CALCULUS



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MIT & University of Wisconsin-Stevens Point Calculus

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Licensing

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CHAPTER OVERVIEW

1: Functions and Graphs

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

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Thumbnail: The graph of $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$. (CC BY; OpenStax)

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1.1: Prelude to Functions and Graphs

In the past few years, major earthquakes have occurred in several countries around the world. In January 2010, an earthquake of magnitude 7.3 hit Haiti. A magnitude 9 earthquake shook northeastern Japan in March 2011. In April 2014, an 8.2-magnitude earthquake struck off the coast of northern Chile. What do these numbers mean? In particular, how does a magnitude 9 earthquake compare with an earthquake of magnitude 8.2? Or 7.3? Later in this chapter, we show how logarithmic functions are used to compare the relative intensity of two earthquakes based on the magnitude of each earthquake.



Figure **1.1.1**: A portion of the San Andreas Fault in California. Major faults like this are the sites of most of the strongest earthquakes ever recorded. (credit: modification of work by Robb Hannawacker, NPS)

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

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1.2: Review of Functions

Learning Objectives

- Use functional notation to evaluate a function.
- Determine the domain and range of a function.
- Draw the graph of a function.
- Find the zeros of a function.
- Recognize a function from a table of values.
- Make new functions from two or more given functions.
- Describe the symmetry properties of a function.

In this section, we provide a formal definition of a function and examine several ways in which functions are represented—namely, through tables, formulas, and graphs. We study formal notation and terms related to functions. We also define composition of functions and symmetry properties. Most of this material will be a review for you, but it serves as a handy reference to remind you of some of the algebraic techniques useful for working with functions.

Functions

Given two sets A and B a set with elements that are ordered pairs (x, y) where x is an element of A and y is an element of B, is a relation from A to B. A relation from A to B defines a relationship between those two sets. A function is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the ***input***; the element of the second set is called the ***output***. Functions are used all the time in mathematics to describe relationships between two sets. For any function, when we know the input, the output is determined, so we say that the output is a function of the input. For example, the area of a square is determined by its side length, so we say that the area (the output) is a function of its side length (the input). The velocity of a ball thrown in the air can be described as a function of the amount of time the ball is in the air. The cost of mailing a package is a function of the weight of the package. Since functions have so many uses, it is important to have precise definitions and terminology to study them.



Figure 1.2.1: A function can be visualized as an input/output device

Definition: Functions

A *function* f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the *domain* of the function. The set of outputs is called the *range* of the *function*.

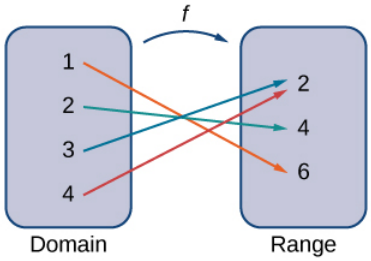


Figure 1.2.2: A function maps every element in the domain to exactly one element in the range. Although each input can be sent to only one output, two different inputs can be sent to the same output.

For example, consider the function f, where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9$.

Since every nonnegative real number has a real-value square root, every nonnegative number is an element of the range of this function. Since there is no real number with a square that is negative, the negative real numbers are not elements of the range. We conclude that the range is the set of nonnegative real numbers.

For a general function f with domain D, we often use x to denote the input and y to denote the output associated with x. When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on x. Using function notation, we write $y = f(x)$, and we read this equation as "y equals f of x." For the squaring function described earlier, we write $f(x) = x^2$.

The concept of a function can be visualized using Figures **1.2.1** - **1.2.3**.

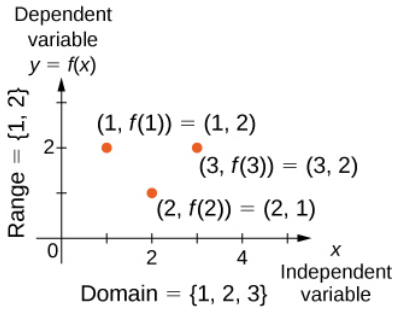


Figure **1.2.3**: In this case, a graph of a function f has a domain of $\{1,2,3\}$ and a range of $\{1,2\}$. The independent variable is x and the dependent variable is y.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The graph of a function is the set of all these points. For example, consider the function f, where the domain is the set $D = \{1,2,3\}$ and the rule is $f(x) = 3 - x$. In Figure **1.2.4**, we plot a graph of this function.

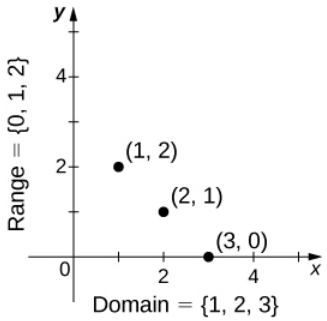


Figure **1.2.4**: Here we see a graph of the function f with domain $\{1,2,3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number. For example, since any real number can be squared, if no other domain is specified, we consider the domain of $f(x) = x^2$ to be the set of all real numbers. On the other hand, the square root function $f(x) = \sqrt{x}$ only gives a real output if x is nonnegative. Therefore, the domain of the function $f(x) = \sqrt{x}$ is the set of nonnegative real numbers, sometimes called the natural domain.

For the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$, the domains are sets with an infinite number of elements. Clearly we cannot list all these elements. When describing a set with an infinite number of elements, it is often helpful to use set-builder or interval notation. When using set-builder notation to describe a subset of all real numbers, denoted R, we write

$$\{x \mid x \text{ has some property}\}.$$

We read this as the set of real numbers x such that x has some property. For example, if we were interested in the set of real numbers that are greater than one but less than five, we could denote this set using set-builder notation by writing

$$\{x \mid 1 < x < 5\}.$$

A set such as this, which contains all numbers greater than a and less than b, can also be denoted using the *interval notation* (a, b). Therefore,

$$(1,5) = \{x \mid 1 < x < 5\}.$$

The numbers 1 and 5 are called the endpoints of this set. If we want to consider the set that includes the endpoints, we would denote this set by writing

$$[1,5] = \{x \mid 1 \leq x \leq 5\}.$$

We can use similar notation if we want to include one of the endpoints, but not the other. To denote the set of nonnegative real numbers, we would use the set-builder notation

$$\{x \mid x \geq 0\}.$$

The smallest number in this set is zero, but this set does not have a largest number. Using interval notation, we would use the symbol ∞, which refers to positive infinity, and we would write the set as

$$[0,\infty) = \{x \mid x \geq 0\}.$$

It is important to note that ∞ is not a real number. It is used symbolically here to indicate that this set includes all real numbers greater than or equal to zero. Similarly, if we wanted to describe the set of all nonpositive numbers, we could write

$$(-\infty, 0] = \{x \mid x \leq 0\}.$$

Here, the notation $-\infty$ refers to negative infinity, and it indicates that we are including all numbers less than or equal to zero, no matter how small. The set

$$(-\infty,\infty) = \{x \mid x \text{ is any real number}\}$$

refers to the set of all real numbers. Some functions are defined using different equations for different parts of their domain. These types of functions are known as piecewise-defined functions. For example, suppose we want to define a function f with a domain that is the set of all real numbers such that $f(x) = 3x + 1$ for $x \geq 2$ and $f(x) = x^2$ for $x < 2$. We denote this function by writing

$$f(x) = \begin{cases} 3x+1, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$$

When evaluating this function for an input x, the equation to use depends on whether $x \geq 2$ or $x < 2$. For example, since $5 > 2$, we use the fact that $f(x) = 3x + 1$ for $x \geq 2$ and see that $f(5) = 3(5) + 1 = 16$. On the other hand, for $x = -1$, we use the fact that $f(x) = x^2$ for $x < 2$ and see that $f(-1) = 1$.

