2)(x+1)(x-4)$.

## Example 3.3.9: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x)=x^4-4x^2-45$, determine the $y$- and $x$-intercepts.

**Solution**

The $y$-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4-4(0)^2-45 \\ &= -45 \end{aligned}$$

The $y$-intercept is $(0,-45)$.

The $x$-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4-4x^2-45 \\ &= (x^2-9)(x^2+5) \\ &= (x-3)(x+3)(x^2+5) \end{aligned}$$

$$0=(x-3)(x+3)(x^2+5)$$

$$\begin{array}{rcrcr} x-3=0 & \text{or} & x+3=0 & \text{or} & x^2+5=0 \\ x=3 & \text{or} & x=-3 & \text{or} & (\text{no real solution}) \end{array}$$

The $x$-intercepts are $(3,0)$ and $(-3,0)$.

We can see these intercepts on the graph of the function shown in Figure 3.3.12. We can see that the function is even because $f(x)=f(-x)$.

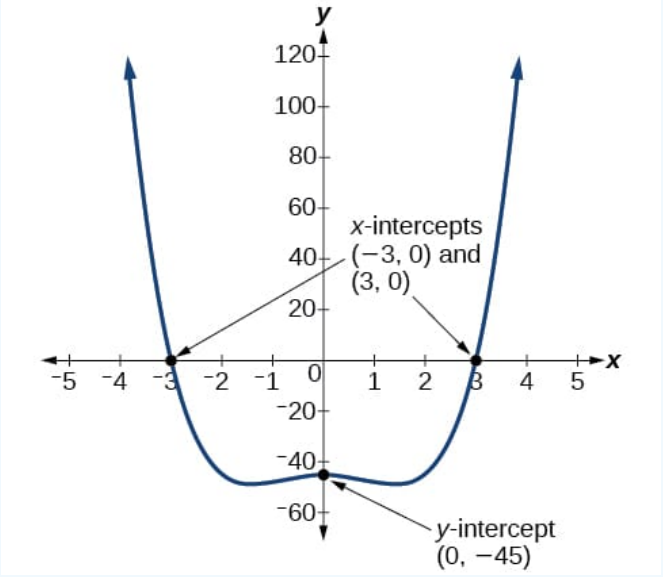



Figure 3.3.12: Graph of $f(x) = x^4 - 4x^2 - 45$.

### Exercise 3.3.5

3.3.5**:** Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the $y$- and $x$-intercepts.

*Solution available at the end of this section.*

## Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of $x$-intercepts and the number of turning points. A polynomial function of $n^{\text{th}}$ degree is the product of $n$ factors, so it will have at most $n$ roots or zeros, or $x$-intercepts. The graph of the polynomial function of degree $n$ must have at most $n–1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A smooth curve is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

### Intercepts and Turning Points of Polynomials

A polynomial of degree $n$ will have, at most, $n$ $x$-intercepts and $n - 1$ turning points.

### Example 3.3.10: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of $x$-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

**Solution**

The polynomial has a degree of **10**, so there are at most $n$ $x$-intercepts and at most $n - 1$ turning points.

## Exercise 3.3.6

Without graphing the function, determine the maximum number of $x$-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$

*Answer available at the end of this section.*

## Example 3.3.11: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 3.3.12 based on its intercepts and turning points?

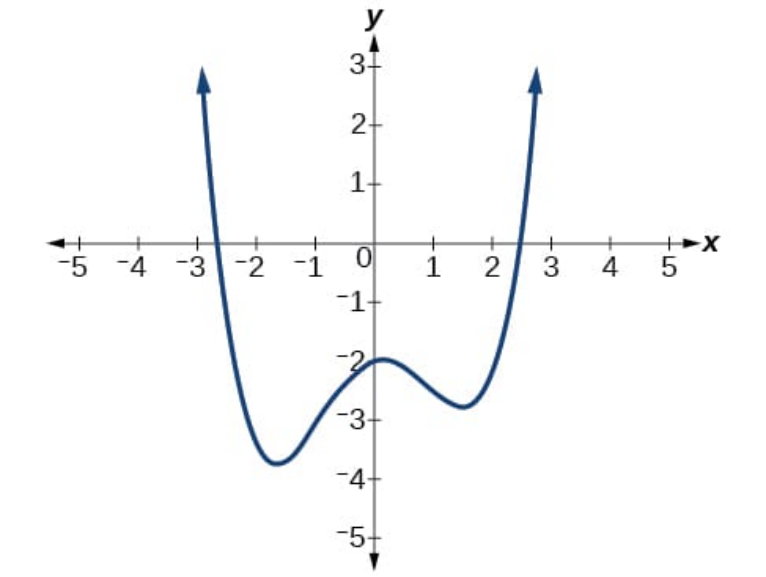



Figure 3.3.13.

**Solution**

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure 3.3.14.

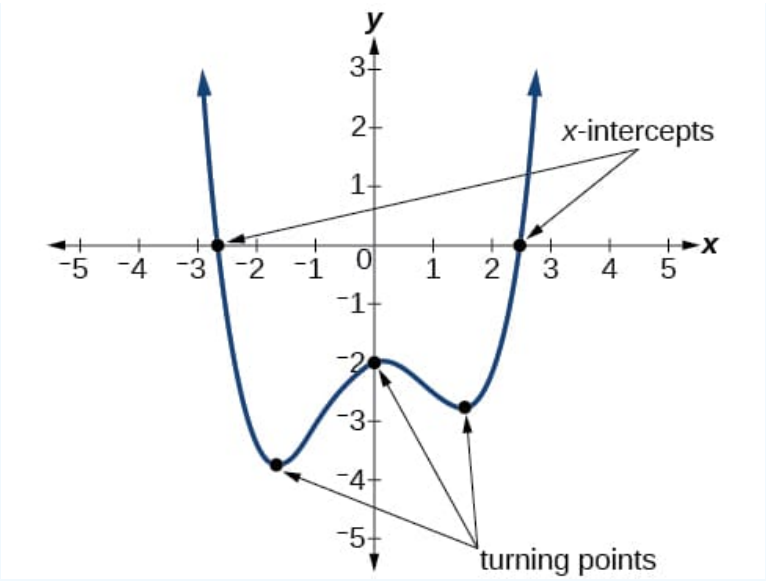



Figure 3.3.14: Graph of an even-degree polynomial.

The graph has 2 $x$-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

## Exercise 3.3.7

What can we conclude about the polynomial represented by the graph shown in Figure 3.3.15 based on its intercepts and turning points?

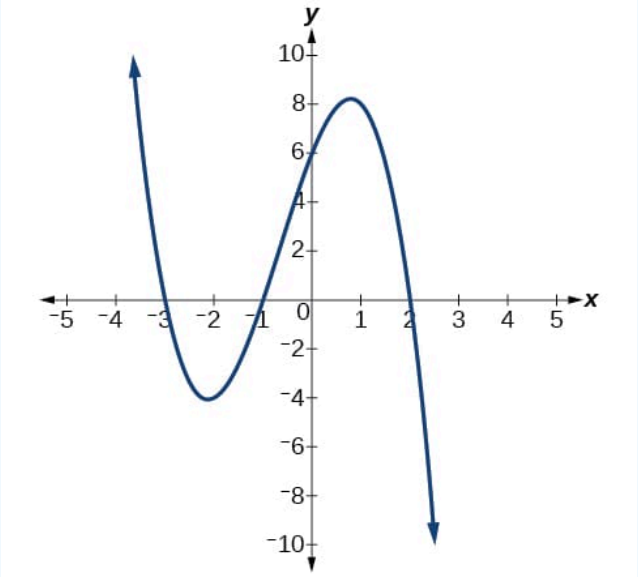



**Figure 3.3.15.**

*Answer available at the end of this section.*

**Solution**

The end behavior indicates an odd-degree polynomial function; there are 3 $x$-intercepts and 2 turning points, so the degree is odd and at least 3. Because of the end behavior, we know that the lead coefficient must be negative.

### Example 3.3.12: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

**Solution**

The $y$-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The $y$-intercept is $(0, 0)$.

The $x$-intercepts are found by determining the zeros of the function.

$$\begin{aligned} 0 &= -4x(x+3)(x-4) \\ x &= 0 \quad \text{or} \quad x+3=0 \quad \text{or} \quad x-4=0 \\ x &= 0 \quad \text{or} \quad x=-3 \quad \text{or} \quad x=4 \end{aligned}$$

The $x$-intercepts are $(0, 0)$,$(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

### Exercise 3.3.8

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

*Answer available at the end of this section.*

## Key Equations

- general form of a polynomial function: $f(x) = a_n x^n + a_{n-1}x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

## Key Concepts

- A power function is a variable base raised to a number power.
- The behavior of a graph as the input decreases without bound and increases without bound is called the end behavior.
- The end behavior depends on whether the power is even or odd.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function.
- A polynomial of degree $n$ will have at most $n$ $x$-intercepts and at most $n-1$ turning points.

## Glossary

**coefficient**

a nonzero real number that is multiplied by a variable raised to an exponent (only the number factor is the coefficient)

**continuous function**

a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

**degree**

the highest power of the variable that occurs in a polynomial

**end behavior**

the behavior of the graph of a function as the input decreases without bound and increases without bound

**leading coefficient**

the coefficient of the leading term

**leading term**

the term containing the highest power of the variable

**polynomial function**

a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

**power function**

a function that can be represented in the form $f(x) = kx^p$ where $k$ is a constant, the base is a variable, and the exponent, $p$, is a constant

**smooth curve**

a graph with no sharp corners

**term of a polynomial function**

any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} ... + a_2 x^2 + a_1 x + a_0$

**turning point**

the location at which the graph of a function changes direction

## Answers

**Exercise** $3.3.1$

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

**Exercise** $3.3.2$

As $x$ approaches positive or negative infinity, $f(x)$ decreases without bound: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the negative coefficient.

**Exercise** $3.3.3$

The degree is $6$. The leading term is $-x^6$. The leading coefficient is $-1$.

**Exercise** $3.3.1$

As $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to -\infty$. It has the shape of an even degree power function with a negative coefficient.

**Exercise** $3.3.7$

The leading term is $0.2x^3$, so it is a degree 3 polynomial. As $x$ approaches positive infinity, $f(x)$ increases without bound; as $x$ approaches negative infinity, $f(x)$ decreases without bound.

**Exercise** $3.3.5$

$y$-intercept $(0, 0)$; $x$-intercepts $(0, 0)$,$(-2, 0)$, and $(5, 0)$

**Exercise** $3.3.6$

There are at most 12 $x$-intercepts and at most 11 turning points.

**Exercise** $3.3.7$

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**Exercise** $3.3.8$

The $x$-intercepts are $(2,0)$, $(-1,0)$, and $(5,0)$, the $y$-intercept is $(0,2)$, and the graph has at most 2 turning points.

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# 3.4: Graphs of Polynomial Functions

**Learning Objectives**

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in Table 3.4.1.

*Table* 3.4.1

| **Year** | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | 2013 |
|---|---|---|---|---|---|---|---|---|
| **Revenues** | 52.4 | 52.8 | 51.2 | 49.5 | 48.6 | 48.6 | 48.7 | 47.1 |

The revenue can be modeled by the polynomial function

$$R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332 \tag{3.4.1}$$

where $R$ represents the revenue in millions of dollars and $t$ represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing? These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

## Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure 3.4.1 shows a graph that represents a **polynomial function** and a graph that represents a function that is not a polynomial.

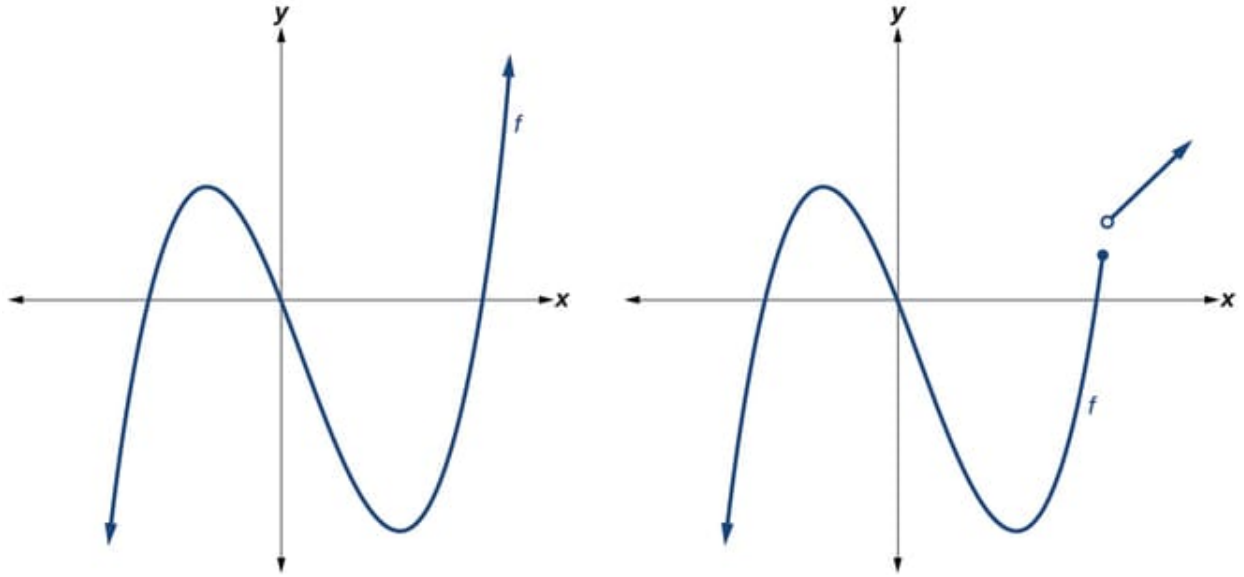



Figure 3.4.1: Graph of $f(x) = x^3 - 0.01x$.

## Example 3.4.1: Recognizing Polynomial Functions

Which of the graphs in Figure 3.4.2 represents a polynomial function?

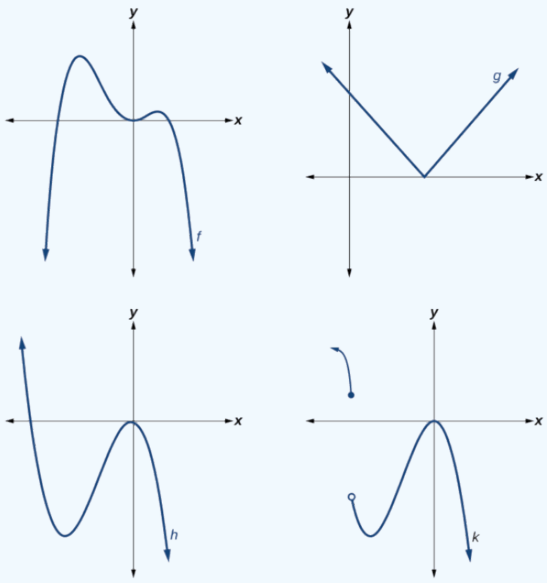



***Figure*** 3.4.2

**Solution**

- The graphs of $f$ and $h$ are graphs of polynomial functions. They are smooth and **continuous**.
- The graphs of $g$ and $k$ are graphs of functions that are not polynomials. The graph of function $g$ has a sharp corner. The graph of function $k$ is not continuous.

## Q&A

Do all polynomial functions have as their domain all real numbers?

- Yes. Any real number is a valid input for a polynomial function.

## Using Factoring to Find Zeros of Polynomial Functions

Recall that if $f$ is a polynomial function, the values of $x$ for which $f(x) = 0$ are called **zeros** of $f$. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
The polynomial is given in factored form.
Technology is used to determine the intercepts.

### HOwTO: Given a polynomial function $f$, find the x-intercepts by factoring

1. Set $f(x) = 0$.
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

### Example 3.4.2: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

**Solution**

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

$$\begin{aligned} x^6 - 3x^4 + 2x^2 &= 0 && \text{Factor out the greatest common factor.} \\ x^2(x^4 - 3x^2 + 2) &= 0 && \text{Factor the trinomial.} \\ x^2(x^2 - 1)(x^2 - 2) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x^2 &= 0 \\ x^2 &= 0 \\ x &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} (x^2 - 1) &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned} \quad \text{or} \quad \begin{aligned} (x^2 - 2) &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

.

This gives us five x-intercepts: $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(\sqrt{2}, 0)$,and $(-\sqrt{2}, 0)$ (Figure 3.4.3). We can see that this is an even function.

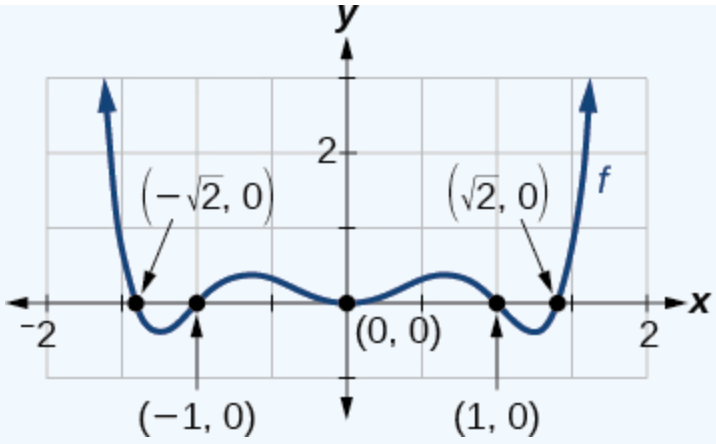



Figure 3.4.3.

## Example 3.4.3: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

**Solution**

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 && \text{Factor by grouping.} \\ x^2(x-5) - (x-5) &= 0 && \text{Factor out the common factor.} \\ (x^2-1)(x-5) &= 0 && \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x+1 &= 0 \quad \text{or} \quad x-1 = 0 \quad \text{or} \quad x-5 = 0 \\ x &= -1 \qquad\quad x = 1 \qquad\quad x = 5 \end{aligned}$$

There are three x-intercepts: $(-1, 0)$, $(1, 0)$, and $(5, 0)$ (Figure 3.4.4).

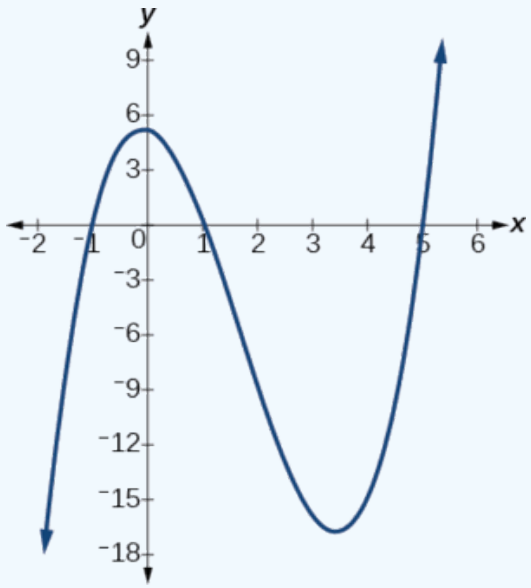



Figure 3.4.4: Graph of $f(x)$.

## Example 3.4.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x-2)^2(2x+3)$.

**Solution**

The y-intercept can be found by evaluating $g(0)$.

$$\begin{aligned} g(0) &= (0-2)^2(2(0)+3) \\ &= 12 \end{aligned}$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0 \tag{3.4.2}$$

$$\begin{aligned} (x-2)^2 = 0 \quad & \text{or} \quad (2x+3) = 0 \\ x-2 = 0 \quad & \text{or} \quad x = -\frac{3}{2} \\ x = 2 \quad & \end{aligned}$$

So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

**Analysis**

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in Figure 3.4.5.

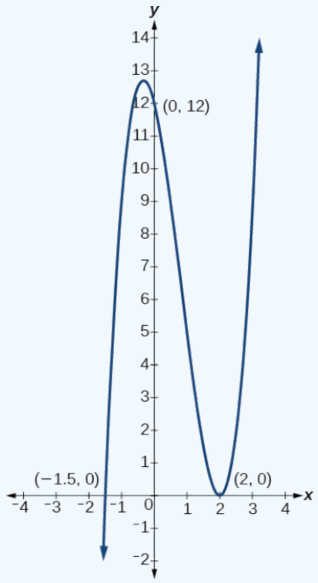



Figure 3.4.5: Graph of $g(x)$.

## Example 3.4.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

**Solution**

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 3.4.6, it appears that there are x-intercepts at $x = -3, -2, \text{ and } 1$.

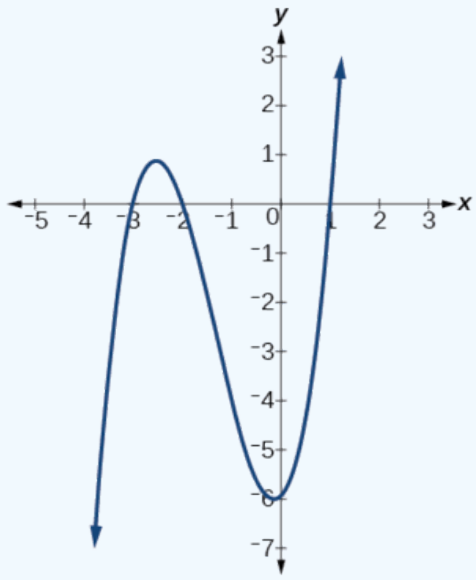



Figure 3.4.6: Graph of $h(x)$.

We can check whether these are correct by substituting these values for $x$ and verifying that

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6$, we have:

$$\begin{aligned} h(-3) &= (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0 \\ h(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0 \\ h(1) &= (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0 \end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$\begin{aligned} h(x) &= x^3 + 4x^2 + x - 6 \\ &= (x+3)(x+2)(x-1) \end{aligned}$$

**Exercise 3.4.1**

Find the y-and x-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

*Answer available at the end of this section.*

## Identifying Zeros and Their Multiplicities

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off. Suppose, for example, we graph the function

$$f(x) = (x+3)(x-2)^2(x+1)^3. \tag{3.4.3}$$

Notice in Figure 3.4.7 that the behavior of the function at each of the x-intercepts is different.

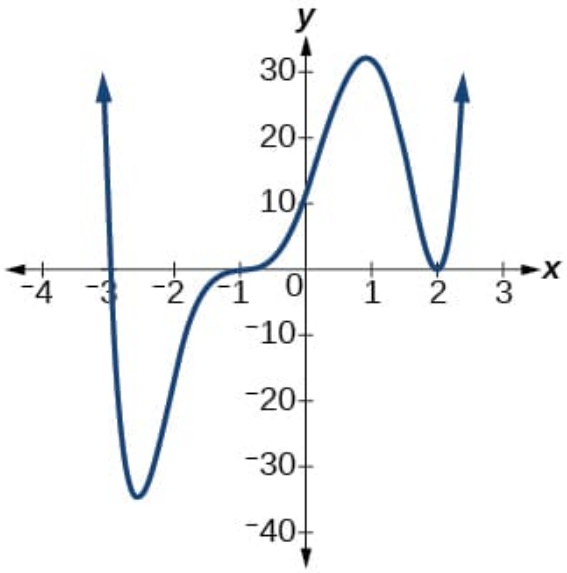



Figure 3.4.7: Identifying the behavior of the graph at an x-intercept by examining the multiplicity of the zero.

The x-intercept –3 is the solution of equation $(x+3)=0$. The graph passes directly through thex-intercept at $x=-3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The x-intercept 2 is the repeated solution of equation $(x-2)^2=0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.

$$(x-2)^2=(x-2)(x-2) \tag{3.4.4}$$

The factor is repeated, that is, the factor $(x-2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x=2$, has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept –1 is the repeated solution of factor $(x+1)^3=0$.The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near the intercept as the toolkit function $f(x)=x^3$. We call this a triple zero, or a zero with multiplicity 3.

For **zeros** with even multiplicities, the graphs touch or are tangent to the x-axis. For zeros with odd multiplicities, the graphs cross or intersect the x-axis. See Figure 3.4.8 for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

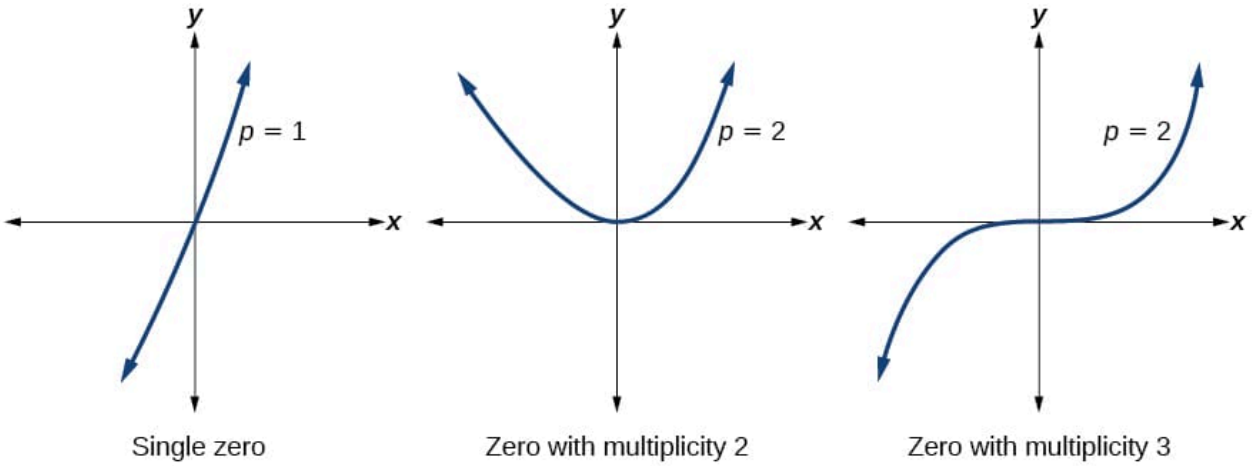



Figure 3.4.8: Three graphs showing three different polynomial functions with multiplicity 1, 2, and 3.

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

### Graphical Behavior of Polynomials at x-Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercepth is determined by the power $p$. We say that $x=h$ is a zero of **multiplicity** $p$.

The graph of a polynomial function will touch the x-axis at zeros with even multiplicities. The graph will cross the x-axis at zeros with odd multiplicities.

The sum of the multiplicities is the degree of the polynomial function.

### HOWTO: Given a graph of a polynomial function of degree $n$, identify the zeros and their multiplicities

1. If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is $n$.

### Example 3.4.6: Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure 3.4.9 to identify the zeros of the function and their possible multiplicities.

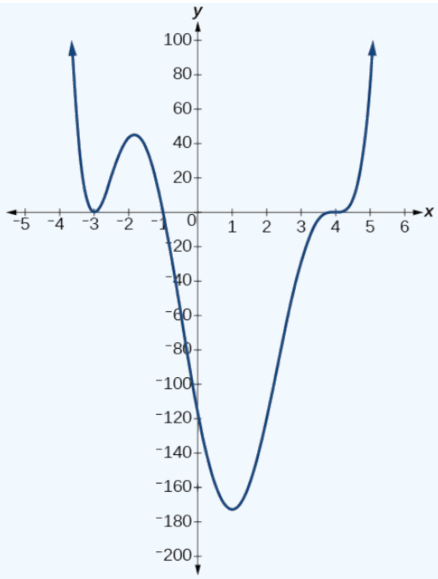



Figure 3.4.9: Graph of a polynomial function with degree 5.

**Solution**

The polynomial function is of degree $n$. The sum of the multiplicities must be $n$.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of –3 has multiplicity 2.

The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 4$.The graph crosses the x-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3 and that the sum of the multiplicities is likely 6.

## Exercise 3.4.2

Use the graph of the function of degree 5 in Figure 3.4.10 to identify the zeros of the function and their multiplicities.

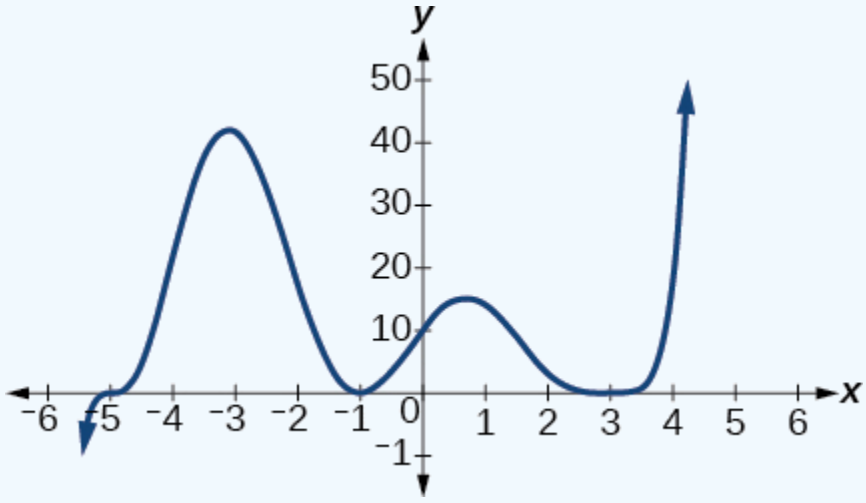



***Figure** 3.4.10**:** Graph of a polynomial function with degree 5.*

*Answer available at the end of this section.*

## Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{3.4.5}$$

will either ultimately rise or fall as $x$ increases without bound and will either rise or fall as $x$ decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as $x$ increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as $x$ decreases without bound, $f(x)$ also decreases without bound; as $x$ increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure 3.4.11 summarizes all four cases.

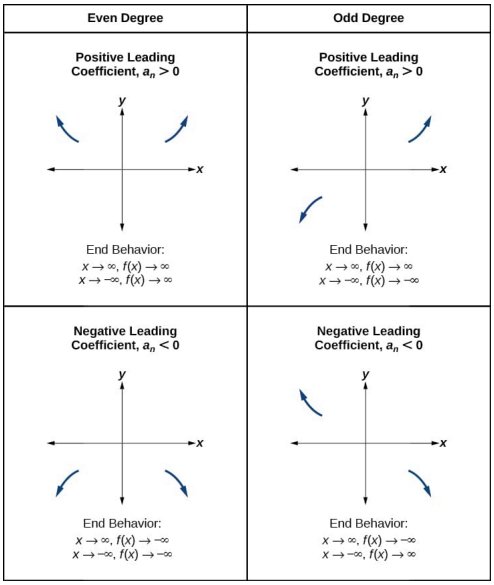



Figure 3.4.11.

## Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure 3.4.12. The graph has three turning points.

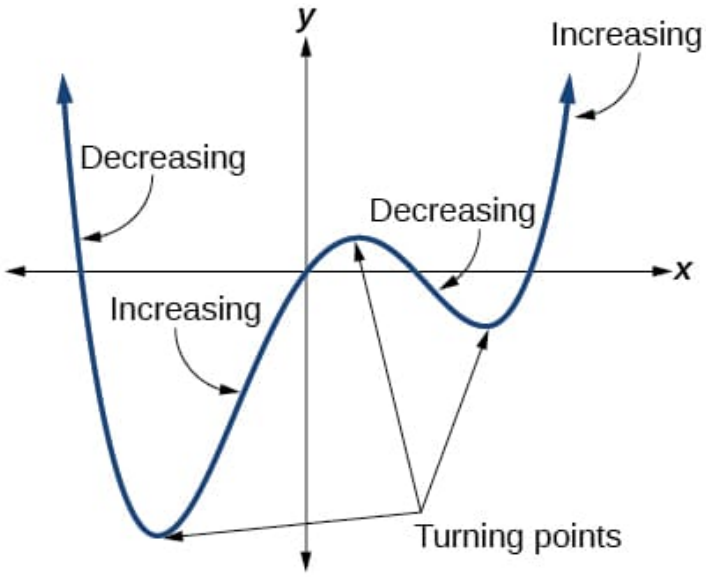



Figure 3.4.12: Graph of $f(x) = x^4 - x^3 - 4x^2 + 4x$

This function $f$ is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

### Definition: Interpreting Turning Points

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). A polynomial of degree $n$ will have at most $n - 1$ turning points.

### Example 3.4.7: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
b. $f(x) = -(x - 1)^2(1 + 2x^2)$

**Solution**

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$

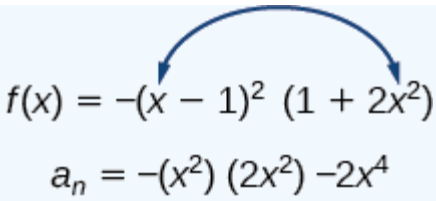
First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum number of turning points is $5 - 1 = 4$.

b. $f(x) = -(x - 1)^2(1 + 2x^2)$

First, identify the leading term of the polynomial function if the function were expanded.




Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4 - 1 = 3$.

## Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

### Howto: Given a polynomial function, sketch the graph

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

### Example $3.4.8$: Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

**Solution**

This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$,the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The y-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure $3.4.13$.

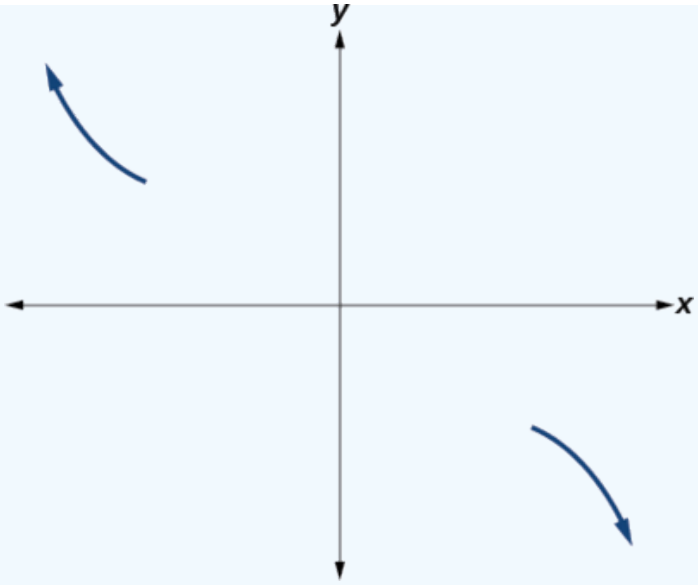



Figure 3.4.13: Showing the distribution for the leading term.

To sketch this, we consider that:

- As $x \to -\infty$ the function $f(x) \to \infty$,so we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of thex-axis, so the function must start increasing.
- At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure 3.4.14.

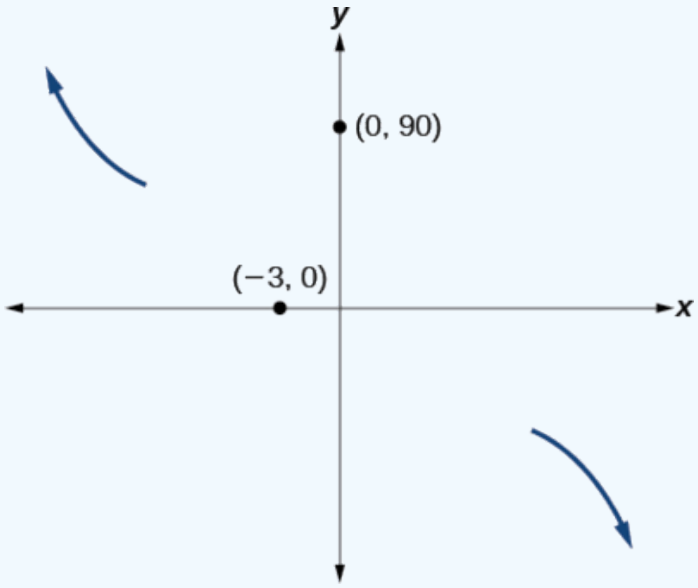



Figure 3.4.14: Graph of the end behavior and intercepts, $(-3, 0)$ and $(0, 90)$, for the function $f(x) = -2(x+3)^2(x-5)$.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5, 0)$. See Figure 3.4.15.

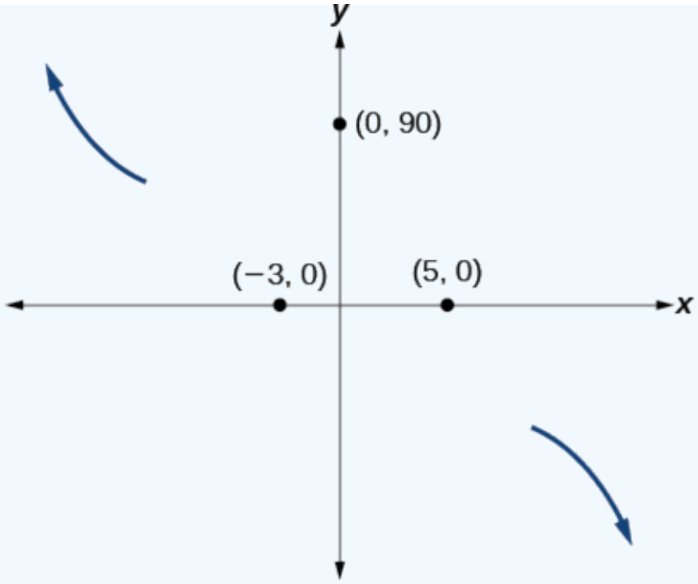



Figure 3.4.15: Graph of the end behavior and intercepts, $(-3,0)$, $(0,90)$ and $(5,0)$, for the function $f(x)=-2(x+3)^2(x-5)$.

As $x \to \infty$ the function $f(x) \to -\infty$,

so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure 3.4.16, and verify that the resulting graph looks like our sketch in Figure 3.4.15.

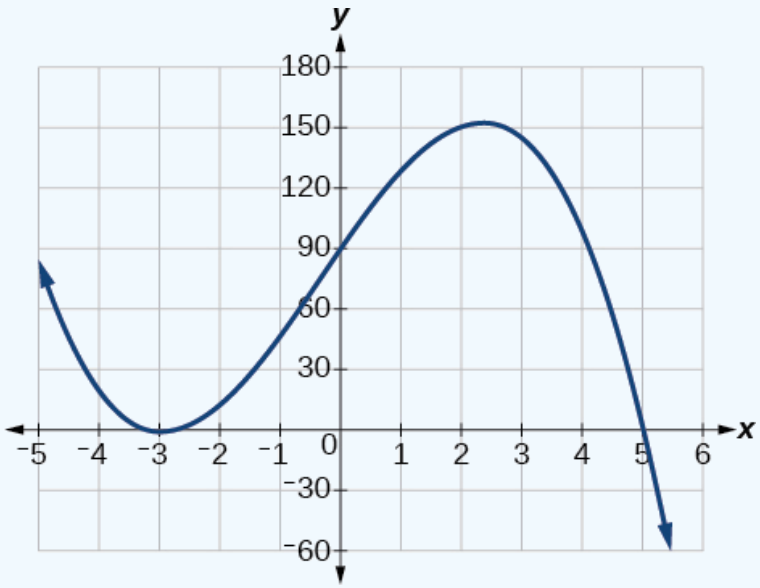



***Figure*** 3.4.16***:*** *The complete graph of the polynomial function* $f(x)=-2(x+3)^2(x-5)$.

## Exercise 3.4.8

Sketch a graph of $f(x)=\frac{1}{4}x(x-1)^4(x+3)^3$.

*Answer available at the end of this section.*

# Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function $f$ whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers $a$ and $b$ in the domain of $f$,if $a < b$ and $f(a) \neq f(b)$,then the function $f$ takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function $f$ at $x = a$ lies above the x-axis and another point at $x = b$ lies below thex-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. Call this point $(c, f(c))$.This means that we are assured there is a solution $c$ where $f(c) = 0$.

In other words, the **Intermediate Value Theorem** tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. Figure $3.4.18$ shows that there is a zero between $a$ and $b$.

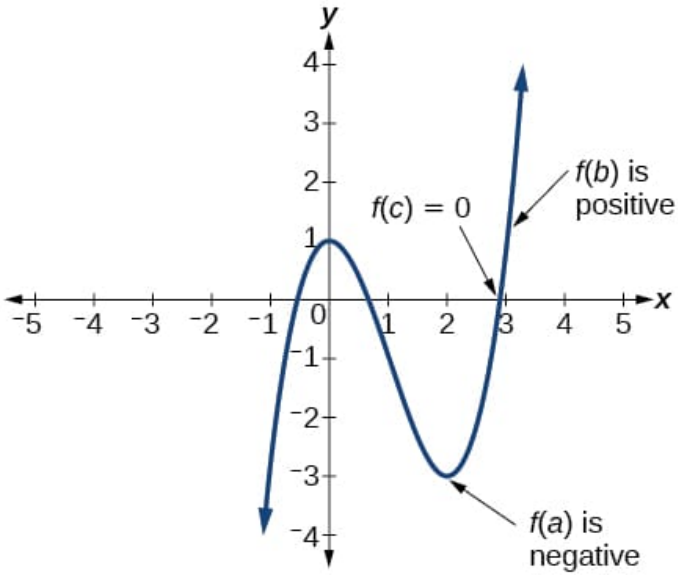



Figure $3.4.18$: Using the Intermediate Value Theorem to show there exists a zero.

## Definition: Intermediate Value Theorem

Let $f$ be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value $c$ between $a$ and $b$ for which $f(c) = 0$.

## Example $3.4.9$: Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

**Solution**

As a start, evaluate $f(x)$ at the integer values $x = 1,\ 2,\ 3,\ \text{and}\ 4$ (Table $3.4.2$).

*Table* $3.4.2$

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 5 | 0 | -3 | 2 |

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.

We have shown that there are at least two real zeros between $x = 1$ and $x = 4$.

**Analysis**

We can also see on the graph of the function in Figure $3.4.19$ that there are two real zeros between $x = 1$ and $x = 4$.

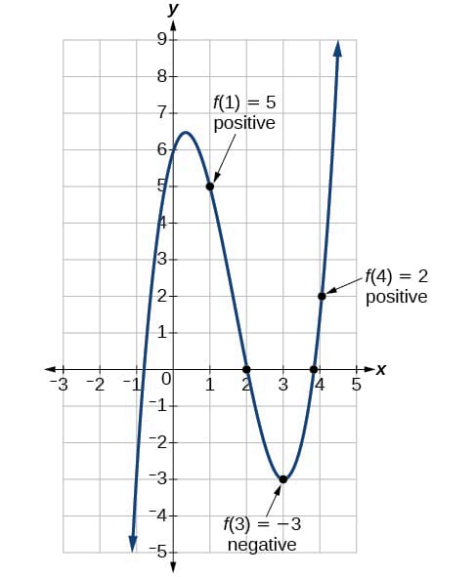



Figure $3.4.19$.

**Exercise $3.4.4$**

Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

*Answer available at the end of this section.*

## Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a **polynomial function** written in factored form will have an x-intercept where each factor is equal to zero, we can form a function that will pass through a set of x-intercepts by introducing a corresponding set of factors.

**Note: Factored Form of Polynomials**

If a polynomial of lowest degree $p$ has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form: $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$ where the powers $p_i$ on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor $a$ can be determined given a value of the function other than the x-intercept.

**How To: Given a graph of a polynomial function, write a formula for the function.**

1. Identify the x-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the x-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

## Example 3.4.10: Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 3.4.20.

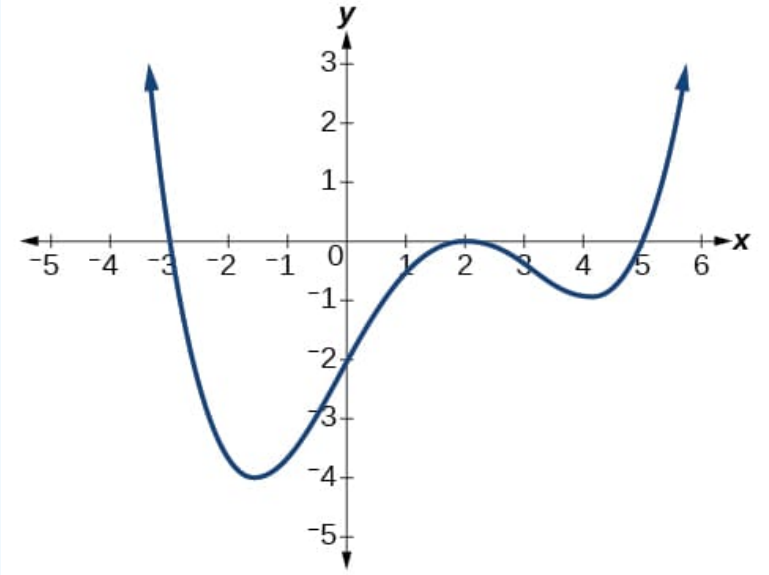



Figure 3.4.20.

**Solution**

This graph has three x-intercepts: $x = -3,\ 2,\ \text{and}\ 5$. The y-intercept is located at $(0, 2)$.At $x = -3$ and $x = 5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x) = a(x + 3)(x - 2)^2(x - 5) \tag{3.4.6}$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, -2)$, to solve for $a$.

$$\begin{aligned} f(0) &= a(0 + 3)(0 - 2)^2(0 - 5) \\ -2 &= a(0 + 3)(0 - 2)^2(0 - 5) \\ -2 &= -60a \\ a &= \frac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x + 3)(x - 2)^2(x - 5)$.

## Exercise 3.4.5

Given the graph shown in Figure 3.4.21, write a formula for the function shown.

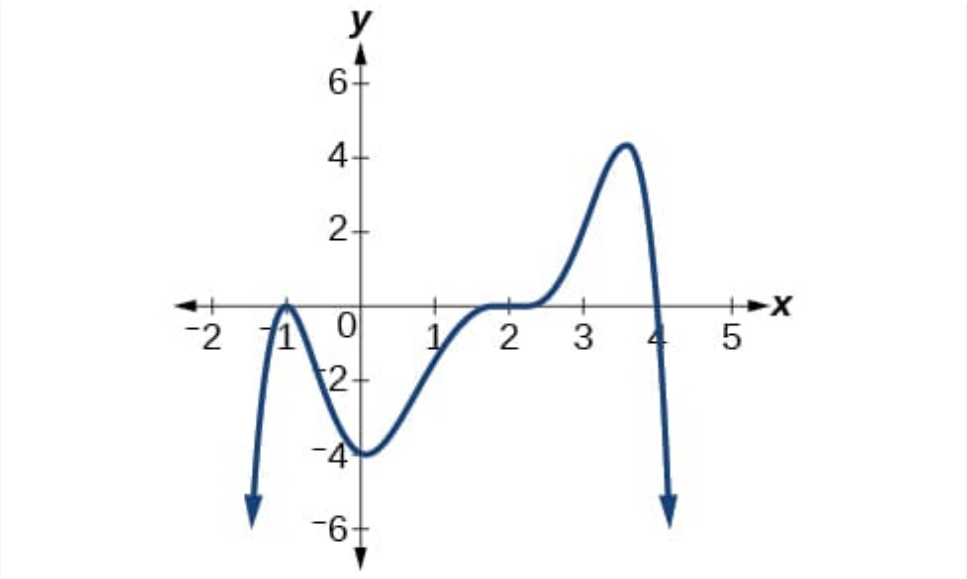



Figure 3.4.21.

*Answer available at the end of this section.*

## Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

### Note: Local and Global Extrema

A **local maximum** or **local minimum** at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$.If a function has a local maximum at $a$, then $f(a) \geq f(x)$for all $x$ in an open interval around $x = a$. If a function has a local minimum at $a$, then $f(a) \leq f(x)$for all $x$ in an open interval around $x = a$.

A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at $a$, then $f(a) \geq f(x)$ for all $x$. If a function has a global minimum at $a$, then $f(a) \leq f(x)$ for all $x$.

We can see the difference between local and global extrema in Figure 3.4.22.

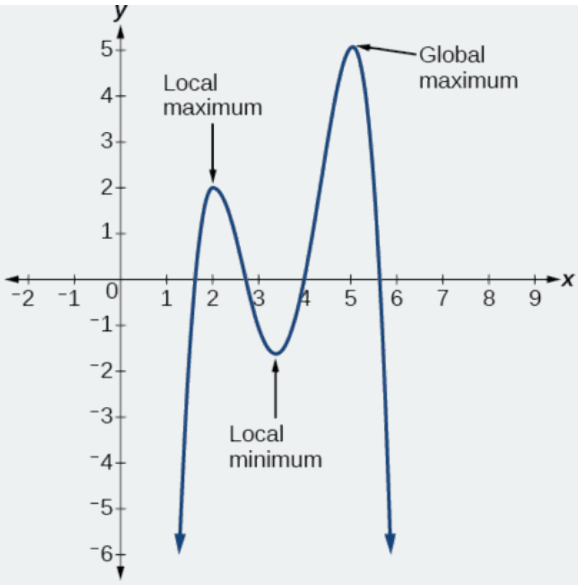



Figure 3.4.22: Graph of an even-degree polynomial that denotes the local maximum and minimum and the global maximum.

**Do all polynomial functions have a global minimum or maximum?**

*No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.*

## Example 3.4.11: Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

**Solution**

We will start this problem by drawing a picture like that in Figure 3.4.23, labeling the width of the cut-out squares with a variable, $w$.

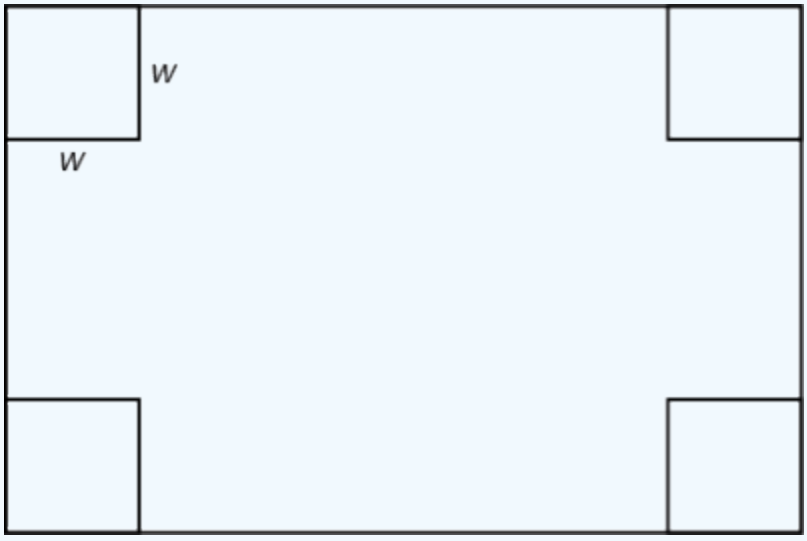



Figure 3.4.23: Diagram of a rectangle with four squares at the corners.

Notice that after a square is cut out from each end, it leaves $a(14-2w)$ cm by $(20-2w)$ cm rectangle for the base of the box, and the box will be $w$ cm tall. This gives the volume

$$\begin{aligned} V(w) &= (20-2w)(14-2w)w \\ &= 280w - 68w^2 + 4w^3 \end{aligned}$$

Notice, since the factors are $20–2w$, $14–2w$, and $w$, the three zeros are $10,\ 7$, and $0,)respectively.Becauseaheightof\(0$ cm is not reasonable, we consider the only the zeros $10$ and $7$. The shortest side is $14$ and we are cutting off two squares, so values $w$ may take on are greater than zero or less than $7$. This means we will restrict the domain of this function to $0 < w < 7$.Using technology to sketch the graph of $V(w)$ on this reasonable domain, we get a graph like that in Figure $3.4.24$. We can use this graph to estimate the maximum value for the volume, restricted to values for $w$ that are reasonable for this problem—values from 0 to 7.

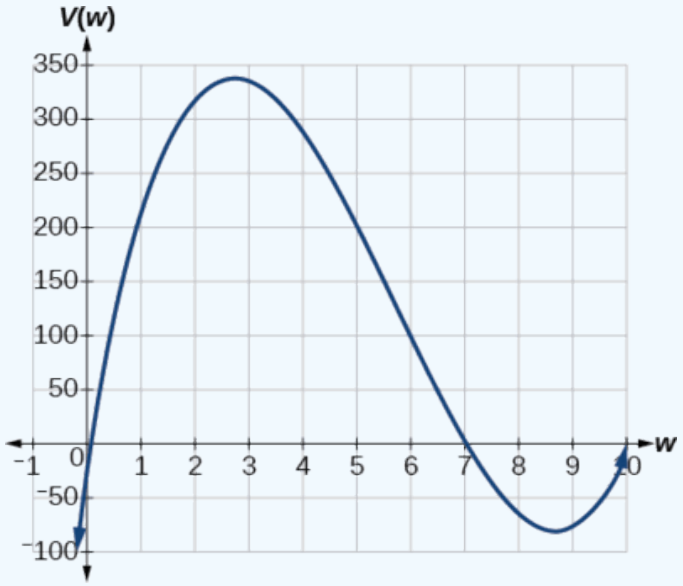



Figure $3.4.24$: Graph of $V(w) = (20-2w)(14-2w)w$

From this graph, we turn our focus to only the portion on the reasonable domain, $[0, 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce Figure $3.4.25$.

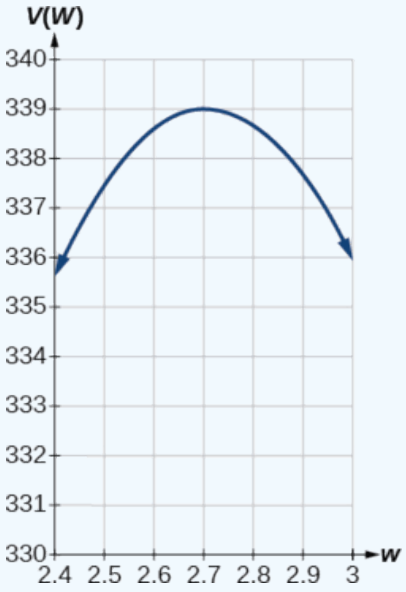



Figure $3.4.25$: Graph of $V(w) = (20-2w)(14-2w)w$.

From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

**Exercise 3.4.1**

Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

*Answer available at the end of this section.*

## Key Concepts

- Polynomial functions of degree 2 or more are smooth, continuous functions.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero.
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis.
- The multiplicity of a zero determines how the graph behaves at the x-intercepts.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree $n$ has at most $n-1$ turning points.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points.
- Graphing a polynomial function helps to estimate local and global extremas.
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value $c$ between $a$ and $b$ for which $f(c) = 0$.

## Glossary

**global maximum**
highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all $x$.

**global minimum**
lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all $x$.

**Intermediate Value Theorem**
for two numbers $a$ and $b$ in the domain of $f$, if $a < b$ and $f(a) \neq f(b)$, then the functionf takes on every value between $f(a)$ and $f(b)$ ; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis

**multiplicity**
the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x-h)^p$, $x = h$ is a zero of multiplicity $p$.

## Answers

**Exercise** 3.4.1

- y-intercept $(0, 0)$;
- x-intercepts $(0, 0)$, $(-5, 0)$, $(2, 0)$, and $(3, 0)$

**Exercise** 3.4.2

The graph has a zero of –5 with multiplicity 1, a zero of –1 with multiplicity 2, and a zero of 3 with even multiplicity.

**Exercise** 3.4.8

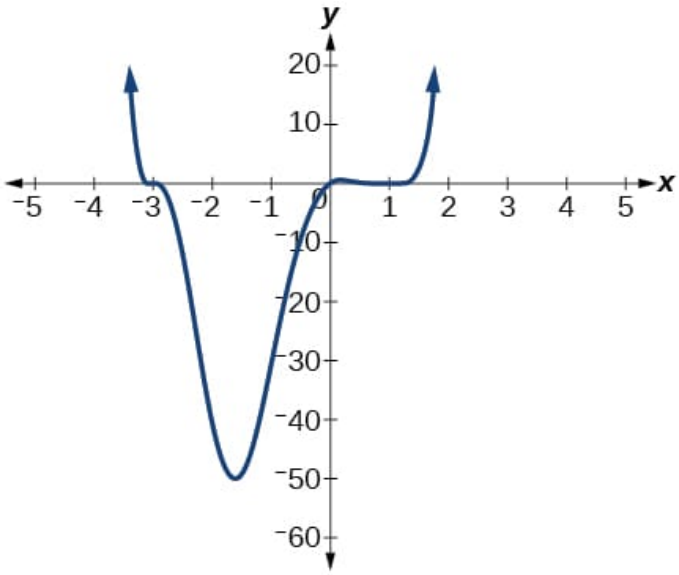



Figure 3.4.17: Graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$

**Exercise** 3.4.4

Because $f$ is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x = 1$ and $x = 2$.

**Exercise** 3.4.5

$f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

**Exercise** 3.4.1

The minimum occurs at approximately the point $(0, -6.5)$,
and the maximum occurs at approximately the point $(3.5, 7)$.

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# 3.5: Dividing Polynomials

## Learning Objectives

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length $61.5$ meters (m), width $40$ m, and height $30$ m.[1]

We can easily find the volume using elementary geometry.

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 61.5 \cdot 40 \cdot 30 \\ &= 73,800 \end{aligned}$$

So the volume is $73,800$ cubic meters ($m^3$).



Figure $3.5.1$: Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

Suppose we knew the volume, length, and width. We could divide to find the height.

$$\begin{aligned} h &= \frac{V}{l \cdot w} \\ &= \frac{73,800}{61.5 \cdot 40} \\ &= 30 \end{aligned}$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$; the width is given by $x - 2$.

To find the height of the solid, we can use polynomial division, which is the focus of this section.

# Using Long Division to Divide Polynomials

We are familiar with the **long division** algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

**Long Division**

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ -15\downarrow \\ \hline 28 \\ -27 \\ \hline 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor} \cdot \text{quotient}) + \text{remainder} \\ 178 &= (3 \cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$x+2\overline{)2x^3-3x^2+4x+5}$ Set up the division problem.

$2x^2$ $2x^3$ divided by $x$ is $2x^2$.
$x+2\overline{)2x^3-3x^2+4x+5}$

$2x^2$
$x+2\overline{)2x^3-3x^2+4x+5}$
$\underline{-(2x^3+4x^2)}$ Multiply $x+2$ by $2x^2$ and subtract.
$-7x^2+4x$ Bring down the next term.

$2x^2-7x$ $-7x^2$ divided by $x$ is $-7x$.
$x+2\overline{)2x^3-3x^2+4x+5}$
$\underline{-(2x^3+4x^2)}$
$-7x^2+4x$
$\underline{-(-7x^2-14x)}$ Multiply $x+2$ by $-7x$.
$18x+5$ Subtract and bring down the next term.

$2x^2-7x+18$ $18x$ divided by $x$ is 18.
$x+2\overline{)2x^3-3x^2+4x+5}$
$\underline{-(2x^3+4x^2)}$
$-7x^2+4x$
$\underline{-(-7x^2-14x)}$
$18x+5$
$\underline{-(18x+36)}$ Multiply $x+2$ by 18.
$-31$ Subtract.

We have found

$$\frac{2x^3-3x^2+4x+5}{x+2}=2x^2-7x+18-\frac{31}{x+2}$$

or

$$2x^3-3x^2+4x+5=(x+2)(2x^2-7x+18)-31$$

We can identify the dividend, the **divisor**, the **quotient**, and the **remainder**.

Writing the result in this manner illustrates the Division Algorithm.

## The Division Algorithm

The Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x)=d(x)q(x)+r(x) \tag{3.5.1}$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

If $r(x)=0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

## Given a polynomial and a binomial, use long division to divide the polynomial by the binomial

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.

4. Subtract the bottom **binomial** from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

## Example 3.5.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

**Solution**

$$
\begin{array}{rl}
x+1\,\overline{)\,5x^2+3x-2} & \text{Set up division problem.} \\
\quad 5x & \\
x+1\,\overline{)\,5x^2+3x-2} & 5x^2 \text{ divided by } x \text{ is } 5x. \\
\quad 5x & \\
x+1\,\overline{)\,5x^2+3x-2} & \text{Multiply } x+1 \text{ by } 5x. \\
\quad \underline{-(5x^2+5x)} & \text{Subtract.} \\
\quad -2x-2 & \\
\quad 5x-2 & \text{Bring down the next term.} \\
x+1\,\overline{)\,5x^2+3x-2} & -2x \text{ divided by } x \text{ is } -2. \\
\quad \underline{-(5x^2+5x)} & \\
\quad -2x-2 & \\
\quad \underline{-(-2x-2)} & \text{Multiply } x+1 \text{ by } -2. \\
\quad 0 & \text{Subtract.}
\end{array}
$$

The quotient is $5x - 2$. The remainder is $0$. We write the result as

$$\frac{5x^2 + 3x - 2}{x + 1} = 5x - 2$$

or

$$5x^2 + 3x - 2 = (x + 1)(5x - 2)$$

**Analysis**

This division problem had a remainder of $0$. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

## Example 3.5.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

**Solution**

$$
\begin{array}{rl}
2x^2+\ 5x-\ 7 & 6x^3 \text{ divided by } 3x \text{ is } 2x^2. \\
3x-2\overline{)6x^3+11x^2-31x+15} & \\
\underline{-(6x^3-4x^2)} & \text{Multiply } 3x-2 \text{ by } 2x^2. \\
15x^2-31x & \text{Subtract. Bring down next term. } 15x^2 \text{ divided by } 3x \text{ is } 5x. \\
\underline{-(15x^2-10x)} & \text{Multiply } 3x-2 \text{ by } 5x. \\
-21x+15 & \text{Subtract. Bring down next term. } -21x \text{ divided by } 3x \text{ is } -7. \\
\underline{-(-21x+14)} & \text{Multiply } 3x-2 \text{ by } -7. \\
1 & \text{Subtract. The remainder is } 1.
\end{array}
$$

There is a remainder of 1. We can express the result as:

$$\frac{6x^3+11x^2-31x+15}{3x-2}=2x^2+5x-7+\frac{1}{3x-2}$$

**Analysis**

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x-2)(2x^2+5x-7)+1=6x^3+11x^2-31x+15$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is $1$

**Try It! 3.5.2**

Divide $16x^3-12x^2+20x-3$ by $4x+5$.

*Solution available at the end of this section.*

## Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is $1$.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm.

The final form of the process looked like this:

$$
\begin{array}{r}
2x^2-7x+18 \\
x+2\overline{)2x^3-3x^2+4x+5} \\
\underline{-(2x^3+4x^2)} \\
-7x^2+4x \\
\underline{-(-7x^2-14x)} \\
18x+\ 5 \\
\underline{-(18x+36)} \\
-31
\end{array}
$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & 5 \\ & -2 & -4 & & \\ \hline & & -7 & 14 & \\ & & & 18 & -36 \\ \hline & & & & -31 \end{array}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2–7x+18$ and the remainder is $–31$. The process will be made more clear in Example $3.5.3$.

## Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x-k$. In *synthetic division*, only the coefficients are used in the division process.

## Given two polynomials, use synthetic division to divide

1. Write $k$ for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by $k$. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by $k$. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

## Example $3.5.3$: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2-3x-36$ by $x-3$.

**Solution**

Begin by setting up the synthetic division. Write $k$ and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ \hline & & & \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by $k$.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by $k$.Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x + 12$. The remainder is 0. So $x - 3$ is a factor of the original polynomial.

**Analysis**

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x - 3)(5x + 12) + 0 = 5x^2 - 3x - 36$$

## Example 3.5.4: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

**Solution**

The binomial divisor is $x + 2$ so $k = -2$. Add each column, multiply the result by –2, and repeat until the last column is reached.

Erroneous nesting of equation structures

The result is $4x^2 + 2x - 10$.

The remainder is 0. Thus, $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

**Analysis**

The graph of the polynomial function $f(x) = 4x^3 + 10x^2 - 6x - 20$ in Figure 3.5.2 shows a zero at $x = k = -2$. This confirms that $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

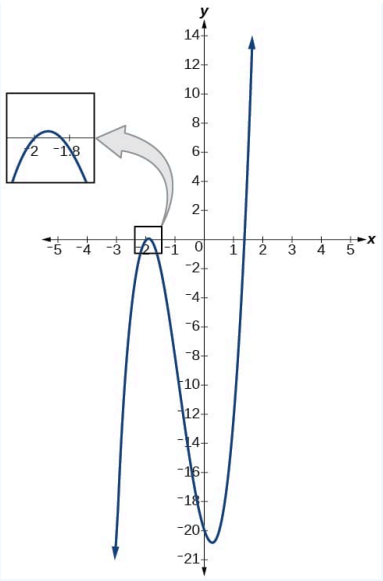



Figure 3.5.2

## Example 3.5.5: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

**Solution**

Notice there is no $x$-term. We will use a zero as the coefficient for that term.

Erroneous nesting of equation structures

The result is $-9x^3 + x^2 + 8x + 8 + \dfrac{2}{x-1}$.

## Try It! 3.5.3

Use synthetic division to divide $3x^4 + 18x^3 - 3x + 40$ by $x + 7$.

*Solution available at the end of this section.*

# Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

## Example 3.5.6: Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. Find the height $h$ of the solid.

**Solution**

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in Figure 3.5.3. Let $h$ equal the height of the box.

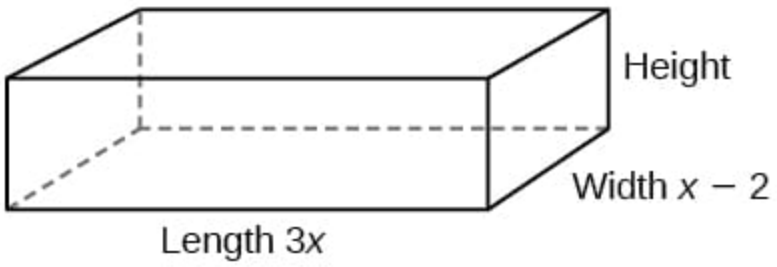



Figure 3.5.3

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$V = l \cdot w \cdot h$$
$$3x^4 - 3x^3 - 33x^2 + 54x = 3x \cdot (x-2) \cdot h$$

To solve for $h$, first divide both sides by $3x$.

$$\frac{3x \cdot (x-2) \cdot h}{3x} = \frac{3x^4 - 3x^3 - 33x^2 + 54x}{3x}$$
$$(x-2)h = \frac{x^3 - x^2 - 11x + 18}{x-2}$$

Now solve for $h$ using synthetic division.

$$h = \frac{x^3 - x^2 - 11x + 18}{x-2}$$

Erroneous nesting of equation structures

The quotient is $x^2 + x - 9$ and the remainder is $0$. The height of the solid is $x^2 + x - 9$.

**Try It! 3.5.4**

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

*Solution available at the end of this section.*

## Key Equations

Division Algorithm $f(x) = d(x)q(x) + r(x)$ where $q(x) \neq 0$

## Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x - k$.
- Polynomial division can be used to solve application problems, including area and volume.

## Footnotes

[1] National Park Service. "Lincoln Memorial Building Statistics." www.nps.gov/linc/historycultu...statistics.htm. Accessed 4/3/2014

# Glossary

**Division Algorithm**

given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

**synthetic division**

a shortcut method that can be used to divide a polynomial by a binomial of the form $x - k$

## Answers

**Try It!** 3.5.2

$$4x^2 - 8x + 15 - \frac{78}{4x + 5}$$

**Try It!** 3.5.3

$$3x^3 - 3x^2 + 21x - 150 + \frac{1090}{x + 7}$$

**Try It!** 3.5.4

$$3x^2 - 4x + 1$$

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# 3.6: Zeros of Polynomial Functions

**Learning Objectives**

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zero Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs.
- Solve real-world applications of polynomial equations

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

## Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x–k$, the remainder may be found quickly by evaluating the polynomial function at $k$, that is, $f(k)$. Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$,there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

If the divisor, $d(x)$, is $x - k$, this takes the form

$$f(x) = (x - k)q(x) + r$$

Since the divisor $x - k$

is linear, the remainder will be a constant, $r$. And, if we evaluate this for $x = k$, we have

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= 0 \cdot q(k) + r \\ &= r \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$by $x - k$.

**The Remainder Theorem**

If a polynomial $f(x)$ is divided by $x - k$,then the remainder is the value $f(k)$.



Given a polynomial function $f$, evaluate $f(x)$ at $x = k$ using the Remainder Theorem.

1. Use synthetic division to divide the polynomial by $x-k$.
2. The remainder is the value $f(k)$.

### Example 3.6.1: Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x)=6x^4-x^3-15x^2+2x-7$ at $x=2$.

**Solution**

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x-2$.

$$\begin{array}{r|rrrrr} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array} \tag{3.6.1}$$

The remainder is 25. Therefore, $f(2)=25$.

**Analysis**

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4-x^3-15x^2+2x-7 \\ f(2) &= 6(2)^4-(2)^3-15(2)^2+2(2)-7 \\ &= 25 \end{aligned}$$

### Exercise 3.6.1

se the Remainder Theorem to evaluate $f(x)=2x^5-3x^4-9x^3+8x^2+2$ at $x=-3$.

*Answer available at the end of this section.*

## Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm.

$$f(x)=(x-k)q(x)+r \tag{3.6.2}$$

If $k$ is a zero, then the remainder $r$ is $f(k)=0$ and $f(x)=(x-k)q(x)+0$ or $f(x)=(x-k)q(x)$.

Notice, written in this form, $x-k$ is a factor of $f(x)$. We can conclude if $k$ is a zero of $f(x)$, then $x-k$ is a factor of $f(x)$.

Similarly, if $x-k$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x)=(x-k)q(x)+r$ is $0$. This tells us that $k$ is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree $n$ in the complex number system will have $n$ zeros. We can use the Factor Theorem to completely factor a polynomial into the product of $n$ factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

### THE FACTOR THEOREM

According to the Factor Theorem, $k$ is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$.

**How to: Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial**

1. Use synthetic division to divide the polynomial by $(x-k)$.
2. Confirm that the remainder is $0$.
3. Write the polynomial as the product of $(x-k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

**Example 3.6.2: Using the Factor Theorem to Solve a Polynomial Equation**

Show that $(x+2)$ is a factor of $x^3-6x^2-x+30$. Find the remaining factors. Use the factors to determine the zeros of the **polynomial**.

**Solution**

We can use synthetic division to show that $(x+2)$ is a factor of the polynomial.

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array} \tag{3.6.3}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x+2)(x^2-8x+15) \tag{3.6.4}$$

We can factor the quadratic factor to write the polynomial as

$$(x+2)(x-3)(x-5) \tag{3.6.5}$$

By the Factor Theorem, the zeros of $x^3-6x^2-x+30$ are –2, 3, and 5.

**Exercise 3.6.2**

Use the Factor Theorem to find the zeros of $f(x)=x^3+4x^2-4x-16$ given that $(x-2)$ is a factor of the polynomial.

*Answer available at the end of this section.*

## Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading **coefficient** of the polynomial

Consider a quadratic function with two zeros, $x=\frac{2}{5}$ and $x=\frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.

| | |
|---|---|
| $x - \frac{2}{5} = 0$ or $x - \frac{3}{4} = 0$ | Set each factor equal to 0. |
| $5x - 2 = 0$ or $4x - 3 = 0$ | Multiply both sides of the equation to eliminate fractions. |
| $f(x) = (5x - 2)(4x - 3)$ | Create the quadratic function, multiplying the factors. |
| $f(x) = 20x^2 - 23x + 6$ | Expand the polynomial. |
| $f(x) = (5 \cdot 4)x^2 - 23x + (2 \cdot 3)$ | |

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

## THE RATIONAL ZERO THEOREM

The Rational Zero Theorem states that, if the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where $p$ is a factor of the constant term $a_0$ and $q$ is a factor of the leading coefficient $a_n$.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

## How to: Given a polynomial function $f(x)$, use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where $p$ is a factor of the constant term and $q$ is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f(\frac{p}{q})$.

## Example 3.6.3: Listing All Possible Rational Zeros

List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

**Solution**

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, −4, and the factors of the leading coefficient, 2.

The constant term is −4; the factors of −4 are $p = \pm 1, \pm 2, \pm 4$.

The leading coefficient is 2; the factors of 2 are $q = \pm 1, \pm 2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of −4 divided by one of the factors of 2.

$$\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{1}{2} \qquad \frac{p}{q} = \pm\frac{2}{1}, \pm\frac{2}{2} \qquad \frac{p}{q} = \pm\frac{4}{1}, \pm\frac{4}{2}$$

Note that $\frac{2}{2} = 1$ and $\frac{4}{2} = 2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q} = \frac{\text{Factors of the last}}{\text{Factors of the first}} = \pm 1, \pm 2, \pm 4, \pm\frac{1}{2}$$

### Example 3.6.4: Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x) = 2x^3 + x^2 - 4x + 1$.

**Solution**

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then $p$ is a factor of 1 and $q$ is a factor of 2.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$= \frac{factor\ of\ 1}{factor\ of\ 2}$$

The factors of 1 are ±1 and the factors of 2 are ±1 and ±2. The possible values for $\frac{p}{q}$ are ±1 and $\pm\frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for $x$ in $f(x)$.

$$f(-1) = 2(-1)^3 + (-1)^2 - 4(-1) + 1 = 4 \tag{3.6.6}$$

$$f(1) = 1(1)^3 + (1)^2 - 4(1) + 1 = 0 \tag{3.6.7}$$

$$f(-\frac{1}{2}) = 2(-\frac{1}{2})^3 + (-\frac{1}{2})^2 - 4(-\frac{1}{2}) + 1 = 3 \tag{3.6.8}$$

$$f(\frac{1}{2}) = 2(\frac{1}{2})^3 + (\frac{1}{2})^2 - 4(\frac{1}{2}) + 1 = -\frac{1}{2} \tag{3.6.9}$$

Of those, $-1$,$-\frac{1}{2}$, and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

### Exercise 3.6.3

Use the Rational Zero Theorem to find the rational zeros of $f(x) = x^3 - 5x^2 + 2x + 1$.

*Answer available at the end of this section.*

## Finding the Zeros of Polynomial Functions

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use **synthetic division** repeatedly to determine all of the **zeros** of a polynomial function.

### How to: Given a polynomial function $f$, use synthetic division to find its zeros.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.
4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

### Example 3.6.5: Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x) = 4x^3 - 3x - 1$.

**Solution**

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then $p$ is a factor of –1 and $q$ is a factor of 4.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$= \frac{factor\ of\ -1}{factor\ of\ 4}$$

The factors of –1 are ±1 and the factors of 4 are ±1,±2, and ±4. The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}$, and $\pm\frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x-1)(4x^2+4x+1)$$

The quadratic is a perfect square. $f(x)$ can be written as

$$(x-1)(2x+1)^2$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$2x+1=0$$
$$x=-\frac{1}{2}$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

**Analysis**

Look at the graph of the function $f$ in Figure 3.6.1. Notice, at $x=-0.5$, the graph bounces off the x-axis, indicating the even multiplicity (2,4,6...) for the zero –0.5. At $x=1$, the graph crosses the x-axis, indicating the odd multiplicity (1,3,5...) for the zero $x=1$.

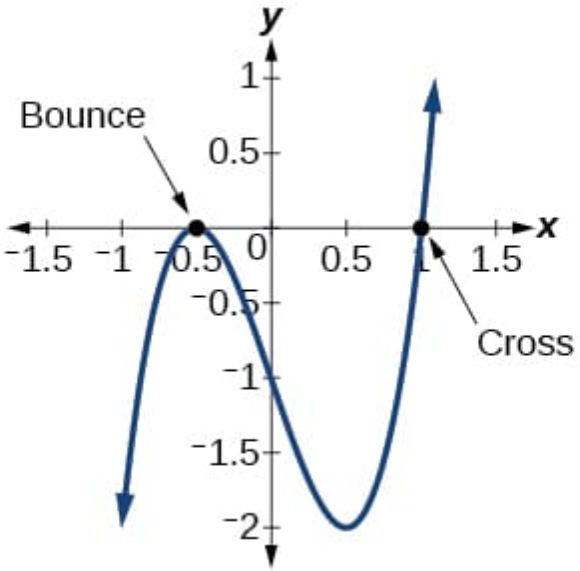



Figure 3.6.1.

# Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose $f$ is a polynomial function of degree four, and $f(x) = 0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it $c_1$. By the Factor Theorem, we can write $f(x)$ as a product of $x - c_1$ and a polynomial quotient. Since $x - c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it $c_2$. So we can write the polynomial quotient as a product of $x - c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

## THE FUNDAMENTAL THEOREM OF ALGEBRA

The *Fundamental Theorem of Algebra* states that, if $f(x)$ is a polynomial of degree $n > 0$, then $f(x)$ has at least one complex zero.

We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n > 0$, and a is a non-zero real number, then $f(x)$ has exactly $n$ linear factors

$$f(x) = a(x - c_1)(x - c_2)...(x - c_n) \tag{3.6.10}$$

where $c_1, c_2,...,c_n$ are complex numbers. Therefore, $f(x)$ has $n$ roots if we allow for multiplicities.

## Q&A: Does every polynomial have at least one imaginary zero?

No. Real numbers are a subset of complex numbers, but not the other way around. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

## Example 3.6.6: Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of $f(x) = 3x^3 + 9x^2 + x + 3$.

**Solution**

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then $p$ is a factor of 3 and $q$ is a factor of 3.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$= \frac{factor\ of\ 3}{factor\ of\ 3}$$

The factors of 3 are ±1 and ±3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are ±3,±1, and $\pm\frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –3.

Dividing by $(x + 3)$ gives a remainder of 0, so –3 is a zero of the function. The polynomial can be written as

$$(x + 3)(3x^2 + 1)$$

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$3x^2 + 1 = 0$$
$$x^2 = -\frac{1}{3}$$
$$x = \pm - \sqrt{\frac{1}{3}}$$
$$= \pm \frac{i\sqrt{3}}{3}$$

The zeros of $f(x)$ are $-3$ and $\pm\frac{i\sqrt{3}}{3}$.

**Analysis**

Look at the graph of the function $f$ in Figure 3.6.2. Notice that, at $x = -3$, the graph crosses the x-axis, indicating an odd multiplicity (1) for the zero $x = -3$. Also note the presence of the two turning points. This means that, since there is a $3^{rd}$ degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the x-intercepts for the function are shown. So either the multiplicity of $x = -3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x = -3$ is three. Either way, our result is correct.

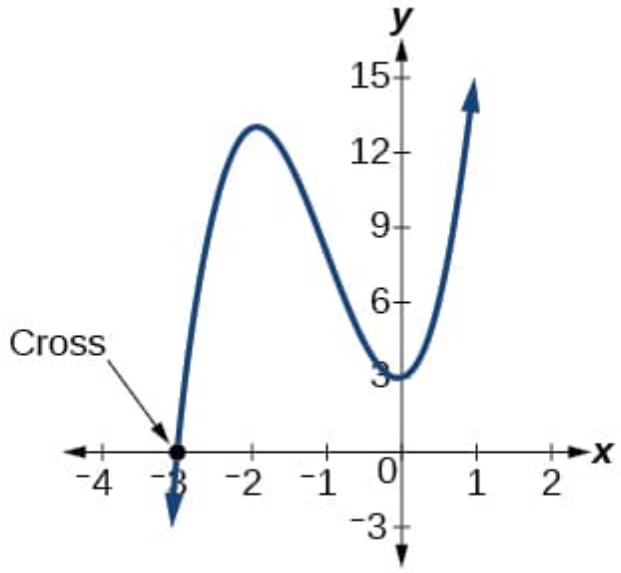



Figure 3.6.2.

3.6.4

Find the zeros of $f(x) = 2x^3 + 5x^2 - 11x + 4$.

**Solution**

The zeros are $-4$, $\frac{1}{2}$, and $1$.

## Using the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the **Fundamental Theorem of Algebra**, as we stated above, is that a polynomial function of degree n will have $n$ zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into $n$ factors. The **Linear Factorization Theorem** tells us that a polynomial function will have the same number of factors as its

degree, and that each factor will be in the form $(x-c)$, where c is a complex number.

Let $f$ be a polynomial function with real coefficients, and suppose $a+bi$, $b \neq 0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x-(a+bi)$ is a factor of $f(x)$. For $f$ to have real coefficients, $x-(a-bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x-(a-bi)$, when multiplied by $x-(a+bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function $f$ with real coefficients has a complex zero $a+bi$, then the complex conjugate $a-bi$ must also be a zero of $f(x)$. This is called the **Complex Conjugate Theorem**.

### COMPLEX CONJUGATE THEOREM

According to the Linear Factorization Theorem, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where $c$ is a complex number.

If the polynomial function $f$ has real coefficients and a complex zero in the form $a+bi$, then the complex conjugate of the zero, $a-bi$, is also a zero.

### How to

**Given the zeros of a polynomial function $f$ and a point $(c, f(c))$ on the graph of $f$, use the Linear Factorization Theorem to find the polynomial function.**

1. Use the zeros to construct the linear factors of the polynomial.
2. Multiply the linear factors to expand the polynomial.
3. Substitute $(c, f(c))$ into the function to determine the leading coefficient.
4. Simplify.

### Example 3.6.7: Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of –3, 2, $i$, such that $f(-2)=100$.

**Solution**

Because $x=i$ is a zero, by the Complex Conjugate Theorem $x=-i$ is also a zero. The polynomial must have factors of $(x+3)$, $(x-2)$, $(x-i)$, and $(x+i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$f(x)=a(x+3)(x-2)(x-i)(x+i) \tag{3.6.11}$$
$$f(x)=a(x^2+x-6)(x^2+1) \tag{3.6.12}$$
$$f(x)=a(x^4+x^3-5x^2+x-6) \tag{3.6.13}$$

We need to find $a$ to ensure $f(-2)=100$. Substitute $x=-2$ and $f(-2)=100$ into $f(x)$.

$$100=a((-2)^4+(-2)^3-5(-2)^2+(-2)-6) \tag{3.6.14}$$
$$100=a(-20) \tag{3.6.15}$$
$$-5=a \tag{3.6.16}$$

So the polynomial function is

$$f(x)=-5(x^4+x^3-5x^2+x-6) \tag{3.6.17}$$

or

$$f(x)=-5x^4-5x^3+25x^2-5x+30 \tag{3.6.18}$$

Analysis

We found that both $i$ and $-i$ were zeros, but only one of these zeros needed to be given. If $i$ is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of $i$.

Q&A

**If $2+3i$ were given as a zero of a polynomial with real coefficients, would $2-3i$ also need to be a zero?**

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

3.6.5

Find a third degree polynomial with real coefficients that has zeros of $5$ and $-2i$ such that $f(1)=10$.

**Solution**

$f(x)=-\frac{1}{2}x^3+\frac{5}{2}x^2-2x+10$

## Using Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x)=x^4+x^3+x^2+x-1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x)=(-x)^4+(-x)^3+(-x)^2+(-x)-1$$
$$f(-x)=+x^4-x^3+x^2-x-1$$

In this case, $f(-x)$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

DESCARTES' RULE OF SIGNS

According to *Descartes' Rule of Signs*, if we let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.

## Example 3.6.8: Using Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$.

**Solution**

Begin by determining the number of sign changes.

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine $f(-x)$ to determine the number of negative real roots.

$$f(-x) = -(-x)^4 - 3(-x)^3 + 6(-x)^2 - 4(-x) - 12 \tag{3.6.19}$$
$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12 \tag{3.6.20}$$

$$f(-x) = \underbrace{-x^4}_{} + 3x^3 + 6x^2 \underbrace{+4x}_{} - 12$$

Again, there are two sign changes, so there are either 2 or 0 negative real roots.

There are four possibilities, as we can see in Table 3.6.1.

*Table* 3.6.1

| Positive Real Zeros | Negative Real Zeros | Complex Zeros | Total Zeros |
|---|---|---|---|
| 2 | 2 | 0 | 4 |
| 2 | 0 | 2 | 4 |
| 0 | 2 | 2 | 4 |
| 0 | 0 | 4 | 4 |

**Analysis**

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See Figure 3.6.3. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

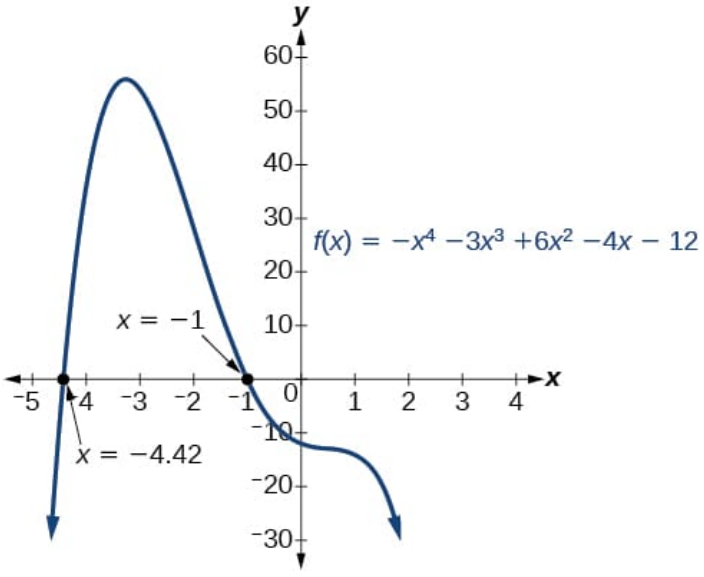



Figure 3.6.3.

3.6.6

Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12$. Use a graph to verify the numbers of positive and negative real zeros for the function.

**Solution**

There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

## Solving Real-World Applications

We have now introduced a variety of tools for solving polynomial equations. Let's use these tools to solve the bakery problem from the beginning of the section.

Example 3.6.9

Solving Polynomial Equations

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

**Solution**

Begin by writing an equation for the volume of the cake. The volume of a rectangular solid is given by $V = lwh$. We were given that the length must be four inches longer than the width, so we can express the length of the cake as $l = w + 4$. We were given that the height of the cake is one-third of the width, so we can express the height of the cake as $h = \frac{1}{3}w$. Let's write the volume of the cake in terms of width of the cake.

$$V = (w + 4)(w)(\frac{1}{3}w) \tag{3.6.21}$$

$$V = \frac{1}{3}w^3 + \frac{4}{3}w^2 \tag{3.6.22}$$

Substitute the given volume into this equation.

$351 = 13w^3 + 43w^2$ Substitute 351 for V. $1053 = w^3 + 4w^2$ Multiply both sides by 3. $0 = w^3 + 7w^2 - 1053$ Subtract 1053 from both sides.

Descartes' rule of signs tells us there is one positive solution. The Rational Zero Theorem tells us that the possible rational zeros are $\pm 1, \pm 3, \pm 9, \pm 13, \pm 27, \pm 39, \pm 81, \pm 117, \pm 351$, and $\pm 1053$. We can use synthetic division to test these possible zeros. Only positive numbers make sense as dimensions for a cake, so we need not test any negative values. Let's begin by testing values that make the most sense as dimensions for a small sheet cake. Use synthetic division to check $x = 1$.

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 4 | 0 | −1053 |
| | | 1 | 5 | 5 |
| | 1 | 5 | 5 | −1048 |

Since 1 is not a solution, we will check $x = 3$.

$$\begin{array}{r|rrrr} 3 & 1 & 4 & 0 & -1053 \\ & & 3 & 21 & 63 \\ \hline & 1 & 7 & 21 & -990 \end{array}$$

Since 3 is not a solution either, we will test $x = 9$.

$$\begin{array}{r|rrrr} 9 & 1 & 4 & 0 & -1053 \\ & & 9 & 117 & 1053 \\ \hline & 1 & 13 & 117 & 0 \end{array}$$

Synthetic division gives a remainder of 0, so 9 is a solution to the equation. We can use the relationships between the width and the other dimensions to determine the length and height of the sheet cake pan.

$l = w + 4 = 9 + 4 = 13$ and $h = \frac{1}{3}w = \frac{1}{3}(9) = 3$

The sheet cake pan should have dimensions 13 inches by 9 inches by 3 inches

**3.6.7**

A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?

**Solution**

3 meters by 4 meters by 7 meters

**Media**

Access these online resources for additional instruction and practice with zeros of polynomial functions.

- Real Zeros, Factors, and Graphs of Polynomial Functions
- Complex Factorization Theorem
- Find the Zeros of a Polynomial Function
- Find the Zeros of a Polynomial Function 2
- Find the Zeros of a Polynomial Function 3

## Key Concepts

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x - k$. This is known as the Remainder Theorem. See Example 3.6.1.
- According to the Factor Theorem, $k$ is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$. See Example3.6.2.
- According to the Rational Zero Theorem, each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See Example 3.6.3 and Example 3.6.4.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See Example 3.6.5.
- According to the Fundamental Theorem, every polynomial function with degree greater than 0 has at least one complex zero. See Example 3.6.6.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x - c)$, where $c$ is a complex number. See Example 3.6.7.

- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less than the number of sign changes by an even integer. See Example 3.6.8.
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division. See Example 3.6.9.

## Glossary

**Descartes' Rule of Signs**

a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

**Factor Theorem**

$k$ is a zero of polynomial function $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$

**Fundamental Theorem of Algebra**

a polynomial function with degree greater than 0 has at least one complex zero

**Linear Factorization Theorem**

allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where $c$ is a complex number

**Rational Zero Theorem**

the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where $p$ is a factor of the constant term and $q$ is a factor of the leading coefficient.

**Remainder Theorem**

if a polynomial $f(x)$ is divided by $x-k$,then the remainder is equal to the value $f(k)$

## Answers

**Exercise** 3.6.1

$f(-3)=-412$

**Exercise** 3.6.2

The zeros are 2, –2, and –4.

**Exercise** 3.6.3

There are no rational zeros.

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# 3.7: Rational Functions

**Learning Objectives**

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, $x$, produced. This is given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing $x$ items, we would divide the cost function by the number of items, $x$. The average cost function, which yields the average cost per item for $x$ items produced, is

$$f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}$$

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

## Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure $3.7.1$, and notice some of their features.

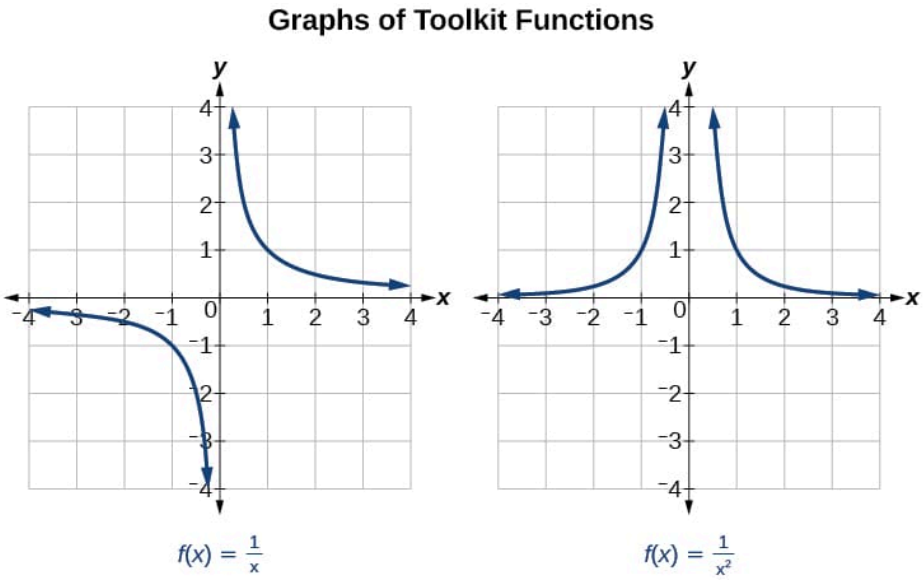



Figure $3.7.1$

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

- On the left branch of the graph, the curve approaches the $x$-axis $(y = 0)$ as $x \to -\infty$.
- As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.

- Finally, on the right branch of the graph, the curves approaches the $x$-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that $x$ or $f(x)$ is approaching a particular value (Table 3.7.1).

*Table* 3.7.1

| Symbol | Meaning |
|---|---|
| $x \to a^-$ | $x$ approaches a from the left ($x < a$ but close to $a$ ) |
| $x \to a^+$ | $x$ approaches a from the right ($x > a$ but close to $a$ ) |
| $x \to \infty$ | $x$ approaches infinity ($x$ increases without bound) |
| $x \to -\infty$ | $x$ approaches negative infinity ($x$ decreases without bound) |
| $f(x) \to \infty$ | the output approaches infinity (the output increases without bound) |
| $f(x) \to -\infty$ | the output approaches negative infinity (the output decreases without bound) |
| $f(x) \to a$ | the output approaches $a$ |

## Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 3.7.2.

*Table* 3.7.2

| $x$ | -0.1 | -0.01 | -0.001 | -0.0001 |
|---|---|---|---|---|
| $f(x) = \frac{1}{x}$ | -10 | -100 | -1000 | -10,000 |

We write in arrow notation

$$\text{as } x \to 0^-, f(x) \to -\infty$$

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 3.7.3.

*Table* 3.7.3

| $x$ | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|
| $f(x) = \frac{1}{x}$ | 10 | 100 | 1000 | 10,000 |

We write in arrow notation

$$\text{As } x \to 0^+, f(x) \to \infty.$$

See Figure 3.7.2.

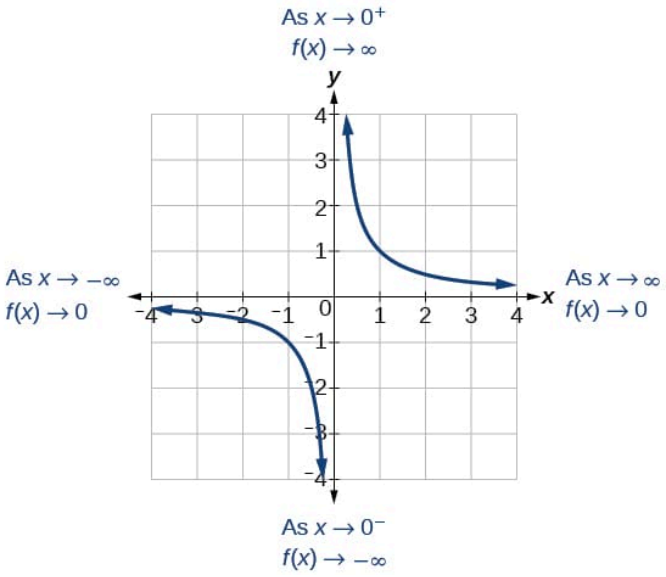



Figure 3.7.2.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero (Figure 3.7.3).

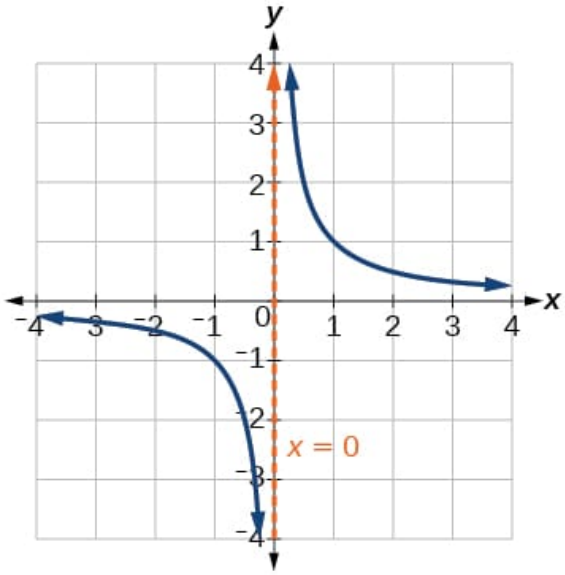



Figure 3.7.3.

**Definition: VERTICAL ASYMPTOTE**

A vertical asymptote of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach $a$. We write

As $x \to a$, $f(x) \to \infty$, or as $x \to a$, $f(x) \to -\infty$.

## End Behavior of $f(x) = \frac{1}{x}$

As the values of $x$ approach infinity, the function values approach $0$. As the values of $x$ approach negative infinity, the function values approach $0$ (Figure 3.7.4). Symbolically, using arrow notation

As $x \to \infty$, $f(x) \to 0$,and as $x \to -\infty$, $f(x) \to 0$.

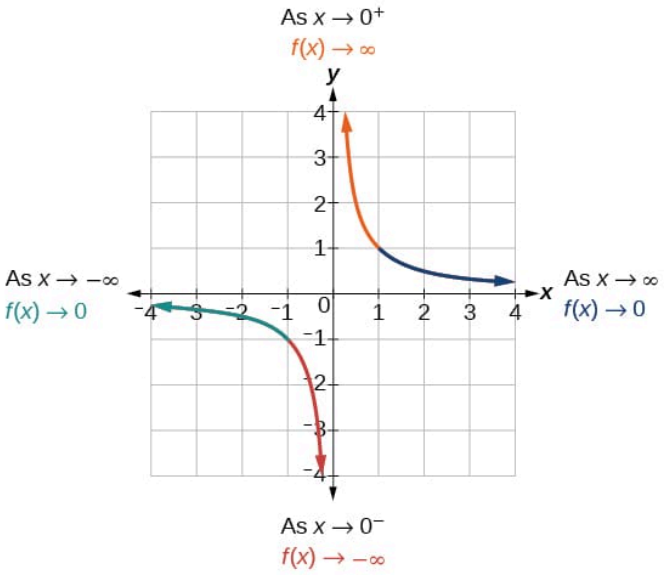



Figure 3.7.4.

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure 3.7.5.

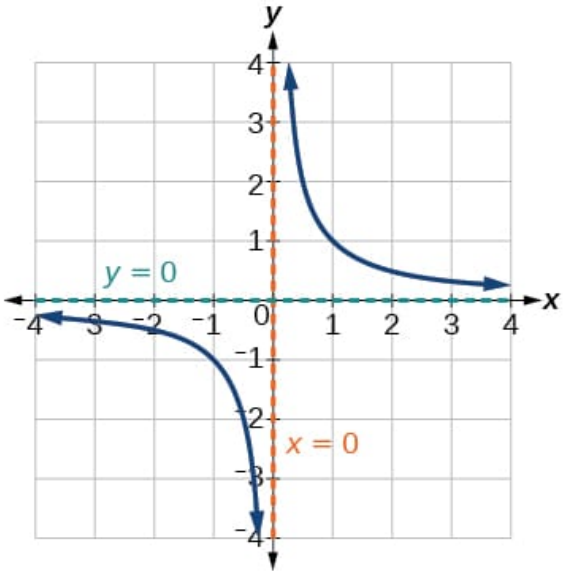



Figure 3.7.5.

## Definition: HORIZONTAL ASYMPTOTE

A *horizontal asymptote* of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty, f(x) \to b.$$

## Example 3.7.1: Using Arrow Notation.

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 3.7.6.

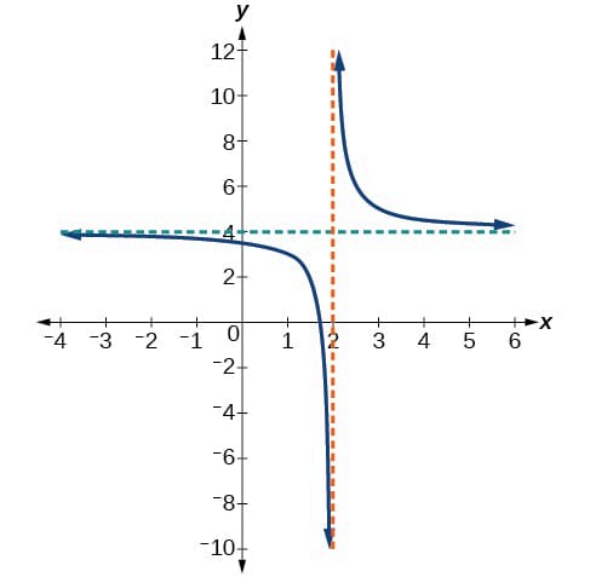



Figure 3.7.6.

**Solution**

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

$$\text{As } x \to 2^-,\ f(x) \to -\infty, \text{ and as } x \to 2^+,\ f(x) \to \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of $4$, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph levels off at $4$.

$$\text{As } x \to \infty,\ f(x) \to 4 \text{ and as } x \to -\infty,\ f(x) \to 4.$$

## Exercises 3.7.1

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

*Answer available at the end of this section.*

## Example 3.7.2: Using Transformations to Graph a Rational Function.

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

**Solution**

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3 \tag{3.7.1}$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2} \tag{3.7.2}$$

The graph of the shifted function is displayed in Figure 3.7.7.

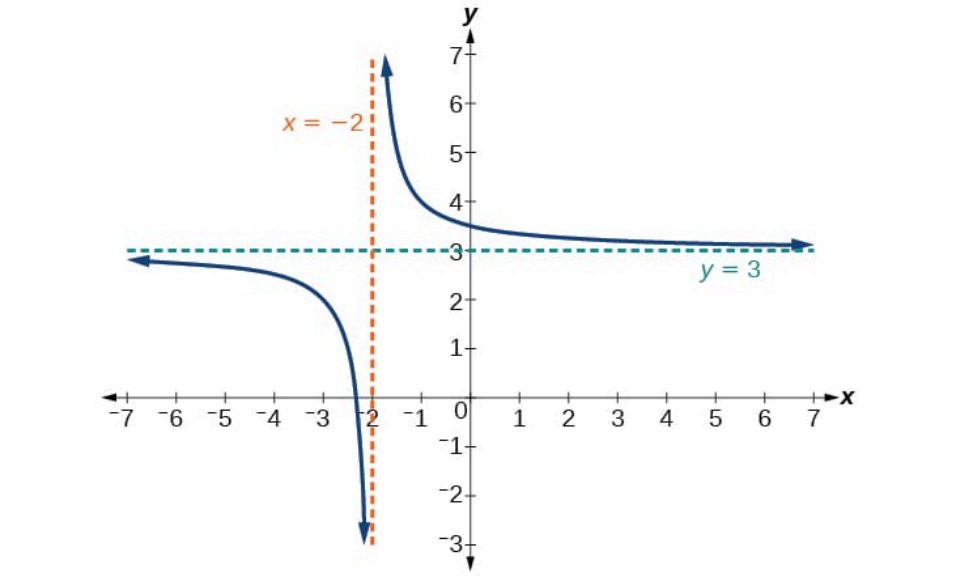



Figure 3.7.7.

Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \to -2^{-},\ f(x) \to -\infty, \text{ and as } x \to -2^{+},\ f(x) \to \infty.$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \to \pm\infty,\ f(x) \to 3.$$

**Analysis**

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

## Exercise 3.7.2

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

**Solution**

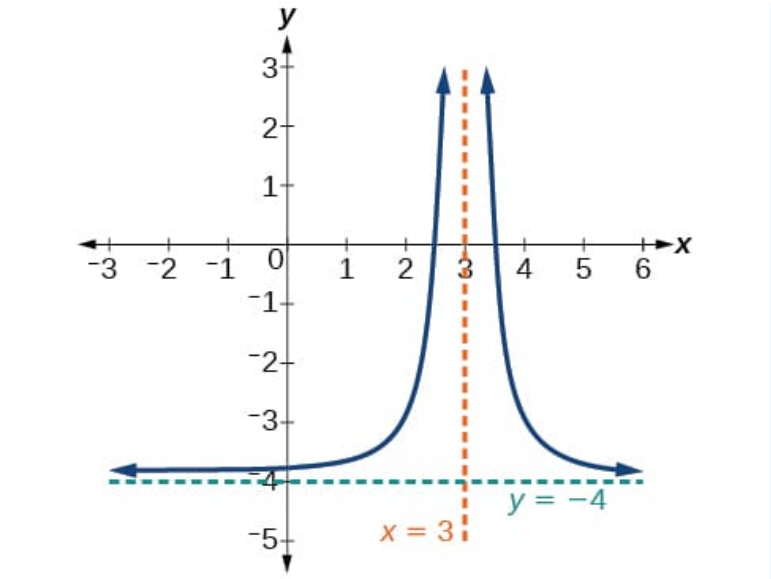



Figure 3.7.8.

The function and the asymptotes are shifted 3 units right and 4 units down. As $x \to 3$, $f(x) \to \infty$, and as $x \to \pm\infty$, $f(x) \to -4$.

The function is $f(x) = \frac{1}{(x-3)^2} - 4$.

## Solving Applied Problems Involving Rational Functions

In Example 3.7.2, we shifted a toolkit function in a way that resulted in the function $f(x) = \frac{3x+7}{x+2}$. This is an example of a rational function. A *rational function* is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

### Definition: RATIONAL FUNCTION

A ***rational function*** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0},\ Q(x) \neq 0 \tag{3.7.3}$$

### Example 3.7.3: Solving an Applied Problem Involving a Rational Function

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

**Solution**

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

water: $W(t) = 100 + 10t$ in gallons
sugar: $S(t) = 5 + 1t$ in pounds

The concentration, $C$, will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5+t}{100+10t} \tag{3.7.4}$$

The concentration after 12 minutes is given by evaluating $C(t)$ at $t = 12$.

$$C(12) = \frac{5+12}{100+10(12)} \tag{3.7.5}$$

$$= \frac{17}{220} \tag{3.7.6}$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

$$C(0) = \frac{5+0}{100+10(0)} \tag{3.7.7}$$

$$= \frac{1}{20} \tag{3.7.8}$$

Since $\frac{17}{220} \approx 0.08 > \frac{1}{20} = 0.05$, the concentration is greater after 12 minutes than at the beginning.

**Analysis**

To find the horizontal asymptote, divide the leading coefficient in the numerator by the leading coefficient in the denominator:

$$\frac{1}{10} = 0.1 \tag{3.7.9}$$

Notice the horizontal asymptote is $y = 0.1$. This means the concentration, $C$, the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

### Exercise 3.7.3

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

*Answer available at the end of this section.*

## Finding the Domains of Rational Functions

A *vertical asymptote* represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

### Definition: DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

### How To: Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

## Example 3.7.4: Finding the Domain of a Rational Function

Find the domain of $f(x) = \dfrac{x+3}{x^2-9}$.

**Solution**

Begin by setting the denominator equal to zero and solving.

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

**Analysis**

A graph of this function, as shown in Figure 3.7.9, confirms that the function is not defined when $x = \pm 3$.

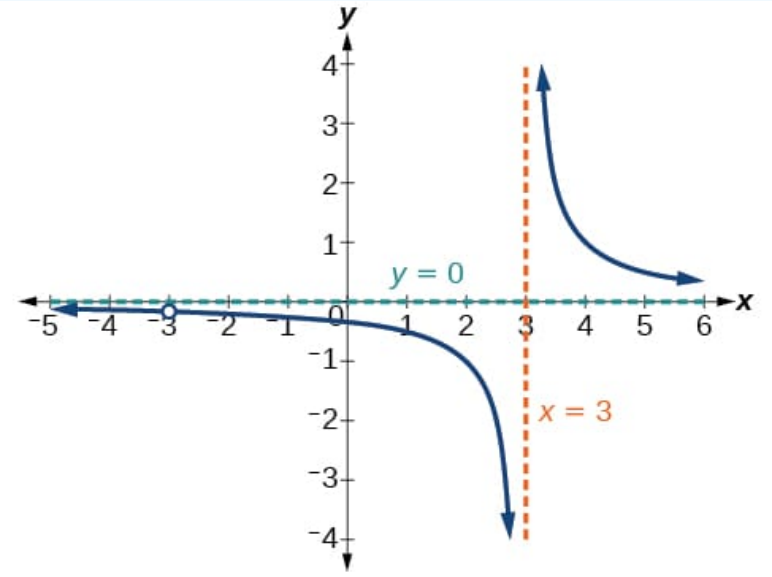



Figure 3.7.9.

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

## Exercise 3.7.4

Find the domain of $f(x) = \dfrac{4x}{5(x-1)(x-5)}$.

*Answer available at the end of this section.*

# Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

# Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

## How To: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or “holes.”

## Example 3.7.5: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \dfrac{5+2x^2}{2-x-x^2}$.

**Solution**

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$

$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$

$$x = -2,\ x = 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 3.7.10 confirms the location of the two vertical asymptotes.

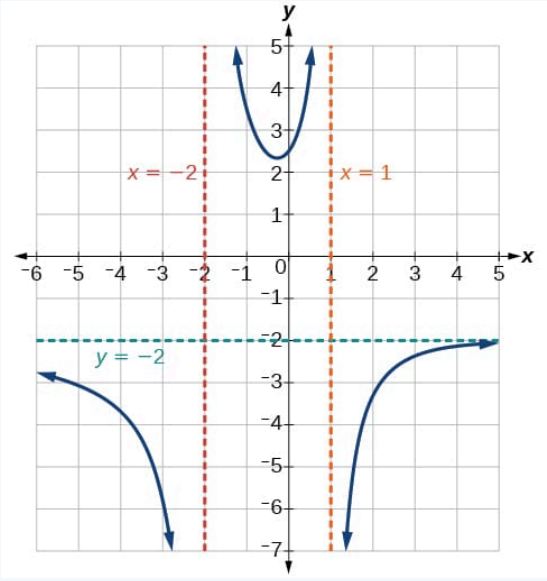


***Figure*** 3.7.10.

## Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x + 1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $(x–3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x = 3$, is the vertical asymptote. See Figure 3.7.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

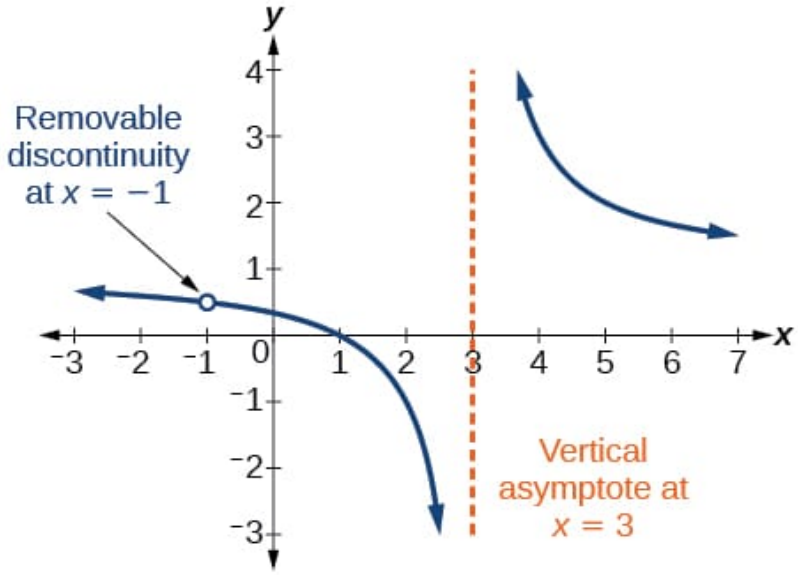



Figure 3.7.11.

### REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x = a$ if $a$ is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

### Example 3.7.6: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \dfrac{x - 2}{x^2 - 4}$.

**Solution**

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x$–2. The zero for this factor is $x = 2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x + 2$. The zero for this factor is $x = -2$. The vertical asymptote is $x = -2$. See Figure 3.7.12.

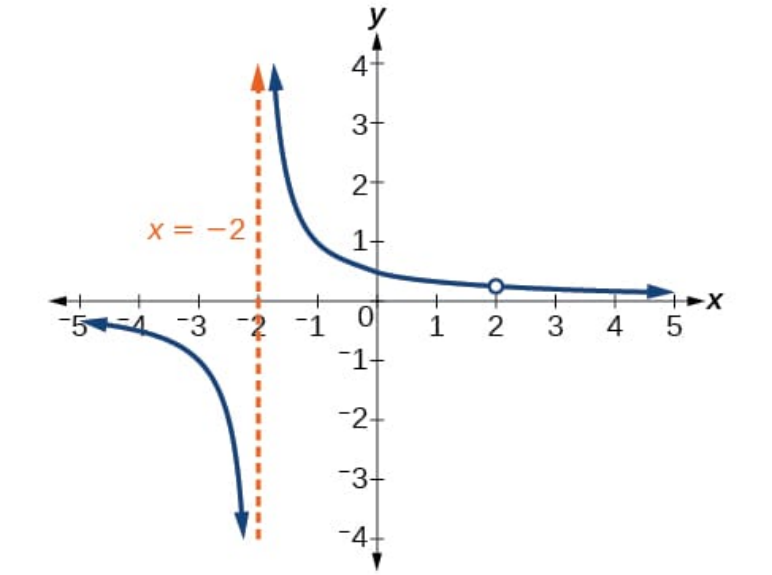



Figure 3.7.12.

The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole.

**Exercise 3.7.5**

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2 - 25}{x^3 - 6x^2 + 5x}$.

*Answer available at the end of this section.*

## Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

**Case 1**: If the degree of the denominator > degree of the numerator, there is a **horizontal asymptote** at $y = 0$.

**Example**: $f(x) = \frac{4x + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \frac{4x}{x^2} = \frac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = \frac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$. See **Figure** 3.7.13. Note that this graph crosses the horizontal asymptote.

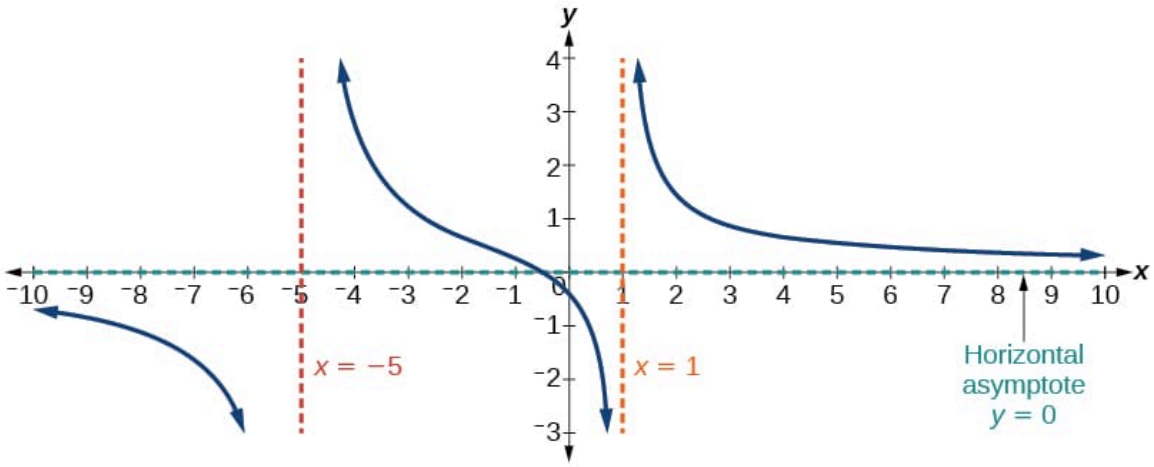



Figure 3.7.13: Horizontal asymptote $y = 0$ when $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p$ < degree of $q$.

**Case 2**: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

**Example**: $f(x) = \dfrac{3x^2 - 2x + 1}{x - 1}$

In this case, the end behavior is $f(x) \approx \dfrac{3x^2}{x} = 3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ looks like a diagonal line, and since $f$ will behave similarly to $g$, it will approach a line close to $y = 3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\dfrac{3x^2 - 2x + 1}{x - 1}$. The quotient is $3x + 1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x) = 3x + 1$. See Figure 3.7.14.

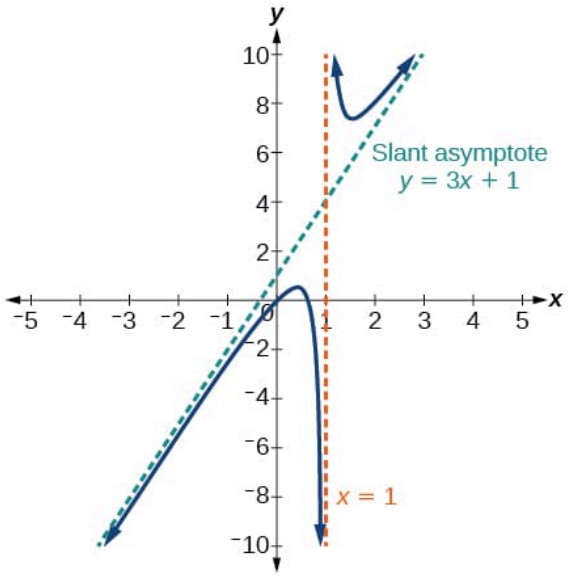



Figure 3.7.14: Slant asymptote when $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p$ >degree of $q$ by 1.

**Case 3**: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where $a_n$ and $b_n$ are respectively the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$.

**Example**: $f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See Figure 3.7.15. Note that this graph crosses the horizontal asymptote.

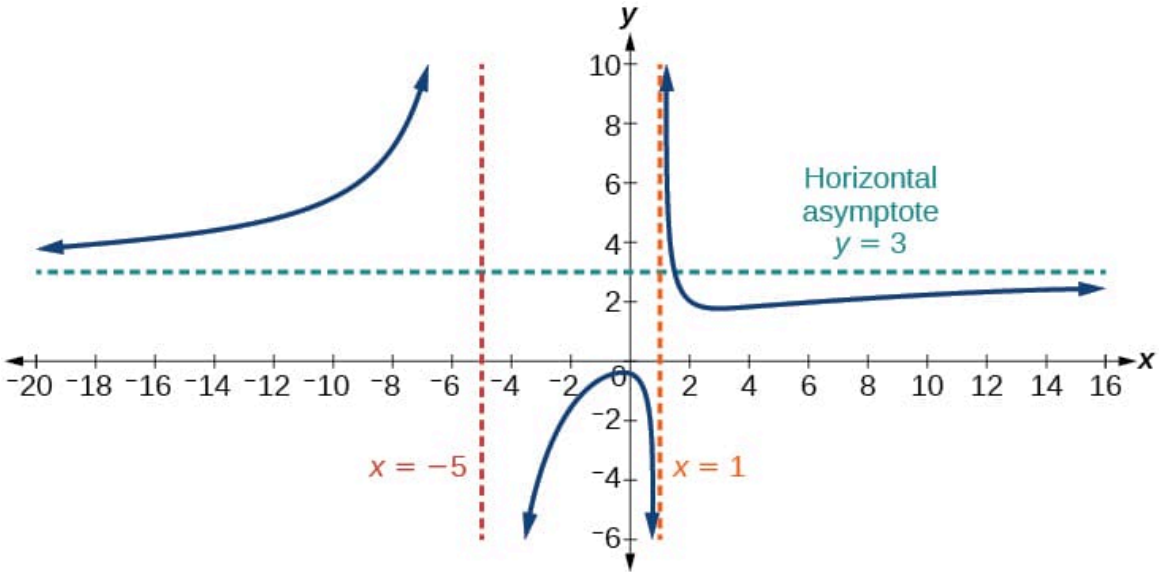



Figure 3.7.15: Horizontal asymptote when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p$ = degree of $q$.

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, f(x) \to \infty$$

## HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

## Example 3.7.7: Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$
3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$

**Solution**

For these solutions, we will use $f(x) = \dfrac{p(x)}{q(x)},\ q(x) \neq 0$.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of $p =$ degree of $q = 3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$: The degree of $p = 2$ and degree of $q = 1$. Since $p > q$ by 1, there is a slant asymptote found at $\dfrac{x^2 - 4x + 1}{x + 2}$.

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$ : The degree of $p = 2$ < degree of $q = 3$, so there is a horizontal asymptote $y = 0$.

## Example 3.7.8 Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \dfrac{5 + t}{100 + 10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

**Solution**

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is $t$, with coefficient 1. In the denominator, the leading term is 10t, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty,\ C(t) \to \tfrac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of $t$ increase, the values of $C$ will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

## Example 3.7.9: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

**Solution**

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2,$and $5$, indicating vertical asymptotes at these values.

The numerator has degree $2$, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 3.7.16.

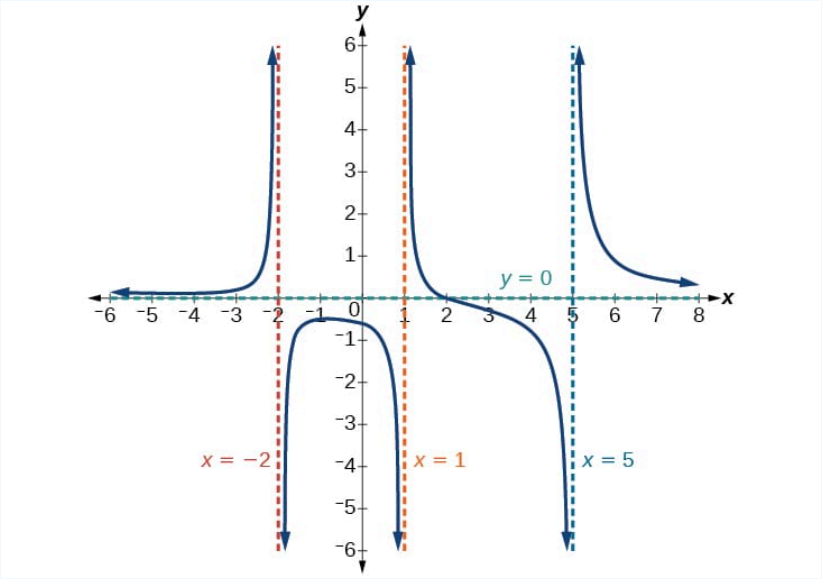



Figure 3.7.16.

## Exercise 3.7.6

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

*Answer available at the end of this section.*

## INTERCEPTS OF RATIONAL FUNCTIONS

A *rational function* will have a $y$-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a $y$ -intercept if the function is not defined at zero.

Likewise, a rational function will have $x$-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

## Example 3.7.10: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

**Solution**

We can find the y-intercept by evaluating the function at zero

$$\begin{aligned} f(0) &= \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\ &= -\frac{6}{10} \\ &= -\frac{3}{5} \\ &= -0.6 \end{aligned}$$

The x-intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \quad \text{This is zero when the numerator is zero.}$$

$$0 = (x-2)(x+3)$$

$$x = 2, x = -3$$

The y-intercept is $(0, -0.6)$, the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 3.7.17.

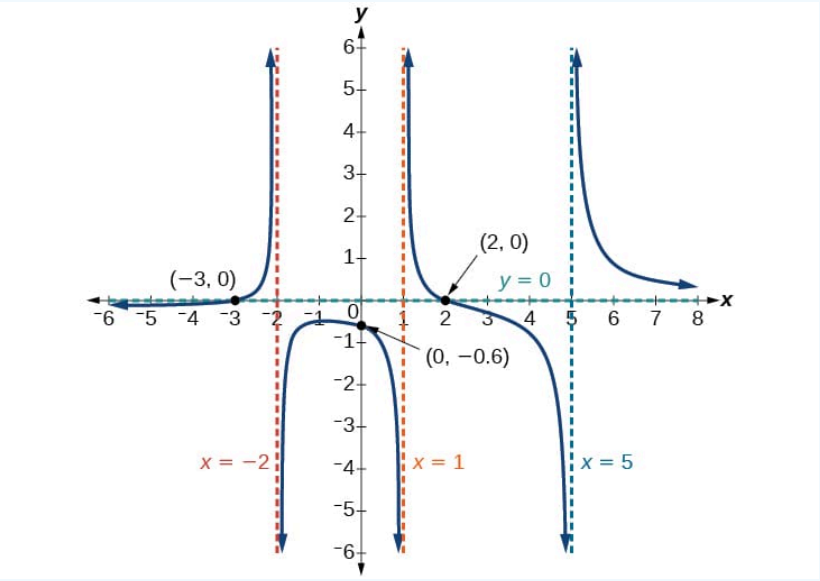



Figure 3.7.17.

## Exercise 3.7.7

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

*Answer available at the end of this section.*

## Graphing Rational Functions

In Example 3.7.10, we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 3.7.18.

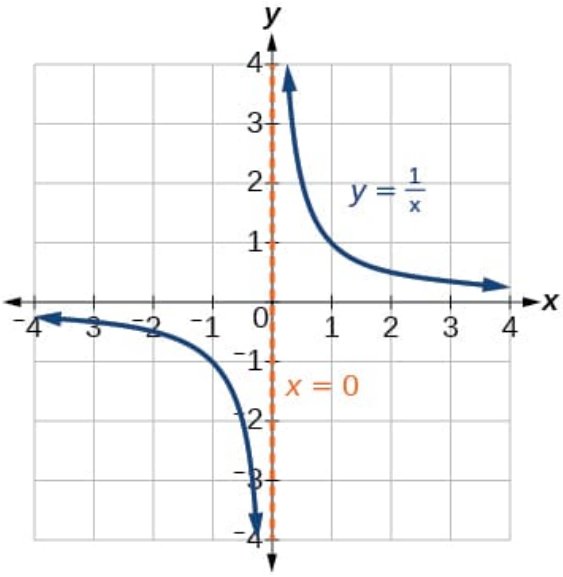



Figure 3.7.18.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 3.7.19.

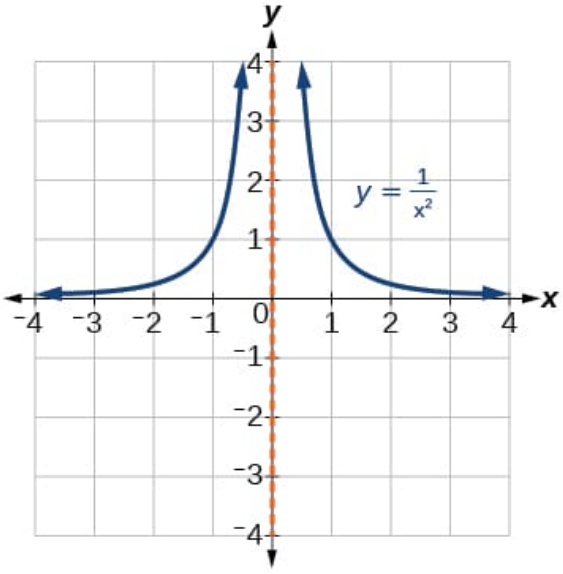



Figure 3.7.19.

For example, the graph of $f(x) = \dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 3.7.20.

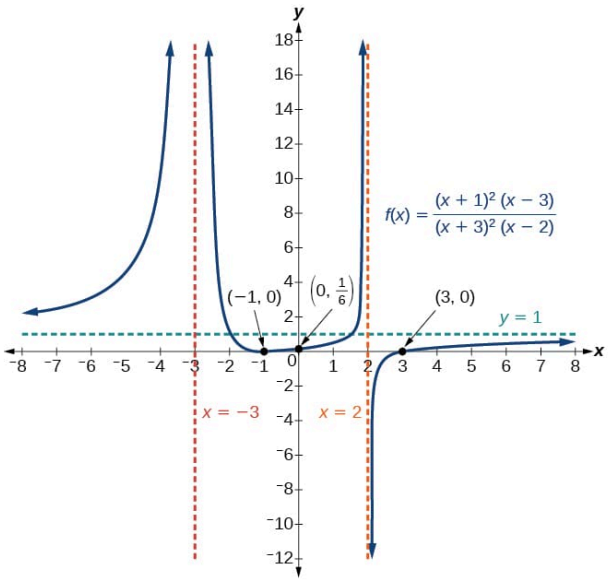



Figure 3.7.20.

- At the x-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the x-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \dfrac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x) = \dfrac{1}{x}$.

## Howto: Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

## Example 3.7.11: Graphing a Rational Function

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

**Solution**

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)} = 3$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0, 3)$ and x-intercepts at $(-2, 0)$ and $(3, 0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x + 1 = 0$ and when $x - 2 = 0$, giving us vertical asymptotes at $x = -1$ and $x = 2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 3.7.21.

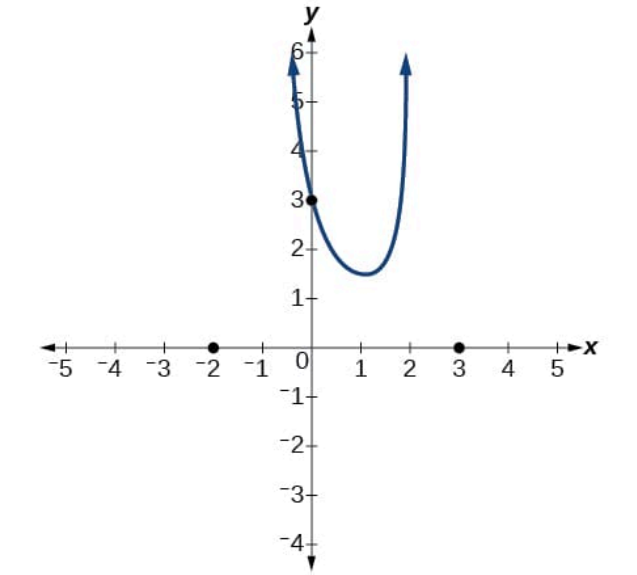



Figure 3.7.21.

The factor associated with the vertical asymptote at $x = -1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 3.7.22. After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

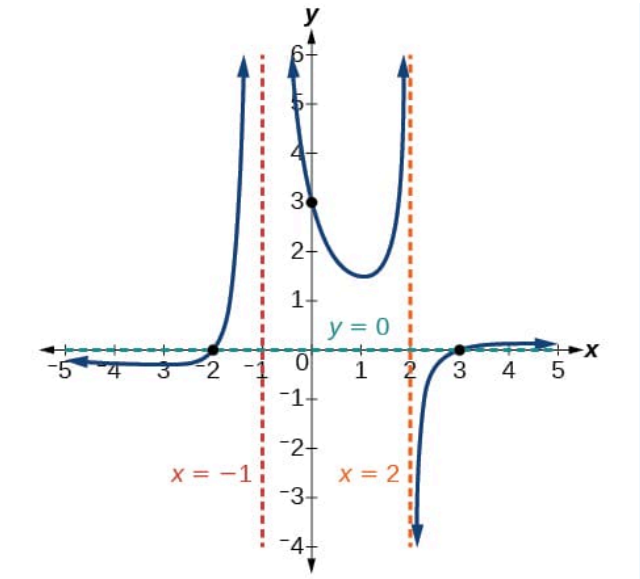



Figure 3.7.22.

**Exercise 3.7.8**

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

*Answer available at the end of this section.*

## Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an *x*-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of *x*-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

**WRITING RATIONAL FUNCTIONS FROM INTERCEPTS AND ASYMPTOTES**

If a **rational function** has x-intercepts at $x = x_1, x_2, ..., x_n$, vertical asymptotes at $x = v_1, v_2, ..., v_m$, and no $x_i =$ any $v_j$, then the function can be written in the form:

$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$$

where the powers $p_i$ or $q_i$ on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor $a$ can be determined given a value of the function other than the x-intercept or by the horizontal asymptote if it is nonzero.

## Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the *x*-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the "simplest" function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

## Example 3.7.12: Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 3.7.24.

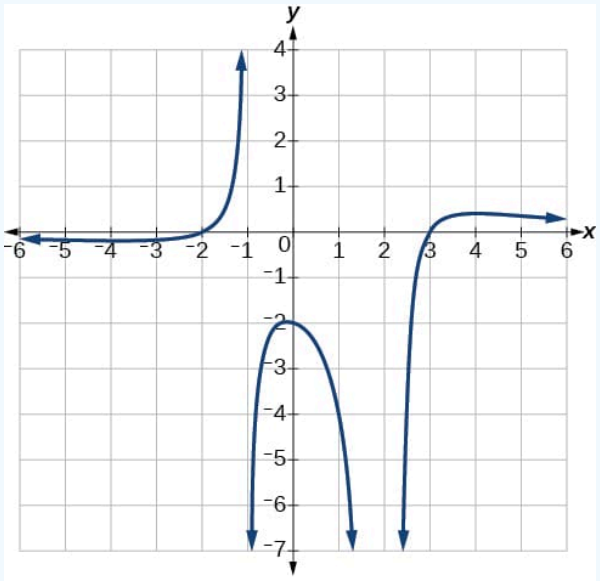



Figure 3.7.24.

**Solution**

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure 3.7.25.

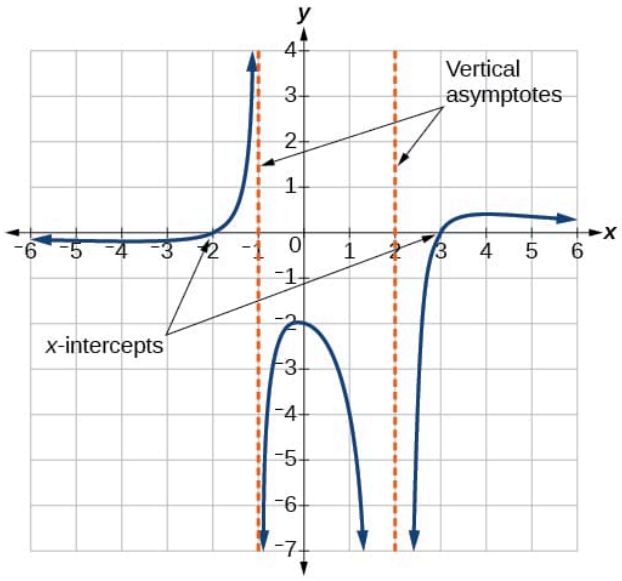



Figure 3.7.25.

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0,-2)$.

$$-2 = a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$
$$-2 = a\frac{-6}{4}$$
$$a = \frac{-8}{-6} = \frac{4}{3}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

## Media

Access these online resources for additional instruction and practice with rational functions.

- Graphing Rational Functions
- Find the Equation of a Rational Function
- Determining Vertical and Horizontal Asymptotes
- Find the Intercepts, Asymptotes, and Hole of a Rational Function

# Key Equations

| | |
|---|---|
| Rational Function | $f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0},\ Q(x) \neq 0$ |

# Key Concepts

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See Example 3.7.1.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See Example.
- Application problems involving rates and concentrations often involve rational functions. See Example.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See Example.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See Example.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See Example.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See Example, Example, Example, and Example.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See Example.
- If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any $v_j$, then the function can be written in the form

## Answers

**Exercises** 3.7.1

End behavior: as $x \to \pm\infty$, $f(x) \to 0$;

Local behavior: as $x \to 0$, $f(x) \to \infty$ (there are no x- or y-intercepts)

**Exercise** 3.7.3

$\frac{12}{11}$

**Exercise** 3.7.4

The domain is all real numbers except $x = 1$ and $x = 5$.

**Exercise** 3.7.5

Removable discontinuity at $x = 5$.

Vertical asymptotes: $x = 0$, $x = 1$.

**Exercise** 3.7.6

Vertical asymptotes at $x = 2$ and $x = -3$

horizontal asymptote at $y = 4$.

**Exercise** 3.7.7

For the transformed reciprocal squared function, we find the rational form.

$$f(x) = \frac{1}{(x-3)^2} - 4 = \frac{1-4(x-3)^2}{(x-3)^2} = \frac{1-4(x^2-6x+9)}{(x-3)(x-3)} = \frac{-4x^2+24x-35}{x^2-6x+9}$$

Because the numerator is the same degree as the denominator we know that as $x \to \pm\infty$, $f(x) \to -4$; so $y = -4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x = 3$, because as $x \to 3$, $f(x) \to \infty$. We then set the numerator equal to $0$ and find the x-intercepts are at $(2.5, 0)$ and $(3.5, 0)$. Finally, we evaluate the function at $0$ and find the y-intercept to be at $(0, -\frac{35}{9})$.

**Exercise** 3.7.8

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $(0, \frac{4}{3})$.

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity $2$, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.

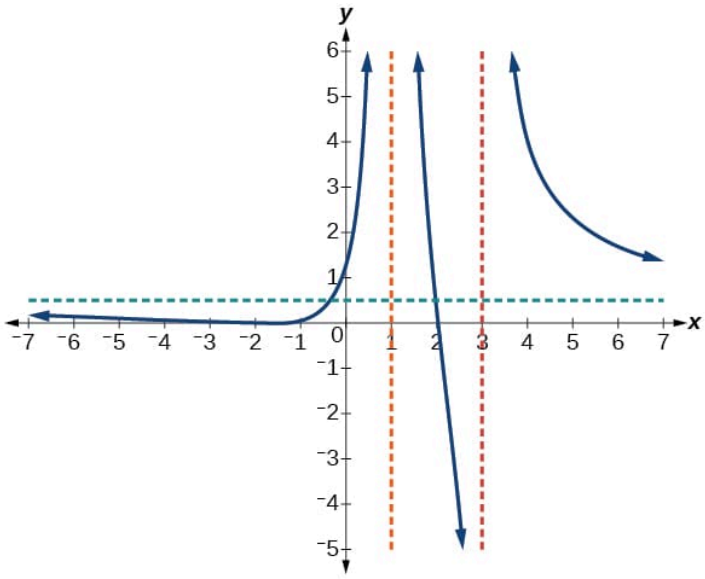



Figure 3.7.23.

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# 3.8: Inverses and Radical Functions

## Learning Objectives

In this section, you will:

- Find the inverse of an invertible polynomial function.
- Restrict the domain to find the inverse of a polynomial function.

A mound of gravel is in the shape of a cone with the height equal to twice the radius.



Figure 3.8.1

The volume is found using a formula from elementary geometry.

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi r^2 (2r) \\ &= \frac{2}{3}\pi r^3 \end{aligned}$$

We have written the volume $V$ in terms of the radius $r$. However, in some cases, we may start out with the volume and want to find the radius. For example: A customer purchases 100 cubic feet of gravel to construct a cone shape mound with a height twice the radius. What are the radius and height of the new cone? To answer this question, we use the formula

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This function is the inverse of the formula for $V$ in terms of $r$.

In this section, we will explore the inverses of polynomial and rational functions and in particular the radical functions we encounter in the process.

# Finding the Inverse of a Polynomial Function

Two functions $f$ and $g$ are inverse functions if for every coordinate pair in $f$, $(a, b)$, there exists a corresponding coordinate pair in the inverse function, $g$, $(b, a)$. In other words, the coordinate pairs of the inverse functions have the input and output interchanged. Only one-to-one functions have inverses. Recall that a one-to-one function has a unique output value for each input value and passes the horizontal line test.

For example, suppose a water runoff collector is built in the shape of a parabolic trough as shown in Figure $3.8.2$. We can use the information in the figure to find the surface area of the water in the trough as a function of the depth of the water.

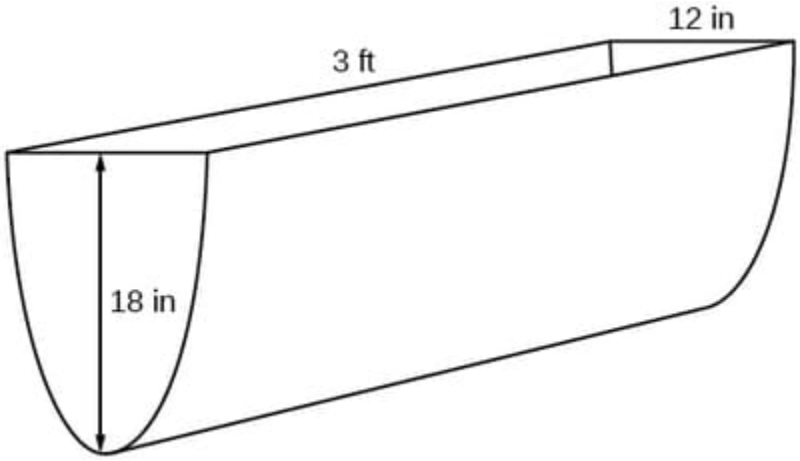



Figure $3.8.2$

Because it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with $x$ measured horizontally and $y$ measured vertically, with the origin at the vertex of the parabola (Figure $3.8.3$).

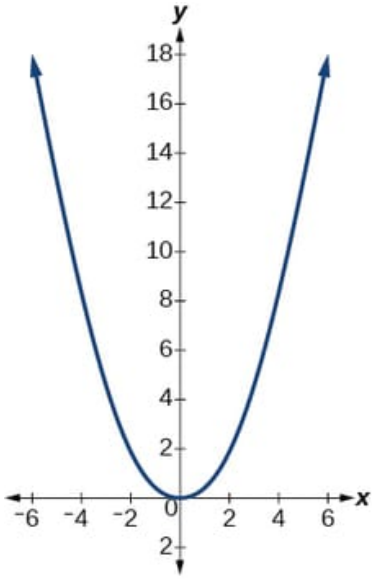



Figure $3.8.3$

From this we find an equation for the parabolic shape. We placed the origin at the vertex of the parabola, so we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point $(6, 18)$, from which we can solve for the stretch factor $a$.

$$\begin{aligned} 18 &= a6^2 \\ a &= \frac{18}{36} \\ &= \frac{1}{2} \end{aligned}$$

Our parabolic cross section has the equation

$$y(x) = \frac{1}{2}x^2$$

We are interested in the *surface area* of the water, so we must determine the width at the top of the water as a function of the water depth. For any depth $y$, the width will be given by $2x$, so we need to solve the equation above for $x$ and find the inverse function. However, notice that the original function is not one-to-one, and indeed, given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive $x$ values. On this domain, we can find an inverse by solving for the input variable:

$$\begin{aligned} y &= \frac{1}{2}x^2 \\ 2y &= x^2 \\ x &= \pm\sqrt{2y} \end{aligned}$$

This is not a function as written. We are limiting ourselves to positive $x$ values, so we eliminate the negative solution, giving us the inverse function we're looking for.

$$y = \frac{x^2}{2},\ x > 0$$

Because $x$ is the distance from the center of the parabola to either side, the entire width of the water at the top will be $2x$. The trough is 3 feet (36 inches) long, so the surface area will then be:

$$\begin{aligned} \text{Area} &= l \cdot w \\ &= 36 \cdot 2x \\ &= 72x \\ &= 72\sqrt{2y} \end{aligned}$$

This example illustrates two important points:

1. When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2. The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called **radical functions**. While it is not possible to find an inverse of most polynomial functions, some basic polynomials do have inverses. Such functions are called **invertible functions**, and we use the notation $f^{-1}(x)$.

Warning: $f^{-1}(x)$ is not the same as the reciprocal of the function $f(x)$. This use of "–1" is reserved to denote inverse functions. To denote the reciprocal of a function $f(x)$, we would need to write:

$$(f(x))^{-1} = \frac{1}{f(x)}. \tag{3.8.1}$$

An important relationship between inverse functions is that they "undo" each other. If $f^{-1}$ is the inverse of a function $f$, then $f$ is the inverse of the function $f^{-1}$. In other words, whatever the function $f$ does to $x$, $f^{-1}$ undoes it—and vice-versa.

$$f^{-1}(f(x)) = x, \text{ for all } x \text{ in the domain of } f$$

and

$f(f^{-1}(x)) = x$, for all $x$ in the domain of $f^{-1}$

Note that the inverse switches the domain and range of the original function.

## VERIFYING TWO FUNCTIONS ARE INVERSES OF ONE ANOTHER

Two functions, $f$ and $g$, are inverses of one another if for all $x$ in the domain of $f$ and $g$,

$$g(f(x)) = f(g(x)) = x$$

## Howto: Given a polynomial function, find the inverse of the function by restricting the domain in such a way that the new function is one-to-one

1. Replace $f(x)$ with $y$.
2. Interchange $x$ and $y$.
3. Solve for $y$, and rename the function $f^{-1}(x)$.

## Example 3.8.1: Verifying Inverse Functions

Show that $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses, for $x \neq 0, -1$.

**Solution**

We must show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{x+1}\right) \\ &= \frac{1}{\frac{1}{x+1}} - 1 \\ &= (x+1) - 1 \\ &= x \end{aligned}$$

and

$$\begin{aligned} f(f^{-1}(x)) &= f(\frac{1}{x-1}) \\ &= \frac{1}{\left(\frac{1}{x-1}\right)+1} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

Therefore, $f(x) = \dfrac{1}{x+1}$ and $f^{-1}(x) = \dfrac{1}{x} - 1$ are inverses.

## Exercise 3.8.1

Show that $f(x) = \frac{x+5}{3}$ and $f^{-1}(x) = 3x - 5$ are inverses.

*Answer a available at the end of this section.*

## Example 3.8.2: Finding the Inverse of a Cubic Function

Find the inverse of the function $f(x) = 5x^3 + 1$.

**Solution**

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for $x$.

$$y = 5x^3 + 1$$

$$x = 5y^3 + 1$$

$$x - 1 = 5y^3$$

$$\frac{x-1}{5} = y^3$$

$$f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$$

**Analysis**

Look at the graph of $f$ and $f^{-1}$. Notice that one graph is the reflection of the other about the line $y = x$. This is always the case when graphing a function and its inverse function.

Also, since the method involved interchanging $x$ and $y$, notice corresponding points. If $(a, b)$ is on the graph of $f$,then $(b, a)$ is on the graph of $f^{-1}$. Since $(0, 1)$ is on the graph of $f$, then $(1, 0)$ is on the graph of $f^{-1}$. Similarly, since $(1, 6)$ is on the graph of $f$,then $(6, 1)$ is on the graph of $f^{-1}$ (Figure 3.8.9).

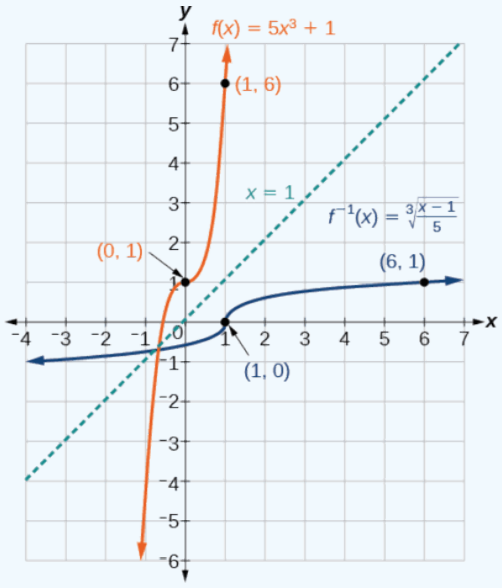



Figure 3.8.4

## Exercise 3.8.2

Find the inverse function of $f(x) = \sqrt[3]{x+4}$.

*Answer available at the end of this section.*

# Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of **cubic functions** without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an **inverse function**. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

## RESTRICTING THE DOMAIN

If a function is not one-to-one, it cannot have an inverse. If we restrict the domain of the function so that it becomes one-to-one, thus creating a new function, this new function will have an inverse.

## How to: Given a polynomial function, restrict the domain of a function that is not one-to-one and then find the inverse

1. Restrict the domain by determining a domain on which the original function is one-to-one.
2. Replace $f(x)$ with $y$.
3. Interchange $x$ and $y$.
4. Solve for $y$, and rename the function or pair of function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

## Example 3.8.3: Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of $f$:

1. $f(x) = (x-4)^2, x \geq 4$
2. $f(x) = (x-4)^2, x \leq 4$

**Solution**

The original function $f(x) = (x-4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ or $x \leq 4$ on which it is one-to-one (Figure 3.8.6).

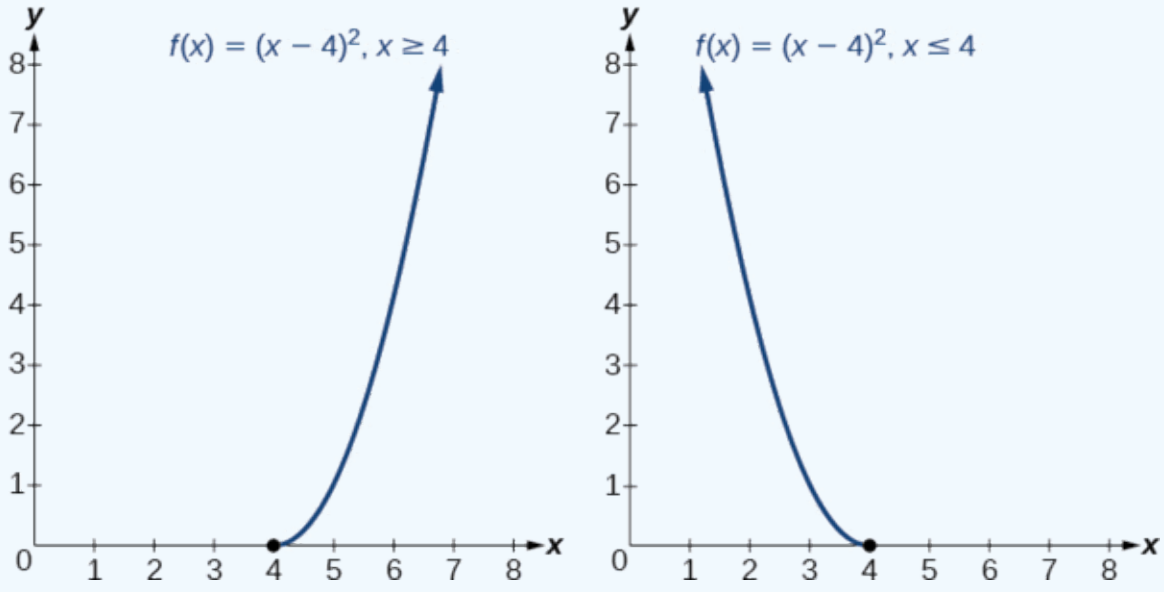



Figure 3.8.5

To find the inverse, start by replacing $f(x)$ with the simple variable $y$.

$$y = (x-4)^2 \text{ Interchange } x \text{ and } y.$$

$$x = (y-4)^2 \text{ Take the square root.}$$

$$\pm\sqrt{x} = y - 4 \text{ Add } 4 \text{ to both sides.}$$

$$4 \pm \sqrt{x} = y$$

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse. Since we reversed the roles of $x$ and $y$ for the original $f(x)$, we looked at the domain: the values $x$ could assume. When we reversed the roles of $x$ and $y$, this gave us the values $y$ could assume. For this function, $x \geq 4$, so for the inverse, we should have $y \geq 4$, which is what our inverse function gives.

1. The domain of the original function was restricted to $x \geq 4$, so the outputs of the inverse need to be the same, $f(x) \geq 4$, and we must use the + case:

$$f^{-1}(x) = 4 + \sqrt{x}$$

2. The domain of the original function was restricted to $x \leq 4$, so the outputs of the inverse need to be the same, $f(x) \leq 4$, and we must use the – case:

$$f^{-1}(x) = 4 - \sqrt{x}$$

**Analysis**

On the graphs in Figure 3.8.6, we see the original function graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y = x$. The coordinate pair $(4, 0)$ is on the graph off f and the coordinate pair $(0, 4)$ is on the graph of $f^{-1}$. For any coordinate pair, if $(a, b)$ is on the graph of $f$, then $(b, a)$ is on the graph of $f^{-1}$. Finally, observe that the graph of $f$ intersects the graph of $f^{-1}$ on the line $y = x$. Points of intersection for the graphs of $f$ and $f^{-1}$ will always lie on the line $y = x$.

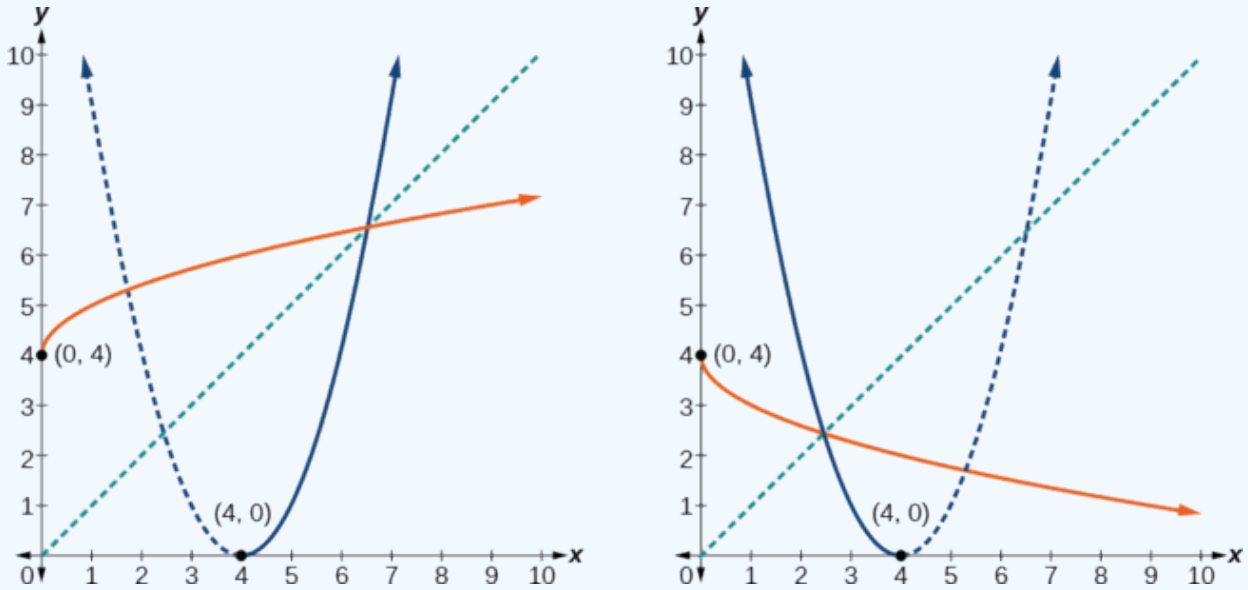



Figure 3.8.6

## Example 3.8.4: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of

$$f(x) = (x-2)^2 - 3.$$

**Solution**

We can see this is a parabola with vertex at $(2,-3)$ that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we will use the vertex form of the quadratic. We start by replacing $f(x)$ with a simple variable, $y$, then solve for $x$.

$$y = (x-2)^2 - 3 \text{ Interchange } x \text{ and } y.$$

$$x = (y-2)^2 - 3 \text{ Add 3 to both sides.}$$

$$x + 3 = (y-2)^2 \text{ Take the square root.}$$

$$\pm\sqrt{x+3} = y - 2 \text{ Add 2 to both sides.}$$

$$2 \pm \sqrt{x+3} = y \text{ Rename the function.}$$

$$f^{-1}(x) = 2 \pm \sqrt{x+3}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the + case

$$f^{-1}(x) = 2 + \sqrt{x+3}$$

If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

Analysis

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just as easily have opted to restrict the domain to $x \leq 2$, in which case $f^{-1}(x) = 2 - \sqrt{x+3}$. Observe the original function graphed on the same set of axes as its inverse function in Figure 3.8.7. Notice that both graphs show symmetry about the line $y = x$. The coordinate pair $(2,-3)$ is on the graph of $f$ and the coordinate pair $(-3,2)$ is on the graph of $f^{-1}$. Observe from the graph of both functions on the same set of axes that

$$\text{domain of } f = \text{range of } f^{-1} = [2,\infty)$$

and

$$\text{domain of } f^{-1} = \text{range of } f = [-3,\infty).$$

Finally, observe that the graph of $f$ intersects the graph of $f^{-1}$ along the line $y = x$.

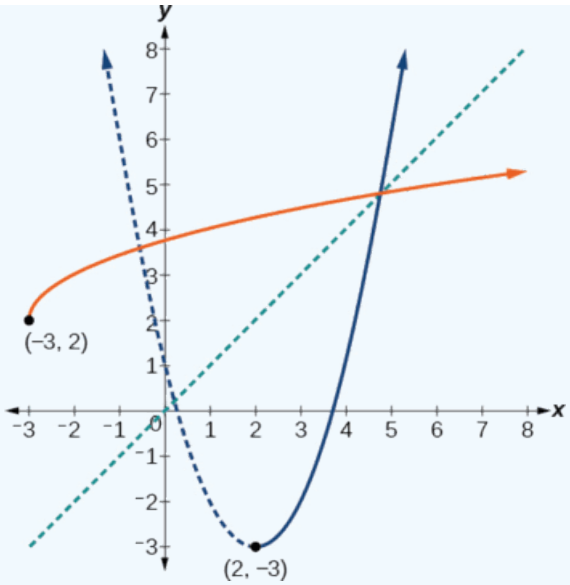



Figure 3.8.7

**Exercise 3.8.4**

Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

*Answer available at the end of this section.*

## Finding Inverses

Notice that the functions from previous examples were all polynomials, and their inverses were radical functions. If we want to find the inverse of a radical function, we will need to restrict the domain of the answer because the range of the original function is limited.

**How to: Given a radical function, find the inverse**

1. Determine the range of the original function.
2. Replace $f(x)$ with $y$, then solve for $x$.
3. If necessary, restrict the domain of the inverse function to the range of the original function.

**Example 3.8.5: Finding the Inverse of a Radical Function**

Restrict the domain of the function $f(x) = \sqrt{x-4}$ and then find the inverse.

**Solution**

Note that the original function has range $f(x) \geq 0$. Replace $f(x)$ with $y$, then solve for $x$.

$$y = \sqrt{x-4} \text{ Replace } f(x) \text{ with } y.$$

$$x = \sqrt{y-4} \text{ Interchange } x \text{ and } y.$$

$$x = \sqrt{y-4} \text{ Square each side.}$$

$$x^2 = y - 4 \text{ Add 4.}$$

$x^2 + 4 = y$ Rename the function $f^{-1}(x)$.

$$f^{-1}(x) = x^2 + 4$$

Recall that the domain of this function must be limited to the range of the original function.

$$f^{-1}(x) = x^2 + 4,\ x \geq 0$$

**Analysis**

Notice in Figure $3.8.8$ that the inverse is a reflection of the original function over the line $y = x$. Because the original function has only positive outputs, the inverse function has only nonnegative inputs.

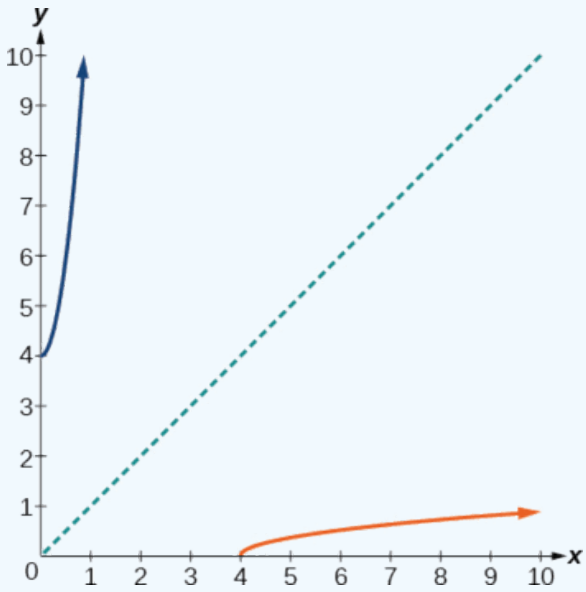



Figure $3.8.8$

## Exercise $3.8.5$

Restrict the domain and then find the inverse of the function $f(x) = \sqrt{2x + 3}$.

*Answer available at the end of this section.*

## Solving Applications of Radical Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

## Example $3.8.6$: Solving an Application with a Cubic Function

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3$$

Find the inverse of the function $V = \frac{2}{3}\pi r^3$ that determines the volume $V$ of a cone and is a function of the radius $r$. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi = 3.14$.

**Solution**

Start with the given function for $V$. Notice that the meaningful domain for the function is $r > 0$ since negative radii would not make sense in this context nor would a radius of $0$. Also note the range of the function (hence, the domain of the inverse function) is $V > 0$. Solve for $r$ in terms of $V$, using the method outlined previously. Note that in real-world applications, we do not swap the variables when finding inverses. Instead, we change which variable is considered to be the independent variable.

$$V = \frac{2}{3}\pi r^3$$

Solve for $r^3$.

$$r^3 = \frac{3V}{2\pi}$$

Solve for $r$.

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi = 3.14$.

$$\begin{aligned} r &= \sqrt[3]{\frac{3V}{2\pi}} \\ &= \sqrt[3]{\frac{3 \cdot 100}{2 \cdot 3.14}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft.

## Determining the Domain of a Radical Function Composed with Other Functions

When radical functions are composed with other functions, determining domain can become more complicated.

### Example 3.8.7: Finding the Domain of a Radical Function Composed with a Rational Function

Find the domain of the function:

$$f(x) = \sqrt{\frac{(x+2)(x-3)}{(x-1)}}.$$

**Solution**

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where

$$\frac{(x+2)(x-3)}{(x-1)} \geq 0.$$

The output of a rational function can change signs (change from positive to negative or vice versa) at *x*-intercepts and at vertical asymptotes. For this equation, the graph could change signs at $x = -2$, $1$, and $3$.

To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, for this example we will use a graph as shown in Figure $3.8.9$.

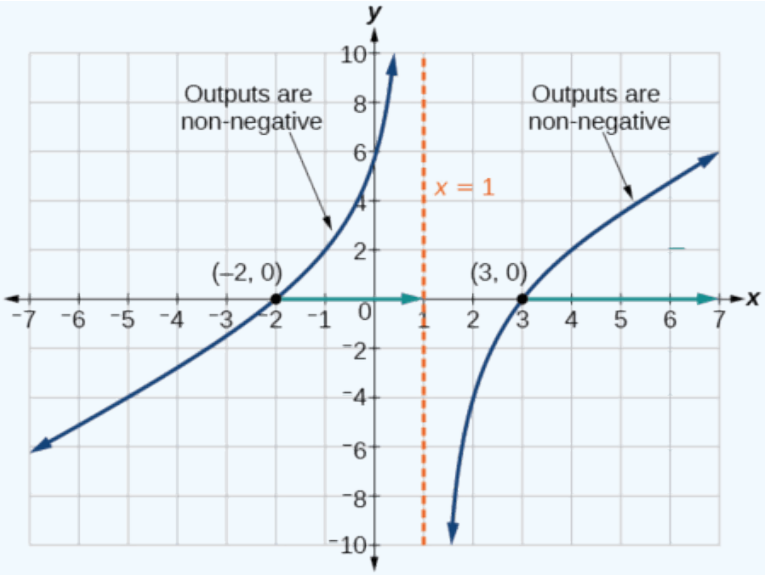



Figure 3.8.9

This function has two *x*-intercepts, both of which exhibit linear behavior near the *x*-intercepts. There is one vertical asymptote, corresponding to a linear factor; this behavior is similar to the basic reciprocal toolkit function, and there is no horizontal asymptote because the degree of the numerator is larger than the degree of the denominator. There is a *y*-intercept at $(0, \sqrt{6})$ .

From the *y*-intercept and *x*-intercept at $x = -2$, we can sketch the left side of the graph. From the behavior at the asymptote, we can sketch the right side of the graph.

From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-2 \leq x < 1$ or $x \geq 3$, or in interval notation, $[-2, 1) \cup [3, \infty)$.

## Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the *inverse of a rational function*, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

### Example 3.8.8: Finding the Inverse of a Rational Function

The function

$$C = \frac{20 + 0.4n}{100 + n} \tag{3.8.2}$$

represents the concentration $C$ of an acid solution after $n$ mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for $n$ in terms of $C$. Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

**Solution**

We first want the inverse of the function in order to determine how many mL we need for a given concentration. We will solve for $n$ in terms of $C$.

$$C = \frac{20 + 0.4n}{100 + n}$$
$$C(100 + n) = 20 + 0.4n$$
$$100C + Cn = 20 + 0.4n$$
$$100C - 20 = 0.4n - Cn$$
$$100C - 20 = (0.4 - C)n$$
$$n = \frac{100C - 20}{0.4 - C}$$

Now evaluate this function at 35%, which is $C = 0.35$.

$$n = \frac{100(0.35) - 20}{0.4 - 0.35}$$
$$= \frac{15}{0.05}$$
$$= 300$$

We can conclude that 300 mL of the 40% solution should be added.

### Exercise 3.8.8

Find the inverse of the function $f(x) = \frac{x+3}{x-2}$.

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with inverses and radical functions.

- Graphing the Basic Square Root Function
- Find the Inverse of a Square Root Function
- Find the Inverse of a Rational Function
- Find the Inverse of a Rational Function and an Inverse Function Value
- Inverse Functions

## Key Concepts

- The inverse of a quadratic function is a square root function.
- If $f^{-1}$ is the inverse of a function $f$, then $f$ is the inverse of the function $f^{-1}$. See Example 3.8.1.
- While it is not possible to find an inverse of most polynomial functions, some basic polynomials are invertible. See Example 3.8.2.
- To find the inverse of certain functions, we must restrict the function to a domain on which it will be one-to-one. See Examples 3.8.3 and 3.8.4
- When finding the inverse of a radical function, we need a restriction on the domain of the answer. See Example 3.8.5 and 3.8.7.
- Inverse and radical and functions can be used to solve application problems. See Examples 3.8.6 and 3.8.8.

### Answers

**Exercise** 3.8.1

$f^{-1}(f(x)) = f^{-1}(\frac{x+5}{3}) = 3(\frac{x+5}{3}) - 5 = (x-5) + 5 = x$

$f(f^{-1}(x)) = f(3x-5) = \frac{(3x-5)+5}{3} = \frac{3x}{3} = x$

**Exercise** 3.8.2

$f^{-1}(x) = x^3 - 4$

**Exercise** 3.8.4

$f^{-1}(x) = \sqrt{x-1}$

**Exercise** 3.8.5

$f^{-1}(x) = \frac{x^2-3}{2}$, $x \geq 0$

**Exercise** 3.8.8

$f^{-1}(x) = \frac{2x+3}{x-1}$

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# 3.9: Modeling Using Variation

**Learning Objectives**

In this section, you will:

- Solve direct variation problems.
- Solve inverse variation problems.
- Solve problems involving joint variation.

A used-car company has just offered their best candidate, Nicole, a position in sales. The position offers 16% commission on her sales. Her earnings depend on the amount of her sales. For instance, if she sells a vehicle for $4,600, she will earn $736. She wants to evaluate the offer, but she is not sure how. In this section, we will look at relationships, such as this one, between earnings, sales, and commission rate.

## Solving Direct Variation Problems

In the example above, Nicole's earnings can be found by multiplying her sales by her commission. The formula $e = 0.16s$ tells us her earnings, $e$, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See **Table 5.8.1**.

| $s$, sales price | $e = 0.16s$ | Interpretation |
|---|---|---|
| $9,200 | $e = 0.16(9,200) = 1,472$ | A sale of a $9,200 vehicle results in $1472 earnings. |
| $4,600 | $e = 0.16(4,600) = 736$ | A sale of a $4,600 vehicle results in $736 earnings. |
| $18,400 | $e = 0.16(18,400) = 2,944$ | A sale of a $18,400 vehicle results in $2944 earnings. |

**Table 5.8.1**

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from $4,600 to $9,200, and we double the earnings from $736 to $1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship **varies directly** with the other.

Figure 5.8.1 represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the car. The formula $y = kx^n$ is used for direct variation. The value $k$ is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$ and $n = 1$. We saw functions like this one when we discussed power functions.

Figure has invalid source: image visible until saved.... src="/@api/deki/pages/=Bookshelves%252FPrecalculus%252FBook%25253A_Precalculus_(OpenStax)%252F03%25253A_Polynomial_and_Ratio files/CNX_Precalc_Figure_03_09_001.jpg

Figure 5.8.1

**A General Note: DIRECT VARIATION**

If $x$ and $y$ are related by an equation of the form

$$y = kx^n$$

then we say that the relationship is **direct variation** and $y$ **varies directly** with, or is proportional to, the $n$th power of $x$. In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x^n}$, where $k$ is called the **constant of variation**, which help defines the relationship between the variables.

## Given a description of a direct variation problem, solve for an unknown.

1. Identify the input, $x$,and the output, $y$.
2. Determine the constant of variation. You may need to divide $y$ by the specified power of $x$ to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

## Example

Solving a Direct Variation Problem

The quantity $y$ varies directly with the cube of $x$. If $y = 25$ when $x = 2$, find $y$ when $x$ is $6$.

**Solution**

The general formula for direct variation with a cube is $y = kx^3$. The constant can be found by dividing $y$ by the cube of $x$.

$$k = \frac{y}{x^3}$$

$$= \frac{25}{2^3}$$

$$= \frac{25}{8}$$

Now use the constant to write an equation that represents this relationship.

$$y = \frac{25}{8}x^3$$

Substitute $x = 6$ and solve for $y$.

$$y = \frac{25}{8}(6)^3$$

$$= 675$$

Analysis

The graph of this equation is a simple cubic, as shown in **Figure 5.8.2**.

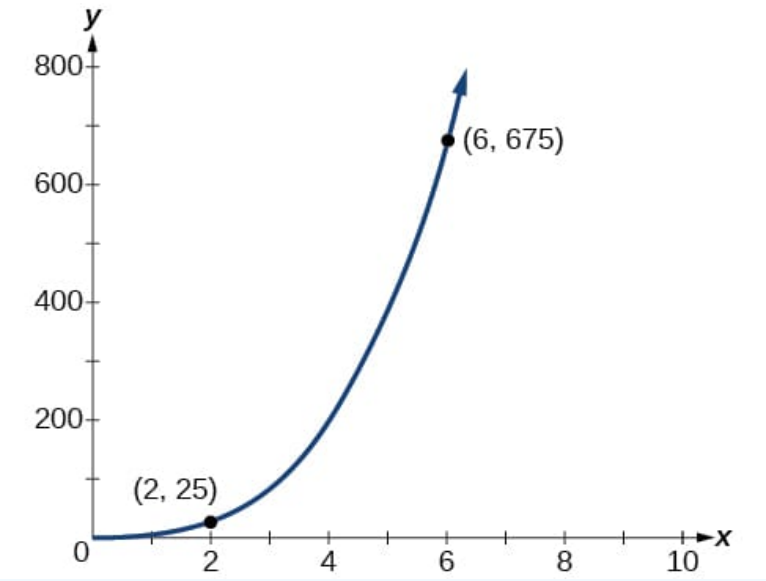



Figure 5.8.2

**Q&A**

**Do the graphs of all direct variation equations look like Example?**

*No. Direct variation equations are power functions—they may be linear, quadratic, cubic, quartic, radical, etc. But all of the graphs pass through* $(0, 0)$.

**Exercise**

The quantity $y$ varies directly with the square of $x$. If $y = 24$ when $x = 3$, find $y$ when $x$ is 4.

**Solution**

$\frac{128}{3}$

## Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water's depth. The formula $T = \frac{14{,}000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers 22% of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F.

If we create **Table 5.8.2**, we observe that, as the depth increases, the water temperature decreases.

| $d$, depth | $T = \frac{14{,}000}{d}$ | Interpretation |
|---|---|---|
| 500 ft | $\frac{14{,}000}{500} = 28$ | At a depth of 500 ft, the water temperature is 28° F. |
| 1000 ft | $\frac{14{,}000}{1000} = 14$ | At a depth of 1,000 ft, the water temperature is 14° F. |
| 2000 ft | $\frac{14{,}000}{2000} = 7$ | At a depth of 2,000 ft, the water temperature is 7° F. |

**Table 5.8.2**

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be **inversely proportional** and each term **varies inversely** with the other. Inversely proportional relationships are also

called **inverse variations**.

For our example, **Figure 5.8.3** depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula $y = \frac{k}{x}$ for inverse variation in this case uses $k = 14,000$.

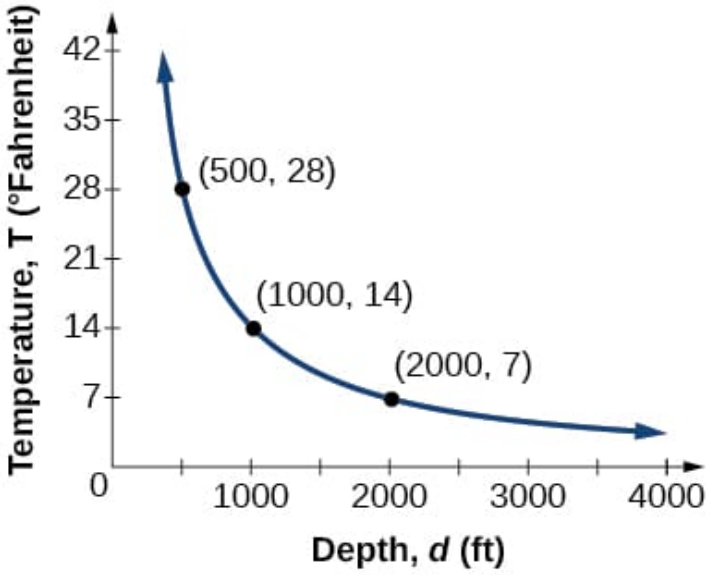



Figure 5.8.3

## A General Note: INVERSE VARIATION

If $x$ and $y$ are related by an equation of the form

$$y = \frac{k}{x^n}$$

where $k$ is a nonzero constant, then we say that $y$ **varies inversely** with the $n$th power of $x$. In **inversely proportional** relationships, or **inverse variations**, there is a constant multiple $k = x^n y$.

## Example

Writing a Formula for an Inversely Proportional Relationship

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

**Solution**

Recall that multiplying speed by time gives distance. If we let $t$ represent the drive time in hours, and $v$ represent the velocity (speed or rate) at which the tourist drives, then $vt$ =distance. Because the distance is fixed at 100 miles, $vt = 100$ so $t = \frac{100}{v}$. Because time is a function of velocity, we can write $t(v)$.

$$t(v) = \frac{100}{v}$$

$$= 100v^{-1}$$

We can see that the constant of variation is 100 and, although we can write the relationship using the negative exponent, it is more common to see it written as a fraction. We say that time varies inversely with velocity.

## Given a description of an indirect variation problem, solve for an unknown.

1. Identify the input, $x$, and the output, $y$.
2. Determine the constant of variation. You may need to multiply $y$ by the specified power of $x$ to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

## Example

Solving an Inverse Variation Problem

A quantity $y$ varies inversely with the cube of $x$. If $y = 25$ when $x = 2$, find $y$ when $x$ is $6$.

**Solution**

The general formula for inverse variation with a cube is $y = \frac{k}{x^3}$. The constant can be found by multiplying $y$ by the cube of $x$.

$$k = x^3 y$$

$$= 2^3 \cdot 25$$

$$= 200$$

Now we use the constant to write an equation that represents this relationship.

$$y = \frac{k}{x^3}, \; k = 200$$

$$y = \frac{200}{x^3}$$

Substitute $x = 6$ and solve for $y$.

$$y = \frac{200}{6^3}$$

$$= \frac{25}{27}$$

Analysis

The graph of this equation is a rational function, as shown in **Figure 5.8.4**.

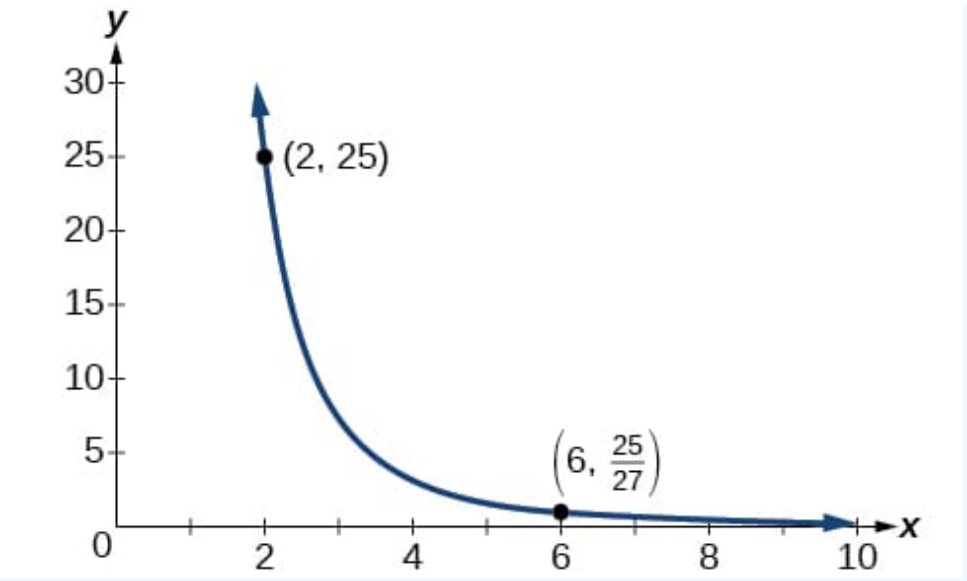



Figure 5.8.4

**Exercise**

A quantity $y$ varies inversely with the square of $x$. If $y = 8$ when $x = 3$, find $y$ when $x$ is $4$.

**Solution**

$\frac{9}{2}$

## Solving Problems Involving Joint Variation

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called **joint variation**. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable $c$,cost, varies jointly with the number of students, $n$,and the distance, $d$.

**A General Note: JOINT VARIATION**

Joint variation occurs when a variable varies directly or inversely with multiple variables.

For instance, if $x$ varies directly with both $y$ and $z$, we have $x = kyz$. If $x$ varies directly with $y$ and inversely with $z$,we have $x = \frac{ky}{z}$. Notice that we only use one constant in a joint variation equation.

**Example**

Solving Problems Involving Joint Variation

A quantity $x$ varies directly with the square of $y$ and inversely with the cube root of $z$. If $x = 6$ when $y = 2$ and $z = 8$, find $x$ when $y = 1$ and $z = 27$.

**Solution**

Begin by writing an equation to show the relationship between the variables.

$$x = \frac{ky^2}{\sqrt[3]{z}}$$

Substitute $x = 6$, $y = 2$, and $z = 8$ to find the value of the constant $k$.

$$6 = \frac{k2^2}{\sqrt[3]{8}}$$

$$6 = \frac{4k}{2}$$

$$3 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$x = \frac{3y^2}{\sqrt[3]{z}}$$

To find $x$ when $y = 1$ and $z = 27$, we will substitute values for $y$ and $z$ into our equation.

$$x = \frac{3(1)^2}{\sqrt[3]{27}}$$

$$= 1$$

## Exercise

A quantity $x$ varies directly with the square of $y$ and inversely with $z$. If $x = 40$ when $y = 4$ and $z = 2$, find $x$ when $y = 10$ and $z = 25$.

**Solution**

$x = 20$

## Media

Access these online resources for additional instruction and practice with direct and inverse variation.

- Direct Variation
- Inverse Variation
- Direct and Inverse Variation

Visit this website for additional practice questions from Learningpod.

# Key Equations

| | |
|---|---|
| Direct variation | $y = kx^n$, $k$ is a nonzero constant. |
| Inverse variation | $y = \frac{k}{x^n}$, $k$ is a nonzero constant. |

# Key Concepts

- A relationship where one quantity is a constant multiplied by another quantity is called direct variation. See Example.
- Two variables that are directly proportional to one another will have a constant ratio.

- A relationship where one quantity is a constant divided by another quantity is called inverse variation. See Example.
- Two variables that are inversely proportional to one another will have a constant multiple. See Example.
- In many problems, a variable varies directly or inversely with multiple variables. We call this type of relationship joint variation. See Example.

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# 3.E: Polynomial and Rational Functions(Exercises)

## 3.1 Complex Numbers

### Verbal

1) Explain how to add complex numbers.

**Answer**

Add the real parts together and the imaginary parts together.

2) What is the basic principle in multiplication of complex numbers?

3) Give an example to show the product of two imaginary numbers is not always imaginary.

**Answer**

$i$ times $i$ equals $-1$, which is not imaginary (answers vary)

4) What is a characteristic of the plot of a real number in the complex plane?

### Algebraic

**For the exercises 5-10, evaluate the algebraic expressions.**

5) If $f(x) = x^2 + x - 4$, evaluate $f(2i)$.

**Answer**

$-8 + 2i$

6) If $f(x) = x^3 - 2$, evaluate $f(i)$.

7) If $f(x) = x^2 + 3x + 5$,evaluate $f(2 + i)$.

**Answer**

$14 + 7i$

8) If $f(x) = 2x^2 + x - 3$, evaluate $f(2 - 3i)$.

9) If $f(x) = \dfrac{x+1}{2-x}$, evaluate $f(5i)$.

**Answer**

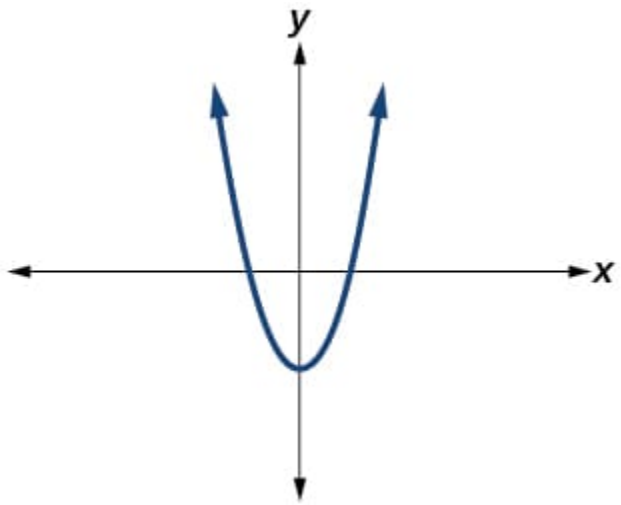
$-\dfrac{23}{29} + \dfrac{15}{29}i$

10) If $f(x) = \dfrac{1+2x}{x+3}$, evaluate $f(4i)$.

### Graphical

**For the exercises 11-12, determine the number of real and nonreal solutions for each quadratic function shown.**

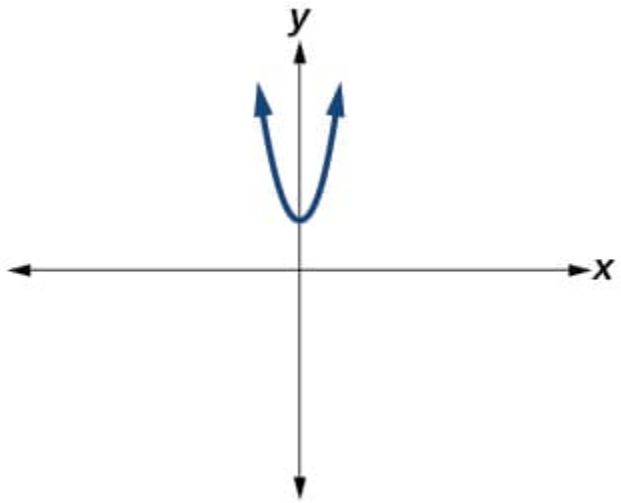
11)




**Answer**

$2$ real and $0$ nonreal

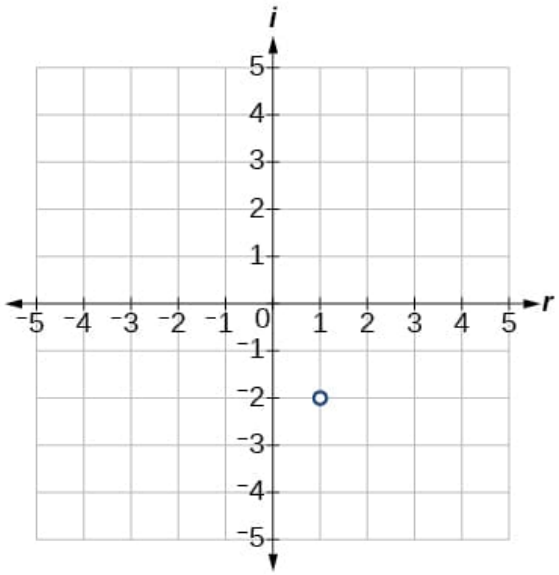
12)




**For the exercises 13-16, plot the complex numbers on the complex plane.**

13) $1-2i$

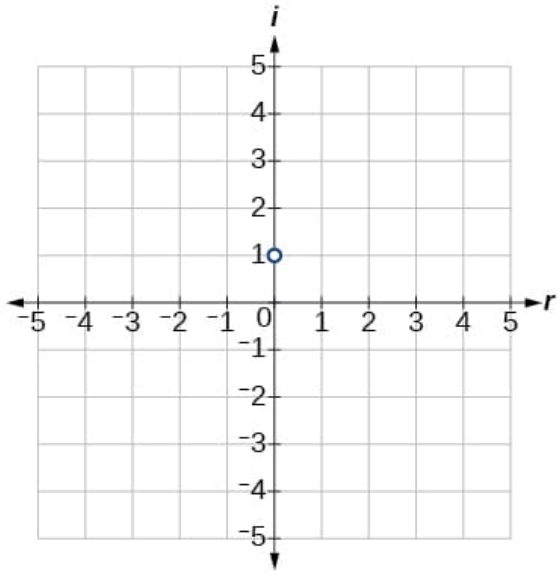
**Answer**




14) $-2+3i$

15) $i$

**Answer**




16) $-3-4i$

## Numeric

**For the exercises 17-43, perform the indicated operation and express the result as a simplified complex number.**

17) $(3+2i)+(5-3i)$

**Answer**

$8 - i$

18) $(-2 - 4i) + (1 + 6i)$

19) $(-5 + 3i) - (6 - i)$

**Answer**

$-11 + 4i$

20) $(2 - 3i) - (3 + 2i)$

21) $(-4 + 4i) - (-6 + 9i)$

**Answer**

$2 - 5i$

22) $(2 + 3i)(4i)$

23) $(5 - 2i)(3i)$

**Answer**

$6 + 15i$

24) $(6 - 2i)(5)$

25) $(-2 + 4i)(8)$

**Answer**

$-16 + 32i$

26) $(2 + 3i)(4 - i)$

27) $(-1 + 2i)(-2 + 3i)$

**Answer**

$-4 - 7i$

28) $(4 - 2i)(4 + 2i)$

29) $(3 + 4i)(3 - 4i)$

**Answer**

$25$

30) $\dfrac{3 + 4i}{2}$

31) $\dfrac{6 - 2i}{3}$

**Answer**

$2 - \dfrac{2}{3}i$

32) $\frac{-5+3i}{2i}$

33) $\frac{6+4i}{i}$

**Answer**

$4-6i$

34) $\frac{2-3i}{4+3i}$

35) $\frac{3+4i}{2-i}$

**Answer**

$\frac{2}{5}+\frac{11}{5}i$

36) $\frac{2+3i}{2-3i}$

37) $\sqrt{-9}+3\sqrt{-16}$

**Answer**

$15i$

38) $-\sqrt{-4}-4\sqrt{-25}$

39) $\frac{2+\sqrt{-12}}{2}$

**Answer**

$1+i\sqrt{3}$

40) $\frac{4+\sqrt{-20}}{2}$

41) $i^8$

**Answer**

$1$

42) $i^{15}$

43) $i^{22}$

**Answer**

$-1$

## Technology

**For the exercises 44-48, use a calculator to help answer the questions.**

44) Evaluate $(1+i)^k$ for $k=4,8$, and $12$.Predict the value if $k=16$.

45) Evaluate $(1-i)^k$ for $k=2,6$, and $10$.Predict the value if $k=14$.

**Answer**

$128i$

46) Evaluate $(1+i)^k-(1-i)^k$ for $k=4$, $8$, and $12$. Predict the value for $k=16$.

47) Show that a solution of $x^6+1=0$ is $\frac{\sqrt{3}}{2}+\frac{1}{2}i$.

**Answer**

$(\frac{\sqrt{3}}{2}+\frac{1}{2}i)^6=-1$

48) Show that a solution of $x^8-1=0$ is $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

## Extensions

**For the exercises 49-58, evaluate the expressions, writing the result as a simplified complex number.**

49) $\frac{1}{i}+\frac{4}{i^3}$

**Answer**

$3i$

50) $\frac{1}{i^{11}}-\frac{1}{i^{21}}$

51) $i^7(1+i^2)$

**Answer**

$0$

52) $i^{-3}+5i^7$

53) $\frac{(2+i)(4-2i)}{(1+i)}$

**Answer**

$5–5i$

54) $\frac{(1+3i)(2-4i)}{(1+2i)}$

55) $\frac{(3+i)^2}{(1+2i)^2}$

**Answer**

$-2i$

56) $\frac{3+2i}{2+i}+(4+3i)$

57) $\frac{4+i}{i}+\frac{3-4i}{1-i}$

**Answer**

$$\frac{9}{2}-\frac{9}{2}i$$

58) $\frac{3+2i}{1+2i}-\frac{2-3i}{3+i}$

# 3.2 Quadratic Functions

## Verbal

1) Explain the advantage of writing a quadratic function in standard form.

**Answer**

When written in that form, the vertex can be easily identified.

2) How can the vertex of a parabola be used in solving real world problems?

3) Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

**Answer**

If $a = 0$ then the function becomes a linear function.

4) What is another name for the standard form of a quadratic function?

5) What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

**Answer**

If possible, we can use factoring. Otherwise, we can use the quadratic formula.

## Algebraic

**For the exercises 6-13, rewrite the quadratic functions in standard form and give the vertex.**

6) $f(x) = x^2 - 12x + 32$

7) $g(x) = x^2 + 2x - 3$

**Answer**

$g(x) = (x+1)^2 - 4$, Vertex $(-1, -4)$

8) $f(x) = x^2 - x$

9) $f(x) = x^2 + 5x - 2$

**Answer**

$f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$, Vertex $\left(-\frac{5}{2}, -\frac{33}{4}\right)$

10) $h(x) = 2x^2 + 8x - 10$

11) $k(x) = 3x^2 - 6x - 9$

**Answer**

$k(x) = 3(x-1)^2 - 12$, Vertex $(1, -12)$

12) $f(x) = 2x^2 - 6x$

13) $f(x) = 3x^2 - 5x - 1$

**Answer**

$f(x) = 3\left(x - \frac{5}{6}\right)^2 - \frac{37}{12}$, Vertex $\left(\frac{5}{6}, -\frac{37}{12}\right)$

**For the exercises 14-20, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.**

14) $y(x) = 2x^2 + 10x + 12$

15) $f(x) = 2x^2 - 10x + 4$

**Answer**

Minimum is $-\frac{17}{2}$ and occurs at $\frac{5}{2}$. Axis of symmetry is $x = \frac{5}{2}$.

16) $f(x) = -x^2 + 4x + 3$

17) $f(x) = 4x^2 + x - 1$

**Answer**

Minimum is $-\frac{17}{16}$ and occurs at $-\frac{1}{8}$. Axis of symmetry is $x = -\frac{1}{8}$.

18) $h(t) = -4t^2 + 6t - 1$

19) $f(x) = \frac{1}{2}x^2 + 3x + 1$

**Answer**

Minimum is $-\frac{7}{2}$ and occurs at $-3$. Axis of symmetry is $x = -3$.

20) $f(x) = -\frac{1}{3}x^2 - 2x + 3$

**For the exercises 21-25, determine the domain and range of the quadratic function.**

21) $f(x) = (x-3)^2 + 2$

**Answer**

Domain is $(-\infty, \infty)$. Range is $[2, \infty)$.

22) $f(x) = -2(x+3)^2 - 6$

23) $f(x) = x^2 + 6x + 4$

**Answer**

Domain is $(-\infty, \infty)$. Range is $[-5, \infty)$.

24) $f(x) = 2x^2 - 4x + 2$

25) $k(x) = 3x^2 - 6x - 9$

**Answer**

Domain is $(-\infty, \infty)$. Range is $[-12, \infty)$.

**For the exercises 26-44, solve the equations over the complex numbers.**

26) $x^2 = -25$

27) $x^2 = -8$

**Answer**

$2i\sqrt{2}, -2i\sqrt{2}$

28) $x^2 + 36 = 0$

29) $x^2 + 27 = 0$

**Answer**

$3i\sqrt{3}, -3i\sqrt{3}$

30) $x^2 + 2x + 5 = 0$

31) $x^2 - 4x + 5 = 0$

**Answer**

$2 + i, 2 - i$

32) $x^2 + 8x + 25 = 0$

33) $x^2 - 4x + 13 = 0$

**Answer**

$2 + 3i, 2 - 3i$

34) $x^2 + 6x + 25 = 0$

35) $x^2 - 10x + 26 = 0$

**Answer**

$5 + i, 5 - i$

36) $x^2 - 6x + 10 = 0$

37) $x(x - 4) = 20$

**Answer**

$2 + 2\sqrt{6}, 2 - 2\sqrt{6}$

38) $x(x - 2) = 10$

39) $2x^2 + 2x + 5 = 0$

**Answer**

$$\left\{-\frac{1}{2}+\frac{3}{2}i,-\frac{1}{2}-\frac{3}{2}i\right\}$$

40) $5x^2-8x+5=0$

41) $5x^2+6x+2=0$

**Answer**

$$\left\{-\frac{3}{5}+\frac{1}{5}i,-\frac{3}{5}-\frac{1}{5}i\right\}$$

42) $2x^2-6x+5=0$

43) $x^2+x+2=0$

**Answer**

$$\left\{-\frac{1}{2}+\frac{1}{2}i\sqrt{7},-\frac{1}{2}-\frac{1}{2}i\sqrt{7}\right\}$$

44) $x^2-2x+4=0$

**For the exercises 45-52, use the vertex $(h,k)$ and a point on the graph $(x,y)$ to find the general form of the equation of the quadratic function.**

45) $(h,k)=(2,0),(x,y)=(4,4)$

**Answer**

$$f(x)=x^2-4x+4$$

46) $(h,k)=(-2,-1),(x,y)=(-4,3)$

47) $(h,k)=(0,1),(x,y)=(2,5)$

**Answer**

$$f(x)=x^2+1$$

48) $(h,k)=(2,3),(x,y)=(5,12)$

49) $(h,k)=(-5,3),(x,y)=(2,9)$

**Answer**

$$f(x)=\frac{6}{49}x^2+\frac{60}{49}x+\frac{297}{49}$$

50) $(h,k)=(3,2),(x,y)=(10,1)$

51) $(h,k)=(0,1),(x,y)=(1,0)$

**Answer**

$$f(x)=-x^2+1$$

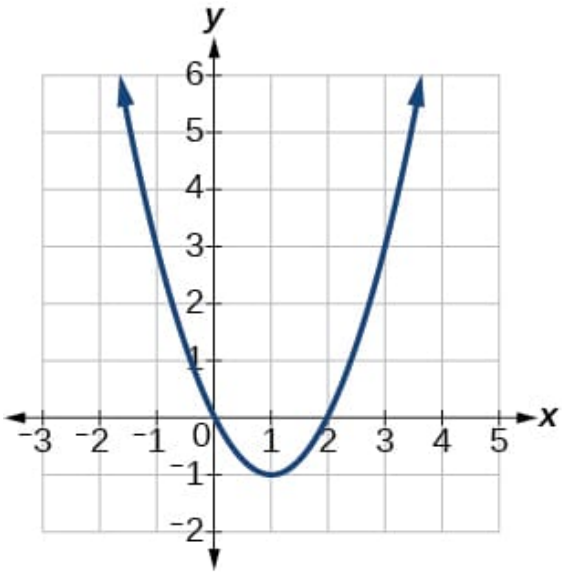
52) $(h,k)=(1,0),(x,y)=(0,1)$

## Graphical

**For the exercises 53-58, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.**

53) $f(x) = x^2 - 2x$

**Answer**

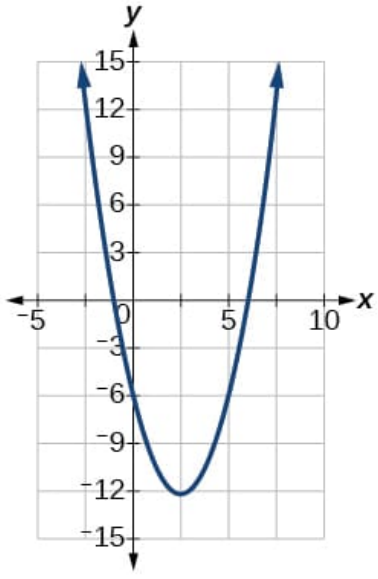



Vertex $(1, -1)$, Axis of symmetry is $x = 1$. Intercepts are $(0, 0), (2, 0)$.

54) $f(x) = x^2 - 6x - 1$

55) $f(x) = x^2 - 5x - 6$

**Answer**

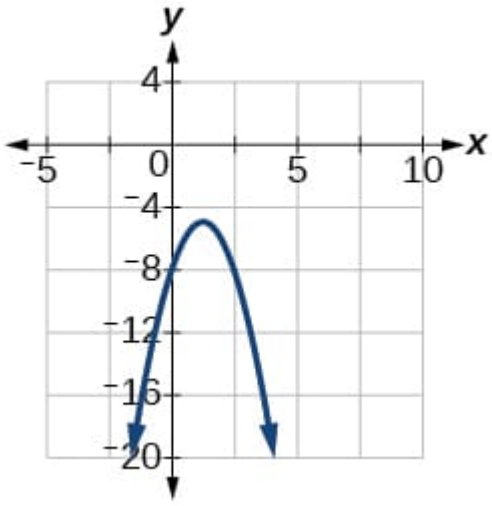



Vertex $\left(\frac{5}{2}, \frac{-49}{4}\right)$, Axis of symmetry is $x = \frac{5}{2}$. Intercepts are $(0, -6), (-1, 0), (6, 0)$.

56) $f(x) = x^2 - 7x + 3$

57) $f(x) = -2x^2 + 5x - 8$

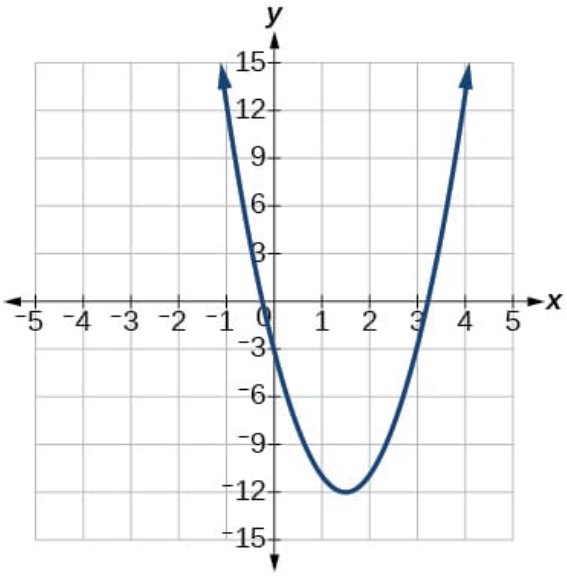
**Answer**




Vertex $\left(\frac{5}{4}, -\frac{39}{8}\right)$, Axis of symmetry is $x = \frac{5}{4}$. Intercepts are $(0, -8)$.

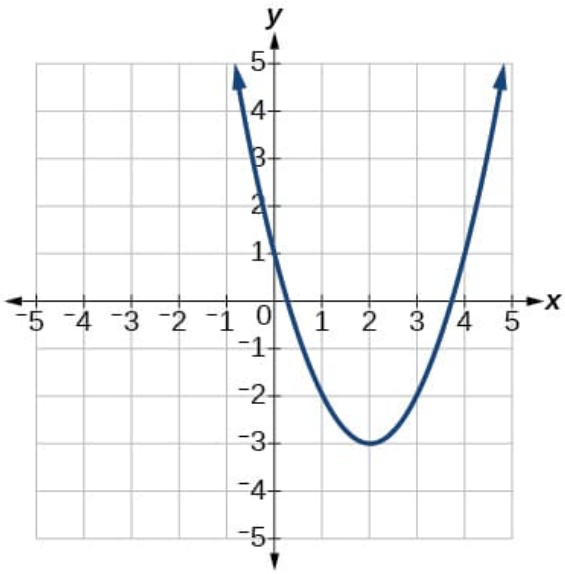
58) $f(x) = 4x^2 - 12x - 3$

**Answer**




**For the exercises 59-64, write the equation for the graphed function.**

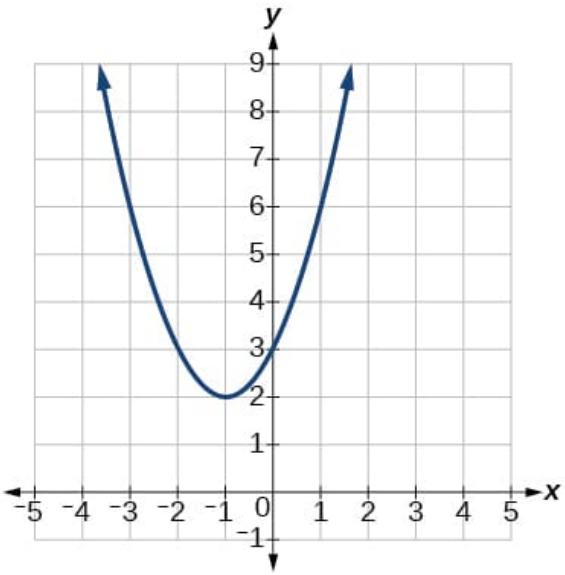
59)



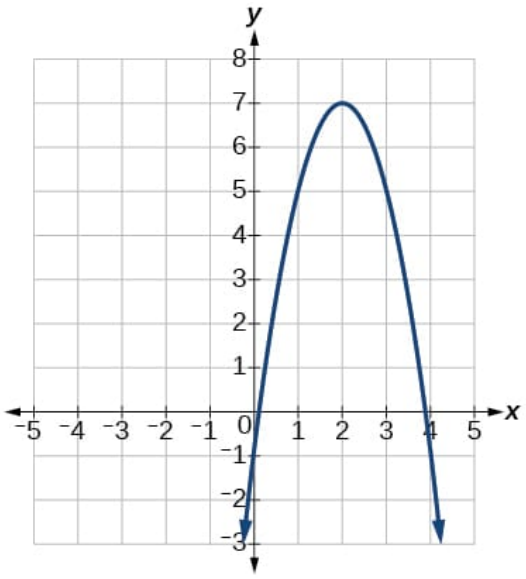

**Answer**

$f(x) = x^2 - 4x + 1$

60)



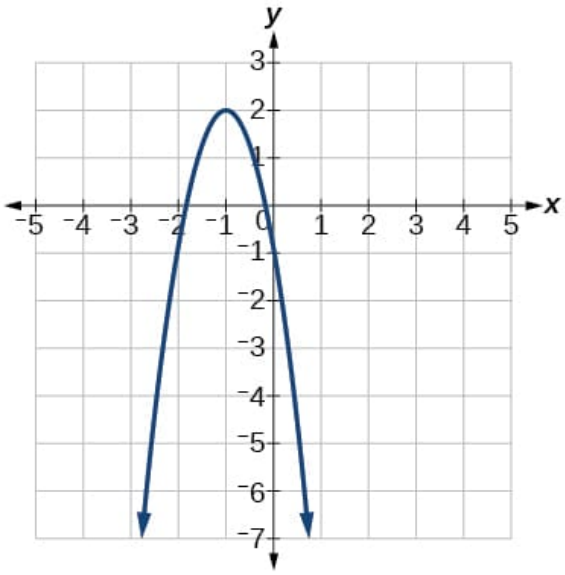

61)



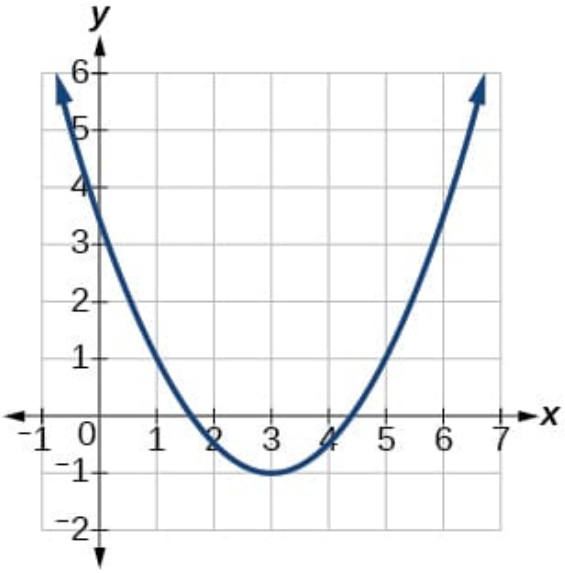

**Answer**

$f(x) = -2x^2 + 8x - 1$

62)



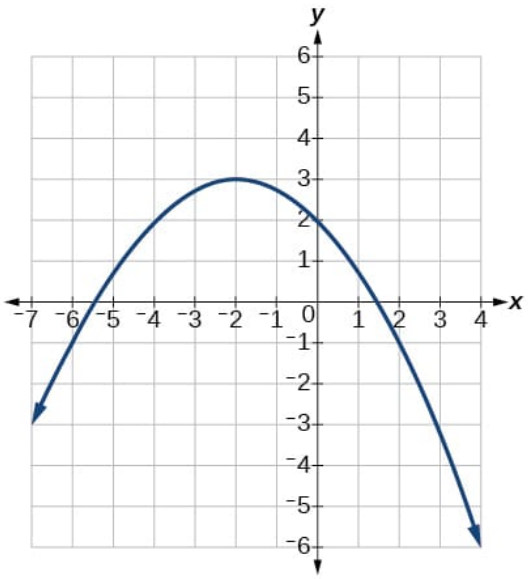

63)




**Answer**

$$f(x) = \frac{1}{2}x^2 - 3x + \frac{7}{2}$$

64)




## Numeric

**For the exercises 65-69, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.**

65)

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 5 | 2 | 1 | 2 | 5 |

**Answer**

$f(x) = x^2 + 1$

66)

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 1 | 0 | 1 | 4 | 9 |

67)

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | –2 | 1 | 2 | 1 | –2 |

**Answer**

$f(x) = 2 - x^2$

68)

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | –8 | –3 | 0 | 1 | 0 |

69)

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 8 | 2 | 0 | 2 | 8 |

**Answer**

$f(x) = 2x^2$

## Technology

**For the exercises 70-74, use a calculator to find the answer.**

70) Graph on the same set of axes the functions $f(x) = x^2, f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$. What appears to be the effect of changing the coefficient?

71) Graph on the same set of axes $f(x) = x^2, f(x) = x^2 + 2$ and $f(x) = x^2, f(x) = x^2 + 5$ and $f(x) = x^2 - 3$. What appears to be the effect of adding a constant?

**Answer**

The graph is shifted up or down (a vertical shift).

72) Graph on the same set of axes $f(x) = x^2, f(x) = (x - 2)^2, f(x - 3)2$, and $f(x) = (x + 4)^2$. What appears to be the effect of adding or subtracting those numbers?

73) The path of an object projected at a $45$ degree angle with initial velocity of $80$ feet per second is given by the function $h(x) = -\frac{32}{(80)^2}x^2 + x$ where $x$ is the horizontal distance traveled and $h(x)$ is the height in feet. Use the TRACE feature of your calculator to determine the height of the object when it has traveled $100$ feet away horizontally.

**Answer**

$50$ feet

74) A suspension bridge can be modeled by the quadratic function $h(x) = .0001x^2$ with $-2000 \leq x \leq 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use the TRACE feature of your calculator to estimate how far from the center does the bridge have a height of $100$ feet.

## Extensions

**For the exercises 75-78, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.**

75) Vertex $(1, -2)$, opens up.

**Answer**

Domain is $(-\infty, \infty)$. Range is $[-2, \infty)$.

76) Vertex $(-1, 2)$ opens down.

77) Vertex $(-5, 11)$, opens down.

**Answer**

Domain is $(-\infty, \infty)$ Range is $(-\infty, 11]$.

78) Vertex $(-100, 100)$, opens up.

**For the exercises 79-84, write the equation of the quadratic function that contains the given point and has the same shape as the given function.**

79) Contains $(1, 1)$ and has shape of $f(x) = 2x^2$. Vertex is on the $y$-axis.

**Answer**

$f(x) = 2x^2 - 1$

80) Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the $y$-axis.

81) Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the $y$-axis.

**Answer**

$f(x) = 3x^2 - 9$

82) Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the $y$-axis.

83) Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the $y$-axis.

**Answer**

$f(x) = 5x^2 - 77$

84) Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has $x$-coordinate of $-1$.

## Real-World Applications

85) Find the dimensions of the rectangular corral producing the greatest enclosed area given $200$ feet of fencing.

**Answer**

$50$ feet by $50$ feet. Maximize $f(x) = -x^2 + 100x$.

86) Find the dimensions of the rectangular corral split into $2$ pens of the same size producing the greatest possible enclosed area given $300$ feet of fencing.

87) Find the dimensions of the rectangular corral producing the greatest enclosed area split into $3$ pens of the same size given $500$ feet of fencing.

**Answer**

$125$ feet by $62.5$ feet. Maximize $f(x) = -2x^2 + 250x$.

88) Among all of the pairs of numbers whose sum is $6$, find the pair with the largest product. What is the product?

89) Among all of the pairs of numbers whose difference is $12$, find the pair with the smallest product. What is the product?

**Answer**

$6$ and $-6$; product is $-36$; maximize $f(x) = x^2 + 12x$.

90) Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x$, where $x$ is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.

91) A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.

**Answer**

$2909.56$ meters

92) A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?

93) A soccer stadium holds $62,000$ spectators. With a ticket price of $\$11$, the average attendance has been $26,000$. When the price dropped to $\$9$, the average attendance rose to $31,000$. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

**Answer**

$\$10.70$

94) A farmer finds that if she plants $75$ trees per acre, each tree will yield $20$ bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by $3$ bushels. How many trees should she plant per acre to maximize her harvest?

## 3.3 Power Functions and Polynomial Functions

# Verbal

1) Explain the difference between the coefficient of a power function and its degree.

**Answer**

The coefficient of the power function is the real number that is multiplied by the variable raised to a power. The degree is the highest power appearing in the function.

2) If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?

3) In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.

**Answer**

As $x$ decreases without bound, so does $f(x)$. As $x$ increases without bound, so does $f(x)$.

4) What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?

5) What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty, f(x) \to -\infty$ and as $x \to \infty, f(x) \to -\infty$.

**Answer**

The polynomial function is of even degree and leading coefficient is negative.

# Algebraic

**For the exercises 6-11, identify the function as a power function, a polynomial function, or neither.**

6) $f(x) = x^5$

7) $f(x) = (x^2)^3$

**Answer**

Power function

8) $f(x) = x - x^4$

9) $f(x) = \dfrac{x^2}{x^2 - 1}$

**Answer**

Neither

10) $f(x) = 2x(x + 2)(x - 1)^2$

11) $f(x) = 3^{x+1}$

**Answer**

Neither

**For the exercises 12-16, find the degree and leading coefficient for the given polynomial.**

12) $-3x^4$

13) $7 - 2x^2$

**Answer**

Degree = $2$, Coefficient = $-2$

14) $-2x^2 - 3x^5 + x - 6$

15) $x(4-x^2)(2x+1)$

**Answer**

Degree =4, Coefficient = $-2$

16) $x^2(2x-3)^2$

**For the exercises 17-24, determine the end behavior of the functions.**

17) $f(x)=x^4$

**Answer**

As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to \infty$

18) $f(x)=x^3$

19) $f(x)=-x^4$

**Answer**

As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$

20) $f(x)=-x^9$

21) $f(x)=-2x^4-3x^2+x-1$

**Answer**

As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$

22) $f(x)=3x^2+x-2$

23) $f(x)=x^2(2x^3-x+1)$

**Answer**

As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to -\infty$

24) $f(x)=(2-x)^7$

**For the exercises 25-30, find the intercepts of the functions.**

25) $f(t)=2(t-1)(t+2)(t-3)$

**Answer**

$y$-intercept is $(0,12)$, $t$-intercepts are $(1,0)$; $(-2,0)$; and $(3,0)$.

26) $g(n)=-2(3n-1)(2n+1)$

27) $f(x)=x^4-16$

**Answer**

$y$-intercept is $(0,-16)$, $x$-intercepts are $(2,0)$ and $(-2,0)$.

28) $f(x)=x^3+27$

29) $f(x) = x(x^2 - 2x - 8)$

**Answer**

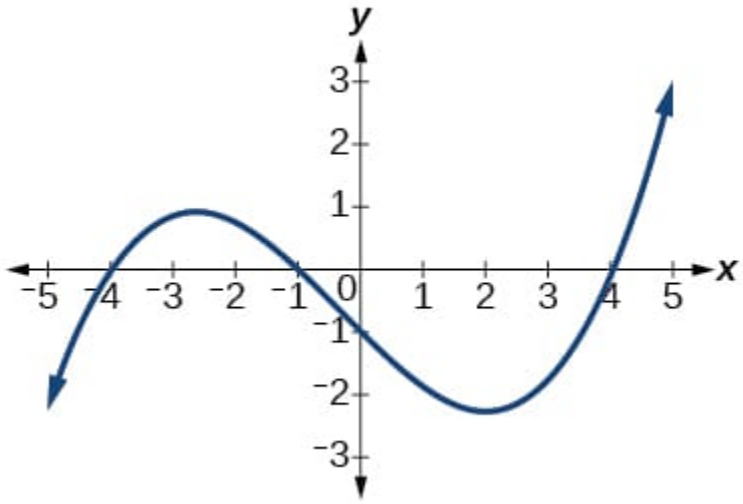
$y$-intercept is $(0, 0)$, $x$-intercepts are $(0, 0), (4, 0)$, and $(-2, 0)$.