Example 1.2.1: Evaluating Functions

For the function $f(x) = 3x^2 + 2x - 1$, evaluate:

a. $f(-2)$
b. $f(\sqrt{2})$
c. $f(a+h)$

Solution

Substitute the given value for x in the formula for $f(x)$.

a. $f(-2) = 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7$
b. $f(\sqrt{2}) = 3(\sqrt{2})^2 + 2\sqrt{2} - 1 = 6 + 2\sqrt{2} - 1 = 5 + 2\sqrt{2}$
c. $f(a+h) = 3(a+h)^2 + 2(a+h) - 1 = 3(a^2 + 2ah + h^2) + 2a + 2h - 1 = 3a^2 + 6ah + 3h^2 + 2a + 2h - 1$

Exercise 1.2.1

For $f(x) = x^2 - 3x + 5$, evaluate $f(1)$ and $f(a+h)$.

Hint

Substitute 1 and $a+h$ for x in the formula for $f(x)$.

Answer

$f(1) = 3$ and $f(a+h) = a^2 + 2ah + h^2 - 3a - 3h + 5$

Example 1.2.2: Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. $f(x) = (x-4)^2 + 5$
b. $f(x) = \sqrt{3x+2} - 1$
c. $f(x) = \dfrac{3}{x-2}$

Solution

a. Consider $f(x) = (x-4)^2 + 5$.

1.Since $f(x) = (x-4)^2 + 5$ is a real number for any real number x, the domain of f is the interval $(-\infty, \infty)$.

2. Since $(x-4)^2 \geq 0$, we know $f(x) = (x-4)^2 + 5 \geq 5$. Therefore, the range must be a subset of $\{y \mid y \geq 5\}$. To show that every element in this set is in the range, we need to show that for a given y in that set, there is a real number x such that $f(x) = (x-4)^2 + 5 = y$. Solving this equation for x, we see that we need x such that

$$(x-4)^2 = y - 5.$$

This equation is satisfied as long as there exists a real number x such that

$$x - 4 = \pm\sqrt{y-5}$$

Since $y \geq 5$, the square root is well-defined. We conclude that for $x = 4 \pm \sqrt{y-5}$, $f(x) = y$, and therefore the range is $\{y \mid y \geq 5\}$.

b. Consider $f(x) = \sqrt{3x+2} - 1$.

1.To find the domain of f, we need the expression $3x + 2 \geq 0$. Solving this inequality, we conclude that the domain is $\{x \mid x \geq -2/3\}$.

2.To find the range of f, we note that since $\sqrt{3x+2} \geq 0$, $f(x) = \sqrt{3x+2} - 1 \geq -1$. Therefore, the range of f must be a subset of the set $\{y \mid y \geq -1\}$. To show that every element in this set is in the range of f, we need to show that for all y in this set, there exists a real number x in the domain such that $f(x) = y$. Let $y \geq -1$. Then, $f(x) = y$ if and only if

$$\sqrt{3x+2} - 1 = y.$$

Solving this equation for x, we see that x must solve the equation

$$\sqrt{3x+2} = y + 1.$$

Since $y \geq -1$, such an x could exist. Squaring both sides of this equation, we have $3x + 2 = (y+1)^2$.

Therefore, we need

$$3x = (y+1)^2 - 2,$$

which implies

$$x = \tfrac{1}{3}(y+1)^2 - \tfrac{2}{3}.$$

We just need to verify that x is in the domain of f. Since the domain of f consists of all real numbers greater than or equal to $\frac{-2}{3}$, and

$$\tfrac{1}{3}(y+1)^2 - \tfrac{2}{3} \geq -\tfrac{2}{3},$$

there does exist an x in the domain of f. We conclude that the range of f is $\{y \mid y \geq -1\}$.

c. Consider $f(x) = \dfrac{3}{x-2}$.

1.Since $3/(x-2)$ is defined when the denominator is nonzero, the domain is $\{x \mid x \neq 2\}$.

2.To find the range of f, we need to find the values of y such that there exists a real number x in the domain with the property that

$$\frac{3}{x-2} = y.$$

Solving this equation for x, we find that

$$x = \frac{3}{y} + 2.$$

Therefore, as long as $y \neq 0$, there exists a real number x in the domain such that $f(x) = y$. Thus, the range is $\{y \mid y \neq 0\}$.

? Exercise 1.2.2

Find the domain and range for $f(x) = \sqrt{4-2x} + 5$.

Hint

Use $4 - 2x \geq 0$.

Answer

Domain = $\{x \mid x \leq 2\}$ and range = $\{y \mid y \geq 5\}$

Representing Functions

Typically, a function is represented using one or more of the following tools:

- A table
- A graph
- A formula

We can identify a function in each form, but we can also use them together. For instance, we can plot on a graph the values from a table or create a table from a formula.

Tables

Functions described using a **table of values** arise frequently in real-world applications. Consider the following simple example. We can describe temperature on a given day as a function of time of day. Suppose we record the temperature every hour for a 24-hour period starting at midnight. We let our input variable x be the time after midnight, measured in hours, and the output variable y be the temperature x hours after midnight, measured in degrees Fahrenheit. We record our data in Table 1.2.1.

Table 1.2.1: Temperature as a Function of Time of Day

Hour After Midnight	Temperature(°F)	Hour After Midnight	Temperature(°F)
0	58	12	84
1	54	13	85
2	53	14	85
3	52	15	83
4	52	16	82
5	55	17	80
6	60	18	77
7	64	19	74
8	72	20	69
9	75	21	65
10	78	22	60
11	80	23	58

We can see from the table that temperature is a function of time, and the temperature decreases, then increases, and then decreases again. However, we cannot get a clear picture of the behavior of the function without graphing it.

Graphs

Given a function f described by a table, we can provide a visual picture of the function in the form of a graph. Graphing the temperatures listed in Table 1.2.1 can give us a better idea of their fluctuation throughout the day. Figure 1.2.5 shows the plot of the temperature function.

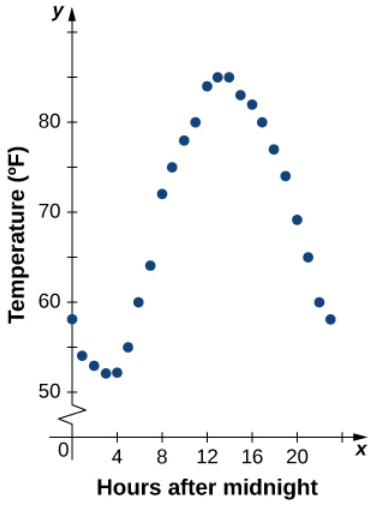


Figure 1.2.5: The graph of the data from Table 1.2.1 shows temperature as a function of time.