30) $f(x) = (x + 3)(4x^2 - 1)$

# Graphical

**For the exercises 31-38, determine the least possible degree of the polynomial function shown.**

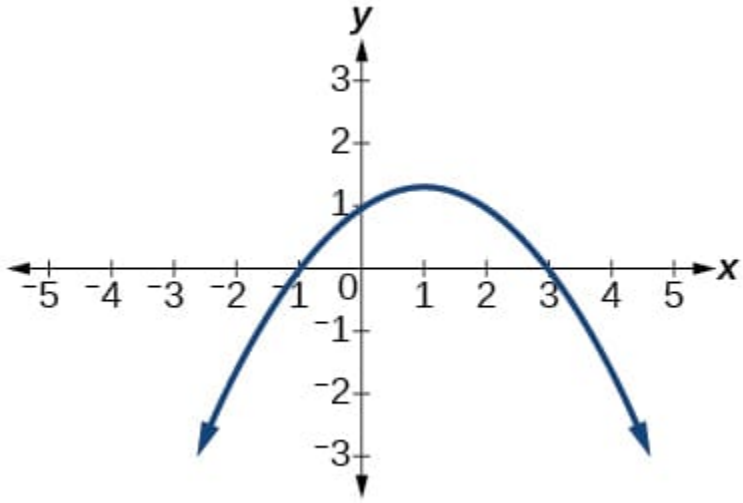
31)



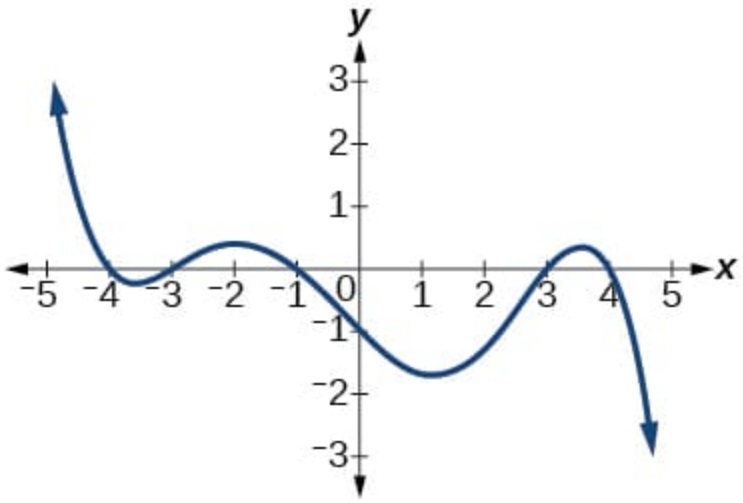

**Answer**

3

32)



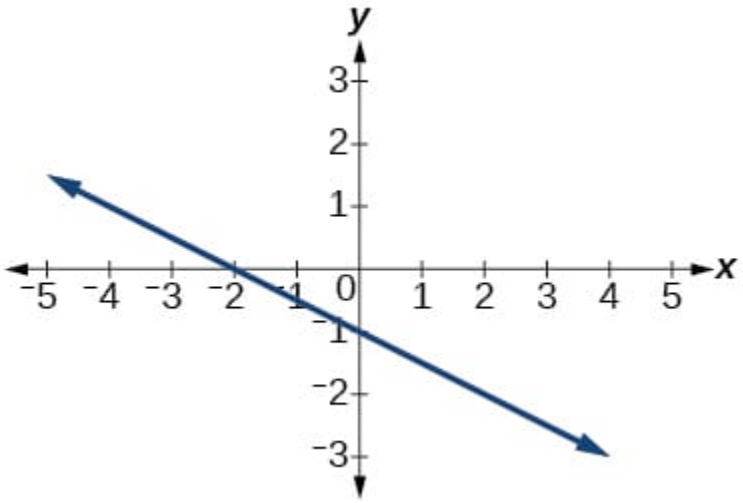

33)



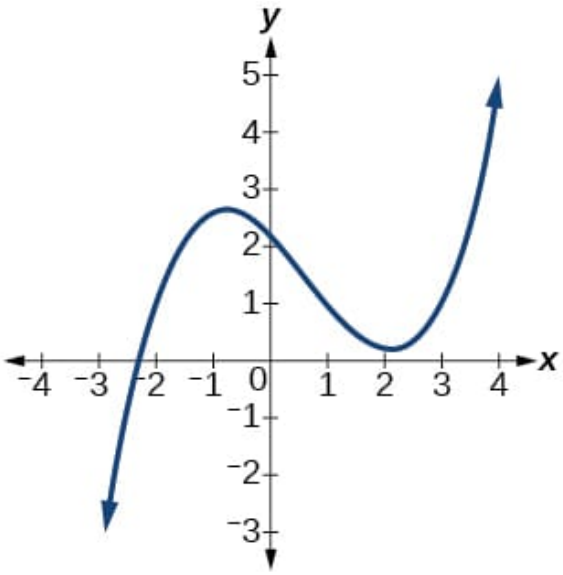

**Answer**

5

34)



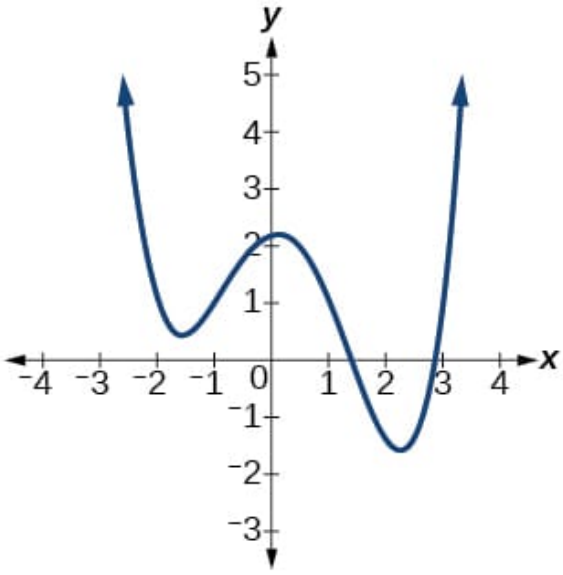

35)



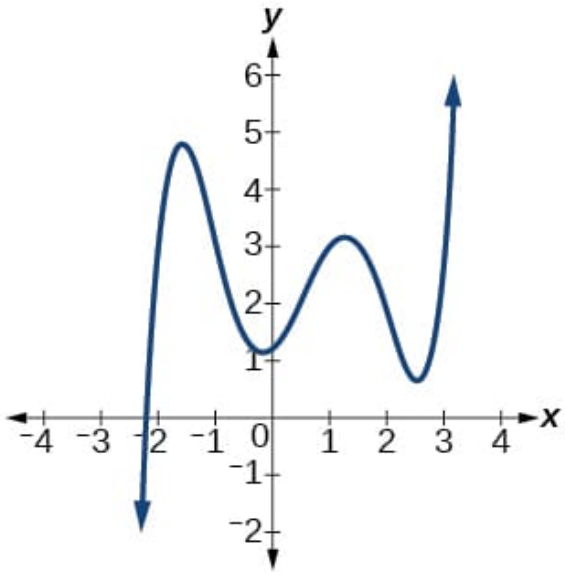

**Answer**

3

36)



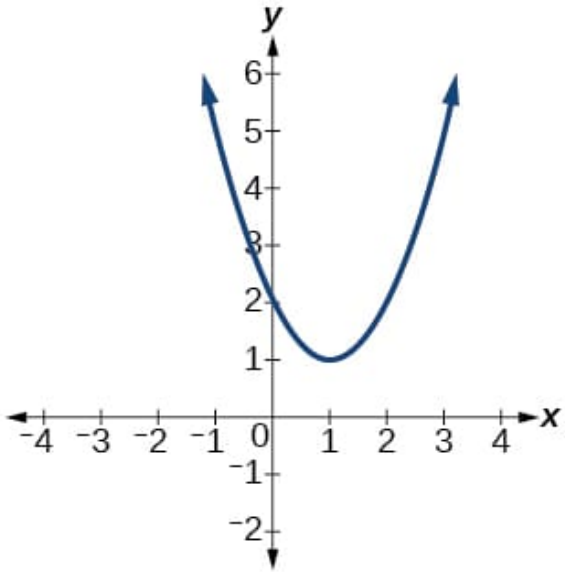

37)




**Answer**

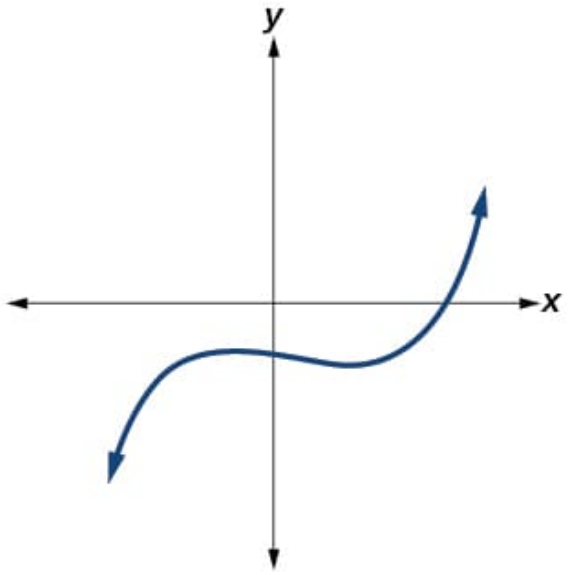
5

38)




**For the exercises 39-45, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.**

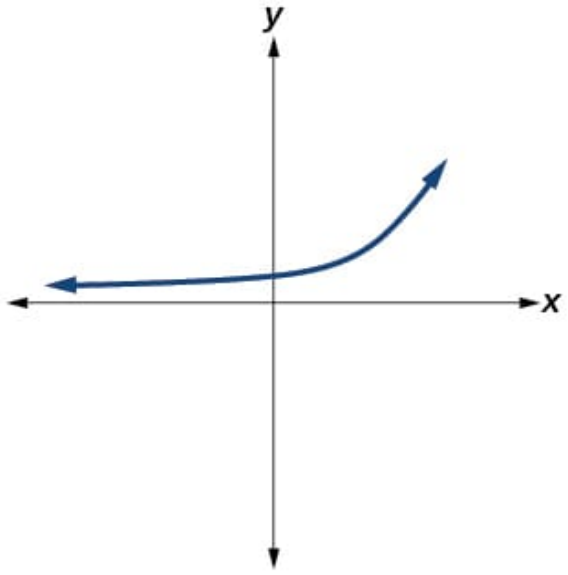
39)



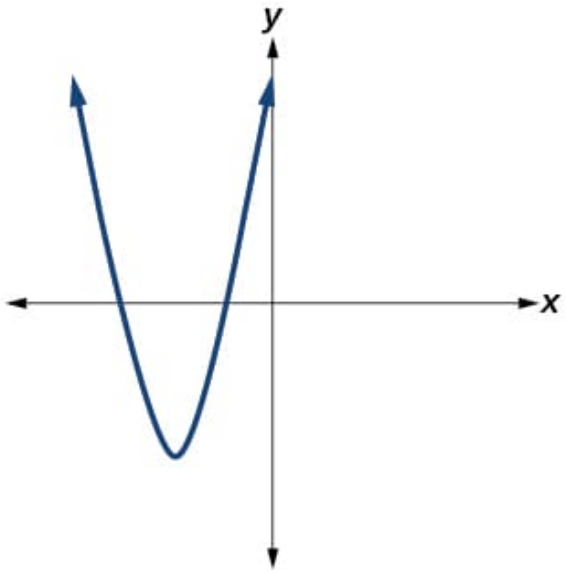

**Answer**

Yes. Number of turning points is $2$. Least possible degree is $3$.

40)



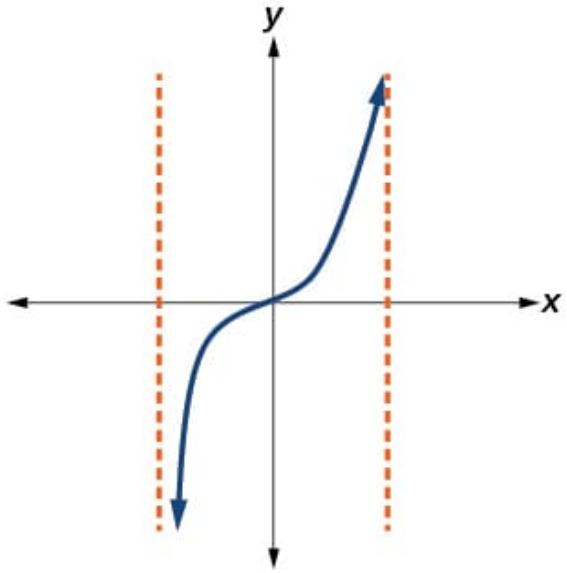

41)



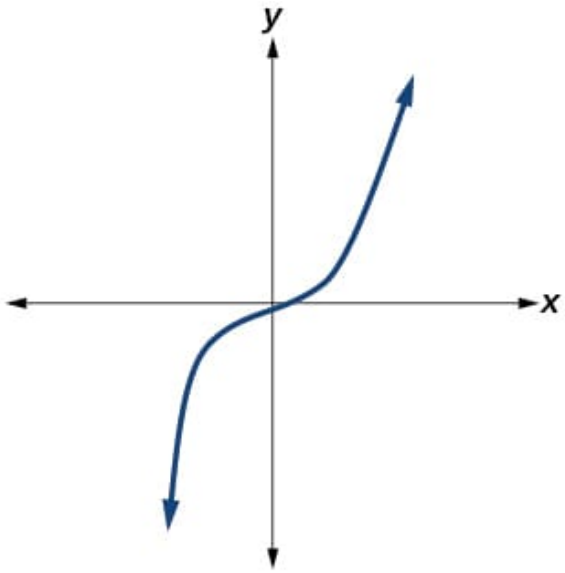

**Answer**

Yes. Number of turning points is $1$. Least possible degree is $2$.

42)



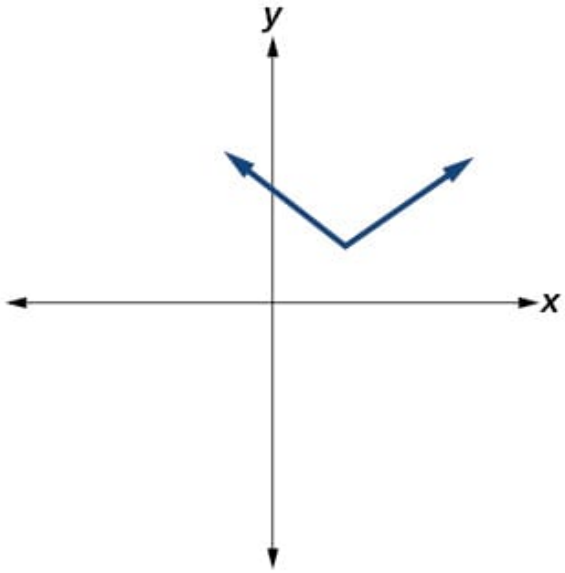

43)




**Answer**

Yes. Number of turning points is $0$. Least possible degree is $1$.

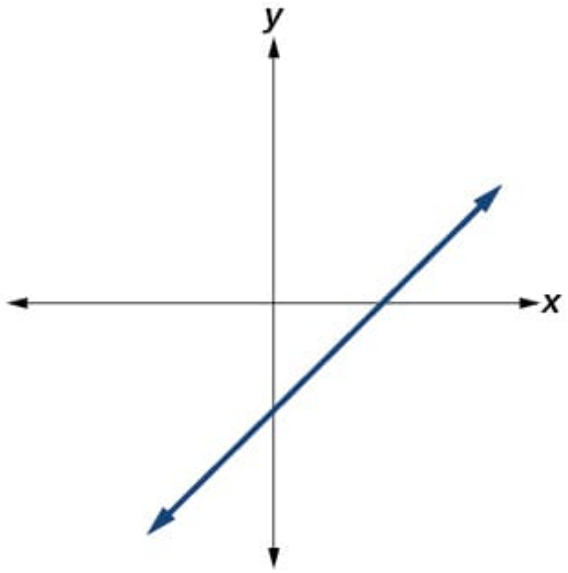
44)




**Answer**

No.

45)




**Answer**

Yes. Number of turning points is $0$. Least possible degree is $1$.

# Numeric

**For the exercises 46-50, make a table to confirm the end behavior of the function.**

46) $f(x) = -x^3$

47) $f(x) = x^4 - 5x^2$

**Answer**

| $x$ | $f(x)$ |
|---|---|
| 10 | 9,500 |
| 100 | 99,950,000 |
| -10 | 9,500 |
| -100 | 99,950,000 |

as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$

48) $f(x) = x^2(1-x)^2$

49) $f(x) = (x-1)(x-2)(3-x)$

**Answer**

| $x$ | $f(x)$ |
|---|---|
| 10 | -504 |
| 100 | -941,094 |
| -10 | 1,716 |
| -100 | 1,061,106 |

as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$

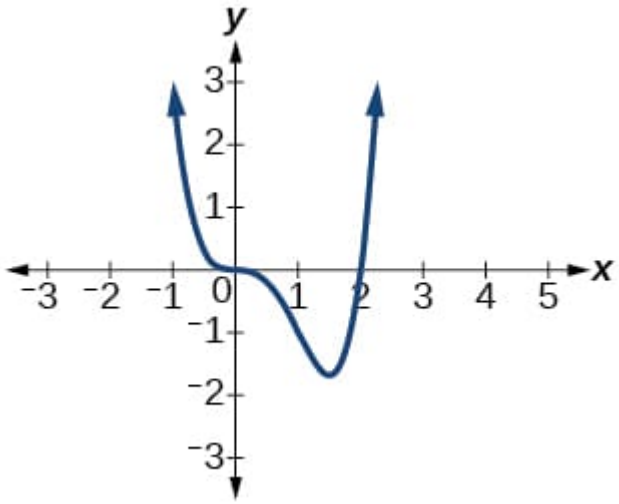
50) $f(x) = \frac{x^5}{10} - x^4$

# Technology

**For the exercises 51-60, graph the polynomial functions using a calculator. Based on the graph, determine the intercepts and the end behavior.**

51) $f(x) = x^3(x-2)$

**Answer**

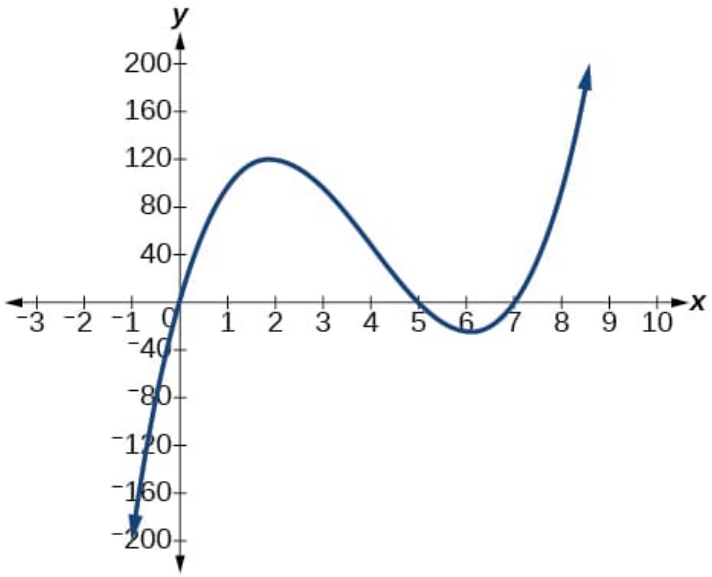



The $y$-intercept is $(0,0)$. The x-intercepts are $(0,0), (2,0)$. As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$

52) $f(x) = x(x-3)(x+3)$

53) $f(x) = x(14-2x)(10-2x)$

**Answer**

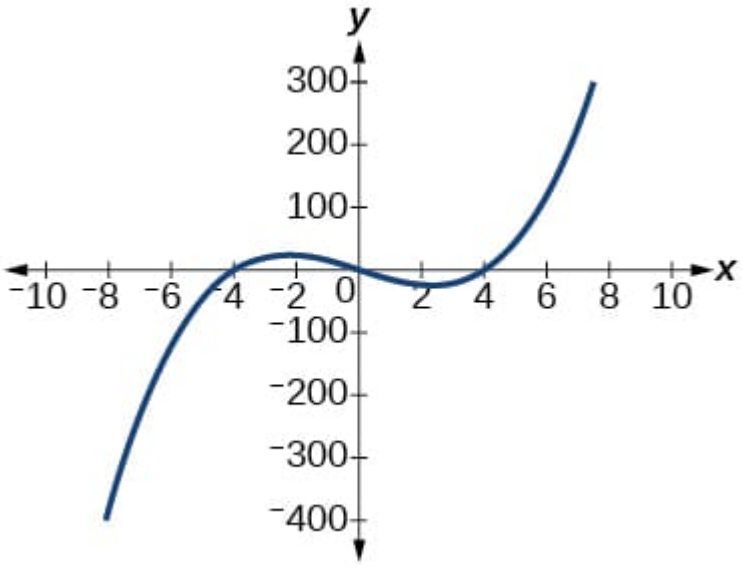



The $y$-intercept is $(0,0)$. The x-intercepts are $(0,0),(5,0),(7,0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$

54) $f(x) = x(14-2x)(10-2x)^2$

55) $f(x) = x^3 - 16x$

**Answer**

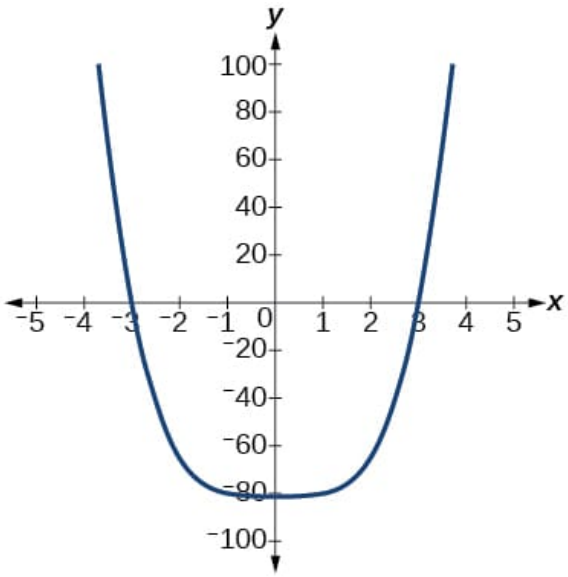



The $y$-intercept is $(0,0)$. The x-intercept is $(-4,0),(0,0),(4,0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$

56) $f(x) = x^3 - 27$

57) $f(x) = x^4 - 81$

**Answer**

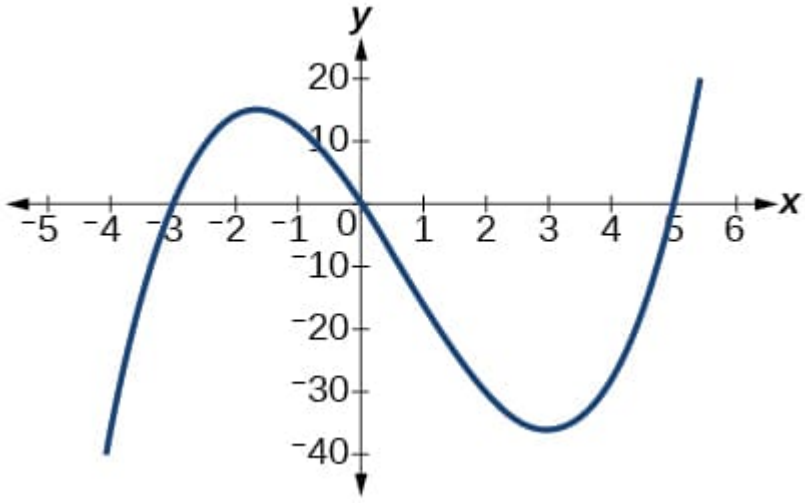



The $y$-intercept is $(0, -81)$. The $x$-intercept are $(3, 0), (-3, 0)$. As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$

58) $f(x) = -x^3 + x^2 + 2x$

59) $f(x) = x^3 - 2x^2 - 15x$

**Answer**




The $y$-intercept is $(0, 0)$. The x-intercepts are $(-3, 0), (0, 0), (5, 0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$

60) $f(x) = x^3 - 0.01x$

# Extensions

**For the exercises 61-65, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is $1$ or $-1$. There may be more than one correct answer.**

61) The $y$-intercept is $(0, -4)$. The $x$-intercepts are $(-2, 0), (2, 0)$. Degree is $2$.

End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$.

**Answer**

$$f(x) = x^2 - 4$$

62) The $y$-intercept is $(0, 9)$. The $x$-intercepts are $(-3, 0), (3, 0)$. Degree is $2$.

End behavior: as $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$.

63) The $y$-intercept is $(0, 0)$. The $x$-intercepts are $(0, 0), (2, 0)$. Degree is $3$.

End behavior: as $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.

**Answer**

$$f(x) = x^3 - 4x^2 + 4x$$

64) The $y$-intercept is $(0, 1)$. The x-intercept is $(1, 0)$. Degree is $3$.

End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$.

65) The $y$-intercept is $(0, 1)$. There is no $x$-intercept. Degree is $4$.

End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$.

**Answer**

$$f(x) = x^4 + 1$$

# Real-World Applications

**For the exercises 66-70, use the written statements to construct a polynomial function that represents the required information.**

66) An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of $20$ meters per day. Express the area of the circle as a function of $d$, the number of days elapsed.

67) A cube has an edge of $3$ feet. The edge is increasing at the rate of $2$ feet per minute. Express the volume of the cube as a function of $m$, the number of minutes elapsed.

**Answer**

$$V(m) = 8m^3 + 36m^2 + 54m + 27$$

68) A rectangle has a length of $10$ inches and a width of $6$ inches. If the length is increased by $x$ inches and the width increased by twice that amount, express the area of the rectangle as a function of $x$.

69) An open box is to be constructed by cutting out square corners of $x$-inch sides from a piece of cardboard $8$ inches by $8$ inches and then folding up the sides. Express the volume of the box as a function of $x$.

**Answer**

$$V(x) = 4x^3 - 32x^2 + 64x$$

70) A rectangle is twice as long as it is wide. Squares of side $2$ feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width ($x$).

## 3.4 Graphs of Polynomial Functions

# Verbal

1) What is the difference between an $x$-intercept and a zero of a polynomial function $f$?

**Answer**

The $x$-intercept is where the graph of the function crosses the $x$-axis, and the zero of the function is the input value for which $f(x) = 0$.

2) If a polynomial function of degree $n$ has $n$ distinct zeros, what do you know about the graph of the function?

3) Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.

**Answer**

If we evaluate the function at $a$ and at $b$ and the sign of the function value changes, then we know a zero exists between $a$ and $b$.

4) Explain how the factored form of the polynomial helps us in graphing it.

5) If the graph of a polynomial just touches the $x$-axis and then changes direction, what can we conclude about the factored form of the polynomial?

**Answer**

There will be a factor raised to an even power.

# Algebraic

**For the exercises 6-23, find the $x$-or $t$-intercepts of the polynomial functions.**

6) $C(t) = 2(t-4)(t+1)(t-6)$

7) $C(t) = 3(t+2)(t-3)(t+5)$

**Answer**

$$(-2, 0), (3, 0), (-5, 0)$$

8) $C(t) = 4t(t-2)^2(t+1)$

9) $C(t) = 2t(t-3)(t+1)^2$

**Answer**

$$(3, 0), (-1, 0), (0, 0)$$

10) $C(t) = 2t^4 - 8t^3 + 6t^2$

11) $C(t) = 4t^4 + 12t^3 - 40t^2$

**Answer**

$$(0,0), (-5,0), (2,0)$$

12) $f(x) = x^4 - x^2$

13) $f(x) = x^3 + x^2 - 20x$

**Answer**

$$(0,0), (-5,0), (4,0)$$

14) $f(x) = x^3 + 6x^2 - 7x$

15) $f(x) = x^3 + x^2 - 4x - 4$

**Answer**

$$(2,0), (-2,0), (-1,0)$$

16) $f(x) = x^3 + 2x^2 - 9x - 18$

17) $f(x) = 2x^3 - x^2 - 8x + 4$

**Answer**

$$(-2,0), (2,0), \left(\frac{1}{2}, 0\right)$$

18) $f(x) = x^6 - 7x^3 - 8$

19) $f(x) = 2x^4 + 6x^2 - 8$

**Answer**

$$(1,0), (-1,0)$$

20) $f(x) = x^3 - 3x^2 - x + 3$

21) $f(x) = x^6 - 2x^4 - 3x^2$

**Answer**

$$(0,0), (\sqrt{3},0), (-\sqrt{3},0)$$

22) $f(x) = x^6 - 3x^4 - 4x^2$

23) $f(x) = x^5 - 5x^3 + 4x$

**Answer**

$$(0,0), (1,0), (-1,0), (2,0), (-2,0)$$

**For the exercises 24-29, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.**

24) $f(x) = x^3 - 9x$, between $x = -4$ and $x = -2$.

25) $f(x) = x^3 - 9x$, between $x = 2$ and $x = 4$.

**Answer**

$f(2) = -10$ and $f(4) = 28$. Sign change confirms.

26) $f(x) = x^5 - 2x$, between $x = 1$ and $x = 2$.

27) $f(x) = -x^4 + 4$, between $x = 1$ and $x = 3$.

**Answer**

$f(1) = 3$ and $f(3) = -77$. Sign change confirms.

28) $f(x) = -2x^3 - x$, between $x = -1$ and $x = 1$.

29) $f(x) = x^3 - 100x + 2$, between $x = 0.01$ and $x = 0.1$

**Answer**

$f(0.01) = 1.000001$ and $f(0.1) = -7.999$. Sign change confirms.

**For the exercises 30-41, find the zeros and give the multiplicity of each.**

30) $f(x) = (x + 2)^3(x - 3)^2$

31) $f(x) = x^2(2x + 3)^5(x - 4)^2$

**Answer**

$0$ with multiplicity $2$, $-\frac{3}{2}$ with multiplicity $5$, $4$ with multiplicity $2$

32) $f(x) = x^3(x - 1)^3(x + 2)$

33) $f(x) = x^2(x^2 + 4x + 4)$

**Answer**

$0$ with multiplicity $2$, $-2$ with multiplicity $2$

34) $f(x) = (2x + 1)^3(9x^2 - 6x + 1)$

35) $f(x) = (3x + 2)^5(x^2 - 10x + 25)$

**Answer**

$-\frac{2}{3}$ with multiplicity $5$, $5$ with multiplicity $2$

36) $f(x) = x(4x^2 - 12x + 9)(x^2 + 8x + 16)$

37) $f(x) = x^6 - x^5 - 2x^4$

**Answer**

$0$ with multiplicity $4$, $2$ with multiplicity $1$, $-1$ with multiplicity $1$

38) $f(x) = 3x^4 + 6x^3 + 3x^2$

39) $f(x) = 4x^5 - 12x^4 + 9x^3$

**Answer**

$\frac{3}{2}$ with multiplicity 2, 0 with multiplicity 3

40) $f(x) = 2x^4(x^3 - 4x^2 + 4x)$

41) $f(x) = 4x^4(9x^4 - 12x^3 + 4x^2)$

**Answer**

0 with multiplicity 6, $\frac{2}{3}$ with multiplicity 2

# Graphical

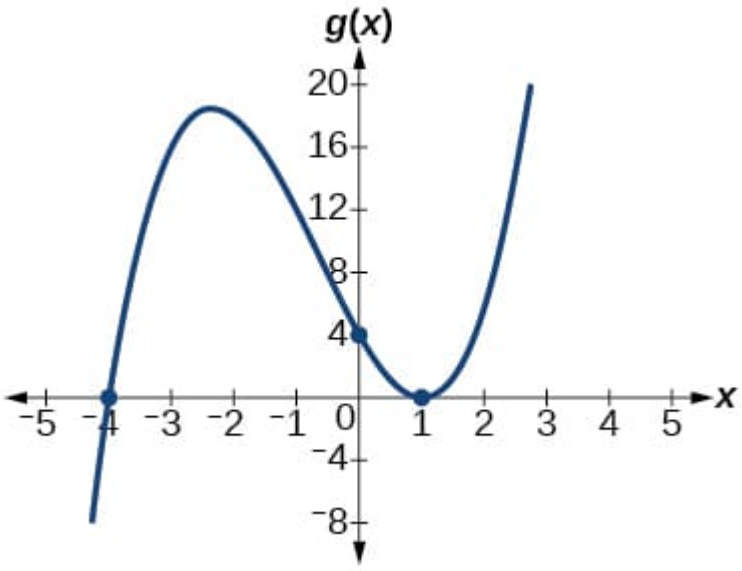
**For the exercises 42-47, graph the polynomial functions. Note $x$- and $y$- intercepts, multiplicity, and end behavior.**

42) $f(x) = (x + 3)^2(x - 2)$

43) $g(x) = (x + 4)(x - 1)^2$

**Answer**

$x$-intercepts, $(1, 0)$ with multiplicity 2, $(-4, 0)$ with multiplicity 1, $y$- intercept $(0, 4)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.

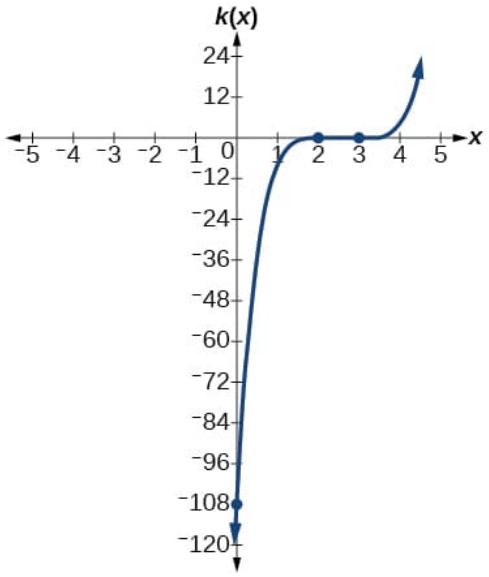



44) $h(x) = (x - 1)^3(x + 3)^2$

45) $k(x) = (x - 3)^3(x - 2)^2$

**Answer**

$x$-intercepts $(3, 0)$ with multiplicity 3, $(2, 0)$ with multiplicity 2, $y$- intercept $(0, -108)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.

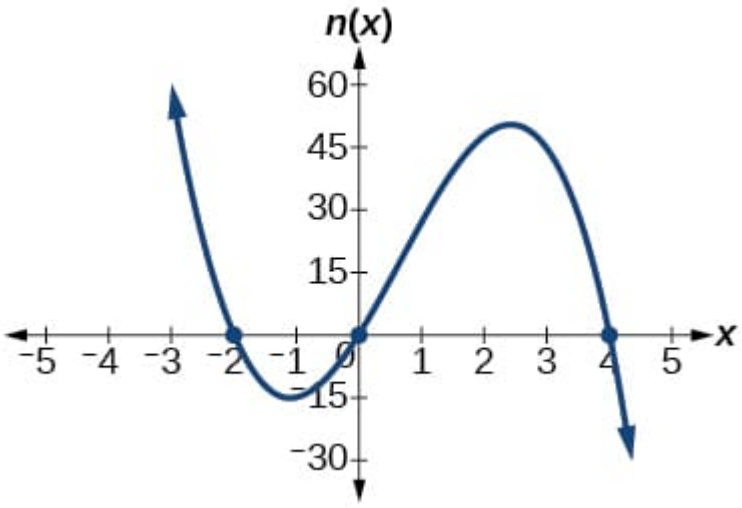



46) $m(x) = -2x(x-1)(x+3)$

47) $n(x) = -3x(x+2)(x-4)$

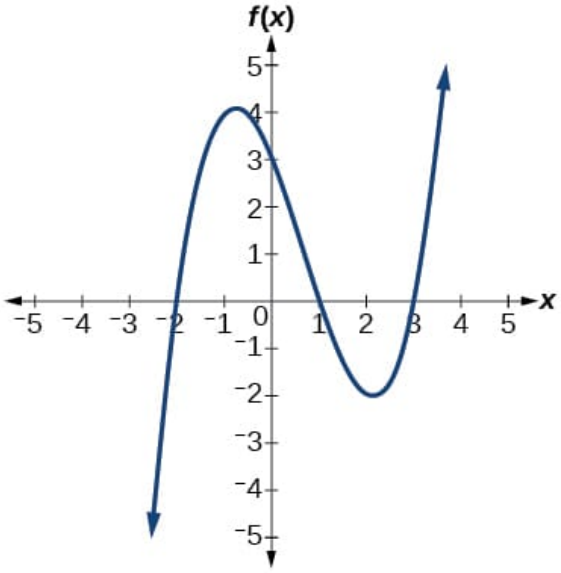
**Answer**

$x$-intercepts $(0,0),(-2,0),(4,0)$ with multiplicity $1$, $y$-intercept (0, 0). As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$.

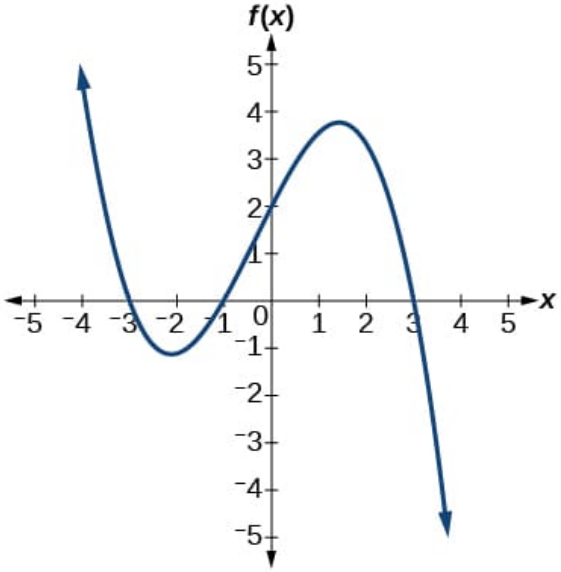



**For the exercises 48-52, use the graphs to write the formula for a polynomial function of least degree.**

48)



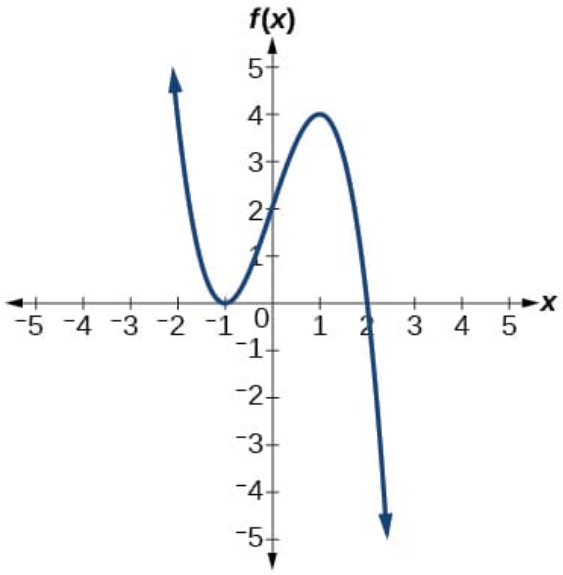

49)



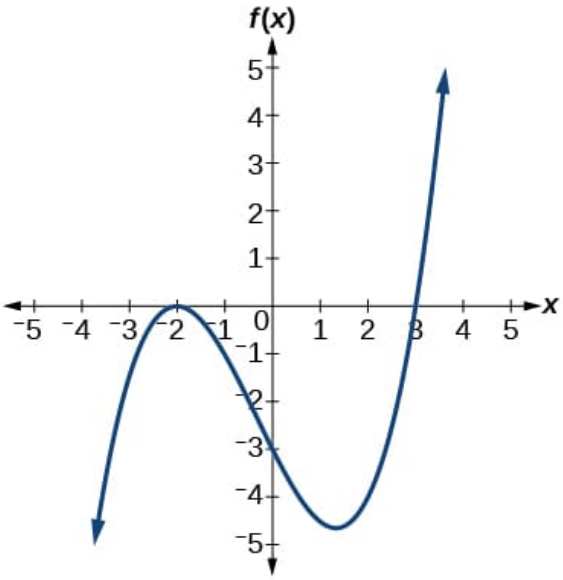

**Answer**

$$f(x) = -\frac{2}{9}(x-3)(x+1)(x+3)$$

50)



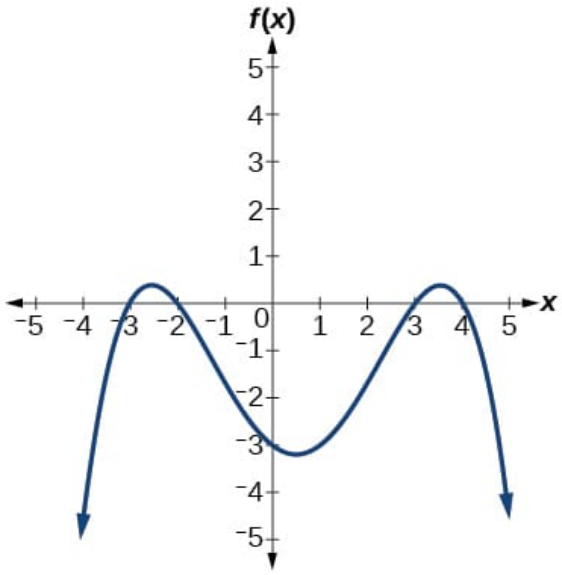

51)




**Answer**

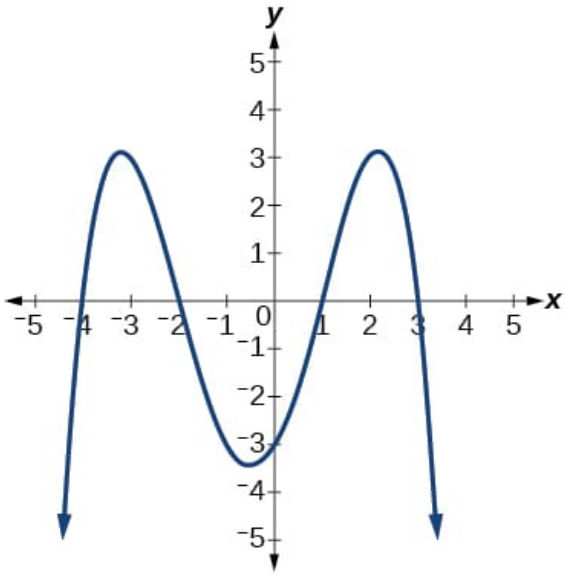
$$f(x) = \frac{1}{4}(x+2)^2(x-3)$$

52)




**For the exercises 53-56, use the graph to identify zeros and multiplicity.**

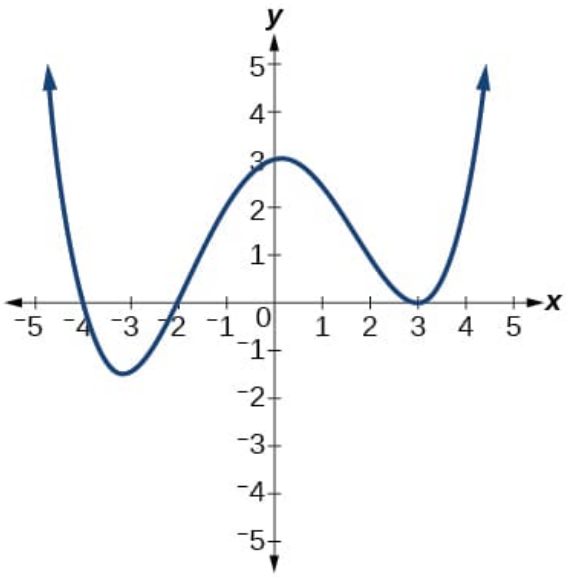
53)



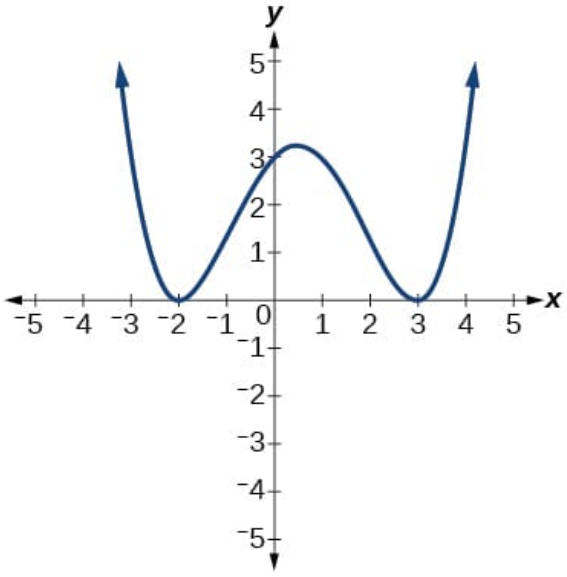

**Answer**

$-4, -2, 1, 3$ with multiplicity $1$

54)



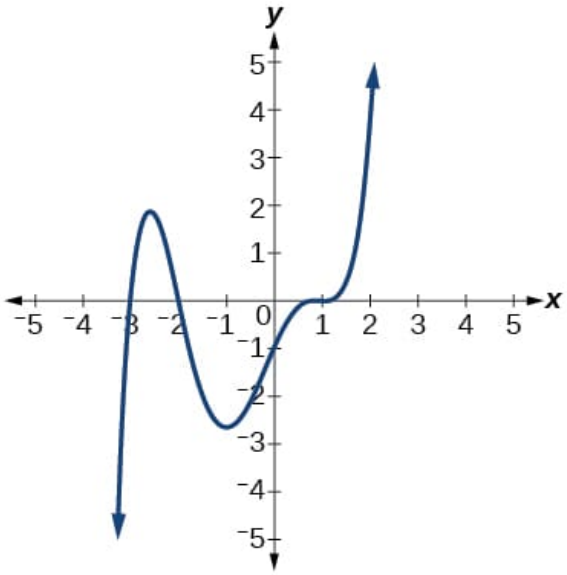

55)




**Answer**

$-2, 3$ each with multiplicity $2$

56)




**For the exercises 57-66, use the given information about the polynomial graph to write the equation.**

57) Degree $3$. Zeros at $x = -2$,$x = 1$, and $x = 3$. $y$-intercept at $(0, -4)$

**Answer**

$$f(x) = -\frac{2}{3}(x+2)(x-1)(x-3)$$

58) Degree $3$. Zeros at $x = -5$, $x = -2$, and $x = 1$. $y$-intercept at $(0, 6)$

59) Degree $5$. Roots of multiplicity $2$ at $x = 3$ and $x = 1$, and a root of multiplicity $1$ at $x = -3$. $y$-intercept at $(0, 9)$

**Answer**

$$f(x) = \frac{1}{3}(x-3)^2(x-1)^2(x+3)$$

60) Degree $4$. Root of multiplicity $2$ at $x = 4$, and a roots of multiplicity $1$ at $x = 1$ and $x = -2$. $y$-intercept at $(0, -3)$

61) Degree $5$. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point $(2, 15)$

**Answer**

$$f(x) = -15(x-1)^2(x-3)^3$$

62) Degree $3$. Zeros at $x = 4$, $x = 3$, and $x = 2$. $y$-intercept at $(0, -24)$

63) Degree $3$. Zeros at $x = -3$, $x = -2$ and $x = 1$. $y$-intercept at $(0, 12)$

**Answer**

$$f(x) = -2(x+3)(x+2)(x-1)$$

64) Degree $5$. Roots of multiplicity $2$ at $x = -3$ and $x = 2$ and a root of multiplicity $1$ at $x = -2$. $y$-intercept at $(0, 4)$.

65) Degree $4$. Roots of multiplicity $2$ at $x = \frac{1}{2}$ and roots of multiplicity $1$ at $x = 6$ and $x = -2$. $y$-intercept at $(0, 18)$

**Answer**

$$f(x) = -\frac{3}{2}(2x-1)^2(x-6)(x+2)$$

66) Double zero at $x = -3$ and triple zero at $x = 0$. Passes through the point $(1, 32)$.

# Technology

**For the exercises 67-71, use a calculator to approximate local minima and maxima or the global minimum and maximum.**

67) $f(x) = x^3 - x - 1$

**Answer**

local max $(-.58, -.62)$, local min $(.58, -1.38)$

68) $f(x) = 2x^3 - 3x - 1$

69) $f(x) = x^4 + x$

**Answer**

global min $(-.63, -.47)$

70) $f(x) = -x^4 + 3x - 2$

71) $f(x) = x^4 - x^3 + 1$

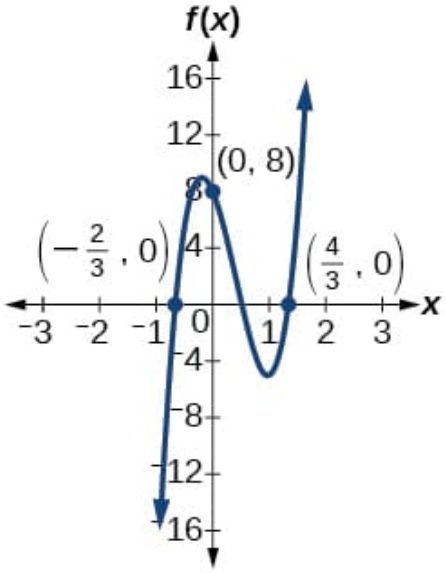
**Answer**

global min $(.75, .89)$

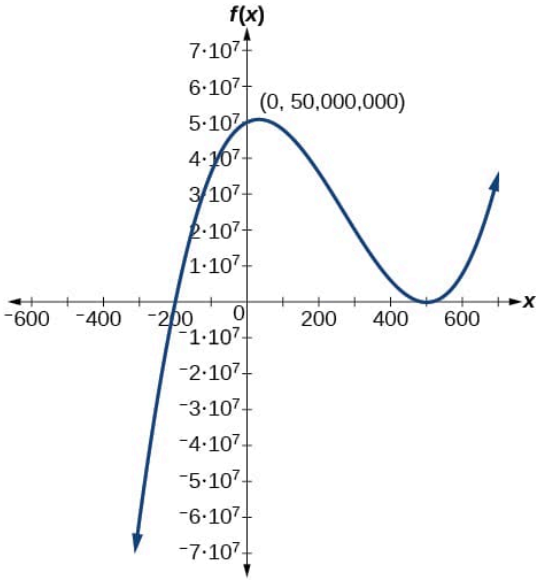
# Extensions

**For the exercises 72-74, use the graphs to write a polynomial function of least degree.**

72)



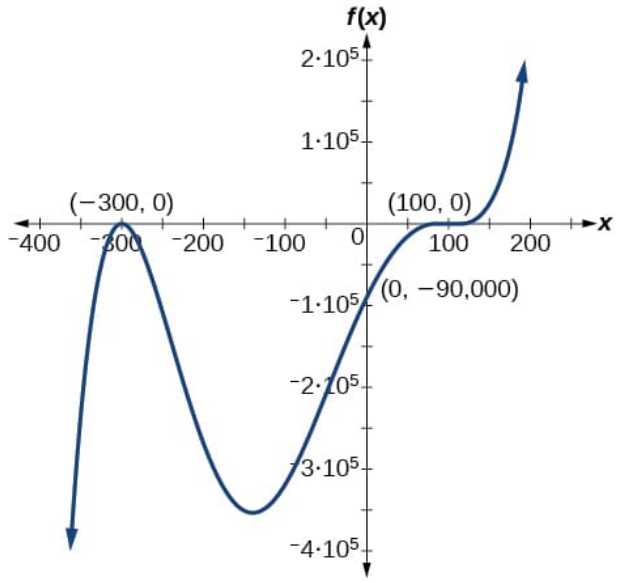

73)




**Answer**

$f(x) = (x - 500)^2(x + 200)$

74)




# Real-World Applications

**For the exercises 75-, write the polynomial function that models the given situation.**

75) A rectangle has a length of $10$ units and a width of 8 units. Squares of $x$ by $x$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of $x$.

**Answer**

$$f(x) = 4x^3 - 36x^2 + 80x$$

76) Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of $x$.

77) A square has sides of $12$ units. Squares $x + 1$ by $x + 1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of $x$.

**Answer**

$$f(x) = 4x^3 - 36x^2 + 60x + 100$$

78) A cylinder has a radius of $x + 2$ units and a height of $3$ units greater. Express the volume of the cylinder as a polynomial function.

79) A right circular cone has a radius of $3x + 6$ and a height $3$ units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ for radius $r$ and height $h$.

**Answer**

$$f(x) = \pi(9x^3 + 45x^2 + 72x + 36)$$

# 3.5 Dividing Polynomials

## Verbal

1) If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?

**Answer**

The binomial is a factor of the polynomial.

2) If a polynomial of degree $n$ is divided by a binomial of degree $1$, what is the degree of the quotient?

## Algebraic

**For the exercises 3-13, use long division to divide. Specify the quotient and the remainder.**

3) $(x^2+5x-1)\div(x-1)$

**Answer**

$$\mathrm{x}+6+\frac{5}{\mathrm{x}-1},\mathrm{quotient}:\mathrm{x}+6,\mathrm{remainder}:5$$

4) $(2x^2-9x-5)\div(x-5)$

5) $(3x^2+23x+14)\div(x+7)$

**Answer**

$$3\mathrm{x}+2,\mathrm{quotient}:3\mathrm{x}+2,\mathrm{remainder}:0$$

6) $(4x^2-10x+6)\div(4x+2)$

7) $(6x^2-25x-25)\div(6x+5)$

**Answer**

$$\mathrm{x}-5,\mathrm{quotient}:\mathrm{x}-5,\mathrm{remainder}:0$$

8) $(-x^2-1)\div(x+1)$

9) $(2x^2-3x+2)\div(x+2)$

**Answer**

$$2\mathrm{x}-7+\frac{16}{\mathrm{x}+2},\mathrm{quotient}:2\mathrm{x}-7,\mathrm{remainder}:16$$

10) $(x^3-126)\div(x-5)$

11) $(3x^2-5x+4)\div(3x+1)$

**Answer**

$$\mathrm{x}-2+\frac{6}{3\mathrm{x}+1},\mathrm{quotient}:\mathrm{x}-2,\mathrm{remainder}:6$$

12) $(x^3 - 3x^2 + 5x - 6) \div (x - 2)$

13) $(2x^3 + 3x^2 - 4x + 15) \div (x + 3)$

**Answer**

$$2x^2 - 3x + 5, \text{quotient} : 2x^2 - 3x + 5, \text{remainder} : 0$$

**For the exercises 14-37, use synthetic division to find the quotient.**

14) $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

15) $(2x^3 - 6x^2 - 7x + 6) \div (x - 4)$

**Answer**

$$2x^2 + 2x + 1 + \frac{10}{x - 4}$$

16) $(6x^3 - 10x^2 - 7x - 15) \div (x + 1)$

17) $(4x^3 - 12x^2 - 5x - 1) \div (2x + 1)$

**Answer**

$$2x^2 - 7x + 1 - \frac{2}{2x + 1}$$

18) $(9x^3 - 9x^2 + 18x + 5) \div (3x - 1)$

19) $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

**Answer**

$$3x^2 - 11x + 34 - \frac{106}{x + 3}$$

20) $(-6x^3 + x^2 - 4) \div (2x - 3)$

21) $(2x^3 + 7x^2 - 13x - 3) \div (2x - 3)$

**Answer**

$$x^2 + 5x + 1$$

22) $(3x^3 - 5x^2 + 2x + 3) \div (x + 2)$

23) $(4x^3 - 5x^2 + 13) \div (x + 4)$

**Answer**

$$4x^2 - 21x + 84 - \frac{323}{x + 4}$$

24) $(x^3 - 3x + 2) \div (x + 2)$

25) $(x^3 - 21x^2 + 147x - 343) \div (x - 7)$

**Answer**

$$x^2 - 14x + 49$$

26) $(x^3 - 15x^2 + 75x - 125) \div (x - 5)$

27) $(9x^3 - x + 2) \div (3x - 1)$

**Answer**

$$3x^2 + x + \frac{2}{3x - 1}$$

28) $(6x^3 - x^2 + 5x + 2) \div (3x + 1)$

29) $(x^4 + x^3 - 3x^2 - 2x + 1) \div (x + 1)$

**Answer**

$$x^3 - 3x + 1$$

30) $(x^4 - 3x^2 + 1) \div (x - 1)$

31) $(x^4 + 2x^3 - 3x^2 + 2x + 6) \div (x + 3)$

**Answer**

$$x^3 - x^2 + 2$$

32) $(x^4 - 10x^3 + 37x^2 - 60x + 36) \div (x - 2)$

33) $(x^4 - 8x^3 + 24x^2 - 32x + 16) \div (x - 2)$

**Answer**

$$x^3 - 6x^2 + 12x - 8$$

34) $(x^4 + 5x^3 - 3x^2 - 13x + 10) \div (x + 5)$

35) $(x^4 - 12x^3 + 54x^2 - 108x + 81) \div (x - 3)$

**Answer**

$$x^3 - 9x^2 + 27x - 27$$

36) $(4x^4 - 2x^3 - 4x + 2) \div (2x - 1)$

37) $(4x^4 + 2x^3 - 4x^2 + 2x + 2) \div (2x + 1)$

**Answer**

$$2x^3 - 2x + 2$$

**For the exercises 38-43, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.**

38) $x - 2, 4x^3 - 3x^2 - 8x + 4$

39) $x - 2, 3x^4 - 6x^3 - 5x + 10$

**Answer**

Yes $(x - 2)(3x^3 - 5)$

40) $x+3, -4x^3+5x^2+8$

41) $x-2, 4x^4-15x^2-4$

**Answer**

Yes $(x-2)(4x^3+8x^2+x+2)$

42) $x-\frac{1}{2}, 2x^4-x^3+2x-1$

43) $x+\frac{1}{3}, 3x^4+x^3-3x+1$

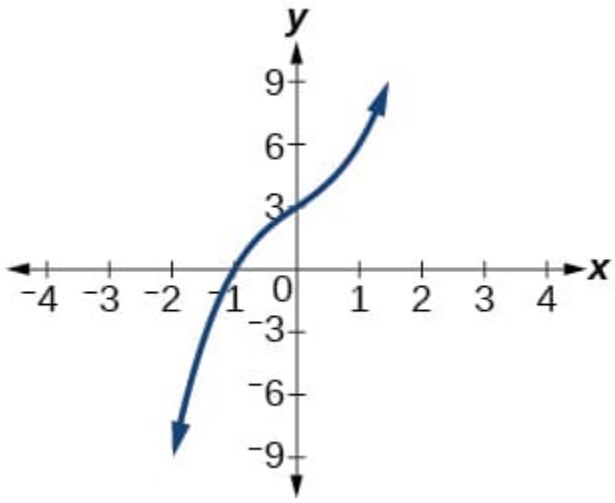
**Answer**

No

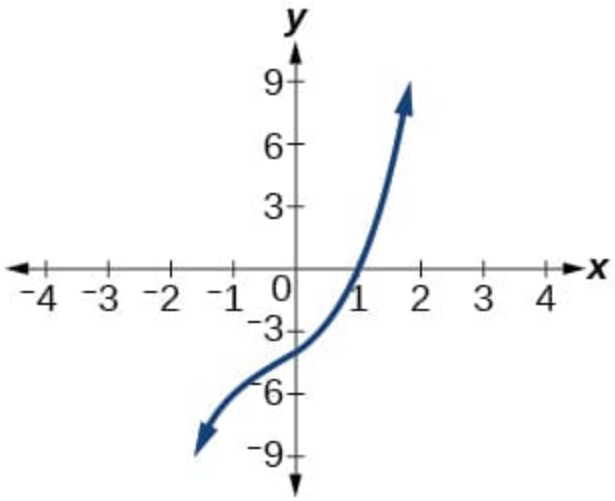
# Graphical

**For the exercises 44-48, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.**

44) Factor is $x^2-x+3$



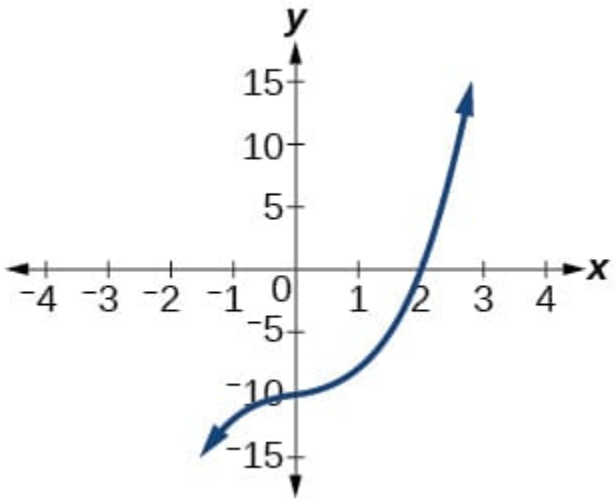

45) Factor is $(x^2+2x+4)$



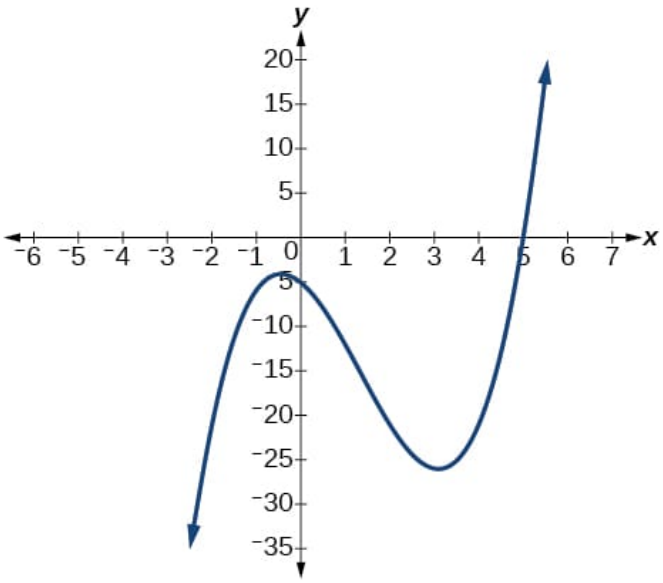

**Answer**

$(x-1)(x^2+2x+4)$

46) Factor is $x^2+2x+5$



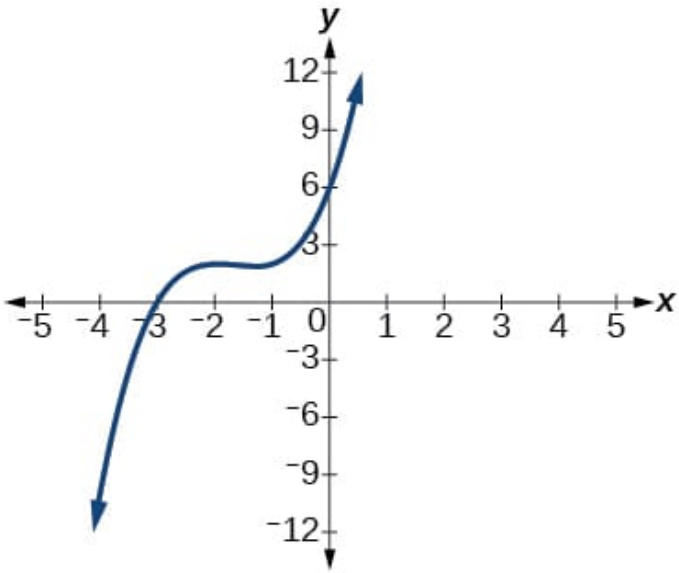

47) Factor is $x^2+x+1$




**Answer**

$(x-5)(x^2+x+1)$

48) Factor is $x^2+2x+2$




**For the exercises 49-53, use synthetic division to find the quotient and remainder.**

49) $\dfrac{4x^3-33}{x-2}$

**Answer**

$\text{Quotient}: 4x^2+8x+16, \text{remainder}: -1$

50) $\dfrac{2x^3+25}{x+3}$

51) $\dfrac{3x^3+2x-5}{x-1}$

**Answer**

$$\text{Quotient}: 3x^2+3x+5, \text{remainder}: 0$$

52) $\dfrac{-4x^3-x^2-12}{x+4}$

53) $\dfrac{x^4-22}{x+2}$

**Answer**

$$\text{Quotient}: x^3-2x^2+4x-8, \text{remainder}: -6$$

# Technology

**For the exercises 54-58, use a calculator with CAS to answer the questions.**

54) Consider $\dfrac{x^k-1}{x-1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

55) Consider $\dfrac{x^k+1}{x+1}$ for $k=1,3,5$. What do you expect the result to be if $k=7$?

**Answer**

$$x^6-x^5+x^4-x^3+x^2-x+1$$

56) Consider $\dfrac{x^4-k^4}{x-k}$ for $k=1,2,3$. What do you expect the result to be if $k=4$?

57) Consider $\dfrac{x^k}{x+1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

**Answer**

$$x^3-x^2+x-1+\frac{1}{x+1}$$

58) Consider $\dfrac{x^k}{x-1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

## Extensions

**For the exercises 59-63, use synthetic division to determine the quotient involving a complex number.**

59) $\dfrac{x+1}{x-i}$

**Answer**

$$1+\frac{1+i}{x-i}$$

60) $\dfrac{x^2+1}{x-i}$

61) $\dfrac{x+1}{x+i}$

**Answer**

$$1+\frac{1-i}{x+i}$$

62) $\dfrac{x^2+1}{x+i}$

63) $\frac{x^3+1}{x-i}$

**Answer**

$$x^2-ix-1+\frac{1-i}{x-i}$$

# Real-World Applications

**For the exercises 64-66, use the given length and area of a rectangle to express the width algebraically.**

64) Length is $x+5$, area is $2x^2+9x-5$.

65) Length is $2x+5$, area is $4x^3+10x^2+6x+15$.

**Answer**

$$2x^2+3$$

66) Length is $3x$–$4$, area is $6x^4-8x^3+9x^2-9x-4$.

**For the exercises 67-70, use the given volume of a box and its length and width to express the height of the box algebraically.**

67) Volume is $12x^3+20x^2-21x-36$, length is $2x+3$, width is $3x-4$.

**Answer**

$$2x+3$$

68) Volume is $18x^3-21x^2-40x+48$, length is $3x$–$4$, width is $3x$–$4$.

69) Volume is $10x^3+27x^2+2x-24$, length is $5x$–$4$, width is $2x+3$.

**Answer**

$$x+2$$

70) Volume is $10x^3+30x^2-8x-24$, length is $2$, width is $x+3$.

**For the exercises 71-73, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.**

71) Volume is $\pi(25x^3-65x^2-29x-3)$, radius is $5x+1$.

**Answer**

$$x-3$$

72) Volume is $\pi(4x^3+12x^2-15x-50)$, radius is $2x+5$.

73) Volume is $\pi(3x^4+24x^3+46x^2-16x-32)$, radius is $x+4$.

**Answer**

$$3x^2-2$$

## 3.6 Zeros of Polynomial Functions

# Verbal

1) Describe a use for the Remainder Theorem.

**Answer**

The theorem can be used to evaluate a polynomial.

2) Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.

3) What is the difference between rational and real zeros?

**Answer**

Rational zeros can be expressed as fractions whereas real zeros include irrational numbers.

4) If Descartes' Rule of Signs reveals a no change of signs or one sign of changes, what specific conclusion can be drawn?

5) If synthetic division reveals a zero, why should we try that value again as a possible solution?

**Answer**

Polynomial functions can have repeated zeros, so the fact that number is a zero doesn't preclude it being a zero again.

# Algebraic

**For the exercises 6-13, use the Remainder Theorem to find the remainder.**

6) $(x^4 - 9x^2 + 14) \div (x - 2)$

7) $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

**Answer**

$-106$

8) $(x^4 + 5x^3 - 4x - 17) \div (x + 1)$

9) $(-3x^2 + 6x + 24) \div (x - 4)$

**Answer**

$0$

10) $(5x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1) \div (x + 6)$

11) $(x^4 - 1) \div (x - 4)$

**Answer**

$255$

12) $(3x^3 + 4x^2 - 8x + 2) \div (x - 3)$

13) $(4x^3 + 5x^2 - 2x + 7) \div (x + 2)$

**Answer**

$-1$

**For the exercises 14-21, use the Factor Theorem to find all real zeros for the given polynomial function and one factor.**

14) $f(x) = 2x^3 - 9x^2 + 13x - 6; x - 1$

15) $f(x) = 2x^3 + x^2 - 5x + 2; x + 2$

**Answer**

$-2, 1, \frac{1}{2}$

16) $f(x) = 3x^3 + x^2 - 20x + 12; x + 3$

17) $f(x) = 2x^3 + 3x^2 + x + 6; x + 2$

**Answer**

$-2$

18) $f(x) = -5x^3 + 16x^2 - 9; x - 3$

19) $x^3 + 3x^2 + 4x + 12; x + 3$

**Answer**

$-3$

20) $4x^3 - 7x + 3; x - 1$

21) $2x^3 + 5x^2 - 12x - 30, 2x + 5$

**Answer**

$-\frac{5}{2}, \sqrt{6}, -\sqrt{6}$

**For the exercises 22-39, use the Rational Zero Theorem to find all real zeros.**

22) $x^3 - 3x^2 - 10x + 24 = 0$

23) $2x^3 + 7x^2 - 10x - 24 = 0$

**Answer**

$2, -4, -\frac{3}{2}$

24) $x^3 + 2x^2 - 9x - 18 = 0$

25) $x^3 + 5x^2 - 16x - 80 = 0$

**Answer**

$4, -4, -5$

26) $x^3 - 3x^2 - 25x + 75 = 0$

27) $2x^3 - 3x^2 - 32x - 15 = 0$

**Answer**

$$5, -3, -\frac{1}{2}$$

28) $2x^3 + x^2 - 7x - 6 = 0$

29) $2x^3 - 3x^2 - x + 1 = 0$

**Answer**

$$\frac{1}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

30) $3x^3 - x^2 - 11x - 6 = 0$

31) $2x^3 - 5x^2 + 9x - 9 = 0$

**Answer**

$$\frac{3}{2}$$

32) $2x^3 - 3x^2 + 4x + 3 = 0$

33) $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$

**Answer**

$$2, 3, -1, -2$$

34) $x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$

35) $4x^4 + 4x^3 - 25x^2 - x + 6 = 0$

**Answer**

$$\frac{1}{2}, -\frac{1}{2}, 2, -3$$

36) $2x^4 - 3x^3 - 15x^2 + 32x - 12 = 0$

37) $x^4 + 2x^3 - 4x^2 - 10x - 5 = 0$

**Answer**

$$-1, -1, \sqrt{5}, -\sqrt{5}$$

38) $4x^3 - 3x + 1 = 0$

39) $8x^4 + 26x^3 + 39x^2 + 26x + 6$

**Answer**

$$-\frac{3}{4}, -\frac{1}{2}$$

**For the exercises 40-45, find all complex solutions (real and non-real).**

40) $x^3 + x^2 + x + 1 = 0$

41) $x^3 - 8x^2 + 25x - 26 = 0$

**Answer**

$$2, 3 + 2i, 3 - 2i$$

42) $x^3 + 13x^2 + 57x + 85 = 0$

43) $3x^3 - 4x^2 + 11x + 10 = 0$

**Answer**

$$-\frac{2}{3}, 1 + 2i, 1 - 2i$$

44) $x^4 + 2x^3 + 22x^2 + 50x - 75 = 0$

45) $2x^3 - 3x^2 + 32x + 17 = 0$

**Answer**

$$-\frac{1}{2}, 1 + 4i, 1 - 4i$$

# Graphical

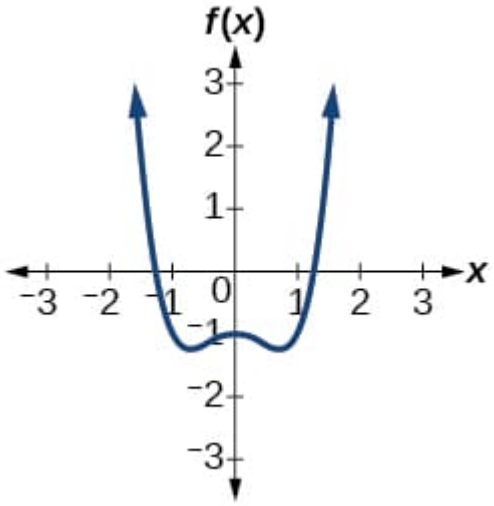
**For the exercises 46-55, use Descartes' Rule to determine the possible number of positive and negative solutions. Confirm with the given graph.**

46) $f(x) = x^3 - 1$

47) $f(x) = x^4 - x^2 - 1$

**Answer**

1 positive, 1 negative

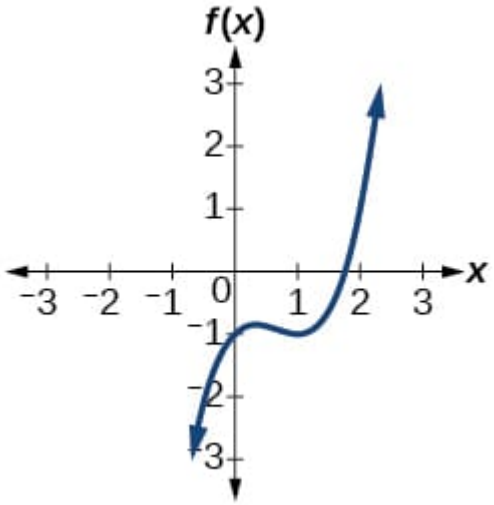


48) $f(x) = x^3 - 2x^2 - 5x + 6$

49) $f(x) = x^3 - 2x^2 + x - 1$

**Answer**

3 or 1 positive, 0 negative

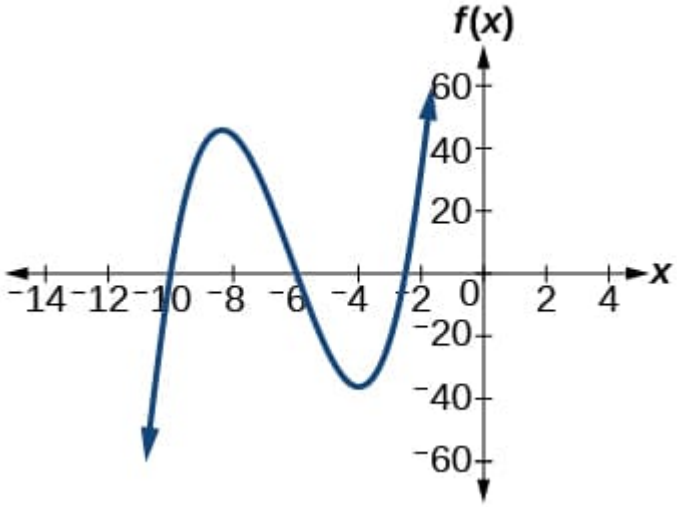



50) $f(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

51) $f(x) = 2x^3 + 37x^2 + 200x + 300$

**Answer**

0 positive, 3 or 1 negative

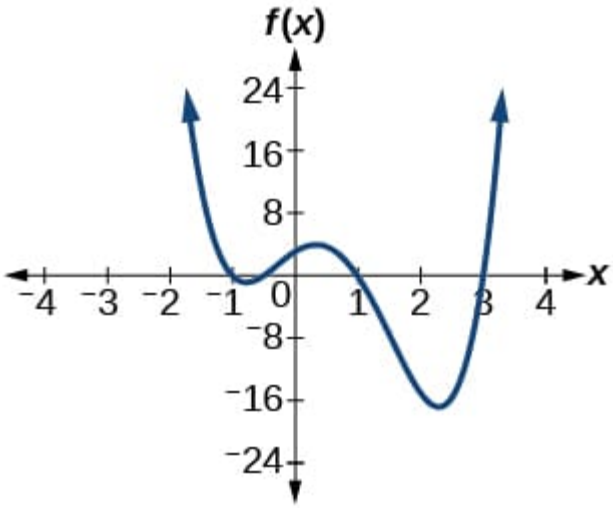



52) $f(x) = x^3 - 2x^2 - 16x + 32$

53) $f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$

**Answer**

2 or 0 positive, 2 or 0 negative

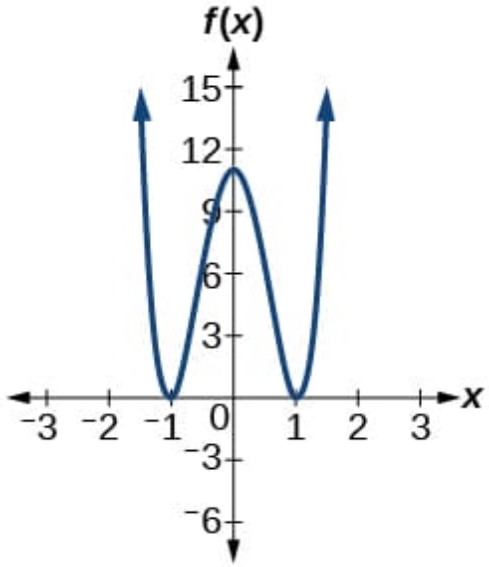



54) $f(x) = 2x^4 - 5x^3 - 14x^2 + 20x + 8$

55) $f(x) = 10x^4 - 21x^2 + 11$

**Answer**

2 or 0 positive, 2 or 0 negative




# Numeric

**For the exercises 56-60, list all possible rational zeros for the functions.**

56) $f(x) = x^4 + 3x^3 - 4x + 4$

57) $f(x) = 2x^3 + 3x^2 - 8x + 5$

**Answer**

$$\pm 5, \pm 1, \pm \frac{5}{2}$$

58) $f(x) = 3x^3 + 5x^2 - 5x + 4$

59) $f(x) = 6x^4 - 10x^2 + 13x + 1$

**Answer**

$$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}$$

60) $f(x) = 4x^5 - 10x^4 + 8x^3 + x^2 - 8$

# Technology

**For the exercises 61-65, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.**

61) $f(x) = 6x^3 - 7x^2 + 1$

**Answer**

$$1, \frac{1}{2}, -\frac{1}{3}$$

62) $f(x) = 4x^3 - 4x^2 - 13x - 5$

63) $f(x) = 8x^3 - 6x^2 - 23x + 6$

**Answer**

$$2, \frac{1}{4}, -\frac{3}{2}$$

64) $f(x) = 12x^4 + 55x^3 + 12x^2 - 117x + 54$

65) $f(x) = 16x^4 - 24x^3 + x^2 - 15x + 25$

**Answer**

$$\frac{5}{4}$$

# Extensions

**For the exercises 66-70, construct a polynomial function of least degree possible using the given information.**

66) Real roots: $-1, 1, 3$ and $(2, f(2)) = (2, 4)$

67) Real roots: $-1$ (with multiplicity 2 and 1) and $(2, f(2)) = (2, 4)$

**Answer**

$$f(x) = \frac{4}{9}(x^3 + x^2 - x - 1)$$

68) Real roots: $-2, \frac{1}{2}$ (with multiplicity 2) and $(-3, f(-3)) = (-3, 5)$

69) Real roots: $-\frac{1}{2}, 0, \frac{1}{2}$ and $(-2, f(-2)) = (-2, 6)$

**Answer**

$$f(x) = -\frac{1}{5}(4x^3 - x)$$

70) Real roots: $-4, -1, 1, 4$ and $(-2, f(-2)) = (-2, 10)$

# Real-World Applications

**For the exercises 71-75, find the dimensions of the box described.**

71) The length is twice as long as the width. The height is $2$ inches greater than the width. The volume is $192$ cubic inches.

**Answer**

$8$ by $4$ by $6$ inches

72) The length, width, and height are consecutive whole numbers. The volume is $120$cubic inches.

73) The length is one inch more than the width, which is one inch more than the height. The volume is $86.625$ cubic inches.

**Answer**

$5.5$ by $4.5$ by $3.5$ inches

74) The length is three times the height and the height is one inch less than the width. The volume is $108$ cubic inches.

75) The length is $3$ inches more than the width. The width is $2$ inches more than the height. The volume is $120$ cubic inches.

**Answer**

$8$ by $5$ by $3$ inches

**For the exercises 76-80, find the dimensions of the right circular cylinder described.**

76) The radius is $3$ inches more than the height. The volume is $16\pi$ cubic meters.

77) The height is one less than one half the radius. The volume is $72\pi$ cubic meters.

**Answer**

Radius = $6$ meters, Height = $2$ meters

78) The radius and height differ by one meter. The radius is larger and the volume is $48\pi$ cubic meters.

79) The radius and height differ by two meters. The height is greater and the volume is $28.125\pi$ cubic meters.

**Answer**

Radius = $2.5$ meters, Height = $4.5$ meters

80) The radius is $\frac{1}{3}$ meter greater than the height. The volume is $\frac{98}{9}\pi$ cubic meters.

# 3.7 Rational Functions

## Verbal

1) What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?

**Answer**

The rational function will be represented by a quotient of polynomial functions.

2) What is the fundamental difference in the graphs of polynomial functions and rational functions?

3) If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?

**Answer**

The numerator and denominator must have a common factor.

4) Can a graph of a rational function have no vertical asymptote? If so, how?

5) Can a graph of a rational function have no x-intercepts? If so, how?

**Answer**

Yes. The numerator of the formula of the functions would have only complex roots and/or factors common to both the numerator and denominator.

## Algebraic

**For the exercises 6-9, find the domain of the rational functions.**

6) $f(x) = \dfrac{x-1}{x+2}$

7) $f(x) = \dfrac{x+1}{x^2-1}$

**Answer**

All reals $x \neq -1, 1$

8) $f(x) = \dfrac{x^2+4}{x^2-2x-8}$

9) $f(x) = \dfrac{x^2+4x-3}{x^4-5x^2+4}$

**Answer**

All reals $x \neq -1, -2, 1, 2$

**For the exercises 10-19, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.**

10) $f(x) = \dfrac{4}{x-1}$

11) $f(x) = \dfrac{2}{5x+2}$

**Answer**

V.A. at $x = -\frac{2}{5}$; H.A. at $y = 0$; Domain is all reals $x \neq -\frac{2}{5}$

12) $f(x) = \frac{x}{x^2 - 9}$

13) $f(x) = \frac{x}{x^2 + 5x - 36}$

**Answer**

V.A. at $x = 4, -9$; H.A. at $y = 0$; Domain is all reals $x \neq 4, -9$

14) $f(x) = \frac{3 + x}{x^3 - 27}$

15) $f(x) = \frac{3x - 4}{x^3 - 16x}$

**Answer**

V.A. at $x = 0, 4, -4$; H.A. at $y = 0$; Domain is all reals $x \neq 0, 4, -4$

16) $f(x) = \frac{x^2 - 1}{x^3 + 9x^2 + 14x}$

17) $f(x) = \frac{x + 5}{x^2 - 25}$

**Answer**

V.A. at $x = -5$; H.A. at $y = 0$; Domain is all reals $x \neq 5, -5$

18) $f(x) = \frac{x - 4}{x - 6}$

19) $f(x) = \frac{4 - 2x}{3x - 1}$

**Answer**

V.A. at $x = \frac{1}{3}$; H.A. at $y = -\frac{2}{3}$; Domain is all reals $x \neq \frac{1}{3}$

**For the exercises 20-24, find the $x$- and $y$-intercepts for the functions.**

20) $f(x) = \frac{x + 5}{x^2 + 4}$

21) $f(x) = \frac{x}{x^2 - x}$

**Answer**

none

22) $f(x) = \frac{x^2 + 8x + 7}{x^2 + 11x + 30}$

23) $f(x) = \frac{x^2 + x + 6}{x^2 - 10x + 24}$

**Answer**

$x$ -intercepts none, $y$ -intercept $\left(0, \frac{1}{4}\right)$

24) $f(x) = \frac{94 - 2x^2}{3x^2 - 12}$

**For the exercises 25-29, describe the local and end behavior of the functions.**

25) $f(x) = \frac{x}{2x + 1}$

**Answer**

Local behavior: $x \to -\frac{1}{2}^{+}, f(x) \to -\infty, x \to -\frac{1}{2}^{-}, f(x) \to \infty$

End behavior: $x \to \pm\infty, f(x) \to \frac{1}{2}$

26) $f(x) = \frac{2x}{x - 6}$

27) $f(x) = \frac{-2x}{x - 6}$

**Answer**

Local behavior: $x \to 6^{+}, f(x) \to -\infty, x \to 6^{-}, f(x) \to \infty$

End behavior: $x \to \pm\infty, f(x) \to -2$

28) $f(x) = \frac{x^2 - 4x + 3}{x^2 - 4x - 5}$

29) $f(x) = \frac{2x^2 - 32}{6x^2 + 13x - 5}$

**Answer**

Local behavior: $x \to -\frac{1}{3}, f(x) \to \infty, x \to -\frac{1}{3}, f(x) \to -\infty, x \to \frac{5}{2}, f(x) \to \infty, x \to \frac{5}{2} + f(x) \to -\infty$

End behavior: $x \to \pm\infty, f(x) \to \frac{1}{3}$

**For the exercises 30-34, find the slant asymptote of the functions.**

30) $f(x) = \frac{24x^2 + 6x}{2x + 1}$

31) $f(x) = \frac{4x^2 - 10}{2x - 4}$

**Answer**

$y = 2x + 4$

32) $f(x) = \frac{81x^2 - 18}{3x - 2}$

33) $f(x) = \frac{6x^3 - 5x}{3x^2 + 4}$

**Answer**

$$y = 2x$$

34) $f(x) = \dfrac{x^2 + 5x + 4}{x - 1}$

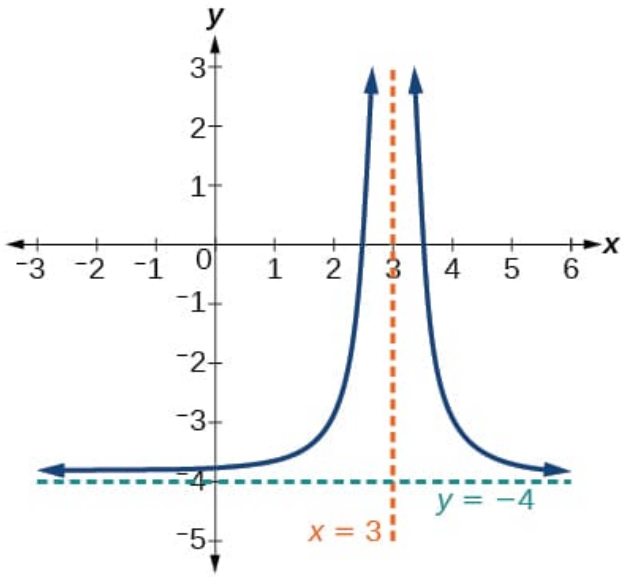
## Graphical

**For the exercises 35-38, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.**

35) The reciprocal function shifted up two units.

**Answer**

V.A. $x = 0$, H.A. $y = 2$

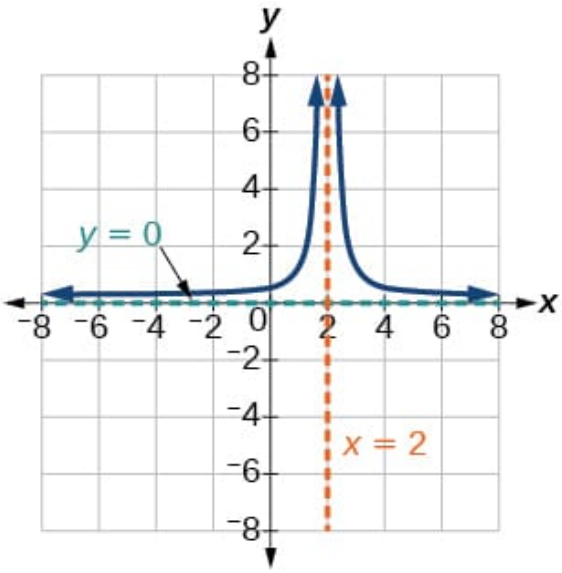



36) The reciprocal function shifted down one unit and left three units.

37) The reciprocal squared function shifted to the right $2$ units.

**Answer**

V.A. $x = 2$, H.A. $y = 0$



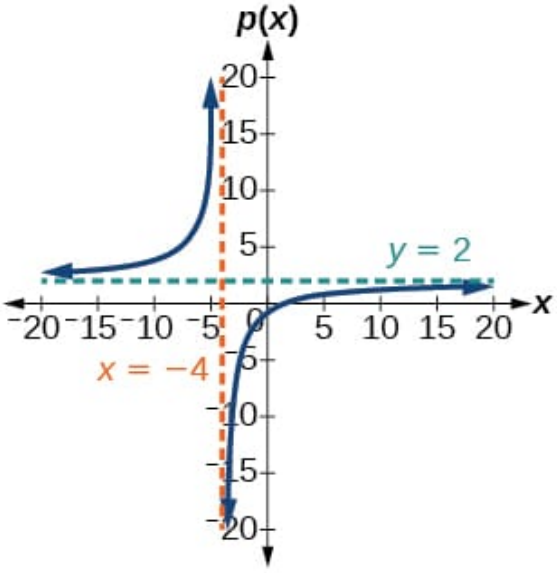

38) The reciprocal squared function shifted down $2$ units and right $1$ unit.

**For the exercises 39-50, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.**

39) $p(x) = \dfrac{2x-3}{x+4}$

**Answer**

V.A. $x = -4$, H.A. $y = 2$; $\left(\frac{3}{2}, 0\right)$; $\left(0, -\frac{3}{4}\right)$

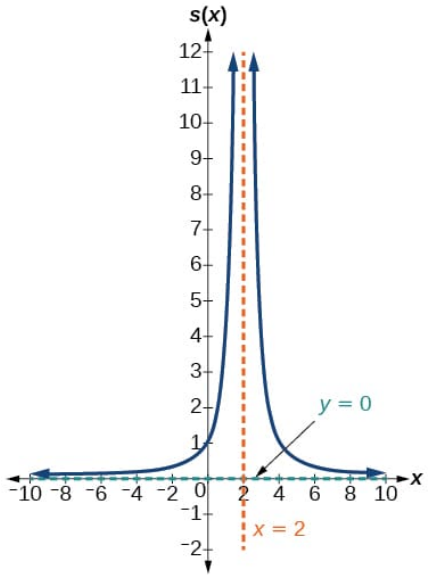



40) $q(x) = \dfrac{x-5}{3x-1}$

41) $s(x) = \dfrac{4}{(x-2)^2}$

**Answer**

V.A. $x = 2$, H.A. $y = 0$, $(0, 1)$

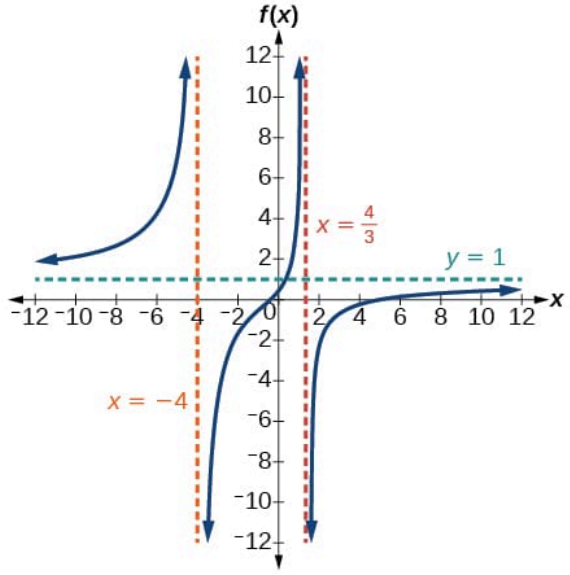



42) $r(x) = \dfrac{5}{(x+1)^2}$

43) $f(x) = \dfrac{3x^2 - 14x - 5}{3x^2 + 8x - 16}$

**Answer**

V.A. $x = -4$, $x = \dfrac{4}{3}$, H.A. $y = 1$; $(5, 0)$; $\left(-\dfrac{1}{3}, 0\right)$; $\left(0, \dfrac{5}{16}\right)$

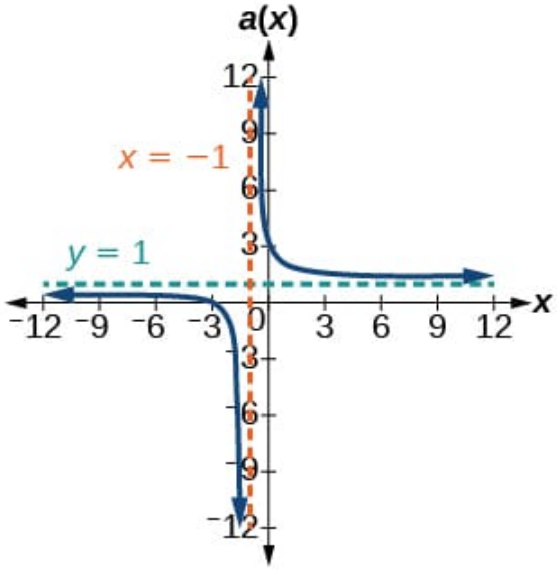



44) $g(x) = \dfrac{2x^2 + 7x - 15}{3x^2 - 14 + 15}$

45) $a(x) = \dfrac{x^2 + 2x - 3}{x^2 - 1}$

**Answer**

V.A. $x = -1$, H.A. $y = 1$; $(-3, 0)$; $(0, 3)$

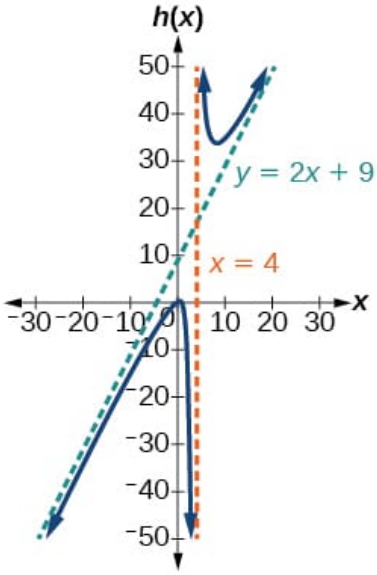



46) $b(x) = \dfrac{x^2 - x - 6}{x^2 - 4}$

47) $h(x) = \dfrac{2x^2 + x - 1}{x - 4}$

**Answer**

V.A. $x = 4$, S.A. $y = 2x + 9$; $(-1, 0)$; $\left(\frac{1}{2}, 0\right)$; $\left(0, \frac{1}{4}\right)$

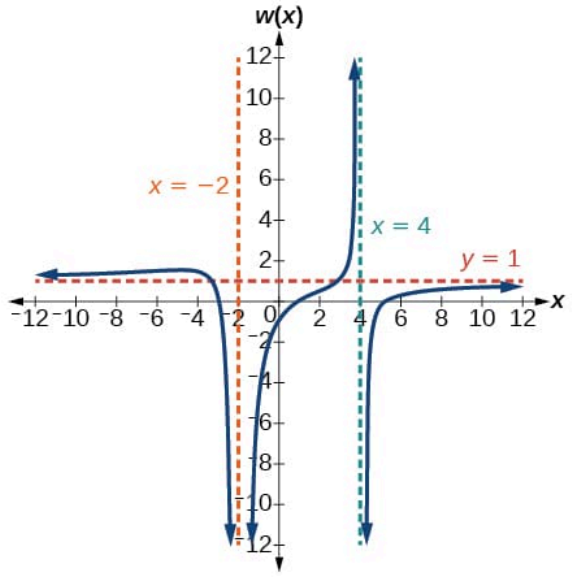



48) $k(x) = \dfrac{2x^2 - 3x - 20}{x - 5}$

49) $w(x) = \dfrac{(x - 1)(x + 3)(x - 5)}{(x + 2)^2(x - 4)}$

**Answer**

V.A. $x=-2$, $x=4$, H.A. $y=1$, $(1,0)$; $(5,0)$; $(-3,0)$; $\left(0,-\frac{15}{16}\right)$




50) $z(x)=\dfrac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

**For the exercises 51-56, write an equation for a rational function with the given characteristics.**

51) Vertical asymptotes at $x=5$, $x$ -intercepts at $(2,0)$ and $(-1,0)$, $y$ -intercept at $(0,4)$

**Answer**

$$y=50\frac{x^2-x-2}{x^2-25}$$

52) Vertical asymptotes at $x=-4$ and $x=-1$, $x$ -intercepts at $(1,0)$ and $(5,0)$, $y$ -intercept at $(0,7)$

53) Vertical asymptotes at $x=-4$ and $x=-5$, $x$ -intercepts at $(4,0)$ and $(-6,0)$, Horizontal asymptote at $y=7$

**Answer**

$$y=7\frac{x^2+2x-24}{x^2+9x+20}$$

54) Vertical asymptotes at $x=-3$ and $x=6$, $x$ -intercepts at $(-2,0)$ and $(1,0)$, Horizontal asymptote at $y=-2$

55) Vertical asymptote at $x=-1$, Double zero at $x=2$, $y$ -intercept at $(0,2)$

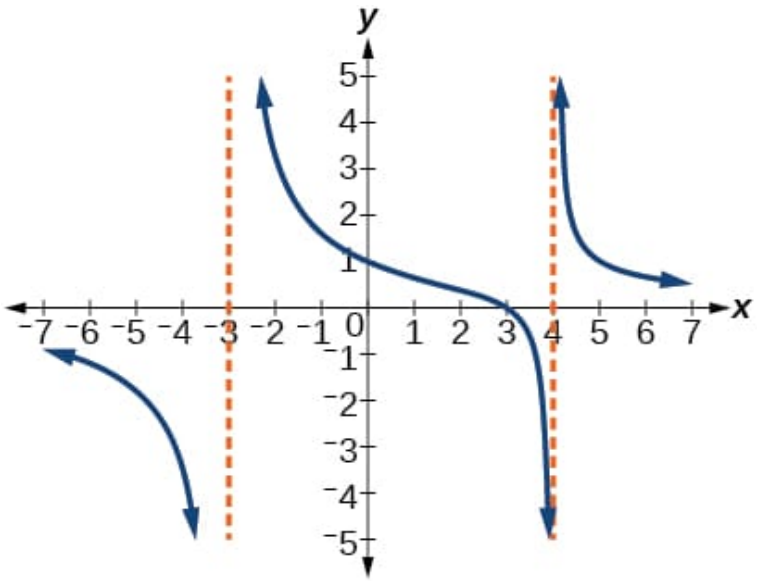
**Answer**

$$y=\frac{1}{2}\frac{x^2-4x+4}{x+1}$$

56) Vertical asymptote at $x=3$, Double zero at $x=1$, $y$ -intercept at $(0,4)$

**For the exercises 57-,64 use the graphs to write an equation for the function.**

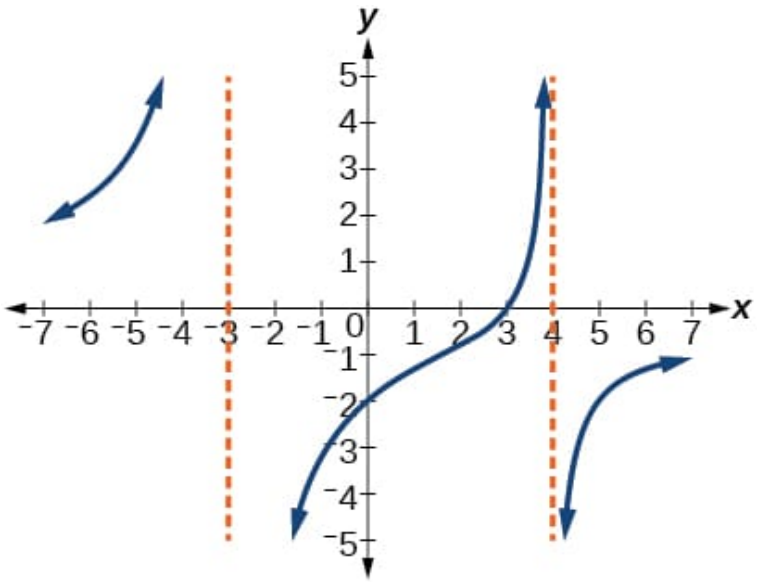
57)



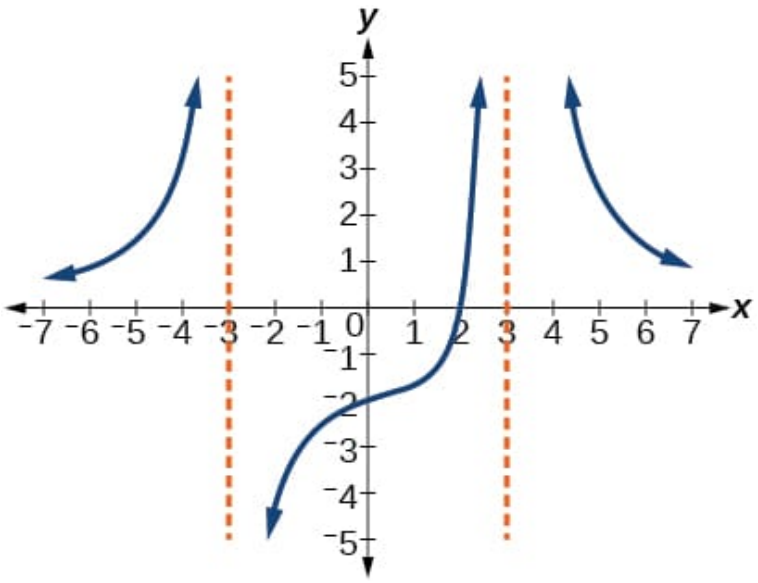

**Answer**

$$y = 4\frac{x-3}{x^2-x-12}$$

58)



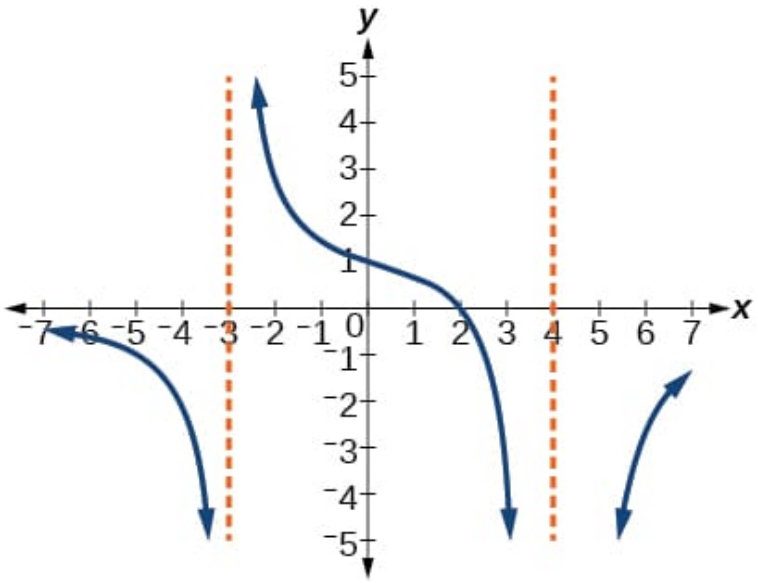

59)



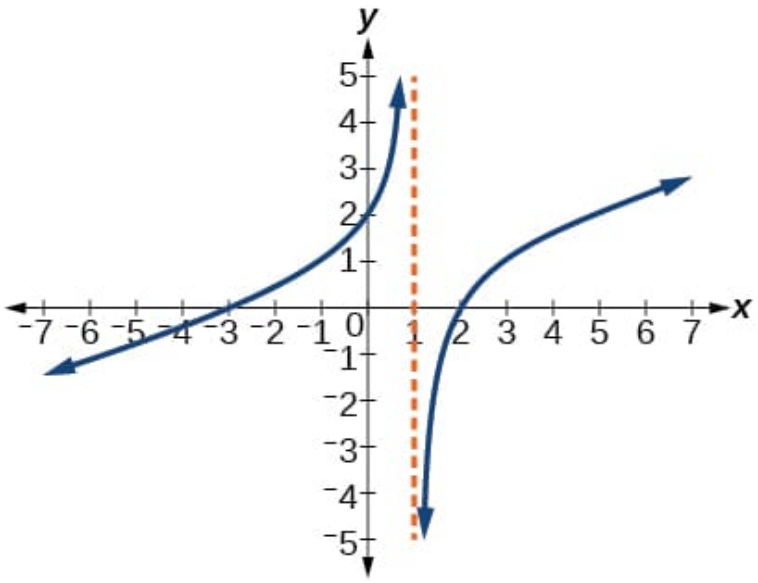

**Answer**

$$y = -9\frac{x-2}{x^2-9}$$

60)



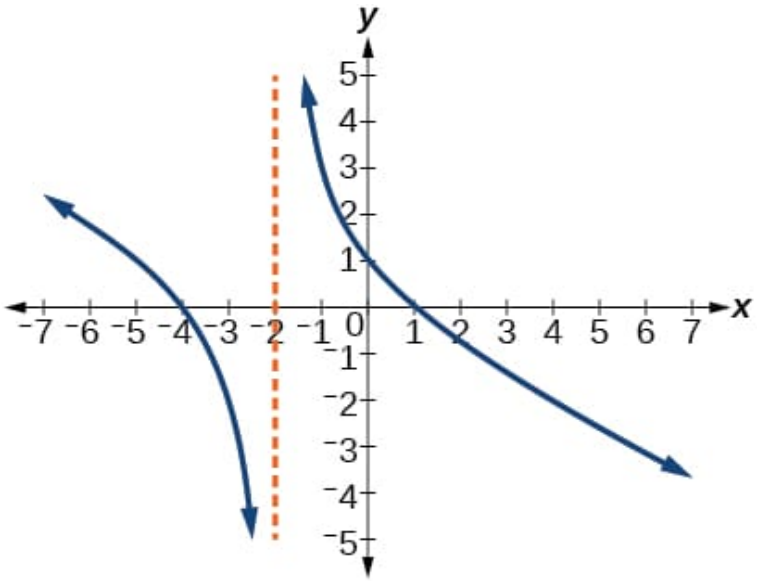

61)



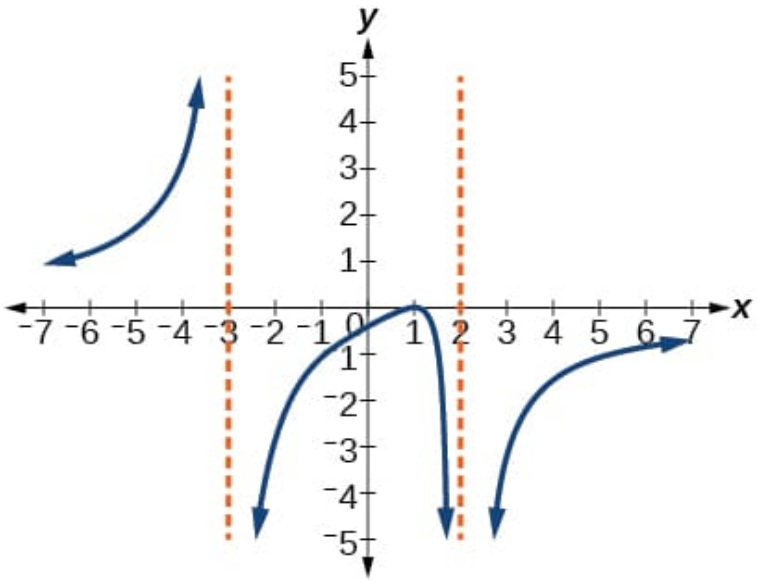

**Answer**

$$y = \frac{1}{3}\frac{x^2 + x - 6}{x - 1}$$

62)



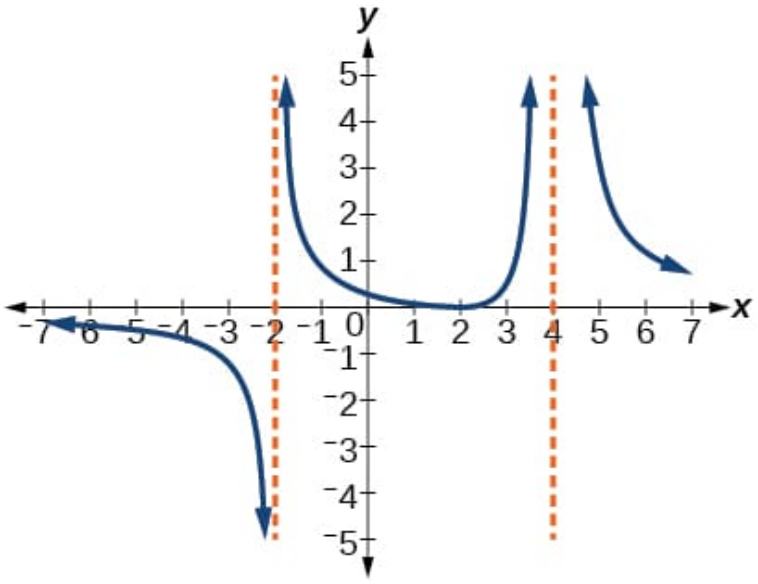

63)




**Answer**

$$y = -6\frac{(x-1)^2}{(x+3)(x-2)^2}$$

64)




## Numeric

**For the exercises 65-69, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote**

65) $f(x) = \dfrac{1}{x-2}$

**Answer**

| $x$ | 2.01 | 2.001 | 2.0001 | 1.99 | 1.999 |
|---|---|---|---|---|---|
| $y$ | 100 | 1,000 | 10,000 | −100 | −1,000 |

| $x$ | 10 | 100 | 1,000 | 10,000 | 100,000 |
|---|---|---|---|---|---|
| $y$ | .125 | .0102 | .001 | .0001 | .00001 |

Vertical asymptote $x = 2$, Horizontal asymptote $y = 0$

66) $f(x) = \dfrac{x}{x-3}$

67) $f(x) = \dfrac{2x}{x+4}$

**Answer**

| $x$ | −4.1 | −4.01 | −4.001 | −3.99 | −3.999 |
|---|---|---|---|---|---|
| $y$ | 82 | 802 | 8,002 | −798 | −7998 |

| $x$ | 10 | 100 | 1,000 | 10,000 | 100,000 |
|---|---|---|---|---|---|
| $y$ | 1.4286 | 1.9331 | 1.992 | 1.9992 | 1.999992 |

Vertical asymptote $x = -4$, Horizontal asymptote $y = 2$

68) $f(x) = \dfrac{2x}{(x-3)^2}$

69) $f(x) = \dfrac{x^2}{x^2+2x+1}$

**Answer**

| $x$ | −.9 | −.99 | −.999 | −1.1 | −1.01 |
|---|---|---|---|---|---|
| $y$ | 81 | 9,801 | 998,001 | 121 | 10,201 |

| $x$ | 10 | 100 | 1,000 | 10,000 | 100,000 |
|---|---|---|---|---|---|
| $y$ | .82645 | .9803 | .998 | .9998 | |

Vertical asymptote $x = -1$, Horizontal asymptote $y = 1$

## Technology

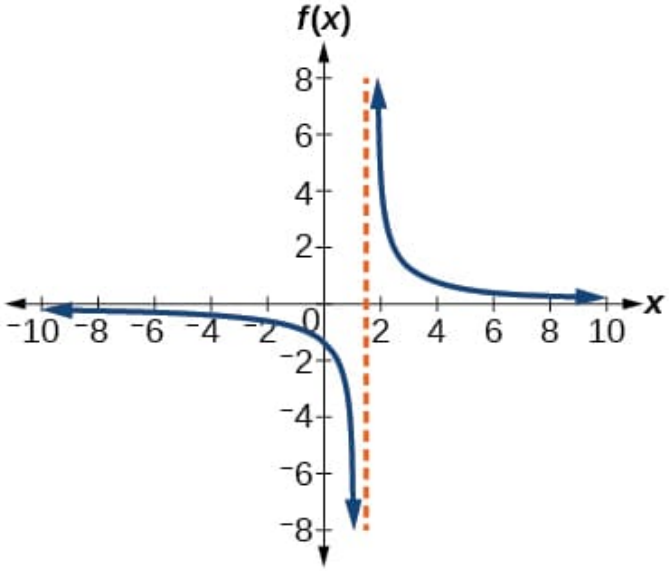
**For the exercises 70-74, use a calculator to graph $f(x)$. Use the graph to solve $f(x) > 0$.**

70) $f(x) = \dfrac{2}{x+1}$

71) $f(x) = \dfrac{4}{2x-3}$

**Answer**

$\left(\frac{3}{2}, \infty\right)$

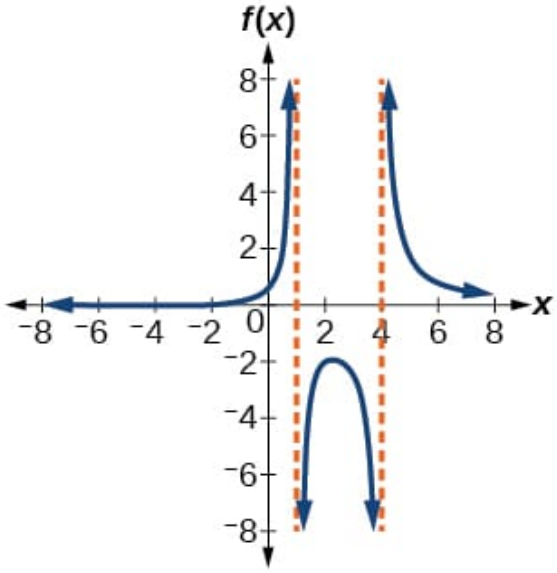



72) $f(x) = \dfrac{2}{(x-1)(x+2)}$

73) $f(x) = \dfrac{x+2}{(x-1)(x-4)}$

**Answer**

$(-2, 1) \cup (4, \infty)$




74) $f(x) = \dfrac{(x+3)^2}{(x-1)^2(x+1)}$

## Extensions

**For the exercises 75-79, identify the removable discontinuity.**

75) $f(x) = \dfrac{x^2 - 4}{x - 2}$

**Answer**

$(2, 4)$

76) $f(x) = \dfrac{x^3 + 1}{x + 1}$

77) $f(x) = \dfrac{x^2 + x - 6}{x - 2}$

**Answer**

$(2, 5)$

78) $f(x) = \dfrac{2x^2 + 5x - 3}{x + 3}$

79) $f(x) = \dfrac{x^3 + x^2}{x + 1}$

**Answer**

$(-1, 1)$

## Real-World Applications

**For the exercises 80-81, express a rational function that describes the situation.**

80) A large mixing tank currently contains $200$ gallons of water, into which $10$ pounds of sugar have been mixed. A tap will open, pouring $10$ gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of $3$ pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after $t$ minutes.

81) A large mixing tank currently contains $300$ gallons of water, into which $8$ pounds of sugar have been mixed. A tap will open, pouring $20$ gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of $2$ pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after $t$ minutes.

**Answer**

$$C(t) = \frac{8 + 2t}{300 + 20t}$$

**For the exercises 82-83, use the given rational function to answer the question.**

82) The concentration $C$ of a drug in a patient's bloodstream $t$ hours after injection in given by $C(t) = \dfrac{2t}{3 + t^2}$. What happens to the concentration of the drug as $t$ increases?

83) The concentration $C$ of a drug in a patients bloodstream, hours after injection is given by $C(t) = \dfrac{100t}{2t^2 + 75}$. Use a calculator to approximate the time when the concentration is highest.

**Answer**

After about $6.12$ hours.

**For the exercises 84-88, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.**

84) An open box with a square base is to have a volume of $108$ cubic inches. Find the dimensions of the box that will have minimum surface area. Let $x$ = length of the side of the base.

85) A rectangular box with a square base is to have a volume of $20$ cubic feet. The material for the base costs $30$ cents/ square foot. The material for the sides costs $10$ cents/square foot. The material for the top costs $20$ cents/square foot. Determine the dimensions that will yield minimum cost. Let $x$ = length of the side of the base.

**Answer**

$A(x) = 50x^2 + \frac{800}{x}$ · 2 by 2 by $5$ feet.

86) A right circular cylinder has volume of $100$ cubic inches. Find the radius and height that will yield minimum surface area. Let $x$ = radius.

87) A right circular cylinder with no top has a volume of $50$ cubic meters. Find the radius that will yield minimum surface area. Let $x$ = radius.

**Answer**

$A(x) = \pi x^2 + \frac{100}{x}$. Radius = $2.52$ meters.

88) A right circular cylinder is to have a volume of $40$ cubic inches. It costs $4$ cents/square inch to construct the top and bottom and $1$ cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let $x$ = radius.

# 3.8 Inverses and Radical Functions

## Verbal

1) Explain why we cannot find inverse functions for all polynomial functions.

**Answer**

It can be too difficult or impossible to solve for in terms of $y$.

2) Why must we restrict the domain of a quadratic function when finding its inverse?

3) When finding the inverse of a radical function, what restriction will we need to make?

**Answer**

We will need a restriction on the domain of the answer.

4) The inverse of a quadratic function will always take what form?

## Algebraic

**For the exercises 5-12, find the inverse of the function on the given domain.**

5) $f(x) = (x-4)^2, [4, \infty)$

**Answer**

$f^{-1}(x) = \sqrt{x} + 4$

6) $f(x) = (x+2)^2, [-2, \infty)$

7) $f(x) = (x+1)^2 - 3, [-1, \infty)$

**Answer**

$$f^{-1}(x) = \sqrt{x+3} - 1$$

8) $f(x) = 2 - \sqrt{3+x}$

9) $f(x) = 3x^2 + 5, (-\infty, 0]$

**Answer**

$$f^{-1}(x) = -\sqrt{\frac{x-5}{3}}$$

10) $f(x) = 12 - x^2, [0, \infty)$

11) $f(x) = 9 - x^2, [0, \infty)$

**Answer**

$$f(x) = \sqrt{9-x}$$

12) $f(x) = 2x^2 + 4, [0, \infty)$

**For the exercises 13-16, find the inverse of the functions.**

13) $f(x) = x^3 + 5$

**Answer**

$$f^{-1}(x) = \sqrt[3]{x-5}$$

14) $f(x) = 3x^3 + 1$

15) $f(x) = 4 - x^3$

**Answer**

$$f^{-1}(x) = \sqrt[3]{4-x}$$

16) $f(x) = 4 - 2x^3$

**For the exercises 17-31, find the inverse of the functions.**

17) $f(x) = \sqrt{2x+1}$

**Answer**

$$f^{-1}(x) = \frac{x^2-1}{2}, [0, \infty)$$

18) $f(x) = \sqrt{3-4x}$

19) $f(x) = 9 + \sqrt{4x-4}$

**Answer**

$$f^{-1}(x) = \frac{(x-9)^2+4}{4}, [9, \infty)$$

20) $f(x) = \sqrt{6x - 8} + 5$

21) $f(x) = 9 + 2\sqrt[3]{x}$

**Answer**

$$f^{-1}(x) = \left(\frac{x - 9}{2}\right)^3$$

22) $f(x) = 3 - \sqrt[3]{x}$

23) $f(x) = \dfrac{2}{x + 8}$

**Answer**

$$f^{-1}(x) = \frac{2 - 8x}{x}$$

24) $f(x) = \dfrac{3}{x - 4}$

25) $f(x) = \dfrac{x + 3}{x + 7}$

**Answer**

$$f^{-1}(x) = \frac{7x - 3}{1 - x}$$

26) $f(x) = \dfrac{x - 2}{x + 7}$

27) $f(x) = \dfrac{3x + 4}{5}$

**Answer**

$$f^{-1}(x) = \frac{5x - 4}{4x + 3}$$

28) $f(x) = \dfrac{5x + 1}{2 - 5x}$

29) $f(x) = x^2 + 2x, [-1, \infty)$

**Answer**

$$f^{-1}(x) = \sqrt{x + 1} - 1$$

30) $f(x) = x^2 + 4x + 1, [-2, \infty)$

31) $f(x) = x^2 - 6x + 3, [3, \infty)$

**Answer**

$$f^{-1}(x) = \sqrt{x + 6} + 3$$

## Graphical

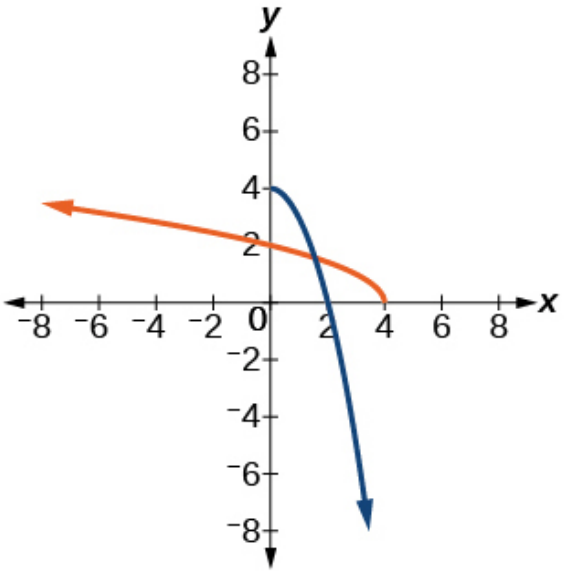
**For the exercises 32-41, find the inverse of the function and graph both the function and its inverse.**

32) $f(x) = x^2 + 2, x \geq 0$

33) $f(x) = 4 - x^2, x \geq 0$

**Answer**

$f^{-1}(x) = \sqrt{4 - x}$

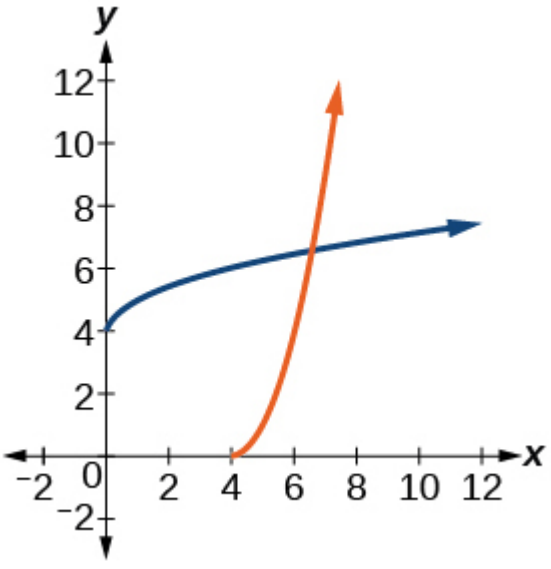



34) $f(x) = (x + 3)^2, x \geq -3$

35) $f(x) = (x - 4)^2, x \geq 4$

**Answer**

$f^{-1}(x) = \sqrt{x} + 4$

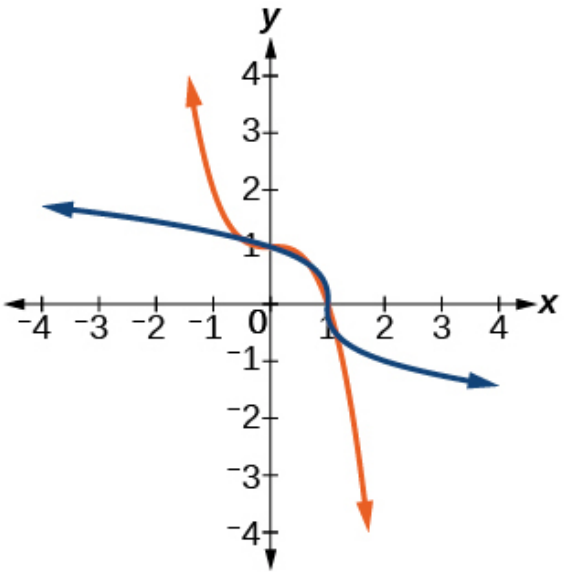



36) $f(x) = x^3 + 3$

37) $f(x) = 1 - x^3$

**Answer**

$f^{-1}(x) = \sqrt[3]{1-x}$

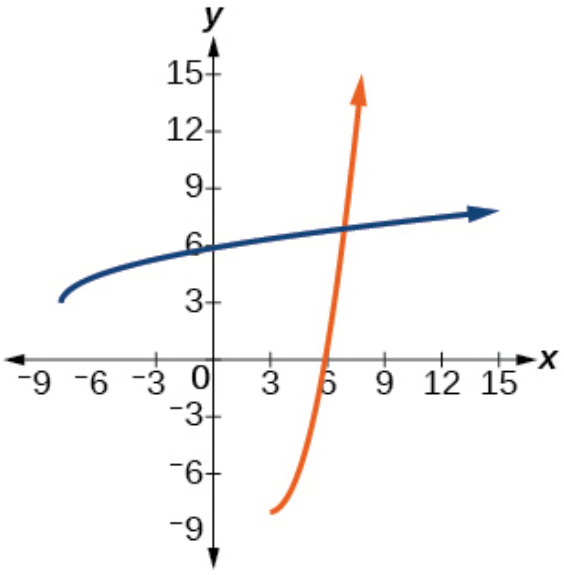



38) $f(x) = x^2 + 4x, x \geq -2$

39) $f(x) = x^2 - 6x + 1, x \geq 3$

**Answer**

$f^{-1}(x) = \sqrt{x+8} + 3$

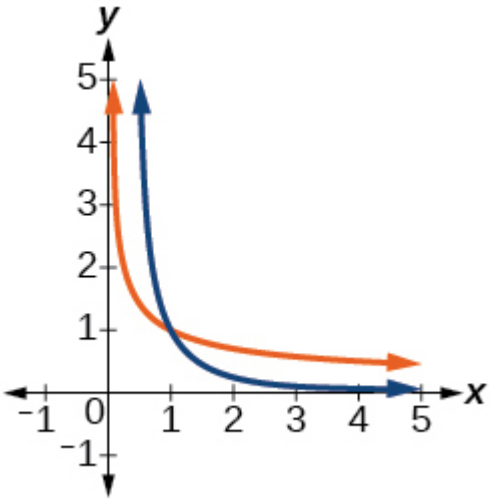



40) $f(x) = \dfrac{2}{x}$

41) $f(x) = \dfrac{1}{x^2}, x \geq 0$

**Answer**

$$f^{-1}(x) = \sqrt{\frac{1}{x}}$$



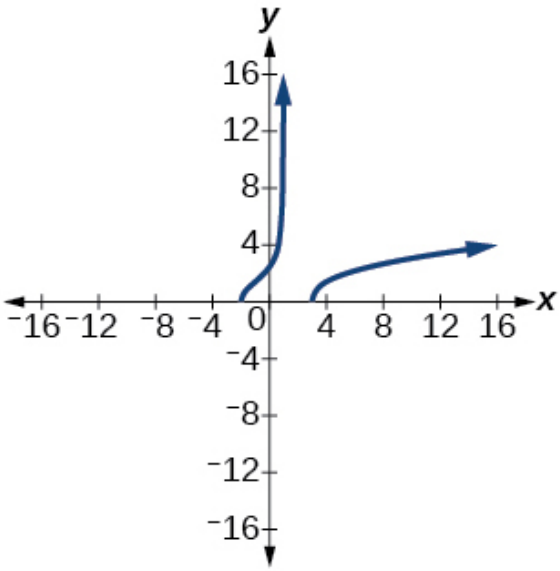

**For the exercises 42-46, use a graph to help determine the domain of the functions.**

42) $f(x) = \sqrt{\frac{(x+1)(x-1)}{x}}$

43) $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$

**Answer**

$$[-2, 1) \cup [3, \infty)$$

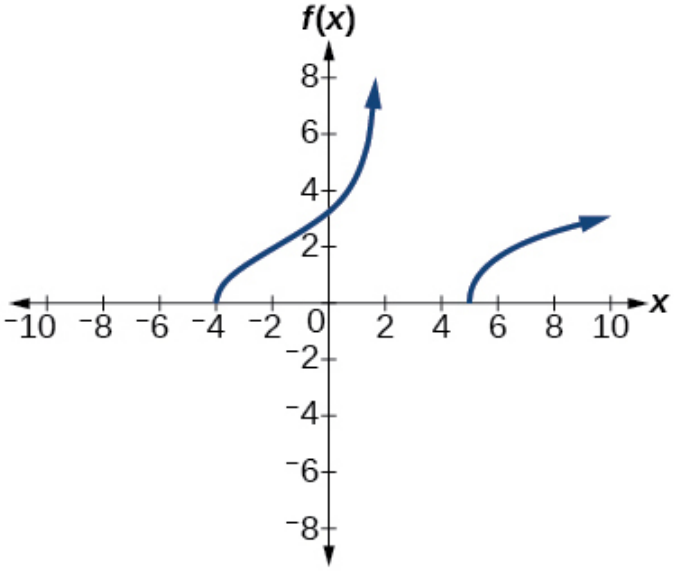



44) $f(x) = \sqrt{\frac{x(x+3)}{x-4}}$

45) $f(x) = \sqrt{\dfrac{x^2 - x - 20}{x - 2}}$

**Answer**

$[-4, 2) \cup [5, \infty)$




46) $f(x) = \sqrt{\dfrac{9x^2}{x + 4}}$

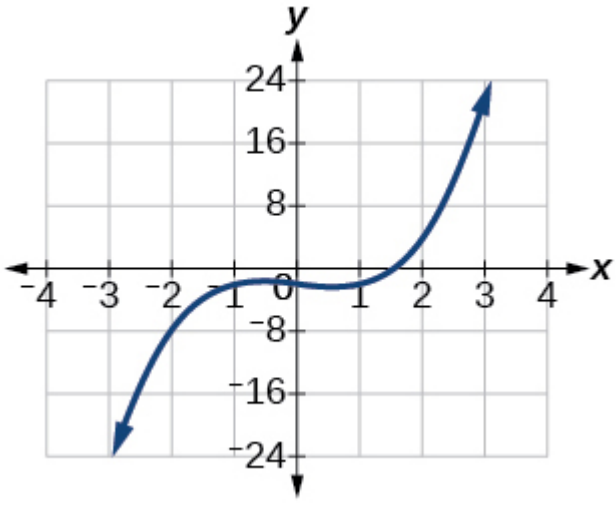
## Technology

**For the exercises 47-51, use a calculator to graph the function. Then, using the graph, give three points on the graph of the inverse with $y$-coordinates given.**

47) $f(x) = x^3 - x - 2, y = 1, 2, 3$

**Answer**

$(-2, 0); (4, 2); (22, 3)$

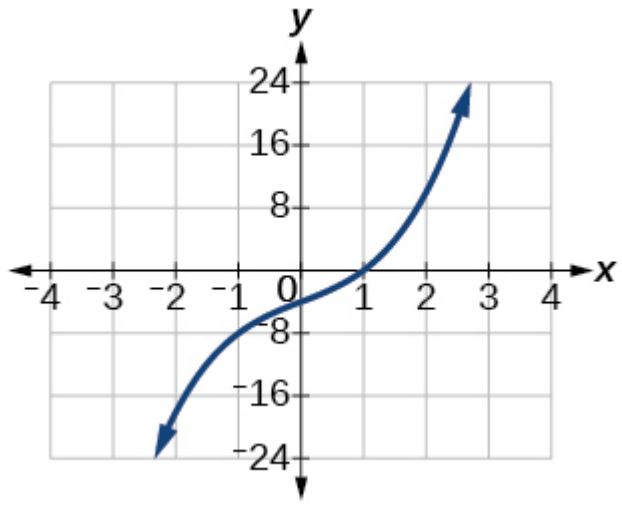


48) $f(x) = x^3 + x - 2, y = 0, 1, 2$

49) $f(x) = x^3 + 3x - 4, y = 0, 1, 2$

**Answer**

$(-4, 0); (0, 1); (10, 2)$



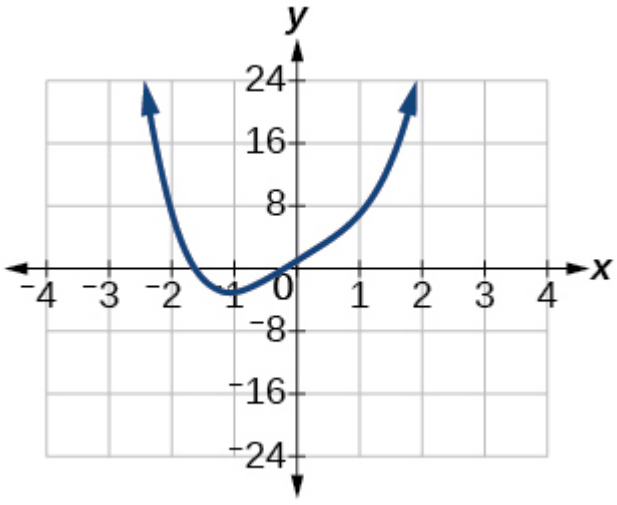

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50) $f(x) = x^3 + 8x - 4, y = -1, 0, 1$

51) $f(x) = x^4 + 5x + 1, y = -1, 0, 1$

**Answer**

$(-3, -1); (1, 0); (7, 1)$




## Extensions

**For the exercises 52-56, find the inverse of the functions with $a, b, c$ positive real numbers.**

52) $f(x) = ax^3 + b$

53) $f(x) = x^2 + bx$

**Answer**

$$f^{-1}(x) = \sqrt{x + \frac{b^2}{4}} - \frac{b}{2}$$

54) $f(x) = \sqrt{ax^2 + b}$

55) $f(x) = \sqrt[3]{ax + b}$

**Answer**

$$f^{-1}(x) = \frac{x^2 - b}{a}$$

56) $f(x) = \frac{ax + b}{x + c}$

## Real-World Applications

**For the exercises 57-66, determine the function described and then use it to answer the question.**

57) An object dropped from a height of $200$ meters has a height, $h(t)$, in meters after $t$ seconds have lapsed, such that $h(t) = 200 - 4.9t^2$. Express $t$ as a function of height, $h$, and find the time to reach a height of $50$ meters.

**Answer**

$t(h) = \sqrt{\frac{200 - h}{4.9}}$, $5.53$ seconds

58) An object dropped from a height of $600$ feet has a height, $h(t)$, in feet after $t$ seconds have elapsed, such that $h(t) = 600 - 16t^2$. Express $t$ as a function of height $h$, and find the time to reach a height of $400$ feet.

59) The volume, $V$, of a sphere in terms of its radius, $r$, is given by $V(r) = \frac{4}{3}\pi r^3$. Express $r$ as a function of $V$, and find the radius of a sphere with volume of $200$ cubic feet.

**Answer**

$r(V) = \sqrt[3]{\frac{3V}{4\pi}}$, $3.63$ feet

60) The surface area, $A$, of a sphere in terms of its radius, $r$, is given by $A(r) = 4\pi r^2$. Express $r$ as a function of $V$, and find the radius of a sphere with a surface area of $1000$ square inches.

61) A container holds $100$ ml of a solution that is $25$ ml acid. If $n$ ml of a solution that is $60\%$ acid is added, the function $C(n) = \frac{25 + .6n}{100 + n}$ gives the concentration, $C$, as a function of the number of ml added, $n$. Express $n$ as a function of $C$ and determine the number of mL that need to be added to have a solution that is $50\%$ acid.

**Answer**

$n(C) = \frac{100C - 25}{6 - C}, 250\text{mL}$

62) The period $T$, in seconds, of a simple pendulum as a function of its length $l$, in feet, is given by $T(l) = 2\pi\sqrt{\frac{l}{322}}$. Express $l$ as a function of $T$ and determine the length of a pendulum with period of $2$ seconds.

63) The volume of a cylinder, $V$, in terms of radius, $r$, and height, $h$, is given by $V = \pi r^2 h$. If a cylinder has a height of $6$ meters, express the radius as a function of $V$ and find the radius of a cylinder with volume of $300$ cubic meters.

**Answer**

$$r(V) = \sqrt{\frac{V}{6\pi}}, 3.99 \text{ meters}$$

64) The surface area, $A$, of a cylinder in terms of its radius, $r$, and height, $h$, is given by $A = 2\pi r^2 + 2\pi rh$. If the height of the cylinder is $4$ feet, express the radius as a function of $V$ and find the radius if the surface area is $200$ square feet.

65) The volume of a right circular cone, $V$, in terms of its radius, $r$, and its height, $h$, is given by $V = \frac{1}{3}\pi r^2 h$. Express $r$ in terms of $V$ if the height of the cone is $12$ feet and find the radius of a cone with volume of $50$ cubic inches.

**Answer**

$$r(V) = \sqrt{\frac{V}{4\pi}}, 1.99 \text{ inches}$$

66) Consider a cone with height of $30$ feet. Express the radius, $r$, in terms of the volume, $V$, and find the radius of a cone with volume of $1000$ cubic feet.

# 3.9 Modeling Using Variation

## Verbal

1) What is true of the appearance of graphs that reflect a direct variation between two variables?

**Answer**

The graph will have the appearance of a power function.

2) If two variables vary inversely, what will an equation representing their relationship look like?

3) Is there a limit to the number of variables that can jointly vary? Explain.

**Answer**

No. Multiple variables may jointly vary.

## Algebraic

**For the exercises 4-23, write an equation describing the relationship of the given variables.**

4) $y$ varies directly as $x$ and when $x = 6, y = 12$

5) $y$ varies directly as the square of $x$ and when $x = 4, y = 80$

**Answer**

$$y = 5x^2$$

6) $y$ varies directly as the square root of $x$ and when $x = 36, y = 24$

7) $y$ varies directly as the cube of $x$ and when $x = 36, y = 24$

**Answer**

$$y = 10x^3$$

8) $y$ varies directly as the cube root of $x$ and when $x = 27, y = 15$

9) $y$ varies directly as the fourth power of $x$ and when $x = 1, y = 6$

**Answer**

$$y = 6x^4$$

10) $y$ varies inversely as $x$ and when $x = 4, y = 2$

11) $y$ varies inversely as the square of $x$ and when $x = 3, y = 2$

**Answer**

$$y = \frac{18}{x^2}$$

12) $y$ varies inversely as the cube of $x$ and when $x = 2, y = 5$

13) $y$ varies inversely as the fourth power of $x$ and when $x = 3, y = 1$

**Answer**

$$y = \frac{81}{x^4}$$

14) $y$ varies inversely as the square root of $x$ and when $x = 25, y = 3$

15) $y$ varies inversely as the cube root of $x$ and when $x = 64, y = 5$

**Answer**

$$y = \frac{20}{\sqrt[3]{x}}$$

16) $y$ varies jointly with $x$ and $z$ and when $x = 2$ and $z = 3, y = 36$

17) $y$ varies jointly as $x, z$, and $w$ and when $x = 1, z = 2, w = 5$, then $y = 100$

**Answer**

$$y = 10xzw$$

18) $y$ varies jointly as the square of $x$ and the square of $z$ and when $x = 3$ and $z = 4$, then $y = 72$

19) $y$ varies jointly as $x$ and the square root of $z$ and when $x = 2$ and $z = 25$, then $y = 100$

**Answer**

$$y = 10x\sqrt{z}$$

20) $y$ varies jointly as the square of $x$ the cube of $z$ and the square root of $w$. When $x = 1, z = 2$, and $w = 36$, then $y = 48$

21) $y$ varies jointly as $x$ and $z$ and inversely as $w$. When $x = 3, z = 5$, and $w = 6$, then $y = 10$

**Answer**

$$y = 4\frac{xz}{w}$$

22) $y$ varies jointly as the square of $x$ and the square root of $z$ and inversely as the cube of $w$. When $x = 3, z = 4$, and $w = 3$, then $y = 6$

23) $y$ varies jointly as $x$ and $z$ and inversely as the square root of wand the square of $t$. When $x = 3, z = 1, w = 25$, and $t = 2$, then $y$

$= 6$

**Answer**

$$y = 40\frac{xz}{\sqrt{w}t^2}$$

## Numeric

**For the exercises 24-40, use the given information to find the unknown value.**

24) $y$ varies directly as $x$. When $x = 3$, then $y = 12$. Find $y$ when $x = 20$.

25) $y$ varies directly as the square of $x$. When $x = 2$, then $y = 16$. Find $y$ when $x = 8$.

**Answer**

$$y = 256$$

26) $y$ varies directly as the cube of $x$. When $x = 3$, then $y = 5$. Find $y$ when $x = 4$.

27) $y$ varies directly as the square root of $x$. When $x = 16$, then $y = 4$. Find $y$ when $x = 36$.

**Answer**

$$y = 6$$

28) $y$ varies directly as the cube root of $x$. When $x = 125$, then $y = 15$. Find $y$ when $x = 1,000$.

29) $y$ varies inversely with $x$. When $x = 3$, then $y = 2$. Find $y$ when $x = 1$.

**Answer**

$$y = 6$$

30) $y$ varies inversely with the square of $x$. When $x = 4$, then $y = 3$. Find $y$ when $x = 2$.

31) $y$ varies inversely with the cube of $x$. When $x = 3$, then $y = 1$. Find $y$ when $x = 1$.

**Answer**

$$y = 27$$

32) $y$ varies inversely with the square root of $x$. When $x = 64$, then $y = 12$. Find $y$ when $x = 36$.

33) $y$ varies inversely with the cube root of $x$. When $x = 27$, then $y = 5$. Find $y$ when $x = 125$.

**Answer**

$$y = 3$$

34) $y$ varies jointly as $x$ and $z$. When $x = 4$ and $z = 2$, then $y = 16$. Find $y$ when $x = 3$ and $z = 3$.

35) $y$ varies jointly as $x, z$, and $w$. When $x = 2, z = 1$, and $w = 12$, then $y = 72$. Find $y$ when $x = 1, z = 2$, and $w = 3$.

**Answer**

$$y = 18$$

36) $y$ varies jointly as $x$ and the square of $z$. When $x = 2$ and $z = 4$, then $y = 144$. Find $y$ when $x = 4$ and $z = 5$.

37) $y$ varies jointly as the square of $x$ and the square root of $z$. When $x = 9$, then $y = 24$. Find $y$ when $x = 3$ and $z = 25$.

**Answer**

$$y = 90$$

38) $y$ varies jointly as $x$ and inversely as $w$. When $x = 5, z = 2$, and $w = 20$, then $y = 4$. Find $y$ when $x = 3$ and $z = 8$ and $w = 48$.

39) $y$ varies jointly as the square of $x$ and the cube of $z$ and inversely as the square root of $w$. When $x = 2, z = 2$, and $w = 64$ then $y = 12$. Find $y$ when $x = 1, z = 3$, and $w = 4$.

**Answer**

$$y = \frac{81}{2}$$

40) $y$ varies jointly as the square of $x$ and of $z$ and inversely as the square root of wand of $t$. When $x = 2, z = 3, w = 16$, and $t = 3$, then $y = 1$. Find $y$ when $x = 3, z = 2, w = 36$, and $t = 5$.

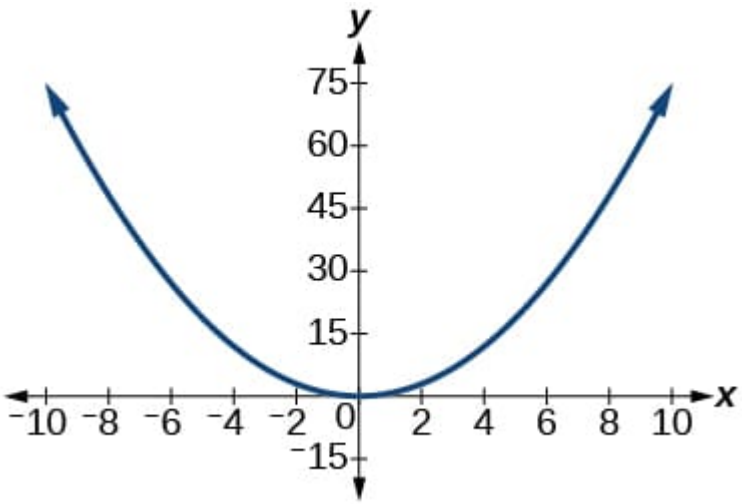
## Technology

**For the exercises 41-45, use a calculator to graph the equation implied by the given variation.**

41) $y$ varies directly with the square of $x$ and when $x = 2, y = 3$

**Answer**

$$y = \frac{3}{4}x^2$$

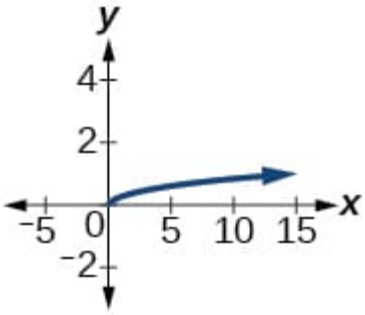



42) $y$ varies directly as the cube of $x$ and when $x = 2, y = 4$.

43) $y$ varies directly as the square root of $x$ and when $x = 36, y = 2$.

**Answer**

$$y = \frac{1}{3}\sqrt{x}$$

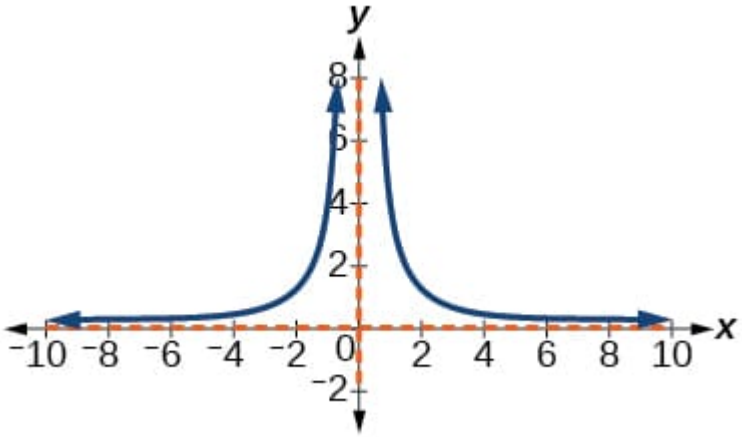



44) $y$ varies inversely with $x$ and when $x = 6, y = 2$.

45) $y$ varies inversely as the square of $x$ and when $x = 1, y = 4$.

**Answer**

$$y = \frac{4}{x^2}$$




## Extensions

**For the exercises 46-50, use Kepler's Law, which states that the square of the time, $T$, required for a planet to orbit the Sun varies directly with the cube of the mean distance, $a$, that the planet is from the Sun.**

46) Using the Earth's time of $1$ year and mean distance of 93 million miles, find the equation relating $T$ and $a$.

47) Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is $142$ million miles.

**Answer**

$1.89$ years

48) Using Earth's distance of $150$ million kilometers, find the equation relating $T$ and $a$.

49) Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is $108$ million kilometers.

**Answer**

$0.61$ years

50) Using Earth's distance of $1$ astronomical unit (A.U.), determine the time for Saturn to orbit the Sun if its mean distance is $9.54$ A.U.

## Real-World Applications

**For the exercises 51-60, use the given information to answer the questions.**

51) The distance $s$ that an object falls varies directly with the square of the time, $t$, of the fall. If an object falls $16$ feet in one second, how long for it to fall $144$ feet?

**Answer**

$3$ seconds

52) The velocity $v$ of a falling object varies directly to the time, $t$, of the fall. If after $2$ seconds, the velocity of the object is $64$ feet per second, what is the velocity after $5$ seconds?

53) The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is $24$ inches long and vibrates $128$ times per second, what is the length of a string that vibrates $64$ times per second?

**Answer**

$48$ inches

54) The volume of a gas held at constant temperature varies indirectly as the pressure of the gas. If the volume of a gas is $1200$ cubic centimeters when the pressure is $200$ millimeters of mercury, what is the volume when the pressure is $300$ millimeters of mercury?

55) The weight of an object above the surface of the Earth varies inversely with the square of the distance from the center of the Earth. If a body weighs $50$ pounds when it is $3960$ miles from Earth's center, what would it weigh it were $3970$ miles from Earth's center?

**Answer**

$49.75$ pounds

56) The intensity of light measured in foot-candles varies inversely with the square of the distance from the light source. Suppose the intensity of a light bulb is $0.08$ foot-candles at a distance of $3$ meters. Find the intensity level at $8$ meters.

57) The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is $40$ amperes, the resistance is $10$ ohms. Find the current if the resistance is $12$ ohms.

**Answer**

$33.33$ amperes

58) The force exerted by the wind on a plane surface varies jointly with the square of the velocity of the wind and with the area of the plane surface. If the area of the surface is $40$ square feet surface and the wind velocity is $20$ miles per hour, the resulting force is $15$ pounds. Find the force on a surface of $65$ square feet with a velocity of $30$ miles per hour.

59) The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm)) and the cube of the diameter. If the shaft of a certain material $3$ inches in diameter can transmit $45$ hp at $100$ rpm, what must the diameter be in order to transmit $60$ hp at $150$ rpm?

**Answer**

$2.88$ inches

60) The kinetic energy $K$ of a moving object varies jointly with its mass $m$ and the square of its velocity $v$. If an object weighing $40$ kilograms with a velocity of $15$ meters per second has a kinetic energy of $1000$ joules, find the kinetic energy if the velocity is

increased to $20$ meters per second.

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# 3.R: Polynomial and Rational Functions(Review)

## 3.1 Complex Numbers

**Perform the indicated operation with complex numbers.**

1) $(4+3i)+(-2-5i)$

**Answer**

$2-2i$

2) $(6-5i)-(10+3i)$

3) $(2-3i)(3+6i)$

**Answer**

$24+3i$

4) $\dfrac{2-i}{2+i}$

**Solve the following equations over the complex number system.**

5) $x^2-4x+5=0$

**Answer**

$\{2+i, 2-i\}$

6) $x^2+2x+10=0$

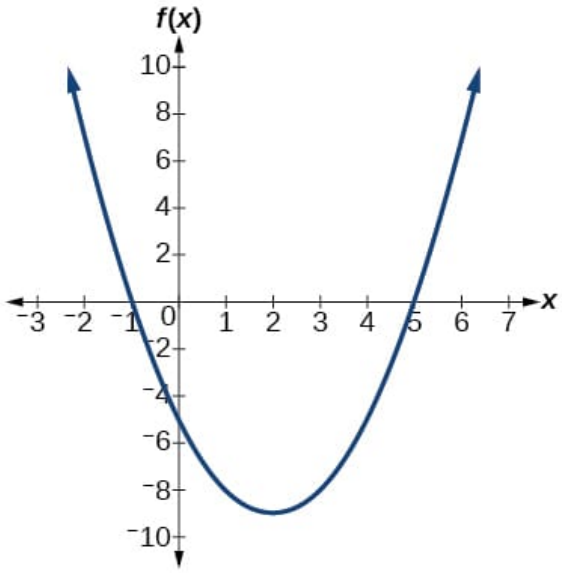
## 3.2 Quadratic Functions

**For the exercises 1-2, write the quadratic function in standard form. Then, give the vertex and axes intercepts. Finally, graph the function.**

1) $f(x)=x^2-4x-5$

**Answer**

$f(x)=(x-2)^2-9$ vertex $(2,-9)$, intercepts $(5,0);(-1,0);(0,-5)$




2) $f(x) = -2x^2 - 4x$

**For the problems 3-4, find the equation of the quadratic function using the given information.**

3) The vertex is $(-2, 3)$ and a point on the graph is $(3, 6)$.

**Answer**

$$f(x) = \frac{3}{25}(x+2)^2 + 3$$

4) The vertex is $(-3, 6.5)$ and a point on the graph is $(2, 6)$.

**Answer the following questions.**

5) A rectangular plot of land is to be enclosed by fencing. One side is along a river and so needs no fence. If the total fencing available is $600$ meters, find the dimensions of the plot to have maximum area.

**Answer**

$300$ meters by $150$ meters, the longer side parallel to river.

6) An object projected from the ground at a $45$ degree angle with initial velocity of $120$ feet per second has height, $h$, in terms of horizontal distance traveled, $x$, given by $h(x) = \frac{-32}{(120)^2}x^2 + x$. Find the maximum height the object attains.

## 3.3 Power Functions and Polynomial Functions

**For the exercises 1-3, determine if the function is a polynomial function and, if so, give the degree and leading coefficient.**

1) $f(x) = 4x^5 - 3x^3 + 2x - 1$

**Answer**

Yes, $\text{degree} = 5$, $\text{leading coefficient} = 4$

2) $f(x) = 5^{x+1} - x^2$

3) $f(x) = x^2\left(3 - 6x + x^2\right)$

**Answer**

Yes, degree $= 4$, leading coefficient $= 1$

**For the exercises 4-6, determine end behavior of the polynomial function.**

4) $f(x) = 2x^4 + 3x^3 - 5x^2 + 7$

5) $f(x) = 4x^3 - 6x^2 + 2$

**Answer**

As $x \to -\infty, f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$

6) $f(x) = 2x^2\left(1 + 3x - x^2\right)$

## 3.4 Graphs of Polynomial Functions

**For the exercises 1-3, find all zeros of the polynomial function, noting multiplicities.**

1) $f(x) = (x+3)^2(2x-1)(x+1)^3$

**Answer**

$-3$ with multiplicity $2$, $-\frac{1}{2}$ with multiplicity $1$, $-1$ with multiplicity $3$

2) $f(x) = x^5 + 4x^4 + 4x^3$

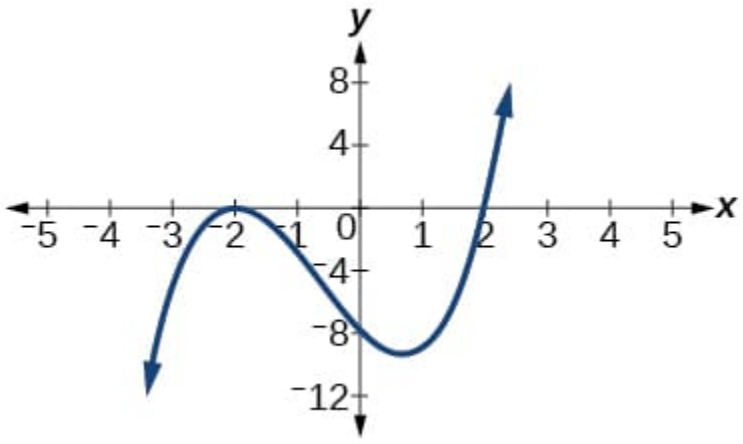
3) $f(x) = x^3 - 4x^2 + x - 4$

**Answer**

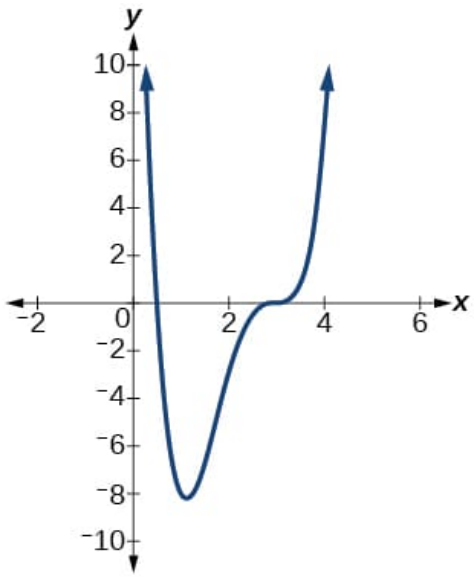
$4$ with multiplicity $1$

**For the exercises 4-5, based on the given graph, determine the zeros of the function and note multiplicity.**

4)




5)




**Answer**

$\frac{1}{2}$ with multiplicity $1$, $3$ with multiplicity $3$

6) Use the Intermediate Value Theorem to show that at least one zero lies between $2$ and $3$ for the function $f(x) = x^3 - 5x + 1$

## 3.5 Dividing Polynomials

**For the exercises 1-2, use long division to find the quotient and remainder.**

1) $\frac{x^3 - 2x^2 + 4x + 4}{x - 2}$

**Answer**

$x^2 + 4$ with remainder $12$

2) $\frac{3x^4 - 4x^2 + 4x + 8}{x + 1}$

**For the exercises 3-6, use synthetic division to find the quotient. If the divisor is a factor, then write the factored form.**

3) $\frac{x^2 - 2x^2 + 5x - 1}{x + 3}$

**Answer**

$x^2 - 5x + 20 - \frac{61}{x + 3}$

4) $\frac{x^2 + 4x + 10}{x - 3}$

5) $\frac{2x^3 + 6x^2 - 11x - 12}{x + 4}$

**Answer**

$2x^2 - 2x - 3$, so factored form is $(x + 4)(2x^2 - 2x - 3)$

6) $\dfrac{3x^4+3x^3+2x+2}{x+1}$

## 3.6 Zeros of Polynomial Functions

**For the exercises 1-4, use the Rational Zero Theorem to help you solve the polynomial equation.**

1) $2x^3-3x^2-18x-8=0$

**Answer**

$$\left\{-2,4,-\frac{1}{2}\right\}$$

2) $3x^3+11x^2+8x-4=0$

3) $2x^4-17x^3+46x^2-43x+12=0$

**Answer**

$$\left\{1,3,4,\frac{1}{2}\right\}$$

4) $4x^4+8x^3+19x^2+32x+12=0$

**For the exercises 5-6, use Descartes' Rule of Signs to find the possible number of positive and negative solutions.**

5) $x^3-3x^2-2x+4=0$

**Answer**

$0$ or $2$ positive, $1$ negative

6) $2x^4-x^3+4x^2-5x+1=0$

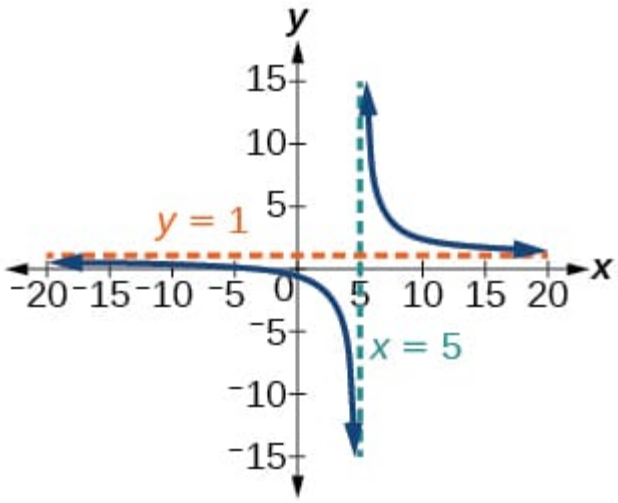
## 3.7 Rational Functions

**For the following rational functions 1-4, find the intercepts and the vertical and horizontal asymptotes, and then use them to sketch a graph.**

1) $f(x)=\dfrac{x+2}{x-5}$

**Answer**

Intercepts $(-2,0)$ and $\left(0,-\dfrac{2}{5}\right)$, Asymptotes $x=5$ and $y=1$

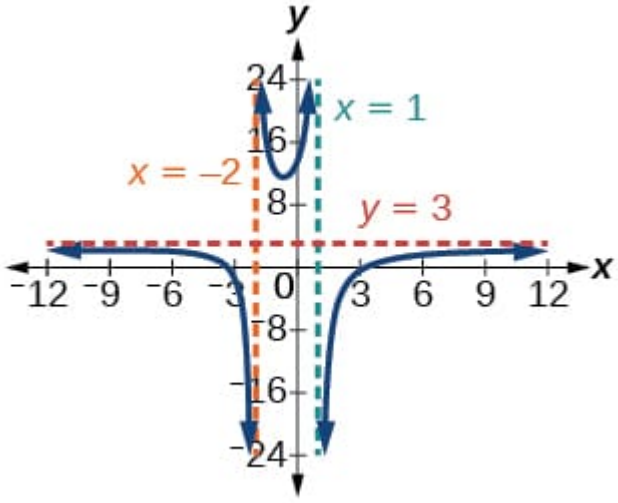



2) $f(x) = \dfrac{x^2+1}{x^2-4}$

3) $f(x) = \dfrac{3x^2-27}{x^2-9}$

**Answer**

Intercepts $(3,0), (-3,0)$, and $\left(0, \dfrac{27}{2}\right)$, Asymptotes $x = 1, x = -2, y = 3$




4) $f(x) = \dfrac{x+2}{x^2-9}$

**For the exercises 5-6, find the slant asymptote.**

5) $f(x) = \dfrac{x^2-1}{x+2}$

**Answer**

$y = x - 2$

6) $f(x) = \dfrac{2x^3-x^2+4}{x^2+1}$

# 3.8 Inverses and Radical Functions

**For the exercises 1-6, find the inverse of the function with the domain given.**

1) $f(x) = (x-2)^2, x \geq 2$

**Answer**

$$f^{-1}(x) = \sqrt{x} + 2$$

2) $f(x) = (x+4)^2 - 3, x \geq -4$

3) $f(x) = x^2 + 6x - 2, x \geq -3$

**Answer**

$$f^{-1}(x) = \sqrt{x+11} - 3$$

4) $f(x) = 2x^3 - 3$

5) $f(x) = \sqrt{4x+5} - 3$

**Answer**

$$f^{-1}(x) = \frac{(x+3)^2 - 5}{4}, x \geq -3$$

6) $f(x) = \dfrac{x-3}{2x+1}$

# 3.9 Modeling Using Variation

**For the exercises 1-4, find the unknown value.**

1) $y$ varies directly as the square of $x$. If when $x = 3, y = 36$, find $y$ if $x = 4$.

**Answer**

$$y = 64$$

2) $y$ varies inversely as the square root of $x$. If when $x = 25, y = 2$, find $y$ if $x = 4$.

3) $y$ varies jointly as the cube of $x$ and as $z$. If when $x = 1$ and $z = 2, y = 6$, find $y$ if $x = 2$ and $z = 3$.

**Answer**

$$y = 72$$

4) $y$ varies jointly as $x$ and the square of $z$ and inversely as the cube of $w$. If when $x = 3, z = 4$, and $w = 2, y = 48$, ind $y$ if $x = 4, z = 5$, and $w = 3$.

**For the exercises 5-6, solve the application problem.**

5) The weight of an object above the surface of the earth varies inversely with the distance from the center of the earth. If a person weighs $150$ pounds when he is on the surface of the earth $(3,960$ miles from center), find the weight of the person if he is $20$ miles above the surface.

**Answer**

$148.5$ pounds

6) The volume $V$ of an ideal gas varies directly with the temperature $T$ and inversely with the pressure$P$. A cylinder contains oxygen at a temperature of $310$ degrees K and a pressure of $18$ atmospheres in a volume of $120$ liters. Find the pressure if the volume is decreased to $100$ liters and the temperature is increased to $320$ degrees K.

# Practice Test

**Perform the indicated operation or solve the equation.**

1) $(3-4i)(4+2i)$

**Answer**

$20-10i$

2) $\dfrac{1-4i}{3+4i}$

3) $x^2-4x+13=0$

**Answer**

$\{2+3i, 2-3i\}$

4) Give the degree and leading coefficient of the following polynomial function.

$$f(x)=x^3\left(3-6x^2-2x^2\right)$$

**Determine the end behavior of the polynomial function.**

5) $f(x)=8x^3-3x^2+2x-4$

**Answer**

As $x\to-\infty, f(x)\to-\infty$, as $x\to\infty, f(x)\to\infty$

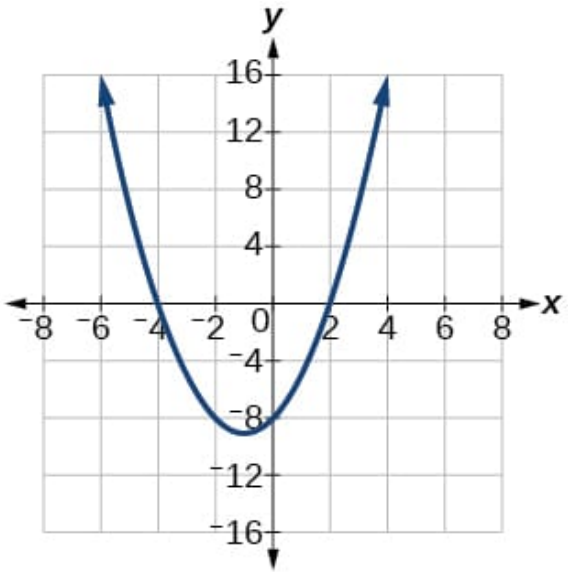
6) $f(x)=-2x^2\left(4-3x-5x^2\right)$

7) Write the quadratic function in standard form. Determine the vertex and axes intercepts and graph the function.

$$f(x)=x^2+2x-8$$

**Answer**

$f(x)=(x+1)^2-9$, vertex $(-1,-9)$, intercepts $(2,0);(-4,0);(0,-8)$




8) Given information about the graph of a quadratic function, find its equation: Vertex $(2, 0)$ and point on graph $(4, 12)$

**Solve the following application problem.**

9) A rectangular field is to be enclosed by fencing. In addition to the enclosing fence, another fence is to divide the field into two parts, running parallel to two sides. If $1,200$ feet of fencing is available, find the maximum area that can be enclosed.

**Answer**

$60,000$ square feet

**Find all zeros of the following polynomial functions, noting multiplicities.**

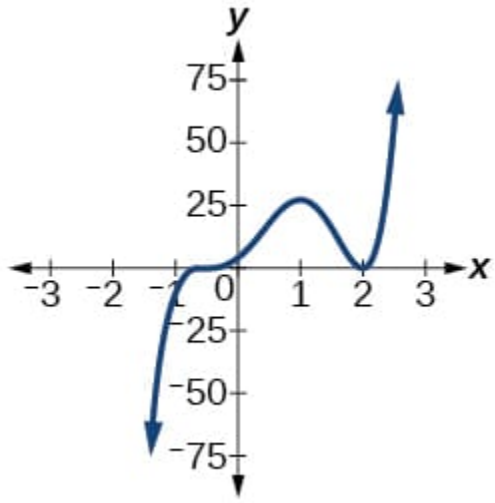
10) $f(x) = (x-3)^3(3x-1)(x-1)^2$

11) $f(x) = 2x^6 - 12x^5 + 18x^4$

**Answer**

$0$ with multiplicity $4$, $3$ with multiplicity $2$

12) Based on the graph, determine the zeros of the function and multiplicities.



13) Use long division to find the quotient:

$$\frac{2x^2+3x-4}{x+2}$$

**Answer**

$$2x^2-4x+11-\frac{26}{x+2}$$

**Use synthetic division to find the quotient. If the divisor is a factor, write the factored form.**

14) $\frac{x^4+3x^2-4}{x-2}$

15) $\frac{2x^3+5x^2-7x-12}{x+3}$

**Answer**

$2x^2-x-4$. So factored form is $(x+3)\left(2x^2-x-4\right)$

**Use the Rational Zero Theorem to help you find the zeros of the polynomial functions.**

16) $f(x)=2x^3+5x^2-6x-9$

17) $f(x)=4x^4+8x^3+21x^2+17x+4$

**Answer**

$-\frac{1}{2}$ (has multiplicity 2), $\frac{-1+i\sqrt{15}}{2}$

18) $f(x)=4x^4+16x^3+13x^2-15x-18$

19) $f(x)=x^5+6x^4+13x^3+14x^2+12x+8$

**Answer**

$-2$ (has multiplicity 3), $\pm i$

**Given the following information about a polynomial function, find the function.**

20) It has a double zero at $x=3$ and zeroes at $x=1$ and $x=-2$. Its $y$ -intercept is $(0,12)$.

21) It has a zero of multiplicity 3 at $x=\frac{1}{2}$ and another zero at $x=-3$. It contains the point $(1,8)$.

**Answer**

$f(x)=2(2x-1)^3(x+3)$

22) Use Descartes' Rule of Signs to determine the possible number of positive and negative solutions.

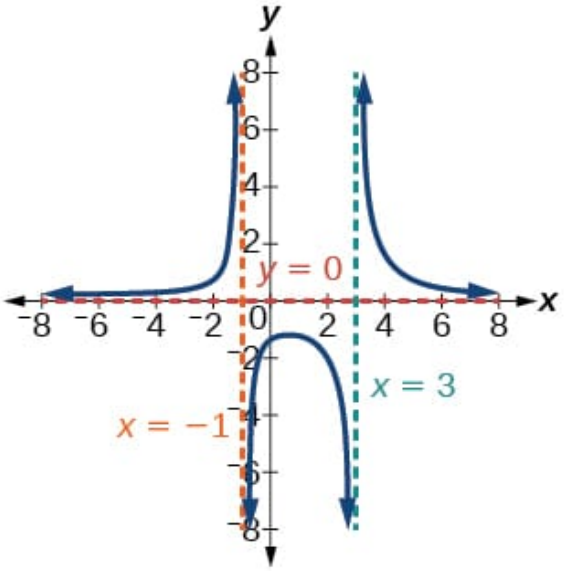
$$8x^3-21x^2+6=0$$

**For the following rational functions, find the intercepts and horizontal and vertical asymptotes, and sketch a graph.**

23) $f(x)=\frac{x+4}{x^2-2x-3}$

**Answer**

Intercepts $(-4, 0)$, $\left(0, -\frac{4}{3}\right)$, Asymptotes $x = 3, x = -1, y = 0$




24) $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - 4}$

25) Find the slant asymptote of the rational function.

$$f(x) = \frac{x^2 + 3x - 3}{x - 1}$$

**Answer**

$y = x + 4$

**Find the inverse of the function.**

26) $f(x) = \sqrt{x - 2} + 4$

27) $f(x) = 3x^3 - 4$

**Answer**

$f^{-1}(x) = \sqrt[3]{\dfrac{x + 4}{3}}$

28) $f(x) = \dfrac{2x + 3}{3x - 1}$

**Find the unknown value.**

29) $y$ varies inversely as the square of $x$ and when $x = 3, y = 2$. Find $y$ if $x = 1$.

**Answer**

$y = 18$

30) $y$ varies jointly with $x$ and the cube root of $z$. If when $x = 27, y = 12$, find $y$ if $x = 5$ and $z = 8$.

**Solve the following application problem.**

31) The distance a body falls varies directly as the square of the time it falls. If an object falls $64$ feet in $2$ seconds, how long will it take to fall $256$ feet?

**Answer**

$4$ seconds

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# 4: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.


4.0: Prelude to Exponential and Logarithmic Functions

4.1: Exponential Functions

4.2: Graphs of Exponential Functions

4.3: Logarithmic Functions

4.4: Graphs of Logarithmic Functions

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# Contributors

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# 4.0: Prelude to Exponential and Logarithmic Functions

Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years. For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. Table 4.0.1 shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

*Table* 4.0.1

| **Hour** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Bacteria** | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

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# 4.1: Exponential Functions

**Learning Objectives**

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base $e$.

India is the second most populous country in the world with a population of about $1.25$ billion people in 2013. The population is growing at a rate of about $.2\%$ each year. If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.

**Identifying Exponential Functions**

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by $3$ each time the input increases by $1$. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

## Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, $2.5\%$ of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up $10\%$ of the U.S. population in 2015, $40\%$ in 2019, and $80\%$ in 2050.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of **exponential growth**, let us contrast exponential growth with **linear growth**. We will construct two functions. The first function is exponential. We will start with an input of $0$, and increase each input by $1$. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of $0$, and increase each input by $1$. We will add $2$ to the corresponding consecutive outputs (Table $4.1.1$).

From Table $4.1.1$ we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

*Table* 4.1.1

| $x$ | $f(x) = 2^x$ | $g(x) = 2x$ |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 2 | 2 |
| 2 | 4 | 4 |
| 3 | 8 | 6 |
| 4 | 16 | 8 |
| 5 | 32 | 10 |
| 6 | 64 | 12 |

Apparently, the difference between “the same percentage” and “the same amount” is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding $2$ to the output whenever the input was increased by one.

The general form of the **exponential function** is $f(x) = ab^x$, where $a$ is any nonzero number, $b$ is a positive real number not equal to $1$.

- If $b > 1$,the function grows at a rate proportional to its size.
- If $0 < b < 1$, the function decays at a rate proportional to its size.

Let’s look at the function $f(x) = 2^x$ from our example. We will create a table (Table $4.1.2$) to determine the corresponding outputs over an interval in the domain from $-3$ to $3$.

*Table* 4.1.2

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 2^x$ | $2^{-3} = \frac{1}{8}$ | $2^{-2} = \frac{1}{4}$ | $2^{-1} = \frac{1}{2}$ | $2^0 = 1$ | $2^1 = 2$ | $2^2 = 4$ | $2^3 = 8$ |

Let us examine the graph of $f$ by plotting the ordered pairs from Table $4.1.2$ and then make a few observations $4.1.1$.

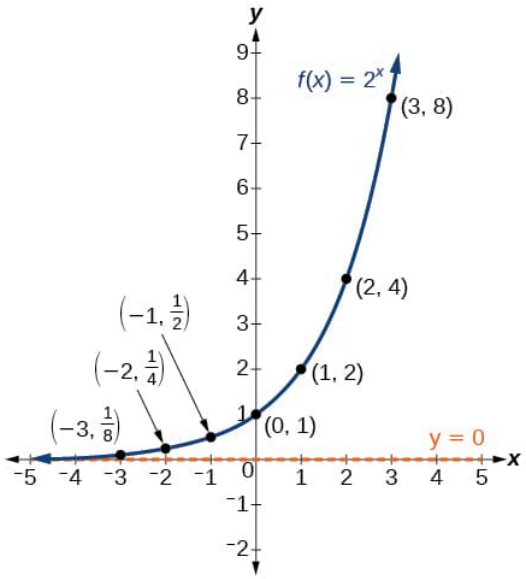



Figure 4.1.1

Let’s define the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some its key characteristics.

- the domain is $(-\infty, \infty)$,
- the range is $(0, \infty)$,
- as $x \to \infty$, $f(x) \to \infty$,
- as $x \to -\infty$, $f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the *y*-intercept is $1$.

## Definition: Exponential Functions

For any real number $x$,an exponential function is a function with the form

$$f(x) = ab^x \tag{4.1.1}$$

where

- $a$ is a non-zero real number called the initial value and
- $b$ is any positive real number such that $b \neq 1$.
- The domain of $f$ is all real numbers.
- The range of $f$ is all positive real numbers if $a > 0$.
- The range of $f$ is all negative real numbers if $a < 0$.
- The *y*-intercept is $(0, a)$,and the horizontal asymptote is $y = 0$.

## Example 4.1.1: Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

**Solution**

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base $b$ of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, $-2$, is less than $0$.

## Exercise 4.1.1

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

*Answer available at the end of this section.*

# Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than $1$.Why do we limit the base bb to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b=-9$ and $x=\frac{1}{2}$. Then $f(x)=f\left(\frac{1}{2}\right)=(-9)^{\frac{1}{2}}=\sqrt{-9}$,which is not a real number.

Why do we limit the base to positive values other than $1$? Because base $1$ results in the constant function. Observe what happens if the base is $1$:

- Let $b=1$. Then $f(x)=1^x=1$ for any value of $x$.

To evaluate an exponential function with the form $f(x)=b^x$,we simply substitute $x$ with the given value, and calculate the resulting power. For example:

Let $f(x)=2^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 2^x \\ f(3) &= 2^3 && \text{Substitute } x=3 \\ &= 8 && \text{Evaluate the power} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x)=30(2)^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 30(2)^x \\ f(3) &= 30(2)^3 && \text{Substitute } x=3 \\ &= 30(8) && \text{Simplify the power first} \\ &= 240 && \text{Multiply} \end{aligned}$$

Note that if the order of operations were not followed, the result would be incorrect:

$$f(3)=30(2)^3 \neq 60^3=216,000$$

## Example $4.1.2$: Evaluating Exponential Functions

Let $f(x)=5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

**Solution**

Follow the order of operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} f(x) &= 5(3)^{x+1} \\ f(2) &= 5(3)^{2+1} && \text{Substitute } x=2 \\ &= 5(3)^3 && \text{Add the exponents} \\ &= 5(27) && \text{Simplify the power} \\ &= 135 && \text{Multiply} \end{aligned}$$

## Exercise $4.1.2$

Let $f(x)=8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

*Answer available at the end of this section.*

# Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term “exponential growth” is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

## Definition: Exponential Growth

A function that models exponential growth grows by a rate proportional to the amount present. For any real number $x$ and any positive real numbers $a$ and $b$ such that $b \neq 1$,an exponential growth function has the form

$$f(x) = ab^x \tag{4.1.2}$$

where

- $a$ is the initial or starting value of the function.
- $b$ is the growth factor or growth multiplier per unit $x$.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let’s consider two companies, A and B. Company A has $100$ stores and expands by opening $50$ new stores a year, so its growth can be represented by the function $A(x) = 100 + 50x$. Company B has $100$ stores and expands by increasing the number of stores by $50\%$ each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in Table $4.1.3$.

*Table* $4.1.3$

| Year, $x$ | Stores, Company A | Stores, Company B |
|---|---|---|
| $0$ | $100 + 50(0) = 100$ | $100(1 + 0.5)^0 = 100$ |
| $1$ | $100 + 50(1) = 150$ | $100(1 + 0.5)^1 = 150$ |
| $2$ | $100 + 50(2) = 200$ | $100(1 + 0.5)^2 = 225$ |
| $3$ | $100 + 50(3) = 250$ | $100(1 + 0.5)^3 = 337.5$ |
| $x$ | $A(x) = 100 + 50x$ | $B(x) = 100(1 + 0.5)^x$ |

The graphs comparing the number of stores for each company over a five-year period are shown in Figure $4.1.2$. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

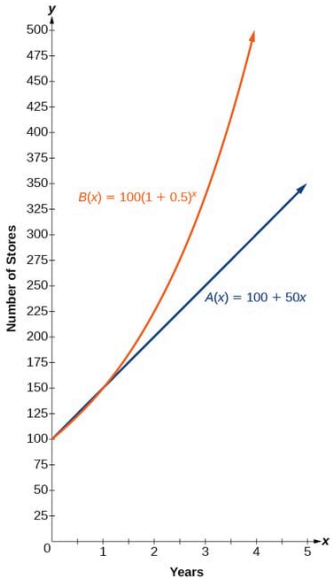



Figure 4.1.2: The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is $[0, \infty)$,and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company $B$, $B(x) = 100(1 + 0.5)^x$. In this exponential function, $100$ represents the initial number of stores, $0.50$ represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$,where $100$ is the initial value, $1.5$ is called the *base*, and $x$ is called the *exponent*.

## Example 4.1.3: Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about $1.25$ billion in the year 2013, with an annual growth rate of about $1.2\%$. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where $t$ is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

**Solution**

To estimate the population in 2031, we evaluate the models for $t = 18$, because 2031 is $18$ years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

There will be about $1.549$ billion people in India in the year 2031.

## Exercise 4.1.3

The population of China was about $1.39$ billion in the year 2013, with an annual growth rate of about $0.6\%$. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where $t$ is the number of years since 2013.To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example 4.1.3?

*Answer available at the end of this section.*

# Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants $a, a$ and $b, b$,and evaluate the function.

## How to: Given two data points, write an exponential model

1. If one of the data points has the form $(0, a)$, then $a$ is the initial value. Using $a$, substitute the second point into the equation $f(x) = a(b)^x$, and solve for $b$.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find $a$ and $b$.
3. Using the $a$ and $b$ found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

## Example 4.1.4: Writing an Exponential Model When the Initial Value Is Known

In 2006, $80$ deer were introduced into a wildlife refuge. By 2012, the population had grown to $180$ deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population $(N)$ of deer over time $t$.

**Solution**

We let our independent variable $t$ be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find $b$:

$$\begin{aligned} N(t) &= 80b^t \\ 180 &= 80b^6 \qquad \text{Substitute using point } (6, 180) \\ \frac{9}{4} &= b^6 \qquad \text{Divide and write in lowest terms} \\ b &= \left(\frac{9}{4}\right)^{\frac{1}{6}} \qquad \text{Isolate b using properties of exponents} \\ b &\approx 1.1447 \qquad \text{Round to 4 decimal places} \end{aligned}$$

> Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure $4.1.3$ passes through the initial points given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$,and the range for the function is $[80, \infty)$.

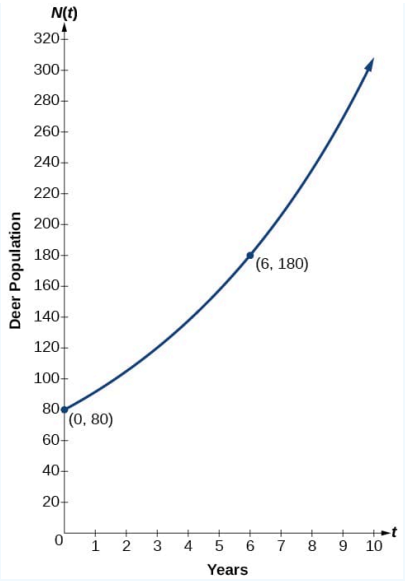



Figure 4.1.3: Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, $t$ years after 2006

## Exercise 4.1.4

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population $N$ of wolves over time $t$.

*Answer available at the end of this section.*

## Example 4.1.5: Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

**Solution**

Because we don't have the initial value, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for $a$ and $b$.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for $a$ in terms of $b$:

$$\begin{aligned} 6 &= ab^{-2} \\ \frac{6}{b^{-2}} &= a \qquad \text{Divide} \\ a &= 6b^2 \qquad \text{Use properties of exponents to rewrite the denominator} \end{aligned}$$

Substitute a in the second equation, and solve for $b$:

$$\begin{aligned} 1 &= ab^2 \\ 1 &= 6b^2b^2 \\ &= 6b^4 \qquad \text{Substitute a} \\ b &= \left(\frac{1}{6}\right)^{\frac{1}{4}} \qquad \text{Round 4 decimal places rewrite the denominator} \\ b &\approx 0.6389 \end{aligned}$$

Use the value of $b$ in the first equation to solve for the value of $a$:

$$\begin{aligned} a &= 6b^2 \\ &\approx 6(0.6389)^2 \\ &\approx 2.4492 \end{aligned}$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in Figure $4.1.4$ passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

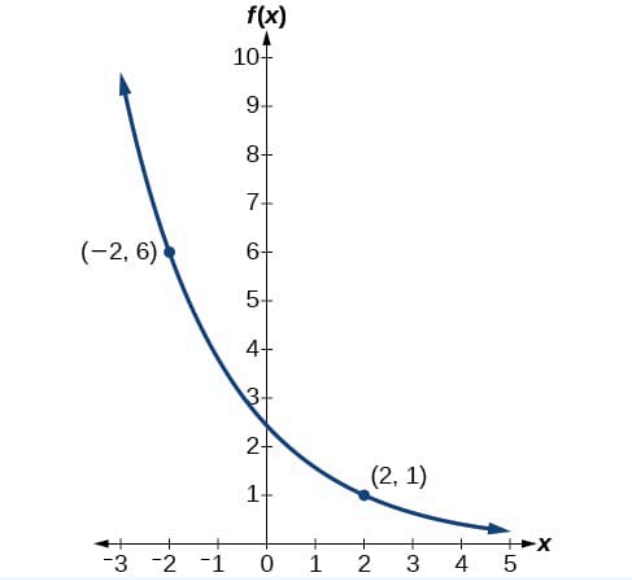



Figure $4.1.4$: The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.

## Exercise $4.1.5$

Given the two points $(1, 3)$ and $(2, 4.5)$,find the equation of the exponential function that passes through these two points.

*Answer available at the end of this section.*

## Q&A: Do two points always determine a unique exponential function?

*Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in $x$, which in many real world cases involves time.*

## How to: Given the graph of an exponential function, write its equation

1. First, identify two points on the graph. Choose the $y$-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the $y$-intercept $(0, a)$, then $a$ is the initial value. Using $a$, substitute the second point into the equation $f(x) = a(b)^x$, and solve for $b$
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find $a$ and $b$.
4. Write the exponential function, $f(x) = a(b)^x$.

## Example 4.1.6: Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in Figure 4.1.5.

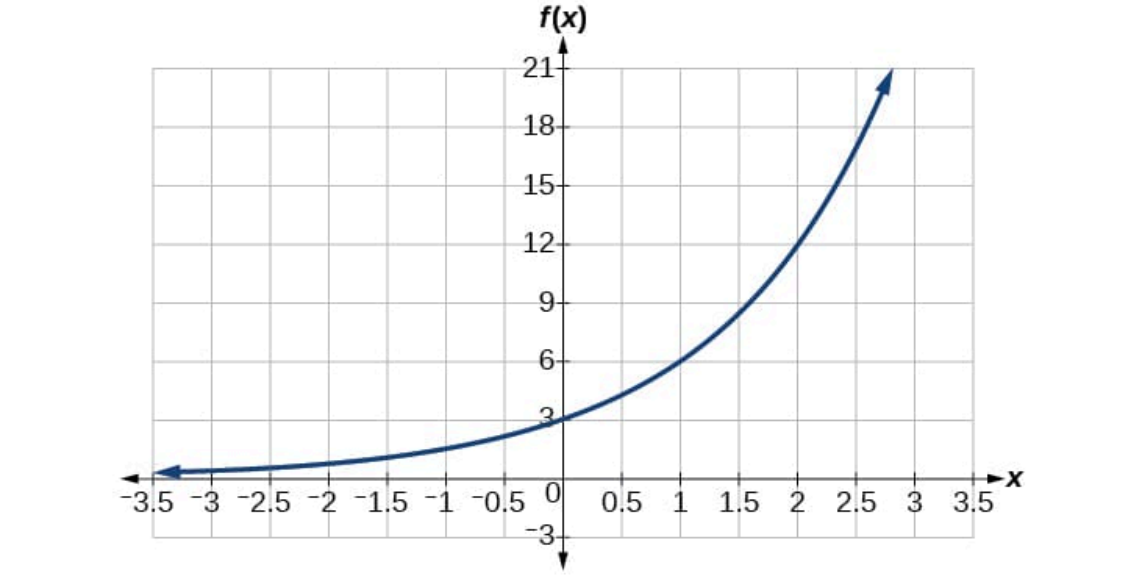



Figure 4.1.5

**Solution**

We can choose the $y$-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$$
\begin{aligned}
y &= ab^x && \text{Write the general form of an exponential equation} \\
y &= 3b^x && \text{Substitute the initial value 3 for } a \\
12 &= 3b^2 && \text{Substitute in 12 for } y \text{ and 2 for } x \\
4 &= b^2 && \text{Divide by 3} \\
b &= \pm 2 && \text{Take the square root}
\end{aligned}
$$

Because we restrict ourselves to positive values of $b$, we will use $b = 2$. Substitute $a$ and $b$ into the standard form to yield the equation $f(x) = 3(2)^x$.

## Exercise 4.1.6

Find an equation for the exponential function graphed in Figure 4.1.6.

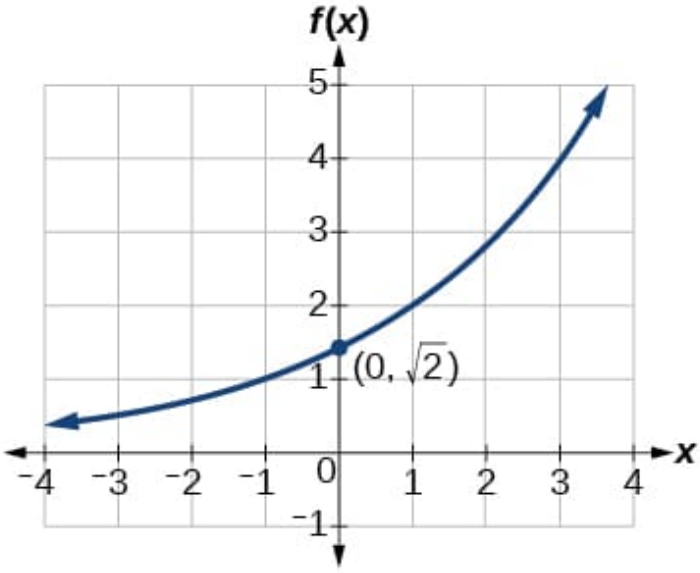



Figure 4.1.6

*Answer available at the end of this section.*

### How to: Given two points on the curve of an exponential function, use a graphing calculator to find the equation

1. Press **[STAT].**
2. Clear any existing entries in columns **L1** or **L2.**
3. In **L1**, enter the *x*-coordinates given.
4. In **L2**, enter the corresponding *y*-coordinates.
5. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential Regression)**, and press **[ENTER].**
6. The screen displays the values of *a* and *b* in the exponential equation $y = a \cdot b^x$.

### Example 4.1.7: Using a Graphing Calculator to Find an Exponential Function

Use a graphing calculator to find the exponential equation that includes the points $(2, 24.8)$ and $(5, 198.4)$.

**Solution**

Follow the guidelines above. First press **[STAT]**, **[EDIT]**, **[1: Edit...],** and clear the lists **L1** and **L2**. Next, in the **L1** column, enter the $x$-coordinates, $2$ and $5$. Do the same in the **L2** column for the $y$-coordinates, $24.8$ and $198.4$.

Now press **[STAT]**, **[CALC]**, **[0: ExpReg]** and press **[ENTER]**. The values $a = 6.2$ and $b = 2$ will be displayed. The exponential equation is $y = 6.2 \cdot 2^x$.

### Exercise 4.1.7

Use a graphing calculator to find the exponential equation that includes the points $(3, 75.98)$ and $(6, 481.07)$.

*Answer available at the end of this section.*

# Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time $t$, principal $P$, $APR$ $r$, and number of compounding periods in a year $n$:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe Table $4.1.4$, which shows the result of investing $\$1,000$ at $10\%$ for one year. Notice how the value of the account increases as the compounding frequency increases.

*Table* $4.1.4$

| Frequency | Value after $1$ year |
|---|---|
| Annually | $\$1100$ |
| Semiannually | $\$1102.50$ |
| Quarterly | $\$1103.81$ |
| Monthly | $\$1104.71$ |
| Daily | $\$1105.16$ |

## Definition: Compound Interest

*Compound interest* can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \tag{4.1.3}$$

where

- $A(t)$ is the account value,
- $t$ is measured in years,
- $P$ is the starting amount of the account, often called the principal, or more generally present value,
- $r$ is the annual percentage rate (APR) expressed as a decimal, and
- $n$ is the number of compounding periods in one year.

## Example $4.1.8$: Calculating Compound Interest

If we invest $\$3,000$ in an investment account paying $3\%$ interest compounded quarterly, how much will the account be worth in $10$ years?

**Solution**

Because we are starting with $\$3,000$, $P = 3000$. Our interest rate is $3\%$, so $r = 0.03$. Because we are compounding quarterly, we are compounding $4$ times per year, so $n = 4$. We want to know the value of the account in $10$ years, so we are looking for $A(10)$,the value when $t = 10$.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula}$$
$$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{(4)\cdot(10)} \quad \text{Substitute using given values}$$
$$\approx \$4045.05 \quad \text{Round to two decimal places}$$

The account will be worth about $\$4,045.05$ in $10$ years.

### Exercise 4.1.8

An initial investment of $\$100,000$ at $12\%$ interest is compounded weekly (use $52$ weeks in a year). What will the investment be worth in $30$ years?

*Answer available at the end of this section.*

### Example 4.1.9: Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $\$40,000$ over $18$ years. She believes the account will earn $6\%$ compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

**Solution**

The nominal interest rate is $6\%$, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, $P$, needed so that the value of the account will be worth $\$40,000$ in $18$ years. Substitute the given values into the compound interest formula, and solve for $P$.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula}$$
$$40,000 = P\left(1 + \frac{0.06}{2}\right)^{2(18)} \quad \text{Substitute using given values } A, r, n, t$$
$$40,000 = P(1.03)^{36} \quad \text{Simplify}$$
$$\frac{40,000}{(1.03)^{36}} = P \quad \text{Isolate } P$$
$$P \approx \$13,801 \quad \text{Divide and round to the nearest dollar}$$

Lily will need to invest $\$13,801$ to have $\$40,000$ in $18$ years.

### Exercise 4.1.9

Refer to Example $4.1.9$. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

*Answer available at the end of this section.*

## Evaluating Functions with Base $e$

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table $4.1.5$ shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $\$1$ invested at $100\%$ interest for $1$ year, compounded at various frequencies, listed in Table $4.1.5$.

*Table* 4.1.5

| Frequency | $A(t)=\left(1+\frac{1}{n}\right)^n$ | Value |
|---|---|---|
| Annually | $\left(1+\frac{1}{1}\right)^1$ | $2 |
| Semiannually | $\left(1+\frac{1}{2}\right)^2$ | $2.25 |
| Quarterly | $\left(1+\frac{1}{4}\right)^4$ | $2.441406 |
| Monthly | $\left(1+\frac{1}{12}\right)^{12}$ | $2.613035 |
| Daily | $\left(1+\frac{1}{365}\right)^{365}$ | $2.714567 |
| Hourly | $\left(1+\frac{1}{8760}\right)^{8760}$ | $2.718127 |
| Once per minute | $\left(1+\frac{1}{525600}\right)^{525600}$ | $2.718279 |
| Once per second | $\left(1+\frac{1}{31536000}\right)^{31536000}$ | $2.718282 |

These values appear to be approaching a limit as $n$ increases without bound. In fact, as $n$ gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter $e$. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

## Definition: The Number e

The letter $e$ represents the irrational number

$$\left(1+\frac{1}{n}\right)^n \tag{4.1.4}$$

as $n$ increases without bound

The letter $e$ is used as a base for many real-world exponential models. To work with base $e$, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

## Example 4.1.10: Using a Calculator to Find Powers of $e$

Calculate $e^{3.14}$. Round to five decimal places.

**Solution**

On a calculator, press the button labeled $[e^x]$. The window shows $[e^(]$. Type $3.14$ and then close parenthesis, $[)]$. Press [ENTER]. Rounding to $5$ decimal places, $e^{3.14} \approx 23.10387$. Caution: Many scientific calculators have an “Exp” button, which is used to enter numbers in scientific notation. It is not used to find powers of $e$.

## Exercise 4.1.10

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

*Answer available at the end of this section.*

## Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, $e$ is used as the base for exponential functions. Exponential models that use $e$ as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

### Definition: The Continuous Growth/decay Formula

For all real numbers $t$,and all positive numbers $a$ and $r$, continuous growth or decay is represented by the formula

$$A(t) = ae^{rt} \tag{4.1.5}$$

where

- $a$ is the initial value,
- $r$ is the continuous growth rate per unit time,
- $t$ is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt} \tag{4.1.6}$$

where

- $P$ is the principal or the initial invested,
- $r$ is the growth or interest rate per unit time,
- $t$ is the period or term of the investment.

### How to: Given the initial value, rate of growth or decay, and time $t$, solve a continuous growth or decay function

1. Use the information in the problem to determine $a$, the initial value of the function.
2. Use the information in the problem to determine the growth rate $r$.
   - If the problem refers to continuous growth, then $r > 0$.
   - If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time $t$.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

### Example 4.1.11: Calculating Continuous Growth

A person invested $\$1,000$ in an account earning a nominal $10\%$ per year compounded continuously. How much was in the account at the end of one year?

**Solution**

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $\$1,000$, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P, r, t \\ &\approx 1105.17 && \text{Use a calculator to approximate} \end{aligned}$$

The account is worth $\$1,105.17$ after one year.

### Exercise 4.1.11

A person invests $\$100,000$ at a nominal $12\%$ interest per year compounded continuously. What will be the value of the investment in $30$ years?

*Answer available at the end of this section.*

### Example 4.1.12: Calculating Continuous Decay

$Radon-222$ decays at a continuous rate of $17.3\%$ per day. How much will $100mg$ of $Radon-222$ decay to in $3$ days?

**Solution**

Since the substance is decaying, the rate, $17.3\%$, is negative. So, $r=-0.173$. The initial amount of $Radon-222$ was $100$ mg, so $a=100$. We use the continuous decay formula to find the value after $t=3$ days:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula} \\ &= 100e6{-}0.173(3) && \text{Substitute known values for } a, r, t \\ &\approx 59.5115 && \text{Use a calculator to approximate} \end{aligned}$$

So $59.5115$ mg of $Radon-222$ will remain.

### Exercise 4.1.12

Using the data in Example $4.1.12$, how much $Radon-222$ will remain after one year?

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with exponential functions.

- Exponential Growth Function
- Compound Interest

## Key Equations

| | |
|---|---|
| definition of the exponential function | $f(x)=b^x$, where $b>0$, $b\neq 1$ |
| definition of exponential growth | $f(x)=ab^x$, where $a>0$, $b>0$, $b\neq 1$ |

| | |
|---|---|
| compound interest formula | $A(t) = P(1 + \frac{r}{n})^{nt}$,<br>where $A(t)$ is the account value at time $t$<br>$t$ is the number of years<br>$P$ is the initial investment, often called the principal<br>$r$ is the annual percentage rate (APR), or nominal rate<br>$n$ is the number of compounding periods in one year |
| continuous growth formula | $A(t) = ae^{rt}$, where $t$ is the number of unit time periods of growth $a$ is the starting amount (in the continuous compounding formula a is replaced with $P$, the principal) $e$ is the mathematical constant, $e \approx 2.718282$ |

## Key Concepts

- An exponential function is defined as a function with a positive constant other than $1$ raised to a variable exponent. See Example.
- A function is evaluated by solving at a specific value. See Example and Example.
- An exponential model can be found when the growth rate and initial value are known. See Example.
- An exponential model can be found when the two data points from the model are known. See Example.
- An exponential model can be found using two data points from the graph of the model. See Example.
- An exponential model can be found using two data points from the graph and a calculator. See Example.
- The value of an account at any time $t$ can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See Example.
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See Example.
- The number $e$ is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282$.
- Scientific and graphing calculators have the key $[ex]$ or $[exp(x)]$ for calculating powers of $e$. See Example.
- Continuous growth or decay models are exponential models that use $e$ as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See Example and Example.

### Answers

**Exercise** $4.1.1$

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions.

**Exercise** $4.1.2$

$5.5556$

**Exercise** $4.1.3$

About $1.548$ billion people; by the year 2031, India's population will exceed China's by about $0.001$ billion, or $1$ million people.

**Exercise** $4.1.4$

$(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

**Exercise** $4.1.5$

$f(x) = 2(1.5)^x$

**Exercise** $4.1.6$

$f(x) = \sqrt{2}(\sqrt{2})^x$. Answers may vary due to round-off error. The answer should be very close to $1.4142(1.4142)^x$.

**Exercise** $4.1.7$

$y \approx 12 \cdot 1.85^x$

**Exercise** $4.1.8$

about $\$3,644,675.88$

**Exercise** $4.1.9$

$\$13,693$

**Exercise** $4.1.10$

$e^{-0.5} \approx 0.60653$

**Exercise** $4.1.11$

$\$3,659,823.44$

**Exercise** $4.1.12$

$3.77E-26$ (This is calculator notation for the number written as $3.77 \times 10^{-26}$ in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

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# 4.2: Graphs of Exponential Functions

**Learning Objectives**

- Graph exponential functions.
- Graph exponential functions using transformations.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

## Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table $4.2.1$ change as the input increases by $1$.

*Table* $4.2.1$

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 2^x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ | $8$ |

Each output value is the product of the previous output and the base, $2$. We call the base $2$ the *constant ratio*. In fact, for any exponential function with the form $f(x) = ab^x$, $b$ is the constant ratio of the function. This means that as the input increases by $1$, the output value will be the product of the base and the previous output, regardless of the value of $a$.

Notice from the table that

- the output values are positive for all values of $x$;
- as $x$ increases, the output values increase without bound; and
- as $x$ decreases, the output values grow smaller, approaching zero.

Figure $4.2.1$ shows the exponential growth function $f(x) = 2^x$.

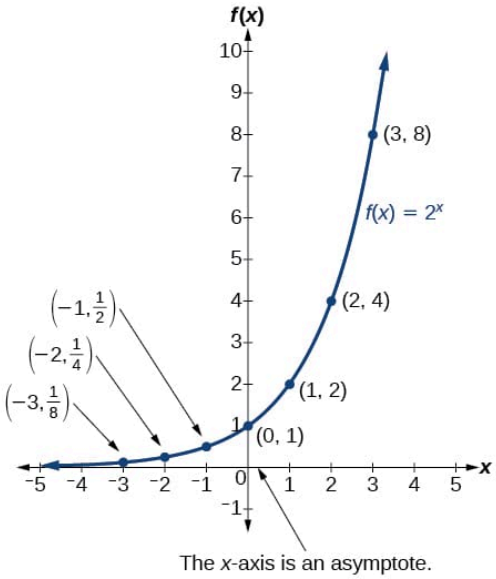



Figure $4.2.1$: Notice that the graph gets close to the x-axis, but never touches it.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table $4.2.2$ change as the input increases by $1$.

*Table* $4.2.2$

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $g(x) = \left(\frac{1}{2}\right)^x$ | $8$ | $4$ | $2$ | $1$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

Again, because the input is increasing by $1$, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of $x$;
- as $x$ increases, the output values grow smaller, approaching zero; and
- as $x$ decreases, the output values grow without bound.

Figure $4.2.2$ shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

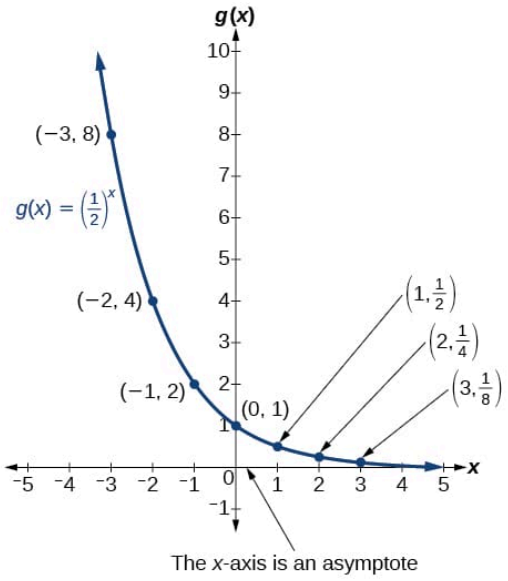



Figure $4.2.2$

The domain of $g(x) = (\frac{1}{2})^{x}$ is all real numbers, the range is $(0, \infty)$,and the horizontal asymptote is $y = 0$.

## CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION $F(X) = b^x$

An exponential function with the form $f(x) = b^x, b > 0, b \neq 1$,has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- *x*-intercept: none
- *y*-intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

Figure $4.2.3$ compares the graphs of exponential growth and decay functions.

Graph of two functions where the first graph is of a function of f(x) = b^x when b
1 and the second graph is of the same function when b is 0

**Figure**
$4.2.3$

## How to: Given an exponential function of the form $f(x) = b^x$,graph the function

1. Create a table of points.
2. Plot at least $3$ point from the table, including the *y*-intercept $(0, 1)$.
3. Draw a smooth curve through the points.
4. State the domain, $(-\infty, \infty)$,the range, $(0, \infty)$,and the horizontal asymptote, $y = 0$.

## Example $4.2.1$: Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

**Solution**

Before graphing, identify the behavior and create a table of points for the graph.

*Table* 4.2.3

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 0.25^x$ | 64 | 16 | 4 | 1 | 0.25 | \(0.0625\0 | 0.015625 |

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in Table 4.2.3.
- Plot the *y*-intercept, $(0, 1)$,along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in Figure 4.2.4.

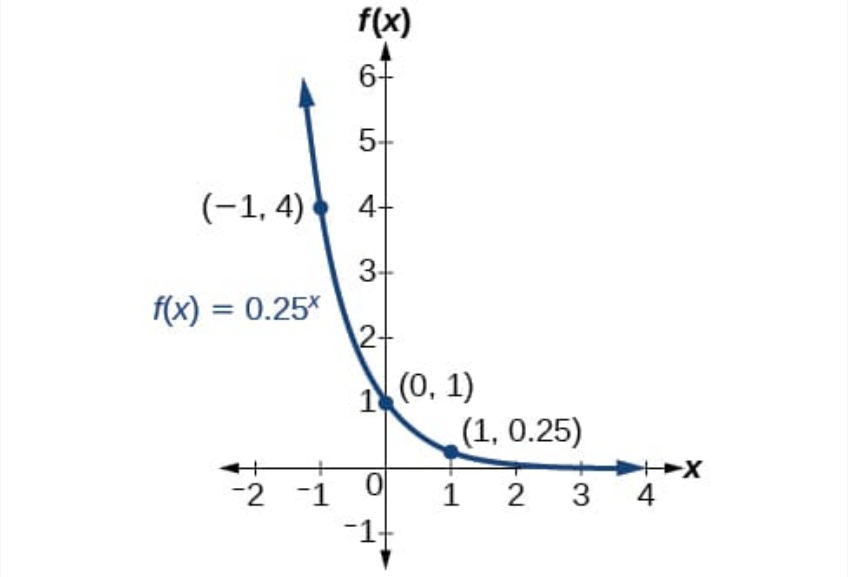



Figure 4.2.4

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

**Exercise 4.2.1**

Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

*Answer available at the end of this section.*

# Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

## Graphing a Vertical Shift

The first transformation occurs when we add a constant $d$ to the parent function $f(x) = b^x$, giving us a vertical shiftd d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown

in Figure $4.2.5$.

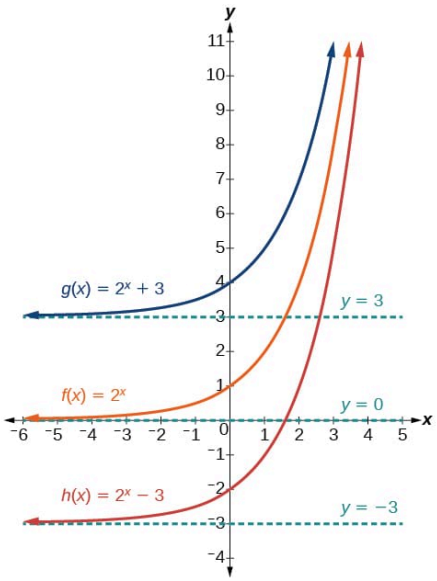



Figure $4.2.5$

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up $3$ units to $g(x) = 2^x + 3$:
  - The *y*-intercept shifts up $3$ units to $(0, 4)$.
  - The asymptote shifts up $3$ units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down $3$ units to $h(x) = 2^x - 3$:
  - The *y*-intercept shifts down $3$ units to $(0, -2)$.
  - The asymptote also shifts down $3$ units to $y = -3$.
  - The range becomes $(-3, \infty)$.

## Graphing a Horizontal Shift

The next transformation occurs when we add a constant $c$ to the input of the parent function $f(x) = b^x$, giving us a horizontal shift $c$ units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure $4.2.6$.

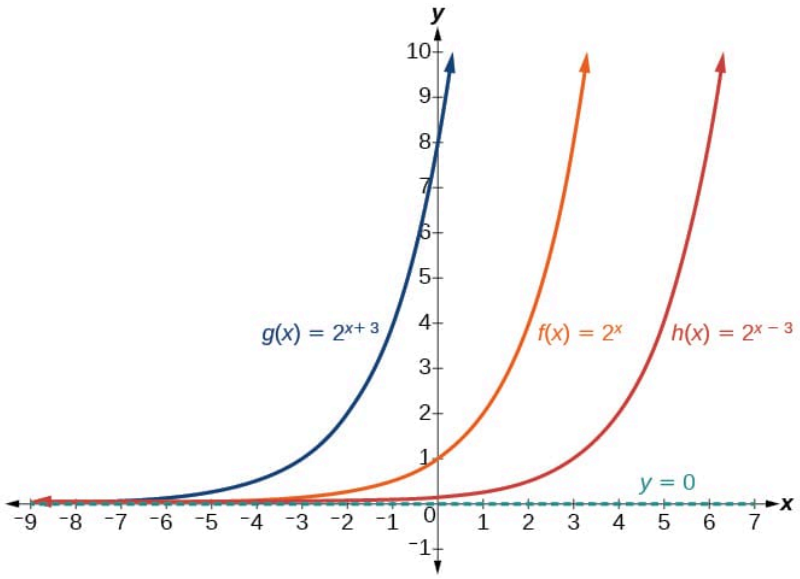



Figure 4.2.6

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$,remains unchanged.
- The asymptote, $y = 0$,remains unchanged.
- The *y*-intercept shifts such that:
  - When the function is shifted left $3$ units to $g(x) = 2^{x+3}$,the *y*-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$,so the initial value of the function is $8$.
  - When the function is shifted right $3$ units to $h(x) = 2^{x-3}$,the *y*-intercept becomes $(0, \frac{1}{8})$. Again, see that $2^{x-3} = (\frac{1}{8})2^x$,so the initial value of the function is $\frac{1}{8}$.

## SHIFTS OF THE PARENT FUNCTION $F(X) = b^x$

For any constants $c$ and $d$,the function $f(x) = b^{x+c} + d$ shifts the parent function $f(x) = b^x$

- vertically $d$ units, in the *same* direction of the sign of $d$.
- horizontally $c$ units, in the *opposite* direction of the sign of $c$.
- The *y*-intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes $(d, \infty)$.
- The domain, $(-\infty, \infty)$,remains unchanged.

## How to: Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left $c$ units if $c$ is positive, and right $c$ units if $c$ is negative.
3. Shift the graph of $f(x) = b^x$ up $d$ units if $d$ is positive, and down $d$ units if $d$ is negative.
4. State the domain, $(-\infty, \infty)$,the range, $(d, \infty)$,and the horizontal asymptote $y = d$.

## Example 4.2.2: Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

**Solution**

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

Draw the horizontal asymptote $y = d$, so draw $y = -3$.

Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.

Shift the graph of $f(x) = b^x$ left $1$ units and down $3$ units.

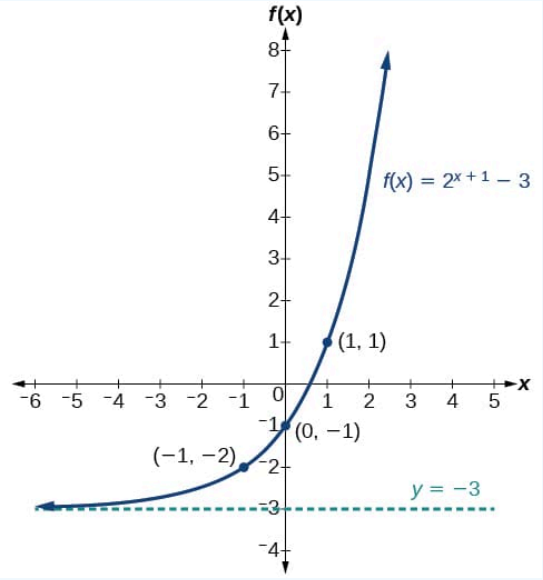



Figure $4.2.7$

The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

## Exercise $4.2.2$

Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

*Answer available at the end of this section.*

## How to: Given an equation of the form $f(x) = b^{x+c} + d$ for $x$, use a graphing calculator to approximate the solution

- Press **[Y=]**. Enter the given exponential equation in the line headed "**$Y_1$=**".
- Enter the given value forf(x) f(x) in the line headed "**$Y_2$=**".
- Press **[WINDOW]**. Adjust the *y*-axis so that it includes the value entered for "**$Y_2$=**".
- Press **[GRAPH]** to observe the graph of the exponential function along with the line for the specified value off(x). f(x).
- To find the value ofx, x,we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of *x* for the indicated value of the function.

## Example $4.2.3$: Approximating the Solution of an Exponential Equation

Solve $42 = 1.2(5)^x + 2.8$ graphically. Round to the nearest thousandth.

**Solution**

Press **[Y=]** and enter $1.2(5)^x + 2.8$ next to **$Y_1$=**. Then enter $42$ next to **Y2=**. For a window, use the values $-3$ to $3$ for $x$ and $-5$ to $55$ for $y$. Press **[GRAPH]**. The graphs should intersect somewhere near $x = 2$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as $2.1661943$. (Your answer may be different if you use a different window or use a different value for **Guess?**) To the nearest thousandth, $x \approx 2.166$.

**Exercise 4.2.3**

Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

*Answer available at the end of this section.*

## Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a **stretch** or **compression** occurs when we multiply the parent function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the parent function $f(x) = 2^x$,we can then graph the stretch, using $a = 3$,to get $g(x) = 3(2)^x$ as shown on the left in Figure 4.2.8, and the compression, using $a = \frac{1}{3}$,to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure 4.2.8.

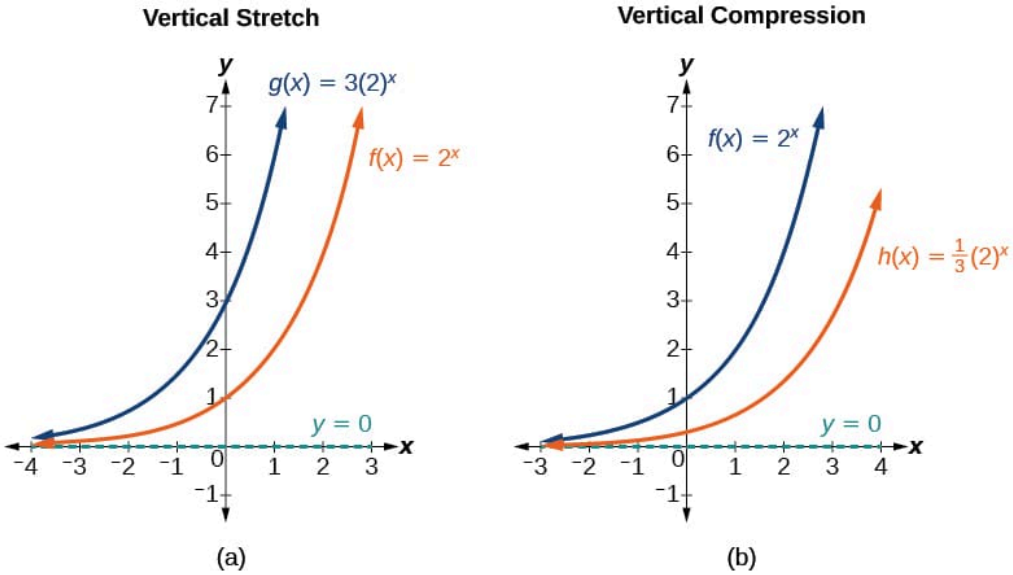



Figure 4.2.8: (a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

### STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $F(X) = B^X$

For any factor $a > 0$,the function $f(x) = a(b)^x$

- is stretched vertically by a factor of $a$ if $|a| > 1$.
- is compressed vertically by a factor of $a$ if $|a| < 1$.
- has a *y*-intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$,which are unchanged from the parent function.

## Example 4.2.4: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4(\frac{1}{2})^x$. State the domain, range, and asymptote.

**Solution**

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as $x$ decreases, and the right tail will approach the *x*-axis as $x$ increases.
- Since $a = 4$,the graph of $f(x) = (\frac{1}{2})^x$ will be stretched by a factor of $4$.
- Create a table of points as shown in Table $4.2.4$.

*Table* $4.2.4$

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 4(\frac{1}{2})^x$ | $32$ | $16$ | $8$ | $4$ | $2$ | $1$ | $0.5$ |

- Plot the *y*-intercept, $(0, 4)$,along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in Figure $4.2.9$.

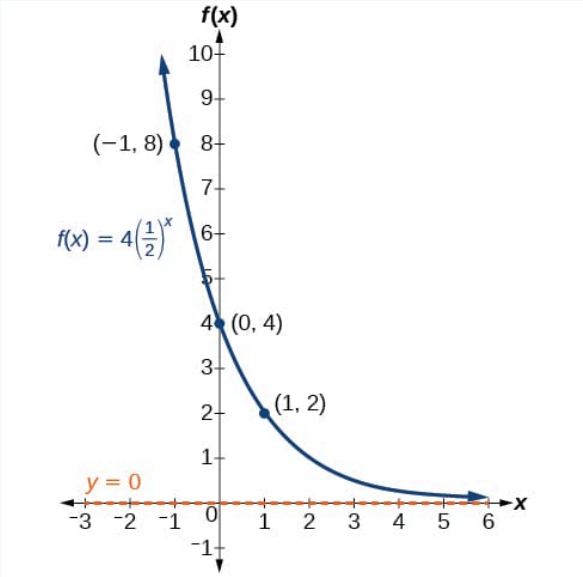



Figure $4.2.9$

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

## Exercise 4.2.4

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the *x*-axis or the *y*-axis. When we multiply the parent function $f(x) = b^x$ by $-1$,we get a reflection about the *x*-axis. When we multiply the input by $-1$,we get a reflection about the *y*-axis. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the *x*-axis, $g(x) = -2^x$,is shown on the left side of Figure $4.2.10$, and the reflection about the y-axis $h(x) = 2^{-x}$ , is shown on the right side of Figure $4.2.10$.

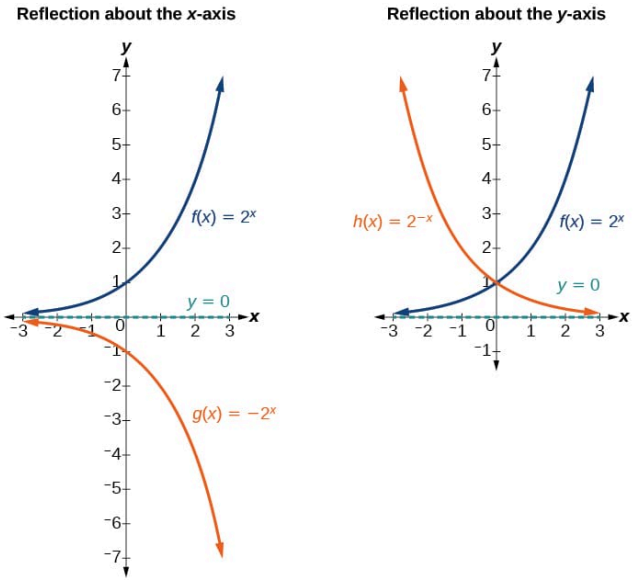



Figure $4.2.10$: (a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.

### REFLECTIONS OF THE PARENT FUNCTION $F(X) = B^X$

The function $f(x) = -b^x$

- reflects the parent function $f(x) = b^x$ about the *x*-axis.
- has a *y*-intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$,which are unchanged from the parent function.

The function $f(x) = b^{-x}$

- reflects the parent function $f(x) = b^x$ about the *y*-axis.
- has a *y*-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

### Example $4.2.5$: Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$,that reflects $f(x) = (\frac{1}{4})^x$ about the *x*-axis. State its domain, range, and asymptote.

**Solution**

Since we want to reflect the parent function $f(x) = (\frac{1}{4})^x$ about the *x*-axis, we multiply $f(x)$ by $-1$ to get, $g(x) = -(\frac{1}{4})^x$. Next we create a table of points as in Table $4.2.5$.

*Table* 4.2.5

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $g(x) = -(\frac{1}{4})^x$ | $-64$ | $-16$ | $-4$ | $-1$ | $-0.25$ | $-0.0625$ | $-0.0156$ |

Plot the *y*-intercept, $(0, -1)$,along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

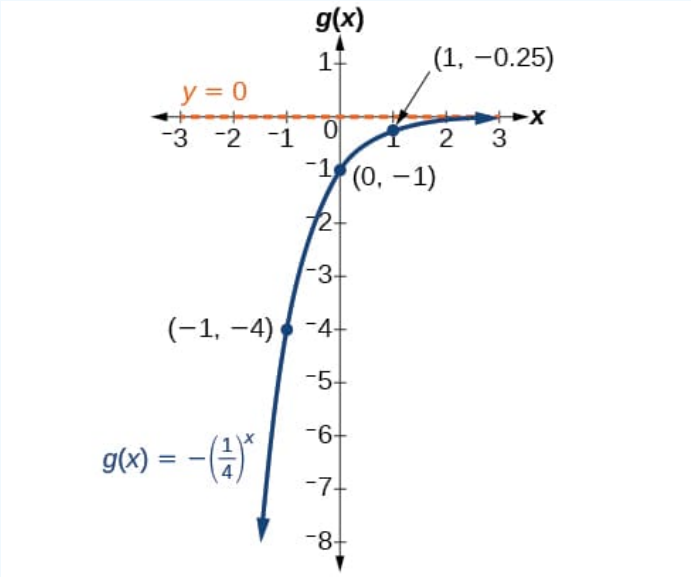



Figure 4.2.11

The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

## Exercise 4.2.5

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the *y*-axis. State its domain, range, and asymptote.

*Answer available at the end of this section.*

## Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in **Table** 4.2.6 to arrive at the general equation for translating exponential functions.

1, and notes the following changes: the reflected function is decreasing as x moves from 0 to infinity, the asymptote remains x=0, the x-intercept remains (1, 0), the key point changes to (b^(-1), 1), the domain remains (0, infinity), and the range remains (-infinity, infinity). The second column shows the left shift of the equation g(x)=log_b(x) when b>1, and notes the following changes: the reflected function is decreasing as x moves from 0 to infinity, the asymptote remains x=0, the x-intercept changes to (-1, 0), the key point changes to (-b, 1), the domain changes to (-infinity, 0), and the range remains (-infinity, infinity).">

*Table 4.2.6: Translations of the Parent Function* $f(x) = b^x$

| Translation | Form |
|---|---|
| Shift<br>• Horizontally $c$ units to the left<br>• Vertically $d$ units up | $f(x) = b^{x+c} + d$ |
| Stretch and Compress<br>• Stretch if $\|a\| > 1$<br>• Compression if $0 < \|a\| < 1$ | $f(x) = ab^x$ |
| Reflect about the x-axis | $f(x) = -b^x$ |
| Reflect about the y-axis | $f(x) = b^{-x} = (\frac{1}{b})^x$ |
| General equation for all translations | $f(x) = ab^{x+c} + d$ |

## TRANSLATIONS OF EXPONENTIAL FUNCTIONS

A translation of an exponential function has the form

$$f(x) = ab^{x+c} + d$$

Where the parent function, $y = b^x, b > 1$,is

- shifted horizontally $c$ units to the left.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- shifted vertically $d$ units.
- reflected about the *x*-axis when $a < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

## Example 4.2.6: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is vertically stretched by a factor of $2$ , reflected across the *y*-axis, and then shifted up $4$ units.

**Solution**

We want to find an equation of the general form f(x)=abx+c+d. f(x)=abx+c+d. We use the description provided to finda, a, b,b, c,c, and d. d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of $2$, so $a = 2$.
- The function is reflected about the *y*-axis. We replace $x$ with $-x$ to get: $e^{-x}$.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$f(x) = ab^{x+c} + d$$

$$= 2e^{-x+0} + 4$$

$$= 2e^{-x} + 4$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

### Exercise 4.2.6

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the *x*-axis and then shifted down $2$ units.

*Answer available at the end of this section.*

### Media

Access this online resource for additional instruction and practice with graphing exponential functions.

- Graph Exponential Functions

## Key Equations

| | |
|---|---|
| General Form for the Translation of the Parent Function $f(x) = b^x$ | $f(x) = ab^{x+c} + d$ |

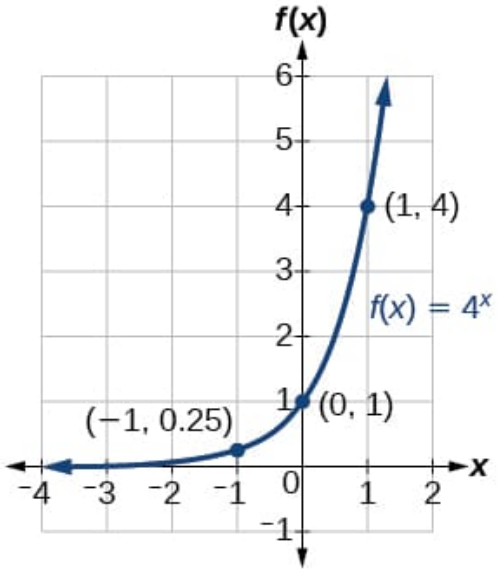
## Key Concepts

- The graph of the function $f(x) = b^x$ has a *y*-intercept at $(0, 1)$,domain $(-\infty, \infty)$,range $(0, \infty)$, and horizontal asymptote $y = 0$. See Example.
- If $b > 1$,the function is increasing. The left tail of the graph will approach the asymptote $y = 0$, and the right tail will increase without bound.
- If $0 < b < 1$, the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- The equation $f(x) = b^x + d$ represents a vertical shift of the parent function $f(x) = b^x$.
- The equation $f(x) = b^{x+c}$ represents a horizontal shift of the parent function $f(x) = b^x$. See Example.
- Approximate solutions of the equation $f(x) = b^{x+c} + d$ can be found using a graphing calculator. See Example.
- The equation $f(x) = ab^x$, where $a > 0$, represents a vertical stretch if $|a| > 1$ or compression if $0 < |a| < 1$ of the parent function $f(x) = b^x$. See Example.
- When the parent function $f(x) = b^x$ is multiplied by $-1$,the result, $f(x) = -b^x$, is a reflection about the *x*-axis. When the input is multiplied by $-1$,the result, $f(x) = b^{-x}$, is a reflection about the *y*-axis. See Example.
- All translations of the exponential function can be summarized by the general equation $f(x) = ab^{x+c} + d$. See Table.
- Using the general equation $f(x) = ab^{x+c} + d$, we can write the equation of a function given its description. See Example.

### Answers

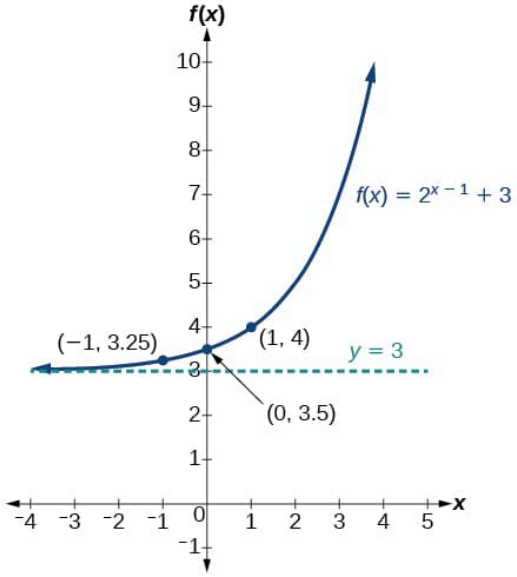
**Exercise** 4.2.1

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.




**Exercise** 4.2.2

The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.

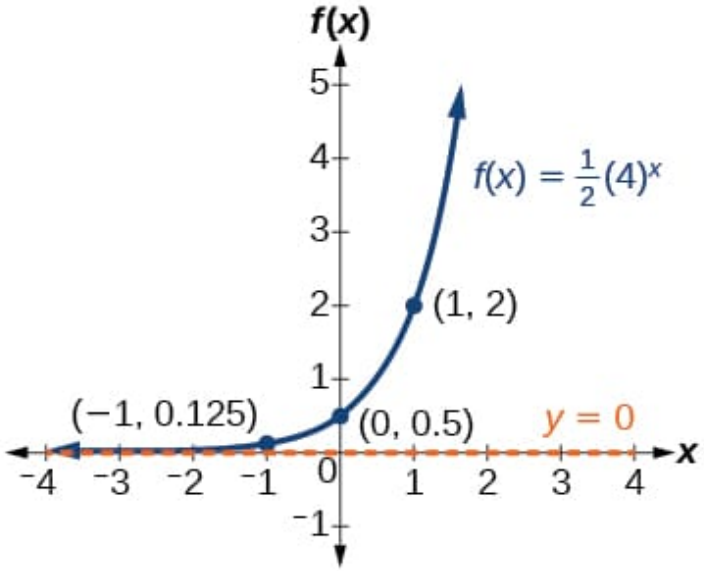



**Exercise** 4.2.3

$x \approx -1.608$

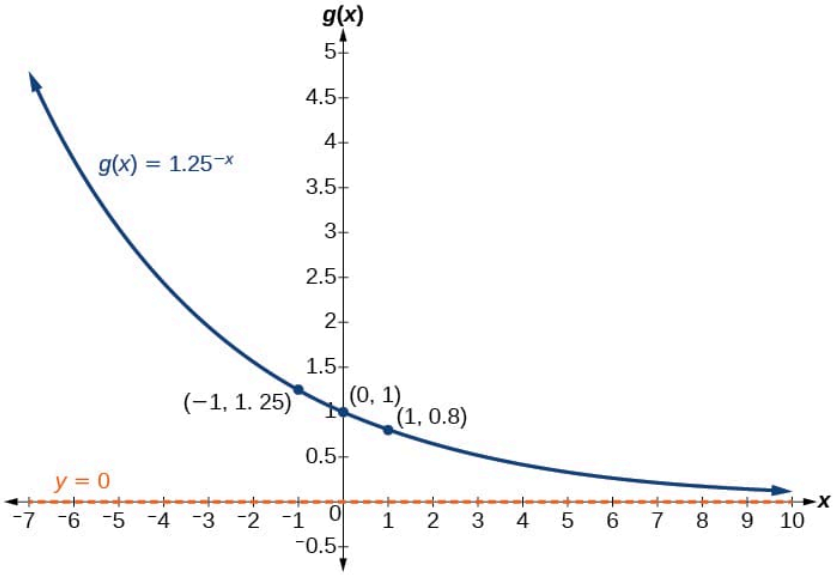
**Exercise** 4.2.4

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.




**Exercise** $4.2.5$

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.




**Exercise** $4.2.6$

$f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

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# 4.3: Logarithmic Functions

**Learning Objectives**

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure $4.3.1$. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.



Figure $4.3.1$: Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce).

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude $8$ is not twice as great as an earthquake of magnitude $4$. It is

$$10^{8-4} = 10^4 = 10,000$$

times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

## Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where $x$ represents the difference in magnitudes on the Richter Scale. How would we solve for $x$?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to

solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that $x$ must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure $4.3.1$, to better estimate the solution.

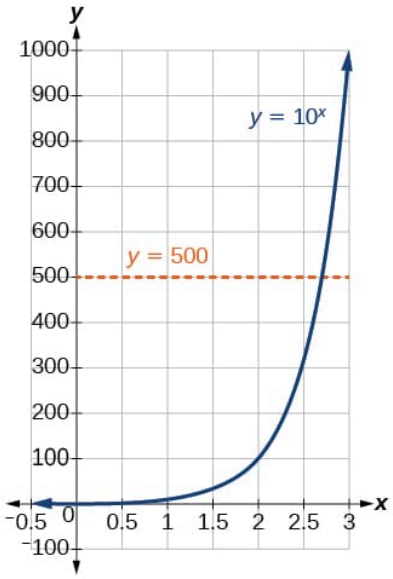



Figure $4.3.2$

Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure $4.3.2$ passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange $x$ and $y$ and solve for $y$ to find the inverse function. To represent $y$ as a function of $x$, we use a logarithmic function of the form $y = \log_b(x)$. The base $b$ **logarithm** of a number is the exponent by which we must raise $b$ to get that number.

We read a logarithmic expression as, "The logarithm with base $b$ of $x$ is equal to $y$," or, simplified, "log base $b$ of $x$ is $y$." We can also say, "$b$ raised to the power of $y$ is $x$," because logs are exponents. For example, the base $2$ logarithm of $32$ is $5$, because $5$ is the exponent we must apply to $2$ to get $32$. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base $2$ of $32$ is $5$."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1 \tag{4.3.1}$$

Note that the base $b$ is always positive.




Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the $x$.

We can illustrate the notation of logarithms as follows:

CNX_Precalc_Figure_04_03_003_gsa7.jpg

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log^b(x)$ and $y = b^x$ are inverse functions.

## DEFINITION OF THE LOGARITHMIC FUNCTION

A logarithm base $b$ of a positive number $x$ satisfies the following definition.

For $x > 0$, $b > 0$, $b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x \tag{4.3.2}$$

where,

- we read $\log_b(x)$ as, "the logarithm with base $b$ of $x$" or the "log base $b$ of $x$."
- the logarithm $y$ is the exponent to which $b$ must be raised to get $x$.

Also, since the logarithmic and exponential functions switch the $x$ and $y$ values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base $b$ is $(0, \infty)$.
- the range of the logarithm function with base $b$ is $(-\infty, \infty)$.

## Q&A: Can we take the logarithm of a negative number?

*No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.*

## How to: Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form

1. Examine the equation $y = \log_b x$ and identify $b$, $y$,and $x$.
2. Rewrite $\log_b x = y$ as $b^y = x$.

## Example 4.3.1: Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$
b. $\log_3(9) = 2$

**Solution**

First, identify the values of $b$, $y$,and $x$. Then, write the equation in the form $b^y = x$.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$

Here, $b = 6$, $y = \frac{1}{2}$,and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to

$$6^{\frac{1}{2}} = \sqrt{6}$$

b. $\log_3(9) = 2$

Here, $b = 3$, $y = 2$,and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to

$$3^2 = 9$$

### Exercise 4.3.1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

*Answer a available at the end of this section.*

*Answer b available at the end of this section.*

## Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base $b$,exponent $x$,and output $y$. Then we write $x = \log_b(y)$.

### Example 4.3.2: Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \dfrac{1}{10,000}$

**Solution**

First, identify the values of $b$, $y$,and $x$. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$

Here, $b = 2$, $x = 3$,and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$

Here, $b = 5$, $x = 2$,and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \dfrac{1}{10,000}$

Here, $b = 10$, $x = -4$,and $y = \dfrac{1}{10,000}$. Therefore, the equation $10^{-4} = \dfrac{1}{10,000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10,000}\right) = -4$.

### Exercise 4.3.2

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \dfrac{1}{2}$

*Answer a available at the end of this section.*

*Answer b available at the end of this section.*

*Answer c available at the end of this section.*

## Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must $2$ be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must $7$ be raised in order to get $49$?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, "To what exponent must $3$ be raised in order to get $27$?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$? " We know $2^2 = 4$ and $3^2 = 9$, so

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Therefore,

$$\log_{\frac{2}{3}} \left(\frac{4}{9}\right) = 2.$$

### How to: Given a logarithm of the form $y = \log_b(x)$,evaluate it mentally

1. Rewrite the argument $x$ as a power of $b$: $b^y = x$.
2. Use previous knowledge of powers of $b$ identify $y$ by asking, "To what exponent should $b$ be raised in order to get $x$?"

### Example 4.3.3: Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

**Solution**

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must $4$ be raised in order to get $64$?"

We know

$$4^3 = 64$$

Therefore,

$$\log_4(64) = 3$$

### Exercise 4.3.3

Solve $y = \log_{121}(11)$ without using a calculator.

*Answer available at the end of this section.*

### Example 4.3.4: Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

**Solution**

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must $3$ be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$,but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

### Exercise 4.3.4

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

*Answer available at the end of this section.*

## Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is $10$. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base $-10$ logarithm a *common logarithm*. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

### DEFINITION OF THE COMMON LOGARITHM

A common logarithm is a logarithm with base $10$. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number $x$ satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x \tag{4.3.3}$$

We read $\log(x)$ as, "the logarithm with base $10$ of $x$ " or "log base $10$ of $x$."

The logarithm $y$ is the exponent to which $10$ must be raised to get $x$.

### How to: Given a common logarithm of the form $y = \log(x)$, evaluate it mentally

1. Rewrite the argument $x$ as a power of $10$: $10^y = x$.
2. Use previous knowledge of powers of $10$ to identify $y$ by asking, "To what exponent must $10$ be raised in order to get $x$?"

### Example 4.3.5: Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

**Solution**

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must $10$ be raised in order to get $1000$?" We know

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

### Exercise 4.3.5

Evaluate $y = \log(1,000,000)$.

*Answer available at the end of this section.*

### How to: Given a common logarithm with the form $y = \log(x)$,evaluate it using a calculator

1. Press **[LOG]**.
2. Enter the value given for $x$,followed by **[ ) ]**.
3. Press **[ENTER]**.

### Example 4.3.6: Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

**Solution**

- Press **[LOG]**.
- Enter 321, followed by **[ ) ]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

**Analysis**

Note that $10^2 = 100$ and that $10^3 = 1000$. Since $321$ is between $100$ and $1000$, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

$$100 < 321 < 1000$$

$$2 < 2.5065 < 3$$

### Exercise 4.3.6

Evaluate $y = \log(123)$ to four decimal places using a calculator.

*Answer available at the end of this section.*

### Example 4.3.7: Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was $500$ times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where $x$ is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

**Solution**

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$

$$\log(500) = x \text{ Use the definition of the common log.}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter $500$,followed by **[ ) ]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about $2.699$.

### Exercise 4.3.7

The amount of energy released from one earthquake was $8,500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where $x$ is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

*Answer available at the end of this section.*

## Using Natural Logarithms

The most frequently used base for logarithms is $e$. Base $e$ logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base $e$ logarithm, $\log_e(x)$, has its own notation,$\ln(x)$. Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of $1$ is always $0$ in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of $e$ using the inverse property of logarithms.

### DEFINITION OF THE NATURAL LOGARITHM

A natural logarithm is a logarithm with base $e$. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number $x$ satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base $e$ of $x$" or "the natural logarithm of $x$."

The logarithm $y$ is the exponent to which $e$ must be raised to get $x$.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all $x$ and $e^{\ln(x)} = x$ for $x > 0$.

### How to: Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator

1. Press **[LN]**.
2. Enter the value given for $x$, followed by **[ ) ]**.
3. Press **[ENTER]**.

### Example 4.3.8: Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

**Solution**

- Press **[LN]**.
- Enter $500$,followed by **[ ) ]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

### Exercise 4.3.8

Evaluate $\ln(-500)$.

*Answer available at the end of this section.*

### Media

Access this online resource for additional instruction and practice with logarithms.

- Introduction to Logarithms

## Key Equations

| | |
|---|---|
| Definition of the logarithmic function | For $x > 0$, $b > 0$, $b \neq 1$, $y = \log_b(x)$ if and only if $b^y = x$. |
| Definition of the common logarithm | For $x > 0$, $y = \log(x)$ if and only if $10^y = x$. |
| Definition of the natural logarithm | For $x > 0$, $y = \ln(x)$ if and only if $e^y = x$. |

## Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See Example $4.3.1$.
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See Example $4.3.2$.
- Logarithmic functions with base $b$ can be evaluated mentally using previous knowledge of powers of $b$. See Example $4.3.3$ and Example $4.3.4$.
- Common logarithms can be evaluated mentally using previous knowledge of powers of $10$. See Example $4.3.5$.
- When common logarithms cannot be evaluated mentally, a calculator can be used. See Example $4.3.6$.
- Real-world exponential problems with base $10$ can be rewritten as a common logarithm and then evaluated using a calculator. See Example $4.3.7$.

- Natural logarithms can be evaluated using a calculator Example $4.3.8$.

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## Answers

**Exercise** $4.3.1$

$\log_{10}(1,000,000) = 6$ is equivalent to $10^6 = 1,000,000$

**Exercise** $4.3.1$

$\log_5(25) = 2$ is equivalent to $5^2 = 25$

**Exercise** $4.3.2$

$3^2 = 9$ is equivalent to $\log_3(9) = 2$

**Exercise** $4.3.2$

$5^3 = 125$ is equivalent to $\log_5(125) = 3$

**Exercise** $4.3.2$

$2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$

**Exercise** $4.3.3$

$\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{\frac{1}{2}} = 11$)

**Exercise** $4.3.4$

$\log_2\left(\frac{1}{32}\right) = -5$

**Exercise** $4.3.5$

$\log(1,000,000) = 6$

**Exercise** $4.3.6$

$\log(123) \approx 2.0899$

**Exercise** $4.3.7$

The difference in magnitudes was about $3.929$.

**Exercise** $4.3.8$

It is not possible to take the logarithm of a negative number in the set of real numbers.

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# 4.4: Graphs of Logarithmic Functions

**Learning Objectives**

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

In the Section on Graphs of Exponential Functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year $t$ can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure $4.4.1$ shows this point on the logarithmic graph.

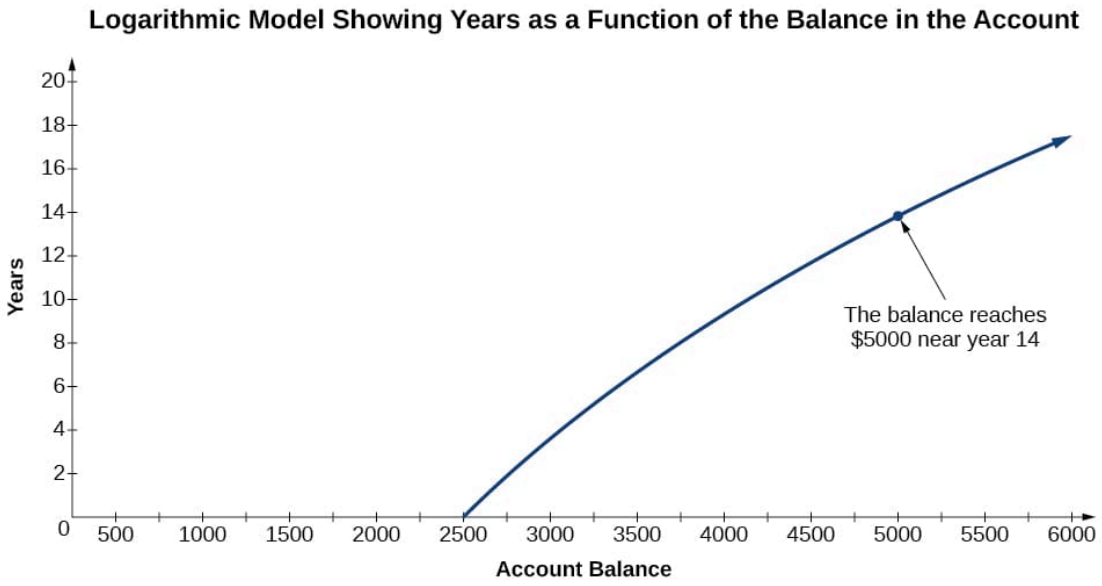



Figure $4.4.1$

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

## Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number $x$ and constant $b > 0$, $b \neq 1$, where

- The domain of $y$ is $(-\infty, \infty)$.
- The range of $y$ is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as

inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of $x$ such that the argument, in this case $2x - 3$ ,is greater than zero. To find the domain, we set up an inequality and solve for $x$:

$$\begin{aligned} 2x - 3 &> 0 && \text{Show the argument greater than zero} \\ 2x &> 3 && \text{Add 3} \\ x &> 1.5 && \text{Divide by 2} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

### Given a logarithmic function, identify the domain

1. Set up an inequality showing the argument greater than zero.
2. Solve for $x$.
3. Write the domain in interval notation.

### Example 4.4.1: Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

**Solution**

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$\begin{aligned} x + 3 &> 0 && \text{The input must be positive} \\ x &> -3 && \text{Subtract 3} \end{aligned}$$

The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

### Exercise 4.4.1

What is the domain of $f(x) = \log_5(x - 2) + 1$?

*Answer available at the end of this section.*

### Example 4.4.2: Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x) = \log(5 - 2x)$?

**Solution**

The logarithmic function is defined only when the input is positive, so this function is defined when $5–2x > 0$. Solving this inequality,

$$\begin{aligned} 5-2x &> 0 && \text{The input must be positive} \\ -2x &> -5 && \text{Subtract 5} \\ x &< \frac{5}{2} && \text{Divide by -2 and switch the inequality} \end{aligned}$$

The domain of $f(x) = \log(5-2x)$ is $\left(-\infty, \frac{5}{2}\right)$.

**Exercise 4.4.2**

What is the domain of $f(x) = \log(x-5)+2$?

*Answer available at the end of this section.*

## Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table 4.4.1.

*Table* 4.4.1

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $2^x = y$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ | $8$ |
| $\log_2(y) = x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |

Using the inputs and outputs from Table 4.4.1, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table 4.4.2.

*Table* 4.4.2

| $f(x) = 2^x$ | $\left(-3, \frac{1}{8}\right)$ | $\left(-2, \frac{1}{4}\right)$ | $\left(-1, \frac{1}{2}\right)$ | $(0,1)$ | $(1,2)$ | $(2,4)$ | $(3,8)$ |
|---|---|---|---|---|---|---|---|
| $g(x) = \log_2(x)$ | $\left(\frac{1}{8}, -3\right)$ | $\left(\frac{1}{4}, -2\right)$ | $\left(\frac{1}{2}, -1\right)$ | $(1,0)$ | $(2,1)$ | $(4,2)$ | $(8,3)$ |

As we'd expect, the $x$- and $y$-coordinates are reversed for the inverse functions. Figure 4.4.2 shows the graph of $f$ and $g$.

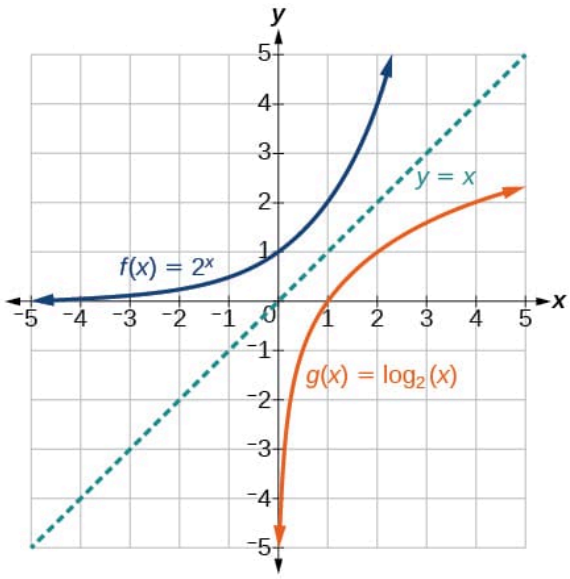



Figure $4.4.2$: Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a $y$-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an $x$- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

## CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION, $F(X) = LOG_B(X)$

For any real number $x$ and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- $x$-intercept: $(1, 0)$ and key point $(b, 1)$
- $y$-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

See Figure $4.4.3$.

Two graphs of the function f(x)=log_b(x) with points (1,0) and (b, 1). The first graph shows the line when b
1, and the second graph shows the line when 0
**<1.">**

**Figure**
$4.4.3$

Figure $4.4.4$ shows how changing the base $b$ in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (Note: recall that the function $\ln(x)$ has base $e \approx 2.718$.)

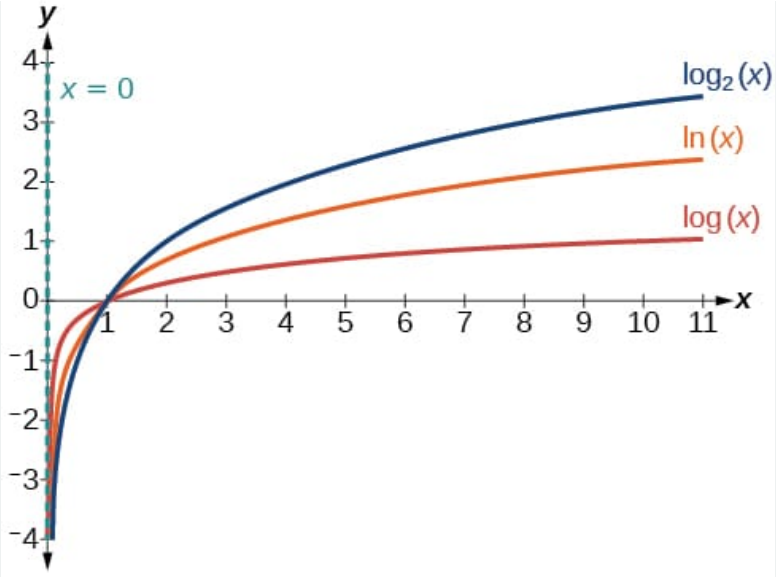



Figure 4.4.4: The graphs of three logarithmic functions with different bases, all greater than 1.

## Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$,the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

## Example 4.4.3: Graphing a Logarithmic Function with the Form $f(x) = log_b(x)$

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

**Solution**

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The $x$-intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure 4.4.5 ).

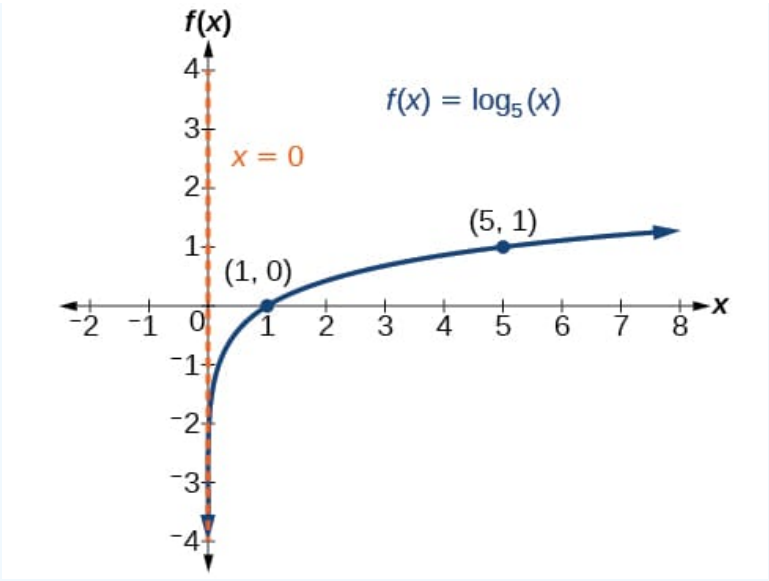



Figure 4.4.5

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

**Exercise 4.4.3**

Graph $f(x) = \log_{\frac{1}{5}}(x)$. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

### Graphing a Horizontal Shift of $f(x) = log_b(x)$

When a constant $c$ is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift $c$ units in the *opposite* direction of the sign on $c$. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 4.4.7.

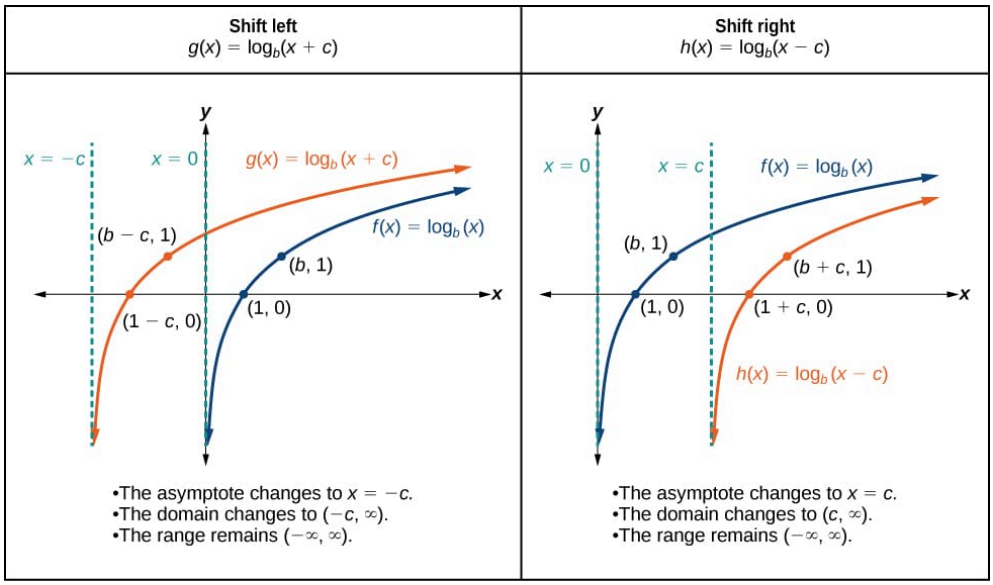



Figure 4.4.7

## HORIZONTAL SHIFTS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant $c$,the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left $c$ units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right $c$ units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

## Given a logarithmic function with the form $f(x) = \log_b(x + c)$, graph the translation.

1. Identify the horizontal shift:
   - If $c > 0$,shift the graph of $f(x) = \log_b(x)$ left $c$ units.
   - If $c < 0$,shift the graph of $f(x) = \log_b(x)$ right $c$ units.
2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting $c$ from the $x$ coordinate.
4. Label the three points.
5. The Domain is $(-c, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

## Example 4.4.4: Graphing a Horizontal Shift of the Parent Function $y = log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x - 2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

**Solution**

Since the function is $f(x) = \log_3(x - 2)$, we notice $x + (-2) = x$–2.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$,and $(3, 1)$.

The new coordinates are found by adding $2$ to the $x$ coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$, $(3, 0)$,and $(5, 1)$.

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

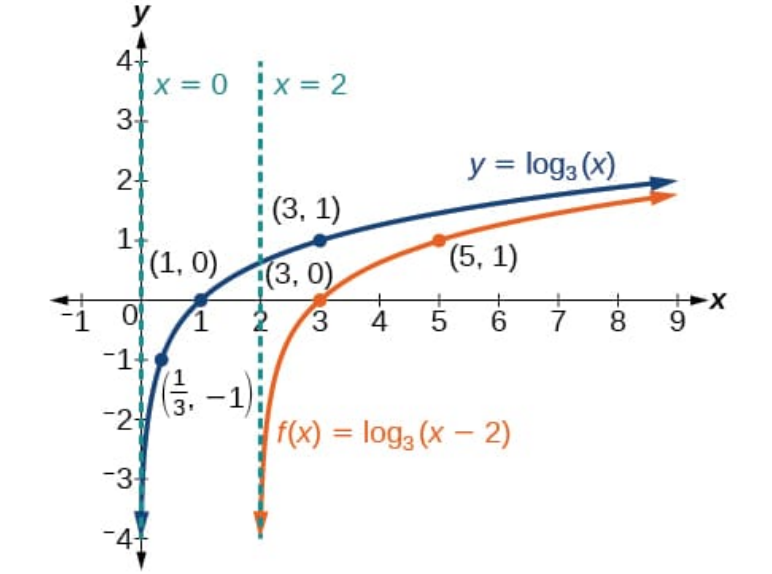



Figure $4.4.8$

**Exercise $4.4.4$**

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing a Vertical Shift of $y = log_b(x)$

When a constant $d$ is added to the parent function $f(x) = \log_b(x)$,the result is a vertical shift $d$ units in the direction of the sign on $d$. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$.See Figure $4.4.10$.

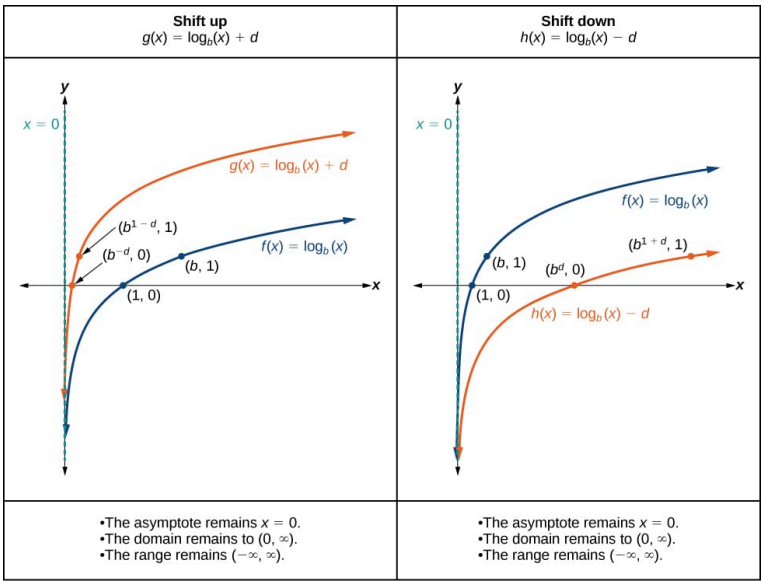



Figure 4.4.10

## VERTICAL SHIFTS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant $d$,the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up $d$ units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down $d$ units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

## Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up $d$ units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$ down $d$ units.
2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding $d$ to the $y$ coordinate.
4. Label the three points.
5. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

## Example 4.4.5: Graphing a Vertical Shift of the Parent Function $y = log_b(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Solution**

Since the function is $f(x) = \log_3(x) - 2$,we will notice $d = -2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$,and $(3, 1)$.

The new coordinates are found by subtracting $2$ from the *y* coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

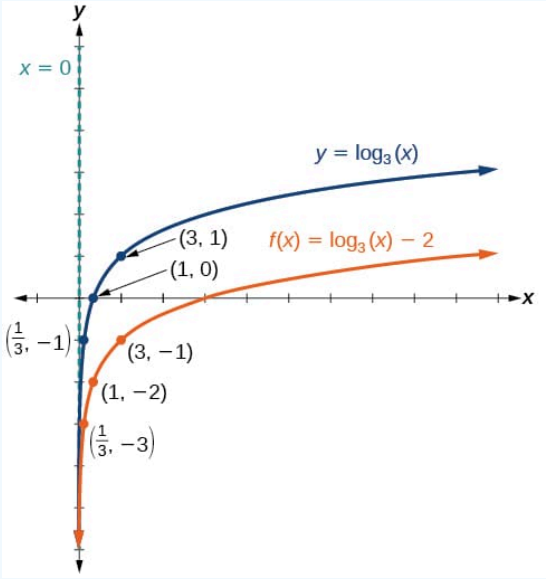



Figure $4.4.11$

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

**Exercise $4.4.5$**

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing Stretches and Compressions of $y = log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$.See Figure $4.4.13$.

Graph of two functions. The parent function is f(x)=log_b(x), with an asymptote at x=0 and g(x)=alog_b(x) when a 1 is the translation function with an asymptote at x=0. The graph note the intersection of the two lines at (1, 0). This shows the translation of a vertical stretch." src="/@api/deki/files/12175/fig_6.5.13.jpg">

Figure $4.4.13$

**VERTICAL STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $Y = LOG_B(X)$**

For any constant $a > 1$,the function $f(x) = a\log_b(x)$

- stretches the parent function $y = \log_b(x)$ vertically by a factor of $a$ if $a > 1$.
- compresses the parent function $y = \log_b(x)$ vertically by a factor of $a$ if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the $x$-intercept $(1, 0)$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

### Given a logarithmic function with the form $f(x) = a\log_b(x)$, $a > 0$,graph the translation.

1. Identify the vertical stretch or compressions:
   - If $|a| > 1$, the graph of $f(x) = \log_b(x)$ is stretched by a factor of $a$ units.
   - If $|a| < 1$, the graph of $f(x) = \log_b(x)$ is compressed by a factor of $a$ units.
2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying they y coordinates by $a$.
4. Label the three points.
5. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

### Example $4.4.6$: Graphing a Stretch or Compression of the Parent Function $y = log_b(x)$

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Solution**

Since the function is $f(x) = 2\log_4(x)$,we will notice $a = 2$.

This means we will stretch the function $f(x) = \log_4(x)$ by a factor of $2$.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{4}, -1\right)$, $(1, 0)$, and $(4, 1)$.

The new coordinates are found by multiplying the $y$ coordinates by $2$.

Label the points $\left(\frac{1}{4}, -2\right)$, $(1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$. See Figure $4.4.14$.

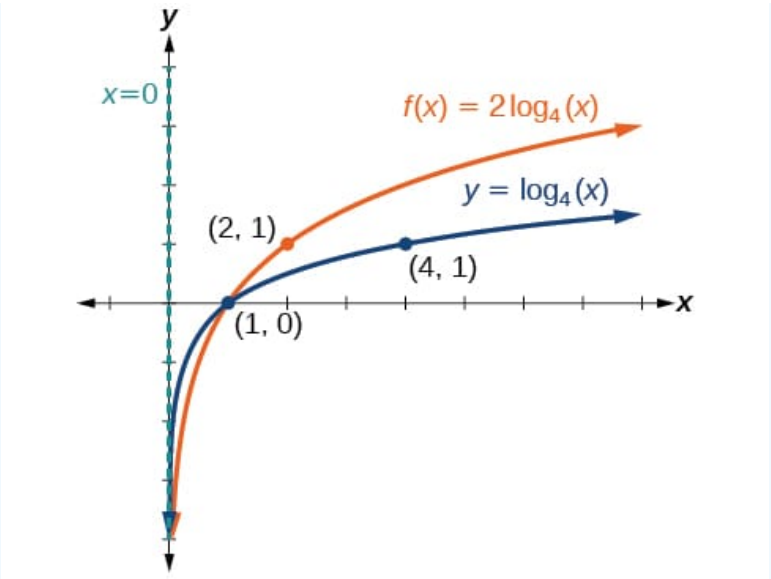



Figure 4.4.14

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

## Exercise 4.4.6

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Example 4.4.7: Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x + 2)$. State the domain, range, and asymptote.

**Solution**

Remember: what happens inside parentheses happens first. First, we move the graph left $2$ units, then stretch the function vertically by a factor of $5$, as in Figure 4.4.16. The vertical asymptote will be shifted to $x = -2$. The $x$-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the $x$-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

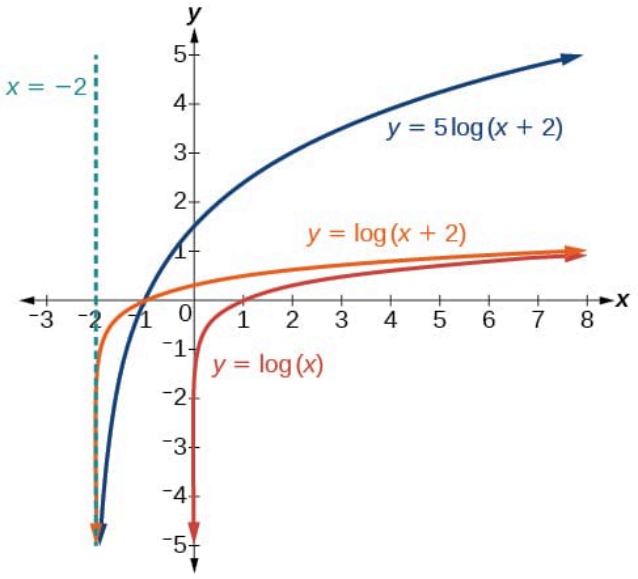



Figure 4.4.16

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = -2$.

**Exercise 4.4.7**

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing Reflections of $f(x) = log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by $-1$,the result is a reflection about the $x$-axis. When the *input* is multiplied by $-1$,the result is a reflection about the $y$-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the $x$-axis, $g(x) = -\log_b(x)$ and the reflection about the $y$-axis, $h(x) = \log_b(-x)$.

Graph of two functions. The parent function is f(x)=log_b(x), with an asymptote at x=0 and g(x)=-log_b(x) when b 1 is the translation function with an asymptote at x=0. The graph note the intersection of the two lines at (1, 0). This shows the translation of a reflection about the x-axis." src="/@api/deki/files/12180/fig_6.5.18.jpg">

Figure 4.4.18

**REFLECTIONS OF THE PARENT FUNCTION $Y = LOG_B(X)$**

The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the $x$-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the $y$-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

## Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

*Table* 4.4.3

| **If** $f(x) = -\log_b(x)$ | **If** $f(x) = \log_b(-x)$ |
|---|---|
| 1. Draw the vertical asymptote, $x = 0$. | 1. Draw the vertical asymptote, $x = 0$. |
| 2. Plot the *x*-intercept, $(1, 0)$. | 2. Plot the *x*-intercept, $(1, 0)$. |
| 3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *x*-axis. | 3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *y*-axis. |
| 4. Draw a smooth curve through the points. | 4. Draw a smooth curve through the points. |
| 5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$. | 5. State the domain, $(-\infty, 0)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$. |

## Example 4.4.8: Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Solution**

Before graphing $f(x) = log(-x)$, $f(x) = log(-x)$,identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by $-1$, $f(x)$ is a reflection of the parent graph about the $y$-axis. Thus, $f(x) = \log(-x)$ will be decreasing as $x$ moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The $x$-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

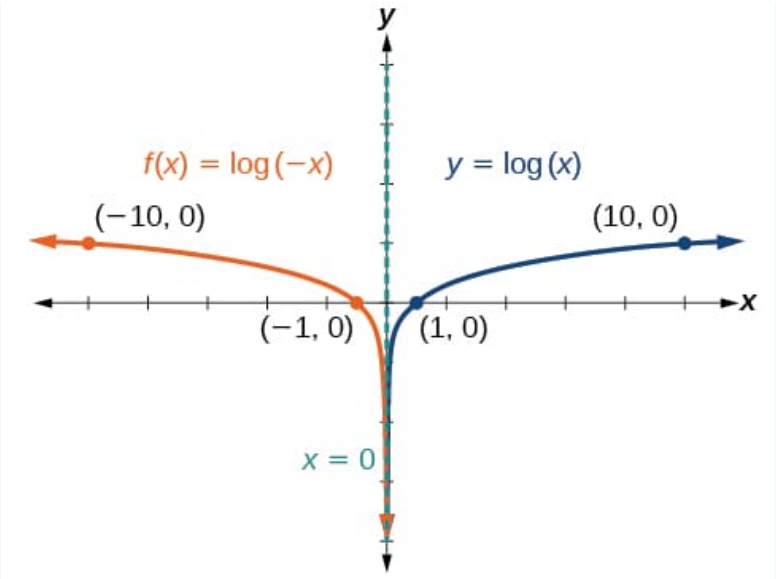



Figure 4.4.19

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

## Exercise 4.4.8

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Given a logarithmic equation, use a graphing calculator to approximate solutions.

1. Press **[Y=]**. Enter the given logarithm equation or equations as **$Y_1$=** and, if needed, **$Y_2$=**.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of $x$, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of $x$,for the point(s) of intersection.

## Example 4.4.9: Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x) + 1 = -2\ln(x - 1)$ graphically. Round to the nearest thousandth.

**Solution**

Press **[Y=]** and enter $4\ln(x) + 1$ next to **$Y_1$=**. Then enter $-2\ln(x - 1)$ next to **$Y_2$=**. For a window, use the values $0$ to $5$ for $x\backslash 0 \; and \backslash (-10$ to $10$ for $y$. Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x = 1$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as $1.3385297$. (Your answer may be different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x \approx 1.339$.

**Exercise 4.4.9**

Solve $5\log(x+2) = 4 - \log(x)$ graphically. Round to the nearest thousandth.

*Answer available at the end of this section.*

## Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in Table 4.4.4 to arrive at the general equation for translating exponential functions.

1 and a compression is 0<|a|<1. Note its form is f(x)=alog_b(x). The third translation is a reflection about the x-axis with the form f(x) = -log_b(x). The fourth translation is a reflection about the y-axis with the form f(x)=log_b(-x). The general equation for all translations is f(x)=alog_b(x+c)+d.">

***Table*** 4.4.4

| Translations of the Parent Function $y = \log_b(x)$ | |
|---|---|
| **Translation** | **Form** |
| Shift<br>Horizontally $c$ units to the left<br>Vertically $d$ units up | $y = \log_b(x + c) + d$ |
| Stretch and Compress<br>Stretch if $\lvert a\rvert > 1$<br>Compression if $\lvert a\rvert < 1$ | $y = a\log_b(x)$ |
| Reflect about the x-axis | $y = -\log_b(x)$ |
| Reflect about the y-axis | $y = \log_b(-x)$ |
| General equation for all translations | $y = a\log_b(x + c) + d$ |

### TRANSLATIONS OF LOGARITHMIC FUNCTIONS

All translations of the parent logarithmic function, $y = \log_b(x)$, have the form

$$f(x) = a\log_b(x + c) + d$$

where the parent function, $y = \log_b(x)$, $b > 1$,is

- shifted vertically up $d$ units.
- shifted horizontally to the left $c$ units.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- reflected about the *x*-axis when $a < 0$.

For $f(x) = \log(-x)$, the graph of the parent function is reflected about the *y*-axis.

## Example 4.4.10: Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x+4)+5$?

**Solution**

The vertical asymptote is at $x = -4$.

**Analysis**

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve $4$ units to the left shifts the vertical asymptote to$x = -4$.

## Exercise 4.4.10

What is the vertical asymptote of $f(x) = 3 + \ln(x-1)$?

*Answer available at the end of this section.*

## Example 4.4.11: Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in Figure 4.4.21.

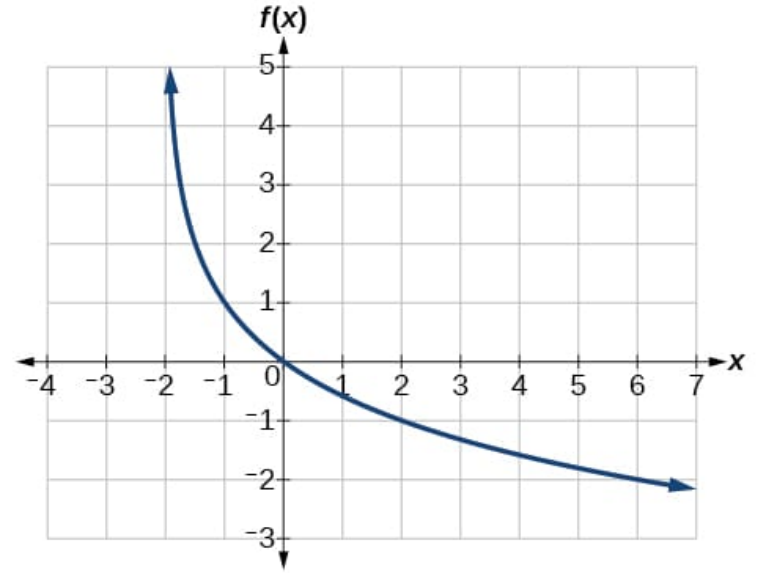



Figure 4.4.21

**Solution**

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$$f(x) = -a\log(x+2)+k$$

It appears the graph passes through the points $(-1,1)$ and $(2,-1)$. Substituting $(-1,1)$,

$$\begin{aligned} 1 &= -a\log(-1+2)+k \quad \text{Substitute}(-1,1) \\ 1 &= -a\log(1)+k \quad \text{Arithmetic} \\ 1 &= k\log(1) \\ &= 0 \end{aligned}$$

Next, substituting in $(2,-1)$,

$$\begin{aligned} -1 &= -a\log(2+2)+1 \quad \text{Substitute}(2,-1) \\ -2 &= -a\log(4) \quad \text{Arithmetic} \\ a &= \frac{2}{\log(4)} \quad \text{Solve for a} \end{aligned}$$

This gives us the equation $f(x) = -\frac{2}{\log(4)}\log(x+2)+1$.

**Analysis**

We can verify this answer by comparing the function values in Table 4.4.5 with the points on the graph in Figure 4.4.21.

*Table* 4.4.5

| $x$ | –1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 0 | –0.58496 | –1 | –1.3219 |
| $x$ | 4 | 5 | 6 | 7 | 8 |
| $f(x)$ | –1.5850 | –1.8074 | –2 | –2.1699 | –2.3219 |

## Exercise 4.4.11

Give the equation of the natural logarithm graphed in Figure 4.4.22.

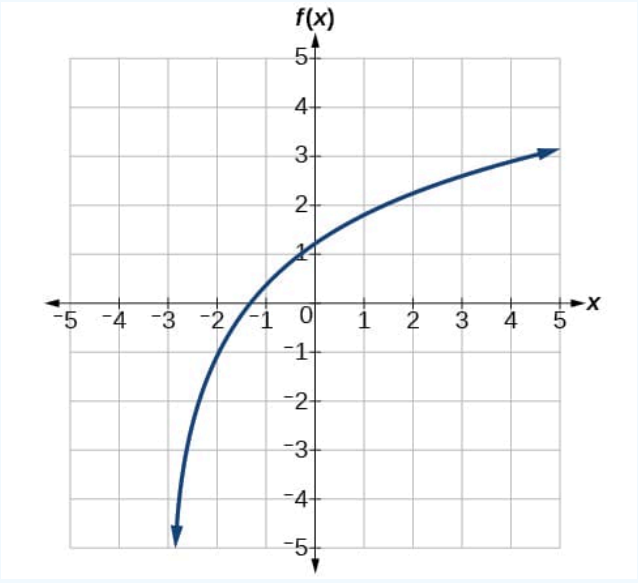



Figure 4.4.22

*Answer available at the end of this section.*

**Media: Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?**

Yes, if we know the function is a general logarithmic function. For example, look at the graph in Figure $4.4.22$. The graph approaches $x = -3$ (or thereabouts) more and more closely, so $x = -3$ is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right, $x|x > -3$. The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

**Media**

Access these online resources for additional instruction and practice with graphing logarithms.

- Graph an Exponential Function and Logarithmic Function
- Match Graphs with Exponential and Logarithmic Functions
- Find the Domain of Logarithmic Functions

## Key Equations

| | |
|---|---|
| General Form for the Translation of the Parent Logarithmic Function $f(x) = \log_b(x)$ | $f(x) = a\log_b(x + c) + d$ |

## Key Concepts

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for $x$. See Example $4.4.1$ and Example $4.4.2$
- The graph of the parent function $f(x) = \log_b(x)$ has an *x*-intercept at $(1, 0)$,domain $(0, \infty)$,range $(-\infty, \infty)$,vertical asymptote $x = 0$,and
- ◦ if $b > 1$,the function is increasing.
  ◦ if $0 < b < 1$, the function is decreasing.
  See Example $4.4.3$.
- The equation $f(x) = \log_b(x + c)$ shifts the parent function $y = \log_b(x)$ horizontally
  ◦ left $c$ units if $c > 0$.
  ◦ right $c$ units if $c < 0$.
  See Example $4.4.4$.
- The equation $f(x) = \log_b(x) + d$ shifts the parent function $y = \log_b(x)$ vertically
  ◦ up $d$ units if $d > 0$.
  ◦ down $d$ units if $d < 0$.
  See Example $4.4.5$.
- For any constant $a > 0$, the equation $f(x) = a\log_b(x)$
  ◦ stretches the parent function $y = \log_b(x)$ vertically by a factor of $a$ if $|a| > 1$.
  ◦ compresses the parent function $y = \log_b(x)$ vertically by a factor of $a$ if $|a| <$1.
  See Example $4.4.6$ and Example $4.4.7$.
- When the parent function $y = \log_b(x)$ is multiplied by $-1$, the result is a reflection about the *x*-axis. When the input is multiplied by $-1$, the result is a reflection about the *y*-axis.
  ◦ The equation $f(x) = -\log_b(x)$ represents a reflection of the parent function about the *x*-axis.
  ◦ The equation $f(x) = \log_b(-x)$ represents a reflection of the parent function about the *y*-axis.
  See Example $4.4.8$.
  ◦ A graphing calculator may be used to approximate solutions to some logarithmic equations See Example $4.4.9$.
- All translations of the logarithmic function can be summarized by the general equation $f(x) = a\log_b(x + c) + d$. See Table $4.4.4$.
- Given an equation with the general form $f(x) = a\log_b(x + c) + d$,we can identify the vertical asymptote $x = -c$ for the transformation. See Example $4.4.10$.
- Using the general equation $f(x) = a\log_b(x + c) + d$, we can write the equation of a logarithmic function given its graph. See

Example 4.4.11.

## Answers

**Exercise** 4.4.1

$(2, \infty)$

**Exercise** 4.4.2

$(5, \infty)$

**Exercise** 4.4.3

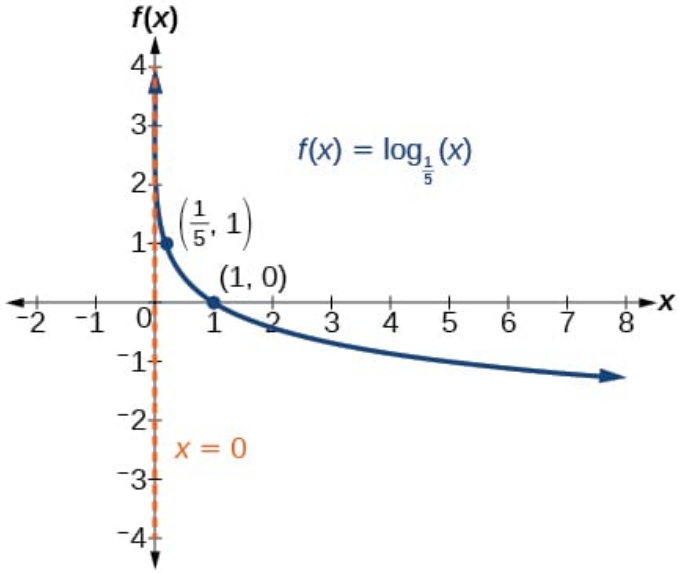



Figure 4.4.6

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

**Exercise** 4.4.4

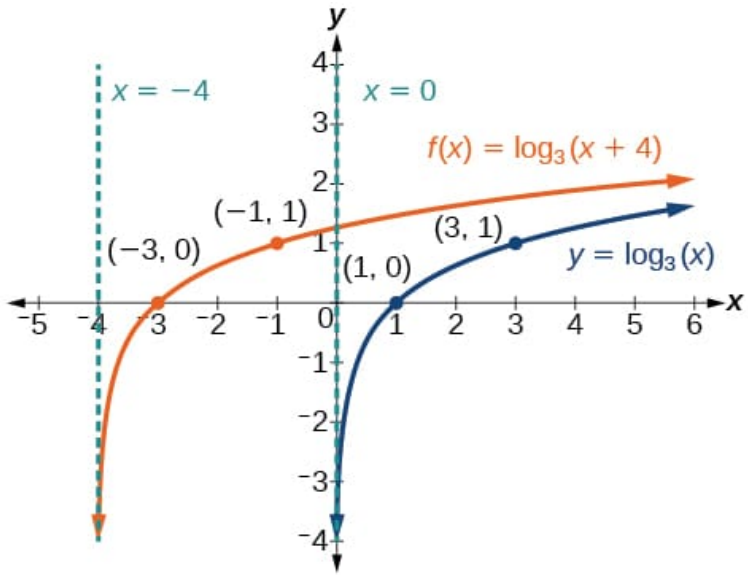



Figure 4.4.9

The domain is $(-4, \infty)$,the range $(-\infty, \infty)$,and the asymptote $x = -4$.

**Exercise** 4.4.5

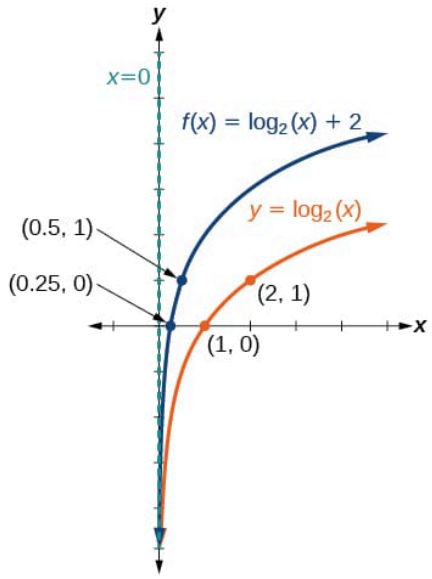



Figure 4.4.12

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

**Exercise** 4.4.6

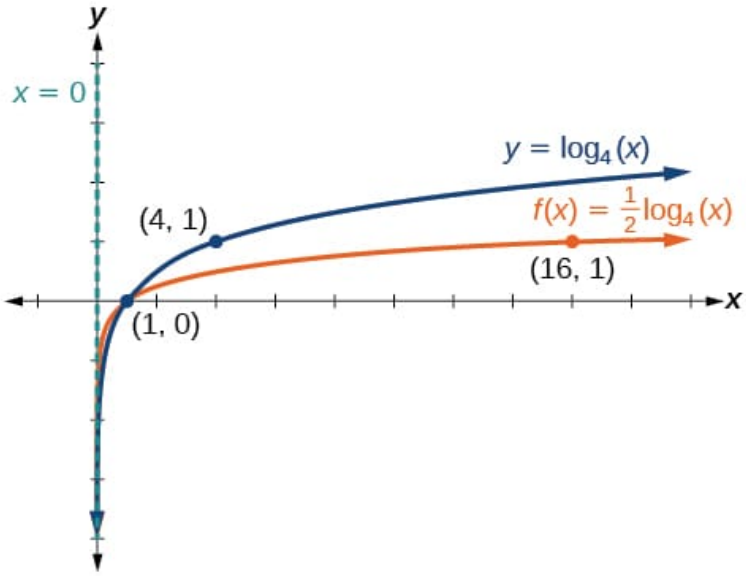



Figure 4.4.15

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

**Exercise** 4.4.7

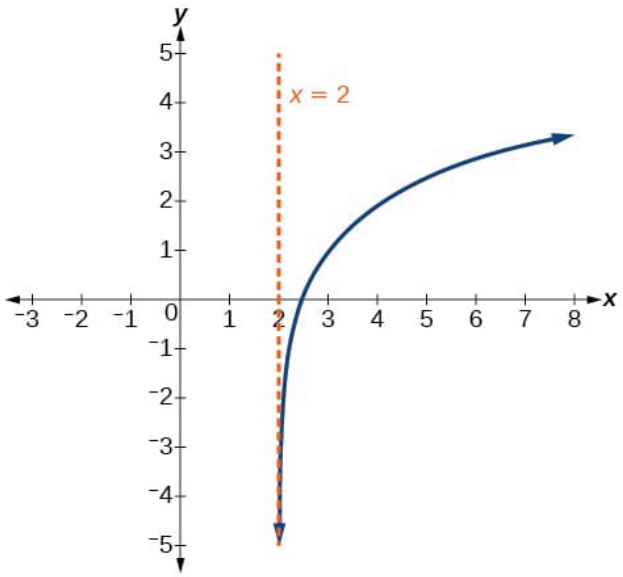



Figure 4.4.17

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

**Exercise** 4.4.8

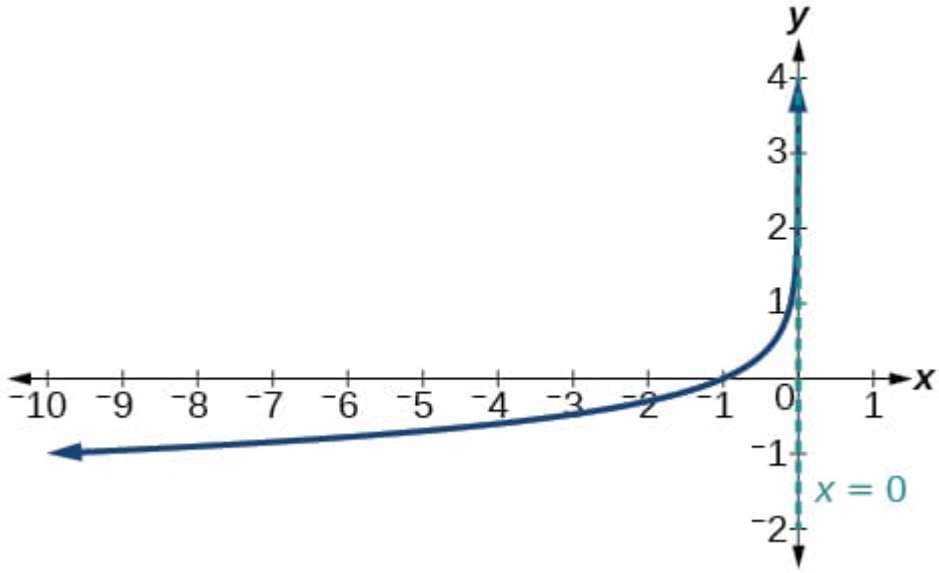



Figure 4.4.20

The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

**Exercise** 4.4.9

$x \approx 3.049$

**Exercise** 4.4.10

$x = 1$

**Exercise** 4.4.11

$f(x) = 2\ln(x + 3) - 1$

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# 4.5: Logarithmic Properties

## Learning Objectives

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

In chemistry, the pH scale is used as a measure of the acidity or alkalinity of a substance. Substances with a pH less than $7$ are considered acidic, and substances with a pH greater than $7$ are said to be alkaline. Our bodies, for instance, must maintain a pH close to $7.35$ in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: $0.8$
- Stomach acid: $2.7$
- Orange juice: $3.3$
- Pure water: $7$ at $25^{\circ}C$
- Human blood: $7.35$
- Fresh coconut: $7.8$
- Sodium hydroxide (lye): $14$

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where $[\mathrm{H}^+]$ is the concentration of hydrogen ions in the solution

$$pH = -\log([\mathrm{H}^+]) \tag{4.5.1}$$

$$= \log\left(\frac{1}{[\mathrm{H}^+]}\right) \tag{4.5.2}$$

The equivalence of Equations 4.5.1 and 4.5.2 is one of the logarithm properties we will examine in this section.



Figure 4.5.1: The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan).

# Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0$$
$$\log_b b = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b(b^x) = x$$
$$b^{\log_b x} = x,\ x > 0$$

For example, to evaluate $\log(100)$, we can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

$$\log_b M = \log_b N \text{ if and only if } M = N \tag{4.5.3}$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x + 5)$ for $x$. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for $x$:

$3x = 2x + 5$ Set the arguments equal.

$x = 5$ Subtract $2x$.

But what about the equation $\log_3(3x) + \log_3(2x + 5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number $x$ and positive real numbers $M$, $N$, and $b$, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}
\log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for M and N} \\
&= \log_b(b^{m+n}) && \text{Apply the product rule for exponents} \\
&= m + n && \text{Apply the inverse property of logs} \\
&= \log_b(M) + \log_b(N) && \text{Substitute for m and n}
\end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

### The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0 \tag{4.5.4}$$

### How to: Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

### Example 4.5.1: Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4))$.

**Solution**

We begin by factoring the argument completely, expressing $30$ as a product of primes.

$$\log_3(30x(3x+4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30x(3x+4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x+4)$$

### Exercise 4.5.1

Expand $\log_b(8k)$.

*Answer available at the end of this section.*

## Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $x^{\frac{a}{b}} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms.

### The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N \tag{4.5.5}$$

Just as with the product rule, we can use the inverse property to derive the quotient rule.

**Proof**

Given any real number $x$ and positive real numbers $M$, $N$, and b, b, where $b \neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}
\log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for M and N} \\
&= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents} \\
&= m - n && \text{Apply the inverse property of logs} \\
&= \log_b(M) - \log_b(N) && \text{Substitute for m and n}
\end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned}
\log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator} \\
&= \log\left(\frac{2x}{3}\right) && \text{Cancel the common factors}
\end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\begin{aligned}
\log\left(\frac{2x}{3}\right) &= \log(2x) - \log(3) \\
&= \log(2) + \log(x) - \log(3)
\end{aligned}$$

**How to: Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms**

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

**Example 4.5.2: Using the Quotient Rule for Logarithms**

Expand $log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

**Solution**

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2(\frac{15x(x-1)}{(3x+4)(2-x)}) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{aligned}
\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] - [\log_2(3x+4) + \log_2(2-x)] \\
&= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log_2(2-x)
\end{aligned}$$

**Analysis**

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

**Exercise 4.5.2**

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

*Answer available at the end of this section.*

## Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as $x^2$? One method is as follows:

$$\begin{aligned}\log_b(x^2) &= \log_b(x \cdot x)\\ &= \log_b x + \log_b x\\ &= 2\log_b x\end{aligned}$$

Notice that we used the **product rule for logarithms** to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \ \sqrt{3} = 3^{\frac{1}{2}} \ \frac{1}{e} = e^{-1}$$

**The Power Rule for Logarithms**

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b M \tag{4.5.6}$$

**How to: Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm**

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

**Example 4.5.3: Expanding a Logarithm with Powers**

Expand $\log_2 x^5$.

**Solution**

The argument is already written as a power, so we identify the exponent, 5, and the base, $x$, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

### Exercise 4.5.3

Expand $\ln x^2$.

*Answer available at the end of this section.*

### Example 4.5.4: Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3(25)$ using the power rule for logs.

**Solution**

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(52) = 2\log_3(5)$$

### Exercise 4.5.4

Expand $\ln\left(\frac{1}{x^2}\right)$.

*Answer available at the end of this section.*

### Example 4.5.5: Using the Power Rule in Reverse

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

**Solution**

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, $x$, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

### Exercise 4.5.5

Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

*Answer available at the end of this section.*

## Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called “laws of logs.” Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\log_b\left(\frac{6x}{y}\right) = \log_b(6x) - \log_b y$$
$$= \log_b 6 + \log_b x - \log_b y$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\log_b\left(\frac{A}{C}\right) = \log_b(AC^{-1})$$
$$= \log_b(A) + \log_b(C^{-1})$$
$$= \log_b A + (-1)\log_b C$$
$$= \log_b A - \log_b C$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

## Example 4.5.6: Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $ln\left(\frac{x^4y}{7}\right)$ as a sum or difference of logs.

**Solution**

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

## Exercise 4.5.6

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

*Answer available at the end of this section.*

## Example 4.5.7: Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log(x)$.

**Solution**

$$\log(\sqrt{x}) = \log x^{\left(\frac{1}{2}\right)}$$
$$= \frac{1}{2}\log x$$

### Exercise 4.5.7

Expand $\ln(\sqrt[3]{x^2})$.

*Answer available at the end of this section.*

### Q&A: Can we expand $\ln(x^2 + y^2)$?

*No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.*

### Example 4.5.8: Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

**Solution**

We can expand by applying the Product and Quotient Rules.

$$\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) = \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \quad \text{Apply the Quotient Rule}$$
$$= \log_6 26 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \quad \text{Simplify by writing 64 as } 2^6$$
$$= 6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1) \quad \text{Apply the Power Rule}$$

### Exercise 4.5.8

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

*Answer available at the end of this section.*

## Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

### How to: Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

## Example 4.5.9: Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

**Solution**

Using the product and quotient rules

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$$

This reduces our original expression to

$$\log_3(40) - \log_3(2)$$

Then, using the quotient rule

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

## Exercise 4.5.9

Condense $\log_3 - \log_4 + \log_5 - \log_6$.

*Answer available at the end of this section.*

## Example 4.5.10: Condensing Complex Logarithmic Expressions

Condense $\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2)$.

**Solution**

We apply the power rule first:

$$\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2) = \log_2(x^2) + \log_2(\sqrt{x-1}) - \log_2((x+3)^6)$$

Next we apply the product rule to the sum:

$$\log_2(x^2) + \log_2(\sqrt{x-1}) - \log_2((x+3)^6) = \log_2(x^2\sqrt{x-1}) - \log_2((x+3)^6)$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2\sqrt{x-1}) - \log_2((x+3)^6) = \log_2\frac{x^2\sqrt{x-1}}{(x+3)^6}$$

## Example 4.5.11: Rewriting as a Single Logarithm

Rewrite $2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5)$ as a single logarithm.

**Solution**

We apply the power rule first:

$$2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5) = \log(x^2) - \log(x+5)^4 + \log((3x+5)^{x^{-1}})$$

Next we rearrange and apply the product rule to the sum:

$$\begin{aligned}\log(x^2) - \log(x+5)^4 + \log((3x+5)^{x^{-1}}) &= \log(x^2) + \log((3x+5)^{x^{-1}} - \log(x+5)^4 \\ &= \log(x^2(3x+5)^{x^{-1}}) - \log(x+5)^4 \\ &= \log\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4} \qquad \text{Apply the quotient rule to the difference}\end{aligned}$$

## Exercise 4.5.10

Rewrite $\log(5) + 0.5\log(x) - \log(7x-1) + 3\log(x-1)$ as a single logarithm.

*Answer available at the end of this section.*

## Exercise 4.5.11

Condense $4(3\log(x) + \log(x+5) - \log(2x+3))$.

*Answer available at the end of this section.*

## Example 4.5.12: Applying of the Laws of Logs

Recall that, in chemistry, $pH = -\log[H+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

**Solution**

Suppose $C$ is the original concentration of hydrogen ions, and $P$ is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log(2C)$$

Using the product rule of logs

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$,the new pH is

$$pH = P - \log(2) \approx P - 0.301$$

## Exercise 4.5.12

When the concentration of hydrogen ions is doubled, the pH decreases by about $0.301$.

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

*Answer available at the end of this section.*

# Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than $10$ ore, e,we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the **one-to-one** property and **power rule for logarithms**.

Given any positive real numbers $M$, $b$, and $n$, where $n \neq 1$ and $b \neq 1$,we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$. By taking the log base $n$ of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{aligned} \log_n(b^y) &= \log_n M && \text{Apply the one-to-one property} \\ y\log_n b &= \log_n M && \text{Apply the power rule for logarithms} \\ y &= \frac{\log_n M}{\log_n b} && \text{Isolate y} \\ \log_b M &= \frac{\log_n M}{\log_n b} && \text{Substitute for y} \end{aligned}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{aligned} \log_5 36 &= \frac{\log(36)}{\log(5)} && \text{Apply the change of base formula using base 10} \\ &\approx 2.2266 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

## THE CHANGE-OF-BASE FORMULA

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers $M$, $b$, and $n$, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b} \tag{4.5.7}$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b} \tag{4.5.8}$$

and

$$\log_b M = \frac{\log M}{\log b} \tag{4.5.9}$$

## How to: Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base $n$, where $n \neq 1$

1. Determine the new base $n$, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$,has base $e$.
2. Rewrite the log as a quotient using the change-of-base formula
   - The numerator of the quotient will be a logarithm with base $n$ and argument $M$.

- The denominator of the quotient will be a logarithm with base $n$ and argument $b$.

### Example 4.5.13: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

**Solution**

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument $3$. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$

$$\log_5 3 = \frac{\ln 3}{\ln 5}$$

### Exercise 4.5.13

Change $\log 0.58$ to a quotient of natural logarithms.

*Answer available at the end of this section.*

### Q&A: Can we change common logarithms to natural logarithms?

*Yes. Remember that* $\log 9$ *means* $\log_{10} 9$. *So,* $\log 9 = \frac{\ln 9}{\ln 10}$.

### Example 4.5.14: Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

**Solution**

According to the change-of-base formula, we can rewrite the log base $2$ as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base $e$.

$$\begin{aligned} \log_2 10 &= \frac{\ln 10}{\ln 2} && \text{Apply the change of base formula using base } e \\ &\approx 3.3219 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

### Exercise 4.5.14

Evaluate $\log_5(100)$ using the change-of-base formula.

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with laws of logarithms.

- The Properties of Logarithms
- Expand Logarithmic Expressions
- Evaluate a Natural Logarithmic Expression

## Key Equations

| | |
|---|---|
| The Product Rule for Logarithms | $\log_b(MN) = \log_b(M) + \log_b(N)$ |
| The Quotient Rule for Logarithms | $\log_b(\frac{M}{N}) = \log_b M - \log_b N$ |
| The Power Rule for Logarithms | $\log_b(M^n) = n\log_b M$ |
| The Change-of-Base Formula | $\log_b M = \frac{\log_n M}{\log_n b}$ $n > 0$, $n \neq 1$, $b \neq 1$ |

## Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See Example $4.5.1$.
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See Example $4.5.2$.
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See Example $4.5.3$, Example $4.5.4$, and Example $4.5.5$.
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See Example $4.5.6$, Example $4.5.7$, and Example $4.5.8$.
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See Example $4.5.9$, Example $4.5.10$, Example $4.5.11$, and Example $4.5.12$.
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See Example $4.5.13$.
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and $e$ as the quotient of natural or common logs. That way a calculator can be used to evaluate. See Example $4.5.14$.

### Answers

**Exercise** $4.5.1$

$\log_b 2 + \log_b 2 + \log_b 2 + \log_b k = 3\log_b 2 + \log_b k$

**Exercise** $4.5.2$

$\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$

**Exercise** $4.5.3$

$2\ln x$

**Exercise** $4.5.4$

$-2\ln(x)$

**Exercise** $4.5.5$

$\log_3 16$

**Exercise** $4.5.6$

$2\log x + 3\log y - 4\log z$

**Exercise** $4.5.7$

$\frac{2}{3}\ln x$

**Exercise** $4.5.8$

$\frac{1}{2}\ln(x-1) + \ln(2x+1) - \ln(x+3) - \ln(x-3)$

**Exercise** $4.5.9$

$\log\left(\frac{3\cdot 5}{4\cdot 6}\right)$; can also be written $\log\left(\frac{5}{8}\right)$ by reducing the fraction to lowest terms.

**Exercise** $4.5.10$

$\log\frac{5(x-1)^3\sqrt{x}}{(7x-1)}$

**Exercise** $4.5.11$

$\log \frac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log\left(\frac{x^3(x+5)}{(2x+3)}\right)^4$

**Exercise** 4.5.12

The pH increases by about $0.301$.

**Exercise** 4.5.13

$\frac{\ln 8}{\ln 0.5}$

**Exercise** 4.5.14

$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$

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# 4.6: Exponential and Logarithmic Equations

**Learning Objectives**

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

In 1859, an Australian landowner named Thomas Austin released $24$ rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.



Figure $4.6.1$: Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

## Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers $b$, $S$, and $T$, where $b > 0$, $b \neq 1$, $b^S = b^T$ if and only if $S = T$.

In other words, when an **exponential equation** has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \dfrac{3^{2x}}{3}$. To solve for $x$, we use the division property of exponents to rewrite the right side so that both sides have the common base, $3$. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for $x$:

$$
\begin{aligned}
3^{4x-7} &= \frac{3^{2x}}{3} \\
3^{4x-7} &= \frac{3^{2x}}{3^1} && \text{Rewrite 3 as } 3^1 \\
3^{4x-7} &= 3^{2x-1} && \text{Use the division property of exponents} \\
4x-7 &= 2x-1 && \text{Apply the one-to-one property of exponents} \\
2x &= 6 && \text{Subtract 2x and add 7 to both sides} \\
x &= 3 && \text{Divide by 3}
\end{aligned}
$$

### USING THE ONE-TO-ONE PROPERTY OF EXPONENTIAL FUNCTIONS TO SOLVE EXPONENTIAL EQUATIONS

For any algebraic expressions $S$ and $T$, and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{4.6.1}$$

### How to: Given an exponential equation with the form $b^S = b^T$, where $S$ and $T$ are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

### Example 4.6.1: Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

**Solution**

$$
\begin{aligned}
2^{x-1} &= 2^{2x-4} && \text{The common base is 2} \\
x-1 &= 2x-4 && \text{By the one-to-one property the exponents must be equal} \\
x &= 3 && \text{Solve for x}
\end{aligned}
$$

### Exercise 4.6.1

Solve $5^{2x} = 5^{3x+2}$.

*Answer available at the end of this section.*

## Rewrite Equations So All Powers Have the Same Base

Sometimes the *common base* for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for $x$:

$$
\begin{aligned}
256 &= 4^{x-5} && \\
2^8 &= (2^2)^{x-5} && \text{Rewrite each side as a power with base 2} \\
2^8 &= 2^{2x-10} && \text{Use the one-to-one property of exponents} \\
8 &= 2x-10 && \text{Apply the one-to-one property of exponents} \\
18 &= 2x && \text{Add 10 to both sides} \\
x &= 9 && \text{Divide by 2}
\end{aligned}
$$

### How to: Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
3. Use the one-to-one property to set the exponents equal.
4. Solve the resulting equation, $S = T$, for the unknown.

### Example 4.6.2: Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

**Solution**

$$
\begin{aligned}
8^{x+2} &= 16^{x+1} && \\
(2^3)^{x+2} &= (2^4)^{x+1} && \text{Write 8 and 16 as powers of 2} \\
2^{3x+6} &= 2^{4x+4} && \text{To take a power of a power, multiply exponents} \\
3x+6 &= 4x+4 && \text{Use the one-to-one property to set the exponents equal} \\
x &= 2 && \text{Solve for x}
\end{aligned}
$$

### Exercise 4.6.2

Solve $5^{2x} = 25^{3x+2}$.

*Answer available at the end of this section.*

### Example 4.6.3: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

**Solution**

$$
\begin{aligned}
2^{5x} &= 2^{\frac{1}{2}} && \text{Write the square root of 2 as a power of 2} \\
5x &= \frac{1}{2} && \text{Use the one-to-one property} \\
x &= \frac{1}{10} && \text{Solve for x}
\end{aligned}
$$

### Exercise 4.6.3

Solve $5^x = \sqrt{5}$.

*Answer available at the end of this section.*

**Q&A: Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?**

*No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.*

**Example 4.6.4: Solving an Equation with Positive and Negative Powers**

Solve $3^{x+1} = -2$.

**Solution**

This equation has no solution. There is no real value of $x$ that will make the equation a true statement because any power of a positive number is positive.

Analysis

Figure 4.6.2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

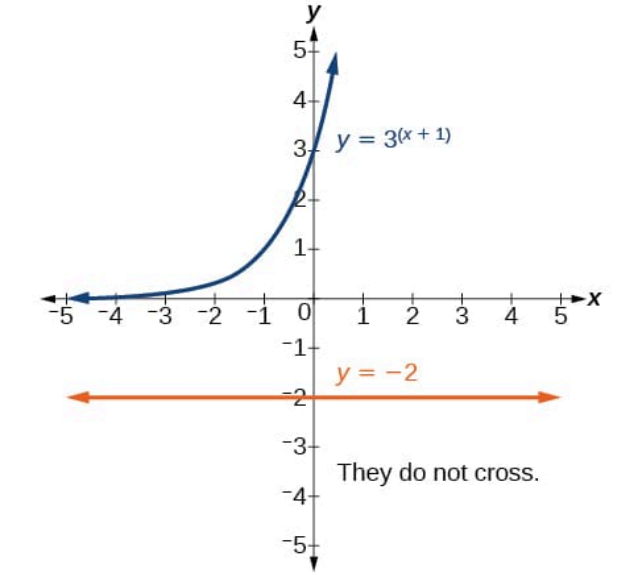



Figure 4.6.2

**Exercise 4.6.4**

Solve $2^x = -100$.

*Answer available at the end of this section.*

## Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

**How to: Given an exponential equation in which a common base cannot be found, solve for the unknown**

1. Apply the logarithm of both sides of the equation.
   - If one of the terms in the equation has base 10, use the common logarithm.
   - If none of the terms in the equation has base 10, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

**Example 4.6.5: Solving an Equation Containing Powers of Different Bases**

Solve $5^{x+2} = 4^x$.

**Solution**

$$
\begin{aligned}
5^{x+2} &= 4^x && \text{There is no easy way to get the powers to have the same base}\\
\ln 5^{x+2} &= \ln 4^x && \text{Take ln of both sides}\\
(x+2)\ln 5 &= x \ln 4 && \text{Use laws of logs}\\
x\ln 5 + 2\ln 5 &= x\ln 4 && \text{Use the distributive law}\\
x\ln 5 - x\ln 4 &= -2\ln 5 && \text{Get terms containing x on one side, terms without x on the other}\\
x(\ln 5 - \ln 4) &= -2\ln 5 && \text{On the left hand side, factor out an x}\\
x\ln\left(\frac{5}{4}\right) &= \ln\left(\frac{1}{25}\right) && \text{Use the laws of logs}\\
x &= \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)} && \text{Divide by the coefficient of x}
\end{aligned}
$$

**Exercise 4.6.5**

Solve $2^x = 3^{x+1}$.

*Answer available at the end of this section.*

**Q&A: Is there any way to solve $2^x = 3^x$?**

*Yes. The solution is* $0$.

## Equations Containing $e$

One common type of exponential equations are those with base $e$. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base $e$ on either side, we can use the natural logarithm to solve it.

**How to: Given an equation of the form $y = Ae^{kt}$, solve for $t$.**

1. Divide both sides of the equation by $A$.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by $k$.

### Example 4.6.6: Solve an Equation of the Form $y = Ae^{kt}$

Solve $100 = 20e^{2t}$.

**Solution**

$$\begin{aligned} 100 &= 20e^{2t} \\ 5 &= e^{2t} && \text{Divide by the coefficient of the power} \\ \ln 5 &= 2t && \text{Take ln of both sides. Use the fact that } ln(x) \text{ and } e^x \text{ are inverse functions} \\ t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of t} \end{aligned}$$

**Analysis**

Using laws of logs, we can also write this answer in the form $t = \ln \sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

### Exercise 4.6.6

Solve $3e^{0.5t} = 11$.

*Answer available at the end of this section.*

### Q&A: Does every equation of the form $y = Ae^{kt}$ have a solution?

No. There is a solution when $k \neq 0$,and when $y$ and $A$ are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2 = -3e^t$.

### Example 4.6.7: Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x} + 5 = 12$.

**Solution**

$$\begin{aligned} 4e^{2x} + 5 &= 12 \\ 4e^{2x} &= 7 && \text{Combine like terms} \\ e^{2x} &= \frac{7}{4} && \text{Divide by the coefficient of the power} \\ 2x &= \ln\left(\frac{7}{4}\right) && \text{Take ln of both sides} \\ x &= \frac{1}{2}\ln\left(\frac{7}{4}\right) && \text{Solve for x} \end{aligned}$$

### Exercise 4.6.7

Solve $3 + e^{2t} = 7e^{2t}$.

*Answer available at the end of this section.*

## Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct

algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

### Example 4.6.8: Solving Exponential Functions in Quadratic Form

Solve $e^{2x} - e^x = 56$.

**Solution**

$$\begin{aligned} e^{2x} - e^x &= 56 \\ e^{2x} - e^x - 56 &= 0 && \text{Get one side of the equation equal to zero} \\ (e^x + 7)(e^x - 8) &= 0 && \text{Factor by the FOIL method} \\ e^x + 7 &= 0 && \text{or} \\ e^x - 8 &= 0 && \text{If a product is zero, then one factor must be zero} \\ e^x &= -7 && \text{or} \\ e^x &= 8 && \text{Isolate the exponentials} \\ e^x &= 8 && \text{Reject the equation in which the power equals a negative number} \\ x &= \ln 8 && \text{Solve the equation in which the power equals a positive number} \end{aligned}$$

**Analysis**

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^x = -7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

### Exercise 4.6.8

Solve $e^{2x} = e^x + 2$.

*Answer available at the end of this section.*

### Q&A: Does every logarithmic equation have a solution?

*No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.*

## Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for $x$:

$$\begin{aligned} \log_2(2) + \log_2(3x - 5) &= 3 \\ \log_2(2(3x - 5)) &= 3 && \text{Apply the product rule of logarithms} \\ \log_2(6x - 10) &= 3 && \text{Distribute} \\ 2^3 &= 6x - 10 && \text{Apply the definition of a logarithm} \\ 8 &= 6x - 10 && \text{Calculate } 2^3 \\ 18 &= 6x && \text{Add 10 to both sides} \\ x &= 3 && \text{Divide by 6} \end{aligned}$$

## USING THE DEFINITION OF A LOGARITHM TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expression $S$ and real numbers $b$ and $c$,where $b > 0$, $b \neq 1$,

$$\log_b(S) = c \text{ if and only if } b^c = S \tag{4.6.2}$$

### Example 4.6.9: Using Algebra to Solve a Logarithmic Equation

Solve $2\ln x + 3 = 7$.

**Solution**

$$\begin{aligned} 2\ln x + 3 &= 7 \\ 2\ln x &= 4 \quad \text{Subtract 3} \\ \ln x &= 2 \quad \text{Divide by 2} \\ x &= e^2 \quad \text{Rewrite in exponential form} \end{aligned}$$

### Exercise 4.6.9

Solve $6 + \ln x = 10$.

*Answer available at the end of this section.*

### Example 4.6.10: Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x) = 7$.

**Solution**

$$\begin{aligned} 2\ln(6x) &= 7 \\ \ln(6x) &= \frac{7}{2} \quad \text{Divide by 2} \\ 6x &= e^{\left(\frac{7}{2}\right)} \quad \text{Use the definition of ln} \\ x &= \frac{1}{6}e^{\left(\frac{7}{2}\right)} \quad \text{Divide by 6} \end{aligned}$$

### Exercise 4.6.10

Solve $2\ln(x + 1) = 10$.

*Answer available at the end of this section.*

### Example 4.6.11: Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x = 3$.

**Solution**

$$\begin{aligned} \ln x &= 3 \\ x &= e^3 \quad \text{Use the definition of the natural logarithm} \end{aligned}$$

Figure $4.6.3$ represents the graph of the equation. On the graph, the *x*-coordinate of the point at which the two graphs intersect is close to $20$. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

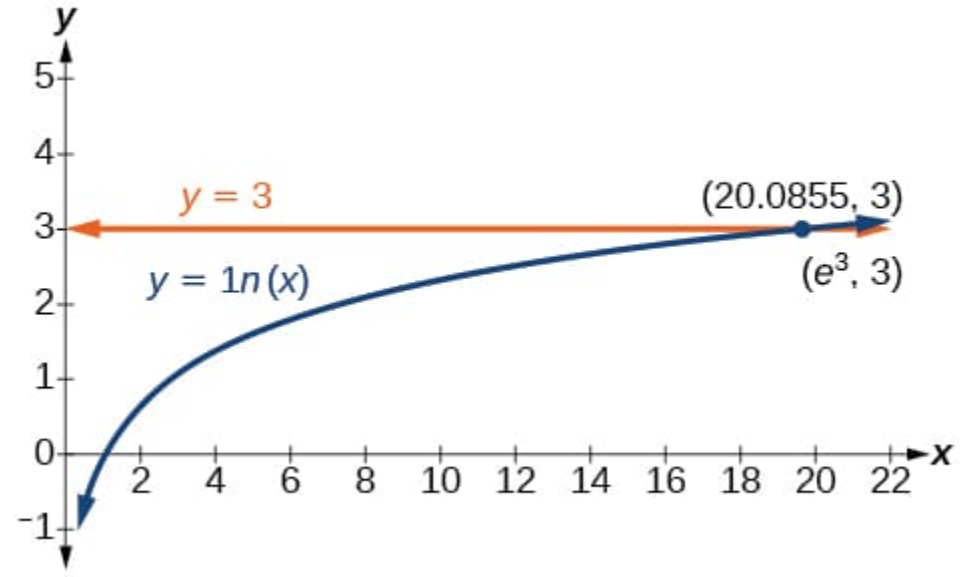



Figure $4.6.3$: The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately $(20.0855, 3)$.

**Exercise $4.6.11$**

Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to $2$ decimal places.

*Answer available at the end of this section.*

## Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number $b$, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for $x$,and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9 - 1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x - 2) - \log(2) = \log(x + 4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for $x$:

$$\begin{aligned}
\log(3x-2)-\log(2) &= \log(x+4)\\
\log\left(\frac{3x-2}{2}\right) &= \log(x+4) \quad \text{Apply the quotient rule of logarithms}\\
\frac{3x-2}{2} &= x+4 \quad \text{Apply the one to one property of a logarithm}\\
3x-2 &= 2x+8 \quad \text{Multiply both sides of the equation by 2}\\
x &= 10 \quad \text{Subtract 2x and add 2}
\end{aligned}$$

To check the result, substitute $x = 10$ into $\log(3x-2)-\log(2) = \log(x+4)$.

$$\begin{aligned}
\log(3(10)-2)-\log(2) &= \log((10)+4)\\
\log(28)-\log(2) &= \log(14)\\
\log\left(\frac{28}{2}\right) &= \log(14) \quad \text{The solution checks}
\end{aligned}$$

## USING THE ONE-TO-ONE PROPERTY OF LOGARITHMS TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expressions $S$ and $T$ and any positive real number $b$, where $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{4.6.3}$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

## How to: Given an equation containing logarithms, solve it using the one-to-one property

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S = \log_b T$ .
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S = T$, for the unknown.

## Example 4.6.12: Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x+3)$.

**Solution**

$$\begin{aligned}
\ln(x^2) &= \ln(2x+3)\\
x^2 &= 2x+3 \quad \text{Use the one-to-one property of the logarithm}\\
x^2-2x-3 &= 0 \quad \text{Get zero on one side before factoring}\\
(x-3)(x+1) &= 0 \quad \text{Factor using FOIL}\\
x-3 &= 0 \quad \text{or } x+1=0 \text{ If a product is zero, one of the factors must be zero}\\
x &= 3 \quad \text{or}\\
x &= -11 \quad \text{Solve for x}
\end{aligned}$$

**Analysis**

There are two solutions: $3$ or $-1$. The solution $-1$ is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

## Exercise 4.6.12

Solve $\ln(x^2) = \ln 1$.

*Answer available at the end of this section.*

## Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its **half-life**. Table $4.6.1$ lists the half-life for several of the more common radioactive substances.

*Table* $4.6.1$

| Substance | Use | Half-life |
|---|---|---|
| *gallium-67* | *nuclear medicine* | *80 hours* |
| *cobalt-60* | *manufacturing* | *5.3 years* |
| *technetium-99m* | *nuclear medicine* | *6 hours* |
| *americium-241* | *construction* | *432 years* |
| *carbon-14* | *archeological dating* | *5,715 years* |
| *uranium-235* | *atomic power* | *703,800,000 years* |

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$A(t) = A_0 e^{\frac{\ln(0.5)}{T}t} \tag{4.6.4}$$

$$A(t) = A_0 e^{\frac{\ln(0.5)t}{T}} \tag{4.6.5}$$

$$A(t) = A_0 (e^{\ln(0.5)})^{\frac{t}{T}} \tag{4.6.6}$$

$$A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}} \tag{4.6.7}$$

where

- $A_0$ is the amount initially present
- $T$ is the half-life of the substance
- $t$ is the time period over which the substance is studied
- $y$ is the amount of the substance present after time $t$

### Example $4.6.13$: Using the Formula for Radioactive Decay to Find the Quantity of a Substance

How long will it take for ten percent of a $1000$-gram sample of uranium-235 to decay?

**Solution**

$$
\begin{aligned}
y &= 1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} \\
900 &= 1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} && \text{After 10\% decays, 900 grams are left} \\
0.9 &= e^{\frac{\ln(0.5)}{703{,}800{,}000}t} && \text{Divide by 1000} \\
\ln(0.9) &= \ln\left(e^{\frac{\ln(0.5)}{703{,}800{,}000}t}\right) && \text{Take ln of both sides} \\
\ln(0.9) &= \frac{\ln(0.5)}{703,800,000}t && \ln(e^M) = M \\
t &= 703,800,000 \times \frac{\ln(0.9)}{\ln(0.5)} && \text{years Solve for t} \\
t &\approx 106,979,777 && \text{years}
\end{aligned}
$$

**Analysis**

Ten percent of $1000$ grams is $100$ grams. If $100$ grams decay, the amount of uranium-235 remaining is $900$ grams.

### Exercise 4.6.13

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Exponential Equations with Logarithms

## Key Equations

| | |
|---|---|
| One-to-one property for exponential functions | For any algebraic expressions $S$ and $T$ and any positive real number $b$, where $b^S = b^T$ if and only if $S = T$. |
| Definition of a logarithm | For any algebraic expression S and positive real numbers $b$ and $c$, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$. |
| One-to-one property for logarithmic functions | For any algebraic expressions $S$ and $T$ and any positive real number $b$, where $b \neq 1$, $\log_b S = \log_b T$ if and only if $S = T$. |

## Key Concepts

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown. See Example $4.6.1$.
- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See Example $4.6.2$, Example $4.6.3$, and Example $4.6.4$.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See Example $4.6.5$.

- We can solve exponential equations with base $e$,by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See Example $4.6.6$ and Example $4.6.7$.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions. See Example $4.6.8$.
- When given an equation of the form $\log_b(S) = c$, where $S$ is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See Example $4.6.9$ and Example $4.6.10$.
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the *x*-value of the intersecting point. See Example $4.6.11$.
- When given an equation of the form $\log_b S = \log_b T$, where $S$ and $T$ are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See Example $4.6.12$.
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See Example $4.6.13$.

## Answers

**Exercise** $4.6.1$

$x = -2$

**Exercise** $4.6.2$

$x = -1$

**Exercise** $4.6.3$

$x = \frac{1}{2}$

**Exercise** $4.6.4$

The equation has no solution.

**Exercise** $4.6.5$

$x = \frac{\ln 3}{\ln\left(\frac{2}{3}\right)}$

**Exercise** $4.6.6$

$t = 2\ln\left(\frac{11}{3}\right)$ or $\ln\left(\frac{11}{3}\right)^2$

**Exercise** $4.6.7$

$t = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2)$

**Exercise** $4.6.8$

$x = \ln 2$

**Exercise** $4.6.9$

$x = e^4$

**Exercise** $4.6.10$

$x = e^5 - 1$

**Exercise** $4.6.11$

$x \approx 9.97$

**Exercise** $4.6.12$

$x = 1$ or $x = -1$

**Exercise** $4.6.13$

$t = 703,800,000 \times \frac{\ln(0.8)}{\ln(0.5)}$years ≈ 226,572,993 years.

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# 4.7: Exponential and Logarithmic Models

**Learning Objectives**

- Model exponential growth and decay.
- Use Newton's Law of Cooling.
- Use logistic-growth models.
- Choose an appropriate model for data.
- Express an exponential model in base $e$.

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton's Law of Cooling.

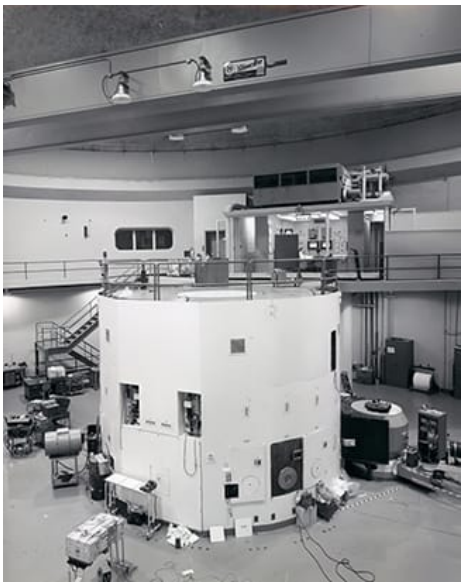


Figure 4.7.1: A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus (credit: Georgia Tech Research Institute)

## Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function:

$$y = A_0 e^{kt} \tag{4.7.1}$$

where $A_0$ is equal to the value at time zero, $e$ is Euler's constant, and $k$ is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double. Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth based on a doubling time. In some applications, however, as we will see when we discuss the logistic equation, the logistic model sometimes fits the data better than the exponential model.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the **exponential decay** model. Again, we have the form $y = A_0 e^{kt}$ where $A_0$ is the starting value, and $e$ is Euler's constant. Now $k$ is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating half-life, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in Figure $4.7.2$ and Figure $4.7.3$. It is important to remember that, although parts of each of the two graphs seem to lie on the $x$-axis, they are really a tiny distance above the $x$-axis.

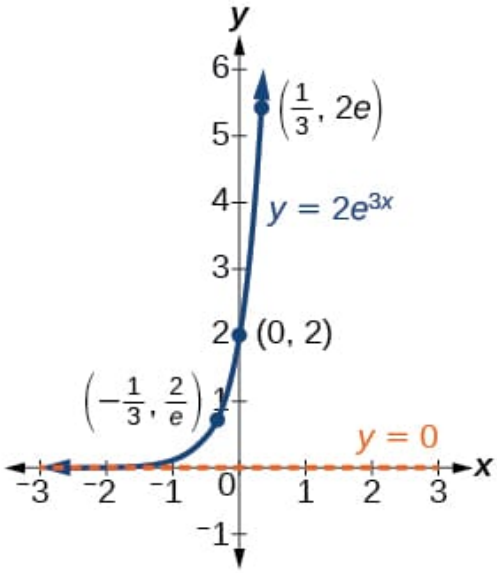



Figure $4.7.2$: A graph showing exponential growth. The equation is $y = 2e^{3x}$.

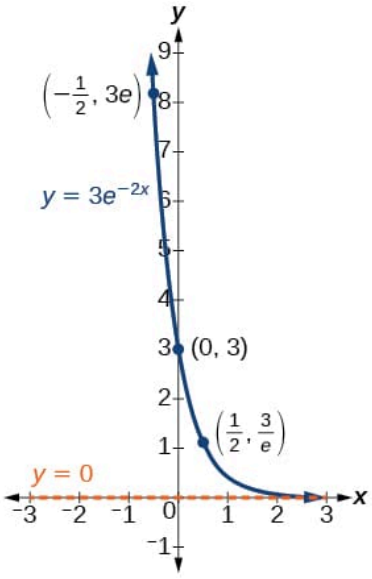



Figure $4.7.3$: A graph showing exponential decay. The equation is $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude. The **order of magnitude** is the power of ten, when the number is expressed in scientific notation, with one digit to the left of the decimal. For example, the distance to the nearest star, **Proxima Centauri**, measured in kilometers, is $40,113,497,200,000$ kilometers. Expressed in scientific notation, this is $4.01134972 \times 1013$. So, we could describe this number as having order of magnitude $1013$.

### $y = a_0 e^{kt}$

An exponential function with the form $y = A_0 e^{kt}$ has the following characteristics:

- one-to-one function

- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- $x$ intercept: none
- $y$-intercept: $(0, A_0)$
- increasing if $k > 0$ (see Figure $4.7.4$)
- decreasing if $k < 0$ (see Figure $4.7.4$)

Two graphs of y=(A_0)(e^(kt)) with the asymptote at y=0. The first graph is of when k
0 and with the labeled points (1/k, (A_0)e), (0, A_0), and (-1/k, (A_0)/e). The second graph is of when k<0 and with the labeled points (-1/k, (A_0)e), (0, A_0), and (1/k, (A_0)/e)." src="/@api/deki/files/12238/fig_6.8.4.jpg">

Figure $4.7.4$: An exponential function models exponential growth when $k > 0$ and exponential decay when $k < 0$.

## Example $4.7.1$: Graphing Exponential Growth

A population of bacteria doubles every hour. If the culture started with $10$ bacteria, graph the population as a function of time.

**Solution**

When an amount grows at a fixed percent per unit time, the growth is exponential. To find $A_0$ we use the fact that $A_0$ is the amount at time zero, so $A_0 = 10$. To find $k$, use the fact that after one hour $(t = 1)$ the population doubles from $10$ to $20$.The formula is derived as follows

$$\begin{aligned} 20 &= 10e^{k \cdot 1} \\ 2 &= e^k \qquad \text{Divide by } 10 \\ \ln 2 &= k \qquad \text{Take the natural logarithm} \end{aligned}$$

so $k = \ln(2)$. Thus the equation we want to graph is $y = 10e^{(\ln 2)t} = 10(e^{\ln 2})^t = 10 \cdot 2^t$. The graph is shown in Figure $4.7.5$.

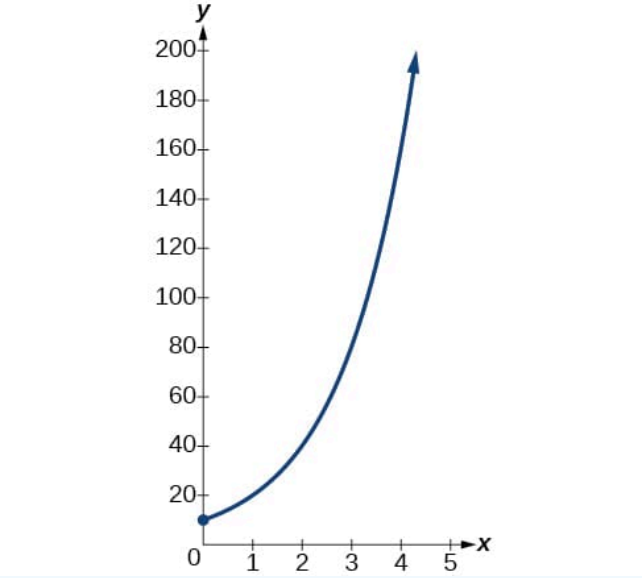



Figure $4.7.5$: The graph of $y = 10e^{(\ln 2)t}$

**Analysis**

The population of bacteria after ten hours is $10,240$. We could describe this amount is being of the order of magnitude $10^4$. The population of bacteria after twenty hours is $10,485,760$ which is of the order of magnitude $10^7$, so we could say that the population has increased by three orders of magnitude in ten hours.

# Half-Life

We now turn to **exponential decay**. One of the common terms associated with exponential decay, as stated above, is **half-life**, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay.

To find the half-life of a function describing exponential decay, solve the following equation:

$$\frac{1}{2}A_0 = A_0 e^{kt}$$

We find that the half-life depends only on the constant $k$ and not on the starting quantity $A_0$.

The formula is derived as follows

$$\begin{aligned} \frac{1}{2}A_0 &= A_0 e^{kt} \\ \frac{1}{2} &= e^{kt} && \text{Divide by } A_0 \\ \ln\left(\frac{1}{2}\right) &= ktv && \text{Take the natural log} \\ -\ln(2) &= kt && \text{Apply laws of logarithms} \\ -\ln(2)k &= t && \text{Divide by k} \end{aligned}$$

Since $t$, the time, is positive, $k$ must, as expected, be negative. This gives us the half-life formula

$$t = -\frac{\ln(2)}{k} \tag{4.7.2}$$

## Given the half-life, find the decay rate.

1. Write $A = A_0 e^{kt}$.
2. Replace $A$ by $\frac{1}{2}A_0$ and replace $t$ by the given half-life.
3. Solve to find $k$. Express $k$ as an exact value (do not round).

Note: It is also possible to find the decay rate using $k = -\ln(2)t$.

## Example 4.7.2: Finding the Function that Describes Radioactive Decay

The half-life of carbon-14 is $5,730$ years. Express the amount of carbon-14 remaining as a function of time, $t$.

**Solution**

This formula is derived as follows.

$$\begin{aligned} A &= A_0 e^{kt} && \text{The continuous growth formula} \\ 0.5A_0 &= A_0 e^{k\cdot 5730} && \text{Substitute the half-life for t and } 0.5A_0 \text{ for } f(t) \\ 0.5 &= e^{5730k} && \text{Divide by } A_0 \\ \ln(0.5) &= 5730k && \text{Take the natural log of both sides} \\ k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of k} \\ A &= A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} && \text{Substitute for r in the continuous growth formula} \end{aligned}$$

The function that describes this continuous decay is $f(t) = A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$. We observe that the coefficient of $t$, $\frac{\ln(0.5)}{5730} \approx -1.2097 \times 10^{-4}$ is negative, as expected in the case of exponential decay.

**Exercise 4.7.1**

The half-life of plutonium-244 is $80,000,000$ years. Find function gives the amount of carbon-14 remaining as a function of time, measured in years.

*Answer available at the end of this section.*

## Radiocarbon Dating

The formula for radioactive decay is important in **radiocarbon dating**, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about $1\%$ error for plants or animals that died within the last $60,000$ years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of $5,730$ years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of $12$ and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last $60,000$ years, using tree rings and other organic samples of known dates—although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is $5,730$ years, the formula for the amount of carbon-14 remaining after $t$ years is

$$A \approx A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$$

where

- $A$ is the amount of carbon-14 remaining
- $A_0$ is the amount of carbon-14 when the plant or animal began decaying.

This formula is derived as follows:

$$\begin{aligned}
A &= A_0 e^{kt} && \text{The continuous growth formula} \\
0.5A_0 &= A_0 e^{k\cdot 5730} && \text{Substitute the half-life for t and } 0.5A_0 \text{ for f(t)} \\
0.5 &= e^{5730k} && \text{Divide by } A_0 \\
\ln(0.5) &= 5730k && \text{Take the natural log of both sides} \\
k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of k} \\
A &= A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} && \text{Substitute for r in the continuous growth formula}
\end{aligned}$$

To find the age of an object, we solve this equation for $t$:

$$t = \frac{\ln\left(\frac{A}{A_0}\right)}{-0.000121}$$

Out of necessity, we neglect here the many details that a scientist takes into consideration when doing carbon-14 dating, and we only look at the basic formula. The ratio of carbon-14 to carbon-12 in the atmosphere is approximately $0.0000000001$. Let $r$ be the ratio of carbon-14 to carbon-12 in the organic artifact or fossil to be dated, determined by a method called liquid scintillation. From the equation $A \approx A_0 e^{-0.000121t}$ we know the ratio of the percentage of carbon-14 in the object we are dating to the percentage of carbon-14 in the atmosphere is $r = \frac{A}{A_0} \approx e^{-0.000121t}$. We solve this equation for $t$, to get

$$t = \frac{\ln(r)}{-0.000121}$$

1. Express the given percentage of carbon-14 as an equivalent decimal, $k$.
2. Substitute for $k$ in the equation $t = \dfrac{\ln(r)}{-0.000121}$ and solve for the age, $t$.

### Example 4.7.3: Finding the Age of a Bone

A bone fragment is found that contains $20\%$ of its original carbon-14. To the nearest year, how old is the bone?

**Solution**

We substitute $20\% = 0.20$ for $k$ in the equation and solve for $t$:

$$\begin{aligned} t &= \frac{\ln(r)}{-0.000121} && \text{Use the general form of the equation} \\ &= \frac{\ln(0.20)}{-0.000121} && \text{Substitute for r} \\ &\approx 13301 && \text{Round to the nearest year} \end{aligned}$$

The bone fragment is about $13,301$ years old.

**Analysis**

The instruments that measure the percentage of carbon-14 are extremely sensitive and, as we mention above, a scientist will need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about $1\%$, so this age should be given as $13,301$ years $\pm 1\%$ or $13,301$ years $\pm 133$ years.

### Exercise 4.7.2

Cesium-137 has a half-life of about $30$ years. If we begin with $200$ mg of cesium-137, will it take more or less than $230$ years until only $1$ milligram remains?

*Answer available at the end of this section.*

## Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the **doubling time**.

Given the basic **exponential growth** equation $A = A_0 e^{kt}$, doubling time can be found by solving for when the original quantity has doubled, that is, by solving $2A_0 = A_0 e^{kt}$.

The formula is derived as follows:

$$\begin{aligned} 2A_0 &= A_0 e^{kt} \\ 2 &= e^{kt} && \text{Divide by } A_0 \\ \ln 2 &= kt && \text{Take the natural logarithm} \\ t &= \frac{\ln 2}{k} && \text{Divide by the coefficient of t} \end{aligned}$$

Thus the doubling time is

$$t = \frac{\ln 2}{k} \tag{4.7.3}$$

### Example 4.7.4: Finding a Function That Describes Exponential Growth

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

**Solution**

The formula is derived as follows:

$$\begin{aligned} t &= \frac{\ln 2}{k} && \text{The doubling time formula} \\ 2 &= \frac{\ln 2}{k} && \text{Use a doubling time of two years} \\ k &= \frac{\ln 2}{2} && \text{Multiply by k and divide by 2} \\ A &= A_0 e^{\frac{\ln 2}{2} t} && \text{Substitute k into the continuous growth formula} \end{aligned}$$

The function is $A = A_0 e^{\frac{\ln 2}{2} t}$.

### Exercise 4.7.3

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

*Answer available at the end of this section.*

## Using Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a **vertical shift** of the generic **exponential decay** function. This translation leads to **Newton's Law of Cooling**, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature

$$T(t) = ae^{kt} + T_s$$

This formula is derived as follows:

$$\begin{aligned} T(t) &= Ab^{ct} + T_s \\ T(t) &= Ae^{\ln(b^{ct})} + T_s && \text{Laws of logarithms} \\ T(t) &= Ae^{ct \ln b} + T_s && \text{Laws of logarithms} \\ T(t) &= Ae^{kt} + T_s && \text{Rename the constant c } lnb\text{, calling it } k \end{aligned}$$

### NEWTON'S LAW OF COOLING

The temperature of an object, $T$, in surrounding air with temperature $T_s$ will behave according to the formula

$$T(t) = Ae^{kt} + T_s \tag{4.7.4}$$

where

- $t$ is time

- $A$ is the difference between the initial temperature of the object and the surroundings
- $k$ is a constant, the continuous rate of cooling of the object

### Given a set of conditions, apply Newton's Law of Cooling.

1. Set $T_s$ equal to the $y$-coordinate of the horizontal asymptote (usually the ambient temperature).
2. Substitute the given values into the continuous growth formula $T(t) = Ae^{kt} + T_s$ to find the parameters $A$ and $k$.
3. Substitute in the desired time to find the temperature or the desired temperature to find the time.

### Example 4.7.5: Using Newton's Law of Cooling

A cheesecake is taken out of the oven with an ideal internal temperature of $165\ °F$, and is placed into a $35\ °F$ refrigerator. After $10$ minutes, the cheesecake has cooled to $150\ °F$. If we must wait until the cheesecake has cooled to $70\ °F$ before we eat it, how long will we have to wait?

**Solution**

Because the surrounding air temperature in the refrigerator is $35$ degrees, the cheesecake's temperature will decay exponentially toward $35$, following the equation

$$T(t) = Ae^{kt} + 35$$

We know the initial temperature was $165$, so $T(0) = 165$.

$$\begin{aligned} 165 &= Ae^{k0} + 35 && \text{Substitute } (0, 165) \\ A &= 130 && \text{Solve for A} \end{aligned}$$

We were given another data point, $T(10) = 150$, which we can use to solve for $k$.

$$\begin{aligned} 150 &= 130e^{k10} + 35 && \text{Substitute } (10, 150) \\ 115 &= 130e^{k10} && \text{Subtract } 35 \\ \frac{115}{130} &= e^{10k} && \text{Divide by } 130 \\ \ln\left(\frac{115}{130}\right) &= 10k && \text{Take the natural log of both sides} \\ k &= \frac{\ln\left(\frac{115}{130}\right)}{10} \\ &= -0.0123 && \text{Divide by the coefficient of k} \end{aligned}$$

This gives us the equation for the cooling of the cheesecake: $T(t) = 130e^{-0.0123t} + 35$.

Now we can solve for the time it will take for the temperature to cool to $70$ degrees.

$$\begin{aligned} 70 &= 130e^{-0.0123t} + 35 && \text{Substitute in } 70 \text{ for } T(t) \\ 35 &= 130e^{-0.0123t} && \text{Subtract } 35 \\ \frac{35}{130} &= e^{-0.0123t} && \text{Divide by } 130 \\ \ln\left(\frac{35}{130}\right) &= -0.0123t && \text{Take the natural log of both sides} \\ t &= \frac{\ln\left(\frac{35}{130}\right)}{-0.0123} \\ &\approx 106.68 && \text{Divide by the coefficient of t} \end{aligned}$$

It will take about $107$ minutes, or one hour and $47$ minutes, for the cheesecake to cool to $70\ °F$.

**Exercise 4.7.4**

A pitcher of water at $40$ degrees Fahrenheit is placed into a $70$ degree room. One hour later, the temperature has risen to $45$ degrees. How long will it take for the temperature to rise to $60$ degrees?

*Answer available at the end of this section.*

## Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over $17$ billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the **exponential growth** model is still useful over a short term, before approaching the limiting value.

The **logistic growth model** is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, called the **carrying capacity**. For constants $a$, $b$, and $c$, the logistic growth of a population over time $x$ is represented by the model

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

The graph in Figure $4.7.6$ shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.

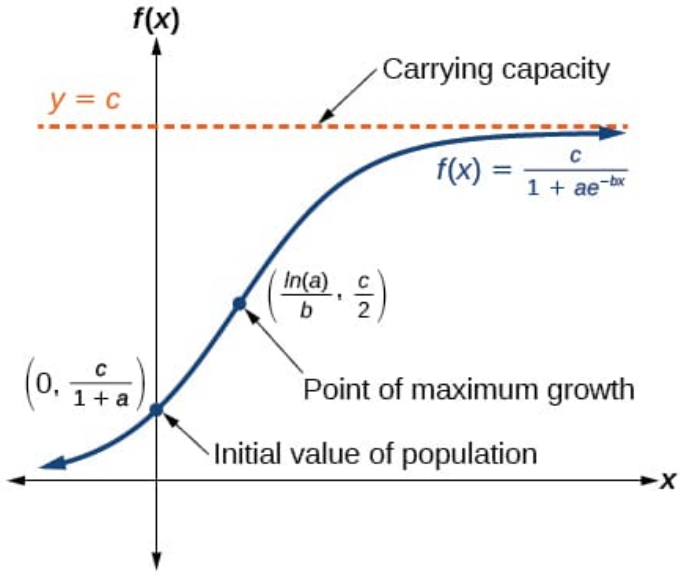



Figure $4.7.6$

**LOGISTIC GROWTH**

The logistic growth model is

$$f(x) = \frac{c}{1 + ae^{-bx}} \tag{4.7.5}$$

where

- $\frac{c}{1+a}$ is the initial value
- $c$ is the *carrying capacity*, or *limiting value*
- $b$ is a constant determined by the rate of growth.

## Example 4.7.6: Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of $1,000$ people who has the flu. So, in that community, at most $1,000$ people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

**Solution**

We substitute the given data into the logistic growth model

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

Because at most $1,000$ people, the entire population of the community, can get the flu, we know the limiting value is $c = 1000$. To find $a$,we use the formula that the number of cases at time $t = 0$ is $\frac{c}{1+a} = 1$, from which it follows that $a = 999$.This model predicts that, after ten days, the number of people who have had the flu is $f(x) = \frac{1000}{1 + 999e^{-0.6030x}} \approx 293.8$. Because the actual number must be a whole number (a person has either had the flu or not) we round to $294$. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000$.

**Analysis**

Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.

The graph in Figure 4.7.7 gives a good picture of how this model fits the data.

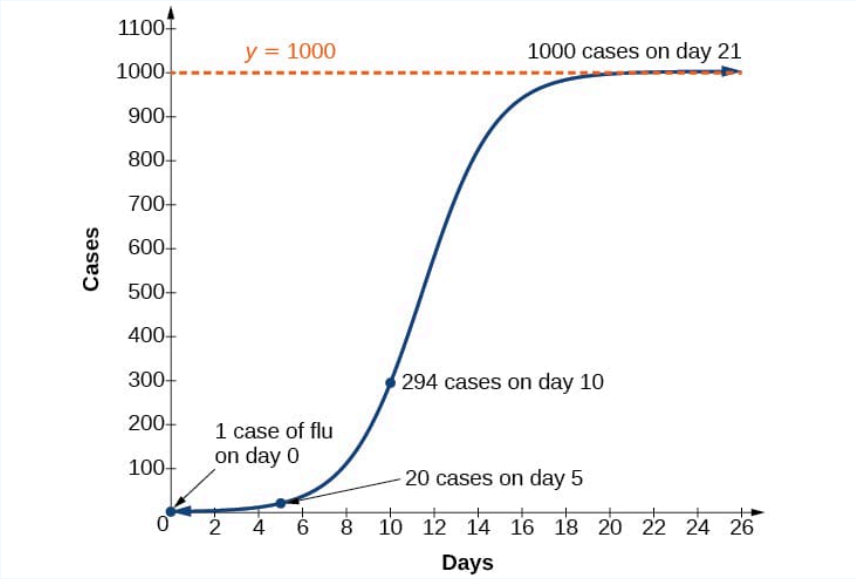



Figure 4.7.7: The graph of $f(x) = \frac{1000}{1 + 999e^{-0.6030x}}$

**Exercise 4.7.5**

Using the model in the Example above, estimate the number of cases of flu on day 15.

*Answer available at the end of this section.*

## Choosing an Appropriate Model for Data

Now that we have discussed various mathematical models, we need to learn how to choose the appropriate model for the raw data we have. Many factors influence the choice of a mathematical model, among which are experience, scientific laws, and patterns in the data itself. Not all data can be described by elementary functions. Sometimes, a function is chosen that approximates the data over a given interval. For instance, suppose data were gathered on the number of homes bought in the United States from the years 1960 to 2013. After plotting these data in a scatter plot, we notice that the shape of the data from the years 2000 to 2013 follow a logarithmic curve. We could restrict the interval from 2000 to 2010, apply regression analysis using a logarithmic model, and use it to predict the number of home buyers for the year 2015.

Three kinds of functions that are often useful in mathematical models are linear functions, exponential functions, and logarithmic functions. If the data lies on a straight line, or seems to lie approximately along a straight line, a linear model may be best. If the data is non-linear, we often consider an exponential or logarithmic model, though other models, such as quadratic models, may also be considered.

In choosing between an exponential model and a logarithmic model, we look at the way the data curves. This is called the concavity. If we draw a line between two data points, and all (or most) of the data between those two points lies above that line, we say the curve is concave down. We can think of it as a bowl that bends downward and therefore cannot hold water. If all (or most) of the data between those two points lies below the line, we say the curve is concave up. In this case, we can think of a bowl that bends upward and can therefore hold water. An exponential curve, whether rising or falling, whether representing growth or decay, is always concave up away from its horizontal asymptote. A logarithmic curve is always concave away from its vertical asymptote. In the case of positive data, which is the most common case, an exponential curve is always concave up, and a logarithmic curve always concave down.

A logistic curve changes concavity. It starts out concave up and then changes to concave down beyond a certain point, called a point of inflection.

After using the graph to help us choose a type of function to use as a model, we substitute points, and solve to find the parameters. We reduce round-off error by choosing points as far apart as possible.

**Example 4.7.7: Choosing a Mathematical Model**

Does a linear, exponential, logarithmic, or logistic model best fit the values listed in Table 4.7.1? Find the model, and use a graph to check your choice.

*Table* 4.7.1

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0 | 1.386 | 2.197 | 2.773 | 3.219 | 3.584 | 3.892 | 4.159 | 4.394 |

**Solution**

First, plot the data on a graph as in Figure 4.7.8. For the purpose of graphing, round the data to two significant digits.

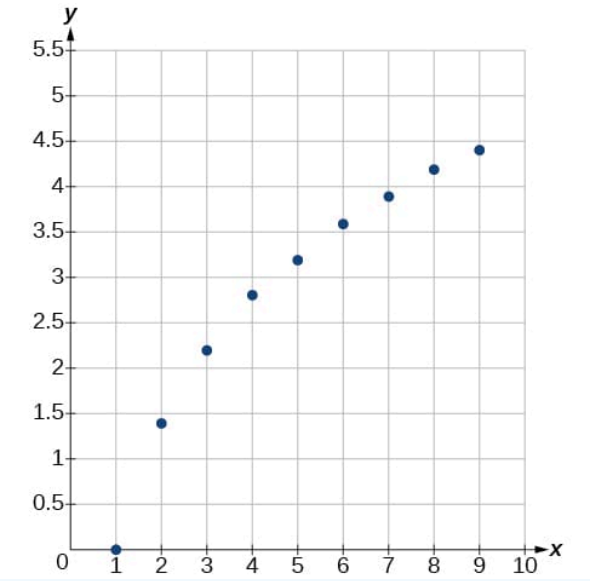



Figure $4.7.8$

Clearly, the points do not lie on a straight line, so we reject a linear model. If we draw a line between any two of the points, most or all of the points between those two points lie above the line, so the graph is concave down, suggesting a logarithmic model. We can try $y = a\ln(bx)$. Plugging in the first point, $(1, 0)$, gives $0 = a\ln b$. We reject the case that $a = 0$ (if it were, all outputs would be $0$), so we know $\ln(b) = 0$. Thus $b = 1$ and $y = a\ln(x)$. Next we can use the point $(9, 4.394)$ to solve for $a$:

$$\begin{aligned} y &= a\ln(x) \\ 4.394 &= a\ln(9) \\ a &= \frac{4.394}{\ln(9)} \end{aligned}$$

Because $a = \dfrac{4.394}{\ln(9)} \approx 2$,an appropriate model for the data is $y = 2\ln(x)$.

To check the accuracy of the model, we graph the function together with the given points as in Figure $4.7.9$.

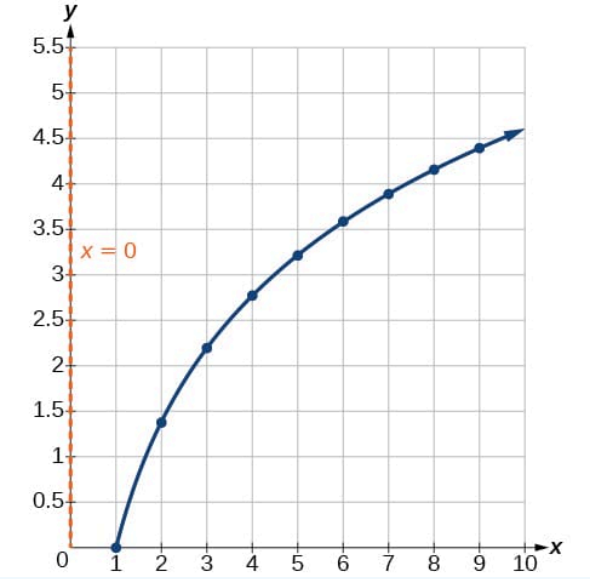



Figure $4.7.9$: The graph of $y = 2\ln x$.

We can conclude that the model is a good fit to the data.

Compare Figure 4.7.9 to the graph of $y = \ln(x^2)$ shown in Figure 4.7.10.

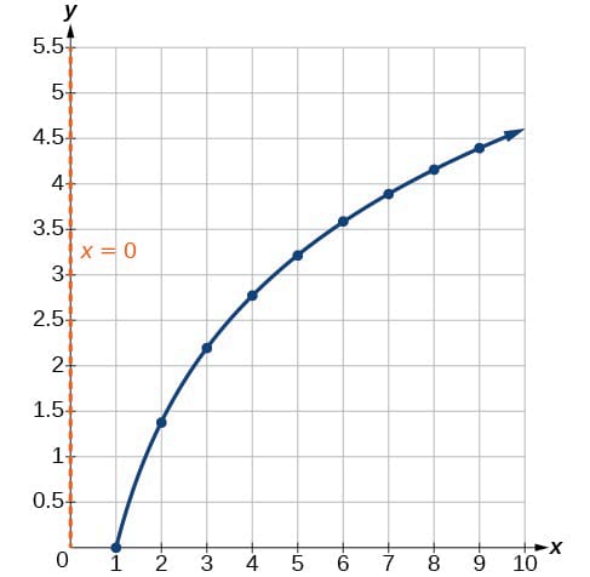



Figure 4.7.10: The graph of $y = \ln(x^2)$

The graphs appear to be identical when $x > 0$. A quick check confirms this conclusion: $y = \ln(x^2) = 2\ln(x)$ for $x > 0$.

However, if $x < 0$, the graph of $y = \ln(x^2)$ includes a "extra" branch, as shown in Figure 4.7.11. This occurs because, while $y = 2\ln(x)$ cannot have negative values in the domain (as such values would force the argument to be negative), the function $y = \ln(x^2)$ can have negative domain values.

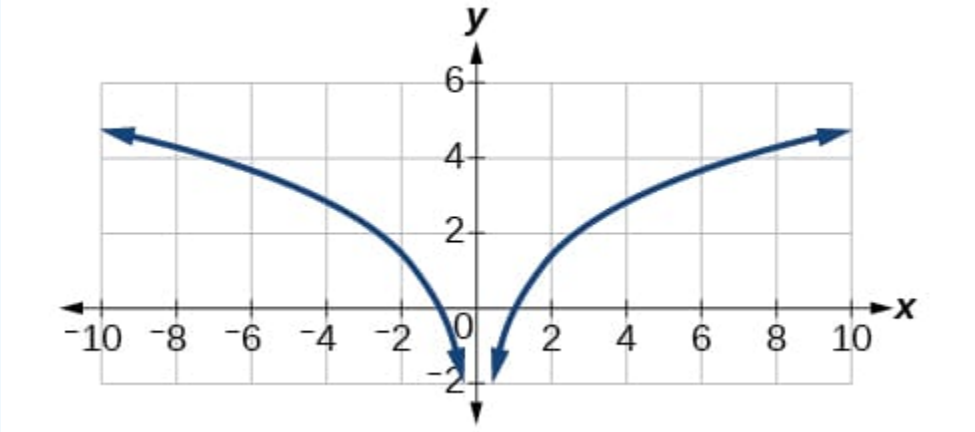



Figure 4.7.11

## Exercise 4.7.6

Does a linear, exponential, or logarithmic model best fit the data in Table 4.7.2? Find the model.

*Table* 4.7.2

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 3.297 | 5.437 | 8.963 | 14.778 | 24.365 | 40.172 | 66.231 | 109.196 | 180.034 |

*Answer available at the end of this section.*

# Expressing an Exponential Model in Base $e$

While powers and logarithms of any base can be used in modeling, the two most common bases are $10$ and $e$. In science and mathematics, the base $e$ is often preferred. We can use laws of exponents and laws of logarithms to change any base to base $e$.

## Given a model with the form $y = ab^x$, change it to the form $y = A_0e^{kx}$.

1. Rewrite $y = ab^x$ as $y = ae^{\ln(b^x)}$.
2. Use the power rule of logarithms to rewrite $y$ as $y = ae^{x\ln(b)} = ae^{\ln(b)^x}$.
3. Note that $a = A_0$ and $k = \ln(b)$ in the equation $y = A_0e^{kx}$.

## Example 4.7.8: Changing to base $e$

Change the function $y = 2.5(3.1)^x$ so that this same function is written in the form $y = A_0e^{kx}$.

**Solution**

The formula is derived as follows

$$\begin{aligned} y &= 2.5(3.1)^x \\ &= 2.5e^{\ln(3.1^x)} && \text{Insert exponential and its inverse} \\ &= 2.5e^{x\ln 3.1} && \text{Laws of logs} \\ &= 2.5e^{(\ln 3.1)x} && \text{Commutative law of multiplication} \end{aligned}$$

## Exercise 4.7.7

Change the function $y = 3(0.5)^x$ to one having $e$ as the base.

*Answer available at the end of this section.*

## Media

Access these online resources for additional instruction and practice with exponential and logarithmic models.

- Logarithm Application – pH
- Exponential Model – Age Using Half-Life
- Newton's Law of Cooling
- Exponential Growth Given Doubling Time
- Exponential Growth – Find Initial Amount Given Doubling Time

# Key Equations

| | |
|---|---|
| Half-life formula | If $A = A_0e^{kt}$, $k < 0$, the half-life is $t = -\dfrac{\ln(2)}{k}$. |
| Carbon-14 dating | $t = \dfrac{\ln(\frac{A}{A_0})}{-0.000121}$.<br>$A_0$ $A$ is the amount of carbon-14 when the plant or animal died<br>$t$ is the amount of carbon-14 remaining today<br>is the age of the fossil in years |

| | |
|---|---|
| Doubling time formula | If $A = A_0e^{kt}$, $k > 0$, the doubling time is $t = \frac{\ln 2}{k}$ |
| Newton's Law of Cooling | $T(t) = Ae^{kt} + T_s$, where $T_s$ is the ambient temperature, $A = T(0) - T_s$, and $k$ is the continuous rate of cooling. |

## Key Concepts

- The basic exponential function is $f(x) = ab^x$. If $b > 1$,we have exponential growth; if $0 < b < 1$, we have exponential decay.
- We can also write this formula in terms of continuous growth as $A = A_0e^{kx}$, where $A_0$ is the starting value. If $A_0$ is positive, then we have exponential growth when $k > 0$ and exponential decay when $k < 0$. See Example $4.7.1$.
- In general, we solve problems involving exponential growth or decay in two steps. First, we set up a model and use the model to find the parameters. Then we use the formula with these parameters to predict growth and decay. See Example $4.7.2$.
- We can find the age, $t$, of an organic artifact by measuring the amount, $k$, of carbon-14 remaining in the artifact and using the formula $t = \frac{\ln(k)}{-0.000121}$ to solve for $t$. See Example $4.7.3$.
- Given a substance's doubling time or half-time, we can find a function that represents its exponential growth or decay. See Example $4.7.4$.
- We can use Newton's Law of Cooling to find how long it will take for a cooling object to reach a desired temperature, or to find what temperature an object will be after a given time. See Example $4.7.5$.
- We can use logistic growth functions to model real-world situations where the rate of growth changes over time, such as population growth, spread of disease, and spread of rumors. See Example $4.7.6$.
- We can use real-world data gathered over time to observe trends. Knowledge of linear, exponential, logarithmic, and logistic graphs help us to develop models that best fit our data. See Example $4.7.7$.
- Any exponential function with the form $y = abx$ can be rewritten as an equivalent exponential function with the form $y = A_0e^{kx}$ where $k = \ln b$. See Example $4.7.8$.

## Answers

**Exercise** $4.7.1$

$f(t) = A_0e^{-0.0000000087t}$

**Exercise** $4.7.2$

less than $230$ years, $229.3157$ to be exact

**Exercise** $4.7.3$

$f(t) = A_0e^{\frac{\ln 2}{3}t}$

**Exercise** $4.7.4$

$6.026$ hours

**Exercise** $4.7.5$

$895$ cases on day $15$

**Exercise** $4.7.6$

Exponential $y = 2e^{0.5x}$.

**Exercise** $4.7.7$

$y = 3e^{(\ln 0.5)x}$

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# 4.8: Fitting Exponential Models to Data

**Learning Objectives**

- Build an exponential model from data.
- Build a logarithmic model from data.
- Build a logistic model from data.

In previous sections of this chapter, we were either given a function explicitly to graph or evaluate, or we were given a set of points that were guaranteed to lie on the curve. Then we used algebra to find the equation that fit the points exactly. In this section, we use a modeling technique called *regression analysis* to find a curve that models data collected from real-world observations. With regression analysis, we don't expect all the points to lie perfectly on the curve. The idea is to find a model that best fits the data. Then we use the model to make predictions about future events.

Do not be confused by the word *model*. In mathematics, we often use the terms *function*, *equation*, and *model* interchangeably, even though they each have their own formal definition. The term *model* is typically used to indicate that the equation or function approximates a real-world situation.

We will concentrate on three types of regression models in this section: exponential, logarithmic, and logistic. Having already worked with each of these functions gives us an advantage. Knowing their formal definitions, the behavior of their graphs, and some of their real-world applications gives us the opportunity to deepen our understanding. As each regression model is presented, key features and definitions of its associated function are included for review. Take a moment to rethink each of these functions, reflect on the work we've done so far, and then explore the ways regression is used to model real-world phenomena.

## Building an Exponential Model from Data

As we've learned, there are a multitude of situations that can be modeled by exponential functions, such as investment growth, radioactive decay, atmospheric pressure changes, and temperatures of a cooling object. What do these phenomena have in common? For one thing, all the models either increase or decrease as time moves forward. But that's not the whole story. It's the *way* data increase or decrease that helps us determine whether it is best modeled by an exponential equation. Knowing the behavior of exponential functions in general allows us to recognize when to use exponential regression, so let's review exponential growth and decay.

Recall that exponential functions have the form $y = ab^x$ or $y = A_0e^{kx}$. When performing regression analysis, we use the form most commonly used on graphing utilities, $y = ab^x$. Take a moment to reflect on the characteristics we've already learned about the exponential function $y = ab^x$ (assume $a > 0$):

- $b$ must be greater than zero and not equal to one.
- The initial value of the model is $y = a$.
  - If $b > 1$, the function models exponential growth. As $x$ increases, the outputs of the model increase slowly at first, but then increase more and more rapidly, without bound.
  - If $0 < b < 1$, the function models **exponential decay**. As $x$ increases, the outputs for the model decrease rapidly at first and then level off to become asymptotic to the *x*-axis. In other words, the outputs never become equal to or less than zero.

As part of the results, your calculator will display a number known as the *correlation coefficient*, labeled by the variable $r$, or $r^2$. (You may have to change the calculator's settings for these to be shown.) The values are an indication of the "goodness of fit" of the regression equation to the data. We more commonly use the value of $r^2$ instead of $r$, but the closer either value is to $1$, the better the regression equation approximates the data.

## EXPONENTIAL REGRESSION

*Exponential regression* is used to model situations in which growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero. We use the command "ExpReg" on a graphing utility to fit an exponential function to a set of data points. This returns an equation of the form,

$$y = ab^x \tag{4.8.1}$$

Note that:

- $b$ must be non-negative.
- when $b > 1$, we have an exponential growth model.
- when $0 < b < 1$, we have an exponential decay model.

## How to: Given a set of data, perform exponential regression using a graphing utility

1. Use the STAT then EDIT menu to enter given data.
   i. Clear any existing data from the lists.
   ii. List the input values in the L1 column.
   iii. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   i. Use ZOOM [9] to adjust axes to fit the data.
   ii. Verify the data follow an exponential pattern.
3. Find the equation that models the data.
   i. Select "ExpReg" from the STAT then CALC menu.
   ii. Use the values returned for *a* and *b* to record the model, $y = ab^x$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

In 2007, a university study was published investigating the crash risk of alcohol impaired driving. Data from $2,871$ crashes were used to measure the association of a person's blood alcohol level (BAC) with the risk of being in an accident. Table $4.8.1$ shows results from the study. The *relative risk* is a measure of how many times more likely a person is to crash. So, for example, a person with a BAC of $0.09$ is $3.54$ times as likely to crash as a person who has not been drinking alcohol.

*Table* $4.8.1$

| **BAC** | 0 | 0.01 | 0.03 | 0.05 | 0.07 | 0.09 |
|---|---|---|---|---|---|---|
| **Relative Risk of Crashing** | 1 | 1.03 | 1.06 | 1.38 | 2.09 | 3.54 |
| **BAC** | 0.11 | 0.13 | 0.15 | 0.17 | 0.19 | 0.21 |
| **Relative Risk of Crashing** | 6.41 | 12.6 | 22.1 | 39.05 | 65.32 | 99.78 |

a. Let $x$ represent the BAC level, and let $y$ represent the corresponding relative risk. Use exponential regression to fit a model to these data.
b. After $6$ drinks, a person weighing $160$ pounds will have a BAC of about $0.16$. How many times more likely is a person with this weight to crash if they drive after having a $6$-pack of beer? Round to the nearest hundredth.

## Solution

a. Using the STAT then EDIT menu on a graphing utility, list the BAC values in L1 and the relative risk values in L2. Then use the STATPLOT feature to verify that the scatterplot follows the exponential pattern shown in Figure $4.8.1$:

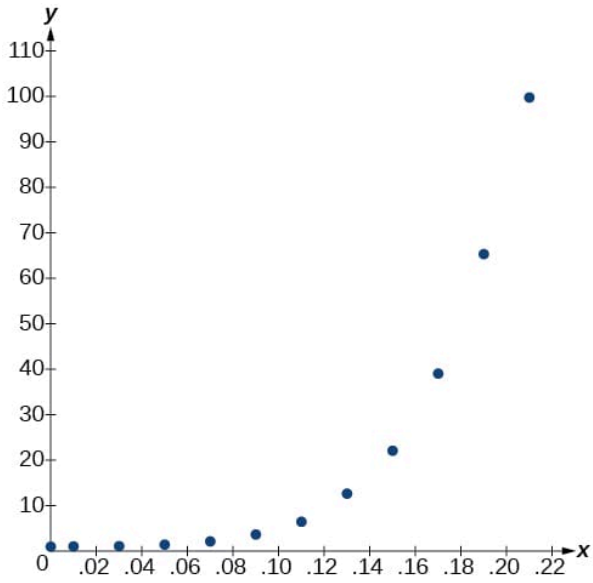



Figure 4.8.1

Use the "ExpReg" command from the STAT then CALC menu to obtain the exponential model,

$$y = 0.58304829(2.20720213E10)^x$$

Converting from scientific notation, we have:

$$y = 0.58304829(22,072,021,300)^x$$

Notice that $r^2 \approx 0.97$ which indicates the model is a good fit to the data. To see this, graph the model in the same window as the scatterplot to verify it is a good fit as shown in Figure 4.8.2:

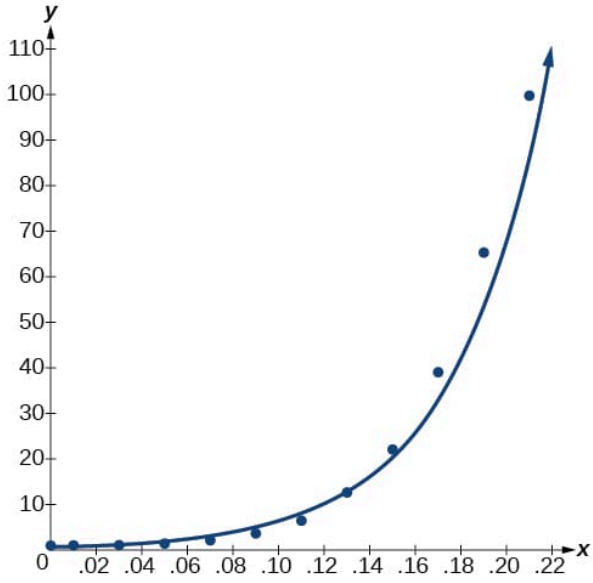



Figure 4.8.2

b. Use the model to estimate the risk associated with a BAC of $0.16$. Substitute $0.16$ for $x$ in the model and solve for $y$.

$$\begin{aligned} y &= 0.58304829(22,072,021,300)^x \quad \text{Use the regression model found in part } (a) \\ &= 0.58304829(22,072,021,300)^{0.16} \quad \text{Substitute 0.16 for x} \\ &\approx 26.35 \quad \text{Round to the nearest hundredth} \end{aligned}$$

If a $160$-pound person drives after having $6$ drinks, he or she is about $26.35$ times more likely to crash than if driving while sober.

### Exercise $4.8.1$

Table $4.8.2$ shows a recent graduate's credit card balance each month after graduation.

*Table* $4.8.2$

| **Month** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **Debt ($)** | 620.00 | 761.88 | 899.80 | 1039.93 | 1270.63 | 1589.04 | 1851.31 | 2154.92 |

a. Use exponential regression to fit a model to these data.
b. If spending continues at this rate, what will the graduate's credit card debt be one year after graduating?

*Answer a available at the end of this section.*

*Answer b available at the end of this section.*

### Q&A: Is it reasonable to assume that an exponential regression model will represent a situation indefinitely?

*No. Remember that models are formed by real-world data gathered for regression. It is usually reasonable to make estimates within the interval of original observation (interpolation). However, when a model is used to make predictions, it is important to use reasoning skills to determine whether the model makes sense for inputs far beyond the original observation interval (extrapolation).*

## Building a Logarithmic Model from Data

Just as with exponential functions, there are many real-world applications for logarithmic functions: intensity of sound, pH levels of solutions, yields of chemical reactions, production of goods, and growth of infants. As with exponential models, data modeled by logarithmic functions are either always increasing or always decreasing as time moves forward. Again, it is the *way* they increase or decrease that helps us determine whether a **logarithmic model** is best.

Recall that logarithmic functions increase or decrease rapidly at first, but then steadily slow as time moves on. By reflecting on the characteristics we've already learned about this function, we can better analyze real world situations that reflect this type of growth or decay. When performing logarithmic **regression analysis**, we use the form of the logarithmic function most commonly used on graphing utilities, $y = a + b\ln(x)$. For this function

- All input values, $x$,must be greater than zero.
- The point $(1, a)$ is on the graph of the model.
- If $b > 0$,the model is increasing. Growth increases rapidly at first and then steadily slows over time.
- If $b < 0$,the model is decreasing. Decay occurs rapidly at first and then steadily slows over time.

### LOGARITHMIC REGRESSION

*Logarithmic regression* is used to model situations where growth or decay accelerates rapidly at first and then slows over time. We use the command "LnReg" on a graphing utility to fit a logarithmic function to a set of data points. This returns an equation of the form,

$$y = a + b\ln(x) \tag{4.8.2}$$

Note that

- all input values, $x$,must be non-negative.
- when $b > 0$, the model is increasing.
- when $b < 0$, the model is decreasing.

## How to: Given a set of data, perform logarithmic regression using a graphing utility

1. Use the STAT then EDIT menu to enter given data.
   i. Clear any existing data from the lists.
   ii. List the input values in the L1 column.
   iii. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   i. Use ZOOM [9] to adjust axes to fit the data.
   ii. Verify the data follow a logarithmic pattern.
3. Find the equation that models the data.
   i. Select "LnReg" from the STAT then CALC menu.
   ii. Use the values returned for *a* and *b* to record the model, $y = a + b\ln(x)$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

## Example 4.8.2: Using Logarithmic Regression to Fit a Model to Data

Due to advances in medicine and higher standards of living, life expectancy has been increasing in most developed countries since the beginning of the 20$^{th}$ century. Table 4.8.3 shows the average life expectancies, in years, of Americans from 1900–2010.

***Table*** 4.8.3

| **Year** | 1900 | 1910 | 1920 | 1930 | 1940 | 1950 |
|---|---|---|---|---|---|---|
| **Life Expectancy(Years)** | 47.3 | 50.0 | 54.1 | 59.7 | 62.9 | 68.2 |
| **Year** | 1960 | 1970 | 1980 | 1990 | 2000 | 2010 |
| **Life Expectancy(Years)** | 69.7 | 70.8 | 73.7 | 75.4 | 76.8 | 78.7 |

a. Let $x$ represent time in decades starting with $x = 1$ for the year 1900, $x = 2$ for the year 1910, and so on. Let $y$ represent the corresponding life expectancy. Use logarithmic regression to fit a model to these data.
b. Use the model to predict the average American life expectancy for the year 2030.

**Solution**

a. Using the STAT then EDIT menu on a graphing utility, list the years using values 1–12 in L1 and the corresponding life expectancy in L2. Then use the STATPLOT feature to verify that the scatterplot follows a logarithmic pattern as shown in Figure 4.8.3:

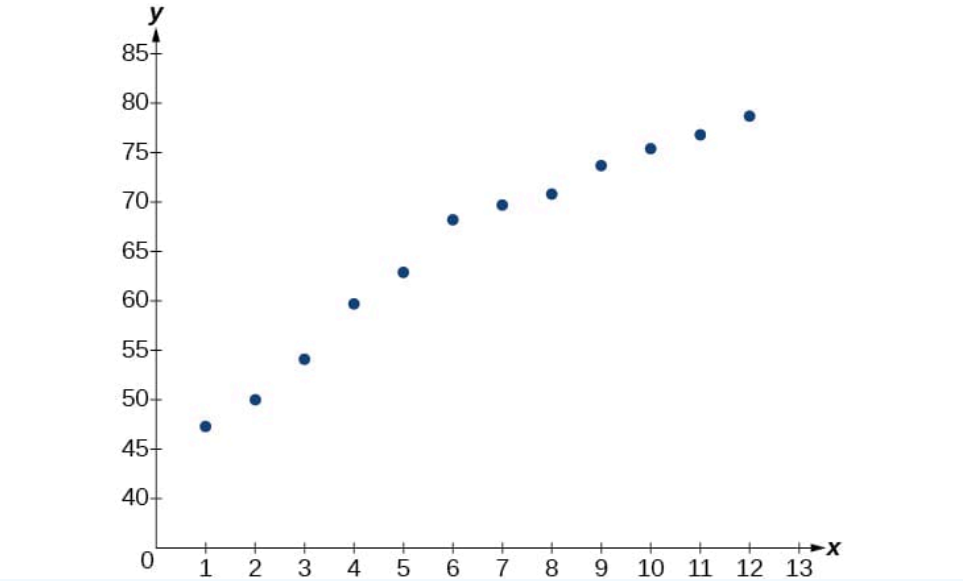



Figure 4.8.3

Use the "LnReg" command from the STAT then CALC menu to obtain the logarithmic model,

$$y = 42.52722583 + 13.85752327 \ln(x)$$

Next, graph the model in the same window as the scatterplot to verify it is a good fit as shown in Figure 4.8.4:

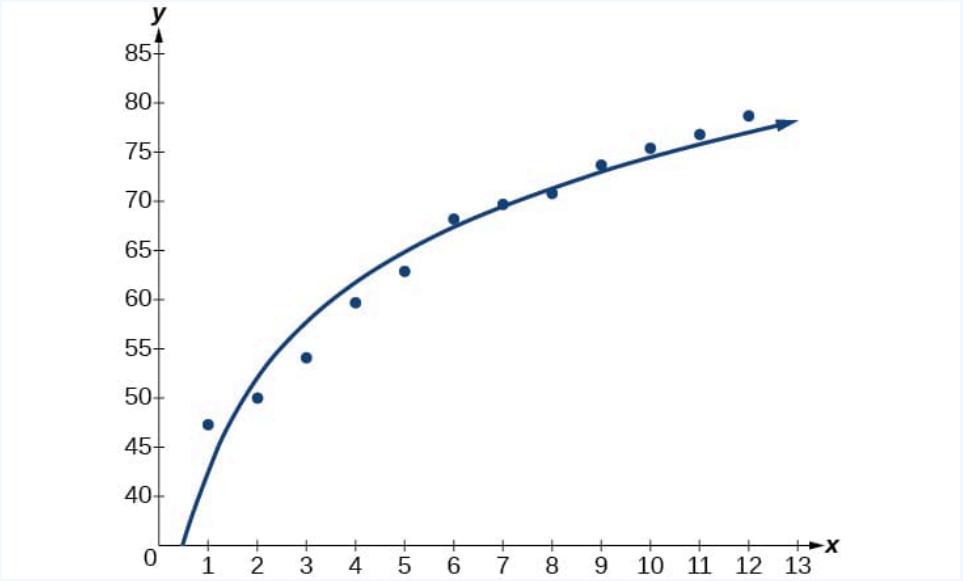



Figure 4.8.4

b. To predict the life expectancy of an American in the year 2030, substitute $x = 14$ for the in the model and solve for $y$:

$$\begin{aligned} y &= 42.52722583 + 13.85752327 \ln(x) && \text{Use the regression model found in part } (a) \\ &= 42.52722583 + 13.85752327 \ln(14) && \text{Substitute 14 for x} \\ &\approx 79.1 && \text{Round to the nearest tenth} \end{aligned}$$

If life expectancy continues to increase at this pace, the average life expectancy of an American will be 79.1 by the year 2030.

**Exercise 4.8.2**

Sales of a video game released in the year 2000 took off at first, but then steadily slowed as time moved on. Table 4.8.4 shows the number of games sold, in thousands, from the years 2000–2010.

*Table* 4.8.4

| **Year** | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
|---|---|---|---|---|---|---|
| **Number Sold (thousands)** | 142 | 149 | 154 | 155 | 159 | 161 |
| **Year** | 2006 | 2007 | 2008 | 2009 | 2010 | - |
| **Number Sold (thousands)** | 163 | 164 | 164 | 166 | 167 | - |

Let $x$ represent time in years starting with $x = 1$ for the year 2000. Let $y$ represent the number of games sold in thousands.

a. Use logarithmic regression to fit a model to these data.
b. If games continue to sell at this rate, how many games will sell in 2015? Round to the nearest thousand.

*Answer a available at the end of this section.*

*Answer b available at the end of this section.*

## Building a Logistic Model from Data

Like exponential and logarithmic growth, logistic growth increases over time. One of the most notable differences with logistic growth models is that, at a certain point, growth steadily slows and the function approaches an upper bound, or *limiting value*. Because of this, logistic regression is best for modeling phenomena where there are limits in expansion, such as availability of living space or nutrients.

It is worth pointing out that logistic functions actually model resource-limited exponential growth. There are many examples of this type of growth in real-world situations, including population growth and spread of disease, rumors, and even stains in fabric. When performing logistic regression analysis, we use the form most commonly used on graphing utilities:

$$y = \frac{c}{1 + ae^{-bx}}$$

Recall that:

- $\frac{c}{1+a}$ is the initial value of the model.
- when $b > 0$, the model increases rapidly at first until it reaches its point of maximum growth rate, $(\frac{\ln(a)}{b}, \frac{c}{2})$. At that point, growth steadily slows and the function becomes asymptotic to the upper bound $y = c$.
- $c$ is the limiting value, sometimes called the *carrying capacity*, of the model.

**LOGISTIC REGRESSION**

*Logistic regression* is used to model situations where growth accelerates rapidly at first and then steadily slows to an upper limit. We use the command “Logistic” on a graphing utility to fit a logistic function to a set of data points. This returns an equation of the form

$$y = \frac{c}{1 + ae^{-bx}} \tag{4.8.3}$$

Note that

- The initial value of the model is $\frac{c}{1+a}$.
- Output values for the model grow closer and closer to $y = c$ as time increases.

## How to: Given a set of data, perform logistic regression using a graphing utility

1. Use the STAT then EDIT menu to enter given data.
   i. Clear any existing data from the lists.
   ii. List the input values in the L1 column.
   iii. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   i. Use ZOOM [9] to adjust axes to fit the data.
   ii. Verify the data follow a logistic pattern.
3. Find the equation that models the data.
   i. Select "Logistic" from the STAT then CALC menu.
   ii. Use the values returned for $a$, $b$, and $c$ to record the model, $y = \frac{c}{1+ae^{-bx}}$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

## Example 4.8.3: Using Logistic Regression to Fit a Model to Data

Mobile telephone service has increased rapidly in America since the mid 1990s. Today, almost all residents have cellular service. Table 4.8.5 shows the percentage of Americans with cellular service between the years 1995 and 2012.

*Table* 4.8.5

| Year | Americans with Cellular Service (%) | Year | Americans with Cellular Service (%) |
|---|---|---|---|
| 1995 | 12.69 | 2004 | 62.852 |
| 1996 | 16.35 | 2005 | 68.63 |
| 1997 | 20.29 | 2006 | 76.64 |
| 1998 | 25.08 | 2007 | 82.47 |
| 1999 | 30.81 | 2008 | 85.68 |
| 2000 | 38.75 | 2009 | 89.14 |
| 2001 | 45.00 | 2010 | 91.86 |
| 2002 | 49.16 | 2011 | 95.28 |
| 2003 | 55.15 | 2012 | 98.17 |

a. Let $x$ represent time in years starting with $x = 0$ for the year 1995. Let $y$ represent the corresponding percentage of residents with cellular service. Use logistic regression to fit a model to these data.
b. Use the model to calculate the percentage of Americans with cell service in the year 2013. Round to the nearest tenth of a percent.
c. Discuss the value returned for the upper limit, $c$. What does this tell you about the model? What would the limiting value be if the model were exact?

**Solution**

a. Using the STAT then EDIT menu on a graphing utility, list the years using values 0–15 in L1 and the corresponding percentage in L2. Then use the STATPLOT feature to verify that the scatterplot follows a logistic pattern as shown in Figure 4.8.5:

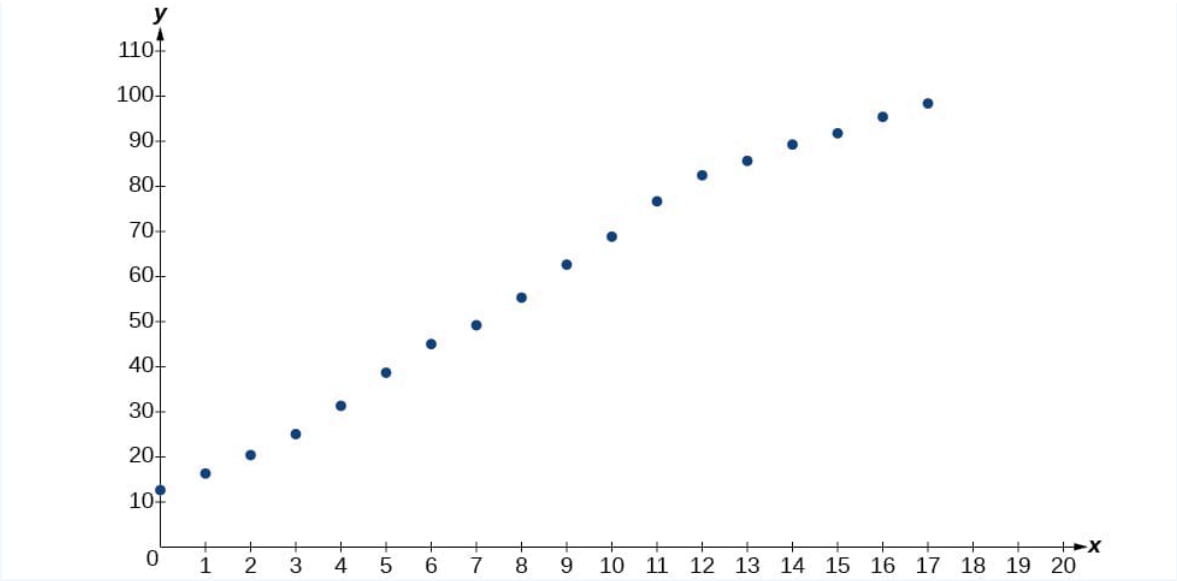



Figure 4.8.5

Use the "Logistic" command from the STAT then CALC menu to obtain the logistic model,

$$y = 105.73795261 + 6.88328979e^{-0.2595440013x} \tag{4.8.4}$$

Next, graph the model in the same window as shown in Figure 4.8.6 the scatterplot to verify it is a good fit:

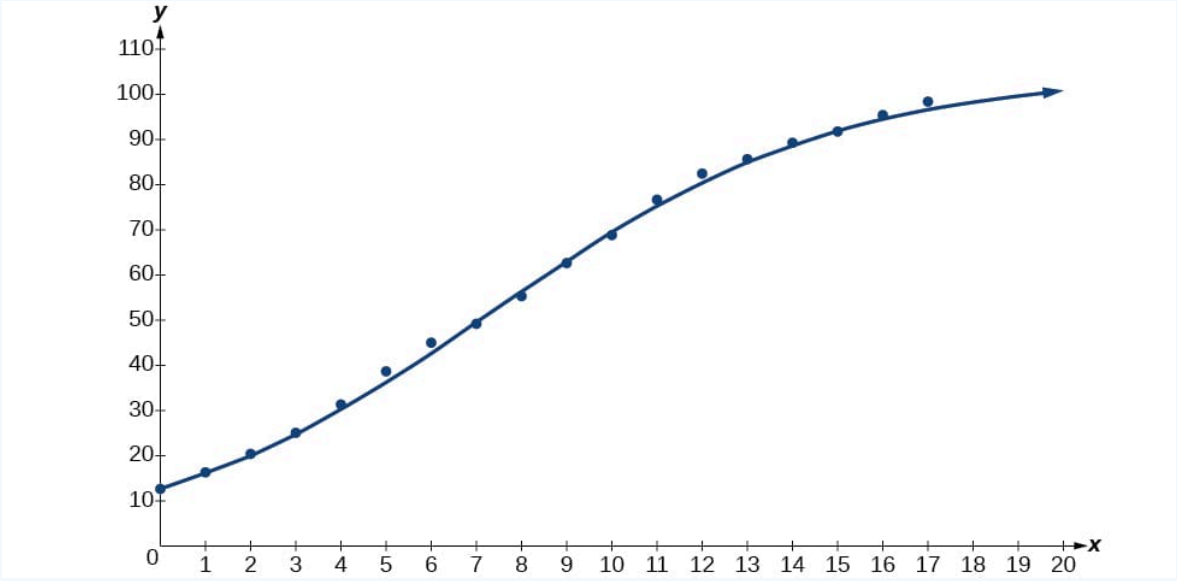



Figure 4.8.6

b. To approximate the percentage of Americans with cellular service in the year 2013, substitute $x = 18$ for the in the model and solve for $y$:

$$\begin{aligned} y &= \frac{105.7379526}{1 + 6.88328979e^{-0.2595440013x}} && \text{Use the regression model found in part } (a) \\ &= \frac{105.7379526}{1 + 6.88328979e^{-0.2595440013(18)}} && \text{Substitute 18 for x} \\ &\approx 99.3 && \text{Round to the nearest tenth} \end{aligned}$$

According to the model, about 98.8% of Americans had cellular service in 2013.

c. The model gives a limiting value of about $105$. This means that the maximum possible percentage of Americans with cellular service would be $105$, which is impossible. (How could over $100$ of a population have cellular service?) If the model were exact, the limiting value would be $c = 100$ and the model's outputs would get very close to, but never actually reach $100$. After all, there will always be someone out there without cellular service!

## Exercise $4.8.3$

Table $4.8.6$ shows the population, in thousands, of harbor seals in the Wadden Sea over the years 1997 to 2012.

*Table* $4.8.6$

| Year | Seal Population (Thousands) | Year | Seal Population (Thousands) |
|---|---|---|---|
| 1997 | 3.493 | 2005 | 19.590 |
| 1998 | 5.282 | 2006 | 21.955 |
| 1999 | 6.357 | 2007 | 22.862 |
| 2000 | 9.201 | 2008 | 23.869 |
| 2001 | 11.224 | 2009 | 24.243 |
| 2002 | 12.964 | 2010 | 24.344 |
| 2003 | 16.226 | 2011 | 24.919 |
| 2004 | 18.137 | 2012 | 25.108 |

Let $x$ represent time in years starting with $x = 0$ for the year 1997. Let $y$ represent the number of seals in thousands.

a. Use logistic regression to fit a model to these data.
b. Use the model to predict the seal population for the year 2020.
c. To the nearest whole number, what is the limiting value of this model?

*Answer a available at the end of this section.*

*Answer b available at the end of this section.*

*Answer c available at the end of this section.*

## Media

Access this online resource for additional instruction and practice with exponential function models.

- Exponential Regression on a Calculator

Visit this website for additional practice questions from Learningpod.

# Key Concepts

- Exponential regression is used to model situations where growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero.
- We use the command "ExpReg" on a graphing utility to fit function of the form $y = ab^x$ to a set of data points. See Example $4.8.1$.
- Logarithmic regression is used to model situations where growth or decay accelerates rapidly at first and then slows over time.
- We use the command "LnReg" on a graphing utility to fit a function of the form $y = a + b\ln(x)$ to a set of data points. See

Example $4.8.2$.

- Logistic regression is used to model situations where growth accelerates rapidly at first and then steadily slows as the function approaches an upper limit.
- We use the command “Logistic” on a graphing utility to fit a function of the form $y = \dfrac{c}{1 + ae^{-bx}}$ to a set of data points. See Example $4.8.3$.

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## Answers

**Exercise** $4.8.1$

The exponential regression model that fits these data is $y = 522.88585984(1.19645256)^x$.

**Exercise** $4.8.1$

If spending continues at this rate, the graduate’s credit card debt will be $\$4,499.38$ after one year.

**Exercise** $4.8.2$

The logarithmic regression model that fits these data is $y = 141.91242949 + 10.45366573\ln(x)$

**Exercise** $4.8.2$

If sales continue at this rate, about $171,000$ games will be sold in the year $2015$.

**Exercise** $4.8.3$

The logistic regression model that fits these data is $y = \dfrac{25.65665979}{1 + 6.113686306e^{-0.3852149008x}}$.

**Exercise** $4.8.3$

If the population continues to grow at this rate, there will be about $25,634$ seals in $2020$.

**Exercise** $4.8.3$

To the nearest whole number, the carrying capacity is $25,657$.


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# 4.E: Exponential and Logarithmic Functions (Exercises)

## 4.1: Exponential Functions

When populations grow rapidly, we often say that the growth is “exponential,” meaning that something is growing very rapidly. To a mathematician, however, the term exponential growth has a very specific meaning. In this section, we will take a look at exponential functions, which model this kind of rapid growth.

### Verbal

1) Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

**Answer:**

Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

2) Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

3)The Oxford Dictionary defines the word nominal as a value that is “stated or expressed but not necessarily corresponding exactly to the real value.” Develop a reasonable argument for why the term nominal rate is used to describe the annual percentage rate of an investment account that compounds interest.

**Answer:**

When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of nominal.

### Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

4) The average annual population increase of a pack of wolves is $25$.

5) A population of bacteria decreases by a factor of $\frac{1}{8}$ every $24$ hours.

**Answer:**

exponential; the population decreases by a proportional rate.

6) The value of a coin collection has increased by $3.25\%$ annually over the last $20$ years.

7) For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.

**Answer:**

not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function.

8) The height of a projectile at time $t$ is represented by the function $h(t) = -4.9t^2 + 18t + 40$

**For the following exercises, consider this scenario: For each year $t$, the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)**

9) Which forest's population is growing at a faster rate?

**Answer:**

The forest represented by the function $B(t) = 82(1.029)^t$.

10) Which forest had a greater number of trees initially? By how many?

11) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after $20$ years? By how many?

**Answer:**

After $t = 20$ years, forest A will have $43$ more trees than forest B.

12) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after $100$ years? By how many?

13) Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

**Answer:**

Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

**For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.**

14) $y = 300(1 - t)^5$

15) $y = 220(1.06)^x$

**Answer:**

exponential growth; The growth factor, $1.06$ is greater than $1$.

16) $y = 16.5(1.025)^{\frac{1}{x}}$

17) $y = 11,701(0.97)^t$

**Answer:**

exponential decay; The decay factor, $0.97$, is between $0$ and $1$.

For the following exercises, find the formula for an exponential function that passes through the two points given.

18) $(0, 6)$ and $(3, 750)$

19) $(0, 2000)$ and $(2, 20)$

**Answer:**

$f(x) = 2000(0.1)^x$

20) $\left(-1, \frac{3}{2}\right)$ and $(3, 24)$

21) $(-2, 6)$ and $(3, 1)$

**Answer:**

$$f(x) = \left(\frac{1}{6}\right)^{-\frac{3}{5}} \left(\frac{1}{6}\right)^{\frac{x}{5}} \approx 2.93(0.699)^x$$

22) $(3, 1)$ and $(5, 4)$

**For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.**

23)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 70 | 40 | 10 | -20 |

**Answer:**

Linear

24)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $h(x)$ | 70 | 49 | 34.3 | 24.01 |

25)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $m(x)$ | 80 | 61 | 42.9 | 25.61 |

**Answer:**

Neither

26)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 10 | 20 | 40 | 80 |

27)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $g(x)$ | -3.25 | 2 | 7.25 | 12.5 |

**Answer:**

Linear

For the following exercises, use the compound interest formula, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

28) After a certain number of years, the value of an investment account is represented by the equation $10,250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the value of the account?

29) What was the initial deposit made to the account in the previous exercise?

**Answer:**

$\$10,250$

30) How many years had the account from the previous exercise been accumulating interest?

31) An account is opened with an initial deposit of $\$6,500$ and earns $3.6\%$ interest compounded semi-annually. What will the account be worth in $20$ years?

**Answer:**

$\$13,268.58$

32) How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

33) Solve the compound interest formula for the principal, $P$.

**Answer:**

$P = A(t) \cdot (1 + \frac{r}{n})^{-nt}$

34) Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning $5.5\%$ interest compounded monthly for $5$ years. (Round to the nearest dollar.)

35) How much more would the account in the previous two exercises be worth if it were earning interest for $5$ more years?

**Answer:**

$\$4,572.56$

36) Use properties of rational exponents to solve the compound interest formula for the interest rate, $r$.

37) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $\$9,000$ and was worth $\$13,373.53$ after $10$ years.

**Answer:**

$4\%$

38) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $\$5,500$, and was worth $\$38,455$ after $30$ years.

**For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.**

39) $y = 3742(e)^{0.75t}$

**Answer:**

continuous growth; the growth rate is greater than $0$.

40) $y = 150(e)^{\frac{3.25}{t}}$

41) $y = 2.25(e)^{-2t}$

**Answer:**

continuous decay; the growth rate is less than $0$.

42) Suppose an investment account is opened with an initial deposit of $\$12,000$ earning $7.2\%$ interest compounded continuously. How much will the account be worth after $30$ years?

43) How much less would the account from Exercise 42 be worth after $30$ years if it were compounded monthly instead?

**Answer:**

$\$669.42$

## Numeric

**For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.**

44) $f(x) = 2(5)^x$ for $f(-3)$

45) $f(x) = -4^{2x+3}$ for $f(-1)$

**Answer:**

$f(-1) = -4$

46) $f(x) = e^x$, for $f(3)$

47) $f(x) = -2e^{x-1}$, for $f(-1)$

**Answer:**

$f(-1) \approx -0.2707$

48) $f(x) = 2.7(4)^{-x+1} + 1.5$, for $f(-2)$

49) $f(x) = 1.2e^{2x} - 0.3$, for $f(3)$

**Answer:**

$f(3) \approx 483.8146$

50) $f(x) = -\frac{3}{2}(3)^{-x} + \frac{3}{2}$, for $f(2)$

## Technology

**For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.**

51) $(0, 3)$ and $(3, 375)$

**Answer:**

$y = 3 \cdot 5^x$

52) $(3, 222.62)$ and $(10, 77.456)$

53) $(20, 29.495)$ and $(150, 730.89)$

**Answer:**

$y \approx 18 \cdot 1.025^x$

54) $(5, 2.909)$ and $(13, 0.005)$

55) ((11,310.035)\) and $(25, 356.3652)$

**Answer:**

$$y \approx 0.2 \cdot 1.95^x$$

## Extensions

56) The annual percentage yield (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $APY = \left(1 + \frac{r}{12}\right)^{12} - 1$

57) Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds $n$ times per year.

**Answer:**

$$\begin{aligned} APY &= \frac{A(t) - a}{a} \\ &= \frac{a\left(1 + \frac{r}{365}\right)^{365(1)} - a}{a} \\ &= \frac{a\left[\left(1 + \frac{r}{365}\right)^{365} - 1\right]}{a} \\ &= \left(1 + \frac{r}{365}\right)^{365} - 1 \end{aligned} \; ; I(n) = \left(1 + \tfrac{r}{n}\right)^n - 1$$

58) Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that $a$ and $b$ are positive numbers and $b \neq 1$. Any positive number $b$ can be written as $b = e^n$ for some value of $n$. Use this fact to rewrite the formula for an exponential function that uses the number $e$ as a base.

59) In an exponential decay function, the base of the exponent is a value between $0$ and $1$. Thus, for some number $b > 1$, the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f(x) = a \cdot (e)^{-nx}$ for some positive number $n$.

**Answer:**

Let $f$ be the exponential decay function $f(x) = a \cdot \left(\frac{1}{b}\right)^x$ such that $b > 1$. Then for some number $n > 0$,

$$\begin{aligned} f(x) &= a \cdot \left(\frac{1}{b}\right)^x \\ &= a\left(b^{-1}\right)^x \\ &= a\left((e^n)^{-1}\right)^x \\ &= a\left(e^{-n}\right)^x \\ &= a(e)^{-nx} \end{aligned}$$

60) The formula for the amount $A$ in an investment account with a nominal interest rate $r$ at any time $t$ is given by $A(t) = a(e)^{rt}$, where $a$ is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time $t$ can be calculated with the formula $I(t) = e^{rt} - 1$

## Real-World Applications

61) The fox population in a certain region has an annual growth rate of $9\%$ per year. In the year 2012, there were $23,900$ fox counted in the area. What is the fox population predicted to be in the year 2020?

**Answer:**

$47,622$ fox

62) A scientist begins with $100$ milligrams of a radioactive substance that decays exponentially. After $35$ hours, $50$mg of the substance remains. How many milligrams will remain after $54$ hours?

63) In the year 1985, a house was valued at $\$110,000$. By the year 2005, the value had appreciated to $\$145,000$. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

**Answer:**

$1.39\%$; $\$155,368.09$

64) A car was valued at $\$38,000$ in the year 2007. By 2013, the value had depreciated to $\$11,000$ If the car's value continues to drop by the same percentage, what will it be worth by 2017?

65) Jamal wants to save $\$54,000$ for a down payment on a home. How much will he need to invest in an account with $8.2\%$ APR, compounding daily, in order to reach his goal in $5$ years?

**Answer:**

$\$35,838.76$

66) Kyoko has $\$10,000$ that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $\$15,000$ by the time she finishes graduate school in $6$ years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (Hint: solve the compound interest formula for the interest rate.)

67) Alyssa opened a retirement account with $7.25\%$ APR in the year 2000. Her initial deposit was $\$13,500$. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

**Answer:**

$\$82,247.78$; $\$449.75$

68) An investment account with an annual interest rate of $7\%$ was opened with an initial deposit of $\$4,000$ Compare the values of the account after $9$ years when the interest is compounded annually, quarterly, monthly, and continuously.

# 4.2: Graphs of Exponential Functions

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool.

## Verbal

1) What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?

**Answer:**

An asymptote is a line that the graph of a function approaches, as $x$ either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

2) What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

## Algebraic

3) The graph of $f(x) = 3^x$ is reflected about the $y$-axis and stretched vertically by a factor of $4$. What is the equation of the new function, $g(x)$. State its $y$-intercept, domain, and range.

**Answer:**

$g(x) = 4(3)^{-x}$; y-intercept: $(0, 4)$; Domain: all real numbers; Range: all real numbers greater than $0$.

4) The graph of $f(x) = \left(\frac{1}{2}\right)^{-x}$ is reflected about the $y$-axis and compressed vertically by a factor of $\left(\frac{1}{5}\right)$. What is the equation of the new function, $g(x)$? State its $y$-intercept, domain, and range.

5) The graph of $f(x) = 10^x$ is reflected about the $x$-axis and shifted upward $7$ units. What is the equation of the new function, $g(x)$ ? State its $y$-intercept, domain, and range.

**Answer:**

$g(x) = -10^x + 7$; $y$-intercept: $(0, 6)$; Domain: all real numbers; Range: all real numbers less than $7$.

6) The graph of $f(x) = (1.68)^x$ is shifted right $3$ units, stretched vertically by a factor of $2$,reflected about the $x$-axis, and then shifted downward $3$ units. What is the equation of the new function, $g(x)$? State its $y$-intercept (to the nearest thousandth), domain, and range.

7) The graph of $f(x) = 2\left(\frac{1}{4}\right)^{x-20}$ is shifted left $2$ units, stretched vertically by a factor of $4$,reflected about the $x$-axis, and then shifted downward $4$ units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

**Answer:**

$g(x) = 2\left(\frac{1}{4}\right)^{x}$; $y$-intercept: $(0, 2)$; Domain: all real numbers; Range: all real numbers greater than $0$.

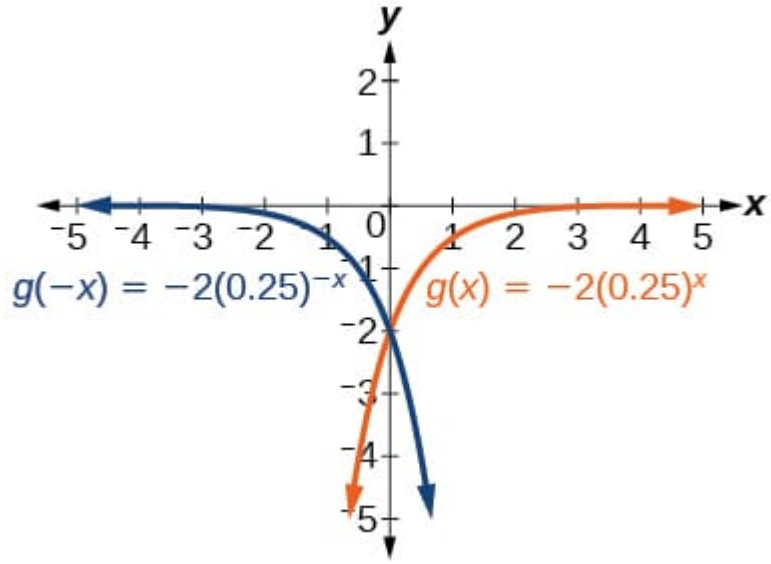
## Graphical

**For the following exercises, graph the function and its reflection about the $y$-axis on the same axes, and give the $y$ -intercept.**

8) $f(x) = 3\left(\frac{1}{2}\right)^{x}$

9) $g(x) = -2(0.25)^x$

**Answer:**



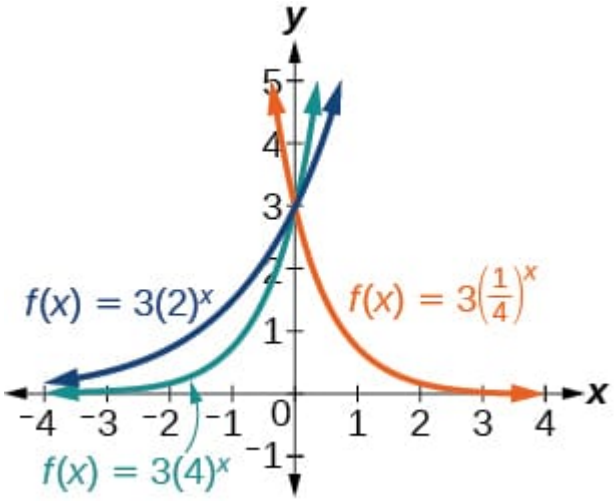

$y$-intercept: $(0, -2)$

10) $h(x) = 6(1.75)^{-x}$

**For the following exercises, graph each set of functions on the same axes.**

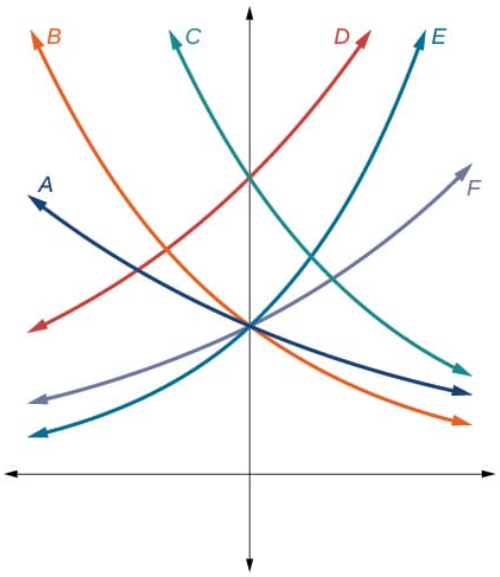
11) $f(x) = 3\left(\frac{1}{4}\right)^x, g(x) = 3(2)^x, h(x) = 3(4)^x$

**Answer:**




12) $f(x) = \frac{1}{4}(3)^x, g(x) = 2(3)^x, h(x) = 4(3)^x$

**For the following exercises, match each function with one of the graphs in Figure below.**




13) $f(x) = 2(0.69)^x$

**Answer:**

B

14) $f(x) = 2(1.28)^x$

15) $f(x) = 2(0.81)^x$

**Answer:**

A

16) $f(x) = 4(1.28)^x$

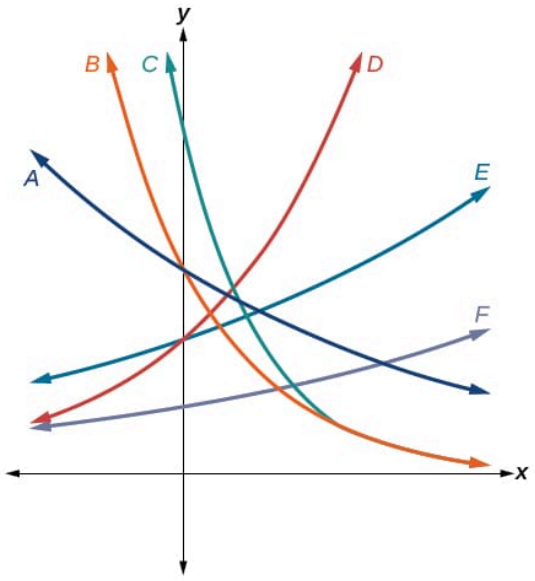
17) $f(x) = 2(1.59)^x$

**Answer:**

E

18) $f(x) = 4(0.69)^x$

**For the following exercises, use the graphs shown in Figure below. All have the form $f(x) = ab^x$.**




19) Which graph has the largest value for $b$?

**Answer:**

D

20) Which graph has the smallest value for $b$?

21) Which graph has the largest value for $a$?

**Answer:**

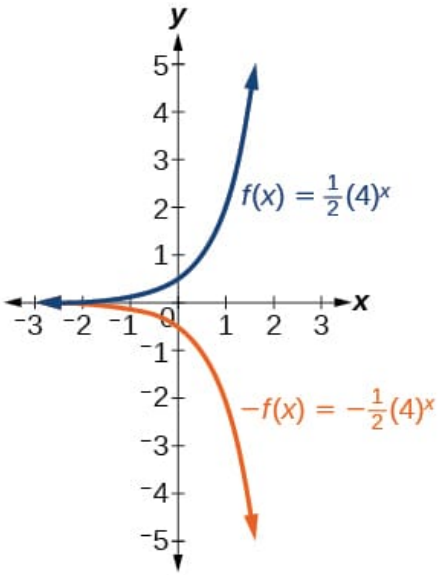
C

22) Which graph has the smallest value for $a$?

**For the following exercises, graph the function and its reflection about the $x$-axis on the same axes.**

23) $f(x) = \frac{1}{2}(4)^x$

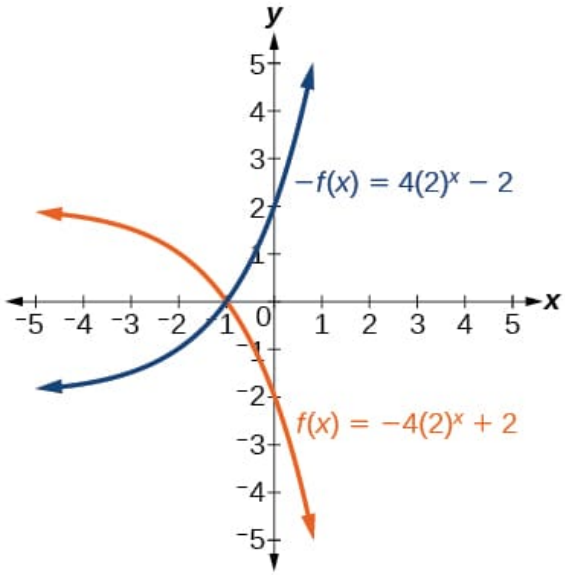
**Answer:**




24) $f(x) = 3(0.75)^x - 1$

25) $f(x) = -4(2)^x + 2$

**Answer:**

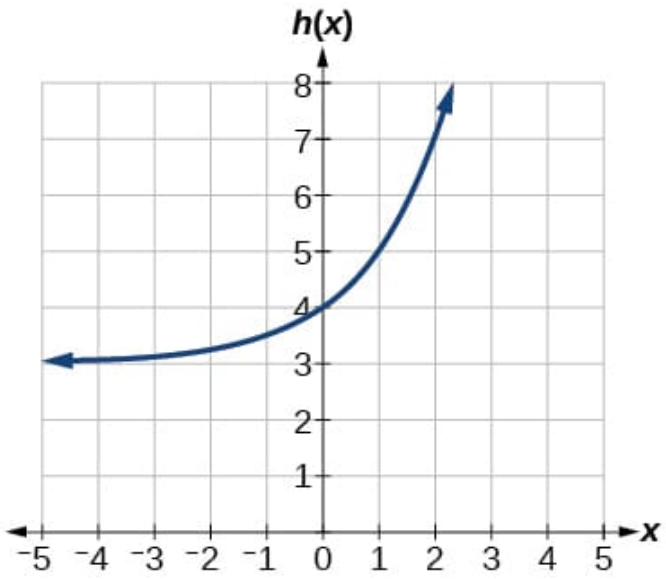



**For the following exercises, graph the transformation of $f(x) = 2^x$. Give the horizontal asymptote, the domain, and the range.**

26) $f(x) = 2^{-x}$

27) $h(x) = 2^x + 3$

**Answer:**




Horizontal asymptote: $h(x) = 3$; Domain: all real numbers; Range: all real numbers strictly greater than $3$.

28) $f(x) = 2^{x-2}$

**For the following exercises, describe the end behavior of the graphs of the functions.**

29) $f(x) = -5(4)^x - 1$

**Answer:**

As $x \to \infty, f(x) \to -\infty$

As $x \to -\infty, f(x) \to -1$

30) $f(x) = 3\left(\frac{1}{2}\right)^x - 2$

31) $f(x) = 3(4)^{-x} + 2$

**Answer:**

As $x \to \infty, f(x) \to 2$

As $x \to -\infty, f(x) \to \infty$

**For the following exercises, start with the graph of $f(x) = 4^x$. Then write a function that results from the given transformation.**

32) Shift $f(x)$ $4$ units upward

33) Shift $f(x)$ $3$ units downward

**Answer:**

$f(x) = 4^x - 3$

34) Shift $f(x)$ $2$ units left

35) Shift $f(x)$ $5$ units right

**Answer:**

$$f(x) = 4^{x-5}$$

36) Reflect $f(x)$ about the $x$-axis

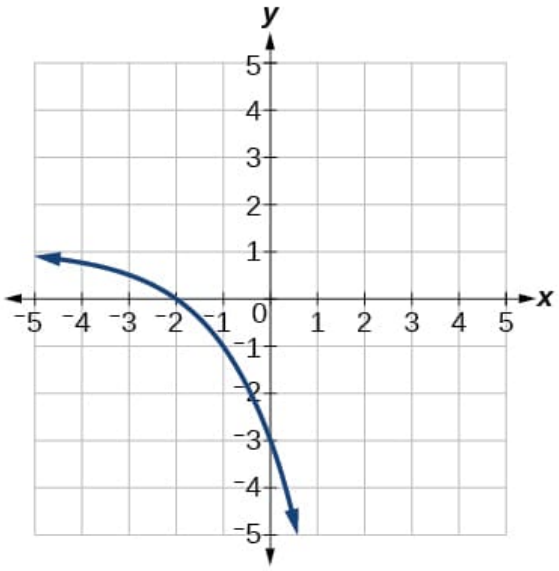
37) Reflect $f(x)$ about the $y$-axis

**Answer:**

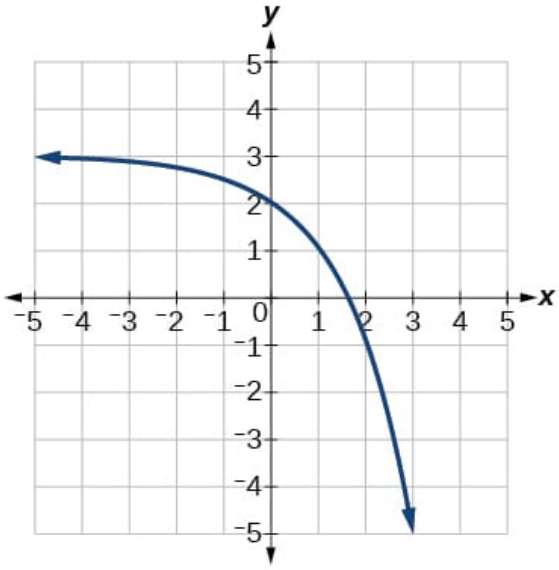
$$f(x) = 4^{-x}$$

**For the following exercises, each graph is a transformation of $f(x) = 2^x$. Write an equation describing the transformation.**

38)



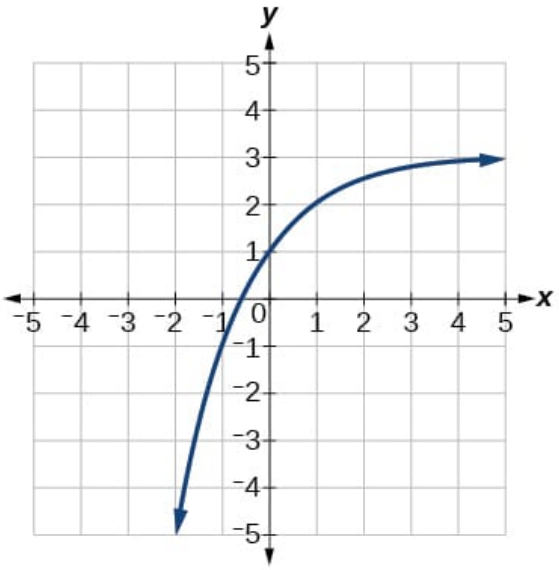

39)



**Answer:**

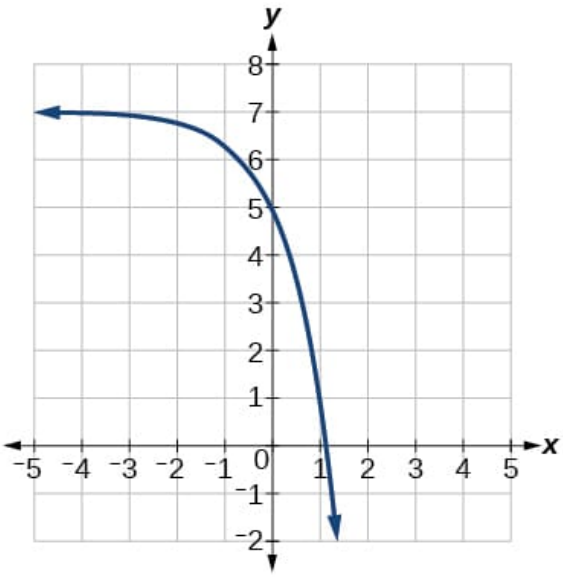
$$y = -2^x + 3$$

40)




**For the following exercises, find an exponential equation for the graph.**

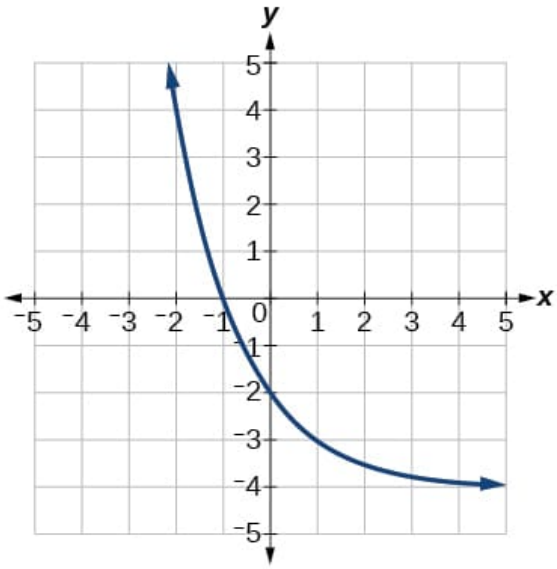
41)




**Answer:**

$$y = -2(3)^x + 7$$

42)




## Numeric

**For the following exercises, evaluate the exponential functions for the indicated value of $x$.**

43) $g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.

**Answer:**

$$g(6) = 800 + \frac{1}{3} \approx 800.3333$$

44) $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.

45) $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^{x} + 6$ for $h(-7)$.

**Answer:**

$$h(-7) = -58$$

## Technology

**For the following exercises, use a graphing calculator to approximate the solutions of the equation. Round to the nearest thousandth.**

46) $-50 = \left(\frac{1}{2}\right)^{-x}$

47) $116 = \left(\frac{1}{4}\right)\left(\frac{1}{8}\right)^{x}$

**Answer:**

$$x \approx -2.953$$

48) $12 = 2(3)^{x} + 1$

49) $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$

**Answer:**

$$x \approx -0.222$$

50) $-30 = -4(2)^{x+2} + 2$

## Extensions

51) Explore and discuss the graphs of $F(x) = (b)^x$ and $G(x) = \left(\frac{1}{b}\right)^x$. Then make a conjecture about the relationship between the graphs of the functions $b^x$ and $\left(\frac{1}{b}\right)^x$ for any real number $b > 0$.

**Answer:**

The graph of $G(x) = \left(\frac{1}{b}\right)^x$ is the refelction about the $y$-axis of the graph of $F(x) = (b)^x$; For any real number $b > 0$ and function $f(x) = (b)^x$, the graph of $\left(\frac{1}{b}\right)^x$ is the the reflection about the $y$-axis, $F(-x)$.

52) Prove the conjecture made in the previous exercise.

53) Explore and discuss the graphs of $f(x) = 4^x$, $g(x) = 4^{x-2}$, and $h(x) = \left(\frac{1}{16}\right)4^x$. Then make a conjecture about the relationship between the graphs of the functions $b^x$ and $\left(\frac{1}{b^n}\right)b^x$ for any real number $n$ and real number $b > 0$.

**Answer:**

The graphs of $g(x)$ and $h(x)$ are the same and are a horizontal shift to the right of the graph of $f(x)$; For any real number $n$, real number $b > 0$, and function $f(x) = b^x$, the graph of $\left(\frac{1}{b^n}\right)b^x$ is the horizontal shift $f(x - n)$.

54) Prove the conjecture made in the previous exercise.

# 4.3: Logarithmic Functions

The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.

## Verbal

1) What is a base $b$ logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1$.

**Answer**

A logarithm is an exponent. Specifically, it is the exponent to which a base $b$ is raised to produce a given value. In the expressions given, the base $b$ has the same value. The exponent, $y$, in the expression $b^y$ can also be written as the logarithm, $\log_b x = y$,and the value of $x$ is the result of raising $b$ to the power of $y$.

2) How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?

3) How can the logarithmic equation $\log_b x = y$ be solved for $x$ using the properties of exponents?

**Answer**

Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$, and then properties of exponents can be applied to solve for $x$.

4) Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base $b$, and how does the notation differ?

5) Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base $b$, and how does the notation differ?

**Answer**

The natural logarithm is a special case of the logarithm with base $b$ in that the natural log always has base $e$. Rather than notating the natural logarithm as $\log_e(x)$,the notation used is $\ln(x)$.

## Algebraic

**For the following exercises, rewrite each equation in exponential form.**

6) $\log_4(q) = m$

7) $\log_a(b) = c$

**Answer**

$$a^c = b$$

8) $\log_{16}(y) = x$

9) $\log_x(64) = y$

**Answer**

$$x^y = 64$$

10) $\log_y(x) = -11$

11) $\log_{15}(a) = b$

**Answer**

$$15^b = a$$

12) $\log_y(137) = x$

13) $\log_{13}(142) = a$

**Answer**

$$13^a = 142$$

14) $\log(v) = t$

15) $\ln(w) = n$

**Answer**

$$e^n = w$$

**For the following exercises, rewrite each equation in logarithmic form.**

16) $4^x = y$

17) $c^d = k$

**Answer**

$$\log_c(k) = d$$

18) $m^{-7} = n$

19) $19^x = y$

**Answer**

$$\log_{19}(y) = x$$

20) $x^{-\frac{10}{13}} = y$

21) $n^4 = 103$

**Answer**

$$\log_n(103) = 4$$

22) $\left(\frac{7}{5}\right)^m = n$

23) $y^x = \frac{39}{100}$

**Answer**

$$\log_y\left(\frac{39}{100}\right) = x$$

24) $10^a = b$

25) $e^k = h$

**Answer**

$$\ln(w) = n$$

**For the following exercises, solve for $x$ by converting the logarithmic equation to exponential form.**

26) $\log_3(x) = 2$

27) $\log_2(x) = -3$

**Answer**

$$x = 2^{-3} = \frac{1}{8}$$

28) $\log_5(x) = 2$

29) $\log_3(x) = 3$

**Answer**

$$x = 3^3 = 27$$

30) $\log_2(x) = 6$

31) $\log_9(x) = \frac{1}{2}$

**Answer**

$$x = 9^{\frac{1}{2}} = 3$$

32) $\log_{18}(x) = 2$

33) $\log_6(x) = -3$

**Answer**

$$x = 6^{-3} = \frac{1}{216}$$

34) $\log(x) = 3$

35) $\ln(x) = 2$

**Answer**

$$x = e^2$$

**For the following exercises, use the definition of common and natural logarithms to simplify.**

36) $\log(100^8)$

37) $10^{\log(32)}$

**Answer**

$$32$$

38) $2\log(.0001)$

39) $e^{\ln(1.06)}$

**Answer**

$$1.06$$

40) $\ln(e^{-5.03})$

41) $e^{\ln(10.125)} + 4$

**Answer**

$$14.125$$

## Numeric

**For the following exercises, evaluate the base $b$ logarithmic expression without using a calculator.**

42) $\log_3\left(\frac{1}{27}\right)$

43) $\log_6(\sqrt{6})$

**Answer**

$$\frac{1}{2}$$

44) $\log_2\left(\frac{1}{8}\right) + 4$

45) $6\log_8(4)$

**Answer**

$$4$$

**For the following exercises, evaluate the common logarithmic expression without using a calculator.**

46) $\log(10,000)$

47) $\log(0.001)$

**Answer**

$-3$

48) $\log(1)+7$

49) $2\log(100^{-3})$

**Answer**

$-12$

**For the following exercises, evaluate the natural logarithmic expression without using a calculator.**

50) $\ln\left(e^{\frac{1}{3}}\right)$

51) $\ln(1)$

**Answer**

$0$

52) $\ln\left(e^{-0.225}\right)-3$

53) $25\ln\left(e^{\frac{2}{5}}\right)$

**Answer**

$10$

## Technology

**For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.**

54) $\log(0.04)$

55) $\ln(15)$

**Answer**

$2.708$

56) $\ln\left(\frac{4}{5}\right)$

57) $\log(\sqrt{2})$

**Answer**

$0.151$

58) $\ln(\sqrt{2})$

## Extensions

59) Is $x = 0$ in the domain of the function $f(x) = \log x$? If so, what is the value of the function when $x = 0$? Verify the result.

**Answer**

No, the function has no defined value for $x = 0$. To verify, suppose $x = 0$ is in the domain of the function $f(x) = \log(x)$. Then there is some number $n$ such that $n = \log(0)$. Rewriting as an exponential equation gives: $10^n = 0$, which is impossible since no such real number $n$ exists. Therefore, $x = 0$ is not the domain of the function $f(x) = \log(x)$.

60) Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of $x$? Verify the result.

61) Is there a number $x$ such that $\ln x = 2$? If so, what is that number? Verify the result.

**Answer**

Yes. Suppose there exists a real number $x$ such that $\ln x = 2$. Rewriting as an exponential equation gives $x = e^2$, which is a real number. To verify, let $x = e^2$. Then, by definition, $\ln(x) = \ln\left(e^2\right) = 2$.

62) Is the following true: $\frac{\log_3(27)}{\log_4(\frac{1}{64})} = -1$ log3(27)log4(164)=–1? log3(27)log4(164)=–1? Verify the result.

63) Is the following true: $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$ Verify the result.

**Answer**

No; $\ln(1) = 0$, so $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$ is undefined.

## Real-World Applications

64) The exposure index $EI$ for a $35$ millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where $f$ is the "f-stop" setting on the camera, and $t$ is the exposure time in seconds. Suppose the f-stop setting is $8$ and the desired exposure time is $2$ seconds. What will the resulting exposure index be?

65) Refer to the previous exercise. Suppose the light meter on a camera indicates an $EI$ of $-2$, and the desired exposure time is $16$ seconds. What should the f-stop setting be?

**Answer**

$2$

66) The intensity levels $I$ of two earthquakes measured on a seismograph can be compared by the formula $\log\left(\frac{I_1}{I_2}\right) = M_1 - M_2$ where $M$ is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude $6.1$ hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of $9.0$. How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

# 4.4: Graphs of Logarithmic Functions

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

## Verbal

1) The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

**Answer**

Since the functions are inverses, their graphs are mirror images about the line $y - x$. So for every point $(a, b)$ on the graph of a logarithmic function, there is a corresponding point $(b, a)$ on the graph of its inverse exponential function.

2) What type(s) of translation(s), if any, affect the range of a logarithmic function?

3) What type(s) of translation(s), if any, affect the domain of a logarithmic function?

**Answer**

Shifting the function right or left and reflecting the function about the $y$-axis will affect its domain.

4) Consider the general logarithmic function $f(x) = \log_b(x)$. Why can't $x$ be zero?

5) Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

**Answer**

No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

## Algebraic

**For the following exercises, state the domain and range of the function.**

6) $f(x) = \log_3(x + 4)$

7) $h(x) = \ln\left(\frac{1}{2} - x\right)$

**Answer**

Domain: $\left(-\infty, \frac{1}{2}\right)$; Range: $(-\infty, \infty)$

8) $g(x) = \log_5(2x + 9) - 2$

9) $h(x) = \ln(4x + 17) - 5$

**Answer**

Domain: $\left(-\frac{17}{4}, \infty\right)$; Range: $(-\infty, \infty)$

10) $f(x) = \log_2(12 - 3x) - 3$

**For the following exercises, state the domain and the vertical asymptote of the function.**

11) $f(x) = \log_b(x - 5)$

**Answer**

Domain: $(5, \infty)$; Vertical asymptote: $x = 5$

12) $g(x) = \ln(3 - x)$

13) $f(x) = \log(3x + 1)$

**Answer**

Domain: $\left(-\frac{1}{3}, \infty\right)$; Vertical asymptote: $x = -\frac{1}{3}$

14) $f(x) = 3\log(-x) + 2$

15) $g(x) = -\ln(3x + 9) - 7$

**Answer**

Domain: $(-3, \infty)$; Vertical asymptote: $x = -3$

**For the following exercises, state the domain, vertical asymptote, and end behavior of the function.**

16) $f(x) = \ln(2 - x)$

17) $f(x) = \log\left(x - \frac{3}{7}\right)$

**Answer**

Domain: $\left(\frac{3}{7}, \infty\right)$

Vertical asymptote: $x = \frac{3}{7}$

End behavior: as $x \to \left(\frac{3}{7}\right)^+$, $f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$

18) $h(x) = -\log(3x - 4) + 3$

19) $g(x) = \ln(2x + 6) - 5$

**Answer**

Domain: $(-3, \infty)$

Vertical asymptote: $x = -3$

End behavior: as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$

20) $f(x) = \log_3(15 - 5x) + 6$

**For the following exercises, state the domain, range, and x- and y-intercepts, if they exist. If they do not exist, write DNE.**

21) $h(x) = \log_4(x - 1) + 1$

**Answer**

Domain: $(1, \infty)$

Range: $-\infty, \infty$

Vertical asymptote: $x = 1$

$x$-intercept: $\left(\frac{5}{4}, 0\right)$

$y$-intercept: DNE

22) $f(x) = \log(5x + 10) + 3$

23) $g(x) = \ln(-x) - 2$

**Answer**

Domain: $(-\infty, 0)$

Range: $-\infty, \infty$

Vertical asymptote: $x = 0$

$x$-intercept: $(-e^2, 0)$

$y$-intercept: DNE

24) $f(x) = \log_2(x + 2) - 5$

25) $h(x) = 3\ln(x) - 9$

**Answer**

Domain: $(0, \infty)$

Range: $-\infty, \infty$

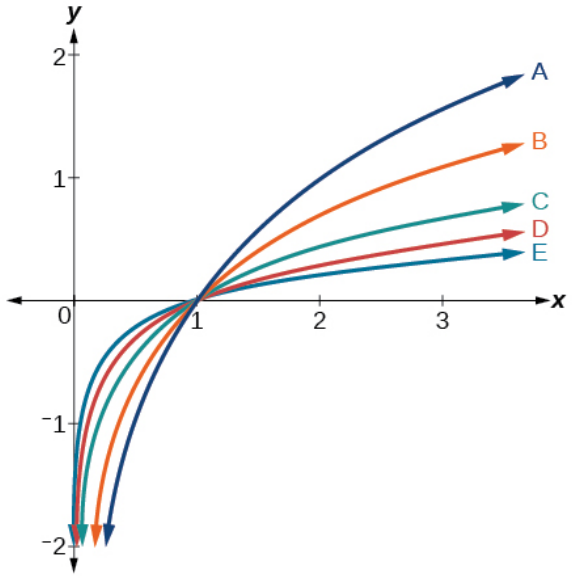
Vertical asymptote: $x = 0$

$x$-intercept: $(e^3, 0)$

$y$-intercept: DNE

## Graphical

**For the following exercises, match each function in Figure below with the letter corresponding to its graph.**




26) $d(x) = \log(x)$

27) $f(x) = \ln(x)$

**Answer**

$B$

28) $g(x) = \log_2(x)$

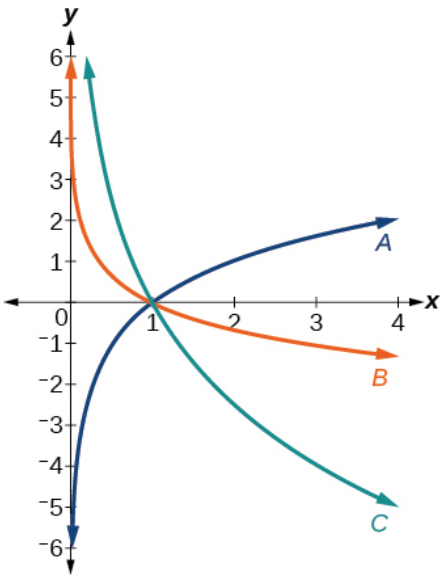
29) $h(x) = \log_5(x)$

**Answer**

$C$

30) $j(x) = \log_{25}(x)$

**For the following exercises, match each function in Figure with the letter corresponding to its graph.**




31) $f(x) = \log_{\frac{1}{3}}(x)$

**Answer**

$B$

32) $g(x) = \log_2(x)$

33) $h(x) = \log_{\frac{3}{4}}(x)$

**Answer**

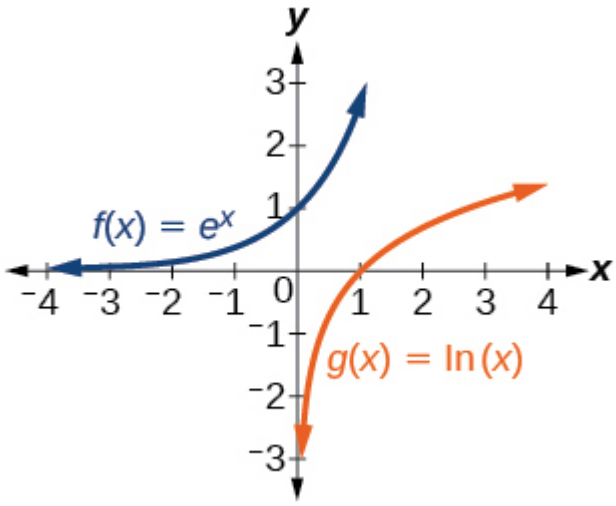
$C$

**For the following exercises, sketch the graphs of each pair of functions on the same axis.**

34) $f(x) = \log(x)$ and $g(x) = 10^x$

35) $f(x) = e^x$ and $g(x) = \ln(x)$

**Answer**




**For the following exercises, match each function in Figure with the letter corresponding to its graph.**

36) $f(x) = \log_4(-x + 2)$

37) $g(x) = -\log_4(x + 2)$

**Answer**

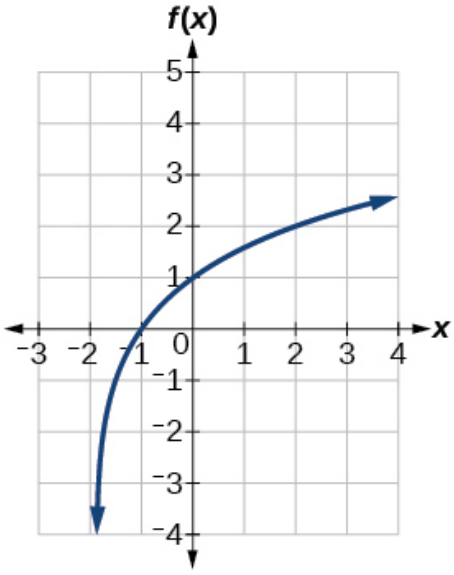
$C$

38) $h(x) = \log_4(x + 2)$

**For the following exercises, sketch the graph of the indicated function.**

39) $f(x) = \log_2(x + 2)$

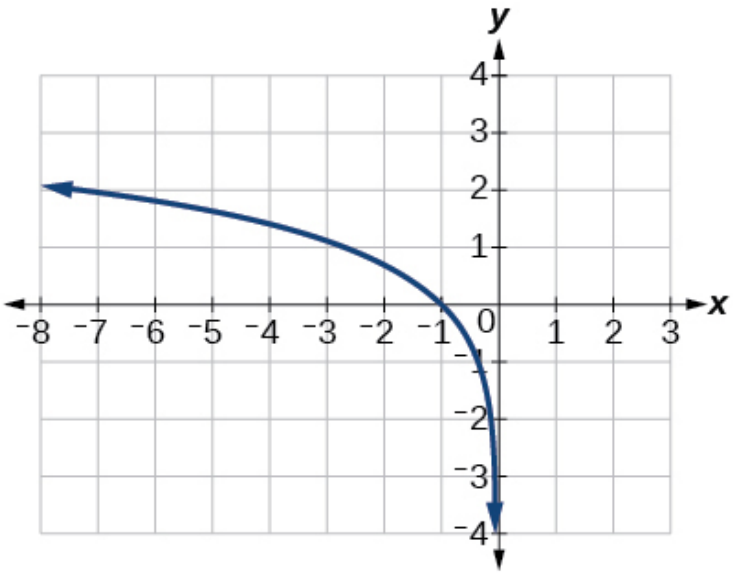
**Answer**




40) $f(x) = 2\log(x)$

41) $f(x) = \ln(-x)$

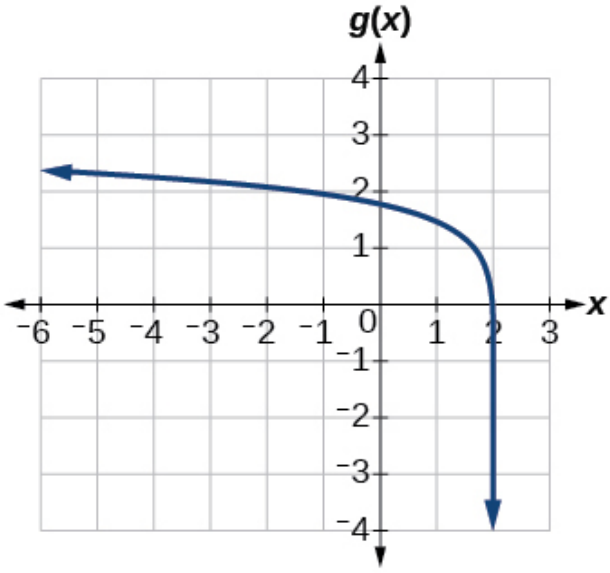
**Answer**




42) $g(x) = \log(4x + 16) + 4$

43) $g(x) = \log(6 - 3x) + 1$

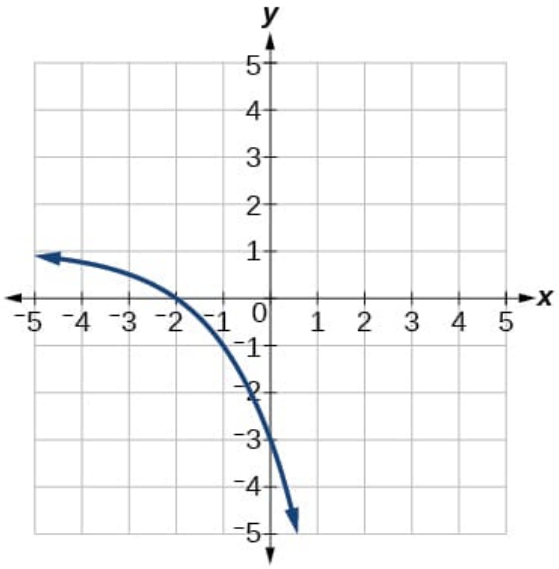
**Answer**




44) $h(x) = -\frac{1}{2}\log(x + 1) - 3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

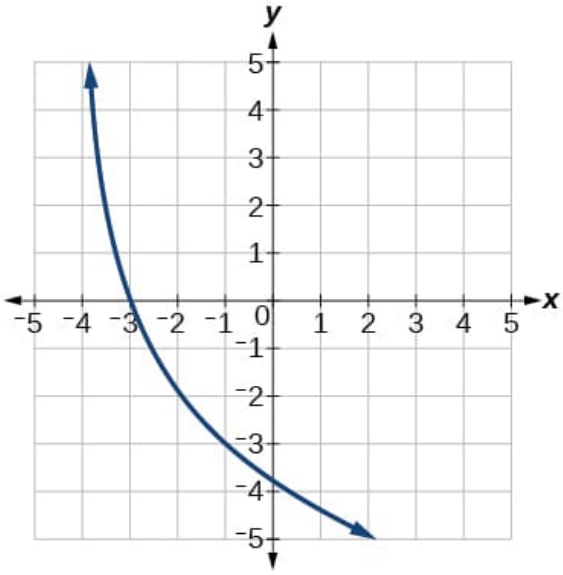
45) Use $y = \log_2(x)$ as the parent function.