From the points plotted on the graph in Figure 1.2.5, we can visualize the general shape of the graph. It is often useful to connect the dots in the graph, which represent the data from the table. In this example, although we cannot make any definitive conclusion regarding what the temperature was at any time for which the temperature was not recorded, given the number of data points collected and the pattern in these points, it is reasonable to suspect that the temperatures at other times followed a similar pattern, as we can see in Figure 1.2.6.

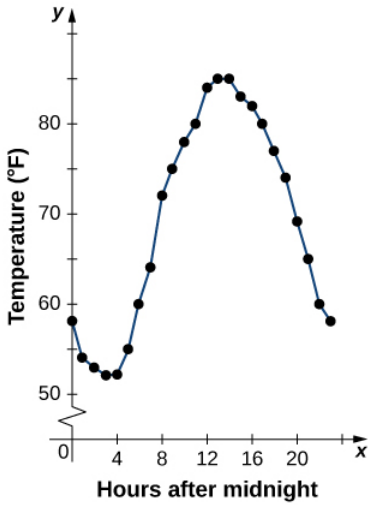


Figure **1.2.6**: Connecting the dots in Figure **1.2.5** shows the general pattern of the data.

Algebraic Formulas

Sometimes we are not given the values of a function in table form, rather we are given the values in an explicit formula. Formulas arise in many applications. For example, the area of a circle of radius r is given by the formula $A(r) = \pi r^2$. When an object is thrown upward from the ground with an initial velocity v_0 ft/s, its height above the ground from the time it is thrown until it hits the ground is given by the formula $s(t) = -16t^2 + v_0 t$. When P dollars are invested in an account at an annual interest rate r compounded continuously, the amount of money after t years is given by the formula $A(t) = Pe^{rt}$. Algebraic formulas are important tools to calculate function values. Often we also represent these functions visually in graph form.

Given an algebraic formula for a function f, the graph of f is the set of points $(x, f(x))$, where x is in the domain of f and $f(x)$ is in the range. To graph a function given by a formula, it is helpful to begin by using the formula to create a table of inputs and outputs. If the domain of f consists of an infinite number of values, we cannot list all of them, but because listing some of the inputs and outputs can be very useful, it is often a good way to begin.

When creating a table of inputs and outputs, we typically check to determine whether zero is an output. Those values of x where $f(x) = 0$ are called the zeros of a function. For example, the zeros of $f(x) = x^2 - 4$ are $x = \pm 2$. The zeros determine where the graph of f intersects the x-axis, which gives us more information about the shape of the graph of the function. The graph of a function may never intersect the x-axis, or it may intersect multiple (or even infinitely many) times.

Another point of interest is the y -intercept, if it exists. The y-intercept is given by $(0, f(0))$.

Since a function has exactly one output for each input, the graph of a function can have, at most, one y-intercept. If $x = 0$ is in the domain of a function f, then f has exactly one y-intercept. If $x = 0$ is not in the domain of f, then f has no y-intercept. Similarly, for any real number c, if c is in the domain of f, there is exactly one output $f(c)$, and the line $x = c$ intersects the graph of f exactly once. On the other hand, if c is not in the domain of f, $f(c)$ is not defined and the line $x = c$ does not intersect the graph of f. This property is summarized in the vertical line test.

Vertical Line Test

Given a function f, every vertical line that may be drawn intersects the graph of f no more than once. If any vertical line intersects a set of points more than once, the set of points does not represent a function.

We can use this test to determine whether a set of plotted points represents the graph of a function (Figure **1.2.7**).

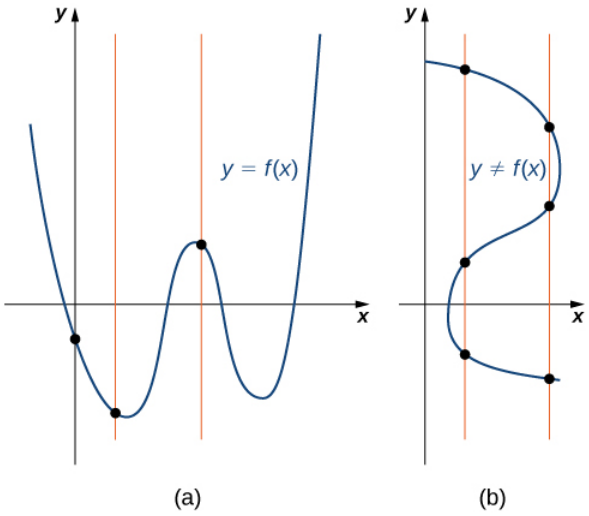


Figure 1.2.7: (a) The set of plotted points represents the graph of a function because every vertical line intersects the set of points, at most, once. (b) The set of plotted points does not represent the graph of a function because some vertical lines intersect the set of points more than once.

Example 1.2.3: Finding Zeros and y-Intercepts of a Function

Consider the function $f(x) = -4x + 2$.

a. Find all zeros of f.
b. Find the y-intercept (if any).
c. Sketch a graph of f.

Solution

1.To find the zeros, solve $f(x) = -4x + 2 = 0$. We discover that f has one zero at $x = 1/2$.

2. The y-intercept is given by $(0, f(0)) = (0, 2)$.

3. Given that f is a linear function of the form $f(x) = mx + b$ that passes through the points $(1/2, 0)$ and $(0, 2)$, we can sketch the graph of f (Figure 1.2.8).

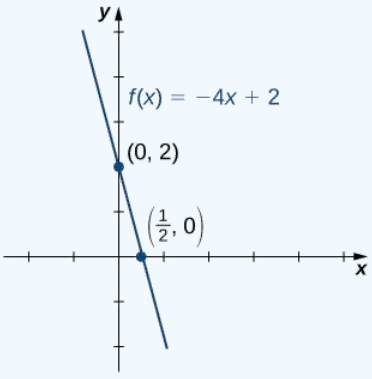


Figure 1.2.8: The function $f(x) = -4x + 2$ is a line with x-intercept $(1/2, 0)$ and y-intercept $(0, 2)$.

Example 1.2.4: Using Zeros and y-Intercepts to Sketch a Graph

Consider the function $f(x) = \sqrt{x + 3} + 1$.

a. Find all zeros of f.
b. Find the y-intercept (if any).
c. Sketch a graph of f.

Solution

1.To find the zeros, solve $\sqrt{x+3}+1=0$. This equation implies $\sqrt{x+3}=-1$. Since $\sqrt{x+3}\geq 0$ for all x, this equation has no solutions, and therefore f has no zeros.

2.The y-intercept is given by $(0, f(0)) = (0, \sqrt{3}+1)$.

3.To graph this function, we make a table of values. Since we need $x+3\geq 0$, we need to choose values of $x\geq -3$. We choose values that make the square-root function easy to evaluate.

x	-3	-2	1
$f(x)$	1	2	3

Making use of the table and knowing that, since the function is a square root, the graph of f should be similar to the graph of $y=\sqrt{x}$, we sketch the graph (Figure 1.2.9).

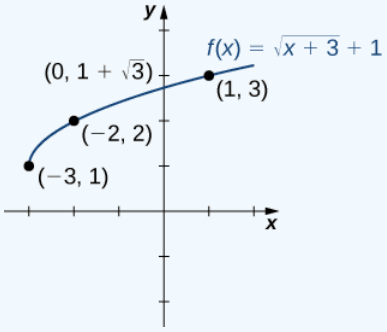


Figure 1.2.9: The graph of $f(x)=\sqrt{x+3}+1$ has a y-intercept but no x-intercepts.