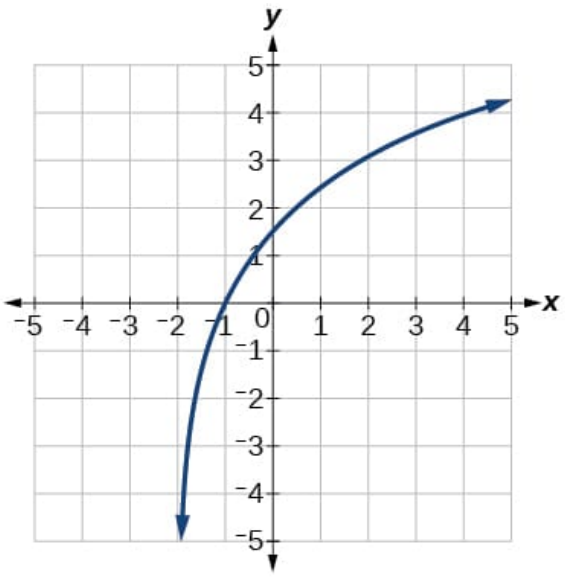

**Answer**

$$f(x) = \log_2(-(x-1))$$

46) Use $f(x) = \log_3(x)$ as the parent function.



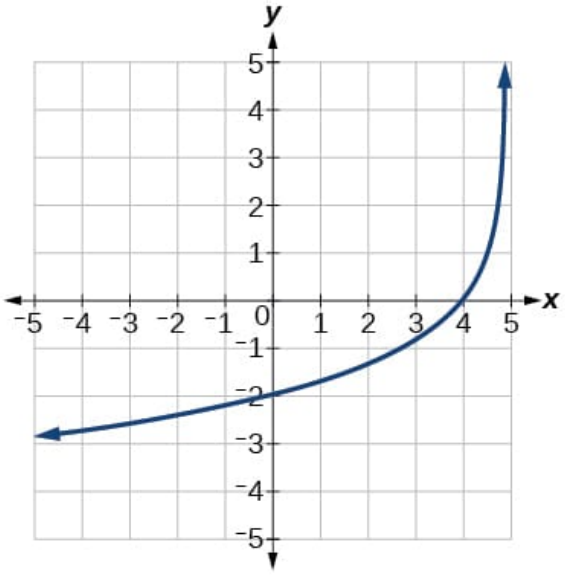

47) Use $f(x) = \log_4(x)$ as the parent function.




**Answer**

$$f(x) = 3\log_4(x+2)$$

48) Use $f(x) = \log_5(x)$ as the parent function.




## Technology

**For the following exercises, use a graphing calculator to find approximate solutions to each equation.**

49) $\log(x-1)+2 = \ln(x-1)+2$

**Answer**

$$x = 2$$

50) $\log(2x-3)+2 = -\log(2x-3)+5$

51) $\ln(x-2)+2=-\ln(x+1)$

**Answer**

$x \approx 2.303$

52) $2\ln(5x+1)=\frac{1}{2}\ln(-5x)+1$

53) $\frac{1}{3}\log(1-x)=\log(x+1)+\frac{1}{3}$

**Answer**

$x \approx -0.472$

## Extensions

54) Let $b$ be any positive real number such that $b \neq 1$. What must $\log_b 1$be equal to? Verify the result.

55) Explore and discuss the graphs of $f(x)=\log_{\frac{1}{2}}(x)$ and $g(x)=-\log_2(x)$. Make a conjecture based on the result.

**Answer**

The graphs of $f(x)=\log_{\frac{1}{2}}(x)$ and $g(x)=-\log_2(x)$ appear to be the same;

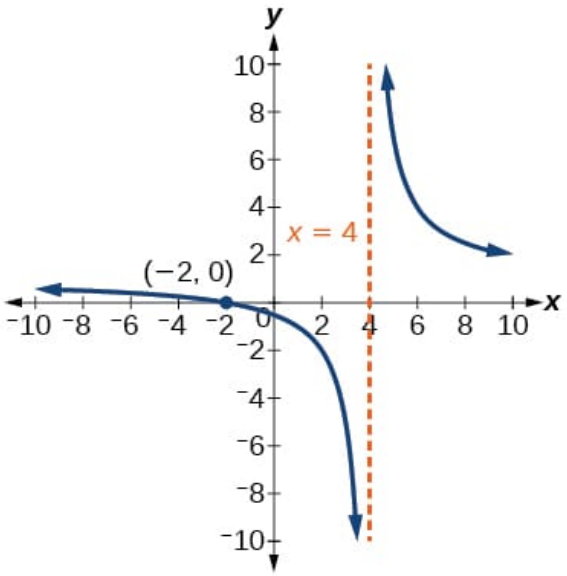
Conjecture: for any positive base $b \neq 1$, $\log_b(x)=\log_{\frac{1}{b}}(x)$

56) Prove the conjecture made in the previous exercise.

57) What is the domain of the function $f(x)=\ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.

**Answer**

Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4}>0$. From the graph of the function $f(x)=\frac{x+2}{x-4}$, note that the graph lies above the $x$-axis on the interval $(-\infty,-2)$ and again to the right of the vertical asymptote, that is $(4,\infty)$. Therefore, the domain is $(-\infty,-2)\cup(4,\infty)$.




58) Use properties of exponents to find the $x$-intercepts of the function $f(x)=\log\left(x^2+4x+4\right)$ algebraically. Show the steps for

solving, and then verify the result by graphing the function.

# 4.5: Logarithmic Properties

Recall that the logarithmic and exponential functions “undo” each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here.

## Verbal

1) How does the power rule for logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?

**Answer**

Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_b(x)$.

2) What does the change-of-base formula do? Why is it useful when using a calculator?

## Algebraic

**For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.**

3) $\log_b(7x \cdot 2y)$

**Answer**

$\log_b(2) + \log_b(7) + \log_b(x) + \log_b(y)$

4) $\ln(3ab \cdot 5c)$

5) $\log_b\left(\frac{13}{17}\right)$

**Answer**

$\log_b(13) - \log_b(17)$

6) $\log_4\left(\frac{\frac{x}{z}}{w}\right)$

7) $\ln\left(\frac{1}{4^k}\right)$

**Answer**

$-k\ln(4)$

8) $\log_2(y^x)$

**For the following exercises, condense to a single logarithm if possible.**

9) $\ln(7) + \ln(x) + \ln(y)$

**Answer**

$\ln(7xy)$

10) $\log_3(2) + \log_3(a) + \log_3(11) + \log_3(b)$

11) $\log_b(28) - \log_b(7)$

**Answer**

$\log_b(4)$

12) $\ln(a) - \ln(d) - \ln(c)$

13) $-\log_b\left(\frac{1}{7}\right)$

**Answer**

$\log_b(7)$

14) $\frac{1}{3}\ln(8)$

**For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.**

15) $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

**Answer**

$15\log(x) + 13\log(y) - 19\log(z)$

16) $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$

17) $\log\left(\sqrt{x^3y^{-4}}\right)$

**Answer**

$\frac{3}{2}\log(x) - 2\log(y)$

18) $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

19) $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$

**Answer**

$\frac{8}{3}\log(x) + \frac{14}{3}\log(y)$

**For the following exercises, condense each expression to a single logarithm using the properties of logarithms.**

20) $\log\left(2x^4\right) + \log\left(3x^5\right)$

21) $\ln\left(6x^9\right) - \ln\left(3x^2\right)$

**Answer**

$\ln\left(2x^7\right)$

22) $2\log(x) + 3\log(x+1)$

23) $\log(x) - \frac{1}{2}\log(y) + 3\log(z)$

**Answer**

$$\log\left(\frac{xz^3}{\sqrt{y}}\right)$$

24) $4\log_7(c)+\frac{\log_7(a)}{3}+\frac{\log_7(b)}{3}$

**For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.**

25) $\log_7(15)$ to base $e$

**Answer**

$$\log_7(15)=\frac{\ln(15)}{\ln(7)}$$

**For the following exercises, suppose $\log_5(6)=a$ and $\log_5(11)=b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of $a$ and $b$. Show the steps for solving.**

27) $\log_{11}(5)$

**Answer**

$$\log_{11}(5)=\frac{\log_5(5)}{\log_5(11)}=\frac{1}{b}$$

28) $\log_6(55)$

29) $\log_{11}\left(\frac{6}{11}\right)$

**Answer**

$$\log_{11}\left(\frac{6}{11}\right)=\frac{\log_{11}\left(\frac{6}{11}\right)}{\log_5(11)}=\frac{\log_5(6)-\log_5(11)}{\log_5(11)}=\frac{a-b}{b}=\frac{a}{b}-1$$

## Numeric

**For the following exercises, use properties of logarithms to evaluate without using a calculator.**

30) $\log_3\left(\frac{1}{9}\right)-3\log_3(3)$

31) $6\log_8(2)+\frac{\log_8(64)}{3\log_8(4)}$

**Answer**

$$3$$

32) $2\log_9(3)-4\log_9(3)+\log_9\left(\frac{1}{729}\right)$

**For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.**

33) $\log_3(22)$

**Answer**

$$2.81359$$

34) $\log_8(65)$

35) $\log_6(5.38)$

**Answer**

$0.93913$

36) $\log_4\left(\frac{15}{2}\right)$

37) $\log_{\frac{1}{2}}(4.7)$

**Answer**

$-2.23266$

## Extensions

38) Use the product rule for logarithms to find all $x$ values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2$. Show the steps for solving.

39) Use the quotient rule for logarithms to find all $x$ values such that $\log_6(x+2)-\log_6(x-3)=1$. Show the steps for solving.

**Answer**

Rewriting as an exponential equation and solving for $x$:

$$\begin{aligned} 6^1 &= \frac{x+2}{x-3} \\ 0 &= \frac{x+2}{x-3}-6 \\ 0 &= \frac{x+2}{x-3}-\frac{6(x-3)}{(x-3)} \\ 0 &= \frac{x+2-6x+18}{x-3} \\ 0 &= \frac{x-4}{x-3} \\ x &= 4 \end{aligned}$$

Checking, we find that $\log_6(4+2)-\log_6(4-3)=\log_6(6)-\log_6(1)$ is defined, so $x=4$

40) Can the power property of logarithms be derived from the power property of exponents using the equation $b^x=m$? If not, explain why. If so, show the derivation.

41) Prove that $\log_b(n)=\frac{1}{\log_b(n)}$ for any positive integers $b>1$ and $n>1$.

**Answer**

Let $b$ and $n$ be positive integers greater than $1$. Then, by the change-of-base formula, $\log_b(n)=\frac{\log_n(n)}{\log_n(b)}=\frac{1}{\log_n(b)}$

42) Does $\log_{81}(2401)=\log_3(7)$? Verify the claim algebraically.

# 4.6: Exponential and Logarithmic Equations

Uncontrolled population growth can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

## Verbal

1) How can an exponential equation be solved?

**Answer**

Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

2) When does an extraneous solution occur? How can an extraneous solution be recognized?

3) When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

**Answer**

The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

## Algebraic

**For the following exercises, use like bases to solve the exponential equation.**

4) $4^{-3v-2} = 4^{-v}$

5) $64 \cdot 4^{3x} = 16$

**Answer**

$$x = -\frac{1}{3}$$

6) $3^{2x+1} \cdot 3^{x} = 243$

7) $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$

**Answer**

$$n = -1$$

8) $625 \cdot 5^{3x+3} = 125$

9) $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$

**Answer**

$$b = \frac{6}{5}$$

10) $\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^{6}$

**For the following exercises, use logarithms to solve.**

11) $9^{x-10} = 1$

**Answer**

$$x = 10$$

12) $2e^{6x} = 13$

13) $e^{r+10} - 10 = -42$

**Answer**

No solution

14) $2 \cdot 10^{9a} = 29$

15) $-8 \cdot 10^{p+7} - 7 = -24$

**Answer**

$$p = \log\left(\frac{17}{8}\right) - 7$$

16) $7e^{3n-5} + 5 = -89$

17) $e^{-3k} + 6 = 44$

**Answer**

$$k = -\frac{\ln(38)}{3}$$

18) $-5e^{9x-8} - 8 = -62$

19) $-6e^{9x+8} + 2 = -74$

**Answer**

$$x = \frac{\frac{\ln(38)}{3} - 8}{9}$$

20) $2^{x+1} = 5^{2x-1}$

21) $e^{2x} - e^{x} - 132 = 0$

**Answer**

$$x = \ln 12$$

22) $7e^{8x+8} - 5 = -95$

23) $10e^{8x+3} + 2 = 8$

**Answer**

$$x = \frac{\frac{\ln(3)}{5} - 3}{8}$$

24) $4e^{3x+3} - 7 = 53$

25) $8e^{-5x-2} - 4 = -90$

**Answer**

No solution

26) $3^{2x+1} = 7^{x-2}$

27) $e^{2x} - e^{x} - 6 = 0$

**Answer**

$$x = \ln 3$$

28) $3e^{3-3x} + 6 = -31$

**For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.**

29) $\log\left(\frac{1}{100}\right) = -2$

**Answer**

$$10^{-2} = \frac{1}{100}$$

30) $\log_{324}(18) = \frac{1}{2}$

**For the following exercises, use the definition of a logarithm to solve the equation.**

31) $5\log_7 n = 10$

**Answer**

$$n = 49$$

32) $-8\log_9 x = 16$

33) $4 + \log_2(9k) = 2$

**Answer**

$$k = \frac{1}{36}$$

34) $2\log(8n + 4) + 6 = 10$

35) $10 - 4\ln(9 - 8x) = 6$

**Answer**

$$x = \frac{9 - e}{8}$$

**For the following exercises, use the one-to-one property of logarithms to solve.**

36) $\ln(10 - 3x) = \ln(-4x)$

37) $\log_{13}(5n - 2) = \log_{13}(8 - 5n)$

**Answer**

$$n = 1$$

38) $\log(x + 3) - \log(x) = \log(74)$

39) $\ln(-3x) = \ln(x^2 - 6x)$

**Answer**

No solution

40) $\log_4(6-m)=\log_4(3m)$

41) $\ln(x-2)-\ln(x)=\ln(54)$

**Answer**

No solution

42) $\log_9(2n^2-14n)=\log_9(-45+n^2)$

43) $\ln(x^2-10)+\ln(9)=\ln(10)$

**Answer**

$x=\pm\frac{10}{3}$

For the following exercises, solve each equation for $x$.

44) $\log(x+12)=\log(x)+\log(12)$

45) $\ln(x)+\ln(x-3)=\ln(7x)$

**Answer**

$x=10$

46) $\log_2(7x+6)=3$

47) $\ln(7)+\ln(2-4x^2)=\ln(14)$

**Answer**

$x=0$

48) $\log_8(x+6)-\log_8(x)=\log_8(58)$

49) $\ln(3)-\ln(3-3x)=\ln(4)$

**Answer**

$x=\dfrac{3}{4}$

50) $\log_3(3x)-\log_3(6)=\log_3(77)$

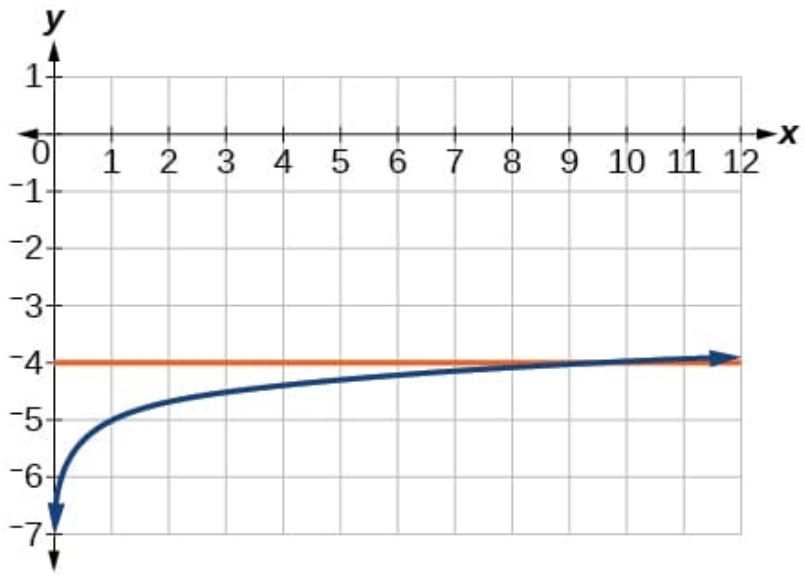
## Graphical

**For the following exercises, solve the equation for $x$, if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.**

51) $\log_9(x)-5=-4$

**Answer**

$x=9$

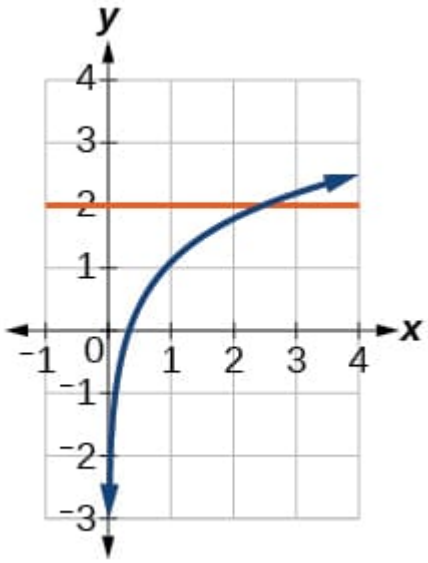



52) $\log_3(x) + 3 = 2$

53) $\ln(3x) = 2$

**Answer**

$$x = \frac{e^2}{3} \approx 2.5$$

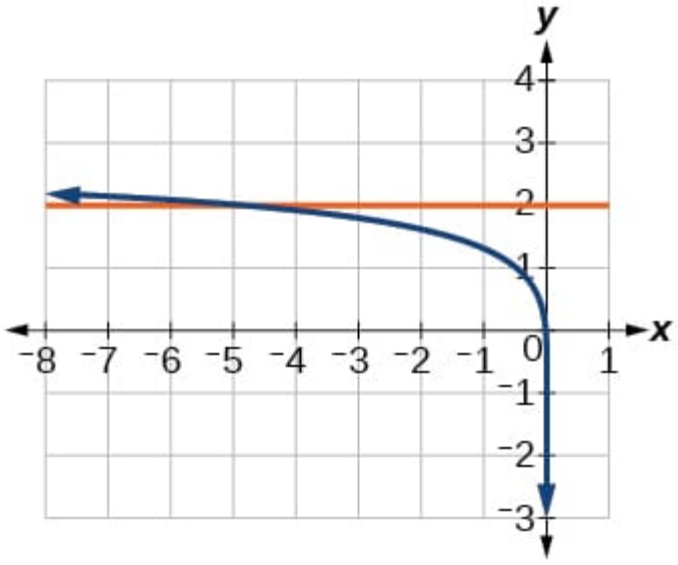



54) $\ln(x - 5) = 1$

55) $\log(4) + \log(-5x) = 2$

**Answer**

$$x = -5$$

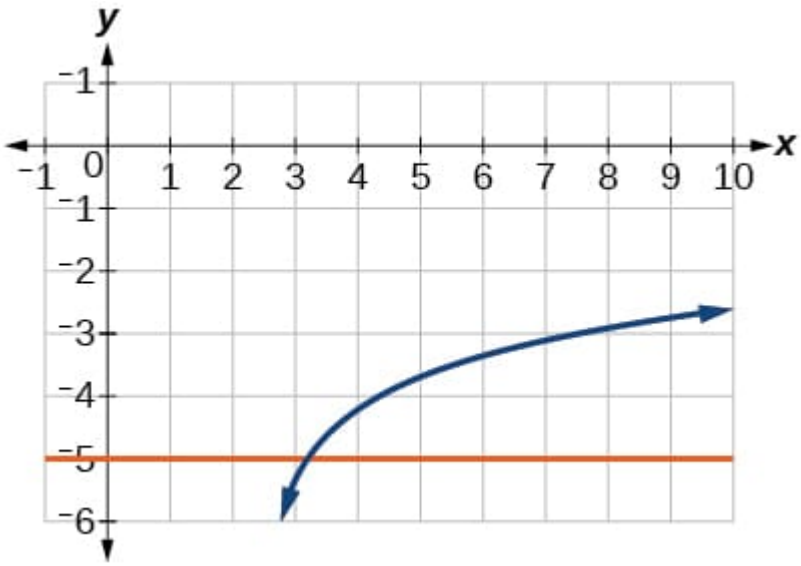



56) $-7 + \log_3(4 - x) = -6$

57) $\ln(4x - 10) - 6 = -5$

**Answer**

$$x = \frac{e + 10}{4} \approx 3.2$$

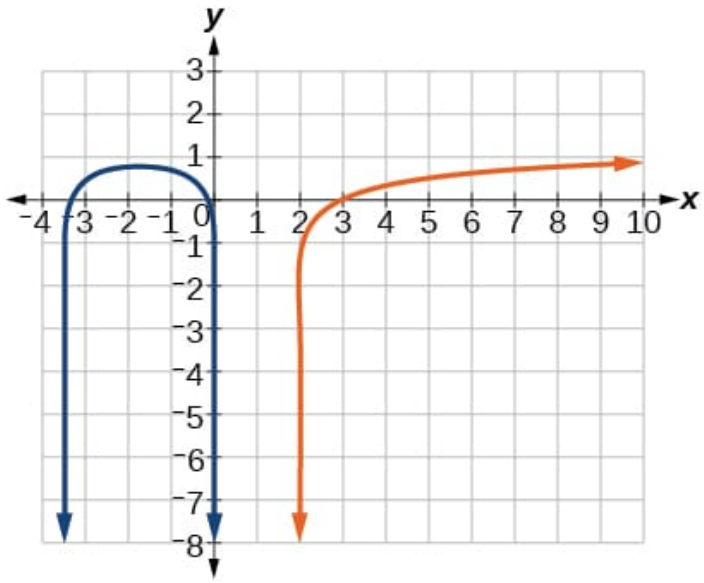



58) $\log(4 - 2x) = \log(-4x)$

59) $\log_{11}(-2x^2 - 7x) = \log_{11}(x - 2)$

**Answer**

No solution

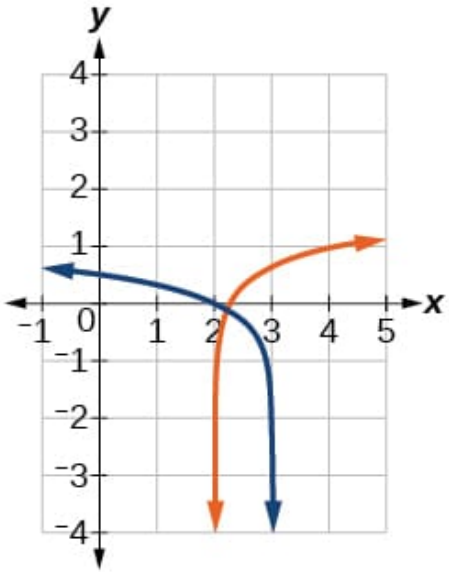



60) $\ln(2x+9) = \ln(-5x)$

61) $\log_9(3-x) = \log_9(4x-8)$

**Answer**

$$x = \frac{11}{5} \approx 2.2$$

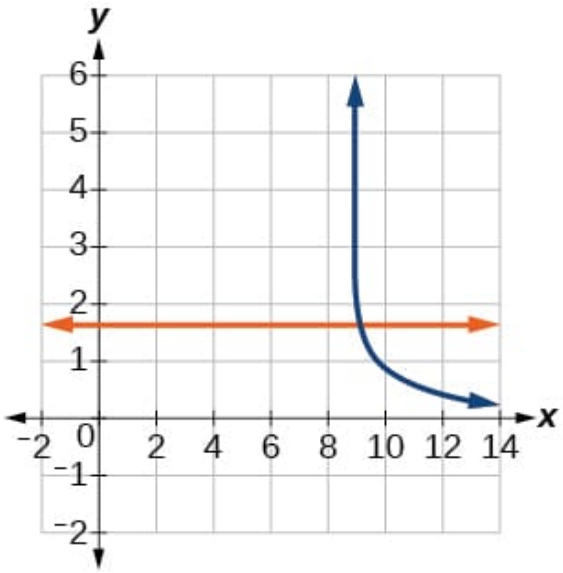



62) $\log(x^2+13) = \log(7x+3)$

63) $\dfrac{3}{\log_2(10)} - \log(x-9) = \log(44)$

**Answer**

$$x = \frac{101}{11} \approx 9.2$$



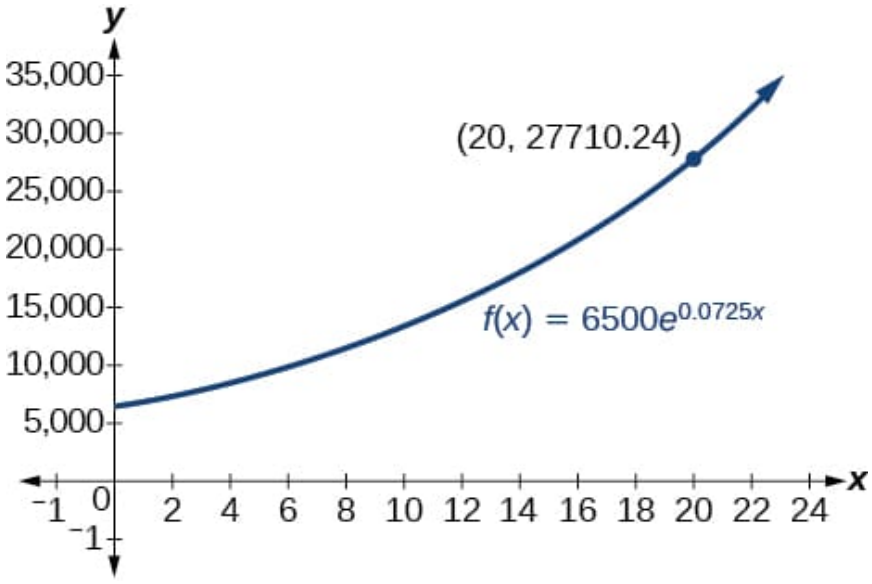

64) $\ln(x) - \ln(x+3) = \ln(6)$

**For the following exercises, solve for the indicated value, and graph the situation showing the solution point.**

65) An account with an initial deposit of $\$6,500$ earns $7.25\%$ annual interest, compounded continuously. How much will the account be worth after $20$ years?

**Answer**

about $\$27,710.24$

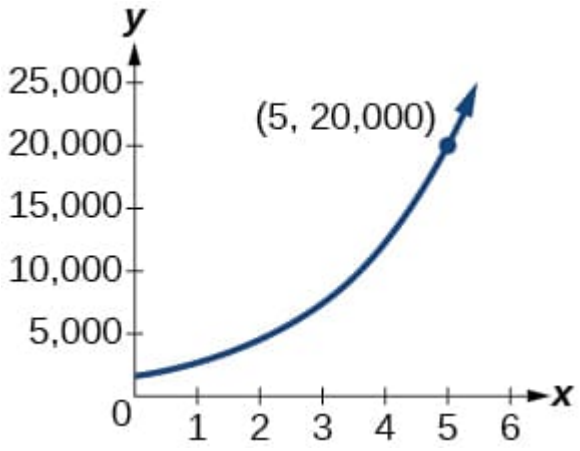



66) The formula for measuring sound intensity in decibels $D$ is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where $I$ is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8 \cdot 3 \cdot 10^2$ watts per square meter?

67) The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where $t$ is measured in years. In approximately how many years will the town's population reach $20,000$?

**Answer**

about $5$ years




## Technology

**For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to $3$ decimal places.**

68) $1000(1.03)^t = 5000$ using the common log.

69) $e^{5x} = 17$ using the natural log.

**Answer**

$$\frac{\ln(17)}{5} \approx 0.567$$

70) $3(1.04)^{3t} = 8$ using the common log

71) $3^{4x-5} = 38$ using the common log

**Answer**

$$x = \frac{\log(38 + 5\log(3))}{4\log(3)} \approx 2.078$$

72) $50e^{-0.12t} = 10$ using the natural log

**For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.**

73) $7e^{3x-5} + 7.9 = 47$

**Answer**

$$x \approx 2.2401$$

74) $\ln(3) + \ln(4.4x + 6.8) = 2$

75) $\log(-0.7x - 9) = 1 + 5\log(5)$

**Answer**

$x \approx -44655.7143$

76) Atmospheric pressure $P$ in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where $x$ is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of $8.369$ pounds per square inch? (Hint: there are $5280$ feet in a mile)

77) The magnitude $M$ of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where $E$ is the amount of energy released by the earthquake in joules $E_0 = 10^{4.4}$ and is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

**Answer**

about $5.83$

## Extensions

78) Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b x} = x$.

79) Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time $t$ such that $t$ is equal to a single logarithm.

**Answer**

$t = \ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$

80) Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time $t$.

81) Newton's Law of Cooling states that the temperature $T$ of an object at any time $t$ can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$, where $T_s$ is the temperature of the surrounding environment, $T_0$ is the initial temperature of the object, and $k$ is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time $t$ such that $t$ is equal to a single logarithm.

**Answer**

$t = \ln\left(\left(\frac{T - T_s}{T_0 - T_s}\right)^{-\frac{1}{k}}\right)$

# 4.7: Exponential and Logarithmic Models

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton's Law of Cooling.

## Verbal

1) With what kind of exponential model would *half-life* be associated? What role does half-life play in these models?

**Answer**

Half-life is a measure of decay and is thus associated with exponential decay models. The half-life of a substance or quantity is the amount of time it takes for half of the initial amount of that substance or quantity to decay.

2) What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.

3) With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?

**Answer**

Doubling time is a measure of growth and is thus associated with exponential growth models. The doubling time of a substance or quantity is the amount of time it takes for the initial amount of that substance or quantity to double in size.

4) Define Newton's Law of Cooling. Then name at least three real-world situations where Newton's Law of Cooling would be applied.

5) What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

**Answer**

An order of magnitude is the nearest power of ten by which a quantity exponentially grows. It is also an approximate position on a logarithmic scale; Sample response: Orders of magnitude are useful when making comparisons between numbers that differ by a great amount. For example, the mass of Saturn is $95$ times greater than the mass of Earth. This is the same as saying that the mass of Saturn is about $10^2$ times, or $2$ orders of magnitude greater, than the mass of Earth.

## Numeric

6) The temperature of an object in degrees Fahrenheit after $t$ minutes is represented by the equation $T(t) = 68e^{-0.0174t} + 72$. To the nearest degree, what is the temperature of the object after one and a half hours?

**For the following exercises, use the logistic growth model** $f(x) = \dfrac{150}{1 + 8e^{-2x}}$

7) Find and interpret $f(0)$. Round to the nearest tenth.

**Answer**

$f(0) \approx 16.7$; The amount initially present is about $16.7$ units.

8) Find and interpret $f(4)$. Round to the nearest tenth.

9) Find the carrying capacity.

**Answer**

$150$

10) Graph the model.

11) Determine whether the data from the table could best be represented as a function that is linear, exponential, or logarithmic. Then write a formula for a model that represents the data.

| $x$ | $f(x)$ |
|---|---|
| –2 | 0.694 |
| –1 | 0.833 |
| 0 | 1 |
| 1 | 1.2 |
| 2 | 1.44 |
| 3 | 1.728 |

| $x$ | $f(x)$ |
|---|---|
| 4 | 2.074 |
| 5 | 2.488 |

**Answer**

exponential; $f(x) = 1.2^x$

12) Rewrite $f(x) = 1.68(0.65)^x$ as an exponential equation with base $e$ to five significant digits.

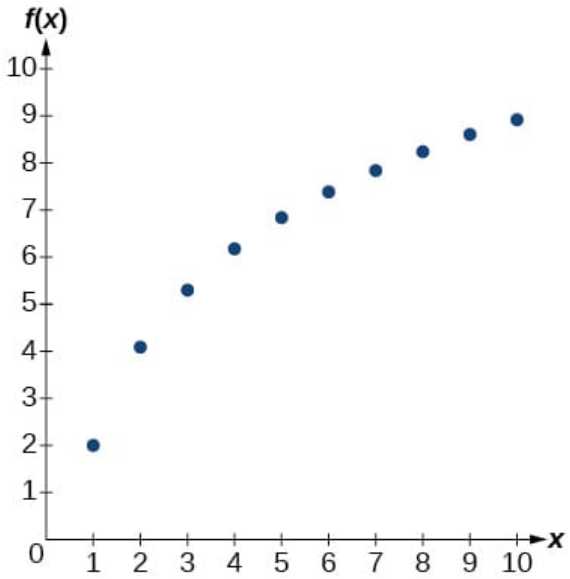
## Technology

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table could represent a function that is linear, exponential, or logarithmic.

13)

| $x$ | $f(x)$ |
|---|---|
| 1 | 2 |
| 2 | 4.079 |
| 3 | 5.296 |
| 4 | 6.159 |
| 5 | 6.828 |
| 6 | 7.375 |
| 7 | 7.838 |
| 8 | 8.238 |
| 9 | 8.592 |
| 10 | 8.908 |

**Answer**



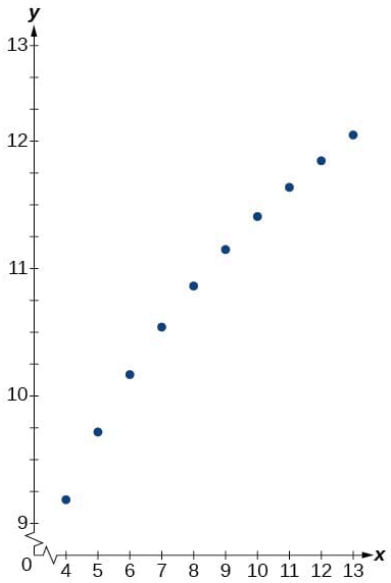
14)

| $x$ | $f(x)$ |
|---|---|
| 1 | 2.4 |
| 2 | 2.88 |
| 3 | 3.456 |
| 4 | 4.147 |
| 5 | 4.977 |
| 6 | 5.972 |
| 7 | 7.166 |
| 8 | 8.6 |
| 9 | 10.32 |
| 10 | 12.383 |

15)

| $x$ | $f(x)$ |
|---|---|
| 4 | 9.429 |
| 5 | 9.972 |
| 6 | 10.415 |
| 7 | 10.79 |
| 8 | 11.115 |
| 9 | 11.401 |
| 10 | 11.657 |
| 11 | 11.889 |
| 12 | 12.101 |
| 13 | 12.295 |

**Answer**



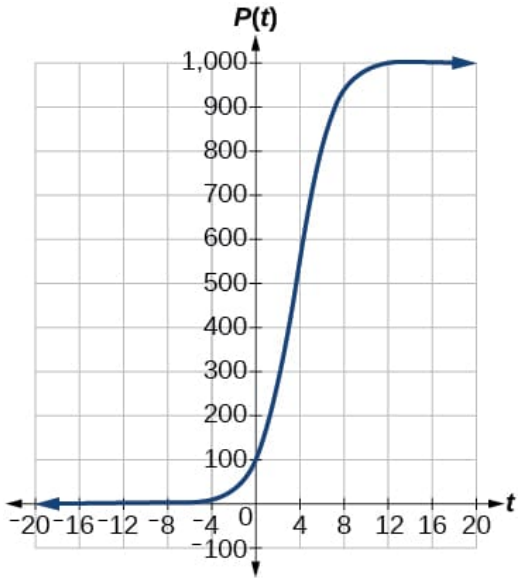

16)

| $x$ | $f(x)$ |
|---|---|
| 1.25 | 5.75 |
| 2.25 | 8.75 |
| 3.56 | 12.68 |
| 4.2 | 14.6 |
| 5.65 | 18.95 |
| 6.75 | 22.25 |
| 7.25 | 23.75 |
| 8.6 | 27.8 |
| 9.25 | 29.75 |
| 10.5 | 33.5 |

**For the following exercises, use a graphing calculator and this scenario: the population of a fish farm in $t$ years is modeled by the equation $P(t) = \frac{1000}{1 + 9e^{-0.6t}}$**

17) Graph the function.

**Answer**




18) What is the initial population of fish?

19) To the nearest tenth, what is the doubling time for the fish population?

**Answer**

about $1.4$ years

20) To the nearest whole number, what will the fish population be after $2$ years?

21) To the nearest tenth, how long will it take for the population to reach $900$?

**Answer**

about $7.3$ years

22) What is the carrying capacity for the fish population? Justify your answer using the graph of $P$.

## Extensions

23) A substance has a half-life of $2.045$ minutes. If the initial amount of the substance was $132.8$ grams, how many half-lives will have passed before the substance decays to $8.3$ grams? What is the total time of decay?

**Answer**

$4$ half-lives; $8.18$ minutes

24) The formula for an increasing population is given by $P(t) = P_0 e^{rt}$ where $P_0$ is the initial population and $r > 0$. Derive a general formula for the time $t$ it takes for the population to increase by a factor of $M$.

25) Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. Show each step for solving this equation algebraically for the seismic moment $S$.

**Answer**

$$M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$
$$\log\left(\frac{S}{S_0}\right) = \frac{3}{2}M$$
$$\frac{S}{S_0} = 10^{\frac{3M}{2}}$$
$$S = S_0 10^{\frac{3M}{2}}$$

26) What is the $y$-intercept of the logistic growth model $y = \frac{c}{1+ae^{-rx}}$ ? Show the steps for calculation. What does this point tell us about the population?

27) Prove that $b^x = e^{x\ln(b)}$ for positive $b \neq 1$.

**Answer**

Let $y = b^x$ for some non-negative real number $b$ such that $b \neq 1$. Then,

$$\ln(y) = \ln(b^x)$$
$$\ln(y) = x\ln(b)$$
$$e^{\ln(y)} = e^{x\ln(b)}$$
$$y = e^{x\ln(b)}$$

## Real-World Applications

**For the following exercises, use this scenario: A doctor prescribes $125$ milligrams of a therapeutic drug that decays by about $30\%$ each hour.**

28) To the nearest hour, what is the half-life of the drug?

29) Write an exponential model representing the amount of the drug remaining in the patient's system after $t$ hours. Then use the formula to find the amount of the drug that would remain in the patient's system after $3$ hours. Round to the nearest milligram.

**Answer**

$A = 125e^{(-0.3567t)}$; $A \approx 43$ mg

30) Using the model found in the previous exercise, find $f(10$ and interpret the result. Round to the nearest hundredth.

**For the following exercises, use this scenario: A tumor is injected with $0.5$ grams of Iodine-125, which has a decay rate of $1.15\%$ per day.**

31) To the nearest day, how long will it take for half of the Iodine-125 to decay?

**Answer**

about $60$ days

32) Write an exponential model representing the amount of Iodine-125 remaining in the tumor after $t$ days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after $60$ days. Round to the nearest tenth of a gram.

33) A scientist begins with $250$ grams of a radioactive substance. After $250$ minutes, the sample has decayed to $32$ grams. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?

**Answer**

$f(t) = 250e^{(-0.00914t)}$; half-life: about $76$ minutes

34) The half-life of Radium-226 is $1590$ years. What is the annual decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

35) The half-life of Erbium-165 is $10.4$ hours. What is the hourly decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

**Answer**

$r \approx -0.0667$, So the hourly decay rate is about $6.67\%$

36) A wooden artifact from an archeological dig contains $60$ percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is $5730$ years.)

37) A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was $1350$ bacteria. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest whole number, what is the population size after $3$ hours?

**Answer**

$f(t) = 1350e^{(0.03466t)}$; after $3$ hours: $P(180) \approx 691,200$

**For the following exercises, use this scenario: A biologist recorded a count of $360$ bacteria present in a culture after $5$ minutes and $1000$ bacteria present after $20$ minutes.**

38) To the nearest whole number, what was the initial population in the culture?

39) Rounding to six significant digits, write an exponential equation representing this situation. To the nearest minute, how long did it take the population to double?

**Answer**

$f(t) = 256e^{(0.068110t)}$; doubling time: about $10$ minutes

**For the following exercises, use this scenario: A pot of boiling soup with an internal temperature of $100^\circ$ Fahrenheit was taken off the stove to cool in a $69^\circ$ F room. After fifteen minutes, the internal temperature of the soup was $95^\circ$ F.**

40) Use Newton's Law of Cooling to write a formula that models this situation.

41) To the nearest minute, how long will it take the soup to cool to $80^\circ$ F?

**Answer**

about $88$ minutes

42) To the nearest degree, what will the temperature be after $2$ and a half hours?

**For the following exercises, use this scenario: A turkey is taken out of the oven with an internal temperature of $165^\circ$ F and is allowed to cool in a $75^\circ$ F room. After half an hour, the internal temperature of the turkey is $145^\circ$ F.**

43) Write a formula that models this situation.

**Answer**

$T(t) = 90e^{(-0.008377t)} + 75$, where $t$ is in minutes.

44) To the nearest degree, what will the temperature be after $50$ minutes?

45) To the nearest minute, how long will it take the turkey to cool to $110^\circ$ F?

**Answer**

about $113$ minutes

**For the following exercises, find the value of the number shown on each logarithmic scale. Round all answers to the nearest thousandth.**

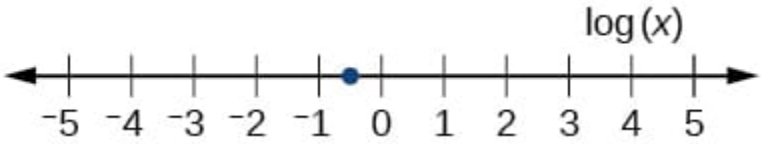



Figure_04_07_207.jpg" src="/@api/deki/files/13777/CNX_PreCalc_Figure_04_07_207.jpg" />

**Answer**

$\log(x) = 1.5; x \approx 31.623$

48) Plot each set of approximate values of intensity of sounds on a logarithmic scale: Whisper: $10^{-10}\frac{W}{m^2}$, Vacuum: $10^{-4}\frac{W}{m^2}$, Jet: $10^{2}\frac{W}{m^2}$

49) Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. One earthquake has magnitude $3.9$ on the MMS scale. If a second earthquake has $750$ times as much energy as the first, find the magnitude of the second quake. Round to the nearest hundredth.

**Answer**

MMS magnitude: $5.82$

**For the following exercises, use this scenario: The equation $N(t) = \frac{500}{1+49e^{-0.7t}}$ models the number of people in a town who have heard a rumor after $t$ days.**

50) How many people started the rumor?

51) To the nearest whole number, how many people will have heard the rumor after $3$ days?

**Answer**

$N(3) \approx 71$

52) As t t t increases without bound, what value does $N(t)$ approach? Interpret your answer.

**For the following exercise, choose the correct answer choice.**

53) A doctor and injects a patient with $13$ milligrams of radioactive dye that decays exponentially. After $12$ minutes, there are $4.75$ milligrams of dye remaining in the patient's system. Which is an appropriate model for this situation?

a. $f(t) = 13(0.0805)^t$
b. $f(t) = 13e^{0.9195t}$
c. $f(t) = 13e^{(-0.0839t)}$
d. $f(t) = \frac{4.75}{1+13e^{-0.83925t}}$

**Answer**

c

# 4.8: Fitting Exponential Models to Data

We will concentrate on three types of regression models in this section: exponential, logarithmic, and logistic. Having already worked with each of these functions gives us an advantage. Knowing their formal definitions, the behavior of their graphs, and some of their real-world applications gives us the opportunity to deepen our understanding. As each regression model is presented, key features and definitions of its associated function are included for review.

## Verbal

1) What situations are best modeled by a logistic equation? Give an example, and state a case for why the example is a good fit.

**Answer**

Logistic models are best used for situations that have limited values. For example, populations cannot grow indefinitely since resources such as food, water, and space are limited, so a logistic model best describes populations.

2) What is a carrying capacity? What kind of model has a carrying capacity built into its formula? Why does this make sense?

3) What is regression analysis? Describe the process of performing regression analysis on a graphing utility.

**Answer**

Regression analysis is the process of finding an equation that best fits a given set of data points. To perform a regression analysis on a graphing utility, first list the given points using the STAT then EDIT menu. Next graph the scatter plot using the STAT PLOT feature. The shape of the data points on the scatter graph can help determine which regression feature to use. Once this is determined, select the appropriate regression analysis command from the STAT then CALC menu.

4) What might a scatterplot of data points look like if it were best described by a logarithmic model?

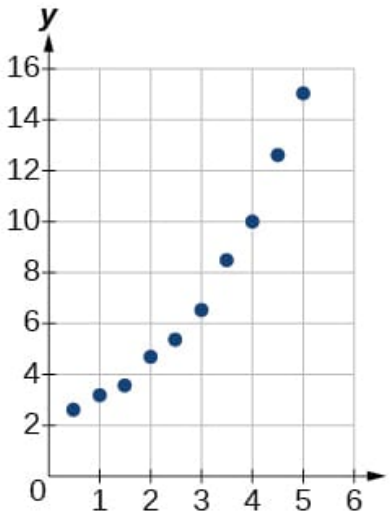
5) What does the y-intercept on the graph of a logistic equation correspond to for a population modeled by that equation?

**Answer**

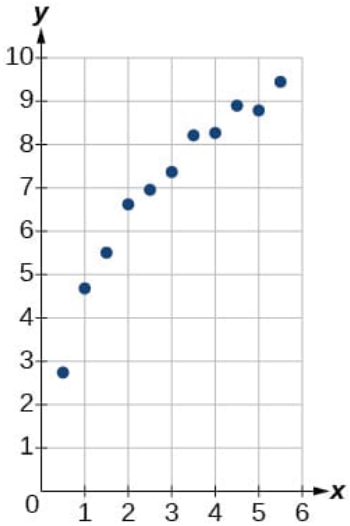
The $y$-intercept on the graph of a logistic equation corresponds to the initial population for the population model.

## Graphical

**For the following exercises, match the given function of best fit with the appropriate scatterplot in Figure (a) through Figure (e). Answer using the letter beneath the matching graph.**

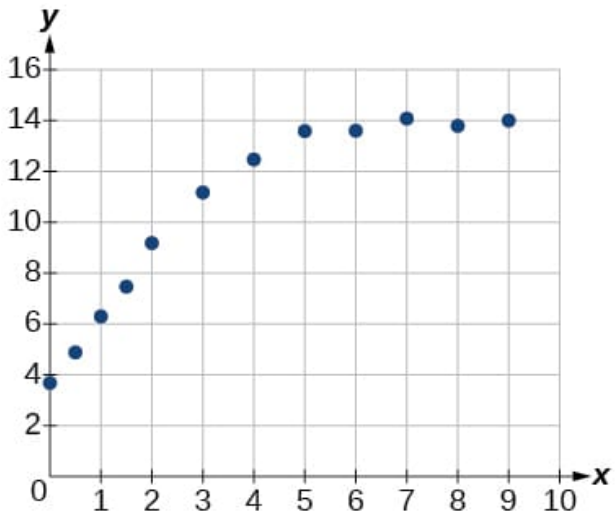



(a)

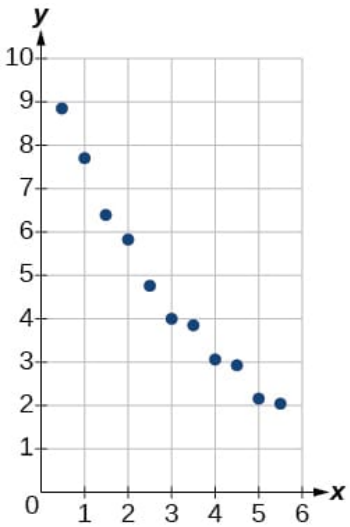



(b)

Figure_04_08_203.jpg" src="/@api/deki/files/13803/CNX_PreCalc_Figure_04_08_203.jpg" />




(d)




(e)

6) $y = 10.209e^{-0.294x}$

7) $y = 5.598 - 1.912\ln(x)$

**Answer**

c

8) $y = 2.104(1.479)^x$

9) $y = 4.607 + 2.733\ln(x)$

**Answer**

b

10) $y = \dfrac{14.005}{1 + 2.79e^{-0.812x}}$

## Numeric

11) To the nearest whole number, what is the initial value of a population modeled by the logistic equation $P(t) = \dfrac{175}{1 + 6.995e^{-0.68t}}$ ? What is the carrying capacity?

**Answer**

$P(0) = 22$; 175

12) Rewrite the exponential model $A(t) = 1550(1.085)^x$ as an equivalent model with base $e$. Express the exponent to four significant digits.

13) A logarithmic model is given by the equation $h(p) = 67.682 - 5.792\ln(p)$. To the nearest hundredth, for what value of $p$ does $h(p) = 62$?

**Answer**

$p \approx 2.67$

14) A logistic model is given by the equation $P(t) = \frac{90}{1 + 5e^{-0.42t}}$. To the nearest hundredth, for what value of $t$ does $P(t) = 45$?

15) What is the $y$-intercept on the graph of the logistic model given in the previous exercise?

**Answer**

$y$-intercept: $(0, 15)$

## Technology

**For the following exercises, use this scenario: The population $P$ of a koi pond over $x$ months is modeled by the function $P(x) = \frac{68}{1 + 16e^{-0.28x}}$.**

16) Graph the population model to show the population over a span of $3$ years.

17) What was the initial population of koi?

**Answer**

\(4)\ koi

18) How many koi will the pond have after one and a half years?

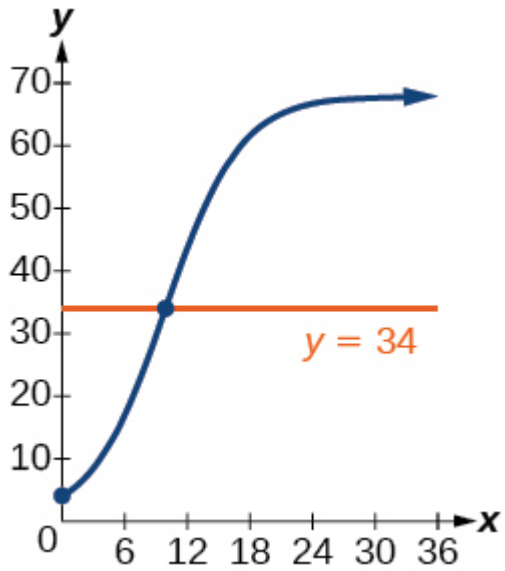
19) How many months will it take before there are $20$ koi in the pond?

**Answer**

about $6.8$ months

20) Use the intersect feature to approximate the number of months it will take before the population of the pond reaches half its carrying capacity.

**Answer**




**For the following exercises, use this scenario: The population $P$ of an endangered species habitat for wolves is modeled by the function $P(x) = \frac{558}{1 + 54.8e^{-0.462x}}$, where $x$ is given in years.**

21) Graph the population model to show the population over a span of $10$ years.

22) What was the initial population of wolves transported to the habitat?

**Answer**

\(10\)\ wolves

23) How many wolves will the habitat have after $3$ years?

24) How many years will it take before there are $100$ wolves in the habitat?

**Answer**

about $5.4$ years

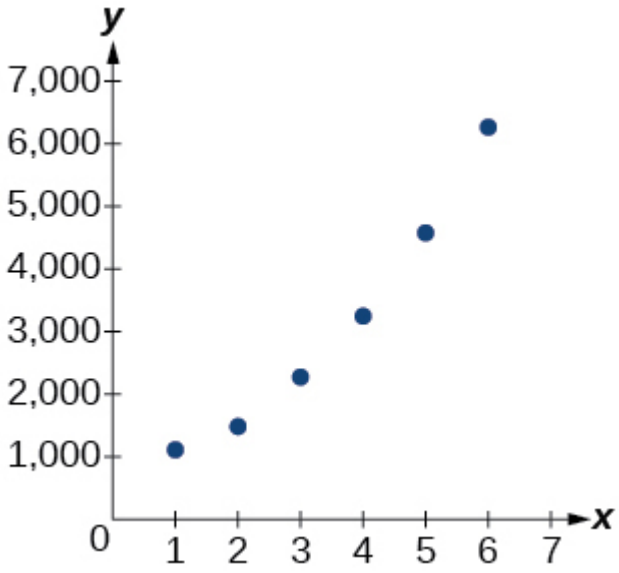
25) Use the intersect feature to approximate the number of years it will take before the population of the habitat reaches half its carrying capacity.

**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 1 | 1125 |
| 2 | 1495 |
| 3 | 2310 |
| 4 | 3294 |
| 5 | 4650 |
| 6 | 6361 |

26) Use a graphing calculator to create a scatter diagram of the data.

**Answer**




27) Use the regression feature to find an exponential function that best fits the data in the table.

28) Write the exponential function as an exponential equation with base $e$.

**Answer**

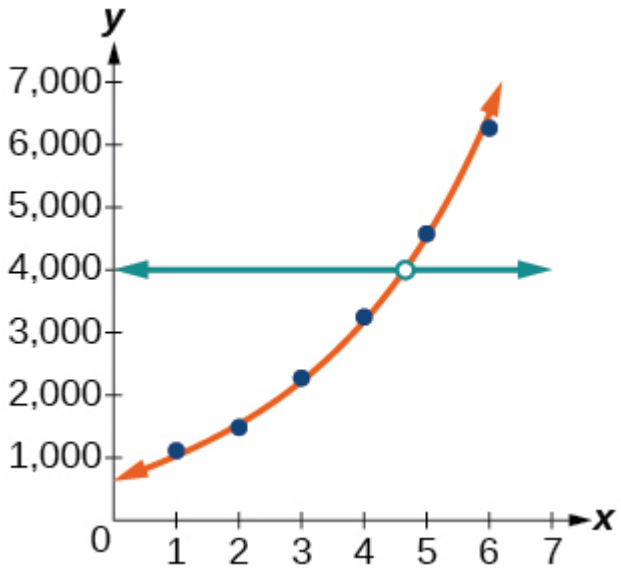
$$f(x) = 776.682e^{0.3549x}$$

29) Graph the exponential equation on the scatter diagram.

30) Use the intersect feature to find the value of $x$ for which $f(x) = 4000$.

**Answer**

When $f(x) = 4000$, $x \approx 4.6$




**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 1 | 555 |
| 2 | 383 |
| 3 | 307 |
| 4 | 210 |
| 5 | 158 |
| 6 | 122 |

31) Use a graphing calculator to create a scatter diagram of the data.

32) Use the regression feature to find an exponential function that best fits the data in the table.

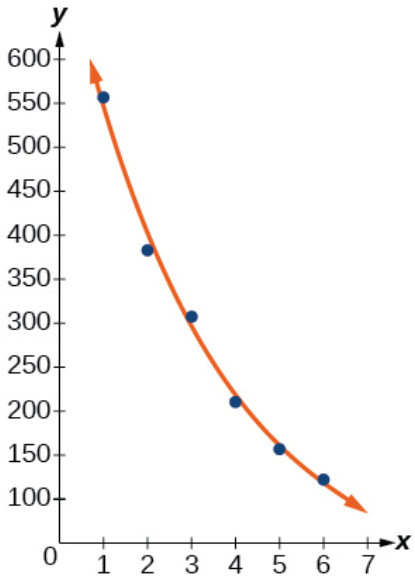
**Answer**

$$f(x) = 731.92(0.738)^x$$

33) Write the exponential function as an exponential equation with base $e$.

34) Graph the exponential equation on the scatter diagram.

**Answer**



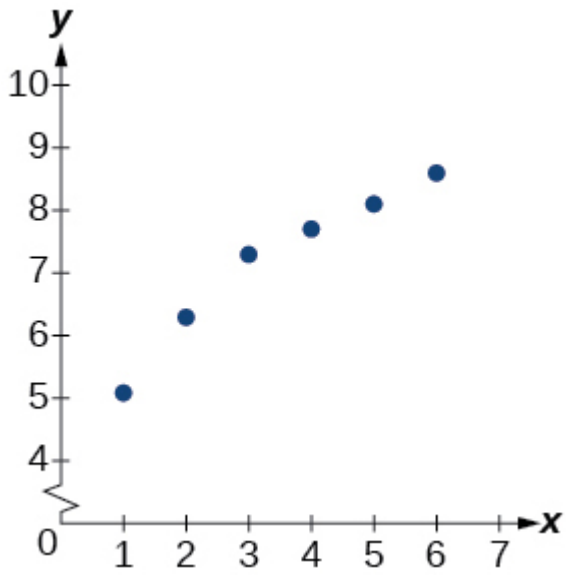

35) Use the intersect feature to find the value of $x$ for which $f(x) = 250$.

**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 1 | 5.1 |
| 2 | 6.3 |
| 3 | 7.3 |
| 4 | 7.7 |
| 5 | 8.1 |
| 6 | 8.6 |

36) Use a graphing calculator to create a scatter diagram of the data.

**Answer**




37) Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

38) Use the logarithmic function to find the value of the function when $x = 10$.

**Answer**

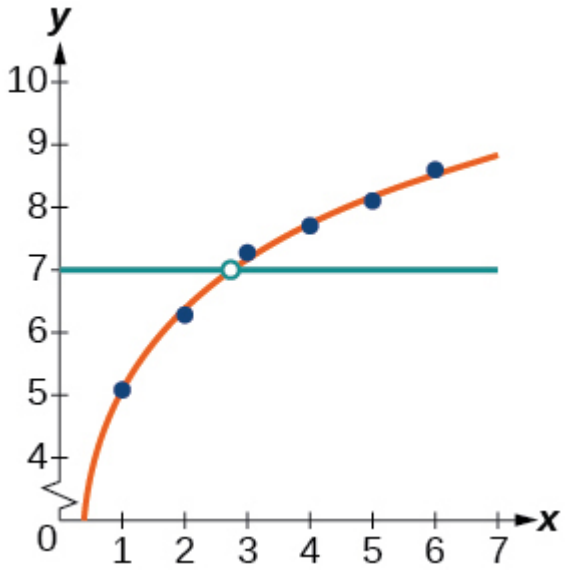
$f(10) \approx 9.5$

39) Graph the logarithmic equation on the scatter diagram.

40) Use the intersect feature to find the value of $x$ for which $f(x) = 7$.

**Answer**

When $f(x) = 7$, $x \approx 2.7$



**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 1 | 7.5 |
| 2 | 6 |
| 3 | 5.2 |
| 4 | 4.3 |
| 5 | 3.9 |
| 6 | 3.4 |
| 7 | 3.1 |
| 8 | 2.9 |

41) Use a graphing calculator to create a scatter diagram of the data.

42) Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

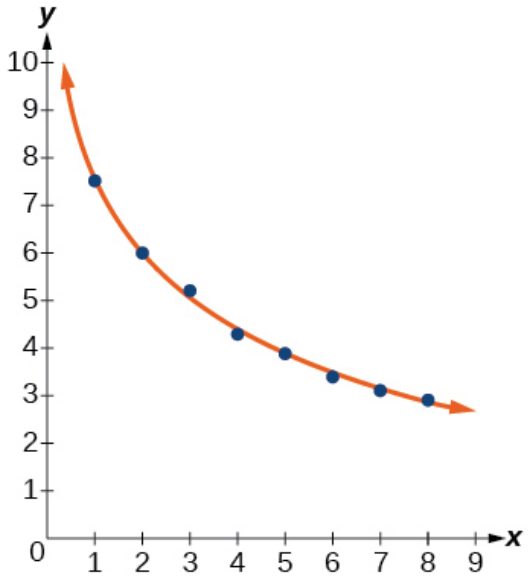
**Answer**

$$f(x) = 7.544 - 2.268\ln(x)$$

43) Use the logarithmic function to find the value of the function when $x = 10$.

44) Graph the logarithmic equation on the scatter diagram.

**Answer**




45) Use the intersect feature to find the value of $x$ for which $f(x) = 8$.

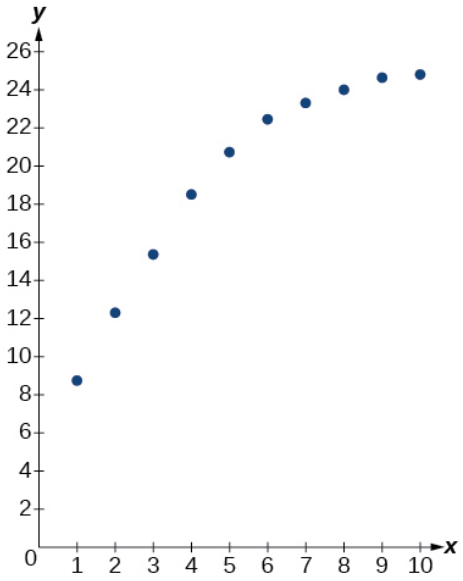
**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 1 | 8.7 |

| $x$ | $f(x)$ |
|---|---|
| 2 | 12.3 |
| 3 | 15.4 |
| 4 | 18.5 |
| 5 | 20.7 |
| 6 | 22.5 |
| 7 | 23.3 |
| 8 | 24 |
| 9 | 24.6 |
| 10 | 24.8 |

46) Use a graphing calculator to create a scatter diagram of the data.

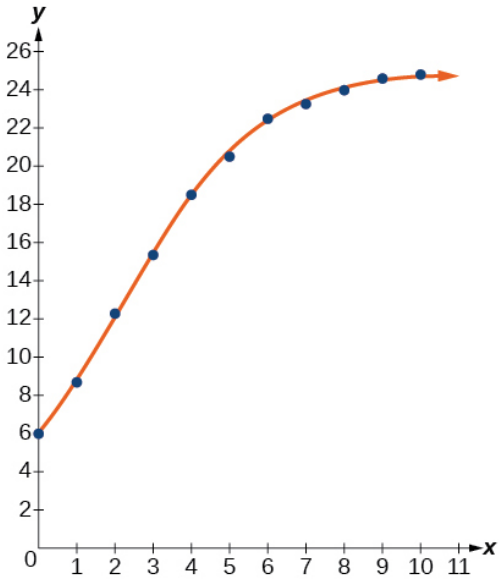
**Answer**




47) Use the LOGISTIC regression option to find a logistic growth model of the form $y = \dfrac{c}{1 + ae^{-bx}}$ that best fits the data in the table.

48) Graph the logistic equation on the scatter diagram.

**Answer**

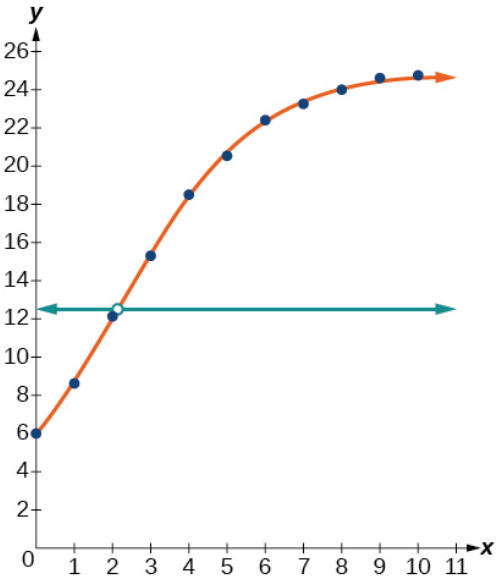



49) To the nearest whole number, what is the predicted carrying capacity of the model?

50) Use the intersect feature to find the value of $x$ for which the model reaches half its carrying capacity.

**Answer**

When $f(x) = 12.5$, $x \approx 2.1$




**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 0 | 12 |
| 2 | 28.6 |
| 4 | 52.8 |

| $x$ | $f(x)$ |
|---|---|
| 5 | 70.3 |
| 7 | 99.9 |
| 8 | 112.5 |
| 10 | 125.8 |
| 11 | 127.9 |
| 15 | 135.1 |
| 17 | 135.9 |

51) Use a graphing calculator to create a scatter diagram of the data.

52) Use the LOGISTIC regression option to find a logistic growth model of the form $y = \dfrac{c}{1 + ae^{-bx}}$ that best fits the data in the table.

**Answer**

$$y = \frac{136.068}{1 + 10.324e^{-0.480x}}$$

53) Graph the logistic equation on the scatter diagram.

54) To the nearest whole number, what is the predicted carrying capacity of the model?

**Answer**

about $136$

55) Use the intersect feature to find the value of $x$ for which the model reaches half its carrying capacity.

## Extensions

56) Recall that the general form of a logistic equation for a population is given by $P(t) = \dfrac{c}{1 + ae^{-bt}}$, such that the initial population at time $t = 0$ is $P(0) = P_0$. Show algebraically that $\dfrac{c - P(t)}{P(t)} = \dfrac{c - P_0}{P_0} e^{-bt}$.

**Answer**

Working with the left side of the equation, we see that it can be rewritten as $ae^{-bt}$:

$$\begin{aligned}
\frac{c - P(t)}{P(t)} &= \frac{c - \frac{c}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}} \\
&= \frac{\frac{c(1+ae^{-bt}) - c}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}} \\
&= \frac{\frac{c(1+ae^{-bt} - 1)}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}} \\
&= 1 + ae^{-bt} - 1 \\
&= ae^{-bt}
\end{aligned}$$

$$\begin{aligned}
P_0 &= \frac{c}{1 + ae^{-b(0)}} \\
&= \frac{c}{1 + a}
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{c-P_0}{P_0}e^{-bt} &= \frac{c-\frac{c}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= \frac{\frac{c(1+a)-c}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= \frac{\frac{c(1+a-1)}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= (1+a-1)e^{-bt}\\ &= ae^{-bt}\end{aligned}$$

Thus,

$$\frac{c-P(t)}{P(t)} = \frac{c-P_0}{P_0}e^{-bt}$$

57) Use a graphing utility to find an exponential regression formula $f(x)$ and a logarithmic regression formula $g(x)$ for the points $(1.5, 1.5)$ and $(8.5, 8.5)$. Round all numbers to $6$ decimal places. Graph the points and both formulas along with the line $y = x$ on the same axis. Make a conjecture about the relationship of the regression formulas.

58) Verify the conjecture made in the previous exercise. Round all numbers to six decimal places when necessary.

**Answer**

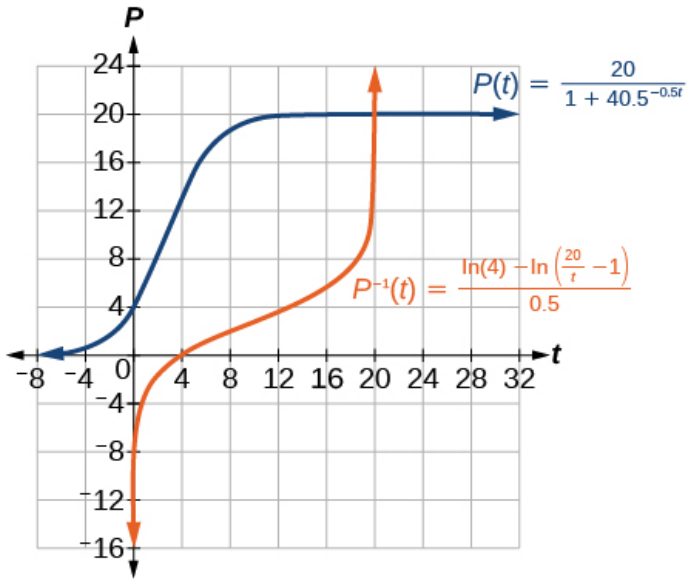
First rewrite the exponential with base e: $f(x) = 1.034341e^{0.247800x}$. Then test to verify that $f(g(x)) = x$,taking rounding error into consideration:

$$\begin{aligned}g(f(x)) &= 4.035510\ln\left(1.034341e^{0.247800x}\right) - 0.136259\\ &= 4.03551(\ln(1.034341) + \ln\left(e^{0.2478x}\right)) - 0.136259\\ &= 4.03551(\ln(1.034341) + 0.2478x) - 0.136259\\ &= 0.136257 + 0.999999x - 0.136259\\ &= -0.000002 + 0.999999x\\ &\approx 0 + x\\ &= x\end{aligned}$$

59) Find the inverse function $f^{-1}(x)$ for the logistic function $f(x) = \dfrac{c}{1+ae^{-bx}}$. Show all steps.

60) Use the result from the previous exercise to graph the logistic model $P(t) = \dfrac{20}{1+4e^{-0.5t}}$ along with its inverse on the same axis. What are the intercepts and asymptotes of each function?

**Answer**




The graph of $P(t)$ has a $y$-intercept at $(0, 4)$ and horizontal asymptotes at $y = 0$ and $y = 20$. The graph of $P^{-1}(t)$ has an $x$- intercept at $(4, 0)$ and vertical asymptotes at $x = 0$ and $x = 20$.

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# 4.R: Exponential and Logarithmic Functions (Review)

4.R: Exponential and Logarithmic Functions (Review)

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# 5: SYSTEMS OF EQUATIONS AND INEQUALITIES

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions.


5.0: Prelude to Systems of Equations and Inequalities

5.1: Systems of Linear Equations - Two Variables

5.2: Systems of Linear Equations - Three Variables

5.3: Systems of Nonlinear Equations and Inequalities - Two Variables

5.4: Partial Fractions

5.5: Matrices and Matrix Operations

5.6: Solving Systems with Gaussian Elimination

5.7: Solving Systems with Inverses

5.8: Solving Systems with Cramer's Rule

5.E: Systems of Equations and Inequalities (Exercises)

5.R: Systems of Equations and Inequalities (Review)


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# 5.0: Prelude to Systems of Equations and Inequalities



Figure 5.0.1: Enigma machines like this one, once owned by Italian dictator Benito Mussolini, were used by government and military officials for enciphering and deciphering top-secret communications during World War II. (credit: Dave Addey, Flickr)

By 1943, it was obvious to the Nazi regime that defeat was imminent unless it could build a weapon with unlimited destructive power, one that had never been seen before in the history of the world. In September, Adolf Hitler ordered German scientists to begin building an atomic bomb. Rumors and whispers began to spread from across the ocean. Refugees and diplomats told of the experiments happening in Norway. However, Franklin D. Roosevelt wasn't sold, and even doubted British Prime Minister Winston Churchill's warning. Roosevelt wanted undeniable proof. Fortunately, he soon received the proof he wanted when a group of mathematicians cracked the "Enigma" code, proving beyond a doubt that Hitler was building an atomic bomb. The next day, Roosevelt gave the order that the United States begin work on the same.

The Enigma is perhaps the most famous cryptographic device ever known. It stands as an example of the pivotal role cryptography has played in society. Now, technology has moved cryptanalysis to the digital world.

Many ciphers are designed using invertible matrices as the method of message transference, as finding the inverse of a matrix is generally part of the process of decoding. In addition to knowing the matrix and its inverse, the receiver must also know the key that, when used with the matrix inverse, will allow the message to be read.

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions. We will not be breaking any secret codes here, but we will lay the foundation for future courses.

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# 5.1: Systems of Linear Equations - Two Variables

**Learning Objectives**

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.



Figure 5.1.1

## Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. **A system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x+y &= 15 \\ 3x–y &= 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair $(4, 7)$ is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$2(4)+(7)=15 \text{ True}$$
$$3(4)-(7)=5 \text{ True}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A consistent system of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same *y*-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different *y*-intercepts. There are no points common to both lines; hence, there is no solution to the system.

## TYPES OF LINEAR SYSTEMS

There are three types of systems of linear equations in two variables, and three types of solutions.

- An independent system has exactly one solution pair $(x, y)$. The point where the two lines intersect is the only solution.
- An inconsistent system has no solution. Notice that the two lines are parallel and will never intersect.
- A dependent system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Figure $5.1.2$ compares graphical representations of each type of system.

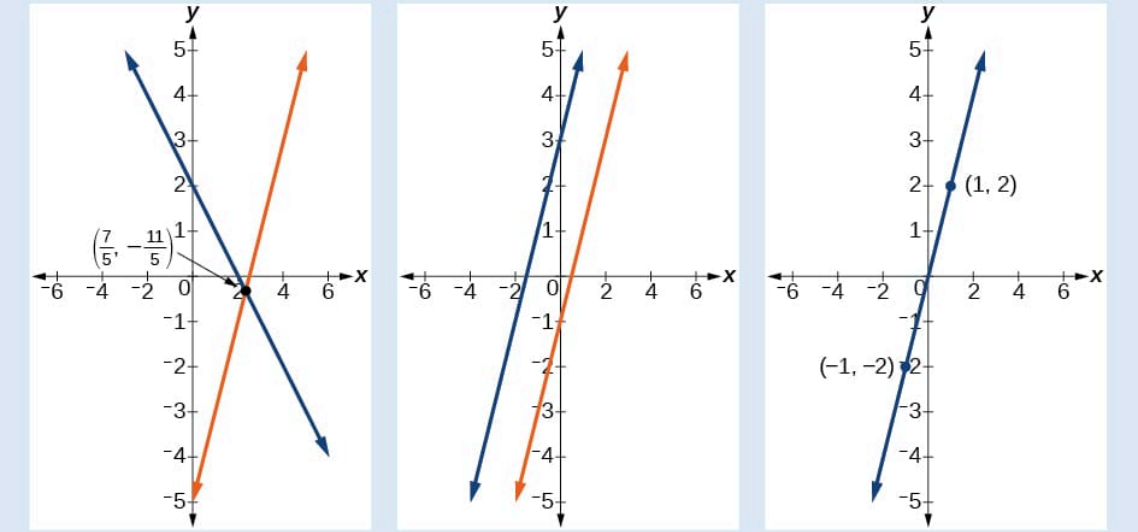



Figure $5.1.2$

## Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

## Example $5.1.1$: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$x + 3y = 8$$
$$2x - 9 = y$$

**Solution**

Substitute the ordered pair $(5, 1)$ into both equations.

$$\begin{aligned} (5) + 3(1) &= 8 \\ 8 &= 8 \text{ True} \\ 2(5) - 9 &= (1) \\ 1 &= 1 \text{ True} \end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

**Analysis**

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure 5.1.3.

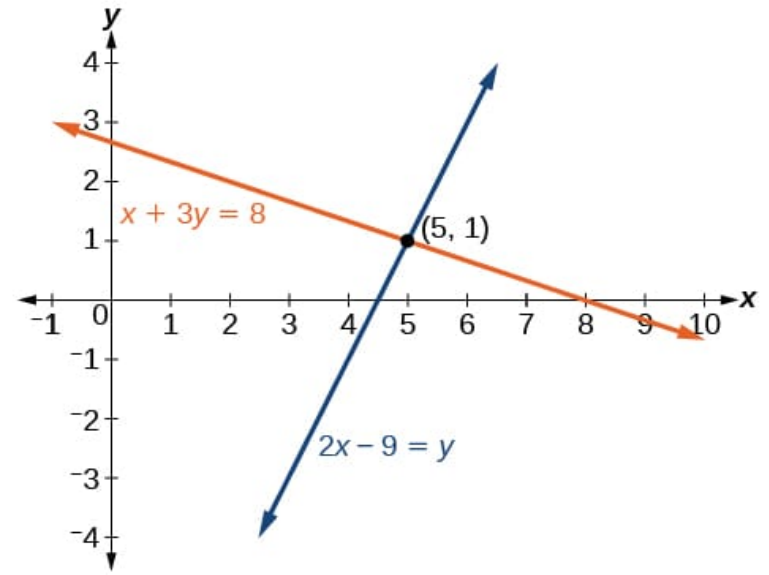



Figure 5.1.3

## Exercise 5.1.1

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$5x - 4y = 20$$
$$2x + 1 = 3y$$

*Answer available at the end of this section.*

## Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

## Example 5.1.2: Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

**Solution**

Solve the first equation for $y$.

$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for $y$.

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in Figure 5.1.4.

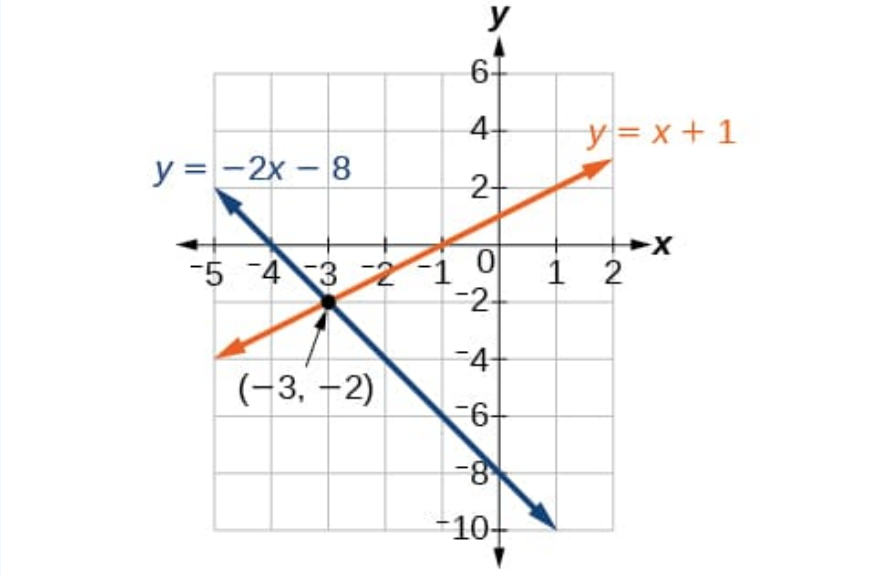



Figure 5.1.4

The lines appear to intersect at the point $(-3, -2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3) + (-2) &= -8 \\ -8 &= -8 \text{ True} \\ (-3) - (-2) &= -1 \\ -1 &= -1 \text{ True} \end{aligned}$$

The solution to the system is the ordered pair $(-3, -2)$,so the system is independent.

## Exercise 5.1.2

Solve the following system of equations by graphing.

$$\begin{aligned} 2x - 5y &= -25 \\ -4x + 5y &= 35 \end{aligned}$$

*Answer available at the end of this section.*

Q&A

**Can graphing be used if the system is inconsistent or dependent?**

*Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.*

## Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a **system of linear equations** that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

How to: Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example $5.1.3$: Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{aligned} -x+y&=-5 \\ 2x-5y&=1 \end{aligned}$$

**Solution**

First, we will solve the first equation for $y$.

$$\begin{aligned} -x+y&=-5 \\ y&=x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for $y$ in the second equation.

$$\begin{aligned} 2x-5y&=1 \\ 2x-5(x-5)&=1 \\ 2x-5x+25&=1 \\ -3x&=-24 \\ x&=8 \end{aligned}$$

Now, we substitute $x=8$ into the first equation and solve for $y$.

$$\begin{aligned} -(8)+y&=-5 \\ y&=3 \end{aligned}$$

Our solution is $(8, 3)$.

Check the solution by substituting $(8, 3)$ into both equations.

$$-x + y = -5$$
$$-(8) + (3) = -5 \text{ True}$$
$$2x - 5y = 1$$
$$2(8) - 5(3) = 1 \text{ True}$$

### Exercise 5.1.3

Solve the following system of equations by substitution.

$$x = y + 3$$
$$4 = 3x - 2y$$

*Answer available at the end of this section.*

### Q&A

**Can the substitution method be used to solve any linear system in two variables?**

*Yes, but the method works best if one of the equations contains a coefficient of $1$ or $-1$ so that we do not have to deal with fractions.*

## Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the addition method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

### How to: Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

### Example 5.1.4: Solving a System by the Addition Method

Solve the given system of equations by addition.

$$x + 2y = -1$$
$$-x + y = 3$$

**Solution**

Both equations are already set equal to a constant. Notice that the coefficient of $x$ in the second equation, $-1$, is the opposite of the coefficient of $x$ in the first equation, $1$. We can add the two equations to eliminate $x$ without needing to multiply by a constant.

$$\begin{aligned} x+2y &= -1 \\ \underline{-x+y &= 3} \\ 3y &= 2 \end{aligned}$$

Now that we have eliminated $x$, we can solve the resulting equation for $y$.

$$\begin{aligned} 3y &= 2 \\ y &= \frac{2}{3} \end{aligned}$$

Then, we substitute this value for $y$ into one of the original equations and solve for $x$.

$$\begin{aligned} -x+y &= 3 \\ -x+\frac{2}{3} &= 3 \\ -x &= 3-\frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3},\frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} x+2y &= -1 \\ \left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) &= \\ -\frac{7}{3}+\frac{4}{3} &= -\frac{3}{3} \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

**Analysis**

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure $5.1.6$ to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

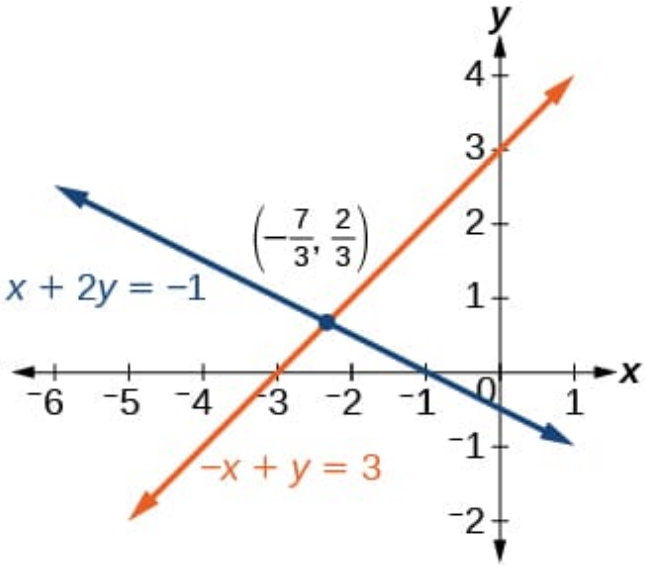



Figure 5.1.6

## Example 5.1.5: Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$$\begin{aligned} 3x+5y &= -11 \\ x-2y &= 11 \end{aligned}$$

**Solution**

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has $x$. So if we multiply the second equation by $-3$,the *x*-terms will add to zero.

$$\begin{aligned} x-2y &= 11 \\ -3(x-2y) &= -3(11) \qquad \text{Multiply both sides by } -3. \\ -3x+6y &= -33 \qquad \text{Use the distributive property.} \end{aligned}$$

Now, let's add them.

$$\begin{aligned} 3x+5y &= -11 \\ \underline{-3x+6y} &= \underline{-33} \\ 11y &= -44 \\ y &= -4 \end{aligned}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for $x$.

$$\begin{aligned} 3x+5y &= -11 \\ 3x+5(-4) &= -11 \\ 3x-20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3,-4)$. See Figure 5.1.7. Check the solution in the original second equation.

$$\begin{aligned} x-2y &= 11 \\ (3)-2(-4) &= 3+8 \\ &= 11 \qquad \text{True} \end{aligned}$$

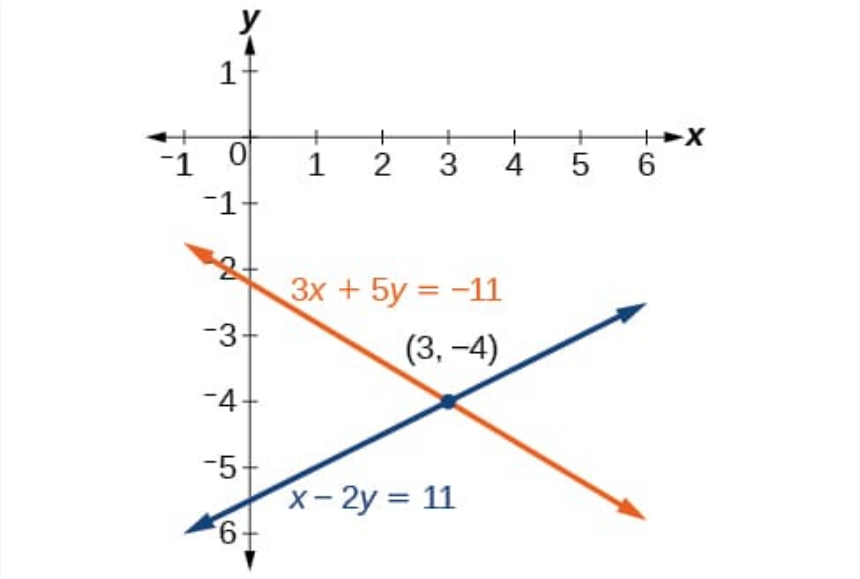



Figure 5.1.7

## Exercise 5.1.4

Solve the system of equations by addition.

$$\begin{aligned} 2x - 7y &= 2 \\ 3x + y &= -20 \end{aligned}$$

*Answer available at the end of this section.*

## Example 5.1.6: Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$$\begin{aligned} 2x + 3y &= -16 \\ 5x - 10y &= 30 \end{aligned}$$

**Solution**

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate $x$ by multiplying the first equation by $-5$ and the second equation by $2$.

$$\begin{aligned} -5(2x + 3y) &= -5(-16) \\ -10x - 15y &= 80 \\ 2(5x - 10y) &= 2(30) \\ 10x - 20y &= 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{aligned} -10x - 15y &= 80 \\ \underline{10x - 20y} &= \underline{60} \\ -35y &= 140 \\ y &= -4 \end{aligned}$$

Substitute $y = -4$ into the original first equation.

$$2x + 3(-4) = -16$$
$$2x - 12 = -16$$
$$2x = -4$$
$$x = -2$$

The solution is $(-2, -4)$. Check it in the other equation.

$$5x - 10y = 30$$
$$5(-2) - 10(-4) = 30$$
$$-10 + 40 = 30$$
$$30 = 30$$

See Figure 5.1.8.

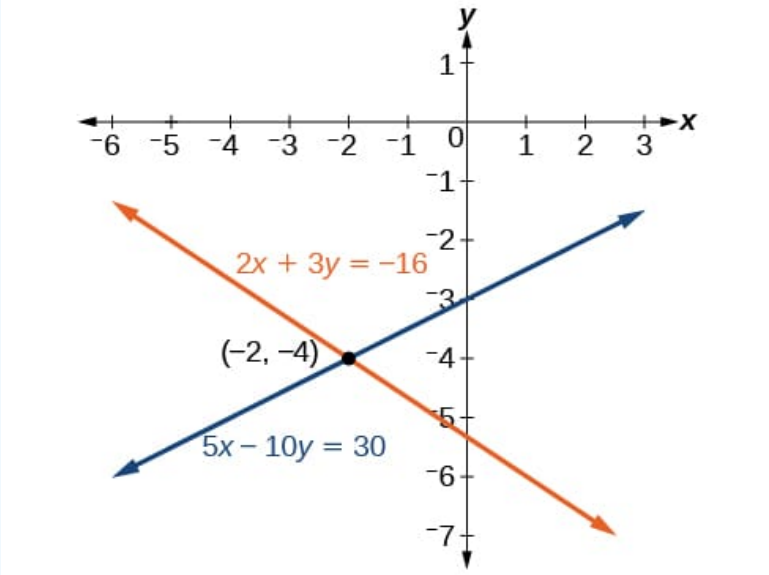



Figure 5.1.8

## Example 5.1.7: Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

$$\frac{x}{3} + \frac{y}{6} = 3$$
$$\frac{x}{2} - \frac{y}{4} = 1$$

**Solution**

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$6\left(\frac{x}{3} + \frac{y}{6}\right) = 6(3)$$
$$2x + y = 18$$
$$4\left(\frac{x}{2} - \frac{y}{4}\right) = 4(1)$$
$$2x - y = 4$$

Now multiply the second equation by $-1$ so that we can eliminate the *x*-variable.

$$-1(2x - y) = -1(4)$$
$$-2x + y = -4$$

Add the two equations to eliminate the $x$-variable and solve the resulting equation.

$$\begin{aligned} 2x+y &= 18 \\ -2x+y &= -4 \\ 2y &= 14 \\ y &= 7 \end{aligned}$$

Substitute $y=7$ into the first equation.

$$\begin{aligned} 2x+(7) &= 18 \\ 2x &= 11 \\ x &= \frac{11}{2} \\ &= 7.5 \end{aligned}$$

The solution is $\left(\frac{11}{2}, 7\right)$. Check it in the other equation.

$$\begin{aligned} \frac{x}{2}-\frac{y}{4} &= 1 \\ \frac{\frac{11}{2}}{2}-\frac{7}{4} &= 1 \\ \frac{11}{4}-\frac{7}{4} &= 1 \\ \frac{4}{4} &= 1 \end{aligned}$$

## Exercise 5.1.5

Solve the system of equations by addition.

$$\begin{aligned} 2x+3y &= 8 \\ 3x+5y &= 10 \end{aligned}$$

*Answer available at the end of this section.*

## Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12=0$.

## Example 5.1.8: Solving an Inconsistent System of Equations

Solve the following system of equations.

$$\begin{aligned} x &= 9-2y \\ x+2y &= 13 \end{aligned}$$

**Solution**

We can approach this problem in two ways. Because one equation is already solved for $x$, the most obvious step is to use substitution.

$$\begin{aligned} x + 2y &= 13 \\ (9 - 2y) + 2y &= 13 \\ 9 + 0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9 - 2y \\ 2y &= -x + 9 \\ y &= -\frac{1}{2}x + \frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x + 2y &= 13 \\ 2y &= -x + 13 \\ y &= -\frac{1}{2}x + \frac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different $y$-intercepts. Therefore, the lines are parallel and do not intersect.

$$\begin{aligned} y &= -\frac{1}{2}x + \frac{9}{2} \\ y &= -\frac{1}{2}x + \frac{13}{2} \end{aligned}$$

**Analysis**

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure $5.1.9$.

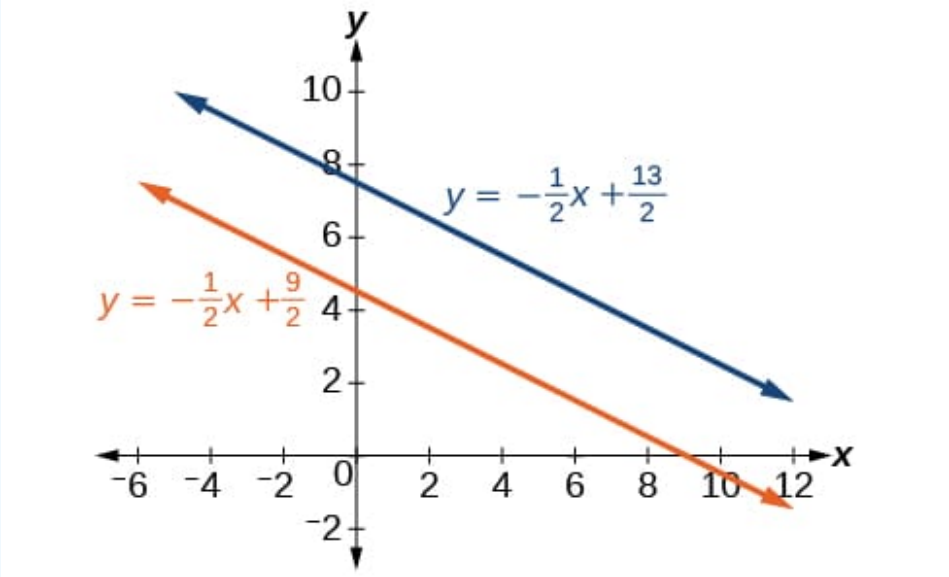



Figure $5.1.9$

### Exercise 5.1.6

Solve the following system of equations in two variables.

$$2y - 2x = 2$$
$$2y - 2x = 6$$

*Answer available at the end of this section.*

## Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0 = 0$.

### Example 5.1.9: Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the **addition method**.

$$x + 3y = 2$$
$$3x + 9y = 6$$

**Solution**

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating $x$. If we multiply both sides of the first equation by $-3$, then we will be able to eliminate the x-variable.

$$x + 3y = 2$$
$$(-3)(x + 3y) = (-3)(2)$$
$$-3x - 9y = -6$$

Now add the equations.

$$\begin{aligned} -3x - 9y &= -6 \\ +\ 3x + 9y &= \underline{6} \\ 0 &= 0 \end{aligned}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

**Analysis**

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x + 3y &= 2 \\ 3y &= -x + 2 \\ y &= -\frac{1}{3}x + \frac{2}{3} \\ 3x + 9y &= 6 \\ 9y &= -3x + 6 \\ y &= -\frac{3}{9}x + \frac{6}{9} \\ y &= -\frac{1}{3}x + \frac{2}{3} \end{aligned}$$

See Figure 5.1.10. Notice the results are the same. The general solution to the system is $\left(x, -\frac{1}{3}x + \frac{2}{3}\right)$.

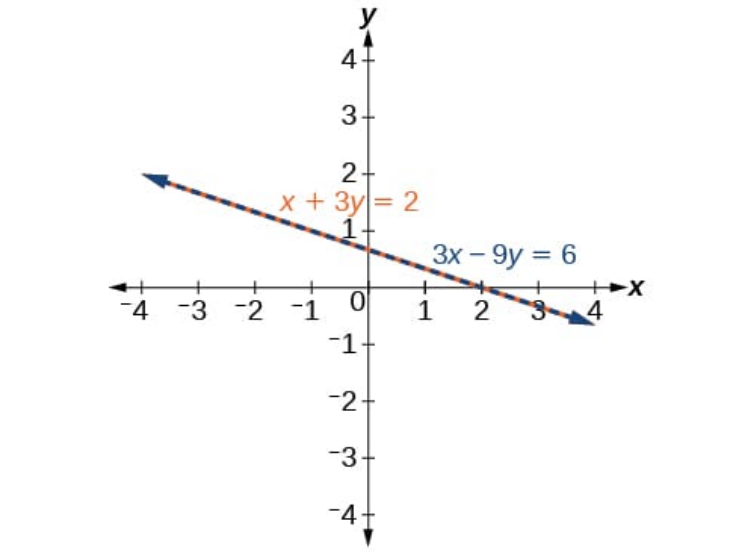



Figure 5.1.10

**Exercise 5.1.7**

Solve the following system of equations in two variables.

$$y - 2x = 5$$
$$-3y + 6x = -15$$

*Answer available at the end of this section.*

## Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where $x$=quantity and $p$=price. The revenue function is shown in orange in Figure 5.1.11.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 5.1.11. The $x$-axis represents quantity in hundreds of units. The $y$-axis represents either cost or revenue in hundreds of dollars.

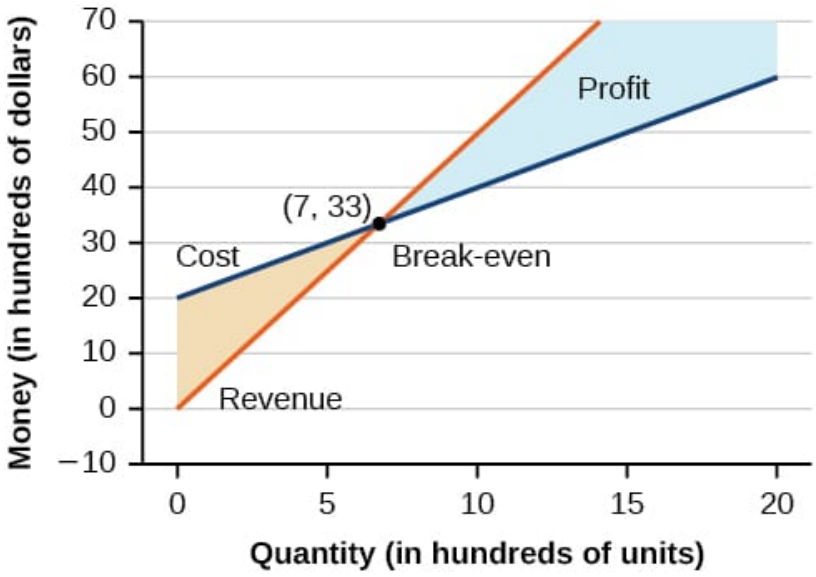



Figure 5.1.11

The point at which the two lines intersect is called the break-even point. We can see from the graph that if $700$ units are produced, the cost is $\$3,300$ and the revenue is also $\$3,300$. In other words, the company breaks even if they produce and sell $700$ units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

## Example 5.1.10: Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35,000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

**Solution**

Write the system of equations using $y$ to replace function notation.

$$\begin{aligned} y &= 0.85x + 35,000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35,000$ from the first equation into the second equation and solve for $x$.

$$\begin{aligned} 0.85x + 35,000 &= 1.55x \\ 35,000 &= 0.7x \\ 50,000 &= x \end{aligned}$$

Then, we substitute $x = 50,000$ into either the cost function or the revenue function.

$$1.55(50,000) = 77,500$$

The break-even point is $(50,000, 77,500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$\begin{aligned} P(x) &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

The profit function is $P(x) = 0.7x - 35,000$.

**Analysis**

The cost to produce $50,000$ units is $\$77,500$, and the revenue from the sales of $50,000$ units is also $\$77,500$. To make a profit, the business must produce and sell more than $50,000$ units. See Figure $5.1.12$.

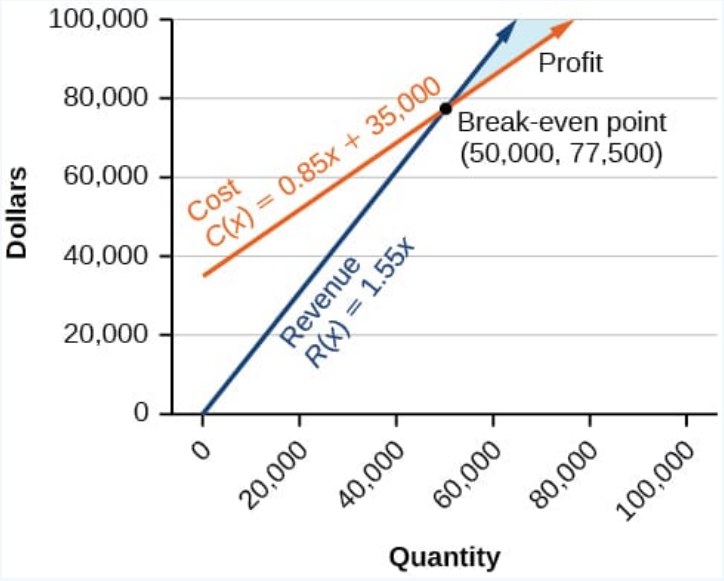



Figure $5.1.12$

We see from the graph in Figure $5.1.13$ that the profit function has a negative value until $x = 50,000$, when the graph crosses the $x$-axis. Then, the graph emerges into positive $y$-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is $0$. The area to the left of the break-even point represents operating at a loss.

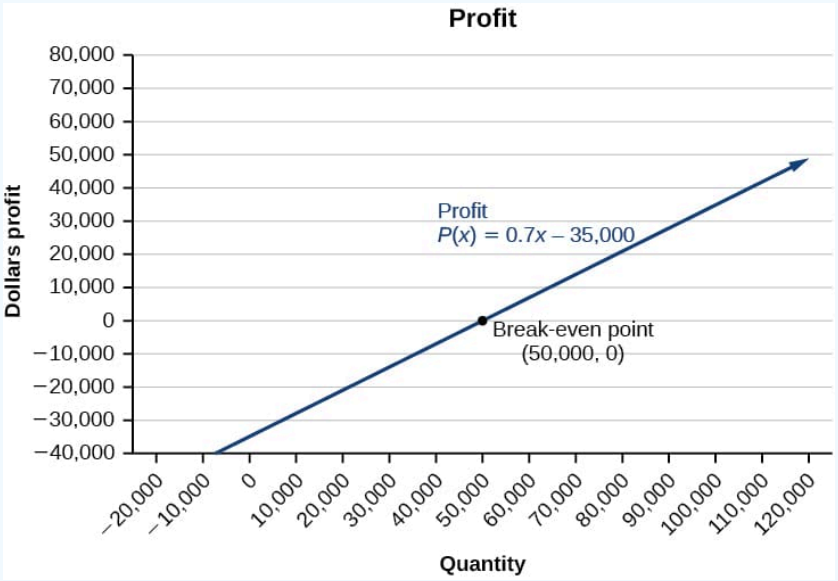



Figure $5.1.13$

## Example 5.1.11: Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000$. How many children and how many adults bought tickets?

**Solution**

Let $c$ = the number of children and $a$ = the number of adults in attendance.

The total number of people is $2,000$. We can use this to write an equation for the number of people at the circus that day.

$$c + a = 2,000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is $\$70,000$. We can use this to write an equation for the revenue.

$$25c + 50a = 70,000$$

We now have a system of linear equations in two variables.

$$c + a = 2,000$$

$$25c + 50a = 70,000$$

In the first equation, the coefficient of both variables is $1$. We can quickly solve the first equation for either $c$ or $a$. We will solve for $a$.

$$\begin{aligned} c + a &= 2,000 \\ a &= 2,000 - c \end{aligned}$$

Substitute the expression $2,000 - c$ in the second equation fora a and solve for $c$.

$$\begin{aligned} 25c + 50(2,000 - c) &= 70,000 \\ 25c + 100,000 - 50c &= 70,000 \\ -25c &= -30,000 \\ c &= 1,200 \end{aligned}$$

Substitute $c = 1,200$ into the first equation to solve for $a$.

$$\begin{aligned} 1,200 + a &= 2,000 \\ a &= 800 \end{aligned}$$

We find that $1,200$ children and $800$ adults bought tickets to the circus that day.

## Exercise 5.1.8

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1,650$ meal tickets were bought for a total of, $\$14,200$ , how many children and how many adults bought meal tickets?

*Answer available at the end of this section.*

## Media

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using Elimination
- Applications of Systems of Equations


## Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See Example $5.1.1$.
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See Example $5.1.2$.
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See Example $5.1.3$.
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See Example $5.1.4$.
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See Example $5.1.5$, Example $5.1.6$, and Example $5.1.7$.
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See Example $5.1.8$.
- The solution to a system of dependent equations will always be true because both equations describe the same line. See Example $5.1.9$.
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See Example $5.1.10$ and Example $5.1.11$.

## Answers

**Exercise** $5.1.1$

Not a solution.

**Exercise** $5.1.2$

The solution to the system is the ordered pair $(-5, 3)$.

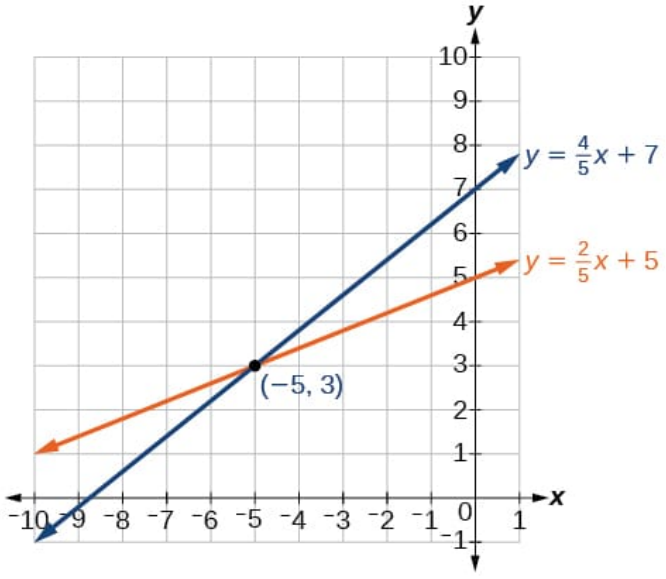



Figure $5.1.5$

**Exercise** $5.1.3$

$(-2, -5)$

**Exercise** $5.1.4$

$(-6, -2)$

**Exercise** $5.1.5$

$(10, -4)$

**Exercise** $5.1.6$

No solution. It is an inconsistent system.

**Exercise** $5.1.7$

The system is dependent so there are infinite solutions of the form $(x, 2x + 5)$.

**Exercise** $5.1.8$

$700$ children, $950$ adults

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# 5.2: Systems of Linear Equations - Three Variables

**Learning Objectives**

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

John received an inheritance of $\$12,000$ that he divided into three parts and invested in three ways: in a money-market fund paying $3\%$ annual interest; in municipal bonds paying $4\%$ annual interest; and in mutual funds paying $7\%$ annual interest. John invested $\$4,000$ more in municipal funds than in municipal bonds. He earned $\$670$ in interest the first year. How much did John invest in each type of fund?



Figure 5.2.1: (credit: "Elembis," Wikimedia Commons)

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

## Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution $(x, y, z)$, which we call an ordered triple. A system in upper triangular form looks like the following:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for $z$,and then we back-substitute to find $y$ and $x$. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The **solution set** to a three-by-three system is an ordered triple $(x, y, z)$. Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

## NUMBER OF POSSIBLE SolutionS

Figure $5.2.2$ and Figure $5.2.3$ illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple $(x, y, z)$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

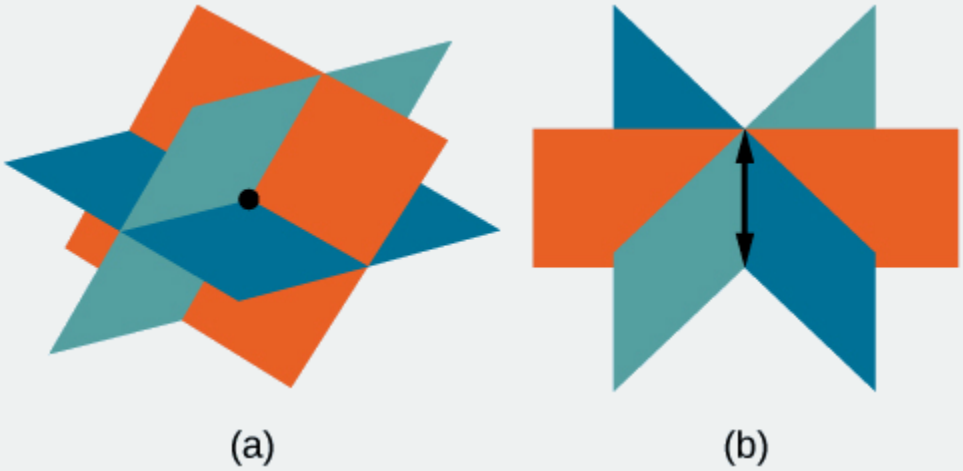



Figure $5.2.2$: (a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.

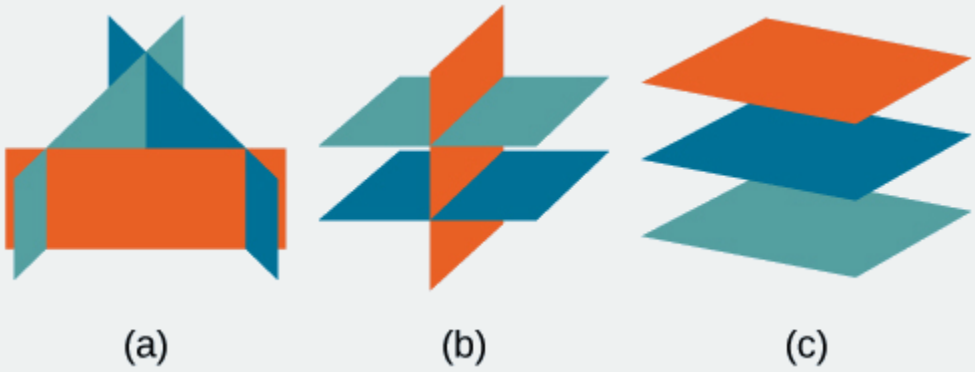



Figure $5.2.3$: All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

## Example 5.2.1: Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

**Solution**

We will check each equation by substituting in the values of the ordered triple for $x, y$, and $z$.

$$\begin{array}{rrr} 6x - 4y + 5z = 31 & 5x + 2y + 2z = 13 \\ x + y + z = 2 \quad 6(3) - 4(-2) + 5(1) = 31 & 5(3) + 2(-2) + 2(1) = 13 \\ (3) + (-2) + (1) = 2 \quad 18 + 8 + 5 = 31 & 15 - 4 + 2 = 13 \\ \text{True} \quad \text{True} & \text{True} \end{array}$$

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

## How to: Given a linear system of three equations, solve for three unknowns

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

## Example 5.2.2: Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{aligned} x - 2y + 3z &= 9 \ (1) \\ -x + 3y - z &= -6 \ (2) \\ 2x - 5y + 5z &= 17 \ (3) \end{aligned}$$

**Solution**

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate $x$ by adding equations (1) and (2).

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ \underline{-x + 3y - z} &\underline{= -6 \quad (2)} \\ y + 2z &= 3 \quad (4) \end{aligned}$$

The second step is multiplying equation (1) by $-2$ and adding the result to equation (3). These two steps will eliminate the variable $x$.

$$\begin{aligned} -2x + 4y - 6z &= -18 \ (1) \quad \text{multiplied by } -2 \\ \underline{2x - 5y + 5z} &\underline{= 17 \ (3)} \\ -y - z &= -1 \ (5) \end{aligned}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for $z$ by adding the two equations.

$$y + 2z = 3 \ (4)$$
$$\underline{-y - z = -1 \ (5)}$$
$$z = 2 \ (6)$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$x - 2y + 3z = 9 \ (1)$$
$$y + 2z = 3 \ (4)$$
$$z = 2 \ (6)$$

Next, we back-substitute $z = 2$ into equation (4) and solve for $y$.

$$y + 2(2) = 3$$
$$y + 4 = 3$$
$$y = -1$$

Finally, we can back-substitute $z = 2$ and $y = -1$ into equation (1). This will yield the solution for $x$.

$$x - 2(-1) + 3(2) = 9$$
$$x + 2 + 6 = 9$$
$$x = 1$$

The solution is the ordered triple $(1, -1, 2)$. See Figure $5.2.4$.

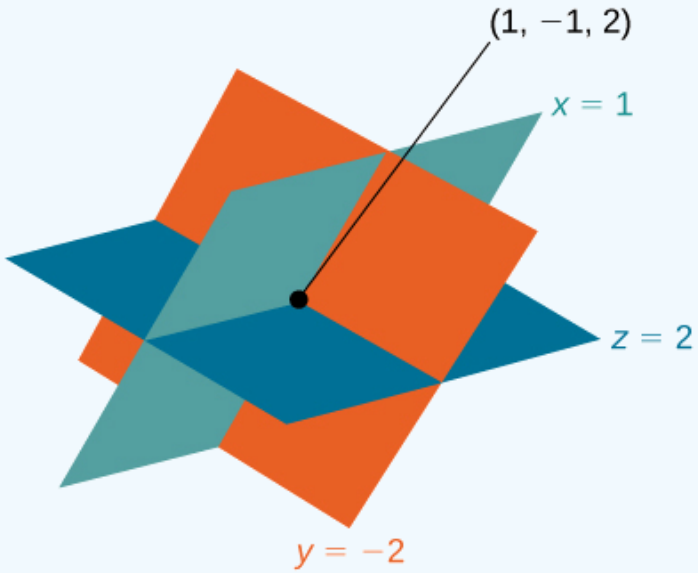



Figure $5.2.4$

## Example $5.2.3$: Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $\$12,000$ in three different funds: part in a money-market fund paying $3\%$ interest annually; part in municipal bonds paying $4\%$ annually; and the rest in mutual funds paying $7\%$ annually. John invested $\$4,000$ more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $\$670$. How much did he invest in each type of fund?

**Solution**

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$x = \text{amount invested in money-market fund}$$
$$y = \text{amount invested in municipal bonds}$$
$$z = \text{amount invested in mutual funds}$$

The first equation indicates that the sum of the three principal amounts is $\$12,000$.

$$x + y + z = 12,000$$

We form the second equation according to the information that John invested $\$4,000$ more in mutual funds than he invested in municipal bonds.

$$z = y + 4,000$$

The third equation shows that the total amount of interest earned from each fund equals $\$670$.

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.

$$\begin{aligned} x + y + z &= 12,000 \\ -y + z &= 4,000 \\ 0.03x + 0.04y + 0.07z &= 670 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by $100$. Thus,

$$\begin{aligned} x + y + z &= 12,000 \quad (1) \\ -y + z &= 4,000 \quad (2) \\ 3x + 4y + 7z &= 67,000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

$$\begin{aligned} x + y + z &= 12,000 \\ 3x + 4y + 7z &= 67,000 \\ -y + z &= 4,000 \end{aligned}$$

Step 2. Multiply equation (1) by $-3$ and add to equation (2). Write the result as row 2.

$$\begin{aligned} x + y + z &= 12,000 \\ y + 4z &= 31,000 \\ -y + z &= 4,000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

$$\begin{aligned} x + y + z &= 12,000 \\ y + 4z &= 31,000 \\ 5z &= 35,000 \end{aligned}$$

Step 4. Solve for $z$ in equation (3). Back-substitute that value in equation (2) and solve for $y$. Then, back-substitute the values for $z$ and $y$ into equation (1) and solve for $x$.

$$5z = 35,000$$
$$z = 7,000$$

$$y + 4(7,000) = 31,000$$
$$y = 3,000$$

$$x + 3,000 + 7,000 = 12,000$$
$$x = 2,000$$

John invested $\$2,000$ in a money-market fund, $\$3,000$ in municipal bonds, and $\$7,000$ in mutual funds.

### Exercise 5.2.1

Solve the system of equations in three variables.

$$2x + y - 2z = -1$$
$$3x - 3y - z = 5$$
$$x - 2y + 3z = 6$$

*Answer available at the end of this section.*

## Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3 = 7$ or some other contradiction.

### Example 5.2.4: Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$$x - 3y + z = 4 \tag{5.2.1}$$
$$-x + 2y - 5z = 3 \tag{5.2.2}$$
$$5x - 13y + 13z = 8 \tag{5.2.3}$$

**Solution**

Looking at the coefficients of $x$, we can see that we can eliminate $x$ by adding Equation 5.2.1 to Equation 5.2.2.

$$\begin{aligned} x - 3y + z &= 4 \quad (1) \\ \underline{-x + 2y - 5z} &\underline{= 3 \quad (2)} \\ -y - 4z &= 7 \quad (4) \end{aligned}$$

Next, we multiply equation (1) by $-5$ and add it to equation (3).

$$\begin{aligned} -5x + 15y - 5z &= -20 \quad (1) \quad \text{multiplied by } -5 \\ \underline{5x - 13y + 13z} &\underline{= 8 \quad (3)} \\ 2y + 8z &= -12 \quad (5) \end{aligned}$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$\begin{aligned} -2y-8z &= 14 \quad (4) \quad \text{multiplied by } 2 \\ \underline{2y+8z} &= \underline{-12} \quad (5) \\ 0 &= 2 \end{aligned}$$

The final equation $0 = 2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

**Analysis**

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

**Exercise 5.2.2**

Solve the system of three equations in three variables.

$$\begin{aligned} x+y+z &= 2 \\ y-3z &= 1 \\ 2x+y+5z &= 0 \end{aligned}$$

*Answer available at the end of this section.*

## Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

**Example 5.2.5: Finding the Solution to a Dependent System of Equations**

Find the solution to the given system of three equations in three variables.

$$\begin{aligned} 2x+y-3z &= 0 \quad (1) \\ 4x+2y-6z &= 0 \quad (2) \\ x-y+z &= 0 \quad (3) \end{aligned}$$

**Solution**

First, we can multiply equation (1) by $-2$ and add it to equation (2).

$$\begin{aligned} -4x-2y+6z &= 0 \quad (1) \quad \text{multiplied by } -2 \\ \underline{4x+2y-6z} &= \underline{0} \quad (2) \\ 0 &= 0 \end{aligned}$$

We do not need to proceed any further. The result we get is an identity, $0 = 0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by $-2$, and adding it to equation (1). We then perform the same steps as above and find the same result, $0 = 0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$$2x + y - 3z = 0$$
$$x - y + z = 0$$
$$3x - 2z = 0$$

We then solve the resulting equation for $z$.

$$3x - 2z = 0$$
$$z = \frac{3}{2}x$$

We back-substitute the expression for $z$ into one of the equations and solve for $y$.

$$2x + y - 3(\frac{3}{2}x) = 0$$
$$2x + y - \frac{9}{2}x = 0$$
$$y = \frac{9}{2}x - 2x$$
$$y = \frac{5}{2}x$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, $x$ can be any real number. The values of $y$ and $z$ are dependent on the value selected for $x$.

**Analysis**

As shown in Figure 5.2.5, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

CNX_Precalc_Figure_09_02_009_d0ee.jpg

Figure 5.2.5

### Q&A: Does the generic solution to a dependent system always have to be written in terms of $x$?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of $x$ and if needed $x$ and $y$.

### Exercise 5.2.3:

Solve the following system.

$$x + y + z = 7$$
$$3x - 2y - z = 4$$
$$x + 6y + 5z = 24$$

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

## Key Concepts

- A solution set is an ordered triple {(x,y,z)} that represents the intersection of three planes in space. See Example $5.2.1$.
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See Example $5.2.2$.
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See Example $5.2.3$.
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See Example $5.2.4$.
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See Example $5.2.5$.
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

## Contributors

- Jay Abramson (Arizona State University) with contributing authors. Textbook content produced by OpenStax College is licensed under a Creative Commons Attribution License 4.0 license. Download for free at https://openstax.org/details/books/precalculus.

## Answers

**Exercise** $5.2.1$

$(1, -1, 1)$

**Exercise** $5.2.2$

No solution.

**Exercise** $5.2.3$**:**

Infinite number of solutions of the form $(x, 4x - 11, -5x + 18)$.

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# 5.3: Systems of Nonlinear Equations and Inequalities - Two Variables

**Learning Objectives**

- Solve a system of nonlinear equations using substitution.
- Solve a system of nonlinear equations using elimination.
- Graph a nonlinear inequality.
- Graph a system of nonlinear inequalities.

Halley's Comet (Figure $5.3.1$) orbits the sun about once every $75$ years. Its path can be considered to be a very elongated ellipse. Other comets follow similar paths in space. These orbital paths can be studied using systems of equations. These systems, however, are different from the ones we considered in the previous section because the equations are not linear.



Figure $5.3.1$: Halley's Comet (credit: "NASA Blueshift"/Flickr)

In this section, we will consider the intersection of a parabola and a line, a circle and a line, and a circle and an ellipse. The methods for solving systems of nonlinear equations are similar to those for linear equations.

## Solving a System of Nonlinear Equations Using Substitution

A system of nonlinear equations is a system of two or more equations in two or more variables containing at least one equation that is not linear. Recall that a linear equation can take the form $Ax + By + C = 0$. Any equation that cannot be written in this form in nonlinear. The substitution method we used for linear systems is the same method we will use for nonlinear systems. We solve one equation for one variable and then substitute the result into the second equation to solve for another variable, and so on. There is, however, a variation in the possible outcomes.

### Intersection of a Parabola and a Line

There are three possible types of solutions for a system of nonlinear equations involving a parabola and a line.

## Possible Types of Solutions For Points of Intersection of a Parabola and a Line

Figure 5.3.2 illustrates possible solution sets for a system of equations involving a parabola and a line.

- No solution - The line will never intersect the parabola.
- One solution - The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions - The line crosses on the inside of the parabola and intersects the parabola at two points.

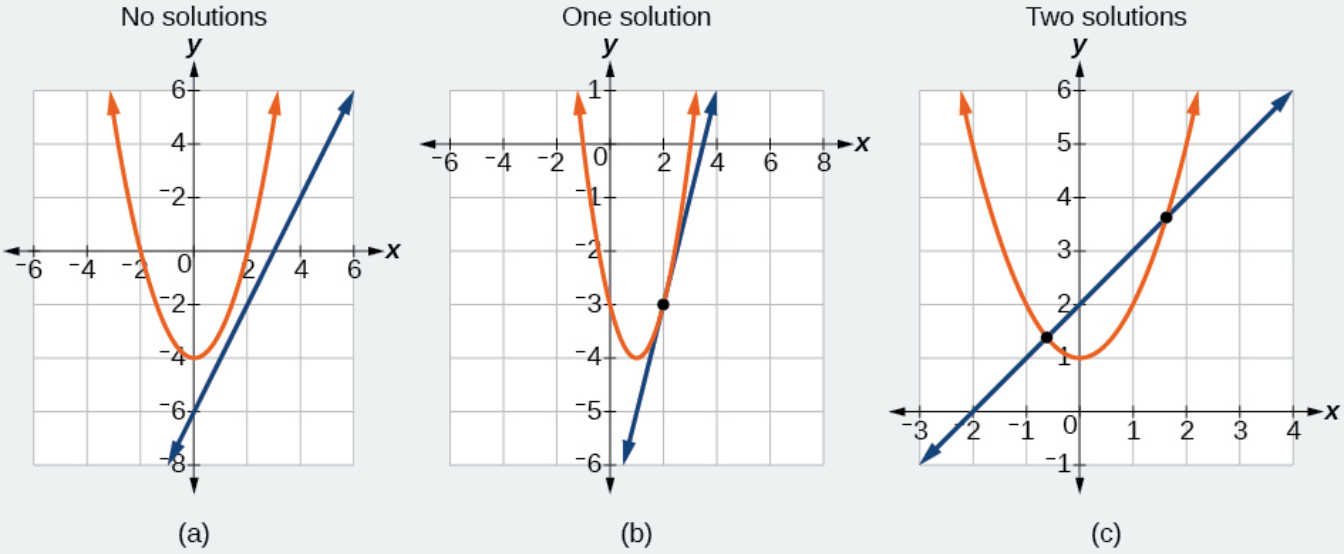



Figure 5.3.2

## How to: Given a system of equations containing a line and a parabola, find the solution

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the parabola equation.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

## Example 5.3.1: Solving a System of Nonlinear Equations Representing a Parabola and a Line

Solve the system of equations.

$$\begin{aligned} x - y &= -1 \\ y &= x^2 + 1 \end{aligned}$$

**Solution**

Solve the first equation for $x$ and then substitute the resulting expression into the second equation.

$$\begin{aligned} x - y &= -1 \\ x &= y - 1 \qquad \text{Solve for } x. \end{aligned}$$

$$\begin{aligned} y &= x^2 + 1 \\ y &= (y-1)^2 + 1 \qquad \text{Substitute expression for } x. \end{aligned}$$

Expand the equation and set it equal to zero.

$$\begin{aligned} y &= (y-1)^2 + 1 \\ &= (y^2 - 2y + 1) + 1 \\ &= y^2 - 2y + 2 \\ 0 &= y^2 - 3y + 2 \\ &= (y-2)(y-1) \end{aligned}$$

Solving for $y$ gives $y = 2$ and $y = 1$. Next, substitute each value for $y$ into the first equation to solve for $x$. Always substitute the value into the linear equation to check for extraneous solutions.

$$\begin{aligned} x - y &= -1 \\ x - (2) &= -1 \\ x &= 1 \\ x - (1) &= -1 \\ x &= 0 \end{aligned}$$

The solutions are $(1, 2)$ and $(0, 1)$,which can be verified by substituting these $(x, y)$ values into both of the original equations (Figure $5.3.3$).

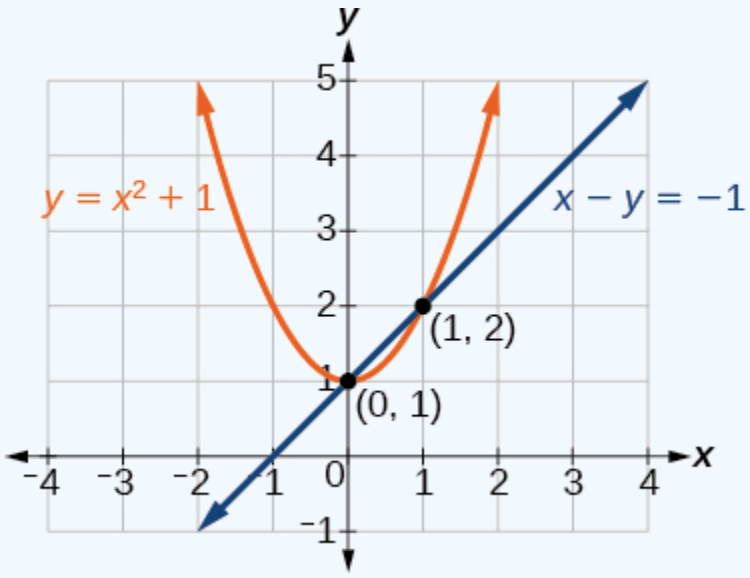



Figure $5.3.3$

**Q&A: Could we have substituted values for $y$ into the second equation to solve for $x$ in last example?**

Yes, but because $x$ is squared in the second equation this could give us extraneous solutions for $x$.

For $y = 1$

$$\begin{aligned} y &= x^2 + 1 \\ y &= x^2 + 1 \\ x^2 &= 0 \\ x &= \pm\sqrt{0} = 0 \end{aligned}$$

This gives us the same value as in the solution.

For $y = 2$

$$y = x^2 + 1$$
$$2 = x^2 + 1$$
$$x^2 = 1$$
$$x = \pm\sqrt{1} = \pm 1$$

Notice that $-1$ is an extraneous solution.

**Exercise 5.3.1**

Solve the given system of equations by substitution.

$$3x - y = -2$$
$$2x^2 - y = 0$$

*Answer available at the end of this section.*

## Intersection of a Circle and a Line

Just as with a parabola and a line, there are three possible outcomes when solving a system of equations representing a circle and a line.

**POSSIBLE TYPES OF SOLUTIONS FOR THE POINTS OF INTERSECTION OF A CIRCLE AND A LINE**

Figure 5.3.4 illustrates possible solution sets for a system of equations involving a circle and a line.

- No solution - The line does not intersect the circle.
- One solution - The line is tangent to the circle and intersects the circle at exactly one point.
- Two solutions - The line crosses the circle and intersects it at two points.

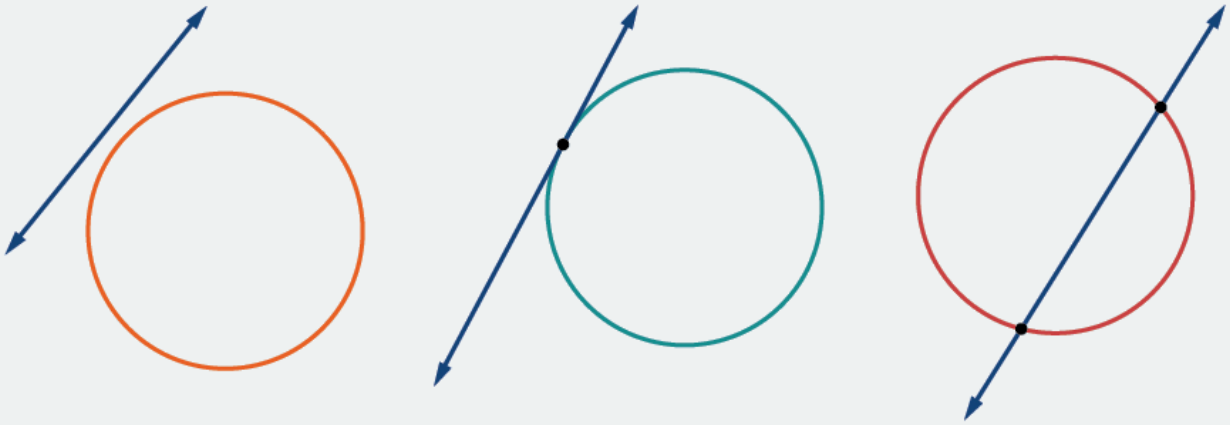



Figure 5.3.4

**How to: Given a system of equations containing a line and a circle, find the solution**

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the equation for the circle.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

## Example 5.3.2: Finding the Intersection of a Circle and a Line by Substitution

Find the intersection of the given circle and the given line by substitution.

$$\begin{aligned} x^2 + y^2 &= 5 \\ y &= 3x - 5 \end{aligned}$$

**Solution**

One of the equations has already been solved for $y$. We will substitute $y = 3x - 5$ into the equation for the circle.

$$\begin{aligned} x^2 + (3x - 5)^2 &= 5 \\ x^2 + 9x^2 - 30x + 25 &= 5 \\ 10x^2 - 30x + 20 &= 0 \end{aligned}$$

Now, we factor and solve for $x$.

$$\begin{aligned} 10(x2 - 3x + 2) &= 0 \\ 10(x - 2)(x - 1) &= 0 \\ x &= 2 \\ x &= 1 \end{aligned}$$

Substitute the two $x$-values into the original linear equation to solve for $y$.

$$\begin{aligned} y &= 3(2) - 5 \\ &= 1 \\ y &= 3(1) - 5 \\ &= -2 \end{aligned}$$

The line intersects the circle at $(2, 1)$ and $(1, -2)$,which can be verified by substituting these $(x, y)$ values into both of the original equations (Figure 5.3.5).

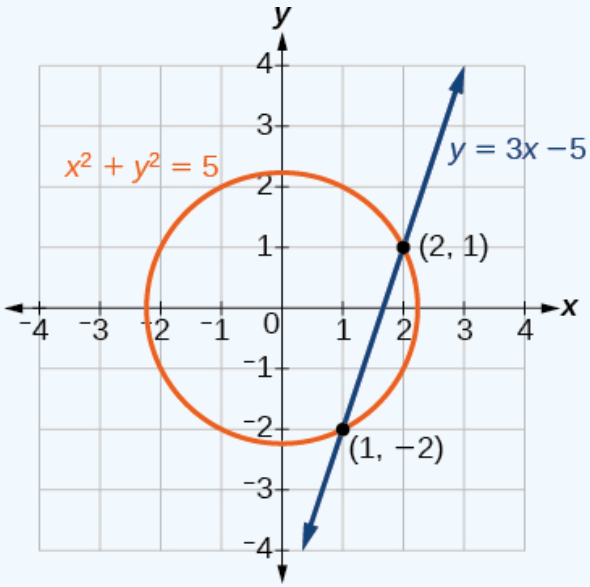



Figure 5.3.5

## Exercise 5.3.2

Solve the system of nonlinear equations.

$$x^2 + y^2 = 10$$
$$x - 3y = -10$$

*Answer available at the end of this section.*

## Solving a System of Nonlinear Equations Using Elimination

We have seen that substitution is often the preferred method when a system of equations includes a linear equation and a nonlinear equation. However, when both equations in the system have like variables of the second degree, solving them using elimination by addition is often easier than substitution. Generally, elimination is a far simpler method when the system involves only two equations in two variables (a two-by-two system), rather than a three-by-three system, as there are fewer steps. As an example, we will investigate the possible types of solutions when solving a system of equations representing a circle and an ellipse.

### POSSIBLE TYPES OF SOLUTIONS FOR THE POINTS OF INTERSECTION OF A CIRCLE AND AN ELLIPSE

Figure 5.3.6 illustrates possible solution sets for a system of equations involving a circle and an ellipse.

- No solution - The circle and ellipse do not intersect. One shape is inside the other or the circle and the ellipse are a distance away from the other.
- One solution - The circle and ellipse are tangent to each other, and intersect at exactly one point.
- Two solutions - The circle and the ellipse intersect at two points.
- Three solutions - The circle and the ellipse intersect at three points.
- Four solutions - The circle and the ellipse intersect at four points.

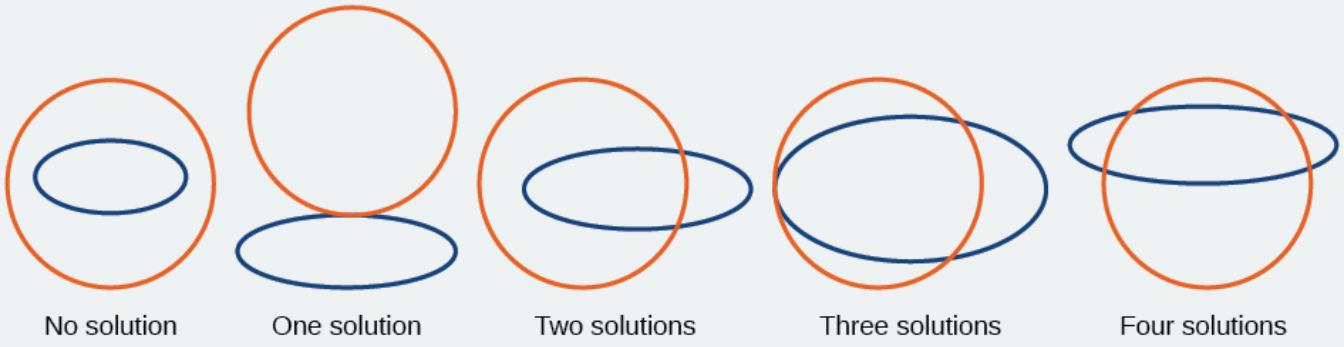



Figure 5.3.6

### Example 5.3.3: Solving a System of Nonlinear Equations Representing a Circle and an Ellipse

Solve the system of nonlinear equations.

$$x^2 + y^2 = 26 \quad (1)$$
$$3x^2 + 25y^2 = 100 \quad (2)$$

**Solution**

Let's begin by multiplying equation (1) by $-3$,and adding it to equation (2).

$$\begin{aligned} (-3)(x^2 + y^2) &= (-3)(26) \\ -3x^2 - 3y^2 &= -78 \\ 3x^2 + 25y^2 &= 100 \\ \hline 22y^2 &= 22 \end{aligned}$$

After we add the two equations together, we solve for $y$.

$$\begin{aligned} y^2 &= 1 \\ y &= \pm\sqrt{1} = \pm 1 \end{aligned}$$

Substitute $y = \pm 1$ into one of the equations and solve for $x$.

$$\begin{aligned} x^2 + (1)^2 &= 26 \\ x^2 + 1 &= 26 \\ x^2 &= 25 \\ x &= \pm\sqrt{25} = \pm 5 \\ x^2 + (-1)^2 &= 26 \\ x^2 + 1 &= 26 \\ x^2 &= \pm\sqrt{25} = \pm 5 \end{aligned}$$

There are four solutions: $(5, 1)$, $(-5, 1)$, $(5, -1)$,and $(-5, -1)$. See Figure $5.3.7$.

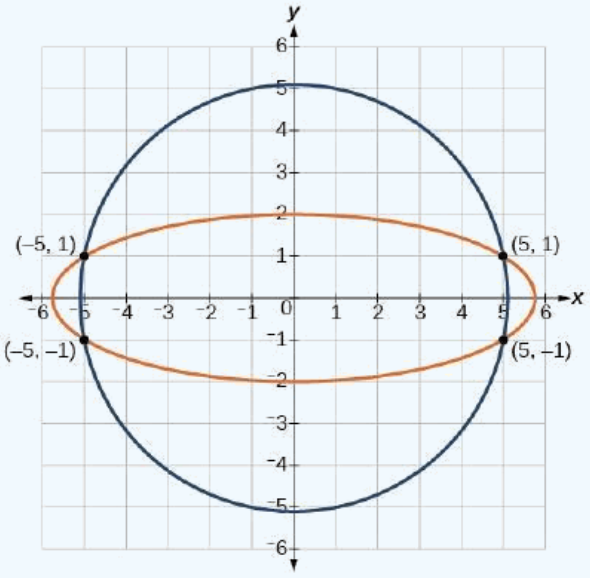



Figure $5.3.7$

## Exercise $5.3.3$

Find the solution set for the given system of nonlinear equations.

$$\begin{aligned} 4x^2 + y^2 &= 13 \\ x^2 + y^2 &= 10 \end{aligned}$$

*Answer available at the end of this section.*

# Graphing a Nonlinear Inequality

All of the equations in the systems that we have encountered so far have involved equalities, but we may also encounter systems that involve inequalities. We have already learned to graph linear inequalities by graphing the corresponding equation, and then shading the region represented by the inequality symbol. Now, we will follow similar steps to graph a nonlinear **inequality** so that we can learn to solve systems of nonlinear inequalities. A **nonlinear inequality** is an inequality containing a nonlinear expression. Graphing a nonlinear inequality is much like graphing a linear inequality.

Recall that when the inequality is greater than, $y > a$,or less than, $y < a$,the graph is drawn with a dashed line. When the inequality

is greater than or equal to, $y \geq a$,or less than or equal to, $y \leq a$,the graph is drawn with a solid line. The graphs will create regions in the plane, and we will test each region for a solution. If one point in the region works, the whole region works. That is the region we shade (Figure $5.3.8$).

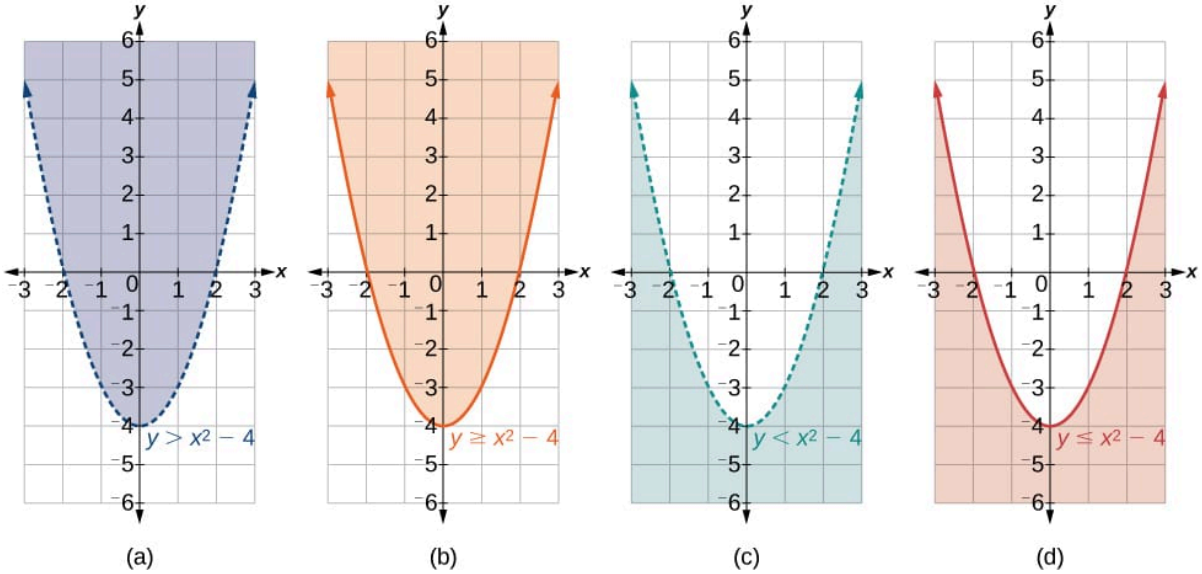



Figure $5.3.8$: (a) an example of $y > a$; (b) an example of $y \geq a$; (c) an example of $y < a$; (d) an example of $y \leq a$

## How to: Given an inequality bounded by a parabola, sketch a graph

1. Graph the parabola as if it were an equation. This is the boundary for the region that is the solution set.
2. If the boundary is included in the region (the operator is $\leq$ or $\geq$), the parabola is graphed as a solid line.
3. If the boundary is not included in the region (the operator is $<$ or $>$), the parabola is graphed as a dashed line.
4. Test a point in one of the regions to determine whether it satisfies the inequality statement. If the statement is true, the solution set is the region including the point. If the statement is false, the solution set is the region on the other side of the boundary line.
5. Shade the region representing the solution set.

## Example $5.3.4$: Graphing an Inequality for a Parabola

Graph the inequality $y > x^2 + 1$.

**Solution**

First, graph the corresponding equation $y = x^2 + 1$. Since $y > x^2 + 1$ has a greater than symbol, we draw the graph with a dashed line. Then we choose points to test both inside and outside the parabola. Let's test the points

$(0, 2)$ and $(2, 0)$. One point is clearly inside the parabola and the other point is clearly outside.

$$\begin{aligned} y &> x^2 + 1 \\ 2 &> (0)^2 + 1 \\ 2 &> 1 \qquad \text{True} \end{aligned}$$

$$\begin{aligned} 0 &> (2)^2 + 1 \\ 0 &> 5 \qquad \text{False} \end{aligned}$$

The graph is shown in Figure $5.3.9$. We can see that the solution set consists of all points inside the parabola, but not on the graph itself.

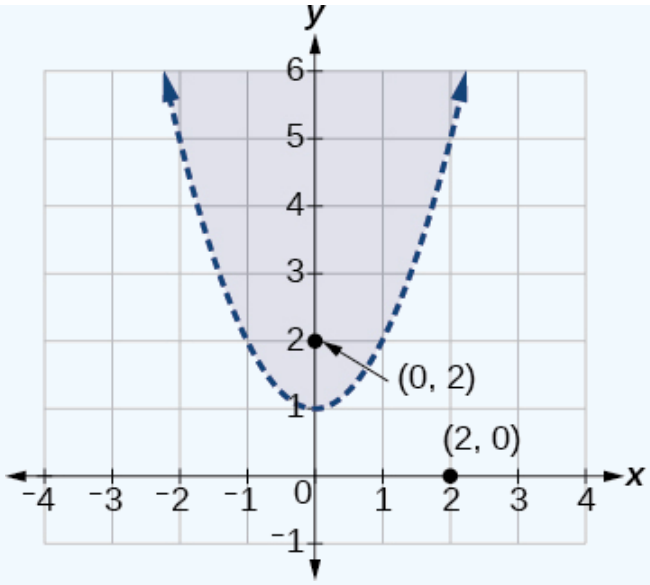



Figure 5.3.9

## Graphing a System of Nonlinear Inequalities

Now that we have learned to graph nonlinear inequalities, we can learn how to graph systems of nonlinear inequalities. A **system of nonlinear inequalities** is a system of two or more inequalities in two or more variables containing at least one inequality that is not linear. Graphing a system of nonlinear inequalities is similar to graphing a system of linear inequalities. The difference is that our graph may result in more shaded regions that represent a solution than we find in a system of linear inequalities. The solution to a nonlinear system of inequalities is the region of the graph where the shaded regions of the graph of each inequality overlap, or where the regions intersect, called the **feasible region**.

### How to: Given a system of nonlinear inequalities, sketch a graph

1. Find the intersection points by solving the corresponding system of nonlinear equations.
2. Graph the nonlinear equations.
3. Find the shaded regions of each inequality.
4. Identify the feasible region as the intersection of the shaded regions of each inequality or the set of points common to each inequality.

### Example 5.3.5: Graphing a System of Inequalities

Graph the given system of inequalities.

$$x^2 - y \leq 0$$
$$2x^2 + y \leq 12$$

**Solution**

These two equations are clearly parabolas. We can find the points of intersection by the elimination process: Add both equations and the variable $y$ will be eliminated. Then we solve for $x$.

$$\begin{aligned} x^2 - y &= 0 \\ \underline{2x^2 + y} &= \underline{12} \\ 3x^2 &= 12 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Substitute the $x$-values into one of the equations and solve for $y$.

$$\begin{aligned} x^2 - y &= 0 \\ (2)^2 - y &= 0 \\ 4 - y &= 0 \\ y &= 4 \end{aligned}$$

$$\begin{aligned} (-2)^2 - y &= 0 \\ 4 - y &= 0 \\ y &= 4 \end{aligned}$$

The two points of intersection are $(2, 4)$ and $(-2, 4)$. Notice that the equations can be rewritten as follows.

$$\begin{aligned} x^2 - y &\leq 0 \\ x^2 &\leq y \\ y &\geq x^2 \end{aligned}$$

$$\begin{aligned} 2x^2 + y &\leq 12 \\ y &\leq -2x^2 + 12 \end{aligned}$$

Graph each inequality. See Figure 5.3.10. The feasible region is the region between the two equations bounded by $2x^2 + y \leq 12$ on the top and $x^2 - y \leq 0$ on the bottom.

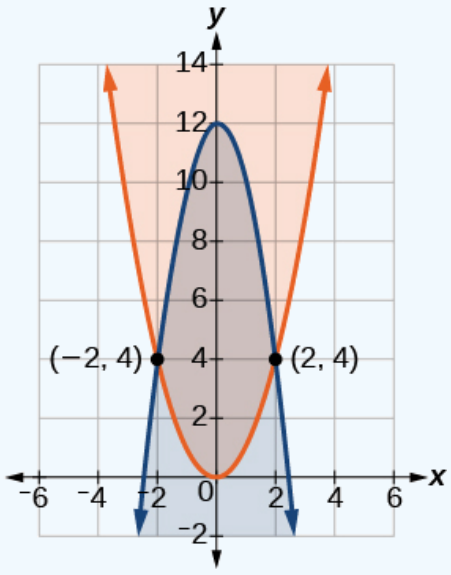



Figure 5.3.10

## Exercise 5.3.5

Graph the given system of inequalities.

$$y \geq x^2 - 1$$
$$x - y \geq -1$$

*Answer available at the end of this section.*

**Media**

Access these online resources for additional instruction and practice with nonlinear equations.

- Solve a System of Nonlinear Equations Using Substitution
- Solve a System of Nonlinear Equations Using Elimination

## Key Concepts

- There are three possible types of solutions to a system of equations representing a line and a parabola: (1) no solution, the line does not intersect the parabola; (2) one solution, the line is tangent to the parabola; and (3) two solutions, the line intersects the parabola in two points. See Example $5.3.1$.
- There are three possible types of solutions to a system of equations representing a circle and a line: (1) no solution, the line does not intersect the circle; (2) one solution, the line is tangent to the parabola; (3) two solutions, the line intersects the circle in two points. See Example $5.3.2$.
- There are five possible types of solutions to the system of nonlinear equations representing an ellipse and a circle: (1) no solution, the circle and the ellipse do not intersect; (2) one solution, the circle and the ellipse are tangent to each other; (3) two solutions, the circle and the ellipse intersect in two points; (4) three solutions, the circle and ellipse intersect in three places; (5) four solutions, the circle and the ellipse intersect in four points. See Example $5.3.3$.
- An inequality is graphed in much the same way as an equation, except for > or <, we draw a dashed line and shade the region containing the solution set. See Example $5.3.4$.
- Inequalities are solved the same way as equalities, but solutions to systems of inequalities must satisfy both inequalities. See Example $5.3.5$.

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### Answers

**Exercise** $5.3.1$

$\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(2, 8)$

**Exercise** $5.3.2$

$(-1, 3)$

**Exercise** $5.3.3$

$(1, 3), (1, -3), (-1, 3), (-1, -3)$

**Exercise** $5.3.5$

Shade the area bounded by the two curves, above the quadratic and below the line.

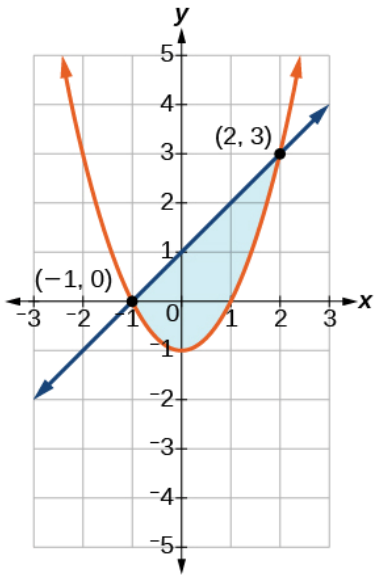



Figure 5.3.11

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# 5.4: Partial Fractions

**Learning Objectives**

Decompose $\frac{P(x)}{Q(x)}$, where

- $Q(x)$ has only nonrepeated linear factors.
- $Q(x)$ has repeated linear factors.
- $Q(x)$ has a nonrepeated irreducible quadratic factor.
- $Q(x)$ has a repeated irreducible quadratic factor.

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized—the decomposition of rational expressions. Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

## Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Only Nonrepeated Linear Factors

Recall the algebra regarding adding and subtracting rational expressions. These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at **partial fraction decomposition**, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified **rational expression** to the original expressions, called the **partial fractions**.

For example, suppose we add the following fractions:

$$\frac{2}{x-3}+\frac{-1}{x+2}$$

We would first need to find a common denominator: $(x+2)(x-3)$.

Next, we would write each expression with this common denominator and find the sum of the terms.

$$\begin{aligned}\frac{2}{x-3}\left(\frac{x+2}{x+2}\right)+\frac{-1}{x+2}\left(\frac{x-3}{x-3}\right)&=\frac{2x+4-x+3}{(x+2)(x-3)}\\&=\frac{x+7}{x^2-x-6}\end{aligned}$$

Partial fraction **decomposition** is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.

$$\underbrace{\frac{x+7}{x^2-x-6}}_{\text{Simplified sum}}=\underbrace{\frac{2}{x-3}+\frac{-1}{x+2}}_{\text{Partial fraction decomposition}}$$

We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of $x^2-x-6$ are $(x-3)(x+2)$, the denominators of the decomposed rational expression. So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.

## PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS NONREPEATED LINEAR FACTORS

The *partial fraction decomposition* of $\frac{P(x)}{Q(x)}$ when $Q(x)$ has nonrepeated linear factors and the degree of $P(x)$ is less than the degree of $Q(x)$ is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{5.4.1}$$

### How to: Given a rational expression with distinct linear factors in the denominator, decompose it

1. Use a variable for the original numerators, usually $A$, $B$, or $C$, depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use $A_n$ for each numerator

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

### Example 5.4.1: Decomposing a Rational Function with Distinct Linear Factors

Decompose the given rational expression with distinct linear factors.

$$\frac{3x}{(x+2)(x-1)}$$

**Solution**

We will separate the denominator factors and give each numerator a symbolic label, like $A$, $B$ ,or $C$.

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$$

Multiply both sides of the equation by the common denominator to eliminate the fractions:

$$(x+2)(x-1)\left[\frac{3x}{(x+2)(x-1)}\right] = (x+2)(x-1)\left[\frac{A}{(x+2)}\right] + (x+2)(x-1)\left[\frac{B}{(x-1)}\right]$$

The resulting equation is

$$3x = A(x-1) + B(x+2)$$

Expand the right side of the equation and collect like terms.

$$3x = Ax - A + Bx + 2B$$
$$3x = (A+B)x - A + 2B$$

Set up a system of equations associating corresponding coefficients.

$$3 = A + B$$
$$0 = -A + 2B$$

Add the two equations and solve for $B$.

$$3 = A + B$$
$$\underline{0 = -A + 2B}$$
$$3 = 0 + 3B$$
$$1 = B$$

Substitute $B = 1$ into one of the original equations in the system.

$$3 = A + 1$$
$$2 = A$$

Thus, the partial fraction decomposition is

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Another method to use to solve for $A$ or $B$ is by considering the equation that resulted from eliminating the fractions and substituting a value for $x$ that will make either the $A-$ or $B-$term equal 0. If we let $x = 1$, the

$A-$ term becomes 0 and we can simply solve for $B$.

$$3x = A(x-1) + B(x+2)$$
$$3(1) = A[(1)-1] + B[(1)+2]$$
$$3 = 0 + 3B$$
$$1 = B$$

Next, either substitute $B = 1$ into the equation and solve for $A$, or make the $B-$term $0$ by substituting $x = -2$ into the equation.

$$3x = A(x-1) + B(x+2)$$
$$3(-2) = A[(-2)-1] + B[(-2)+2]$$
$$-6 = -3A + 0$$
$$\frac{-6}{-3} = A$$
$$2 = A$$

We obtain the same values for $A$ and $B$ using either method, so the decompositions are the same using either method.

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Although this method is not seen very often in textbooks, we present it here as an alternative that may make some partial fraction decompositions easier. It is known as the **Heaviside method**, named after Charles Heaviside, a pioneer in the study of electronics.

## Exercise 5.4.1

Find the partial fraction decomposition of the following expression.

$$\frac{x}{(x-3)(x-2)}$$

*Answer available at the end of this section.*

# Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Repeated Linear Factors

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors. We must remember that we account for repeated factors by writing each factor in increasing powers.

## PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS REPEATED LINEAR FACTORS

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated linear factor occurringn n times and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{5.4.2}$$

Write the denominator powers in increasing order.

## How to: decompose a rational expression with repeated linear factors

1. Use a variable like $A$, $B$, or $C$ for the numerators and account for increasing powers of the denominators.
$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{5.4.3}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

## Example 5.4.2: Decomposing with Repeated Linear Factors

Decompose the given rational expression with repeated linear factors.

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}$$

**Solution**

The denominator factors are $x(x-2)^2$. To allow for the repeated factor of $(x-2)$, the decomposition will include three denominators: $x$, $(x-2)$, and $(x-2)^2$. Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

Next, we multiply both sides by the common denominator.

$$x(x-2)^2\left[\frac{-x^2+2x+4}{x(x-2)^2}\right] = \left[\frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}\right]x(x-2)^2$$
$$-x^2+2x+4 = A(x-2)^2 + Bx(x-2) + Cx$$

On the right side of the equation, we expand and collect like terms.

$$\begin{aligned} -x^2+2x+4 &= A(x^2-4x+4) + B(x^2-2x) + Cx \\ &= Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx \\ &= (A+B)x^2 + (-4A-2B+C)x + 4A \end{aligned}$$

Next, we compare the coefficients of both sides. This will give the system of equations in three variables:

$$-x^2+2x+4=(A+B)x^2+(-4A-2B+C)x+4A \tag{5.4.4}$$
$$A+B=-1 \tag{5.4.5}$$
$$-4A-2B+C=2 \tag{5.4.6}$$
$$4A=4 \tag{5.4.7}$$

Solving for $A$ in Equation 5.4.7, we have

$$\begin{aligned} 4A&=4\\ A&=1 \end{aligned}$$

Substitute $A=1$ into Equation 5.4.5.

$$\begin{aligned} A+B&=-1\\ (1)+B&=-1\\ B&=-2 \end{aligned}$$

Then, to solve for $C$, substitute the values for $A$ and $B$ into Equation 5.4.6.

$$\begin{aligned} -4A-2B+C&=2\\ -4(1)-2(-2)+C&=2\\ -4+4+C&=2\\ C&=2 \end{aligned}$$

Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}=\frac{1}{x}-\frac{2}{(x-2)}+\frac{2}{(x-2)^2}$$

**Exercise 5.4.2**

Find the partial fraction decomposition of the expression with repeated linear factors.

$$\frac{6x-11}{(x-1)^2}$$

*Answer available at the end of this section.*

## Decomposing $\frac{P(x)}{Q(x)}$, where $Q(x)$ Has a Nonrepeated Irreducible Quadratic Factor

So far, we have performed partial fraction decomposition with expressions that have had linear factors in the denominator, and we applied numerators $A$, $B$, or $C$ representing constants. Now we will look at an example where one of the factors in the denominator is a quadratic expression that does not factor. This is referred to as an irreducible quadratic factor. In cases like this, we use a linear numerator such as $Ax+B$, $Bx+C$, etc.

## DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS A NONREPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ such that $Q(x)$ has a nonrepeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$ is written as

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{(a_1x^2 + b1_x + c_1)} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)} + \cdots + \frac{A_nx + B_n}{(a_nx^2 + b_nx + c_n)} \tag{5.4.8}$$

The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: $A$, $B$, $C$, and so on.

## Howto: decompose a rational expression where the factors of the denominator are distinct, irreducible quadratic factors

1. Use variables such as $A$, $B$, or $C$ for the constant numerators over linear factors, and linear expressions such as $A_1x + B_1$ , $A_2x + B_2$, etc., for the numerators of each quadratic factor in the denominator.

$$\frac{P(x)}{Q(x)} = \frac{A}{ax + b} + \frac{A_1x + B_1}{(a_1x^2 + b1_x + c_1)} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)} + \cdots + \frac{A_nx + B_n}{(a_nx^2 + b_nx + c_n)}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

## Example 5.4.3: Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Contains a Nonrepeated Irreducible Quadratic Factor

Find a partial fraction decomposition of the given expression.

$$\frac{8x^2 + 12x - 20}{(x + 3)(x^2 + x + 2)}$$

**Solution**

We have one linear factor and one irreducible quadratic factor in the denominator, so one numerator will be a constant and the other numerator will be a linear expression. Thus,

$$\frac{8x^2 + 12x - 20}{(x + 3)(x^2 + x + 2)} = \frac{A}{(x + 3)} + \frac{Bx + C}{(x^2 + x + 2)}$$

We follow the same steps as in previous problems. First, clear the fractions by multiplying both sides of the equation by the common denominator.

$$(x + 3)(x^2 + x + 2)\left[\frac{8x^2 + 12x - 20}{(x + 3)(x^2 + x + 2)}\right] = \left[\frac{A}{(x + 3)} + \frac{Bx + C}{(x^2 + x + 2)}\right](x + 3)(x^2 + x + 2)$$

$$8x^2 + 12x - 20 = A(x^2 + x + 2) + (Bx + C)(x + 3)$$

Notice we could easily solve for $A$ by choosing a value for $x$ that will make the $Bx + C$ term equal $0$. Let $x = -3$ and substitute it into the equation.

$$8x^2+12x-20=A(x^2+x+2)+(Bx+C)(x+3)$$
$$8(-3)^2+12(-3)-20=A((-3)^2+(-3)+2)+(B(-3)+C)((-3)+3)$$
$$16=8A$$
$$A=2$$

Now that we know the value of $A$, substitute it back into the equation. Then expand the right side and collect like terms.

$$8x^2+12x-20=2(x^2+x+2)+(Bx+C)(x+3)$$
$$8x^2+12x-20=2x^2+2x+4+Bx^2+3B+Cx+3C$$
$$8x^2+12x-20=(2+B)x^2+(2+3B+C)x+(4+3C)$$

Setting the coefficients of terms on the right side equal to the coefficients of terms on the left side gives the system of equations.

$$2+B=8 \tag{5.4.9}$$
$$2+3B+C=12 \tag{5.4.10}$$
$$4+3C=-20 \tag{5.4.11}$$

Solve for $B$ using Equation 5.4.9

$$\[$$
$$2+B=8$$
$$B=6$$

and solve for $C$ using Equation 5.4.11.

$$4+3C=-20$$
$$3C=-24$$
$$C=-8$$

Thus, the partial fraction decomposition of the expression is

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{2}{(x+3)}+\frac{6x-8}{(x^2+x+2)}$$

### Q&A: Could we have just set up a system of equations to solve the example above?

Yes, we could have solved it by setting up a system of equations without solving for $A$ first. The expansion on the right would be:

$$8x^2+12x-20=Ax^2+Ax+2A+Bx^2+3B+Cx+3C$$
$$8x^2+12x-20=(A+B)x^2+(A+3B+C)x+(2A+3C)$$

So the system of equations would be:

$$A+B=8$$
$$A+3B+C=12$$
$$2A+3C=-20$$

### Exercise 5.4.3

Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.

$$\frac{5x^2-6x+7}{(x-1)(x^2+1)}$$

*Answer available at the end of this section.*

## Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Has a Repeated Irreducible Quadratic Factor

Now that we can decompose a simplified rational expression with an irreducible quadratic factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.

### DECOMPOSITION OF $\frac{P(x)}{Q(x)}$ WHEN $Q(X)$ HAS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{(ax^2+bx+c)^n}=\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n} \tag{5.4.12}$$

Write the denominators in increasing powers.

### How to: decompose a rational expression that has a repeated irreducible factor

1. Use variables like $A$, $B$, or $C$ for the constant numerators over linear factors, and linear expressions such as $A_1x+B_1$, $A_2x+B_2$, etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as
$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

### Example 5.4.4: Decomposing a Rational Function with a Repeated Irreducible Quadratic Factor in the Denominator

Decompose the given expression that has a repeated irreducible factor in the denominator.

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}$$

**Solution**

The factors of the denominator are $x$, $(x^2+1)$, and $(x^2+1)^2$. Recall that, when a factor in the denominator is a quadratic that includes at least two terms, the numerator must be of the linear form $Ax+B$. So, let's begin the decomposition.

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{(x^2+1)}+\frac{Dx+E}{(x^2+1)^2}$$

We eliminate the denominators by multiplying each term by $x(x^2+1)^2$. Thus,

$$\begin{aligned} x^4+x^3+x^2-x+1 &= A(x^2+1)^2+(Bx+C)(x)(x^2+1)+(Dx+E)(x) \\ x^4+x^3+x^2-x+1 &= A(x^4+2x^2+1)+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex \qquad \text{Expand the right side.} \\ &= Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex \end{aligned}$$

Now we will collect like terms.

$$x^4+x^3+x^2-x+1=(A+B)x^4+(C)x^3+(2A+B+D)x^2+(C+E)x+A$$

Set up the system of equations matching corresponding coefficients on each side of the equal sign.

$$\begin{aligned} A+B &= 1 \\ C &= 1 \\ 2A+B+D &= 1 \\ C+E &= -1 \\ A &= 1 \end{aligned}$$

We can use substitution from this point. Substitute $A=1$ into the first equation.

$$\begin{aligned} 1+B &= 1 \\ B &= 0 \end{aligned}$$

Substitute $A=1$ and $B=0$ into the third equation.

$$\begin{aligned} 2(1)+0+D &= 1 \\ D &= -1 \end{aligned}$$

Substitute $C=1$ into the fourth equation.

$$\begin{aligned} 1+E &= -1 \\ E &= -2 \end{aligned}$$

Now we have solved for all of the unknowns on the right side of the equal sign. We have $A=1$, $B=0, C=1$, $D=-1$, and $E=-2$. We can write the decomposition as follows:

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}=\frac{1}{x}+\frac{1}{(x^2+1)}-\frac{x+2}{(x^2+1)^2}$$

## Exercise 5.4.4

Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.

$$\frac{x^3-4x^2+9x-5}{(x^2-2x+3)^2}$$

*Answer available at the end of this section.*

## Media

Access these online resources for additional instruction and practice with partial fractions.

- Partial Fraction Decomposition
- Partial Fraction Decomposition With Repeated Linear Factors

- Partial Fraction Decomposition With Linear and Quadratic Factors

## Key Concepts

- Decompose $\frac{P(x)}{Q(x)}$ by writing the partial fractions as

$$\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}.$$

Solve by clearing the fractions, expanding the right side, collecting like terms, and setting corresponding coefficients equal to each other, then setting up and solving a system of equations (see Example $5.4.1$).

- The decomposition of $\frac{P(x)}{Q(x)}$ with repeated linear factors must account for the factors of the denominator in increasing powers (see Example $5.4.2$).
- The decomposition of $\frac{P(x)}{Q(x)}$ with a nonrepeated irreducible quadratic factor needs a linear numerator over the quadratic factor, as in $\frac{A}{x}+\frac{Bx+C}{(ax^2+bx+c)}$ (see Example $5.4.3$).
- In the decomposition of $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a repeated irreducible quadratic factor, when the irreducible quadratic factors are repeated, powers of the denominator factors must be represented in increasing powers as

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

See Example $5.4.4$.

### Answers

**Exercise** $5.4.1$

$\frac{3}{x-3}-\frac{2}{x-2}$

**Exercise** $5.4.2$

$$\frac{6}{x-1}-\frac{5}{(x-1)^2}$$

**Exercise** $5.4.3$

$\frac{3}{x-1}+\frac{2x-4}{x^2+1}$

**Exercise** $5.4.4$

$$\frac{x-2}{x^2-2x+3}+\frac{2x+1}{(x^2-2x+3)^2}$$

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# 5.5: Matrices and Matrix Operations

**Learning Objectives**

- Find the sum and difference of two matrices.
- Find scalar multiples of a matrix.
- Find the product of two matrices.

Two club soccer teams, the Wildcats and the Mud Cats, are hoping to obtain new equipment for an upcoming season. Table 5.5.1 shows the needs of both teams.

*Table* 5.5.1

| | Wildcats | Mud Cats |
|---|---|---|
| Goals | 6 | 10 |
| Balls | 30 | 24 |
| Jerseys | 14 | 20 |

A goal costs $300; a ball costs $10; and a jersey costs $30. How can we find the total cost for the equipment needed for each team? In this section, we discover a method in which the data in the soccer equipment table can be displayed and used for calculating other information. Then, we will be able to calculate the cost of the equipment.



Figure 5.5.1

## Finding the Sum and Difference of Two Matrices

To solve a problem like the one described for the soccer teams, we can use a matrix, which is a rectangular array of numbers. A row in a matrix is a set of numbers that are aligned horizontally. A column in a matrix is a set of numbers that are aligned vertically. Each number is an entry, sometimes called an element, of the matrix. Matrices (plural) are enclosed in [ ] or ( ), and are usually named with capital letters. For example, three matrices named $A$, $B$, and $C$ are shown below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -5 & 6 \\ 7 & 8 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 3 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$$

A matrix is often referred to by its size or dimensions: $m \times n$ indicating $m$ rows and $n$ columns. Matrix entries are defined first by row and then by column. For example, to locate the entry in matrix $A$ identified as $a_{ij}$,we look for the entry in row $i$,column $j$. In matrix $A$, shown below, the entry in row $2$, column $3$ is $a_{23}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- A **square matrix** is a matrix with dimensions $n \times n$, meaning that it has the same number of rows as columns. The $3 \times 3$ matrix above is an example of a square matrix.
- A **row matrix** is a matrix consisting of one row with dimensions $1 \times n$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

- A **column matrix** is a matrix consisting of one column with dimensions $m \times 1$.

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

A matrix may be used to represent a system of equations. In these cases, the numbers represent the coefficients of the variables in the system. Matrices often make solving systems of equations easier because they are not encumbered with variables. We will investigate this idea further in the next section, but first we will look at basic **matrix operations**.

### Definition: MATRICES

A *matrix* is a rectangular array of numbers that is usually named by a capital letter: $A$, $B$, $C$,and so on. Each entry in a matrix is referred to as $a_{ij}$,such that $i$ represents the row and $j$ represents the column. Matrices are often referred to by their dimensions: $m \times n$ indicating $m$ rows and $n$ columns.

### Example 5.5.1: Finding the Dimensions of the Given Matrix and Locating Entries

Given matrix $A$:

a. What are the dimensions of matrix $A$?
b. What are the entries at $a_{31}$ and $a_{22}$?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 7 \\ 3 & 1 & -2 \end{bmatrix}$$

**Solution**

a. The dimensions are $3 \times 3$ because there are three rows and three columns.
b. Entry $a_{31}$ is the number at row 3, column 1, which is $3$. The entry $a_{22}$ is the number at row 2, column 2, which is $4$. Remember, the row comes first, then the column.

## Adding and Subtracting Matrices

We use matrices to list data or to represent systems. Because the entries are numbers, we can perform operations on matrices. We add or subtract matrices by adding or subtracting corresponding entries. To do this, the entries must correspond. Therefore, *addition and subtraction of matrices is only possible when the matrices have the same dimensions*. We can add or subtract a $3 \times 3$ matrix and another $3 \times 3$ matrix, but we cannot add or subtract a $2 \times 3$ matrix and a $3 \times 3$ matrix because some entries in one matrix will not have a corresponding entry in the other matrix.

### ADDING AND SUBTRACTING MATRICES

Given matrices $A$ and $B$ of like dimensions, addition and subtraction of $A$ and $B$ will produce matrix $C$ or matrix $D$ of the same dimension.

$$A + B = C \tag{5.5.1}$$

such that $a_{ij} + b_{ij} = c_{ij}$

$$A - B = D \tag{5.5.2}$$

such that $a_{ij} - b_{ij} = d_{ij}$

Matrix addition is *commutative*.

$$A + B = B + A \tag{5.5.3}$$

It is also *associative*.

$$(A + B) + C = A + (B + C) \tag{5.5.4}$$

### Example $5.5.2A$: Finding the Sum of Matrices

Find the sum of $A$ and $B$, given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

**Solution**

Add corresponding entries.

$$\begin{aligned} A + B &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix} \end{aligned}$$

### Example $5.5.2B$: Adding Matrix $A$ and Matrix $B$

Find the sum of $A$ and $B$.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix}$$

**Solution**

Add corresponding entries. Add the entry in row 1, column 1, $a_{11}$, of matrix $A$ to the entry in row 1, column 1, $b_{11}$, of $B$. Continue the pattern until all entries have been added.

$$\begin{aligned} A + B &= \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 4+5 & 1+9 \\ 3+0 & 2+7 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 10 \\ 3 & 9 \end{bmatrix} \end{aligned}$$

## Example $5.5.2C$: Finding the Difference of Two Matrices

Find the difference of $A$ and $B$.

$$A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

**Solution**

We subtract the corresponding entries of each matrix.

$$\begin{aligned} A - B &= \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -2-8 & 3-1 \\ 0-5 & 1-4 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 2 \\ -5 & -3 \end{bmatrix} \end{aligned}$$

## Example $5.5.2D$: Finding the Sum and Difference of Two 3 x 3 Matrices

Given $A$ and $B$:

a. Find the sum.
b. Find the difference.

$$A = \begin{bmatrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{bmatrix}$$

**Solution**

a. Add the corresponding entries.

$$A+B=\begin{bmatrix}2&-10&-2\\14&12&10\\4&-2&2\end{bmatrix}+\begin{bmatrix}6&10&-2\\0&-12&-4\\-5&2&-2\end{bmatrix}$$
$$=\begin{bmatrix}2+6&-10+10&-2-2\\14+0&12-12&10-4\\4-5&-2+2&2-2\end{bmatrix}$$
$$=\begin{bmatrix}8&0&-4\\14&0&6\\-1&0&0\end{bmatrix}$$

b. Subtract the corresponding entries.

$$A-B=\begin{bmatrix}2&-10&-2\\14&12&10\\4&-2&2\end{bmatrix}-\begin{bmatrix}6&10&-2\\0&-12&-4\\-5&2&-2\end{bmatrix}$$
$$=\begin{bmatrix}2-6&-10-10&-2+2\\14-0&12+12&10+4\\4+5&-2-2&2+2\end{bmatrix}$$
$$=\begin{bmatrix}-4&-20&0\\14&24&14\\9&-4&4\end{bmatrix}$$

**Exercise 5.5.1**

Add matrix $A$ and matrix $B$.

$$A=\begin{bmatrix}2&6\\1&0\\1&-3\end{bmatrix}$$

and

$$B=\begin{bmatrix}3&-2\\1&5\\-4&3\end{bmatrix}$$

*Answer available at the end of this section.*

## Finding Scalar Multiples of a Matrix

Besides adding and subtracting whole matrices, there are many situations in which we need to multiply a matrix by a constant called a scalar. Recall that a scalar is a real number quantity that has magnitude, but not direction. For example, time, temperature, and distance are scalar quantities. The process of scalar multiplication involves multiplying each entry in a matrix by a scalar. A scalar multiple is any entry of a matrix that results from scalar multiplication.

Consider a real-world scenario in which a university needs to add to its inventory of computers, computer tables, and chairs in two of the campus labs due to increased enrollment. They estimate that 15 more equipment is needed in both labs. The school's current inventory is displayed in Table 5.5.2.

*Table* 5.5.2

| | Lab A | Lab B |
|---|---|---|
| **Computers** | 15 | 27 |

| | Lab A | Lab B |
|---|---|---|
| **Computer Tables** | 16 | 34 |
| **Chairs** | 16 | 34 |

Converting the data to a matrix, we have

$$C_{2013} = \begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix}$$

To calculate how much computer equipment will be needed, we multiply all entries in matrix $C$ by $0.15$.

$$(0.15)C_{2013} = \begin{bmatrix} (0.15)15 & (0.15)27 \\ (0.15)16 & (0.15)34 \\ (0.15)16 & (0.15)34 \end{bmatrix} = \begin{bmatrix} 2.25 & 4.05 \\ 2.4 & 5.1 \\ 2.4 & 5.1 \end{bmatrix}$$

We must round up to the next integer, so the amount of new equipment needed is

$$\begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix}$$

Adding the two matrices as shown below, we see the new inventory amounts.

$$\begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix}$$

This means

$$C_{2014} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix}$$

Thus, Lab A will have $18$ computers, $19$ computer tables, and $19$ chairs; Lab B will have $32$ computers, $40$ computer tables, and $40$ chairs.

## SCALAR MULTIPLICATION

Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the scalar multiple $cA$ is

$$cA = c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

Scalar multiplication is distributive. For the matrices $A$, $B$,and $C$ with scalars $a$ and $b$,

$$a(A+B) = aA + aB \tag{5.5.5}$$

$$(a+b)A = aA + bA \tag{5.5.6}$$

## Example 5.5.3: Multiplying the Matrix by a Scalar

Multiply matrix $A$ by the scalar $3$.

$$A = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

**Solution**

Multiply each entry in $A$ by the scalar $3$.

$$\begin{aligned} 3A &= 3\begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 8 & 3 \cdot 1 \\ 3 \cdot 5 & 3 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 24 & 3 \\ 15 & 12 \end{bmatrix} \end{aligned}$$

## Exercise 5.5.2

Given matrix $B$, find $-2B$ where

$$B = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

*Answer available at the end of this section.*

## Example 5.5.4: Finding the Sum of Scalar Multiples

Find the sum $3A + 2B$.

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & -4 \end{bmatrix}$$

**Solution**

First, find $3A$, then $2B$.

$$\begin{aligned} 3A &= \begin{bmatrix} 3 \cdot 1 & 3(-2) & 3 \cdot 0 \\ 3 \cdot 0 & 3(-1) & 3 \cdot 2 \\ 3 \cdot 4 & 3 \cdot 3 & 3(-6) \end{bmatrix} \\ &= \begin{bmatrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{bmatrix} \end{aligned}$$

$$2B = \begin{bmatrix} 2(-1) & 2\cdot 2 & 2\cdot 1 \\ 2\cdot 0 & 2(-3) & 2\cdot 2 \\ 2\cdot 0 & 2\cdot 1 & 2(-4) \end{bmatrix} = \begin{bmatrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{bmatrix}$$

Now, add $3A + 2B$.

$$3A + 2B = \begin{bmatrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{bmatrix} + \begin{bmatrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{bmatrix} = \begin{bmatrix} 3-2 & -6+4 & 0+2 \\ 0+0 & -3-6 & 6+4 \\ 12+0 & 9+2 & -18-8 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -9 & 10 \\ 12 & 11 & -26 \end{bmatrix}$$

## Finding the Product of Two Matrices

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices is only possible when the inner dimensions are the same, meaning that the number of columns of the first matrix is equal to the number of rows of the second matrix. If $A$ is an $m \times r$ matrix and $B$ is an $r \times n$ matrix, then the product matrix $AB$ is an $m \times n$ matrix. For example, the product $AB$ is possible because the number of columns in $A$ is the same as the number of rows in $B$. If the inner dimensions do not match, the product is not defined.

$$\underset{2\times 3}{A} \cdot \underbrace{\underset{3\times 3}{B}}_{\text{same}}$$

We multiply entries of $A$ with entries of $B$ according to a specific pattern as outlined below. The process of **matrix multiplication** becomes clearer when working a problem with real numbers.

To obtain the entries in row $i$ of $AB$, we multiply the entries in row $i$ of $A$ by column $j$ in $B$ and add. For example, given matrices $A$ and $B$, where the dimensions of $A$ are $2 \times 3$ and the dimensions of $B$ are $3 \times 3$,the product of $AB$ will be a $2 \times 3$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Multiply and add as follows to obtain the first entry of the product matrix $AB$.

1. To obtain the entry in row 1, column 1 of $AB$,multiply the first row in $A$ by the first column in $B$,and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}\cdot b_{11} + a_{12}\cdot b_{21} + a_{13}\cdot b_{31}$$

2. To obtain the entry in row 1, column 2 of $AB$,multiply the first row of $A$ by the second column in $B$,and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}$$

3. To obtain the entry in row 1, column 3 of $AB$,multiply the first row of $A$ by the third column in $B$,and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33}$$

We proceed the same way to obtain the second row of $AB$. In other words, row 2 of $A$ times column 1 of $B$; row 2 of $A$ times column 2 of $B$; row 2 of $A$ times column 3 of $B$. When complete, the product matrix will be

$$AB = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \end{bmatrix}$$

## PROPERTIES OF MATRIX MULTIPLICATION

For the matrice $A, B$,and $C$ the following properties hold.

- Matrix multiplication is *associative*:

$$(AB)C = A(BC). \tag{5.5.7}$$

- Matrix multiplication is *distributive*:

$$C(A + B) = CA + CB \tag{5.5.8}$$

$$(A + B)C = AC + BC. \tag{5.5.9}$$

Note that matrix multiplication is not commutative.

## Example $5.5.5A$: Multiplying Two Matrices

Multiply matrix $A$ and matrix $B$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

**Solution**

First, we check the dimensions of the matrices. Matrix $A$ has dimensions $2 \times 2$ and matrix $B$ has dimensions $2 \times 2$. The inner dimensions are the same so we can perform the multiplication. The product will have the dimensions $2 \times 2$.

We perform the operations outlined previously.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix}$$
$$= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

## Example $5.5.5B$: Multiplying Two Matrices

Given $A$ and $B$:

a. Find $AB$.
b. Find $BA$.

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix}$$

**Solution**

a. As the dimensions of $A$ are $2 \times 3$ and the dimensions of $B$ are $3 \times 2$,these matrices can be multiplied together because the number of columns in $A$ matches the number of rows in $B$. The resulting product will be a $2 \times 2$ matrix, the number of rows in $A$ by the number of columns in $B$.

$$AB = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -1(5) + 2(-4) + 3(2) & -1(-1) + 2(0) + 3(3) \\ 4(5) + 0(-4) + 5(2) & 4(-1) + 0(0) + 5(3) \end{bmatrix}$$
$$= \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix}$$

b. The dimensions of $B$ are $3 \times 2$ and the dimensions of $A$ are $2 \times 3$. The inner dimensions match so the product is defined and will be a $3 \times 3$ matrix.

$$BA = \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 5(-1)+-1(4) & 5(2)+-1(0) & 5(3)+-1(5) \\ -4(-1)+0(4) & -4(2)+0(0) & -4(3)+0(5) \\ 2(-1)+3(4) & 2(2)+3(0) & 2(3)+3(5) \end{bmatrix}$$
$$= \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix}$$

**Analysis**

Notice that the products $AB$ and $BA$ are not equal.

$$AB = \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \neq \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} = BA$$

This illustrates the fact that matrix multiplication is **not** commutative.

### Q&A: Is it possible for AB to be defined but not BA?

*Yes, consider a matrix $A$ with dimension $3 \times 4$ and matrix $B$ with dimension $4 \times 2$. For the product $AB$ the inner dimensions are $4$ and the product is defined, but for the product $BA$ the inner dimensions are $2$ and $3$ so the product is undefined.*

### Example 5.5.6: Using Matrices in Real-World Problems

Let's return to the problem presented at the opening of this section. We have Table 5.5.3, representing the equipment needs of two soccer teams.

*Table* 5.5.3

| | Wildcats | Mud Cats |
|---|---|---|
| **Goals** | 6 | 10 |
| **Balls** | 30 | 24 |
| **Jerseys** | 14 | 20 |

We are also given the prices of the equipment, as shown in Table 5.5.4.

*Table* 5.5.4

| | |
|---|---|
| **Goal** | $300 |
| **Ball** | $10 |
| **Jersey** | $30 |

We will convert the data to matrices. Thus, the equipment need matrix is written as

$$E = \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix}$$

The cost matrix is written as

$$C = [300 \quad 10 \quad 30]$$

We perform matrix multiplication to obtain costs for the equipment.

$$\begin{aligned} CE &= [300 \quad 10 \quad 30] \cdot \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix} \\ &= [300(6) + 10(30) + 30(14) \quad 300(10) + 10(24) + 30(20)] \\ &= [2,520 \quad 3,840] \end{aligned}$$

The total cost for equipment for the Wildcats is $\$2,520$, and the total cost for equipment for the Mud Cats is $\$3,840$.

## How to: Given a matrix operation, evaluate using a calculator

1. Save each matrix as a matrix variable $[A], [B], [C], \ldots$
2. Enter the operation into the calculator, calling up each matrix variable as needed.
3. If the operation is defined, the calculator will present the solution matrix; if the operation is undefined, it will display an error message.

## Example 5.5.7: Using a Calculator to Perform Matrix Operations

Find $AB - C$ given

$$A = \begin{bmatrix} -15 & 25 & 32 \\ 41 & -7 & -28 \\ 10 & 34 & -2 \end{bmatrix}, B = \begin{bmatrix} 45 & 21 & -37 \\ -24 & 52 & 19 \\ 6 & -48 & -31 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -100 & -89 & -98 \\ 25 & -56 & 74 \\ -67 & 42 & -75 \end{bmatrix}$$

**Solution**

On the matrix page of the calculator, we enter matrix $A$ above as the matrix variable $[A]$, matrix $B$ above as the matrix variable $[B]$, and matrix $C$ above as the matrix variable $[C]$.

On the home screen of the calculator, we type in the problem and call up each matrix variable as needed.

$$[A] \times [B] - [C]$$

The calculator gives us the following matrix.

$$\begin{bmatrix} -983 & -462 & 136 \\ 1,820 & 1,897 & -856 \\ -311 & 2,032 & 413 \end{bmatrix}$$

## Media

Access these online resources for additional instruction and practice with matrices and matrix operations.

- Dimensions of a Matrix
- Matrix Addition and Subtraction
- Matrix Operations
- Matrix Multiplication

# Key Concepts

- A matrix is a rectangular array of numbers. Entries are arranged in rows and columns.
- The dimensions of a matrix refer to the number of rows and the number of columns. A $3 \times 2$ matrix has three rows and two columns. See Example $5.5.1$.
- We add and subtract matrices of equal dimensions by adding and subtracting corresponding entries of each matrix. See Example $5.5.2$, Example $5.5.3$, Example $5.5.4$, and Example $5.5.5$.
- Scalar multiplication involves multiplying each entry in a matrix by a constant. See Example $5.5.6$.
- Scalar multiplication is often required before addition or subtraction can occur. See Example $5.5.7$.
- Multiplying matrices is possible when inner dimensions are the same—the number of columns in the first matrix must match the number of rows in the second.
- The product of two matrices, $A$ and $B$,is obtained by multiplying each entry in row 1 of $A$ by each entry in column 1 of $B$; then multiply each entry of row 1 of $A$ by each entry in columns 2 of $B$,and so on. See Example $5.5.8$ and Example $5.5.9$.
- Many real-world problems can often be solved using matrices. See Example $5.5.10$.
- We can use a calculator to perform matrix operations after saving each matrix as a matrix variable. See Example $5.5.11$.

## Answers

**Exercise** $5.5.1$

$$\begin{aligned} A+B &= \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2+3 & 6+(-2) \\ 1+1 & 0+5 \\ 1+(-4) & -3+3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 4 \\ 2 & 5 \\ -3 & 0 \end{bmatrix} \end{aligned}$$

**Exercise** $5.5.2$

$$-2B = \begin{bmatrix} -8 & -2 \\ -6 & -4 \end{bmatrix}$$

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# 5.6: Solving Systems with Gaussian Elimination

**Learning Objectives**

- Write the augmented matrix of a system of equations.
- Write the system of equations from an augmented matrix.
- Perform row operations on a matrix.
- Solve a system of linear equations using matrices.

Carl Friedrich Gauss lived during the late $18^{th}$ century and early $19^{th}$ century, but he is still considered one of the most prolific mathematicians in history. His contributions to the science of mathematics and physics span fields such as algebra, number theory, analysis, differential geometry, astronomy, and optics, among others. His discoveries regarding matrix theory changed the way mathematicians have worked for the last two centuries.



Figure 5.6.1: German mathematician Carl Friedrich Gauss (1777–1855).

We first encountered Gaussian elimination in Systems of Linear Equations: Two Variables. In this section, we will revisit this technique for solving systems, this time using matrices.

## Writing the Augmented Matrix of a System of Equations

A matrix can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an **augmented matrix**.

For example, consider the following $2 \times 2$ system of equations.

$$\begin{aligned} 3x + 4y &= 7 \\ 4x - 2y &= 5 \end{aligned}$$

We can write this system as an augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 4 & 7 \\ 4 & -2 & 5 \end{array}\right]$$

We can also write a matrix containing just the coefficients. This is called the **coefficient matrix.**

$$\begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}$$

A three-by-three **system of equations** such as

$$\begin{aligned} 3x - y - z &= 0 \\ x + y &= 5 \\ 2x - 3z &= 2 \end{aligned}$$

has a coefficient matrix

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

and is represented by the augmented matrix

$$\left[\begin{array}{rrrr} 3 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2 \end{array}\right]$$

Notice that the matrix is written so that the variables line up in their own columns: $x$-terms go in the first column, $y$-terms in the second column, and $z$-terms in the third column. It is very important that each equation is written in standard form $ax + by + cz = d$ so that the variables line up. When there is a missing variable term in an equation, the coefficient is $0$.

### How to: Given a system of equations, write an augmented matrix

1. Write the coefficients of the $x$-terms as the numbers down the first column.
2. Write the coefficients of the $y$-terms as the numbers down the second column.
3. If there are $z$-terms, write the coefficients as the numbers down the third column.
4. Draw a vertical line and write the constants to the right of the line.

### Example 5.6.1: Writing the Augmented Matrix for a System of Equations

Write the augmented matrix for the given system of equations.

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - y + 2z &= 6 \\ x - 3y + 3z &= 4 \end{aligned}$$

**Solution**

The augmented matrix displays the coefficients of the variables, and an additional column for the constants.

$$\left[\begin{array}{rrrr} 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 6 \\ 1 & -3 & 3 & 4 \end{array}\right]$$

### Exercise 5.6.1

Write the augmented matrix of the given system of equations.

$$\begin{aligned} 4x - 3y &= 11 \\ 3x + 2y &= 4 \end{aligned}$$

*Answer available at the end of this section.*

## Writing a System of Equations from an Augmented Matrix

We can use augmented matrices to help us solve systems of equations because they simplify operations when the systems are not encumbered by the variables. However, it is important to understand how to move back and forth between formats in order to make finding solutions smoother and more intuitive. Here, we will use the information in an augmented matrix to write the system of equations in standard form.

### Example 5.6.2: Writing a System of Equations from an Augmented Matrix Form

Find the system of equations from the augmented matrix.

$$\begin{bmatrix} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{bmatrix}$$

**Solution**

When the columns represent the variables $x$, $y$, and $z$,

Erroneous nesting of equation structures

### Exercise 5.6.2

Write the system of equations from the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & -1 & 3 & 1 \\ 0 & 1 & 1 & -9 \end{bmatrix}$$

*Answer available at the end of this section.*

# Performing Row Operations on a Matrix

Now that we can write systems of equations in augmented matrix form, we will examine the various **row operations** that can be performed on a matrix, such as addition, multiplication by a constant, and interchanging rows.

Performing row operations on a matrix is the method we use for solving a system of equations. In order to solve the system of equations, we want to convert the matrix to **row-echelon** form, in which there are ones down the **main diagonal** from the upper left corner to the lower right corner, and zeros in every position below the main diagonal as shown.

$$\text{Row-echelon form } \begin{bmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$$

We use row operations corresponding to equation operations to obtain a new matrix that is **row-equivalent** in a simpler form. Here are the guidelines to obtaining row-echelon form.

1. In any nonzero row, the first nonzero number is a $1$. It is called a *leading* $1$.
2. Any all-zero rows are placed at the bottom on the matrix.
3. Any leading $1$ is below and to the right of a previous leading $1$.
4. Any column containing a leading $1$ has zeros in all other positions in the column.

To solve a system of equations we can perform the following row operations to convert the **coefficient matrix** to row-echelon form and do back-substitution to find the solution.

1. Interchange rows. (Notation: $R_i \leftrightarrow R_j$)
2. Multiply a row by a constant. (Notation: $cR_i$)
3. Add the product of a row multiplied by a constant to another row. (Notation: $R_i + cR_j$)

Each of the row operations corresponds to the operations we have already learned to solve systems of equations in three variables. With these operations, there are some key moves that will quickly achieve the goal of writing a matrix in row-echelon form. To obtain a matrix in row-echelon form for finding solutions, we use Gaussian elimination, a method that uses row operations to obtain a $1$ as the first entry so that row $1$ can be used to convert the remaining rows.

## GAUSSIAN ELIMINATION

The Gaussian elimination method refers to a strategy used to obtain the row-echelon form of a matrix. The goal is to write matrix $A$ with the number $1$ as the entry down the main diagonal and have all zeros below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{After\ Gaussian\ elimination} A = \begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

The first step of the Gaussian strategy includes obtaining a $1$ as the first entry, so that row $1$ may be used to alter the rows below.

## How to: Given an augmented matrix, perform row operations to achieve row-echelon form

1. The first equation should have a leading coefficient of $1$. Interchange rows or multiply by a constant, if necessary.
2. Use row operations to obtain zeros down the first column below the first entry of $1$.
3. Use row operations to obtain a $1$ in row 2, column 2.
4. Use row operations to obtain zeros down column 2, below the entry of 1.
5. Use row operations to obtain a $1$ in row 3, column 3.
6. Continue this process for all rows until there is a \(1 in every entry down the main diagonal and there are only zeros below.
7. If any rows contain all zeros, place them at the bottom.

## Example $5.6.3$: Solving a $2 \times 2$ System by Gaussian Elimination

Solve the given system by Gaussian elimination.

$$\begin{aligned} 2x + 3y &= 6 \\ x - y &= \frac{1}{2} \end{aligned}$$

**Solution**

First, we write this as an augmented matrix.

$$\begin{bmatrix} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{bmatrix}$$

We want a $1$ in row 1, column 1. This can be accomplished by interchanging row 1 and row 2.

$$R_1 \leftrightarrow R_2 \rightarrow \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{bmatrix}$$

We now have a $1$ as the first entry in row 1, column 1. Now let's obtain a $0$ in row 2, column 1. This can be accomplished by multiplying row 1 by $-2$, and then adding the result to row 2.

$$-2R_1 + R_2 = R_2 \rightarrow \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{bmatrix}$$

We only have one more step, to multiply row 2 by $\frac{1}{5}$.

$$\frac{1}{5}R_2 = R_2 \rightarrow \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

Use back-substitution. The second row of the matrix represents $y = 1$. Back-substitute $y = 1$ into the first equation.

$$\begin{aligned} x - (1) &= \frac{1}{2} \\ x &= \frac{3}{2} \end{aligned}$$

The solution is the point $\left(\frac{3}{2}, 1\right)$.

## Exercise 5.6.3

Solve the given system by Gaussian elimination.

$$\begin{aligned} 4x + 3y &= 11 \\ x - 3y &= -1 \end{aligned}$$

*Answer available at the end of this section.*

## Example 5.6.4: Using Gaussian Elimination to Solve a System of Equations

Use **Gaussian elimination** to solve the given $2 \times 2$ **system of equations**.

$$\begin{aligned} 2x + y &= 1 \\ 4x + 2y &= 6 \end{aligned}$$

**Solution**

Write the system as an **augmented matrix**.

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 6 \end{bmatrix}$$

Obtain a $1$ in row 1, column 1. This can be accomplished by multiplying the first row by $\frac{1}{2}$.

$$\frac{1}{2}R_1 = R_1 \rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 4 & 2 & 6 \end{bmatrix}$$

Next, we want a $0$ in row 2, column 1. Multiply row 1 by $-4$ and add row 1 to row 2.

$$-4R_1 + R_2 = R_2 \rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 4 \end{bmatrix}$$

The second row represents the equation $0 = 4$. Therefore, the system is inconsistent and has no solution.

## Example 5.6.5: Solving a Dependent System

Solve the system of equations.

$$\begin{aligned} 3x + 4y &= 12 \\ 6x + 8y &= 24 \end{aligned}$$

**Solution**

Perform **row operations** on the augmented matrix to try and achieve **row-echelon form**.

$$A = \begin{bmatrix} 3 & 4 & 12 \\ 6 & 8 & 24 \end{bmatrix}$$

$$-\frac{1}{2}R_2 + R_1 = R_1 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 6 & 8 & 24 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 = \begin{bmatrix} 6 & 8 & 24 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix ends up with all zeros in the last row: $0y = 0$. Thus, there are an infinite number of solutions and the system is classified as dependent. To find the generic solution, return to one of the original equations and solve for $y$.

$$\begin{aligned} 3x + 4y &= 12 \\ 4y &= 12 - 3x \\ y &= 3 - \frac{3}{4}x \end{aligned}$$

So the solution to this system is $\left(x, 3 - \frac{3}{4}x\right)$.

## Example 5.6.6: Performing Row Operations on a $3 \times 3$ Augmented Matrix to Obtain Row-Echelon Form

Perform row operations on the given matrix to obtain row-echelon form.

$$\begin{bmatrix} 1 & -3 & 4 & 3 \\ 2 & -5 & 6 & 6 \\ -3 & 3 & 4 & 6 \end{bmatrix}$$

**Solution**

The first row already has a $1$ in row 1, column 1. The next step is to multiply row 1 by $-2$ and add it to row 2. Then replace row 2 with the result.

$$-2R_1 + R_2 = R_2 \begin{bmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ -3 & 3 & 4 & 6 \end{bmatrix}$$

Next, obtain a zero in row 3, column 1.

$$3R_1 + R_3 = R_3 \begin{bmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -6 & 16 & 15 \end{bmatrix}$$

Next, obtain a zero in row 3, column 2.

$$6R_2 + R_3 = R_3 \begin{bmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 15 \end{bmatrix}$$

The last step is to obtain a 1 in row 3, column 3.

$$\frac{1}{4}R_3 = R_3 \begin{bmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & \frac{15}{4} \end{bmatrix}$$

**Exercise 5.6.4**

Write the system of equations in row-echelon form.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

*Answer available at the end of this section.*

## Solving a System of Linear Equations Using Matrices

We have seen how to write a **system of equations** with an **augmented matrix**, and then how to use row operations and back-substitution to obtain row-echelon form. Now, we will take **row-echelon form** a step farther to solve a $3$ by $3$ system of linear equations. The general idea is to eliminate all but one variable using row operations and then back-substitute to solve for the other variables.

**Example 5.6.7: Solving a System of Linear Equations Using Matrices**

Solve the system of linear equations using matrices.

$$\begin{aligned} x - y + z &= 8 \\ 2x + 3y - z &= -2 \\ 3x - 2y - 9z &= 9 \end{aligned}$$

**Solution**

First, we write the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{bmatrix}$$

Next, we perform row operations to obtain row-echelon form.

$$-2R_1 + R_2 = R_2 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 3 & -2 & -9 & 9 \end{bmatrix}$$

$$-3R_1 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{bmatrix}$$

The easiest way to obtain a $1$ in row 2 of column 1 is to interchange $R_2$ and $R_3$.

$$Interchange\ R_2\ and\ R_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{bmatrix}$$

Then

$$-5R_2 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{bmatrix}$$

$$-\frac{1}{57}R_3 = R_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The last matrix represents the equivalent system.

$$\begin{aligned} x - y + z &= 8 \\ y - 12z &= -15 \\ z &= 1 \end{aligned}$$

Using back-substitution, we obtain the solution as $(4, -3, 1)$.

## Example 5.6.8: Solving a Dependent System of Linear Equations Using Matrices

Solve the following system of linear equations using matrices.

$$\begin{aligned} -x - 2y + z &= -1 \\ 2x + 3y &= 2 \\ y - 2z &= 0 \end{aligned}$$

**Solution**

Write the augmented matrix.

$$\begin{bmatrix} -1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

First, multiply row 1 by $-1$ to get a $1$ in row 1, column 1. Then, perform **row operations** to obtain row-echelon form.

$$-R_1 \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 2 & 3 & 0 & 2 \end{bmatrix}$$

$$-2R_1 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

$$R_2 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix represents the following system.

$$\begin{aligned} x+2y-z &= 1 \\ y-2z &= 0 \\ 0 &= 0 \end{aligned}$$

We see by the identity $0=0$ that this is a dependent system with an infinite number of solutions. We then find the generic solution. By solving the second equation for $y$ and substituting it into the first equation we can solve for $z$ in terms of $x$.

$$\begin{aligned} x+2y-z &= 1 \\ y &= 2z \\ x+2(2z)-z &= 1 \\ x+3z &= 1 \\ z &= \frac{1-x}{3} \end{aligned}$$

Now we substitute the expression for $z$ into the second equation to solve for $y$ in terms of $x$.

$$\begin{aligned} y-2z &= 0 \\ z &= \frac{1-x}{3} \\ y-2\left(\frac{1-x}{3}\right) &= 0 \\ y &= \frac{2-2x}{3} \end{aligned}$$

The generic solution is $\left(x, \frac{2-2x}{3}, \frac{1-x}{3}\right)$.

## Exercise 5.6.5

Solve the system using matrices.

$$\begin{aligned} x+4y-z &= 4 \\ 2x+5y+8z &= 1 \\ 5x+3y-3z &= 1 \end{aligned}$$

*Answer available at the end of this section.*

## Q&A: Can any system of linear equations be solved by Gaussian elimination?

*Yes, a system of linear equations of any size can be solved by Gaussian elimination.*

## How to: Given a system of equations, solve with matrices using a calculator

1. Save the augmented matrix as a matrix variable $[A], [B], [C], \ldots$.
2. Use the **ref(** function in the calculator, calling up each matrix variable as needed.

## Example 5.6.9A: Solving Systems of Equations with Matrices Using a Calculator

Solve the system of equations.

$$\begin{aligned} 5x+3y+9z &= -1 \\ -2x+3y-z &= -2 \\ -x-4y+5z &= 1 \end{aligned}$$

**Solution**

Write the augmented matrix for the system of equations.

$$\left[\begin{array}{cccc} 5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & 1 \end{array}\right]$$

On the matrix page of the calculator, enter the augmented matrix above as the matrix variable $[A]$.

$$[A]=\left[\begin{array}{cccc} 5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & 1 \end{array}\right]$$

Use the **ref(** function in the calculator, calling up the matrix variable $[A]$.

**ref([A])**

**Evaluate**

Erroneous nesting of equation structures

Using back-substitution, the solution is $\left(\frac{61}{187},-\frac{92}{187},-\frac{24}{187}\right)$.

## Example $5.6.9B$: Applying $2\times 2$ Matrices to Finance

Carolyn invests a total of $\$12,000$ in two municipal bonds, one paying $10.5$ interest and the other paying $12$ interest. The annual interest earned on the two investments last year was $\$1,335$. How much was invested at each rate?

**Solution**

We have a system of two equations in two variables. Let $x=$ the amount invested at $10.5$ interest, and $y=$ the amount invested at $12$ interest.

$$\begin{aligned} x+y &= 12,000 \\ 0.105x+0.12y &= 1,335 \end{aligned}$$

As a matrix, we have

$$\left[\begin{array}{ccc} 1 & 1 & 12,000 \\ 0.105 & 0.12 & 1,335 \end{array}\right]$$

Multiply row 1 by $-0.105$ and add the result to row 2.

$$\left[\begin{array}{ccc} 1 & 1 & 12,000 \\ 0 & 0.015 & 75 \end{array}\right]$$

Then,

$$\begin{aligned} 0.015y &= 75 \\ y &= 5,000 \end{aligned}$$

So $12,000-5,000=7,000$.

Thus, $\$5,000$ was invested at $12$ interest and $\$7,000$ at $10.5$ interest.

## Example 5.6.10: Applying $3 \times 3$ Matrices to Finance

Ava invests a total of $\$10,000$ in three accounts, one paying $5$ interest, another paying $8$ interest, and the third paying $9$ interest. The annual interest earned on the three investments last year was $\$770$. The amount invested at $9$ was twice the amount invested at $5$. How much was invested at each rate?

**Solution**

We have a system of three equations in three variables. Let $x$ be the amount invested at $5$ interest, let $y$ be the amount invested at $8$ interest, and let $z$ be the amount invested at $9$ interest. Thus,

$$\begin{aligned} x + y + z &= 10,000 \\ 0.05x + 0.08y + 0.09z &= 770 \\ 2x - z &= 0 \end{aligned}$$

As a matrix, we have

$$\begin{bmatrix} 1 & 1 & 1 & 10,000 \\ 0.05 & 0.08 & 0.09 & 770 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

Now, we perform Gaussian elimination to achieve row-echelon form.

$$-0.05R_1 + R_2 = R_2 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 10,000 \\ 0 & 0.03 & 0.04 & 270 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

$$-2R_1 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 10,000 \\ 0 & 0.03 & 0.04 & 270 \\ 0 & -2 & -3 & -20,000 \end{bmatrix}$$

$$\frac{1}{0.03}R_2 = R_2 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 10,000 \\ 0 & 1 & \frac{4}{3} & 9,000 \\ 0 & -2 & -3 & -20,000 \end{bmatrix}$$

$$2R_2 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 10,000 \\ 0 & 1 & \frac{4}{3} & 9,000 \\ 0 & 0 & -\frac{1}{3} & -2,000 \end{bmatrix}$$

The third row tells us $-\frac{1}{3}z = -2,000$; thus $z = 6,000$.

The second row tells us $y + \frac{4}{3}z = 9,000$. Substituting $z = 6,000$,we get

$$\begin{aligned} y + \frac{4}{3}(6,000) &= 9,000 \\ y + 8,000 &= 9,000 \\ y &= 1,000 \end{aligned}$$

The first row tells us $x + y + z = 10,000$. Substituting $y = 1,000$ and $z = 6,000$,we get

$$\begin{aligned} x + 1,000 + 6,000 &= 10,000 \\ x &= 3,000 \end{aligned}$$

The answer is $\$3,000$ invested at $5$ interest, $\$1,000$ invested at $8$, and $\$6,000$ invested at $9$ interest.

### Exercise 5.6.6

A small shoe company took out a loan of $\$1,500,000$ to expand their inventory. Part of the money was borrowed at 7, part was borrowed at 8, and part was borrowed at 10. The amount borrowed at 10 was four times the amount borrowed at 7, and the annual interest on all three loans was $\$130,500$. Use matrices to find the amount borrowed at each rate.

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with solving systems of linear equations using Gaussian elimination.

- Solve a System of Two Equations Using an Augmented Matrix
- Solve a System of Three Equations Using an Augmented Matrix
- Augmented Matrices on the Calculator

## Key Concepts

- An augmented matrix is one that contains the coefficients and constants of a system of equations. See Example 5.6.1.
- A matrix augmented with the constant column can be represented as the original system of equations. See Example 5.6.2.
- Row operations include multiplying a row by a constant, adding one row to another row, and interchanging rows.
- We can use Gaussian elimination to solve a system of equations. See Example 5.6.3, Example 5.6.4, and Example 5.6.5.
- Row operations are performed on matrices to obtain row-echelon form. See Example 5.6.6.
- To solve a system of equations, write it in augmented matrix form. Perform row operations to obtain row-echelon form. Back-substitute to find the solutions. See Example 5.6.7 and Example 5.6.8.
- A calculator can be used to solve systems of equations using matrices. See Example 5.6.9.
- Many real-world problems can be solved using augmented matrices. See Example 5.6.10 and Example 5.6.11.

### Answers

**Exercise** 5.6.1

$$\begin{bmatrix} 4 & -3 & 11 \\ 3 & 2 & 4 \end{bmatrix}$$

**Exercise** 5.6.2

$$\begin{aligned} x - y + z &= 5 \\ 2x - y + 3z &= 1 \\ y + z &= -9 \end{aligned}$$

**Exercise** 5.6.3

$(2, 1)$

**Exercise** 5.6.4

$$\begin{bmatrix} 1 & -\frac{5}{2} & \frac{5}{2} & \frac{17}{2} \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

**Exercise** 5.6.5

$(1, 1, 1)$

**Exercise** 5.6.6

$\$150,000$ at 7, $\$750,000$ at 8, $\$600,000$ at 10

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# 5.7: Solving Systems with Inverses

**Learning Objectives**

- Find the inverse of a matrix.
- Solve a system of linear equations using an inverse matrix

Nancy plans to invest $\$10,500$ into two different bonds to spread out her risk. The first bond has an annual return of $10$, and the second bond has an annual return of $6$. To receive an $8.5$ return from the two bonds, how much should Nancy invest in each bond? What is the best method to solve this problem? There are several ways we can solve this problem. As we have seen in previous sections, systems of equations and matrices are useful in solving real-world problems involving finance. After studying this section, we will have the tools to solve the bond problem using the inverse of a matrix.

## Finding the Inverse of a Matrix

We know that the multiplicative inverse of a real number $a$ is $a^{-1}$, so

$$aa^{-1} = a^{-1}a = \left(\frac{1}{a}\right)a = 1 \tag{5.7.1}$$

For example, consider the scalar multiplication situation

$$2^{-1} = \frac{1}{2}$$

therefore from Equation 5.7.1

$$\left(\frac{1}{2}\right)2 = 1.$$

The **multiplicative inverse of a matrix** is similar in concept, except that the product of matrix $A$ and its inverse $A^{-1}$ equals the **identity matrix.** The identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by $I_n$ where $n$ represents the dimension of the matrix. Equations 5.7.2 and 5.7.3 are the identity matrices for a $2 \times 2$ matrix and a $3 \times 3$ matrix, respectively:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.7.2}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.7.3}$$

The identity matrix acts as a $1$ in matrix algebra. For example,

$$AI = IA = A$$

A matrix that has a multiplicative inverse has the properties

$$AA^{-1} = I \tag{5.7.4}$$

$$A^{-1}A = I \tag{5.7.5}$$

A matrix that has a multiplicative inverse is called an **invertible matrix**. Only a square matrix may have a multiplicative inverse, as the reversibility,

$$AA^{-1} = A^{-1}A = I \tag{5.7.6}$$

is a requirement. Not all square matrices have an inverse, but if $A$ is invertible, then $A^{-1}$ is unique. We will look at two methods for finding the inverse of a $2 \times 2$ matrix and a third method that can be used on both $2 \times 2$ and $3 \times 3$ matrices.

## Definitions: THE IDENTITY MATRIX AND MULTIPLICATIVE INVERSE

The *identity matrix*, $I_n$, is a square matrix containing ones down the main diagonal and zeros everywhere else.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

as for the $2 \times 2$ identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as for the $3 \times 3$ identity matrix

If $A$ is an $n \times n$ matrix and $B$ is an $n \times n$ matrix such that $AB = BA = I_n$, then $B = A - 1$, the **multiplicative inverse of a matrix $A$.**

## Example 5.7.1: Showing That the Identity Matrix Acts as a 1

Given matrix $A$, show that $AI = IA = A$.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

**Solution**

Use matrix multiplication to show that the product of $A$ and the identity matric is equal to the product of the identity matrix and $A$.

$$\begin{aligned} AI &= \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot 1 \\ -2 \cdot 1 + 5 \cdot 0 & -2 \cdot 0 + 5 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} AI &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 3 + 0 \cdot (-2) & 1 \cdot 4 + 0 \cdot 5 \\ 0 \cdot 3 + 1 \cdot (-2) & 0 \cdot 4 + 1 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \end{aligned}$$

## How to: Given two matrices, show that one is the multiplicative inverse of the other

- Given matrix $A$ of order $n \times n$ and matrix $B$ of order $n \times n$ multiply $AB$.
- If $AB = I$, then find the product $BA$. If $BA = I$, then $B = A^{-1}$ and $A = B^{-1}$.

### Example 5.7.2: Showing That Matrix $A$ Is the Multiplicative Inverse of Matrix $B$

Show that the given matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix}$$

**Solution**

Multiply $AB$ and $BA$. If both products equal the identity, then the two matrices are inverses of each other.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \cdot \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-9)+5(2) & 1(-5)+5(1) \\ -2(-9)-9(2) & -2(-5)-9(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} BA &= \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \\ &= \begin{bmatrix} -9(1)-5(-2) & -9(5)-5(-9) \\ 2(1)+1(-2) & 2(-5)+1(-9) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$A$ and $B$ are inverses of each other.

### Exercise 5.7.1

Show that the following two matrices are inverses of each other.

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

*Answer available at the end of this section.*

## Finding the Multiplicative Inverse Using Matrix Multiplication

We can now determine whether two matrices are inverses, but how would we find the inverse of a given matrix? Since we know that the product of a matrix and its inverse is the identity matrix, we can find the inverse of a matrix by setting up an equation using **matrix multiplication**.

## Example 5.7.3: Finding the Multiplicative Inverse Using Matrix Multiplication

Use matrix multiplication to find the inverse of the given matrix.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

**Solution**

For this method, we multiply $A$ by a matrix containing unknown constants and set it equal to the identity.

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find the product of the two matrices on the left side of the equal sign.

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a - 2c & 1b - 2d \\ 2a - 3c & 2b - 3d \end{bmatrix}$$

Next, set up a system of equations with the entry in row 1, column 1 of the new matrix equal to the first entry of the identity, $1$. Set the entry in row 2, column 1 of the new matrix equal to the corresponding entry of the identity, which is $0$.

$$1a - 2c = 1 \; R_1$$

$$2a - 3c = 0 \; R_2$$

Using row operations, multiply and add as follows: $(-2)R_1 + R_2 \to R_2$. Add the equations, and solve for $c$.

$$\begin{aligned} 1a - 2c &= 1 \\ 0 + 1c &= -2 \\ c &= -2 \end{aligned}$$

Back-substitute to solve for $a$.

$$\begin{aligned} a - 2(-2) &= 1 \\ a + 4 &= 1 \\ a &= -3 \end{aligned}$$

Write another system of equations setting the entry in row 1, column 2 of the new matrix equal to the corresponding entry of the identity, $0$. Set the entry in row 2, column 2 equal to the corresponding entry of the identity.

$$1b - 2d = 0 \; R_1$$

$$2b - 3d = 1 \; R_2$$

Using row operations, multiply and add as follows: $(-2)R_1 + R_2 = R_2$. Add the two equations and solve for $d$.

$$\begin{aligned} 1b - 2d &= 0 \\ 0 + 1d &= 1 \\ d &= 1 \end{aligned}$$

Once more, back-substitute and solve for $b$.

$$\begin{aligned} b - 2(1) &= 0 \\ b - 2 &= 0 \\ b &= 2 \end{aligned}$$

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

## Finding the Multiplicative Inverse by Augmenting with the Identity

Another way to find the **multiplicative inverse** is by augmenting with the identity. When matrix $A$ is transformed into $I$, the augmented matrix $I$ transforms into $A^{-1}$.

For example, given

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

augment $A$ with the identity

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right]$$

Perform row operations with the goal of turning A into the identity.

1. Switch row 1 and row 2.

$$\left[\begin{array}{cc|cc} 5 & 3 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

2. Multiply row 2 by –2 and add to row 1.

$$\left[\begin{array}{cc|cc} 1 & 1 & -2 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

3. Multiply row 1 by –2 and add to row 2.

$$\left[\begin{array}{cc|cc} 1 & 1 & -2 & 1 \\ 0 & -1 & 5 & -2 \end{array}\right]$$

4. Add row 2 to row 1.

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & -1 & 5 & -2 \end{array}\right]$$

5. Multiply row 2 by–1. –1.

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

The matrix we have found is $A^{-1}$.

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

## Finding the Multiplicative Inverse of $2 \times 2$ Matrices Using a Formula

When we need to find the **multiplicative inverse** of a $2 \times 2$ matrix, we can use a special formula instead of using matrix multiplication or augmenting with the identity.

If $A$ is a $2 \times 2$ matrix, such as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the multiplicative inverse of $A$ is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $ad - bc \neq 0$. If $ad - bc = 0$, then $A$ has no inverse.

## Example 5.7.4: Using the Formula to Find the Multiplicative Inverse of Matrix $A$

Use the formula to find the multiplicative inverse of

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

**Solution**

We can check that our formula works by using one of the other methods to calculate the inverse. Let's augment $A$ with the identity.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix}$$

Perform row operations with the goal of turning $A$ into the identity.

1. Multiply row 1 by $-2$ and add to row 2.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

2. Multiply row 1 by $2$ and add to row 1.

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

So, we have verified our original solution.

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

## Exercise 5.7.2

Use the formula to find the inverse of matrix $A$. Verify your answer by augmenting with the identity matrix.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

*Answer available at the end of this section.*

### Example 5.7.5: Finding the Inverse of the Matrix, If It Exists

Find the inverse, if it exists, of the given matrix.

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

**Solution**

We will use the method of augmenting with the identity.

$$\left[\begin{array}{cccc} 3 & 6 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

1. Switch row 1 and row 2.

$$\left[\begin{array}{cccc} 1 & 3 & 0 & 1 \\ 3 & 6 & 1 & 0 \end{array}\right]$$

2. Multiply row 1 by –3 and add it to row 2.

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]$$

3. There is nothing further we can do. The zeros in row 2 indicate that this matrix has no inverse.

## Finding the Multiplicative Inverse of $3 \times 3$ Matrices

Unfortunately, we do not have a formula similar to the one for a $2 \times 2$ matrix to find the inverse of a $3 \times 3$ matrix. Instead, we will augment the original matrix with the identity matrix and use row operations to obtain the inverse.

Given a $3 \times 3$ matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

augment $A$ with the identity matrix

$$\begin{array}{cc} A & I \end{array} = \left[\begin{array}{cccccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

To begin, we write the **augmented matrix** with the identity on the right and $A$ on the left. Performing elementary **row operations** so that the **identity matrix** appears on the left, we will obtain the **inverse matrix** on the right. We will find the inverse of this matrix in the next example.

### How to: Given a $3 \times 3$ matrix, find the inverse

1. Write the original matrix augmented with the identity matrix on the right.
2. Use elementary row operations so that the identity appears on the left.
3. What is obtained on the right is the inverse of the original matrix.
4. Use matrix multiplication to show that $AA^{-1} = I$ and $A^{-1}A = I$.

## Example 5.7.6: Finding the Inverse of a $3 \times 3$ Matrix

Given the $3 \times 3$ matrix $A$, find the inverse.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

**Solution**

Augment $A$ with the identity matrix, and then begin row operations until the identity matrix replaces $A$. The matrix on the right will be the inverse of $A$.

$$\left[\begin{array}{cccccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Interchange } R_2 \text{ and } R_1} \left[\begin{array}{cccccc} 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-R_2 + R_1 = R_1 \rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$-2R_1 + R_3 = R_3 \rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array}\right]$$

$$-3R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

Thus,

$$A^{-1} = B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

**Analysis**

To prove that $B = A^{-1}$, let's multiply the two matrices together to see if the product equals the identity, if $AA^{-1} = I$ and $A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)+3(-1)+1(6) & 2(1)+3(0)+1(-2) & 2(0)+3(1)+1(-3) \\ 3(-1)+3(-1)+1(6) & 3(1)+3(0)+1(-2) & 3(0)+3(1)+1(-3) \\ 2(-1)+4(-1)+1(6) & 2(1)+4(0)+1(-2) & 2(0)+4(1)+1(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & & & \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} & 2 & 31 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(2)+1(3)+0(2) & -1(3)+1(3)+0(4) & -1(1)+1(1)+0(1) \\ -1(2)+0(3)+1(2) & -1(3)+0(3)+1(4) & -1(1)+0(1)+1(1) \\ 6(2)+-2(3)+-3(2) & 6(3)+-2(3)+-3(4) & 6(1)+-2(1)+-3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Exercise 5.7.3**

Find the inverse of the $3 \times 3$ matrix.

$$A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$$

*Answer available at the end of this section.*

## Solving a System of Linear Equations Using the Inverse of a Matrix

Solving a system of linear equations using the inverse of a matrix requires the definition of two new matrices: $X$ is the matrix representing the variables of the system, and $B$ is the matrix representing the constants. Using **matrix multiplication**, we may define a system of equations with the same number of equations as variables as

$$AX = B$$

To solve a system of linear equations using an inverse matrix, let $A$ be the coefficient matrix, let $X$ be the variable matrix, and let $B$ be the constant matrix. Thus, we want to solve a system $AX = B$. For example, look at the following system of equations.

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

From this system, the coefficient matrix is

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The variable matrix is

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

And the constant matrix is

$$B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then $AX = B$ looks like

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Recall the discussion earlier in this section regarding multiplying a real number by its inverse, $(2^{-1})2 = \left(\frac{1}{2}\right) 2 = 1$. To solve a single linear equation $ax = b$ for $x$, we would simply multiply both sides of the equation by the multiplicative inverse (reciprocal) of $a$. Thus,

$$\begin{aligned} ax &= b \\ \left(\frac{1}{a}\right) ax &= \left(\frac{1}{a}\right) b \\ \left(a^{-1}\right) ax &= \left(a^{-1}\right) b \\ \left[\left(a^{-1}\right) a\right] x &= \left(a^{-1}\right) b \\ 1x &= \left(a^{-1}\right) b \\ x &= \left(a^{-1}\right) b \end{aligned}$$

The only difference between a solving a linear equation and a system of equations written in matrix form is that finding the inverse of a matrix is more complicated, and matrix multiplication is a longer process. However, the goal is the same—to isolate the variable.

We will investigate this idea in detail, but it is helpful to begin with a $2 \times 2$ system and then move on to a $3 \times 3$ system.

### SOLVING A SYSTEM OF EQUATIONS USING THE INVERSE OF A MATRIX

Given a system of equations, write the coefficient matrix $A$, the variable matrix $X$, and the constant matrix $B$. Then

$$AX = B$$

Multiply both sides by the inverse of $A$ to obtain the solution.

$$\begin{aligned} \left(A^{-1}\right) AX &= \left(A^{-1}\right) B \\ \left[\left(A^{-1}\right) A\right] X &= \left(A^{-1}\right) B \\ IX &= \left(A^{-1}\right) B \\ X &= \left(A^{-1}\right) B \end{aligned}$$

### Q&A: If the coefficient matrix does not have an inverse, does that mean the system has no solution?

*No, if the coefficient matrix is not invertible, the system could be inconsistent and have no solution, or be dependent and have infinitely many solutions.*

### Example 5.7.7: Solving a $2 \times 2$ System Using the Inverse of a Matrix

Solve the given system of equations using the inverse of a matrix.

$$3x + 8y = 5$$
$$4x + 11y = 7$$

**Solution**

Write the system in terms of a coefficient matrix, a variable matrix, and a constant matrix.

$$A = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

First, we need to calculate $A^{-1}$. Using the formula to calculate the inverse of a $2$ by $2$ matrix, we have:

$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3(11) - 8(4)} \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \\ &= \frac{1}{1} \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \end{aligned}$$

So,

$$A^{-1} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

Now we are ready to solve. Multiply both sides of the equation by $A^{-1}$.

$$\left(A^{-1}\right) AX = \left(A^{-1}\right) B$$
$$\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11(5) + (-8)7 \\ -4(5) + 3(7) \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The solution is $(-1, 1)$.

## Q&A: Can we solve for $X$ by finding the product $BA^{-1}$?

*No, recall that matrix multiplication is not commutative, so $A^{-1}B \neq BA^{-1}$. Consider our steps for solving the matrix equation.*

$$\left(A^{-1}\right) AX = \left(A^{-1}\right) B$$
$$\left[\left(A^{-1}\right) A\right] X = \left(A^{-1}\right) B$$
$$IX = \left(A^{-1}\right) B$$
$$X = \left(A^{-1}\right) B$$

*Notice in the first step we multiplied both sides of the equation by $A^{-1}$, but the $A^{-1}$ was to the left of $A$ on the left side and to the left of $B$ on the right side. Because matrix multiplication is not commutative, order matters.*

## Example 5.7.8: Solving a 3 × 3 System Using the Inverse of a Matrix

Solve the following system using the inverse of a matrix.

$$\begin{aligned} 5x+15y+56z &= 35 \\ -4x-11y-41z &= -26 \\ -x-3y-11z &= -7 \end{aligned}$$

**Solution**

Write the equation $AX = B$.

$$\begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

First, we will find the inverse of $A$ by augmenting with the identity.

$$\left[\begin{array}{ccc|ccc} 5 & 15 & 56 & 1 & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by $\frac{1}{5}$.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by $4$ and add to row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Add row 1 to row 3.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array}\right]$$

Multiply row 2 by $-3$ and add to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array}\right]$$

Multiply row 3 by $5$.

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{bmatrix}$$

Multiply row 3 by $\frac{1}{5}$ and add to row 1.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{bmatrix}$$

Multiply row 3 by $-\frac{19}{5}$ and add to row 2.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & 1 & -19 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{bmatrix}$$

So,

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix}$$

Multiply both sides of the equation by $A^{-1}$. We want $A^{-1}AX = A^{-1}B$:

$$\begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

Thus,

$$A^{-1}B = \begin{bmatrix} -70 + 78 - 7 \\ -105 - 26 + 133 \\ 35 + 0 - 35 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The solution is $(1, 2, 0)$.

## Exercise 5.7.4

Solve the system using the inverse of the coefficient matrix.

$$\begin{aligned} 2x - 17y + 11z &= 0 \\ -x + 11y - 7z &= 8 \\ 3y - 2z &= -2 \end{aligned}$$

*Answer available at the end of this section.*

### How to: Given a system of equations, solve with matrix inverses using a calculator

1. Save the coefficient matrix and the constant matrix as matrix variables $[A]$ and $[B]$.
2. Enter the multiplication into the calculator, calling up each matrix variable as needed.
3. If the coefficient matrix is invertible, the calculator will present the solution matrix; if the coefficient matrix is not invertible, the calculator will present an error message.

### Example 5.7.9: Using a Calculator to Solve a System of Equations with Matrix Inverses

Solve the system of equations with matrix inverses using a calculator

$$\begin{aligned} 2x+3y+z&=32\\ 3x+3y+z&=-27\\ 2x+4y+z&=-2 \end{aligned}$$

**Solution**

On the matrix page of the calculator, enter the coefficient matrix as the matrix variable $[A]$, and enter the constant matrix as the matrix variable $[B]$.

$$[A]=\begin{bmatrix}2&3&1\\3&3&1\\2&4&1\end{bmatrix},\ [B]=\begin{bmatrix}32\\-27\\-2\end{bmatrix}$$

On the home screen of the calculator, type in the multiplication to solve for $X$, calling up each matrix variable as needed.

$$[A]^{-1}\times[B]$$

Evaluate the expression.

$$\begin{bmatrix}-59\\-34\\252\end{bmatrix}$$

### Media

Access these online resources for additional instruction and practice with solving systems with inverses.

- The Identity Matrix
- Determining Inverse Matrices
- Using a Matrix Equation to Solve a System of Equations

## Key Equations

| | |
|---|---|
| Identity matrix for a $2\times 2$ matrix | $I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ |
| Identity matrix for a $3\times 3$ matrix | $I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ |

| | |
|---|---|
| Multiplicative inverse of a $2 \times 2$ matrix | $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$ |

## Key Concepts

- An identity matrix has the property $AI = IA = A$. See Example $5.7.1$.
- An invertible matrix has the property $AA^{-1} = A^{-1}A = I$. See Example $5.7.2$.
- Use matrix multiplication and the identity to find the inverse of a $2 \times 2$ matrix. See Example $5.7.3$.
- The multiplicative inverse can be found using a formula. See Example $5.7.4$.
- Another method of finding the inverse is by augmenting with the identity. See Example $5.7.5$.
- We can augment a $3 \times 3$ matrix with the identity on the right and use row operations to turn the original matrix into the identity, and the matrix on the right becomes the inverse. See Example $5.7.6$.
- Write the system of equations as $AX = B$, and multiply both sides by the inverse of $A$: $A^{-1}AX = A^{-1}B$. See Example $5.7.7$ and Example $5.7.8$.
- We can also use a calculator to solve a system of equations with matrix inverses. See Example $5.7.9$.

## Answers

**Exercise** $5.7.1$

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-3)+4(1) & 1(-4)+4(1) \\ -1(-3)+-3(1) & -1(-4)+-3(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -3(1)+-4(-1) & -3(4)+-4(-3) \\ 1(1)+1(-1) & 1(4)+1(-3) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

**Exercise** $5.7.2$

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

**Exercise** $5.7.3$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

**Exercise** $5.7.4$

$$X = \begin{bmatrix} 4 \\ 38 \\ 58 \end{bmatrix}$$

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# 5.8: Solving Systems with Cramer's Rule

**Learning Objectives**

- Evaluate 2 × 2 determinants.
- Use Cramer's Rule to solve a system of equations in two variables.
- Evaluate 3 × 3 determinants.
- Use Cramer's Rule to solve a system of three equations in three variables.
- Know the properties of determinants.

We have learned how to solve systems of equations in two variables and three variables, and by multiple methods: substitution, addition, Gaussian elimination, using the inverse of a matrix, and graphing. Some of these methods are easier to apply than others and are more appropriate in certain situations. In this section, we will study two more strategies for solving systems of equations.

## Evaluating the Determinant of a 2 × 2 Matrix

A determinant is a real number that can be very useful in mathematics because it has multiple applications, such as calculating area, volume, and other quantities. Here, we will use determinants to reveal whether a matrix is invertible by using the entries of a square matrix to determine whether there is a solution to the system of equations. Perhaps one of the more interesting applications, however, is their use in cryptography. Secure signals or messages are sometimes sent encoded in a matrix. The data can only be decrypted with an invertible matrix and the determinant. For our purposes, we focus on the determinant as an indication of the invertibility of the matrix. Calculating the determinant of a matrix involves following the specific patterns that are outlined in this section.

**FIND THE DETERMINANT OF A 2 × 2 MATRIX**

The *determinant* of a 2 × 2 matrix, given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Notice the change in notation. There are several ways to indicate the determinant, including $\det(A)$ and replacing the brackets in a matrix with straight lines, $|A|$.

**Example 5.8.1: Finding the Determinant of a $2 \times 2$ Matrix**

Find the determinant of the given matrix.

$$A = \begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$$

**Solution**

$$\begin{aligned}\det(A) &= \begin{vmatrix} 5 & 2 \\ -6 & 3 \end{vmatrix} \\ &= 5(3) - (-6)(2) \\ &= 27\end{aligned}$$

## Using Cramer's Rule to Solve a System of Two Equations in Two Variables

We will now introduce a final method for solving systems of equations that uses determinants. Known as **Cramer's Rule**, this technique dates back to the middle of the 18th century and is named for its innovator, the Swiss mathematician Gabriel Cramer (1704–1752), who introduced it in 1750 in *Introduction à l'Analyse des lignes Courbes algébrique*s. Cramer's Rule is a viable and efficient method for finding solutions to systems with an arbitrary number of unknowns, provided that we have the same number of equations as unknowns.

Cramer's Rule will give us the unique solution to a system of equations, if it exists. However, if the system has no solution or an infinite number of solutions, this will be indicated by a determinant of zero. To find out if the system is inconsistent or dependent, another method, such as elimination, will have to be used.

To understand Cramer's Rule, let's look closely at how we solve systems of linear equations using basic row operations. Consider a system of two equations in two variables.

$$a_1x + b_1y = c_1 \quad (1) \tag{5.8.1}$$
$$a_2x + b_2y = c_2 \quad (2) \tag{5.8.2}$$

We eliminate one variable using row operations and solve for the other. Say that we wish to solve for $x$. If Equation 5.8.2 is multiplied by the opposite of the coefficient of $y$ in Equation 5.8.1, Equation 5.8.1 is multiplied by the coefficient of $y$ in Equation 5.8.2, and we add the two equations, the variable $y$ will be eliminated.

$$\begin{aligned} b_2a_1x + b_2b_1y &= b_2c_1 && \text{Multiply } R_1 \text{ by } b_2 \\ -b_1a_2x - b_1b_2y &= -b_1c_2 && \text{Multiply } R_2 \text{ by } -b_1 \\ \hline b_2a_1x - b_1a_2x &= b_2c_1 - b_1c_2 \end{aligned}$$

Now, solve for $x$.

$$\begin{aligned} b_2a_1x - b_1a_2x &= b_2c_1 - b_1c_2 \\ x(b_2a_1 - b_1a_2) &= b_2c_1 - b_1c_2 \\ x &= \frac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2} = \frac{\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}} \end{aligned}$$

Similarly, to solve for $y$,we will eliminate $x$.

$$\begin{aligned} a_2a_1x + a_2b_1y &= a_2c_1 && \text{Multiply } R_1 \text{ by } a_2 \\ -a_1a_2x - a_1b_2y &= -a_1c_2 && \text{Multiply } R_2 \text{ by } -a_1 \\ \hline a_2b_1y - a_1b_2y &= a_2c_1 - a_1c_2 \end{aligned}$$

Solving for $y$ gives

$$\begin{aligned} a_2b_1y - a_1b_2y &= a_2c_1 - a_1c_2 \\ y(a_2b_1 - a_1b_2) &= a_2c_1 - a_1c_2 \\ y &= \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}} \end{aligned}$$

Notice that the denominator for both $x$ and $y$ is the determinant of the coefficient matrix.

We can use these formulas to solve for $x$ and $y$, but Cramer's Rule also introduces new notation:

- $D$:determinant of the coefficient matrix
- $D_x$:determinant of the numerator in the solution of $x$

$$x = \frac{D_x}{D} \tag{5.8.3}$$

- $D_y$:determinant of the numerator in the solution of $y$

$$y = \frac{D_y}{D} \tag{5.8.4}$$

The key to Cramer's Rule is replacing the variable column of interest with the constant column and calculating the determinants. We can then express $x$ and $y$ as a quotient of two determinants.

## CRAMER'S RULE FOR $2 \times 2$ SYSTEMS

Cramer's Rule is a method that uses determinants to solve systems of equations that have the same number of equations as variables.

Consider a system of two linear equations in two variables.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

The solution using Cramer's Rule is given as

$$x = \frac{D_x}{D} = \frac{\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}, D \neq 0 \tag{5.8.5}$$

$$y = \frac{D_y}{D} = \frac{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}, D \neq 0 \tag{5.8.6}$$

If we are solving for $x$, the $x$ column is replaced with the constant column. If we are solving for $y$, the $y$ column is replaced with the constant column.

## Example 5.8.2: Using Cramer's Rule to Solve a $2 \times 2$ System

Solve the following $2 \times 2$ system using Cramer's Rule.

$$\begin{aligned} 12x + 3y &= 15 \\ 2x - 3y &= 13 \end{aligned}$$

**Solution**

Solve for $x$.

$$\begin{aligned} x &= \frac{D_x}{D} \\ &= \frac{\begin{bmatrix} 15 & 3 \\ 13 & -3 \end{bmatrix}}{\begin{bmatrix} 12 & 3 \\ 2 & -3 \end{bmatrix}} \\ &= \frac{-45-39}{-36-6} \\ &= \frac{-84}{-42} \\ &= 2 \end{aligned}$$

Solve for $y$.

$$\begin{aligned} y &= \frac{D_y}{D} \\ &= \frac{\begin{bmatrix} 12 & 15 \\ 2 & 13 \end{bmatrix}}{\begin{bmatrix} 12 & 3 \\ 2 & -3 \end{bmatrix}} \\ &= \frac{156-30}{-36-6} \\ &= -\frac{126}{42} \\ &= -3 \end{aligned}$$

The solution is $(2, -3)$.

**Exercise 5.8.1**

Use Cramer's Rule to solve the $2 \times 2$ system of equations.

$$\begin{aligned} x + 2y &= -11 \\ -2x + y &= -13 \end{aligned}$$

*Answer available at the end of this section.*

## Evaluating the Determinant of a 3 × 3 Matrix

Finding the determinant of a 2×2 matrix is straightforward, but finding the determinant of a 3×3 matrix is more complicated. One method is to augment the 3×3 matrix with a repetition of the first two columns, giving a 3×5 matrix. Then we calculate the sum of the products of entries *down* each of the three diagonals (upper left to lower right), and subtract the products of entries *up* each of the three diagonals (lower left to upper right). This is more easily understood with a visual and an example.

Find the determinant of the 3×3 matrix.

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

1. Augment $A$ with the first two columns.

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{vmatrix}$$

2. From upper left to lower right: Multiply the entries down the first diagonal. Add the result to the product of entries down the

second diagonal. Add this result to the product of the entries down the third diagonal.

3. From lower left to upper right: Subtract the product of entries up the first diagonal. From this result subtract the product of entries up the second diagonal. From this result, subtract the product of entries up the third diagonal.

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{vmatrix}$$

The algebra is as follows:

$$|A| = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

### Example 5.8.3: Finding the Determinant of a 3 × 3 Matrix

Find the determinant of the $3 \times 3$ matrix given

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

**Solution**

Augment the matrix with the first two columns and then follow the formula. Thus,

$$\begin{aligned} |A| &= \begin{vmatrix} 0 & 2 & 1 & 0 & 2 \\ 3 & -1 & 1 & 3 & -1 \\ 4 & 0 & 1 & 4 & 0 \end{vmatrix} \\ &= 0(-1)(1) + 2(1)(4) + 1(3)(0) - 4(-1)(1) - 0(1)(0) - 1(3)(2) \\ &= 0 + 8 + 0 + 4 - 0 - 6 \\ &= 6 \end{aligned}$$

### Exercise 5.8.2

Find the determinant of the 3 × 3 matrix.

$$\det(A) = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

*Answer available at the end of this section.*

### Q&A: Can we use the same method to find the determinant of a larger matrix?

*No, this method only works for 2 × 2 and 3 × 3 matrices. For larger matrices it is best to use a graphing utility or computer software.*

## Using Cramer's Rule to Solve a System of Three Equations in Three Variables

Now that we can find the determinant of a $3 \times 3$ matrix, we can apply Cramer's Rule to solve a system of three equations in three

variables. Cramer's Rule is straightforward, following a pattern consistent with Cramer's Rule for $2 \times 2$ matrices. As the order of the matrix increases to $3 \times 3$, however, there are many more calculations required.

When we calculate the determinant to be zero, Cramer's Rule gives no indication as to whether the system has no solution or an infinite number of solutions. To find out, we have to perform elimination on the system.

Consider a $3 \times 3$ system of equations.

$$a_1x + b_1y + c_1z = d_1 \tag{5.8.7}$$
$$a_2x + b_2y + c_2z = d_2 \tag{5.8.8}$$
$$a_3x + b_3y + c_3z = d_3 \tag{5.8.9}$$

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D},\ D \neq 0$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \tag{5.8.10}$$

If we are writing the determinant $D_x$,we replace the $x$ column with the constant column. If we are writing the determinant $D_y$,we replace they y column with the constant column. If we are writing the determinant $D_z$,we replace the $z$ column with the constant column. Always check the answer.

## Example 5.8.4: Solving a $3 \times 3$ System Using Cramer's Rule

Find the solution to the given $3 \times 3$ system using Cramer's Rule.

$$x + y - z = 6$$
$$3x - 2y + z = -5$$
$$x + 3y - 2z = 14$$

**Solution**

Use Cramer's Rule.

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{vmatrix},\ D_x = \begin{vmatrix} 6 & 1 & -1 \\ -5 & -2 & 1 \\ 14 & 3 & -2 \end{vmatrix},\ D_y = \begin{vmatrix} 1 & 6 & -1 \\ 3 & -5 & 1 \\ 1 & 14 & -2 \end{vmatrix},\ D_z = \begin{vmatrix} 1 & 1 & 6 \\ 3 & -2 & -5 \\ 1 & 3 & 14 \end{vmatrix}$$

Then,

$$x = \frac{D_x}{D} = \frac{-3}{-3} = 1$$
$$y = \frac{D_y}{D} = \frac{-9}{-3} = 3$$
$$z = \frac{D_z}{D} = \frac{6}{-3} = -2$$

The solution is $(1, 3, -2)$.

## Exercise 5.8.3

Use Cramer's Rule to solve the $3 \times 3$ matrix.

$$x - 3y + 7z = 13$$
$$x + y + z = 1$$
$$x - 2y + 3z = 4$$

*Answer available at the end of this section.*

## Example 5.8.5A: Using Cramer's Rule to Solve an Inconsistent System

Solve the system of equations using Cramer's Rule.

$$3x - 2y = 4 \tag{5.8.11}$$
$$6x - 4y = 0 \tag{5.8.12}$$

**Solution**

We begin by finding the determinants $D$, $D_x$,and $D_y$.

$$D = \begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = 3(-4) - 6(-2) = 0$$

We know that a determinant of zero means that either the system has no solution or it has an infinite number of solutions. To see which one, we use the process of elimination. Our goal is to eliminate one of the variables.

1. Multiply Equation 5.8.11 by $-2$.
2. Add the result to Equation 5.8.12.

$$\begin{aligned} -6x + 4y &= -8 \\ 6x - 4y &= 0 \\ \hline 0 &= -8 \end{aligned}$$

We obtain the equation $0 = -8$, which is false. Therefore, the system has no solution. Graphing the system reveals two parallel lines. See Figure 5.8.1.

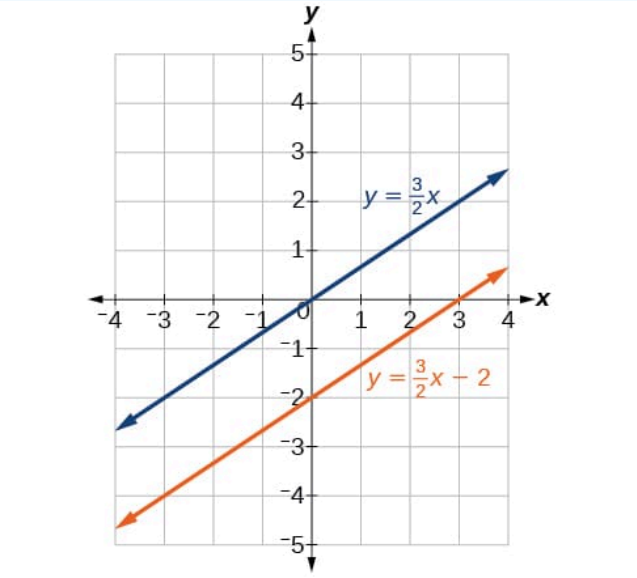



Figure 5.8.1

## Example 5.8.5B: Use Cramer's Rule to Solve a Dependent System

Solve the system with an infinite number of solutions.

$$x - 2y + 3z = 0 \tag{5.8.13}$$
$$3x + y - 2z = 0 \tag{5.8.14}$$
$$2x - 4y + 6z = 0 \tag{5.8.15}$$

**Solution**

Let's find the determinant first. Set up a matrix augmented by the first two columns.

$$\begin{vmatrix} 1 & -2 & 3 & 1 & -2 \\ 3 & 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & 2 & -4 \end{vmatrix}$$

Then,

$$1(1)(6) + (-2)(-2)(2) + 3(3)(-4) - 2(1)(3) - (-4)(-2)(1) - 6(3)(-2) = 0$$

As the determinant equals zero, there is either no solution or an infinite number of solutions. We have to perform elimination to find out.

1. Multiply Equation 5.8.13 by $-2$ and add the result to Equation 5.8.15:

$$\begin{array}{r} -2x + 4y - 6x = 0 \\ \underline{2x - 4y + 6z = 0} \\ 0 = 0 \end{array}$$

2. Obtaining an answer of $0 = 0$, a statement that is always true, means that the system has an infinite number of solutions. Graphing the system, we can see that two of the planes are the same and they both intersect the third plane on a line. See Figure 5.8.2.

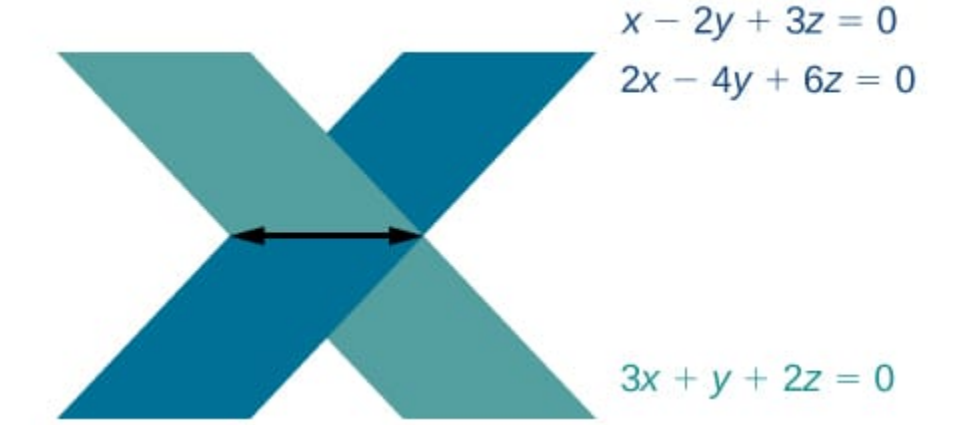



Figure 5.8.2

## Understanding Properties of Determinants

There are many properties of determinants. Listed here are some properties that may be helpful in calculating the determinant of a matrix.

### PROPERTIES OF DETERMINANTS

1. If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.
2. When two rows are interchanged, the determinant changes sign.
3. If either two rows or two columns are identical, the determinant equals zero.
4. If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.

5. The determinant of an inverse matrix $A^{-1}$ is the reciprocal of the determinant of the matrix $A$.
6. If any row or column is multiplied by a constant, the determinant is multiplied by the same factor.

## Example 5.8.6: Illustrating Properties of Determinants

Illustrate each of the properties of determinants.

**Solution**

Property 1 states that if the matrix is in upper triangular form, the determinant is the product of the entries down the main diagonal.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Augment $A$ with the first two columns.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} \det(A) &= 1(2)(-1) + 2(1)(0) + 3(0)(0) - 0(2)(3) - 0(1)(1) + 1(0)(2) \\ &= -2 \end{aligned}$$

Property 2 states that interchanging rows changes the sign. Given

$$\begin{aligned} A &= \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix} \\ \det(A) &= (-1)(-3) - (4)(5) \\ &= 3 - 20 \\ &= -17 \end{aligned}$$

$$\begin{aligned} B &= \begin{bmatrix} 4 & -3 \\ -1 & 5 \end{bmatrix} \\ \det(B) &= (4)(5) - (-1)(-3) \\ &= 20 - 3 \\ &= 17 \end{aligned}$$

Property 3 states that if two rows or two columns are identical, the determinant equals zero.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ -1 & 2 & 2 & -1 & 2 \end{bmatrix} \\ \det(A) &= 1(2)(2) + 2(2)(-1) + 2(2)(2) + 1(2)(2) - 2(2)(1) - 2(2)(2) \\ &= 4 - 4 + 8 + 4 - 4 - 8 \\ &= 0 \end{aligned}$$

Property 4 states that if a row or column equals zero, the determinant equals zero. Thus,

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\ \det(A) &= 1(0) - 2(0) \\ &= 0 \end{aligned}$$

Property 5 states that the determinant of an inverse matrix $A^{-1}$ is the reciprocal of the determinant $A$. Thus,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$\det(A) = 1(4) - 3(2)$$
$$= -2$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$
$$\det(A^{-1}) = -2\left(-\frac{1}{2}\right) - \frac{3}{2}(1)$$
$$= -\frac{1}{2}$$

Property 6 states that if any row or column of a matrix is multiplied by a constant, the determinant is multiplied by the same factor. Thus,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$\det(A) = 1(4) - 2(3)$$
$$= -2$$

$$B = \begin{bmatrix} 2(1) & 2(2) \\ 3 & 4 \end{bmatrix}$$
$$\det(B) = 2(4) - 3(4)$$
$$= -4$$

## Example 5.8.7: Using Cramer's Rule and Determinant Properties to Solve a System

Find the solution to the given $3 \times 3$ system.

$$2x + 4y + 4z = 2 \tag{5.8.16}$$
$$3x + 7y + 7z = -5 \tag{5.8.17}$$
$$x + 2y + 2z = 4 \tag{5.8.18}$$

**Solution**

Using Cramer's Rule, we have

$$D = \begin{bmatrix} 2 & 4 & 4 \\ 3 & 7 & 7 \\ 1 & 2 & 2 \end{bmatrix}$$

Notice that the second and third columns are identical. According to Property 3, the determinant will be zero, so there is either no solution or an infinite number of solutions. We have to perform elimination to find out.

1. Multiply Equation 5.8.18 by –2 and add the result to Equation 5.8.16.

$$-2x - 4y - 4x = -8$$
$$2x + 4y + 4z = 2$$
$$0 = -6$$

Obtaining a statement that is a contradiction means that the system has no solution.

## Media

Access these online resources for additional instruction and practice with Cramer's Rule.

- Solve a System of Two Equations Using Cramer's Rule
- Solve a Systems of Three Equations using Cramer's Rule

## Key Concepts

- The determinant for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. See Example $5.8.1$.
- Cramer's Rule replaces a variable column with the constant column. Solutions are $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$. See Example $5.8.2$.
- To find the determinant of a $3 \times 3$ matrix, augment with the first two columns. Add the three diagonal entries (upper left to lower right) and subtract the three diagonal entries (lower left to upper right). See Example $5.8.3$.
- To solve a system of three equations in three variables using Cramer's Rule, replace a variable column with the constant column for each desired solution: $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$. See Example $5.8.4$.
- Cramer's Rule is also useful for finding the solution of a system of equations with no solution or infinite solutions. See Example $5.8.5$ and Example $5.8.6$.
- Certain properties of determinants are useful for solving problems. For example:
  - If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.
  - When two rows are interchanged, the determinant changes sign.
  - If either two rows or two columns are identical, the determinant equals zero.
  - If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.
  - The determinant of an inverse matrix $A^{-1}$ is the reciprocal of the determinant of the matrix $A$.
  - If any row or column is multiplied by a constant, the determinant is multiplied by the same factor. See Example $5.8.7$ and Example $5.8.8$.

### Answers

**Exercise** $5.8.1$

$(3, -7)$

**Exercise** $5.8.2$

$-10$

**Exercise** $5.8.3$

$\left(-2, \frac{3}{5}, \frac{12}{5}\right)$

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# 5.E: Systems of Equations and Inequalities (Exercises)

## 9.1: Systems of Linear Equations: Two Variables

### Verbal

1) Can a system of linear equations have exactly two solutions? Explain why or why not.

**Answer**

No, you can either have zero, one, or infinitely many. Examine graphs.

2) If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company's profit margins.

3) If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?

**Answer**

This means there is no realistic break-even point. By the time the company produces one unit they are already making profit.

4) If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?

5) Given a system of equations, explain at least two different methods of solving that system.

**Answer**

You can solve by substitution (isolating $x$ or $y$), graphically, or by addition.

### Algebraic

**For the exercises 6-10, determine whether the given ordered pair is a solution to the system of equations.**

6) $\begin{array}{l} 5x-y=4 \\ x+6y=2 \end{array}$ and $(4,0)$

7) $\begin{array}{l} -3x-5y=13 \\ -x+4y=10 \end{array}$ and $(-6,1)$

**Answer**

Yes

8) $\begin{array}{l} 3x+7y=1 \\ 2x+4y=0 \end{array}$ and $(2,3)$

9) $\begin{array}{l} -2x+5y=7 \\ 2x+9y=7 \end{array}$ and $(-1,1)$

**Answer**

Yes

10) $\begin{array}{l} x+8y=43 \\ 3x-2y=-1 \end{array}$ and $(3,5)$

**For the exercises 11-20, solve each system by substitution.**

11) $\begin{array}{l} x+5y=5 \\ 2x+3y=4 \end{array}$

**Answer**

$(-1,2)$

12) $\begin{array}{l} 3x-2y=18 \\ 5x+10y=-10 \end{array}$

13) $\begin{array}{l} 4x+2y=-10 \\ 3x+9y=0 \end{array}$

**Answer**

$(-3,1)$

14) $\begin{array}{l} 2x+4y=-3.8 \\ 9x-5y=1.3 \end{array}$

15) $\begin{array}{l} -2x+3y=1.2 \\ -3x-6y=1.8 \end{array}$

**Answer**

$\left(-\frac{3}{5},0\right)$

16) $\begin{array}{l} x-0.2y=1 \\ -10x+2y=5 \end{array}$

17) $\begin{array}{l} 3x+5y=9 \\ 30x+50y=-90 \end{array}$

**Answer**

No solutions exist

18) $\begin{array}{l} -3x+y=2 \\ 12x-4y=-8 \end{array}$

19) $\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y=16 \\ \frac{1}{6}x+\frac{1}{4}y=9 \end{array}$

**Answer**

$\left(\frac{72}{5},\frac{132}{5}\right)$

20) $\begin{array}{l} -\frac{1}{4}x+\frac{3}{2}y=11 \\ -\frac{1}{8}x+\frac{1}{3}y=3 \end{array}$

**For the exercises 21-30, solve each system by addition.**

21) $\begin{aligned} -2x + 5y &= -42 \\ 7x + 2y &= 30 \end{aligned}$

**Answer**

$(6, -6)$

22) $\begin{aligned} 6x - 5y &= -34 \\ 2x + 6y &= 4 \end{aligned}$

23) $\begin{aligned} 5x - y &= -2.6 \\ -4x - 6y &= 1.4 \end{aligned}$

**Answer**

$\left(-\frac{1}{2}, \frac{1}{10}\right)$

24) $\begin{aligned} 7x - 2y &= 3 \\ 4x + 5y &= 3.25 \end{aligned}$

25) $\begin{aligned} -x + 2y &= -1 \\ 5x - 10y &= 6 \end{aligned}$

**Answer**

No solutions exist

26) $\begin{aligned} 7x + 6y &= 2 \\ -28x - 24y &= -8 \end{aligned}$

27) $\begin{aligned} \frac{5}{6}x + \frac{1}{4}y &= 0 \\ \frac{1}{8}x - \frac{1}{2}y &= -\frac{43}{120} \end{aligned}$

**Answer**

$\left(-\frac{1}{5}, \frac{2}{3}\right)$

28) $\begin{aligned} \frac{1}{3}x + \frac{1}{9}y &= \frac{2}{9} \\ -\frac{1}{2}x + \frac{4}{5}y &= -\frac{1}{3} \end{aligned}$

29) $\begin{aligned} -0.2x + 0.4y &= 0.6 \\ x - 2y &= -3 \end{aligned}$

**Answer**

$\left(x, \frac{x+3}{2}\right)$

30) $\begin{aligned} -0.1x + 0.2y &= 0.6 \\ 5x - 10y &= 1 \end{aligned}$

**For the exercises 31-40, solve each system by any method.**

31) $\begin{aligned} 5x + 9y &= 16 \\ x + 2y &= 4 \end{aligned}$

**Answer**

$(-4, 4)$

32) $\begin{aligned} 6x-8y&=-0.6\\ 3x+2y&=0.9 \end{aligned}$

33) $\begin{aligned} 5x-2y&=2.25\\ 7x-4y&=3 \end{aligned}$

**Answer**

$$\left(\frac{1}{2},\frac{1}{8}\right)$$

34) $\begin{aligned} x-\frac{5}{12}y&=-\frac{55}{12}\\ -6x+\frac{5}{2}y&=\frac{55}{2} \end{aligned}$

35) $\begin{aligned} 7x-4y&=\frac{7}{6}\\ 2x+4y&=\frac{1}{3} \end{aligned}$

**Answer**

$$\left(\frac{1}{6},0\right)$$

36) $\begin{aligned} 3x+6y&=11\\ 2x+4y&=9 \end{aligned}$

37) $\begin{aligned} \frac{7}{3}x-\frac{1}{6}y&=2\\ -\frac{21}{6}x+\frac{3}{12}y&=-3 \end{aligned}$

**Answer**

$$(x,2(7x-6))$$

38) $\begin{aligned} \frac{1}{2}x+\frac{1}{3}y&=\frac{1}{3}\\ \frac{3}{2}x+\frac{1}{4}y&=-\frac{1}{8} \end{aligned}$

39) $\begin{aligned} 2.2x+1.3y&=-0.1\\ 4.2x+4.2y&=2.1 \end{aligned}$

**Answer**

$$\left(-\frac{5}{6},\frac{4}{3}\right)$$

40) $\begin{aligned} 0.1x+0.2y&=2\\ 0.35x-0.3y&=0 \end{aligned}$

## Graphical

**For the exercises 41-45, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.**

41) $\begin{aligned} 3x-y&=0.6\\ x-2y&=1.3 \end{aligned}$

**Answer**

Consistent with one solution

42) $\begin{array}{r} -x+2y=4 \\ 2x-4y=1 \end{array}$

43) $\begin{array}{r} x+2y=7 \\ 2x+6y=12 \end{array}$

**Answer**

Consistent with one solution

44) $\begin{array}{r} 3x-5y=7 \\ x-2y=3 \end{array}$

45) $\begin{array}{r} 3x-2y=5 \\ -9x+6y=-15 \end{array}$

**Answer**

Dependent with infinitely many solutions

## Technology

**For the exercises 46-50, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.**

46) $\begin{array}{r} 0.1x+0.2y=0.3 \\ -0.3x+0.5y=1 \end{array}$

47) $\begin{array}{r} -0.01x+0.12y=0.62 \\ 0.15x+0.20y=0.52 \end{array}$

**Answer**

$(-3.08, 4.91)$

48) $\begin{array}{r} 0.5x+0.3y=4 \\ 0.25x-0.9y=0.46 \end{array}$

49) $\begin{array}{r} 0.15x+0.27y=0.39 \\ -0.34x+0.56y=1.8 \end{array}$

**Answer**

$(-1.52, 2.29)$

50) $\begin{array}{r} -0.71x+0.92y=0.13 \\ 0.83x+0.05y=2.1 \end{array}$

## Extensions

**For the exercises 51-55, solve each system in terms of $A, B, C, D,$ and $F$ where $A-F$ are nonzero numbers. Note that $A \neq B$ and $AE \neq BD$.**

51) $\begin{array}{r} x+y=A \\ x-y=B \end{array}$

**Answer**

$$\left(\frac{A+B}{2}, \frac{A-B}{2}\right)$$

52) $\begin{array}{l} x+Ay=1 \\ x+By=1 \end{array}$

53) $\begin{array}{l} Ax+y=0 \\ Bx+y=1 \end{array}$

**Answer**

$$\left(\frac{-1}{A-B}, \frac{A}{A-B}\right)$$

54) $\begin{array}{l} Ax+By=C \\ x+y=1 \end{array}$

55) $\begin{array}{l} Ax+By=C \\ Dx+Ey=F \end{array}$

**Answer**

$$\left(\frac{CE-BF}{BD-AE}, \frac{AF-CD}{BD-AE}\right)$$

## Real-World Applications

**For the exercises 56-60, solve for the desired quantity.**

56) A stuffed animal business has a total cost of production $C = 12x + 30$ and a revenue function $R = 20x$. Find the break-even point.

57) A fast-food restaurant has a cost of production $C(x) = 11x + 120$ and a revenue function $R(x) = 5x$. When does the company start to turn a profit?

**Answer**

They never turn a profit.

58) A cell phone factory has a cost of productiona $C(x) = 150x + 10,000$ and a revenue function $R(x) = 200x$. What is the break-even point?

59) A musician charges $C(x) = 64x + 20,000$, where $x$ is the total number of attendees at the concert. The venue charges $\$80$ per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

**Answer**

$(1,250,100,000)$

60) A guitar factory has a cost of production $C(x) = 75x + 50,000$. If the company needs to break even after $150$ units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

**For the exercises 61-77, use a system of linear equations with two variables and two equations to solve.**

61) Find two numbers whose sum is $28$ and difference is $13$.

**Answer**

The numbers are $7.5$ and $20.5$

62) A number is $9$ more than another number. Twice the sum of the two numbers is $10$. Find the two numbers.

63) The startup cost for a restaurant is $\$120,000$, and each meal costs $\$10$ for the restaurant to make. If each meal is then sold for $\$15$, after how many meals does the restaurant break even?

**Answer**

$24,000$

64) A moving company charges a flat rate of $\$150$, and an additional $\$5$ for each box. If a taxi service would charge $\$20$ for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

65) A total of $1,595$ first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by $15$. How many freshmen and sophomores were in attendance?

**Answer**

$790$ sophomores, $805$ freshman

66) $276$ students enrolled in a freshman-level chemistry class. By the end of the semester, $5$ times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.

67) There were $130$ faculty at a conference. If there were $18$ more women than men attending, how many of each gender attended the conference?

**Answer**

$56$ men, $74$ women

68) A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway $30$ minutes before the BMW did, and traveled $7$ mph slower than the BMW. After $2$ hours from the time the BMW entered the highway, the cars were $306.5$ miles apart. Find the speed of each car, assuming they were driven on cruise control.

69) If a scientist mixed $10\%$ saline solution with $60\%$ saline solution to get $25$ gallons of $40\%$ saline solution, how many gallons of $10\%$ and $60\%$ solutions were mixed?

**Answer**

$10$ gallons of $10\%$ solution, $15$ gallons of $60\%$ solution

70) An investor earned triple the profits of what she earned last year. If she made $\$500,000.48$ total for both years, how much did she earn in profits each year?

71) An investor who dabbles in real estate invested $1.1$ million dollars into two land investments. On the first investment, Swan Peak, her return was a $110\%$ increase on the money she invested. On the second investment, Riverside Community, she earned $50\%$ over what she invested. If she earned $\$1$ million in profits, how much did she invest in each of the land deals?

**Answer**

Swan Peak: $\$750,000$, Riverside: $\$350,000$

72) If an investor invests a total of $\$25,000$ into two bonds, one that pays $3\%$ simple interest, and the other that pays $2\frac{7}{8}\%$ interest, and the investor earns $\$737.50$ annual interest, how much was invested in each account?

73) If an investor invests $\$23,000$ into two bonds, one that pays $4\%$ in simple interest, and the other paying $2\%$ simple interest, and the investor earns $\$710.00$ annual interest, how much was invested in each account?

**Answer**

$\$12,500$ in the first account, $\$10,500$ in the second account.

74) CDs cost $\$5.96$ more than DVDs at All Bets Are Off Electronics. How much would $6$ CDs and $2$ DVDs cost if $5$ CDs and $2$ DVDs cost $\$127.73$?

75) A store clerk sold $60$ pairs of sneakers. The high-tops sold for $\$98.99$ and the low-tops sold for $\$129.99$. If the receipts for the two types of sales totaled $\$6,404.40$, how many of each type of sneaker were sold?

**Answer**

High-tops: $45$, Low-tops: $15$

76) A concert manager counted $350$ ticket receipts the day after a concert. The price for a student ticket was $\$12.50$, and the price for an adult ticket was $\$16.00$. The register confirms that $\$5,075$ was taken in. How many student tickets and adult tickets were sold?

77) Admission into an amusement park for $4$ children and $2$ adults is $\$116.90$. For $6$ children and $3$ adults, the admission is $\$175.35$. Assuming a different price for children and adults, what is the price of the child's ticket and the price of the adult ticket?

**Answer**

Infinitely many solutions. We need more information.

# 9.2: Systems of Linear Equations: Three Variables

## Verbal

1) Can a linear system of three equations have exactly two solutions? Explain why or why not

**Answer**

No, there can be only one, zero, or infinitely many solutions.

2) If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.

3) If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.

**Answer**

Not necessarily. There could be zero, one, or infinitely many solutions. For example, $(0,0,0)$ is not a solution to the system below, but that does not mean that it has no solution.

$$\begin{aligned} 2x+3y-6z&=1\\ -4x-6y+12z&=-2\\ x+2y+5z&=10 \end{aligned}$$

4) Using the method of addition, is there only one way to solve the system?

5) Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

**Answer**

Every system of equations can be solved graphically, by substitution, and by addition. However, systems of three equations become very complex to solve graphically so other methods are usually preferable.

# Algebraic

**For the exercises 6-10, determine whether the ordered triple given is the solution to the system of equations.**

6) $\begin{array}{l} 2x-6y+6z=-12 \\ x+4y+5z=-1 \\ -x+2y+3z=-1 \end{array}$ and $(0,1,-1)$

7) $\begin{array}{l} 6x-y+3z=6 \\ 3x+5y+2z=0 \\ x+y=0 \end{array}$ and $(3,-3,-5)$

**Answer**

No

8) $\begin{array}{l} 6x-7y+z=2 \\ -x-y+3z=4 \\ 2x+y-z=1 \end{array}$ and $(4,2,-6)$

9) $\begin{array}{l} x-y=0 \\ x-z=5 \\ x-y+z=-1 \end{array}$ and $(4,4,-1)$

**Answer**

Yes

10) $\begin{array}{l} -x-y+2z=3 \\ 5x+8y-3z=4 \\ -x+3y-5z=-5 \end{array}$ and $(4,1,-7)$

**For the exercises 11-16, solve each system by substitution.**

11) $\begin{array}{l} 3x-4y+2z=-15 \\ 2x+4y+z=16 \\ 2x+3y+5z=20 \end{array}$

**Answer**

$(-1,4,2)$

12) $\begin{array}{l} 5x-2y+3z=20 \\ 2x-4y-3z=-9 \\ x+6y-8z=21 \end{array}$

13) $\begin{array}{l} 5x+2y+4z=9 \\ -3x+2y+z=10 \\ 4x-3y+5z=-3 \end{array}$

**Answer**

$\left(-\frac{85}{107},\frac{312}{107},\frac{191}{107}\right)$

14) $\begin{array}{l} 4x-3y+5z=31 \\ -x+2y+4z=20 \\ x+5y-2z=-29 \end{array}$

15) $\begin{array}{l} 5x-2y+3z=4 \\ -4x+6y-7z=-1 \\ 3x+2y-z=4 \end{array}$

**Answer**

$$\left(1, \frac{1}{2}, 0\right)$$

16) $\begin{aligned} 4x + 6y + 9z &= 4 \\ -5x + 2y - 6z &= 3 \\ 7x - 4y + 3z &= -3 \end{aligned}$

**For the exercises 17-45, solve each system by Gaussian elimination.**

17) $\begin{aligned} 2x - y + 3z &= 17 \\ -5x + 4y - 2z &= -46 \\ 2y + 5z &= -7 \end{aligned}$

**Answer**

$$(4, -6, 1)$$

18) $\begin{aligned} 5x - 6y + 3z &= 50 \\ -x + 4y &= 10 \\ 2x - z &= 10 \end{aligned}$

19) $\begin{aligned} 2x + 3y - 6z &= 1 \\ -4x - 6y + 12z &= -2 \\ x + 2y + 5z &= 10 \end{aligned}$

**Answer**

$$\left(x, \frac{1}{27}(65 - 16x), \frac{x + 28}{27}\right)$$

20) $\begin{aligned} 4x + 6y - 2z &= 8 \\ 6x + 9y - 3z &= 12 \\ -2x - 3y + z &= -4 \end{aligned}$

21) $\begin{aligned} 2x + 3y - 4z &= 5 \\ -3x + 2y + z &= 11 \\ -x + 5y + 3z &= 4 \end{aligned}$

**Answer**

$$\left(-\frac{45}{13}, \frac{17}{13}, -2\right)$$

22) $\begin{aligned} 10x + 2y - 14z &= 8 \\ -x - 2y - 4z &= -1 \\ -12x - 6y + 6z &= -12 \end{aligned}$

23) $\begin{aligned} x + y + z &= 14 \\ 2y + 3z &= -14 \\ -16y - 24z &= -112 \end{aligned}$

**Answer**

No solutions exist

24) $\begin{aligned} 5x - 3y + 4z &= -1 \\ -4x + 2y - 3z &= 0 \\ -x + 5y + 7z &= -11 \end{aligned}$

25) $$\begin{aligned} x+y+z&=0\\ 2x-y+3z&=0\\ x-z&=0\end{aligned}$$

**Answer**

$$(0,0,0)$$

26) $$\begin{aligned} 3x+2y-5z&=6\\ 5x-4y+3z&=-12\\ 4x+5y-2z&=15\end{aligned}$$

27) $$\begin{aligned} x+y+z&=0\\ 2x-y+3z&=0\\ x-z&=1\end{aligned}$$

**Answer**

$$\left(\frac{4}{7},-\frac{1}{7},-\frac{3}{7}\right)$$

28) $$\begin{aligned} 3x-\frac{1}{2}y-z&=-\frac{1}{2}\\ 4x+z&=3\\ -x+\frac{3}{2}y&=\frac{5}{2}\end{aligned}$$

29) $$\begin{aligned} 6x-5y+6z&=38\\ \frac{1}{5}x-\frac{1}{2}y+\frac{3}{5}z&=1\\ -4x-\frac{3}{2}y-z&=-74\end{aligned}$$

**Answer**

$$(7,20,16)$$

30) $$\begin{aligned} \frac{1}{2}x-\frac{1}{5}y+\frac{2}{5}z&=-\frac{13}{10}\\ \frac{1}{4}x-\frac{2}{5}y-\frac{1}{5}z&=-\frac{7}{20}\\ -\frac{1}{2}x-\frac{3}{4}y-\frac{1}{2}z&=-\frac{5}{4}\end{aligned}$$

31) $$\begin{aligned} -\frac{1}{3}x-\frac{1}{2}y-\frac{1}{4}z&=\frac{3}{4}\\ -\frac{1}{2}x-\frac{1}{4}y-\frac{1}{2}z&=2\\ -\frac{1}{4}x-\frac{3}{4}y-\frac{1}{2}z&=-\frac{1}{2}\end{aligned}$$

**Answer**

$$(-6,2,1)$$

32) $$\begin{aligned} \frac{1}{2}x-\frac{1}{4}y+\frac{3}{4}z&=0\\ \frac{1}{4}x-\frac{1}{10}y+\frac{2}{5}z&=-2\\ \frac{1}{8}x+\frac{1}{5}y-\frac{1}{8}z&=2\end{aligned}$$

33) $$\begin{aligned} \frac{4}{5}x - \frac{7}{8}y + \frac{1}{2}z &= 1 \\ -\frac{4}{5}x - \frac{3}{4}y + \frac{1}{3}z &= -8 \\ -\frac{2}{5}x - \frac{7}{8}y + \frac{1}{2}z &= -5 \end{aligned}$$

**Answer**

$$(5, 12, 15)$$

34) $$\begin{aligned} -\frac{1}{3}x - \frac{1}{8}y + \frac{1}{6}z &= -\frac{4}{3} \\ -\frac{2}{3}x - \frac{7}{8}y + \frac{1}{3}z &= -\frac{23}{3} \\ -\frac{1}{3}x - \frac{5}{8}y + \frac{5}{6}z &= 0 \end{aligned}$$

35) $$\begin{aligned} -\frac{1}{4}x - \frac{5}{4}y + \frac{5}{2}z &= -5 \\ -\frac{1}{2}x - \frac{5}{3}y + \frac{5}{4}z &= \frac{55}{12} \\ -\frac{1}{3}x - \frac{1}{3}y + \frac{1}{3}z &= \frac{5}{3} \end{aligned}$$

**Answer**

$$(-5, -5, -5)$$

36) $$\begin{aligned} \frac{1}{40}x + \frac{1}{60}y + \frac{1}{80}z &= \frac{1}{100} \\ -\frac{1}{2}x - \frac{1}{3}y - \frac{1}{4}z &= -\frac{1}{5} \\ \frac{3}{8}x + \frac{3}{12}y + \frac{3}{16}z &= \frac{3}{20} \end{aligned}$$

37) $$\begin{aligned} 0.1x - 0.2y + 0.3z &= 2 \\ 0.5x - 0.1y + 0.4z &= 8 \\ 0.7x - 0.2y + 0.3z &= 8 \end{aligned}$$

**Answer**

$$(10, 10, 10)$$

38) $$\begin{aligned} 0.2x + 0.1y - 0.3z &= 0.2 \\ 0.8x + 0.4y - 1.2z &= 0.1 \\ 1.6x + 0.8y - 2.4z &= 0.2 \end{aligned}$$

39) $$\begin{aligned} 1.1x + 0.7y - 3.1z &= -1.79 \\ 2.1x + 0.5y - 1.6z &= -0.13 \\ 0.5x + 0.4y - 0.5z &= -0.07 \end{aligned}$$

**Answer**

$$\left(\frac{1}{2}, \frac{1}{5}, \frac{4}{5}\right)$$

40) $$\begin{aligned} 0.5x - 0.5y + 0.5z &= 10 \\ 0.2x - 0.2y + 0.2z &= 4 \\ 0.1x - 0.1y + 0.1z &= 2 \end{aligned}$$

41) $$\begin{aligned} 0.1x + 0.2y + 0.3z &= 0.37 \\ 0.1x - 0.2y - 0.3z &= -0.27 \\ 0.5x - 0.1y - 0.3z &= -0.03 \end{aligned}$$

**Answer**

$$\left(\frac{1}{2}, \frac{2}{5}, \frac{4}{5}\right)$$

42) $$\begin{aligned} 0.5x - 0.5y - 0.3z &= 0.13 \\ 0.4x - 0.1y - 0.3z &= 0.11 \\ 0.2x - 0.8y - 0.9z &= -0.32 \end{aligned}$$

43) $$\begin{aligned} 0.5x + 0.2y - 0.3z &= 1 \\ 0.4x - 0.6y + 0.7z &= 0.8 \\ 0.3x - 0.1y - 0.9z &= 0.6 \end{aligned}$$

**Answer**

$$(2, 0, 0)$$

44) $$\begin{aligned} 0.3x + 0.3y + 0.5z &= 0.6 \\ 0.4x + 0.4y + 0.4z &= 1.8 \\ 0.4x + 0.2y + 0.1z &= 1.6 \end{aligned}$$

45) $$\begin{aligned} 0.8x + 0.8y + 0.8z &= 2.4 \\ 0.3x - 0.5y + 0.2z &= 0 \\ 0.1x + 0.2y + 0.3z &= 0.6 \end{aligned}$$

**Answer**

$$(1, 1, 1)$$

## Extensions

**For the exercises 46-50, solve the system for $x, y,$ and $z$.**

46) $$\begin{aligned} x + y + z &= 3 \\ \frac{x-1}{2} + \frac{y-3}{2} + \frac{z+1}{2} &= 0 \\ \frac{x-2}{3} + \frac{y+4}{3} + \frac{z-3}{3} &= \frac{2}{3} \end{aligned}$$

47) $$\begin{aligned} 5x - 3y - \frac{z+1}{2} &= \frac{1}{2} \\ 6x + \frac{y-9}{2} + 2z &= -3 \\ \frac{x+8}{2} - 4y + z &= 4 \end{aligned}$$

**Answer**

$$\left(\frac{128}{557}, \frac{23}{557}, \frac{428}{557}\right)$$

48) $$\begin{aligned} \frac{x+4}{7} - \frac{y-1}{6} + \frac{z+2}{3} &= 1 \\ \frac{x-2}{4} + \frac{y+1}{8} - \frac{z+8}{2} &= 0 \\ \frac{x+6}{3} - \frac{y+2}{3} + \frac{z+4}{2} &= 3 \end{aligned}$$

49) $$\begin{aligned} \frac{x-3}{6} + \frac{y+2}{2} - \frac{z-3}{3} &= 2 \\ \frac{x+2}{4} + \frac{y-5}{2} + \frac{z+4}{2} &= 1 \\ \frac{x+6}{2} - \frac{y-3}{3} + z + 1 &= 9 \end{aligned}$$

**Answer**

$(6, -1, 0)$

50) $\begin{aligned} \frac{x-1}{3} + \frac{y+3}{4} + \frac{z+2}{6} &= 1 \\ 4x + 3y - 2z &= 11 \\ 0.02x + 0.015y - 0.01z &= 0.065 \end{aligned}$

## Real-World Applications

51) Three even numbers sum up to $108$. The smaller is half the larger and the middle number is $\frac{3}{4}$ the larger. What are the three numbers?

**Answer**

$24, 36, 48$

52) Three numbers sum up to $147$. The smallest number is half the middle number, which is half the largest number. What are the three numbers?

53) At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were $400$ people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

**Answer**

$70$ grandparents, $140$ parents, $190$ children

54) An animal shelter has a total of $350$ animals comprised of cats, dogs, and rabbits. If the number of rabbits is $5$ less than one-half the number of cats, and there are $20$ more cats than dogs, how many of each animal are at the shelter?

55) Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was $\$82$. She forgot to save the individual receipts but remembered that your groceries were $\$0.05$ cheaper than half of her groceries, and that your other roommate's groceries were $\$2.10$ more than your groceries. How much was each of your share of the groceries?

**Answer**

Your share was $\$19.95$, Sarah's share was $\$40$, and your other roommate's share was $\$22.05$.

56) Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was $\$100.75$. Your supplies were bought with $5\%$ tax, John's with $8\%$ tax, and your third roommate's with $9\%$ sales tax. The total amount of money spent without taxes is $\$93.50$. If your supplies before tax were $\$1$ more than half of what your third roommate's supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.

57) Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $\$82,000$. The office manager makes $\$4,000$ more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $\$78,000$. What is the annual salary of each of the co-workers?

**Answer**

There are infinitely many solutions; we need more information

58) At a carnival, $\$2,914.25$ in receipts were taken at the end of the day. The cost of a child's ticket was $\$20.50$, an adult ticket was $\$29.75$, and a senior citizen ticket was $\$15.25$. There were twice as many senior citizens as adults in attendance, and $20$ more children than senior citizens. How many children, adult, and senior citizen tickets were sold?

59) A local band sells out for their concert. They sell all $1,175$ tickets for a total purse of $\$28,112.50$. The tickets were priced at $\$20$ for student tickets, $\$22.50$ for children, and $\$29$ for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?

**Answer**

$500$ students, $225$ children, and $450$ adults

60) In a bag, a child has $325$ coins worth $\$19.50$. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?

61) Last year, at Haven's Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of $\$140,000$. This year, due to inflation, the same cars would cost $\$151,830$. The cost of the BMW increased by $8\%$, the Jeep by $5\%$, and the Toyota by $12\%$. If the price of last year's Jeep was $\$7,000$ less than the price of last year's BMW, what was the price of each of the three cars last year?

**Answer**

The BMW was $\$49,636$, the Jeep was $\$42,636$, and the Toyota was $\$47,727$.

62) A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested $\$80,500$ into three accounts, one that paid $4\%$ simple interest, one that paid $3\frac{1}{8}\%$ simple interest, and one that paid $2\frac{1}{2}\%$ simple interest. He earned $\$2,670$ interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

63) You inherit one million dollars. You invest it all in three accounts for one year. The first account pays $3\%$ compounded annually, the second account pays $4\%$ compounded annually, and the third account pays $2\%$ compounded annually. After one year, you earn $\$34,000$ in interest. If you invest four times the money into the account that pays $3\%$ compared to $2\%$, how much did you invest in each account?

**Answer**

$\$400,000$ in the account that pays $3\%$ interest, $\$500,000$ in the account that pays $4\%$ interest, and $\$100,000$ in the account that pays $2\%$ interest.

64) You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays $4\%$ compounded annually, the second account pays $3\%$ compounded annually, and the third account pays $2\%$ compounded annually. After one year, you earn $\$3,650$ in interest. If you invest five times the money in the account that pays $4\%$ compared to $3\%$, how much did you invest in each account?

65) The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed $39.8\%$ of the world's consumed oil. The United States consumed $0.7\%$ more than four times China's consumption. The United States consumed $5\%$ more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?

**Answer**

The United States consumed $26.3\%$, Japan $7.1\%$, and China $6.4\%$ of the world's oil.

66) The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced $31.4\%$ of the world's produced oil. Saudi Arabia and the United States combined for $22.1\%$ of the world's production, and Saudi Arabia produced $2\%$ more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?

67) The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for $47\%$ of oil imports. The United States imported $1.8\%$ more from Saudi Arabia than they did from Mexico, and $1.7\%$ more from Saudi Arabia than they did from Canada. What percent of the United States oil imports were from these three countries?

**Answer**

Saudi Arabia imported $16.8\%$, Canada imported $15.1\%$, and Mexico $15.0\%$

68) The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for $64\%$ of the United States oil production. The Gulf of Mexico and Texas combined for $47\%$ of oil production. Texas produced $3\%$ more than Alaska. What percent of United States oil production came from these regions?

69) At one time, in the United States, $398$ species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised $55\%$ of the endangered species. Birds accounted for $0.7\%$ more than fish, and fish accounted for $1.5\%$ more than mammals. What percent of the endangered species came from mammals, birds, and fish?

**Answer**

Birds were $19.3\%$, fish were $18.6\%$, and mammals were $17.1\%$ of endangered species

70) Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up $4\%$ less than one-quarter of poultry consumption, and red meat consumption is $18.2\%$ higher than poultry consumption, what are the percentages of meat consumption?

# 9.3: Systems of Nonlinear Equations and Inequalities: Two Variables

## Verbal

1) Explain whether a system of two nonlinear equations can have exactly two solutions. What about exactly three? If not, explain why not. If so, give an example of such a system, in graph form, and explain why your choice gives two or three answers.

**Answer**

A nonlinear system could be representative of two circles that overlap and intersect in two locations, hence two solutions. A nonlinear system could be representative of a parabola and a circle, where the vertex of the parabola meets the circle and the branches also intersect the circle, hence three solutions.

2) When graphing an inequality, explain why we only need to test one point to determine whether an entire region is the solution?

3) When you graph a system of inequalities, will there always be a feasible region? If so, explain why. If not, give an example of a graph of inequalities that does not have a feasible region. Why does it not have a feasible region?

**Answer**

No. There does not need to be a feasible region. Consider a system that is bounded by two parallel lines. One inequality represents the region above the upper line; the other represents the region below the lower line. In this case, no points in the plane are located in both regions; hence there is no feasible region.

4) If you graph a revenue and cost function, explain how to determine in what regions there is profit.

5) If you perform your break-even analysis and there is more than one solution, explain how you would determine which x-values are profit and which are not.

**Answer**

Choose any number between each solution and plug into $C(x)$ and $R(x)$. If $C(x) < R(x)$, then there is profit.

# Algebraic

**For the exercises 6-10, solve the system of nonlinear equations using substitution.**

6) $\begin{aligned} x+y&=4 \\ x^2+y^2&=9 \end{aligned}$

7) $\begin{aligned} y&=x-3 \\ x^2+y^2&=9 \end{aligned}$

**Answer**

$(0,-3),\ (3,0)$

8) $\begin{aligned} y&=x \\ x^2+y^2&=9 \end{aligned}$

9) $\begin{aligned} y&=-x \\ x^2+y^2&=9 \end{aligned}$

**Answer**

$\left(-\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right),\left(\frac{3\sqrt{2}}{2},-\frac{3\sqrt{2}}{2}\right)$

10) $\begin{aligned} x&=2 \\ x^2-y^2&=9 \end{aligned}$

**For the exercises 11-15, solve the system of nonlinear equations using elimination.**

11) $\begin{aligned} 4x^2-9y^2&=36 \\ 4x^2+9y^2&=36 \end{aligned}$

**Answer**

$(-3,0),\ (3,0)$

12) $\begin{aligned} x^2+y^2&=25 \\ x^2-y^2&=1 \end{aligned}$

13) $\begin{aligned} 2x^2+4y^2&=4 \\ 2x^2-4y^2&=25x-10 \end{aligned}$

**Answer**

$\left(\frac{1}{4},-\frac{\sqrt{62}}{8}\right),\left(\frac{1}{4},\frac{\sqrt{62}}{8}\right)$

14) $\begin{aligned} y^2-x^2&=9 \\ 3x^2+2y^2&=8 \end{aligned}$

15) $\begin{aligned} x^2+y^2+\frac{1}{16}&=2500 \\ y&=2x^2 \end{aligned}$

**Answer**

$\left(-\frac{\sqrt{398}}{4},\frac{199}{4}\right),\left(\frac{\sqrt{398}}{4},\frac{199}{4}\right)$

**For the exercises 16-23, use any method to solve the system of nonlinear equations.**

16) $\begin{aligned} -2x^2+y&=-5\\ 6x-y&=9\end{aligned}$

17) $\begin{aligned} -x^2+y&=2\\ -x+y&=2\end{aligned}$

**Answer**

$(0,2),\,(1,3)$

18) $\begin{aligned} x^2+y^2&=1\\ y&=20x^2-1\end{aligned}$

19) $\begin{aligned} x^2+y^2&=1\\ y&=-x^2\end{aligned}$

**Answer**

$$\left(-\sqrt{\frac{1}{2}(\sqrt{5}-1)},\frac{1}{2}\left(1-\sqrt{5}\right)\right),\left(\sqrt{\frac{1}{2}(\sqrt{5}-1)},\frac{1}{2}\left(1-\sqrt{5}\right)\right)$$

20) $\begin{aligned} 2x^3-x^2&=y\\ y&=\frac{1}{2}-x\end{aligned}$

21) $\begin{aligned} 9x^2+25y^2&=225\\ (x-6)^2+y^2&=1\end{aligned}$

**Answer**

$(5,0)$

22) $\begin{aligned} x^4-x^2&=y\\ x^2+y&=0\end{aligned}$

23) $\begin{aligned} 2x^3-x^2&=y\\ x^2+y&=0\end{aligned}$

**Answer**

$(0,0)$

**For the exercises 24-38, use any method to solve the nonlinear system.**

24) $\begin{aligned} x^2+y^2&=9\\ y&=3-x^2\end{aligned}$

25) $\begin{aligned} x^2-y^2&=9\\ x&=3\end{aligned}$

**Answer**

$(3,0)$

26) $\begin{aligned} x^2-y^2&=9\\ y&=3\end{aligned}$

27) $\begin{aligned} x^2-y^2&=9\\ x-y&=0\end{aligned}$

**Answer**

No Solutions Exist

28) $\begin{aligned} -x^2+y&=2 \\ -4x+y&=-1 \end{aligned}$

29) $\begin{aligned} -x^2+y&=2 \\ 2y&=-x \end{aligned}$

**Answer**

No Solutions Exist

30) $\begin{aligned} x^2+y^2&=25 \\ x^2-y^2&=36 \end{aligned}$

31) $\begin{aligned} x^2+y^2&=1 \\ y^2&=x^2 \end{aligned}$

**Answer**

$$\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right),\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right),\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right),\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$

32) $\begin{aligned} 16x^2-9y^2+144&=0 \\ y^2+x^2&=16 \end{aligned}$

33) $\begin{aligned} 3x^2-y^2&=12 \\ (x-1)^2+y^2&=1 \end{aligned}$

**Answer**

$$(2,0)$$

34) $\begin{aligned} 3x^2-y^2&=12 \\ (x-1)^2+y^2&=4 \end{aligned}$

35) $\begin{aligned} 3x^2-y^2&=12 \\ x^2+y^2&=16 \end{aligned}$

**Answer**

$$(-\sqrt{7},-3),(-\sqrt{7},3),(\sqrt{7},-3),(\sqrt{7},3)$$

36) $\begin{aligned} x^2-y^2-6x-4y-11&=0 \\ -x^2+y^2&=5 \end{aligned}$

37) $\begin{aligned} x^2+y^2-6y&=7 \\ x^2+y&=1 \end{aligned}$

**Answer**

$$\left(-\sqrt{\frac{1}{2}(\sqrt{73}-5)},\frac{1}{2}\left(7-\sqrt{73}\right)\right),\left(\sqrt{\frac{1}{2}(\sqrt{73}-5)},\frac{1}{2}\left(7-\sqrt{73}\right)\right)$$

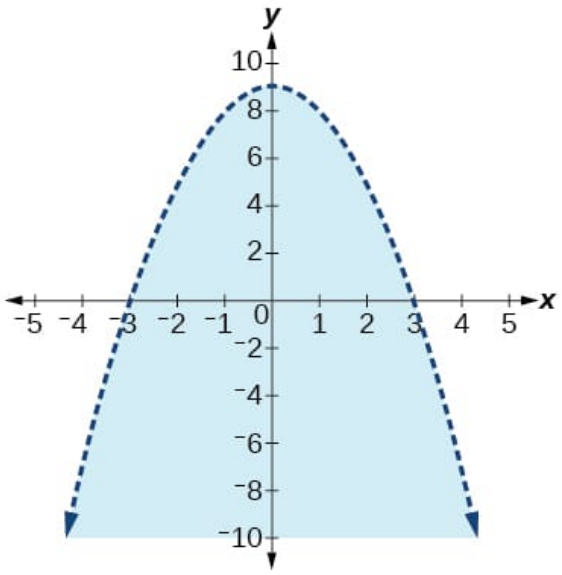
38) $\begin{aligned} x^2+y^2&=6 \\ xy&=1 \end{aligned}$

## Graphical

**For the exercises 39-40, graph the inequality.**

39) $x^2 + y < 9$

**Answer**

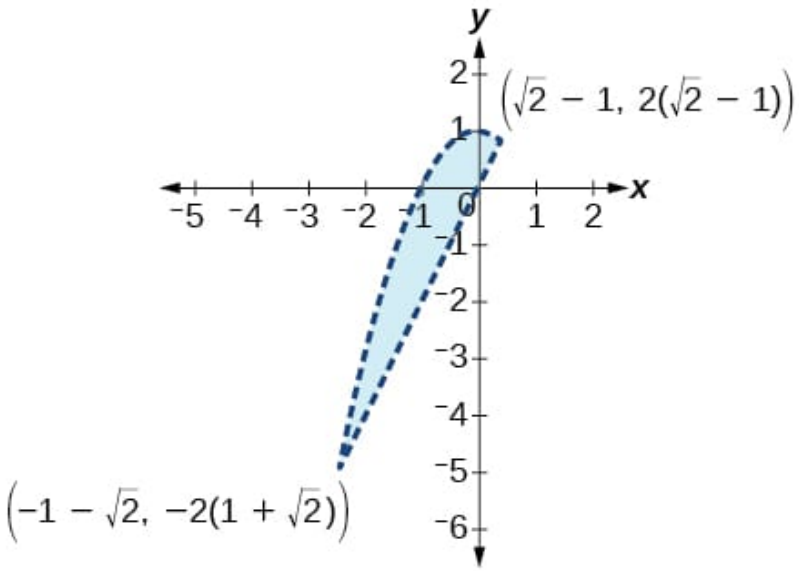



40) $x^2 + y^2 < 4$

**For the exercises 41-45, graph the system of inequalities. Label all points of intersection.**

41) $x^2 + y < 1$
$y > 2x$

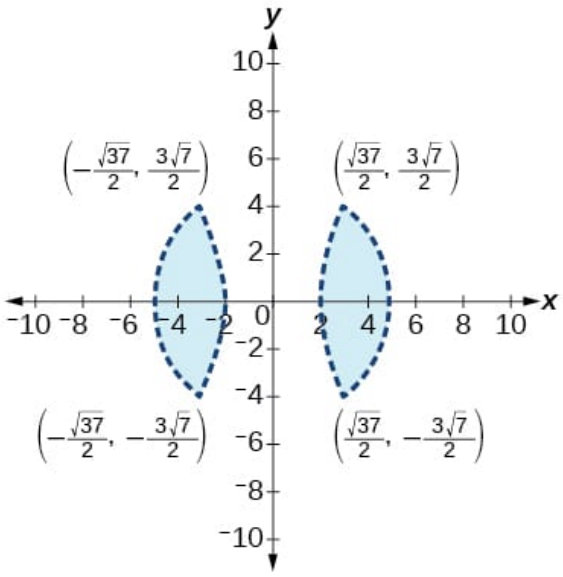
**Answer**




42) $x^2 + y < -5$
$y > 5x + 10$

43) $\begin{array}{l} x^2+y^2<25 \\ 3x^2-y^2>12 \end{array}$

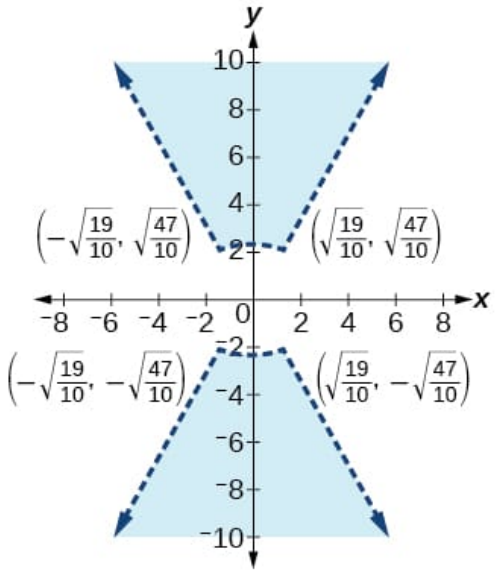
**Answer**




44) $\begin{array}{l} x^2-y^2>-4 \\ x^2+y^2<12 \end{array}$

45) $\begin{array}{l} x^2+3y^2>16 \\ 3x^2-y^2<1 \end{array}$

**Answer**



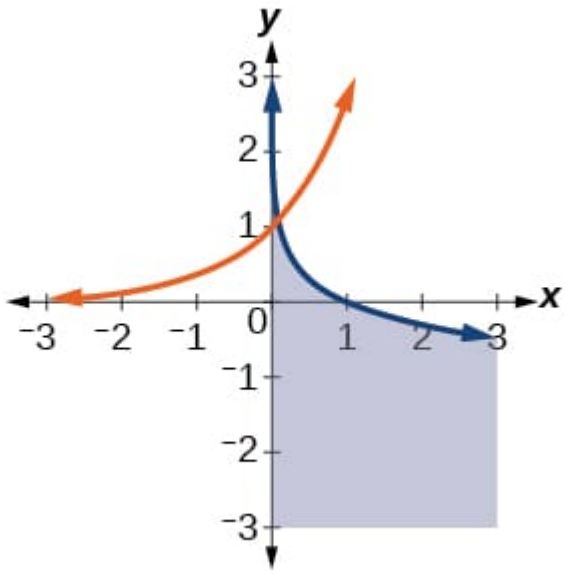

## Extensions

**For the exercises 46-47, graph the inequality.**

46) $y \geq e^x$
$y \leq \ln(x) + 5$

47) $y \leq -\log(x)$
$y \leq e^x$

**Answer**




**For the exercises 48-52, find the solutions to the nonlinear equations with two variables.**

48) $\frac{4}{x^2} + \frac{1}{y^2} = 24$
$\frac{5}{x^2} - \frac{2}{y^2} + 4 = 0$

49) $\frac{6}{x^2} - \frac{1}{y^2} = 8$
$\frac{1}{x^2} - \frac{6}{y^2} = \frac{1}{8}$

**Answer**

$$\left(-2\sqrt{\frac{70}{383}}, -2\sqrt{\frac{35}{29}}\right), \left(-2\sqrt{\frac{70}{383}}, 2\sqrt{\frac{35}{29}}\right), \left(2\sqrt{\frac{70}{383}}, -2\sqrt{\frac{35}{29}}\right), \left(2\sqrt{\frac{70}{383}}, 2\sqrt{\frac{35}{29}}\right)$$

50) $x^2 - xy + y^2 - 2 = 0$
$x + 3y = 4$

51) $x^2 - xy - 2y^2 - 6 = 0$
$x^2 + y^2 = 1$

**Answer**

No Solution Exists

52) $x^2 + 4xy - 2y^2 - 6 = 0$
$x = y + 2$

## Technology

**For the exercises 53-54, solve the system of inequalities. Use a calculator to graph the system to confirm the answer.**

53) $\begin{array}{l} xy < 1 \\ y > \sqrt{x} \end{array}$

**Answer**

$x = 0, y > 0$ and $0 < x < 1, \sqrt{x} < y < \frac{1}{x}$

54) $\begin{array}{l} x^2 + y < 3 \\ y > 2x \end{array}$

## Real-World Applications

**For the exercises 55-, construct a system of nonlinear equations to describe the given behavior, then solve for the requested solutions.**

55) Two numbers add up to $300$. One number is twice the square of the other number. What are the numbers?

**Answer**

$12, 288$

56) The squares of two numbers add to $360$. The second number is half the value of the first number squared. What are the numbers?

57) A laptop company has discovered their cost and revenue functions for each day: $C(x) = 3x^2 - 10x + 200$ and $R(x) = -2x^2 + 100x + 50$. If they want to make a profit, what is the range of laptops per day that they should produce? Round to the nearest number which would generate profit.

**Answer**

$2$ - $20$ computers

58) A cell phone company has the following cost and revenue functions: $C(x) = 8x^2 - 600x + 21,500$ and $R(x) = -3x^2 + 480x$. What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.

# 9.4: Partial Fractions

## Verbal

1) Can any quotient of polynomials be decomposed into at least two partial fractions? If so, explain why, and if not, give an example of such a fraction.

**Answer**

No, a quotient of polynomials can only be decomposed if the denominator can be factored. For example, $\frac{1}{x^2+1}$ cannot be decomposed because the denominator cannot be factored.

2) Can you explain why a partial fraction decomposition is unique? (**Hint**: Think about it as a system of equations.)

3) Can you explain how to verify a partial fraction decomposition graphically?

**Answer**

Graph both sides and ensure they are equal.

4) You are unsure if you correctly decomposed the partial fraction correctly. Explain how you could double-check your answer.

5) Once you have a system of equations generated by the partial fraction decomposition, can you explain another method to solve it? For example if you had $\frac{7x+13}{3x^2+8x+15} = \frac{A}{x+1} + \frac{B}{3x+5}$ we eventually simplify to $7x+13 = A(3x+5) + B(x+1)$. Explain how you could intelligently choose an $x$-value that will eliminate either $A$ or $B$ and solve for $A$ and $B$.

**Answer**

If we choose $x = -1$, then the $B$-term disappears, letting us immediately know that $A = 3$. We could alternatively plug in $x = -\frac{5}{3}$, giving us a $B$-value of $-2$.

## Algebraic

**For the exercises 6-19, find the decomposition of the partial fraction for the nonrepeating linear factors.**

6) $\frac{5x+16}{x^2+10x+24}$

7) $\frac{3x-79}{x^2-5x-24}$

**Answer**

$$\frac{8}{x+3} - \frac{5}{x-8}$$

8) $\frac{-x-24}{x^2-2x-24}$

9) $\frac{10x+47}{x^2+7x+10}$

**Answer**

$$\frac{1}{x+5} + \frac{9}{x+2}$$

10) $\frac{x}{6x^2+25x+25}$

11) $\frac{32x-11}{20x^2-13x+2}$

**Answer**

$$\frac{3}{5x-2} + \frac{4}{4x-1}$$

12) $\frac{x+1}{x^2+7x+10}$

13) $\frac{5x}{x^2-9}$

**Answer**

$$\frac{5}{2(x+3)} + \frac{5}{2(x-3)}$$

14) $\frac{10x}{x^2-25}$

15) $\frac{6x}{x^2-4}$

**Answer**

$$\frac{3}{x+2}+\frac{3}{x-2}$$

16) $\frac{2x-3}{x^2-6x+5}$

17) $\frac{4x-1}{x^2-x-6}$

**Answer**

$$\frac{9}{5(x+2)}+\frac{11}{5(x-3)}$$

18) $\frac{4x+3}{x^2+8x+15}$

19) $\frac{3x-1}{x^2-5x+6}$

**Answer**

$$\frac{8}{x-3}-\frac{5}{x-2}$$

**For the exercises 20-30, find the decomposition of the partial fraction for the repeating linear factors.**

20) $\frac{-5x-19}{(x+4)^2}$

21) $\frac{x}{(x-2)^2}$

**Answer**

$$\frac{1}{x-2}-\frac{2}{(x-2)^2}$$

22) $\frac{7x+14}{(x+3)^2}$

23) $\frac{-24x-27}{(4x+5)^2}$

**Answer**

$$-\frac{6}{4x+5}+\frac{3}{(4x+5)^2}$$

24) $\frac{-24x-27}{(6x-7)^2}$

25) $\frac{5-x}{(x-7)^2}$

**Answer**

$$-\frac{1}{x-7}-\frac{2}{(x-7)^2}$$

26) $\dfrac{5x+14}{2x^2+12x+18}$

27) $\dfrac{5x^2+20x+8}{2x(x+1)^2}$

**Answer**

$$\frac{4}{x}-\frac{3}{2(x+1)}+\frac{7}{2(x+1)^2}$$

28) $\dfrac{4x^2+55x+25}{5x(3x+5)^2}$

29) $\dfrac{54x^3+127x^2+80x+16}{2x^2(3x+2)^2}$

**Answer**

$$\frac{4}{x}+\frac{2}{x^2}-\frac{3}{3x+2}+\frac{7}{2(3x+2)^2}$$

30) $\dfrac{x^3-5x^2+12x+144}{x^2(x^2+12x+36)}$

**For the exercises 31-43, find the decomposition of the partial fraction for the irreducible nonrepeating quadratic factor.**

31) $\dfrac{4x^2+6x+11}{(x+2)(x^2+x+3)}$

**Answer**

$$\frac{x+1}{x^2+x+3}+\frac{3}{(x+2)}$$

32) $\dfrac{4x^2+9x+23}{(x-1)(x^2+6x+11)}$

33) $\dfrac{-2x^2+10x+4}{(x-1)(x^2+3x+8)}$

**Answer**

$$\frac{4-3x}{x^2+3x+8}+\frac{1}{(x-1)}$$

34) $\dfrac{x^2+3x+1}{(x+1)(x^2+5x-2)}$

35) $\dfrac{4x^2+17x-1}{(x+3)(x^2+6x+1)}$

**Answer**

$$\frac{2x-1}{x^2+6x+1}+\frac{2}{(x+3)}$$

36) $\frac{4x^2}{(x+5)(x^2+7x-5)}$

37) $\frac{4x^2+x+3}{x^3-1}$

**Answer**

$$\frac{1}{x^2+x+1}+\frac{4}{(x-1)}$$

38) $\frac{-5x^2+18x-4}{x^3+8}$

39) $\frac{3x^2-7x+33}{x^3+27}$

**Answer**

$$\frac{2}{x^2-3x+9}+\frac{3}{(x+3)}$$

40) $\frac{x^2+2x+40}{x^3-125}$

41) $\frac{4x^2+4x+12}{8x^3-27}$

**Answer**

$$-\frac{1}{4x^2+6x+9}+\frac{1}{(2x-3)}$$

42) $\frac{-50x^2+5x-3}{125x^3-1}$

43) $\frac{-2x^3-30x^2+36x+216}{x^4+216x}$

**Answer**

$$\frac{1}{x}+\frac{1}{x+6}-\frac{4x}{x^2-6x+36}$$

**For the exercises 44-54, find the decomposition of the partial fraction for the irreducible repeating quadratic factor.**

44) $\frac{3x^3+2x^2+14x+15}{(x^2+4)^2}$

45) $\frac{x^3+6x^2+5x+9}{(x^2+1)^2}$

**Answer**

$$\frac{x+6}{x^2+1}+\frac{4x+3}{(x^2+1)^2}$$

46) $\frac{x^3-x^2+x-1}{(x^2-3)^2}$

47) $\frac{x^2+5x+5}{(x+2)^2}$

**Answer**

$$\frac{x+1}{x+2}+\frac{2x+3}{(x+2)^2}$$

48) $\frac{x^3+2x^2+4x}{(x^2+2x+9)^2}$

49) $\frac{x^2+25}{(x^2+3x+25)^2}$

**Answer**

$$\frac{1}{x^2+3x+25}-\frac{3x}{(x^2+3x+25)^2}$$

50) $\frac{2x^3+11x+7x+70}{(2x^2+x+14)^2}$

51) $\frac{5x+2}{x(x^2+4)^2}$

**Answer**

$$\frac{1}{8x}-\frac{x}{8(x^2+4)}+\frac{10-x}{8(x^2+4)^2}$$

52) $\frac{x^4+x^3+8x^2+6x+36}{x(x^2+6)^2}$

53) $\frac{2x-9}{(x^2-x)^2}$

**Answer**

$$-\frac{16}{x}-\frac{9}{x^2}+\frac{16}{x-1}-\frac{7}{(x-1)^2}$$

54) $\frac{5x^3-2x+1}{(x^2+2x)^2}$

## Extensions

**For the exercises 55-56, find the partial fraction expansion.**

55) $\frac{x^2+4}{(x+1)^3}$

**Answer**

$$\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{5}{(x+1)^3}$$

56) $\frac{x^3-4x^2+5x+4}{(x-2)^3}$

**For the exercises 57-59, perform the operation and then find the partial fraction decomposition.**

57) $\frac{7}{x+8}+\frac{5}{x-2}-\frac{x-1}{x^2-6x-16}$

**Answer**

$$\frac{5}{x-2}-\frac{3}{10(x+2)}+\frac{7}{x+8}-\frac{7}{10(x-8)}$$

58) $\frac{1}{x-4}-\frac{3}{x+6}-\frac{2x+7}{x^2+2x-24}$

59) $\frac{2x}{x^2-16}-\frac{1-2x}{x^2+6x+8}-\frac{x-5}{x^2-4x}$

**Answer**

$$-\frac{5}{4x}-\frac{5}{2(x+2)}+\frac{11}{2(x+4)}+\frac{5}{4(x+4)}$$

# 9.5: Matrices and Matrix Operations

## Verbal

1) Can we add any two matrices together? If so, explain why; if not, explain why not and give an example of two matrices that cannot be added together.

**Answer**

No, they must have the same dimensions. An example would include two matrices of different dimensions. One cannot add the following two matrices because the first is a $2 \times 2$ matrix and the second is a $2 \times 3$. $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + \begin{bmatrix}6 & 5 & 4\\3 & 2 & 1\end{bmatrix}$ has no sum.

2) Can we multiply any column matrix by any row matrix? Explain why or why not.

3) Can both the products $AB$ and $BA$ be defined? If so, explain how; if not, explain why.

**Answer**

Yes, if the dimensions of $A$ are $m \times n$ and the dimensions of $B$ are $n \times m$ both products will be defined.

4) Can any two matrices of the same size be multiplied? If so, explain why, and if not, explain why not and give an example of two matrices of the same size that cannot be multiplied together.

5) Does matrix multiplication commute? That is, does $AB = BA$? If so, prove why it does. If not, explain why it does not.

**Answer**

Not necessarily. To find $AB$, we multiply the first row of $A$ by the first column of $B$ to get the first entry of $AB$. To find $BA$, we multiply the first row of $B$ by the first column of $A$ to get the first entry of $BA$. Thus, if those are unequal, then the matrix multiplication does not commute.

## Algebraic

**For the exercises 6-11, use the matrices below and perform the matrix addition or subtraction. Indicate if the operation is undefined.**

$$A=\begin{bmatrix}1 & 3\\0 & 7\end{bmatrix}, B=\begin{bmatrix}2 & 14\\22 & 6\end{bmatrix}, C=\begin{bmatrix}1 & 5\\8 & 92\\12 & 6\end{bmatrix}, D=\begin{bmatrix}10 & 14\\7 & 2\\5 & 61\end{bmatrix}, E=\begin{bmatrix}6 & 12\\14 & 5\end{bmatrix}, F=\begin{bmatrix}0 & 9\\78 & 17\\15 & 4\end{bmatrix}$$

6) $A + B$

7) $C + D$

**Answer**

$$\begin{bmatrix} 11 & 19 \\ 15 & 94 \\ 17 & 67 \end{bmatrix}$$

8) $A + C$

9) $B - E$

**Answer**

$$\begin{bmatrix} -4 & 2 \\ 8 & 1 \end{bmatrix}$$

10) $C + F$

11) $D - B$

**Answer**

Undefined; dimensions do not match

**For the exercises 12-17, use the matrices below to perform scalar multiplication.**

$$A = \begin{bmatrix} 4 & 6 \\ 13 & 12 \end{bmatrix}, B = \begin{bmatrix} 3 & 9 \\ 21 & 12 \\ 0 & 64 \end{bmatrix}, C = \begin{bmatrix} 16 & 3 & 7 & 18 \\ 90 & 5 & 3 & 29 \end{bmatrix}, D = \begin{bmatrix} 18 & 12 & 13 \\ 8 & 14 & 6 \\ 7 & 4 & 21 \end{bmatrix}$$

12) $5A$

13) $3B$

**Answer**

$$\begin{bmatrix} 9 & 27 \\ 63 & 36 \\ 0 & 192 \end{bmatrix}$$

14) $-2B$

15) $-4C$

**Answer**

$$\begin{bmatrix} -64 & -12 & -28 & -72 \\ -360 & -20 & -12 & -116 \end{bmatrix}$$

16) $\frac{1}{2}C$

17) $100D$

**Answer**

$$\begin{bmatrix} 1,800 & 1,200 & 1,300 \\ 800 & 1,400 & 600 \\ 700 & 400 & 2,100 \end{bmatrix}$$

**For the exercises 18-23, use the matrices below to perform matrix multiplication.**

$$A=\begin{bmatrix}-1 & 5\\ 3 & 2\end{bmatrix}, B=\begin{bmatrix}3 & 6 & 4\\ -8 & 0 & 12\end{bmatrix}, C=\begin{bmatrix}4 & 10\\ -2 & 6\\ 5 & 9\end{bmatrix}, D=\begin{bmatrix}2 & -3 & 12\\ 9 & 3 & 1\\ 0 & 8 & -10\end{bmatrix}$$

18) $AB$

19) $BC$

**Answer**

$$\begin{bmatrix}20 & 102\\ 28 & 28\end{bmatrix}$$

20) $CA$

21) $BD$

**Answer**

$$\begin{bmatrix}60 & 41 & 2\\ -16 & 120 & -216\end{bmatrix}$$

22) $DC$

23) $CB$

**Answer**

$$\begin{bmatrix}-68 & 24 & 136\\ -54 & -12 & 64\\ -57 & 30 & 128\end{bmatrix}$$

**For the exercises 24-29, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed.**

$$A=\begin{bmatrix}2 & -5\\ 6 & 7\end{bmatrix}, B=\begin{bmatrix}-9 & 6\\ -4 & 2\end{bmatrix}, C=\begin{bmatrix}0 & 9\\ 7 & 1\end{bmatrix}, D=\begin{bmatrix}-8 & 7 & -5\\ 4 & 3 & 2\\ 0 & 9 & 2\end{bmatrix}, E=\begin{bmatrix}4 & 5 & 3\\ 7 & -6 & -5\\ 1 & 0 & 9\end{bmatrix}$$

24) $A+B-C$

25) $4A+5D$

**Answer**

Undefined; dimensions do not match.

26) $2C+B$

27) $3D+4E$

**Answer**

$$\begin{bmatrix}-8 & 41 & -3\\ 40 & -15 & -14\\ 4 & 27 & 42\end{bmatrix}$$

28) $C-0.5D$

29) $100D - 10E$

**Answer**

$$\begin{bmatrix} -840 & 650 & -530 \\ 330 & 360 & 250 \\ -10 & 900 & 110 \end{bmatrix}$$

**For the exercises 30-40, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2 = A \cdot A$)**

$$A = \begin{bmatrix} -10 & 20 \\ 5 & 25 \end{bmatrix}, B = \begin{bmatrix} 40 & 10 \\ -20 & 30 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

30) $AB$

31) $BA$

**Answer**

$$\begin{bmatrix} -350 & 1,050 \\ 350 & 350 \end{bmatrix}$$

32) $CA$

33) $BC$

**Answer**

Undefined; inner dimensions do not match.

34) $A^2$

35) $B^2$

**Answer**

$$\begin{bmatrix} 1,400 & 700 \\ -1,400 & 700 \end{bmatrix}$$

36) $C^2$

37) $B^2A^2$

**Answer**

$$\begin{bmatrix} 332,500 & 927,500 \\ -227,500 & 87,500 \end{bmatrix}$$

38) $A^2B^2$

39) $(AB)^2$

**Answer**

$$\begin{bmatrix} 490,000 & 0 \\ 0 & 490,000 \end{bmatrix}$$

40) $(BA)^2$

**For the exercises 41-49, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2 = A \cdot A$)**

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 1 & -5 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 0.1 \\ 1 & 0.2 \\ -0.5 & 0.3 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & -1 \\ -6 & 7 & 5 \\ 4 & 2 & 1 \end{bmatrix}$$

41) $AB$

**Answer**

$$\begin{bmatrix} -2 & 3 & 4 \\ -7 & 9 & -7 \end{bmatrix}$$

42) $BA$

43) $BD$

**Answer**

$$\begin{bmatrix} -4 & 29 & 21 \\ -27 & -3 & 1 \end{bmatrix}$$

44) $DC$

45) $D^2$

**Answer**

$$\begin{bmatrix} -3 & -2 & -2 \\ -28 & 59 & 46 \\ -4 & 16 & 7 \end{bmatrix}$$

46) $A^2$

47) $D^3$

**Answer**

$$\begin{bmatrix} 1 & -18 & -9 \\ -198 & 505 & 369 \\ -72 & 126 & 91 \end{bmatrix}$$

48) $(AB)C$

49) $A(BC)$

**Answer**

$$\begin{bmatrix} 0 & 1.6 \\ 9 & -1 \end{bmatrix}$$

## Technology

**For the exercises 50-54, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. Use a calculator to verify your solution.**

$$A = \begin{bmatrix} -2 & 0 & 9 \\ 1 & 8 & -3 \\ 0.5 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 3 & 0 \\ -4 & 1 & 6 \\ 8 & 7 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

50) $AB$

51) $BA$

**Answer**

$$\begin{bmatrix} 2 & 24 & -4.5 \\ 12 & 32 & -9 \\ -8 & 64 & 61 \end{bmatrix}$$

52) $CA$

53) $BC$

**Answer**

$$\begin{bmatrix} 0.5 & 3 & 0.5 \\ 2 & 1 & 2 \\ 10 & 7 & 10 \end{bmatrix}$$

54) $ABC$

## Extensions

**For the exercises 55-, use the matrix below to perform the indicated operation on the given matrix.**

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

55) $B^2$

**Answer**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

56) $B^3$

57) $B^4$

**Answer**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

58) $B^5$

59) Using the above questions, find a formula for $B^n$. Test the formula for $B^{201}$ and $B^{202}$, using a calculator.

**Answer**

$$B^n = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & n \text{ even} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, & n \text{ odd} \end{cases}$$

# 9.6: Solving Systems with Gaussian Elimination

## Verbal

1) Can any system of linear equations be written as an augmented matrix? Explain why or why not. Explain how to write that augmented matrix.

**Answer**

Yes. For each row, the coefficients of the variables are written across the corresponding row, and a vertical bar is placed; then the constants are placed to the right of the vertical bar.

2) Can any matrix be written as a system of linear equations? Explain why or why not. Explain how to write that system of equations.

3) Is there only one correct method of using row operations on a matrix? Try to explain two different row operations possible to solve the augmented matrix $\begin{bmatrix} 9 & 3 & 0 \\ 1 & -2 & 6 \end{bmatrix}$.

**Answer**

No, there are numerous correct methods of using row operations on a matrix. Two possible ways are the following:

i. Interchange rows 1 and 2. Then $R_2 = R_2 - 9R_1$.
ii. $R_2 = R_1 - 9R_2$. Then divide row 1 by $9$.

4) Can a matrix whose entry is $0$ on the diagonal be solved? Explain why or why not. What would you do to remedy the situation?

5) Can a matrix that has $0$ entries for an entire row have one solution? Explain why or why not.

**Answer**

No. A matrix with $0$ entries for an entire row would have either zero or infinitely many solutions.

## Algebraic

**For the exercises 6-10, write the augmented matrix for the linear system.**

6) $\begin{aligned} 8x - 37y &= 8 \\ 2x + 12y &= 3 \end{aligned}$

7) $\begin{aligned} 16y &= 4 \\ 9x - y &= 2 \end{aligned}$

**Answer**

$$\begin{bmatrix} 0 & 16 & 4 \\ 9 & -1 & 2 \end{bmatrix}$$

8) $$\begin{aligned} 3x + 2y + 10z &= 3 \\ -6x + 2y + 5z &= 13 \\ 4x + z &= 18 \end{aligned}$$

9) $$\begin{aligned} x + 5y + 8z &= 19 \\ 12x + 3y &= 4 \\ 3x + 4y + 9z &= -7 \end{aligned}$$

**Answer**

$$\begin{bmatrix} 1 & 5 & 8 & 16 \\ 12 & 3 & 0 & 4 \\ 3 & 4 & 9 & -7 \end{bmatrix}$$

10) $$\begin{aligned} 6x + 12y + 16z &= 4 \\ 19x - 5y + 3z &= -9 \\ x + 2y &= -8 \end{aligned}$$

**For the exercises 11-15, write the linear system from the augmented matrix.**

11) $$\begin{bmatrix} -2 & 5 & 5 \\ 6 & -18 & 26 \end{bmatrix}$$

**Answer**

$$\begin{aligned} -2x + 5y &= 5 \\ 6x - 18y &= 26 \end{aligned}$$

12) $$\begin{bmatrix} 3 & 4 & 10 \\ 10 & 17 & 439 \end{bmatrix}$$

13) $$\begin{bmatrix} 3 & 2 & 0 & 3 \\ -1 & -9 & 4 & -1 \\ 8 & 5 & 7 & 8 \end{bmatrix}$$

**Answer**

$$\begin{aligned} 3x + 2y &= 13 \\ -x - 9y + 4z &= 53 \\ 8x + 5y + 7z &= 80 \end{aligned}$$

14) $$\begin{bmatrix} 8 & 29 & 1 & 43 \\ -1 & 7 & 5 & 38 \\ 0 & 0 & 3 & 10 \end{bmatrix}$$

15) $$\begin{bmatrix} 4 & 5 & -2 & 12 \\ 0 & 1 & 58 & 2 \\ 8 & 7 & -3 & -5 \end{bmatrix}$$

**Answer**

$$\begin{aligned} 4x + 5y - 2z &= 12 \\ y + 58z &= 2 \\ 8x + 7y - 3z &= -5 \end{aligned}$$

**For the exercises 16-46, solve the system by Gaussian elimination.**

16) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

17) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

**Answer**

No solutions

18) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

19) $\begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$

**Answer**

$(-1, -2)$

20) $\begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

21) $\begin{aligned} 2x - 3y &= -9 \\ 5x + 4y &= 58 \end{aligned}$

**Answer**

$(6, 7)$

22) $\begin{aligned} 6x + 2y &= -4 \\ 3x + 4y &= -17 \end{aligned}$

23) $\begin{aligned} 2x + 3y &= 12 \\ 4x + y &= 14 \end{aligned}$

**Answer**

$(3, 2)$

24) $\begin{aligned} -4x - 3y &= -2 \\ 3x - 5y &= -13 \end{aligned}$

25) $\begin{aligned} -5x + 8y &= 3 \\ 10x + 6y &= 5 \end{aligned}$

**Answer**

$\left(\frac{1}{5}, \frac{1}{2}\right)$

26) $\begin{aligned} 3x + 4y &= 12 \\ -6x - 8y &= -24 \end{aligned}$

27) $\begin{aligned} -60x + 45y &= 12 \\ 20x - 15y &= -4 \end{aligned}$

**Answer**

$\left(x, \frac{4}{15}(5x + 1)\right)$

28) $\begin{aligned} 11x + 10y &= 43 \\ 15x + 20y &= 65 \end{aligned}$

29) $\begin{aligned} 2x - y &= 2 \\ 3x + 2y &= 17 \end{aligned}$

**Answer**

$$(3, 4)$$

30) $\begin{aligned} -1.06x - 2.25y &= 5.51 \\ -5.03x - 1.08y &= 5.40 \end{aligned}$

31) $\begin{aligned} \frac{3}{4}x - \frac{3}{5}y &= 4 \\ \frac{1}{4}x + \frac{2}{3}y &= 1 \end{aligned}$

**Answer**

$$\left(\frac{196}{39}, -\frac{5}{13}\right)$$

32) $\begin{aligned} \frac{1}{4}x - \frac{2}{3}y &= -1 \\ \frac{1}{2}x + \frac{1}{3}y &= 3 \end{aligned}$

33) $\begin{bmatrix} 1 & 0 & 0 & 31 \\ 0 & 1 & 1 & 45 \\ 0 & 0 & 1 & 87 \end{bmatrix}$

**Answer**

$$(31, -42, 87)$$

34) $\begin{bmatrix} 1 & 0 & 1 & 50 \\ 1 & 1 & 0 & 20 \\ 0 & 1 & 1 & -90 \end{bmatrix}$

35) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{bmatrix}$

**Answer**

$$\left(\frac{21}{40}, \frac{1}{20}, \frac{9}{8}\right)$$

36) $\begin{bmatrix} -0.1 & 0.3 & -0.1 & 0.2 \\ -0.4 & 0.2 & 0.1 & 0.8 \\ 0.6 & 0.1 & 0.7 & -0.8 \end{bmatrix}$

37) $\begin{aligned} -2x + 3y - 2z &= 3 \\ 4x + 2y - z &= 9 \\ 4x - 8y + 2z &= -6 \end{aligned}$

**Answer**

$$\left(\frac{18}{13}, \frac{15}{13}, -\frac{15}{13}\right)$$

38) $$\begin{aligned} x+y-4z&=-4\\ 5x-3y-2z&=0\\ 2x+6y+7z&=30 \end{aligned}$$

39) $$\begin{aligned} 2x+3y+2z&=1\\ -4x-6y-4z&=-2\\ 10x+15y+10z&=5 \end{aligned}$$

**Answer**

$$\left(x, y, \frac{1}{2}(1-2x-3y)\right)$$

40) $$\begin{aligned} x+2y-z&=1\\ -x-2y+2z&=-2\\ 3x+6y-3z&=5 \end{aligned}$$

41) $$\begin{aligned} x+2y-z&=1\\ -x-2y+2z&=-2\\ 3x+6y-3z&=3 \end{aligned}$$

**Answer**

$$\left(x, -\frac{x}{2}, -1\right)$$

42) $$\begin{aligned} x+y&=2\\ x+z&=1\\ -y-z&=-3 \end{aligned}$$

43) $$\begin{aligned} x+y+z&=100\\ x+2z&=125\\ -y+2z&=25 \end{aligned}$$

**Answer**

$$(125, -25, 0)$$

44) $$\begin{aligned} \frac{1}{4}x-\frac{2}{3}z&=-\frac{1}{2}\\ \frac{1}{5}x+\frac{1}{3}y&=\frac{4}{7}\\ \frac{1}{5}y-\frac{1}{3}z&=\frac{2}{9} \end{aligned}$$

45) $$\begin{aligned} -\frac{1}{2}x+\frac{1}{2}y+\frac{1}{7}z&=-\frac{53}{14}\\ \frac{1}{2}x-\frac{1}{2}y+\frac{1}{4}z&=3\\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{3}z&=\frac{23}{15} \end{aligned}$$

**Answer**

$$(8, 1, -2)$$

46) $$\begin{aligned} -\frac{1}{2}x-\frac{1}{3}y+\frac{1}{4}z&=-\frac{29}{6}\\ \frac{1}{5}x+\frac{1}{6}y-\frac{1}{7}z&=\frac{431}{210}\\ -\frac{1}{8}x+\frac{1}{9}y+\frac{1}{10}z&=-\frac{49}{45} \end{aligned}$$

## Extensions

**For the exercises 47-51, use Gaussian elimination to solve the system.**

47) $$\begin{aligned} \frac{x-1}{7}+\frac{y-2}{8}+\frac{z-3}{4}&=0\\ x+y+z&=6\\ \frac{x+2}{3}+2y+\frac{z-3}{3}&=5 \end{aligned}$$

**Answer**

$$(1,2,3)$$

48) $$\begin{aligned} \frac{x-1}{4}-\frac{y+1}{4}+3z&=-1\\ \frac{x+5}{2}+\frac{y+7}{4}-z&=4\\ x+y-\frac{z-2}{2}&=1 \end{aligned}$$

49) $$\begin{aligned} \frac{x-3}{4}-\frac{y-1}{3}+2z&=-1\\ \frac{x+5}{2}+\frac{y+5}{2}+\frac{z+5}{2}&=8\\ x+y+z&=1 \end{aligned}$$

**Answer**

$$\left(x,\frac{31}{28}-\frac{3x}{4},\frac{1}{28}(-7x-3)\right)$$

50) $$\begin{aligned} \frac{x-3}{10}+\frac{y+3}{2}-2z&=3\\ \frac{x+5}{4}-\frac{y-1}{8}+z&=\frac{3}{2}\\ \frac{x-1}{4}+\frac{y+4}{2}+3z&=\frac{3}{2} \end{aligned}$$

51) $$\begin{aligned} \frac{x-3}{4}-\frac{y-1}{3}+2z&=-1\\ \frac{x+5}{2}+\frac{y+5}{2}+\frac{z+5}{2}&=7\\ x+y+z&=1 \end{aligned}$$

**Answer**

No solutions exist.

## Real-World Applications

**For the exercises 52-61, set up the augmented matrix that describes the situation, and solve for the desired solution.**

52) Every day, a cupcake store sells $5,000$ cupcakes in chocolate and vanilla flavors. If the chocolate flavor is $3$ times as popular as the vanilla flavor, how many of each cupcake sell per day?

53) At a competing cupcake store, $\$4,520$ worth of cupcakes are sold daily. The chocolate cupcakes cost $\$2.25$ and the red velvet cupcakes cost $\$1.75$. If the total number of cupcakes sold per day is $2,200$, how many of each flavor are sold each day?

**Answer**

$860$ red velvet, $1,340$ chocolate

54) You invested $\$10,000$ into two accounts: one that has simple $3\%$ interest, the other with $2.5\%$ interest. If your total interest payment after one year was $\$283.50$, how much was in each account after the year passed?

55) You invested $\$2,300$ into account 1, and $\$2,700$ into account 2. If the total amount of interest after one year is $\$254$, and account 2 has $1.5$ times the interest rate of account 1, what are the interest rates? Assume simple interest rates.

**Answer**

$4\%$ for account 1, $6\%$ for account 2

56) Bikes'R'Us manufactures bikes, which sell for $\$250$. It costs the manufacturer $\$180$ per bike, plus a startup fee of $\$3,500$. After how many bikes sold will the manufacturer break even?

57) A major appliance store is considering purchasing vacuums from a small manufacturer. The store would be able to purchase the vacuums for $\$86$ each, with a delivery fee of $\$9,200$, regardless of how many vacuums are sold. If the store needs to start seeing a profit after $230$ units are sold, how much should they charge for the vacuums?

**Answer**

$\$126$

58) The three most popular ice cream flavors are chocolate, strawberry, and vanilla, comprising $83\%$ of the flavors sold at an ice cream shop. If vanilla sells $1\%$ more than twice strawberry, and chocolate sells $11\%$ more than vanilla, how much of the total ice cream consumption are the vanilla, chocolate, and strawberry flavors?

59) At an ice cream shop, three flavors are increasing in demand. Last year, banana, pumpkin, and rocky road ice cream made up $12\%$ of total ice cream sales. This year, the same three ice creams made up $16.9\%$ of ice cream sales. The rocky road sales doubled, the banana sales increased by $50\%$, and the pumpkin sales increased by $20\%$. If the rocky road ice cream had one less percent of sales than the banana ice cream, find out the percentage of ice cream sales each individual ice cream made last year.

**Answer**

Banana was $3\%$, pumpkin was $7\%$, and rocky road was $2\%$

60) A bag of mixed nuts contains cashews, pistachios, and almonds. There are $1,000$ total nuts in the bag, and there are $100$ less almonds than pistachios. The cashews weigh $3$ g, pistachios weigh $4$ g, and almonds weigh $5$ g. If the bag weighs $3.7$ kg, find out how many of each type of nut is in the bag.

61) A bag of mixed nuts contains cashews, pistachios, and almonds. Originally there were $900$ nuts in the bag. $30\%$ of the almonds, $20\%$ of the cashews, and $10\%$ of the pistachios were eaten, and now there are $770$ nuts left in the bag. Originally, there were $100$ more cashews than almonds. Figure out how many of each type of nut was in the bag to begin with.

**Answer**

$100$ almonds, $200$ cashews, $600$ pistachios

# 9.7: Solving Systems with Inverses

## Verbal

1) In a previous section, we showed that matrix multiplication is not commutative, that is, $AB \neq BA$ in most cases. Can you explain why matrix multiplication is commutative for matrix inverses, that is, $A^{-1}A = AA^{-1}$?

**Answer**

If $A^{-1}$ is the inverse of $A$, then $AA^{-1} = I$, the identity matrix. Since $A$ is also the inverse of $A^{-1}$, $A^{-1}A = I$. You can also check by proving this for a $2 \times 2$ matrix.

2) Does every $2 \times 2$ matrix have an inverse? Explain why or why not. Explain what condition is necessary for an inverse to exist.

3) Can you explain whether a $2 \times 2$ matrix with an entire row of zeros can have an inverse?

**Answer**

No, because $ad$ and $bc$ are both $0$, so $ad - bc = 0$, which requires us to divide by $0$ in the formula.

4) Can a matrix with an entire column of zeros have an inverse? Explain why or why not.

5) Can a matrix with zeros on the diagonal have an inverse? If so, find an example. If not, prove why not. For simplicity, assume a $2 \times 2$ matrix.

**Answer**

Yes. Consider the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The inverse is found with the following calculation:

$$A^{-1} = \frac{1}{0(0) - 1(1)} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

## Algebraic

**In the exercises 6-12, show that matrix $A$ is the inverse of matrix $B$.**

6) $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

7) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

**Answer**

$$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

8) $A = \begin{bmatrix} 4 & 5 \\ 7 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{1}{5} & -\frac{4}{35} \end{bmatrix}$

9) $A = \begin{bmatrix} -2 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix}$

**Answer**

$$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

10) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, B = \frac{1}{2} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

11) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, B = \frac{1}{2} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

**Answer**

$$AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

12) $A = \begin{bmatrix} 3 & 8 & 2 \\ 1 & 1 & 1 \\ 5 & 6 & 12 \end{bmatrix}, B = \frac{1}{36}\begin{bmatrix} -6 & 84 & -6 \\ 7 & -26 & 1 \\ -1 & -22 & 5 \end{bmatrix}$

**For the exercises 13-26, find the multiplicative inverse of each matrix, if it exists.**

13) $\begin{bmatrix} 3 & -2 \\ 1 & 9 \end{bmatrix}$

**Answer**

$$\frac{1}{29}\begin{bmatrix} 9 & 2 \\ -1 & 3 \end{bmatrix}$$

14) $\begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}$

15) $\begin{bmatrix} -3 & 7 \\ 9 & 2 \end{bmatrix}$

**Answer**

$$\frac{1}{69}\begin{bmatrix} -2 & 7 \\ 9 & 3 \end{bmatrix}$$

16) $\begin{bmatrix} -4 & -3 \\ -5 & 8 \end{bmatrix}$

17) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

**Answer**

There is no inverse

18) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

19) $\begin{bmatrix} 0.5 & 1.5 \\ 1 & -0.5 \end{bmatrix}$

**Answer**

$$\frac{4}{7}\begin{bmatrix} 0.5 & 1.5 \\ 1 & -0.5 \end{bmatrix}$$

20) $\begin{bmatrix} 1 & 0 & 6 \\ -2 & 1 & 7 \\ 3 & 0 & 2 \end{bmatrix}$

21) $\begin{bmatrix} 0 & 1 & -3 \\ 4 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

**Answer**

$$\frac{1}{17}\begin{bmatrix} -5 & 5 & -3 \\ 20 & -3 & 12 \\ 1 & -1 & 4 \end{bmatrix}$$

22) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 1 \\ -2 & -4 & -5 \end{bmatrix}$

23) $\begin{bmatrix} 1 & 9 & -3 \\ 2 & 5 & 6 \\ 4 & -2 & -7 \end{bmatrix}$

**Answer**

$$\frac{1}{209}\begin{bmatrix} 47 & -57 & 69 \\ 10 & 19 & -12 \\ -24 & 38 & -13 \end{bmatrix}$$

24) $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 8 & -12 \\ 1 & 4 & 2 \end{bmatrix}$

25) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix}$

**Answer**

$$\begin{bmatrix} 18 & 60 & -168 \\ -56 & -140 & 448 \\ 40 & 80 & -280 \end{bmatrix}$$

26) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

**For the exercises 27-34, solve the system using the inverse of a $2 \times 2$ matrix.**

27) $\begin{aligned} 5x - 6y &= -61 \\ 4x + 3y &= -2 \end{aligned}$

**Answer**

$$(-5, 6)$$

28) $\begin{aligned} 8x + 4y &= -100 \\ 3x - 4y &= 1 \end{aligned}$

29) $\begin{aligned} 3x - 2y &= 6 \\ -x + 5y &= -2 \end{aligned}$

**Answer**

$$(2, 0)$$

30) $\begin{aligned} 5x - 4y &= -5 \\ 4x + y &= 2.3 \end{aligned}$

31) $\begin{aligned} -3x - 4y &= 9 \\ 12x + 4y &= -6 \end{aligned}$

**Answer**

$$\left(\frac{1}{3}, -\frac{5}{2}\right)$$

32) $$\begin{aligned} -2x+3y &= \frac{3}{10} \\ -x+5y &= \frac{1}{2} \end{aligned}$$

33) $$\begin{aligned} \frac{8}{5}x-\frac{4}{5}y &= \frac{2}{5} \\ -\frac{8}{5}x+\frac{1}{5}y &= \frac{7}{10} \end{aligned}$$

**Answer**

$$\left(-\frac{2}{3},-\frac{11}{6}\right)$$

34) $$\begin{aligned} \frac{1}{2}x+\frac{1}{5}y &= -\frac{1}{4} \\ \frac{1}{2}x-\frac{3}{5}y &= -\frac{9}{4} \end{aligned}$$

**For the exercises 35-42, solve a system using the inverse of a $3\times 3$ matrix.**

35) $$\begin{aligned} 3x-2y+5z &= 21 \\ 5x+4y &= 37 \\ x-2y-5z &= 5 \end{aligned}$$

**Answer**

$$\left(7,\frac{1}{2},\frac{1}{5}\right)$$

36) $$\begin{aligned} 4x+4y+4z &= 40 \\ 2x-3y+4z &= -12 \\ -x+3y+4z &= 9 \end{aligned}$$

37) $$\begin{aligned} 6x-5y-z &= 31 \\ -x+2y+z &= -6 \\ 3x+3y+2z &= 13 \end{aligned}$$

**Answer**

$$(5,0,-1)$$

38) $$\begin{aligned} 6x-5y+2z &= -4 \\ 2x+5y-z &= 12 \\ 2x+5y+z &= 12 \end{aligned}$$

39) $$\begin{aligned} 4x-2y+3z &= -12 \\ 2x+2y-9z &= 33 \\ 6y-4z &= 1 \end{aligned}$$

**Answer**

$$\frac{1}{34}(-35,-97,-154)$$

40) $$\begin{aligned} \frac{1}{10}x-\frac{1}{5}y+4z &= \frac{-41}{2} \\ \frac{1}{5}x-20y+\frac{2}{5}z &= -101 \\ \frac{3}{10}x+4y-\frac{3}{10}z &= 23 \end{aligned}$$

41) $$\begin{aligned} \frac{1}{2}x - \frac{1}{5}y + \frac{1}{5}z &= \frac{31}{100} \\ -\frac{3}{4}x - \frac{1}{4}y + \frac{1}{2}z &= \frac{7}{40} \\ -\frac{4}{5}x - \frac{1}{2}y + \frac{3}{2}z &= \frac{1}{4} \end{aligned}$$

**Answer**

$$\frac{1}{690}(65, -1136, -229)$$

42) $$\begin{aligned} 0.1x + 0.2y + 0.3z &= -1.4 \\ 0.1x - 0.2y + 0.3z &= 0.6 \\ 0.4y + 0.9z &= -2 \end{aligned}$$

## Technology

**For the exercises 43-46, use a calculator to solve the system of equations with matrix inverses.**

43) $$\begin{aligned} 2x - y &= -3 \\ -x + 2y &= 2.3 \end{aligned}$$

**Answer**

$$\left(-\frac{37}{30}, \frac{8}{15}\right)$$

44) $$\begin{aligned} -\frac{1}{2}x - \frac{3}{2}y &= -\frac{43}{20} \\ \frac{5}{2}x + \frac{11}{5}y &= \frac{31}{4} \end{aligned}$$

45) $$\begin{aligned} 12.3x - 2y - 2.5z &= 2 \\ 36.9x + 7y - 7.5z &= -7 \\ 8y - 5z &= -10 \end{aligned}$$

**Answer**

$$\left(\frac{10}{123}, -1, \frac{2}{5}\right)$$

46) $$\begin{aligned} 0.5x - 3y + 6z &= -0.8 \\ 0.7x - 2y &= -0.06 \\ 0.5x + 4y + 5z &= 0 \end{aligned}$$

## Extensions

**For the exercises 47-51, find the inverse of the given matrix.**

47) $$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

**Answer**

$$\frac{1}{2}\begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

48) $\begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & -1 & 0 \\ 1 & -3 & 0 & 1 \end{bmatrix}$

49) $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 4 & -2 & 3 \\ -5 & 0 & 1 & 1 \end{bmatrix}$

**Answer**

$$\frac{1}{39}\begin{bmatrix} 3 & 2 & 1 & -7 \\ 18 & -53 & 32 & 10 \\ 24 & -36 & 21 & 9 \\ -9 & 46 & -16 & -5 \end{bmatrix}$$

50) $\begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$

51) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

**Answer**

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

## Real-World Applications

**For the exercises 52-61, write a system of equations that represents the situation. Then, solve the system using the inverse of a matrix.**

52) $2,400$ tickets were sold for a basketball game. If the prices for floor 1 and floor 2 were different, and the total amount of money brought in is $\$64,000$, how much was the price of each ticket?

53) In the previous exercise, if you were told there were $400$ more tickets sold for floor 2 than floor 1, how much was the price of each ticket?

**Answer**

Infinite solutions.

54) A food drive collected two different types of canned goods, green beans and kidney beans. The total number of collected cans was $350$ and the total weight of all donated food was $348$ lb, $12$ oz. If the green bean cans weigh $2$ oz less than the kidney bean cans, how many of each can was donated?

55) Students were asked to bring their favorite fruit to class. $95\%$ of the fruits consisted of banana, apple, and oranges. If oranges were twice as popular as bananas, and apples were $5\%$ less popular than bananas, what are the percentages of each individual fruit?

**Answer**

$50\%$ oranges, $25\%$ bananas, $20\%$ apples

56) A sorority held a bake sale to raise money and sold brownies and chocolate chip cookies. They priced the brownies at $\$1$ and the chocolate chip cookies at $\$0.75$. They raised $\$700$ and sold $850$ items. How many brownies and how many cookies were sold?

57) A clothing store needs to order new inventory. It has three different types of hats for sale: straw hats, beanies, and cowboy hats. The straw hat is priced at $\$13.99$, the beanie at $\$7.99$, and the cowboy hat at $\$14.49$. If $100$ hats were sold this past quarter, $\$1,119$ was taken in by sales, and the amount of beanies sold was $10$ more than cowboy hats, how many of each should the clothing store order to replace those already sold?

**Answer**

$10$ straw hats, $50$ beanies, $40$ cowboy hats

58) Anna, Ashley, and Andrea weigh a combined $370$ lb. If Andrea weighs $20$ lb more than Ashley, and Anna weighs $1.5$ times as much as Ashley, how much does each girl weigh?

59) Three roommates shared a package of $12$ ice cream bars, but no one remembers who ate how many. If Tom ate twice as many ice cream bars as Joe, and Albert ate three less than Tom, how many ice cream bars did each roommate eat?

**Answer**

Tom ate $6$, Joe ate $3$, and Albert ate $3$.

60) A farmer constructed a chicken coop out of chicken wire, wood, and plywood. The chicken wire cost $\$2$ per square foot, the wood $\$10$ per square foot, and the plywood $\$5$ per square foot. The farmer spent a total of $\$51$, and the total amount of materials used was $14$ ft$^2$. He used $3$ ft$^2$ more chicken wire than plywood. How much of each material in did the farmer use?

61) Jay has lemon, orange, and pomegranate trees in his backyard. An orange weighs $8$ oz, a lemon $5$ oz, and a pomegranate $11$ oz. Jay picked $142$ pieces of fruit weighing a total of $70$ lb, $10$ oz. He picked $15.5$ times more oranges than pomegranates. How many of each fruit did Jay pick?

**Answer**

$124$ oranges, $10$ lemons, $8$ pomegranates

# 9.8: Solving Systems with Cramer's Rule

## Verbal

1) Explain why we can always evaluate the determinant of a square matrix.

**Answer**

A determinant is the sum and products of the entries in the matrix, so you can always evaluate that product—even if it does end up being $0$.

2) Examining Cramer's Rule, explain why there is no unique solution to the system when the determinant of your matrix is $0$. For simplicity, use a $2 \times 2$ matrix.

3) Explain what it means in terms of an inverse for a matrix to have a $0$ determinant.

**Answer**

The inverse does not exist.

4) The determinant of $2 \times 2$ matrix $A$ is $3$. If you switch the rows and multiply the first row by $6$ and the second row by $2$, explain how to find the determinant and provide the answer.

## Algebraic

**For the exercises 5-24, find the determinant.**

5) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

**Answer**

$-2$

6) $\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}$

7) $\begin{vmatrix} 2 & -5 \\ -1 & 6 \end{vmatrix}$

**Answer**

$7$

8) $\begin{vmatrix} -8 & 4 \\ -1 & 5 \end{vmatrix}$

9) $\begin{vmatrix} 1 & 0 \\ 3 & -4 \end{vmatrix}$

**Answer**

$-4$

10) $\begin{vmatrix} 10 & 20 \\ 0 & -10 \end{vmatrix}$

11) $\begin{vmatrix} 10 & 0.2 \\ 5 & 0.1 \end{vmatrix}$

**Answer**

$0$

12) $\begin{vmatrix} 6 & -3 \\ 8 & 4 \end{vmatrix}$

13) $\begin{vmatrix} -2 & -3 \\ 3.1 & 4,000 \end{vmatrix}$

**Answer**

$-7,990.7$

14) $\begin{vmatrix} -1.1 & 0.6 \\ 7.2 & -0.5 \end{vmatrix}$

15) $\begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix}$

**Answer**

$3$

16) $\begin{vmatrix} -1 & 4 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$

17) $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

**Answer**

$-1$

18) $\begin{vmatrix} 2 & -3 & 1 \\ 3 & -4 & 1 \\ -5 & 6 & 1 \end{vmatrix}$

19) $\begin{vmatrix} -2 & 1 & 4 \\ -4 & 2 & -8 \\ 2 & -8 & -3 \end{vmatrix}$

**Answer**

$224$

20) $\begin{vmatrix} 6 & -1 & 2 \\ -4 & -3 & 5 \\ 1 & 9 & -1 \end{vmatrix}$

21) $\begin{vmatrix} 5 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -6 & -3 \end{vmatrix}$

**Answer**

$15$

22) $\begin{vmatrix} 1.1 & 2 & -1 \\ -4 & 0 & 0 \\ 4.1 & -0.4 & 2.5 \end{vmatrix}$

23) $\begin{vmatrix} 2 & -1.6 & 3.1 \\ 1.1 & 3 & -8 \\ -9.3 & 0 & 2 \end{vmatrix}$

**Answer**

$-17.03$

24) $\begin{vmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & -\frac{1}{6} & \frac{1}{7} \\ 0 & 0 & \frac{1}{8} \end{vmatrix}$

**For the exercises 25-34, solve the system of linear equations using Cramer's Rule.**

25) $\begin{array}{l} 2x-3y=-1 \\ 4x+5y=9 \end{array}$

**Answer**

$(1,1)$

26) $\begin{array}{l} 5x-4y=2 \\ -4x+7y=6 \end{array}$

27) $\begin{array}{l} 6x-3y=2 \\ -8x+9y=-1 \end{array}$

**Answer**

$\left(\frac{1}{2},\frac{1}{3}\right)$

28) $\begin{array}{l} 2x+6y=12 \\ 5x-2y=13 \end{array}$

29) $\begin{array}{l} 4x+3y=23 \\ 2x-y=-1 \end{array}$

**Answer**

$(2,5)$

30) $\begin{array}{l} 10x-6y=2 \\ -5x+8y=-1 \end{array}$

31) $\begin{array}{l} 4x-3y=-3 \\ 2x+6y=-4 \end{array}$

**Answer**

$\left(-1,-\frac{1}{3}\right)$

32) $\begin{array}{l} 4x-5y=7 \\ -3x+9y=0 \end{array}$

33) $\begin{array}{l} 4x+10y=180 \\ -3x-5y=-105 \end{array}$

**Answer**

$(15,12)$

34) $\begin{array}{l} 8x-2y=-3 \\ -4x+6y=4 \end{array}$

**For the exercises 35-44, solve the system of linear equations using Cramer's Rule.**

35) $\begin{array}{l} x+2y-4z=-1 \\ 7x+3y+5z=26 \\ -2x-6y+7z=-6 \end{array}$

**Answer**

$(1,3,2)$

36) $$\begin{aligned} -5x + 2y - 4z &= -47 \\ 4x - 3y - z &= -94 \\ 3x - 3y + 2z &= 94 \end{aligned}$$

37) $$\begin{aligned} 4x + 5y - z &= -7 \\ -2x - 9y + 2z &= 8 \\ 5y + 7z &= 21 \end{aligned}$$

**Answer**

$$(-1, 0, 3)$$

38) $$\begin{aligned} 4x - 3y + 4z &= 10 \\ 5x - 2z &= -2 \\ 3x + 2y - 5z &= -9 \end{aligned}$$

39) $$\begin{aligned} 4x - 2y + 3z &= 6 \\ -6x + y &= -2 \\ 2x + 7y + 8z &= 24 \end{aligned}$$

**Answer**

$$\left(\frac{1}{2}, 1, 2\right)$$

40) $$\begin{aligned} 5x + 2y - z &= 1 \\ -7x - 8y + 3z &= 1.5 \\ 6x - 12y + z &= 7 \end{aligned}$$

41) $$\begin{aligned} 13x - 17y + 16z &= 73 \\ -11x + 15y + 17z &= 61 \\ 46x + 10y - 30z &= -18 \end{aligned}$$

**Answer**

$$(2, 1, 4)$$

42) $$\begin{aligned} -4x - 3y - 8z &= -7 \\ 2x - 9y + 5z &= 0.5 \\ 5x - 6y - 5z &= -2 \end{aligned}$$

43) $$\begin{aligned} 4x - 6y + 8z &= 10 \\ -2x + 3y - 4z &= -5 \\ x + y + z &= 1 \end{aligned}$$

**Answer**

Infinite solutions

44) $$\begin{aligned} 4x - 6y + 8z &= 10 \\ -2x + 3y - 4z &= -5 \\ 12x + 18y - 24z &= -30 \end{aligned}$$

## Technology

**For the exercises 45-48, use the determinant function on a graphing utility.**

45) $$\begin{vmatrix} 1 & 0 & 8 & 9 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 4 & 3 \end{vmatrix}$$

**Answer**

$24$

46) $\begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & -9 & 1 & 3 \\ 3 & 0 & -2 & -1 \\ 0 & 1 & 1 & -2 \end{vmatrix}$

47) $\begin{vmatrix} \frac{1}{2} & 1 & 7 & 4 \\ 0 & \frac{1}{2} & 100 & 5 \\ 0 & 0 & 2 & 2,000 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

**Answer**

$1$

48) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{vmatrix}$

## Real-World Applications

**For the exercises 49-52, create a system of linear equations to describe the behavior. Then, calculate the determinant. Will there be a unique solution? If so, find the unique solution.**

49) Two numbers add up to $56$. One number is $20$ less than the other.

**Answer**

Yes; $18$, $38$

50) Two numbers add up to $104$. If you add two times the first number plus two times the second number, your total is $208$

51) Three numbers add up to $106$. The first number is $3$ less than the second number. The third number is $4$ more than the first number.

**Answer**

Yes; $33$, $36$, $37$

52) Three numbers add to $216$. The sum of the first two numbers is $112$. The third number is 8 less than the first two numbers combined.

**For the exercises 53-65, create a system of linear equations to describe the behavior. Then, solve the system for all solutions using Cramer's Rule.**

53) You invest $\$10,000$ into two accounts, which receive $8\%$ interest and $5\%$ interest. At the end of a year, you had $\$10,710$ in your combined accounts. How much was invested in each account?

**Answer**

$\$7,000$ in first account, $\$3,000$ in second account.

54) You invest $\$80,000$ into two accounts, $\$22,000$ in one account, and $\$58,000$ in the other account. At the end of one year, assuming simple interest, you have earned $\$2,470$ in interest. The second account receives half a percent less than twice the interest on the first account. What are the interest rates for your accounts?

55) A movie theater needs to know how many adult tickets and children tickets were sold out of the $1,200$ total tickets. If children's tickets are $\$5.95$, adult tickets are $\$11.15$, and the total amount of revenue was $\$12,756$, how many children's tickets and adult tickets were sold?

**Answer**

$120$ children, $1,080$ adult

56) A concert venue sells single tickets for $\$40$ each and couple's tickets for $\$65$. If the total revenue was $\$18,090$ and the $321$ tickets were sold, how many single tickets and how many couple's tickets were sold?

57) You decide to paint your kitchen green. You create the color of paint by mixing yellow and blue paints. You cannot remember how many gallons of each color went into your mix, but you know there were $10$ gal total. Additionally, you kept your receipt, and know the total amount spent was $\$29.50$. If each gallon of yellow costs $\$2.59$, and each gallon of blue costs $\$3.19$, how many gallons of each color go into your green mix?

**Answer**

$4$ gal yellow, $6$ gal blue

58) You sold two types of scarves at a farmers' market and would like to know which one was more popular. The total number of scarves sold was $56$, the yellow scarf cost $\$10$, and the purple scarf cost $\$11$. If you had total revenue of $\$583$, how many yellow scarves and how many purple scarves were sold?

59) Your garden produced two types of tomatoes, one green and one red. The red weigh $10$ oz, and the green weigh $4$ oz. You have $30$ tomatoes, and a total weight of $13$ lb, $14$ oz. How many of each type of tomato do you have?

**Answer**

$13$ green tomatoes, $17$ red tomatoes

60) At a market, the three most popular vegetables make up $53\%$ of vegetable sales. Corn has $4\%$ higher sales than broccoli, which has $5\%$ more sales than onions. What percentage does each vegetable have in the market share?

61) At the same market, the three most popular fruits make up $37\%$ of the total fruit sold. Strawberries sell twice as much as oranges, and kiwis sell one more percentage point than oranges. For each fruit, find the percentage of total fruit sold.

**Answer**

Strawberries $18\%$, oranges $9\%$, kiwi $10\%$

62) Three bands performed at a concert venue. The first band charged $\$15$ per ticket, the second band charged $\$45$ per ticket, and the final band charged $\$22$ per ticket. There were $510$ tickets sold, for a total of $\$12,700$. If the first band had $40$ more audience members than the second band, how many tickets were sold for each band?

63) A movie theatre sold tickets to three movies. The tickets to the first movie were $\$5$, the tickets to the second movie were $\$11$, and the third movie was $\$12$. $100$ tickets were sold to the first movie. The total number of tickets sold was $642$, for a total revenue of $\$6,774$. How many tickets for each movie were sold?

**Answer**

$100$ for movie 1, $230$ for movie 2, $312$ for movie 3

64) Men aged $20$–$29$, $30$–$39$, and $40$–$49$ made up $78\%$ of the population at a prison last year. This year, the same age groups made up $82.08\%$ of the population. The $20$–$29$ age group increased by $20$, the $30$–$39$ age group increased by $2\%$, and the $40$–$49$ age group decreased to $\frac{3}{4}$ of their previous population. Originally, the $30$–$39$ age group had $2\%$ more prisoners than the $20$–$29$ age group. Determine the prison population percentage for each age group last year.

65) At a women's prison down the road, the total number of inmates aged $20$–$49$ totaled $5,525$. This year, the $20$–$29$ age group increased by $10\%$, the $30$–$39$ age group decreased by $20\%$, and the $40$–$49$ age group doubled. There are now $6,040$ prisoners.

Originally, there were $500$ more in the $30$–$39$ age group than the $20$–$29$ age group. Determine the prison population for each age group last year.

**Answer**

$$20\text{–}29 : 2,100,\ 30\text{–}39 : 2,600,\ 40\text{–}49 : 825$$

**For the exercises 66-68, use this scenario: A health-conscious company decides to make a trail mix out of almonds, dried cranberries, and chocolate-covered cashews. The nutritional information for these items is shown in the Table below.**

| | Fat (g) | Protein (g) | Carbohydrates (g) |
|---|---|---|---|
| Almonds (10) | 6 | 2 | 3 |
| Cranberries (10) | 0.02 | 0 | 8 |
| Cashews (10) | 7 | 3.5 | 5.5 |

66) For the special “low-carb”trail mix, there are $1,000$ pieces of mix. The total number of carbohydrates is $425$ g, and the total amount of fat is $570.2$ g. If there are $200$ more pieces of cashews than cranberries, how many of each item is in the trail mix?

67) For the “hiking” mix, there are $1,000$ pieces in the mix, containing $390.8$ g of fat, and $165$ g of protein. If there is the same amount of almonds as cashews, how many of each item is in the trail mix?

**Answer**

$300$ almonds, $400$ cranberries, $300$ cashews

68) For the “energy-booster” mix, there are $1,000$ pieces in the mix, containing $145$ g of protein and $625$ g of carbohydrates. If the number of almonds and cashews summed together is equivalent to the amount of cranberries, how many of each item is in the trail mix?

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# 5.R: Systems of Equations and Inequalities (Review)

## 9.1: Systems of Linear Equations: Two Variables

**For the exercises 1-2, determine whether the ordered pair is a solution to the system of equations.**

1) $\begin{array}{l} 3x-y=4 \\ x+4y=-3 \end{array}$ and $(-1,1)$

**Answer**

No

2) $\begin{array}{l} 6x-2y=24 \\ -3x+3y=18 \end{array}$ and $(9,15)$

**For the exercises 3-5, use substitution to solve the system of equations.**

3) $\begin{array}{l} 10x+5y=-5 \\ 3x-2y=-12 \end{array}$

**Answer**

$(-2,3)$

4) $\begin{array}{l} \frac{4}{7}x+\frac{1}{5}y=\frac{43}{70} \\ \frac{5}{6}x-\frac{1}{3}y=-\frac{2}{3} \end{array}$

5) $\begin{array}{l} 5x+6y=14 \\ 4x+8y=8 \end{array}$

**Answer**

$(4,-1)$

**For the exercises 6-8, use addition to solve the system of equations.**

6) $\begin{array}{l} 3x+2y=-7 \\ 2x+4y=6 \end{array}$

7) $\begin{array}{l} 3x+4y=2 \\ 9x+12y=3 \end{array}$

**Answer**

No solutions exist.

8) $\begin{array}{l} 8x+4y=2 \\ 6x-5y=0.7 \end{array}$

**For the exercises 9-10, write a system of equations to solve each problem. Solve the system of equations.**

9) A factory has a cost of production $C(x)=150x+15,000$ and a revenue function $R(x)=200x$. What is the break-even point?

**Answer**

$(300,60,000)$

10) A performer charges $C(x) = 50x + 10,000$, where $x$ is the total number of attendees at a show. The venue charges $\$75$ per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

**Answer**

$(400, 30,000)$

# 9.2: Systems of Linear Equations: Three Variables

**For the exercises 1-8, solve the system of three equations using substitution or addition.**

1) $\begin{aligned} 0.5x - 0.5y &= 10 \\ -0.2y + 0.2x &= 4 \\ 0.1x + 0.1z &= 2 \end{aligned}$

**Answer**

$(10, -10, 10)$

2) $\begin{aligned} 5x + 3y - z &= 5 \\ 3x - 2y + 4z &= 13 \\ 4x + 3y + 5z &= 22 \end{aligned}$

3) $\begin{aligned} x + y + z &= 1 \\ 2x + 2y + 2z &= 1 \\ 3x + 3y &= 2 \end{aligned}$

**Answer**

No solutions exist.

4) $\begin{aligned} 2x - 3y + z &= -1 \\ x + y + z &= -4 \\ 4x + 2y - 3z &= 33 \end{aligned}$

5) $\begin{aligned} 3x + 2y - z &= -10 \\ x - y + 2z &= 7 \\ -x + 3y + z &= -2 \end{aligned}$

**Answer**

$(-1, -2, 3)$

6) $\begin{aligned} 3x + 4z &= -11 \\ x - 2y &= 5 \\ 4y - z &= -10 \end{aligned}$

7) $\begin{aligned} 2x - 3y + z &= 0 \\ 2x + 4y - 3z &= 0 \\ 6x - 2y - z &= 0 \end{aligned}$

**Answer**

$\left(x, \frac{8x}{5}, \frac{14x}{5}\right)$

$$\begin{aligned} 6x - 4y - 2z &= 2 \\ 3x + 2y - 5z &= 4 \\ 6y - 7z &= 5 \end{aligned}$$

8)

**For the exercises 9-10, write a system of equations to solve each problem. Solve the system of equations.**

9) Three odd numbers sum up to $61$. The smaller is one-third the larger and the middle number is $16$ less than the larger. What are the three numbers?

**Answer**

$11, 17, 33$

10) A local theatre sells out for their show. They sell all $500$ tickets for a total purse of $\$8,070.00$. The tickets were priced at $\$15$ for students, $\$12$ for children, and $\$18$ for adults. If the band sold three times as many adult tickets as children's tickets, how many of each type was sold?

# 9.3: Systems of Nonlinear Equations and Inequalities: Two Variables

**For the exercises 1-5, solve the system of nonlinear equations.**

1) $\begin{aligned} y &= x^2 - 7 \\ y &= 5x - 13 \end{aligned}$

**Answer**

$(2, -3), (3, 2)$

2) $\begin{aligned} y &= x^2 - 4 \\ y &= 5x + 10 \end{aligned}$

3) $\begin{aligned} x^2 + y^2 &= 16 \\ y &= x - 8 \end{aligned}$

**Answer**

No solution

4) $\begin{aligned} x^2 + y^2 &= 25 \\ y &= x^2 + 5 \end{aligned}$

5) $\begin{aligned} x^2 + y^2 &= 4 \\ y - x^2 &= 3 \end{aligned}$

**Answer**

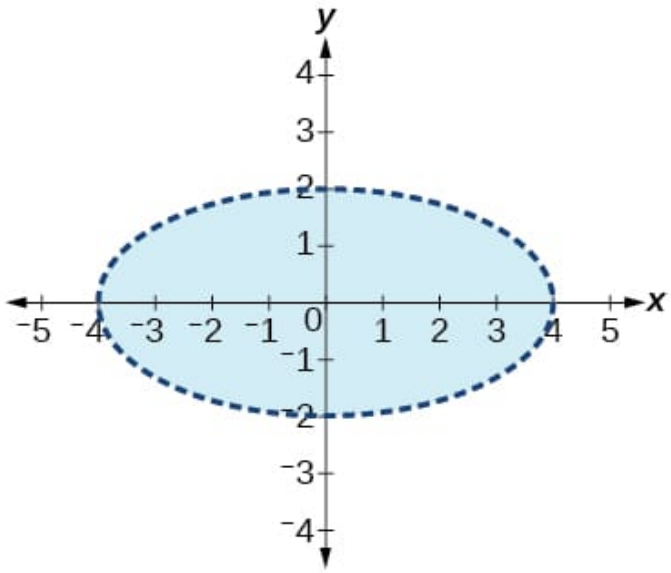
No solution

**For the exercises 6-7, graph the inequality.**

6) $y > x^2 - 1$

7) $\frac{1}{4}x^2 + y^2 < 4$

**Answer**

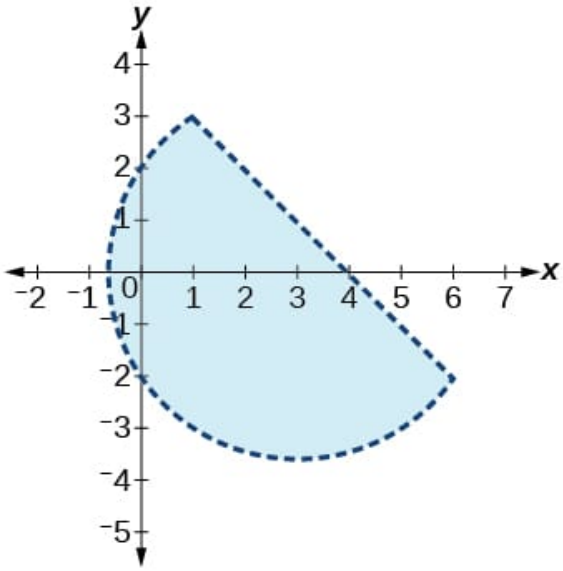



**For the exercises 8-10, graph the system of inequalities.**

8) $x^2 + y^2 + 2x < 3$
$y > -x^2 - 3$

9) $x^2 - 2x + y^2 - 4x < 4$
$y < -x + 4$

**Answer**




10) $x^2 + y^2 < 1$
$y^2 < x$

# 9.4: Partial Fractions

**For the exercises 1-8, decompose into partial fractions.**

1) $\frac{-2x+6}{x^2+3x+2}$

**Answer**

$$\frac{2}{x+2}, \frac{-4}{x+1}$$

2) $\frac{10x+2}{4x^2+4x+1}$

3) $\frac{7x+20}{x^2+10x+25}$

**Answer**

$$\frac{7}{x+5}, \frac{-15}{(x+5)^2}$$

4) $\frac{x-18}{x^2-12x+36}$

5) $\frac{-x^2+36x+70}{x^3-125}$

**Answer**

$$\frac{3}{x-5}, \frac{-4x+1}{x^2+5x+25}$$

6) $\frac{-5x^2+6x-2}{x^3+27}$

7) $\frac{x^3-4x^2+3x+11}{(x^2-2)^2}$

**Answer**

$$\frac{x-4}{(x^2-2)}, \frac{5x+3}{(x^2-2)^2}$$

8) $\frac{4x^4-2x^3+22x^2-6x+48}{x(x^2+4)^2}$

# 9.5: Matrices and Matrix Operations

**For the exercises 1-12, perform the requested operations on the given matrices.**

$$A=\begin{bmatrix}4 & -2\\ 1 & 3\end{bmatrix}, \begin{bmatrix}6 & 7 & -3\\ 11 & -2 & 4\end{bmatrix}, C=\begin{bmatrix}6 & 7\\ 11 & -2\\ 14 & 0\end{bmatrix} D=\begin{bmatrix}1 & -4 & 9\\ 10 & 5 & -7\\ 2 & 8 & 5\end{bmatrix} E=\begin{bmatrix}7 & -14 & 3\\ 2 & -1 & 3\\ 0 & 1 & 9\end{bmatrix}$$

1) $-4A$

**Answer**

$$\begin{bmatrix} -16 & 8 \\ -4 & -12 \end{bmatrix}$$

2) $10D - 6E$

3) $B + C$

**Answer**

undefined; dimensions do not match

4) $AB$

5) $BA$

**Answer**

undefined; inner dimensions do not match

6) $BC$

7) $CB$

**Answer**

$$\begin{bmatrix} 113 & 28 & 10 \\ 44 & 81 & -41 \\ 84 & 98 & -42 \end{bmatrix}$$

8) $DE$

9) $ED$

**Answer**

$$\begin{bmatrix} -127 & -74 & 176 \\ -2 & 11 & 40 \\ 28 & 77 & 38 \end{bmatrix}$$

10) $EC$

11) $CE$

**Answer**

undefined; inner dimensions do not match

12) $A^3$

# 9.6: Solving Systems with Gaussian Elimination

**For the exercises 1-2, write the system of linear equations from the augmented matrix. Indicate whether there will be a unique solution.**

1) $\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**Answer**

$\begin{aligned} x - 3z &= 7 \\ y + 2z &= -5 \end{aligned}$ with infinite solutions

2) $\begin{bmatrix} 1 & 0 & 5 & -9 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

**For the exercises 3-5, write the augmented matrix from the system of linear equations.**

3) $\begin{aligned} -2x + 2y + z &= 7 \\ 2x - 8y + 5z &= 0 \\ 19x - 10y + 22z &= 3 \end{aligned}$

**Answer**

$\begin{bmatrix} -2 & 2 & 1 & 7 \\ 2 & -8 & 5 & 0 \\ 19 & -10 & 22 & 3 \end{bmatrix}$

4) $\begin{aligned} 4x + 2y - 3z &= 14 \\ -12x + 3y + z &= 100 \\ 9x - 6y + 2z &= 31 \end{aligned}$

5) $\begin{aligned} x + 3z &= 12 \\ -x + 4y &= 0 \\ y + 2z &= -7 \end{aligned}$

**Answer**

$\begin{bmatrix} 1 & 0 & 3 & 12 \\ -1 & 4 & 0 & 0 \\ 0 & 1 & 2 & -7 \end{bmatrix}$

**For the exercises 6-10, solve the system of linear equations using Gaussian elimination.**

6) $\begin{aligned} 3x - 4y &= -7 \\ -6x + 8y &= 14 \end{aligned}$

7) $\begin{aligned} 3x - 4y &= 1 \\ -6x + 8y &= 6 \end{aligned}$

**Answer**

No solutions exist.

8) $\begin{aligned} -1.1x - 2.3y &= 6.2 \\ -5.2x - 4.1y &= 4.3 \end{aligned}$

9) $\begin{aligned} 2x + 3y + 2z &= 1 \\ -4x - 6y - 4z &= -2 \\ 10x + 15y + 10z &= 0 \end{aligned}$

**Answer**

No solutions exist.

10) $$\begin{aligned} -x + 2y - 4z &= 8 \\ 3y + 8z &= -4 \\ -7x + y + 2z &= 1 \end{aligned}$$

# 9.7: Solving Systems with Inverses

**For the exercises 1-4, find the inverse of the matrix.**

1) $\begin{bmatrix} -0.2 & 1.4 \\ 1.2 & -0.4 \end{bmatrix}$

**Answer**

$$\frac{1}{8}\begin{bmatrix} 2 & 7 \\ 6 & 1 \end{bmatrix}$$

2) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$

3) $\begin{bmatrix} 12 & 9 & -6 \\ -1 & 3 & 2 \\ -4 & -3 & 2 \end{bmatrix}$

**Answer**

No inverse exists.

4) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

**For the exercises 5-8, find the solutions by computing the inverse of the matrix.**

5) $$\begin{aligned} 0.3x - 0.1y &= -10 \\ -0.1x + 0.3y &= 14 \end{aligned}$$

**Answer**

$$(-20, 40)$$

6) $$\begin{aligned} 0.4x - 0.2y &= -0.6 \\ -0.1x + 0.05y &= 0.3 \end{aligned}$$

7) $$\begin{aligned} 4x + 3y - 3z &= -4.3 \\ 5x - 4y - z &= -6.1 \\ x + z &= -0.7 \end{aligned}$$

**Answer**

$$(-1, 0.2, 0.3)$$

8) $$\begin{aligned} -2x - 3y + 2z &= 3 \\ -x + 2y + 4z &= -5 \\ -2y + 5z &= -3 \end{aligned}$$

**For the exercises 9-10, write a system of equations to solve each problem. Solve the system of equations.**

9) Students were asked to bring their favorite fruit to class. $90\%$ of the fruits consisted of banana, apple, and oranges. If oranges were half as popular as bananas and apples were $5\%$ more popular than bananas, what are the percentages of each individual

fruit?

**Answer**

$17\%$ oranges, $34\%$ bananas, $39\%$ apples

10) A sorority held a bake sale to raise money and sold brownies and chocolate chip cookies. They priced the brownies at $\$2$ and the chocolate chip cookies at $\$1$. They raised $\$250$ and sold $175$ items. How many brownies and how many cookies were sold?

# 9.8: Solving Systems with Cramer's Rule

**For the exercises 1-4, find the determinant.**

1) $\begin{vmatrix} 100 & 0 \\ 0 & 0 \end{vmatrix}$

**Answer**

$0$

2) $\begin{vmatrix} 0.2 & -0.6 \\ 0.7 & -1.1 \end{vmatrix}$

3) $\begin{vmatrix} -1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$

**Answer**

$6$

4) $\begin{vmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{vmatrix}$

**For the exercises 5-10, use Cramer's Rule to solve the linear systems of equations.**

5) $\begin{array}{r} 4x - 2y = 23 \\ -5x - 10y = -35 \end{array}$

**Answer**

$\left(6, \frac{1}{2}\right)$

6) $\begin{array}{r} 0.2x - 0.1y = 0 \\ -0.3x + 0.3y = 2.5 \end{array}$

7) $\begin{array}{r} -0.5x + 0.1y = 0.3 \\ -0.25x + 0.05y = 0.15 \end{array}$

**Answer**

$x, 5x + 3$

8) $\begin{array}{r} x + 6y + 3z = 4 \\ 2x + y + 2z = 3 \\ 3x - 2y + z = 0 \end{array}$

9) $$\begin{aligned} 4x - 3y + 5z &= -\frac{5}{2} \\ 7x - 9y - 3z &= \frac{3}{2} \\ x - 5y - 5z &= \frac{5}{2} \end{aligned}$$

**Answer**

$$\left(0, 0, -\frac{1}{2}\right)$$

10) $$\begin{aligned} \frac{3}{10}x - \frac{1}{5}y - \frac{3}{10}z &= -\frac{1}{50} \\ \frac{1}{10}x - \frac{1}{10}y - \frac{1}{2}z &= -\frac{9}{50} \\ \frac{2}{5}x - \frac{1}{2}y - \frac{3}{5}z &= -\frac{1}{5} \end{aligned}$$

# Practice Test

1) Is the following ordered pair a solution to the system of equations?

$$\begin{aligned} -5x - y &= 12 \text{ with } (-3, 3) \\ x + 4y &= 9 \end{aligned}$$

**Answer**

Yes

**For the exercises 2-9, solve the systems of linear and nonlinear equations using substitution or elimination. Indicate if no solution exists.**

2) $$\begin{aligned} \frac{1}{2}x - \frac{1}{3}y &= 4 \\ \frac{3}{2}x - y &= 0 \end{aligned}$$

3) $$\begin{aligned} -\frac{1}{2}x - 4y &= 4 \\ 2x + 16y &= 2 \end{aligned}$$

**Answer**

No solutions exist.

4) $$\begin{aligned} 5x - y &= 1 \\ -10x + 2y &= -2 \end{aligned}$$

5) $$\begin{aligned} 4x - 6y - 2z &= \frac{1}{10} \\ x - 7y + 5z &= -\frac{1}{4} \\ 3x + 6y - 9z &= \frac{6}{5} \end{aligned}$$

**Answer**

$$\frac{1}{20}(10, 5, 4)$$

6) $$\begin{aligned} x + z &= 20 \\ x + y + z &= 20 \\ x + 2y + z &= 10 \end{aligned}$$

$$\begin{aligned} 5x-4y-3z&=0\\ 2x+y+2z&=0\\ x-6y-7z&=0 \end{aligned}$$

7)

**Answer**

$$\left(x, \frac{16x}{5}-\frac{13x}{5}\right)$$

8) $\begin{aligned} y&=x^2+2x-3\\ y&=x-1 \end{aligned}$

9) $\begin{aligned} y^2+x^2&=25\\ y^2-2x^2&=1 \end{aligned}$

**Answer**

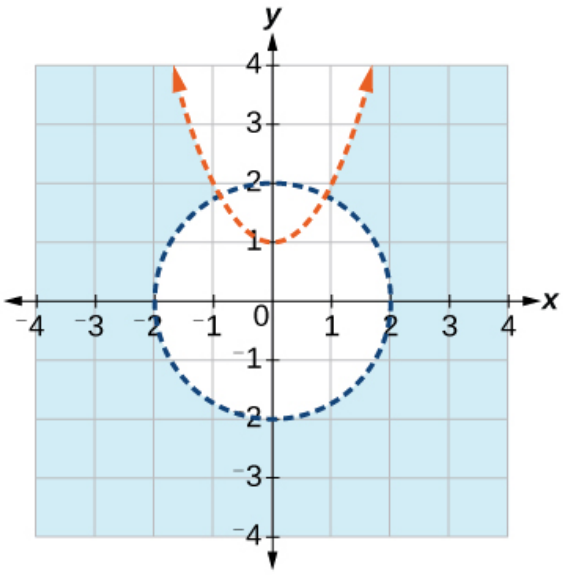
$$(-2\sqrt{2},-\sqrt{17}),(-2\sqrt{2},\sqrt{17}),(2\sqrt{2},-\sqrt{17}),(2\sqrt{2},\sqrt{17})$$

**For the exercises 10-11, graph the following inequalities.**

10) $y<x^2+9$

11) $\begin{aligned} x^2+y^2&>4\\ y&<x^2+1 \end{aligned}$

**Answer**




**For the exercises 12-14, write the partial fraction decomposition.**

12) $\dfrac{-8x-30}{x^2+10x+25}$

13) $\dfrac{13x+2}{(3x+1)^2}$

**Answer**

$$\frac{5}{3x+1}-\frac{2x+3}{(3x+1)^2}$$

14) $\dfrac{x^4 - x^3 + 2x - 1}{x(x^2+1)^2}$

**For the exercises 15-21, perform the given matrix operations.**

15) $5\begin{bmatrix} 4 & 9 \\ -2 & 3 \end{bmatrix} + \dfrac{1}{2}\begin{bmatrix} -6 & 12 \\ 4 & -8 \end{bmatrix}$

**Answer**

$$\begin{bmatrix} 17 & 51 \\ -8 & 11 \end{bmatrix}$$

16) $\begin{bmatrix} 1 & 4 & -7 \\ -2 & 9 & 5 \\ 12 & 0 & -4 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 3 \\ 5 & 10 \end{bmatrix}$

17) $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}^{-1}$

**Answer**

$$\begin{bmatrix} 12 & -20 \\ -15 & 30 \end{bmatrix}$$

18) $\det\begin{vmatrix} 0 & 0 \\ 400 & 4,000 \end{vmatrix}$

19) $\det\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{vmatrix}$

**Answer**

$$-\frac{1}{8}$$

20) If $\det(A) = -6$, what would be the determinant if you switched rows 1 and 3, multiplied the second row by $12$, and took the inverse?

21) Rewrite the system of linear equations as an augmented matrix.

$$\begin{aligned} 14x - 2y - 13z &= 140 \\ -2x + 3y - 6z &= -1 \\ x - 5y + 12z &= 11 \end{aligned}$$

**Answer**

$$\begin{bmatrix} 14 & -2 & 13 & 140 \\ -2 & 3 & -6 & -1 \\ 1 & -5 & 12 & 11 \end{bmatrix}$$

22) Rewrite the augmented matrix as a system of linear equations.

$$\begin{bmatrix} 1 & 0 & 3 & 12 \\ -2 & 4 & 9 & -5 \\ -6 & 1 & 2 & 8 \end{bmatrix}$$

**For the exercises 23-24, use Gaussian elimination to solve the systems of equations.**

23) $\begin{array}{l} x-6y=4 \\ 2x-12y=0 \end{array}$

**Answer**

No solutions exist.

24) $\begin{array}{l} 2x+y+z=-3 \\ x-2y+3z=6 \\ x-y-z=6 \end{array}$

**For the exercises 25-26, use the inverse of a matrix to solve the systems of equations.**

25) $\begin{array}{l} 4x-5y=-50 \\ -x+2y=80 \end{array}$

**Answer**

$(100, 90)$

26) $\begin{array}{l} \frac{1}{100}x-\frac{3}{100}y+\frac{1}{20}z=-49 \\ \frac{3}{100}x-\frac{7}{100}y-\frac{1}{100}z=13 \\ \frac{9}{100}x-\frac{9}{100}y-\frac{9}{100}z=99 \end{array}$

**For the exercises 27-28, use Cramer's Rule to solve the systems of equations.**

27) $\begin{array}{l} 200x-300y=2 \\ 400x+715y=4 \end{array}$

**Answer**

$\left(\frac{1}{100}, 0\right)$

28) $\begin{array}{l} 0.1x+0.1y-0.1z=-1.2 \\ 0.1x-0.2y+0.4z=-1.2 \\ 0.5x-0.3y+0.8z=-5.9 \end{array}$

**For the exercises 29-30, solve using a system of linear equations.**

29) A factory producing cell phones has the following cost and revenue functions: $C(x)=x^2+75x+2,688$ and $R(x)=x^2+160x$. What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.

**Answer**

$32$ or more cell phones per day

30) A small fair charges $\$1.50$ for students, $\$1$ for children, and $\$2$ for adults. In one day, three times as many children as adults attended. A total of $800$ tickets were sold for a total revenue of $\$1,050$. How many of each type of ticket was sold?

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# 6: ANALYTIC GEOMETRY

In this chapter, we will investigate the two-dimensional figures that are formed when a right circular cone is intersected by a plane. We will begin by studying each of three figures created in this manner. We will develop defining equations for each figure and then learn how to use these equations to solve a variety of problems. Conic sections are formed when a plane intersects two right circular cones aligned tip to tip and extending infinitely far in opposite directions, which we also call a cone. The way in which we slice the cone will determine the type of conic section formed at the intersection. A circle is formed by slicing a cone with a plane perpendicular to the axis of symmetry of the cone. An ellipse is formed by slicing a single cone with a slanted plane not perpendicular to the axis of symmetry.


6.0: Prelude to Analytic Geometry

6.1: The Ellipse

6.2: The Hyperbola

6.3: The Parabola

6.4: Rotation of Axes

6.5: Conic Sections in Polar Coordinates

6.E: Analytic Geometry (Exercises)

6.R: Analytic Geometry (Review)


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# 6.0: Prelude to Analytic Geometry

The Greek mathematician Menaechmus (c. 380–c. 320 BCE) is generally credited with discovering the shapes formed by the intersection of a plane and a right circular cone. Depending on how he tilted the plane when it intersected the cone, he formed different shapes at the intersection–beautiful shapes with near-perfect symmetry. It was also said that Aristotle may have had an intuitive understanding of these shapes, as he observed the orbit of the planet to be circular. He presumed that the planets moved in circular orbits around Earth, and for nearly $2000$ years this was the commonly held belief.



(a) (b)

Figure 6.0.1: *(a) Greek philosopher Aristotle (384–322 BCE) (b) German mathematician and astronomer Johannes Kepler (1571–1630)*

It was not until the Renaissance movement that Johannes Kepler noticed that the orbits of the planet were not circular in nature. His published law of planetary motion in the 1600s changed our view of the solar system forever. He claimed that the sun was at one end of the orbits, and the planets revolved around the sun in an oval-shaped path. In this chapter, we will investigate the two-dimensional figures that are formed when a right circular cone is intersected by a plane. We will begin by studying each of three figures created in this manner. We will develop defining equations for each figure and then learn how to use these equations to solve a variety of problems.

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# 6.1: The Ellipse

**Learning Objectives**

- Write equations of ellipses in standard form.
- Graph ellipses centered at the origin.
- Graph ellipses not centered at the origin.
- Solve applied problems involving ellipses.

Can you imagine standing at one end of a large room and still being able to hear a whisper from a person standing at the other end? The National Statuary Hall in Washington, D.C., shown in Figure 6.1.1, is such a room. It is an oval-shaped room called a *whispering chamber* because the shape makes it possible for sound to travel along the walls. In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.



Figure 6.1.1: The National Statuary Hall in Washington, D.C. (credit: Greg Palmer, Flickr)

## Writing Equations of Ellipses in Standard Form

A conic section, or conic, is a shape resulting from intersecting a right circular cone with a plane. The angle at which the plane intersects the cone determines the shape, as shown in Figure 6.1.2.

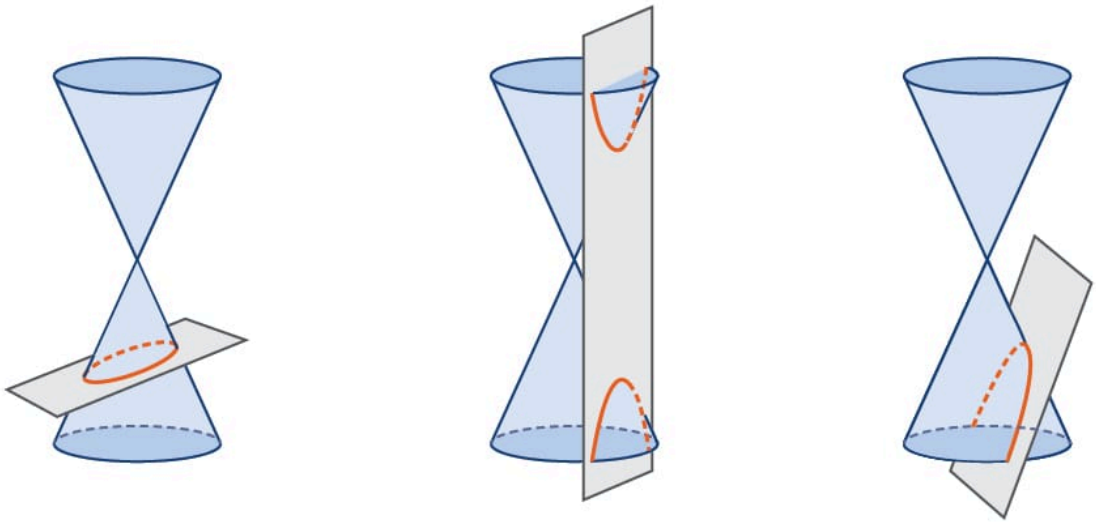



Figure 6.1.2

Conic sections can also be described by a set of points in the coordinate plane. Later in this chapter, we will see that the graph of any quadratic equation in two variables is a conic section. The signs of the equations and the coefficients of the variable terms determine the shape. This section focuses on the four variations of the standard form of the equation for the ellipse. An ellipse is the set of all points $(x, y)$ in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus(plural: foci).

We can draw an ellipse using a piece of cardboard, two thumbtacks, a pencil, and string. Place the thumbtacks in the cardboard to form the foci of the ellipse. Cut a piece of string longer than the distance between the two thumbtacks (the length of the string represents the constant in the definition). Tack each end of the string to the cardboard, and trace a curve with a pencil held taut against the string. The result is an ellipse. See Figure 6.1.3.

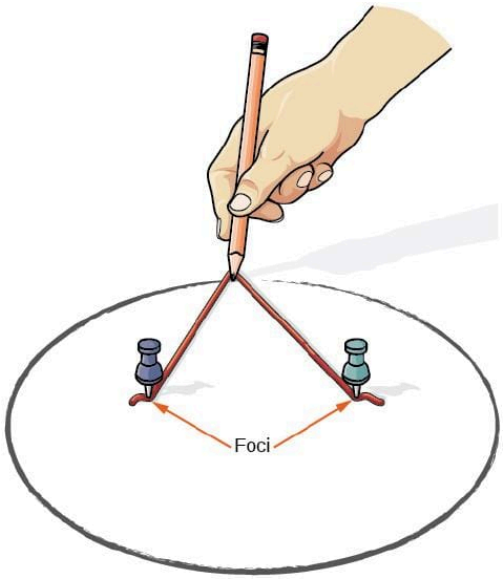



Figure 6.1.3

Every ellipse has two axes of symmetry. The longer axis is called the major axis, and the shorter axis is called the minor axis. Each endpoint of the major axis is the vertex of the ellipse (plural: vertices), and each endpoint of the minor axis is a co-vertex of the ellipse. The center of an ellipse is the midpoint of both the major and minor axes. The axes are perpendicular at the center. The foci always lie on the major axis, and the sum of the distances from the foci to any point on the ellipse (the constant sum) is greater than the distance between the foci (Figure 6.1.4).

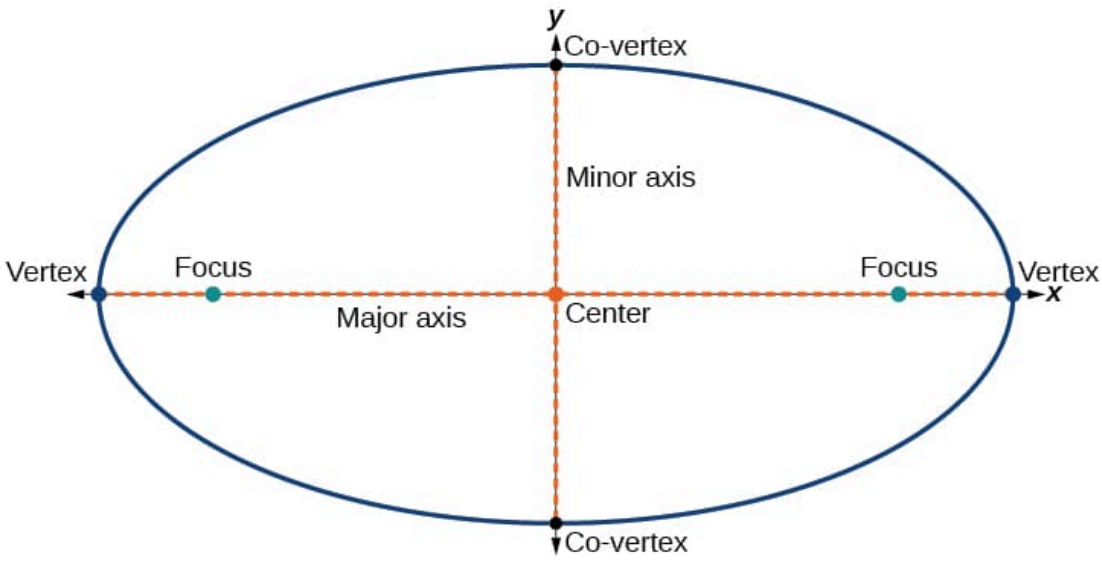



Figure 6.1.4

In this section, we restrict ellipses to those that are positioned vertically or horizontally in the coordinate plane. That is, the axes will either lie on or be parallel to the $x$- and $y$-axes. Later in the chapter, we will see ellipses that are rotated in the coordinate plane.

To work with horizontal and vertical ellipses in the coordinate plane, we consider two cases: those that are centered at the origin and those that are centered at a point other than the origin. First we will learn to derive the equations of ellipses, and then we will learn how to write the equations of ellipses in standard form. Later we will use what we learn to draw the graphs.

## Deriving the Equation of an Ellipse Centered at the Origin

To derive the equation of an **ellipse** centered at the origin, we begin with the foci $(-c, 0)$ and $(c, 0)$. The ellipse is the set of all points $(x, y)$ such that the sum of the distances from $(x, y)$ to the foci is constant, as shown in Figure 6.1.5.

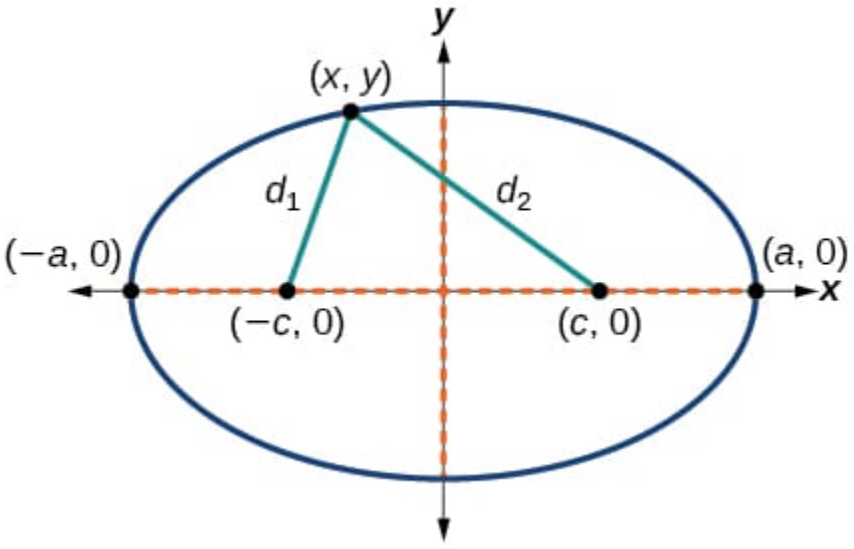



Figure 6.1.5

If $(a, 0)$ is a vertex of the ellipse, the distance from $(-c, 0)$ to $(a, 0)$ is $a - (-c) = a + c$. The distance from $(c, 0)$ to $(a, 0)$ is $a - c$. The sum of the distances from the foci to the vertex is

$$(a + c) + (a - c) = 2a$$

If $(x, y)$ is a point on the ellipse, then we can define the following variables:

- $d_1 =$ the distance from $(-c, 0)$ to $(x, y)$
- $d_2 =$ the distance from $(c, 0)$ to $(x, y)$

By the definition of an ellipse, $d_1 + d_2$ is constant for any point $(x, y)$ on the ellipse. We know that the sum of these distances is $2a$ for the vertex $(a, 0)$. It follows that $d_1 + d_2 = 2a$ for any point on the ellipse. We will begin the derivation by applying the distance formula. The rest of the derivation is algebraic.

$$
\begin{aligned}
d_1 + d_2 &= 2a \\
\sqrt{(x-(-c))^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} &= 2a \qquad \text{Distance formula} \\
\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} &= 2a \qquad \text{Simplify expressions.} \\
\sqrt{(x+c)^2 + y^2} &= 2a - \sqrt{(x-c)^2 + y^2} \qquad \text{Move radical to opposite side.} \\
(x+c)^2 + y^2 &= \left[2a - \sqrt{(x-c)^2 + y^2}\right]^2 \qquad \text{Square both sides.} \\
x^2 + 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 \qquad \text{Expand the squares.} \\
x^2 + 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2 \qquad \text{Expand remaining squares.} \\
2cx &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} - 2cx \qquad \text{Combine like terms.} \\
4cx - 4a^2 &= -4a\sqrt{(x-c)^2 + y^2} \qquad \text{Isolate the radical.} \\
cx - a^2 &= -a\sqrt{(x-c)^2 + y^2} \qquad \text{Divide by 4.} \\
\left[cx - a^2\right]^2 &= a^2\left[\sqrt{(x-c)^2 + y^2}\right]^2 \qquad \text{Square both sides.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2(x^2 - 2cx + c^2 + y^2) \qquad \text{Expand the squares.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 \qquad \text{Distribute } a^2 \\
a^2x^2 - c^2x^2 + a^2y^2 &= a^4 - a^2c^2 \qquad \text{Rewrite.} \\
x^2(a^2 - c^2) + a^2y^2 &= a^2(a^2 - c^2) \qquad \text{Factor common terms.} \\
x^2b^2 + a^2y^2 &= a^2b^2 \qquad \text{Set } b^2 = a^2 - c^2 \\
\frac{x^2b^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} \qquad \text{Divide both sides by } a^2b^2 \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \qquad \text{Simplify}
\end{aligned}
$$

Thus, the standard equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.This equation defines an ellipse centered at the origin. If $a > b$,the ellipse is stretched further in the horizontal direction, and if $b > a$, the ellipse is stretched further in the vertical direction.

## Writing Equations of Ellipses Centered at the Origin in Standard Form

Standard forms of equations tell us about key features of graphs. Take a moment to recall some of the standard forms of equations we've worked with in the past: linear, quadratic, cubic, exponential, logarithmic, and so on. By learning to interpret standard forms of equations, we are bridging the relationship between algebraic and geometric representations of mathematical phenomena.

The key features of the **ellipse** are its center, **vertices**, **co-vertices**, **foci**, and lengths and positions of the **major and minor axes**. Just as with other equations, we can identify all of these features just by looking at the standard form of the equation. There are four variations of the standard form of the ellipse. These variations are categorized first by the location of the center (the origin or not the origin), and then by the position (horizontal or vertical). Each is presented along with a description of how the parts of the equation relate to the graph. Interpreting these parts allows us to form a mental picture of the ellipse.

### STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER $(0, 0)$

The standard form of the equation of an ellipse with center $(0, 0)$ and major axis on the $x$-*axis* is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{6.1.1}$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(\pm a,0)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(0,\pm b)$
- the coordinates of the foci are $(\pm c,0)$, where $c^2=a^2-b^2$. See Figure $6.1.6a$.

The standard form of the equation of an ellipse with center $(0,0)$ and major axis on the $y$-axis is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \tag{6.1.2}$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(0,\pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(\pm b,0)$
- the coordinates of the foci are $(0,\pm c)$, where $c^2=a^2-b^2$. See Figure $6.1.6b$.

Note that the vertices, co-vertices, and foci are related by the equation $c^2=a^2-b^2$. When we are given the coordinates of the foci and vertices of an ellipse, we can use this relationship to find the equation of the ellipse in standard form.

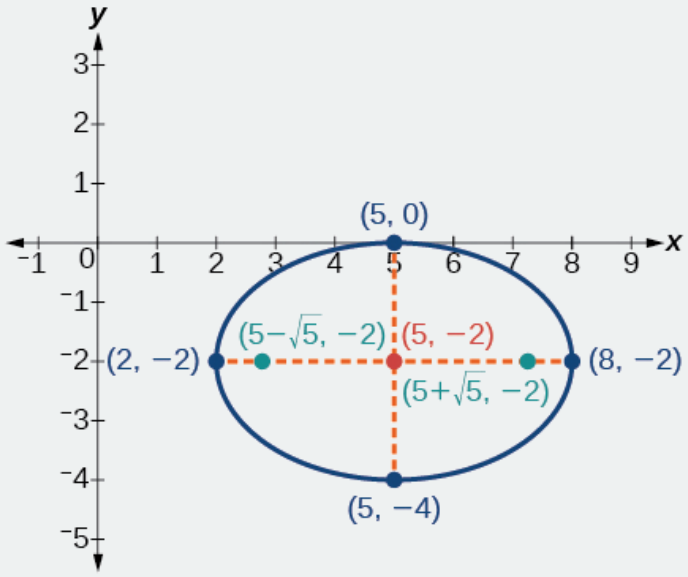



Figure $6.1.6$: (a) Horizontal ellipse with center $(0,0)$ (b) Vertical ellipse with center $(0,0)$

## How to: Given the vertices and foci of an ellipse centered at the origin, write its equation in standard form

1. Determine whether the major axis lies on the *x*- or *y*-axis.
   - If the given coordinates of the vertices and foci have the form $(\pm a,0)$ and $(\pm c,0)$ respectively, then the major axis is the *x*-axis. Use the standard form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

- If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(\pm c, 0)$,respectively, then the major axis is the *y*-axis. Use the standard form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

2. Use the equation $c^2 = a^2 - b^2$, along with the given coordinates of the vertices and foci, to solve for $b^2$.
3. Substitute the values for $a^2$ and $b^2$ into the standard form of the equation determined in Step 1.

### Example 6.1.1: Writing the Equation of an Ellipse Centered at the Origin in Standard Form

What is the standard form equation of the ellipse that has vertices $(\pm 8, 0)$ and foci $(\pm 5, 0)$?

**Solution**

The foci are on the $x$-axis, so the major axis is the $x$-axis. Thus, the equation will have the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The vertices are $(\pm 8, 0)$,so $a = 8$ and $a^2 = 64$.

The foci are $(\pm 5, 0)$,so $c = 5$ and $c^2 = 25$.

We know that the vertices and foci are related by the equation $c^2 = a^2 - b^2$. Solving for $b^2$, we have:

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 25 &= 64 - b^2 \qquad \text{Substitute for } c^2 \text{ and } a^2 \\ b^2 &= 39 \qquad \text{Solve for } b^2 \end{aligned}$$

Now we need only substitute $a^2 = 64$ and $b^2 = 39$ into the standard form of the equation. The equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{39} = 1$.

### Exercise 6.1.1

What is the standard form equation of the ellipse that has vertices $(0, \pm 4)$ and foci $(0, \pm\sqrt{15})$?

*Answer available at the end of this section.*

### Q&A

**Can we write the equation of an ellipse centered at the origin given coordinates of just one focus and vertex?**

*Yes. Ellipses are symmetrical, so the coordinates of the vertices of an ellipse centered around the origin will always have the form $(\pm a, 0)$ or $(0, \pm a)$. Similarly, the coordinates of the foci will always have the form $(\pm c, 0)$ or $(0, \pm c)$. Knowing this, we can use $a$ and $c$ from the given points, along with the equation $c^2 = a^2 - b^2$, to find $b^2$.*

## Writing Equations of Ellipses Not Centered at the Origin

Like the graphs of other equations, the graph of an **ellipse** can be translated. If an ellipse is translated $h$ units horizontally and $k$ units vertically, the center of the ellipse will be $(h, k)$. This **translation** results in the standard form of the equation we saw previously, with $x$ replaced by $(x - h)$ and *y* replaced by $(y - k)$.

### STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER $(H, K)$

The standard form of the equation of an ellipse with center $(h, k)$ and major axis parallel to the $x$-axis is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \tag{6.1.3}$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h\pm a,k)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h,k\pm b)$
- the coordinates of the foci are $(h\pm c,k)$,where $c^2=a^2-b^2$. See Figure $6.1.7a$.

The standard form of the equation of an ellipse with center $(h,k)$ and major axis parallel to the $y$-axis is

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \tag{6.1.4}$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h,k\pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h\pm b,k)$
- the coordinates of the foci are $(h,k\pm c)$, where $c^2=a^2-b^2$. See Figure $6.1.7b$.

Just as with ellipses centered at the origin, ellipses that are centered at a point $(h,k)$ have vertices, co-vertices, and foci that are related by the equation $c^2=a^2-b^2$. We can use this relationship along with the midpoint and distance formulas to find the equation of the ellipse in standard form when the vertices and foci are given.

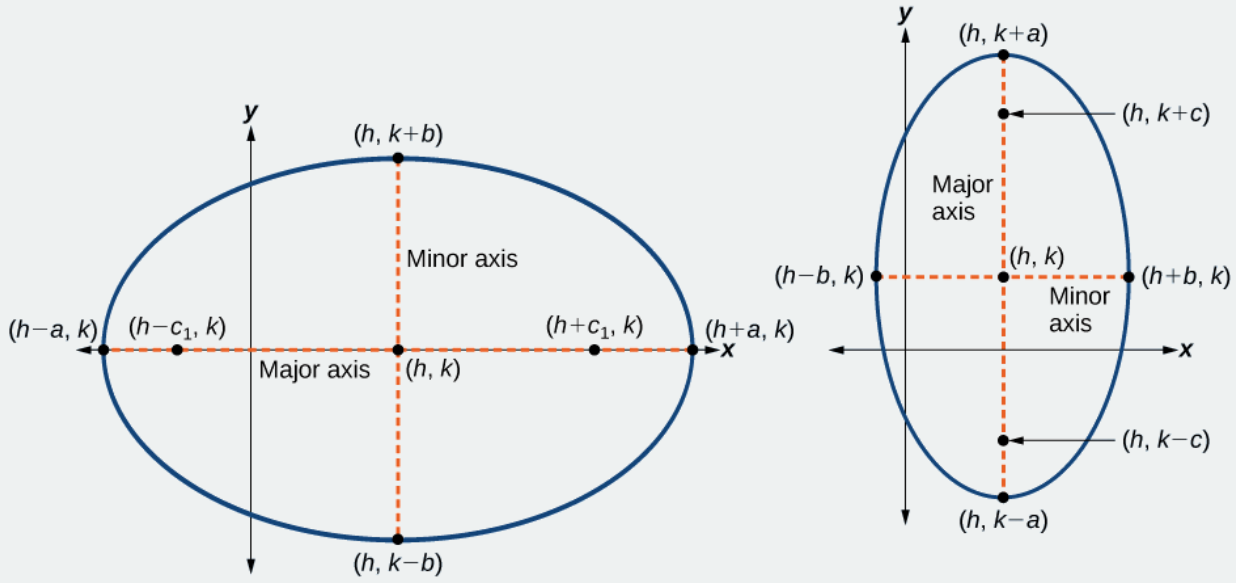



Figure $6.1.7$: (a) Horizontal ellipse with center $(h,k)$ (b) Vertical ellipse with center $(h,k)$

## How to: Given the vertices and foci of an ellipse not centered at the origin, write its equation in standard form

1. Determine whether the major axis is parallel to the $x$- or $y$-axis.
   - If the *y*-coordinates of the given vertices and foci are the same, then the major axis is parallel to the $x$-axis. Use the standard form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

- If the *x*-coordinates of the given vertices and foci are the same, then the major axis is parallel to the *y*-axis. Use the standard form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$

2. Identify the center of the ellipse $(h,k)$ using the midpoint formula and the given coordinates for the vertices.
3. Find $a^2$ by solving for the length of the major axis, $2a$, which is the distance between the given vertices.
4. Find $c^2$ using $h$ and $k$, found in Step 2, along with the given coordinates for the foci.
5. Solve for $b^2$ using the equation $c^2=a^2-b^2$.
6. Substitute the values for $h$, $k$, $a^2$, and $b^2$ into the standard form of the equation determined in Step 1.

### Example 6.1.2: Writing the Equation of an Ellipse Centered at a Point Other Than the Origin

What is the standard form equation of the ellipse that has vertices $(-2,-8)$ and $(-2,2)$ and foci $(-2,-7)$ and $(-2,1)$?

## Solution

The $x$-coordinates of the vertices and foci are the same, so the major axis is parallel to the $y$-axis. Thus, the equation of the ellipse will have the form

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

First, we identify the center, $(h,k)$. The center is halfway between the vertices, $(-2,-8)$ and $(-2,2)$. Applying the midpoint formula, we have:

$$\begin{aligned}(h,k)&=\left(\frac{-2+(-2)}{2},\frac{-8+2}{2}\right)\\&=(-2,-3)\end{aligned}$$

Next, we find $a^2$. The length of the major axis, $2a$, is bounded by the vertices. We solve for $a$ by finding the distance between the *y*-coordinates of the vertices.

$$\begin{aligned}2a&=2-(-8)\\2a&=10\\a&=5\end{aligned}$$

So $a^2=25$.

Now we find $c^2$. The foci are given by $(h,k\pm c)$. So, $(h,k-c)=(-2,-7)$ and $(h,k+c)=(-2,1)$. We substitute $k=-3$ using either of these points to solve for $c$.

$$\begin{aligned}k+c&=1\\-3+c&=1\\c&=4\end{aligned}$$

So $c^2=16$.

Next, we solve for $b^2$ using the equation $c^2=a^2-b^2$.

$$\begin{aligned}c^2&=a^2-b^2\\16&=25-b^2\\b^2&=9\end{aligned}$$

Finally, we substitute the values found for $h$, $k$, $a^2$, and $b^2$ into the standard form equation for an ellipse:

$$\frac{(x+2)^2}{9}+\frac{(y+3)^2}{25}=1$$

**Exercise 6.1.2**

What is the standard form equation of the ellipse that has vertices $(-3, 3)$ and $(5, 3)$ and foci $(1 - 2\sqrt{3}, 3)$ and $(1 + 2\sqrt{3}, 3)$?

*Answer available at the end of this section.*

## Graphing Ellipses Centered at the Origin

Just as we can write the equation for an ellipse given its graph, we can graph an ellipse given its equation. To graph ellipses centered at the origin, we use the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b \text{ for horizontal ellipses}$$

and

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b \text{ for vertical ellipses}$$

**How to: Given the standard form of an equation for an ellipse centered at $(0, 0)$, sketch the graph.**

1. Use the standard forms of the equations of an ellipse to determine the major axis, vertices, co-vertices, and foci.
   - If the equation is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, then
     - the major axis is the $x$-axis
     - the coordinates of the vertices are $(\pm a, 0)$
     - the coordinates of the co-vertices are $(0, \pm b)$
     - the coordinates of the foci are $(\pm c, 0)$
   - If the equation is in the form $x^2b^2 + y^2a^2 = 1$,where $a > b$, then
     - the major axis is the $y$-axis
     - the coordinates of the vertices are $(0, \pm a)$
     - the coordinates of the co-vertices are $(\pm b, 0)$
     - the coordinates of the foci are $(0, \pm c)$
2. Solve for $c$ using the equation $c^2 = a^2 - b^2$.
3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

**Example 6.1.3: Graphing an Ellipse Centered at the Origin**

Graph the ellipse given by the equation, $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Identify and label the center, vertices, co-vertices, and foci.

**Solution**

First, we determine the position of the major axis. Because $25 > 9$,the major axis is on the $y$-axis. Therefore, the equation is in the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$,where $b^2 = 9$ and $a^2 = 25$. It follows that:

- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(0, \pm a) = (0, \pm\sqrt{25}) = (0, \pm 5)$
- the coordinates of the co-vertices are $(\pm b, 0) = (\pm 9, 0) = (\pm 3, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = a^2 - b^2$ Solving for $c$, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{25 - 9} \\ &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned}$$

Therefore, the coordinates of the foci are $(0, \pm 4)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse. See Figure 6.1.8.

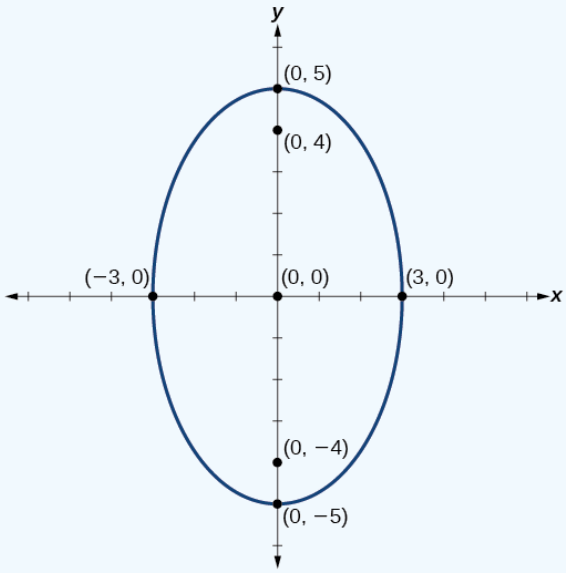



Figure 6.1.8

## Exercise 6.1.3

Graph the ellipse given by the equation $\frac{x^2}{36} + \frac{y^2}{4} = 1$. Identify and label the center, vertices, co-vertices, and foci.

*Answer available at the end of this section.*

## Example 6.1.4: Graphing an Ellipse Centered at the Origin from an Equation Not in Standard Form

Graph the ellipse given by the equation $4x^2 + 25y^2 = 100$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

**Solution**

First, use algebra to rewrite the equation in standard form.

$$\begin{aligned} 4x^2 + 25y^2 &= 100 \\ \frac{4x^2}{100} + \frac{25y^2}{100} &= \frac{100}{100} \\ \frac{x^2}{25} + \frac{y^2}{4} &= 1 \end{aligned}$$

Next, we determine the position of the major axis. Because $25 > 4$, the major axis is on the $x$-axis. Therefore, the equation is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 = 25$ and $b^2 = 4$. It follows that:

- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(\pm a, 0) = (\pm\sqrt{25}, 0) = (\pm 5, 0)$
- the coordinates of the co-vertices are $(0, \pm b) = (0, \pm\sqrt{4}) = (0, \pm 2)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = a^2 - b^2$. Solving for $c$, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{25 - 4} \\ &= \pm\sqrt{21} \end{aligned}$$

Therefore the coordinates of the foci are $(\pm\sqrt{21}, 0)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

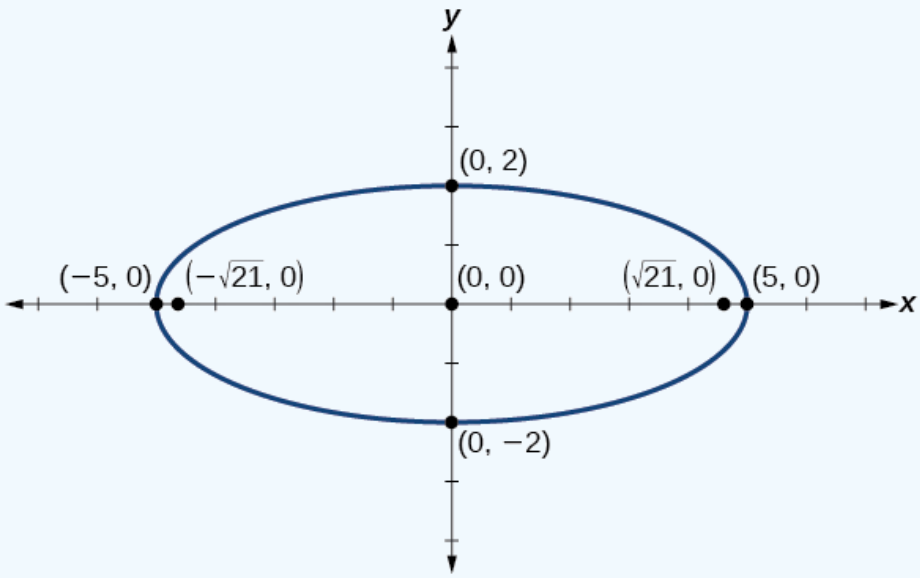



Figure 6.1.10

**Exercise 6.1.4**

Graph the ellipse given by the equation $49x^2 + 16y^2 = 784$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

*Answer available at the end of this section.*

## Graphing Ellipses Not Centered at the Origin

When an **ellipse** is not centered at the origin, we can still use the standard forms to find the key features of the graph. When the ellipse is centered at some point, $(h, k)$,we use the standard forms $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a > b$ for horizontal ellipses and $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, $a > b$ for vertical ellipses. From these standard equations, we can easily determine the center, vertices, co-vertices, foci, and positions of the major and minor axes.

**How to: Given the standard form of an equation for an ellipse centered at $(h, k)$, sketch the graph.**

1. Use the standard forms of the equations of an ellipse to determine the center, position of the major axis, vertices, co-vertices, and foci.

- If the equation is in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a>b$, then
  - the center is $(h,k)$
  - the major axis is parallel to the $x$-axis
  - the coordinates of the vertices are $(h\pm a,k)$
  - the coordinates of the co-vertices are $(h,k\pm b)$
  - the coordinates of the foci are $(h\pm c,k)$
- If the equation is in the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, where $a>b$, then
  - the center is $(h,k)$
  - the major axis is parallel to the $y$-axis
  - the coordinates of the vertices are $(h,k\pm a)$
  - the coordinates of the co-vertices are $(h\pm b,k)$
  - the coordinates of the foci are $(h,k\pm c)$

2. Solve for $c$ using the equation $c^2=a^2-b^2$.
3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

## Example 6.1.5: Graphing an Ellipse Centered at $(h,k)$

Graph the ellipse given by the equation, $\frac{(x+2)^2}{4}+\frac{(y-5)^2}{9}=1$. Identify and label the center, vertices, co-vertices, and foci.

**Solution**

First, we determine the position of the major axis. Because $9>4$, the major axis is parallel to the $y$-axis. Therefore, the equation is in the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, where $b^2=4$ and $a^2=9$. It follows that:

- the center of the ellipse is $(h,k)=(-2,5)$
- the coordinates of the vertices are $(h,k\pm a)=(-2,5\pm\sqrt{9})=(-2,5\pm 3)$, or $(-2,2)$ and $(-2,8)$
- the coordinates of the co-vertices are $(h\pm b,k)=(-2\pm\sqrt{4},5)=(-2\pm 2,5)$, or $(-4,5)$ and $(0,5)$
- the coordinates of the foci are $(h,k\pm c)$, where $c^2=a^2-b^2$. Solving for $c$,we have:

$$\begin{aligned} c &= \pm\sqrt{a^2-b^2} \\ &= \pm\sqrt{9-4} \\ &= \pm\sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $(-2,5-\sqrt{5})$ and $(-2,5+\sqrt{5})$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

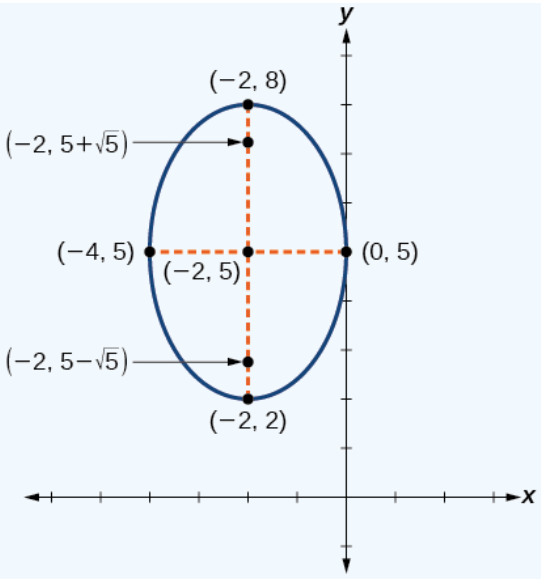



Figure 6.1.12

## Exercise 6.1.5

Graph the ellipse given by the equation $\frac{(x-4)^2}{36}+\frac{(y-2)^2}{20}=1$. Identify and label the center, vertices, co-vertices, and foci.

*Answer available at the end of this section.*

## How to: Given the general form of an equation for an ellipse centered at $(h,k)$, express the equation in standard form.

1. Recognize that an ellipse described by an equation in the form $ax^2+by^2+cx+dy+e=0$ is in general form.
2. Rearrange the equation by grouping terms that contain the same variable. Move the constant term to the opposite side of the equation.
3. Factor out the coefficients of the $x^2$ and $y^2$ terms in preparation for completing the square.
4. Complete the square for each variable to rewrite the equation in the form of the sum of multiples of two binomials squared set equal to a constant, $m_1(x-h)^2+m_2(y-k)^2=m_3$, where $m_1$, $m_2$,and $m_3$ are constants.
5. Divide both sides of the equation by the constant term to express the equation in standard form.

## Example 6.1.6: Graphing an Ellipse Centered at $(h,k)$ by First Writing It in Standard Form

Graph the ellipse given by the equation $4x^2+9y^2-40x+36y+100=0$. Identify and label the center, vertices, co-vertices, and foci.

**Solution**

We must begin by rewriting the equation in standard form.

$$4x^2+9y^2-40x+36y+100=0$$

Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$(4x^2-40x)+(9y^2+36y)=-100$$

Factor out the coefficients of the squared terms.

$$4(x^2 - 10x) + 9(y^2 + 4y) = -100$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$4(x^2 - 10x + 25) + 9(y^2 + 4y + 4) = -100 + 100 + 36$$

Rewrite as perfect squares.

$$4(x - 5)^2 + 9(y + 2)^2 = 36$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x - 5)^2}{9} + \frac{(y + 2)^2}{4} = 1$$

Now that the equation is in standard form, we can determine the position of the major axis. Because $9 > 4$, the major axis is parallel to the $x$-axis. Therefore, the equation is in the form $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$, where $a^2 = 9$ and $b^2 = 4$. It follows that:

- the center of the ellipse is $(h, k) = (5, -2)$
- the coordinates of the vertices are $(h \pm a, k) = (5 \pm \sqrt{9}, -2) = (5 \pm 3, -2)$, or $(2, -2)$ and $(8, -2)$
- the coordinates of the co-vertices are $(h, k \pm b) = (5, -2 \pm \sqrt{4}) = (5, -2 \pm 2)$, or $(5, -4)$ and $(5, 0)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$. Solving for $c$, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{9 - 4} \\ &= \pm\sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $(5 - \sqrt{5}, -2)$ and $(5 + \sqrt{5}, -2)$.

Next we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse as shown in Figure 6.1.14.

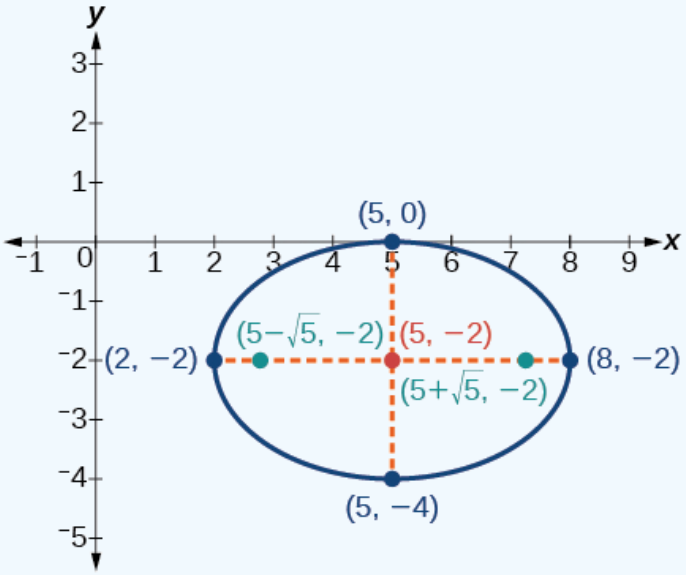



Figure 6.1.14

## Exercise 6.1.6

Express the equation of the ellipse given in standard form. Identify the center, vertices, co-vertices, and foci of the ellipse.

$$4x^2 + y^2 - 24x + 2y + 21 = 0$$

*Answer available at the end of this section.*

## Solving Applied Problems Involving Ellipses

Many real-world situations can be represented by ellipses, including orbits of planets, satellites, moons and comets, and shapes of boat keels, rudders, and some airplane wings. A medical device called a lithotripter uses elliptical reflectors to break up kidney stones by generating sound waves. Some buildings, called whispering chambers, are designed with elliptical domes so that a person whispering at one focus can easily be heard by someone standing at the other focus. This occurs because of the acoustic properties of an ellipse. When a sound wave originates at one focus of a whispering chamber, the sound wave will be reflected off the elliptical dome and back to the other focus (Figure $6.1.15$). In the whisper chamber at the Museum of Science and Industry in Chicago, two people standing at the foci—about $43$ feet apart—can hear each other whisper.

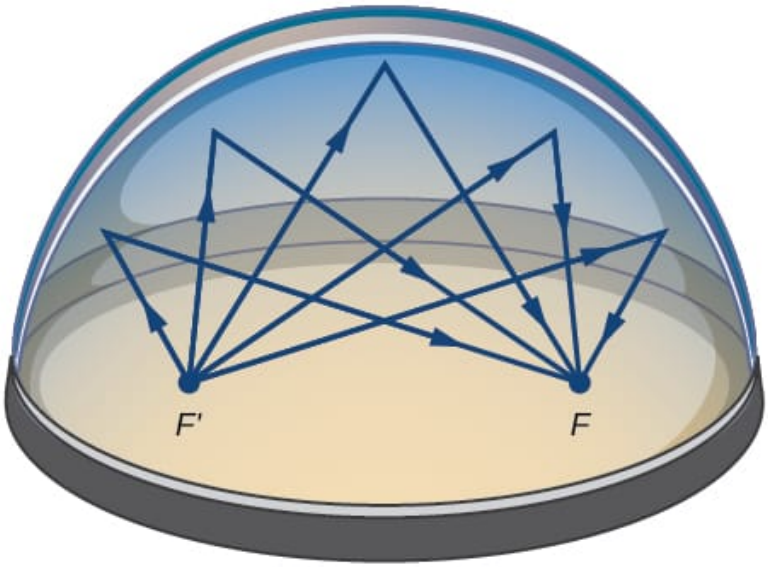



Figure $6.1.15$: Sound waves are reflected between foci in an elliptical room, called a whispering chamber.

### Example $6.1.7$: Locating the Foci of a Whispering Chamber

The Statuary Hall in the Capitol Building in Washington, D.C. is a whispering chamber. Its dimensions are $46$ feet wide by $96$ feet long as shown in Figure $6.1.16$.

a. What is the standard form of the equation of the ellipse representing the outline of the room? ***Hint:*** assume a horizontal ellipse, and let the center of the room be the point $(0, 0)$.
b. If two senators standing at the foci of this room can hear each other whisper, how far apart are the senators? Round to the nearest foot.

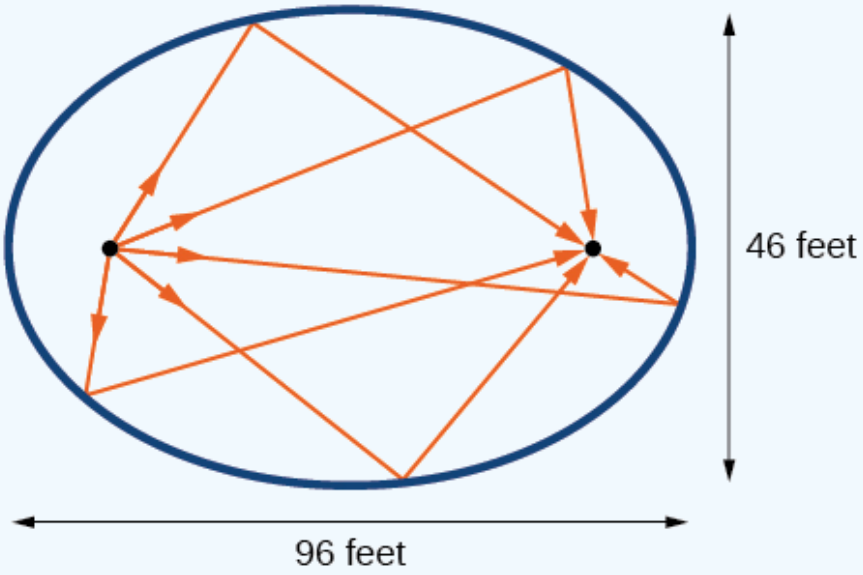



Figure 6.1.16

**Solution**

a. We are assuming a horizontal ellipse with center $(0,0)$, so we need to find an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$. We know that the length of the major axis, $2a$, is longer than the length of the minor axis, $2b$. So the length of the room, 96, is represented by the major axis, and the width of the room, 46, is represented by the minor axis.

   Therefore, the equation of the ellipse is \[(dfrac{x^2}{2304}+\dfrac{y^2}{529}=1\)

   - Solving for $a$, we have $2a=96$, so $a=48$, and $a^2=2304$.
   - Solving for $b$, we have $2b=46$, so $b=23$, and $b^2=529$.

b. To find the distance between the senators, we must find the distance between the foci, $(\pm c,0)$, where $c^2=a^2-b^2$. Solving for $c$,we have:

$$\begin{aligned} c^2 &= a^2-b^2 \\ c^2 &= 2304-529 \quad \text{Substitute using the values found in part } (a) \\ c &= \pm\sqrt{2304-529} \quad \text{Take the square root of both sides.} \\ c &= \pm\sqrt{1775} \quad \text{Subtract.} \\ c &\approx \pm 42 \quad \text{Round to the nearest foot.} \end{aligned}$$

The points $(\pm 42,0)$ represent the foci. Thus, the distance between the senators is $2(42)=84$ feet.

## Exercise 6.1.7

Suppose a whispering chamber is $480$ feet long and $320$ feet wide.

a. What is the standard form of the equation of the ellipse representing the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0,0)$.
b. If two people are standing at the foci of this room and can hear each other whisper, how far apart are the people? Round to the nearest foot.

*Answer a available at the end of this section.*

*Answer b available at the end of this section.*

## Media

Access these online resources for additional instruction and practice with ellipses.

- Conic Sections: The Ellipse
- Graph an Ellipse with Center at the Origin
- Graph an Ellipse with Center Not at the Origin

## Key Equations

| | |
|---|---|
| Horizontal ellipse, center at origin | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=$ |
| Vertical ellipse, center at origin | $\frac{x^2}{b^2}+\frac{y^2}{a^2}=$ |
| Horizontal ellipse, center $(h,k)$ | $\frac{(x-h)^2}{a^2}+$ |
| Vertical ellipse, center $(h,k)$ | $\frac{(x-h)^2}{b^2}+$ |

## Key Concepts

- An ellipse is the set of all points $(x,y)$ in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus (plural: foci).
- When given the coordinates of the foci and vertices of an ellipse, we can write the equation of the ellipse in standard form. See Example $6.1.1$ and Example $6.1.2$.
- When given an equation for an ellipse centered at the origin in standard form, we can identify its vertices, co-vertices, foci, and the lengths and positions of the major and minor axes in order to graph the ellipse. See Example $6.1.3$ and Example $6.1.4$.
- When given the equation for an ellipse centered at some point other than the origin, we can identify its key features and graph the ellipse. See Example $6.1.5$ and Example $6.1.6$.
- Real-world situations can be modeled using the standard equations of ellipses and then evaluated to find key features, such as lengths of axes and distance between foci. See Example $6.1.7$.

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### Answers

**Exercise** $6.1.1$

$x^2+\frac{y^2}{16}=1$

**Exercise** $6.1.2$

$\frac{(x-1)^2}{16}+\frac{(y-3)^2}{4}=1$

**Exercise** $6.1.3$

center: $(0,0)$; vertices: $(\pm 6,0)$; co-vertices: $(0,\pm 2)$; foci: $(\pm 4\sqrt{2},0)$

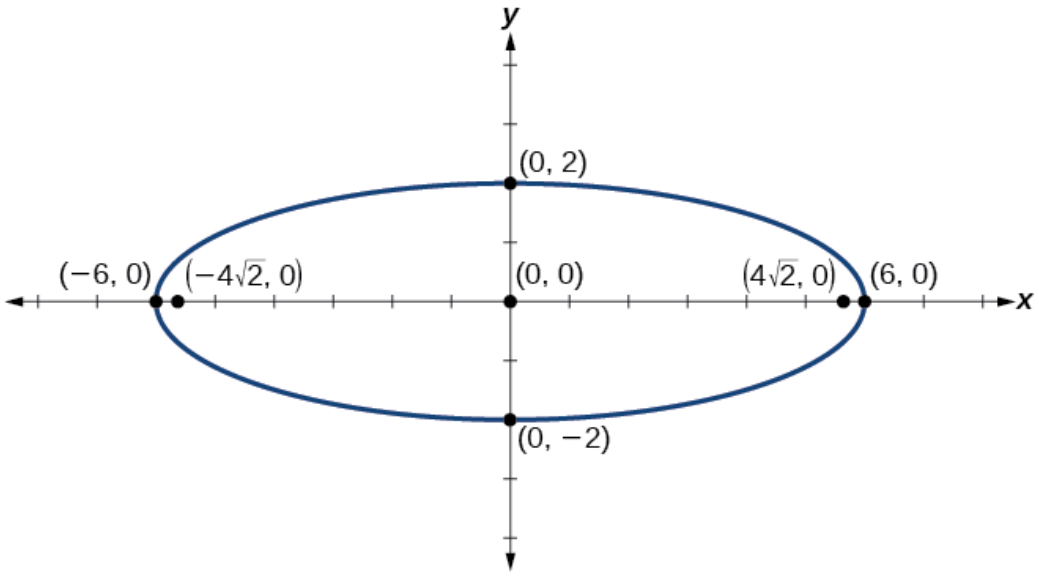



Figure 6.1.9

**Exercise** 6.1.4

Standard form: $\frac{x^2}{16}+\frac{y^2}{49}=1$; center: $(0,0)$; vertices: $(0,\pm 7)$; co-vertices: $(\pm 4,0)$; foci: $(0,\pm\sqrt{33})$

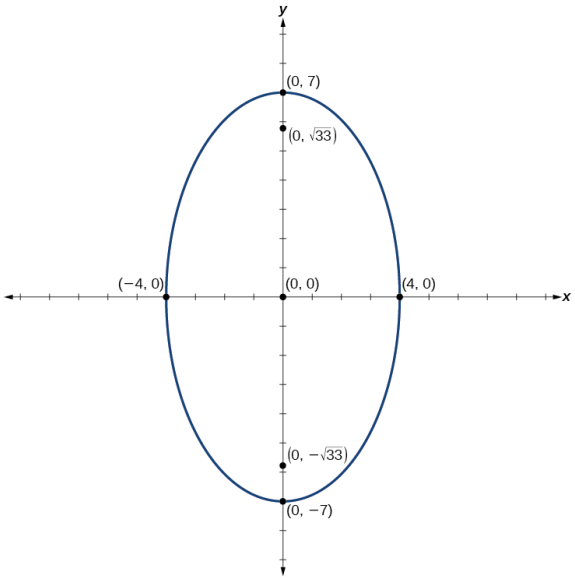



Figure 6.1.11

**Exercise** 6.1.5

Center: $(4,2)$; vertices: $(-2,2)$ and $(10,2)$; co-vertices: $(4,2-2\sqrt{5})$ and $(4,2+2\sqrt{5})$; foci: $(0,2)$ and $(8,2)$

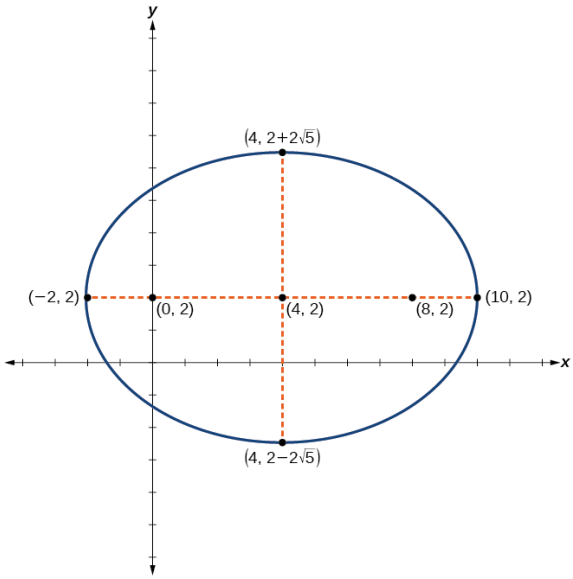



Figure 6.1.13

**Exercise** 6.1.6

$\frac{(x-3)^2}{4}+\frac{(y+1)^2}{16}=1$; center: $(3,-1)$; vertices: $(3,-5)$ and $(3,3)$; co-vertices: $(1,-1)$ and $(5,-1)$; foci: $(3,-1-2\sqrt{3})$ and $(3,-1+2\sqrt{3})$

**Exercise** 6.1.7

$\frac{x^2}{57,600}+\frac{y^2}{25,600}=1$

**Exercise** 6.1.7

The people are standing $358$ feet apart.