Exercise 1.2.4

Find the zeros of $f(x)=x^3-5x^2+6x$.

Hint

Factor the polynomial.

Answer

$x=0,2,3$

Example 1.2.5: Finding the Height of a Free-Falling Object

If a ball is dropped from a height of 100 ft, its height s at time t is given by the function $s(t)=-16t^2+100$, where s is measured in feet and t is measured in seconds. The domain is restricted to the interval $[0,c]$, where $t=0$ is the time when the ball is dropped and $t=c$ is the time when the ball hits the ground.

a. Create a table showing the height s(t) when $t=0,\ 0.5,\ 1,\ 1.5,\ 2,$ and 2.5. Using the data from the table, determine the domain for this function. That is, find the time c when the ball hits the ground.

b. Sketch a graph of s.

Solution

t	0	0.5	1	1.5	2	2.5
$s(t)$	100	96	84	64	36	0

Since the ball hits the ground when $t=2.5$, the domain of this function is the interval $[0,2.5]$.

2.

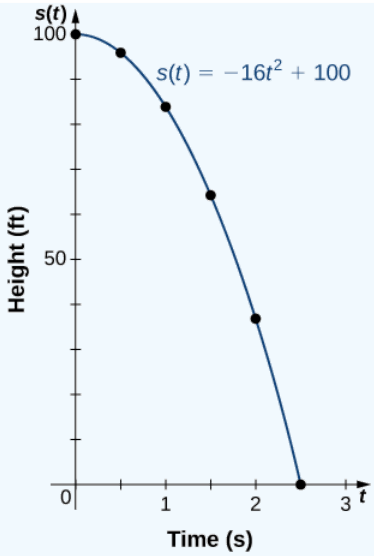


Figure 1.2.8, the values of $f(x)$ are getting smaller as x is getting larger. A function with this property is said to be decreasing. On the other hand, for the function $f(x) = \sqrt{x+3}+1$ graphed in Figure 1.2.9, the values of $f(x)$ are getting larger as the values of x are getting larger. A function with this property is said to be increasing. It is important to note, however, that a function can be increasing on some interval or intervals and decreasing over a different interval or intervals. For example, using our temperature function plotted above, we can see that the function is decreasing on the interval $(0, 4)$, increasing on the interval $(4, 14)$, and then decreasing on the interval $(14, 23)$. We make the idea of a function increasing or decreasing over a particular interval more precise in the next definition.

Definition: Increasing and Decreasing on an Interval

We say that a function f is ***increasing on the interval*** I if for all $x_1, x_2 \in I$,

$$f(x_1) \leq f(x_2) \text{ when } x_1 < x_2.$$

We say f is strictly increasing on the interval I if for all $x_1, x_2 \in I$,

$$f(x_1) < f(x_2) \text{ when } x_1 < x_2.$$

We say that a function f is ***decreasing on the interval*** I if for all $x_1, x_2 \in I$,

$$f(x_1) \geq f(x_2) \text{ if } x_1 < x_2.$$

We say that a function f is strictly decreasing on the interval I if for all $x_1, x_2 \in I$,

$$f(x_1) > f(x_2) \text{ if } x_1 < x_2.$$

For example, the function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$ because $3x_1 < 3x_2$ whenever $x_1 < x_2$. On the other hand, the function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$ because $-x_1^3 > -x_2^3$ whenever $x_1 < x_2$ (Figure 1.2.10).

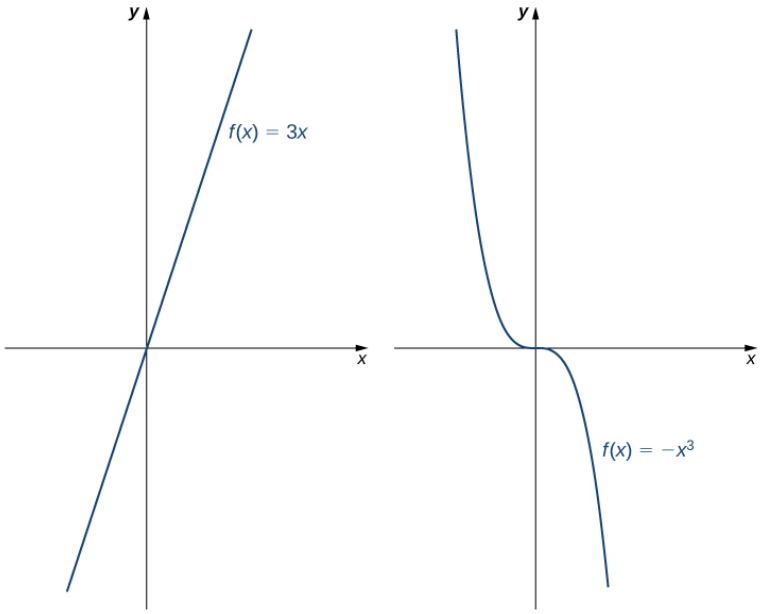


Figure **1.2.10**: (a) The function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$. (b) The function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$.

Combining Functions

Now that we have reviewed the basic characteristics of functions, we can see what happens to these properties when we combine functions in different ways, using basic mathematical operations to create new functions. For example, if the cost for a company to manufacture x items is described by the function $C(x)$ and the revenue created by the sale of x items is described by the function $R(x)$, then the profit on the manufacture and sale of x items is defined as $P(x) = R(x) - C(x)$. Using the difference between two functions, we created a new function.

Alternatively, we can create a new function by composing two functions. For example, given the functions $f(x) = x^2$ and $g(x) = 3x + 1$, the composite function $f \circ g$ is defined such that

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (3x + 1)^2.$$

The composite function $g \circ f$ is defined such that

$$(g \circ f)(x) = g(f(x)) = 3f(x) + 1 = 3x^2 + 1.$$

Note that these two new functions are different from each other.

Combining Functions with Mathematical Operators

To combine functions using mathematical operators, we simply write the functions with the operator and simplify. Given two functions f and g, we can define four new functions:

$(f + g)(x) = f(x) + g(x)$	Sum
$(f - g)(x) = f(x) - g(x)$	Difference
$(f \cdot g)(x) = f(x)g(x)$	Product
$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ for $g(x) \neq 0$	Quotient

Example 1.2.6: Combining Functions Using Mathematical Operations

Given the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, find each of the following functions and state its domain.

a. $(f + g)(x)$
b. $(f - g)(x)$
c. $(f \cdot g)(x)$
d. $\left(\frac{f}{g}\right)(x)$

Solution

1. $(f + g)(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4.$

The domain of this function is the interval $(-\infty, \infty)$.

2. $(f - g)(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2.$

The domain of this function is the interval $(-\infty, \infty)$.

3. $(f \cdot g)(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3.$

The domain of this function is the interval $(-\infty, \infty)$.

4. $\left(\frac{f}{g}\right)(x) = \frac{2x - 3}{x^2 - 1}$.

The domain of this function is $\{x \mid x \neq \pm 1\}$.

Exercise 1.2.6

For $f(x) = x^2 + 3$ and $g(x) = 2x - 5$, find $(f/g)(x)$ and state its domain.

Hint

The new function $(f/g)(x)$ is a quotient of two functions. For what values of x is the denominator zero?

Answer

$\left(\frac{f}{g}\right)(x) = \frac{x^2+3}{2x-5}$. The domain is $\{x \mid x \neq \frac{5}{2}\}$.

Function Composition

When we compose functions, we take a function of a function. For example, suppose the temperature T on a given day is described as a function of time t (measured in hours after midnight) as in Table 1.2.1. Suppose the cost C, to heat or cool a building for 1 hour, can be described as a function of the temperature T. Combining these two functions, we can describe the cost of heating or cooling a building as a function of time by evaluating $C(T(t))$. We have defined a new function, denoted $C \circ T$, which is defined such that $(C \circ T)(t) = C(T(t))$ for all t in the domain of T. This new function is called a composite function. We note that since cost is a function of temperature and temperature is a function of time, it makes sense to define this new function $(C \circ T)(t)$. It does not make sense to consider $(T \circ C)(t)$, because temperature is not a function of cost.

Definition: Composite Functions

Consider the function f with domain A and range B, and the function g with domain D and range E. If B is a subset of D, then the composite function $(g \circ f)(x)$ is the function with domain A such that

$$(g \circ f)(x) = g(f(x))$$

A composite function $g \circ f$ can be viewed in two steps. First, the function f maps each input x in the domain of f to its output $f(x)$ in the range of f. Second, since the range of f is a subset of the domain of g, the output $f(x)$ is an element in the domain of g, and therefore it is mapped to an output $g(f(x))$ in the range of g. In Figure 1.2.11, we see a visual image of a composite function.

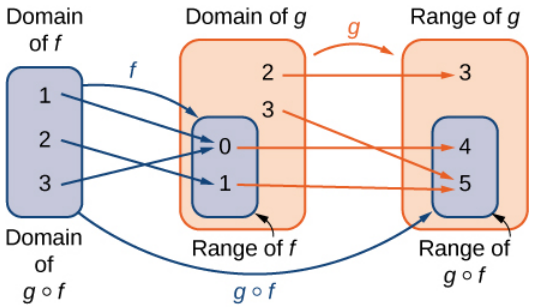


Figure 1.2.11: For the composite function $g \circ f$, we have $(g \circ f)(1) = 4$, $(g \circ f)(2) = 5$, and $(g \circ f)(3) = 4$.

Example 1.2.7: Compositions of Functions Defined by Formulas

Consider the functions $f(x) = x^2 + 1$ and $g(x) = 1/x$.

a. Find $(g \circ f)(x)$ and state its domain and range.
b. Evaluate $(g \circ f)(4)$, $(g \circ f)(-1/2)$.
c. Find $(f \circ g)(x)$ and state its domain and range.
d. Evaluate $(f \circ g)(4)$, $(f \circ g)(-1/2)$.

Solution

1. We can find the formula for $(g \circ f)(x)$ in two different ways. We could write

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = \frac{1}{x^2 + 1}.$$

Alternatively, we could write

$$(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{x^2 + 1}.$$

Since $x^2 + 1 \neq 0$ for all real numbers x, the domain of $(g \circ f)(x)$ is the set of all real numbers. Since $0 < 1/(x^2 + 1) \leq 1$, the range is, at most, the interval $(0, 1]$. To show that the range is this entire interval, we let $y = 1/(x^2 + 1)$ and solve this equation for x to show that for all y in the interval $(0, 1]$, there exists a real number x such that $y = 1/(x^2 + 1)$. Solving this equation for x, we see that $x^2 + 1 = 1/y$, which implies that

$$x = \pm\sqrt{\tfrac{1}{y} - 1}$$

If y is in the interval $(0, 1]$, the expression under the radical is nonnegative, and therefore there exists a real number x such that $1/(x^2 + 1) = y$. We conclude that the range of $g \circ f$ is the interval $(0, 1]$.

2. $(g \circ f)(4) = g(f(4)) = g(4^2 + 1) = g(17) = \frac{1}{17}$

$(g \circ f)(-\frac{1}{2}) = g(f(-\frac{1}{2})) = g((-\frac{1}{2})^2 + 1) = g(\frac{5}{4}) = \frac{4}{5}$

3. We can find a formula for $(f \circ g)(x)$ in two ways. First, we could write

$$(f \circ g)(x) = f(g(x)) = f(\tfrac{1}{x}) = (\tfrac{1}{x})^2 + 1.$$

Alternatively, we could write

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 + 1 = (\tfrac{1}{x})^2 + 1.$$

The domain of $f \circ g$ is the set of all real numbers x such that $x \neq 0$. To find the range of f, we need to find all values y for which there exists a real number $x \neq 0$ such that

$$\left(\frac{1}{x}\right)^2 + 1 = y.$$

Solving this equation for x, we see that we need x to satisfy

$$\left(\frac{1}{x}\right)^2 = y - 1,$$

which simplifies to

$$\frac{1}{x} = \pm\sqrt{y-1}$$

Finally, we obtain

$$x = \pm\frac{1}{\sqrt{y-1}}.$$

Since $1/\sqrt{y-1}$ is a real number if and only if $y > 1$, the range of f is the set $\{y \mid y \geq 1\}$.

4.$(f \circ g)(4) = f(g(4)) = f(\frac{1}{4}) = (\frac{1}{4})^2 + 1 = \frac{17}{16}$

$(f \circ g)(-\frac{1}{2}) = f(g(-\frac{1}{2})) = f(-2) = (-2)^2 + 1 = 5$

In Example 1.2.7, we can see that $(f \circ g)(x) \neq (g \circ f)(x)$. This tells us, in general terms, that the order in which we compose functions matters.

? Exercise 1.2.7

Let $f(x) = 2 - 5x$. Let $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$.

Solution

$(f \circ g)(x) = 2 - 5\sqrt{x}$.

✓ Example 1.2.8: Composition of Functions Defined by Tables

Consider the functions f and g described by

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	4	2	4	-2	0	-2	4

x	-4	-2	0	2	4
$g(x)$	1	0	3	0	5

a. Evaluate $(g \circ f)(3), (g \circ f)(0)$.
b. State the domain and range of $(g \circ f)(x)$.
c. Evaluate $(f \circ f)(3), (f \circ f)(1)$.
d. State the domain and range of $(f \circ f)(x)$.

Solution:

1. $(g \circ f)(3) = g(f(3)) = g(-2) = 0$

$(g \circ f)(0) = g(4) = 5$

2.The domain of $g \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $g \circ f$ is the set $\{0, 3, 5\}$.

3. $(f \circ f)(3) = f(f(3)) = f(-2) = 4$

$(f \circ f)(1) = f(f(1)) = f(-2) = 4$

4.The domain of $f \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $f \circ f$ is the set $\{0, 4\}$.

Example 1.2.9: Application Involving a Composite Function

A store is advertising a sale of 20% off all merchandise. Caroline has a coupon that entitles her to an additional 15% off any item, including sale merchandise. If Caroline decides to purchase an item with an original price of x dollars, how much will she end up paying if she applies her coupon to the sale price? Solve this problem by using a composite function.