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# 6.2: The Hyperbola

**Learning Objectives**

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Solve applied problems involving hyperbolas.

What do paths of comets, supersonic booms, ancient Grecian pillars, and natural draft cooling towers have in common? They can all be modeled by the same type of **conic**. For instance, when something moves faster than the speed of sound, a shock wave in the form of a cone is created. A portion of a conic is formed when the wave intersects the ground, resulting in a sonic boom (Figure 6.2.1).

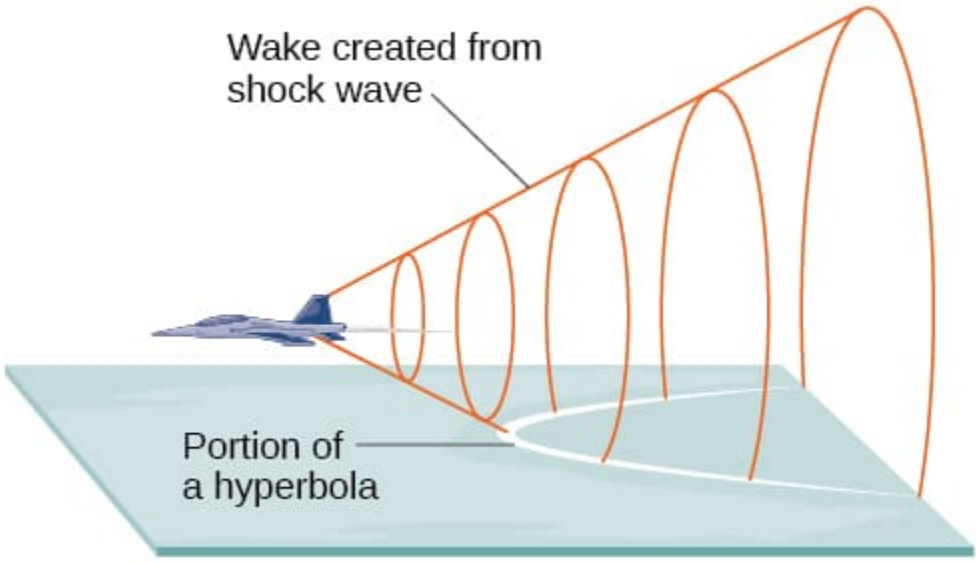



Figure 6.2.1: A shock wave intersecting the ground forms a portion of a conic and results in a sonic boom.

Most people are familiar with the sonic boom created by supersonic aircraft, but humans were breaking the sound barrier long before the first supersonic flight. The crack of a whip occurs because the tip is exceeding the speed of sound. The bullets shot from many firearms also break the sound barrier, although the bang of the gun usually supersedes the sound of the sonic boom.

## Locating the Vertices and Foci of a Hyperbola

In analytic geometry, a **hyperbola** is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other (Figure 6.2.2).

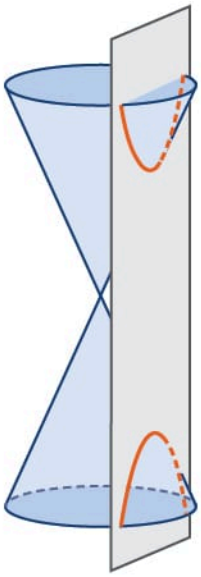


Figure 6.2.2: A hyperbola

Like the ellipse, the hyperbola can also be defined as a set of points in the coordinate plane. A hyperbola is the set of all points $(x, y)$ in a plane such that the difference of the distances between $(x, y)$ and the foci is a positive constant.

Notice that the definition of a hyperbola is very similar to that of an ellipse. The distinction is that the hyperbola is defined in terms of the *difference* of two distances, whereas the ellipse is defined in terms of the *sum* of two distances.

As with the ellipse, every hyperbola has two **axes of symmetry**. The **transverse axis** is a line segment that passes through the center of the hyperbola and has vertices as its endpoints. The foci lie on the line that contains the transverse axis. The **conjugate axis** is perpendicular to the transverse axis and has the co-vertices as its endpoints. The **center of a hyperbola** is the midpoint of both the transverse and conjugate axes, where they intersect. Every hyperbola also has two **asymptotes** that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes. The **central rectangle** of the hyperbola is centered at the origin with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle (Figure 6.2.3).

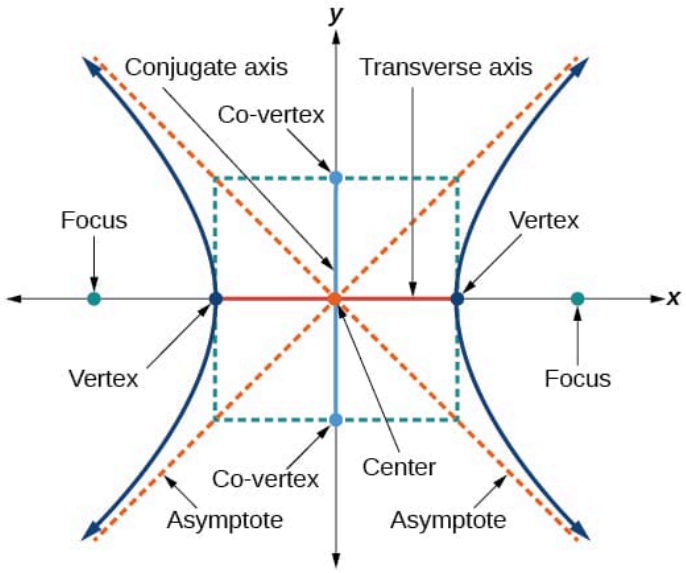



Figure 6.2.3: Key features of the hyperbola

In this section, we will limit our discussion to hyperbolas that are positioned vertically or horizontally in the coordinate plane; the axes will either lie on or be parallel to the $x$- and $y$-axes. We will consider two cases: those that are centered at the origin, and

those that are centered at a point other than the origin.

## Deriving the Equation of an Ellipse Centered at the Origin

Let $(-c, 0)$ and $(c, 0)$ be the foci of a hyperbola centered at the origin. The hyperbola is the set of all points $(x, y)$ such that the difference of the distances from $(x, y)$ to the foci is constant. See Figure $6.2.4$.

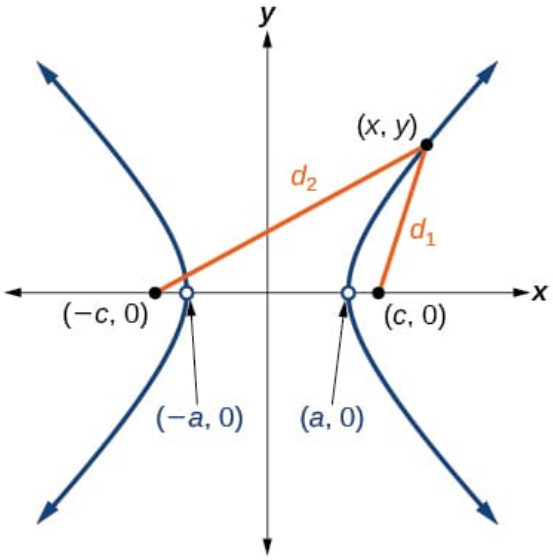



Figure $6.2.4$

If $(a, 0)$ is a vertex of the hyperbola, the distance from $(-c, 0)$ to $(a, 0)$ is $a - (-c) = a + c$. The distance from $(c, 0)$ to $(a, 0)$ is $c - a$. The sum of the distances from the foci to the vertex is

$$(a + c) - (c - a) = 2a$$

If $(x, y)$ is a point on the hyperbola, we can define the following variables:

$$d_2 = \text{the distance from } (-c, 0) \text{ to } (x, y)$$

$$d_1 = \text{the distance from } (c, 0) \text{ to } (x, y)$$

By definition of a hyperbola, $d_2 - d_1$ is constant for any point $(x, y)$ on the hyperbola. We know that the difference of these distances is $2a$ for the vertex $(a, 0)$. It follows that $d_2 - d_1 = 2a$ for any point on the hyperbola. As with the derivation of the equation of an ellipse, we will begin by applying the **distance formula**. The rest of the derivation is algebraic. Compare this derivation with the one from the previous section for ellipses.

$$
\begin{aligned}
d_2 - d_1 &= 2a \\
\sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} &= 2a \quad \text{Distance Formula} \\
\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} &= 2a \quad \text{Simplify expressions.} \\
\sqrt{(x+c)^2+y^2} &= 2a + \sqrt{(x-c)^2+y^2} \quad \text{Move radical to opposite side.} \\
(x+c)^2+y^2 &= \left(2a + \sqrt{(x-c)^2+y^2}\right)^2 \quad \text{Square both sides.} \\
x^2+2cx+c^2+y^2 &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2 \quad \text{Expand the squares.} \\
x^2+2cx+c^2+y^2 &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2 \quad \text{Expand remaining square.} \\
2cx &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} - 2cx \quad \text{Combine like terms.} \\
4cx - 4a^2 &= 4a\sqrt{(x-c)^2+y^2} \quad \text{Isolate the radical.} \\
cx - a^2 &= a\sqrt{(x-c)^2+y^2} \quad \text{Divide by 4.} \\
(cx-a^2)^2 &= a^2\left[\sqrt{(x-c)^2+y^2}\right]^2 \quad \text{Square both sides.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2(x^2 - 2cx + c^2 + y^2) \quad \text{Expand the squares.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 \quad \text{Distribute } a^2 \\
a^4 + c^2x^2 &= a^2x^2 + a^2c^2 + a^2y^2 \quad \text{Combine like terms.} \\
c^2x^2 - a^2x^2 - a^2y^2 &= a^2c^2 - a^4 \quad \text{Rearrange terms.} \\
x^2(c^2-a^2) - a^2y^2 &= a^2(c^2-a^2) \quad \text{Factor common terms.} \\
x^2b^2 - a^2y^2 &= a^2b^2 \quad \text{Set } b^2 = c^2 - a^2 \\
\frac{x^2b^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} \quad \text{Divide both sides by } a^2b^2 \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1
\end{aligned}
$$

This equation defines a hyperbola centered at the origin with vertices $(\pm a, 0)$ and co-vertices $(0, \pm b)$.

## STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER $(0,0)$

The standard form of the equation of a hyperbola with center $(0,0)$ and transverse axis on the $x$-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(\pm c, 0)$
- the equations of the asymptotes are $y = \pm \frac{b}{a}x$

See Figure $6.2.5a$.

The standard form of the equation of a hyperbola with center $(0,0)$ and transverse axis on the $y$-axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(\pm b, 0)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y = \pm \frac{a}{b}x$

See Figure $6.2.5b$.

Note that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. When we are given the equation of a hyperbola, we can use this relationship to identify its vertices and foci.

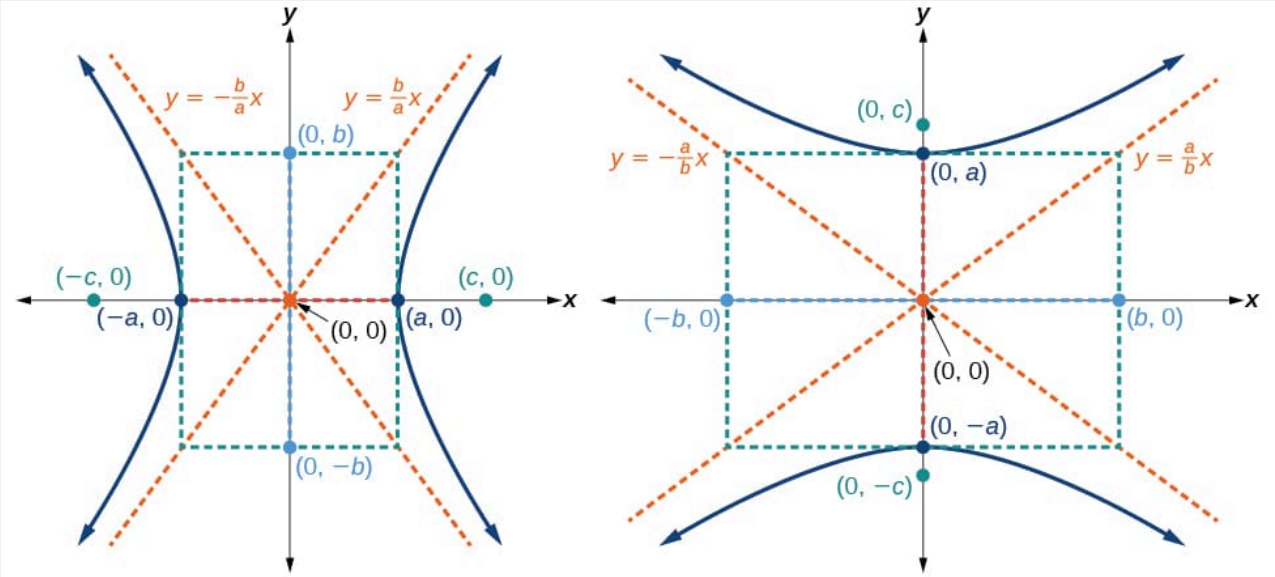



Figure $6.2.5$: (a) Horizontal hyperbola with center $(0, 0)$ (b) Vertical hyperbola with center $(0, 0)$

## How to: Given the equation of a hyperbola in standard form, locate its vertices and foci

1. Determine whether the transverse axis lies on the $x$- or $y$-axis. Notice that $a^2$ is always under the variable with the positive coefficient. So, if you set the other variable equal to zero, you can easily find the intercepts. In the case where the hyperbola is centered at the origin, the intercepts coincide with the vertices.
   - If the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the transverse axis lies on the $x$-axis. The vertices are located at $(\pm a, 0)$, and the foci are located at $(\pm c, 0)$.
   - If the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then the transverse axis lies on the $y$-axis. The vertices are located at $(0, \pm a)$, and the foci are located at $(0, \pm c)$.
2. Solve for $a$ using the equation $a = \sqrt{a^2}$.
3. Solve for $c$ using the equation $c = \sqrt{a^2 + b^2}$.

## Example $6.2.1$: Locating a Hyperbola's Vertices and Foci

Identify the vertices and foci of the hyperbola with equation $\frac{y^2}{49} - \frac{x^2}{32} = 1$.

**Solution**

The equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, so the transverse axis lies on the $y$-axis. The hyperbola is centered at the origin, so the vertices serve as the $y$-intercepts of the graph. To find the vertices, set $x = 0$, and solve for $y$.

$$\begin{aligned} 1 &= \frac{y^2}{49} - \frac{x^2}{32} \\ 1 &= \frac{y^2}{49} - \frac{0^2}{32} \\ 1 &= \frac{y^2}{49} \\ y^2 &= 49 \\ y &= \pm\sqrt{49} \\ &= \pm 7 \end{aligned}$$

The foci are located at $(0, \pm c)$. Solving for $c$,

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{49 + 32} \\ &= \sqrt{81} \\ &= 9 \end{aligned}$$

Therefore, the vertices are located at $(0, \pm 7)$, and the foci are located at $(0, 9)$.

**Exercise 6.2.1**

Identify the vertices and foci of the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{25} = 1$.

*Answer available at the end of this section.*

## Writing Equations of Hyperbolas in Standard Form

Just as with ellipses, writing the equation for a hyperbola in standard form allows us to calculate the key features: its center, vertices, co-vertices, foci, asymptotes, and the lengths and positions of the transverse and conjugate axes. Conversely, an equation for a hyperbola can be found given its key features. We begin by finding standard equations for hyperbolas centered at the origin. Then we will turn our attention to finding standard equations for hyperbolas centered at some point other than the origin.

## Hyperbolas Centered at the Origin

Reviewing the standard forms given for hyperbolas centered at $(0, 0)$,we see that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. Note that this equation can also be rewritten as $b^2 = c^2 - a^2$. This relationship is used to write the equation for a hyperbola when given the coordinates of its foci and vertices.

**How to: Given the vertices and foci of a hyperbola centered at $(0, 0)$, write its equation in standard form**

1. Determine whether the transverse axis lies on the $x$- or $y$-axis.
   - If the given coordinates of the vertices and foci have the form $(\pm a, 0)$ and $(\pm c, 0)$, respectively, then the transverse axis is the $x$-axis. Use the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
   - If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(0, \pm c)$, respectively, then the transverse axis is the $y$-axis. Use the standard form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

2. Find $b^2$ using the equation $b^2 = c^2 - a^2$.
3. Substitute the values for $a^2$ and $b^2$ into the standard form of the equation determined in Step 1.

### Example 6.2.2: Finding the Equation of a Hyperbola Centered at $(0, 0)$ Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices $(\pm 6, 0)$ and foci $(\pm 2\sqrt{10}, 0)$?

**Solution**

The vertices and foci are on the $x$-axis. Thus, the equation for the hyperbola will have the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The vertices are $(\pm 6, 0)$, so $a = 6$ and $a^2 = 36$.

The foci are $(\pm 2\sqrt{10}, 0)$, so $c = 2\sqrt{10}$ and $c^2 = 40$.

Solving for $b^2$, we have

$$\begin{aligned} b^2 &= c^2 - a^2 \\ b^2 &= 40 - 36 \qquad \text{Substitute for } c^2 \text{ and } a^2 \\ b^2 &= 4 \qquad \text{Subtract.} \end{aligned}$$

Finally, we substitute $a^2 = 36$ and $b^2 = 4$ into the standard form of the equation, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The equation of the hyperbola is $\frac{x^2}{36} - \frac{y^2}{4} = 1$, as shown in Figure 6.2.6.

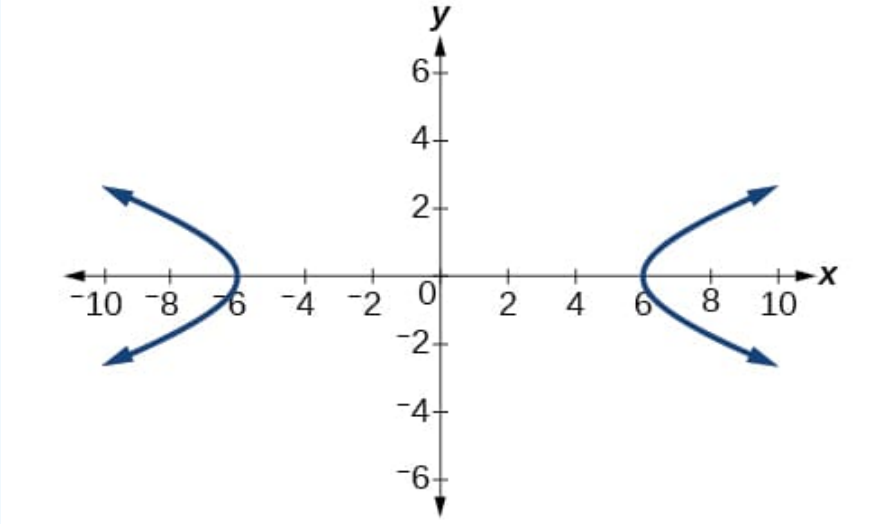



Figure 6.2.6

### Exercise 6.2.2

What is the standard form equation of the hyperbola that has vertices $(0, \pm 2)$ and foci $(0, \pm 2\sqrt{5})$?

*Answer available at the end of this section.*

## Hyperbolas Not Centered at the Origin

Like the graphs for other equations, the graph of a hyperbola can be translated. If a hyperbola is translated $h$ units horizontally and $k$ units vertically, the center of the hyperbola will be $(h, k)$. This translation results in the standard form of the equation we

saw previously, with $x$ replaced by $(x-h)$ and $y$ replaced by $(y-k)$.

## STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER $(H, K)$

The standard form of the equation of a hyperbola with center $(h, k)$ and transverse axis parallel to the $x$-axis is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{6.2.1}$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h \pm c, k)$

The asymptotes of the hyperbola coincide with the diagonals of the central rectangle. The length of the rectangle is $2a$ and its width is $2b$. The slopes of the diagonals are $\pm\frac{b}{a}$,and each diagonal passes through the center $(h, k)$. Using the point-slope formula, it is simple to show that the equations of the asymptotes are $y = \pm\frac{b}{a}(x-h) + k$. See Figure $6.2.7a$.

The standard form of the equation of a hyperbola with center $(h, k)$ and transverse axis parallel to the $y$-axis is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \tag{6.2.2}$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h, k \pm c)$

Using the reasoning above, the equations of the asymptotes are $y = \pm\frac{a}{b}(x-h) + k$. See Figure $6.2.7b$.

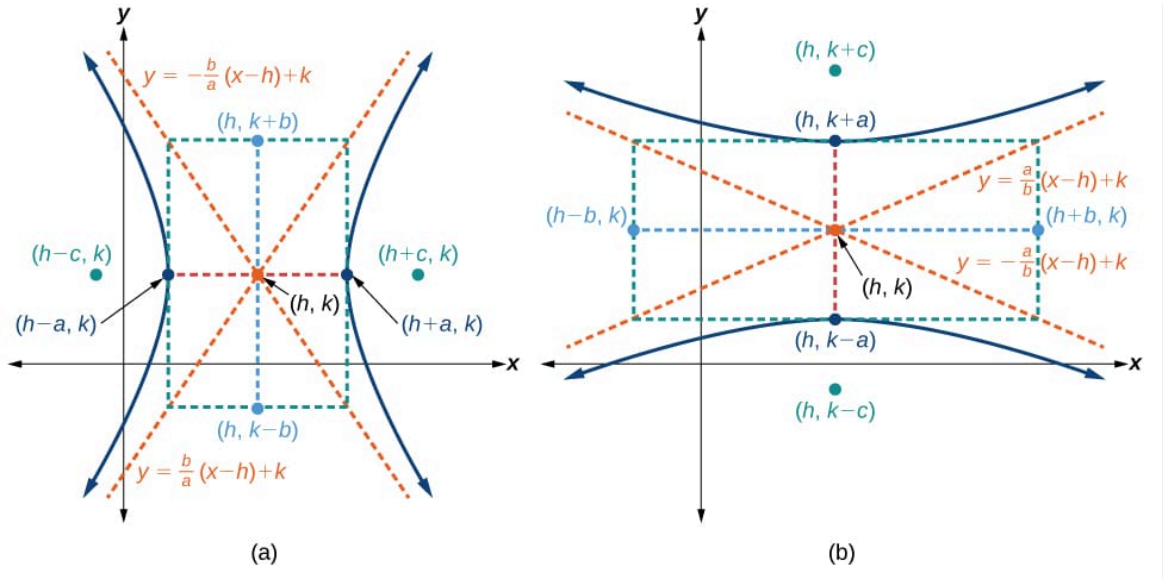



Figure 6.2.7: (a) Horizontal hyperbola with center $(h, k)$ (b) Vertical hyperbola with center $(h, k)$

Like hyperbolas centered at the origin, hyperbolas centered at a point $(h, k)$ have vertices, co-vertices, and foci that are related by the equation $c^2 = a^2 + b^2$. We can use this relationship along with the midpoint and distance formulas to find the standard equation of a hyperbola when the vertices and foci are given.

### How to: Given the vertices and foci of a hyperbola centered at $(h, k)$,write its equation in standard form

1. Determine whether the transverse axis is parallel to the $x$- or $y$-axis.
   - If the $y$-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the $x$-axis. Use the standard form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.
   - If the $x$-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the $y$-axis. Use the standard form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.
2. Identify the center of the hyperbola, $(h, k)$,using the midpoint formula and the given coordinates for the vertices.
3. Find $a^2$ by solving for the length of the transverse axis, $2a$, which is the distance between the given vertices.
4. Find $c^2$ using $h$ and $k$ found in Step 2 along with the given coordinates for the foci.
5. Solve for $b^2$ using the equation $b^2 = c^2 - a^2$.
6. Substitute the values for $h$, $k$, $a^2$, and $b^2$ into the standard form of the equation determined in Step 1.

### Example 6.2.3: Finding the Equation of a Hyperbola Centered at $(h, k)$ Given its Foci and Vertices

What is the standard form equation of the **hyperbola** that has vertices at $(0, -2)$ and $(6, -2)$ and foci at $(-2, -2)$ and $(8, -2)$?

**Solution**

The $y$-coordinates of the vertices and foci are the same, so the transverse axis is parallel to the $x$-axis. Thus, the equation of the hyperbola will have the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

First, we identify the center, $(h, k)$. The center is halfway between the vertices $(0, -2)$ and $(6, -2)$. Applying the midpoint formula, we have

$$(h,k)=(\frac{0+6}{2},\frac{-2+(-2)}{2})=(3,-2)$$

Next, we find $a^2$. The length of the transverse axis, $2a$,is bounded by the vertices. So, we can find $a^2$ by finding the distance between the $x$-coordinates of the vertices.

$$\begin{aligned}2a&=|0-6|\\2a&=6\\a&=3\\a^2&=9\end{aligned}$$

Now we need to find $c^2$. The coordinates of the foci are $(h\pm c,k)$. So $(h-c,k)=(-2,-2)$ and $(h+c,k)=(8,-2)$. We can use the $x$-coordinate from either of these points to solve for $c$. Using the point $(8,-2)$, and substituting $h=3$,

$$\begin{aligned}h+c&=8\\3+c&=8\\c&=5\\c^2&=25\end{aligned}$$

Next, solve for $b^2$ using the equation $b^2=c^2-a^2$:

$$\begin{aligned}b^2&=c^2-a^2\\&=25-9\\&=16\end{aligned}$$

Finally, substitute the values found for $h$, $k$, $a^2$,and $b^2$ into the standard form of the equation.

$$\frac{(x-3)^2}{9}-\frac{(y+2)^2}{16}=1$$

**Exercise 6.2.3**

What is the standard form equation of the hyperbola that has vertices $(1,-2)$ and $(1,8)$ and foci $(1,-10)$ and $(1,16)$?

*Answer available at the end of this section.*

## Graphing Hyperbolas Centered at the Origin

When we have an equation in standard form for a hyperbola centered at the origin, we can interpret its parts to identify the key features of its graph: the center, vertices, co-vertices, asymptotes, foci, and lengths and positions of the transverse and conjugate axes. To graph hyperbolas centered at the origin, we use the standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for horizontal hyperbolas and the standard form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ for vertical hyperbolas.

**How to: Given a standard form equation for a hyperbola centered at $(0,0)$, sketch the graph**

1. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the vertices, co-vertices, and foci; and the equations for the asymptotes.
   - If the equation is in the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then
     - the transverse axis is on the $x$-axis
     - the coordinates of the vertices are \((\pm a,0)\0
     - the coordinates of the co-vertices are $(0,\pm b)$
     - the coordinates of the foci are $(\pm c,0)$

- the equations of the asymptotes are $y = \pm\frac{b}{a}x$

◦ If the equation is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then

- the transverse axis is on the $y$-axis
- the coordinates of the vertices are $(0, \pm a)$
- the coordinates of the co-vertices are $(\pm b, 0)$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y = \pm\frac{a}{b}x$

3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2 + b^2}$.
4. Plot the vertices, co-vertices, foci, and asymptotes in the coordinate plane, and draw a smooth curve to form the hyperbola.

### Example 6.2.4: Graphing a Hyperbola Centered at $(0, 0)$ Given an Equation in Standard Form

Graph the hyperbola given by the equation $\frac{y^2}{64} - \frac{x^2}{36} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

**Solution**

The standard form that applies to the given equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Thus, the transverse axis is on the $y$-axis

The coordinates of the vertices are $(0, \pm a) = (0, \pm\sqrt{64}) = (0, \pm 8)$

The coordinates of the co-vertices are $(\pm b, 0) = (\pm\sqrt{36}, 0) = (\pm 6, 0)$

The coordinates of the foci are $(0, \pm c)$, where $c = \pm\sqrt{a^2 + b^2}$. Solving for $c$, we have

$$c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{64 + 36} = \pm\sqrt{100} = \pm 10$$

Therefore, the coordinates of the foci are $(0, \pm 10)$

The equations of the asymptotes are $y = \pm\frac{a}{b}x = \pm\frac{8}{6}x = \pm\frac{4}{3}x$

Plot and label the vertices and co-vertices, and then sketch the central rectangle. Sides of the rectangle are parallel to the axes and pass through the vertices and co-vertices. Sketch and extend the diagonals of the central rectangle to show the asymptotes. The central rectangle and asymptotes provide the framework needed to sketch an accurate graph of the hyperbola. Label the foci and asymptotes, and draw a smooth curve to form the hyperbola, as shown in Figure 6.2.8.

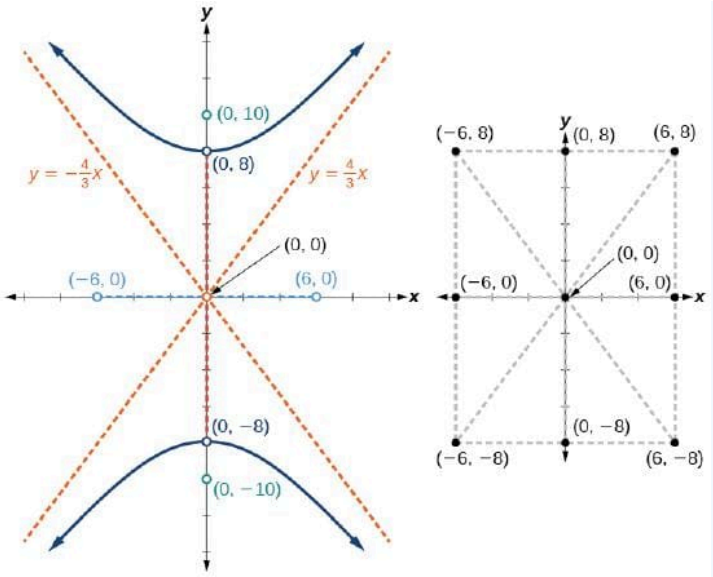



Figure 6.2.8

## Exercise 6.2.4

Graph the hyperbola given by the equation $\frac{x^2}{144} - \frac{y^2}{81} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

*Answer available at the end of this section.*

## Graphing Hyperbolas Not Centered at the Origin

Graphing hyperbolas centered at a point $(h, k)$ other than the origin is similar to graphing ellipses centered at a point other than the origin. We use the standard forms $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ for horizontal hyperbolas, and $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ for vertical hyperbolas. From these standard form equations we can easily calculate and plot key features of the graph: the coordinates of its center, vertices, co-vertices, and foci; the equations of its asymptotes; and the positions of the transverse and conjugate axes.

### How to: Given a general form for a hyperbola centered at $(h, k)$, sketch the graph

1. Convert the general form to that standard form. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the center, vertices, co-vertices, foci; and equations for the asymptotes.
    1. If the equation is in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, then
        - the transverse axis is parallel to the $x$-axis
        - the center is $(h, k)$
        - the coordinates of the vertices are $(h \pm a, k)$
        - the coordinates of the co-vertices are $(h, k \pm b)$
        - the coordinates of the foci are $(h \pm c, k)$
        - the equations of the asymptotes are $y = \pm\frac{b}{a}(x-h) + k$
    2. If the equation is in the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, then
        - the transverse axis is parallel to the $y$-axis
        - the center is $(h, k)$
        - the coordinates of the vertices are $(h, k \pm a)$

- the coordinates of the co-vertices are $(h \pm b, k)$
- the coordinates of the foci are $(h, k \pm c)$
- the equations of the asymptotes are $y = \pm\frac{a}{b}(x-h)+k$

3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2+b^2}$.
4. Plot the center, vertices, co-vertices, foci, and asymptotes in the coordinate plane and draw a smooth curve to form the hyperbola.

## Example 6.2.5: Graphing a Hyperbola Centered at $(h, k)$ Given an Equation in General Form

Graph the hyperbola given by the equation $9x^2 - 4y^2 - 36x - 40y - 388 = 0$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

**Solution**

Start by expressing the equation in standard form. Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$(9x^2 - 36x) - (4y^2 + 40y) = 388$$

Factor the leading coefficient of each expression.

$$9(x^2 - 4x) - 4(y^2 + 10y) = 388$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$9(x^2 - 4x + 4) - 4(y^2 + 10y + 25) = 388 + 36 - 100$$

Rewrite as perfect squares.

$$9(x-2)^2 - 4(y+5)^2 = 324$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-2)^2}{36} - \frac{(y+5)^2}{81} = 1$$

The standard form that applies to the given equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where $a^2 = 36$ and $b^2 = 81$,or $a = 6$ and $b = 9$. Thus, the transverse axis is parallel to the $x$-axis. It follows that:

the center of the ellipse is $(h, k) = (2, -5)$

the coordinates of the vertices are $(h \pm a, k) = (2 \pm 6, -5)$, or $(-4, -5)$ and $(8, -5)$

the coordinates of the co-vertices are $(h, k \pm b) = (2, -5 \pm 9)$, or $(2, -14)$ and $(2, 4)$

the coordinates of the foci are $(h \pm c, k)$, where $c = \pm\sqrt{a^2+b^2}$. Solving for $c$,we have

$$c = \pm\sqrt{36+81} = \pm\sqrt{117} = \pm 3\sqrt{13}$$

Therefore, the coordinates of the foci are $(2 - 3\sqrt{13}, -5)$ and $(2 + 3\sqrt{13}, -5)$.

The equations of the asymptotes are $y = \pm\frac{b}{a}(x-h)+k = \pm\frac{3}{2}(x-2)-5$.

Next, we plot and label the center, vertices, co-vertices, foci, and asymptotes and draw smooth curves to form the hyperbola, as shown in Figure 6.2.10.

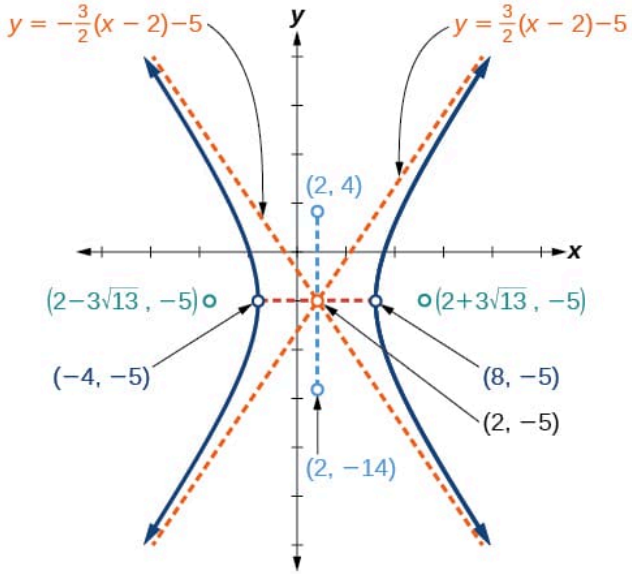



Figure 6.2.10

**Exercise 6.2.5**

Graph the hyperbola given by the standard form of an equation $\frac{(y+4)^2}{100} - \frac{(x-3)^2}{64} = 1$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

*Answer available at the end of this section.*

## Solving Applied Problems Involving Hyperbolas

As we discussed at the beginning of this section, hyperbolas have real-world applications in many fields, such as astronomy, physics, engineering, and architecture. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere and are often touted for their ability to generate power efficiently. Because of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength (Figure 6.2.12). For example, a 500-foot tower can be made of a reinforced concrete shell only 6 or 8 inches wide!



Figure 6.2.12: Cooling towers at the Drax power station in North Yorkshire, United Kingdom (credit: Les Haines, Flickr)

The first hyperbolic towers were designed in 1914 and were 35 meters high. Today, the tallest cooling towers are in France, standing a remarkable 170 meters tall. In Example 6.2.6 we will use the design layout of a cooling tower to find a hyperbolic equation that models its sides.

## Example 6.2.6: Solving Applied Problems Involving Hyperbolas

The design layout of a cooling tower is shown in Figure 6.2.13. The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart.

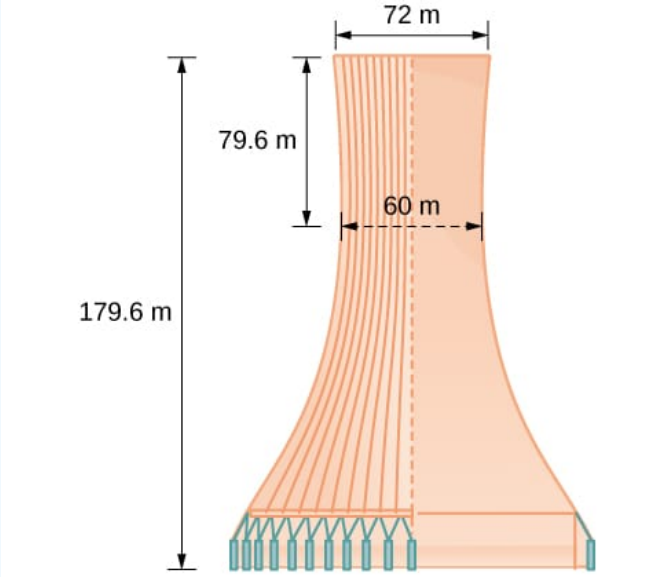



Figure 6.2.13: Project design for a natural draft cooling tower

Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

**Solution**

We are assuming the center of the tower is at the origin, so we can use the standard form of a horizontal hyperbola centered at the origin: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the branches of the hyperbola form the sides of the cooling tower. We must find the values of $a^2$ and $b^2$ to complete the model.

First, we find $a^2$. Recall that the length of the transverse axis of a hyperbola is $2a$. This length is represented by the distance where the sides are closest, which is given as $65.3$ meters. So, $2a = 60$. Therefore, $a = 30$ and $a^2 = 900$.

To solve for $b^2$,we need to substitute for $x$ and $y$ in our equation using a known point. To do this, we can use the dimensions of the tower to find some point $(x, y)$ that lies on the hyperbola. We will use the top right corner of the tower to represent that point. Since the $y$-axis bisects the tower, our $x$-value can be represented by the radius of the top, or $36$ meters. The *y*-value is represented by the distance from the origin to the top, which is given as $79.6$ meters. Therefore,

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 && \text{Standard form of horizontal hyperbola.} \\ b^2 &= \frac{y^2}{\frac{x^2}{a^2} - 1} && \text{Isolate } b^2 \\ &= \frac{(79.6)^2}{\frac{(36)^2}{900} - 1} && \text{Substitute for } a^2,\ x,\ \text{and } y \\ &\approx 14400.3636 && \text{Round to four decimal places} \end{aligned}$$

The sides of the tower can be modeled by the hyperbolic equation

$$\frac{x^2}{900} - \frac{y^2}{14400.3636} = 1, \text{or } \frac{x^2}{30^2} - \frac{y^2}{120.0015^2} = 1$$

## Exercise 6.2.6

A design for a cooling tower project is shown in Figure 6.2.14. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

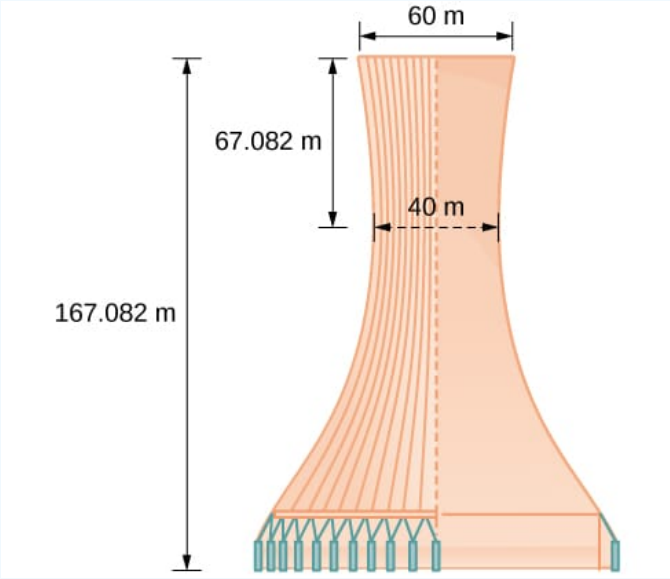



Figure 6.2.14

*Answer available at the end of this section.*

## Media

Access these online resources for additional instruction and practice with hyperbolas.

- Conic Sections: The Hyperbola Part 1 of 2
- Conic Sections: The Hyperbola Part 2 of 2
- Graph a Hyperbola with Center at Origin
- Graph a Hyperbola with Center not at Origin

# Key Equations

| | |
|---|---|
| Hyperbola, center at origin, transverse axis on *x*-axis | $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ |
| Hyperbola, center at origin, transverse axis on *y*-axis | $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ |
| Hyperbola, center at $(h,k)$,transverse axis parallel to *x*-axis | $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ |
| Hyperbola, center at $(h,k)$,transverse axis parallel to *y*-axis | $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ |

# Key Concepts

- A hyperbola is the set of all points $(x,y)$ in a plane such that the difference of the distances between $(x,y)$ and the foci is a positive constant.
- The standard form of a hyperbola can be used to locate its vertices and foci. See Example $6.2.1$.
- When given the coordinates of the foci and vertices of a hyperbola, we can write the equation of the hyperbola in standard form. See Example $6.2.2$ and Example $6.2.3$.
- When given an equation for a hyperbola, we can identify its vertices, co-vertices, foci, asymptotes, and lengths and positions of the transverse and conjugate axes in order to graph the hyperbola. See Example $6.2.4$ and Example $6.2.5$.
- Real-world situations can be modeled using the standard equations of hyperbolas. For instance, given the dimensions of a natural draft cooling tower, we can find a hyperbolic equation that models its sides. See Example $6.2.6$.

## Answers

**Exercise** $6.2.1$

Vertices: $(\pm 3,0)$; Foci: $(\pm\sqrt{34},0)$

**Exercise** $6.2.2$

$\frac{y^2}{4}-\frac{x^2}{16}=1$

**Exercise** $6.2.3$

$\frac{(y-3)^2}{25}+\frac{(x-1)^2}{144}=1$

**Exercise** $6.2.4$

vertices: $(\pm 12,0)$; co-vertices: $(0,\pm 9)$; foci: $(\pm 15,0)$; asymptotes: $y=\pm\frac{3}{4}x$;

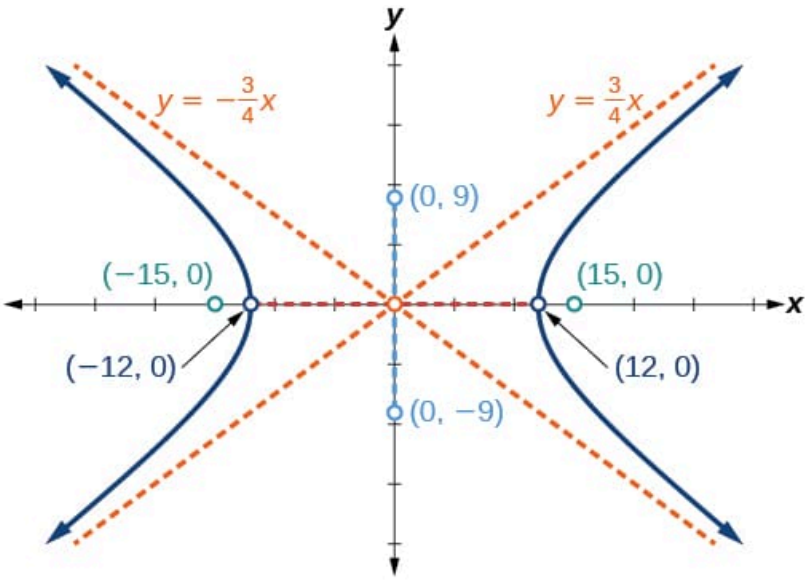



Figure 6.2.9

**Exercise** 6.2.5

center: $(3, -4)$; vertices: $(3, -14)$ and $(3, 6)$; co-vertices: $(-5, -4)$; and $(11, -4)$; foci: $(3, -4 - 2\sqrt{41})$ and $(3, -4 + 2\sqrt{41})$; asymptotes: $y = \pm\frac{5}{4}(x - 3) - 4$

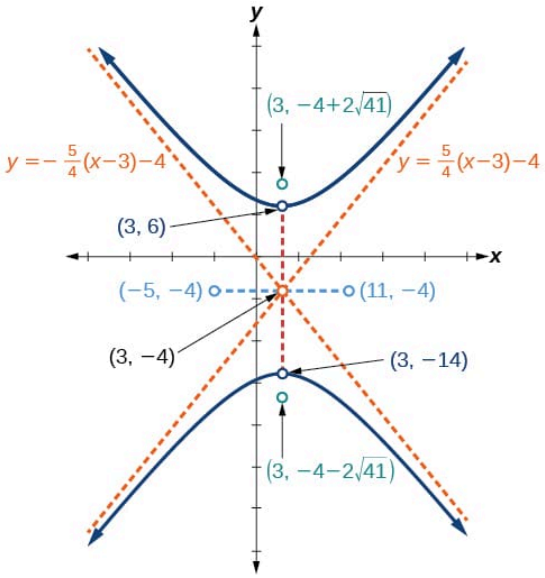



Figure 6.2.11

**Exercise** 6.2.6

The sides of the tower can be modeled by the hyperbolic equation. $\frac{x^2}{400} - \frac{y^2}{3600} = 1$ or $\frac{x^2}{20^2} - \frac{y^2}{60^2} = 1$.

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# 6.3: The Parabola

**Learning Objectives**

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Solve applied problems involving parabolas.

Did you know that the Olympic torch is lit several months before the start of the games? The ceremonial method for lighting the flame is the same as in ancient times. The ceremony takes place at the Temple of Hera in Olympia, Greece, and is rooted in Greek mythology, paying tribute to Prometheus, who stole fire from Zeus to give to all humans. One of eleven acting priestesses places the torch at the focus of a parabolic mirror (Figure 6.3.1), which focuses light rays from the sun to ignite the flame.



Figure 6.3.1: The Olympic torch concludes its journey around the world when it is used to light the Olympic cauldron during the opening ceremony. (credit: Ken Hackman, U.S. Air Force)

Parabolic mirrors (or reflectors) are able to capture energy and focus it to a single point. The advantages of this property are evidenced by the vast list of parabolic objects we use every day: satellite dishes, suspension bridges, telescopes, microphones, spotlights, and car headlights, to name a few. Parabolic reflectors are also used in alternative energy devices, such as solar cookers and water heaters, because they are inexpensive to manufacture and need little maintenance. In this section we will explore the parabola and its uses, including low-cost, energy-efficient solar designs.

## Graphing Parabolas with Vertices at the Origin

Previously, we saw that an ellipse is formed when a plane cuts through a right circular cone. If the plane is parallel to the edge of the cone, an unbounded curve is formed. This curve is a *parabola* (Figure 6.3.2).

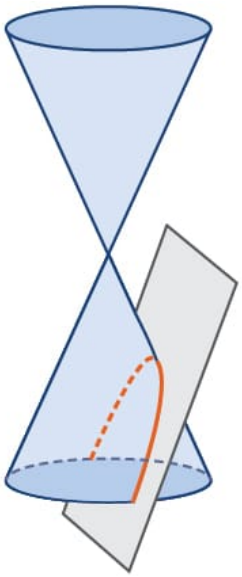


Figure 6.3.2: Parabola

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points $(x, y)$ in a plane that are the same distance from a fixed line, called the *directrix*, and a fixed point (the *focus*) not on the directrix.

Previously, we learned about a parabola's vertex and axis of symmetry. Now we extend the discussion to include other key features of the parabola (Figure 6.3.3). Notice that the axis of symmetry passes through the focus and vertex and is perpendicular to the directrix. The vertex is the midpoint between the directrix and the focus. The line segment that passes through the focus and is parallel to the directrix is called the *latus rectum*. The endpoints of the latus rectum lie on the curve. By definition, the distanced d from the focus to any point $P$ on the parabola is equal to the distance from $P$ to the directrix.

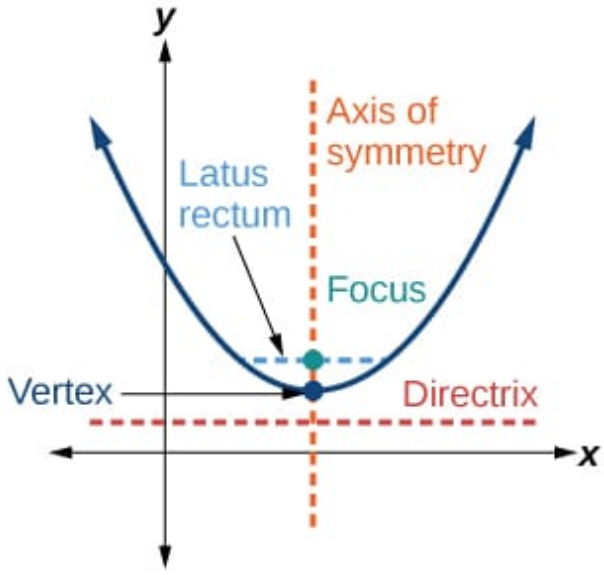



Figure 6.3.3: Key features of the parabola

To work with parabolas in the *coordinate plane*, we consider two cases: those with a vertex at the origin and those with a *vertex* at a point other than the origin. We begin with the former.

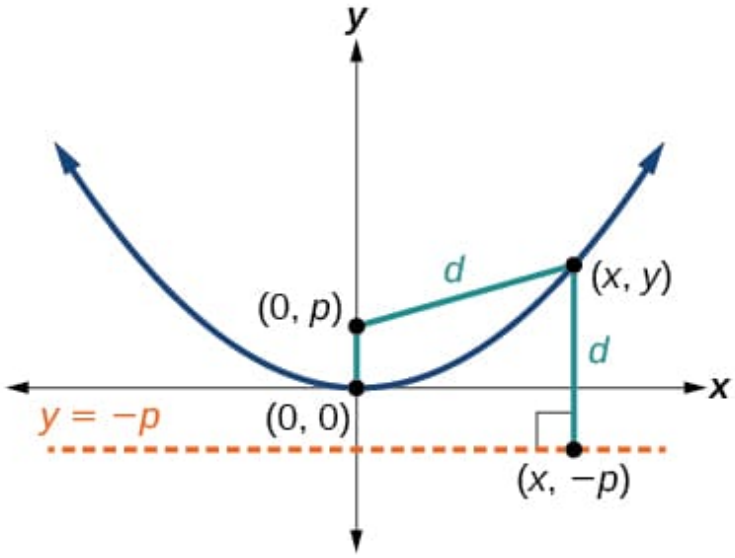



Figure 6.3.4

Let $(x, y)$ be a point on the parabola with vertex $(0, 0)$, focus $(0, p)$,and directrix $y = -p$ as shown in Figure 6.3.4. The distanced d from point $(x, y)$ to point $(x, -p)$ on the directrix is the difference of the *y*-values: $d = y + p$. The distance from the focus $(0, p)$ to the point $(x, y)$ is also equal to $d$ and can be expressed using the distance formula.

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-p)^2} \\ &= \sqrt{x^2 + (y-p)^2} \end{aligned}$$

Set the two expressions for $d$ equal to each other and solve for $y$ to derive the equation of the parabola. We do this because the distance from $(x, y)$ to $(0, p)$ equals the distance from $(x, y)$ to $(x, -p)$.

$$\sqrt{x^2 + (y-p)^2} = y + p \tag{6.3.1}$$

We then square both sides of the equation, expand the squared terms, and simplify by combining like terms.

$$\begin{aligned} x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 - 2py &= 2py \\ x^2 &= 4py \end{aligned}$$

The equations of parabolas with vertex $(0, 0)$ are $y^2 = 4px$ when the *x*-axis is the axis of symmetry and $x^2 = 4py$ when the *y*-axis is the axis of symmetry. These standard forms are given below, along with their general graphs and key features.

## STANDARD FORMS OF PARABOLAS WITH VERTEX $(0, 0)$

Table 6.3.1 and Figure 6.3.5 summarize the standard features of parabolas with a vertex at the origin.

*Table* 6.3.1

| Axis of Symmetry | Equation | Focus | Directrix | Endpoints of Latus Rectum |
|---|---|---|---|---|
| *x*-axis | $y^2 = 4px$ | $(p, 0)$ | $x = -p$ | $(p, \pm 2p)$ |
| *y*-axis | $x^2 = 4py$ | $(0, p)$ | $y = -p$ | $(\pm 2p, p)$ |

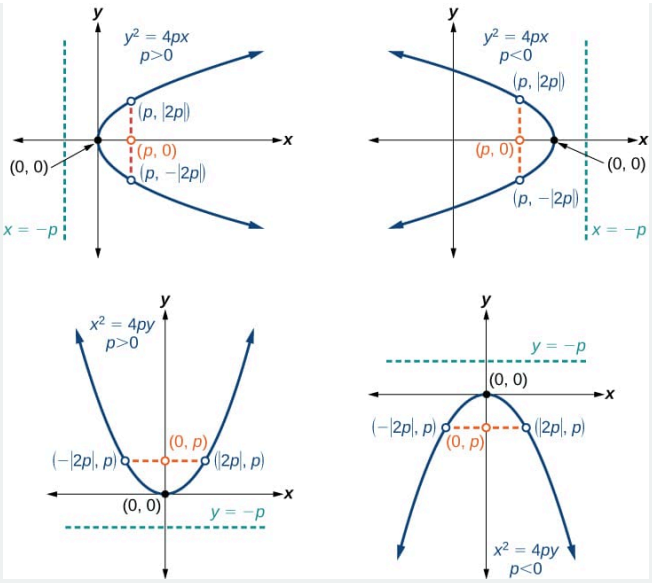



Figure $6.3.5$: (a) When $p > 0$ and the axis of symmetry is the x-axis, the parabola opens right. (b) When $p < 0$ and the axis of symmetry is the x-axis, the parabola opens left. (c) When $p < 0$ and the axis of symmetry is the y-axis, the parabola opens up. (d) When $p < 0$ and the axis of symmetry is the y-axis, the parabola opens down.

The key features of a parabola are its vertex, axis of symmetry, focus, directrix, and *latus rectum* (Figure $6.3.5$). When given a standard equation for a parabola centered at the origin, we can easily identify the key features to graph the parabola. A line is said to be tangent to a curve if it intersects the curve at exactly one point. If we sketch lines tangent to the parabola at the endpoints of the *latus rectum*, these lines intersect on the axis of symmetry, as shown in Figure $6.3.6$.

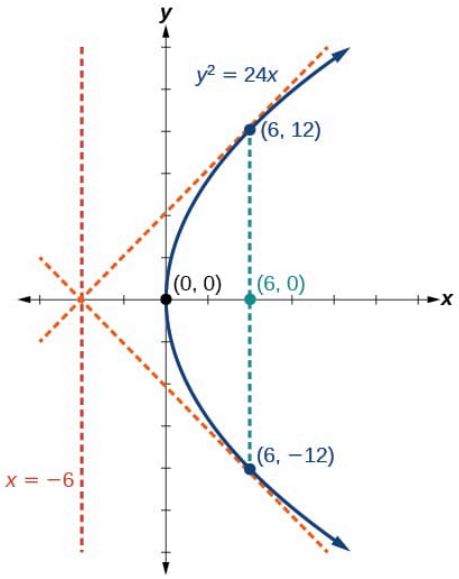



Figure $6.3.6$

## How to: Given a standard form equation for a parabola centered at $(0, 0)$, sketch the graph

1. Determine which of the standard forms applies to the given equation: $y^2 = 4px$ or $x^2 = 4py$.
2. Use the standard form identified in Step 1 to determine the axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   - If the equation is in the form $y^2 = 4px$, then
     - the axis of symmetry is the $x$-axis, $y = 0$

- set $4p$ equal to the coefficient of $x$ in the given equation to solve for $p$. If $p > 0$, the parabola opens right. If $p < 0$ , the parabola opens left.
- use $p$ to find the coordinates of the focus, $(p, 0)$
- use $p$ to find the equation of the directrix, $x = -p$
- use $p$ to find the endpoints of the latus rectum, $(p, \pm 2p)$. Alternately, substitute $x = p$ into the original equation.

◦ If the equation is in the form $x^2 = 4py$,then

- the axis of symmetry is the $y$-axis, $x = 0$
- set $4p$ equal to the coefficient of $y$ in the given equation to solve for $p$. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down.
- use $p$ to find the coordinates of the focus, $(0, p)$
- use $p$ to find equation of the directrix, $y = -p$
- use $p$ to find the endpoints of the latus rectum, $(\pm 2p, p)$

3. Plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

## -axis as the Axis of Symmetry

Graph $y^2 = 24x$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

**Solution**

The standard form that applies to the given equation is $y^2 = 4px$. Thus, the axis of symmetry is the *x*-axis. It follows that:

- $24 = 4p$, so $p = 6$. Since $p > 0$, the parabola opens right
- the coordinates of the focus are $(p, 0) = (6, 0)$
- the equation of the directrix is $x = -p = -6$
- the endpoints of the latus rectum have the same *x*-coordinate at the focus. To find the endpoints, substitute $x = 6$ into the original equation: $(6, \pm 12)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 6.3.7).

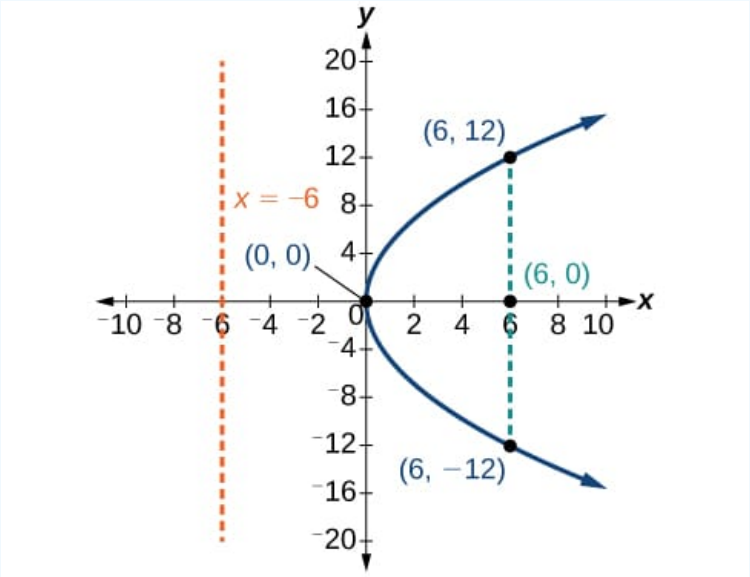



Figure 6.3.7

## Exercise 6.3.1

Graph $y^2 = -16x$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

*Answer available at the end of this section.*

## -axis as the Axis of Symmetry

Graph $x^2 = -6y$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

**Solution**

The standard form that applies to the given equation is $x^2 = 4py$. Thus, the axis of symmetry is the $y$-axis. It follows that:

- $-6 = 4p$,so $p = -\frac{3}{2}$. Since $p < 0$, the parabola opens down.
- the coordinates of the focus are $(0, p) = (0, -\frac{3}{2})$
- the equation of the directrix is $y = -p = \frac{3}{2}$
- the endpoints of the latus rectum can be found by substituting $y = \frac{3}{2}$ into the original equation, $(\pm 3, -\frac{3}{2})$

Next we plot the focus, directrix, and *latus rectum*, and draw a smooth curve to form the parabola.

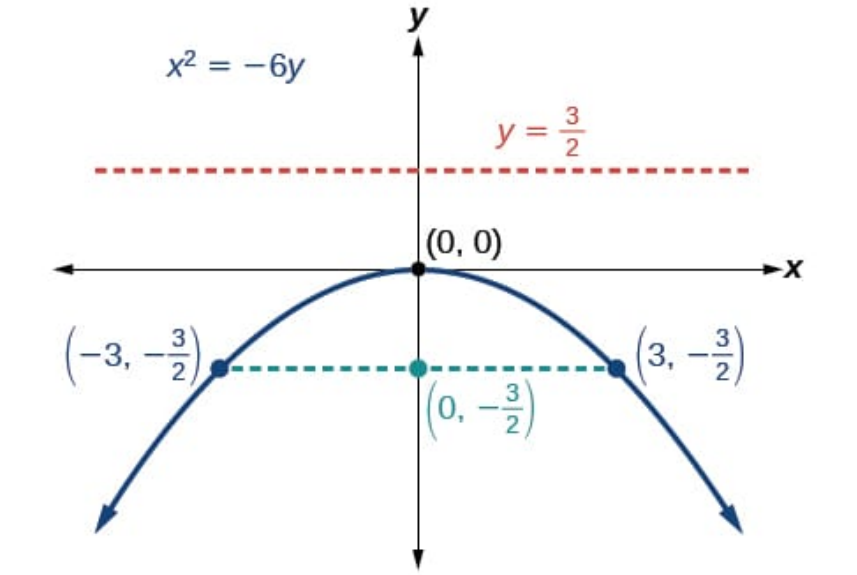



Figure 6.3.9

## Exercise 6.3.2

Graph $x^2 = 8y$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

*Answer available at the end of this section.*

# Writing Equations of Parabolas in Standard Form

In the previous examples, we used the standard form equation of a parabola to calculate the locations of its key features. We can also use the calculations in reverse to write an equation for a parabola when given its key features.

### How to: Given its focus and directrix, write the equation for a parabola in standard form

1. Determine whether the axis of symmetry is the $x$- or $y$-axis.
   i. If the given coordinates of the focus have the form $(p, 0)$, then the axis of symmetry is the $x$-axis. Use the standard form $y^2 = 4px$.

ii. If the given coordinates of the focus have the form $(0, p)$, then the axis of symmetry is the $y$-axis. Use the standard form $x^2 = 4py$.
2. Multiply $4p$.
3. Substitute the value from Step 2 into the equation determined in Step 1.

### Example 6.3.3: Writing the Equation of a Parabola in Standard Form Given its Focus and Directrix

What is the equation for the parabola with focus $(-\frac{1}{2}, 0)$ and directrix $x = \frac{1}{2}$?

**Solution**

The focus has the form $(p, 0)$, so the equation will have the form $y^2 = 4px$.

- Multiplying $4p$, we have $4p = 4(-\frac{1}{2}) = -2$.
- Substituting for $4p$, we have $y^2 = 4px = -2x$.=

Therefore, the equation for the parabola is $y^2 = -2x$.

### Exercise 6.3.3

What is the equation for the parabola with focus $\left(0, \frac{7}{2}\right)$ and directrix $y = -\frac{7}{2}$?

*Answer available at the end of this section.*

## Graphing Parabolas with Vertices Not at the Origin

Like other graphs we've worked with, the graph of a parabola can be translated. If a parabola is translated $h$ units horizontally and $k$ units vertically, the vertex will be $(h, k)$. This translation results in the standard form of the equation we saw previously with $x$ replaced by $(x - h)$ and $y$ replaced by $(y - k)$.

To graph parabolas with a vertex $(h, k)$ other than the origin, we use the standard form $(y - k)^2 = 4p(x - h)$ for parabolas that have an axis of symmetry parallel to the $x$-axis, and $(x - h)^2 = 4p(y - k)$ for parabolas that have an axis of symmetry parallel to the $y$ -axis. These standard forms are given below, along with their general graphs and key features.

### STANDARD FORMS OF PARABOLAS WITH VERTEX $(H, K)$

Table 6.3.2 and Figure 6.3.11 summarize the standard features of parabolas with a vertex at a point $(h, k)$.

*Table* 6.3.2

| Axis of Symmetry | Equation | Focus | Directrix | Endpoints of Latus Rectum |
|---|---|---|---|---|
| $y = k$ | $(y - k)^2 = 4p(x - h)$ | $(h + p, k)$ | $x = h - p$ | $(h + p, k \pm 2p)$ |
| $x = h$ | $(x - h)^2 = 4p(y - k)$ | $(h, k + p)$ | $y = k - p$ | $(h \pm 2p, k + p)$ |

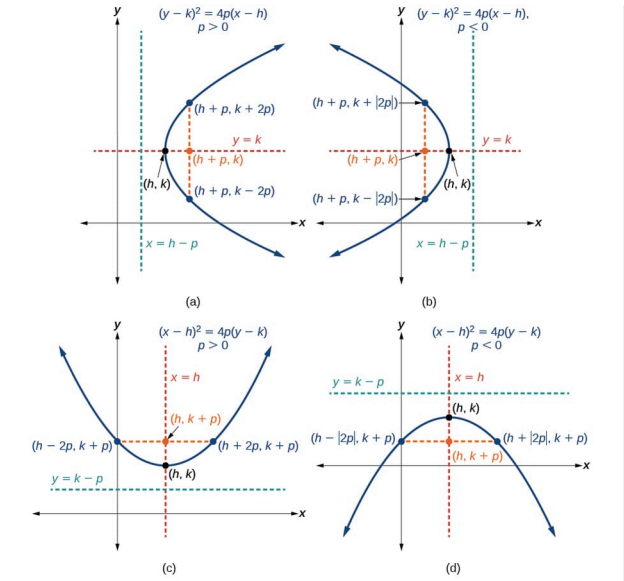



Figure 6.3.11: (a) When $p > 0$, the parabola opens right. (b) When $p < 0$, the parabola opens left. (c) When $p > 0$,the parabola opens up. (d) When $p < 0$, the parabola opens down.

## How to: Given a standard form equation for a parabola centered at $(h, k)$, sketch the graph

1. Determine which of the standard forms applies to the given equation: $(y-k)^2 = 4p(x-h)$ or $(x-h)^2 = 4p(y-k)$.
2. Use the standard form identified in Step 1 to determine the vertex, axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   - If the equation is in the form $(y-k)^2 = 4p(x-h)$,then:
     - use the given equation to identify $h$ and $k$ for the vertex, $(h, k)$
     - use the value of $k$ to determine the axis of symmetry, $y = k$
     - set $4p$ equal to the coefficient of $(x-h)$ in the given equation to solve for $p$. If $p > 0$,the parabola opens right. If $p < 0$, the parabola opens left.
     - use $h$, $k$, and $p$ to find the coordinates of the focus, $(h+p, k)$
     - use $h$ andp p to find the equation of the directrix, $x = h - p$
     - use $h$, $k$, and $p$ to find the endpoints of the latus rectum, $(h+p, k \pm 2p)$
   - If the equation is in the form $(x-h)^2 = 4p(y-k)$,then:
     - use the given equation to identify $h$ and $k$ for the vertex, $(h, k)$
     - use the value of $h$ to determine the axis of symmetry, $x = h$
     - set $4p$ equal to the coefficient of $(y-k)$ in the given equation to solve for $p$. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down.
     - use $h$, $k$, and $p$ to find the coordinates of the focus, $(h, k+p)$
     - use $k$ and $p$ to find the equation of the directrix, $y = k - p$
     - use $h$, $k$, and $p$ to find the endpoints of the latus rectum, $(h \pm 2p, k+p)$
3. Plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

## Example 6.3.4: Graphing a Parabola with Vertex $(h, k)$ and Axis of Symmetry Parallel to the $x$-axis

Graph $(y-1)^2 = -16(x+3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

**Solution**

The standard form that applies to the given equation is $(y-k)^2 = 4p(x-h)$. Thus, the axis of symmetry is parallel to the $x$ -axis. It follows that:

- the vertex is $(h, k) = (-3, 1)$
- the axis of symmetry is $y = k = 1$

- $-16 = 4p$,so $p = -4$. Since $p < 0$, the parabola opens left.
- the coordinates of the focus are $(h + p, k) = (-3 + (-4), 1) = (-7, 1)$
- the equation of the directrix is $x = h - p = -3 - (-4) = 1$
- the endpoints of the latus rectum are $(h + p, k \pm 2p) = (-3 + (-4), 1 \pm 2(-4))$, or $(-7, -7)$ and $(-7, 9)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 6.3.10).

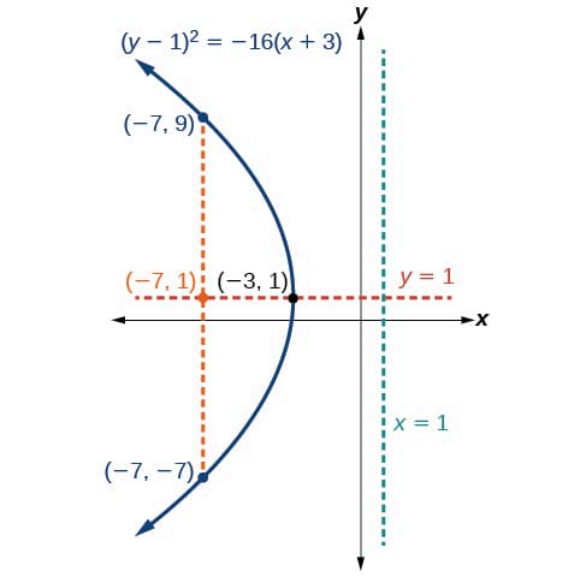



Figure 6.3.12

## Exercise 6.3.4

Graph $(y + 1)^2 = 4(x - 8)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

*Answer available at the end of this section.*

## Example 6.3.5: Graphing a Parabola from an Equation Given in General Form

Graph $x^2 - 8x - 28y - 208 = 0$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

**Solution**

Start by writing the equation of the parabola in standard form. The standard form that applies to the given equation is $(x - h)^2 = 4p(y - k)$. Thus, the axis of symmetry is parallel to the $y$-axis. To express the equation of the parabola in this form, we begin by isolating the terms that contain the variable $x$ in order to complete the square.

$$\begin{aligned} x^2 - 8x - 28y - 208 &= 0 \\ x^2 - 8x &= 28y + 208 \\ x^2 - 8x + 16 &= 28y + 208 + 16 \\ (x - 4)^2 &= 28y + 224 \\ (x - 4)^2 &= 28(y + 8) \\ (x - 4)^2 &= 4 \cdot 7 \cdot (y + 8) \end{aligned}$$

It follows that:

- the vertex is $(h, k) = (4, -8)$
- the axis of symmetry is $x = h = 4$
- since $p = 7$, $p > 0$ and so the parabola opens up
- the coordinates of the focus are $(h, k + p) = (4, -8 + 7) = (4, -1)$
- the equation of the directrix is $y = k - p = -8 - 7 = -15$
- the endpoints of the latus rectum are $(h \pm 2p, k + p) = (4 \pm 2(7), -8 + 7)$, or $(-10, -1)$ and $(18, -1)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 6.3.14).

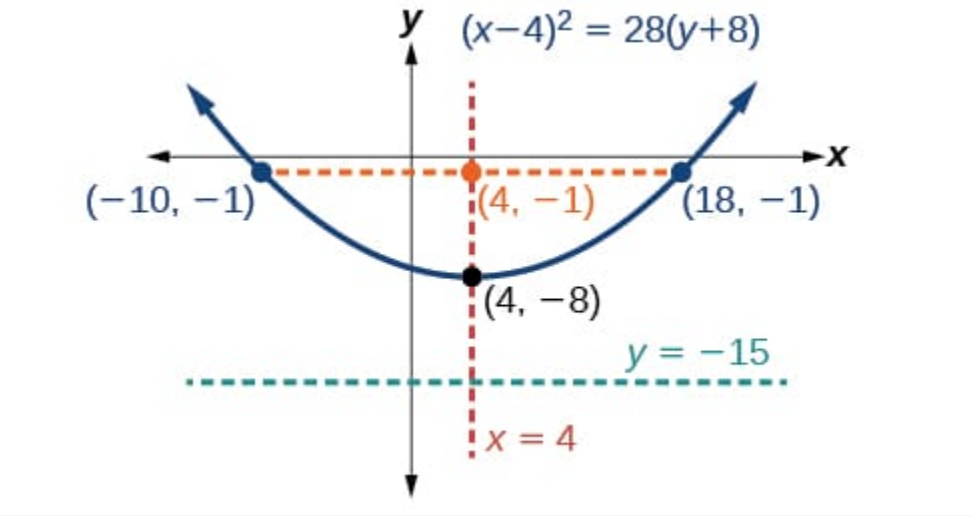



Figure 6.3.14

**Exercise 6.3.5**

Graph $(x + 2)^2 = -20(y - 3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

*Answer available at the end of this section.*

## Solving Applied Problems Involving Parabolas

As we mentioned at the beginning of the section, parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones, and radar equipment. Parabolic mirrors, such as the one used to light the Olympic torch, have a very unique reflecting property. When rays of light parallel to the parabola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus (Figure 6.3.16). This is why the Olympic torch is ignited when it is held at the focus of the parabolic mirror.

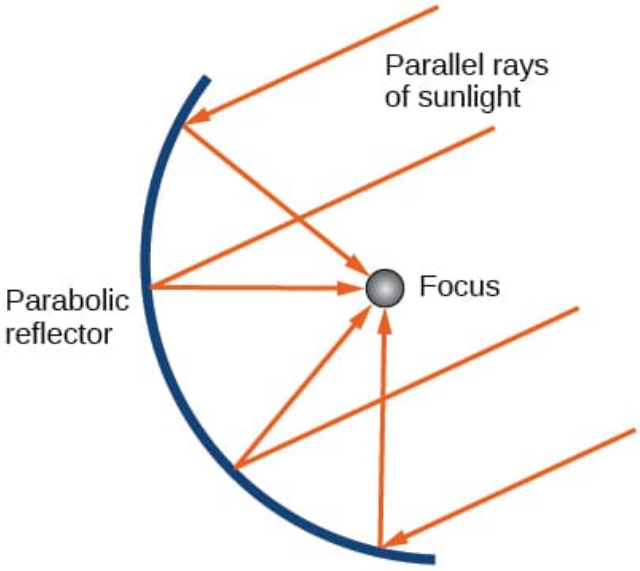



Figure 6.3.16: Reflecting property of parabolas

Parabolic mirrors have the ability to focus the sun's energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus, parabolic mirrors are featured in many low-cost, energy efficient solar products, such as solar cookers, solar heaters, and even travel-sized fire starters.

## Example 6.3.6: Solving Applied Problems Involving Parabolas

A cross-section of a design for a travel-sized solar fire starter is shown in Figure 6.3.17. The sun's rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds.

a. Find the equation of the parabola that models the fire starter. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane.
b. Use the equation found in part (a) to find the depth of the fire starter.

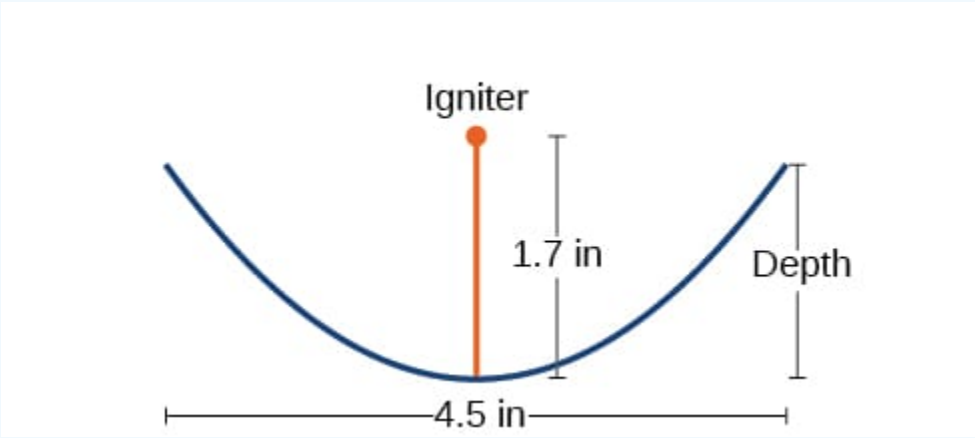



Figure 6.3.17 Cross-section of a travel-sized solar fire starter

**Solution**

a. The vertex of the dish is the origin of the coordinate plane, so the parabola will take the standard form $x^2 = 4py$,where $p > 0$. The igniter, which is the focus, is $1.7$ inches above the vertex of the dish. Thus we have $p = 1.7$.

$$
\begin{aligned}
x^2 &= 4py && \text{Standard form of upward-facing parabola with vertex } (0,0)\\
x^2 &= 4(1.7)y && \text{Substitute } 1.7 \text{ for } p\\
x^2 &= 6.8y && \text{Multiply.}
\end{aligned}
$$

b. The dish extends $\frac{4.5}{2} = 2.25$ inches on either side of the origin. We can substitute $2.25$ for $x$ in the equation from part (a) to find the depth of the dish.

$$
\begin{aligned}
x^2 &= 6.8y && \text{Equation found in part } (a)\\
(2.25)^2 &= 6.8y && \text{Substitute } 2.25 \text{ for } x\\
y &\approx 0.74 && \text{Solve for } y
\end{aligned}
$$

The dish is about $0.74$ inches deep.

### Exercise 6.3.6

Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of $1600$ mm. The sun's rays reflect off the parabolic mirror toward the "cooker," which is placed $320$ mm from the base.

a. Find an equation that models a cross-section of the solar cooker. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane, and that the parabola opens to the right (i.e., has the *x*-axis as its axis of symmetry).
b. Use the equation found in part (a) to find the depth of the cooker.

*Answer a available at the end of this section.*

## Key Equations

| | |
|---|---|
| Parabola, vertex at origin, axis of symmetry on *x*-axis | $y^2 = 4px$ |
| Parabola, vertex at origin, axis of symmetry on *y*-axis | $x^2 = 4py$ |
| Parabola, vertex at $(h,k)$,axis of symmetry on *x*-axis | $(y-k)^2 = 4p(x-h)$ |
| Parabola, vertex at $(h,k)$,axis of symmetry on *y*-axis | $(x-h)^2 = 4p(y-k)$ |

## Key Concepts

- A parabola is the set of all points $(x,y)$ in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix.
- The standard form of a parabola with vertex $(0,0)$ and the *x*-axis as its axis of symmetry can be used to graph the parabola. If $p>0$, the parabola opens right. If $p<0$, the parabola opens left. See Example $6.3.1$.
- The standard form of a parabola with vertex $(0,0)$ and the *y*-axis as its axis of symmetry can be used to graph the parabola. If $p>0$, the parabola opens up. If $p<0$, the parabola opens down. See Example $6.3.2$.
- When given the focus and directrix of a parabola, we can write its equation in standard form. See Example $6.3.3$.
- The standard form of a parabola with vertex $(h,k)$ and axis of symmetry parallel to the $x$-axis can be used to graph the parabola. If $p>0$, the parabola opens right. If $p<0$, the parabola opens left. See Example $6.3.4$.
- The standard form of a parabola with vertex $(h,k)$ and axis of symmetry parallel to the $y$-axis can be used to graph the parabola. If $p>0$, the parabola opens up. If $p<0$, the parabola opens down. See Example $6.3.5$.
- Real-world situations can be modeled using the standard equations of parabolas. For instance, given the diameter and focus of a cross-section of a parabolic reflector, we can find an equation that models its sides. See Example $6.3.6$.

### Answers

**Exercise** $6.3.1$

- Focus: $(-4,0)$
- Directrix: $x=4$

- Endpoints of the latus rectum: $(-4, \pm 8)$

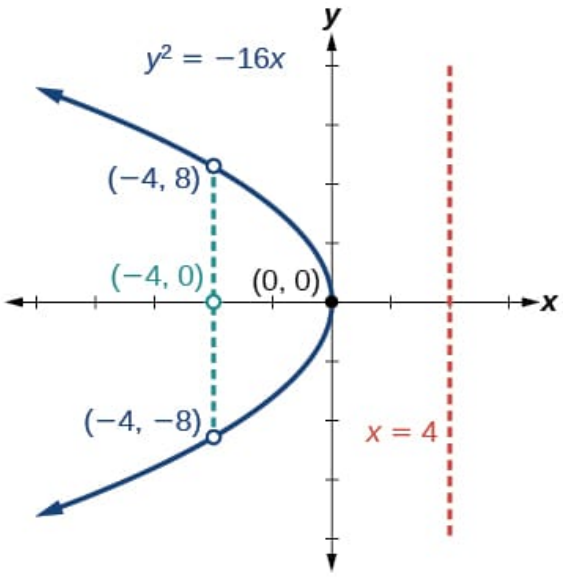



Figure 6.3.8

**Exercise** 6.3.2

- Focus: $(0, 2)$
- Directrix: $y = -2$
- Endpoints of the latus rectum: $(\pm 4, 2)$.

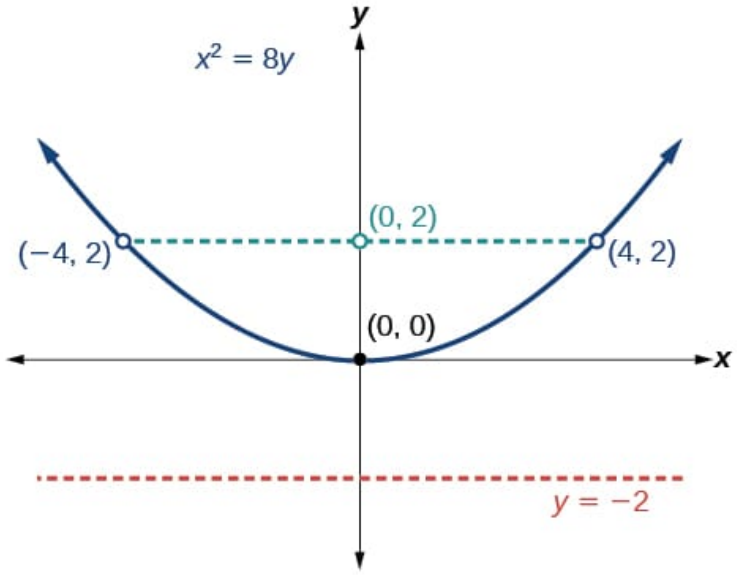



Figure 6.3.10

**Exercise** 6.3.3

$x^2 = 14y$.

**Exercise** 6.3.4

- Vertex: $(8, -1)$
- Axis of symmetry: $y = -1$
- Focus: $(9, -1)$
- Directrix: $x = 7$
- Endpoints of the *latus rectum*: $(9, -3)$ and $(9, 1)$.

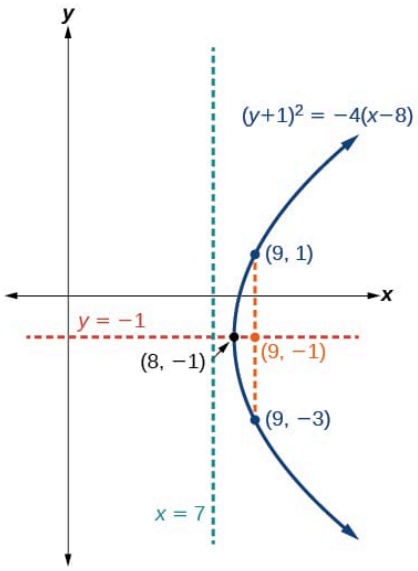



Figure 6.3.13

**Exercise** 6.3.5

- Vertex: $(-2, 3)$
- Axis of symmetry: $x = -2$
- Focus: $(-2, -2)$
- Directrix: $y = 8$
- Endpoints of the *latus rectum*: $(-12, -2)$ and $(8, -2)$.

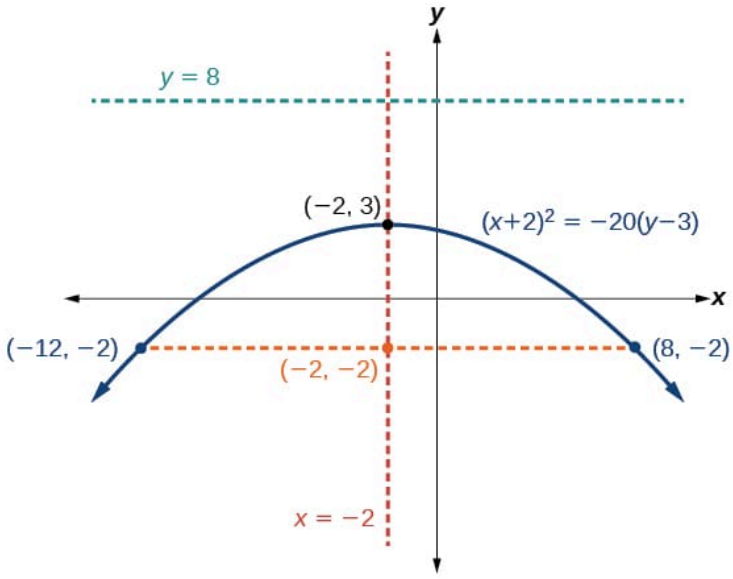



Figure 6.3.15

**Exercise** 6.3.6

$y^2 = 1280x$

The depth of the cooker is $500$ mm

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# 6.4: Rotation of Axes

## Learning Objectives

- Identify nondegenerate conic sections given their general form equations.
- Use rotation of axes formulas.
- Write equations of rotated conics in standard form.
- Identify conics without rotating axes.

As we have seen, conic sections are formed when a plane intersects two right circular cones aligned tip to tip and extending infinitely far in opposite directions, which we also call a *cone*. The way in which we slice the cone will determine the type of conic section formed at the intersection. A circle is formed by slicing a cone with a plane perpendicular to the axis of symmetry of the cone. An ellipse is formed by slicing a single cone with a slanted plane not perpendicular to the axis of symmetry. A parabola is formed by slicing the plane through the top or bottom of the double-cone, whereas a hyperbola is formed when the plane slices both the top and bottom of the cone (Figure 6.4.1).

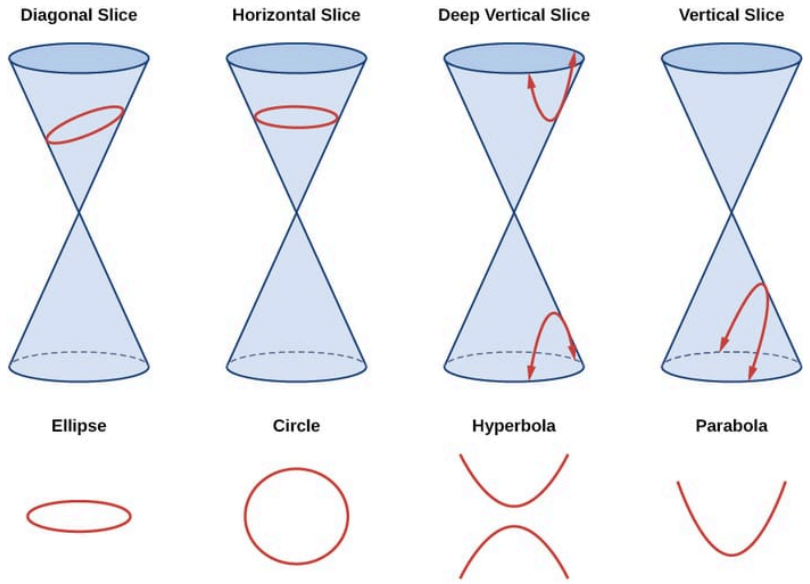



Figure 6.4.1: The nondegenerate conic sections

Ellipses, circles, hyperbolas, and parabolas are sometimes called the nondegenerate conic sections, in contrast to the degenerate conic sections, which are shown in Figure 6.4.2. A degenerate conic results when a plane intersects the double cone and passes through the apex. Depending on the angle of the plane, three types of degenerate conic sections are possible: a point, a line, or two intersecting lines.

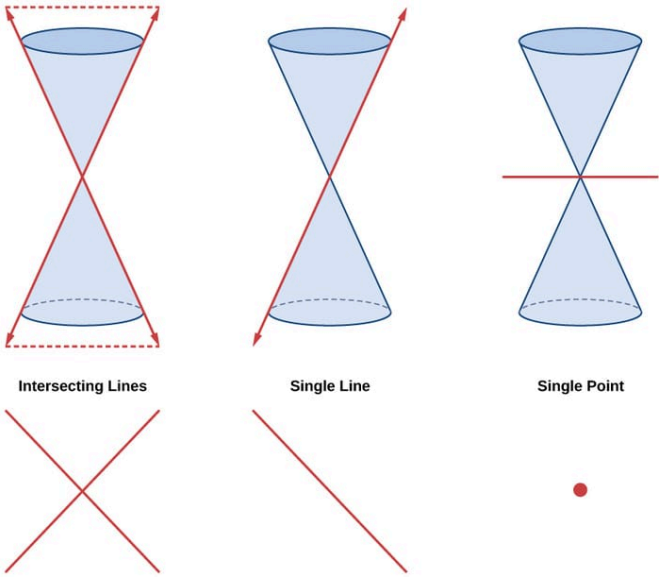



Figure 6.4.2: Degenerate conic sections

## Identifying Nondegenerate Conics in General Form

In previous sections of this chapter, we have focused on the standard form equations for nondegenerate conic sections. In this section, we will shift our focus to the general form equation, which can be used for any conic. The general form is set equal to zero, and the terms and coefficients are given in a particular order, as shown below.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where $A$, $B$, and $C$ are not all zero. We can use the values of the coefficients to identify which type conic is represented by a given equation.

You may notice that the general form equation has an $xy$ term that we have not seen in any of the standard form equations. As we will discuss later, the $xy$ term rotates the conic whenever $B$ is not equal to zero.

***Table*** 6.4.1

| Conic Sections | Example |
|---|---|
| ellipse | $4x^2 + 9y^2 = 1$ |
| circle | $4x^2 + 4y^2 = 1$ |
| hyperbola | $4x^2 - 9y^2 = 1$ |
| parabola | $4x^2 = 9y$ or $4y^2 = 9x$ |
| one line | $4x + 9y = 1$ |
| intersecting lines | $(x - 4)(y + 4) = 0$ |
| parallel lines | $(x - 4)(x - 9) = 0$ |
| a point | $4x^2 + 4y^2 = 0$ |
| no graph | $4x^2 + 4y^2 = -1$ |

### GENERAL FORM OF CONIC SECTIONS

A *conic section* has the general form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{6.4.1}$$

where $A$, $B$, and $C$ are not all zero. Table 6.4.2 summarizes the different conic sections where $B = 0$, and $A$ and $C$ are nonzero real numbers. This indicates that the conic has not been rotated.

***Table*** 6.4.2

| | |
|---|---|
| **ellipse** | $Ax^2 + Cy^2 + Dx + Ey + F = 0$, $A \neq C$ and $AC > 0$ |
| **circle** | $Ax^2 + Cy^2 + Dx + Ey + F = 0$, $A = C$ |
| **hyperbola** | $Ax^2 - Cy^2 + Dx + Ey + F = 0$ or $-Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $A$ and $C$ are positive |
| **parabola** | $Ax^2 + Dx + Ey + F = 0$ or $Cy^2 + Dx + Ey + F = 0$ |

### How to: Given the equation of a conic, identify the type of conic

1. Rewrite the equation in the general form (Equation 6.4.1), $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
2. Identify the values of $A$ and $C$ from the general form.
   - If $A$ and $C$ are nonzero, have the same sign, and are not equal to each other, then the graph may be an ellipse.
   - If $A$ and $C$ are equal and nonzero and have the same sign, then the graph may be a circle.
   - If $A$ and $C$ are nonzero and have opposite signs, then the graph may be a hyperbola.
   - If either $A$ or $C$ is zero, then the graph may be a parabola.

If $B = 0$, the conic section will have a vertical and/or horizontal axes. If $B$ does not equal 0, as shown below, the conic section is rotated. Notice the phrase "may be" in the definitions. That is because the equation may not represent a conic section at all, depending on the values of $A$, $B$, $C$, $D$, $E$, and $F$. For example, the degenerate case of a circle or an ellipse is a point:

$$Ax^2 + By^2 = 0, \tag{6.4.2}$$

when $A$ and $B$ have the same sign.

The degenerate case of a hyperbola is two intersecting straight lines: $Ax^2 + By^2 = 0$, when $A$ and $B$ have opposite signs.

On the other hand, the equation, $Ax^2 + By^2 + 1 = 0$, when $A$ and $B$ are positive does not represent a graph at all, since there are no real ordered pairs which satisfy it.

### Example 6.4.1: Identifying a Conic from Its General Form

Identify the graph of each of the following nondegenerate conic sections.

a. $4x^2 - 9y^2 + 36x + 36y - 125 = 0$
b. $9y^2 + 16x + 36y - 10 = 0$
c. $3x^2 + 3y^2 - 2x - 6y - 4 = 0$
d. $-25x^2 - 4y^2 + 100x + 16y + 20 = 0$

## Solution

1. Rewriting the general form (Equation 6.4.1), we have

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$
$$4x^2 + 0xy + (-9)y^2 + 36x + 36y + (-125) = 0$$

   with $A = 4$ and $C = -9$, so we observe that $A$ and $C$ have opposite signs. The graph of this equation is a hyperbola.
2. Rewriting the general form (Equation 6.4.1), we have

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$
$$0x^2 + 0xy + 9y^2 + 16x + 36y + (-10) = 0$$

with $A = 0$ and $C = 9$. We can determine that the equation is a parabola, since $A$ is zero.

3. Rewriting the general form (Equation 6.4.1), we have

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$
$$3x^2 + 0xy + 3y^2 + (-2)x + (-6)y + (-4) = 0$$

with $A = 3$ and $C = 3$. Because $A = C$, the graph of this equation is a circle.

4. Rewriting the general form (Equation 6.4.1), we have

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$
$$(-25)x^2 + 0xy + (-4)y^2 + 100x + 16y + 20 = 0$$

with $A = -25$ and $C = -4$. Because $AC > 0$ and $A \neq C$, the graph of this equation is an ellipse.

**Exercise 6.4.1**

Identify the graph of each of the following nondegenerate conic sections.

a. $16y^2 - x^2 + x - 4y - 9 = 0$
b. $16x^2 + 4y^2 + 16x + 49y - 81 = 0$

*Answer a available at the end of this section.*

## Finding a New Representation of the Given Equation after Rotating through a Given Angle

Until now, we have looked at equations of conic sections without an $xy$ term, which aligns the graphs with the *x*- and *y*-axes. When we add an $xy$ term, we are rotating the conic about the origin. If the *x*- and *y*-axes are rotated through an angle, say $\theta$,then every point on the plane may be thought of as having two representations: $(x, y)$ on the Cartesian plane with the original *x*-axis and *y*-axis, and $(x', y')$ on the new plane defined by the new, rotated axes, called the *x*'-axis and *y*'-axis (Figure 6.4.3).

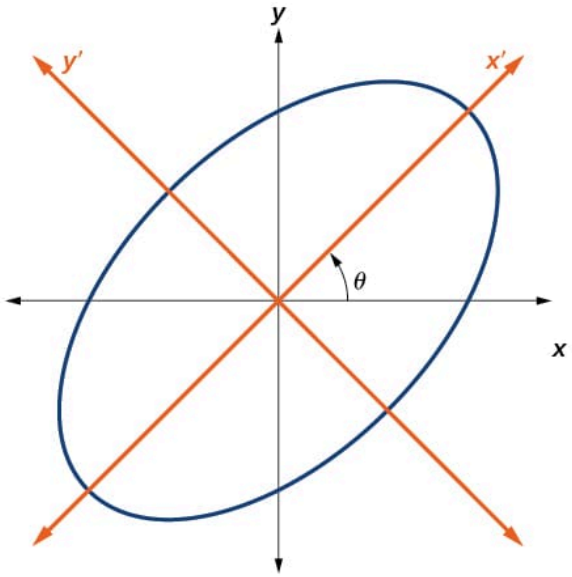



Figure 6.4.3: The graph of the rotated ellipse $x^2 + y^2 – xy – 15 = 0$

We will find the relationships between $x$ and $y$ on the Cartesian plane with $x'$ and $y'$ on the new rotated plane (Figure 6.4.4).

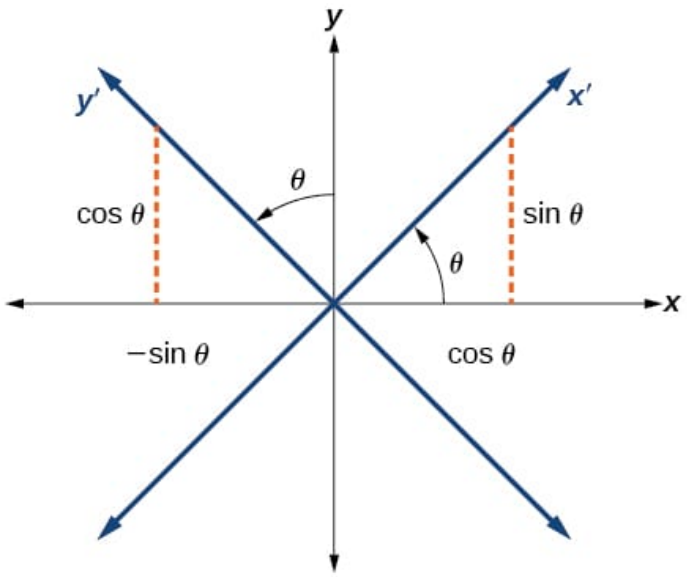



Figure 6.4.4: The Cartesian plane with $x$- and $y$-axes and the resulting $x'$– and $y'$–axes formed by a rotation by an angle $\theta$.

The original coordinate *x*- and *y*-axes have unit vectors $\hat{i}$ and $\hat{j}$. The rotated coordinate axes have unit vectors $\hat{i}'$ and $\hat{j}'$.The angle $\theta$ is known as the angle of rotation (Figure 6.4.5). We may write the new unit vectors in terms of the original ones.

$$\hat{i}' = \cos\theta\hat{i} + \sin\theta\hat{j} \tag{6.4.3}$$

$$\hat{j}' = -\sin\theta\hat{i} + \cos\theta\hat{j} \tag{6.4.4}$$

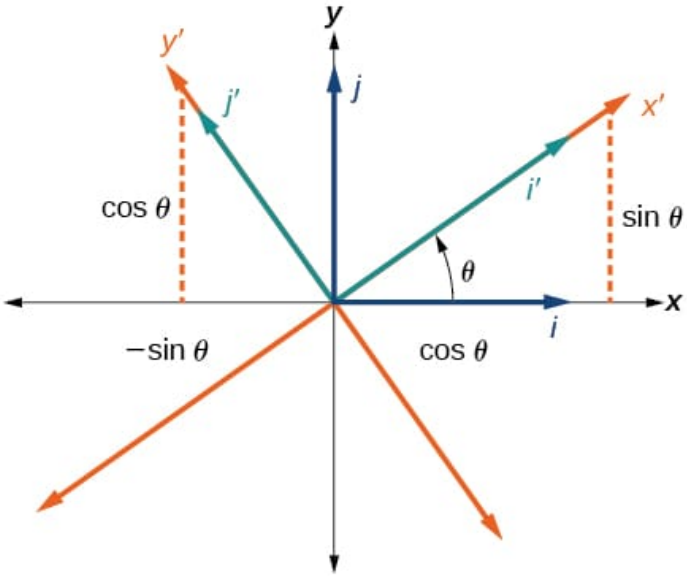



Figure 6.4.5: Relationship between the old and new coordinate planes.

Consider a vector $\vec{u}$ in the new coordinate plane. It may be represented in terms of its coordinate axes.

$$\begin{aligned}
\vec{u} &= x'i' + y'j' \\
&= x'(i\cos\theta + j\sin\theta) + y'(-i\sin\theta + j\cos\theta) && \text{Substitute.} \\
&= ix'\cos\theta + jx'\sin\theta - iy'\sin\theta + jy'\cos\theta && \text{Distribute.} \\
&= ix'\cos\theta - iy'\sin\theta + jx'\sin\theta + jy'\cos\theta && \text{Apply commutative property.} \\
&= (x'\cos\theta - y'\sin\theta)i + (x'\sin\theta + y'\cos\theta)j && \text{Factor by grouping.}
\end{aligned}$$

Because $\vec{u} = x'i' + y'j'$, we have representations of $x$ and $y$ in terms of the new coordinate system.

$$x = x' \cos\theta - y' \sin\theta$$

and

$$y = x' \sin\theta + y' \cos\theta$$

## EQUATIONS OF ROTATION

If a point $(x, y)$ on the Cartesian plane is represented on a new coordinate plane where the axes of rotation are formed by rotating an angle $\theta$ from the positive *x*-axis, then the coordinates of the point with respect to the new axes are $(x', y')$. We can use the following equations of rotation to define the relationship between $(x, y)$ and $(x', y')$:

$$x = x' \cos\theta - y' \sin\theta \tag{6.4.5}$$

and

$$y = x' \sin\theta + y' \cos\theta \tag{6.4.6}$$

## How to: Given the equation of a conic, find a new representation after rotating through an angle

1. Find $x$ and $y$ where $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$.
2. Substitute the expression for $x$ and $y$ into in the given equation, then simplify.
3. Write the equations with $x'$ and $y'$ in standard form.

## Example 6.4.2: Finding a New Representation of an Equation after Rotating through a Given Angle

Find a new representation of the equation $2x^2 - xy + 2y^2 - 30 = 0$ after rotating through an angle of $\theta = 45°$.

**Solution**

Find $x$ and $y$, where $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$.

Because $\theta = 45°$,

$$\begin{aligned} x &= x' \cos(45°) - y' \sin(45°) \\ x &= x' \left(\frac{1}{\sqrt{2}}\right) - y' \left(\frac{1}{\sqrt{2}}\right) \\ x &= \frac{x' - y'}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} y &= x' \sin(45°) + y' \cos(45°) \\ y &= x' \left(\frac{1}{\sqrt{2}}\right) + y' \left(\frac{1}{\sqrt{2}}\right) \\ y &= \frac{x' + y'}{\sqrt{2}} \end{aligned}$$

Substitute $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$ into $2x^2 - xy + 2y^2 - 30 = 0$.

$$2\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - \left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 2\left(\frac{x' + y'}{\sqrt{2}}\right)^2 - 30 = 0$$

**Simplify**.

$$2\frac{(x'-y')(x'-y')}{2} - \frac{(x'-y')(x'+y')}{2} + 2\frac{(x'+y')(x'+y')}{2} - 30 = 0 \quad \text{FOIL method}$$

$$x'^2 - 2x'y' + y'^2 - \frac{(x'^2 - y'^2)}{2} + x'^2 + 2x'y' + y'^2 - 30 = 0 \quad \text{Combine like terms.}$$

$$2x'^2 + 2y'^2 - \frac{(x'^2 - y'^2)}{2} = 30 \quad \text{Combine like terms.}$$

$$2(2x'^2 + 2y'^2 - \frac{(x'^2 - y'^2)}{2}) = 2(30) \quad \text{Multiply both sides by 2.}$$

$$4x'^2 + 4y'^2 - (x'^2 - y'^2) = 60 \quad \text{Simplify.}$$

$$4x'^2 + 4y'^2 - x'^2 + y'2 = 60 \quad \text{Distribute.}$$

$$\frac{3x'^2}{60} + \frac{5y'^2}{60} = \frac{60}{60} \quad \text{Set equal to 1.}$$

Write the equations with $x'$ and $y'$ in the standard form.

$$\frac{x'^2}{20} + \frac{y'^2}{12} = 1$$

This equation is an ellipse. Figure 6.4.6 shows the graph.

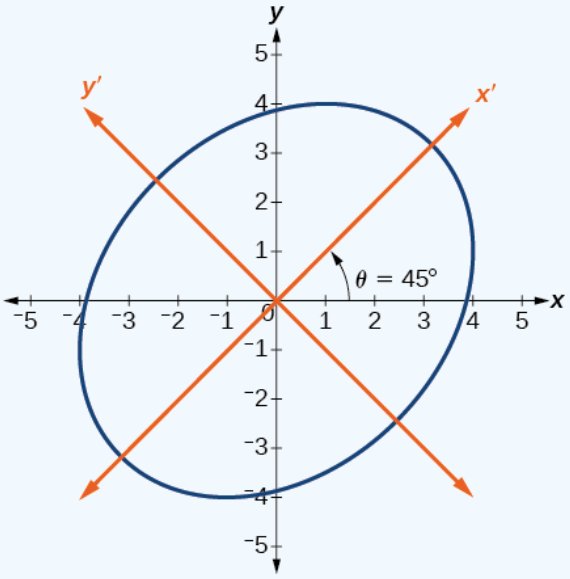



Figure 6.4.6

## Writing Equations of Rotated Conics in Standard Form

Now that we can find the standard form of a conic when we are given an angle of rotation, we will learn how to transform the equation of a conic given in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ into standard form by rotating the axes. To do so, we will rewrite the general form as an equation in the $x'$ and $y'$ coordinate system without the $x'y'$ term, by rotating the axes by a measure of $\theta$ that satisfies

$$\cot(2\theta) = \frac{A - C}{B} \tag{6.4.7}$$

We have learned already that any conic may be represented by the second degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where $A$, $B$,and $C$ are not all zero. However, if $B \neq 0$, then we have an $xy$ term that prevents us from rewriting the equation in

standard form. To eliminate it, we can rotate the axes by an acute angle $\theta$ where $\cot(2\theta) = \dfrac{A-C}{B}$.

- If $\cot(2\theta) > 0$, then $2\theta$ is in the first quadrant, and $\theta$ is between $(0\,°, 45\,°)$.
- If $\cot(2\theta) < 0$, then $2\theta$ is in the second quadrant, and $\theta$ is between $(45\,°, 90\,°)$.
- If $A = C$, then $\theta = 45\,°$.

**How to: Given an equation for a conic in the $x'y'$ system, rewrite the equation without the $x'y'$ term in terms of $x'$ and $y'$,where the $x'$ and $y'$ axes are rotations of the standard axes by $\theta$ degrees**

1. Find $\cot(2\theta)$.
2. Find $\sin\theta$ and $\cos\theta$.
3. Substitute $\sin\theta$ and $\cos\theta$ into $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.
4. Substitute the expression for $x$ and $y$ into in the given equation, and then simplify.
5. Write the equations with $x'$ and $y'$ in the standard form with respect to the rotated axes.

**Example 6.4.3: Rewriting an Equation with respect to the $x'$ and $y'$ axes without the $x'y'$ Term**

Rewrite the equation $8x^2 - 12xy + 17y^2 = 20$ in the $x'y'$ system without an $x'y'$ term.

**Solution**

First, we find $\cot(2\theta)$.

$$8x^2 - 12xy + 17y^2 = 20 \rightarrow A = 8,\ B = -12 \text{ and } C = 17$$

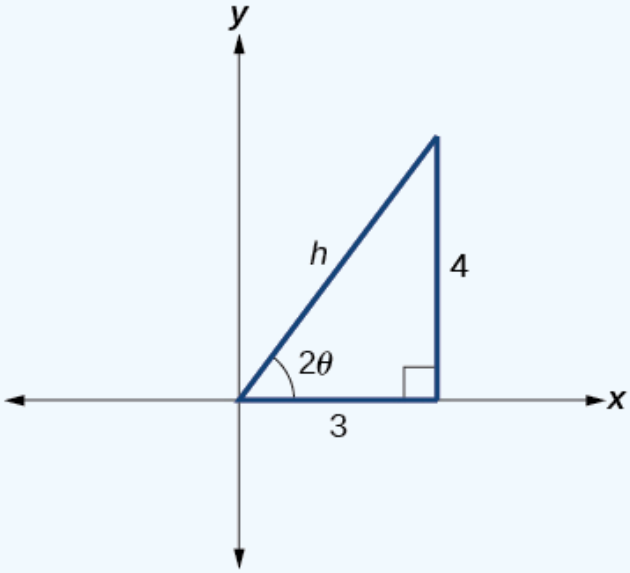



Figure 6.4.7

From Figure 6.4.7:

$$\cot(2\theta) = \frac{A-C}{B} = \frac{8-17}{-12}$$
$$= \frac{-9}{-12} = \frac{3}{4}$$

$$\cot(2\theta) = \frac{3}{4} = \frac{\text{adjacent}}{\text{opposite}}$$

So the hypotenuse is

$$\begin{aligned} 3^2+4^2 &= h^2 \\ 9+16 &= h^2 \\ 25 &= h^2 \\ h &= 5 \end{aligned}$$

Next, we find $\sin\theta$ and $\cos\theta$.

$$\sin\theta = \sqrt{\frac{1-\cos(2\theta)}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{\frac{5}{5}-\frac{3}{5}}{2}} = \sqrt{\frac{5-3}{5}\cdot\frac{1}{2}} = \sqrt{\frac{2}{10}} = \sqrt{\frac{1}{5}}$$

$$\sin\theta = \frac{1}{\sqrt{5}}$$

$$\cos\theta = \sqrt{\frac{1+\cos(2\theta)}{2}} = \sqrt{\frac{1+\frac{3}{5}}{2}} = \sqrt{\frac{\frac{5}{5}+\frac{3}{5}}{2}} = \sqrt{\frac{5+3}{5}\cdot\frac{1}{2}} = \sqrt{\frac{8}{10}} = \sqrt{\frac{4}{5}}$$

$$\cos\theta = \frac{2}{\sqrt{5}}$$

Substitute the values of $\sin\theta$ and $\cos\theta$ into $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ &= x'\left(\frac{2}{\sqrt{5}}\right) - y'\left(\frac{1}{\sqrt{5}}\right) \\ &= \frac{2x'-y'}{\sqrt{5}} \end{aligned}$$

and

$$\begin{aligned} y &= x'\sin\theta + y'\cos\theta \\ &= x'\left(\frac{1}{\sqrt{5}}\right) + y'\left(\frac{2}{\sqrt{5}}\right) \\ &= \frac{x'+2y'}{\sqrt{5}} \end{aligned}$$

Substitute the expressions for $x$ and $y$ into in the given equation, and then simplify.

$$8\left(\frac{2x'-y'}{\sqrt{5}}\right)^2 - 12\left(\frac{2x'-y'}{\sqrt{5}}\right)\left(\frac{x'+2y'}{\sqrt{5}}\right) + 17\left(\frac{x'+2y'}{\sqrt{5}}\right)^2 = 20$$

$$8\left(\frac{(2x'-y')(2x'-y')}{5}\right) - 12\left(\frac{(2x'-y')(x'+2y')}{5}\right) + 17\left(\frac{(x'+2y')(x'+2y')}{5}\right) = 20$$

$$8(4x'^2 - 4x'y' + y'^2) - 12(2x'^2 + 3x'y' - 2y'^2) + 17(x'^2 + 4x'y' + 4y'^2) = 100$$

$$32x'^2 - 32x'y' + 8y'^2 - 24x'^2 - 36x'y' + 24y'^2 + 17x'^2 + 68x'y' + 68y'^2 = 100$$

$$25x'^2 + 100y'^2 = 100$$

$$\frac{25}{100}x'^2 + \frac{100}{100}y'^2 = \frac{100}{100}$$

Write the equations with $x'$ and $y'$ in the standard form with respect to the new coordinate system.

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1$$

Figure $6.4.8$ shows the graph of the ellipse.

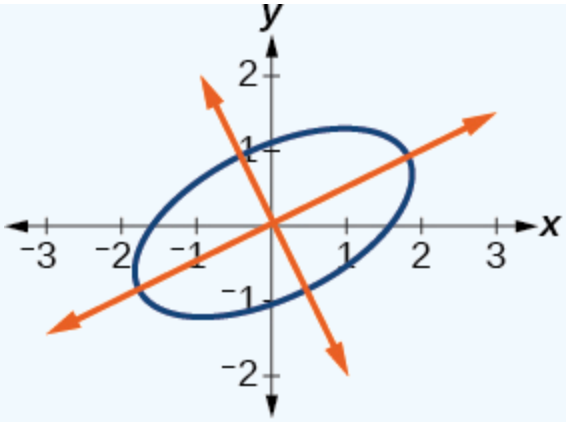



Figure 6.4.8

## Exercise 6.4.2

Rewrite the $13x^2 - 6\sqrt{3}xy + 7y^2 = 16$ in the $x'y'$ system without the $x'y'$ term.

*Answer available at the end of this section.*

## Example 6.4.4 :Graphing an Equation That Has No $x'y'$ Terms

Graph the following equation relative to the $x'y'$ system:

$$x^2 + 12xy - 4y^2 = 30$$

**Solution**

First, we find $\cot(2\theta)$.

$$x^2 + 12xy - 4y^2 = 20 \rightarrow A = 1,\ B = 12, \text{and } C = -4$$

$$\cot(2\theta) = \frac{A - C}{B}$$
$$\cot(2\theta) = \frac{1 - (-4)}{12}$$
$$\cot(2\theta) = \frac{5}{12}$$

Because $\cot(2\theta) = \frac{5}{12}$, we can draw a reference triangle as in Figure 6.4.9.

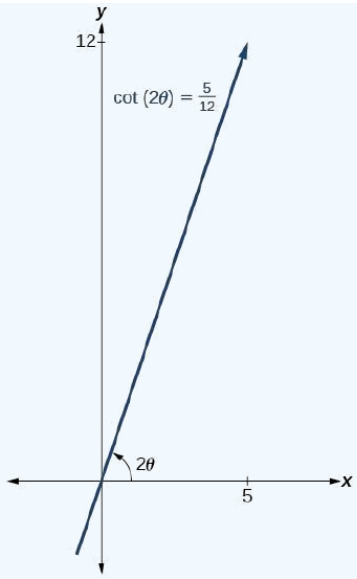



Figure 6.4.9

$$\cot(2\theta) = \frac{5}{12} = \frac{adjacent}{opposite}$$

Thus, the hypotenuse is

$$\begin{aligned} 5^2 + 12^2 &= h^2 \\ 25 + 144 &= h^2 \\ 169 &= h^2 \\ h &= 13 \end{aligned}$$

Next, we find $\sin\theta$ and $\cos\theta$. We will use half-angle identities.

$$\sin\theta = \sqrt{\frac{1-\cos(2\theta)}{2}} = \sqrt{\frac{1-\frac{5}{13}}{2}} = \sqrt{\frac{\frac{13}{13}-\frac{5}{13}}{2}} = \sqrt{\frac{8}{13}\cdot\frac{1}{2}} = \frac{2}{\sqrt{13}}$$

$$\cos\theta = \sqrt{\frac{1+\cos(2\theta)}{2}} = \sqrt{\frac{1+\frac{5}{13}}{2}} = \sqrt{\frac{\frac{13}{13}+\frac{5}{13}}{2}} = \sqrt{\frac{18}{13}\cdot\frac{1}{2}} = \frac{3}{\sqrt{13}}$$

Now we find $x$ and $y$.

$$x = x'\cos\theta - y'\sin\theta$$

$$x = x'\left(\frac{3}{\sqrt{13}}\right) - y'\left(\frac{2}{\sqrt{13}}\right)$$

$$x = \frac{3x' - 2y'}{\sqrt{13}}$$

and

$$y = x'\sin\theta + y'\cos\theta$$

$$y = x'\left(\frac{2}{\sqrt{13}}\right) + y'\left(\frac{3}{\sqrt{13}}\right)$$

$$y = \frac{2x' + 3y'}{\sqrt{13}}$$

Now we substitute $x = \frac{3x' - 2y'}{\sqrt{13}}$ and $y = \frac{2x' + 3y'}{\sqrt{13}}$ into $x^2 + 12xy - 4y^2 = 30$.

$$\left(\frac{3x' - 2y'}{\sqrt{13}}\right)^2 + 12\left(\frac{3x' - 2y'}{\sqrt{13}}\right)\left(\frac{2x' + 3y'}{\sqrt{13}}\right) - 4\left(\frac{2x' + 3y'}{\sqrt{13}}\right)^2 = 30$$

$$\left(\frac{1}{13}\right)[(3x' - 2y')^2 + 12(3x' - 2y')(2x' + 3y') - 4(2x' + 3y')^2] = 30 \quad \text{Factor.}$$

$$\left(\frac{1}{13}\right)[9x'^2 - 12x'y' + 4y'^2 + 12(6x'^2 + 5x'y' - 6y'^2) - 4(4x'^2 + 12x'y' + 9y'^2)] = 30 \quad \text{Multiply.}$$

$$\left(\frac{1}{13}\right)[9x'^2 - 12x'y' + 4y'^2 + 72x'^2 + 60x'y' - 72y'^2 - 16x'^2 - 48x'y' - 36y'^2] = 30 \quad \text{Distribute.}$$

$$\left(\frac{1}{13}\right)[65x'^2 - 104y'^2] = 30 \quad \text{Combine like terms.}$$

$$65x'^2 - 104y'^2 = 390 \quad \text{Multiply.}$$

$$\frac{x'^2}{6} - \frac{4y'^2}{15} = 1 \quad \text{Divide by 390.}$$

Figure 6.4.10 shows the graph of the hyperbola $\frac{x'^2}{6} - \frac{4y'^2}{15} = 1$

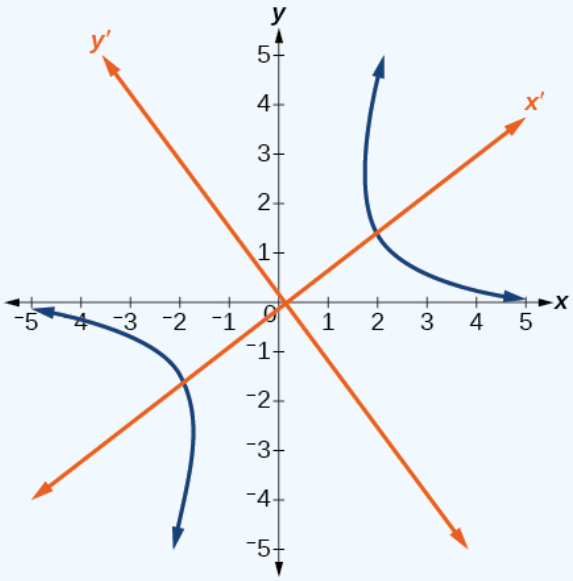



Figure 6.4.10

## Identifying Conics without Rotating Axes

Now we have come full circle. How do we identify the type of conic described by an equation? What happens when the axes are rotated? Recall, the general form of a conic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

If we apply the rotation formulas to this equation we get the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

It may be shown that

$$B^2 - 4AC = B'^2 - 4A'C'$$

The expression does not vary after rotation, so we call the expression *invariant***.** The discriminant, $B^2-4AC$, is invariant and remains unchanged after rotation. Because the discriminant remains unchanged, observing the discriminant enables us to identify the conic section.

## HOWTO: USING THE DISCRIMINANT TO IDENTIFY A CONIC

If the equation

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \tag{6.4.8}$$

is transformed by rotating axes into the equation

$$A'x'^2+B'x'y'+C'y'^2+D'x'+E'y'+F'=0 \tag{6.4.9}$$

then

$$B^2-4AC=B'^2-4A'C' \tag{6.4.10}$$

The equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ is an ellipse, a parabola, or a hyperbola, or a degenerate case of one of these. If the discriminant, $B^2-4AC$, is

- $<0$, the conic section is an ellipse
- $=0$, the conic section is a parabola
- $>0$, the conic section is a hyperbola

## Example 6.4.5: Identifying the Conic without Rotating Axes

Identify the conic for each of the following without rotating axes.

a. $5x^2+2\sqrt{3}xy+2y^2-5=0$
b. $5x^2+2\sqrt{3}xy+12y^2-5=0$

**Solution**

a. Let's begin by determining $A$, $B$, and $C$.

$$\underbrace{5}_{A}x^2+\underbrace{2\sqrt{3}}_{B}xy+\underbrace{2}_{C}y^2-5=0$$

Now, we find the discriminant.

$$\begin{aligned}B^2-4AC&=(2\sqrt{3})^2-4(5)(2)\\&=4(3)-40\\&=12-40\\&=-28<0\end{aligned}$$

Therefore, $5x^2+2\sqrt{3}xy+2y^2-5=0$ represents an ellipse.

b. Again, let's begin by determining $A$,$B$, and $C$.

$$\underbrace{5}_{A}x^2+\underbrace{2\sqrt{3}}_{B}xy+\underbrace{12}_{C}y^2-5=0$$

Now, we find the discriminant.

$$\begin{aligned} B^2-4AC &= \left(2\sqrt{3}\right)^2-4(5)(12) \\ &= 4(3)-240 \\ &= 12-240 \\ &= -228<0 \end{aligned}$$

Therefore, $5x^2+2\sqrt{3}xy+12y^2-5=0$ represents an ellipse.

**Exercise 6.4.3**

Identify the conic for each of the following without rotating axes.

a. $x^2-9xy+3y^2-12=0$
b. $10x^2-9xy+4y^2-4=0$

*Answer a available at the end of this section.*

## Key Equations

| | |
|---|---|
| General Form equation of a conic section | $Ax^2+Bxy+Cy^2+$ |
| Rotation of a conic section | $x=x'\cos\theta-y'\sin$<br>$y=x'\sin\theta+y'\cos$ |
| Angle of rotation | $\theta$, where $\cot(2\theta)=$ |

## Key Concepts

- Four basic shapes can result from the intersection of a plane with a pair of right circular cones connected tail to tail. They include an ellipse, a circle, a hyperbola, and a parabola.
- A nondegenerate conic section has the general form $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ where $A$, $B$ and $C$ are not all zero. The values of $A$, $B$, and $C$ determine the type of conic. See Example 6.4.1.
- Equations of conic sections with an $xy$ term have been rotated about the origin. See Example 6.4.2.
- The general form can be transformed into an equation in the $x'$ and $y'$ coordinate system without the $x'y'$ term. See Example 6.4.3 and Example 6.4.4.
- An expression is described as invariant if it remains unchanged after rotating. Because the discriminant is invariant, observing it enables us to identify the conic section. See Example 6.4.5.

### Answers

**Exercise** 6.4.1
hyperbola
ellipse

**Exercise** 6.4.2
$\frac{x'^2}{4}+\frac{y'^2}{1}=1$

**Exercise** 6.4.3
hyperbola
ellipse

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# 6.5: Conic Sections in Polar Coordinates

**Learning Objectives**

- Identify a conic in polar form.
- Graph the polar equations of conics.
- Define conics in terms of a focus and a directrix.

Most of us are familiar with orbital motion, such as the motion of a planet around the sun or an electron around an atomic nucleus. Within the planetary system, orbits of planets, asteroids, and comets around a larger celestial body are often elliptical. Comets, however, may take on a parabolic or hyperbolic orbit instead. And, in reality, the characteristics of the planets' orbits may vary over time. Each orbit is tied to the location of the celestial body being orbited and the distance and direction of the planet or other object from that body. As a result, we tend to use polar coordinates to represent these orbits.

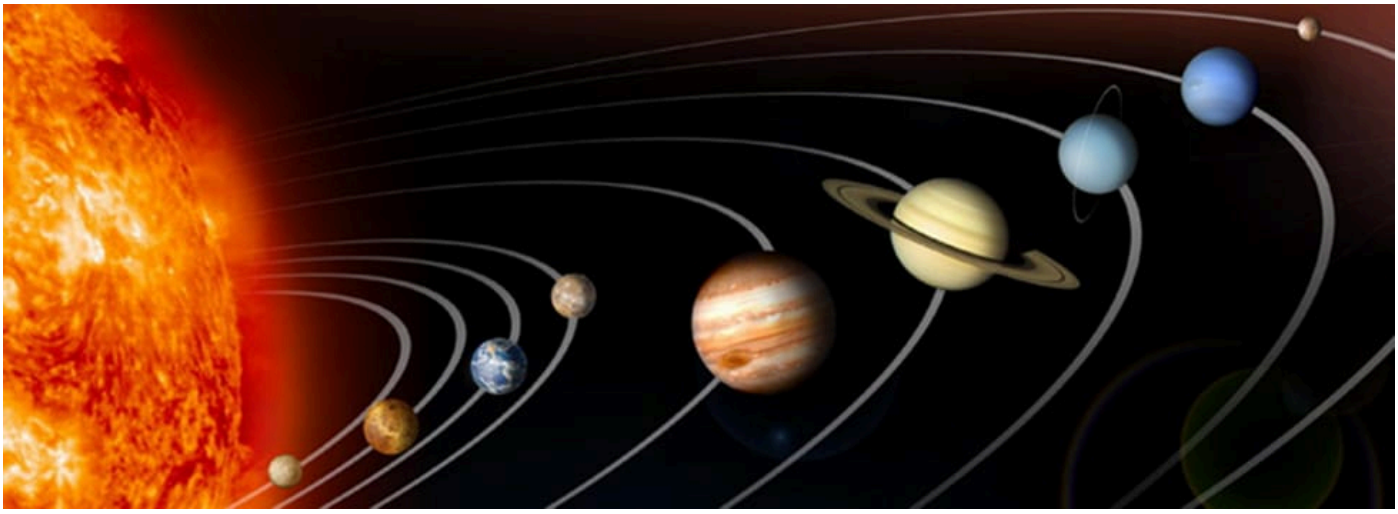


Figure $6.5.1$: Planets orbiting the sun follow elliptical paths. (credit: NASA Blueshift, Flickr)

In an elliptical orbit, the **periapsis** is the point at which the two objects are closest, and the **apoapsis** is the point at which they are farthest apart. Generally, the velocity of the orbiting body tends to increase as it approaches the periapsis and decrease as it approaches the apoapsis. Some objects reach an escape velocity, which results in an infinite orbit. These bodies exhibit either a parabolic or a hyperbolic orbit about a body; the orbiting body breaks free of the celestial body's gravitational pull and fires off into space. Each of these orbits can be modeled by a conic section in the polar coordinate system.

## Identifying a Conic in Polar Form

Any conic may be determined by three characteristics: a single **focus**, a fixed line called the **directrix**, and the ratio of the distances of each to a point on the graph. Consider the **parabola** $x = 2 + y^2$ shown in Figure $6.5.2$.

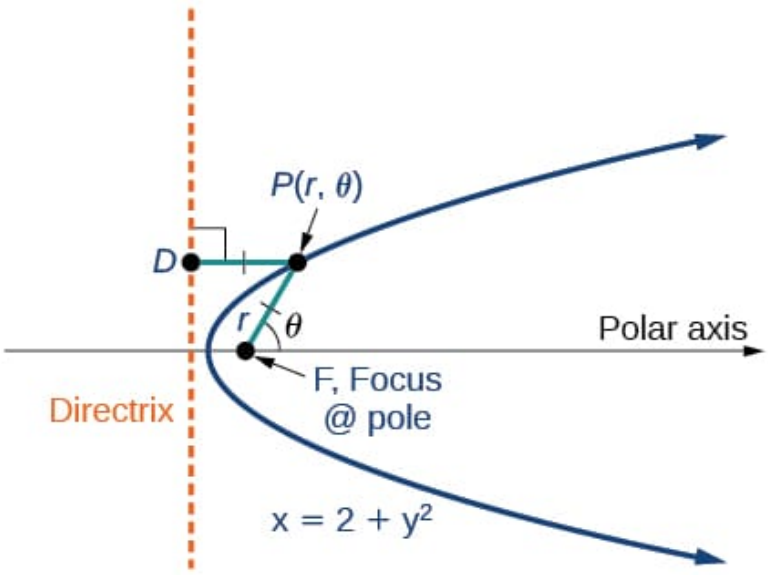



Figure 6.5.2

We previously learned how a parabola is defined by the focus (a fixed point) and the directrix (a fixed line). In this section, we will learn how to define any conic in the polar coordinate system in terms of a fixed point, the focus $P(r,\theta)$ at the pole, and a line, the directrix, which is perpendicular to the polar axis.

If $F$ is a fixed point, the focus, and $D$ is a fixed line, the directrix, then we can let $e$ be a fixed positive number, called the **eccentricity**, which we can define as the ratio of the distances from a point on the graph to the focus and the point on the graph to the directrix. Then the set of all points $P$ such that $e = \dfrac{PF}{PD}$ is a conic. In other words, we can define a conic as the set of all points $P$ with the property that the ratio of the distance from $P$ to $F$ to the distance from $P$ to $D$ is equal to the constant $e$.

For a conic with eccentricity $e$,

- if $0 \leq e < 1$, the conic is an ellipse
- if $e = 1$, the conic is a parabola
- if $e > 1$, the conic is an hyperbola

With this definition, we may now define a conic in terms of the directrix, $x = \pm p$, the eccentricity $e$, and the angle $\theta$. Thus, each conic may be written as a **polar equation**, an equation written in terms of $r$ and $\theta$.

## THE POLAR EQUATION FOR A CONIC

For a conic with a focus at the origin, if the directrix is $x = \pm p$, where $p$ is a positive real number, and the eccentricity is a positive real number $e$, the conic has a polar equation

$$r = \frac{ep}{1 \pm e\cos\theta} \tag{6.5.1}$$

For a conic with a focus at the origin, if the directrix is $y = \pm p$, where $p$ is a positive real number, and the eccentricity is a positive real number $e$, the conic has a polar equation

$$r = \frac{ep}{1 \pm e\sin\theta} \tag{6.5.2}$$

## How to: Given the polar equation for a conic, identify the type of conic, the directrix, and the eccentricity.

1. Multiply the numerator and denominator by the reciprocal of the constant in the denominator to rewrite the equation in standard form.
2. Identify the eccentricity $e$ as the coefficient of the trigonometric function in the denominator.
3. Compare $e$ with $1$ to determine the shape of the conic.
4. Determine the directrix as $x = p$ if cosine is in the denominator and $y = p$ if sine is in the denominator. Set $ep$ equal to the numerator in standard form to solve for $x$ or $y$.

## Example 6.5.1: Identifying a Conic Given the Polar Form

For each of the following equations, identify the conic with focus at the origin, the directrix, and the eccentricity.

a. $r = \dfrac{6}{3 + 2\sin\theta}$

b. $r = \dfrac{12}{4 + 5\cos\theta}$

c. $r = \dfrac{7}{2 - 2\sin\theta}$

**Solution**

For each of the three conics, we will rewrite the equation in standard form. Standard form has a $1$ as the constant in the denominator. Therefore, in all three parts, the first step will be to multiply the numerator and denominator by the reciprocal of the constant of the original equation, $\dfrac{1}{c}$, where $c$ is that constant.

a. Multiply the numerator and denominator by $\dfrac{1}{3}$.

$$r = \frac{6}{3 + 2\sin\theta} \cdot \frac{\left(\frac{1}{3}\right)}{\left(\frac{1}{3}\right)} = \frac{6\left(\frac{1}{3}\right)}{3\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\sin\theta} = \frac{2}{1 + \frac{2}{3}\sin\theta}$$

Because $\sin\theta$ is in the denominator, the directrix is $y = p$. Comparing to standard form, note that $e = \dfrac{2}{3}$.Therefore, from the numerator,

$$\begin{aligned} 2 &= ep \\ 2 &= \frac{2}{3}p \\ \left(\frac{3}{2}\right)2 &= \left(\frac{3}{2}\right)\frac{2}{3}p \\ 3 &= p \end{aligned}$$

Since $e < 1$, the conic is an **ellipse**. The eccentricity is $e = \dfrac{2}{3}$ and the directrix is $y = 3$.

b. Multiply the numerator and denominator by $\dfrac{1}{4}$.

$$r=\frac{12}{4+5\cos\theta}\cdot\frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{4}\right)}$$
$$r=\frac{12\left(\frac{1}{4}\right)}{4\left(\frac{1}{4}\right)+5\left(\frac{1}{4}\right)\cos\theta}$$
$$r=\frac{3}{1+\frac{5}{4}\cos\theta}$$

Because $\cos\theta$ is in the denominator, the directrix is $x=p$. Comparing to standard form, $e=\frac{5}{4}$. Therefore, from the numerator,

$$3=ep$$
$$3=\frac{5}{4}p$$
$$\left(\frac{4}{5}\right)3=\left(\frac{4}{5}\right)\frac{5}{4}p$$
$$\frac{12}{5}=p$$

Since $e>1$, the conic is a **hyperbola**. The eccentricity is $e=\frac{5}{4}$ and the directrix is $x=\frac{12}{5}=2.4$.

c. Multiply the numerator and denominator by $\frac{1}{2}$.

$$r=\frac{7}{2-2\sin\theta}\cdot\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}$$
$$r=\frac{7\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}\right)-2\left(\frac{1}{2}\right)\sin\theta}$$
$$r=\frac{\frac{7}{2}}{1-\sin\theta}$$

Because sine is in the denominator, the directrix is $y=-p$. Comparing to standard form, $e=1$. Therefore, from the numerator,

$$\frac{7}{2}=ep$$
$$\frac{7}{2}=(1)p$$
$$\frac{7}{2}=p$$

Because $e=1$, the conic is a parabola. The eccentricity is $e=1$ and the directrix is $y=-\frac{7}{2}=-3.5$.

## Exercise 6.5.1

Identify the conic with focus at the origin, the directrix, and the eccentricity for $r=\frac{2}{3-\cos\theta}$.

*Answer available at the end of this section.*

# Graphing the Polar Equations of Conics

When graphing in Cartesian coordinates, each conic section has a unique equation. This is not the case when graphing in polar coordinates. We must use the eccentricity of a conic section to determine which type of curve to graph, and then determine its specific characteristics. The first step is to rewrite the conic in standard form as we have done in the previous example. In other words, we need to rewrite the equation so that the denominator begins with $1$. This enables us to determine $e$ and, therefore, the shape of the curve. The next step is to substitute values for $\theta$ and solve for $r$ to plot a few key points. Setting $\theta$ equal to $0$, $\frac{\pi}{2}$, $\pi$, and $\frac{3\pi}{2}$ provides the vertices so we can create a rough sketch of the graph.

## Example $6.5.2A$: Graphing a Parabola in Polar Form

Graph $r = \dfrac{5}{3 + 3\cos\theta}$.

**Solution**

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of $3$, which is $\frac{1}{3}$.

$$r = \frac{5}{3 + 3\cos\theta} = \frac{5\left(\frac{1}{3}\right)}{3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)\cos\theta}$$

$$r = \frac{\frac{5}{3}}{1 + \cos\theta}$$

Because $e = 1$,we will graph a **parabola** with a focus at the origin. The function has a $\cos\theta$, and there is an addition sign in the denominator, so the directrix is $x = p$.

$$\frac{5}{3} = ep$$
$$\frac{5}{3} = (1)p$$
$$\frac{5}{3} = p$$

The directrix is $x = \frac{5}{3}$.

Plotting a few key points as in Table $6.5.1$ will enable us to see the vertices. See Figure $6.5.3$.

*Table* $6.5.1$

| | A | B | C | D |
|---|---|---|---|---|
| $\theta$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
| $r = \frac{5}{3 + 3\cos\theta}$ | $\frac{5}{6} \approx 0.83$ | $\frac{5}{3} \approx 1.67$ | undefined | $\frac{5}{3} \approx 1.67$ |

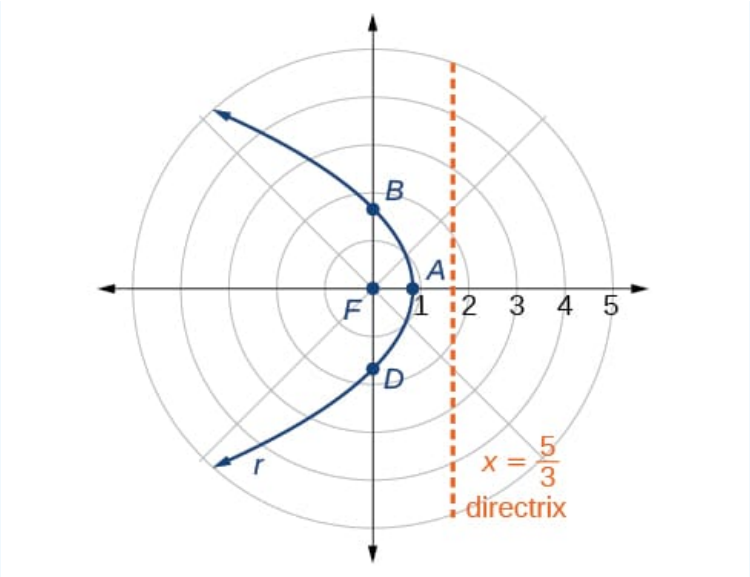



Figure $6.5.3$

We can check our result with a graphing utility. See Figure $6.5.4$.

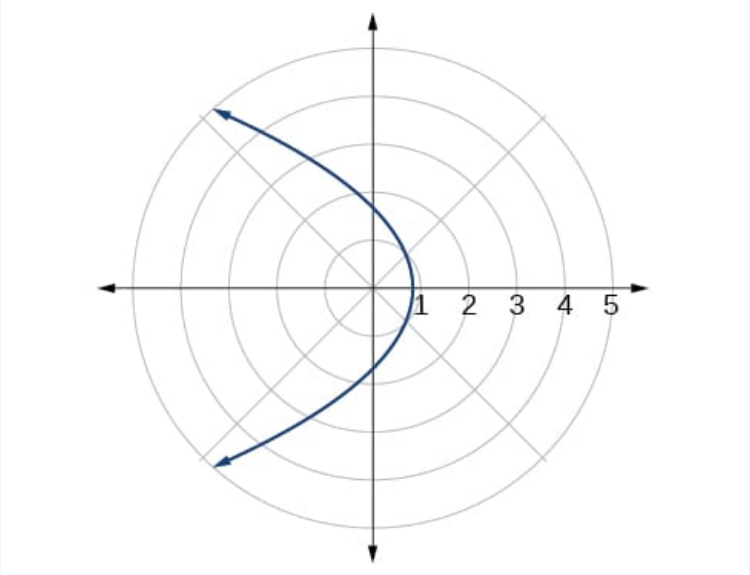



Figure $6.5.4$

## Example $6.5.2B$: Graphing a Hyperbola in Polar Form

Graph $r = \dfrac{8}{2 - 3\sin\theta}$.

**Solution**

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of $2$, which is $\frac{1}{2}$.

$$r=\frac{8}{2-3\sin\theta}=\frac{8\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}\right)-3\left(\frac{1}{2}\right)\sin\theta}$$

$$r=\frac{4}{1-\frac{3}{2}\sin\theta}$$

Because $e=\frac{3}{2}$, $e>1$, so we will graph a hyperbola with a focus at the origin. The function has a $\sin\theta$ term and there is a subtraction sign in the denominator, so the directrix is $y=-p$.

$$\begin{aligned}4&=ep\\4&=\left(\frac{3}{2}\right)p\\4\left(\frac{2}{3}\right)&=p\\\frac{8}{3}&=p\end{aligned}$$

The directrix is $y=-\frac{8}{3}$.

Plotting a few key points as in Table $6.5.2$ will enable us to see the vertices. See Figure $6.5.5$.

*Table* $6.5.2$

| | A | B | C | D |
|---|---|---|---|---|
| $\theta$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
| $r=\frac{8}{2-3\sin\theta}$ | $4$ | $-8$ | $4$ | $\frac{8}{5}=1.6$ |

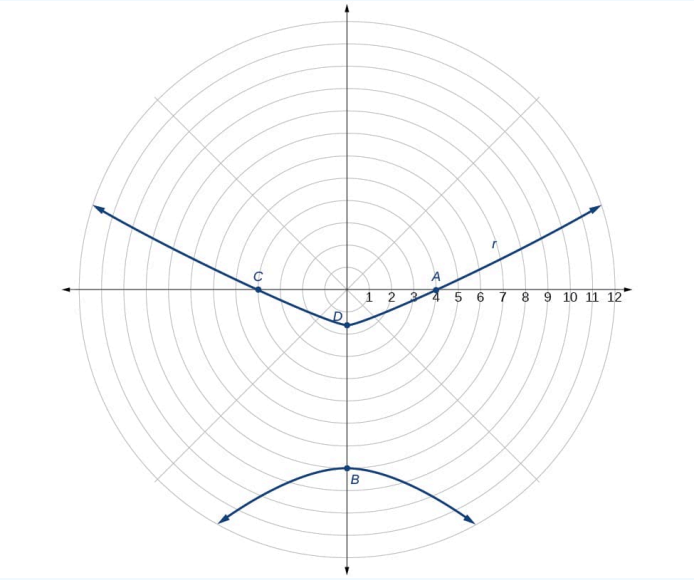



Figure $6.5.5$

## Example $6.5.2C$: Graphing an Ellipse in Polar Form

Graph $r=\frac{10}{5-4\cos\theta}$.

**Solution**

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 5, which is $\frac{1}{5}$.

$$r = \frac{10}{5 - 4\cos\theta} = \frac{10\left(\frac{1}{5}\right)}{5\left(\frac{1}{5}\right) - 4\left(\frac{1}{5}\right)\cos\theta}$$
$$r = \frac{2}{1 - \frac{4}{5}\cos\theta}$$

Because $e = \frac{4}{5}$, $e < 1$, so we will graph an **ellipse** with a **focus** at the origin. The function has a $\cos\theta$, and there is a subtraction sign in the denominator, so the **directrix** is $x = -p$.

$$2 = ep$$
$$2 = \left(\frac{4}{5}\right)p$$
$$2\left(\frac{5}{4}\right) = p$$
$$\frac{5}{2} = p$$

The directrix is $x = -\frac{5}{2}$.

Plotting a few key points as in Table 6.5.3 will enable us to see the vertices. See Figure 6.5.6.

*Table* 6.5.3

| | A | B | C | D |
|---|---|---|---|---|
| $\theta$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
| $r = \frac{10}{5 - 4\cos\theta}$ | $10$ | $2$ | $\frac{10}{9} \approx 1.1$ | $2$ |

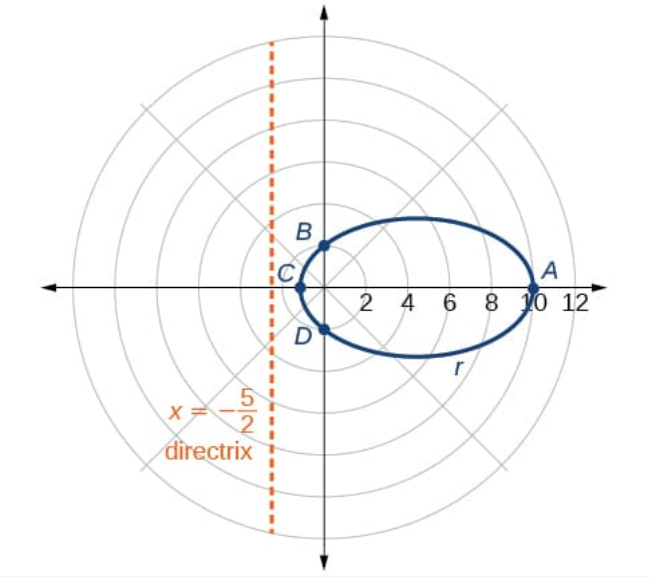



Figure 6.5.6

**Analysis**

We can check our result using a graphing utility. See Figure 6.5.7.

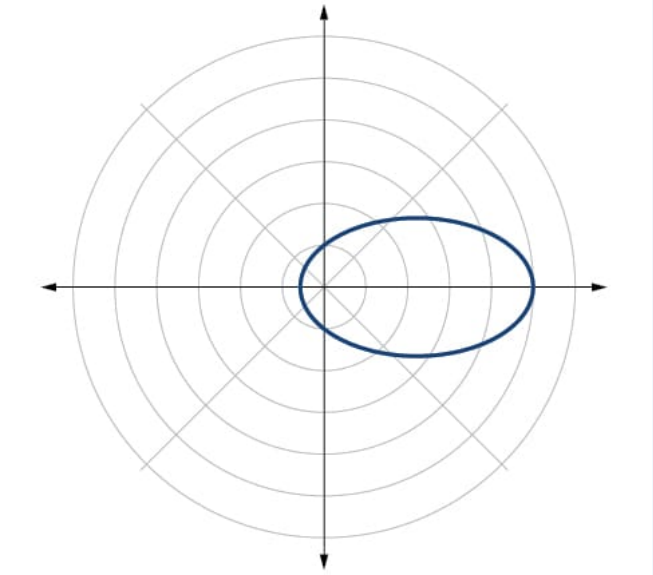


Figure 6.5.6: $r = \dfrac{10}{5 - 4\cos\theta}$ graphed on a viewing window of $[-3, 12, 1]$ by $[-4, 4, 1]$, $\theta min = 0$ and $\theta max = 2\pi$.

**Exercise 6.5.2**

Graph $r = \dfrac{2}{4 - \cos\theta}$.

*Answer available at the end of this section.*

## Defining Conics in Terms of a Focus and a Directrix

So far we have been using polar equations of conics to describe and graph the curve. Now we will work in reverse; we will use information about the origin, eccentricity, and directrix to determine the polar equation.

**How to: Given the focus, eccentricity, and directrix of a conic, determine the polar equation**

1. Determine whether the directrix is horizontal or vertical. If the directrix is given in terms of $y$, we use the general polar form in terms of sine. If the directrix is given in terms of $x$, we use the general polar form in terms of cosine.
2. Determine the sign in the denominator. If $p < 0$, use subtraction. If $p > 0$, use addition.
3. Write the coefficient of the trigonometric function as the given eccentricity.
4. Write the absolute value of $p$ in the numerator, and simplify the equation.

**Example 6.5.3A: Finding the Polar Form of a Vertical Conic Given a Focus at the Origin and the Eccentricity and Directrix**

Find the polar form of the conic given a focus at the origin, $e = 3$ and directrix $y = -2$.

**Solution**

The directrix is $y = -p$, so we know the trigonometric function in the denominator is sine.

Because $y = -2$, $-2 < 0$, so we know there is a subtraction sign in the denominator. We use the standard form of

$$r = \frac{ep}{1 - e\sin\theta}$$

and $e = 3$ and $|-2| = 2 = p$.

Therefore,

$$r = \frac{(3)(2)}{1 - 3\sin\theta}$$
$$r = \frac{6}{1 - 3\sin\theta}$$

## Example $6.5.3B$: Finding the Polar Form of a Horizontal Conic Given a Focus at the Origin and the Eccentricity and Directrix

Find the polar form of a conic given a focus at the origin, $e = \frac{3}{5}$, and directrix $x = 4$.

**Solution**

Because the directrix is $x = p$, we know the function in the denominator is cosine. Because $x = 4$, $4 > 0$, so we know there is an addition sign in the denominator. We use the standard form of

$$r = \frac{ep}{1 + e\cos\theta}$$

and $e = \frac{3}{5}$ and $|4| = 4 = p$.

Therefore,

$$r = \frac{\left(\frac{3}{5}\right)(4)}{1 + \frac{3}{5}\cos\theta}$$
$$r = \frac{\frac{12}{5}}{1 + \frac{3}{5}\cos\theta}$$
$$r = \frac{\frac{12}{5}}{1\left(\frac{5}{5}\right) + \frac{3}{5}\cos\theta}$$
$$r = \frac{\frac{12}{5}}{\frac{5}{5} + \frac{3}{5}\cos\theta}$$
$$r = \frac{12}{5} \cdot \frac{5}{5 + 3\cos\theta}$$
$$r = \frac{12}{5 + 3\cos\theta}$$

## Exercise $6.5.3$

Find the polar form of the conic given a focus at the origin, $e = 1$, and directrix $x = -1$.

*Answer available at the end of this section.*

### Example 6.5.4: Converting a Conic in Polar Form to Rectangular Form

Convert the conic $r = \dfrac{1}{5 - 5\sin\theta}$ to rectangular form.

**Solution**

We will rearrange the formula to use the identities $r = \sqrt{x^2 + y^2}$, $x = r\cos\theta$,and $y = r\sin\theta$.

$$\begin{aligned} r &= \frac{1}{5 - 5\sin\theta} \\ r \cdot (5 - 5\sin\theta) &= \frac{1}{5 - 5\sin\theta} \cdot (5 - 5\sin\theta) \quad \text{Eliminate the fraction.} \\ 5r - 5r\sin\theta &= 1 \quad \text{Distribute.} \\ 5r &= 1 + 5r\sin\theta \quad \text{Isolate } 5r. \\ 25r^2 &= (1 + 5r\sin\theta)^2 \quad \text{Square both sides.} \\ 25(x^2 + y^2) &= (1 + 5y)^2 \quad \text{Substitute } r = \sqrt{x^2 + y^2} \text{ and } y = r\sin\theta. \\ 25x^2 + 25y^2 &= 1 + 10y + 25y^2 \quad \text{Distribute and use FOIL.} \\ 25x^2 - 10y &= 1 \quad \text{Rearrange terms and set equal to 1.} \end{aligned}$$

### Exercise 6.5.4

Convert the conic $r = \dfrac{2}{1 + 2\cos\theta}$ to rectangular form.

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with conics in polar coordinates.

- Polar Equations of Conic Sections
- Graphing Polar Equations of Conics - 1
- Graphing Polar Equations of Conics - 2

Visit this website for additional practice questions from Learningpod.

## Key Concepts

- Any conic may be determined by a single focus, the corresponding eccentricity, and the directrix. We can also define a conic in terms of a fixed point, the focus $P(r, \theta)$ at the pole, and a line, the directrix, which is perpendicular to the polar axis.
- A conic is the set of all points $e = \dfrac{PF}{PD}$, where eccentricity $e$ is a positive real number. Each conic may be written in terms of its polar equation. See Example 6.5.1.
- The polar equations of conics can be graphed. See Example 6.5.2, Example 6.5.3, and Example 6.5.4.
- Conics can be defined in terms of a focus, a directrix, and eccentricity. See Example 6.5.5 and Example 6.5.6.
- We can use the identities $r = \sqrt{x^2 + y^2}$, $x = r\cos\theta$,and $y = r\sin\theta$ to convert the equation for a conic from polar to rectangular form. See Example 6.5.7.

### Answers

**Exercise** 6.5.1

ellipse; $e = \dfrac{1}{3}$; $x = -2$

**Exercise** 6.5.2

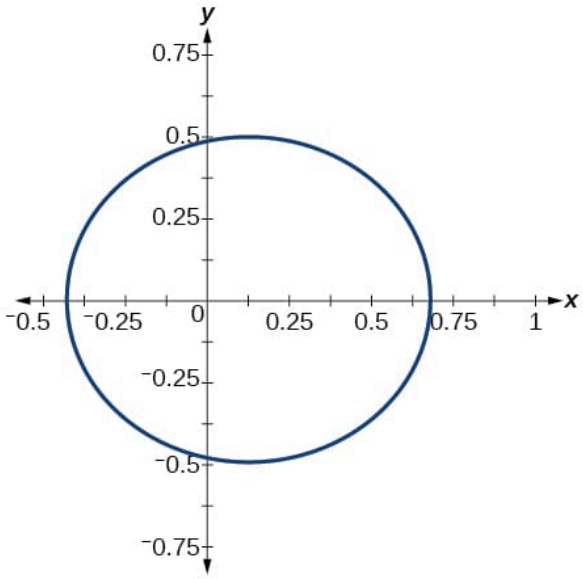



Figure 6.5.7

**Exercise** 6.5.3

$$r = \frac{1}{1 - \cos\theta}$$

**Exercise** 6.5.4

$$4 - 8x + 3x^2 - y^2 = 0$$

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# 6.E: Analytic Geometry (Exercises)

## 10.1: The Ellipse

In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.

### Verbal

1) Define an ellipse in terms of its foci.

**Answer**

An ellipse is the set of all points in the plane the sum of whose distances from two fixed points, called the foci, is a constant.

2) Where must the foci of an ellipse lie?

3) What special case of the ellipse do we have when the major and minor axis are of the same length?

**Answer**

This special case would be a circle.

4) For the special case mentioned above, what would be true about the foci of that ellipse?

5) What can be said about the symmetry of the graph of an ellipse with center at the origin and foci along the $y$-axis?

**Answer**

It is symmetric about the $x$-axis, $y$-axis, and the origin.

### Algebraic

**For the exercises 6-10, determine whether the given equations represent ellipses. If yes, write in standard form.**

6) $2x^2 + y = 4$

7) $4x^2 + 9y^2 = 36$

**Answer**

yes; $\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$

8) $4x^2 - y^2 = 4$

9) $4x^2 + 9y^2 = 1$

**Answer**

yes; $\dfrac{x^2}{\left(\frac{1}{2}\right)^2} + \dfrac{y^2}{\left(\frac{1}{2}\right)^2} = 1$

10) $4x^2 - 8x + 9y^2 - 72y + 112 = 0$

**For the exercises 11-26, write the equation of an ellipse in standard form, and identify the end points of the major and minor axes as well as the foci.**

11) $\frac{x^2}{4}+\frac{y^2}{49}=1$

**Answer**

$\frac{x^2}{2^2}+\frac{y^2}{7^2}=1$; Endpoints of major axis $(0,7)$ and $(0,-7)$. Endpoints of minor axis $(2,0)$ and $(-2,0)$. Foci at $(0,3\sqrt{5})$, $(0,-3\sqrt{5})$.

12) $\frac{x^2}{100}+\frac{y^2}{64}=1$

13) $x^2+9y^2=1$

**Answer**

$\frac{x^2}{(1)^2}+\frac{y^2}{\left(\frac{1}{3}\right)^2}=1$; Endpoints of major axis $(1,0)$ and $(-1,0)$. Endpoints of minor axis $\left(0,\frac{1}{3}\right)$, $\left(0,-\frac{1}{3}\right)$. Foci at $\left(\frac{2\sqrt{2}}{3},0\right)$, $\left(-\frac{2\sqrt{2}}{3},0\right)$.

14) $4x^2+16y^2=1$

15) $\frac{(x-2)^2}{49}+\frac{(y-4)^2}{25}=1$

**Answer**

$\frac{(x-2)^2}{7^2}+\frac{(y-4)^2}{5^2}=1$; Endpoints of major axis $(9,4)$, $(-5,4)$. Endpoints of minor axis $(2,9)$, $(2,-1)$. Foci at $(2+2\sqrt{6},4)$, $(2-2\sqrt{6},4)$

16) $\frac{(x-2)^2}{81}+\frac{(y+1)^2}{16}=1$

17) $\frac{(x+5)^2}{4}+\frac{(y-7)^2}{9}=1$

**Answer**

$\frac{(x+5)^2}{2^2}+\frac{(y-7)^2}{3^2}=1$; Endpoints of major axis $(-5,10)$, $(-5,4)$. Endpoints of minor axis $(-3,7)$, $(-7,7)$. Foci at $(-5,7+\sqrt{5})$, $(-5,7-\sqrt{5})$

18) $\frac{(x-7)^2}{49}+\frac{(y-7)^2}{49}=1$

19) $4x^2-8x+9y^2-72y+112=0$

**Answer**

$\frac{(x-1)^2}{3^2}+\frac{(y-4)^2}{2^2}=1$; Endpoints of major axis $(4,4)$, $(-2,4)$. Endpoints of minor axis $(1,6)$, $(1,2)$. Foci at $(1+\sqrt{5},4)$, $(1-\sqrt{5},4)$

20) $9x^2-54x+9y^2-54y+81=0$

21) $4x^2-24x+36y^2-360y+864=0$

**Answer**

$\frac{(x-3)^2}{(3\sqrt{2})^2}+\frac{(y-5)^2}{\sqrt{2}^2}=1$; Endpoints of major axis $(3+3\sqrt{2},5)$, $(3-3\sqrt{2},5)$. Endpoints of minor axis $(3,5+\sqrt{2})$, $(3,5-\sqrt{2})$. Foci at $(7,5)$, $(-1,5)$

22) $4x^2+24x+16y^2-128y+228=0$

23) $4x^2+40x+25y^2-100y+100=0$

**Answer**

$\frac{(x+5)^2}{(5)^2}+\frac{(y-2)^2}{(2)^2}=1$; Endpoints of major axis $(0,2)$, $(-10,2)$. Endpoints of minor axis $(-5,4)$, $(-5,0)$. Foci at $(-5+\sqrt{21},2)$, $(-5-\sqrt{21},2)$

24) $x^2+2x+100y^2-1000y+2401=0$

25) $4x^2+24x+25y^2+200y+336=0$

**Answer**

$\frac{(x+3)^2}{(5)^2}+\frac{(y+4)^2}{(2)^2}=1$; Endpoints of major axis $(2,-4)$, $(-8,-4)$. Endpoints of minor axis $(-3,-2)$, $(-3,-6)$. Foci at $(-3+\sqrt{21},-4)$, $(-3-\sqrt{21},-4)$

26) $9x^2+72x+16y^2+16y+4=0$

**For the exercises 27-31, find the foci for the given ellipses.**

27) $\frac{(x+3)^2}{25}+\frac{(y+1)^2}{36}=1$

**Answer**

Foci $(-3,-1+\sqrt{11})$, $(-3,-1-\sqrt{11})$

28) $\frac{(x+1)^2}{100}+\frac{(y-2)^2}{4}=1$

29) $x^2+y^2=1$

**Answer**

Focus $(0,0)$

30) $x^2+4y^2+4x+8y=1$

31) $10x^2+y^2+200x=0$

**Answer**

Foci $(-10,30)$, $(-10,-30)$

## Graphical

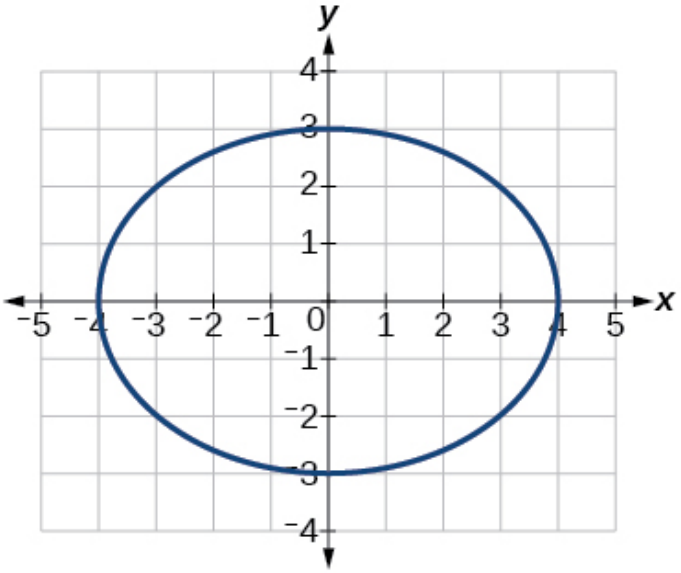
**For the exercises 32-45, graph the given ellipses, noting center, vertices, and foci.**

32) $\frac{x^2}{25}+\frac{y^2}{36}=1$

33) $\frac{x^2}{16}+\frac{y^2}{9}=1$

**Answer**

Center $(0,0)$, Vertices $(4,0)$, $(-4,0)$, $(0,3)$, $(0,-3)$. Foci $(\sqrt{7},0)$, $(-\sqrt{7},0)$

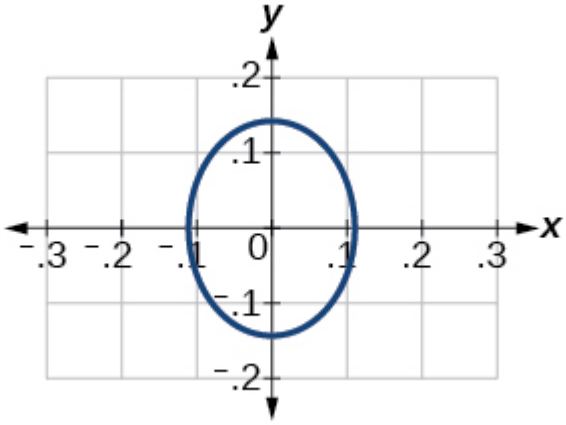



34) $4x^2+9y^2=1$

35) $81x^2+49y^2=1$

**Answer**

Center $(0,0)$, Vertices $\left(\frac{1}{9},0\right)$, $\left(-\frac{1}{9},0\right)$, $\left(0,\frac{1}{7}\right)$, $\left(0,\frac{1}{7}\right)$. Foci $\left(0,\frac{4\sqrt{2}}{63}\right)$, $\left(0,-\frac{4\sqrt{2}}{63}\right)$



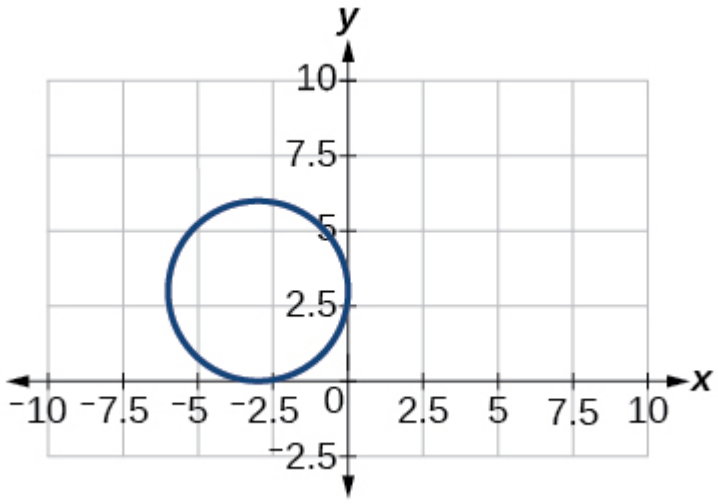

36) $\frac{(x-2)^2}{64}+\frac{(y-4)^2}{16}=1$

37) $\frac{(x+3)^2}{9}+\frac{(y-3)^2}{9}=1$

**Answer**

Center $(-3, 3)$, Vertices $(0, 3)$, $(-6, 3)$, $(-3, 0)$, $(-3, 6)$. Focus $(-3, 3)$

Note that this ellipse is a circle. The circle has only one focus, which coincides with the center.

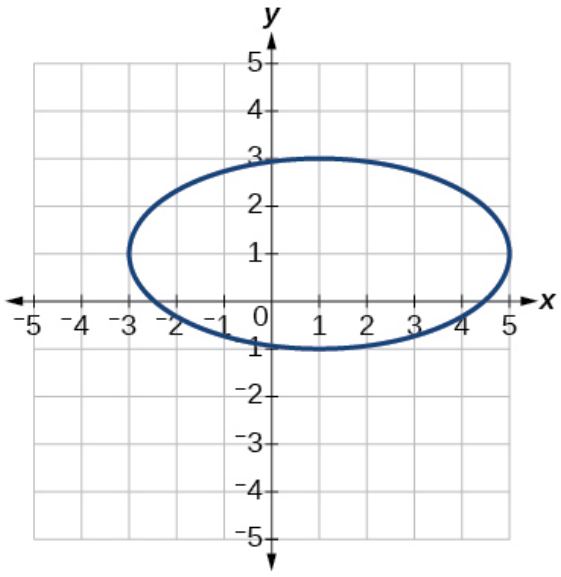



38) $\frac{x^2}{2} + \frac{(y+1)^2}{5} = 1$

39) $4x^2 - 8x + 16y^2 - 32y - 44 = 0$

**Answer**

Center $(1, 1)$, Vertices $(5, 1)$, $(-3, 1)$, $(1, 3)$, $(1, -1)$. Foci $(1, 1 + 4\sqrt{3})$, $(1, 1 - 4\sqrt{3})$

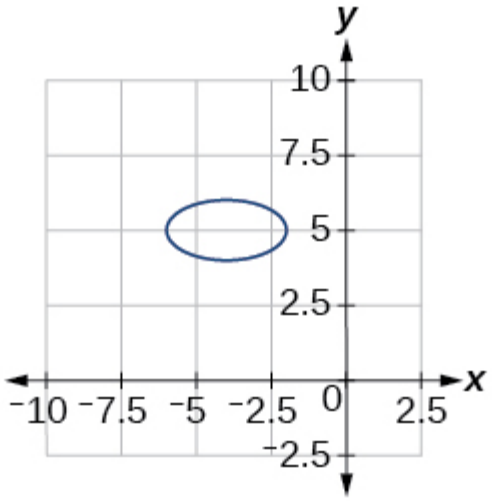



40) $x^2 - 8x + 25y^2 - 100y + 91 = 0$

41) $x^2 + 8x + 4y^2 - 40y + 112 = 0$

**Answer**

Center $(-4, 5)$, Vertices $(-2, 5)$, $(-6, 4)$, $(-4, 6)$, $(-4, 4)$. Foci $(-4 + \sqrt{3}, 5)$, $(-4 - 4\sqrt{3}, 5)$

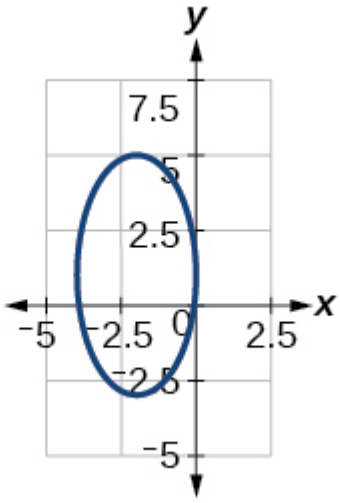



42) $64x^2 + 128x + 9y^2 - 72y - 368 = 0$

43) $16x^2 + 64x + 4y^2 - 8y + 4 = 0$

**Answer**

Center $(-2, 1)$, Vertices $(0, 1)$, $(-4, 1)$, $(-2, 5)$, $(-2, -3)$. Foci $(-2, 1 + 2\sqrt{3})$, $(-2, 1 - 2\sqrt{3})$

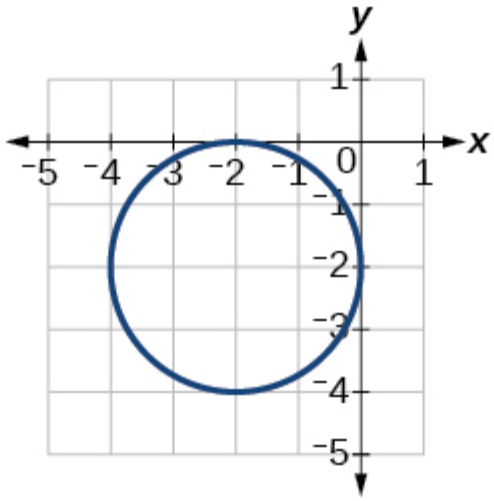



44) $100x^2 + 1000x + y^2 - 10y + 2425 = 0$

45) $4x^2 + 16x + 4y^2 + 16y + 16 = 0$

**Answer**

Center $(-2, 2)$, Vertices $(0, -2)$, $(-4, -2)$, $(-2, 0)$, $(-2, -4)$. Focus $(-2, -2)$




**For the exercises 46-51, use the given information about the graph of each ellipse to determine its equation.**

46) Center at the origin, symmetric with respect to the $x$- and $y$-axes, focus at $(4, 0)$ and point on graph $(0, 3)$.

47) Center at the origin, symmetric with respect to the $x$- and $y$-axes, focus at $(0, -2)$ and point on graph $(5, 0)$.

**Answer**

$$\frac{x^2}{25} + \frac{y^2}{29} = 1$$

48) Center at the origin, symmetric with respect to the $x$- and $y$-axes, focus at $(3, 0)$, and major axis is twice as long as minor axis.

49) Center $(4, 2)$; vertex $(9, 2)$; one focus: $(4 + 2\sqrt{6}, 2)$

**Answer**

$$\frac{(x-4)^2}{25} + \frac{(y-2)^2}{1} = 1$$

50) Center $(3, 5)$; vertex $(3, 11)$; one focus: $(3, 5 + 4\sqrt{2})$

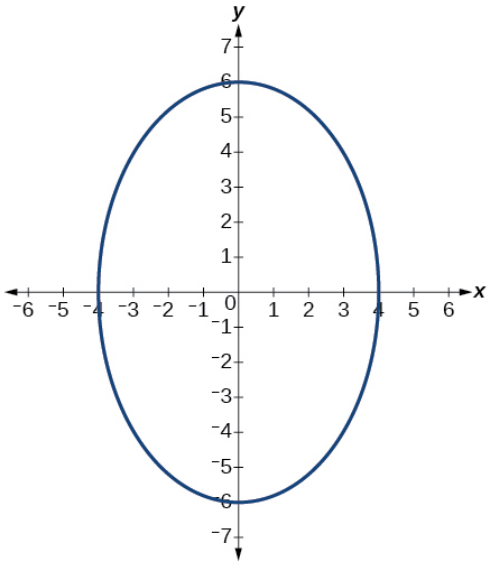
51) Center $(-3, 4)$; vertex $(1, 4)$; one focus: $(-3 + 2\sqrt{3}, 4)$

**Answer**

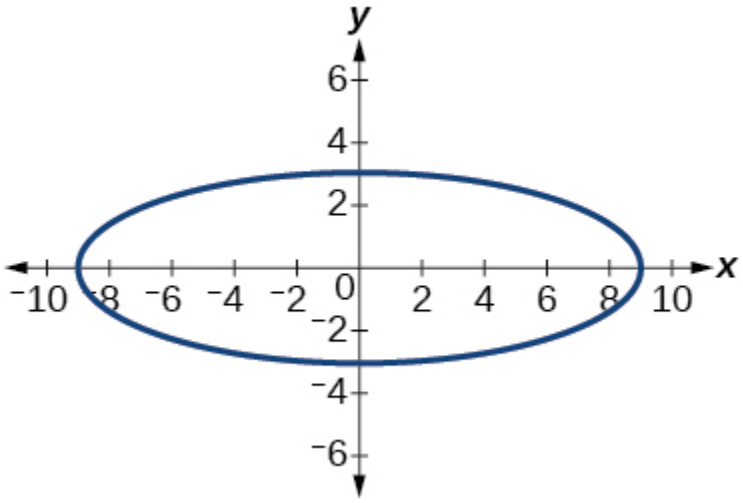
$$\frac{(x+3)^2}{16} + \frac{(y-4)^2}{4} = 1$$

**For the exercises 52-56, given the graph of the ellipse, determine its equation.**

52)



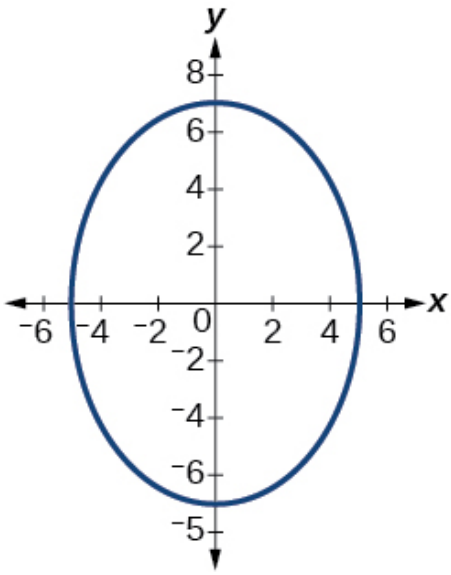

53)



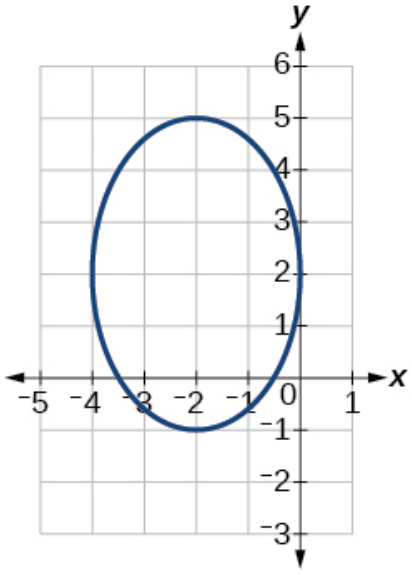

**Answer**

$$\frac{x^2}{81} + \frac{y^2}{9} = 1$$

54)



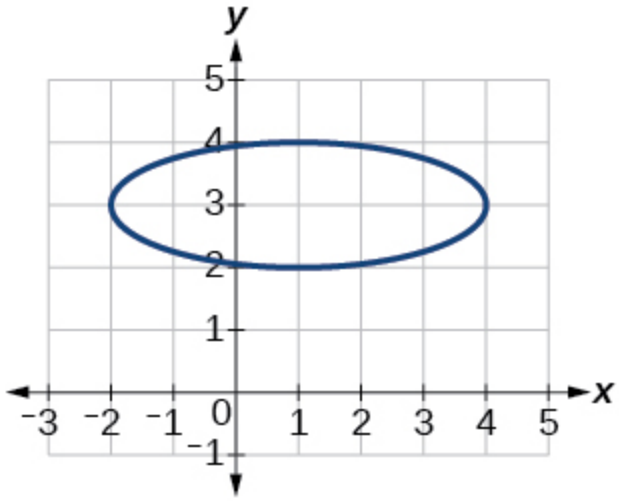

55)




**Answer**

$$\frac{(x+2)^2}{4}+\frac{(y-2)^2}{9}=1$$

56)




## Extensions

**For the exercises 57-61, find the area of the ellipse. The area of an ellipse is given by the formula** $\text{Area} = a \cdot b \cdot \pi$

57) $\frac{(x-3)^2}{9} + \frac{(y-3)^2}{16} = 1$

**Answer**

$\text{Area} = 12\pi$ square units

58) $\frac{(x+6)^2}{16} + \frac{(y-6)^2}{36} = 1$

59) $\frac{(x+1)^2}{4} + \frac{(y-2)^2}{5} = 1$

**Answer**

$\text{Area} = 2\sqrt{5}\pi$ square units

60) $4x^2 - 8x + 9y^2 - 72y + 112 = 0$

61) $9x^2 - 54x + 9y^2 - 54y + 81 = 0$

**Answer**

$\text{Area} = 9\pi$ square units

## Real-World Applications

62) Find the equation of the ellipse that will just fit inside a box that is $8$ units wide and $4$ units high.

63) Find the equation of the ellipse that will just fit inside a box that is four times as wide as it is high. Express in terms of $h$, the height.

**Answer**

$\frac{x^2}{4h^2} + \frac{y^2}{\frac{1}{4}h^2} = 1$

64) An arch has the shape of a semi-ellipse (the top half of an ellipse). The arch has a height of $8$ feet and a span of $20$ feet. Find an equation for the ellipse, and use that to find the height to the nearest $0.01$ foot of the arch at a distance of $4$ feet from the center.

65) An arch has the shape of a semi-ellipse. The arch has a height of $12$ feet and a span of $40$ feet. Find an equation for the ellipse, and use that to find the distance from the center to a point at which the height is $6$ feet. Round to the nearest hundredth.

**Answer**

$\frac{x^2}{400}+\frac{y^2}{144}=1$. Distance = $17.32$ feet

66) A bridge is to be built in the shape of a semi-elliptical arch and is to have a span of $120$ feet. The height of the arch at a distance of $40$ feet from the center is to be $8$ feet. Find the height of the arch at its center.

67) A person in a whispering gallery standing at one focus of the ellipse can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of $120$ feet, and the foci are located $30$ feet from the center, find the height of the ceiling at the center.

**Answer**

Approximately $51.96$ feet

68) A person is standing $8$ feet from the nearest wall in a whispering gallery. If that person is at one focus, and the other focus is $80$ feet away, what is the length and height at the center of the gallery?

# 10.2: The Hyperbola

In analytic geometry, a hyperbola is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other.

## Verbal

1) Define a hyperbola in terms of its foci.

**Answer**

A hyperbola is the set of points in a plane the difference of whose distances from two fixed points (foci) is a positive constant.

2) What can we conclude about a hyperbola if its asymptotes intersect at the origin?

3) What must be true of the foci of a hyperbola?

**Answer**

The foci must lie on the transverse axis and be in the interior of the hyperbola.

4) If the transverse axis of a hyperbola is vertical, what do we know about the graph?

5) Where must the center of hyperbola be relative to its foci?

**Answer**

The center must be the midpoint of the line segment joining the foci.

## Algebraic

**For the exercises 6-10, determine whether the following equations represent hyperbolas. If so, write in standard form.**

6) $3y^2 + 2x = 6$

7) $\frac{x^2}{36} - \frac{y^2}{9} = 1$

**Answer**

yes $\frac{x^2}{6^2} - \frac{y^2}{3^2} = 1$

8) $5y^2 + 4x^2 = 6x$

9) $25x^2 - 16y^2 = 400$

**Answer**

yes $\frac{x^2}{4^2} - \frac{y^2}{5^2} = 1$

10) $-9x^2 + 18x + y^2 + 4y - 14 = 0$

**For the exercises 11-25, write the equation for the hyperbola in standard form if it is not already, and identify the vertices and foci, and write equations of asymptotes.**

11) $\frac{x^2}{25} - \frac{y^2}{36} = 1$

**Answer**

$\frac{x^2}{5^2} - \frac{y^2}{6^2} = 1$; vertices: $(5, 0)$, $(-5, 0)$; foci: $(\sqrt{61}, 0)$, $(-\sqrt{61}, 0)$; asymptotes: $y = \frac{6}{5}x$, $y = -\frac{6}{5}x$

12) $\frac{x^2}{100} - \frac{y^2}{9} = 1$

13) $\frac{y^2}{4} - \frac{x^2}{81} = 1$

**Answer**

$\frac{y^2}{2^2} - \frac{x^2}{9^2} = 1$; vertices: $(0, 2)$, $(0, -2)$; foci: $(0, \sqrt{85})$, $(0, -\sqrt{85})$; asymptotes: $y = \frac{2}{9}x$, $y = -\frac{2}{9}x$

14) $9y^2 - 4x^2 = 1$

15) $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$

**Answer**

$\frac{(x-1)^2}{3^2} - \frac{(y-2)^2}{4^2} = 1$; vertices: $(4, 2)$, $(-2, 2)$; foci: $(6, 2)$, $(-4, 2)$; asymptotes: $y = \frac{4}{3}(x-1) + 2$, $y = -\frac{4}{3}(x-1) + 2$

16) $\frac{(y-6)^2}{36} - \frac{(x+1)^2}{16} = 1$

17) $\frac{(x-2)^2}{49} - \frac{(y+7)^2}{49} = 1$

**Answer**

$\frac{(x-2)^2}{7^2} - \frac{(y+7)^2}{7^2} = 1$; vertices: $(9,-7)$, $(-5,-7)$; foci: $(2+7\sqrt{2},-7)$, $(2-7\sqrt{2},-7)$; asymptotes: $y = x-9$, $y = -x-5$

18) $4x^2 - 8x - 9y^2 - 72y + 112 = 0$

19) $-9x^2 - 54x + 9y^2 - 54y + 81 = 0$

**Answer**

$\frac{(x+3)^2}{3^2} - \frac{(y-3)^2}{3^2} = 1$; vertices: $(0,3)$, $(-6,3)$; foci: $(-3+3\sqrt{2},1)$, $(-3-3\sqrt{2},1)$; asymptotes: $y = x+6$, $y = -x$

20) $4x^2 - 24x - 36y^2 - 360y + 864 = 0$

21) $-4x^2 + 24x + 16y^2 - 128y + 156 = 0$

**Answer**

$\frac{(y-4)^2}{2^2} - \frac{(x-3)^2}{4^2} = 1$; vertices: $(3,6)$, $(3,2)$; foci: $(3,4+2\sqrt{5})$, $(3,4-2\sqrt{5})$; asymptotes: $y = \frac{1}{2}(x-3)+4$, $y = -\frac{1}{2}(x-3)+4$

22) $-4x^2 + 40x + 25y^2 - 100y + 100 = 0$

23) $x^2 + 2x - 100y^2 - 1000y + 2401 = 0$

**Answer**

$\frac{(y+5)^2}{7^2} - \frac{(x+1)^2}{70^2} = 1$; vertices: $(-1,2)$, $(-1,-12)$; foci: $(-1,-5+7\sqrt{101})$, $(-1,-5-7\sqrt{101})$; asymptotes: $y = \frac{1}{10}(x+1)-5$, $y = -\frac{1}{10}(x+1)-5$

24) $-9x^2 + 72x + 16y^2 + 16y + 4 = 0$

25) $4x^2 + 24x - 25y^2 + 200y - 464 = 0$

**Answer**

$\frac{(x+3)^2}{5^2} - \frac{(y-4)^2}{2^2} = 1$; vertices: $(2,4)$, $(-8,4)$; foci: $(-3+\sqrt{29},4)$, $(-3-\sqrt{29},4)$; asymptotes: $y = \frac{2}{5}(x+3)+4$, $y = -\frac{2}{5}(x+3)+4$

**For the exercises 26-30, find the equations of the asymptotes for each hyperbola.**

26) $\frac{y^2}{3^2} - \frac{x^2}{3^2} = 1$

27) $\frac{(x-3)^2}{5^2} - \frac{(y+4)^2}{2^2} = 1$

**Answer**

$y = \frac{2}{5}(x-3)-4$, $y = -\frac{2}{5}(x-3)-4$

28) $\frac{(y-3)^2}{3^2} - \frac{(x+5)^2}{6^2} = 1$

29) $9x^2 - 18x - 16y^2 + 32y - 151 = 0$

**Answer**

$$y = \frac{3}{4}(x-1) + 1,\ y = -\frac{3}{4}(x-1) + 1$$

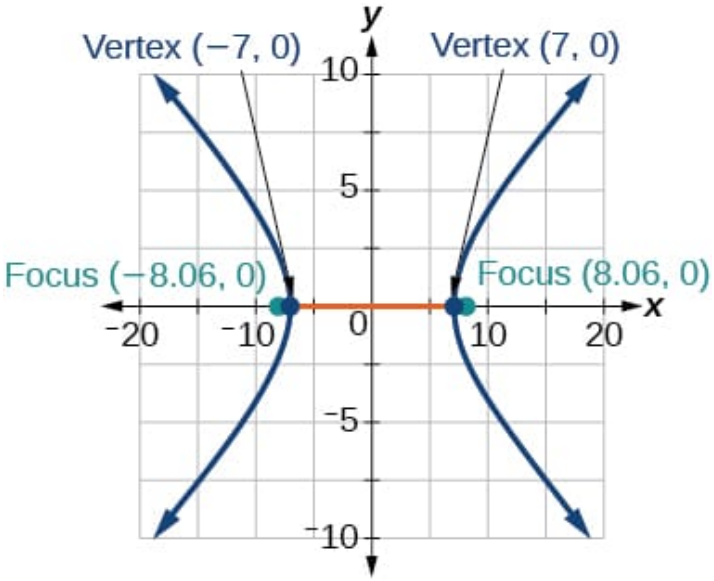
30) $16y^2 + 96y - 4x^2 + 16x + 112 = 0$

## Graphical

**For the exercises 31-44, sketch a graph of the hyperbola, labeling vertices and foci.**

31) $\frac{x^2}{49} - \frac{y^2}{16} = 1$

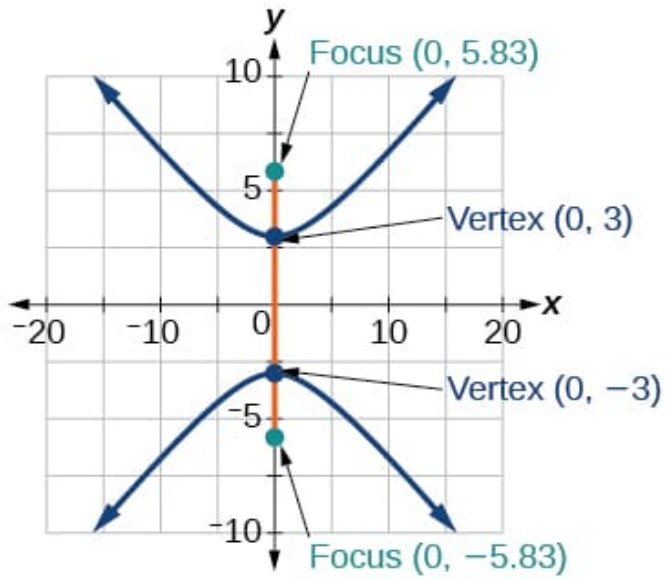
**Answer**




32) $\frac{x^2}{64} - \frac{y^2}{4} = 1$

33) $\frac{y^2}{9} - \frac{x^2}{25} = 1$

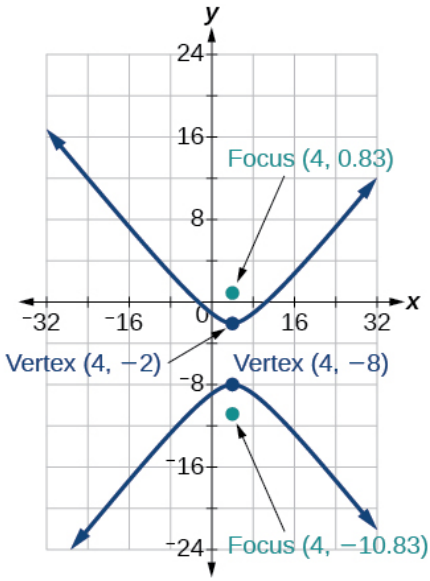
**Answer**




34) $81x^2 - 9y^2 = 1$

35) $\frac{(y+5)^2}{9} - \frac{(x-4)^2}{25} = 1$

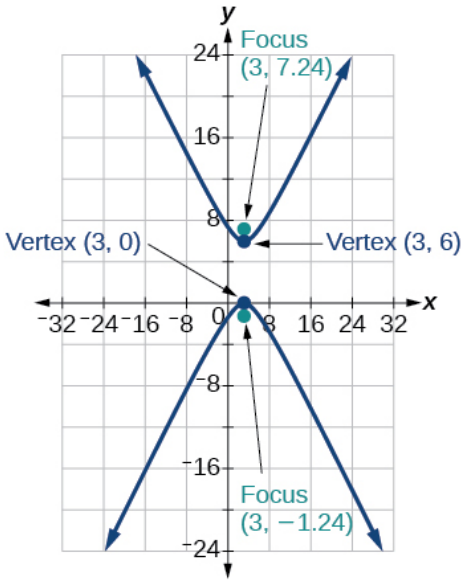
**Answer**




36) $\frac{(x-2)^2}{8} - \frac{(y+3)^2}{27} = 1$

37) $\frac{(y-3)^2}{9} - \frac{(x-3)^2}{9} = 1$

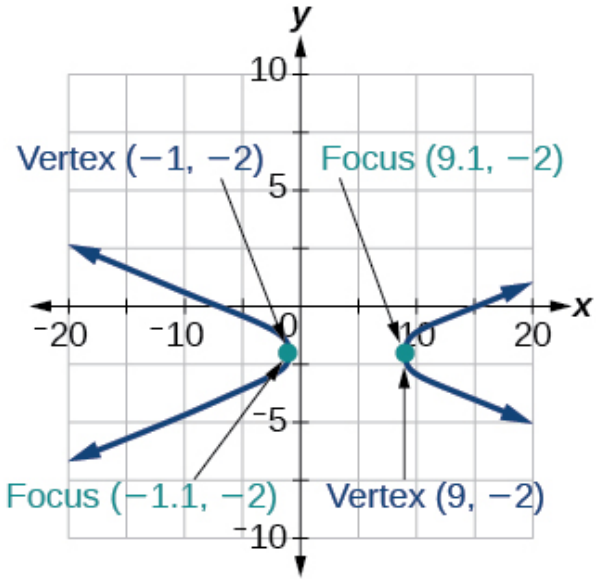
**Answer**




38) $-4x^2 - 8x + 16y^2 - 32y - 52 = 0$

39) $x^2 - 8x - 25y^2 - 100y - 109 = 0$

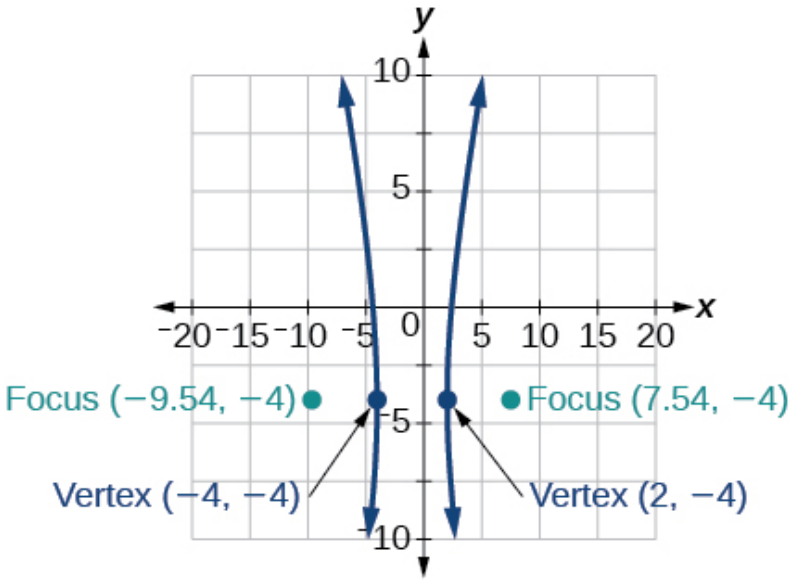
**Answer**




40) $-x^2 + 8x + 4y^2 - 40y + 88 = 0$

41) $64x^2 + 128x - 9y^2 - 72y - 656 = 0$

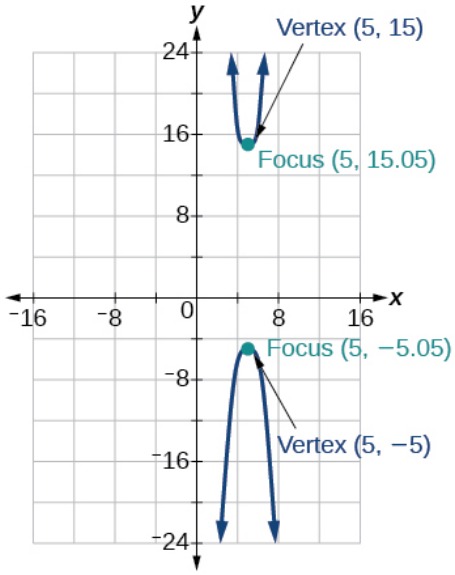
**Answer**




42) $16x^2 + 64x - 4y^2 - 8y - 4 = 0$

43) $-100x^2 + 1000x + y^2 - 10y - 2575 = 0$

**Answer**




44) $4x^2 + 16x - 4y^2 + 16y + 16 = 0$

**For the exercises 45-50, given information about the graph of the hyperbola, find its equation.**

45) Vertices at $(3, 0)$ and $(-3, 0)$ and one focus at $(5, 0)$.

**Answer**

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

46) Vertices at $(0, 6)$ and $(0, -6)$ and one focus at $(0, -8)$.

47) Vertices at $(1, 1)$ and $(11, 1)$ and one focus at $(12, 1)$.

**Answer**

$$\frac{(x-6)^2}{25} - \frac{(y-1)^2}{11} = 1$$

48) Center: $(0, 0)$; vertex: $(0, -13)$; one focus: $(0, \sqrt{313})$.

49) Center: $(4, 2)$; vertex: $(9, 2)$; one focus: $(4 + \sqrt{26}, 2)$.

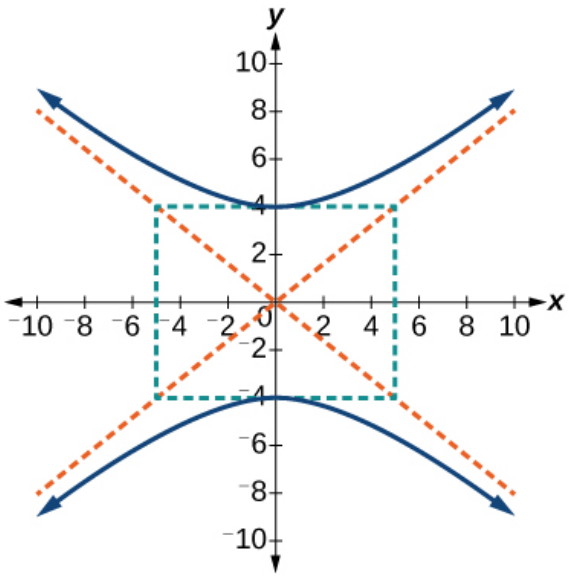
**Answer**

$$\frac{(x-4)^2}{25} - \frac{(y-2)^2}{1} = 1$$

50) Center: $(3, 5)$; vertex: $(3, 11)$ ; one focus: $(3, 5 + 2\sqrt{10})$.

**For the exercises 51-,55 given the graph of the hyperbola, find its equation.**

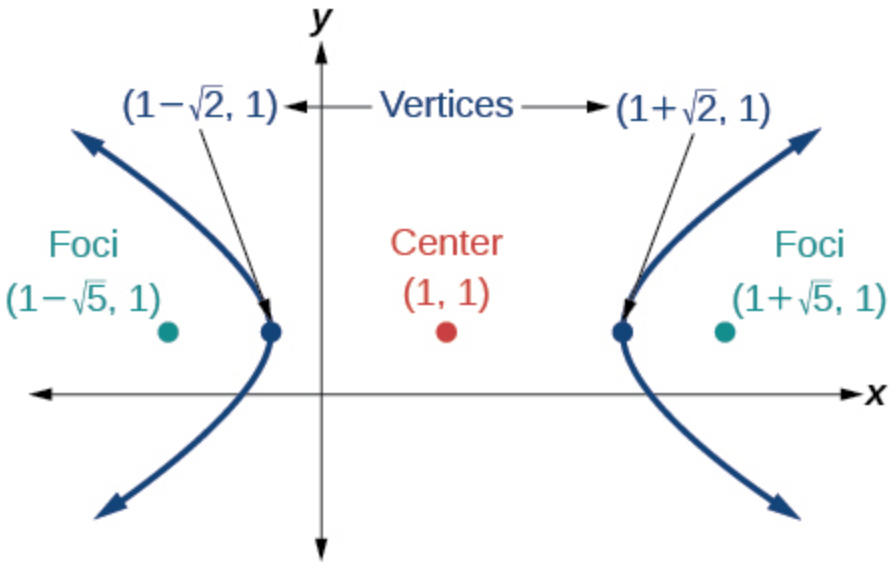
51)



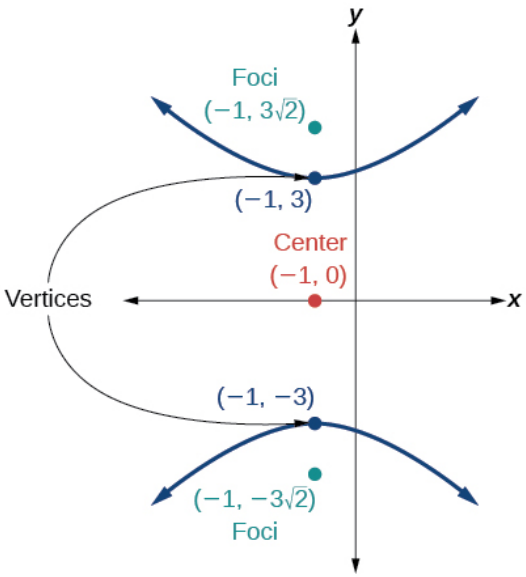

**Answer**

$$\frac{y^2}{16} - \frac{x^2}{25} = 1$$

52)



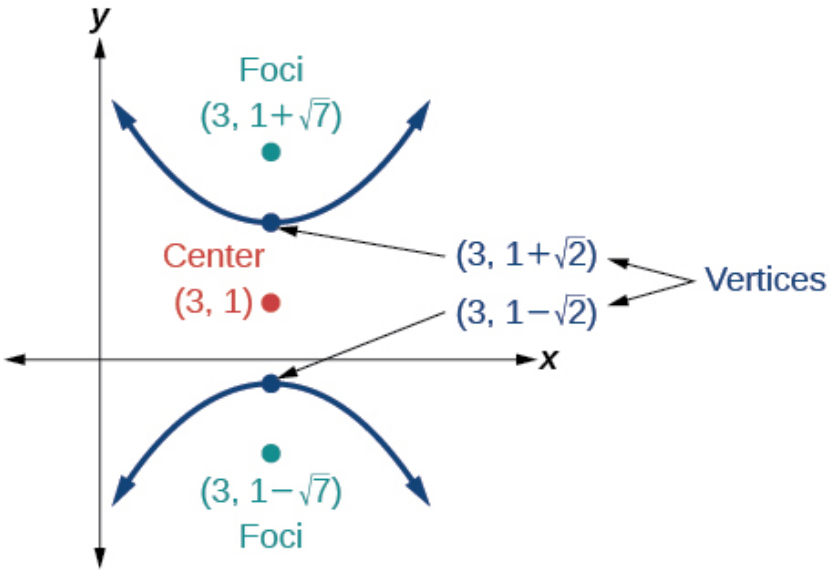

53)



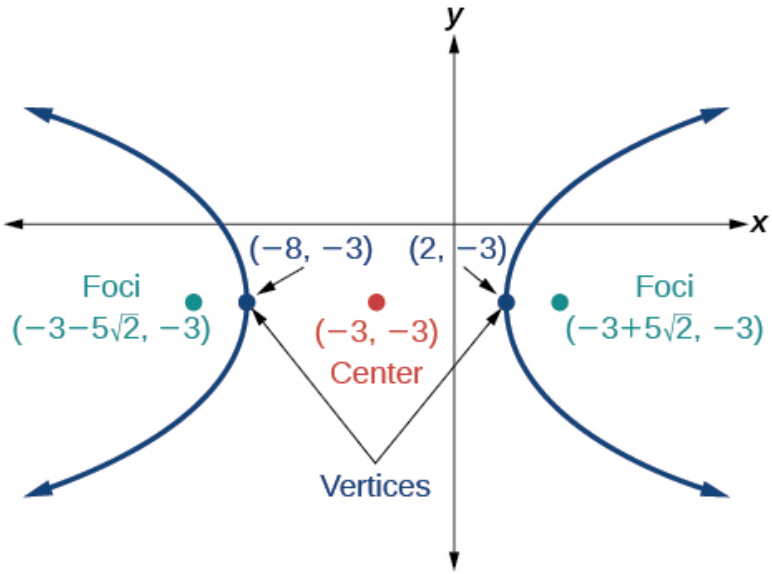

**Answer**

$$\frac{y^2}{9} - \frac{(x+1)^2}{9} = 1$$

54)




55)




**Answer**

$$\frac{(x+3)^2}{25} - \frac{(y+3)^2}{25} = 1$$

## Extensions

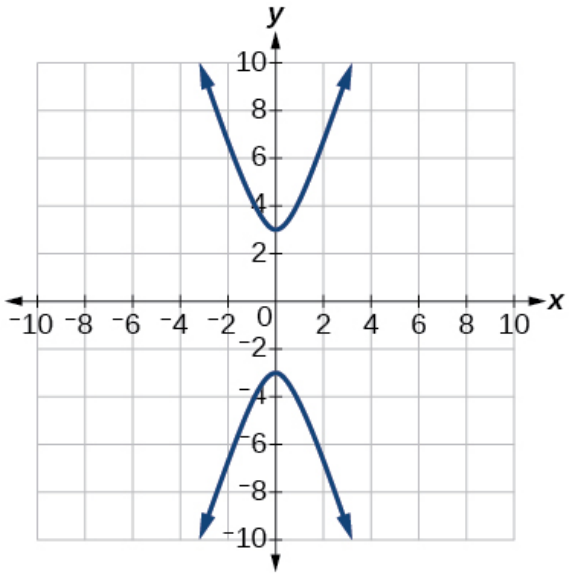
**For the exercises 56-60, express the equation for the hyperbola as two functions, with $y$ as a function of $x$. Express as simply as possible. Use a graphing calculator to sketch the graph of the two functions on the same axes.**

56) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

57) $\frac{y^2}{9} - \frac{x^2}{1} = 1$

**Answer**

$y(x) = 3\sqrt{x^2+1}$, $y(x) = -3\sqrt{x^2+1}$

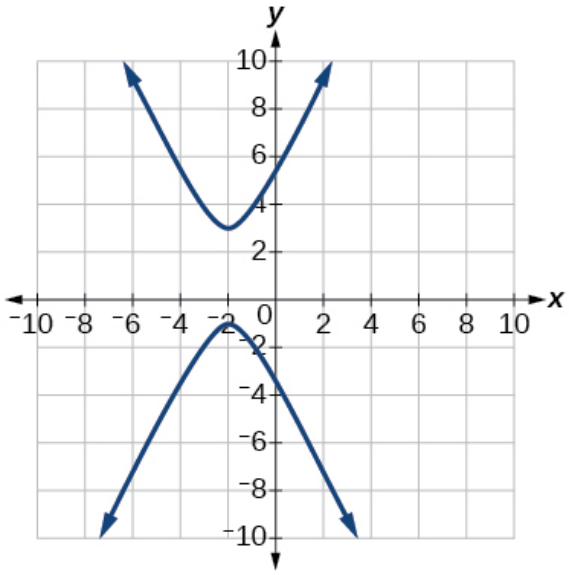



58) $\frac{(x-2)^2}{16} - \frac{(y+3)^2}{25} = 1$

59) $-4x^2 - 16x + y^2 - 2y - 19 = 0$

**Answer**

$y(x) = 1 + 2\sqrt{x^2+4x+5}$, $y(x) = 1 - 2\sqrt{x^2+4x+5}$




60) $4x^2 - 24x - y^2 - 4y + 16 = 0$

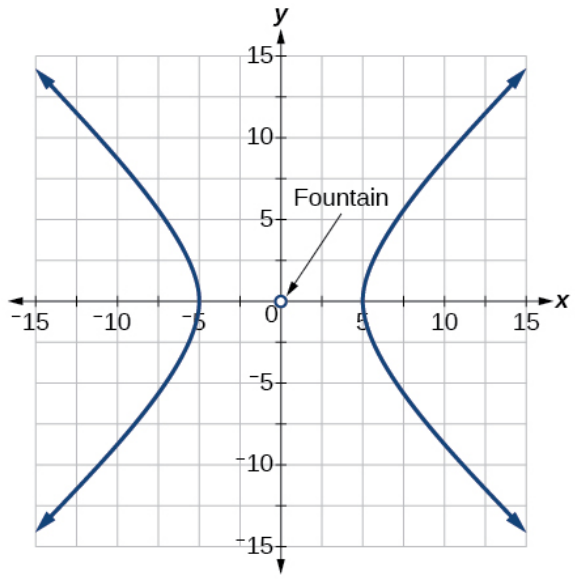
## Real-World Applications

**For the exercises 61-65, a hedge is to be constructed in the shape of a hyperbola near a fountain at the center of the yard. Find the equation of the hyperbola and sketch the graph.**

61) The hedge will follow the asymptotes $y = x$ and $y = -x$, and its closest distance to the center fountain is $5$ yards.

**Answer**

$$\frac{x^2}{25} - \frac{y^2}{25} = 1$$

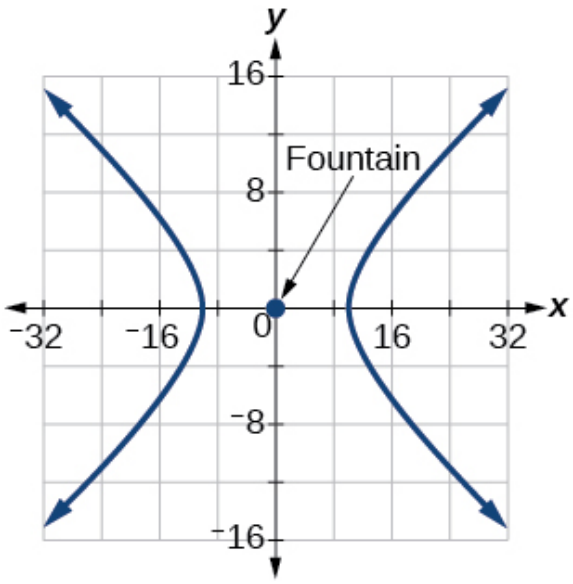



62) The hedge will follow the asymptotes $y = 2x$ and $y = -2x$, and its closest distance to the center fountain is $6$ yards.

63) The hedge will follow the asymptotes $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$, and its closest distance to the center fountain is $10$ yards.

**Answer**

$$\frac{x^2}{100} - \frac{y^2}{25} = 1$$

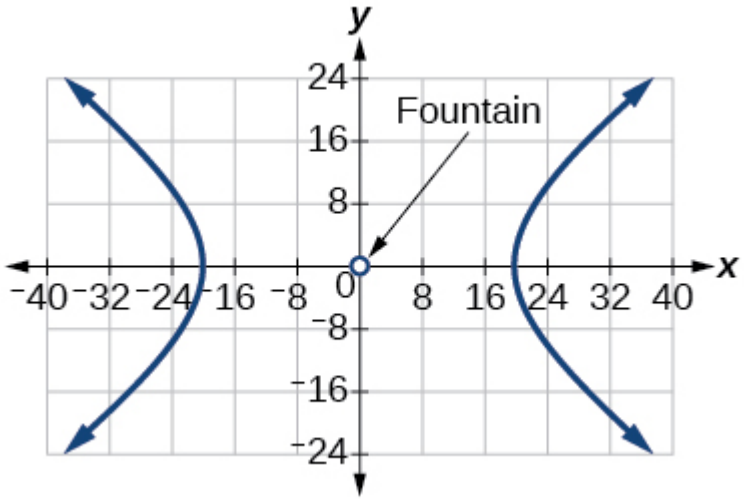



64) The hedge will follow the asymptotes $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$, and its closest distance to the center fountain is $12$ yards.

65) The hedge will follow the asymptotes $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$, and its closest distance to the center fountain is $20$ yards.

**Answer**

$$\frac{x^2}{400} - \frac{y^2}{225} = 1$$




**For the exercises 66-70, assume an object enters our solar system and we want to graph its path on a coordinate system with the sun at the origin and the $x$-axis as the axis of symmetry for the object's path. Give the equation of the flight path of each object using the given information.**

66) The object enters along a path approximated by the line $y = x - 2$ and passes within $1$ au (astronomical unit) of the sun at its closest approach, so that the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -x + 2$.

67) The object enters along a path approximated by the line $y = 2x - 2$ and passes within $0.5$ au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -2x + 2$.

**Answer**

$$\frac{(x-1)^2}{0.25} - \frac{y^2}{0.75} = 1$$

68) The object enters along a path approximated by the line $y = 0.5x + 2$ and passes within $1$ au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -0.5x - 2$.

69) The object enters along a path approximated by the line $y = \frac{1}{3}x - 1$ and passes within $1$ au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -\frac{1}{3}x + 1$.

**Answer**

$$(x-3)^2 - 9y^2 = 4$$

70) The object enters along a path approximated by the line $y = 3x - 9$ and passes within $1$ au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -3x + 9$.

# 10.3: The Parabola

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the

directrix.

## Verbal

1) Define a parabola in terms of its focus and directrix.

**Answer**

A parabola is the set of points in the plane that lie equidistant from a fixed point, the focus, and a fixed line, the directrix.

2) If the equation of a parabola is written in standard form and $p$ is positive and the directrix is a vertical line, then what can we conclude about its graph?

3) If the equation of a parabola is written in standard form and $p$ is negative and the directrix is a horizontal line, then what can we conclude about its graph?

**Answer**

The graph will open down.

4) What is the effect on the graph of a parabola if its equation in standard form has increasing values of $p$?

5) As the graph of a parabola becomes wider, what will happen to the distance between the focus and directrix?

**Answer**

The distance between the focus and directrix will increase.

## Algebraic

**For the exercises 6-10, determine whether the given equation is a parabola. If so, rewrite the equation in standard form.**

6) $y^2 = 4 - x^2$

7) $y = 4x^2$

**Answer**

yes $y = 4(1)x^2$

8) $3x^2 - 6y^2 = 12$

9) $(y-3)^2 = 8(x-2)$

**Answer**

yes $(y-3)^2 = 4(2)(x-2)$

10) $y^2 + 12x - 6y - 51 = 0$

**For the exercises 11-30, rewrite the given equation in standard form, and then determine the vertex $(V)$, focus $(F)$, and directrix $(d)$ of the parabola.**

11) $x = 8y^2$

**Answer**

$y^2 = \frac{1}{8}x$, $V : (0,0)$, $F : \left(\frac{1}{32}, 0\right)$, $d : x = -\frac{1}{32}$

12) $y = \frac{1}{4}x^2$

13) $y = -4x^2$

**Answer**

$$x^2 = -\frac{1}{4}y,\ V : (0,0),\ F : \left(0, -\frac{1}{16}\right),\ d : y = -\frac{1}{16}$$

14) $x = \frac{1}{8}y^2$

15) $x = 36y^2$

**Answer**

$$y^2 = \frac{1}{36}x,\ V : (0,0),\ F : \left(\frac{1}{144}, 0\right),\ d : x = -\frac{1}{144}$$

16) $x = \frac{1}{36}y^2$

17) $(x-1)^2 = 4(y-1)$

**Answer**

$$(x-1)^2 = 4(y-1),\ V : (1,1),\ F : (1,2),\ d : y = 0$$

18) $(y-2)^2 = \frac{4}{5}(x+4)$

19) $(y-4)^2 = 2(x+3)$

**Answer**

$$(y-4)^2 = 2(x+3),\ V : (-3,4),\ F : \left(-\frac{5}{2}, 4\right),\ d : x = -\frac{7}{2}$$

20) $(x+1)^2 = 2(y+4)$

21) $(x+4)^2 = 24(y+1)$

**Answer**

$$(x+4)^2 = 24(y+1),\ V : (-4,-1),\ F : (-4,5),\ d : y = -7$$

22) $(y+4)^2 = 16(x+4)$

23) $y^2 + 12x - 6y + 21 = 0$

**Answer**

$$(y-3)^2 = -12(x+1),\ V : (-1,3),\ F : (-4,3),\ d : x = 2$$

24) $x^2 - 4x - 24y + 28 = 0$

25) $5x^2 - 50x - 4y + 113 = 0$

**Answer**

$$(x-5)^2 = \frac{4}{5}(y+3),\ V : (5,-3),\ F : \left(5, -\frac{14}{5}\right),\ d : y = -\frac{16}{5}$$

26) $y^2 - 24x + 4y - 68 = 0$

27) $x^2 - 4x + 2y - 6 = 0$

**Answer**

$$(x-2)^2 = -2(y-5),\ V:(2,5),\ F:\left(2,\frac{9}{2}\right),\ d: y = \frac{11}{2}$$

28) $y^2 - 6y + 12x - 3 = 0$

29) $3y^2 - 4x - 6y + 23 = 0$

**Answer**

$$(y-1)^2 = \frac{4}{3}(x-5),\ V:(5,1),\ F:\left(\frac{16}{3},1\right),\ d: x = \frac{14}{3}$$

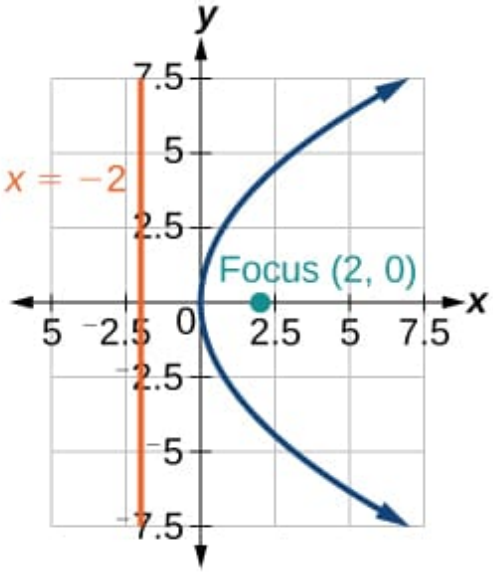
30) $x^2 + 4x + 8y - 4 = 0$

## Graphical

**For the exercises 31-44, graph the parabola, labeling the focus and the directrix.**

31) $x = \frac{1}{8}y^2$

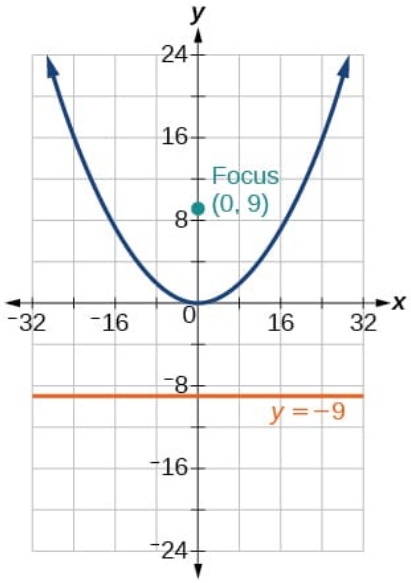
**Answer**




32) $y = 36x^2$

33) $y = \frac{1}{36}x^2$

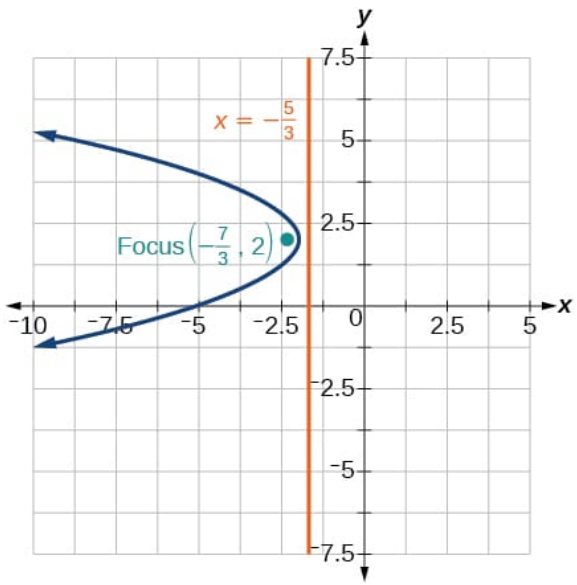
**Answer**




34) $y = -9x^2$

35) $(y-2)^2 = -\frac{4}{3}(x+2)$

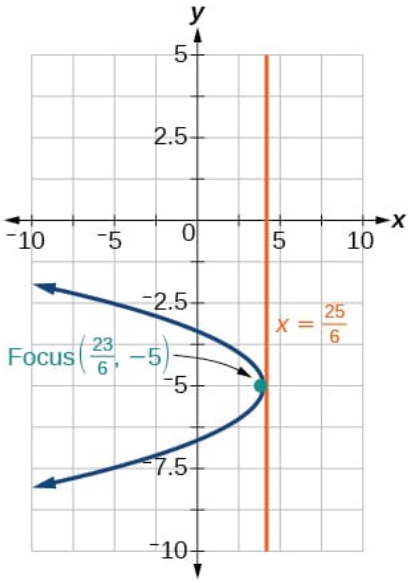
**Answer**




36) $-5(x+5)^2 = 4(y+5)$

37) $-6(y+5)^2 = 4(x-4)$

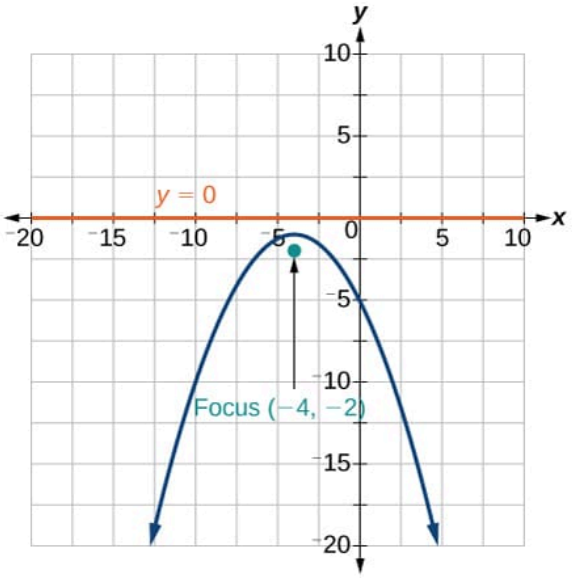
**Answer**




38) $y^2 - 6y - 8x + 1 = 0$

39) $x^2 + 8x + 4y + 20 = 0$

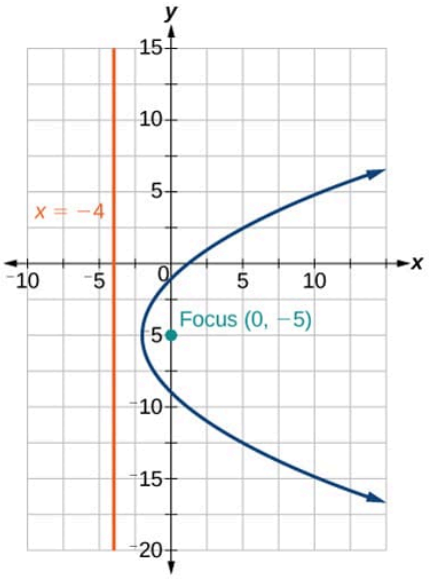
**Answer**




40) $3x^2 + 30x - 4y + 95 = 0$

41) $y^2 - 8x + 10y + 9 = 0$

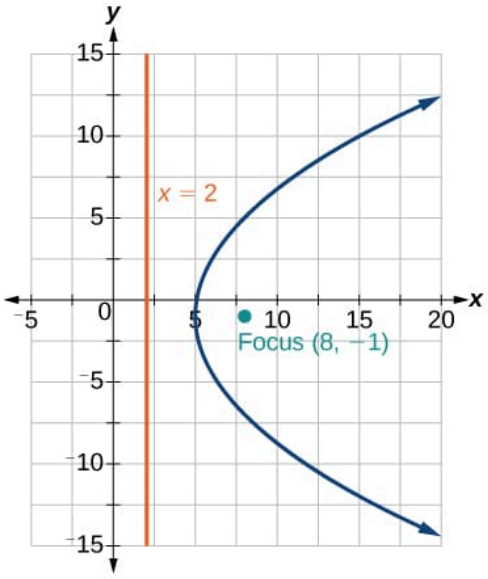
**Answer**




42) $x^2 + 4x + 2y + 2 = 0$

43) $y^2 + 2y - 12x + 61 = 0$

**Answer**




44) $-2x^2 + 8x - 4y - 24 = 0$

**For the exercises 45-50, find the equation of the parabola given information about its graph.**

45) Vertex is $(0, 0)$; directrix is $y = 4$, focus is $(0, -4)$.

**Answer**

$$x^2 = -16y$$

46) Vertex is $(0, 0)$; directrix is $x = 4$, focus is $(-4, 0)$.

47) Vertex is $(2, 2)$; directrix is $x = 2 - \sqrt{2}$, focus is $(2 + \sqrt{2}, 2)$.

**Answer**

$$(y-2)^2 = 4\sqrt{2}(x-2)$$

48) Vertex is $(-2, 3)$; directrix is $x = -\frac{7}{2}$, focus is $\left(-\frac{1}{2}, 3\right)$.

49) Vertex is $(\sqrt{2}, -\sqrt{3})$; directrix is $x = 2\sqrt{2}$, focus is $(0, -\sqrt{3})$.

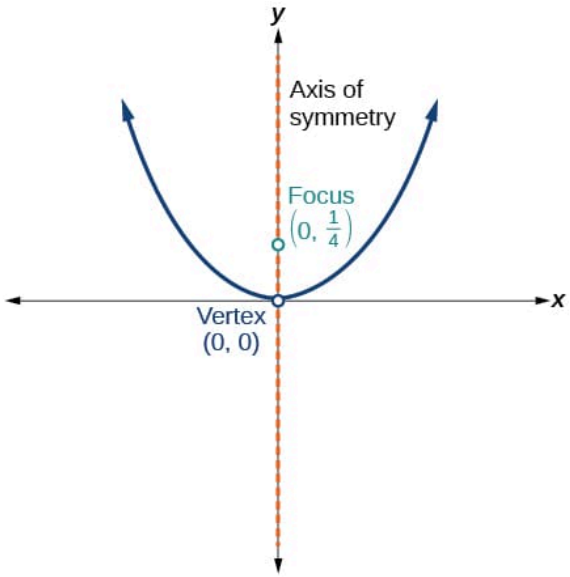
**Answer**

$$(y+\sqrt{3})^2 = -4\sqrt{2}(x-\sqrt{2})$$

50) Vertex is $(-, 21)$; directrix is $y = \frac{11}{3}$, focus is $\left(1, \frac{1}{3}\right)$.

**For the exercises 51-55, determine the equation for the parabola from its graph.**

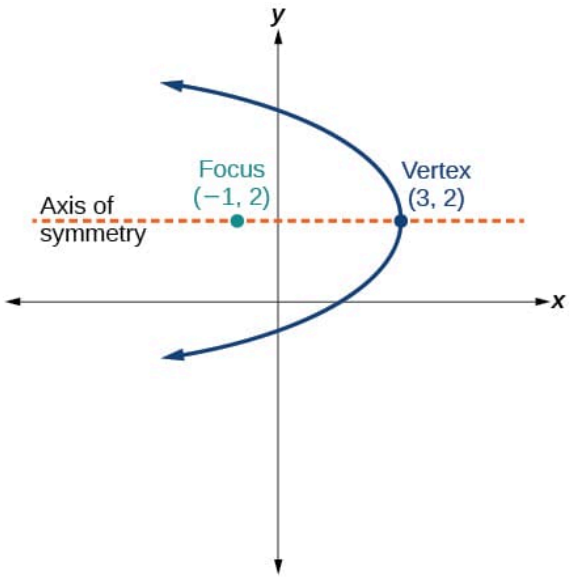
51)



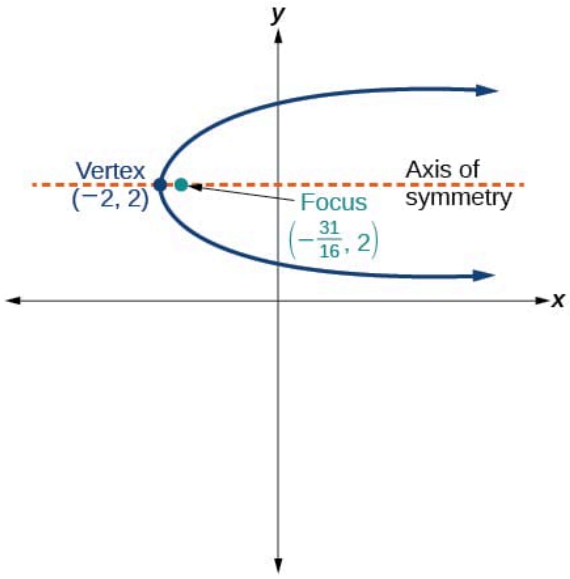

**Answer**

$$x^2 = y$$

52)



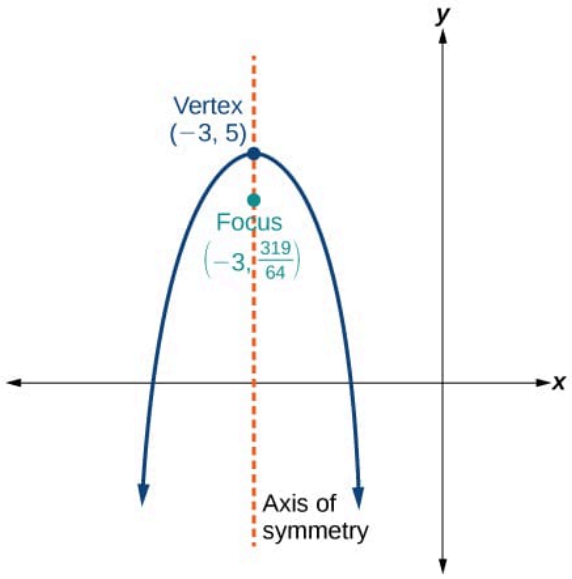

53)



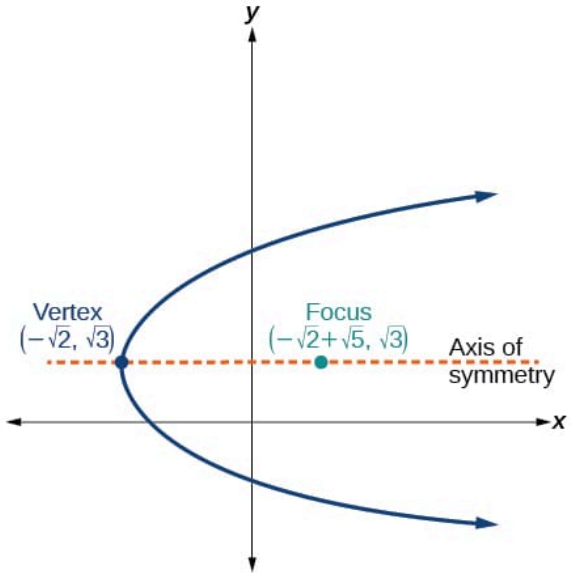

**Answer**

$$(y-2)^2 = \frac{1}{4}(x+2)$$

54)




55)




**Answer**

$$(y-\sqrt{2})^2=4\sqrt{5}(x+\sqrt{2})$$

## Extensions

**For the exercises 56-60, the vertex and endpoints of the latus rectum of a parabola are given. Find the equation.**

56) $V(0,0)$, Endpoints $(2,1),(-2,1)$

57) $V(0,0)$, Endpoints $(-2,4),(-2,-4)$

**Answer**

$$y^2=-8x$$

58) $V(1,2)$, Endpoints $(-5,5),(7,5)$

59) $V(-3,-1)$, Endpoints $(0,5),(0,-7)$

**Answer**

$$(y+1)^2=12(x+3)$$

60) $V(4,-3)$, Endpoints $\left(5,-\frac{7}{2}\right),\left(3,-\frac{7}{2}\right)$

## Real-World Applications

61) The mirror in an automobile headlight has a parabolic cross-section with the light bulb at the focus. On a schematic, the equation of the parabola is given as $x^2=4y$. At what coordinates should you place the light bulb?

**Answer**

$$(0,1)$$

62) If we want to construct the mirror from the previous exercise such that the focus is located at $(0,0.25)$ what should the equation of the parabola be?

63) A satellite dish is shaped like a paraboloid of revolution. This means that it can be formed by rotating a parabola around its axis of symmetry. The receiver is to be located at the focus. If the dish is $12$ feet across at its opening and $4$ feet deep at its center, where should the receiver be placed?

**Answer**

At the point $2.25$ feet above the vertex.

64) Consider the satellite dish from the previous exercise. If the dish is $8$ feet across at the opening and $2$ feet deep, where should we place the receiver?

65) A searchlight is shaped like a paraboloid of revolution. A light source is located $1$ foot from the base along the axis of symmetry. If the opening of the searchlight is $3$ feet across, find the depth.

**Answer**

$0.5625$ feet

66) If the searchlight from the previous exercise has the light source located $6$ inches from the base along the axis of symmetry and the opening is $4$ feet, find the depth.

67) An arch is in the shape of a parabola. It has a span of $100$ feet and a maximum height of $20$ feet. Find the equation of the parabola, and determine the height of the arch $40$ feet from the center.

**Answer**

$x^2=-125(y-20)$, height is $7.2$ feet

68) If the arch from the previous exercise has a span of $160$ feet and a maximum height of $40$ feet, find the equation of the parabola, and determine the distance from the center at which the height is $20$ feet.

69) An object is projected so as to follow a parabolic path given by $y=-x^2+96x$ where $x$ is the horizontal distance traveled in feet and $y$ is the height. Determine the maximum height the object reaches.

**Answer**

$2304$ feet

70) For the object from the previous exercise, assume the path followed is given by $y = -0.5x^2 + 80x$. Determine how far along the horizontal the object traveled to reach maximum height.

# 10.4: Rotation of Axes

In previous sections of this chapter, we have focused on the standard form equations for nondegenerate conic sections. In this section, we will shift our focus to the general form equation, which can be used for any conic. The general form is set equal to zero, and the terms and coefficients are given in a particular order, as shown below.

## Verbal

1) What effect does the $xy$ term have on the graph of a conic section?

**Answer**

The $xy$ term causes a rotation of the graph to occur.

2) If the equation of a conic section is written in the form $Ax^2 + By^2 + Cx + Dy + E = 0$ and $AB = 0$ what can we conclude?

3) If the equation of a conic section is written in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, and $B^2 - 4AC > 0$, what can we conclude?

**Answer**

The conic section is a hyperbola.

4) Given the equation $ax^2 + 4x + 3y^2 - 12 = 0$, what can we conclude if $a > 0$?

5) For the equation \(Ax^2+Bxy+Cy^2+Dx+Ey+F=0\) the value of $\theta$ that satisfies $\cot(2\theta) = \dfrac{A-C}{B}$ gives us what information?

**Answer**

It gives the angle of rotation of the axes in order to eliminate the $xy$ term.

## Algebraic

**For the exercises 6-17, determine which conic section is represented based on the given equation.**

6) $9x^2 + 4y^2 + 72x + 36y - 500 = 0$

7) $x^2 - 10x + 4y - 10 = 0$

**Answer**

$AB = 0$, parabola

8) $2x^2 - 2y^2 + 4x - 6y - 2 = 0$

9) $4x^2 - y^2 + 8x - 1 = 0$

**Answer**

$AB = -4 < 0$, hyperbola

10) $4y^2 - 5x + 9y + 1 = 0$

11) $2x^2 + 3y^2 - 8x - 12y + 2 = 0$

**Answer**

$AB = 6 > 0$, ellipse

12) $4x^2 + 9xy + 4y^2 - 36y - 125 = 0$

13) $3x^2 + 6xy + 3y^2 - 36y - 125 = 0$

**Answer**

$B^2 - 4AC = 0$, parabola

14) $-3x^2 + 3\sqrt{3}xy - 4y^2 + 9 = 0$

15) $2x^2 + 4\sqrt{3}xy + 6y^2 - 6x - 3 = 0$

**Answer**

$B^2 - 4AC = 0$, parabola

16) $-x^2 + 4\sqrt{2}xy + 2y^2 - 2y + 1 = 0$

17) $8x^2 + 4\sqrt{2}xy + 4y^2 - 10x + 1 = 0$

**Answer**

$B^2 - 4AC = -96 < 0$, ellipse

**For the exercises 18-22, find a new representation of the given equation after rotating through the given angle.**

18) $3x^2 + xy + 3y^2 - 5 = 0, \theta = 45^\circ$

19) $4x^2 - xy + 4y^2 - 2 = 0, \theta = 45^\circ$

**Answer**

$7x'^2 + 9y'^2 - 4 = 0$

20) $2x^2 + 8xy - 1 = 0, \theta = 30^\circ$

21) $-2x^2 + 8xy + 1 = 0, \theta = 45^\circ$

**Answer**

$3x'^2 + 2x'y' - 5y'^2 + 1 = 0$

22) $4x^2 + \sqrt{2}xy + 4y^2 + y + 2 = 0, \theta = 45^\circ$

**For the exercises 23-30, determine the angle $\theta$ that will eliminate the $xy$ term and write the corresponding equation without the $xy$ term.**

23) $x^2 + 3\sqrt{3}xy + 4y^2 + y - 2 = 0$

**Answer**

$\theta = 60^\circ, 11x'^2 - y'2 + \sqrt{3}x' + y' - 4 = 0$

24) $4x^2 + 2\sqrt{3}xy + 6y^2 + y - 2 = 0$

25) $9x^2 - 3\sqrt{3}xy + 6y^2 + 4y - 3 = 0$

**Answer**

$$\theta = 150^\circ, 21x'^2 + 9y'^2 + 4x' - 4\sqrt{3}y' - 6 = 0$$

26) $-3x^2 - \sqrt{3}xy - 2y^2 - x = 0$

27) $16x^2 + 24xy + 9y^2 + 6x - 6y + 2 = 0$

**Answer**

$$\theta \approx 36.9^\circ, 125x'^2 + 6x' - 42y' + 10 = 0$$

28) $x^2 + 4xy + 4y^2 + 3x - 2 = 0$

29) $x^2 + 4xy + y^2 - 2x + 1 = 0$

**Answer**

$$\theta = 45^\circ, 3x'^2 - y'^2 - \sqrt{2}x' + \sqrt{2}y' + 1 = 0$$

30) $4x^2 - 2\sqrt{3}xy + 6y^2 - 1 = 0$

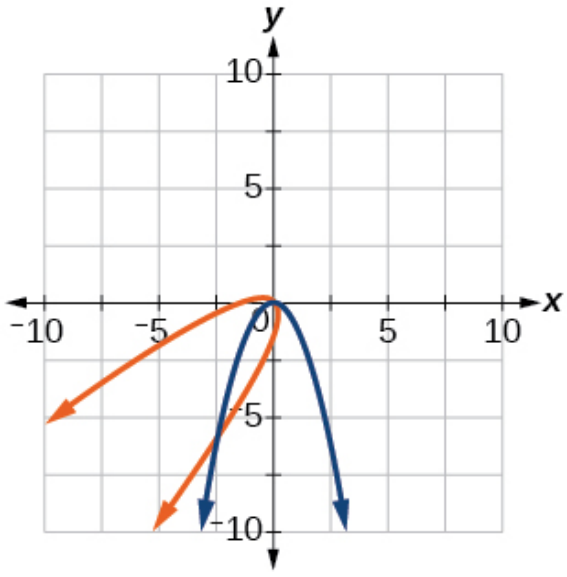
## Graphical

**For the exercises 31-38, rotate through the given angle based on the given equation. Give the new equation and graph the original and rotated equation.**

31) $y = -x^2, \theta = -45^\circ$

**Answer**

$$\frac{\sqrt{2}}{2}(x' + y') = \frac{1}{2}(x' - y')^2$$

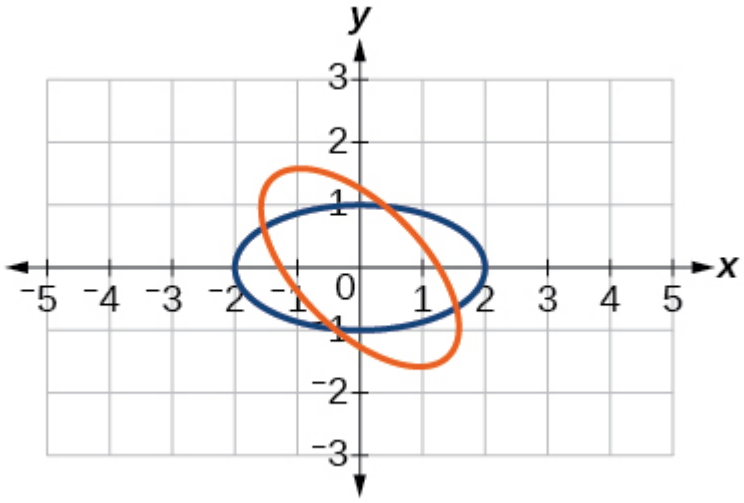



32) $x = y^2, \theta = 45^\circ$

33) $\frac{x^2}{4}+\frac{y^2}{1}=1, \theta=45^\circ$

**Answer**

$$\frac{(x'-y')^2}{8}+\frac{(x'+y')^2}{2}=1$$

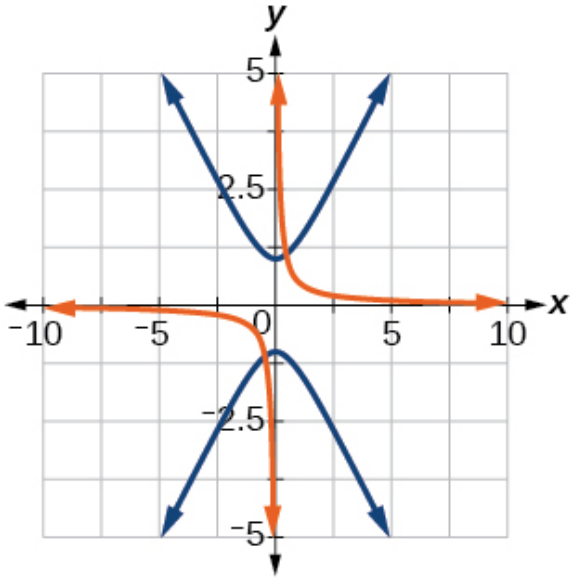



34) $\frac{y^2}{16}+\frac{x^2}{9}=1, \theta=45^\circ$

35) $y^2-x^2=1, \theta=45^\circ$

**Answer**

$$\frac{(x'+y')^2}{2}-\frac{(x'-y')^2}{2}=1$$

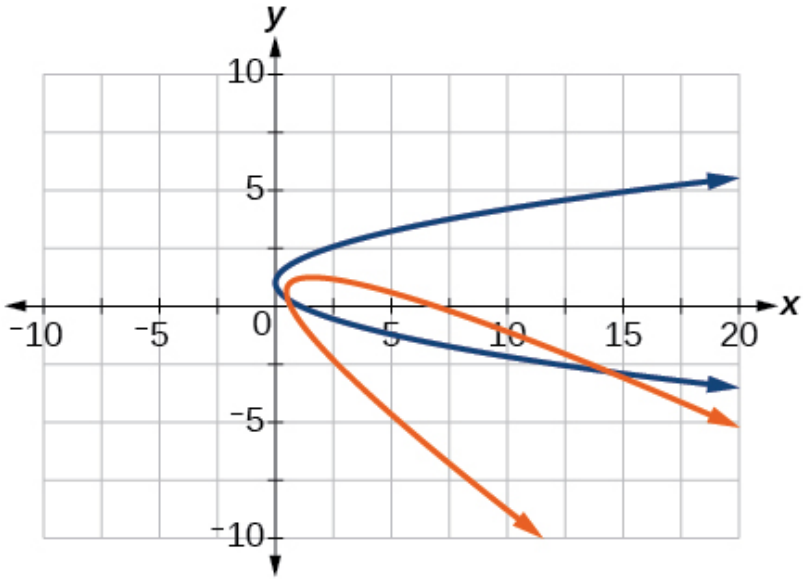



36) $y=\frac{x^2}{2}, \theta=30^\circ$

37) $x=(y-1)^2, \theta=30^\circ$

**Answer**

$$\frac{\sqrt{3}}{2}x' - \frac{1}{2}y' = \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}x' - 1\right)^2$$

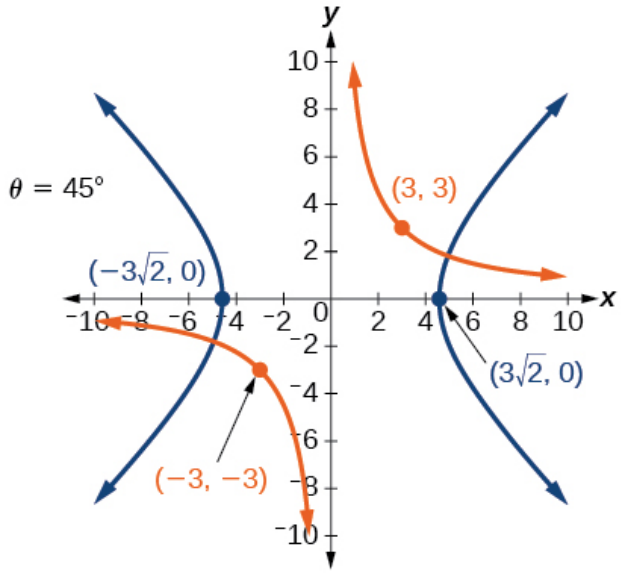



38) $\frac{x^2}{9} + \frac{y^2}{4} = 1, \theta = 30°$

**For the exercises 39-49, graph the equation relative to the $x'y'$ system in which the equation has no $x'y'$ term.**

39) $xy = 9$

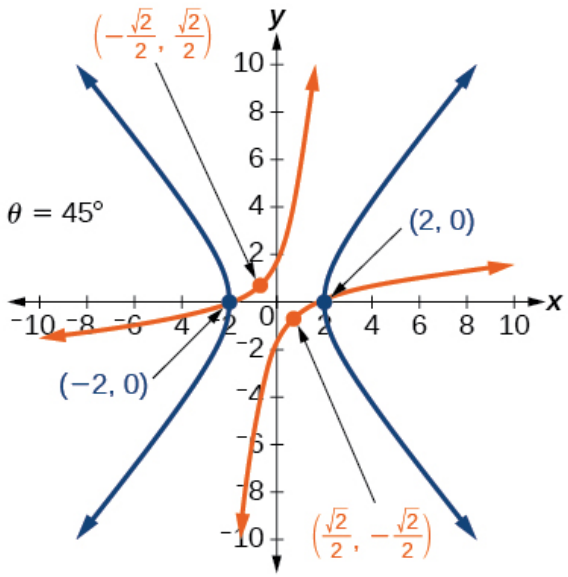
**Answer**




40) $x^2 + 10xy + y^2 - 6 = 0$

41) $x^2 - 10xy + y^2 - 24 = 0$

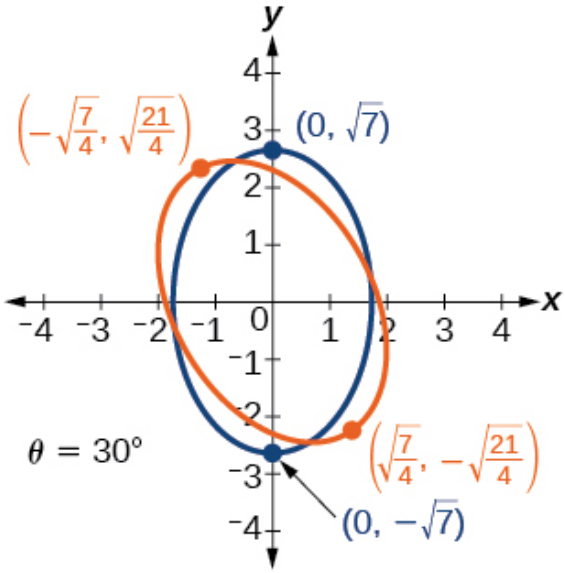
**Answer**




42) $4x^2 - 3\sqrt{3}xy + y^2 - 22 = 0$

43) $6x^2 + 2\sqrt{3}xy + 4y^2 - 21 = 0$

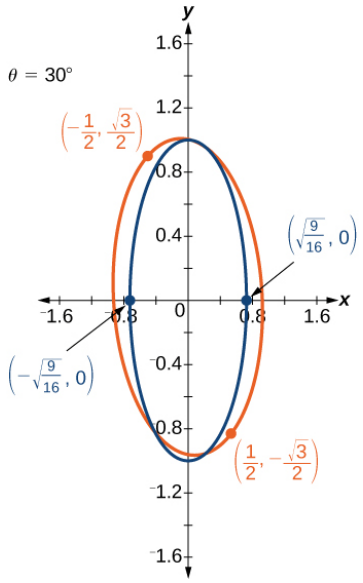
**Answer**




44) $11x^2 + 10\sqrt{3}xy + y^2 - 64 = 0$

45) $21x^2 + 2\sqrt{3}xy + 19y^2 - 18 = 0$

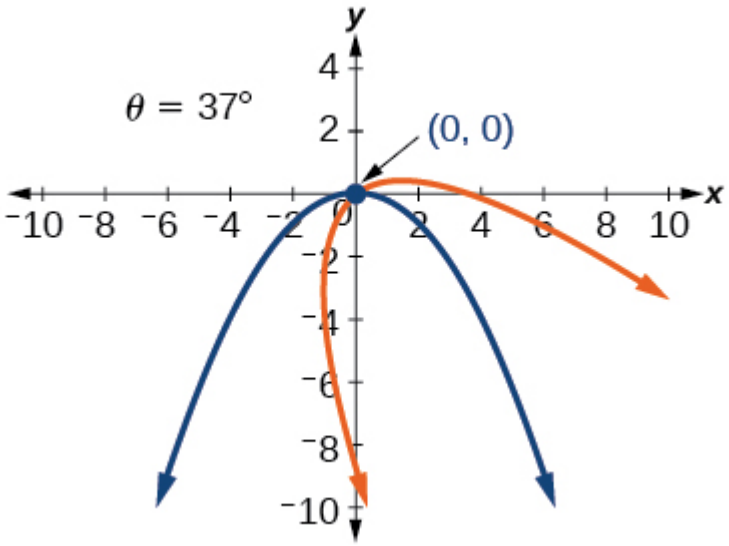
**Answer**




46) $16x^2 + 24xy + 9y^2 - 130x + 90y = 0$

47) $16x^2 + 24xy + 9y^2 - 60x + 80y = 0$

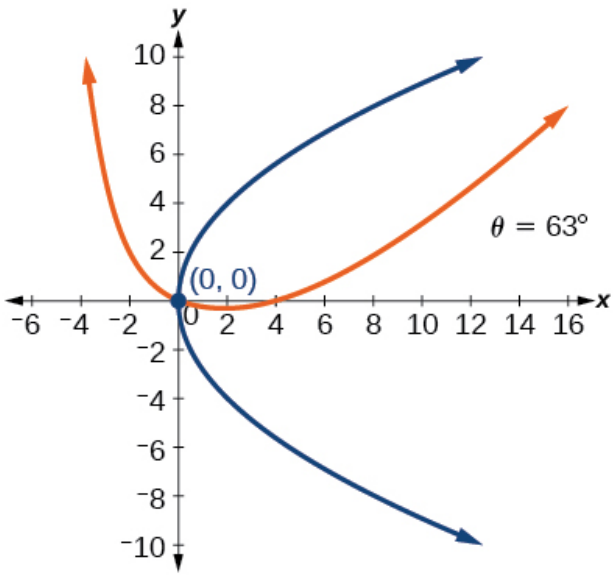
**Answer**




48) $13x^2 - 6\sqrt{3}xy + 7y^2 - 16 = 0$

49) $4x^2 - 4xy + y^2 - 8\sqrt{5}x - 16\sqrt{5}y = 0$

**Answer**



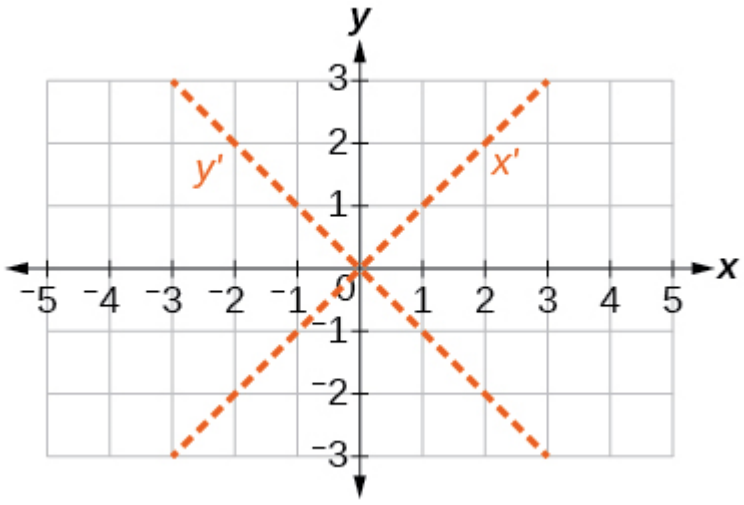

**For the exercises 50-55, determine the angle of rotation in order to eliminate the $xy$ term. Then graph the new set of axes.**

50) $6x^2 - 5\sqrt{3}xy + y^2 + 10x - 12y = 0$

51) $6x^2 - 5xy + 6y^2 + 20x - y = 0$

**Answer**

$\theta = 45^\circ$

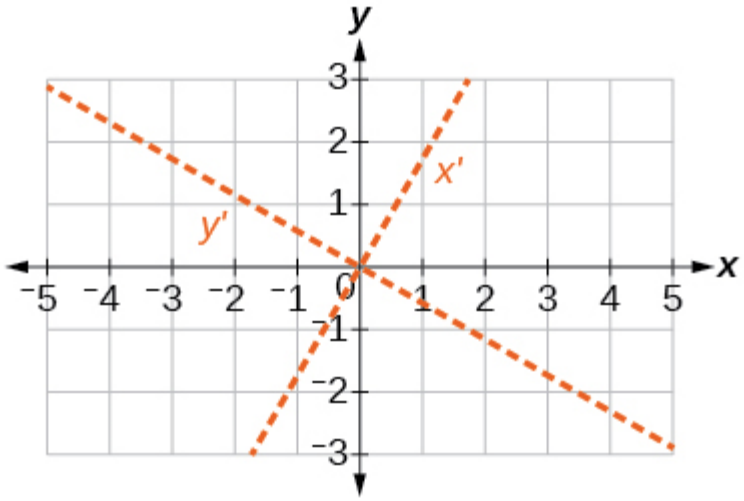



52) $6x^2 - 8\sqrt{3}xy + 14y^2 + 10x - 3y = 0$

53) $4x^2 + 6\sqrt{3}xy + 10y^2 + 20x - 40y = 0$

**Answer**

$\theta = 60^\circ$

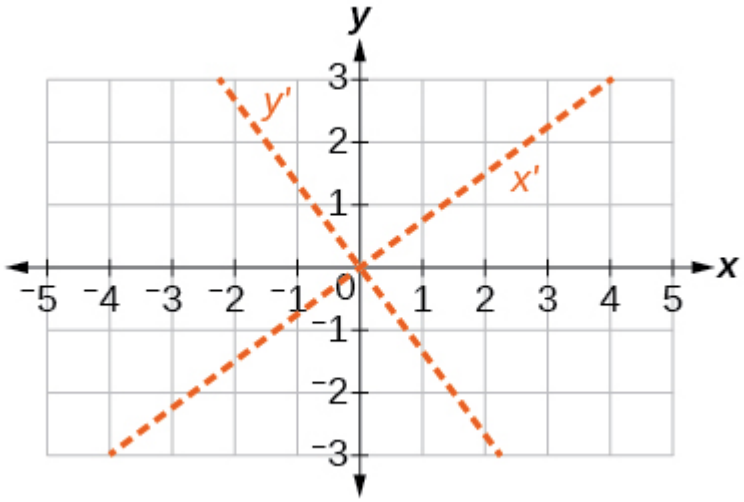



54) $8x^2 + 3xy + 4y^2 + 2x - 4 = 0$

55) $16x^2 + 24xy + 9y^2 + 20x - 44y = 0$

**Answer**

$\theta \approx 36.9^\circ$




**For the exercises 56-60, determine the value of $k$ based on the given equation.**

56) Given $4x^2 + kxy + 16y^2 + 8x + 24y - 48 = 0$, find $k$ for the graph to be a parabola.

57) Given $2x^2 + kxy + 12y^2 + 10x - 16y + 28 = 0$, find $k$ for the graph to be an ellipse.

**Answer**

$-4\sqrt{6} < k < 4\sqrt{6}$

58) Given $3x^2 + kxy + 4y^2 - 6x + 20y + 128 = 0$, find $k$ for the graph to be a hyperbola.

59) Given $kx^2 + 8xy + 8y^2 - 12x + 16y + 18 = 0$ find $k$ for the graph to be a parabola.

**Answer**

$k = 2$

60) Given $6x^2 + 12xy + ky^2 + 16x + 10y + 4 = 0$ find $k$ for the graph to be an ellipse.

# 10.5: Conic Sections in Polar Coordinates

In this section, we will learn how to define any conic in the polar coordinate system in terms of a fixed point, the focus at the pole, and a line, the directrix, which is perpendicular to the polar axis.

## Verbal

1) Explain how eccentricity determines which conic section is given.

**Answer**

If eccentricity is less than $1$, it is an ellipse. If eccentricity is equal to $1$, it is a parabola. If eccentricity is greater than $1$, it is a hyperbola.

2) If a conic section is written as a polar equation, what must be true of the denominator?

3) If a conic section is written as a polar equation, and the denominator involves $\sin\theta$ what conclusion can be drawn about the directrix?

**Answer**

The directrix will be parallel to the polar axis.

4) If the directrix of a conic section is perpendicular to the polar axis, what do we know about the equation of the graph?

5) What do we know about the focus/foci of a conic section if it is written as a polar equation?

**Answer**

One of the foci will be located at the origin.

## Algebraic

**For the exercises 6-17, identify the conic with a focus at the origin, and then give the directrix and eccentricity.**

6) $r = \dfrac{6}{1 - 2\cos\theta}$

7) $r = \dfrac{3}{4 - 4\sin\theta}$

**Answer**

Parabola with $e = 1$ and directrix $\dfrac{3}{4}$ units below the pole.

8) $r = \dfrac{8}{4 - 3\cos\theta}$

9) $r = \dfrac{5}{1 + 2\sin\theta}$

**Answer**

Hyperbola with $e = 2$ and directrix $\frac{5}{2}$ units above the pole.

10) $r = \frac{15}{4 + 3\cos\theta}$

11) $r = \frac{3}{10 + 10\cos\theta}$

**Answer**

Parabola with $e = 1$ and directrix $\frac{3}{10}$ units to the right of the pole.

12) $r = \frac{2}{1 - \cos\theta}$

13) $r = \frac{4}{7 + 2\cos\theta}$

**Answer**

Ellipse with $e = \frac{2}{7}$ and directrix $2$ units to the right of the pole.

14) $r(1 - \cos\theta) = 3$

15) $r(3 + 5\sin\theta) = 11$

**Answer**

Hyperbola with $e = \frac{5}{3}$ and directrix $\frac{11}{5}$ units above the pole.

16) $r(4 - 5\sin\theta) = 1$

17) $r(7 + 8\sin\theta) = 7$

**Answer**

Hyperbola with $e = \frac{8}{7}$ and directrix $\frac{7}{8}$ units to the right of the pole.

**For the exercises 18-30, convert the polar equation of a conic section to a rectangular equation.**

18) $r = \frac{4}{1 + 3\sin\theta}$

19) $r = \frac{2}{5 - 3\sin\theta}$

**Answer**

$25x^2 + 16y^2 - 12y - 4 = 0$

20) $r = \frac{8}{3 - 2\cos\theta}$

21) $r = \frac{3}{2 + 5\cos\theta}$

**Answer**

$21x^2 - 4y^2 - 30x + 9 = 0$

22) $r = \dfrac{4}{2 + 2\sin\theta}$

23) $r = \dfrac{3}{8 - 8\cos\theta}$

**Answer**

$$64y^2 = 48x + 9$$

24) $r = \dfrac{2}{6 + 7\cos\theta}$

25) $r = \dfrac{5}{5 - 11\sin\theta}$

**Answer**

$$96y^2 - 25x^2 + 110y + 25 = 0$$

26) $r(5 + 2\cos\theta) = 6$

27) $r(2 - \cos\theta) = 1$

**Answer**

$$3x^2 + 4y^2 - 2x - 1 = 0$$

28) $r(2.5 - 2.5\sin\theta) = 5$

29) $r = \dfrac{6\sec\theta}{-2 + 3\sec\theta}$

**Answer**

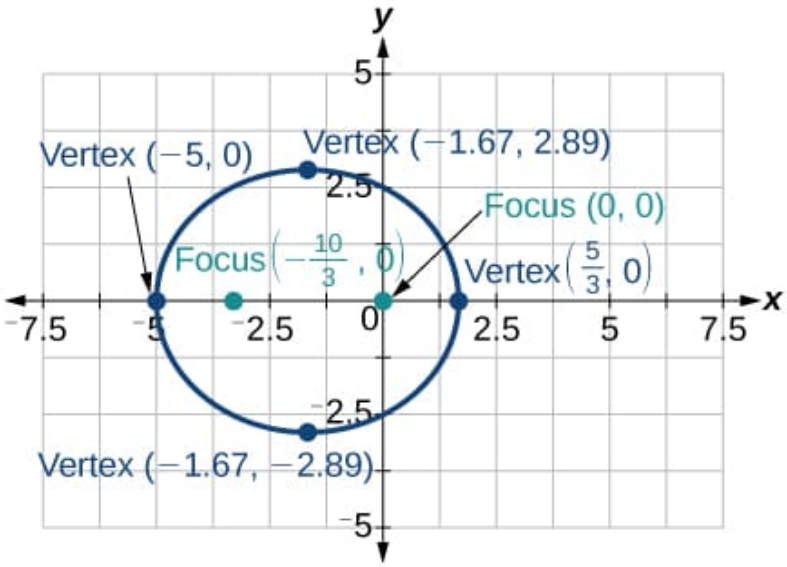
$$5x^2 + 9y^2 - 24x - 36 = 0$$

30) $r = \dfrac{6\csc\theta}{3 + 2\csc\theta}$

**For the exercises 31-42, graph the given conic section. If it is a parabola, label the vertex, focus, and directrix. If it is an ellipse, label the vertices and foci. If it is a hyperbola, label the vertices and foci.**

31) $r = \dfrac{5}{2 + \cos\theta}$

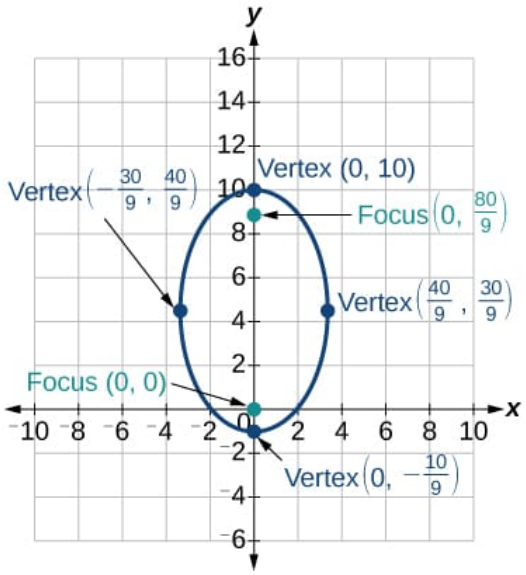
**Answer**




32) $r = \dfrac{2}{3 + 3\sin\theta}$

33) $r = \dfrac{10}{5 - 4\sin\theta}$

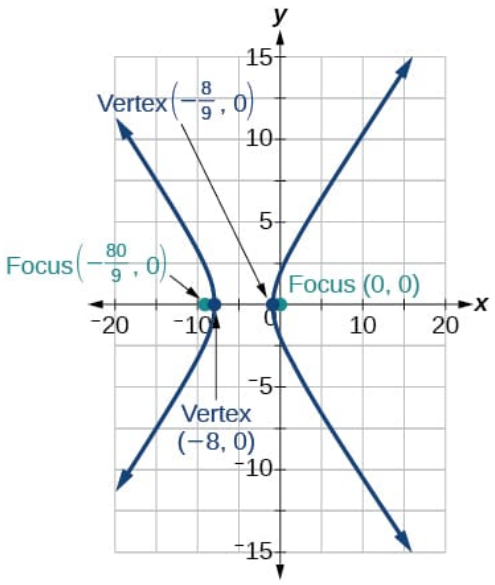
**Answer**




34) $r = \dfrac{3}{1 + 2\cos\theta}$

35) $r = \dfrac{8}{4 - 5\cos\theta}$

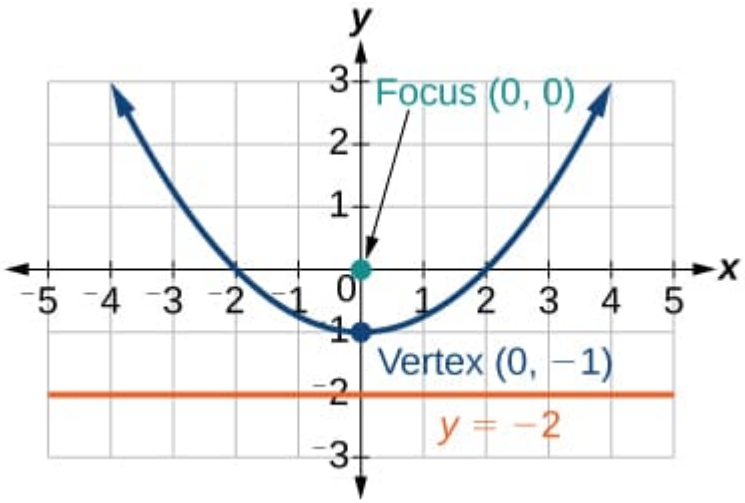
**Answer**




36) $r = \dfrac{3}{4 - 4\cos\theta}$

37) $r = \dfrac{2}{1 - \sin\theta}$

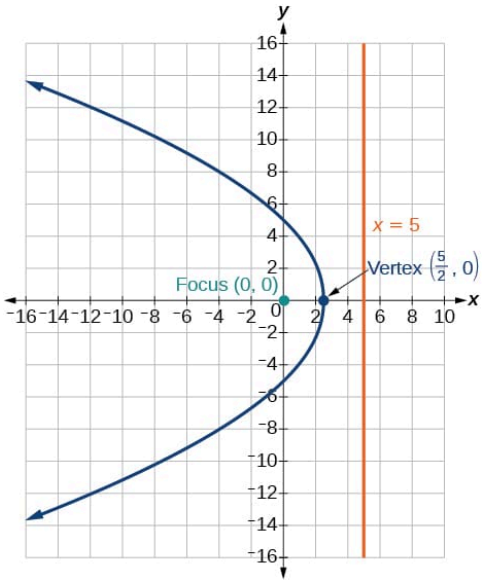
**Answer**




38) $r = \dfrac{6}{3 + 2\sin\theta}$

39) $r(1 + \cos\theta) = 5$

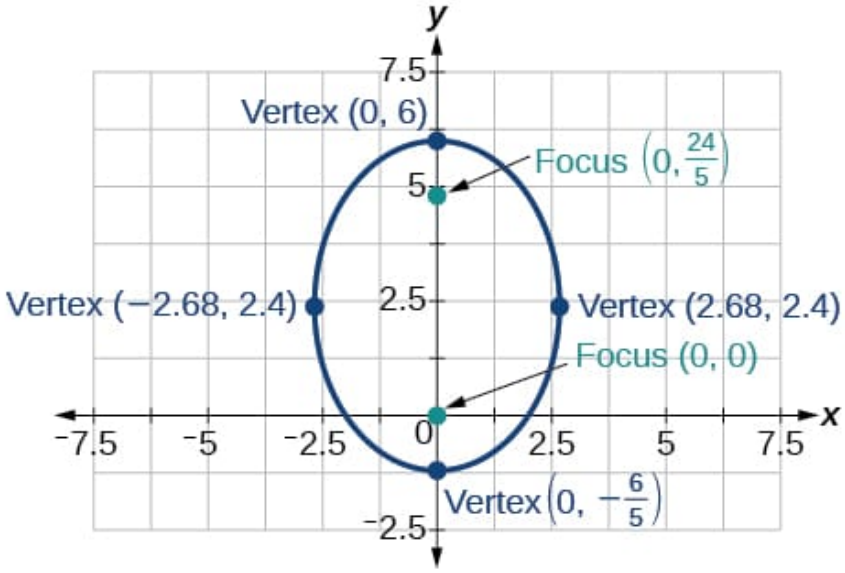
**Answer**




40) $r(3 - 4\sin\theta) = 9$

41) $r(3 - 2\sin\theta) = 6$

**Answer**




42) $r(4 - 6\cos\theta) = 5$

**For the exercises 43-, find the polar equation of the conic with focus at the origin and the given eccentricity and directrix.**

43) Directrix: $x = 4$; $e = \frac{1}{5}$

**Answer**

$$r = \frac{4}{5 + \cos\theta}$$

44) Directrix: $x = -4$; $e = 5$

45) Directrix: $y = 2$; $e = 2$

**Answer**

$$r = \frac{4}{1 + 2\sin\theta}$$

46) Directrix: $y = -2$; $e = \frac{1}{2}$

47) Directrix: $x = 1$; $e = 1$

**Answer**

$$r = \frac{1}{1 + \cos\theta}$$

48) Directrix: $x = -1$; $e = 1$

49) Directrix: $x = -\frac{1}{4}$; $e = \frac{7}{2}$

**Answer**

$$r = \frac{7}{8 - 28\cos\theta}$$

50) Directrix: $y = -\frac{2}{5}$; $e = \frac{7}{2}$

51) Directrix: $y = 4$; $e = \frac{3}{2}$

**Answer**

$$r = \frac{12}{2 + 3\sin\theta}$$

52) Directrix: $x = -2$; $e = \frac{8}{3}$

53) Directrix: $x = -5$; $e = \frac{3}{4}$

**Answer**

$$r = \frac{15}{4 - 3\cos\theta}$$

54) Directrix: $y = 2$; $e = 2.5$

55) Directrix: $x = -3$; $e = \frac{1}{3}$

**Answer**

$$r = \frac{3}{3 - 3\cos\theta}$$

## Extensions

**Recall from Rotation of Axes that equations of conics with an $xy$ term have rotated graphs. For the following exercises, express each equation in polar form with $r$ as a function of $\theta$.**

56) $xy = 2$

57) $x^2 + xy + y^2 = 4$

**Answer**

$$r = \pm \frac{2}{\sqrt{1 + \sin\theta\cos\theta}}$$

58) $2x^2 + 4xy + 2y^2 = 9$

59) $16x^2 + 24xy + 9y^2 = 4$

**Answer**

$$r = \pm \frac{2}{4\cos\theta + 3\sin\theta}$$

60) $2xy + y = 1$

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# 6.R: Analytic Geometry (Review)

## 10.1: The Ellipse

In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.

**For the exercises 1-4, write the equation of the ellipse in standard form. Then identify the center, vertices, and foci.**

1) $\frac{x^2}{25}+\frac{y^2}{64}=1$

**Answer**

$\frac{x^2}{5^2}+\frac{y^2}{8^2}=1$; center: $(0,0)$; Vertices: $(5,0)$, $(-5,0)$, $(0,8)$, $(0,-8)$; foci: $(0,\sqrt{39})$, $(0,-\sqrt{39})$

2) $\frac{(x-2)^2}{100}+\frac{(y+3)^2}{36}=1$

3) $9x^2+y^2+54x-4y+76=0$

**Answer**

$\frac{(x+3)^2}{1^2}+\frac{(y-2)^2}{3^2}=1$; center: $(-3,2)$; Vertices: $(-2,2)$, $(-4,2)$, $(-3,5)$, $(-3,-1)$; foci: $(-3,2+2\sqrt{2})$, $(-3,2-2\sqrt{2})$

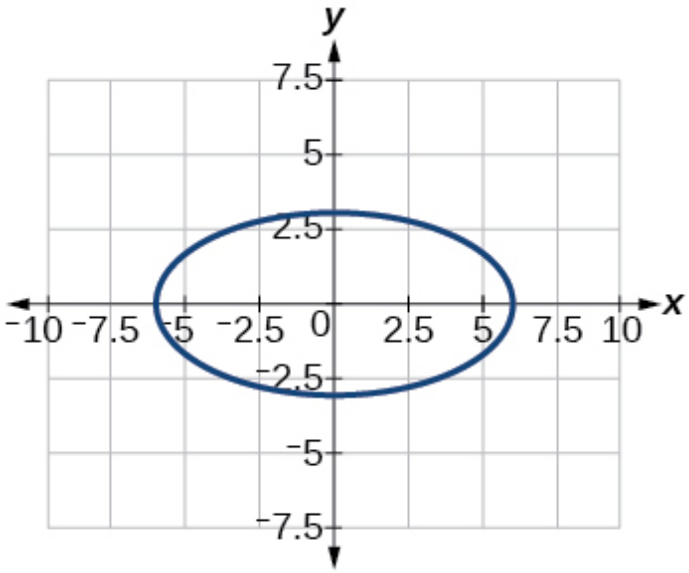
4) $9x^2+36y^2-36x+72y+36=0$

**For the exercises 5-8, graph the ellipse, noting center, vertices, and foci.**

5) $\frac{x^2}{36}+\frac{y^2}{9}=1$

**Answer**

center: $(0,0)$; Vertices: $(6,0$, $(-6,0)$, $(0,3)$, $(0,-3)$; foci: $(3\sqrt{3},0)$, $(-3\sqrt{3},0)$

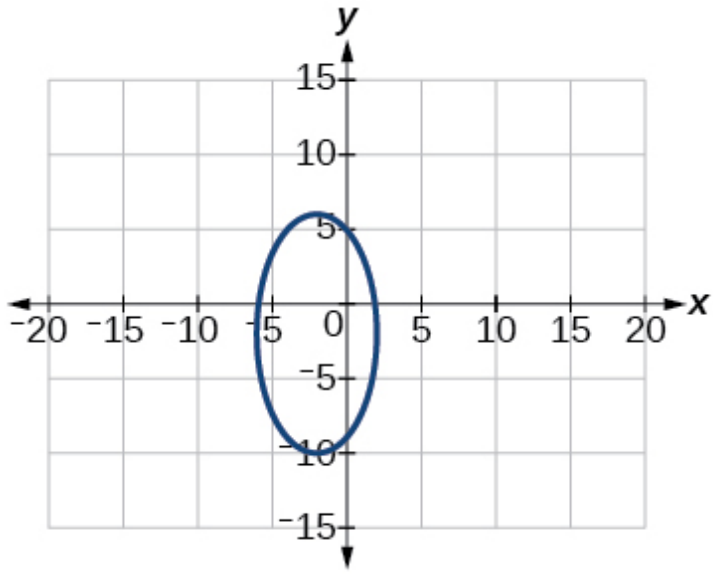


6) $\frac{(x-4)^2}{25}+\frac{(y+3)^2}{49}=1$

7) $4x^2+y^2+16x+4y-44=0$

**Answer**

center: $(-2,-2)$; Vertices: $(2,-2)$, $(-6,-2)$, $(-2,6)$, $(-2,-10)$; foci: $(-2,-2+4\sqrt{3})$, $(-2,-2-4\sqrt{3})$




8) $2x^2+3y^2-20x+12y+38=0$

**For the exercises 9-11, use the given information to find the equation for the ellipse.**

9) Center at $(0,0)$, focus at $(3,0)$, vertex at $(-5,0)$

**Answer**

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

10) Center at $(2,-2)$, vertex at $(7,-2)$, focus at $(4,-2)$

11) A whispering gallery is to be constructed such that the foci are located $35$ feet from the center. If the length of the gallery is to be $100$ feet, what should the height of the ceiling be?

**Answer**

Approximately $35.71$ feet

# 10.2: The Hyperbola

In analytic geometry, a hyperbola is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other.

**For the exercises 1-4, write the equation of the hyperbola in standard form. Then give the center, vertices, and foci.**

1) $\frac{x^2}{81}-\frac{y^2}{9}=1$

2) $\frac{(y+1)^2}{16} - \frac{(x-4)^2}{36} = 1$

**Answer**

$\frac{(y+1)^2}{4^2} - \frac{(x-4)^2}{6^2} = 1$; center: $(4,-1)$; Vertices: $(4,3)$, $(4,-5)$; foci: $(4,-1+2\sqrt{13})$, $(4,-1-2\sqrt{13})$

3) $9y^2 - 4x^2 + 54y - 16x + 29 = 0$

4) $3x^2 - y^2 - 12x - 6y - 9 = 0$

**Answer**

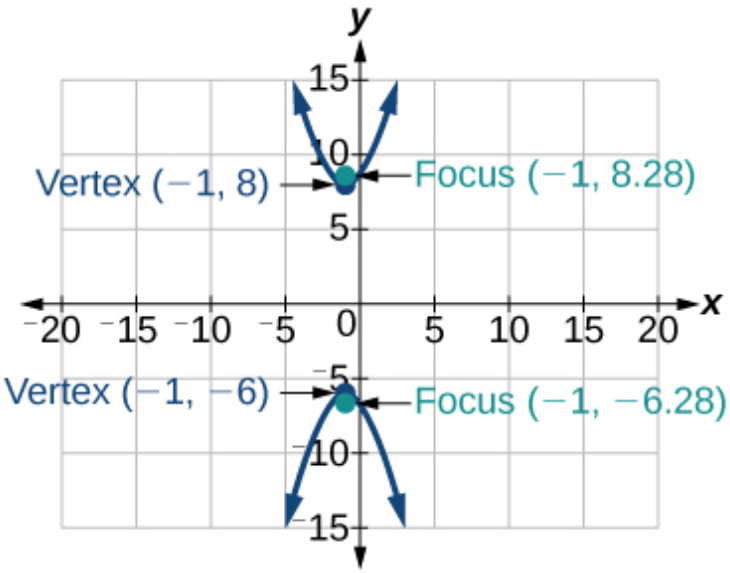
$\frac{(x-2)^2}{2^2} - \frac{(y+3)^2}{(2\sqrt{3})^2} = 1$; center: $(2,-3)$; Vertices: $(4,-3)$, $(0,-3)$; foci: $(6,-3)$, $(-2,-3)$

**For the exercises 5-8, graph the hyperbola, labeling vertices and foci.**

5) $\frac{x^2}{9} - \frac{y^2}{16} = 1$

6) $\frac{(y-1)^2}{49} - \frac{(x+1)^2}{4} = 1$

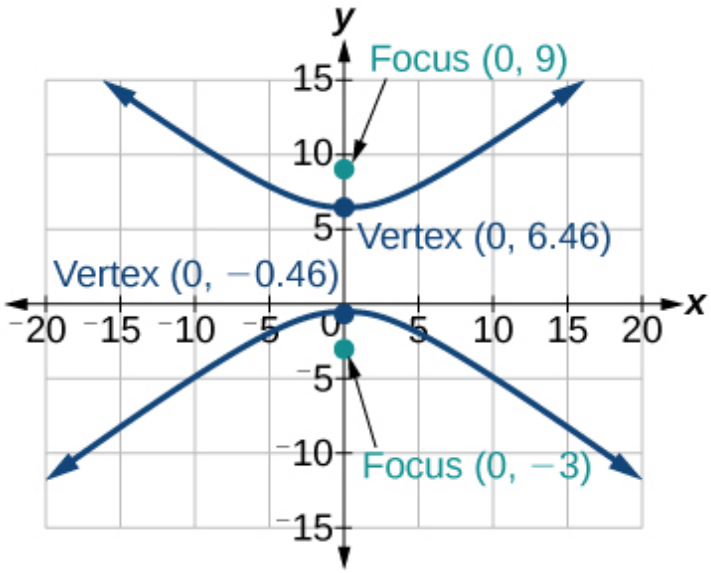
**Answer**




7) $x^2 - 4y^2 + 6x + 32y - 91 = 0$

8) $2y^2 - x^2 - 12y - 6 = 0$

**Answer**




**For the exercises 9-10, find the equation of the hyperbola.**

9) Center at $(0, 0)$, vertex at $(0, 4)$, focus at $(0, -6)$

10) Foci at $(3, 7)$ and $(7, 7)$, vertex at $(6, 7$

**Answer**

$$\frac{(x-5)^2}{1} - \frac{(y-7)^2}{3} = 1$$

# 10.3: The Parabola

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix.

**For the exercises 1-4, write the equation of the parabola in standard form. Then give the vertex, focus, and directrix.**

1) $y^2 = 12x$

2) $(x+2)^2 = \frac{1}{2}(y-1)$

**Answer**

$(x+2)^2 = \frac{1}{2}(y-1)$; vertex: $(-2, 1)$; focus: $\left(-2, \frac{9}{8}\right)$; directrix: $y = \frac{7}{8}$

3) $y^2 - 6y - 6x - 3 = 0$

4) $x^2 + 10x - y + 23 = 0$

**Answer**

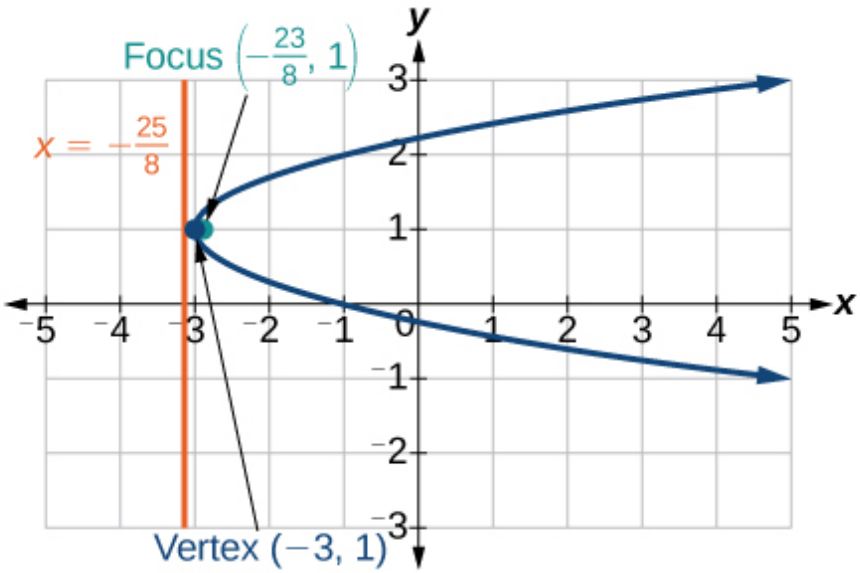
$(x+5)^2 = (y+2)$; vertex: $(-5, -2)$; focus: $\left(-5, -\frac{7}{4}\right)$; directrix: $y = -\frac{9}{4}$

**For the exercises 5-8, graph the parabola, labeling vertex, focus, and directrix.**

5) $x^2 + 4y = 0$

6) $(y-1)^2 = \frac{1}{2}(x+3)$

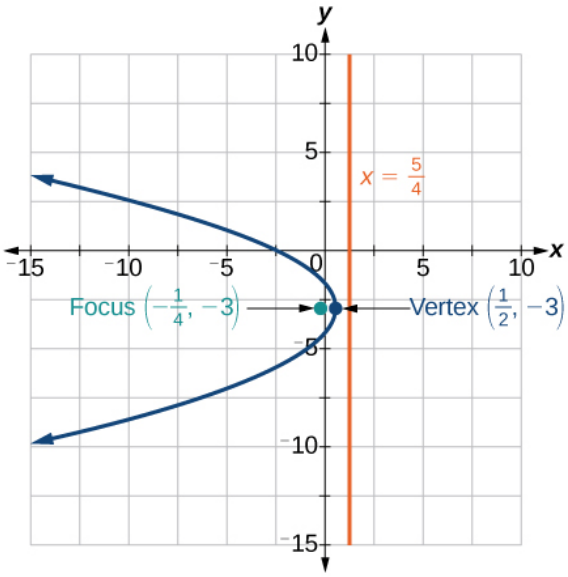
**Answer**




7) $x^2 - 8x - 10y + 46 = 0$

8) $2y^2 + 12y + 6x + 15 = 0$

**Answer**




**For the exercises 9-11, write the equation of the parabola using the given information.**

9) Focus at $(-4, 0)$; directrix is $x = 4$

10) Focus at $\left(2, \frac{9}{8}\right)$; directrix is $y = \frac{7}{8}$

**Answer**

$$(x-2)^2 = \left(\frac{1}{2}\right)(y-1)$$

11) A cable TV receiving dish is the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is $5$ feet across at its opening and $1.5$ feet deep.

# 10.4: Rotation of Axes

In previous sections of this chapter, we have focused on the standard form equations for nondegenerate conic sections. In this section, we will shift our focus to the general form equation, which can be used for any conic. The general form is set equal to zero, and the terms and coefficients are given in a particular order, as shown below.

**For the exercises 1-3, determine which of the conic sections is represented.**

1) $16x^2 + 24xy + 9y^2 + 24x - 60y - 60 = 0$

**Answer**

$B^2 - 4AC = 0$, parabola

2) $4x^2 + 14xy + 5y^2 + 18x - 6y + 30 = 0$

3) $4x^2 + xy + 2y^2 + 8x - 26y + 9 = 0$

**Answer**

$B^2 - 4AC = -31 < 0$, ellipse

**For the exercises 4-5, determine the angle $\theta$ that will eliminate the $xy$ term, and write the corresponding equation without the $xy$ term.**

4) $x^2 + 4xy - 2y^2 - 6 = 0$

5) $x^2 - xy + y^2 - 6 = 0$

**Answer**

$\theta = 45^\circ, x'^2 + 3y'^2 - 12 = 0$

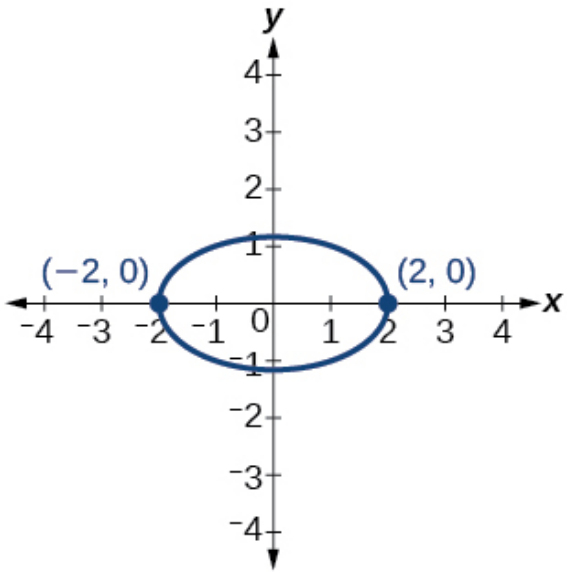
**For the exercises 6-8, graph the equation relative to the $x'y'$ system in which the equation has no $x'y'$ term.**

6) $9x^2 - 24xy + 16y^2 - 80x - 60y + 100 = 0$

7) $x^2 - xy + y^2 - 2 = 0$

**Answer**

$\theta = 45^\circ$




8) $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$

# 10.5: Conic Sections in Polar Coordinates

In this section, we will learn how to define any conic in the polar coordinate system in terms of a fixed point, the focus at the pole, and a line, the directrix, which is perpendicular to the polar axis.

**For the exercises 1-4, given the polar equation of the conic with focus at the origin, identify the eccentricity and directrix.**

1) $r = \dfrac{10}{1 - 5\cos\theta}$

**Answer**

Hyperbola with $e = 5$ and directrix $2$ units to the left of the pole.

2) $r = \dfrac{6}{3 + 2\cos\theta}$

3) $r = \dfrac{1}{4 + 3\sin\theta}$

**Answer**

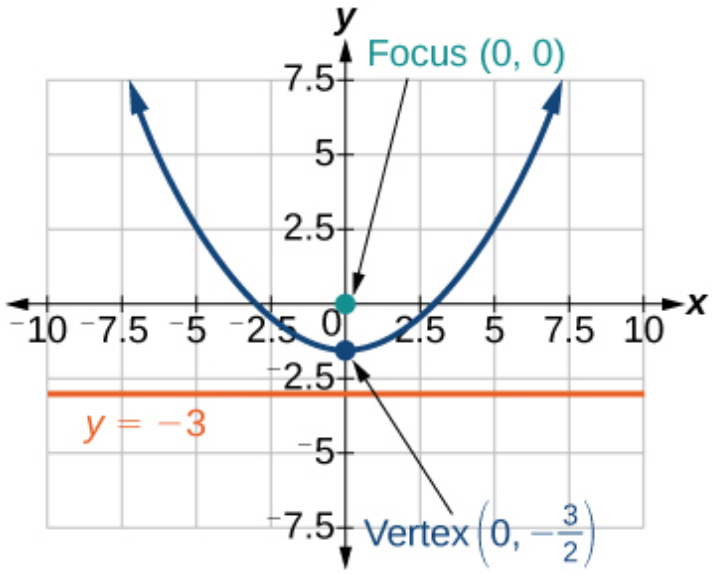
Ellipse with $e = \dfrac{3}{4}$ and directrix $\dfrac{1}{3}$ unit above the pole.

4) $r = \dfrac{3}{5 - 5\sin\theta}$

**For the exercises 5-8, graph the conic given in polar form. If it is a parabola, label the vertex, focus, and directrix. If it is an ellipse or a hyperbola, label the vertices and foci.**

5) $r = \dfrac{3}{1 - \sin\theta}$

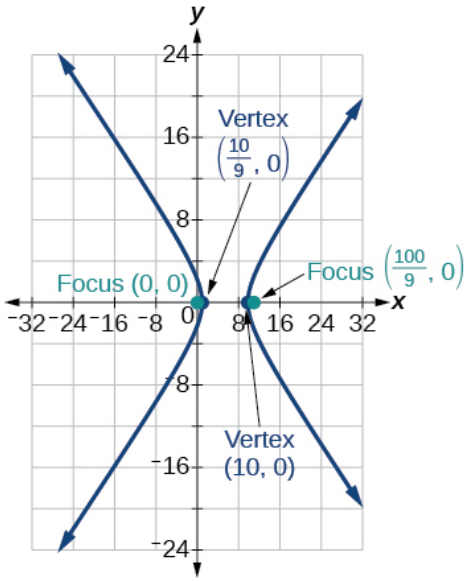
**Answer**




6) $r = \dfrac{8}{4 + 3\sin\theta}$

7) $r = \dfrac{10}{4 + 5\cos\theta}$

**Answer**




8) $r = \dfrac{9}{3 - 6\cos\theta}$

**For the exercises 9-10, given information about the graph of a conic with focus at the origin, find the equation in polar form.**

9) Directrix is $x = 3$ and eccentricity $e = 1$

**Answer**

$$r = \frac{3}{1 + \cos\theta}$$

10) Directrix is $y = -2$ and eccentricity $e = 4$

# Practice Test

**For the exercises 1-2, write the equation in standard form and state the center, vertices, and foci.**

1) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

**Answer**

$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$; center: $(0, 0)$; vertices: $(3, 0)$, $(-3, 0)$, $(0, 2)$, $(0, -2)$; foci: $(\sqrt{5}, 0)$, $(-\sqrt{5}, 0)$

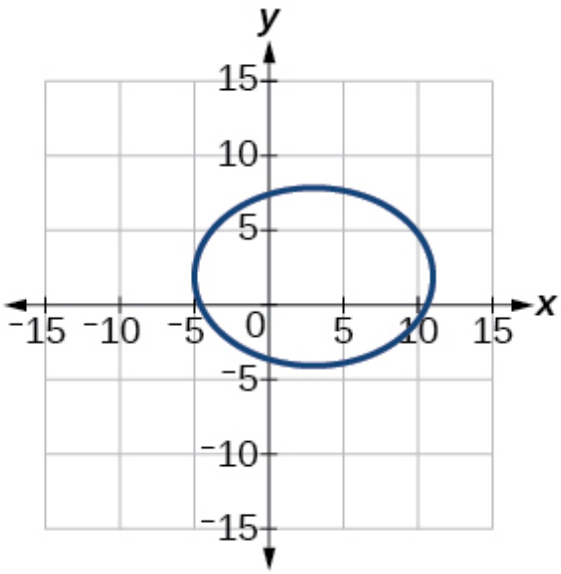
2) $9y^2 + 16x^2 - 36y + 32x - 92 = 0$

**For the exercises 3-6, sketch the graph, identifying the center, vertices, and foci.**

3) $\frac{(x-3)^2}{64} + \frac{(y-2)^2}{36} = 1$

**Answer**

center: $(3, 2)$; vertices: $(11, 2)$, $(-5, 2)$, $(3, 8)$, $(3, -4)$; foci: $(3 + 2\sqrt{7}, 2)$, $(3 - 2\sqrt{7}, 2)$




4) $2x^2 + y^2 + 8x - 6y - 7 = 0$

5) Write the standard form equation of an ellipse with a center at $(1, 2)$, vertex at $(7, 2)$, and focus at $(4, 2)$.

**Answer**

$$\frac{(x-1)^2}{36} + \frac{(y-2)^2}{27} = 1$$

6) A whispering gallery is to be constructed with a length of $150$ feet. If the foci are to be located $20$ feet away from the wall, how

high should the ceiling be?

**For the exercises 7-8, write the equation of the hyperbola in standard form, and give the center, vertices, foci, and asymptotes.**

7) $\frac{x^2}{49} - \frac{y^2}{81} = 1$

**Answer**

$\frac{x^2}{7^2} - \frac{y^2}{9^2} = 1$; center: $(0, 0)$; vertices: $(7, 0)$, $(-7, 0)$; foci: $(\sqrt{130}, 0)$, $(-\sqrt{130}, 0)$; asymptotes: $y = \pm\frac{9}{7}x$

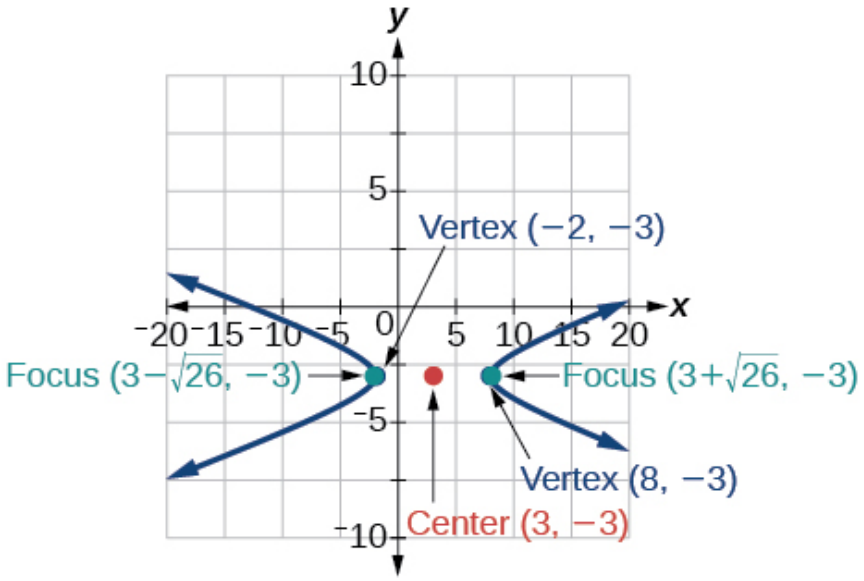
8) $16y^2 - 9x^2 + 128y + 112 = 0$

**For the exercises 9-11, graph the hyperbola, noting its center, vertices, and foci. State the equations of the asymptotes.**

9) $\frac{(x-3)^2}{25} - \frac{(y+3)^2}{1} = 1$

**Answer**

center: $(3, -3)$; vertices: $(8, -3)$, $(-2, -3)$; foci: $(3 + \sqrt{26}, -3)$, $(3 - \sqrt{26}, -3)$; asymptotes: $y = \pm\frac{1}{5}(x - 3) - 3$




10) $y^2 - x^2 + 4y - 4x - 18 = 0$

11) Write the standard form equation of a hyperbola with foci at $(1, 0)$, and $(1, 6)$, and a vertex at $(1, 2)$.

**Answer**

$\frac{(y-3)^2}{1} - \frac{(x-1)^2}{8} = 1$

**For the exercises 12-13, write the equation of the parabola in standard form, and give the vertex, focus, and equation of the directrix.**

12) $y^2 + 10x = 0$

13) $3x^2 - 12x - y + 11 = 0$

**Answer**

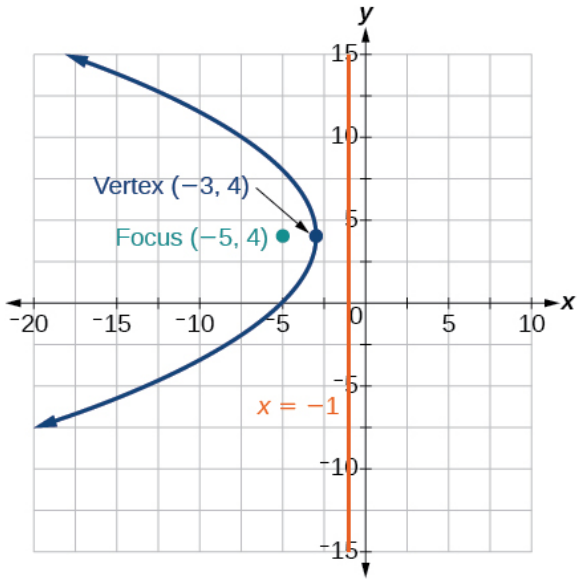
$(x-2)^2=\frac{1}{3}(y+1)$; vertex: $(2,-1)$; focus: $(2,-\frac{11}{12})$; directrix: $y=-\frac{13}{12}$

**For the exercises 14-17, graph the parabola, labeling the vertex, focus, and directrix.**

14) $(x-1)^2=-4(y+3)$

15) $y^2+8x-8y+40=0$

**Answer**




16) Write the equation of a parabola with a focus at $(2,3)$ and directrix $y=-1$.

17) A searchlight is shaped like a paraboloid of revolution. If the light source is located $1.5$ feet from the base along the axis of symmetry, and the depth of the searchlight is $3$ feet, what should the width of the opening be?

**Answer**

Approximately $8.49$ feet

**For the exercises 18-19, determine which conic section is represented by the given equation, and then determine the angle $\theta$ that will eliminate the $xy$ term.**

18) $3x^2-2xy+3y^2=4$

19) $x^2+4xy+4y^2+6x-8y=0$

**Answer**

parabola; $\theta\approx 63.4^\circ$

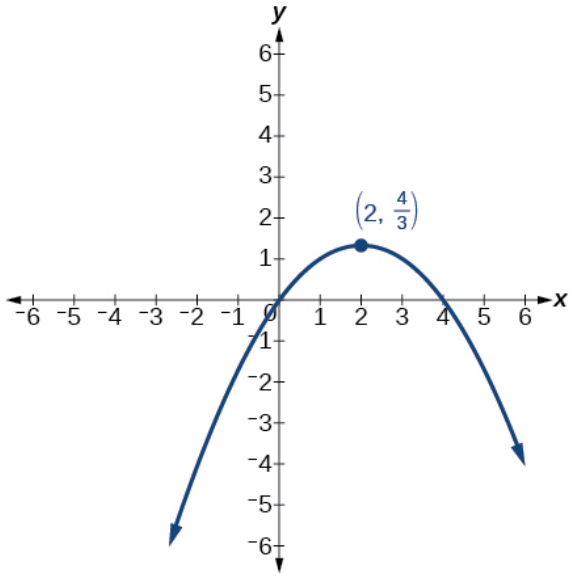
**For the exercises 20-21, rewrite in the $x'y'$ system without the $x'y'$ term, and graph the rotated graph.**

20) $11x^2+10\sqrt{3}xy+y^2=4$

21) $16x^2+24xy+9y^2-125x=0$

**Answer**

$$x'^2 - 4x' + 3y' = 0$$




**For the exercises 22-23, identify the conic with focus at the origin, and then give the directrix and eccentricity.**

22) $r = \dfrac{3}{2 - \sin\theta}$

23) $r = \dfrac{5}{4 + 6\cos\theta}$

**Answer**

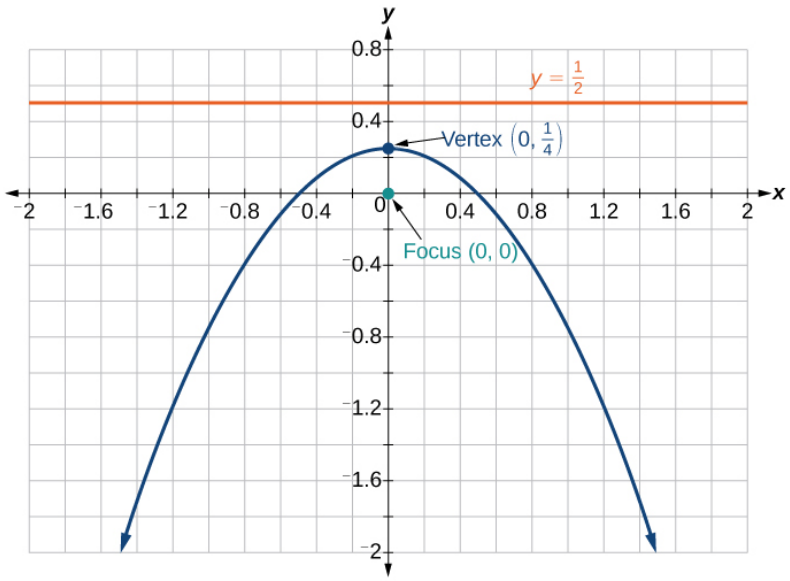
Hyperbola with $e = \dfrac{3}{2}$ and directrix $\dfrac{5}{6}$ units to the right of the pole.

**For the exercises 24-26, graph the given conic section. If it is a parabola, label vertex, focus, and directrix. If it is an ellipse or a hyperbola, label vertices and foci.**

24) $r = \dfrac{12}{4 - 8\sin\theta}$

25) $r = \dfrac{2}{4 + 4\sin\theta}$

**Answer**




26) Find a polar equation of the conic with focus at the origin, eccentricity of $e = 2$, and directrix: $x = 3$.

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# 7: INTRODUCTION TO CALCULUS

Calculus is the broad area of mathematics dealing with such topics as instantaneous rates of change, areas under curves, and sequences and series. Underlying all of these topics is the concept of a limit, which consists of analyzing the behavior of a function at points ever closer to a particular point, but without ever actually reaching that point. Calculus has two basic applications: differential calculus and integral calculus.


7.0: Prelude to Calculus

**7.1: A Preview of Calculus**

7.1E: Exercises for Section 2.1

7.2: Finding Limits - Numerical and Graphical Approaches

7.3: Finding Limits - Properties of Limits

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7.7: Derivatives

7.E: Introduction to Calculus (Exercises)

7.R: Introduction to Calculus (Review)


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# 7.0: Prelude to Calculus

The eight-time world champion and winner of six Olympic gold medals in sprinting, Usain Bolt has truly earned his nickname as the “fastest man on Earth.” Also known as the “lightning bolt,” he set the track on fire by running at a top speed of 27.79 mph—the fastest time ever recorded by a human runner.




Like the fastest land animal, a cheetah, Bolt does not run at his top speed at every instant. How then, do we approximate his speed at any given instant? We will find the answer to this and many related questions in this chapter.

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# 7.1: A PREVIEW OF CALCULUS

**Learning Objectives**

- Describe the tangent problem and how it led to the idea of a derivative.
- Explain how the idea of a limit is involved in solving the tangent problem.
- Recognize a tangent to a curve at a point as the limit of secant lines.
- Identify instantaneous velocity as the limit of average velocity over a small time interval.
- Describe the area problem and how it was solved by the integral.
- Explain how the idea of a limit is involved in solving the area problem.
- Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

## The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3$, $g(x) = \frac{1}{2}x + 1$, and $h(x) = 2$, shown in Figure 7.1.1.

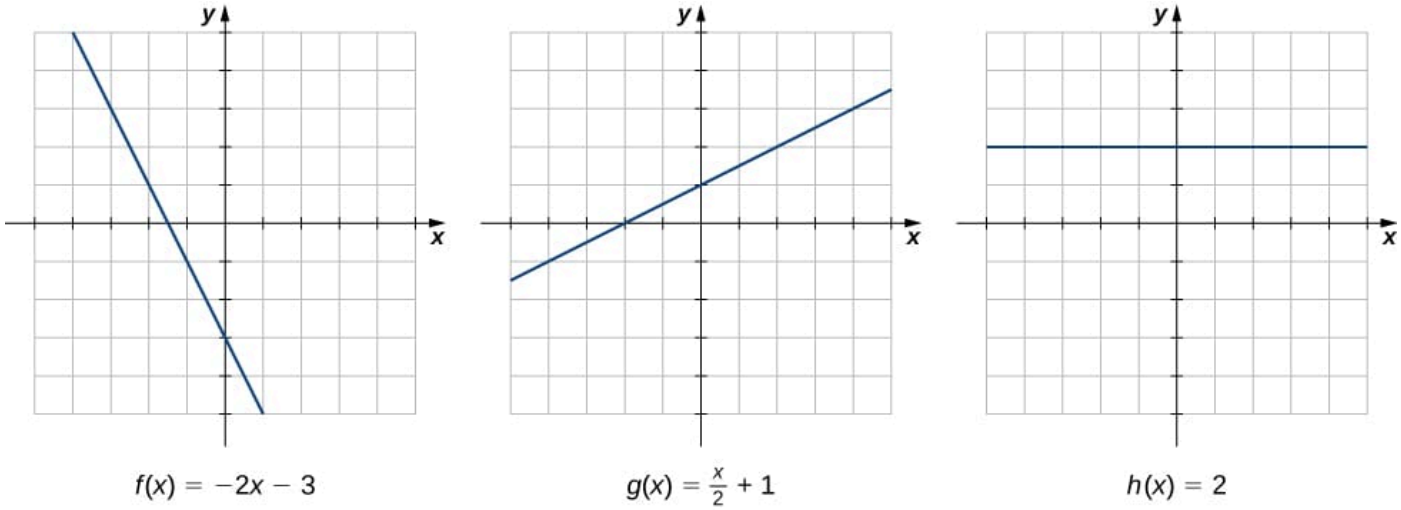



Figure 7.1.1: The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3$, we see that the graph decreases at a constant rate. For every $1$ unit we move to the right along the $x$-axis, the $y$-coordinate decreases by $2$ units. This **rate of change** is determined by the slope $(-2)$ of the line. Similarly, the slope of $1/2$ in the function $g(x)$ tells us that for every change in $x$ of $1$ unit there is a corresponding change in $y$ of $1/2$ unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (Figure 7.1.2). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?

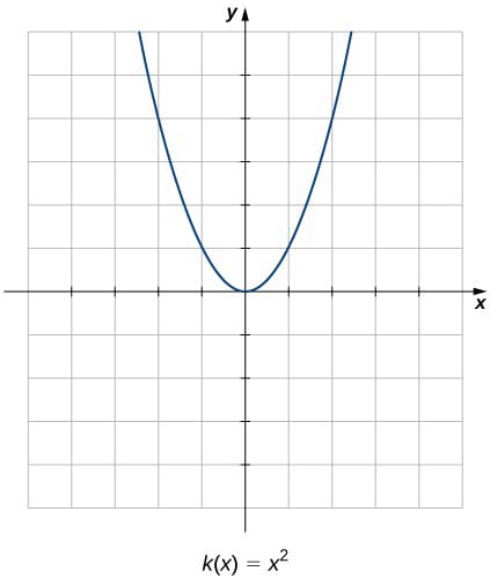



Figure 7.1.2: The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a secant line. Figure 7.1.3 shows a *secant line* to a function $f(x)$ at a point $(a, f(a))$.

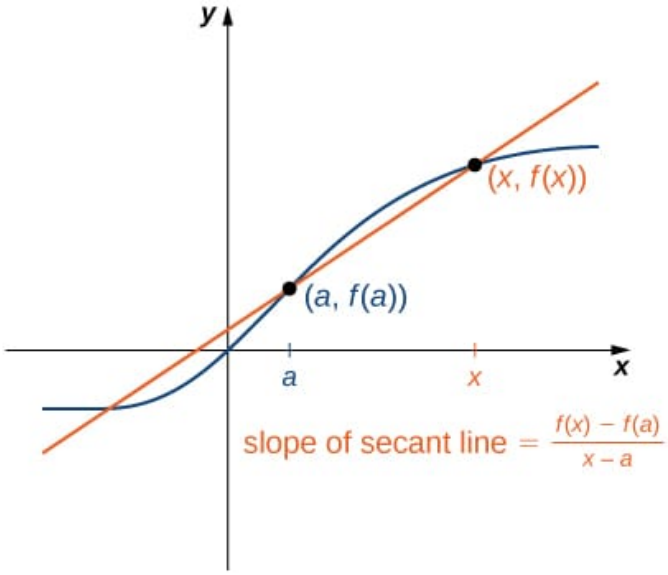



Figure 7.1.3: The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

**Definition: Secant Line**

The *secant* to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{sec} = \frac{f(x) - f(a)}{x - a}. \tag{7.1.1}$$

The accuracy of approximating the rate of change of the function with a secant line depends on how close $x$ is to $a$. As we see in

Figure 7.1.4, if $x$ is closer to $a$, the slope of the secant line is a better measure of the rate of change of $f(x)$ at $a$.

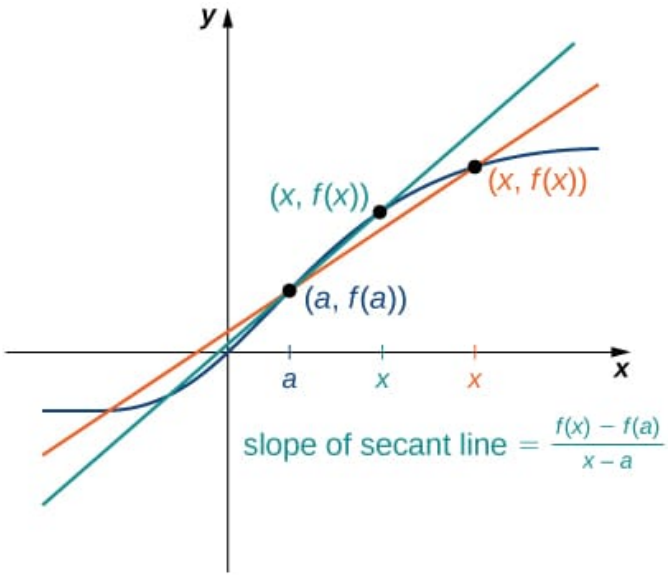



Figure 7.1.4: As $x$ gets closer to $a$, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at $a$.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at $a$ (Figure 7.1.5). The slope of the tangent line to the graph at $a$ measures the rate of change of the function at $a$. This value also represents the derivative of the function $f(x)$ at $a$, or the rate of change of the function at $a$. This derivative is denoted by $f\,'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

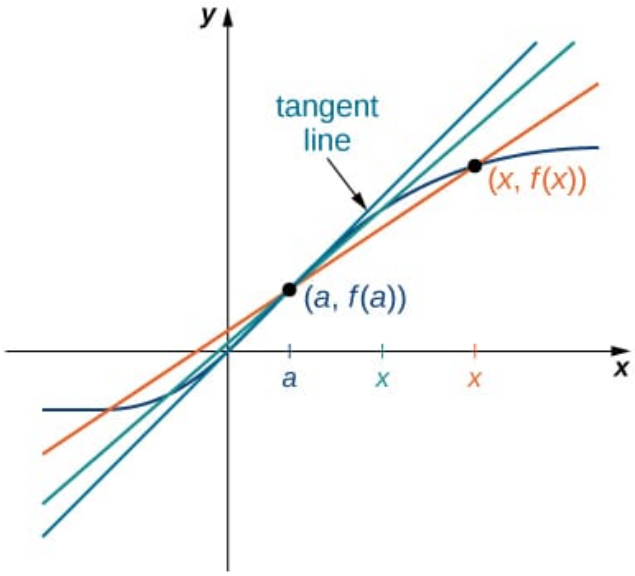



Figure 7.1.5: Solving the Tangent Problem: As $x$ approaches $a$, the secant lines approach the tangent line.

Example 7.1.1 illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

## Example 7.1.1: Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2,4)$
b. $\left(\frac{3}{2},\frac{9}{4}\right)$

**Solution**:

Use the formula for the slope of a secant line (Equation 7.1.1).

a. $m_{sec}=\frac{4-1}{2-1}=3$
b. $m_{sec}=\frac{\frac{9}{4}-1}{\frac{3}{2}-1}=\frac{5}{2}=2.5$

The point in part b. is closer to the point $(1,1)$, so the slope of $2.5$ is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of $2$ to $2.5$ (Figure 7.1.6).

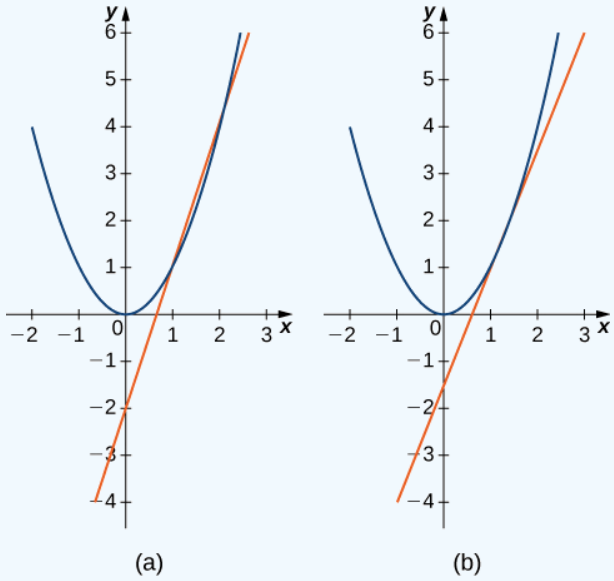



Figure 7.1.6: The secant lines to $f(x)=x^2$ at $(1,1)$ through (a) $(2,4)$ and (b) $(\frac{3}{2},\frac{9}{4})$ provide successively closer approximations to the tangent line to $f(x)=x^2$ at $(1,1)$.

## Exercise 7.1.1

Estimate the slope of the tangent line (rate of change) to $f(x)=x^2$ at $x=1$ by finding slopes of secant lines through $(1,1)$ and the point $(\frac{5}{4},\frac{25}{16})$ on the graph of $f(x)=x^2$.

*Answer available at the end of this section.*

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time $t$. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t=a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the *average velocity* of an object over a time period to be the change in its position divided by the length of the time period.

### Definition: Average Velocity

Let $s(t)$ be the position of an object moving along a coordinate axis at time $t$. The *average velocity* of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$v_{ave} = \frac{s(t) - s(a)}{t - a}. \tag{7.1.2}$$

As $t$ is chosen closer to $a$, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point $a$, we let the $x$-values approach $a$ in the slope of the secant line. Similarly, to find the instantaneous velocity at time $a$, we let the $t$-values approach $a$ in the average velocity. This process of letting $x$ or $t$ approach $a$ in an expression is called taking a **limit**. Thus, we may define the *instantaneous velocity* as follows.

### Definition: Instantaneous Velocity

For a position function $s(t)$, the *instantaneous velocity* at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of $t$ become closer to $a$, provided such a value exists.

Example 7.1.2 illustrates this concept of limits and average velocity.

### Example 7.1.2: Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground t seconds later (for $0 \le t \le 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$
b. $[0.5, 0.51]$

**Solution**

Substitute the data into Equation 7.1.2 for the definition of average velocity.

a. $$v_{ave} = \frac{s(0.49) - s(0.5)}{0.49 - 0.5} = -15.84$$

b. $$v_{ave} = \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.016$$

The instantaneous velocity is somewhere between –15.84 and –16.16 ft/sec. A good guess might be –16 ft/sec.

### Exercise 7.1.2

An object moves along a coordinate axis so that its position at time $t$ is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may

be interpreted as the area under a curve. This leads us to ask the question: How can we find the area between the graph of a function and the $x$-axis over an interval (Figure 7.1.7)?

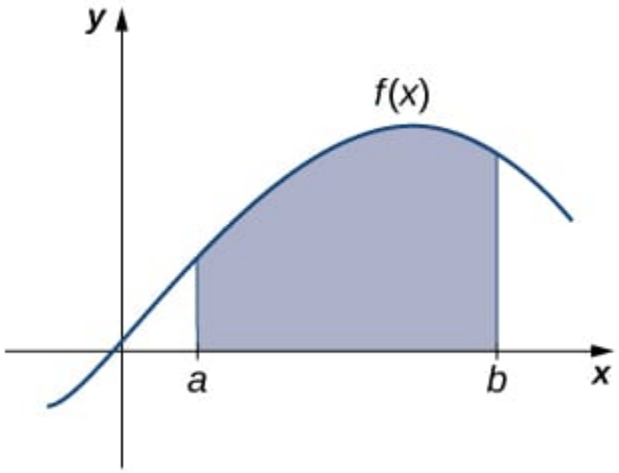



Figure 7.1.7: The Area Problem: How do we find the area of the shaded region?

As in the answer to our previous questions on velocity, we first try to approximate the solution. We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up the areas of these rectangles (Figure 7.1.8).

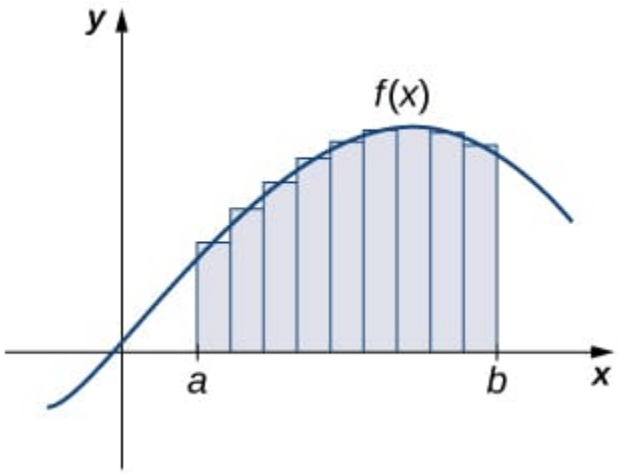



Figure 7.1.8: The area of the region under the curve is approximated by summing the areas of thin rectangles.

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the $x$-axis over the interval $[a, b]$. Once again, we find ourselves taking a limit. Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

### Example 7.1.3: Estimation Using Rectangles

Estimate the area between the $x$-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in Figure 7.1.9.

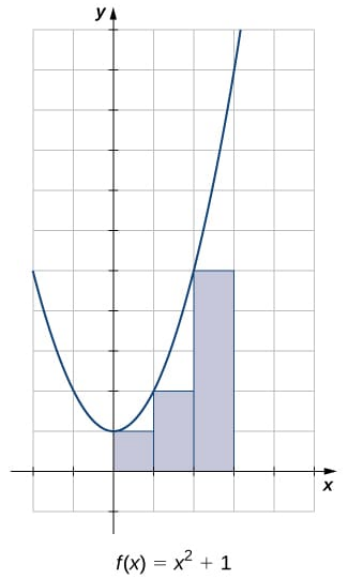



Figure 7.1.9: The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

**Solution**

The areas of the three rectangles are 1 unit$^2$, 2 unit$^2$, and 5 unit$^2$. Using these rectangles, our area estimate is 8 unit$^2$.

## Exercise 7.1.3

Estimate the area between the $x$-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in Figure 7.1.10.

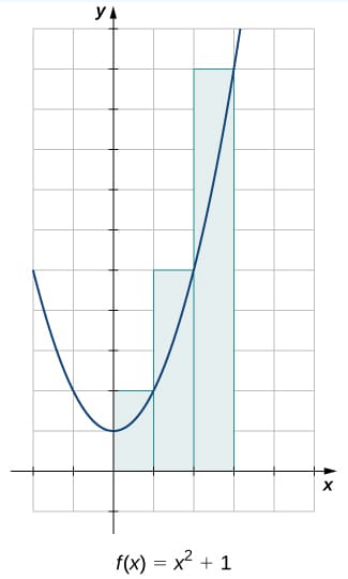



Figure 7.1.10: The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

# Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions. Suppose, for example, that instead of determining the velocity of an object moving along a coordinate axis, we want to determine the velocity of a rock fired from a catapult at a given time, or of an airplane moving in three dimensions. We might want to graph real-value functions of two variables or determine volumes of solids of the type shown in Figure 7.1.11. These are only a few of the types of questions that can be asked and answered using *multivariable calculus*. Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables. However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.

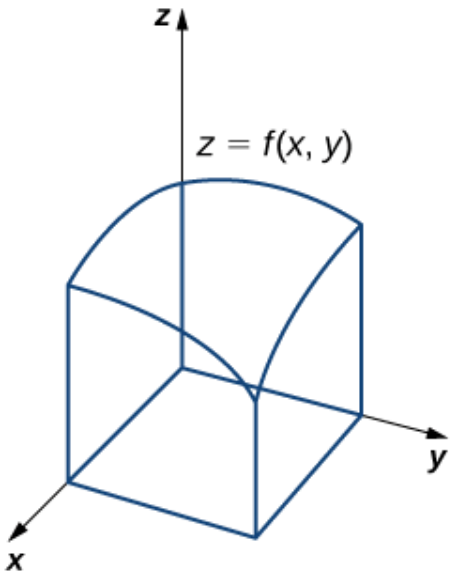



Figure 7.1.11: We can use multivariable calculus to find the volume between a surface defined by a function of two variables and a plane.

# Key Concepts

- Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the rate of change of the function, also called the *derivative*. Calculating a derivative requires finding a limit.
- Integral calculus arose from trying to solve the problem of finding the area of a region between the graph of a function and the $x$-axis. We can approximate the area by dividing it into thin rectangles and summing the areas of these rectangles. This summation leads to the value of a function called the *integral*. The integral is also calculated by finding a limit and, in fact, is related to the derivative of a function.
- Multivariable calculus enables us to solve problems in three-dimensional space, including determining motion in space and finding volumes of solids.

# Key Equations

- **Slope of a Secant Line**

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

- **Average Velocity over Interval [a,t]**

$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$

# Glossary

**average velocity**
: the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval $[t, a]$ (if $t < a$ or $[a, t]$ if $t > a$), with a position given by $s(t)$, that is $v_{ave} = \dfrac{s(t) - s(a)}{t - a}$

**differential calculus**
: the field of calculus concerned with the study of derivatives and their applications

**instantaneous velocity**
: The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t, a]$ and $[a, t]$ approach as the values of $t$ move closer to $a$, provided such a value exists

**integral calculus**
: the study of integrals and their applications

**limit**
: the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as $x$ approaches $a$ is the value that $f(x)$ approaches as $x$ approaches $a$

**multivariable calculus**
: the study of the calculus of functions of two or more variables

**secant**
: A secant line to a function $f(x)$ at $a$ is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{sec} = \dfrac{f(x) - f(a)}{x - a}$

**tangent**
: A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with $x$-values that approach $a$; the slope of the tangent line to a graph at $a$ measures the rate of change of the function at $a$

## Answers

**Exercise** 7.1.1

2.25

**Exercise** 7.1.2

Use Equation 7.1.2 with $v_{ave} = \dfrac{s(2.001) - s(2)}{2.001 - 2}$.

**Exercise** 7.1.2

12.006001

**Exercise** 7.1.3

Use Example 7.1.3 as a guide

**Exercise** 7.1.3

17 $\text{unit}^2$

## 7.1E: Exercises for Section 2.1

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# 7.1E: Exercises for Section 2.1

**For exercises 1 - 3 , points $P(1,2)$ and $Q(x,y)$ are on the graph of the function $f(x)=x^2+1$.**

1) [T] Complete the following table with the appropriate values: $y$-coordinate of $Q$, the point $Q(x,y)$, and the slope of the secant line passing through points $P$ and $Q$. Round your answer to eight significant digits.

| $x$ | $y$ | $Q(x,y)$ | $m_{sec}$ |
|---|---|---|---|
| 1.1 | a. | e. | i. |
| 1.01 | b. | f. | j. |
| 1.001 | c. | g. | k. |
| 1.0001 | d. | h. | l. |

**Answer**

a. 2.21000000
b. 2.02010000
c. 2.00200010
d. 2.00020000
e. (1.10000000, 2.21000000)
f. (1.01000000, 2.02010000)
g. (1.00100000, 2.00200010)
h. (1.00010000, 2.00020000)
i. 2.10000000
j. 2.01000000
k. 2.00100000
l. 2.00010000

2) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to $f$ at $x=1$.

3) Use the value in the preceding exercise to find the equation of the tangent line at point $P$. Graph $f(x)$ and the tangent line.

**Answer**

$y=2x$

**For the exercises 4-6, points $P(1,1)$ and $Q(x,y)$ are on the graph of the function $f(x)=x^3$.**

4) [T] Complete the following table with the appropriate values: $y$-coordinate of $Q$, the point $Q(x,y)$, and the slope of the secant line passing through points $P$ and $Q$. Round your answer to eight significant digits.

| $x$ | $y$ | $Q(x,y)$ | $m_{sec}$ |
|---|---|---|---|
| 1.1 | a. | e. | i. |
| 1.01 | b. | f. | j. |
| 1.001 | c. | g. | k. |
| 1.0001 | d. | h. | l.2 |

5) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to $f$ at $x=1$.

**Answer**

3

6) Use the value in the preceding exercise to find the equation of the tangent line at point $P$. Graph $f(x)$ and the tangent line.

**For the exercises 7 - 9, points $P(4,2)$ and $Q(x,y)$ are on the graph of the function $f(x)=\sqrt{x}$.**

7) [T] Complete the following table with the appropriate values: $y$-coordinate of $Q$, the point $Q(x,y)$, and the slope of the secant line passing through points $P$ and $Q$. Round your answer to eight significant digits.

| $x$ | $y$ | $Q(x,y)$ | $m_{sec}$ |
|---|---|---|---|
| 4.1 | a. | e. | i. |
| 4.01 | b. | f. | j. |
| 4.001 | c. | g. | k. |
| 4.0001 | d. | h. | l. |

**Answer**

a. 2.0248457
b. 2.0024984
c. 2.0002500
d. 2.0000250
e. (4.1000000,2.0248457)
f. (4.0100000,2.0024984)
g. (4.0010000,2.0002500)
h. (4.00010000,2.0000250)
i. 0.24845673
j. 0.24984395
k. 0.24998438
l. 0.24999844

8) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to $f$ at $x=4$.

9) Use the value in the preceding exercise to find the equation of the tangent line at point $P$.

**Answer**

$y=\frac{x}{4}+1$

**For exercises 10 - 12, points $P(1.5,0)$ and $Q(\phi,y)$ are on the graph of the function $f(\phi)=\cos(\pi\phi)$.**

10) [T] Complete the following table with the appropriate values: $y$-coordinate of $Q$, the point $Q(\phi,y)$, and the slope of the secant line passing through points $P$ and $Q$. Round your answer to eight significant digits.

| $x$ | $y$ | $Q(\phi,y)$ | $m_{sec}$ |
|---|---|---|---|
| 1.4 | a. | e. | i. |
| 1.49 | b. | f. | j. |
| 1.499 | c. | g. | k. |
| 1.4999 | d. | h. | l. |

11) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $\phi=1.5$.

**Answer**

$\pi$

12) Use the value in the preceding exercise to find the equation of the tangent line at point $P$.

**For exercises 13 - 15, points $P(-1,-1)$ and $Q(x,y)$ are on the graph of the function $f(x)=\frac{1}{x}$.**

13) [T] Complete the following table with the appropriate values: $y$-coordinate of $Q$, the point $Q(x,y)$, and the slope of the secant line passing through points $P$ and $Q$. Round your answer to eight significant digits.

| $x$ | $y$ | $Q(x,y)$ | $m_{sec}$ |
|---|---|---|---|
| -1.05 | a. | e. | i. |
| -1.01 | b. | f. | j. |
| -1.005 | c. | g. | k. |
| -1.001 | d. | h. | l. |

**Answer**

a. −0.95238095
b. −0.99009901
c. −0.99502488
d. −0.99900100
e. (−1;.0500000,−0;.95238095)
f. (−1;.0100000,−0;.9909901)
g. (−1;.0050000,−0;.99502488)
h. (1.0010000,−0;.99900100)
i. −0.95238095
j. −0.99009901
k. −0.99502488
l. −0.99900100

14) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to $f$ at $x=-1$.

15) Use the value in the preceding exercise to find the equation of the tangent line at point $P$.

**Answer**

$y=-x-2$

**For exercises 16 - 17, the position function of a ball dropped from the top of a 200-meter tall building is given by $s(t)=200-4.9t^2$, where position $s$ is measured in meters and time $t$ is measured in seconds. Round your answer to eight significant digits.**

16) [T] Compute the average velocity of the ball over the given time intervals.

a. [4.99,5]

b. [5,5.01]

c. [4.999,5]

d. [5,5.001]

17) Use the preceding exercise to guess the instantaneous velocity of the ball at $t=5$ sec.

**Answer**

$-49$ m/sec (velocity of the ball is 49 m/sec downward)

**For exercises 18 - 19, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time $t$ seconds is $h(t)=15t-4.9t^2$.**

18) [T] Compute the average velocity of the stone over the given time intervals.

a. [1,1.05]

b. [1,1.01]

c. [1,1.005]

d. [1,1.001]

19) Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

**Answer**

$5.2$ m/sec

**For exercises 20 - 21, consider a rocket shot into the air that then returns to Earth. The height of the rocket in meters is given by $h(t) = 600 + 78.4t - 4.9t^2$, where $t$ is measured in seconds.**

20) [T] Compute the average velocity of the rocket over the given time intervals.

a. [9,9.01]

b. [8.99,9]

c. [9,9.001]

d. [8.999,9]

21) Use the preceding exercise to guess the instantaneous velocity of the rocket at $t = 9$ sec.

**Answer**

$-9.8$ m/sec

**For exercises, consider an athlete running a 40-m dash. The position of the athlete is given by $d(t) = \frac{t^3}{6} + 4t$, where $d$ is the position in meters and $t$ is the time elapsed, measured in seconds.**

22) [T] Compute the average velocity of the runner over the given time intervals.

a. [1.95,2.05]

b. [1.995,2.005]

c. [1.9995,2.0005]

d. [2,2.00001]

23) Use the preceding exercise to guess the instantaneous velocity of the runner at $t = 2$ sec.

**Answer**

$6$ m/sec

**For exercises 24 - 25, consider the function $f(x) = |x|$.**

24) Sketch the graph of $f$ over the interval $[-1, 2]$ and shade the region above the $x$-axis.

25) Use the preceding exercise to find the exact value of the area between the $x$-axis and the graph of $f$ over the interval $[-1, 2]$ using rectangles. For the rectangles, use the square units, and approximate both above and below the lines. Use geometry to find the exact answer.

**Answer**

Under, 1 $unit^2$; over: 4 $unit^2$.
The exact area of the two triangles is $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = 2.5 units^2$.

**For exercises 26 - 27, consider the function $f(x) = \sqrt{1 - x^2}$. (Hint: This is the upper half of a circle of radius 1 positioned at $(0, 0)$.)**

26) Sketch the graph of f over the interval $[-1, 1]$.

27) Use the preceding exercise to find the exact area between the $x$-axis and the graph of $f$ over the interval $[-1, 1]$ using rectangles. For the rectangles, use squares 0.4 by 0.4 units, and approximate both above and below the lines. Use geometry to find the exact answer.

**Answer**

Under, $0.96 \text{ units}^2$; over, $1.92 \text{ units}^2$).
The exact area of the semicircle with radius 1 is $\frac{\pi(1)^2}{2} = \frac{\pi}{2} \text{ units}^2$

**For exercises 28 - 29, consider the function $f(x) = -x^2 + 1$.**

28) Sketch the graph of $f$ over the interval $[-1, 1]$.

29) Approximate the area of the region between the $x$-axis and the graph of $f$ over the interval $[-1, 1]$.

**Answer**

Approximately $1.3333333 \text{ units}^2$

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# 7.2: Finding Limits - Numerical and Graphical Approaches

Intuitively, we know what a **limit** is. A car can go only so fast and no faster. A trash can might hold 33 gallons and no more. It is natural for measured amounts to have limits. What, for instance, is the limit to the height of a woman? The tallest woman on record was Jinlian Zeng from China, who was 8 ft 1 in.[1] Is this the limit of the height to which women can grow? Perhaps not, but there is likely a limit that we might describe in inches if we were able to determine what it was.

To put it mathematically, the function whose input is a woman and whose output is a measured height in inches has a limit. In this section, we will examine numerical and graphical approaches to identifying limits.

## Understanding Limit Notation

We have seen how a **sequence** can have a limit, a value that the sequence of terms moves toward as the nu mber of terms increases. For example, the terms of the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}...$$

gets closer and closer to 0. A sequence is one type of function, but functions that are not sequences can also have limits. We can describe the behavior of the function as the input values get close to a specific value. If the limit of a function $f(x) = L$, then as the input $x$ gets closer and closer to $a$, the output *y*-coordinate gets closer and closer to $L$. We say that the output "approaches" $L$.

Figure $7.2.1$ provides a visual representation of the mathematical concept of limit. As the input value $x$ approaches $a$, the output value $f(x)$ approaches L.

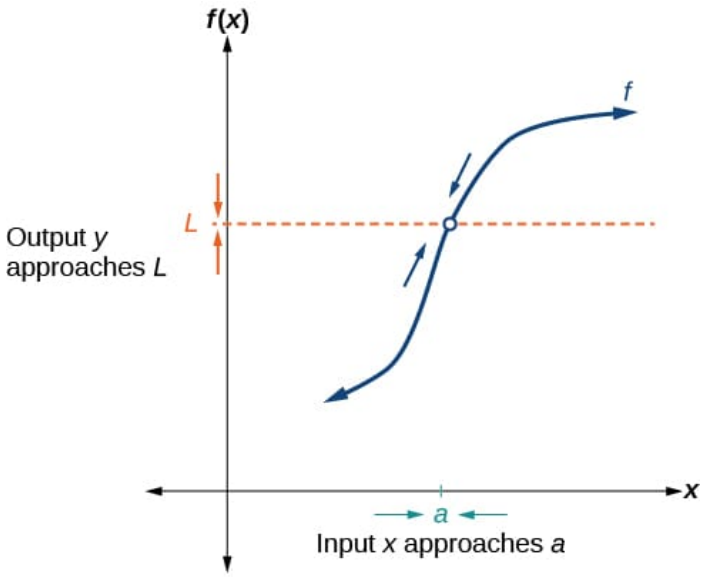



Figure $7.2.1$: The output (*y*--coordinate) approaches $L$ as the input (*x*-coordinate) approaches $a$.

We write the equation of a limit as

$$\lim_{x \to a} f(x) = L.$$

This notation indicates that as $x$ approaches $a$ both from the left of $x = a$ and the right of $x = a$, the output value approaches $L$.

Consider the function

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}. \tag{7.2.1}$$

We can factor the function in Equation 7.2.1 as shown.

$$f(x) = \frac{\cancel{(x-7)}\,(x+1)}{\cancel{x-7}} \quad \text{Cancel like factors in numerator and denominator.}$$

$$f(x) = x + 1, x \neq 7 \quad \text{Simplify.}$$

Notice that $x$ cannot be 7, or we would be dividing by 0, so 7 is not in the domain of the function in Equation 7.2.1. To avoid changing the function when we simplify, we set the same condition, $x \neq 7$, for the simplified function. We can represent the function graphically as shown in Figure 7.2.2.

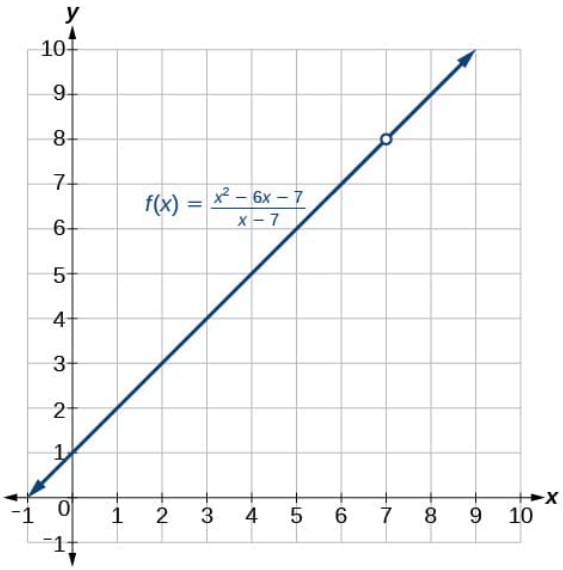



Figure 7.2.2: Because 7 is not allowed as an input, there is no point at $x = 7$.

What happens at $x = 7$ is completely different from what happens at points close to $x = 7$ on either side. The notation

$$\lim_{x \to 7} f(x) = 8$$

indicates that as the input $x$ approaches 7 from either the left or the right, the output approaches 8. The output can get as close to 8 as we like if the input is sufficiently near 7.

What happens at $x = 7$? When $x = 7$, there is no corresponding output. We write this as

$$f(7) \text{does not exist.}$$

This notation indicates that 7 is not in the domain of the function. We had already indicated this when we wrote the function as

$$f(x) = x + 1, x \neq 7.$$

Notice that the limit of a function can exist even when $f(x)$ is not defined at $x = a$. Much of our subsequent work will be determining limits of functions as $x$ nears $a$, even though the output at $x = a$ does not exist.

## THE LIMIT OF A FUNCTION

A quantity $L$ is the limit of a function $f(x)$ as $x$ approaches $a$ if, as the input values of $x$ approach $a$ (but do not equal $a$),the corresponding output values of $f(x)$ get closer to $L$. Note that the value of the limit is not affected by the output value of $f(x)$ at $a$. Both $a$ and $L$ must be real numbers. We write it as

$$\lim_{x \to a} f(x) = L$$

### Example 7.2.1: Understanding the Limit of a Function

For the following limit, define $a$, $f(x)$, and $L$.

$$\lim_{x \to 2}(3x + 5) = 11$$

**Solution**

First, we recognize the notation of a limit. If the limit exists, as $x$ approaches $a$, we write

$$\lim_{x \to a} f(x) = L.$$

We are given

$$\lim_{x \to 2}(3x + 5) = 11.$$

This means that $a = 2$, $f(x) = 3x + 5$, and $L = 11$.

**Analysis**

Recall that $y = 3x + 5$ is a line with no breaks. As the input values approach 2, the output values will get close to 11. This may be phrased with the equation $\lim_{x \to 2}(3x + 5) = 11$, which means that as $x$ nears 2 (but is not exactly 2), the output of the function $f(x) = 3x + 5$ gets as close as we want to $3(2) + 5$, or 11, which is the limit $L$, as we take values of $x$ sufficiently near 2 but not at $x = 2$.

### Exercise 7.2.1

For the following limit, define $a$, $f(x)$, and $L$.

$$\lim_{x \to 5}(2x^2 - 4) = 46$$

**Solution**

$a = 5$, $f(x) = 2x^2 - 4$, and $L = 46$.

## Understanding Left-Hand Limits and Right-Hand Limits

We can approach the input of a function from either side of a value—from the left or the right. Figure 7.2.3 shows the values of

$$f(x) = x + 1, x \neq 7$$

as described earlier and depicted in Figure 7.2.3.

| | Values of $x$ approach 7 from the left ($x < 7$) | | | $x = 7$ | Values of $x$ approach 7 from the right ($x > 7$) | | |
|---|---|---|---|---|---|---|---|
| $x$ | 6.9 | 6.99 | 6.999 | 7 | 7.001 | 7.01 | 7.1 |
| $f(x)$ | 7.9 | 7.99 | 7.999 | Undefined | 8.001 | 8.01 | 8.1 |
| | Values of output approach the limit, 8 | | | | Values of output approach the limit, 8 | | |

Figure 7.2.3 are $6.9, 6.99$, and $6.999$. The corresponding outputs are $7.9, 7.99$, and $7.999$. These values are getting closer to 8. The limit of values of $f(x)$ as $x$ approaches from the left is known as the left-hand limit. For this function, 8 is the left-hand limit of the function $f(x) = x + 1, x \neq 7$ as $x$ approaches 7.

Values described as "from the right" are greater than the input value 7 and would therefore appear to the right of the value on a number line. The input values that approach 7 from the right in Figure 7.2.3 are $7.1, 7.01$, and $7.001$. The corresponding outputs are $8.1, 8.01$, and $8.001$. These values are getting closer to 8. The limit of values of $f(x)$ as $x$ approaches from the right is known as the right-hand limit. For this function, 8 is also the right-hand limit of the function $f(x) = x + 1, x \neq 7$ as $x$ approaches 7.

Figure 7.2.3 shows that we can get the output of the function within a distance of 0.1 from 8 by using an input within a distance of 0.1 from 7. In other words, we need an input $x$ within the interval $6.9 < x < 7.1$ to produce an output value of $f(x)$ within the interval $7.9 < f(x) < 8.1$.

We also see that we can get output values of $f(x)$ successively closer to 8 by selecting input values closer to 7. In fact, we can obtain output values within any specified interval if we choose appropriate input values.

Figure 7.2.4 provides a visual representation of the left- and right-hand limits of the function. From the graph of $f(x)$, we observe the output can get infinitesimally close to $L = 8$ as $x$ approaches 7 from the left and as $x$ approaches 7 from the right.

To indicate the left-hand limit, we write

$$\lim_{x \to 7^-} f(x) = 8.$$

To indicate the right-hand limit, we write

$$\lim_{x \to 7^+} f(x) = 8.$$

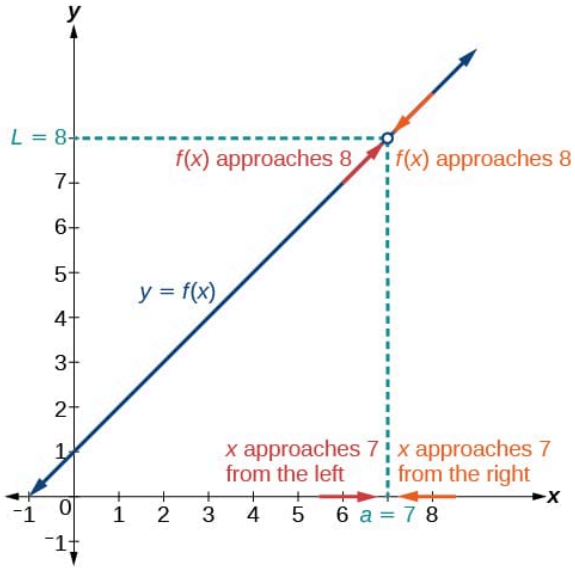



Figure 7.2.4: The left- and right-hand limits are the same for this function.

**LEFT- AND RIGHT-HAND LIMITS**

The left-hand limit of a function $f(x)$ as $x$ approaches $a$ from the left is equal to $L$, denoted by

$$\lim_{x \to a^-} f(x) = L.$$

The values of $f(x)$ can get as close to the limit $L$ as we like by taking values of $x$ sufficiently close to $a$ such that $x < a$ and $x \neq a$.

The right-hand limit of a function $f(x)$, as $x$ approaches $a$ from the right, is equal to $L$, denoted by

$$\lim_{x \to a^+} f(x) = L.$$

The values of $f(x)$ can get as close to the limit $L$ as we like by taking values of $x$ sufficiently close to $a$ but greater than $a$. Both $a$ and $L$ are real numbers.

## Understanding Two-Sided Limits

In the previous example, the left-hand limit and right-hand limit as $x$ approaches $a$ are equal. If the left- and right-hand limits are equal, we say that the function $f(x)$ has a two-sided limit as $x$ approaches $a$. More commonly, we simply refer to a **two-sided limit** as a limit. If the left-hand limit does not equal the right-hand limit, or if one of them does not exist, we say the limit does not exist.

**A**

The limit of a function $f(x)$, as $x$ approaches $a$, is equal to $L$, that is,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

In other words, the left-hand limit of a function $f(x)$ as $x$ approaches $a$ is equal to the right-hand limit of the same function as $x$ approaches $a$. If such a limit exists, we refer to the limit as a two-sided limit. Otherwise we say the limit does not exist.

# Finding a Limit Using a Graph

To visually determine if a limit exists as $x$ approaches $a$, we observe the graph of the function when $x$ is very near to $x = a$. In Figure $7.2.5$ we observe the behavior of the graph on both sides of a.

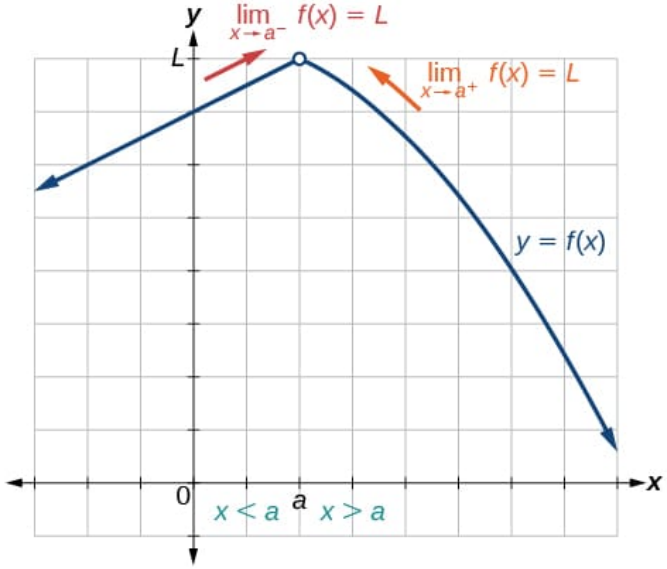



Figure 7.2.5

To determine if a left-hand limit exists, we observe the branch of the graph to the left of $x = a$, but near $x = a$. This is where $x < a$. We see that the outputs are getting close to some real number $L$ so there is a left-hand limit.

To determine if a right-hand limit exists, observe the branch of the graph to the right of $x = a$, but near $x = a$. This is where $x > a$ . We see that the outputs are getting close to some real number $L$, so there is a right-hand limit.

If the left-hand limit and the right-hand limit are the same, as they are in Figure 7.2.5, then we know that the function has a two-sided limit. Normally, when we refer to a "limit," we mean a two-sided limit, unless we call it a one-sided limit.

Finally, we can look for an output value for the function $f(x)$ when the input value $x$ is equal to $a$. The coordinate pair of the point would be $(a, f(a))$. If such a point exists, then $f(a)$ has a value. If the point does not exist, as in Figure 7.2.5, then we say that $f(a)$ does not exist.

### HOW TO: Given a function $f(x)$, use a graph to find the limits and a function value as $x$ approaches $a$.

1. Examine the graph to determine whether a left-hand limit exists.
2. Examine the graph to determine whether a right-hand limit exists.
3. If the two one-sided limits exist and are equal, then there is a two-sided limit—what we normally call a "limit."
4. If there is a point at $x = a$, then $f(a)$ is the corresponding function value.

### Example 7.2.2: Finding a Limit Using a Graph

Determine the following limits and function value for the function $f$ shown in Figure 7.2.6.

a. $\lim_{x\to 2^-} f(x)$
b. $\lim_{x\to 2^+} f(x)$
c. $\lim_{x\to 2} f(x)$
d. $f(2)$

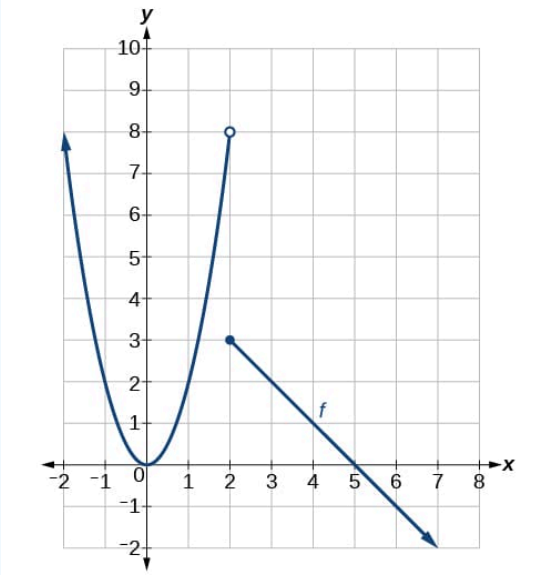



Figure 7.2.6

Determine the following limits and function value for the function f f shown in Figure.

i. $\lim_{x\to 2^-} f(x)$
ii. $\lim_{x\to 2^+} f(x)$
iii. $\lim_{x\to 2} f(x)$
iv. $f(2)$

a. Looking at Figure:
   i. $\lim_{x\to 2^-} f(x) = 8$; when $x < 2$,but infinitesimally close to 2, the output values get close to $y = 8$.
   ii. $\lim_{x\to 2^+} f(x) = 3$; when $x > 2$,but infinitesimally close to 2, the output values approach $y = 3$.
   iii. $\lim_{x\to 2} f(x)$ does not exist because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$; the left and right-hand limits are not equal.
   iv. $f(2) = 3$ because the graph of the function $f$ passes through the point $(2, f(2))$ or $(2, 3)$.

b. Looking at Figure:
   i. $\lim_{x\to 2^-} f(x) = 8$; when $x < 2$ but infinitesimally close to 2, the output values approach $y = 8$.
   ii. $\lim_{x\to 2^+} f(x) = 8$; when $x > 2$ but infinitesimally close to 2, the output values approach $y = 8$.
   iii. $\lim_{x\to 2} f(x) = 8$; because $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 8$; the left and right-hand limits are equal.
   iv. $f(2) = 4$ because the graph of the function $f$ passes through the point $(2, f(2))$ or $(2, 4)$.

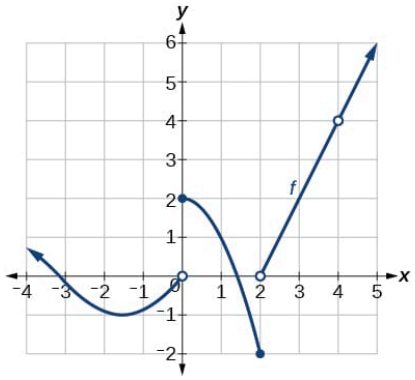
## Exercise 7.2.2:

Using the graph of the function y=f(x) y=f(x) shown in Figure, estimate the following limits.

a. $\lim_{x \to 0^-} f(x)$ b. $\lim_{x \to 0^+} f(x)$ c. $\lim_{x \to 0} f(x)$

d. $\lim_{x \to 2^-} f(x)$ e. $\lim_{x \to 2^+} f(x)$ f. $\lim_{x \to 2} f(x)$

g. $\lim_{x \to 4^-} f(x)$ h. $\lim_{x \to 4^+} f(x)$ i. $\lim_{x \to 4} f(x)$




**Solution**

a. 0; b. 2; c. does not exist; d.–2; –2; e. 0; f. does not exist; g. 4; h. 4; i. 4

## Finding a Limit Using a Table

Creating a table is a way to determine limits using numeric information. We create a table of values in which the input values of $x$ approach $a$ from both sides. Then we determine if the output values get closer and closer to some real value, the limit $L$.

Let's consider an example using the following function:

$$\lim_{x \to 5} \left( \frac{x^3 - 125}{x - 5} \right)$$

To create the table, we evaluate the function at values close to $x = 5$. We use some input values less than 5 and some values greater than 5 as in Figure. The table values show that when $x > 5$ but nearing 5, the corresponding output gets close to 75. When $x > 5$ but nearing 5, the corresponding output also gets close to 75.

| *x* | 4.9 | 4.99 | 4.999 | 5 | 5.001 | 5.01 | 5.1 |
|---|---|---|---|---|---|---|---|
| ***f(x)*** | 73.51 | 74.8501 | 74.985001 | Undefined | 75.015001 | 75.1501 | 76.51 |

$\lim_{x \to 5^-} f(x) = 75$ $\lim_{x \to 5^+} f(x) = 75$

Because

$$\lim_{x \to 5^-} f(x) = 75 = \lim_{x \to 5^+} f(x),$$

then

$$\lim_{x \to 5} f(x) = 75.$$

Remember that $f(5)$ does not exist.

**How to: Given a function $f$,use a table to find the limit as $x$ approaches $a$ and the value of $f(a)$, if it exists.**

1. Choose several input values that approach a a from both the left and right. Record them in a table.
2. Evaluate the function at each input value. Record them in the table.
3. Determine if the table values indicate a left-hand limit and a right-hand limit.
4. If the left-hand and right-hand limits exist and are equal, there is a two-sided limit.
5. Replace $x$ with $a$ to find the value of $f(a)$.

## Example 7.2.3: Finding a Limit Using a Table

Numerically estimate the limit of the following expression by setting up a table of values on both sides of the limit.

$$\lim_{x\to 0}\left(\frac{5\sin(x)}{3x}\right)$$

**Solution**

We can estimate the value of a limit, if it exists, by evaluating the function at values near $x = 0$. We cannot find a function value for $x = 0$ directly because the result would have a denominator equal to 0, and thus would be undefined.

$$f(x) = \frac{5\sin(x)}{3x}$$

We create Figure by choosing several input values close to $x = 0$, with half of them less than $x = 0$ and half of them greater than $x = 0$. Note that we need to be sure we are using radian mode. We evaluate the function at each input value to complete the table.

The table values indicate that when $x < 0$ but approaching 0, the corresponding output nears $\frac{5}{3}$.

When $x > 0$ but approaching 0, the corresponding output also nears $\frac{5}{3}$.

| $x$ | −0.1 | −0.01 | −0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1.66389 | 1.666639 | 1.666666 | Undefined | 1.666666 | 1.666639 | 1.66389 |

$\lim_{x\to 0^-} f(x) = \frac{5}{3}$ $\qquad$ $\lim_{x\to 0^+} f(x) = \frac{5}{3}$

Because

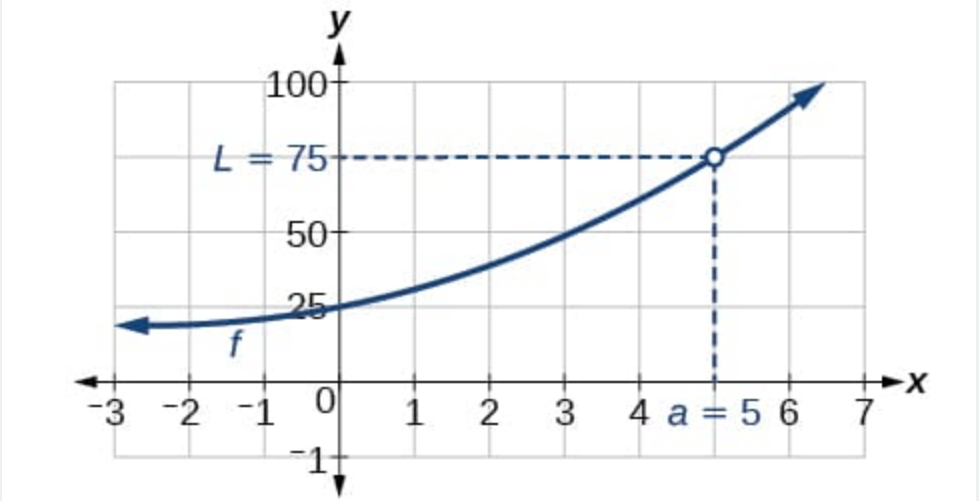
$$\lim_{x\to 0^-} f(x) = \frac{5}{3} = \lim_{x\to 0^+} f(x),$$

then

$$\lim_{x\to 0} f(x) = \frac{5}{3}.$$

## Q & A: Is it possible to check our answer using a graphing utility?

*Yes. We previously used a table to find a limit of 75 for the function $f(x) = \frac{x^3-125}{x-5}$ as $x$ approaches 5. To check, we graph the function on a viewing window as shown in Figure. A graphical check shows both branches of the graph of the function get close to the output 75 as $x$ nears 5. Furthermore, we can use the 'trace' feature of a graphing calculator. By appraoching $x = 5$ we may numerically observe the corresponding outputs getting close to 75.*




## Exercise 7.2.3:

Numerically estimate the limit of the following function by making a table:

$$\lim_{x \to 0} \left( \frac{20 \sin(x)}{4x} \right)$$

**Solution**

$$\lim_{x \to 0} \left( \frac{20 \sin(x)}{4x} \right) = 5$$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 4.9916708 | 4.9999167 | 4.9999992 | Error | 4.9999992 | 4.9999167 | 4.9916708 |

$\lim_{x \to 0^-} \left( \frac{20 \sin(x)}{4x} \right) \longrightarrow 5$ $\qquad$ $5 \longleftarrow \lim_{x \to 0^+} \left( \frac{20 \sin(x)}{4x} \right)$

## Q & A

**Is one method for determining a limit better than the other?**

*No. Both methods have advantages. Graphing allows for quick inspection. Tables can be used when graphical utilities aren't available, and they can be calculated to a higher precision than could be seen with an unaided eye inspecting a graph.*

## Example 7.2.4: Using a Graphing Utility to Determine a Limit

With the use of a graphing utility, if possible, determine the left- and right-hand limits of the following function as $x$ approaches 0. If the function has a limit as $x$ approaches 0, state it. If not, discuss why there is no limit.

$$f(x) = 3 \sin\left(\frac{\pi}{x}\right)$$

**Solution**

We can use a graphing utility to investigate the behavior of the graph close to $x = 0$. Centering around $x = 0$, we choose two viewing windows such that the second one is zoomed in closer to $x = 0$ than the first one. The result would resemble Figure for $[-2, 2]$ by $[-3, 3]$.

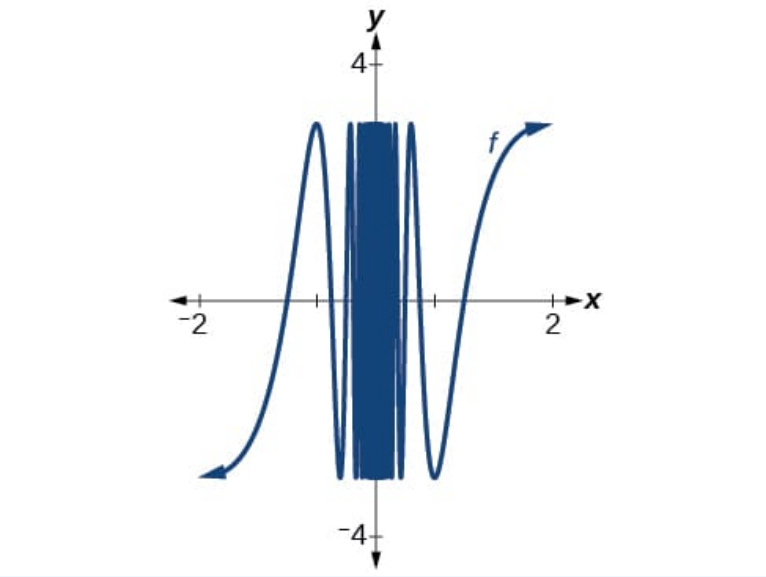



Figure_12_01_012">Figure for$[-0.1, 0.1]$ by $[-3, 3]$.

Even closer to zero, we are even less able to distinguish any limits.

The closer we get to 0, the greater the swings in the output values are. That is not the behavior of a function with either a left-hand limit or a right-hand limit. And if there is no left-hand limit or right-hand limit, there certainly is no limit to the function $f(x)$ as $x$ approaches 0.

We write

$$\lim_{x \to 0^-} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

$$\lim_{x \to 0^+} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

$$\lim_{x \to 0} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

## Exercise 7.2.4:

Numerically estimate the following limit: $\lim_{x \to 0}(\sin(\frac{2}{x}))$.

**Solution**

does not exist

Access these online resources for additional instruction and practice with finding limits.

- Introduction to Limits
- Formal Definition of a Limit

## Key Concepts

- A function has a limit if the output values approach some value $L$ as the input values approach some quantity a. a. See Example.
- A shorthand notation is used to describe the limit of a function according to the form $\lim_{x \to a} f(x) = L$, which indicates that as $x$ approaches $a$, both from the left of $x = a$ and the right of $x = a$, the output value gets close to $L$.
- A function has a left-hand limit if $f(x)$ approaches $L$ as $x$ approaches a a where $x < a$. A function has a right-hand limit if $f(x)$ approaches $L$ as $x$ approaches $a$ where $x > a$.
- A two-sided limit exists if the left-hand limit and the right-hand limit of a function are the same. A function is said to have a limit if it has a two-sided limit.
- A graph provides a visual method of determining the limit of a function.
- If the function has a limit as $x$ approaches $a$, the branches of the graph will approach the same $y-$ coordinate near $x = a$ from the left and the right. See Example.
- A table can be used to determine if a function has a limit. The table should show input values that approach $a$ from both directions so that the resulting output values can be evaluated. If the output values approach some number, the function has a limit. See Example.
- A graphing utility can also be used to find a limit. See Example.

## Footnotes

- 1 https://en.Wikipedia.org/wiki/Human_height and http://en.Wikipedia.org/wiki/List_of_tallest_people

## Glossary

**left-hand limit**

the limit of values of $f(x)$ as $x$ approaches from $a$ the left, denoted

$$\lim_{x \to a^-} f(x) = L.$$

The values of $f(x)$ can get as close to the limit $L$ as we like by taking values of $x$ sufficiently close to a a such that $x < a$ and $x \neq a$.. Both $a$ and $L$ are real numbers.

**limit**

when it exists, the value, $L$,that the output of a function $f(x)$ approaches as the input $x$ gets closer and closer to $a$ but does not equal $a$. The value of the output, $f(x)$, can get as close to $L$ as we choose to make it by using input values of $x$ sufficiently near to $x = a$,but not necessarily at $x = a$. Both $a$ and $L$ are real numbers, and $L$ is denoted

$$\lim_{x \to a} f(x) = L.$$

**right-hand limit**

the limit of values of $f(x)$ as $x$ approaches $a$ from the right, denoted

$$\lim_{x \to a^+} f(x) = L.$$

The values of $f(x)$ can get as close to the limit $L$ as we like by taking values of $x$ sufficiently close to $a$ where $x > a$, and $x \neq a$. Both $a$ and $L$ are real numbers.

**two-sided limit**

the limit of a function

$$f(x),$$

as $x$ approaches $a$,is equal to $L$,that is,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

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# 7.3: Finding Limits - Properties of Limits

Consider the *rational function*

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

The function can be factored as follows:

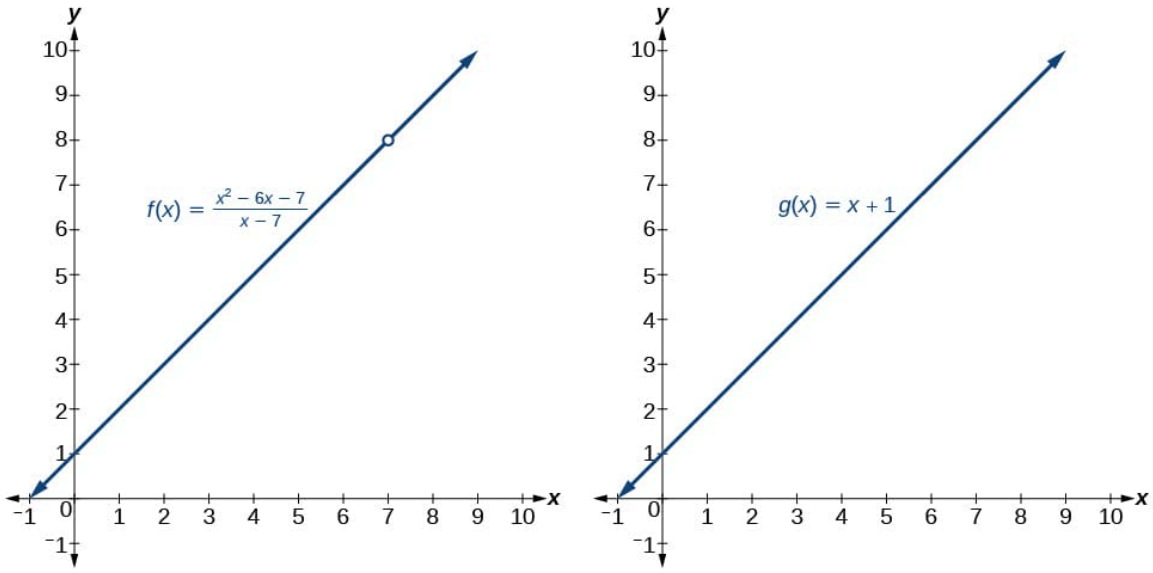
$$f(x) = \frac{\cancel{(x-7)}\,(x+1)}{\cancel{x-7}}$$

which gives us

$$f(x) = x + 1, x \neq 7.$$

Does this mean the function $f(x)$ is the same as the function $g(x) = x + 1$?

The answer is no. Function $f(x)$ does not have $x = 7$ in its domain, but $g(x)$ does. Graphically, we observe there is a hole in the graph of $f(x)$ at $x = 7$, as shown in Figure and no such hole in the graph of $g(x)$, as shown in Figure.




*(left) The graph of function $f$ contains a break at $x = 7$ and is therefore not continuous at $x = 7$. (Right)The graph of function $g$ is continuous.*

So, do these two different functions also have different limits as $x$ approaches 7? Not necessarily. Remember, in determining a limit of a function as $x$ approaches $a$, what matters is whether the output approaches a real number as we get close to $x = a$. The existence of a limit does not depend on what happens when $x$ equals $a$.

Look again at Figure and Figure. Notice that in both graphs, as $x$ approaches 7, the output values approach 8. This means

$$\lim_{x \to 7} f(x) = \lim_{x \to 7} g(x).$$

Remember that when determining a limit, the concern is what occurs near $x = a$, not at $x = a$. In this section, we will use a variety of methods, such as rewriting functions by factoring, to evaluate the limit. These methods will give us formal verification for what we formerly accomplished by intuition.

# Finding the Limit of a Sum, a Difference, and a Product

Graphing a function or exploring a table of values to determine a limit can be cumbersome and time-consuming. When possible, it is more efficient to use the **properties of limits**, which is a collection of theorems for finding limits.

Knowing the properties of limits allows us to compute limits directly. We can add, subtract, multiply, and divide the limits of functions as if we were performing the operations on the functions themselves to find the limit of the result. Similarly, we can find the limit of a function raised to a power by raising the limit to that power. We can also find the limit of the root of a function by taking the root of the limit. Using these operations on limits, we can find the limits of more complex functions by finding the limits of their simpler component functions.

## properties of limits

Let $a, k, A,$ and $B$ represent real numbers, and $f$ and $g$ be functions, such that $\lim\limits_{x\to a} f(x) = A$ and $\lim\limits_{x\to a} g(x) = B$. For limits that exist and are finite, the properties of limits are summarized in Table

| | |
|---|---|
| Constant, *k* | $\lim\limits_{x\to a} k = k$ |
| Constant times a function | $\lim\limits_{x\to a}[k \cdot f(x)] = k \lim\limits_{x\to a} f(x) = kA$ |
| Sum of functions | $\lim\limits_{x\to a}[f(x) + g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{xtoa} g(x) = A + B$ |
| Difference of functions | $\lim\limits_{x\to a}[f(x) - g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = A - B$ |
| Product of functions | $\lim\limits_{x\to a}[f(x) \cdot g(x)] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = A \cdot B$ |
| Quotient of functions | $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{A}{B}, B \neq 0$ |
| Function raised to an exponent | $\lim\limits_{x\to a}[f(x)]^n = [\lim\limits_{x\to\infty} f(x)]^n = A^n$, where $n$ is a positive integer |
| *n*th root of a function, where n is a positive integer | $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a}[f(x)]} = \sqrt[n]{A}$ |
| Polynomial function | $\lim\limits_{x\to a} p(x) = p(a)$ |

## Example 7.3.1: Evaluating the Limit of a Function Algebraically

Evaluate

$$\lim_{x\to 3}(2x + 5).$$

**Solution**

$$\begin{aligned} \lim_{x\to 3}(2x + 5) &= \lim_{x\to 3}(2x) + \lim_{x\to 3}(5) && \text{Sum of functions property} && (7.3.1)\\ &= 2\lim_{x\to 3}(x) + \lim_{x\to 3}(5) && \text{Constant times a function property} && (7.3.2)\\ &= 2(3) + 5 && \text{Evaluate} && (7.3.3)\\ &= 11 && && (7.3.4) \end{aligned}$$

## Exercise 7.3.1:

Evaluate the following limit:

$$\lim_{x\to -12}(-2x + 2).$$

**Solution**

26

## Finding the Limit of a Polynomial

Not all functions or their limits involve simple addition, subtraction, or multiplication. Some may include polynomials. Recall that a polynomial is an expression consisting of the sum of two or more terms, each of which consists of a constant and a variable raised to a nonnegative integral power. To find the limit of a polynomial function, we can find the limits of the individual terms of the function, and then add them together. Also, the limit of a polynomial function as $x$ approaches $a$ is equivalent to simply evaluating the function for $a$.

### how to: Given a function containing a polynomial, find its limit

1. Use the properties of limits to break up the polynomial into individual terms.
2. Find the limits of the individual terms.
3. Add the limits together.
4. Alternatively, evaluate the function for $a$.

### Example 7.3.1: Evaluating the Limit of a Function Algebraically

Evaluate

$$\lim_{x\to 3}(5x^2).$$

**Solution**

$$\begin{aligned} \lim_{x\to 3}(5x^2) &= 5\lim_{x\to 3}(x^2) && \text{Constant times a function property} && (7.3.5)\\ &= 5(3^2) && \text{Function raised to an exponent property} && (7.3.6)\\ &= 45 && && (7.3.7)\end{aligned}$$

### Exercise 7.3.1:

Evaluate

$$\lim_{x\to 4}(x^3-5).$$

**Solution**

59

### Example 7.3.2: Evaluating the Limit of a Polynomial Algebraically

Evaluate

$$\lim_{x\to 5}(2x^3-3x+1).$$

**Solution**

$$\begin{aligned} \lim_{x\to 5}(2x^3-3x+1) &= \lim_{x\to 5}(2x3)-\lim_{x\to 5}(3x)+\lim_{x\to 5}(1) && \text{Sum of functions} && (7.3.8)\\ &= 2\lim_{x\to 5}(x^3)-3\lim_{x\to 5}(x)+\lim_{x\to 5}(1) && \text{Constant times a function} && (7.3.9)\\ &= 2(5^3)-3(5)+1 && \text{Function raised to an exponent} && (7.3.10)\\ &= 236 && \text{Evaluate} && (7.3.11)\end{aligned}$$

**Exercise 7.3.2:**

Evaluate the following limit:

$$\lim_{x\to -1}(x^4-4x^3+5).$$

**Solution**

10

## Finding the Limit of a Power or a Root

When a limit includes a power or a root, we need another property to help us evaluate it. The square of the limit of a function equals the limit of the square of the function; the same goes for higher powers. Likewise, the square root of the limit of a function equals the limit of the square root of the function; the same holds true for higher roots.

**Example 7.3.3: Evaluating a Limit of a Power**

Evaluate

$$\lim_{x\to 2}(3x+1)^5.$$

**Solution**

We will take the limit of the function as $x$ approaches 2 and raise the result to the 5th power.

$$\begin{aligned}\lim_{x\to 2}(3x+1)^5 &= (\lim_{x\to 2}(3x+1))^5 && (7.3.12)\\ &= (3(2)+1)^5 && (7.3.13)\\ &= 7^5 && (7.3.14)\\ &= 16,807 && (7.3.15)\end{aligned}$$

**Exercise 7.3.3:**

Evaluate the following limit: $\lim_{x\to -4}(10x+36)^3$.

**Solution**

−64

**Q & A: If we can't directly apply the properties of a limit, for example in $\lim_{x\to 2}(\frac{x^2+6x+8}{x-2})$, can we still determine the limit of the function as $x$ approaches $a$?**

*Yes. Some functions may be algebraically rearranged so that one can evaluate the limit of a simplified equivalent form of the function.*

## Finding the Limit of a Quotient

Finding the limit of a function expressed as a quotient can be more complicated. We often need to rewrite the function algebraically before applying the properties of a limit. If the denominator evaluates to 0 when we apply the properties of a limit directly, we must rewrite the quotient in a different form. One approach is to write the quotient in factored form and simplify.

## Example 7.3.4: Evaluating the Limit of a Quotient by Factoring

Evaluate

$$\lim_{x\to 2}\left(\frac{x^2-6x+8}{x-2}\right).$$

**Solution**

Factor where possible, and simplify.

$$\lim_{x\to 2}\left(\frac{x^2-6x+8}{x-2}\right)=\lim_{x\to 2}\left(\frac{(x-2)(x-4)}{x-2}\right) \quad \text{Factor the numerator.} \tag{7.3.16}$$

$$=\lim_{x\to 2}\left(\frac{\cancel{(x-2)}\,(x-4)}{\cancel{x-2}}\right) \quad \text{Cancel the common factors.} \tag{7.3.17}$$

$$=\lim_{x\to 2}(x-4) \quad \text{Evaluate.} \tag{7.3.18}$$

$$=2-4=-2 \tag{7.3.19}$$

**Analysis**

When the limit of a rational function cannot be evaluated directly, factored forms of the numerator and denominator may simplify to a result that can be evaluated.

Notice, the function

$$f(x)=\frac{x^2-6x+8}{x-2}$$

is equivalent to the function

$$f(x)=x-4, x\neq 2.$$

Notice that the limit exists even though the function is not defined at $x=2$.

## Exercise 7.3.4

Evaluate the following limit:

$$\lim_{x\to 7}\left(\frac{x^2-11x+28}{7-x}\right).$$

**Solution**

$-3$

## Example 7.3.5: Evaluating the Limit of a Quotient by Finding the LCD

Evaluate

$$\lim_{x\to 5}\left(\frac{\frac{1}{x}-\frac{1}{5}}{x-5}\right).$$

**Solution**

Find the LCD for the denominators of the two terms in the numerator, and convert both fractions to have the LCD as their denominator.

**Analysis**

When determining the limit of a rational function that has terms added or subtracted in either the numerator or denominator, the first step is to find the common denominator of the added or subtracted terms; then, convert both terms to have that denominator, or simplify the rational function by multiplying numerator and denominator by the least common denominator. Then check to see if the resulting numerator and denominator have any common factors.

## Exercise 7.3.5:

Evaluate

$$\lim_{x\to -5}\left(\frac{\frac{1}{5}+\frac{1}{x}}{10+2x}\right).$$

**Solution**

$-\frac{1}{50}$

## how to: Given a limit of a function containing a root, use a conjugate to evaluate

1. If the quotient as given is not in indeterminate $(\frac{0}{0})$ form, evaluate directly.
2. Otherwise, rewrite the sum (or difference) of two quotients as a single quotient, using the **least common denominator (LCD).**
3. If the numerator includes a root, rationalize the numerator; multiply the numerator and denominator by the **conjugate** of the numerator. Recall that $a\pm\sqrt{b}$ are conjugates.
4. Simplify.
5. Evaluate the resulting limit.

## Example 7.3.6: Evaluating a Limit Containing a Root Using a Conjugate

Evaluate

$$\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right).$$

**Solution**

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right) &= \lim_{x\to 0}\left(\frac{(\sqrt{25-x}-5)}{x}\cdot\frac{(\sqrt{25-x}+5)}{(\sqrt{25-x}+5)}\right) && \text{Multiply numerator and denominator by the conjugat}\\
&= \lim_{x\to 0}\left(\frac{(25-x)-25}{x(\sqrt{25-x}+5)}\right) && \text{Multiply: } (\sqrt{25-x}-5)\cdot(\sqrt{25-x}+5)=(25-x)-\\
&= \lim_{x\to 0}\left(\frac{-\cancel{x}}{\cancel{x}(25-x+5)}\right) && \text{Combine like terms.}\\
&= \lim_{x\to 0}\left(\frac{-\cancel{x}}{\cancel{x}(\sqrt{25-x}+5)}\right) && \text{Simplify } \frac{-x}{x}=-1.\\
&= \frac{-1}{\sqrt{25-0}+5} && \text{Evaluate.}\\
&= \frac{-1}{5+5}=-\frac{1}{10}
\end{aligned}$$

**Analysis**

When determining a **limit** of a function with a root as one of two terms where we cannot evaluate directly, think about multiplying the numerator and denominator by the conjugate of the terms.

## Exercise 7.3.6

Evaluate the following limit: $\lim_{h\to 0}\left(\frac{\sqrt{16-h}-4}{h}\right)$.

**Solution**

$-\frac{1}{8}$

## Example 7.3.7: Evaluating the Limit of a Quotient of a Function by Factoring

Evaluate

$$\lim_{x\to 4}\left(\frac{4-x}{\sqrt{x}-2}\right).$$

**Solution**

$$\lim_{x\to 4}\left(\frac{4-x}{\sqrt{x}-2}\right)=\lim_{x\to 4}\left(\frac{(2+\sqrt{x})(2-x)}{\sqrt{x}-2}\right) \quad \text{Factor.} \tag{7.3.26}$$

$$=\lim_{x\to 4}\left(\frac{(2+\sqrt{x})(\cancel{2-\sqrt{x}})}{-(\cancel{2-\sqrt{x}})}\right) \quad \text{Factor } -1 \text{ out of the denominator. Simplify.} \tag{7.3.27}$$

$$=\lim_{x\to 4}-(2+x) \quad \text{Evaluate.} \tag{7.3.28}$$

$$=-(2+\sqrt{4}) \tag{7.3.29}$$

$$=-4 \tag{7.3.30}$$

**Analysis**

Multiplying by a conjugate would expand the numerator; look instead for factors in the numerator. Four is a perfect square so that the numerator is in the form

$$a^2-b^2$$

and may be factored as

$$(a+b)(a-b).$$

## Exercise 7.3.7

Evaluate the following limit:

$$\lim_{x\to 3}\left(\frac{x-3}{\sqrt{x}-\sqrt{3}}\right).$$

**Solution**

$2\sqrt{3}$

**how to: Given a quotient with absolute values, evaluate its limit**

1. Try factoring or finding the LCD.
2. If the **limit** cannot be found, choose several values close to and on either side of the input where the function is undefined.
3. Use the numeric evidence to estimate the limits on both sides.

**Example 7.3.8: Evaluating the Limit of a Quotient with Absolute Values**

Evaluate

$$\lim_{x\to 7}\frac{|x-7|}{x-7}.$$

**Solution**

The function is undefined at $x=7$, so we will try values close to 7 from the left and the right.

Left-hand limit:

$$\frac{|6.9-7|}{6.9-7}=\frac{|6.99-7|}{6.99-7}=\frac{|6.999-7|}{6.999-7}=-1$$

Right-hand limit:

$$\frac{|7.1-7|}{7.1-7}=\frac{|7.01-7|}{7.01-7}=\frac{|7.001-7|}{7.001-7}=1$$

Since the left- and right-hand limits are not equal, there is no limit.

**Exercise 7.3.8**

Evaluate

$$\lim_{x\to 6^+}\frac{6-x}{|x-6|}.$$

**Solution**

## Key Concepts

- The properties of limits can be used to perform operations on the limits of functions rather than the functions themselves. See Example.
- The limit of a polynomial function can be found by finding the sum of the limits of the individual terms. See Example and Example.
- The limit of a function that has been raised to a power equals the same power of the limit of the function. Another method is direct substitution. See Example.
- The limit of the root of a function equals the corresponding root of the limit of the function.
- One way to find the limit of a function expressed as a quotient is to write the quotient in factored form and simplify. See Example.
- Another method of finding the limit of a complex fraction is to find the LCD. See Example.
- A limit containing a function containing a root may be evaluated using a conjugate. See Example.
- The limits of some functions expressed as quotients can be found by factoring. See Example.
- One way to evaluate the limit of a quotient containing absolute values is by using numeric evidence. Setting it up piecewise can also be useful. See Example.

# Glossary

**properties of limits**
: a collection of theorems for finding limits of functions by performing mathematical operations on the limits

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# 7.4: THE LIMIT OF A FUNCTION

## Learning Objectives

- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- Define one-sided limits and provide examples.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

- $f(x) = \dfrac{x^2 - 4}{x - 2}$,
- $g(x) = \dfrac{|x - 2|}{x - 2}$, and
- $h(x) = \dfrac{1}{(x - 2)^2}$,

which are shown in Figure $7.4.1$. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

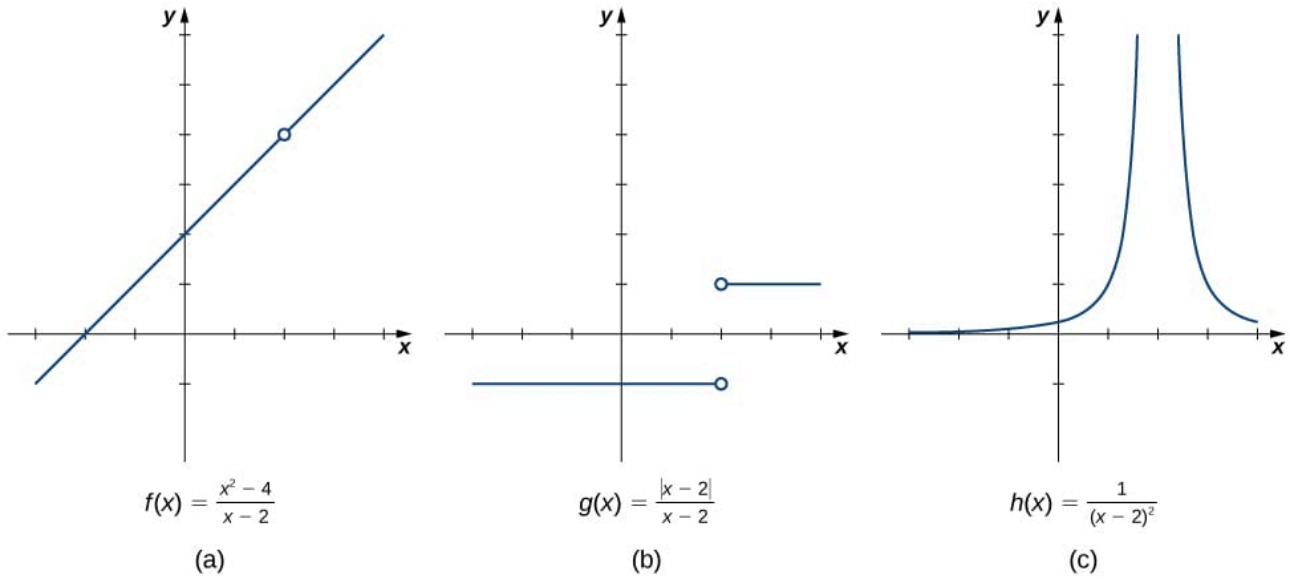



Figure $7.4.1$: These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of $2$ more completely, we need to introduce the concept of a limit.

# Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in Figure 7.4.1. As the values of $x$ approach $2$ from either side of $2$, the values of $y = f(x)$ approach $4$. Mathematically, we say that the limit of $f(x)$ as $x$ approaches $2$ is $4$. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4$$

From this very brief informal look at one limit, let's start to develop an *intuitive definition of the limit*. We can think of the limit of a function at a number $a$ as being the one real number $L$ that the functional values approach as the $x$-values approach $a$, provided such a real number $L$ exists. Stated more carefully, we have the following definition:

## Definition (Intuitive): Limit

Let $f(x)$ be a function defined at all values in an open interval containing $a$, with the possible exception of $a$ itself, and let $L$ be a real number. If all values of the function $f(x)$ approach the real number $L$ as the values of $x (\neq a)$ approach the number $a$, then we say that the limit of $f(x)$ as $x$ approaches $a$ is $L$. (More succinctly, as $x$ gets closer to $a$, $f(x)$ gets closer and stays close to $L$.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L. \tag{7.4.1}$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

## Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of $x$-values—one set of values approaching $a$ and less than $a$, and another set of values approaching $a$ and greater than $a$. Table 7.4.1 demonstrates what your tables might look like.

*Table* 7.4.1

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| $a - 0.1$ | $f(a - 0.1)$ | $a + 0.1$ | $f(a + 0.1)$ |
| $a - 0.01$ | $f(a - 0.01)$ | $a + 0.01$ | $f(a + 0.01)$ |
| $a - 0.001$ | $f(a - 0.001)$ | $a + 0.001$ | $f(a + 0.001)$ |
| $a - 0.0001$ | $f(a - 0.0001)$ | $a + 0.0001$ | $f(a + 0.0001)$ |
| Use additional values as necessary. | | Use additional values as necessary. | |

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a - 0.1)$, $f(a - 0.01)$, $f(a - 0.001)$, $f(a - 0.0001)$, and so on, and $f(a + 0.1)$, $f(a + 0.01)$, $f(a + 0.001)$, $f(a + 0.0001)$, and so on. (Note: Although we have chosen the $x$-values $a \pm 0.1$, $a \pm 0.01$, $a \pm 0.001$, $a \pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)

3. If both columns approach a common $y$-value $L$, we state $\lim_{x \to a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for $x$-values near $a$ are in our window. We can use the trace feature to move along the graph of the function and watch the $y$-value readout as the $x$-values approach $a$. If the $y$-values approach $L$ as our $x$-values approach $a$ from both directions, then $\lim_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in Examples $7.4.1A$ and $7.4.1B$.

## Example $7.4.1A$: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim_{x \to 0} \dfrac{\sin x}{x}$ using a table of functional values.

**Solution**

We have calculated the values of $f(x) = \dfrac{\sin x}{x}$ for the values of $x$ listed in Table $7.4.2$.

*Table* $7.4.2$

| $x$ | $\frac{\sin x}{x}$ | $x$ | $\frac{\sin x}{x}$ |
|---|---|---|---|
| -0.1 | 0.998334166468 | 0.1 | 0.998334166468 |
| -0.01 | 0.999983333417 | 0.01 | 0.999983333417 |
| -0.001 | 0.999999833333 | 0.001 | 0.999999833333 |
| -0.0001 | 0.999999998333 | 0.0001 | 0.999999998333 |

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\dfrac{\sin x}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$. A calculator-or computer-generated graph of $f(x) = \dfrac{\sin x}{x}$ would be similar to that shown in Figure $7.4.2$, and it confirms our estimate.

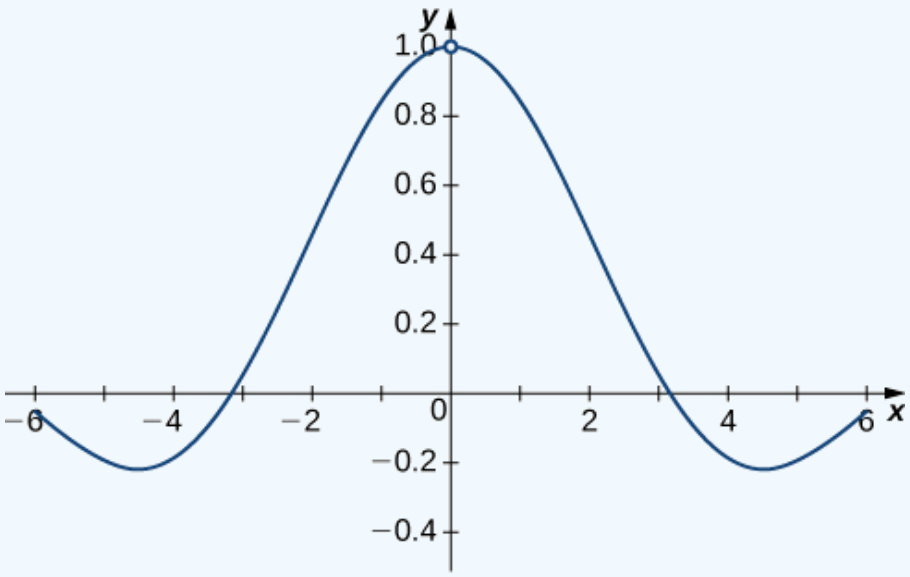



Figure $7.4.2$: The graph of $f(x) = (\sin x)/x$ confirms the estimate from Table $7.4.2$.

## Example 7.4.1$B$: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$ using a table of functional values.

**Solution**

As before, we use a table—in this case, Table 7.4.3—to list the values of the function for the given values of $x$.

*Table* 7.4.3

| $x$ | $\frac{\sqrt{x}-2}{x-4}$ | $x$ | $\frac{\sqrt{x}-2}{x-4}$ |
|---|---|---|---|
| 3.9 | 0.251582341869 | 4.1 | 0.248456731317 |
| 3.99 | 0.25015644562 | 4.01 | 0.24984394501 |
| 3.999 | 0.250015627 | 4.001 | 0.249984377 |
| 3.9999 | 0.250001563 | 4.0001 | 0.249998438 |
| 3.99999 | 0.25000016 | 4.00001 | 0.24999984 |

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}=0.25$. We confirm this estimate using the graph of $f(x)=\dfrac{\sqrt{x}-2}{x-4}$ shown in Figure 7.4.3.

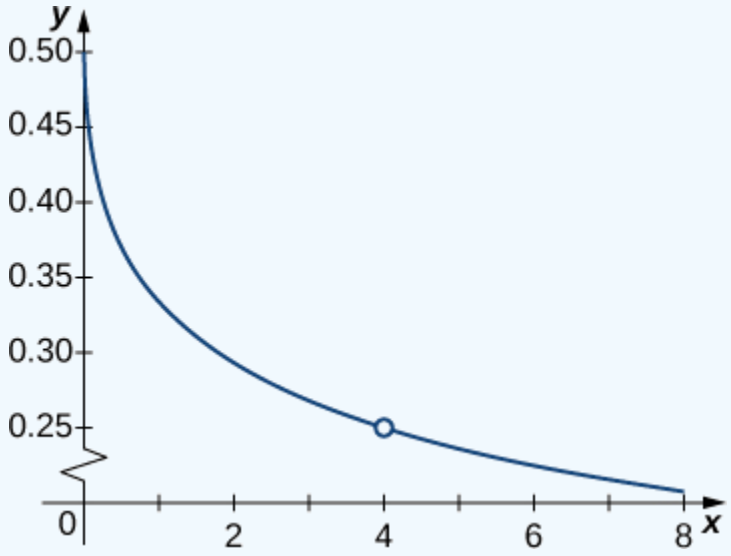



Figure 7.4.3: The graph of $\frac{\sqrt{x}-2}{x-4}$ confirms the estimate from Table 7.4.3.

## Exercise 7.4.1

Estimate $\lim\limits_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1}$ using a table of functional values. Use a graph to confirm your estimate.

*Hint available at the end of this section.*

At this point, we see from Examples 7.4.1$A$ and 7.4.1$B$ that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In Example 7.4.2, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

## Example 7.4.2: Evaluating a Limit Using a Graph

For $g(x)$ shown in Figure 7.4.4, evaluate $\lim_{x \to -1} g(x)$.

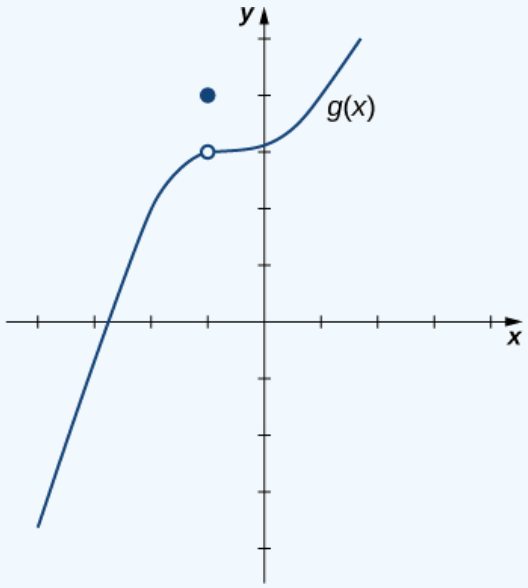



Figure 7.4.4: The graph of $g(x)$ includes one value not on a smooth curve.

**Solution**:

Despite the fact that $g(-1) = 4$, as the $x$-values approach $-1$ from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x \to -1} g(x) = 3$. Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on Example 7.4.2, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

## Exercise 7.4.2

Use the graph of $h(x)$ in Figure 7.4.5 to evaluate $\lim_{x \to 2} h(x)$, if possible.

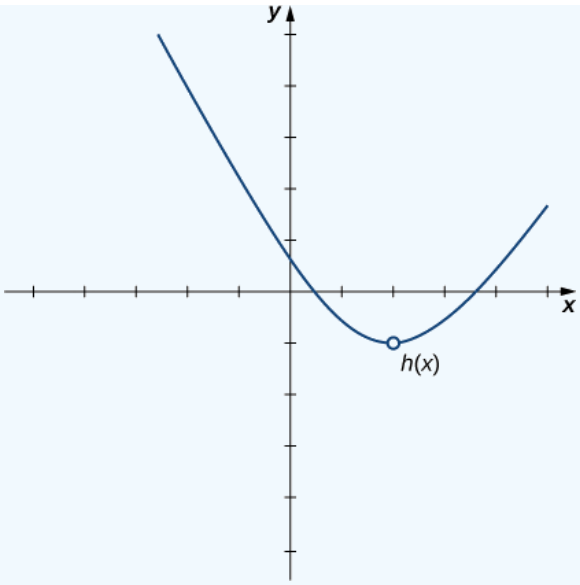



Figure 7.4.5:

*Hint available at the end of this section.*

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

## Two Important Limits

Let $a$ be a real number and $c$ be a constant.

i. $\lim\limits_{x\to a} x = a$
ii. $\lim\limits_{x\to a} c = c$

We can make the following observations about these two limits.

i. For the first limit, observe that as $x$ approaches $a$, so does $f(x)$, because $f(x) = x$. Consequently, $\lim\limits_{x\to a} x = a$.
ii. For the second limit, consider Table 7.4.4.

*Table* 7.4.4

| $x$ | $f(x) = c$ | $x$ | $f(x) = c$ |
|---|---|---|---|
| $a - 0.1$ | $c$ | $a + 0.1$ | $c$ |
| $a - 0.01$ | $c$ | $a + 0.01$ | $c$ |
| $a - 0.001$ | $c$ | $a + 0.001$ | $c$ |
| $a - 0.0001$ | $c$ | $a + 0.0001$ | $c$ |

Observe that for all values of $x$ (regardless of whether they are approaching $a$), the values $f(x)$ remain constant at $c$. We have no choice but to conclude $\lim\limits_{x\to a} c = c$.

# The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

## Example 7.4.3: Evaluating a Limit That Fails to Exist

Evaluate $\lim_{x\to 0} \sin(1/x)$ using a table of values.

**Solution**

Table 7.4.5 lists values for the function $\sin(1/x)$ for the given values of $x$.

*Table* 7.4.5

| $x$ | $\sin(1/x)$ | $x$ | $\sin(1/x)$ |
|---|---|---|---|
| -0.1 | 0.544021110889 | 0.1 | −0.544021110889 |
| -0.01 | 0.50636564111 | 0.01 | −0.50636564111 |
| -0.001 | −0.8268795405312 | 0.001 | 0.8268795405312 |
| -0.0001 | 0.305614388888 | 0.0001 | −0.305614388888 |
| -0.00001 | −0.035748797987 | 0.00001 | 0.035748797987 |
| -0.000001 | 0.349993504187 | 0.000001 | −0.349993504187 |

After examining the table of functional values, we can see that the $y$-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of $x$-values approaching $0$:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding $y$-values are

$$1, -1, 1, -1, 1, -1, \ldots.$$

At this point we can indeed conclude that $\lim_{x\to 0} \sin(1/x)$ does not exist. (Mathematicians frequently abbreviate "does not exist" as DNE. Thus, we would write $\lim_{x\to 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in Figure 7.4.6 and it gives a clearer picture of the behavior of $\sin(1/x)$ as $x$ approaches $0$. You can see that $\sin(1/x)$ oscillates ever more wildly between $-1$ and $1$ as $x$ approaches $0$.

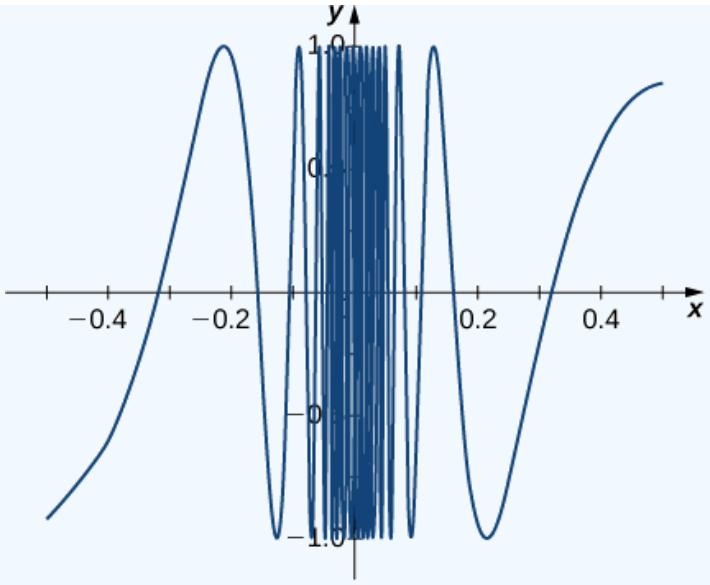



Figure 7.4.6: The graph of $f(x) = \sin(1/x)$ oscillates rapidly between $-1$ and $1$ as $x$ approaches $0$.

**Exercise 7.4.3**

Use a table of functional values to evaluate $\lim_{x \to 2} \dfrac{|\, x^2 - 4\,|}{x - 2}$, if possible.

*Hint available at the end of this section.*

## One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x) = |x - 2|/(x - 2)$ introduced at the beginning of the section (see Figure 7.4.1(b)). As we pick values of $x$ close to $2$, $g(x)$ does not approach a single value, so the limit as $x$ approaches $2$ does not exist—that is, $\lim_{x \to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the $x$-value $2$. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of $2$ (or the negative side of $2$), $g(x) = -1$. Thus, as $x$ approaches $2$ from the left, $g(x)$ approaches $-1$. Mathematically, we say that the limit as $x$ approaches $2$ from the left is $-1$. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x) = -1.$$

Similarly, as $x$ approaches $2$ from the right (or from the positive side), $g(x)$ approaches $1$. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

**Definition: One-sided Limits**

We define two types of one-sided limits.

**Limit from the left:**

Let $f(x)$ be a function defined at all values in an open interval of the form $(z, a)$, and let $L$ be a real number. If the values of the function $f(x)$ approach the real number $L$ as the values of $x$ (where $x < a$) approach the number $a$, then we say that $L$ is the limit of $f(x)$ as $x$ approaches $a$ from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L.$$

**Limit from the right:**

Let $f(x)$ be a function defined at all values in an open interval of the form $(a, c)$, and let $L$ be a real number. If the values of the function $f(x)$ approach the real number $L$ as the values of $x$ (where $x > a$) approach the number $a$, then we say that $L$ is the limit of $f(x)$ as $x$ approaches $a$ from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L.$$

## Example 7.4.4: Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$

**Solution**

We can use tables of functional values again. Observe in Table 7.4.6 that for values of $x$ less than 2, we use $f(x) = x + 1$ and for values of $x$ greater than 2, we use $f(x) = x^2 - 4$.

*Table* 7.4.6

| $x$ | $f(x) = x + 1$ | $x$ | $f(x) = x^2 - 4$ |
|---|---|---|---|
| 1.9 | 2.9 | 2.1 | 0.41 |
| 1.99 | 2.99 | 2.01 | 0.0401 |
| 1.999 | 2.999 | 2.001 | 0.004001 |
| 1.9999 | 2.9999 | 2.0001 | 0.00040001 |
| 1.99999 | 2.99999 | 2.00001 | 0.0000400001 |

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$. Figure 7.4.7 shows a graph of $f(x)$ and reinforces our conclusion about these limits.

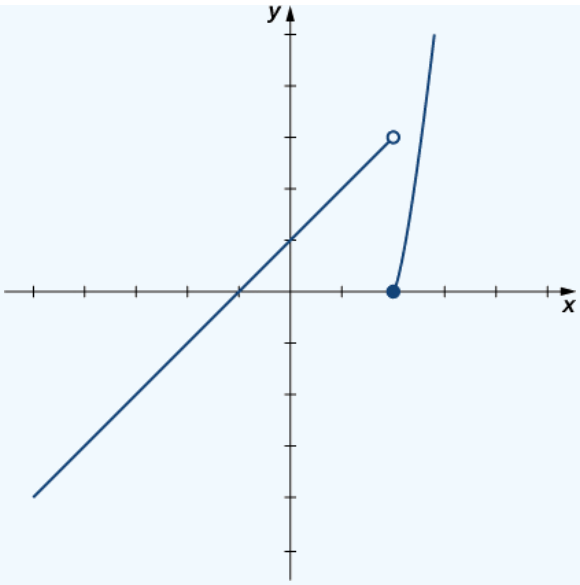



Figure 7.4.7: The graph of $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$ has a break at $x = 2$.

### Exercise 7.4.4

Use a table of functional values to estimate the following limits, if possible.

a. $\lim\limits_{x \to 2^-} \dfrac{|\, x^2 - 4 \,|}{x - 2}$

b. $\lim\limits_{x \to 2^+} \dfrac{|\, x^2 - 4 \,|}{x - 2}$

*Hint available at the end of this section.*

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in Note.

### Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing $a$, with the possible exception of $a$ itself, and let $L$ be a real number. Then,

$$\lim_{x \to a} f(x) = L$$

if and only if $\lim\limits_{x \to a^-} f(x) = L$ and $\lim\limits_{x \to a^+} f(x) = L$.

## Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see Figure 7.4.1(c)). From its graph we see that as the values of $x$ approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in

fact, become infinite. Mathematically, we say that the limit of $h(x)$ as $x$ approaches $2$ is positive infinity. Symbolically, we express this idea as

$$\lim_{x\to 2} h(x) = +\infty.$$

More generally, we define *infinite limits* as follows:

## Definitions: Infinite Limits

We define three types of *infinite limits*.

**Infinite limits from the left:** Let $f(x)$ be a function defined at all values in an open interval of the form $(b, a)$.

i. If the values of $f(x)$ increase without bound as the values of $x$ (where $x < a$) approach the number $a$, then we say that the limit as $x$ approaches $a$ from the left is positive infinity and we write

$$\lim_{x\to a^-} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of $x$ (where $x < a$) approach the number $a$, then we say that the limit as $x$ approaches $a$ from the left is negative infinity and we write

$$\lim_{x\to a^-} f(x) = -\infty.$$

**Infinite limits from the right:** Let $f(x)$ be a function defined at all values in an open interval of the form $(a, c)$.

i. If the values of $f(x)$ increase without bound as the values of $x$ (where $x > a$) approach the number $a$, then we say that the limit as $x$ approaches $a$ from the right is positive infinity and we write

$$\lim_{x\to a^+} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of $x$ (where $x > a$) approach the number $a$, then we say that the limit as $x$ approaches $a$ from the right is negative infinity and we write

$$\lim_{x\to a^+} f(x) = -\infty.$$

**Two-sided infinite limit:** Let $f(x)$ be defined for all $x \neq a$ in an open interval containing $a$

i. If the values of $f(x)$ increase without bound as the values of $x$ (where $x \neq a$) approach the number $a$, then we say that the limit as $x$ approaches $a$ is positive infinity and we write

$$\lim_{x\to a} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of $x$ (where $x \neq a$) approach the number $a$, then we say that the limit as $x$ approaches $a$ is negative infinity and we write

$$\lim_{x\to a} f(x) = -\infty.$$

It is important to understand that when we write statements such as $\lim_{x\to a} f(x) = +\infty$ or $\lim_{x\to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at $a$, it must approach a real number $L$ as $x$ approaches $a$. That said, if, for example, $\lim_{x\to a} f(x) = +\infty$, we always write $\lim_{x\to a} f(x) = +\infty$ rather than $\lim_{x\to a} f(x)$ DNE.

## Example 7.4.5: Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim_{x\to 0^-} \frac{1}{x}$
b. $\lim_{x\to 0^+} \frac{1}{x}$
c. $\lim_{x\to 0} \frac{1}{x}$

**Solution**

Begin by constructing a table of functional values.

*Table* 7.4.7

| $x$ | $\frac{1}{x}$ | $x$ | $\frac{1}{x}$ |
|---|---|---|---|
| -0.1 | -10 | 0.1 | 10 |
| -0.01 | -100 | 0.01 | 100 |
| -0.001 | -1000 | 0.001 | 1000 |
| -0.0001 | -10,000 | 0.0001 | 10,000 |
| -0.00001 | -100,000 | 0.00001 | 100,000 |
| -0.000001 | -1,000,000 | 0.000001 | 1,000,000 |

a. The values of $1/x$ decrease without bound as $x$ approaches $0$ from the left. We conclude that

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as $x$ approaches $0$ from the right. We conclude that

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x\to 0} \frac{1}{x} \quad \text{DNE.}$$

The graph of $f(x) = 1/x$ in Figure 7.4.8 confirms these conclusions.

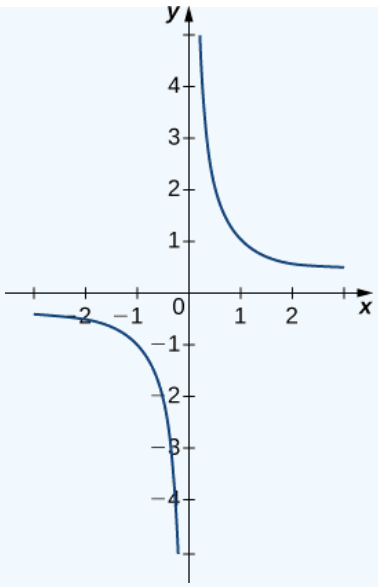



Figure 7.4.8: The graph of $f(x) = 1/x$ confirms that the limit as $x$ approaches $0$ does not exist.

## Exercise 7.4.5

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \frac{1}{x^2}$
b. $\lim_{x \to 0^+} \frac{1}{x^2}$
c. $\lim_{x \to 0} \frac{1}{x^2}$

*Hint available at the end of this section.*

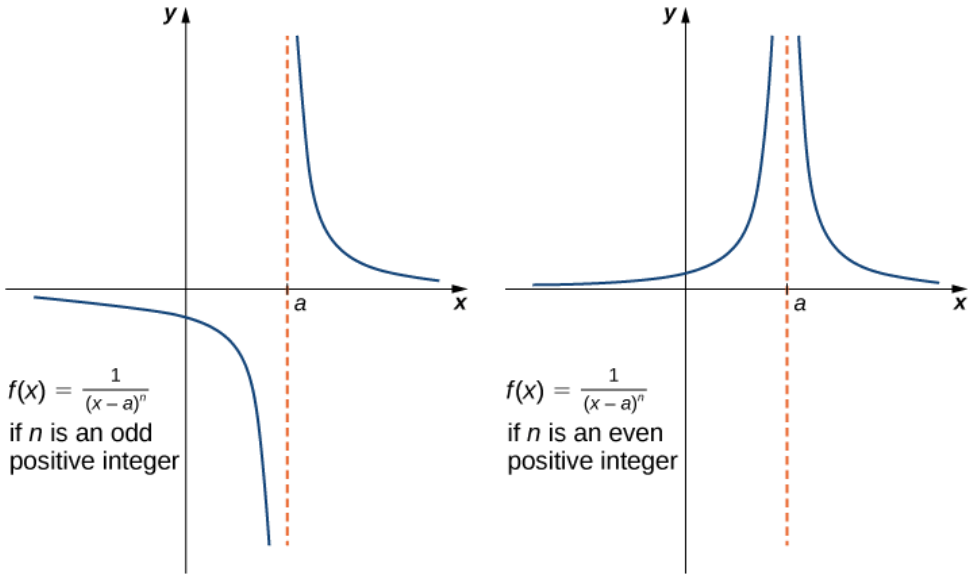



Figure 7.4.9: The function $f(x) = 1/(x-a)^n$ has infinite limits at $a$.

# Infinite Limits from Positive Integers

If $n$ is a positive even integer, then

$$\lim_{x\to a}\frac{1}{(x-a)^n}=+\infty. \tag{7.4.2}$$

If $n$ is a positive odd integer, then

$$\lim_{x\to a^+}\frac{1}{(x-a)^n}=+\infty \tag{7.4.3}$$

and

$$\lim_{x\to a^-}\frac{1}{(x-a)^n}=-\infty. \tag{7.4.4}$$

We should also point out that in the graphs of $f(x)=1/(x-a)^n$, points on the graph having $x$-coordinates very near to $a$ are very close to the vertical line $x=a$. That is, as $x$ approaches $a$, the points on the graph of $f(x)$ are closer to the line $x=a$. The line $x=a$ is called a vertical asymptote of the graph. We formally define a *vertical asymptote* as follows:

## Definition: Vertical Asymptotes

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x=a$ is a vertical asymptote of $f(x)$.

$$\lim_{x\to a^-}f(x)=+\infty$$

$$\lim_{x\to a^-}f(x)=-\infty$$

$$\lim_{x\to a^+}f(x)=+\infty$$

$$\lim_{x\to a^+}f(x)=-\infty$$

$$\lim_{x\to a}f(x)=+\infty$$

$$\lim_{x\to a}f(x)=-\infty$$

## Example 7.4.6: Finding a Vertical Asymptote

Evaluate each of the following limits using Equations 7.4.2, 7.4.3, and 7.4.4 above. Identify any vertical asymptotes of the function $f(x)=1/(x+3)^4$.

a. $\lim_{x\to -3^-}\frac{1}{(x+3)^4}$
b. $\lim_{x\to -3^+}\frac{1}{(x+3)^4}$
c. $\lim_{x\to -3}\frac{1}{(x+3)^4}$

**Solution**

We can use the above equations directly.

a. $\lim_{x\to -3^-}\frac{1}{(x+3)^4}=+\infty$
b. $\lim_{x\to -3^+}\frac{1}{(x+3)^4}=+\infty$

c. $\lim_{x\to -3} \dfrac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

## Exercise 7.4.6

Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \dfrac{1}{(x-2)^3}$.

a. $\lim_{x\to 2^-} \dfrac{1}{(x-2)^3}$
b. $\lim_{x\to 2^+} \dfrac{1}{(x-2)^3}$
c. $\lim_{x\to 2} \dfrac{1}{(x-2)^3}$

*Answer a available at the end of this section.*

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

## Example 7.4.7: Behavior of a Function at Different Points

Use the graph of $f(x)$ in Figure 7.4.10 to determine each of the following values:

a. $\lim_{x\to -4^-} f(x)$; $\lim_{x\to -4^+} f(x)$; $\lim_{x\to -4} f(x)$; $f(-4)$
b. $\lim_{x\to -2^-} f(x)$; $\lim_{x\to -2^+} f(x)$; $\lim_{x\to -2} f(x)$; $f(-2)$
c. $\lim_{x\to 1^-} f(x)$; $\lim_{x\to 1^+} f(x)$; $\lim_{x\to 1} f(x)$; $f(1)$
d. $\lim_{x\to 3^-} f(x)$; $\lim_{x\to 3^+} f(x)$; $\lim_{x\to 3} f(x)$; $f(3)$

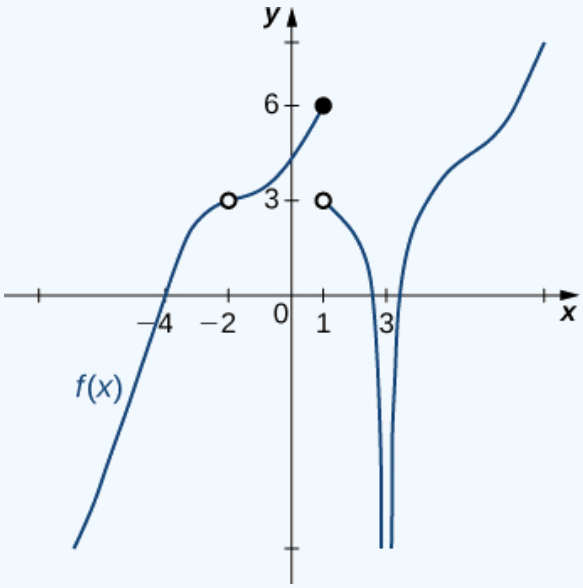



Figure 7.4.10: The graph shows $f(x)$.

**Solution**

Using the definitions above and the graph for reference, we arrive at the following values:

a. $\lim_{x\to -4^-} f(x) = 0$; $\lim_{x\to -4^+} f(x) = 0$; $\lim_{x\to -4} f(x) = 0$; $f(-4) = 0$
b. $\lim_{x\to -2^-} f(x) = 3$; $\lim_{x\to -2^+} f(x) = 3$; $\lim_{x\to -2} f(x) = 3$; $f(-2)$ is undefined
c. $\lim_{x\to 1^-} f(x) = 6$; $\lim_{x\to 1^+} f(x) = 3$; $\lim_{x\to 1} f(x)$ DNE; $f(1) = 6$
d. $\lim_{x\to 3^-} f(x) = -\infty$; $\lim_{x\to 3^+} f(x) = -\infty$; $\lim_{x\to 3} f(x) = -\infty$; $f(3)$ is undefined

## Exercise 7.4.7

Evaluate $\lim_{x\to 1} f(x)$ for $f(x)$ shown here:

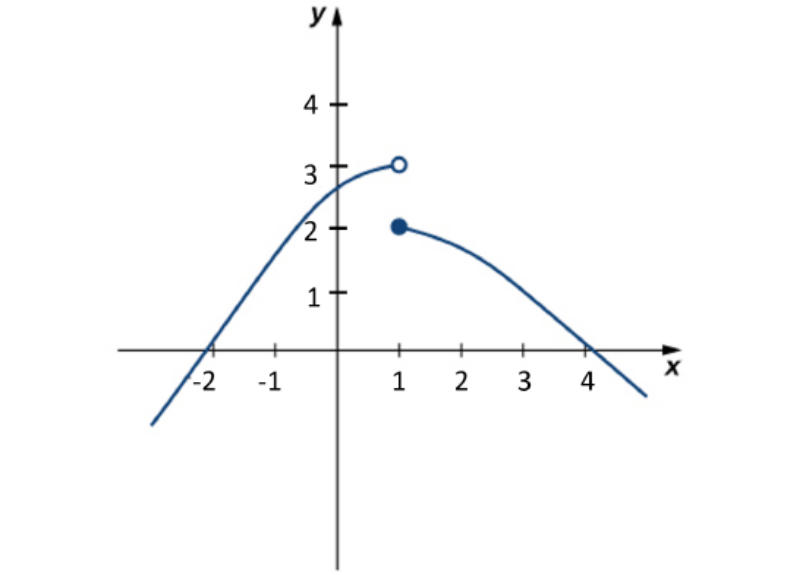



Figure 7.4.11. The graph of a piecewise function $f$.

*Hint available at the end of this section.*

## Example 7.4.8: Einstein's Equation

In the Chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

what is the value of this bound?

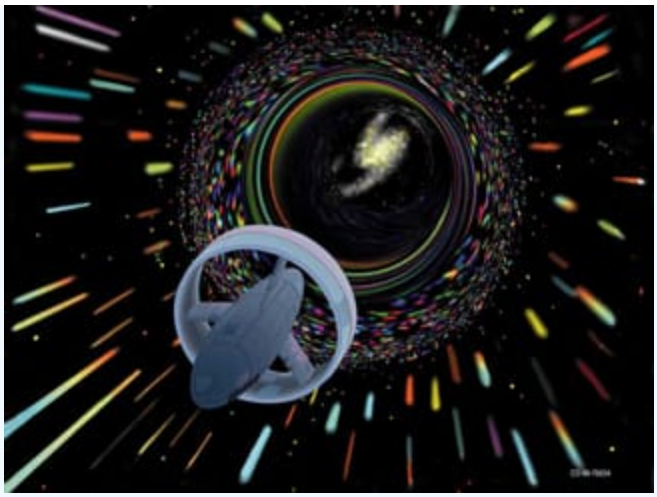


Figure 7.4.12. (Credit:NASA)

**Solution**

Our starting point is Einstein's equation for the mass of a moving object,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where $m_0$ is the object's mass at rest, $v$ is its speed, and $c$ is the speed of light. To see how the mass changes at high speeds, we can graph the ratio of masses $m/m_0$ as a function of the ratio of speeds, $v/c$ (Figure 7.4.13).

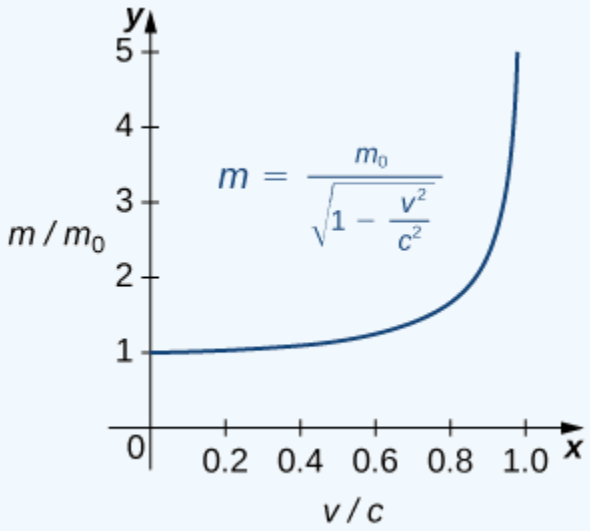



Figure 7.4.13: This graph shows the ratio of masses as a function of the ratio of speeds in Einstein's equation for the mass of a moving object.

We can see that as the ratio of speeds approaches 1—that is, as the speed of the object approaches the speed of light—the ratio of masses increases without bound. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

*Table* 7.4.8

| $v/c$ | $\sqrt{1 - \frac{v^2}{c^2}}$ | $m/m_o$ |
|---|---|---|
| 0.99 | 0.1411 | 7.089 |
| 0.999 | 0.0447 | 22.37 |

| $v/c$ | $\sqrt{1-\frac{v^2}{c^2}}$ | $m/m_o$ |
|---|---|---|
| 0.9999 | 0.0141 | 70.7 |

Thus, according to Table 7.4.8:, if an object with mass 100 kg is traveling at 0.9999c, its mass becomes 7071 kg. Since no object can have an infinite mass, we conclude that no object can travel at or more than the speed of light.

## Key Concepts

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

## Key Equations

- **Intuitive Definition of the Limit**

$$\lim_{x\to a} f(x) = L$$

- **Two Important Limits**

$$\lim_{x\to a} x = a \qquad \lim_{x\to a} c = c$$

- **One-Sided Limits**

$$\lim_{x\to a^-} f(x) = L \qquad \lim_{x\to a^+} f(x) = L$$

- **Infinite Limits from the Left**

$$\lim_{x\to a^-} f(x) = +\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

- **Infinite Limits from the Right**

$$\lim_{x\to a^+} f(x) = +\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

- **Two-Sided Infinite Limits**

$$\lim_{x\to a} f(x) = +\infty\text{: } \lim_{x\to a^-} f(x) = +\infty \text{ and } \lim_{x\to a^+} f(x) = +\infty$$

$$\lim_{x\to a} f(x) = -\infty\text{: } \lim_{x\to a^-} f(x) = -\infty \text{ and } \lim_{x\to a^+} f(x) = -\infty$$

## Glossary

**infinite limit**
A function has an infinite limit at a point $a$ if it either increases or decreases without bound as it approaches $a$

**intuitive definition of the limit**
If all values of the function $f(x)$ approach the real number $L$ as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

**one-sided limit**
A one-sided limit of a function is a limit taken from either the left or the right

**vertical asymptote**
A function has a vertical asymptote at $x = a$ if the limit as $x$ approaches $a$ from the right or left is infinite

## Answers

**Exercise** 7.4.1

Use 0.9, 0.99, 0.999, 0.9999, 0.99999 and 1.1, 1.01, 1.001, 1.0001, 1.00001 as your table values.

$$\lim_{x\to 1} \frac{\frac{1}{x} - 1}{x - 1} = -1$$

**Exercise** 7.4.2

What $y$-value does the function approach as the $x$-values approach 2?

$\lim_{x\to 2} h(x) = -1$.

**Exercise** 7.4.3

Use $x$-values 1.9, 1.99, 1.999, 1.9999, 1.99999 and 2.1, 2.01, 2.001, 2.0001, 2.00001 in your table.

$\lim_{x\to 2} \frac{|\, x^2 - 4\,|}{x - 2}$ does not exist.

**Exercise** 7.4.4

Use $x$-values 1.9, 1.99, 1.999, 1.9999, 1.99999 to estimate $\lim_{x\to 2^-} \frac{|\, x^2 - 4\,|}{x - 2}$.

Use $x$-values 2.1, 2.01, 2.001, 2.0001, 2.00001 to estimate $\lim_{x\to 2^+} \frac{|\, x^2 - 4\,|}{x - 2}$.

(These tables are available from a previous Checkpoint problem.)

a. $\lim_{x\to 2^-} \frac{|\, x^2 - 4\,|}{x - 2} = -4$

$\lim_{x\to 2^+} \frac{|\, x^2 - 4\,|}{x - 2} = 4$

**Exercise** 7.4.5

Follow the procedures from Example 7.4.5.

a. $\lim_{x\to 0^-} \frac{1}{x^2} = +\infty$;

b. $\lim_{x\to 0^+} \frac{1}{x^2} = +\infty$;

c. $\lim_{x\to 0} \frac{1}{x^2} = +\infty$

It is useful to point out that functions of the form $f(x) = 1/(x - a)^n$, where n is a positive integer, have infinite limits as $x$ approaches $a$ from either the left or right (Figure 7.4.9). These limits are summarized in the above definitions.

**Exercise** 7.4.6

$\lim_{x\to 2^-} \frac{1}{(x-2)^3} = -\infty$

$\lim_{x\to 2^+} \frac{1}{(x-2)^3} = +\infty$

$\lim_{x\to 2} \frac{1}{(x-2)^3}$ DNE. The line $x = 2$ is the vertical asymptote of $f(x) = 1/(x - 2)^3$.

**Exercise** 7.4.7

Compare the limit from the right with the limit from the left.

$\lim_{x\to 1} f(x)$ does not exist

## 7.4E: Exercises for Section 2.2

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# 7.4E: Exercises for Section 2.2

## Intuitive Definition of Limits

**For exercises 1 - 2, consider the function $f(x) = \dfrac{x^2 - 1}{|x - 1|}$.**

1) [T] Complete the following table for the function. Round your solutions to four decimal places.

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 0.9 | a. | 1.1 | e. |
| 0.99 | b. | 1.01 | f. |
| 0.999 | c. | 1.001 | g. |
| 0.9999 | d. | 1.0001 | h. |

2) What do your results in the preceding exercise indicate about the two-sided limit $\lim\limits_{x \to 1} f(x)$? Explain your response.

**Answer**

$\lim\limits_{x \to 1} f(x)$ does not exist because $\lim\limits_{x \to 1^-} f(x) = -2 \neq \lim\limits_{x \to 1^+} f(x) = 2.$

**For exercises 3 - 5, consider the function $f(x) = (1 + x)^{1/x}$.**

3) [T] Make a table showing the values of $f$ for $x = -0.01,\ -0.001,\ -0.0001,\ -0.00001$ and for $x = 0.01,\ 0.001,\ 0.0001,\ 0.00001$. Round your solutions to five decimal places.

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| -0.01 | a. | 0.01 | e. |
| -0.001 | b. | 0.001 | f. |
| -0.0001 | c. | 0.0001 | g. |
| -0.00001 | d. | 0.00001 | h. |

4) What does the table of values in the preceding exercise indicate about the function $f(x) = (1 + x)^{1/x}$?

**Answer**

$\lim\limits_{x \to 0} (1 + x)^{1/x} \approx 2.7183.$

5) To which mathematical constant do the values in the preceding exercise appear to be approaching? This is the actual limit here.

**In exercises 6 - 8, use the given values to set up a table to evaluate the limits. Round your solutions to eight decimal places.**

6) [T] $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$; $\pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm .0001$

| $x$ | $\frac{\sin 2x}{x}$ | $x$ | $\frac{\sin 2x}{x}$ |
|---|---|---|---|
| -0.1 | a. | 0.1 | e. |

| $x$ | $\frac{\sin 2x}{x}$ | $x$ | $\frac{\sin 2x}{x}$ |
|---|---|---|---|
| -0.01 | b. | 0.01 | f. |
| -0.001 | c. | 0.001 | g. |
| -0.0001 | d. | 0.0001 | h. |

**Answer**

a. 1.98669331; b. 1.99986667; c. 1.99999867; d. 1.99999999; e. 1.98669331; f. 1.99986667; g. 1.99999867; h. 1.99999999;

$\lim_{x\to 0}\frac{\sin 2x}{x}=2$

7) [T] $\lim_{x\to 0}\frac{\sin 3x}{x}\ \pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm 0.0001$

| $x$ | $\frac{\sin 3x}{x}$ | $x$ | $\frac{\sin 3x}{x}$ |
|---|---|---|---|
| -0.1 | a. | 0.1 | e. |
| -0.01 | b. | 0.01 | f. |
| -0.001 | c. | 0.001 | g. |
| -0.0001 | d. | 0.0001 | h. |

8) Use the preceding two exercises to conjecture (guess) the value of the following limit: $\lim_{x\to 0}\frac{\sin ax}{x}$ for $a$, a positive real value.

**Answer**

$\lim_{x\to 0}\frac{\sin ax}{x}=a$

**[T] In exercises 9 - 14, set up a table of values to find the indicated limit. Round to eight significant digits.**

9) $\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}$

| $x$ | $\frac{x^2-4}{x^2+x-6}$ | $x$ | $\frac{x^2-4}{x^2+x-6}$ |
|---|---|---|---|
| 1.9 | a. | 2.1 | e. |
| 1.99 | b. | 2.01 | f. |
| 1.999 | c. | 2.001 | g. |
| 1.9999 | d. | 2.0001 | h. |

10) $\lim_{x\to 1}(1-2x)$

| $x$ | $1-2x$ | $x$ | $1-2x$ |
|---|---|---|---|
| 0.9 | a. | 1.1 | e. |
| 0.99 | b. | 1.01 | f. |
| 0.999 | c. | 1.001 | g. |
| 0.9999 | d. | 1.0001 | h. |

**Answer**

a. −0.80000000; b. −0.98000000; c. −0.99800000; d. −0.99980000; e. −1.2000000; f. −1.0200000; g. −1.0020000; h. −1.0002000;

$\lim_{x\to 1}(1-2x)=-1$

11) $\lim_{x\to 0} \dfrac{5}{1-e^{1/x}}$

| $x$ | $\frac{5}{1-e^{1/x}}$ | $x$ | $\frac{5}{1-e^{1/x}}$ |
|---|---|---|---|
| -0.1 | a. | 0.1 | e. |
| -0.01 | b. | 0.01 | f. |
| -0.001 | c. | 0.001 | g. |
| -0.0001 | d. | 0.0001 | h. |

12) $\lim_{z\to 0} \dfrac{z-1}{z^2(z+3)}$

| $z$ | $\frac{z-1}{z^2(z+3)}$ | $z$ | $\frac{z-1}{z^2(z+3)}$ |
|---|---|---|---|
| -0.1 | a. | 0.1 | e. |
| -0.01 | b. | 0.01 | f. |
| -0.001 | c. | 0.001 | g. |
| -0.0001 | d. | 0.0001 | h. |

**Answer**

a. −37.931034; b. −3377.9264; c. −333,777.93; d. −33,337,778; e. −29.032258; f. −3289.0365; g. −332,889.04; h. −33,328,889

$\lim_{x\to 0} \dfrac{z-1}{z^2(z+3)} = -\infty$

13) $\lim_{t\to 0^+} \dfrac{\cos t}{t}$

| $t$ | $\frac{\cos t}{t}$ |
|---|---|
| 0.1 | a. |
| 0.01 | b. |
| 0.001 | c. |
| 0.0001 | d. |

14) $\lim_{x\to 2} \dfrac{1-\frac{2}{x}}{x^2-4}$

| $x$ | $\frac{1-\frac{2}{x}}{x^2-4}$ | $x$ | $\frac{1-\frac{2}{x}}{x^2-4}$ |
|---|---|---|---|
| 1.9 | a. | 2.1 | e. |
| 1.99 | b. | 2.01 | f. |
| 1.999 | c. | 2.001 | g. |
| 1.9999 | d. | 2.0001 | h. |

**Answer**

a. 0.13495277; b. 0.12594300; c. 0.12509381; d. 0.12500938; e. 0.11614402; f. 0.12406794; g. 0.12490631; h. 0.12499063;

$\therefore \lim_{x\to 2} \dfrac{1-\frac{2}{x}}{x^2-4} = 0.1250 = \dfrac{1}{8}$

**[T] In exercises 15 - 16, set up a table of values and round to eight significant digits. Based on the table of values,**

**make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?**

15) $\lim_{\theta \to 0} \sin\left(\frac{\pi}{\theta}\right)$

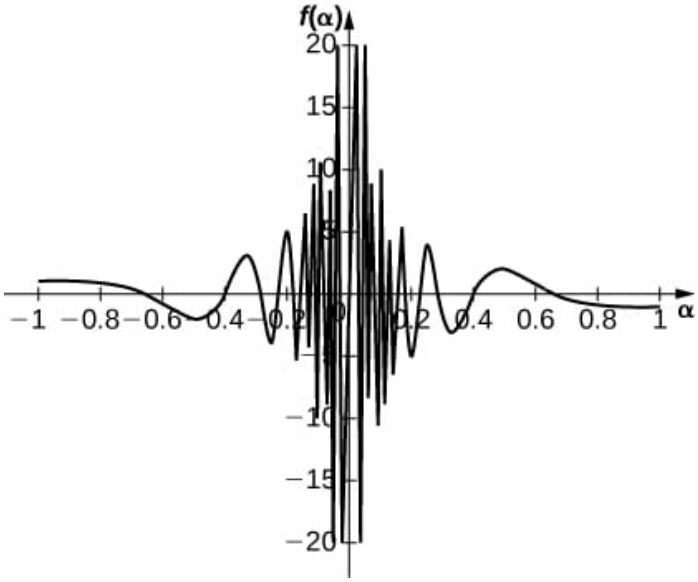
| $\theta$ | $\sin\left(\frac{\pi}{\theta}\right)$ | $\theta$ | $\sin\left(\frac{\pi}{\theta}\right)$ |
|---|---|---|---|
| -0.1 | a. | 0.1 | e. |
| -0.01 | b. | 0.01 | f. |
| -0.001 | c. | 0.001 | g. |
| -0.0001 | d. | 0.0001 | h. |

16) $\lim_{\alpha \to 0^+} \frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$

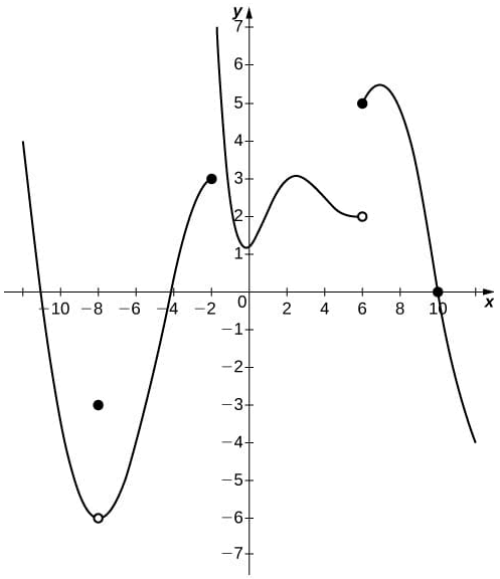
| $a$ | $\frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$ |
|---|---|
| 0.1 | a. |
| 0.01 | b. |
| 0.001 | c. |
| 0.0001 | d. |

**Answer**

a. 10.00000; b. 100.00000; c. 1000.0000; d. 10,000.000;
Guess: $\lim_{\alpha \to 0^+} \frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right) = \infty$;
Actual: DNE , since the graph shows the function oscillates wildly between values approaching positive infinity and values approaching negative infinity, as the value of $\alpha$ approaches $0$ from the positive side.