Solution

Since the sale price is 20% off the original price, if an item is x dollars, its sale price is given by $f(x) = 0.80x$. Since the coupon entitles an individual to 15% off the price of any item, if an item is y dollars, the price, after applying the coupon, is given by g(y)=0.85y. Therefore, if the price is originally x dollars, its sale price will be $f(x) = 0.80x$ and then its final price after the coupon will be $g(f(x)) = 0.85(0.80x) = 0.68x$.

Exercise 1.2.9

If items are on sale for 10% off their original price, and a customer has a coupon for an additional 30% off, what will be the final price for an item that is originally x dollars, after applying the coupon to the sale price?

Hint

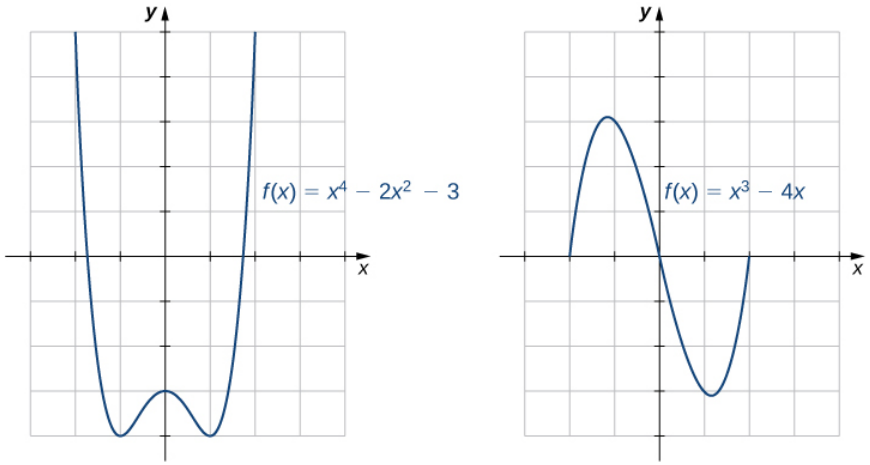
The sale price of an item with an original price of x dollars is $f(x) = 0.90x$. The coupon price for an item that is y dollars is $g(y) = 0.70y$.

Solution

$(g \circ f)(x) = 0.63x$

Symmetry of Functions

The graphs of certain functions have symmetry properties that help us understand the function and the shape of its graph. For example, consider the function $f(x) = x^4 - 2x^2 - 3$ shown in Figure 1.2.12*a*. If we take the part of the curve that lies to the right of the y-axis and flip it over the y-axis, it lays exactly on top of the curve to the left of the y-axis. In this case, we say the function has **symmetry about the y-axis**. On the other hand, consider the function $f(x) = x^3 - 4x$ shown in Figure 1.2.12*b*. If we take the graph and rotate it 180° about the origin, the new graph will look exactly the same. In this case, we say the function has **symmetry about the origin**.



(a) Symmetry about the y-axis (b) Symmetry about the origin

Figure 1.2.12: (a) A graph that is symmetric about the y-axis. (b) A graph that is symmetric about the origin.

If we are given the graph of a function, it is easy to see whether the graph has one of these symmetry properties. But without a graph, how can we determine algebraically whether a function f has symmetry? Looking at Figure 1.2.12*a* again, we see that since f is symmetric about the y-axis, if the point (x, y) is on the graph, the point $(-x, y)$ is on the graph. In other words, $f(-x) = f(x)$. If

a function f has this property, we say f is an even function, which has symmetry about the y-axis. For example, $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

In contrast, looking at Figure 1.2.12*b* again, if a function f is symmetric about the origin, then whenever the point (x, y) is on the graph, the point $(-x, -y)$ is also on the graph. In other words, $f(-x) = -f(x)$. If f has this property, we say f is an odd function, which has symmetry about the origin. For example, $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Definition: Even and Odd Functions

- If $f(x) = f(-x)$ for all x in the domain of f, then f is an *even* function. An even function is symmetric about the y-axis.
- If $f(-x) = -f(x)$ for all x in the domain of f, then f is an *odd* function. An odd function is symmetric about the origin.

Example 1.2.10: Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither.

a. $f(x) = -5x^4 + 7x^2 - 2$
b. $f(x) = 2x^5 - 4x + 5$
c. $f(x) = \frac{3x}{x^2+1}$

Solution

To determine whether a function is even or odd, we evaluate $f(-x)$ and compare it to $f(x)$ and $-f(x)$.

1. $f(-x) = -5(-x)^4 + 7(-x)^2 - 2 = -5x^4 + 7x^2 - 2 = f(x)$. Therefore, f is even.

2. $f(-x) = 2(-x)^5 - 4(-x) + 5 = -2x^5 + 4x + 5$. Now, $f(-x) \neq f(x)$. Furthermore, noting that $-f(x) = -2x^5 + 4x - 5$, we see that $f(-x) \neq -f(x)$. Therefore, f is neither even nor odd.

3. $f(-x) = 3(-x)/((-x)2 + 1) = -3x/(x^2 + 1) = -[3x/(x^2 + 1)] = -f(x)$. Therefore, f is odd.

Exercise 1.2.10

Determine whether $f(x) = 4x^3 - 5x$ is even, odd, or neither.

Hint

Compare $f(-x)$ with $f(x)$ and $-f(x)$.

Answer

$f(x)$ is odd.

One symmetric function that arises frequently is the **absolute value function**, written as $|x|$. The absolute value function is defined as

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Some students describe this function by stating that it "makes everything positive." By the definition of the absolute value function, we see that if $x < 0$, then $|x| = -x > 0$, and if $x > 0$, then $|x| = x > 0$. However, for $x = 0$, $|x| = 0$. Therefore, it is more accurate to say that for all nonzero inputs, the output is positive, but if $x = 0$, the output $|x| = 0$. We conclude that the range of the absolute value function is $\{y \mid y \geq 0\}$. In Figure 1.2.13, we see that the absolute value function is symmetric about the y-axis and is therefore an even function.

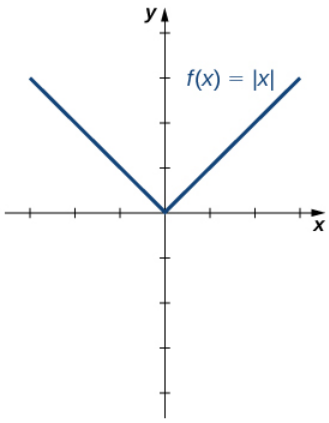


Figure 1.2.13: The graph of $f(x) = |x|$ is symmetric about the y-axis.

Example 1.2.11: Working with the Absolute Value Function

Find the domain and range of the function $f(x) = 2|x - 3| + 4$.

Solution

Since the absolute value function is defined for all real numbers, the domain of this function is $(-\infty, \infty)$. Since $|x - 3| \geq 0$ for all x, the function $f(x) = 2|x - 3| + 4 \geq 4$. Therefore, the range is, at most, the set $\{y \mid y \geq 4\}$. To see that the range is, in fact, this whole set, we need to show that for $y \geq 4$ there exists a real number x such that

$$2|x - 3| + 4 = y$$

A real number x satisfies this equation as long as

$$|x - 3| = \tfrac{1}{2}(y - 4)$$

Since $y \geq 4$, we know $y - 4 \geq 0$, and thus the right-hand side of the equation is nonnegative, so it is possible that there is a solution. Furthermore,

$$|x - 3| = \begin{cases} -(x - 3), & \text{if } x < 3 \\ x - 3, & \text{if } x \geq 3 \end{cases}$$

Therefore, we see there are two solutions:

$$x = \pm\tfrac{1}{2}(y - 4) + 3.$$

The range of this function is $\{y \mid y \geq 4\}$.