**In exercises 17 - 20, consider the graph of the function $y = f(x)$ shown here. Which of the statements about $y = f(x)$ are true and which are false? Explain why a statement is false.**




17) $\lim_{x \to 10} f(x) = 0$

18) $\lim_{x \to -2^+} f(x) = 3$

**Answer**

False; $\lim_{x \to -2^+} f(x) = +\infty$

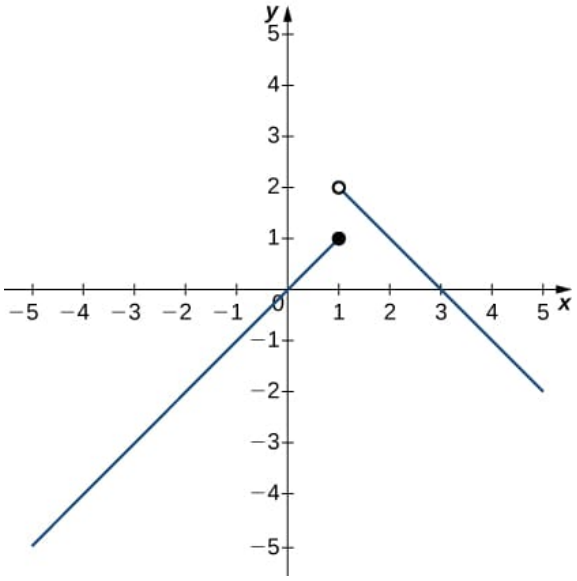
19) $\lim_{x \to -8} f(x) = f(-8)$

20) $\lim_{x \to 6} f(x) = 5$

**Answer**

False; $\lim_{x \to 6} f(x)$ DNE since $\lim_{x \to 6^-} f(x) = 2$ and $\lim_{x \to 6^+} f(x) = 5$.

**In exercises 21 - 25, use the following graph of the function $y = f(x)$ to find the values, if possible. Estimate when necessary.**



21) $\lim_{x \to 1^-} f(x)$

22) $\lim_{x \to 1^+} f(x)$

**Answer**

2

23) $\lim_{x \to 1} f(x)$

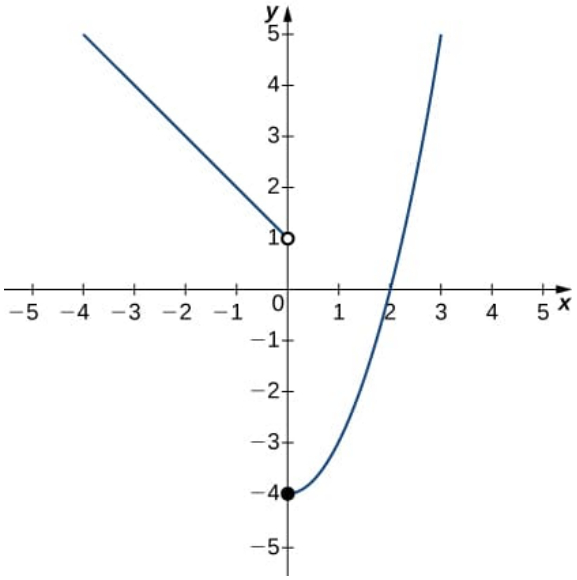
24) $\lim_{x \to 2} f(x)$

**Answer**

1

25) $f(1)$

**In exercises 26 - 29, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.**




26) $\lim_{x \to 0^-} f(x)$

**Answer**

1

27) $\lim_{x \to 0^+} f(x)$

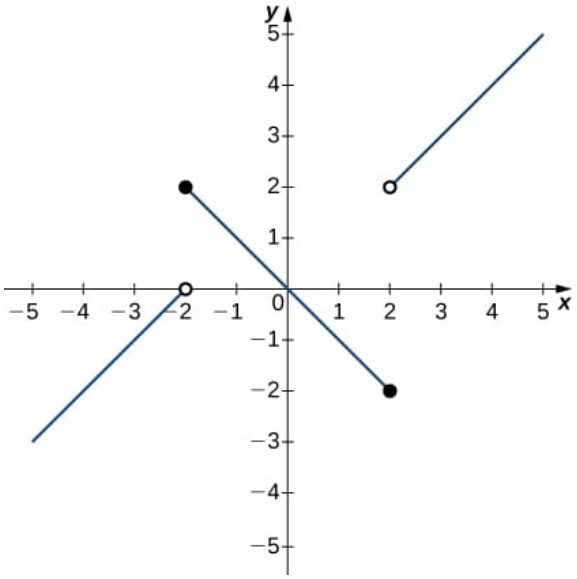
28) $\lim_{x \to 0} f(x)$

**Answer**

DNE

29) $\lim_{x \to 2} f(x)$

**In exercises 30 - 35, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.**




30) $\lim_{x \to -2^-} f(x)$

**Answer**

$0$

31) $\lim_{x \to -2^+} f(x)$

32) $\lim_{x \to -2} f(x)$

**Answer**

DNE

33) $\lim_{x \to 2^-} f(x)$

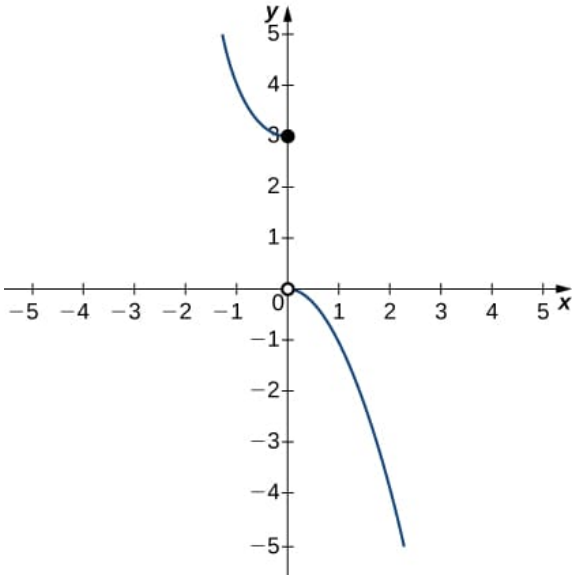
34) $\lim_{x \to 2^+} f(x)$

**Answer**

$2$

35) $\lim_{x \to 2} f(x)$

**In exercises 36 - 38, use the graph of the function $y = g(x)$ shown here to find the values, if possible. Estimate when necessary.**




36) $\lim_{x \to 0^-} g(x)$

**Answer**

$3$

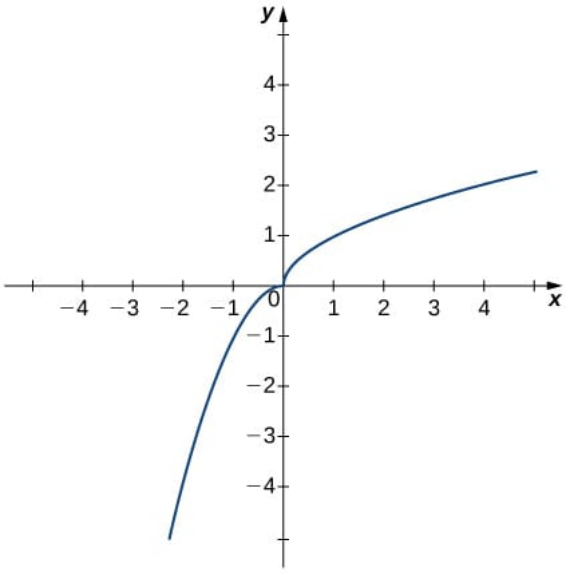
37) $\lim_{x \to 0^+} g(x)$

38) $\lim_{x \to 0} g(x)$

**Answer**

DNE

**In exercises 39 - 41, use the graph of the function $y = h(x)$ shown here to find the values, if possible. Estimate when necessary.**




39) $\lim_{x \to 0^-} h(x)$

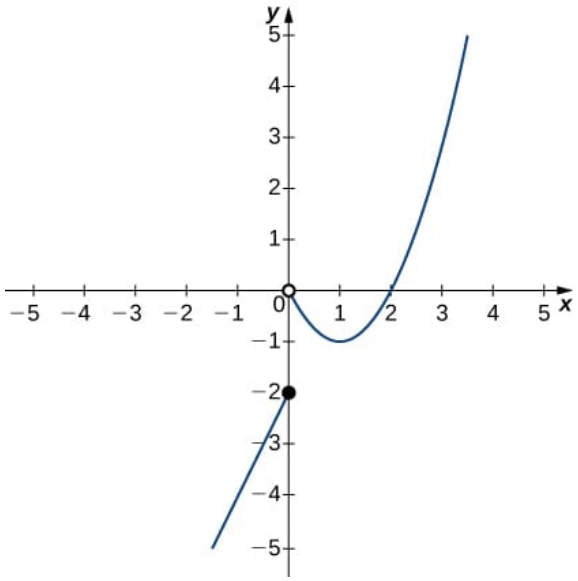
40) $\lim_{x \to 0^+} h(x)$

**Answer**

$0$

41) $\lim_{x \to 0} h(x)$

**In exercises 42 - 46, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.**




42) $\lim_{x \to 0^-} f(x)$

**Answer**

$-2$

43) $\lim_{x \to 0^+} f(x)$

44) $\lim_{x \to 0} f(x)$

**Answer**

DNE

45) $\lim_{x \to 1} f(x)$

46) $\lim_{x \to 2} f(x)$

**Answer**

$0$

# Infinite Limits

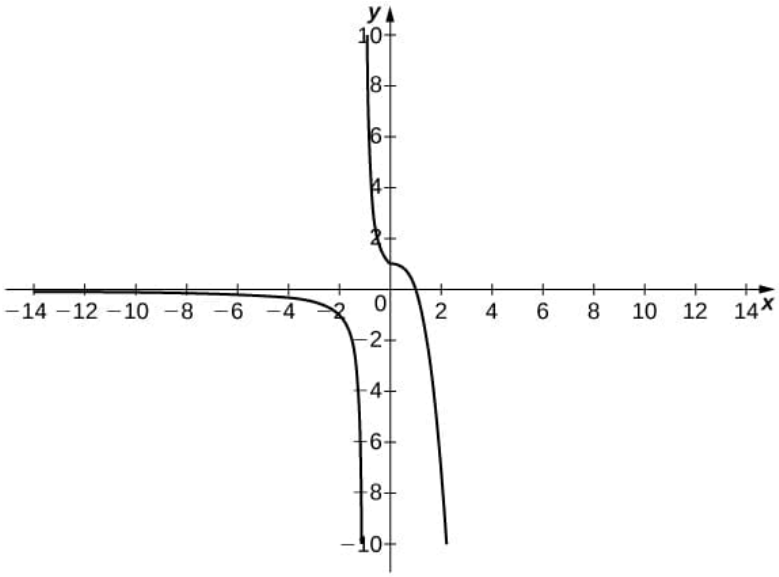
**In exercises 47 - 51, sketch the graph of a function with the given properties.**

47) $\lim_{x \to 2} f(x) = 1, \quad \lim_{x \to 4^-} f(x) = 3, \quad \lim_{x \to 4^+} f(x) = 6, \quad x = 4$ is not defined.

48) $\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to-1^-} f(x) = -\infty, \quad \lim_{x\to-1^+} f(x) = \infty, \quad \lim_{x\to 0} f(x) = f(0), \quad f(0) = 1, \quad \lim_{x\to\infty} f(x) = -\infty$

**Answer**

Answers may vary

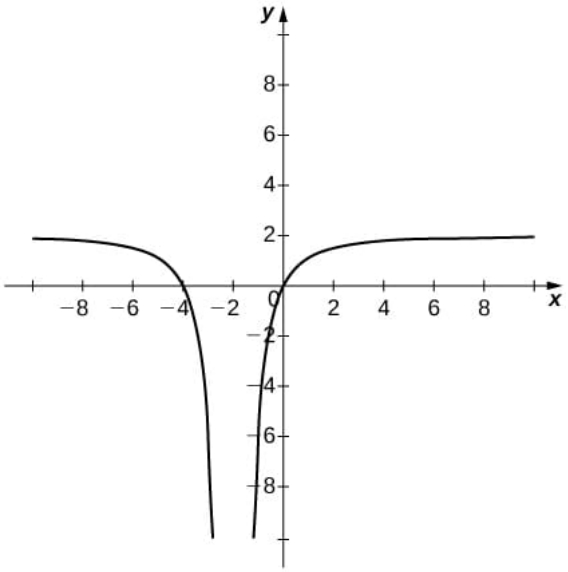



49) $\lim_{x\to-\infty} f(x) = 2, \quad \lim_{x\to 3^-} f(x) = -\infty, \quad \lim_{x\to 3^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = 2, \quad f(0) = -\frac{1}{3}$

50) $\lim_{x\to-\infty} f(x) = 2, \quad \lim_{x\to-2} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = 2, \quad f(0) = 0$

**Answer**

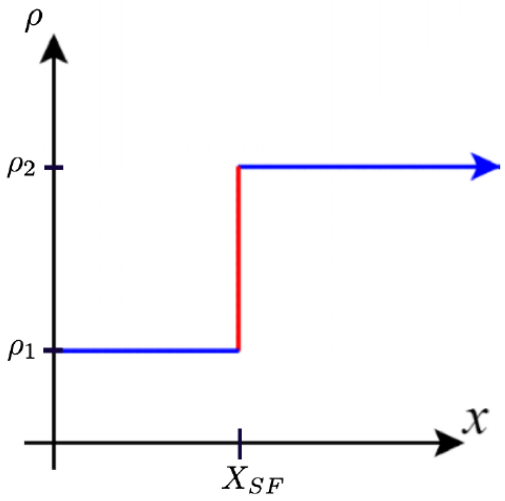
Answer may vary




51) $\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to-1^-} f(x) = \infty, \quad \lim_{x\to-1^+} f(x) = -\infty, \quad f(0) = -1, \quad \lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to 1^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = 0$

52) Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a

shock wave with respect to distance, $x$, is shown here. We are mainly interested in the location of the front of the shock, labeled $X_{SF}$ in the diagram.




a. Evaluate $\lim_{x \to X_{SF}^{+}} \rho(x)$.

b. Evaluate $\lim_{x \to X_{SF}^{-}} \rho(x)$.

c. Evaluate $\lim_{x \to X_{SF}} \rho(x)$. Explain the physical meanings behind your answers.

**Answer**

a. $\rho_2$ b. $\rho_1$ c. DNE unless $\rho_1 = \rho_2$. As you approach $X_{SF}$ from the right, you are in the high-density area of the shock. When you approach from the left, you have not experienced the "shock" yet and are at a lower density.

53) A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where $x$ is the position in meters of the runner and $t$ is time in seconds. What is $\lim_{t \to 2} x(t)$? What does it mean physically?

| $t(sec)$ | $x(m)$ |
|---|---|
| 1.75 | 4.5 |
| 1.95 | 6.1 |
| 1.99 | 6.42 |
| 2.01 | 6.58 |
| 2.05 | 6.9 |
| 2.25 | 8.5 |

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# 7.5: Continuity

Arizona is known for its dry heat. On a particular day, the temperature might rise as high as $118^\circ F$ and drop down only to a brisk $95^\circ F$. Figure $7.5.1$ shows the function $T$, where the output of $T(x)$ is the temperature in Fahrenheit degrees and the input $x$ is the time of day, using a 24-hour clock on a particular summer day.

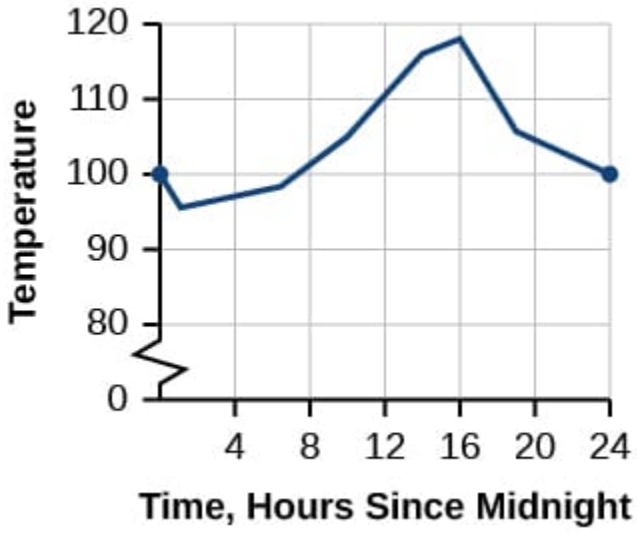



***Figure*** $7.5.1$*: Temperature as a function of time forms a continuous function.*

When we analyze this graph, we notice a specific characteristic. There are no breaks in the graph. We could trace the graph without picking up our pencil. This single observation tells us a great deal about the function. In this section, we will investigate functions with and without breaks.

## Determining Whether a Function Is Continuous at a Number

Let's consider a specific example of temperature in terms of date and location, such as June 27, 2013, in Phoenix, AZ. The graph in Figure $7.5.1$ indicates that, at 2 a.m., the temperature was $96^\circ F$. By 2 p.m. the temperature had risen to $116^\circ F$, and by 4 p.m. it was $118^\circ F$. Sometime between 2 a.m. and 4 p.m., the temperature outside must have been exactly $110.5^\circ F$. In fact, any temperature between $96^\circ F$ and $118^\circ F$ occurred at some point that day. This means all real numbers in the output between $96^\circ F$ and $118^\circ F$ are generated at some point by the function according to the intermediate value theorem,

Look again at Figure $7.5.1$. There are no breaks in the function's graph for this 24-hour period. At no point did the temperature cease to exist, nor was there a point at which the temperature jumped instantaneously by several degrees. A function that has no holes or breaks in its graph is known as a continuous function. Temperature as a function of time is an example of a continuous function.

If temperature represents a continuous function, what kind of function would not be continuous? Consider an example of dollars expressed as a function of hours of parking. Let's create the function $D$, where $D(x)$ is the output representing cost in dollars for parking $x$ number of hours (Figure $7.5.2$).

Suppose a parking garage charges $4.00 per hour or fraction of an hour, with a $25 per day maximum charge. Park for two hours and five minutes and the charge is $12. Park an additional hour and the charge is $16. We can never be charged $13, $14, or $15. There are real numbers between 12 and 16 that the function never outputs. There are breaks in the function's graph for this 24-hour period, points at which the price of parking jumps instantaneously by several dollars.

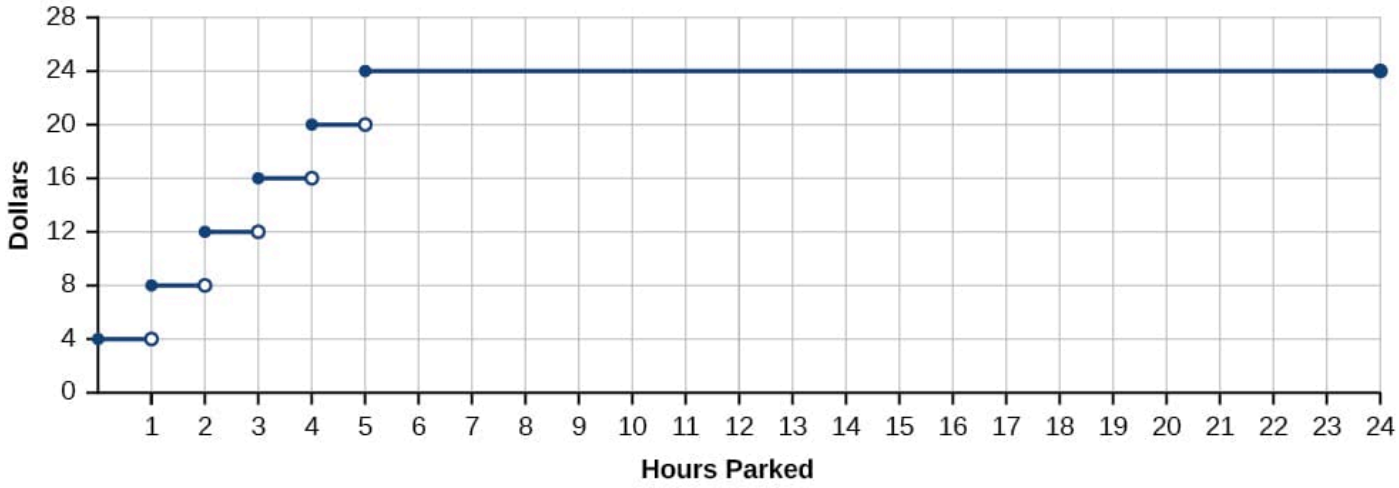



Figure 7.5.2: Parking-garage charges form a discontinuous function.

A function that remains level for an interval and then jumps instantaneously to a higher value is called a **stepwise** function. This function is an example.

A function that has any hole or break in its graph is known as a **discontinuous function**. A stepwise function, such as parking-garage charges as a function of hours parked, is an example of a discontinuous function.

So how can we decide if a function is continuous at a particular number? We can check three different conditions. Let's use the function $y = f(x)$ represented in Figure as an example.

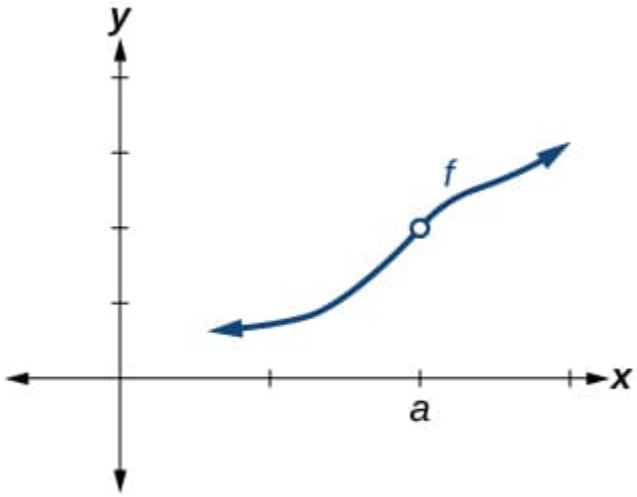



Figure_12_03_004">Figure.

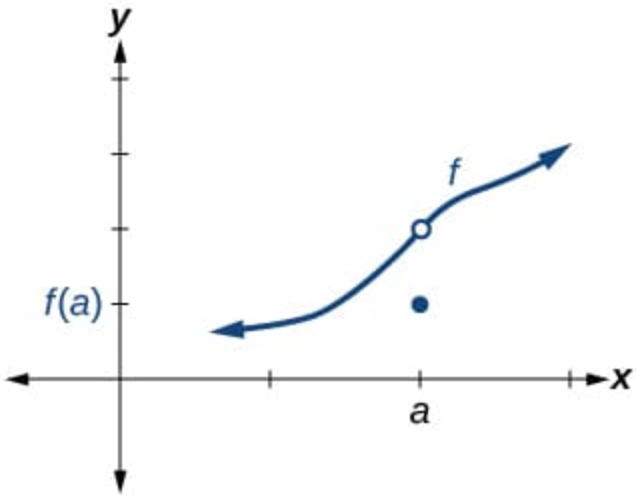



Figure_12_03_003">Figure approaches $x = a$ from the left and right, the same *y*-coordinate is approached. Therefore, Condition 2 is satisfied. However, there could still be a hole in the graph at $x = a$.

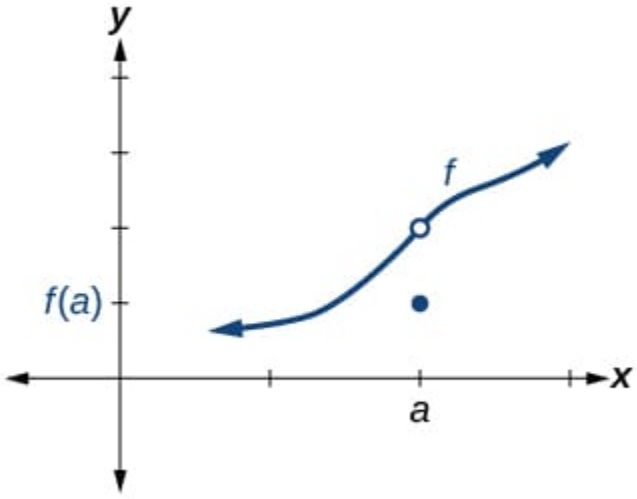
**Condition 3** According to Condition 3, the corresponding y y coordinate at $x = a$ fills in the hole in the graph of $f$. This is written $\lim_{x \to a} f(x) = f(a)$.

Satisfying all three conditions means that the function is continuous. All three conditions are satisfied for the function represented in Figure so the function is continuous as $x = a$.

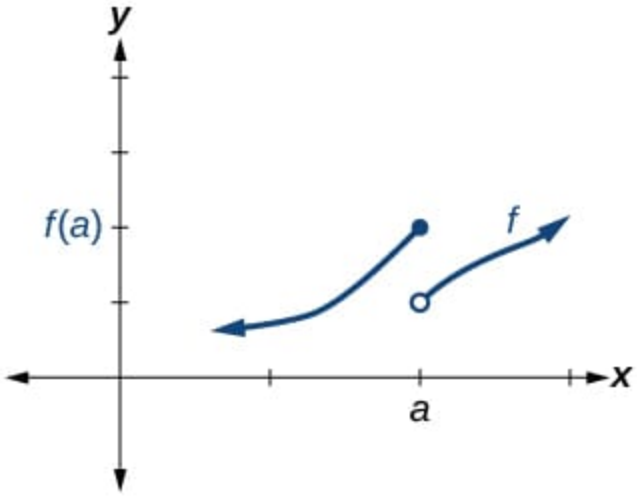
All three conditions are satisfied. The function is continuous at $x = a$.

Figure through Figure provide several examples of graphs of functions that are not continuous at $x = a$ and the condition or conditions that fail.

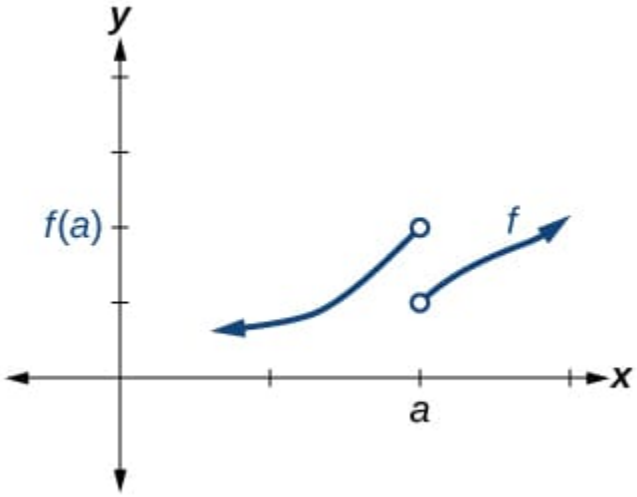
Condition 2 is satisfied. Conditions 1 and 3 both fail.




Conditions 1 and 2 are both satisfied. Condition 3 fails.




Condition 1 is satisfied. Conditions 2 and 3 fail.




Conditions 1, 2, and 3 all fail.

**Definition of continuity**

A function $f(x)$ is **continuous** at $x = a$ provided all three of the following conditions hold true:

- Condition 1: $f(a)$ exists.
- Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
- Condition 3: $\lim_{x \to a} f(x) = f(a)$

If a function $f(x)$ is not continuous at $x = a$,the function is **discontinuous** at $x = a$.

## Identifying a Jump Discontinuity

Discontinuity can occur in different ways. We saw in the previous section that a function could have a left-hand limit and a right-hand limit even if they are not equal. If the left- and right-hand limits exist but are different, the graph "jumps" at $x = a$. The function is said to have a jump discontinuity.

As an example, look at the graph of the function $y = f(x)$ in Figure. Notice as $x$ approaches $a$ how the output approaches different

values from the left and from the right.

Graph of a function with a jump discontinuity.

**JUMP DISCONTINUITY**

A function $f(x)$ has a jump discontinuity at $x = a$ if the left- and right-hand limits both exist but are not equal: $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

## Identifying Removable Discontinuity

Some functions have a discontinuity, but it is possible to redefine the function at that point to make it continuous. This type of function is said to have a removable discontinuity. Let's look at the function $y = f(x)$ represented by the graph in Figure. The function has a limit. However, there is a hole at $x = a$. The hole can be filled by extending the domain to include the input $x = a$ and defining the corresponding output of the function at that value as the limit of the function at $x = a$.

Graph of function $f$ with a removable discontinuity at $x = a$.

**removable discontinuity**

A function f(x) f(x) has a removable discontinuity at $x = a$ if the limit, $\lim_{x \to a} f(x)$, exists, but either

1. $f(a)$ does not exist *or*
2. $f(a)$, the value of the function at $x = a$ does not equal the limit, $f(a) \neq \lim_{x \to a} f(x)$.

**Example 7.5.1: Identifying Discontinuities**

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x) = \frac{x^2 - 2x - 15}{x - 5}$

b. $$g(x) = \begin{cases} x + 1, & x < 2 \\ -x, & x \geq 2 \end{cases} \tag{7.5.1}$$

a. Notice that the function is defined everywhere except at $x = 5$.

Thus, $f(5)$ does not exist, Condition 2 is not satisfied. Since Condition 1 is satisfied, the limit as $x$ approaches 5 is 8, and Condition 2 is not satisfied.This means there is a removable discontinuity at $x = 5$.

b. Condition 2 is satisfied because $g(2) = -2$.

Notice that the function is a **piecewise function**, and for each piece, the function is defined everywhere on its domain. Let's examine Condition 1 by determining the left- and right-hand limits as $x$ approaches 2.

Left-hand limit: $\lim_{x \to 2^-} (x + 1) = 2 + 1 = 3$. The left-hand limit exists.

Right-hand limit: $\lim_{x \to 2^+} (-x) = -2$. The right-hand limit exists. But

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x). \tag{7.5.2}$$

So, $\lim_{x \to 2} f(x)$ does not exist, and Condition 2 fails: There is no removable discontinuity. However, since both left- and right-hand limits exist but are not equal, the conditions are satisfied for a jump discontinuity at $x = 2$.

**Exercise 7.5.1:**

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x) = \frac{x^2-6x}{x-6}$
b. $g(x) = \begin{cases} \sqrt{x}, & 0 \le x < 4 \\ 2x, & x \ge 4 \end{cases}$

a. removable discontinuity at $x = 6$;
b. jump discontinuity at $x = 4$

## Recognizing Continuous and Discontinuous Real-Number Functions

Many of the functions we have encountered in earlier chapters are continuous everywhere. They never have a hole in them, and they never jump from one value to the next. For all of these functions, the limit of $f(x)$ as $x$ approaches a a is the same as the value of $f(x)$ when $x = a$. So $\lim_{x \to a} f(x) = f(a)$. There are some functions that are continuous everywhere and some that are only continuous where they are defined on their domain because they are not defined for all real numbers.

**EXAMPLES OF CONTINUOUS FUNCTIONS**

The following functions are continuous everywhere:

| | |
|---|---|
| Polynomial functions | Ex: $f(x) = x^4 - 9x^2$ |
| Exponential functions | Ex: $f(x) = 4^{x+2} - 5$ |
| Sine functions | Ex: $f(x) = \sin(2x) - 4$ |
| Cosine functions | Ex: $f(x) = -\cos(x + \frac{\pi}{3})$ |

The following functions are continuous everywhere they are defined on their domain:

| | |
|---|---|
| Logarithmic functions | Ex: $f(x) = 2\ln(x), x > 0$ |
| Tangent functions | Ex: $f(x) = \tan(x) + 2, x \ne \frac{\pi}{2} + k\pi, k$ is an integer |
| Rational functions | Ex: $f(x) = \frac{x^2-25}{x-7}, x \ne 7$ |

**how to:Given a function $f(x)$, determine if the function is continuous at $x = a$.**

1. Check Condition 1: $f(a)$ exists.
2. Check Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
3. Check Condition 3: $\lim_{x \to a} f(x) = f(a)$.
4. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions is not satisfied, the function is not continuous at $x = a$.

**Example 7.5.2: Determining Whether a Piecewise Function is Continuous at a Given Number**

Determine whether the function $f(x) = \begin{cases} 4x, & x \le 3 \\ 8 + x, & x > 3 \end{cases}$ is continuous at

a. $x = 3$
b. $x = \frac{8}{3}$

To determine if the function $f$ is continuous at $x = a$, we will determine if the three conditions of continuity are satisfied at $x = a$.

a. Condition 1: Does $f(a)$ exist?

$$f(3) = 4(3) = 12 \tag{7.5.3}$$
$$\Rightarrow \text{Condition 1 is satisfied.} \tag{7.5.4}$$

Condition 2: Does $\lim_{x\to 3} f(x)$ exist?

To the left of $x = 3, f(x) = 4x$; to the right of $x = 3, f(x) = 8 + x$. We need to evaluate the left- and right-hand limits as $x$ approaches 1.

- Left-hand limit: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 4(3) = 12$
- Right-hand limit: $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (8 + x) = 8 + 3 = 11$

Because $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x), \lim_{x\to 1} f(x)$ does not exist.

$$\Rightarrow \text{Condition 2 fails.} \tag{7.5.5}$$

There is no need to proceed further. Condition 2 fails at $x = 3$. If any of the conditions of continuity are not satisfied at $x = 3$, the function $f(x)$ is not continuous at $x = 3$.

b. $x = \frac{8}{3}$

Condition 1: Does $f(\frac{8}{3})$ exist?

$$f(\frac{8}{3}) = 4(\frac{8}{3}) = \frac{32}{3} \tag{7.5.6}$$
$$\Rightarrow \text{Condition 1 is satisfied.} \tag{7.5.7}$$

Condition 2: Does $\lim_{x\to \frac{8}{3}} f(x)$ exist?

To the left of $x = \frac{8}{3}$, $f(x) = 4x$; to the right of $x = \frac{8}{3}, f(x) = 8 + x$. We need to evaluate the left- and right-hand limits as $x$ approaches $\frac{8}{3}$.

- Left-hand limit: $\lim_{x\to \frac{8}{3}^-} f(x) = \lim_{x\to \frac{8}{3}^-} 4(\frac{8}{3}) = \frac{32}{3}$
- Right-hand limit: $\lim_{x\to \frac{8}{3}^+} f(x) = \lim_{x\to \frac{8}{3}^+} (8 + x) = 8 + \frac{8}{3} = \frac{32}{3}$

Because $\lim_{x\to \frac{8}{3}} f(x)$ exists,

$$\Rightarrow \text{Condition 2 is satisfied.} \tag{7.5.8}$$

Condition 3: Is $f(\frac{8}{3}) = \lim_{x\to \frac{8}{3}} f(x)$?

$$f(\frac{32}{3}) = \frac{32}{3} = \lim_{x\to \frac{8}{3}} f(x) \tag{7.5.9}$$
$$\Rightarrow \text{Condition 3 is satisfied.} \tag{7.5.10}$$

Because all three conditions of continuity are satisfied at $x = \frac{8}{3}$, the function $f(x)$ is continuous at $x = \frac{8}{3}$.

## Exercise 7.5.2:

Determine whether the function $f(x) = \begin{cases} \frac{1}{x}, & x \leq 2 \\ 9x - 11.5, & x > 2 \end{cases}$ is continuous at $x = 2$.

yes

## Example 7.5.3: Determining Whether a Rational Function is Continuous at a Given Number

Determine whether the function $f(x) = \frac{x^2-25}{x-5}$ is continuous at $x = 5$.

To determine if the function $f$ is continuous at $x = 5$, we will determine if the three conditions of continuity are satisfied at $x = 5$.

Condition 1:

$$f(5) \text{ does not exist.} \tag{7.5.11}$$
$$\Rightarrow \text{Condition 1 fails.} \tag{7.5.12}$$

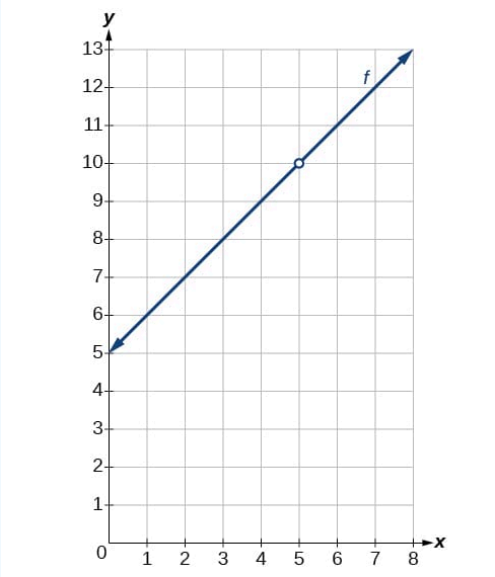
There is no need to proceed further. Condition 2 fails at $x = 5$. If any of the conditions of continuity are not satisfied at $x = 5$ ,the function f f is not continuous at $x = 5$.

Analysis

See Figure. Notice that for Condition 2 we have

$$\lim_{x\to 5} \frac{x^2 - 25}{x - 5} = \lim_{x\to 3} \frac{\cancel{(x-5)}\,(x+5)}{\cancel{x-5}} \tag{7.5.13}$$
$$= \lim_{x\to 5}(x + 5) \tag{7.5.14}$$
$$= 5 + 5 = 10 \tag{7.5.15}$$
$$\Rightarrow \text{Condition 2 is satisfied.} \tag{7.5.16}$$

At x=5, x=5,there exists a removable discontinuity. See Figure.



**Exercise 7.5.3:**

Determine whether the function $f(x) = \frac{9-x^2}{x^2-3x}$ is continuous at $x = 3$. If not, state the type of discontinuity.

No, the function is not continuous at $x = 3$. There exists a removable discontinuity at $x = 3$.

## Determining the Input Values for Which a Function Is Discontinuous

Now that we can identify continuous functions, jump discontinuities, and removable discontinuities, we will look at more complex functions to find discontinuities. Here, we will analyze a piecewise function to determine if any real numbers exist where the function is not continuous. A **piecewise function** may have discontinuities at the boundary points of the function as well as within the functions that make it up.

To determine the real numbers for which a piecewise function composed of **polynomial** functions is not continuous, recall that polynomial functions themselves are continuous on the set of real numbers. Any discontinuity would be at the boundary points. So we need to explore the three conditions of continuity at the boundary points of the piecewise function.

**how to: Given a piecewise function, determine whether it is continuous at the boundary points**

1. For each boundary point $a$ of the piecewise function, determine the left- and right-hand limits as $x$ approaches $a$, as well as the function value at $a$.
2. Check each condition for each value to determine if all three conditions are satisfied.
3. Determine whether each value satisfies condition 1: $f(a)$ exists.
4. Determine whether each value satisfies condition 2: $\lim_{x \to a} f(x)$ exists.
5. Determine whether each value satisfies condition 3: $\lim_{x \to a} f(x) = f(a)$.
6. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions fails, the function is not continuous at $x = a$.

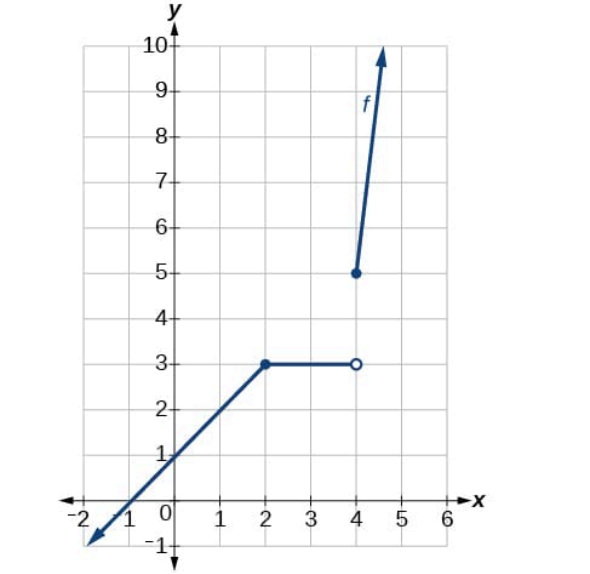
**Example 7.5.4: Determining the Input Values for Which a Piecewise Function Is Discontinuous**

Determine whether the function f f is discontinuous for any real numbers.

$$fx = \begin{cases} x + 1, & x < 2 \\ 3, & 2 \leq x < 4 \\ x^2 - 11, & x \geq 4 \end{cases} \tag{7.5.17}$$

Analysis

See Figure. At $x = 4$,there exists a jump discontinuity. Notice that the function is continuous at $x = 2$.




Graph is continuous at $x = 2$ but shows a jump discontinuity at $x = 4$.

**Exercise 7.5.4:**

Determine where the function $f(x) = \begin{cases} \frac{\pi x}{4}, & x < 2 \\ \frac{\pi}{x}, & 2 \leq x \leq 6 \\ 2\pi x, & x > 6 \end{cases}$ is discontinuous.

$x = 6$

## Determining Whether a Function Is Continuous

To determine whether a **piecewise function** is continuous or discontinuous, in addition to checking the boundary points, we must also check whether each of the functions that make up the piecewise function is continuous.

**how to: Given a piecewise function, determine whether it is continuous.**

1. Determine whether each component function of the piecewise function is continuous. If there are discontinuities, do they occur within the domain where that component function is applied?
2. For each boundary point $x = a$ of the piecewise function, determine if each of the three conditions hold.

**Example 7.5.5: Determining Whether a Piecewise Function Is Continuous**

Determine whether the function below is continuous. If it is not, state the location and type of each discontinuity.

$$fx = \begin{cases} \sin(x), & x < 0 \\ x^3, & x > 0 \end{cases} \tag{7.5.18}$$

The two functions composing this piecewise function are $f(x) = \sin(x)$ on $x < 0$ and $f(x) = x^3$ on $x > 0$. The sine function and all polynomial functions are continuous everywhere. Any discontinuities would be at the boundary point,

At $x = 0$,let us check the three conditions of continuity.

Condition 1:

$$f(0) \text{ does not exist.} \tag{7.5.19}$$
$$\Rightarrow \text{Condition 1 fails.} \tag{7.5.20}$$

Because all three conditions are not satisfied at $x = 0$, the function $f(x)$ is discontinuous at $x = 0$.

Analysis

See Figure. There exists a removable discontinuity at $x = 0$; $\lim_{x \to 0} f(x) = 0$, thus the limit exists and is finite, but $f(a)$ does not exist.

Function has removable discontinuity at 0.

## Media

Access these online resources for additional instruction and practice with continuity.

- Continuity at a Point
- Continuity at a Point: Concept Check

# Key Concepts

- A continuous function can be represented by a graph without holes or breaks.
- A function whose graph has holes is a discontinuous function.
- A function is continuous at a particular number if three conditions are met:
  - Condition 1: $f(a)$ exists.
  - Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
  - Condition 3: $\lim_{x \to a} f(x) = f(a)$.
- A function has a jump discontinuity if the left- and right-hand limits are different, causing the graph to "jump."
- A function has a removable discontinuity if it can be redefined at its discontinuous point to make it continuous. See Example.
- Some functions, such as polynomial functions, are continuous everywhere. Other functions, such as logarithmic functions, are continuous on their domain. See Example and Example.
- For a piecewise function to be continuous each piece must be continuous on its part of the domain and the function as a whole must be continuous at the boundaries. See Example and Example.

# Glossary

**continuous function**
: a function that has no holes or breaks in its graph

**discontinuous function**
: a function that is not continuous at $x = a$

**jump discontinuity**
: a point of discontinuity in a function $f(x)$ at $x = a$ where both the left and right-hand limits exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

**removable discontinuity**
: a point of discontinuity in a function $f(x)$ where the function is discontinuous, but can be redefined to make it continuous

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# 7.6: DEFINING THE DERIVATIVE

## Learning Objectives

- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope of a tangent line.
- Identify the derivative as the limit of a difference quotient.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.
- Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac **Newton (1643–1727)** and the German Gottfried **Leibniz (1646–1716)**, whose images appear in Figure 7.6.1. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.

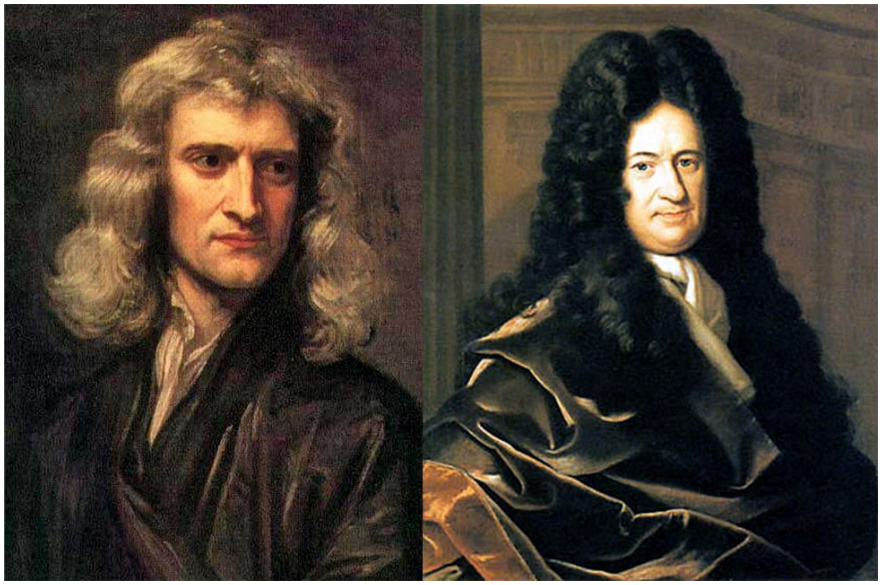


Figure 7.6.1: Newton and Leibniz are credited with developing calculus independently.

## Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in Figure 7.6.2. The slope of this line is given by an equation in the form of a difference quotient:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing $x$ with $a + h$, where

$h$ is a value close to a. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a+h, f(a+h))$. In this case, we find the secant line has a slope given by the following difference quotient with increment $h$:

$$m_{sec} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

**Definition: Difference Quotient**

Let $f$ be a function defined on an interval $I$ containing $a$. If $x \neq a$ is in $I$, then

$$Q = \frac{f(x)-f(a)}{x-a}$$

is a *difference quotient*.

Also, if $h \neq 0$ is chosen so that $a+h$ is in $I$, then

$$Q = \frac{f(a+h)-f(a)}{h}$$

is a difference quotient with increment $h$.

These two expressions for calculating the slope of a secant line are illustrated in Figure 7.6.2. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.

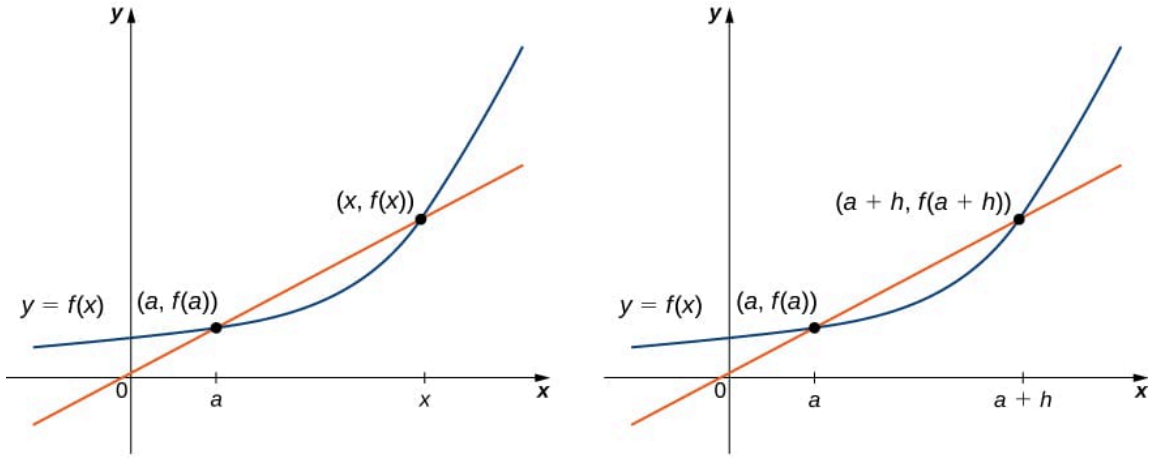



Figure 7.6.2: We can calculate the slope of a secant line in either of two ways.

In Figure 7.6.3$a$ we see that, as the values of $x$ approach $a$, the slopes of the secant lines provide better estimates of the rate of change of the function at $a$. Furthermore, the secant lines themselves approach the tangent line to the function at $a$, which represents the limit of the secant lines. Similarly, Figure 7.6.3$b$ shows that as the values of $h$ get closer to $0$, the secant lines also approach the tangent line. The slope of the tangent line at $a$ is the rate of change of the function at $a$, as shown in Figure 7.6.3$c$.

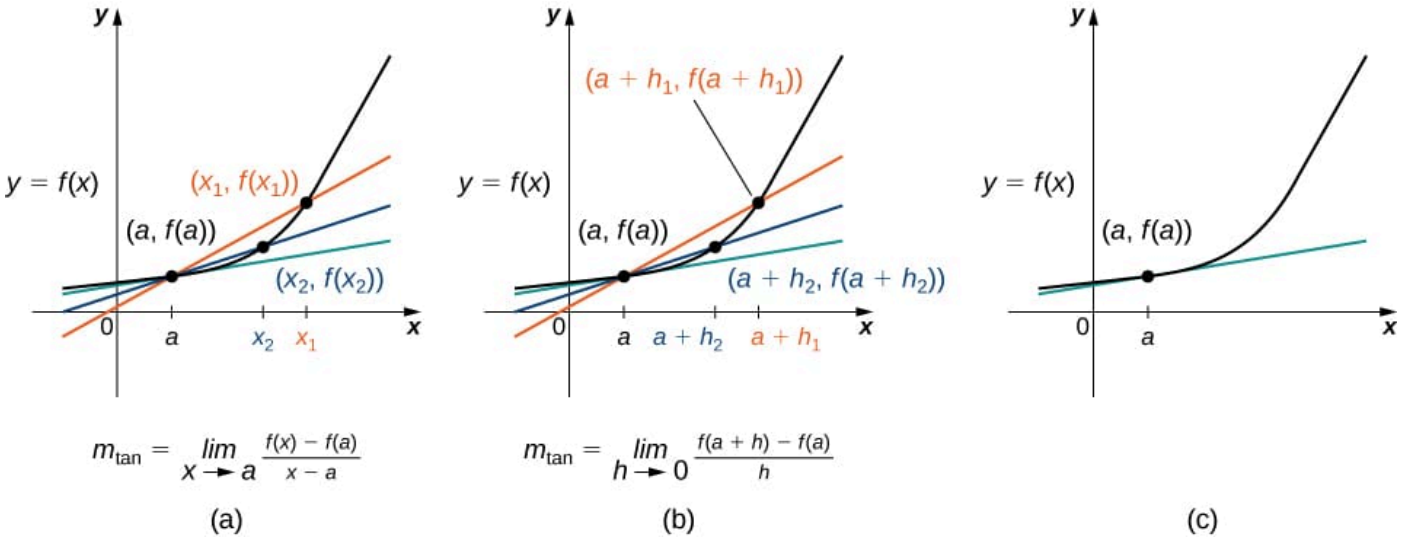



Figure 7.6.3: The secant lines approach the tangent line (shown in green) as the second point approaches the first.

In Figure 7.6.4 we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of $x$ close to 1. In fact, the graph of $f(x)$ itself appears to be **locally linear** in the immediate vicinity of $x = 1$.

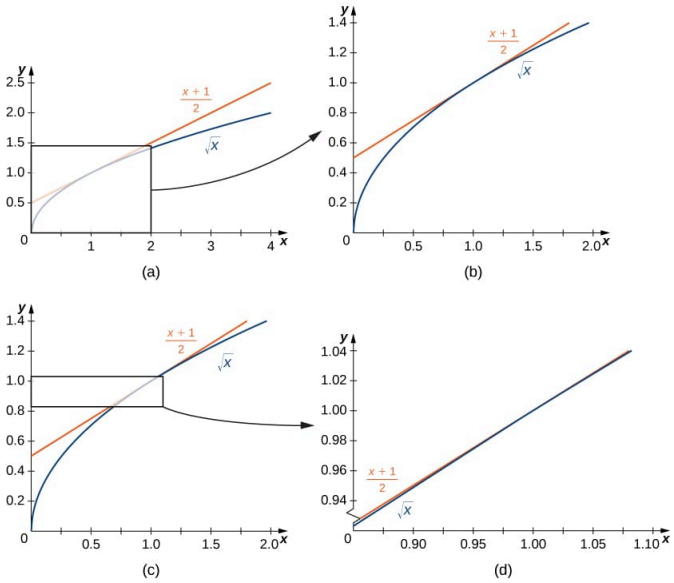



Figure 7.6.4: For values of $x$ close to 1, the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

**Definition: Tangent Line**

Let $f(x)$ be a function defined in an open interval containing $a$. The tangent line to $f(x)$ at $a$ is the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{7.6.1}$$

provided this limit exists.

Equivalently, we may define the *tangent line* to $f(x)$ at $a$ to be the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{7.6.2}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

## Example 7.6.1: Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

**Solution**

First find the slope of the tangent line. In this example, use Equation 7.6.1.

$$\begin{aligned} m_{tan} &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} && \text{Apply the definition.} \\ &= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} && \text{Substitute } f(x) = x^2 \text{ and } f(3) = 9 \\ &= \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3}(x+3) = 6 && \text{Factor the numerator to evaluate the limit.} \end{aligned}$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x = 3$, it passes through the point $(3, f(3))$. We have $f(3) = 9$, so the tangent line passes through the point $(3, 9)$.

Using the point-slope equation of the line with the slope $m = 6$ and the point $(3, 9)$, we obtain the line $y - 9 = 6(x - 3)$. Simplifying, we have $y = 6x - 9$. The graph of $f(x) = x^2$ and its tangent line at 3 are shown in Figure 7.6.5.

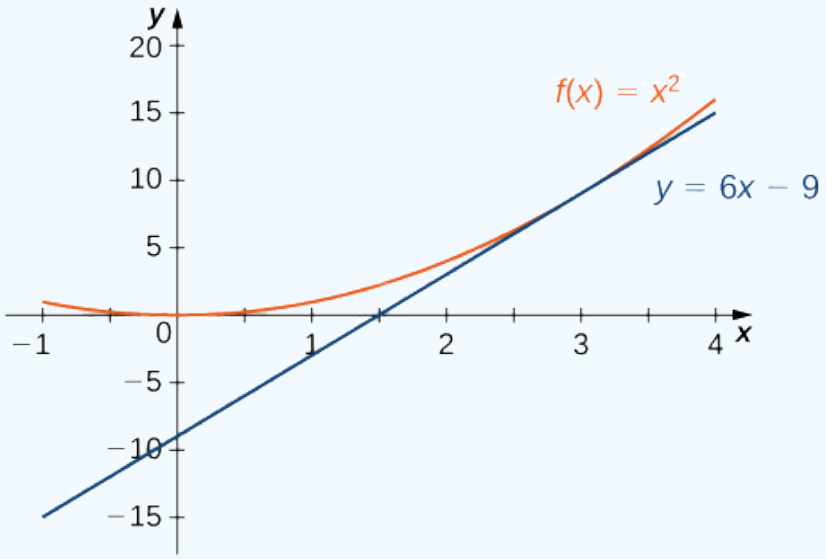



Figure 7.6.5: The tangent line to $f(x)$ at $x = 3$.

## Example 7.6.2: The Slope of a Tangent Line Revisited

Use Equation 7.6.2 to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

**Solution**

The steps are very similar to Example 7.6.1. See Equation 7.6.2 for the definition.

$$
\begin{aligned}
m_{tan} &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} && \text{Apply the definition.} \\
&= \lim_{h\to 0} \frac{(3+h)^2 - 9}{h} && \text{Substitute } f(3+h) = (3+h)^2 \text{ and } f(3) = 9 \\
&= \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h} && \text{Expand and simplify to evaluate the limit.} \\
&= \lim_{h\to 0} \frac{h(6+h)}{h} = \lim_{h\to 0}(6+h) = 6
\end{aligned}
$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

## Example 7.6.3: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

**Solution**

We can use Equation 7.6.1, but as we have seen, the results are the same if we use Equation 7.6.2.

Misplaced &

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f(2) = \frac{1}{2}$. Since the tangent line passes through the point $(2, \frac{1}{2})$ we can use the point-slope equation of a line to find the equation of the tangent line. Thus the tangent line has the equation $y = -\frac{1}{4}x + 1$. The graphs of $f(x) = \frac{1}{x}$ and $y = -\frac{1}{4}x + 1$ are shown in Figure 7.6.6.

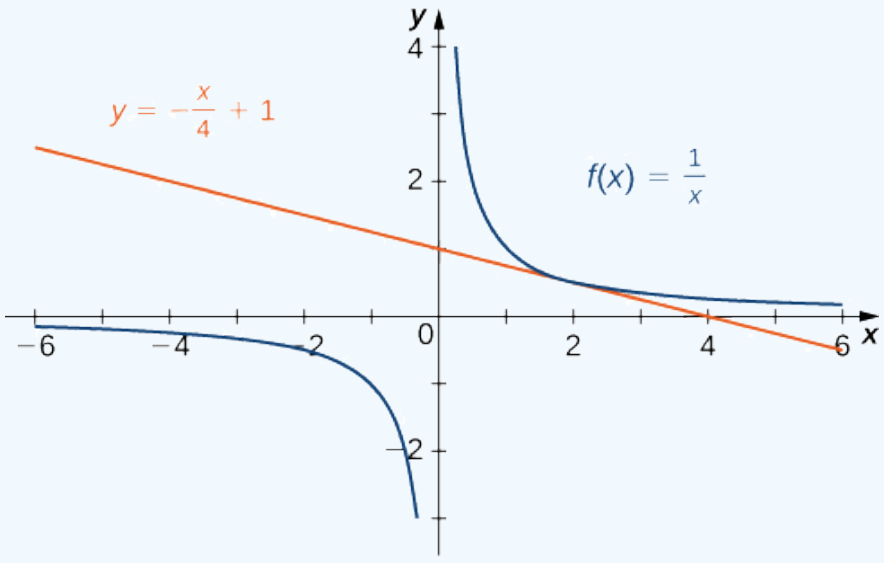



Figure 7.6.6:The line is tangent to $f(x)$ at $x = 2$.

**Exercise 7.6.1**

Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative.** The process of finding a derivative is called **differentiation**.

**Definition: Derivative**

Let $f(x)$ be a function defined in an open interval containing $a$. The **derivative** of the function $f(x)$ at $a$, denoted by $f\,'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{7.6.3}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at $a$ as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{7.6.4}$$

**Example 7.6.4: Estimating a Derivative**

For $f(x) = x^2$, use a table to estimate $f\,'(3)$ using Equation 7.6.3.

**Solution**

Create a table using values of $x$ just below $3$ and just above $3$.

| $x$ | $\frac{x^2 - 9}{x - 3}$ |
|---|---|
| 2.9 | 5.9 |
| 2.99 | 5.99 |
| 2.999 | 5.999 |
| 3.001 | 6.001 |
| 3.01 | 6.01 |
| 3.1 | 6.1 |

After examining the table, we see that a good estimate is $f\,'(3) = 6$.

## Exercise 7.6.2

For $f(x) = x^2$, use a table to estimate $f'(3)$ using Equation 7.6.4.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Example 7.6.6: Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 7.6.3.

**Solution**

Substitute the given function and value directly into the equation.

$$
\begin{aligned}
f'(x) &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} && \text{Apply the definition.} \\
&= \lim_{x \to 2} \frac{(3x^2 - 4x + 1) - 5}{x - 2} && \text{Substitute } f(x) = 3x^2 - 4x + 1 \text{ and } f(2) = 5. \\
&= \lim_{x \to 2} \frac{(x - 2)(3x + 2)}{x - 2} && \text{Simplify and factor the numerator.} \\
&= \lim_{x \to 2} (3x + 2) && \text{Cancel the common factor.} \\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

## Example 7.6.7: Revisiting the Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 7.6.4.

**Solution**

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in Example 7.6.6.

$$
\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} && \text{Apply the definition.} \\
&= \lim_{h \to 0} \frac{(3(2 + h)^2 - 4(2 + h) + 1) - 5}{h} && \text{Substitute } f(2) = 5 \text{ and } f(2 + h) = 3(2 + h)^2 - 4(2 + h) + 1. \\
&= \lim_{h \to 0} \frac{3(4 + 4h + h^2) - 8 - 4h + 1 - 5}{h} && \text{Expand the numerator.} \\
&= \lim_{h \to 0} \frac{12 + 12h + 3h^2 - 12 - 4h}{h} && \text{Distribute and begin simplifying the numerator.} \\
&= \lim_{h \to 0} \frac{3h^2 + 8h}{h} && \text{Finish simplifying the numerator.} \\
&= \lim_{h \to 0} \frac{h(3h + 8)}{h} && \text{Factor the numerator.} \\
&= \lim_{h \to 0} (3h + 8) && \text{Cancel the common factor.} \\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

The results are the same whether we use Equation 7.6.3 or Equation 7.6.4.

**Exercise 7.6.4**

For $f(x) = x^2 + 3x + 2$, find $f\,'(1)$.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the **average velocity** of the object over a time interval $[a, t]$ if $t > a$ or $[t, a]$ if $t < a$ is given by the difference quotient

$$v_{ave} = \frac{s(t) - s(a)}{t - a}. \tag{7.6.5}$$

As the values of $t$ approach $a$, the values of $v_{ave}$ approach the value we call the **instantaneous velocity** at $a$. That is, instantaneous velocity at $a$, denoted $v(a)$, is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}. \tag{7.6.6}$$

To better understand the relationship between average velocity and instantaneous velocity, see Figure 7.6.7. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t = a$ whose position at time $t$ is given by the function $s(t)$. The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a, t]$.

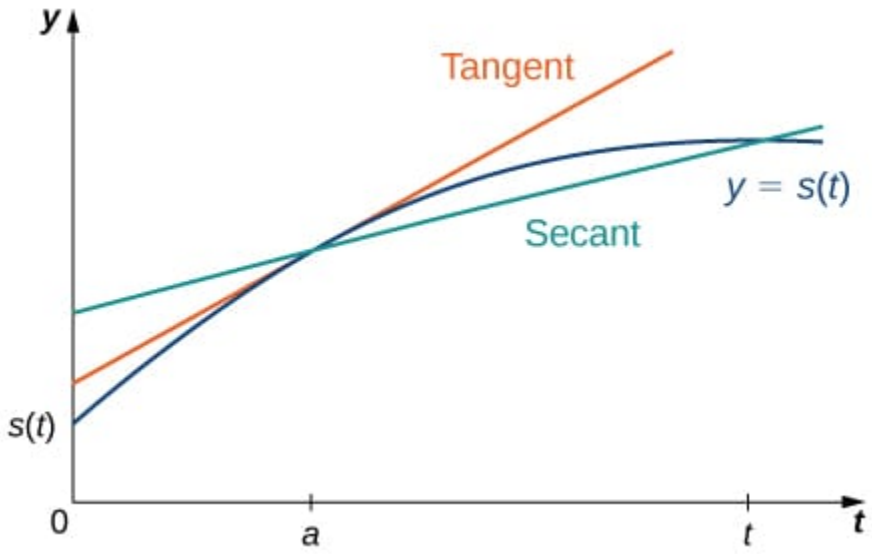



Figure 7.6.7: The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

We can use Equation 7.6.6 to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using Equation 7.6.5.

**Example 7.6.8: Estimating Velocity**

A lead weight on a spring is oscillating up and down. Its position at time $t$ with respect to a fixed horizontal line is given by $s(t) = \sin t$ (Figure 7.6.8). Use a table of values to estimate $v(0)$. Check the estimate by using Equation 7.6.6.

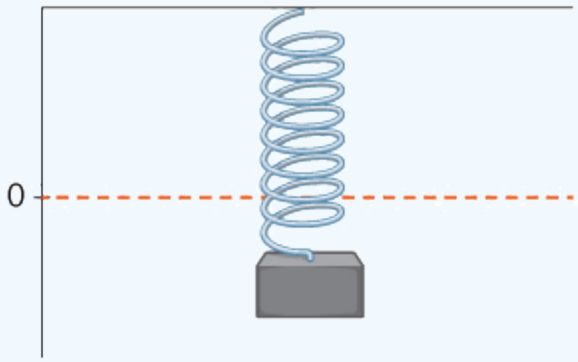



Figure 7.6.8: A lead weight suspended from a spring in vertical oscillatory motion.

**Solution**

We can estimate the instantaneous velocity at $t = 0$ by computing a table of average velocities using values of $t$ approaching $0$, as shown in Table 7.6.2.

*Table 7.6.2: Average velocities using values of $t$ approaching 0*

| $t$ | $\frac{\sin t - \sin 0}{t-0} = \frac{\sin t}{t}$ |
|---|---|
| −0.1 | 0.998334166 |
| −0.01 | 0.9999833333 |
| −0.001 | 0.999999833 |
| 0.001 | 0.999999833 |
| 0.01 | 0.9999833333 |
| 0.1 | 0.998334166 |

From the table we see that the average velocity over the time interval $[-0.1, 0]$ is $0.998334166$, the average velocity over the time interval $[-0.01, 0]$ is $0.9999833333$, and so forth. Using this table of values, it appears that a good estimate is $v(0) = 1$.

By using Equation 7.6.6, we can see that

$$v(0) = s'(0) = \lim_{t \to 0} \frac{\sin t - \sin 0}{t - 0} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$$

Thus, in fact, $v(0) = 1$.

## Exercise 7.6.5

A rock is dropped from a height of $64$ feet. Its height above ground at time $t$ seconds later is given by $s(t) = -16t^2 + 64, \quad 0 \leq t \leq 2$. Find its instantaneous velocity $1$ second after it is dropped, using Equation 7.6.6.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

## Definition: Instantaneous Rate of Change

The **instantaneous rate of change** of a function $f(x)$ at a value $a$ is its derivative $f'(a)$.

## Example 7.6.9: Chapter Opener: Estimating Rate of Change of Velocity

Reaching a top speed of $270.49$ mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from $0$ to $60$ mph in $3.05$ seconds, from $0$ to $100$ mph in $5.88$ seconds, from $0$ to $200$ mph in $14.51$ seconds, and from $0$ to $229.9$ mph in $19.96$ seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its **acceleration**) as it approaches $229.9$ mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?



Figure 7.6.9: (credit: modification of work by Codex41, Flickr)

Solution: First observe that $60$ mph = $88$ ft/s, $100$ mph ≈$146.67$ ft/s, $200$ mph ≈$293.33$ ft/s, and $229.9$ mph ≈$337.19$ ft/s. We can summarize the information in a table.

*Table 7.6.3: $v(t)$ at different values of $t$*

| $t$ | $v(t)$ |
|---|---|
| 0 | 0 |
| 3.05 | 88 |
| 5.88 | 147.67 |
| 14.51 | 293.33 |
| 19.96 | 337.19 |

Now compute the average acceleration of the car in feet per second on intervals of the form $[t, 19.96]$ as $t$ approaches $19.96$, as shown in the following table.

*Average acceleration*

| $t$ | $\frac{v(t)-v(19.96)}{t-19.96}=\frac{v(t)-337.19}{t-19.96}$ |
|---|---|
| 0.0 | 16.89 |
| 3.05 | 14.74 |
| 5.88 | 13.46 |
| 14.51 | 8.05 |

The rate at which the car is accelerating is decreasing as its velocity approaches $229.9$ mph ($337.19$ ft/s).

## Example 7.6.10: Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70\,°F$ at $9$ p.m., reaches a low of $60\,°$ during the night, and rises back to $70\,°$ by $7$ a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10$, where $t$ is the number of hours past $9$ p.m. Find the instantaneous rate of change of the temperature at midnight.

**Solution**

Since midnight is $3$ hours past $9$ p.m., we want to compute $T\,'(3)$. Refer to Equation 7.6.3.

$$\begin{aligned}
T'(3) &= \lim_{t\to 3}\frac{T(t)-T(3)}{t-3} && \text{Apply the definition.}\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+70-61.6}{t-3} && \text{Substitute } T(t)=0.4t^2-4t+70 \text{ and } T(3)=61.6.\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+8.4}{t-3} && \text{Simplify.}\\
&= \lim_{t\to 3}\frac{0.4(t-3)(t-7)}{t-3} &&\\
&= \lim_{t\to 3}0.4(t-7) && \text{Cancel.}\\
&= -1.6 && \text{Evaluate the limit.}
\end{aligned}$$

The instantaneous rate of change of the temperature at midnight is $-1.6\,°F$ per hour.

## Example 7.6.11: Rate of Change of Profit

A toy company can sell $x$ electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing $x$ systems is given by $C(x) = 100x + 10,000$ dollars. Find the rate of change of profit when $10,000$ games are produced. Should the toy company increase or decrease production?

**Solution**

The profit $P(x)$ earned by producing $x$ gaming systems is $R(x) - C(x)$, where $R(x)$ is the revenue obtained from the sale of $x$ games. Since the company can sell $x$ games at $p = -0.01x + 400$ per game,

$$R(x) = xp = x(-0.01x + 400) = -0.01x^2 + 400x.$$

Consequently,

$$P(x) = -0.01x^2 + 300x - 10,000.$$

Therefore, evaluating the rate of change of profit gives

$$\begin{aligned}
P'(10000) &= \lim_{x\to 10000}\frac{P(x)-P(10000)}{x-10000}\\
&= \lim_{x\to 10000}\frac{-0.01x^2+300x-10000-1990000}{x-10000}\\
&= \lim_{x\to 10000}\frac{-0.01x^2+300x-2000000}{x-10000}\\
&= 100
\end{aligned}.$$

Since the rate of change of profit $P'(10,000) > 0$ and $P(10,000) > 0$, the company should increase production.

**Exercise 7.6.6**

A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10$. The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25)$, the rate of change of profit when the price is $\$3.25$ and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Key Concepts

- The slope of the tangent line to a curve measures the instantaneous rate of change of a curve. We can calculate it by finding the limit of the difference quotient or the difference quotient with increment $h$.
- The derivative of a function $f(x)$ at a value $a$ is found using either of the definitions for the slope of the tangent line.
- Velocity is the rate of change of position. As such, the velocity $v(t)$ at time $t$ is the derivative of the position $s(t)$ at time $t$. Average velocity is given by

  $$v_{ave} = \frac{s(t) - s(a)}{t - a}.$$

  Instantaneous velocity is given by

  $$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}.$$

- We may estimate a derivative by using a table of values.

## Key Equations

- **Difference quotient**

  $$Q = \frac{f(x) - f(a)}{x - a}$$

- **Difference quotient with increment h**

  $$Q = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

- **Slope of tangent line**

  $$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  $$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- **Derivative of f(x) at a**

  $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- **Average velocity**

$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$

- **Instantaneous velocity**

$$v(a) = s\,'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

# Glossary

**derivative**
: the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

**difference quotient**
: of a function $f(x)$ at $a$ is given by

$$\frac{f(a+h) - f(a)}{h} \text{ or } \frac{f(x) - f(a)}{x - a}$$

**differentiation**
: the process of taking a derivative

**instantaneous rate of change**
: the rate of change of a function at any point along the function $a$, also called $f\,'(a)$, or the derivative of the function at $a$

## Answers

**Exercise** 7.6.1
Use either Equation 7.6.1 or Equation 7.6.2. Multiply the numerator and the denominator by a conjugate.

**Exercise** 7.6.1
$\frac{1}{4}$

**Exercise** 7.6.2
Evaluate $\frac{(x+h)^2 - x^2}{h}$ at $h = -0.1,\ -0.01,\ -0.001,\ 0.001,\ 0.01,\ 0.1$

**Exercise** 7.6.2
6

**Exercise** 7.6.4
Use either Equation 7.6.3, Equation 7.6.4, or try both.

**Exercise** 7.6.4
$f\,'(1) = 5$

**Exercise** 7.6.5
$v(t) = s\,'(t)$. Follow the earlier examples of the derivative using Equation 7.6.6.

**Exercise** 7.6.5
–32 ft/s

**Exercise** 7.6.6
Use Example 7.6.11 for a guide.

**Exercise** 7.6.6
$P\,'(3.25) = 20 > 0$; raise prices

## 7.6E: Exercises for Section 3.1

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# 7.6E: Exercises for Section 3.1

**For exercises 1 - 10, use the equation $m_{\text{sec}} = \dfrac{f(x) - f(a)}{x - a}$ to find the slope of the secant line between the values $x_1$ and $x_2$ for each function $y = f(x)$.**

1) $f(x) = 4x + 7; \quad x_1 = 2, \quad x_2 = 5$

**Answer**

$m_{\text{sec}} = 4$

2) $f(x) = 8x - 3; \quad x_1 = -1, \quad x_2 = 3$

3) $f(x) = x^2 + 2x + 1; \quad x_1 = 3, \quad x_2 = 3.5$

**Answer**

$m_{\text{sec}} = 8.5$

4) $f(x) = -x^2 + x + 2; \quad x_1 = 0.5, \quad x_2 = 1.5$

5) $f(x) = \dfrac{4}{3x - 1}; \quad x_1 = 1, \quad x_2 = 3$

**Answer**

$m_{\text{sec}} = -\frac{3}{4}$

6) $f(x) = \dfrac{x - 7}{2x + 1}; \quad x_1 = -2, \quad x_2 = 0$

7) $f(x) = \sqrt{x}; \quad x_1 = 1, \quad x_2 = 16$

**Answer**

$m_{\text{sec}} = 0.2$

8) $f(x) = \sqrt{x - 9}; \quad x_1 = 10, \quad x_2 = 13$

9) $f(x) = x^{1/3} + 1; \quad x_1 = 0, \quad x_2 = 8$

**Answer**

$m_{\text{sec}} = 0.25$

10) $f(x) = 6x^{2/3} + 2x^{1/3}; \quad x_1 = 1, \quad x_2 = 27$

**For the functions in exercises 11 - 20,**

**a. use the equation $m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$ to find the slope of the tangent line $m_{\tan} = f\,'(a)$, and**

**b. find the equation of the tangent line to $f$ at $x = a$.**

11) $f(x) = 3 - 4x, \quad a = 2$

**Answer**

a. $m_{\tan} = -4$
b. $y = -4x + 3$

12) $f(x) = \dfrac{x}{5} + 6, \quad a = -1$

13) $f(x) = x^2 + x, \quad a = 1$

**Answer**

a. $m_{\tan} = 3$
b. $y = 3x - 1$

14) $f(x) = 1 - x - x^2, \quad a = 0$

15) $f(x) = \dfrac{7}{x}, \quad a = 3$

**Answer**

a. $m_{\tan} = \frac{-7}{9}$
b. $y = \frac{-7}{9}x + \frac{14}{3}$

16) $f(x) = \sqrt{x+8}, \quad a = 1$

17) $f(x) = 2 - 3x^2, \quad a = -2$

**Answer**

a. $m_{\tan} = 12$
b. $y = 12x + 14$

18) $f(x) = \dfrac{-3}{x-1}, \quad a = 4$

19) $f(x) = \dfrac{2}{x+3}, \quad a = -4$

**Answer**

a. $m_{\tan} = -2$
b. $y = -2x - 10$

20) $f(x) = \dfrac{3}{x^2}, \quad a = 3$

**For the functions $y = f(x)$ in exercises 21 - 30, find $f\,'(a)$ using the equation $f\,'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.**

21) $f(x) = 5x + 4, \quad a = -1$

**Answer**

$f'(-1) = 5$

22) $f(x) = -7x + 1, \quad a = 3$

23) $f(x) = x^2 + 9x, \quad a = 2$

**Answer**

$f'(2) = 13$

24) $f(x) = 3x^2 - x + 2, \quad a = 1$

25) $f(x) = \sqrt{x}, \quad a = 4$

**Answer**

$f'(4) = \frac{1}{4}$

26) $f(x) = \sqrt{x-2}, \quad a = 6$

27) $f(x) = \dfrac{1}{x}, \quad a = 2$

**Answer**

$f'(2) = -\frac{1}{4}$

28) $f(x) = \dfrac{1}{x-3}, \quad a = -1$

29) $f(x) = \dfrac{1}{x^3}, \quad a = 1$

**Answer**

$f'(1) = -3$

30) $f(x) = \dfrac{1}{\sqrt{x}}, \quad a = 4$

**For the following exercises, given the function $y = f(x)$,**

**a. find the slope of the secant line $PQ$ for each point $Q(x, f(x))$ with $x$ value given in the table.**

**b. Use the answers from a. to estimate the value of the slope of the tangent line at $P$.**

**c. Use the answer from b. to find the equation of the tangent line to $f$ at point $P$.**

31) [T] $f(x) = x^2 + 3x + 4, \quad P(1, 8)$ (Round to 6 decimal places.)

| $x$ | Slope $m_{PQ}$ | $x$ | Slope $m_{PQ}$ |
|---|---|---|---|
| 1.1 | (i) | 0.9 | (vii) |
| 1.01 | (ii) | 0.99 | (viii) |
| 1.001 | (iii) | 0.999 | (ix) |
| 1.0001 | (iv) | 0.9999 | (x) |
| 1.00001 | (v) | 0.99999 | (xi) |
| 1.000001 | (vi) | 0.999999 | (xii) |

**Answer**

$a.(i)5.100000, (ii)5.010000, (iii)5.001000, (iv)5.000100, (v)5.000010, (vi)5.000001, (vii)4.900000, (viii)4.990000, (ix)4.999000, (x)4.999900, (xi)4.999990, (x)4.999999$

b. $m_{\tan} = 5$

c. $y = 5x + 3$

32) [T] $f(x) = \dfrac{x+1}{x^2-1}, \quad P(0, -1)$

| $x$ | Slope $m_{PQ}$ | $x$ | Slope $m_{PQ}$ |
|---|---|---|---|
| 0.1 | (i) | −0.1 | (vii) |
| 0.01 | (ii) | −0.01 | (viii) |
| 0.001 | (iii) | −0.001 | (ix) |
| 0.0001 | (iv) | −0.0001 | (x) |
| 0.00001 | (v) | −0.00001 | (xi) |
| 0.000001 | (vi) | −0.000001 | (xii) |

33) [T] $f(x) = 10e^{0.5x}, \quad P(0, 10)$ (Round to 4 decimal places.)

| $x$ | Slope $m_{PQ}$ |
|---|---|

| | |
|---|---|
| −0.1 | (i) |
| −0.01 | (ii) |
| −0.001 | (iii) |
| −0.0001 | (iv) |
| −0.00001 | (v) |
| −0.000001 | (vi) |

**Answer**

a. $(i) 4.8771,\ (ii) 4.9875,\ (iii) 4.9988,\ (iv) 4.9999,\ (v) 4.9999,\ (vi) 4.9999$
b. $m_{\tan} = 5$
c. $y = 5x + 10$

34) [T] $f(x) = \tan(x), \quad P(\pi, 0)$

| $x$ | Slope $m_{PQ}$ |
|---|---|
| 3.1 | (i) |
| 3.14 | (ii) |
| 3.141 | (iii) |
| 3.1415 | (iv) |
| 3.14159 | (v) |
| 3.141592 | (vi) |

[T] For the following position functions $y = s(t)$, an object is moving along a straight line, where $t$ is in seconds and $s$ is in meters. Find

a. the simplified expression for the average velocity from $t = 2$ to $t = 2 + h$;

b. the average velocity between $t = 2$ and $t = 2 + h$, where $(i)\ h = 0.1,\ (ii)\ h = 0.01,\ (iii)\ h = 0.001$, and $(iv)\ h = 0.0001$; and

c. use the answer from a. to estimate the instantaneous velocity at $t = 2$ second.

35) $s(t) = \frac{1}{3}t + 5$

**Answer**

a. $\frac{1}{3}$;
b. $(i)\ \frac{1}{3}$ m/s, $(ii)\ \frac{1}{3}$ m/s, $(iii)\ \frac{1}{3}$ m/s, $(iv)\ \frac{1}{3}$ m/s;
c. $\frac{1}{3}$ m/s

36) $s(t) = t^2 - 2t$

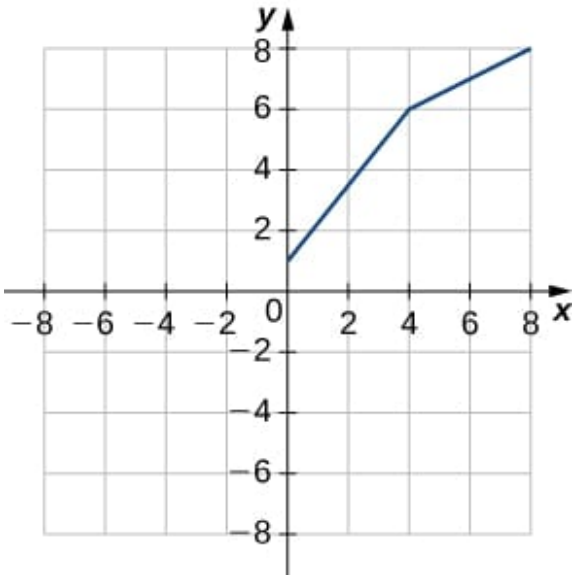
37) $s(t) = 2t^3 + 3$

**Answer**

a. $2(h^2 + 6h + 12)$;
b. $(i)\ 25.22$ m/s, $(ii)\ 24.12$ m/s, $(iii)\ 24.01$ m/s, $(iv)\ 24$ m/s;
c. $24$ m/s

38) $s(t) = \dfrac{16}{t^2} - \dfrac{4}{t}$

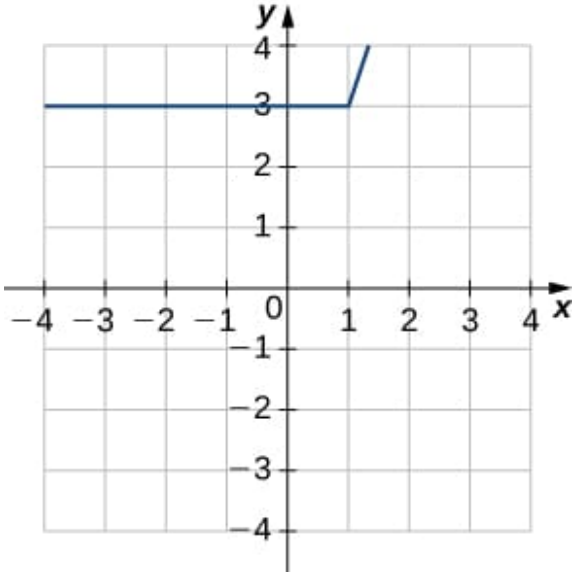
39) Use the following graph to evaluate a. $f\,'(1)$ and b. $f\,'(6)$.




**Answer**

a. $1.25$; b. $0.5$

40) Use the following graph to evaluate a. $f\,'(-3)$ and b. $f\,'(1.5)$.




**For the following exercises, use the limit definition of derivative to show that the derivative does not exist at $x = a$ for each of the given functions.**

41) $f(x) = x^{1/3}, \quad x = 0$

**Answer**

$$\lim_{x \to 0^-} \frac{x^{1/3} - 0}{x - 0} = \lim_{x \to 0^-} \frac{1}{x^{2/3}} = \infty$$

42) $f(x) = x^{2/3}, \quad x = 0$

43) $f(x) = \begin{cases} 1, & \text{if } x < 1 \\ x, & \text{if } x \geq 1 \end{cases}, \quad x = 1$

**Answer**

$$\lim_{x\to 1^-}\frac{1-1}{x-1}=0\neq 1=\lim_{x\to 1^+}\frac{x-1}{x-1}$$

44) $f(x)=\dfrac{|x|}{x},\quad x=0$

45) [T] The position in feet of a race car along a straight track after $t$ seconds is modeled by the function $s(t)=8t^2-\frac{1}{16}t^3$.

a. Find the average velocity of the vehicle over the following time intervals to four decimal places:

i. $[4, 4.1]$

ii. $[4, 4.01]$

iii. $[4, 4.001]$

iv. $[4, 4.0001]$

b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at $t=4$ seconds.

**Answer**

a. $(i)61.7244ft/s,\ \ (ii)61.0725ft/s,\ \ (iii)61.0072ft/s,\ \ (iv)61.0007ft/s$
b. At $4$ seconds the race car is traveling at a rate/velocity of $61$ ft/s.

46) [T] The distance in feet that a ball rolls down an incline is modeled by the function $s(t)=14t^2$,

where t is seconds after the ball begins rolling.

a. Find the average velocity of the ball over the following time intervals:

i. [5, 5.1]

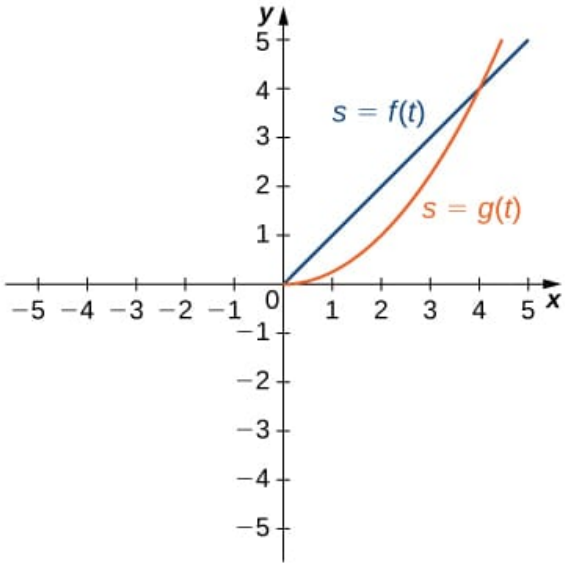
ii. [5, 5.01]

iii. [5, 5.001]

iv. [5, 5.0001]

b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at $t=5$ seconds.

47) Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by $s=f(t)$ and $s=g(t)$, where s is measured in feet and t is measured in seconds.




a. Which vehicle has traveled farther at $t = 2$ seconds?

b. What is the approximate velocity of each vehicle at $t = 3$ seconds?

c. Which vehicle is traveling faster at $t = 4$ seconds?

d. What is true about the positions of the vehicles at $t = 4$ seconds?

**Answer**

a. The vehicle represented by $f(t)$, because it has traveled $2$ feet, whereas $g(t)$ has traveled $1$ foot.
b. The velocity of $f(t)$ is constant at $1$ ft/s, while the velocity of $g(t)$ is approximately $2$ ft/s.
c. The vehicle represented by $g(t)$, with a velocity of approximately $4$ ft/s.
d. Both have traveled $4$ feet in $4$ seconds.

48) [T] The total cost $C(x)$, in hundreds of dollars, to produce $x$ jars of mayonnaise is given by $C(x) = 0.000003x^3 + 4x + 300$.

a. Calculate the average cost per jar over the following intervals:

i. [100, 100.1]

ii. [100, 100.01]

iii. [100, 100.001]

iv. [100, 100.0001]

b. Use the answers from a. to estimate the average cost to produce $100$ jars of mayonnaise.

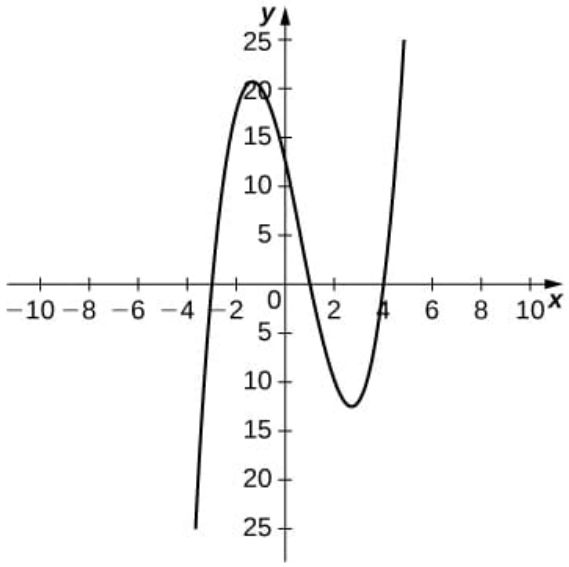
49) [T] For the function $f(x) = x^3 - 2x^2 - 11x + 12$, do the following.

a. Use a graphing calculator to graph $f$ in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the two values of $x = a$ for which $m_{tan} = f\,'(a) = 0$.

**Answer**

a.




b. $a \approx -1.361, \ \ 2.694$

50) [T] For the function $f(x) = \dfrac{x}{1+x^2}$, do the following.

a. Use a graphing calculator to graph $f$ in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the values of $x = a$ for which $m_{\tan} = f\,'(a) = 0$.

51) Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle traveling $x$ miles. Suppose the vehicle gets $30$ mpg.

a. Find a mathematical expression for $N(x)$.

b. What is $N(100)$? Explain the physical meaning.

c. What is $N\,'(100)$? Explain the physical meaning.

**Answer**

a. $N(x) = \dfrac{x}{30}$
b. $\sim 3.3$ gallons. When the vehicle travels $100$ miles, it has used $3.3$ gallons of gas.
c. $\frac{1}{30}$. The rate of gas consumption in gallons per mile that the vehicle is achieving after having traveled $100$ miles.

52) [T] For the function $f(x) = x^4 - 5x^2 + 4$, do the following.

a. Use a graphing calculator to graph $f$ in an appropriate viewing window.

b. Use the $nDeriv$ function, which numerically finds the derivative, on a graphing calculator to estimate $f\,'(-2), \ \ f\,'(-0.5), \ \ f\,'(1.7)$, and $f\,'(2.718)$.

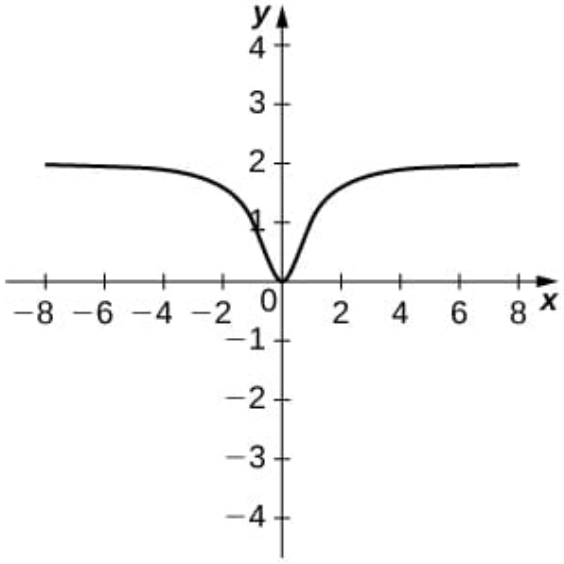
53) [T] For the function $f(x) = \dfrac{x^2}{x^2+1}$, do the following.

a. Use a graphing calculator to graph $f$ in an appropriate viewing window.

b. Use the $nDeriv$ function on a graphing calculator to find $f\,'(-4), \ \ f\,'(-2), \ \ f\,'(2)$, and $f\,'(4)$.

**Answer**

a.




b. $-0.028, -0.16, 0.16, 0.028$

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# 7.7: Derivatives

The average teen in the United States opens a refrigerator door an estimated 25 times per day. Supposedly, this average is up from 10 years ago when the average teenager opened a refrigerator door 20 times per day[1]. It is estimated that a television is on in a home 6.75 hours per day, whereas parents spend an estimated 5.5 minutes per day having a meaningful conversation with their children. These averages, too, are not the same as they were 10 years ago, when the television was on an estimated 6 hours per day in the typical household, and parents spent 12 minutes per day in meaningful conversation with their kids. What do these scenarios have in common? The functions representing them have changed over time. In this section, we will consider methods of computing such changes over time.

## Finding the Average Rate of Change of a Function

The functions describing the examples above involve a change over time. Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

A **tangent line** to a curve is a line that intersects the curve at only a single point but does not cross it there. (The tangent line may intersect the curve at another point away from the point of interest.) If we zoom in on a curve at that point, the curve appears linear, and the **slope of the curve** at that point is close to the slope of the tangent line at that point.

Figure 7.7.1 represents the function $f(x) = x^3 - 4x$. We can see the slope at various points along the curve.

- slope at $x = -2$ is 8
- slope at $x = -1$ is –1
- slope at $x = 2$ is 8

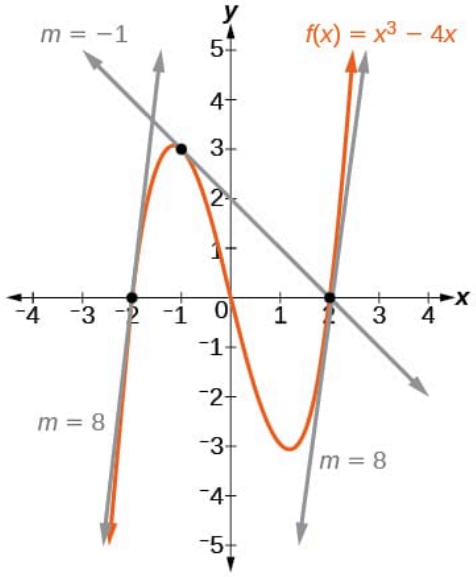



Figure 7.7.1: Graph showing tangents to curve at –2, –1, and 2.

Let's imagine a point on the curve of function $f$ at $x = a$ as shown in Figure 7.7.1. The coordinates of the point are $(a, f(a))$. Connect this point with a second point on the curve a little to the right of $x = a$, with an *x*-value increased by some small real number $h$. The coordinates of this second point are $(a + h, f(a + h))$ for some positive-value $h$.

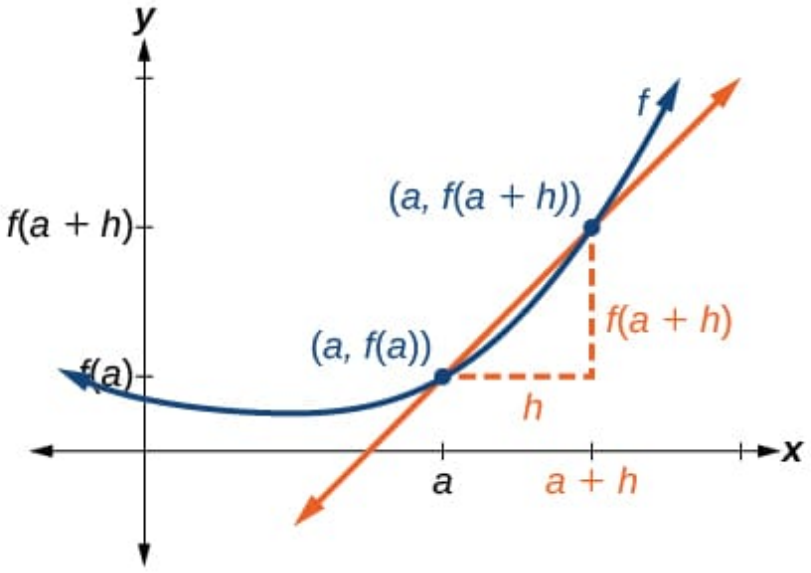



Figure 7.7.2: Connecting point $a$ with a point just beyond allows us to measure a slope close to that of a tangent line at $x = a$.

We can calculate the slope of the line connecting the two points $(a, f(a))$ and $(a + h, f(a + h))$, called a **secant line**, by applying the slope formula,

$$\text{slope} = \frac{\text{change in y}}{\text{change in x}} \tag{7.7.1}$$

We use the notation $m_{sec}$ to represent the slope of the secant line connecting two points.

$$m_{sec} = \frac{f(a+h) - f(a)}{(a+h) - (a)} \tag{7.7.2}$$

$$= \frac{f(a+h) - f(a)}{\cancel{a} + h - \cancel{a}} \tag{7.7.3}$$

The slope $m_{sec}$ equals the *average rate of change* between two points $(a, f(a))$ and $(a + h, f(a + h))$.

$$m_{sec} = \frac{f(a+h) - f(a)}{h} \tag{7.7.4}$$

## the AVERAGE RATE OF CHANGE BETWEEN TWO POINTS ON A CURVE

The **average rate of change** (AROC) between two points $(a, f(a))$ and $(a + h, f(a + h))$ on the curve of $f$ is the slope of the line connecting the two points and is given by

$$\text{AROC} = \frac{f(a+h) - f(a)}{h} \tag{7.7.5}$$

## Example 7.7.1: Finding the Average Rate of Change

Find the average rate of change connecting the points $(2, -6)$ and $(-1, 5)$.

**Solution**

We know the average rate of change connecting two points may be given by

$$\text{AROC} = \frac{f(a+h) - f(a)}{h} \tag{7.7.6}$$

If one point is $(2, -6)$, or $(2, f(2))$, then $f(2) = -6$.

The value $h$ is the displacement from $2$ to $-1$, which equals $-1 - 2 = -3$.

For the other point, $f(a + h)$ is the *y*-coordinate at $a + h$, which is $2 + (-3)$ or $-1$, so $f(a + h) = f(-1) = 5$.

$$\text{AROC} = \frac{f(a+h) - f(a)}{h} \tag{7.7.7}$$
$$= \frac{5 - (-6)}{-3} \tag{7.7.8}$$
$$= \frac{11}{-3} \tag{7.7.9}$$
$$= -\frac{11}{3} \tag{7.7.10}$$

**Exercise 7.7.1**

Find the average rate of change connecting the points $(-5, 1.5)$ and $(-2.5, 9)$

**Solution**

3

## Understanding the Instantaneous Rate of Change

Now that we can find the average rate of change, suppose we make $h$ in Figure 7.7.2 smaller and smaller. Then $a + h$ will approach $a$ as $h$ gets smaller, getting closer and closer to 0. Likewise, the second point $(a + h, f(a + h))$ will approach the first point, $(a, f(a))$. As a consequence, the connecting line between the two points, called the secant line, will get closer and closer to being a tangent to the function at $x = a$, and the slope of the secant line will get closer and closer to the slope of the tangent at $x = a$ (Figure 7.7.3).

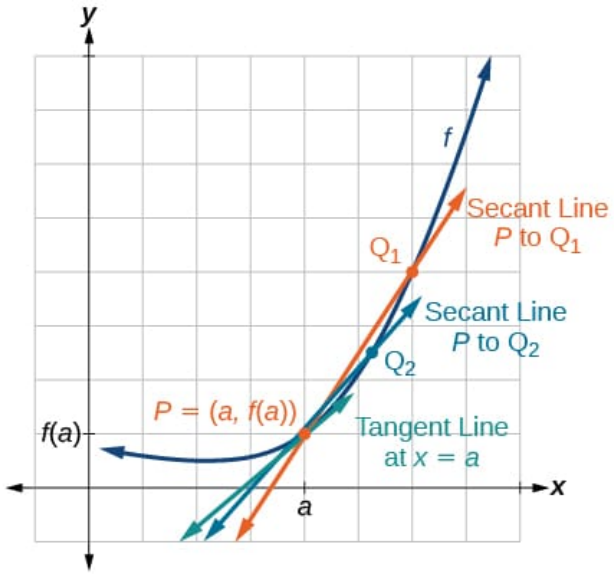



Figure 7.7.3: The connecting line between two points moves closer to being a tangent line at $x = a$.

Because we are looking for the slope of the tangent at $x = a$, we can think of the measure of the slope of the curve of a function $f$ at a given point as the rate of change at a particular instant. We call this slope the **instantaneous rate of change**, or the **derivative** of the function at $x = a$. Both can be found by finding the limit of the slope of a line connecting the point at $x = a$ with a second point infinitesimally close along the curve. For a function $f$ both the instantaneous rate of change of the function and the derivative of the function at $x = a$ are written as $f'(a)$, and we can define them as a two-sided limit that has the same value

whether approached from the left or the right.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{7.7.11}$$

The expression by which the limit is found is known as the **difference quotient**.

### DEFINITION OF INSTANTANEOUS RATE OF CHANGE AND DERIVATIVE

The derivative, or instantaneous rate of change, of a function $f$ at $x = a$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{7.7.12}$$

The expression $\frac{f(a+h)-f(a)}{h}$ is called the difference quotient.

We use the difference quotient to evaluate the limit of the rate of change of the function as $h$ approaches 0.

## Derivatives: Interpretations and Notation

The derivative of a function can be interpreted in different ways. It can be observed as the behavior of a graph of the function or calculated as a numerical rate of change of the function.

- The derivative of a function $f(x)$ at a point $x = a$ is the slope of the tangent line to the curve $f(x)$ at $x = a$. The derivative of $f(x)$ at $x = a$ is written $f\,'(a)$.
- The derivative $f\,'(a)$ measures how the curve changes at the point $(a, f(a))$.
- The derivative $f\,'(a)$ may be thought of as the instantaneous rate of change of the function $f(x)$ at $x = a$.
- If a function measures distance as a function of time, then the derivative measures the instantaneous **velocity** at time $t = a$.

### NOTATIONS FOR THE DERIVATIVE

The equation of the derivative of a function $f(x)$ is written as $y\,' = f\,'(x)$, where $y = f(x)$. The notation $f\,'(x)$ is read as "$f$ prime of $x$." Alternate notations for the derivative include the following:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) \tag{7.7.13}$$

The expression $f\,'(x)$ is now a function of $x$; this function gives the slope of the curve $y = f(x)$ at any value of $x$. The derivative of a function $f(x)$ at a point $x = a$ is denoted $f\,'(a)$.

### how to: Given a function $f$, find the derivative by applying the definition of the derivative.

1. Calculate $f(a+h)$.
2. Calculate $f(a)$.
3. Substitute and simplify $\frac{f(a+h)-f(a)}{h}$.
4. Evaluate the limit if it exists: $f\,'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

### Example 7.7.1: Finding the Derivative of a Polynomial Function

Find the derivative of the function $f(x) = x^2 - 3x + 5$ at $x = a$.

**Solution**

We have:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad \text{Definition of a derivative} \tag{7.7.14}$$

Substitute $f(a+h) = (a+h)^2 - 3(a+h) + 5$ and $f(a) = a^2 - 3a + 5$.

$$f'(a) = \lim_{h\to 0} \frac{(a+h)(a+h) - 3(a+h) + 5 - (a^2 - 3a + 5)}{h} \tag{7.7.15}$$

$$= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 3a - 3h + 5 - a^2 + 3a - 5}{h} \qquad \text{Evaluate to remove parentheses.} \tag{7.7.16}$$

$$= \lim_{h\to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{3a} - 3h + \cancel{5} - \cancel{a^2} + \cancel{3a} - \cancel{5}}{h} \qquad \text{Simplify.} \tag{7.7.17}$$

$$= \lim_{h\to 0} \frac{2ah + h^2 - 3h}{h} \qquad \text{Factor out an } h. \tag{7.7.18}$$

$$= 2a + 0 - 3 \qquad \text{Evaluate the limit.} \tag{7.7.19}$$

$$= 2a - 3 \tag{7.7.20}$$

## Exercise 7.7.1

Find the derivative of the function $f(x) = 3x^2 + 7x$ at $x = a$

**Solution**

$f\,'(a) = 6a + 7$

## Finding Derivatives of Rational Functions

To find the derivative of a rational function, we will sometimes simplify the expression using algebraic techniques we have already learned.

### Example 7.7.1: Finding the Derivative of a Rational Function

Find the derivative of the function $f(x) = \dfrac{3+x}{2-x}$ at $x = a$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3+(a+h)}{2-(a+h)} - (\frac{3+a}{2-a})}{h} \qquad \text{Substitute } f(a+h) \text{ and } f(a)$$

$$= \lim_{h\to 0} \frac{(2-(a+h))(2-a)[\frac{3+(a+h)}{2-(a+h)} - (\frac{3+a}{2-a})]}{(2-(a+h))(2-a)(h)} \qquad \text{Multiply numerator and denominator by (}$$

$$= \lim_{h\to 0} \frac{(\cancel{2-(a+h)})(2-a)(\frac{3+(a+h)}{\cancel{(2-(a+h))}}) - (2-(a+h))\,\cancel{(2-a)}\,(\frac{3+a}{\cancel{2-a}})}{(2-(a+h))(2-a)(h)} \qquad \text{Distribute}$$

$$= \lim_{h\to 0} \frac{6 - 3a + 2a - a^2 + 2h - ah - 6 + 3a + 3h - 2a + a^2 + ah}{(2-(a+h))(2-a)(h)} \qquad \text{Multiply}$$

$$= \lim_{h\to 0} \frac{5\cancel{h}}{(2-(a+h))(2-a)(\cancel{h})} \qquad \text{Combine like terms}$$

$$= \lim_{h\to 0} \frac{5}{(2-(a+h))(2-a)} \qquad \text{Cancel like factors}$$

$$= \frac{5}{(2-(a+0))(2-a)} = \frac{5}{(2-a)(2-a)} = \frac{5}{(2-a)^2} \qquad \text{Evaluate the limit}$$

**Exercise 7.7.1:**

Find the derivative of the function $f(x) = \frac{10x+11}{5x+4}$ at $x = a$.

$$f'(a) = \frac{-15}{(5a+4)^2} \tag{7.7.29}$$

## Finding Derivatives of Functions with Roots

To find derivatives of functions with roots, we use the methods we have learned to find limits of functions with roots, including multiplying by a conjugate.

**Example 7.7.1: Finding the Derivative of a Function with a Root**

Find the derivative of the function $f(x) = 4\sqrt{x}$ at $x = 36$.

We have

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{7.7.30}$$

$$= \lim_{h\to 0} \frac{4\sqrt{a+h} - 4\sqrt{a}}{h} \quad \text{Substitute } f(a+h) \text{ and } f(a) \tag{7.7.31}$$

Multiply the numerator and denominator by the conjugate: $\frac{4\sqrt{a+h}+4\sqrt{a}}{4\sqrt{a+h}+4\sqrt{a}}$.

$$f'(a) = \lim_{h\to 0} \left(\frac{4\sqrt{a+h} - 4\sqrt{a}}{h}\right) \cdot \left(\frac{4\sqrt{a+h} + 4\sqrt{a}}{4\sqrt{a+h} + 4\sqrt{a}}\right) \tag{7.7.32}$$

$$= \lim_{h\to 0} \left(\frac{16(a+h) - 16a}{h4(\sqrt{a+h} + 4\sqrt{a})}\right) \quad \text{Multiply.} \tag{7.7.33}$$

$$= \lim_{h\to 0} \left(\frac{\cancel{16a} + 16h \cancel{- 16a}}{h4(\sqrt{a+h} + 4\sqrt{a})}\right) \quad \text{Distribute and combine like terms.} \tag{7.7.34}$$

$$= \lim_{h\to 0} \left(\frac{16\cancel{h}}{\cancel{h}\,(4\sqrt{a+h} + 4\sqrt{a})}\right) \quad \text{Simplify.} \tag{7.7.35}$$

$$= \lim_{h\to 0} \left(\frac{16}{4\sqrt{a+h} + 4\sqrt{a}}\right) \quad \text{Evaluate the limit by letting } h = 0. \tag{7.7.36}$$

$$= \frac{16}{8\sqrt{a}} = \frac{2}{\sqrt{a}} \tag{7.7.37}$$

$$f'(36) = \frac{2}{\sqrt{36}} \quad \text{Evaluate the derivative at } x = 36. \tag{7.7.38}$$

$$= \frac{2}{6} \tag{7.7.39}$$

$$= \frac{1}{3} \tag{7.7.40}$$

**Exercise 7.7.1:**

Find the derivative of the function $f(x) = 9\sqrt{x}$ at $x = 9$.

$\frac{3}{2}$

# Finding Instantaneous Rates of Change

Many applications of the derivative involve determining the rate of change at a given instant of a function with the independent variable time—which is why the term *instantaneous* is used. Consider the height of a ball tossed upward with an initial velocity of 64 feet per second, given by $s(t) = -16t^2 + 64t + 6$, where $t$ is measured in seconds and $s(t)$ is measured in feet. We know the path is that of a parabola. The derivative will tell us how the height is changing at any given point in time. The height of the ball is shown in Figure as a function of time. In physics, we call this the "s-*t* graph."

## Example 7.7.1: Finding the Instantaneous Rate of Change

Using the function above, $s(t) = -16t^2 + 64t + 6$,what is the instantaneous velocity of the ball at 1 second and 3 seconds into its flight?

The velocity at $t = 1$ and $t = 3$ is the instantaneous rate of change of distance per time, or velocity. Notice that the initial height is 6 feet. To find the instantaneous velocity, we find the **derivative** and evaluate it at $t = 1$ and $t = 3$:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{7.7.41}$$

$$= \lim_{h\to 0} \frac{-16(t+h)^2 + 64(t+h) + 6 - (-16t^2 + 64t + 6)}{h} \quad \text{Substitute } s(t+h) \text{ and } s(t). \tag{7.7.42}$$

$$= \lim_{h\to 0} \frac{-16t^2 - 32ht - h^2 + 64t + 64h + 6 + 16t^2 - 64t - 6}{h} \quad \text{Distribute} \tag{7.7.43}$$

$$= \lim_{h\to 0} \frac{-32ht - h^2 + 64h}{h} \quad \text{Simplify.} \tag{7.7.44}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-32t - h + 64)}{\cancel{h}} \quad \text{Factor the numerator.} \tag{7.7.45}$$

$$= \lim_{h\to 0} -32t - h + 64 \quad \text{Cancel out the common factor } h. \tag{7.7.46}$$

$$s'(t) = -32t + 64 \quad \text{Evaluate the limit by letting } h = 0. \tag{7.7.47}$$

For any value of $t$, $s'(t)$ tells us the velocity at that value of $t$.

Evaluate $t = 1$ and $t = 3$.

$$s'(1) = -32(1) + 64 = 32 \tag{7.7.48}$$

$$s'(3) = -32(3) + 64 = -32 \tag{7.7.49}$$

The velocity of the ball after 1 second is 32 feet per second, as it is on the way up.

The velocity of the ball after 3 seconds is –32 feet per second, as it is on the way down.

## Exercise 7.7.1:

The position of the ball is given by$s(t) = -16t^2 + 64t + 6$. What is its velocity 2 seconds into flight?

0

# Using Graphs to Find Instantaneous Rates of Change

We can estimate an instantaneous rate of change at $x = a$ by observing the slope of the curve of the function $f(x)$ at $x = a$. We do this by drawing a line tangent to the function at $x = a$ and finding its slope.

**how to: Given a graph of a function $f(x)$, find the instantaneous rate of change of the function at $x = a$.**

1. Locate $x = a$ on the graph of the function $f(x)$.
2. Draw a tangent line, a line that goes through $x = a$ at $a$ and at no other point in that section of the curve. Extend the line far enough to calculate its slope as

$$\frac{\text{change in } y}{\text{change in } x.} \tag{7.7.50}$$

## Example 7.7.1: Estimating the Derivative at a Point on the Graph of a Function

From the graph of the function $y = f(x)$ presented in Figure, estimate each of the following:

$f(0); f(2); f'(0); f'(2)$

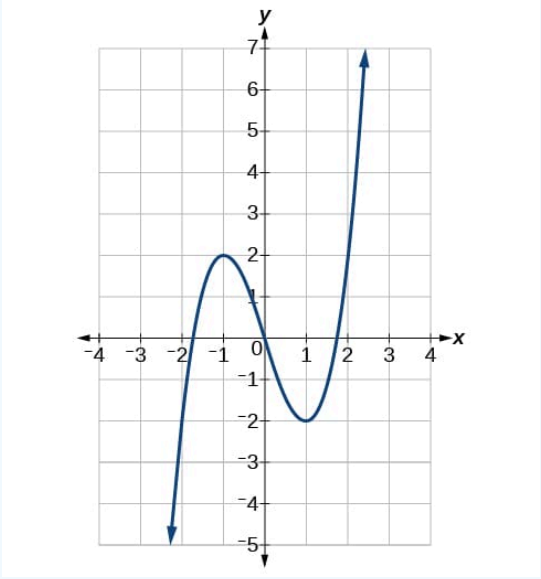
To find the functional value, $f(a)$, find the *y*-coordinate at $x = a$.

To find the **derivative** at $x = a, f\,'(a)$, draw a tangent line at $x = a$, and estimate the slope of that tangent line. See Figure.

- $f(0)$ is the *y*-coordinate at $x = 0$. The point has coordinates $(0, 1)$, thus $f(0) = 1$.
- $f(2)$ is the *y*-coordinate at $x = 2$. The point has coordinates $(2, 1)$, thus $f(2) = 1$.
- $f\,'(0)$ is found by estimating the slope of the tangent line to the curve at $x = 0$. The tangent line to the curve at $x = 0$ appears horizontal. Horizontal lines have a slope of 0, thus $f\,'(0) = 0$.
- $f\,'(2)$ is found by estimating the slope of the tangent line to the curve at $x = 2$. Observe the path of the tangent line to the curve at $x = 2$. As the $x$ value moves one unit to the right, the $y$ value moves up four units to another point on the line. Thus, the slope is 4, so $f\,'(2) = 4$.
-

## Exercise 7.7.1:

Using the graph of the function $f(x) = x^3 - 3x$ shown in Figure, estimate: $f(1), f\,'(1), f(0)$,and $f\,'(0)$.




–2,–2,0, 0, –3

# Using Instantaneous Rates of Change to Solve Real-World Problems

Another way to interpret an instantaneous rate of change at $x = a$ is to observe the function in a real-world context. The unit for the derivative of a function $f(x)$ is

$$\frac{\text{output units}}{\text{input unit}} \tag{7.7.51}$$

Such a unit shows by how many units the output changes for each one-unit change of input. The instantaneous rate of change at a given instant shows the same thing: the units of change of output per one-unit change of input.

One example of an instantaneous rate of change is a marginal cost. For example, suppose the production cost for a company to produce $x$ items is given by $C(x)$, in thousands of dollars. The derivative function tells us how the cost is changing for any value of $x$ in the domain of the function. In other words, $C'(x)$ is interpreted as a **marginal cost**, the additional cost in thousands of dollars of producing one more item when $x$ items have been produced. For example, $C'(11)$ is the approximate additional cost in thousands of dollars of producing the 12$^{\text{th}}$ item after 11 items have been produced. $C'(11) = 2.50$ means that when 11 items have been produced, producing the 12$^{\text{th}}$ item would increase the total cost by approximately $2,500.00.

## Example 7.7.1: Finding a Marginal Cost

The cost in dollars of producing $x$ laptop computers in dollars is $f(x) = x^2 - 100x$. At the point where 200 computers have been produced, what is the approximate cost of producing the 201$^{\text{st}}$unit?

If $f(x) = x^2 - 100x$ describes the cost of producing $x$ computers, $f'(x)$ will describe the marginal cost. We need to find the derivative. For purposes of calculating the derivative, we can use the following functions:

$$f(a+b) = (x+h)^2 - 100(x+h) \tag{7.7.52}$$
$$f(a) = a^2 - 100a \tag{7.7.53}$$

'^' allowed only in math mode

The marginal cost of producing the 201$^{\text{st}}$ unit will be approximately $300.

## Example 7.7.1:Interpreting a Derivative in Context

A car leaves an intersection. The distance it travels in miles is given by the function $f(t)$, where $t$ represents hours. Explain the following notations:

$f(0) = 0 f'(1) = 60 f(1) = 70 f(2.5) = 150$

First we need to evaluate the function $f(t)$ and the derivative of the function $f'(t)$, and distinguish between the two. When we evaluate the function $f(t)$, we are finding the distance the car has traveled in $t$ hours. When we evaluate the derivative f'(t), f′(t), we are finding the speed of the car after $t$ hours.

a. $f(0) = 0$ means that in zero hours, the car has traveled zero miles.
b. $f'(1) = 60$ means that one hour into the trip, the car is traveling 60 miles per hour.
c. $f(1) = 70$ means that one hour into the trip, the car has traveled 70 miles. At some point during the first hour, then, the car must have been traveling faster than it was at the 1-hour mark.
d. $f(2.5) = 150$ means that two hours and thirty minutes into the trip, the car has traveled 150 miles.

## Exercise 7.7.1

A runner runs along a straight east-west road. The function $f(t)$ gives how many feet eastward of her starting point she is after $t$ seconds. Interpret each of the following as it relates to the runner.

$f(0) = 0; f(10) = 150; f\,'(10) = 15; f\,'(20) = -10; f(40) = -100$

a. After zero seconds, she has traveled 0 feet.
b. After 10 seconds, she has traveled 150 feet east.
c. After 10 seconds, she is moving eastward at a rate of 15 ft/sec.
d. After 20 seconds, she is moving westward at a rate of 10 ft/sec.
e. After 40 seconds, she is 100 feet westward of her starting point.

## Finding Points Where a Function's Derivative Does Not Exist

To understand where a function's derivative does not exist, we need to recall what normally happens when a function $f(x)$ has a derivative at $x = a$. Suppose we use a graphing utility to zoom in on $x = a$. If the function $f(x)$ is differentiable, that is, if it is a function that can be differentiated, then the closer one zooms in, the more closely the graph approaches a straight line. This characteristic is called *linearity*.

Look at the graph in Figure. The closer we zoom in on the point, the more linear the curve appears.

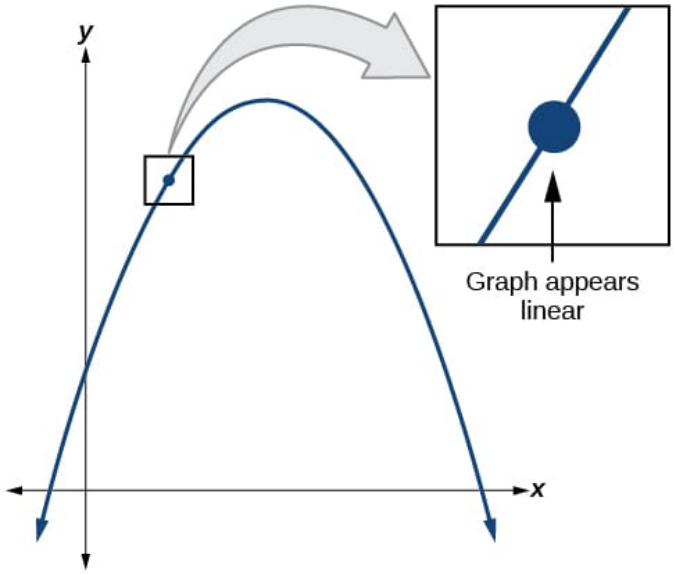



Figure_12_04_009">Figure, the graph does not approach a straight line. No matter how close we zoom in, the graph maintains its sharp corner.

Graph of the function $f(x) = |x|$,with *x*-axis from –0.1 to 0.1 and *y*-axis from –0.1 to 0.1.

What are the characteristics of a graph that is not differentiable at a point? Here are some examples in which function $f(x)$ is not differentiable at $x = a$.

In Figure, we see the graph of

$$f(x) = \begin{cases} x^2, & x \leq 2 \\ 8 - x, & x > 2. \end{cases} \tag{7.7.58}$$

Notice that, as $x$ approaches 2 from the left, the left-hand limit may be observed to be 4, while as $x$ approaches 2 from the right, the right-hand limit may be observed to be 6. We see that it has a discontinuity at $x = 2$.

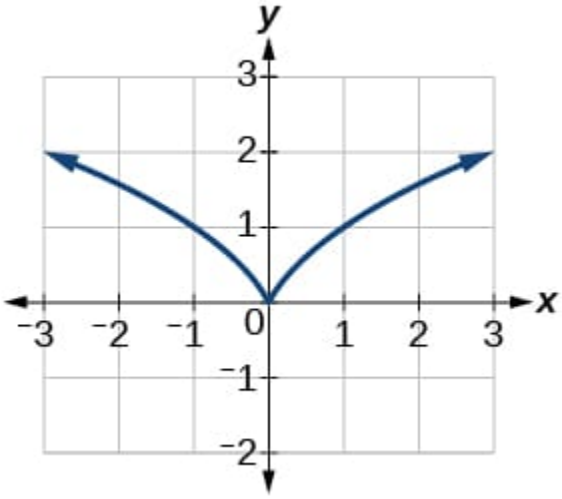
The graph of $f(x)$ has a discontinuity at $x = 2$.

In Figure, we see the graph of $f(x) = |x|$. We see that the graph has a corner point at $x = 0$.

*The graph of $f(x) = |x|$ has a corner point at $x = 0$.*

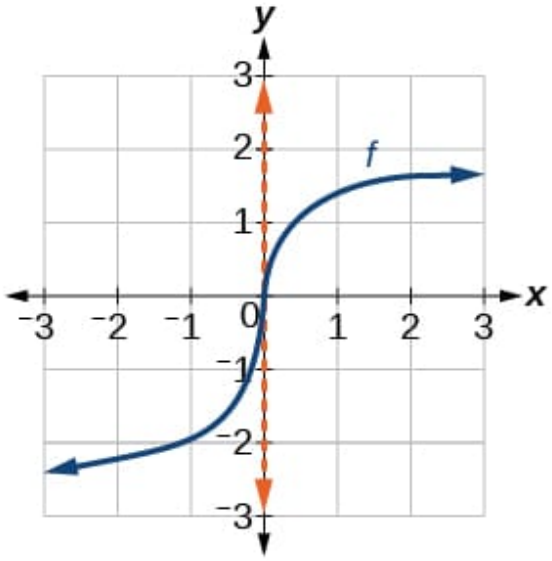
In Figure, we see that the graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$. A cusp has a unique feature. Moving away from the cusp, both

the left-hand and right-hand limits approach either infinity or negative infinity. Notice the tangent lines as $x$ approaches 0 from both the left and the right appear to get increasingly steeper, but one has a negative slope, the other has a positive slope.




*The graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$.*

In Figure, we see that the graph of $f(x) = x^{frac13}$ has a vertical tangent at $x = 0$. Recall that vertical tangents are vertical lines, so where a vertical tangent exists, the slope of the line is undefined. This is why the derivative, which measures the slope, does not exist there.




*The graph of $f(x) = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$.*

## differentiability

A function $f(x)$ is differentiable at $x = a$ if the derivative exists at $x = a$,which means that $f\,'(a)$ exists.

There are four cases for which a function $f(x)$ is not differentiable at a point $x = a$.

1. When there is a discontinuity at $x = a$.
2. When there is a corner point at $x = a$.
3. When there is a cusp at $x = a$.

4. Any other time when there is a vertical tangent at $x = a$.

## Example 7.7.1: Determining Where a Function Is Continuous and Differentiable from a Graph

Using Figure, determine where the function is

a. continuous
b. discontinuous
c. differentiable
d. not differentiable

At the points where the graph is discontinuous or not differentiable, state why.

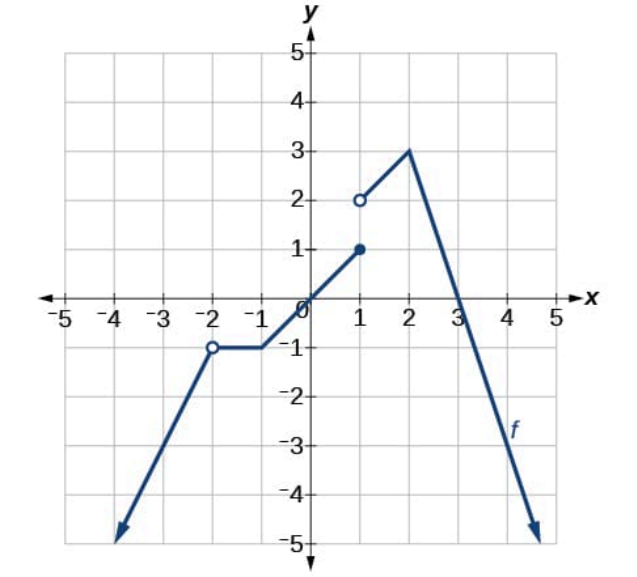



Figure_12_04_016">Figure.

Three intervals where the function is continuous

The graph of is differentiable on $(-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, 2) \cup (2, \infty)$. The graph of $f(x)$ is not differentiable at $x = -2$ because it is a point of discontinuity, at $x = -1$ because of a sharp corner, at $x = 1$ because it is a point of discontinuity, and at $x = 2$ because of a sharp corner. See Figure.

Five intervals where the function is differentiable

## Exercise 7.7.1:

Determine where the function $y = f(x)$ shown in Figure is continuous and differentiable from the graph.

The graph of $f$ is continuous on $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$. The graph of f f is discontinuous at $x = 1$ and $x = 3$. The graph of $f$ is differentiable on $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$. The graph of $f$ is not differentiable at $x = 1$ and $x = 3$.

# Finding an Equation of a Line Tangent to the Graph of a Function

The equation of a tangent line to a curve of the function $f(x)$ at $x = a$ is derived from the point-slope form of a line, $y = m(x - x_1) + y_1$. The slope of the line is the slope of the curve at $x = a$ and is therefore equal to $f\,'(a)$, the derivative of $f(x)$ at $x = a$. The coordinate pair of the point on the line at $x = a$ is $(a, f(a))$.

If we substitute into the point-slope form, we have

The equation of the tangent line is

$$y = f'(a)(x - a) + f(a) \tag{7.7.59}$$

**F**

The equation of a line tangent to the curve of a function $f$ at a point $x = a$ is

$$y = f'(a)(x - a) + f(a) \tag{7.7.60}$$

**how to: Given a function $f$, find the equation of a line tangent to the function at $x = a$.**

1. Find the derivative of $f(x)$ at $x = a$ using $f\,'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.
2. Evaluate the function at $x = a$. This is $f(a)$.
3. Substitute $(a, f(a))$ and $f\,'(a)$ into $y = f'(a)(x - a) + f(a)$.
4. Write the equation of the tangent line in the form $y = mx + b$.

**Example 7.7.1: Finding the Equation of a Line Tangent to a Function at a Point**

Find the equation of a line tangent to the curve $f(x) = x^2 - 4x$ at $x = 3$.

Using:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{7.7.61}$$

Substitute $f(a + h) = (a + h)^2 - 4(a + h)$ and $f(a) = a^2 - 4a$.

$$f'(a) = \lim_{h \to 0} \frac{(a + h)(a + h) - 4(a + h) - (a2 - 4a)}{h} \tag{7.7.62}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 4a - 4h - a^2 + 4a}{h} \quad \text{Remove parentheses.} \tag{7.7.63}$$

$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{4a} - 4h - \cancel{a^2} + \cancel{4a}}{h} \quad \text{Combine like terms.} \tag{7.7.64}$$

$$= \lim_{h \to 0} \frac{2ah + h^2 - 4h}{h} \tag{7.7.65}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2a + h - 4)}{h} \quad \text{Factor out } h. \tag{7.7.66}$$

$$= 2a + 0 - 4 \tag{7.7.67}$$

$$f'(a) = 2a - 4 \quad \text{Evaluate the limit.} \tag{7.7.68}$$

$$f'(3) = 2(3) - 4 = 2 \tag{7.7.69}$$

Equation of tangent line at $x = 3$:

$$y = f'(a)(x - a) + f(a) \tag{7.7.70}$$

$$y = f'(3)(x - 3) + f(3) \tag{7.7.71}$$

$$y = 2(x - 3) + (-3) \tag{7.7.72}$$

$$y = 2x - 9 \tag{7.7.73}$$

Analysis

We can use a graphing utility to graph the function and the tangent line. In so doing, we can observe the point of tangency at $x = 3$ as shown in Figure.

Graph confirms the point of tangency at $x = 3$.

**Exercise 7.7.1:**

Find the equation of a tangent line to the curve of the function $f(x) = 5x^2 - x + 4$ at $x = 2$.

$y = 19x - 16$

## Finding the Instantaneous Speed of a Particle

If a function measures position versus time, the derivative measures displacement versus time, or the speed of the object. A change in speed or direction relative to a change in time is known as ***velocity***. The velocity at a given instant is known as **instantaneous velocity**.

In trying to find the speed or velocity of an object at a given instant, we seem to encounter a contradiction. We normally define speed as the distance traveled divided by the elapsed time. But in an instant, no distance is traveled, and no time elapses. How will we divide zero by zero? The use of a derivative solves this problem. A derivative allows us to say that even while the object's velocity is constantly changing, it has a certain velocity at a given instant. That means that if the object traveled at that exact velocity for a unit of time, it would travel the specified distance.

**INSTANTANEOUS VELOCITY**

Let the function $s(t)$ represent the position of an object at time $t$. The **instantaneous velocity** or velocity of the object at time $t = a$ is given by

$$s'(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} \tag{7.7.74}$$

**Example 7.7.1: Finding the Instantaneous Velocity**

A ball is tossed upward from a height of 200 feet with an initial velocity of 36 ft/sec. If the height of the ball in feet after $t$ seconds is given by $s(t) = -16t^2 + 36t + 200$, find the instantaneous velocity of the ball at $t = 2$.

First, we must find the derivative $s\,'(t)$. Then we evaluate the derivative at $t = 2$, using $s(a+h) = -16(a+h)^2 + 36(a+h) + 200$ and $s(a) = -16a^2 + 36a + 200$.

$$s'(a) = \lim_{h\to 0} \frac{s(a+h)-s(a)}{h} \tag{7.7.75}$$
$$= \lim_{h\to 0} \frac{-16(a+h)^2+36(a+h)+200-(-16a^2+36a+200)}{h} \tag{7.7.76}$$
$$= \lim_{h\to 0} \frac{-16(a^2+2ah+h^2)+36(a+h)+200-(-16a^2+36a+200)}{h} \tag{7.7.77}$$
$$= \lim_{h\to 0} \frac{-16a^2-32ah-16h^2+36a+36h+200+16a^2-36a-200}{h} \tag{7.7.78}$$
$$= \lim_{h\to 0} \frac{\cancel{-16a^2}-32ah-16h^2+\cancel{36a}+36h+\cancel{200}+\cancel{16a^2}-\cancel{36a}-\cancel{200}}{h} \tag{7.7.79}$$
$$= \lim_{h\to 0} \frac{-32ah-16h^2+36h}{h} \tag{7.7.80}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(-32a-16h+36)}{\cancel{h}} \tag{7.7.81}$$
$$= \lim_{h\to 0}(-32a-16h+36) \tag{7.7.82}$$
$$= -32a-16\cdot 0+36 \tag{7.7.83}$$
$$s'(a) = -32a+36 \tag{7.7.84}$$
$$s'(2) = -32(2)+36 \tag{7.7.85}$$
$$= -28 \tag{7.7.86}$$

Analysis

This result means that at time $t = 2$ seconds, the ball is dropping at a rate of 28 ft/sec.

**Exercise 7.7.1:**

A fireworks rocket is shot upward out of a pit 12 ft below the ground at a velocity of 60 ft/sec. Its height in feet after $t$ seconds is given by $s = -16t^2 + 60t - 12$. What is its instantaneous velocity after 4 seconds?

-68 ft/sec, it is dropping back to Earth at a rate of 68 ft/s.

Access these online resources for additional instruction and practice with derivatives.

- Estimate the Derivative
- Estimate the Derivative Ex. 4

Visit this website for additional practice questions from Learningpod.

## Key Equations

| | |
|---|---|
| average rate of change | $\text{AROC} = \frac{f(a+h)-f(a)}{h}$ |
| derivative of a function | $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ |

## Key Concepts

- The slope of the secant line connecting two points is the average rate of change of the function between those points. See Example.
- The derivative, or instantaneous rate of change, is a measure of the slope of the curve of a function at a given point, or the slope of the line tangent to the curve at that point. See Example, Example, and Example.
- The difference quotient is the quotient in the formula for the instantaneous rate of change:

$\frac{f(a+h)-f(a)}{h}$

- Instantaneous rates of change can be used to find solutions to many real-world problems. See Example.
- The instantaneous rate of change can be found by observing the slope of a function at a point on a graph by drawing a line tangent to the function at that point. See Example.
- Instantaneous rates of change can be interpreted to describe real-world situations. See Example and Example.
- Some functions are not differentiable at a point or points. See Example.
- The point-slope form of a line can be used to find the equation of a line tangent to the curve of a function. See Example.
- Velocity is a change in position relative to time. Instantaneous velocity describes the velocity of an object at a given instant. Average velocity describes the velocity maintained over an interval of time.
- Using the derivative makes it possible to calculate instantaneous velocity even though there is no elapsed time. See Example.

# Section Exercises

## Verbal

How is the slope of a linear function similar to the derivative?

The slope of a linear function stays the same. The derivative of a general function varies according to $x$. Both the slope of a line and the derivative at a point measure the rate of change of the function.

What is the difference between the average rate of change of a function on the interval $[x, x+h]$ and the derivative of the function at $x$?

A car traveled 110 miles during the time period from 2:00 P.M. to 4:00 P.M. What was the car's average velocity? At exactly 2:30 P.M., the speed of the car registered exactly 62 miles per hour. What is another name for the speed of the car at 2:30 P.M.? Why does this speed differ from the average velocity?

Average velocity is 55 miles per hour. The instantaneous velocity at 2:30 p.m. is 62 miles per hour. The instantaneous velocity measures the velocity of the car at an instant of time whereas the average velocity gives the velocity of the car over an interval.

Explain the concept of the slope of a curve at point $x$.

Suppose water is flowing into a tank at an average rate of 45 gallons per minute. Translate this statement into the language of mathematics.

The average rate of change of the amount of water in the tank is 45 gallons per minute. If $f(x)$ is the function giving the amount of water in the tank at any time $t$,then the average rate of change of $f(x)$ between $t=a$ and $t=b$ is $f(a)+45(b-a)$.

## Algebraic

For the following exercises, use the definition of derivative $\lim_{hto0} \frac{f(x+h)-f(x)}{h}$ to calculate the derivative of each function.

$f(x)=3x-4$

$f(x)=-2x+1$

$f\,'(x)=-2$

$f(x)=x^2-2x+1$

\(f(x)=2x^2+x−3

$f\,'(x)=4x+1$

$f(x)=2x^2+5$

$f(x)=\frac{-1}{x-2}$

$f\,'(x) = \frac{1}{(x-2)^2}$

$f(x) = \frac{2+x}{1-x}$

$f(x) = \frac{5-2x}{3+2x}$

$\frac{-16(}{3+2x)^2}$

$f(x) = \sqrt{1+3x}$

$f(x) = 3x^3 - x^2 + 2x + 5$

$f\,'(x) = 9x^2 - 2x + 2$

$f(x) = 5$

$f(x) = 5\pi$

$f\,'(x) = 0$

For the following exercises, find the average rate of change between the two points.

$(-2, 0)$ and $(-4, 5)$

$(4, -3)$ and $(-2, -1)$

$-\frac{1}{3}$

$(0, 5)$ and $(6, 5)$

$(7, -2)$ and $(7, 10)$

undefined

For the following polynomial functions, find the derivatives.

$f(x) = x^3 + 1$

$f(x) = -3x^2 - 7x = 6$

$f\,'(x) = -6x - 7$

$f(x) = 7x^2$

$f(x) = 3x^3 + 2x^2 + x - 26$

$f\,'(x) = 9x^2 + 4x + 1$

For the following functions, find the equation of the tangent line to the curve at the given point $x$ on the curve.

$f(x) = 2x^2 - 3x \qquad x = 3$

$f(x) = x^3 + 1 \qquad x = 2$

$y = 12x - 15$

$f(x) = \sqrt{x} \qquad x = 9$

For the following exercise, find $k$ such that the given line is tangent to the graph of the function.

$f(x) = x^2 - kx, \quad y = 4x - 9$

$k = -10$ or $k = 2$

## Graphical

For the following exercises, consider the graph of the function $f$ and determine where the function is continuous/discontinuous and differentiable/not differentiable.

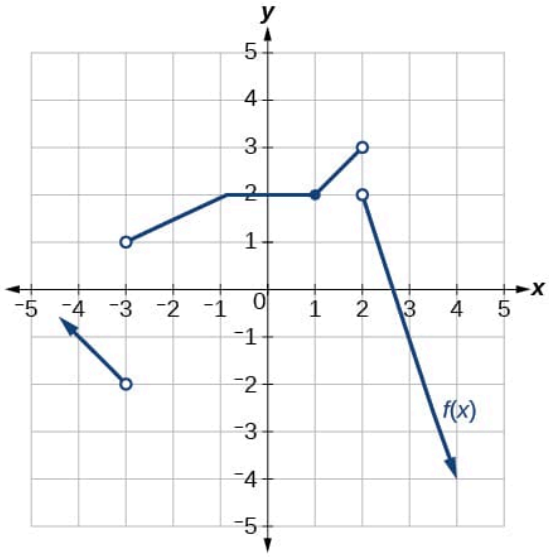



Figure_12_04_202.jpg" style="width: 487px; height: 457px;" width="487px" height="457px" src="/@api/deki/files/7709/CNX_Precalc_Figure_12_04_202.jpg" />

Discontinuous at $x = -2$ and $x = 0$. Not differentiable at –2, 0, 2.

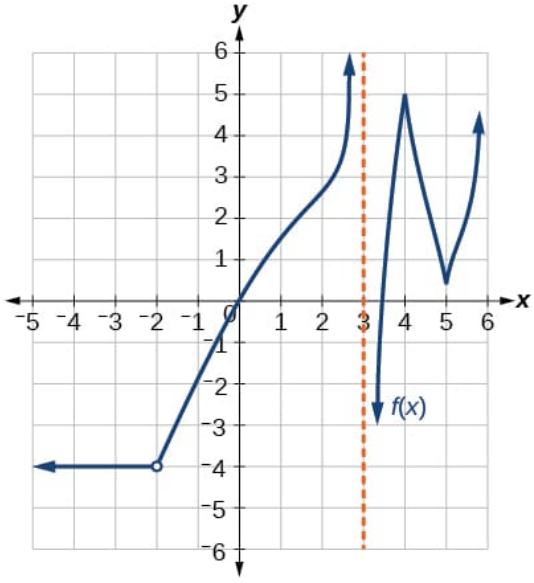
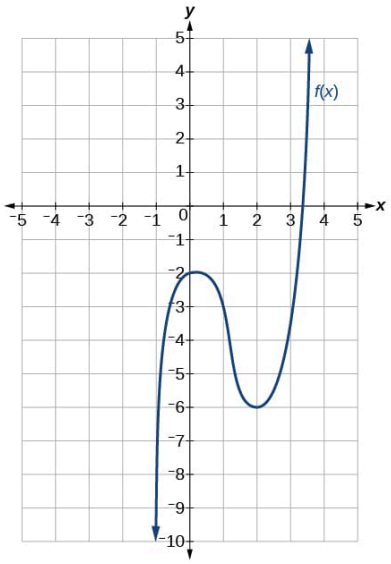



Figure_12_04_204.jpg" style="width: 402px; height: 374px;" width="402px" height="374px" src="/@api/deki/files/7711/CNX_Precalc_Figure_12_04_204.jpg" />

Discontinuous at $x = 5$. Not differentiable at -4, –2, 0, 1, 3, 4, 5.

For the following exercises, use Figure to estimate either the function at a given value of $x$ or the derivative at a given value of $x$, as indicated.




$f(-1)$

$f(0)$

$f(0) = -2$

$f(1)$

$f(2)$

\(f(2)=−6f(2)=−6

$f(3)$

$f\,'(-1)$

$f\,'(-1) = 9$

$f\,'(0)$

$f\,'(1)$

$f\,'(1) = -3$

$f\,'(2)$

$f\,'(3)$

$f\,'(3) = 9$

Sketch the function based on the information below:

$f\,'(x) = 2x, f(2) = 4$

## Technology

Numerically evaluate the derivative. Explore the behavior of the graph of $f(x) = x^2$ around $x = 1$ by graphing the function on the following domains: $[0.9, 1.1], [0.99, 1.01], [0.999, 1.001],$ and $[0.9999, 1.0001]$. We can use the feature on our calculator that automatically sets Ymin and Ymax to the Xmin and Xmax values we preset. (On some of the commonly used graphing calculators,

this feature may be called ZOOM FIT or ZOOM AUTO). By examining the corresponding range values for this viewing window, approximate how the curve changes at $x = 1$, that is, approximate the derivative at $x = 1$.

Answers vary. The slope of the tangent line near $x = 1$ is 2.

## Real-World Applications

For the following exercises, explain the notation in words. The volume $f(t)$ of a tank of gasoline, in gallons, $t$ minutes after noon.

$f(0) = 600$

$f'(30) = -20$

At 12:30 p.m., the rate of change of the number of gallons in the tank is –20 gallons per minute. That is, the tank is losing 20 gallons per minute.

$f(30) = 0$

$f'(200) = 30$

At 200 minutes after noon, the volume of gallons in the tank is changing at the rate of 30 gallons per minute.

$f(240) = 500$

For the following exercises, explain the functions in words. The height, $s$, of a projectile after $t$ seconds is given by $s(t) = -16t^2 + 80t$.

$s(2) = 96$

The height of the projectile after 2 seconds is 96 feet.

$s'(2) = 16$

$s(3) = 96$

The height of the projectile at $t = 3$ seconds is 96 feet.

$s'(3) = -16$

$s(0) = 0, s(5) = 0.$

The height of the projectile is zero at $t = 0$ and again at $t = 5$. In other words, the projectile starts on the ground and falls to earth again after 5 seconds.

For the following exercises, the volume $V$ of a sphere with respect to its radius $r$ is given by $V = \frac{4}{3}\pi r^3$.

Find the average rate of change of $V$ as $r$ changes from 1 cm to 2 cm.

Find the instantaneous rate of change of $V$ when r=3 cm. r=3 cm.

$36\pi$

For the following exercises, the revenue generated by selling $x$ items is given by $R(x) = 2x^2 + 10x$.

Find the average change of the revenue function as $x$ changes from $x = 10$ to $x = 20$.

Find $R'(10)$ and interpret.

$50.00 per unit, which is the instantaneous rate of change of revenue when exactly 10 units are sold.

Find $R'(15)$ and interpret. Compare $R'(15)$ to $R'(10)$, and explain the difference.

For the following exercises, the cost of producing $x$ cellphones is described by the function $C(x) = x^2 - 4x + 1000$.

Find the average rate of change in the total cost as $x$ changes from $x = 10$ to $x = 15$.

$21 per unit

Find the approximate marginal cost, when 15 cellphones have been produced, of producing the 16th cellphone.

Find the approximate marginal cost, when 20 cellphones have been produced, of producing the 21st cellphone.

$36

## Extension

For the following exercises, use the definition for the derivative at a point $x = a$, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, to find the derivative of the functions.

$f(x) = \frac{1}{x^2}$

$f(x) = 5x^2 - x + 4$

$f'(x) = 10a - 1$

$f(x) = -x^2 + 4x + 7$

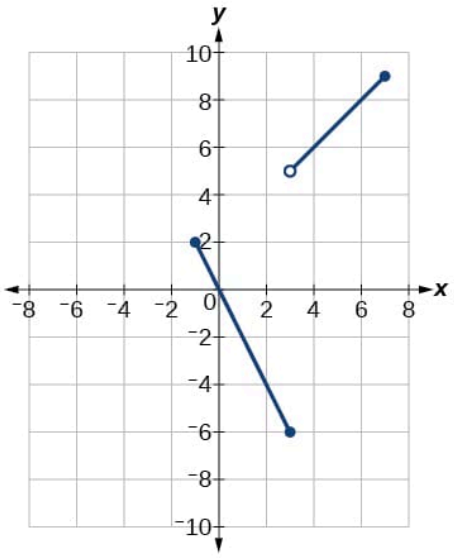
$f(x) = \frac{-4}{3 - x^2}$

$\frac{4}{(3-x)^2}$

# Chapter Review Exercises

## Finding Limits: A Numerical and Graphical Approach

For the following exercises, use Figure.



$\lim_{x \to -1^+} f(x)$

2

$\lim_{x \to -1^-} f(x)$

$\lim_{x \to -1} f(x)$

does not exist

$\lim_{x \to 3} f(x)$

At what values of $x$ is the function discontinuous? What condition of continuity is violated?

Discontinuous at $x = -1$ ($\lim_{x \to a} f(x)$ does not exist), $x = 3$ (jump discontinuity),and $x = 7$ ((\lim \limits_{x \to a} f(x)\) does not exist).

Using Table, estimate $\lim_{x \to 0} f(x)$.

| $x$ | $F(x)$ |
|---|---|
| −0.1 | 2.875 |
| −0.01 | 2.92 |
| −0.001 | 2.998 |
| 0 | Undefined |
| 0.001 | 2.9987 |
| 0.01 | 2.865 |
| 0.1 | 2.78145 |
| 0.15 | 2.678 |

3

For the following exercises, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as $x$ approaches $a$. If the function has limit as $x$ approaches $a$, state it. If not, discuss why there is no limit.

$f(x) = \begin{cases} |x| - 1, & \text{if } x \neq 1 \\ x^3, & \text{if } x = 1 \end{cases} \quad a = 1$

$f(x) = \begin{cases} \frac{1}{x+1}, & \text{if } x = -2 \\ (x+1)^2, & \text{if } x \neq -2 \end{cases} \quad a = -2$

$\lim_{x \to -2} f(x) = 1$

$f(x) = \begin{cases} \sqrt{x+3} & \text{if } x < 1 \\ -\sqrt[3]{x} & \text{if } x > 1 \end{cases} \quad a = 1$

## Finding Limits: Properties of Limits

For the following exercises, find the limits if $\lim_{x \to c} f(x) = -3$ and $\lim_{x \to c} g(x) = 5$.

$\lim_{x \to c}(f(x) + g(x))$

2

$\lim_{x \to c} \frac{f(x)}{g(x)}$

$\lim_{xtoc}(f(x) \cdot g(x))$

$-15$

$\lim_{x \to 0^+} f(x), f(x) = \begin{cases} 3x^2 + 2x + 1 & x > 0 \\ 5x + 3 & x < 0 \end{cases}$

$\lim_{x \to 0^-} f(x), f(x) = \begin{cases} 3x^2 + 2x + 1 & x > 0 \\ 5x + 3 & x < 0 \end{cases}$

3

$\lim_{x \to 3^+} (3x - [\![ \ x]\!] \ )$

For the following exercises, evaluate the limits using algebraic techniques.

$\lim_{h \to 0}(\frac{(h+6)^2-36}{h})$

12

$\lim_{x \to 25}(\frac{x^2-625}{\sqrt{x}-5})$

$\lim_{x \to 1}(\frac{-x^2-9x}{x})$

–10

$\lim_{x \to 4} \frac{7-\sqrt{12x+1}}{x-4}$

$\lim_{x \to -3}(\frac{\frac{1}{3}+\frac{1}{x}}{3+x})$

$-\frac{1}{9}$

## Continuity

For the following exercises, use numerical evidence to determine whether the limit exists at $x = a$. If not, describe the behavior of the graph of the function at $x = a$.

$f(x) = \frac{-2}{x-4}; a = 4$

$f(x) = \frac{-2}{(x-4)^2}; a = 4$

At $x = 4$,the function has a vertical asymptote.

$f(x) = \frac{-x}{x^2-x-6}; a = 3$

$f(x) = \frac{6x^2+23x+20}{4x^2-25}; a = -\frac{5}{2}$

removable discontinuity at $a = -\frac{5}{2}$

$f(x) = \frac{\sqrt{x}-3}{9-x}; a = 9$

For the following exercises, determine where the given function $f(x)$ is continuous. Where it is not continuous, state which conditions fail, and classify any discontinuities.

$f(x) = x^2 - 2x - 15$

continuous on $(-\infty, \infty)$

$f(x) = \frac{x^2-2x-15}{x-5}$

$f(x) = \frac{x^2-2x}{x^2-4x+4}$

removable discontinuity at $x = 2. f(2)$ is not defined, but limits exist.

$f(x) = \frac{x^3-125}{2x^2-12x+10}$

$f(x) = \frac{x^2-\frac{1}{x}}{2-x}$

discontinuity at $x = 0$ and $x = 2$. Both $f(0)$ and $f(2)$ are not defined.

$f(x) = \frac{x+2}{x^2-3x-10}$

$f(x) = \frac{x+2}{x^3+8}$

removable discontinuity at $x = -2. f(-2)$ is not defined.

## Derivatives

For the following exercises, find the average rate of change $\frac{f(x+h)-f(x)}{h}$.

$f(x) = 3x + 2$

$f(x) = 5$

0

$f(x) = \frac{1}{x+1}$

$f(x) = \ln(x)$

$\frac{\ln(x+h)-\ln(x)}{h}$

$f(x) = e^{2x}$

For the following exercises, find the derivative of the function.

$f(x) = 4x - 6$

$= 4$

$f(x) = 5x^2 - 3x$

Find the equation of the tangent line to the graph of $f(x)$ at the indicated $x$ value.

$f(x) = -x^3 + 4x; x = 2.$

$y = -8x + 16$

For the following exercises, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

$f(x) = \frac{x}{|x|}$

Given that the volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$ and that a given cone has a fixed height of 9 cm and variable radius length, find the instantaneous rate of change of volume with respect to radius length when the radius is 2 cm. Give an exact

answer in terms of $\pi$

$12\pi$

# Practice Test

For the following exercises, use the graph of $f$ in Figure.

$f(1)$

3

$\lim_{x\to -1^+} f(x)$

$\lim_{x\to -1^-} f(x)$

0

$\lim_{x\to -1} f(x)$

$\lim_{x\to -2} f(x)$

$-1$

At what values of $x$ is $f$ discontinuous? What property of continuity is violated?

For the following exercises, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as $x$ approaches $a$. If the function has a limit as $x$ approaches $a$,state it. If not, discuss why there is no limit

$f(x)=\begin{cases}\frac{1}{x}-3, & \text{if } x\le 2\\ x^3+1, & \text{if } x>2\end{cases}\ a=2$

$\lim_{x\to 2^-} f(x)=-\frac{5}{2}a$ and $\lim_{x\to 2^+} f(x)=9$ Thus, the limit of the function as $x$ approaches 2 does not exist.

$f(x)=\begin{cases}x^3+1, & \text{if } x<1\\ 3x^2-1, & \text{if } x=1\\ -\sqrt{x+3}+4, & \text{if } x>1\end{cases}\ a=1$

For the following exercises, evaluate each limit using algebraic techniques.

$\lim_{x\to -5}\left(\frac{\frac{1}{5}+\frac{1}{x}}{10+2x}\right)$

$-\frac{1}{50}$

$\lim_{h\to 0}\left(\frac{\sqrt{h^2+25}-5}{h^2}\right)$

$\lim_{h\to 0}\left(\frac{1}{h}-\frac{1}{h^2+h}\right)$

1

For the following exercises, determine whether or not the given function $f$ is continuous. If it is continuous, show why. If it is not continuous, state which conditions fail.

$f(x)=\sqrt{x^2-4}$

$f(x) = \frac{x^3-4x^2-9x+36}{x^3-3x^2+2x-6}$

removable discontinuity at $x = 3$

For the following exercises, use the definition of a derivative to find the derivative of the given function at $x = a$.

$f(x) = \frac{3}{5+2x}$

$f(x) = \frac{3}{\sqrt{x}}$

$f'(x) = -\frac{3}{2a^{\frac{3}{2}}}$

$f(x) = 2x^2 + 9x$

discontinuous at –2,0, not differentiable at –2,0, 2.

For the following exercises, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

$f(x) = |x-2| - |x+2|$

$f(x) = \frac{2}{1+e^{\frac{2}{x}}}$

not differentiable at $x = 0$ (no limit)

For the following exercises, explain the notation in words when the height of a projectile in feet, $s$, is a function of time t t in seconds after launch and is given by the function $s(t)$.

$s(0)$

$s(2)$

the height of the projectile at $t = 2$ seconds

$s'(2)$

$\frac{s(2)-s(1)}{2-1}$

the average velocity from $t = 1$ to $t = 2$

$s(t) = 0$

For the following exercises, use technology to evaluate the limit.

$\lim_{x\to 0} \frac{\sin(x)}{3x}$

$\frac{1}{3}$

$\lim_{x\to 0} \frac{\tan^2(x)}{2x}$

$\lim_{x\to 0} \frac{\sin(x)(1-\cos(x))}{2x^2}$

0

Evaluate the limit by hand.

$\lim_{x\to 1} f(x), \text{where } f(x) = \begin{cases} 4x-7 & x \neq 1 \\ x^2-4 & x = 1 \end{cases}$

At what value(s) of $x$ is the function below discontinuous?

$f(x) = \begin{cases} 4x - 7 & x \neq 1 \\ x^2 - 4 & x = 1 \end{cases}$

For the following exercises, consider the function whose graph appears in Figure.

Find the average rate of change of the function from $x = 1$ to $x = 3$.

2

Find all values of $x$ at which $f'(x) = 0$.

$x = 1$

Find all values of $x$ at which $f'(x)$ does not exist.

Find an equation of the tangent line to the graph of $f$ the indicated point: $f(x) = 3x^2 - 2x - 6, x = -2$

$y = -14x - 18$

For the following exercises, use the function $f(x) = x(1-x)^{\frac{2}{5}}$.

Graph the function $f(x) = x(1-x)^{\frac{2}{5}}$ by entering $f(x) = x((1-x)^2)^{\frac{1}{5}}$ and then by entering $f(x) = x((1-x)^{\frac{1}{5}})^2$.

Explore the behavior of the graph of $f(x)$ around $x = 1$ by graphing the function on the following domains, [0.9, 1.1], [0.99, 1.01], [0.999, 1.001], and [0.9999, 1.0001]. Use this information to determine whether the function appears to be differentiable at $x = 1$ .

The graph is not differentiable at $x = 1$ (cusp).

For the following exercises, find the derivative of each of the functions using the definition: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x) = 2x - 8$

$f(x) = 4x^2 - 7$

$f\,'(x) = 8x$

$f(x) = x - \frac{1}{2}x^2$

$f(x) = \frac{1}{x+2}$

$f'(x) = -\frac{1}{(2+x)^2}$

$f(x) = \frac{3}{x-1}$

$f(x) = -x^3 + 1$

$f\,'(x) = -3^x 2$

$f(x) = x^2 + x^3$

$f(x) = \sqrt{x-1}$

$f'(x) = \frac{1}{2\sqrt{x-1}}$

## Footnotes

- 1 www.csun.edu/science/health/d...tv&health.html Source provided.

## Glossary

**average rate of change**
the slope of the line connecting the two points $(a, f(a))$ and $(a+h, f(a+h))$ on the curve of $f(x)$; it is given by

$$\text{AROC} = \frac{f(a+h)-f(a)}{h}. \tag{7.7.87}$$

**derivative**
the slope of a function at a given point; denoted $f\,'(a)$,at a point $x=a$ it is $f\,'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$,providing the limit exists.

**differentiable**
a function $f(x)$ for which the derivative exists at $x=a$. In other words, if f′(a) f′(a) exists.

**instantaneous rate of change**
the slope of a function at a given point; at $x=a$ it is given by $f\,'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

**instantaneous velocity**
the change in speed or direction at a given instant; a function $s(t)$ represents the position of an object at time $t$ ,and the instantaneous velocity or velocity of the object at time $t=a$ is given by $s\,'(a)=\lim_{h\to 0}\frac{s(a+h)-s(a)}{h}$.

**secant line**
a line that intersects two points on a curve

**tangent line**
a line that intersects a curve at a single point

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# 7.E: Introduction to Calculus (Exercises)

## 12.1: Finding Limits - Numerical and Graphical Approaches

In this section, we will examine numerical and graphical approaches to identifying limits.

### Verbal

1) Explain the difference between a value at $x = a$ and the limit as $x$ approaches $a$.

**Answer**

The value of the function, the output, at $x = a$ is $f(a)$. When the $\lim_{x \to a} f(x)$ is taken, the values of $x$ get infinitely close to $a$ but never equal $a$. As the values of $x$ approach $a$ from the left and right, the limit is the value that the function is approaching.

2) Explain why we say a function does not have a limit as $x$ approaches $a$ if, as $x$ approaches $a$, the left-hand limit is not equal to the right-hand limit.

### Graphical

**For the exercises 3-14, estimate the functional values and the limits from the graph of the function $f$ provided in the Figure below.**

3) $\lim_{x \to -2^-} f(x)$

**Answer**

$-4$

4) $\lim_{x \to -2^+} f(x)$

5) $\lim_{x \to -2f(x)}$

**Answer**

$-4$

6) $f(-2)$

7) $\lim_{x \to -1^- f(x)}$

**Answer**

$2$

8) $\lim_{x \to 1^+} f(x)$

9) $\lim_{x \to 1} f(x)$

**Answer**

does not exist

10) $f(1)$

11) $\lim_{x \to 4^-} f(x)$

**Answer**

4

12) $\lim_{x \to 4^+} f(x)$

13) $\lim_{x \to 4} f(x)$

**Answer**

does not exist

14) $f(4)$

**For the exercises 15-21, draw the graph of a function from the functional values and limits provided.**

15) $\lim_{x \to 0^-} f(x) = 2, \lim_{x \to 0^+} f(x) = -3, \lim_{x \to 2} f(x) = 2, f(0) = 4, f(2) = -1, f(-3)$ does not exist.

**Answer**

Answers will vary.

16) $\lim_{x \to 2^-} f(x) = 0, \lim_{x \to 2^+} = -2, \lim_{x \to 0} f(x) = 3, f(2) = 5, f(0)$

**Answer**

Answers will vary.

17) $\lim_{x \to 2^-} f(x) = 2, \lim_{x \to 2^+} f(x) = -3, \lim_{x \to 0} f(x) = 5, f(0) = 1, f(1) = 0$

**Answer**

Answers will vary.

18) $\lim_{x \to 3^-} f(x) = 0, \lim_{x \to 3^+} f(x) = 5, \lim_{x \to 5} f(x) = 0, f(5) = 4, f(3)$ does not exist.

**Answer**

Answers will vary.

19) $\lim_{x \to 4} f(x) = 6, \lim_{x \to 6^+} f(x) = -1, \lim_{x \to 0} f(x) = 5, f(4) = 6, f(2) = 6$

**Answer**

Answers will vary.

20) $\lim_{x \to -3} f(x) = 2, \lim_{x \to 1^+} f(x) = -2, \lim_{x \to 3} f(x) = -4, f(-3) = 0, f(0) = 0$

**Answer**

Answers will vary.

21) $\lim_{x\to\pi} f(x) = \pi^2$, $\lim_{x\to-\pi} f(x) = \frac{\pi}{2}$, $\lim_{x\to1^-} f(x) = 0, f(\pi) = \sqrt{2}, f(0)$ does not exist.

**Answer**

Answers will vary.

**For the exercises 22-26, use a graphing calculator to determine the limit to $5$ decimal places as $x$ approaches $0$.**

22) $f(x) = (1+x)^{\frac{1}{x}}$

23) $g(x) = (1+x)^{\frac{2}{x}}$

**Answer**

$7.38906$

24) $h(x) = (1+x)^{\frac{3}{x}}$

25) $i(x) = (1+x)^{\frac{4}{x}}$

**Answer**

$54.59815$

26) $j(x) = (1+x)^{\frac{5}{x}}$

27) Based on the pattern you observed in the exercises above, make a conjecture as to the limit of $f(x) = (1+x)^{\frac{6}{x}}, g(x) = (1+x)^{\frac{7}{x}}$, and $h(x) = (1+x)^{\frac{n}{x}}$.

**Answer**

$e^6 \approx 403.428794, e^7 \approx 1096.633158, e^n$

**For the exercises 28-29, use a graphing utility to find graphical evidence to determine the left- and right-hand limits of the function given as $x$ approaches $a$. If the function has a limit as $x$ approaches $a$,state it. If not, discuss why there is no limit.**

28) $(x) = \begin{cases} |x| - 1, & \text{if } x \neq 1 \\ x^3, & \text{if } x = 1 \end{cases} \quad a = 1$

29) $(x) = \begin{cases} \frac{1}{x+1}, & \text{if } x = -2 \\ (x+1)^2, & \text{if } x \neq -2 \end{cases} \quad a = -2$

**Answer**

$\lim_{x\to-2} f(x) = 1$

## Numeric

**For the exercises 30-38, use numerical evidence to determine whether the limit exists at $x = a$. If not, describe the behavior of the graph of the function near $x = a$. Round answers to two decimal places.**

30) $f(x) = \dfrac{x^2 - 4x}{16 - x^2}; a = 4$

31) $f(x) = \dfrac{x^2 - x - 6}{x^2 - 9}; a = 3$

**Answer**

$$\lim_{x \to 3} \left( \frac{x^2 - x - 6}{x^2 - 9} \right) = \frac{5}{6} \approx 0.83$$

32) $f(x) = \dfrac{x^2 - 6x - 7}{x^2 - 7x}; a = 7$

33) $f(x) = \dfrac{x^2 - 1}{x^2 - 3x + 2}; a = 1$

**Answer**

$$\lim_{x \to 1} \left( \frac{x^2 - 1}{x^2 - 3x + 2} \right) = -2.00$$

34) $f(x) = \dfrac{1 - x^2}{x^2 - 3x + 2}; a = 1$

35) $f(x) = \dfrac{10 - 10x^2}{x^2 - 3x + 2}; a = 1$

**Answer**

$$\lim_{x \to 1} \left( \frac{10 - 10x^2}{x^2 - 3x + 2} \right) = 20.00$$

36) $f(x) = \dfrac{x}{6x^2 - 5x - 6}; a = \dfrac{3}{2}$

37) $f(x) = \dfrac{x}{4x^2 + 4x + 1}; a = -\dfrac{1}{2}$

**Answer**

$\lim\limits_{x \to \frac{-1}{2}} \left( \dfrac{x}{4x^2 + 4x + 1} \right)$ does not exist. Function values decrease without bound as $x$ approaches $-0.5$ from either left or right.

38) $f(x) = \frac{2}{x-4}; a = 4$

**For the exercises 39-41, use a calculator to estimate the limit by preparing a table of values. If there is no limit, describe the behavior of the function as $x$ approaches the given value.**

39) $\lim\limits_{x \to 0} \dfrac{7 \tan x}{3x}$

**Answer**

$$\lim_{x \to 0} \frac{7 \tan x}{3x} = \frac{7}{3}$$

| $x$ | $f(x)$ |
|---|---|
| −0.1 | 2.34114234 |
| −0.01 | 2.33341114 |
| −0.001 | 2.33333411 |
| 0 | Error |
| 0.001 | 2.33333411 |
| 0.01 | 2.33341114 |
| 0.1 | 2.34114234 |

$\lim_{x\to 0^-} \frac{7\tan x}{3x}$ ↓ $\frac{7}{3}$

$\frac{7}{3}$ ↑ $\lim_{x\to 0^+} \frac{7\tan x}{3x}$

40) $\lim_{x\to 4} \frac{x^2}{x-4}$

**Answer**

| $x$ | $f(x)$ |
|---|---|
| 3.9 | −152.1 |
| 3.99 | −1592.01 |
| 3.999 | −15992.001 |
| 4 | Error |
| 4.001 | 16008.001 |
| 4.01 | 1608.1 |
| 4.1 | 168.1 |

$\lim_{x\to 4^-} \frac{x^2}{x-4}$ ↓ $-\infty$

$+\infty$ ↑ $\lim_{x\to 4^+} \frac{x^2}{x-4}$

41) $\lim_{x\to 0} \frac{2\sin x}{4\tan x}$

**Answer**

$$\lim_{x\to 0} \frac{2\sin x}{4\tan x} = \frac{1}{2}$$

| $x$ | $f(x)$ |
|---|---|
| $-0.1$ | 0.49750208 |
| $-0.01$ | 0.49997500 |
| $-0.001$ | 0.49999975 |
| 0 | Error |
| 0.001 | 0.49999975 |
| 0.01 | 0.49997500 |
| 0.1 | 0.49750208 |

$\lim_{x \to 0^-} \frac{2 \sin x}{4 \tan x}$ ↓ $\frac{1}{2}$

$\frac{1}{2}$ ↑ $\lim_{x \to 0^+} \frac{2 \sin x}{4 \tan x}$

**For the exercises 42-49, use a graphing utility to find numerical or graphical evidence to determine the left and right-hand limits of the function given as $x$ approaches $a$. If the function has a limit as $x$ approaches $a$, state it. If not, discuss why there is no limit.**

42) $\lim_{x \to 0} e^{e^{\frac{1}{x}}}$

43) $\lim_{x \to 0} e^{e^{-\frac{1}{x^2}}}$

**Answer**

$\lim_{x \to 0} e^{e^{-\frac{1}{x^2}}} = 1.0$

44) $\lim_{x \to 0} \frac{|x|}{x}$

45) $\lim_{x \to -1} \frac{|x+1|}{x+1}$

**Answer**

$\lim_{x \to -1^-} \frac{|x+1|}{x+1} = \frac{-(x+1)}{(x+1)} = -1$ and $\lim_{x \to -1^+} \frac{|x+1|}{x+1} = \frac{(x+1)}{(x+1)} = 1$; since the right-hand limit does not equal the left-hand limit, $\lim_{x \to -1} \frac{|x+1|}{x+1}$ does not exist.

46) $\lim_{x \to 5} \frac{|x-5|}{5-x}$

47) $\lim_{x \to -1} \frac{1}{(x+1)^2}$

**Answer**

$\lim_{x \to -1} \frac{1}{(x+1)^2}$ does not exist. The function increases without bound as $x$ approaches $-1$ from either side.

48) $\lim_{x \to 1} \frac{1}{(x-1)^3}$

49) $\lim_{x\to 0} \dfrac{5}{1-e^{\frac{2}{x}}}$

**Answer**

$\lim_{x\to 0} \dfrac{5}{1-e^{\frac{2}{x}}}$ does not exist. Function values approach $5$ from the left and approach $0$ from the right.

50) Use numerical and graphical evidence to compare and contrast the limits of two functions whose formulas appear similar: $f(x) = \left|\dfrac{1-x}{x}\right|$ and $g(x) = \left|\dfrac{1+x}{x}\right|$ as $x$ approaches $0$. Use a graphing utility, if possible, to determine the left- and right-hand limits of the functions $f(x)$ and $g(x)$ as $x$ approaches $0$. If the functions have a limit as $x$ approaches $0$, state it. If not, discuss why there is no limit.

## Extensions

51) According to the Theory of Relativity, the mass m m of a particle depends on its velocity $v$. That is

$$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$$

where $m_o$ is the mass when the particle is at rest and $c$ is the speed of light. Find the limit of the mass, $m$, as $v$ approaches $c^-$.

**Answer**

Through examination of the postulates and an understanding of relativistic physics, as $v \to c, m \to \infty$.Take this one step further to the solution,

$$\lim_{v\to c^-} m = \lim_{v\to c^-} \frac{m_o}{\sqrt{1-(v^2/c^2)}} = \infty$$

52) Allow the speed of light, $c$, to be equal to $1.0$. If the mass, $m$, is $1$, what occurs to $m$ as $v \to c$? Using the values listed in the Table below, make a conjecture as to what the mass is as $v$ approaches $1.00$.

| $v$ | $m$ |
|---|---|
| 0.5 | 1.15 |
| 0.9 | 2.29 |
| 0.95 | 3.20 |
| 0.99 | 7.09 |
| 0.999 | 22.36 |
| 0.99999 | 223.61 |

# 12.2: Finding Limits - Properties of Limits

Graphing a function or exploring a table of values to determine a limit can be cumbersome and time-consuming. When possible, it is more efficient to use the properties of limits, which is a collection of theorems for finding limits. Knowing the properties of limits allows us to compute limits directly.

## Verbal

1) Give an example of a type of function $f$ whose limit, as $x$ approaches $a$, is $f(a)$.

**Answer**

If $f$ is a polynomial function, the limit of a polynomial function as $x$ approaches $a$ will always be $f(a)$.

2) When direct substitution is used to evaluate the limit of a rational function as $x$ approaches $a$ and the result is $f(a) = \frac{0}{0}$,does this mean that the limit of $f$ does not exist?

3) What does it mean to say the limit of $f(x)$, as $x$ approaches $c$, is undefined?

**Answer**

It could mean either (1) the values of the function increase or decrease without bound as $x$ approaches $c$, or (2) the left and right-hand limits are not equal.

## Algebraic

**For the exercises 4-30, evaluate the limits algebraically.**

4) $\lim_{x \to 0}(3)$

5) $\lim_{x \to 2}\left(\frac{-5x}{x^2-1}\right)$

**Answer**

$$\frac{-10}{3}$$

6) $\lim_{x \to 2}\left(\frac{x^2-5x+6}{x+2}\right)$

7) $\lim_{x \to 3}\left(\frac{x^2-9}{x-3}\right)$

**Answer**

$$6$$

8) $\lim_{x \to -1}\left(\frac{x^2-2x-3}{x+1}\right)$

9) $\lim_{x \to \frac{3}{2}}\left(\frac{6x^2-17x+12}{2x-3}\right)$

**Answer**

$$\frac{1}{2}$$

10) $\lim_{x \to -\frac{7}{2}}\left(\frac{8x^2+18x-35}{2x+7}\right)$

11) $\lim_{x \to 3}\left(\frac{x^2-9}{x-5x+6}\right)$

**Answer**

$$6$$

12) $\lim_{x \to -3}\left(\frac{-7x^4-21x^3}{-12x^4+108x^2}\right)$

13) $\lim_{x \to 3}\left(\frac{x^2+2x-3}{x-3}\right)$

**Answer**

does not exist

14) $\lim_{h\to 0}\left(\frac{(3+h)^3-27}{h}\right)$

15) $\lim_{h\to 0}\left(\frac{(2-h)^3-8}{h}\right)$

**Answer**

$-12$

16) $\lim_{h\to 0}\left(\frac{(h+3)^2-9}{h}\right)$

17) $\lim_{h\to 0}\left(\frac{\sqrt{5-h}-\sqrt{5}}{h}\right)$

**Answer**

$-\frac{\sqrt{5}}{10}$

18) $\lim_{x\to 0}\left(\frac{\sqrt{3-x}-\sqrt{3}}{x}\right)$

19) $\lim_{x\to 9}\left(\frac{x^2-81}{3-x}\right)$

**Answer**

$-108$

20) $\lim_{x\to 1}\left(\frac{\sqrt{x}-x^2}{1-\sqrt{x}}\right)$

21) $\lim_{x\to 0}\left(\frac{x}{\sqrt{1+2x}-1}\right)$

**Answer**

$1$

22) $\lim_{x\to \frac{1}{2}}\left(\frac{x^2-\frac{1}{4}}{2x-1}\right)$

23) $\lim_{x\to 4}\left(\frac{x^3-64}{x^2-16}\right)$

**Answer**

$6$

24) $\lim_{x\to 2^-}\left(\frac{|x-2|}{x-2}\right)$

25) $\lim_{x\to 2^+}\left(\frac{|x-2|}{x-2}\right)$

**Answer**

$1$

26) $\lim_{x\to 2}\left(\dfrac{|x-2|}{x-2}\right)$

27) $\lim_{x\to 4^-}\left(\dfrac{|x-4|}{4-x}\right)$

## Answer

$1$

28) $\lim_{x\to 4^+}\left(\dfrac{|x-4|}{4-x}\right)$

29) $\lim_{x\to 4}\left(\dfrac{|x-4|}{4-x}\right)$

**Answer**

does not exist

30) $\lim_{x\to 2}\left(\dfrac{-8+6x-x^2}{x-2}\right)$

**For the exercises 31-33, use the given information to evaluate the limits:** $\lim_{x\to c} f(x)=3, \lim_{x\to c} g(x)=5$

31) $\lim_{x\to c}[2f(x)+\sqrt{g(x)}]$

**Answer**

$6+\sqrt{5}$

32) $\lim_{x\to c}[3f(x)+\sqrt{g(x)}]$

33) $\lim_{x\to c}\dfrac{f(x)}{g(x)}$

**Answer**

$\dfrac{3}{5}$

**For the exercises 34-43, evaluate the following limits.**

34) $\lim_{x\to 2}\cos(\pi x)$

35) $\lim_{x\to 2}\sin(\pi x)$

**Answer**

$0$

36) $\lim_{x\to 2}\sin\left(\dfrac{\pi}{x}\right)$

37) $f(x) = \begin{cases} 2x^2 + 2x + 1, & x \leq 0 \\ x - 3, & x > 0; \end{cases} \lim_{x \to 0^+} f(x)$

**Answer**

$-3$

38) $f(x) = \begin{cases} 2x^2 + 2x + 1, & x \leq 0 \\ x - 3, & x > 0; \end{cases} \lim_{x \to 0^-} f(x)$

39) $f(x) = \begin{cases} 2x^2 + 2x + 1, & x \leq 0 \\ x - 3, & x > 0; \end{cases} \lim_{x \to 0} f(x)$

**Answer**

does not exist; right-hand limit is not the same as the left-hand limit.

40) $\lim_{x \to 4} \dfrac{\sqrt{x+5} - 3}{x - 4}$

41) $\lim_{x \to 2^+} (2x - [\![ \ x \ ]\!] \ )$

**Answer**

$2$

42) $\lim_{x \to 2} \dfrac{\sqrt{x+7} - 3}{x^2 - x - 2}$

43) $\lim_{x \to 3^+} \dfrac{x^2}{x^2 - 9}$

**Answer**

Limit does not exist; limit approaches infinity.

**For the exercises 44-53, find the average rate of change $\dfrac{f(x+h) - f(x)}{h}$.**

44) $f(x) = x + 1$

45) $f(x) = 2x^2 - 1$

**Answer**

$4x + 2h$

46) $f(x) = x^2 + 3x + 4$

47) $f(x) = x^2 + 4x - 100$

**Answer**

$2x + h + 4$

48) $f(x) = 3x^2 + 1$

49) $f(x) = \cos(x)$

**Answer**

$$\frac{\cos(x+h)-\cos(x)}{h}$$

50) $f(x)=2x^3-4x$

51) $f(x)=\frac{1}{x}$

**Answer**

$$\frac{-1}{x(x+h)}$$

52) $f(x)=\frac{1}{x^2}$

53) $f(x)=\sqrt{x}$

**Answer**

$$\frac{-1}{\sqrt{x+h}+\sqrt{x}}$$

## Graphical

54) Find an equation that could be represented by the Figure below.

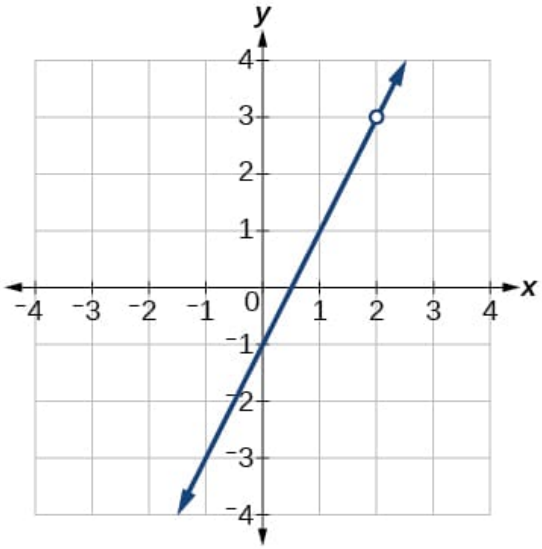



Figure below.

**Answer**

$$f(x)=\frac{x^2+5x+6}{x+3}$$

**For the exercises 56-57, refer to the Figure below.**

56) What is the right-hand limit of the function as $x$ approaches $0$?

57) What is the left-hand limit of the function as $x$ approaches $0$?

**Answer**

does not exist

## Real-World Applications

58) The position function $s(t) = -16t^2 + 144t$ gives the position of a projectile as a function of time. Find the average velocity (average rate of change) on the interval $[1, 2]$.

59) The height of a projectile is given by $s(t) = -64t^2 + 192t$ Find the average rate of change of the height from $t = 1$ second to $t = 1.5$ seconds.

**Answer**

$52$

60) The amount of money in an account after $t$ years compounded continuously at $4.25\%$ interest is given by the formula $A = A_0e^{0.0425t}$,where $A_0$ is the initial amount invested. Find the average rate of change of the balance of the account from $t = 1$ year to $t = 2$ years if the initial amount invested is $\$1,000.00$.

# 12.3: Continuity

A function that remains level for an interval and then jumps instantaneously to a higher value is called a stepwise function. This function is an example. A function that has any hole or break in its graph is known as a discontinuous function. A stepwise function, such as parking-garage charges as a function of hours parked, is an example of a discontinuous function. We can check three different conditions to decide if a function is continuous at a particular number.

## Verbal

1) State in your own words what it means for a function $f$ to be continuous at $x = c$.

**Answer**

Informally, if a function is continuous at $x = c$, then there is no break in the graph of the function at $f(c)$, and $f(c)$ is defined.

2) State in your own words what it means for a function to be continuous on the interval $(a, b)$.

## Algebraic

**For the exercises 3-22, determine why the function $f$ is discontinuous at a given point $a$ on the graph. State which condition fails.**

3) $f(x) = \ln|x + 3|, a = -3$

**Answer**

discontinuous at $a = -3$; $f(-3)$ does not exist

4) $f(x) = \ln|5x - 2|, a = \dfrac{2}{5}$

5) $f(x) = \dfrac{x^2 - 16}{x + 4}, a = -4$

**Answer**

removable discontinuity at $a = -4$; $f(-4)$ is not defined

6) $f(x) = \dfrac{x^2 - 16x}{x}, a = 0$

7) $f(x) = \begin{cases} x, & x \neq 3 \\ 2x, & x = 3 \end{cases} a = 3$

**Answer**

Discontinuous at $a = 3; \lim_{x \to 3} f(x) = 3$, but $f(3) = 6$, which is not equal to the limit.

8) $f(x) = \begin{cases} 5, & x \neq 0 \\ 3, & x = 0 \end{cases} a = 0$

9) $f(x) = \begin{cases} \dfrac{1}{2 - x}, & x \neq 2 \\ 3, & x = 2 \end{cases} a = 2$

**Answer**

$\lim_{x \to 2} f(x)$ does not exist.

10) $f(x) = \begin{cases} \dfrac{1}{x + 6}, & x = -6 \\ x^2, & x \neq -6 \end{cases} a = -6$

11) $f(x) = \begin{cases} 3 + x, & x < 1 \\ x, & x = 1 \\ x^2, & x > 1 \end{cases} a = 1$

**Answer**

$\lim_{x \to 1^-} f(x) = 4; \lim_{x \to 1^+} f(x) = 1$. Therefore, $\lim_{x \to 1} f(x)$ does not exist.

12) $f(x) = \begin{cases} 3 - x, & x < 1 \\ x, & x = 1 \\ 2x^2, & x > 1 \end{cases} a = 1$

13) $f(x) = \begin{cases} 3 + 2x, & x < 1 \\ x, & x = 1 \\ -x^2, & x > 1 \end{cases} a = 1$

**Answer**

$\lim_{x \to 1^-} f(x) = 5 \neq \lim_{x \to 1^+} f(x) = -1$. Thus $\lim_{x \to 1} f(x)$ does not exist.

14) $f(x) = \begin{cases} x^2, & x < -2 \\ 2x + 1, & x = -2 \\ x^3, & x > -2 \end{cases} a = -2$

15) $f(x) = \begin{cases} \dfrac{x^2 - 9}{x + 3}, & x < -3 \\ x - 9, & x = -3 \\ \dfrac{1}{x}, & x > -3 \end{cases} a = -3$

**Answer**

$\lim_{xto-3^+} f(x) = -\dfrac{1}{3}$

Therefore, $\lim_{x \to -3} f(x)$ does not exist.

16) $f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x < -3 \\ x-9, & x = -3 \\ -6, & x > -3 \end{cases}$ $a = 3$

17) $f(x) = \dfrac{x^2-4}{x-2}, a = 2$

**Answer**

$f(2)$ is not defined.

18) $f(x) = \dfrac{25-x^2}{x^2-10x+25}, a = 5$

19) $f(x) = \dfrac{x^3-9x}{x^2+11x+24}, a = -3$

**Answer**

$f(-3)$ is not defined.

20) $f(x) = \dfrac{x^3-27}{x^2-3x}, a = 3$

21) $f(x) = \dfrac{x}{|x|}, a = 0$

**Answer**

$f(0)$ is not defined.

22) $f(x) = \dfrac{2|x+2|}{x+2}, a = -2$

**For the exercises 23-35, determine whether or not the given function $f$ is continuous everywhere. If it is continuous everywhere it is defined, state for what range it is continuous. If it is discontinuous, state where it is discontinuous.**

23) $f(x) = x^3 - 2x - 15$

**Answer**

Continuous on $(-\infty, \infty)$

24) $f(x) = \dfrac{x^2-2x-15}{x-5}$

25) $f(x) = 2 \cdot 3^{x+4}$

**Answer**

Continuous on $(-\infty, \infty)$

26) $f(x) = -\sin(3x)$

27) $f(x) = \dfrac{|x-2|}{x^2-2x}$

**Answer**

Discontinuous at $x = 0$ and $x = 2$

28) $f(x) = \tan(x) + 2$

29) $f(x) = 2x + \dfrac{5}{x}$

**Answer**

Discontinuous at $x = 0$

30) $f(x) = \log_2(x)$

31) $f(x) = \ln x^2$

**Answer**

Continuous on $(0, \infty)$

32) $f(x) = e^{2x}$

33) $f(x) = \sqrt{x - 4}$

**Answer**

Continuous on $[4, \infty)$

34) $f(x) = \sec(x) - 3$

35) $f(x) = x^2 + \sin(x)$

**Answer**

Continuous on $(-\infty, \infty)$.

36) Determine the values of $b$ and $c$ such that the following function is continuous on the entire real number line.

$$f(x) = \begin{cases} x + 1, & 1 < x < 3 \\ x^2 + bx + c, & |x - 2| \geq 1 \end{cases}$$

## Graphical

**For the exercises 37-39, refer to the Figure below. Each square represents one square unit. For each value of $a$, determine which of the three conditions of continuity are satisfied at $x = a$ and which are not.**

37) $x = -3$

**Answer**

1, but not 2 or 3

38) $x = 2$

39) $x = 4$

**Answer**

1 and 2, but not 3

**For the exercises 40-43, use a graphing utility to graph the function $f(x) = \sin\left(\dfrac{12\pi}{x}\right)$ as in Figure. Set the $x$-axis a short distance before and after $0$ to illustrate the point of discontinuity.**

40) Which conditions for continuity fail at the point of discontinuity?

41) Evaluate $f(0)$.

**Answer**

$f(0)$ is undefined.

42) Solve for $x$ if $f(x) = 0$.

43) What is the domain of $f(x)$?

**Answer**

$(-\infty, 0) \cup (0, \infty)$

**For the exercises 44-45, consider the function shown in the Figure below.**

44) At what $x$-coordinates is the function discontinuous?

45) What condition of continuity is violated at these points?

**Answer**

At $x = -1$, the limit does not exist. At $x = 1, f(1)$ does not exist.

At $x = 2$, there appears to be a vertical asymptote, and the limit does not exist.

46) Consider the function shown in the Figure below. At what $x$-coordinates is the function discontinuous? What condition(s) of continuity were violated?

47) Construct a function that passes through the origin with a constant slope of $1$, with removable discontinuities at $x = -7$ and $x = 1$.

**Answer**

$$\frac{x^3 + 6x^2 - 7x}{(x+7)(x-1)}$$

48) The function $f(x) = \dfrac{x^3 - 1}{x - 1}$ is graphed in the Figure below. It appears to be continuous on the interval $[-3, 3]$, but there is an $x$-value on that interval at which the function is discontinuous. Determine the value of $x$ at which the function is discontinuous, and explain the pitfall of utilizing technology when considering continuity of a function by examining its graph.

49) Find the limit $\lim\limits_{x \to 1} f(x)$ and determine if the following function is continuous at $x = 1$:

$$fx = \begin{cases} x^2 + 4 & x \neq 1 \\ 2 & x = 1 \end{cases}$$

**Answer**

The function is discontinuous at $x = 1$ because the limit as $x$ approaches $1$ is $5$ and $f(1) = 2$.

50) The graph of $f(x) = \dfrac{\sin(2x)}{x}$ is shown in the Figure below. Is the function $f(x)$ continuous at $x = 0$? Why or why not?

# 12.4: Derivatives

Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

## Verbal

1) How is the slope of a linear function similar to the derivative?

**Answer**

The slope of a linear function stays the same. The derivative of a general function varies according to $x$. Both the slope of a line and the derivative at a point measure the rate of change of the function.

2) What is the difference between the average rate of change of a function on the interval $[x, x+h]$ and the derivative of the function at $x$?

3) A car traveled $110$ miles during the time period from 2:00 P.M. to 4:00 P.M. What was the car's average velocity? At exactly 2:30 P.M., the speed of the car registered exactly $62$ miles per hour. What is another name for the speed of the car at 2:30 P.M.? Why does this speed differ from the average velocity?

**Answer**

Average velocity is $55$ miles per hour. The instantaneous velocity at 2:30 p.m. is $62$ miles per hour. The instantaneous velocity measures the velocity of the car at an instant of time whereas the average velocity gives the velocity of the car over an interval.

4) Explain the concept of the slope of a curve at point $x$.

5) Suppose water is flowing into a tank at an average rate of $45$ gallons per minute. Translate this statement into the language of mathematics.

**Answer**

The average rate of change of the amount of water in the tank is $45$ gallons per minute. If $f(x)$ is the function giving the amount of water in the tank at any time $t$, then the average rate of change of $f(x)$ between $t=a$ and $t=b$ is $f(a)+45(b-a)$.

## Algebraic

**For the exercises 6-17, use the definition of derivative $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to calculate the derivative of each function.**

6) $f(x)=3x-4$

7) $f(x)=-2x+1$

**Answer**

$$f'(x)=-2$$

8) $f(x)=x^2-2x+1$

9) $f(x)=2x^2+x-3$

**Answer**

$$f'(x)=4x+1$$

10) $f(x)=2x^2+5$

11) $f(x)=\frac{-1}{x-2}$

**Answer**

$$f'(x) = \frac{1}{(x-2)^2}$$

12) $f(x) = \dfrac{2+x}{1-x}$

13) $f(x) = \dfrac{5-2x}{3+2x}$

**Answer**

$$\frac{-16}{(3+2x)^2}$$

14) $f(x) = \sqrt{1+3x}$

15) $f(x) = 3x^3 - x^2 + 2x + 5$

**Answer**

$$f'(x) = 9x^2 - 2x + 2$$

16) $f(x) = 5$

17) $f(x) = 5\pi$

**Answer**

$$f'(x) = 0$$

**For the exercises 18-21, find the average rate of change between the two points.**

18) $(-2, 0)$ and $(-4, 5)$

19) $(4, -3)$ and $(-2, -1)$

**Answer**

$$-\frac{1}{3}$$

20) $(0, 5)$ and $(6, 5)$

21) $(7, -2)$ and $(7, 10)$

**Answer**

undefined

**For the polynomial functions 22-25, find the derivatives.**

22) $f(x) = x^3 + 1$

23) $f(x) = -3x^2 - 7x = 6$

**Answer**

$$f'(x) = -6x - 7$$

24) $f(x) = 7x^2$

25) $f(x) = 3x^3 + 2x^2 + x - 26$

**Answer**

$$f'(x) = 9x^2 + 4x + 1$$

**For the functions 26-28, find the equation of the tangent line to the curve at the given point $x$ on the curve.**

26) $f(x) = 2x^2 - 3x \quad x = 3$

27) $f(x) = x^2 + 1 \quad x = 2$

**Answer**

$$y = 12x - 15$$

28) $f(x) = \sqrt{x} \quad x = 9$

29) For the following exercise, find $k$ such that the given line is tangent to the graph of the function.

$$f(x) = x^2 - kx \quad y = 4x - 9$$

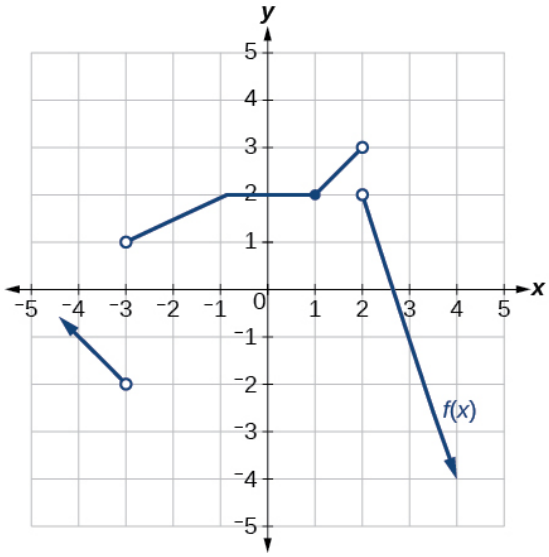
**Answer**

$k = -10$ or $k = 2$

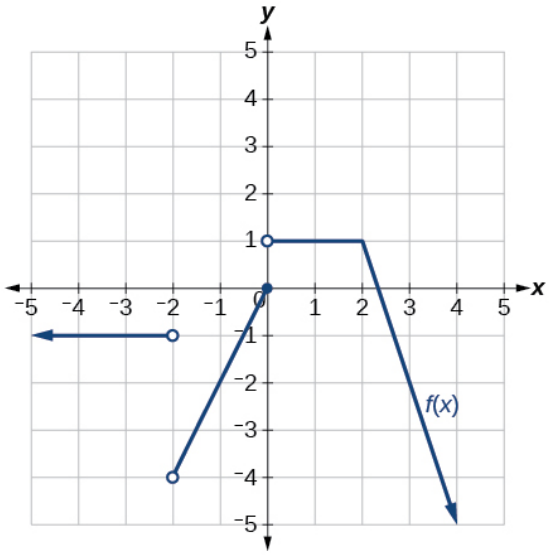
## Graphical

**For the exercises 30-33, consider the graph of the function $f$ and determine where the function is continuous/discontinuous and differentiable/not differentiable.**

30)



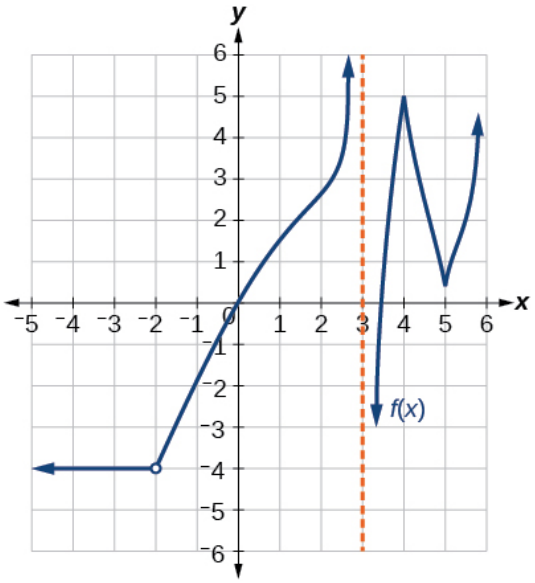

31)



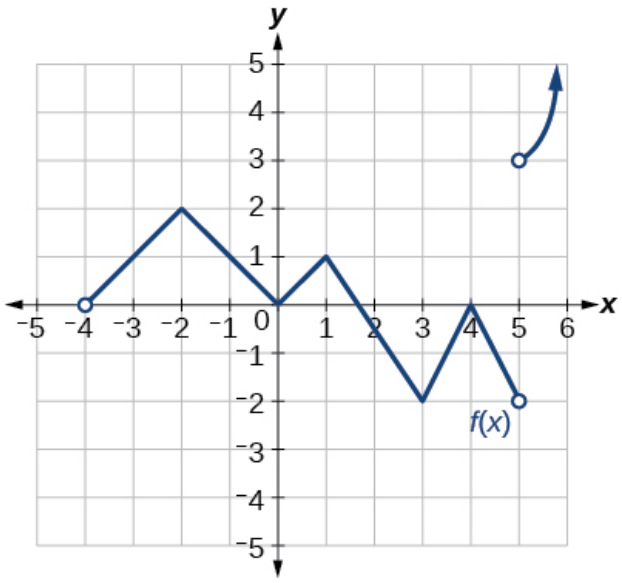

**Answer**

Discontinuous at $x = -2$ and $x = 0$. Not differentiable at $-2, 0, 2$.

32)



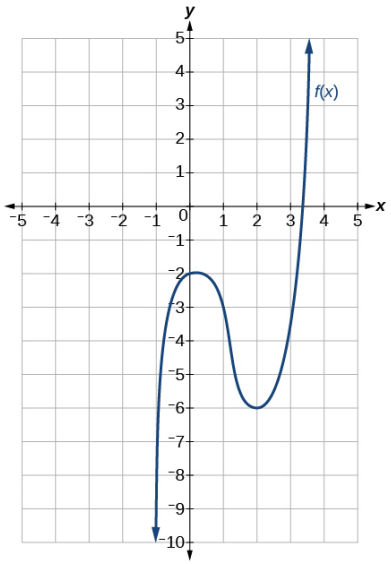

33)




**Answer**

Discontinuous at $x = 5$. Not differentiable at $-4, -2, 0, 1, 3, 4, 5$.

**For the exercises 34-43, use the Figure below to estimate either the function at a given value of $x$ or the derivative at a given value of $x$, as indicated.**




34) $f(-1)$

35) $f(0)$

**Answer**

$f(0) = -2$

36) $f(1)$

37) $f(2)$

**Answer**

$f(2) = -6$

38) $f(3)$

39) $f'(-1)$

**Answer**

$f'(-1) = 9$

40) $f'(0)$

41) $f'(1)$

**Answer**

$f'(1) = -3$

42) $f'(2)$

43) $f'(3)$

**Answer**

$f'(3) = 9$

44) Sketch the function based on the information below:

$$f'(x) = 2x, f(2) = 4$$

## Technology

45) Numerically evaluate the derivative. Explore the behavior of the graph of $f(x) = x^2$ around $x = 1$ by graphing the function on the following domains: $[0.9, 1.1], [0.99, 1.01], [0.999, 1.001], [0.9999, 1.0001]$. We can use the feature on our calculator that automatically sets Ymin and Ymax to the Xmin and Xmax values we preset. (On some of the commonly used graphing calculators, this feature may be called ZOOM FIT or ZOOM AUTO). By examining the corresponding range values for this viewing window, approximate how the curve changes at $x = 1$, that is, approximate the derivative at $x = 1$.

**Answer**

Answers vary. The slope of the tangent line near $x = 1$ is $2$.

## Real-World Applications

**For the exercises 46-50, explain the notation in words. The volume $f(t)$ of a tank of gasoline, in gallons, $t$ minutes after noon.**

46) $f(0) = 600$

47) $f'(30) = -20$

**Answer**

At 12:30 p.m., the rate of change of the number of gallons in the tank is $-20$ gallons per minute. That is, the tank is losing $20$ gallons per minute.

48) $f(30) = 0$

49) $f'(200) = 30$

**Answer**

At $200$ minutes after noon, the volume of gallons in the tank is changing at the rate of $30$ gallons per minute.

50) $f(240) = 500$

**For the exercises 51-55, explain the functions in words. The height, $s$, of a projectile after $t$ seconds is given by $s(t) = -16t^2 + 80t$.**

51) $s(2) = 96$

**Answer**

The height of the projectile after $2$ seconds is $96$ feet.

52) $s'(2) = 16$

53) $s(3) = 96$

**Answer**

The height of the projectile at $t = 3$ seconds is $96$ feet.

54) $s'(3) = -16$

55) $s(0) = 0, s(5) = 0$

**Answer**

The height of the projectile is zero at $t = 0$ and again at $t = 5$. In other words, the projectile starts on the ground and falls to earth again after $5$ seconds.

**For the exercises 56-57, the volume $V$ of a sphere with respect to its radius $r$ is given by $V = \frac{4}{3}\pi r^3$.**

56) Find the average rate of change of $V$ as $r$ changes from $1$ cm to $2$ cm.

57) Find the instantaneous rate of change of $V$ when $r = 3$ cm.

**Answer**

$36\pi$

**For the exercises 58-60, the revenue generated by selling $x$ items is given by $R(x) = 2x^2 + 10x$.**

58) Find the average change of the revenue function as $x$ changes from $x = 10$ to $x = 20$.

59) Find $R'(10)$ and interpret.

**Answer**

$\$50.00$ per unit, which is the instantaneous rate of change of revenue when exactly $10$ units are sold.

60) Find $R'(15)$ and interpret. Compare $R'(15)$ to $R'(10)$, and explain the difference.

**For the exercises 61-63, the cost of producing $x$ cellphones is described by the function $C(x) = x^2 - 4x + 1000$.**

61) Find the average rate of change in the total cost as $x$ changes from $x = 10$ to $x = 15$.

**Answer**

$21 per unit

62) Find the approximate marginal cost, when $15$ cellphones have been produced, of producing the $16^{th}$ cellphone.

63) Find the approximate marginal cost, when $20$ cellphones have been produced, of producing the $21^{st}$ cellphone.

**Answer**

$36

## Extension

**For the exercises 64-67, use the definition for the derivative at a point $x = a$, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, to find the derivative of the functions.**

64) $f(x) = \frac{1}{x^2}$

65) $f(x) = 5x^2 - x + 4$

**Answer**

$f'(x) = 10a - 1$

66) $f(x) = -x^2 + 4x + 7$

67) $f(x) = \frac{-4}{3 - x^2}$

**Answer**

$\frac{4}{(3 - x)^2}$

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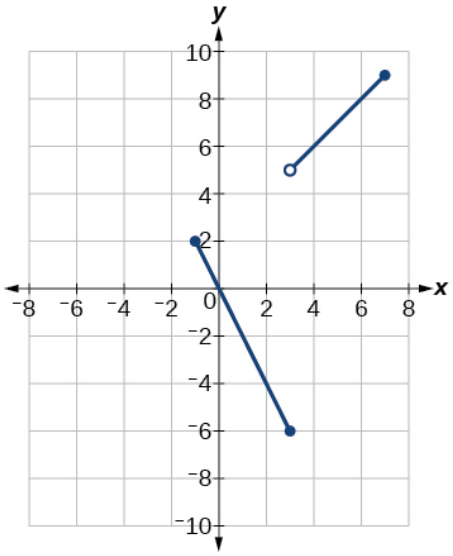
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# 7.R: Introduction to Calculus (Review)

## 12.1: Finding Limits - Numerical and Graphical Approaches

**For the exercises 1-6, use the Figure below.**




1) $\lim_{x \to -1^+} f(x)$

**Answer**

$2$

2) $\lim_{x \to -1^-} f(x)$

3) $\lim_{x \to -1} f(x)$

**Answer**

does not exist

4) $\lim_{x \to 3} f(x)$

5) At what values of $x$ is the function discontinuous? What condition of continuity is violated?

**Answer**

Discontinuous at $x = -1$ $\left(\lim_{x \to a} f(x) \text{ does not exist}\right)$, $x = 3$ ( jump discontinuity), and $x = 7$ $\left(\lim_{x \to a} f(x) \text{ does not exist}\right)$.

6) Using the Table below, estimate $\lim_{x \to 0} f(x)$.

| $x$ | $F(x)$ |
|---|---|
| −0.1 | 2.875 |

| $x$ | $F(x)$ |
|---|---|
| −0.01 | 2.92 |
| −0.001 | 2.998 |
| 0 | Undefined |
| 0.001 | 2.9987 |
| 0.01 | 2.865 |
| 0.1 | 2.78145 |
| 0.15 | 2.678 |

**Answer**

$3$

**For the exercises 7-9, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as $x$ approaches $a$. If the function has limit as $x$ approaches $a$, state it. If not, discuss why there is no limit.**

7) $f(x) = \begin{cases} |x| - 1 & \text{if } x \neq 1 \\ x^3 & \text{if } x = 1 \end{cases} \quad a = 1$

8) $f(x) = \begin{cases} \dfrac{1}{x+1} & \text{if } x = -2 \\ (x+1)^2 & \text{if } x \neq -2 \end{cases} \quad a = -2$

**Answer**

$\lim\limits_{x \to -2} f(x) = 1$

9) $f(x) = \begin{cases} \sqrt{x+3} & \text{if } x < 1 \\ -\sqrt[3]{x} & \text{if } x > 1 \end{cases} \quad a = 1$

## 12.2: Finding Limits - Properties of Limits

**For the exercises 1-6, find the limits if $\lim\limits_{x \to c} f(x) = -3$ and $\lim\limits_{x \to c} g(x) = 5$.**

1) $\lim\limits_{x \to c}(f(x) + g(x))$

**Answer**

$2$

2) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$

3) $\lim\limits_{x \to c}(f(x) \cdot g(x))$

**Answer**

$-15$

4) $\lim\limits_{x \to 0^+} f(x), f(x) = \begin{cases} 3x^2 + 2x + 1 & x > 0 \\ 5x + 3 & x < 0 \end{cases}$

5) $\lim\limits_{x \to 0^-} f(x), f(x) = \begin{cases} 3x^2 + 2x + 1 & x > 0 \\ 5x + 3 & x < 0 \end{cases}$

**Answer**

$3$

6) $\lim_{x\to 3^+}(3x - [\![\ x]\!]\ )$

**For the exercises 7-11, evaluate the limits using algebraic techniques.**

7) $\lim_{h\to 0}\left(\dfrac{(h+6)^2-36}{h}\right)$

**Answer**

$12$

8) $\lim_{x\to 25}\left(\dfrac{x^2-625}{\sqrt{x}-5}\right)$

9) $\lim_{x\to 1}\left(\dfrac{-x^2-9x}{x}\right)$

**Answer**

$-10$

10) $\lim_{x\to 4}\left(\dfrac{7-\sqrt{12x+1}}{x-4}\right)$

11) $\lim_{x\to 3}\left(\dfrac{\frac{1}{3}+\frac{1}{x}}{3+x}\right)$

**Answer**

$-\dfrac{1}{9}$

# 12.3: Continuity

**For the exercises 1-5, use numerical evidence to determine whether the limit exists at $x = a$. If not, describe the behavior of the graph of the function at $x = a$.**

1) $f(x) = \dfrac{-2}{x-4};\ \ a = 4$

2) $f(x) = \dfrac{-2}{(x-4)^2};\ \ a = 4$

**Answer**

At $x = 4$, the function has a vertical asymptote.

3) $f(x) = \dfrac{-x}{x^2-x-6};\ \ a = 3$

4) $f(x) = \dfrac{6x^2+23x+20}{4x^2-25};\ \ a = -\dfrac{5}{2}$

**Answer**

removable discontinuity at $a = -\dfrac{5}{2}$

5) $f(x) = \dfrac{\sqrt{x} - 3}{9 - x}$; $a = 9$

**For the exercises 6-12, determine where the given function $f(x)$ is continuous. Where it is not continuous, state which conditions fail, and classify any discontinuities.**

6) $f(x) = x^2 - 2x - 15$

**Answer**

continuous on $(-\infty, \infty)$

7) $f(x) = \dfrac{x^2 - 2x - 15}{x - 5}$

8) $f(x) = \dfrac{x^2 - 2x}{x^2 - 4x + 4}$

**Answer**

removable discontinuity at $x = 2$. $f(2)$ is not defined, but limits exist.

9) $f(x) = \dfrac{x^3 - 125}{2x^2 - 12x + 10}$

10) $f(x) = \dfrac{x^2 - \frac{1}{x}}{2 - x}$

**Answer**

discontinuity at $x = 0$ and $x = 2$. Both $f(0)$ and $f(2)$ are not defined.

11) $f(x) = \dfrac{x + 2}{x^2 - 3x - 10}$

12) $f(x) = \dfrac{x + 2}{x^3 + 8}$

**Answer**

removable discontinuity at $x = -2$. $f(-2)$ is not defined.

# 12.4: Derivatives

**For the exercises 1-5, find the average rate of change $f(x) = \dfrac{f(x + h) - f(x)}{h}$.**

1) $f(x) = 3x + 2$

2) $f(x) = 5$

**Answer**

$0$

3) $f(x) = \dfrac{1}{x + 1}$

4) $f(x) = \ln(x)$

**Answer**

$$f(x) = \frac{\ln(x+h) - \ln(x)}{h}$$

5) $f(x) = e^{2x}$

**For the exercises 6-7, find the derivative of the function.**

6) $f(x) = 4x - 6$

**Answer**

$4$

7) $f(x) = 5x^2 - 3x$

8) Find the equation of the tangent line to the graph of $f(x)$ at the indicated $x$ value.

$$f(x) = -x^3 + 4x;\ x = 2$$

**Answer**

$y = -8x + 16$

9) For the following exercise, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

$$f(x) = \frac{x}{|x|}$$

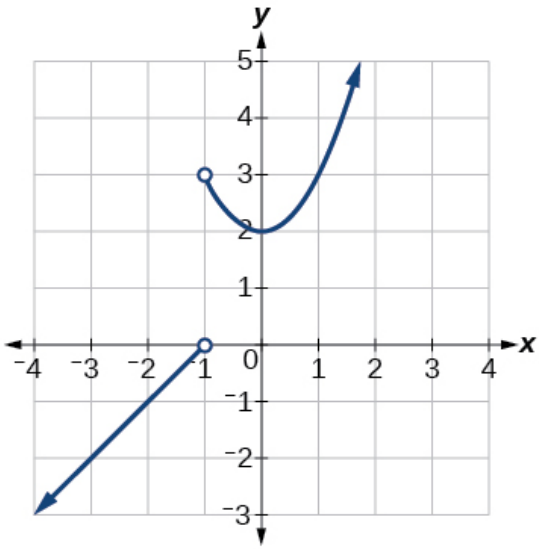
10) Given that the volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$ and that a given cone has a fixed height of $9$ cm and variable radius length, find the instantaneous rate of change of volume with respect to radius length when the radius is $2$ cm. Give an exact answer in terms of $\pi$.

**Answer**

$12\pi$

# Practice Test

**For the exercises 1-6, use the graph of $f$ in the Figure below.**




1) $f(1)$

**Answer**

$3$

2) $\lim_{x \to -1^+} f(x)$

3) $\lim_{x \to -1^-} f(x)$

**Answer**

$0$

4) $\lim_{x \to -1} f(x)$

5) $\lim_{x \to -2} f(x)$

**Answer**

$-1$

6) At what values of $x$ is $f$ discontinuous? What property of continuity is violated?

7) $f(x) = \begin{cases} \frac{1}{3} - 3 & \text{if } x \leq 2 \\ x^3 + 1 & \text{if } x > 2 \end{cases}$ $a = 2$

**Answer**

$\lim_{x \to 2^-} f(x) = -\frac{5}{2}a$ and $\lim_{x \to 2^+} f(x) = 9$

Thus, the limit of the function as $x$ approaches $2$ does not exist.

8) $f(x) = \begin{cases} x^3 + 1 & \text{if } x < 1 \\ 3x^2 - 1 & \text{if } x = 1 \; a = 1 \\ -\sqrt{x+3} + 4 & \text{if } x > 1 \end{cases}$

**For the exercises 9-11, evaluate each limit using algebraic techniques.**

9) $\lim_{x \to -5} \left( \frac{\frac{1}{5} + \frac{1}{x}}{10 + 2x} \right)$

**Answer**

$$-\frac{1}{50}$$

10) $\lim_{h \to 0} \left( \frac{\sqrt{h^2 + 25} - 5}{h^2} \right)$

11) $\lim_{h \to 0} \left( \frac{1}{h} - \frac{1}{h^2 + h} \right)$

**Answer**

$$1$$

**For the exercises 12-13, determine whether or not the given function $f$ is continuous. If it is continuous, show why. If it is not continuous, state which conditions fail.**

12) $f(x) = \sqrt{x^2 - 4}$

13) $f(x) = \frac{x^3 - 4x^2 - 9x + 36}{x^3 - 3x^2 + 2x - 6}$

**Answer**

removable discontinuity at $x = 3$

**For the exercises 14-16, use the definition of a derivative to find the derivative of the given function at $x = a$.**

14) $f(x) = \frac{3}{5 + 2x}$

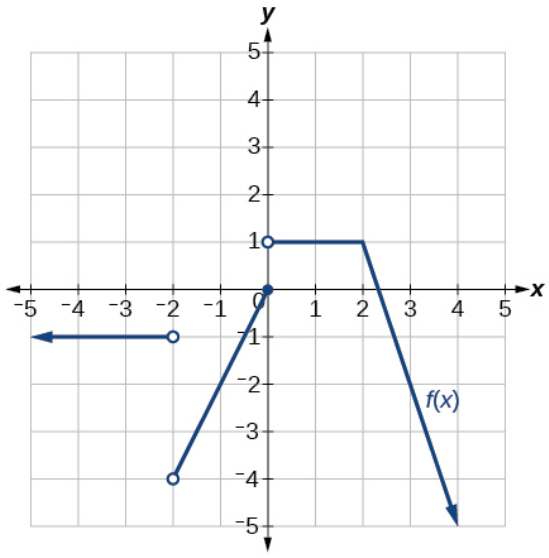
15) $f(x) = \frac{3}{\sqrt{x}}$

**Answer**

$$f'(x) = -\frac{3}{2a^{\frac{3}{2}}}$$

16) $f(x) = 2x^2 + 9x$

17) For the graph in the Figure below, determine where the function is continuous/discontinuous and differentiable/not differentiable.




**Answer**

discontinuous at $-2, 0$, not differentiable at $-2, 0, 2$.

**For the exercises 18-19, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.**

18) $f(x) = |x - 2| - |x + 2|$

19) $f(x) = \dfrac{2}{1 + e^{\frac{2}{x}}}$

**Answer**

not differentiable at $x = 0$ (no limit)

**For the exercises 20-24, explain the notation in words when the height of a projectile in feet, $s$, is a function of time $t$ in seconds after launch and is given by the function $s(t)$.**

20) $s(0)$

21) $s(2)$

**Answer**

the height of the projectile at $t = 2$ seconds

22) $s'(2)$

23) $\dfrac{s(2) - s(1)}{2 - 1}$

**Answer**

the average velocity from $t = 1$ to $t = 2$

24) $s(t) = 0$

**For the exercises 25-28, use technology to evaluate the limit.**

25) $\lim_{x\to 0} \dfrac{\sin(x)}{3x}$

**Answer**

$$\frac{1}{3}$$

26) $\lim_{x\to 0} \dfrac{\tan^2(x)}{2x}$

27) $\lim_{x\to 0} \dfrac{\sin(x)(1-\cos(x))}{2x^2}$

**Answer**

$$0$$

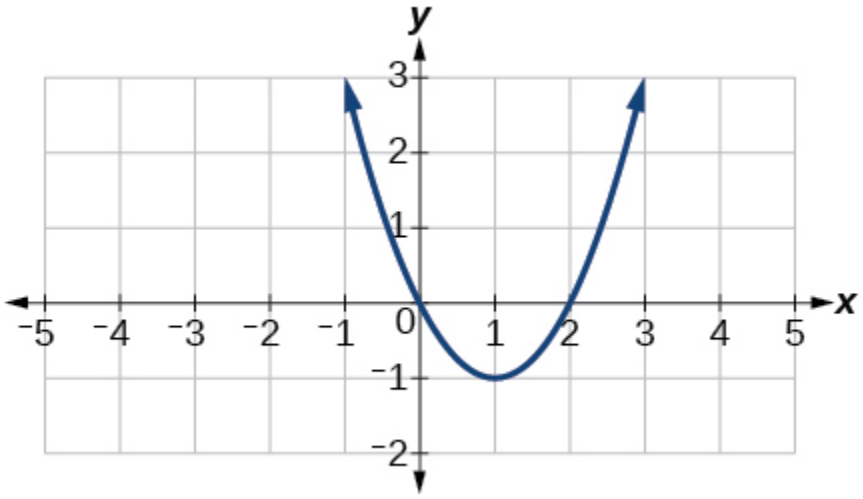
28) Evaluate the limit by hand.

$$\lim_{x\to 1} f(x), \text{ where } f(x) = \begin{cases} 4x-7 & x \neq 1 \\ x^2-4 & x = 1 \end{cases}$$

At what value(s) of $x$ is the function below discontinuous?

$$f(x) = \begin{cases} 4x-7 & x \neq 1 \\ x^2-4 & x = 1 \end{cases}$$

**For the exercises 29-32, consider the function whose graph appears in Figure.**




29) Find the average rate of change of the function from $x = 1$ to $x = 3$.

**Answer**

$$2$$

30) Find all values of $x$ at which $f'(x) = 0$.

**Answer**

$$x = 1$$

31) Find all values of $x$ at which $f'(x)$ does not exist.

32) Find an equation of the tangent line to the graph of $f$ the indicated point: $f(x) = 3x^2 - 2x - 6, \quad x = -2$

**Answer**

$$y = -14x - 18$$

**For the exercises 33-34, use the function** $f(x) = x(1-x)^{\frac{2}{5}}$

33) Graph the function $f(x) = x(1-x)^{\frac{2}{5}}$ by entering $f(x) = x\left((1-x)^2\right)^{\frac{1}{5}}$ and then by entering $f(x) = x\left((1-x)^{\frac{1}{5}}\right)^2$.

34) Explore the behavior of the graph of $f(x)$ around $x = 1$ by graphing the function on the following domains, $[0.9, 1.1], [0.99, 1.01], [0.999, 1.001]$, and $[0.9999, 1.0001]$. Use this information to determine whether the function appears to be differentiable at $x = 1$.

**Answer**

The graph is not differentiable at $x = 1$ (cusp).

**For the exercises 35-42, find the derivative of each of the functions using the definition:** $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

35) $f(x) = 2x - 8$

36) $f(x) = 4x^2 - 7$

**Answer**

$$f'(x) = 8x$$

37) $f(x) = x - \dfrac{1}{2}x^2$

38) $f(x) = \dfrac{1}{x+2}$

**Answer**

$$f'(x) = -\frac{1}{(2+x)^2}$$

39) $f(x) = \dfrac{3}{x-1}$

40) $f(x) = -x^3 + 1$

**Answer**

$$f'(x) = -3x^2$$

41) $f(x) = x^2 + x^3$

42) $f(x) = \sqrt{x-1}$

**Answer**

$$f'(x) = -\frac{1}{2\sqrt{x-1}}$$

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