Exercise 1.2.11: Domain and Range

For the function $f(x) = |x + 2| - 4$, find the domain and range.

Hint

$|x + 2| \geq 0$ for all real numbers x.

Answer

Domain = $(-\infty, \infty)$, range = $\{y \mid y \geq -4\}$.

Key Concepts

- A function is a mapping from a set of inputs to a set of outputs with exactly one output for each input.
- If no domain is stated for a function $y = f(x)$, the domain is considered to be the set of all real numbers x for which the function is defined.
- When sketching the graph of a function f, each vertical line may intersect the graph, at most, once.

- A function may have any number of zeros, but it has, at most, one y-intercept.
- To define the composition $g \circ f$, the range of f must be contained in the domain of g.
- Even functions are symmetric about the y-axis whereas odd functions are symmetric about the origin.

Key Equations

- **Composition of two functions**

$$(g \circ f)(x) = g\big(f(x)\big)$$

- **Absolute value function**

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Glossary

absolute value function

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

composite function

given two functions f and g, a new function, denoted $g \circ f$, such that $(g \circ f)(x) = g(f(x))$

decreasing on the interval I

a function decreasing on the interval I if, for all $x_1, x_2 \in I,\ f(x_1) \geq f(x_2)$ if $x_1 < x_2$

dependent variable

the output variable for a function

domain

the set of inputs for a function

even function

a function is even if $f(-x) = f(x)$ for all x in the domain of f

function

a set of inputs, a set of outputs, and a rule for mapping each input to exactly one output

graph of a function

the set of points (x, y) such that x is in the domain of f and $y = f(x)$

increasing on the interval I

a function increasing on the interval I if for all $x_1,\ x_2 \in I,\ f(x_1) \leq f(x_2)$ if $x_1 < x_2$

independent variable

the input variable for a function

odd function

a function is odd if $f(-x) = -f(x)$ for all x in the domain of f

range

the set of outputs for a function

symmetry about the origin

the graph of a function f is symmetric about the origin if $(-x, -y)$ is on the graph of f whenever (x, y) is on the graph

symmetry about the y-axis

the graph of a function f is symmetric about the y-axis if $(-x, y)$ is on the graph of f whenever (x, y) is on the graph

table of values

a table containing a list of inputs and their corresponding outputs

vertical line test

given the graph of a function, every vertical line intersects the graph, at most, once

zeros of a function

when a real number x is a zero of a function f, $f(x) = 0$

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1.2E: Exercises

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1.3: Basic Classes of Functions

Learning Objectives

- Calculate the slope of a linear function and interpret its meaning.
- Recognize the degree of a polynomial.
- Find the roots of a quadratic polynomial.
- Describe the graphs of basic odd and even polynomial functions.
- Identify a rational function.
- Describe the graphs of power and root functions.
- Explain the difference between algebraic and transcendental functions.
- Graph a piecewise-defined function.
- Sketch the graph of a function that has been shifted, stretched, or reflected from its initial graph position.

We have studied the general characteristics of functions, so now let's examine some specific classes of functions. We begin by reviewing the basic properties of linear and quadratic functions, and then generalize to include higher-degree polynomials. By combining root functions with polynomials, we can define general algebraic functions and distinguish them from the transcendental functions we examine later in this chapter. We finish the section with examples of piecewise-defined functions and take a look at how to sketch the graph of a function that has been shifted, stretched, or reflected from its initial form.

Linear Functions and Slope

The easiest type of function to consider is a **linear function**. Linear functions have the form $f(x) = ax + b$, where a and b are constants. In Figure **1.3.1**, we see examples of linear functions when a is positive, negative, and zero. Note that if $a > 0$, the graph of the line rises as x increases. In other words, $f(x) = ax + b$ is increasing on $(-\infty, \infty)$. If $a < 0$, the graph of the line falls as x increases. In this case, $f(x) = ax + b$ is decreasing on $(-\infty, \infty)$. If $a = 0$, the line is horizontal.

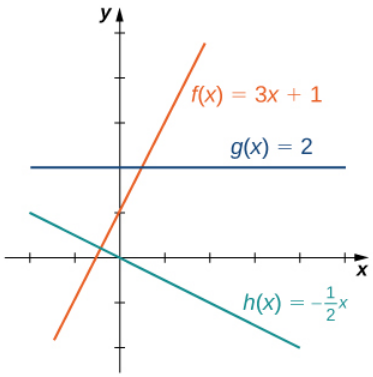


Figure **1.3.1**: These linear functions are increasing or decreasing on (∞, ∞) and one function is a horizontal line.

As suggested by Figure **1.3.1**, the graph of any linear function is a line. One of the distinguishing features of a line is its slope. The **slope** is the change in y for each unit change in x. The slope measures both the steepness and the direction of a line. If the slope is positive, the line points upward when moving from left to right. If the slope is negative, the line points downward when moving from left to right. If the slope is zero, the line is horizontal. To calculate the slope of a line, we need to determine the ratio of the change in y versus the change in x. To do so, we choose any two points (x_1, y_1) and (x_2, y_2) on the line and calculate $\frac{y_2 - y_1}{x_2 - x_1}$. In Figure **1.3.2**, we see this ratio is independent of the points chosen.

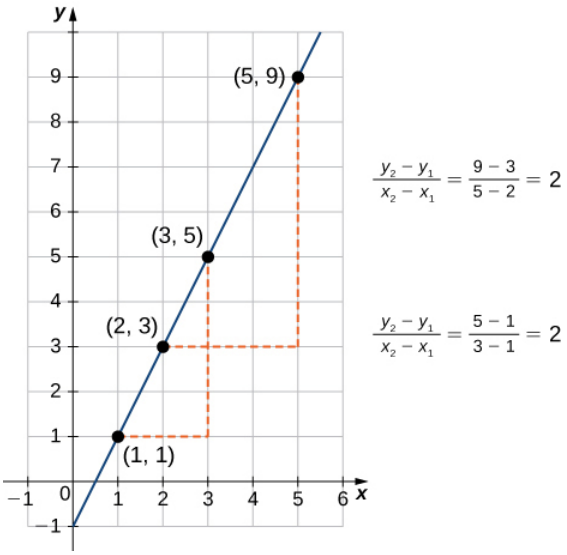


Figure **1.3.2**: For any linear function, the slope $(y_2 - y_1)/(x_2 - x_1)$ is independent of the choice of points (x_1, y_1) and (x_2, y_2) on the line.

Definition: Slope of a Linear Function

Consider line L passing through points (x_1, y_1) and (x_2, y_2). Let $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$ denote the changes in y and x,respectively. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

We now examine the relationship between slope and the formula for a linear function. Consider the linear function given by the formula $f(x) = ax + b$. As discussed earlier, we know the graph of a linear function is given by a line. We can use our definition of slope to calculate the slope of this line. As shown, we can determine the slope by calculating $(y_2 - y_1)/(x_2 - x_1)$ for any points (x_1, y_1) and (x_2, y_2) on the line. Evaluating the function f at $x = 0$, we see that $(0, b)$ is a point on this line. Evaluating this function at $x = 1$, we see that $(1, a + b)$ is also a point on this line. Therefore, the slope of this line is

$$\frac{(a + b) - b}{1 - 0} = a.$$

We have shown that the coefficient a is the slope of the line. We can conclude that the formula $f(x) = ax + b$ describes a line with slope a. Furthermore, because this line intersects the y-axis at the point $(0, b)$, we see that the y-intercept for this linear function is $(0, b)$. We conclude that the formula $f(x) = ax + b$ tells us the slope, a, and the y-intercept, $(0, b)$, for this line. Since we often use the symbol m to denote the slope of a line, we can write

$$\underbrace{f(x) = mx + b}_{\text{slope-intercept form}}$$

to denote the **slope-intercept form** of a linear function.

Sometimes it is convenient to express a linear function in different ways. For example, suppose the graph of a linear function passes through the point (x_1, y_1) and the slope of the line is m. Since any other point $(x, f(x))$ on the graph of f must satisfy the equation

$$m = \frac{f(x) - y_1}{x - x_1},$$

this linear function can be expressed by writing

$$\underbrace{f(x) - y_1 = m(x - x_1)}_{\text{point-slope equation}}.$$

We call this equation the **point-slope equation** for that linear function.

Since every nonvertical line is the graph of a linear function, the points on a nonvertical line can be described using the slope-intercept or point-slope equations. However, a vertical line does not represent the graph of a function and cannot be expressed in either of these forms. Instead, a vertical line is described by the equation $x = k$ for some constant k. Since neither the slope-intercept form nor the point-slope form allows for vertical lines, we use the notation

$$\underbrace{ax + by = c,}_{\text{standard form}}$$

where a, b are both not zero, to denote the **standard form of a line.**

Definition: Point-Slope Equation, and the Slope-Intercept Form and Standard Form of the Equation of a Line

Consider a line passing through the point (x_1, y_1) with slope m. The equation

$$y - y_1 = m(x - x_1)$$

is the *point-slope equation* for that line.

Consider a line with slope m and y-intercept $(0, b)$. The equation

$$y = mx + b$$

is an equation for that line in *slope-intercept form.*

The *standard form of a line* is given by the equation

$$ax + by = c,$$

where a and b are both not zero. This form is more general because it allows for a vertical line, $x = k$.

Example 1.3.1: Finding the Slope and Equations of Lines

Consider the line passing through the points $(11, -4)$ and $(-4, 5)$, as shown in Figure 1.3.3.

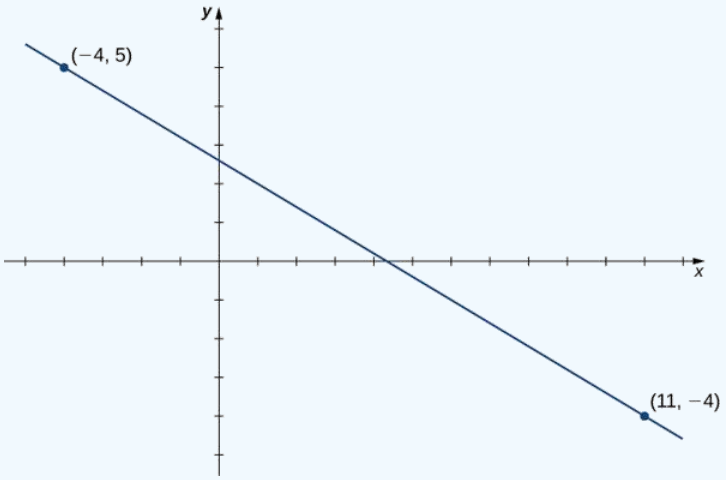


Figure 1.3.3: Finding the equation of a linear function with a graph that is a line between two given points.

1. Find the slope of the line.
2. Find an equation for this linear function in point-slope form.
3. Find an equation for this linear function in slope-intercept form.

Solution

1. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-4)}{-4 - 11} = -\frac{9}{15} = -\frac{3}{5}.$$

2. To find an equation for the linear function in point-slope form, use the slope $m = -3/5$ and choose any point on the line. If we choose the point $(11, -4)$, we get the equation

$$f(x) + 4 = -\frac{3}{5}(x - 11).$$

3. To find an equation for the linear function in slope-intercept form, solve the equation in part b. for $f(x)$. When we do this, we get the equation

$$f(x) = -\frac{3}{5}x + \frac{13}{5}.$$

Exercise 1.3.1

Consider the line passing through points $(-3, 2)$ and $(1, 4)$.

a. Find the slope of the line.
b. Find an equation of that line in point-slope form.
c. Find an equation of that line in slope-intercept form.

Hint

The slope $m = \Delta y/\Delta x$.

Answer a

$m = 1/2$.

Answer b

The point-slope form is $y - 4 = \frac{1}{2}(x - 1)$.

Answer c

The slope-intercept form is $y = \frac{1}{2}x + \frac{7}{2}$.

Example 1.3.2:

Jessica leaves her house at 5:50 a.m. and goes for a 9-mile run. She returns to her house at 7:08 a.m. Answer the following questions, assuming Jessica runs at a constant pace.

a. Describe the distance D (in miles) Jessica runs as a linear function of her run time t (in minutes).
b. Sketch a graph of D.
c. Interpret the meaning of the slope.

Solution

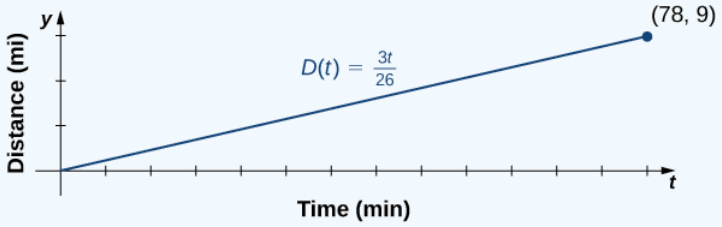
a. At time $t = 0$, Jessica is at her house, so $D(0) = 0$. At time $t = 78$ minutes, Jessica has finished running 9 mi, so $D(78) = 9$. The slope of the linear function is

$$m = \frac{9 - 0}{78 - 0} = \frac{3}{26}.$$

The y-intercept is $(0, 0)$, so the equation for this linear function is

$$D(t) = \frac{3}{26}t.$$

b. To graph D, use the fact that the graph passes through the origin and has slope $m = 3/26$.



c. The slope $m = 3/26 \approx 0.115$ describes the distance (in miles) Jessica runs per minute, or her average velocity.

Polynomials

A linear function is a special type of a more general class of functions: polynomials. A polynomial function is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

for some integer $n \geq 0$ and constants $a_n, a_{n-1}, ..., a_0$, where $a_n \neq 0$. In the case when $n = 0$, we allow for $a_0 = 0$; if $a_0 = 0$, the function $f(x) = 0$ is called the **zero function**. The value n is called the **degree** of the polynomial; the constant a_n is called the **leading coefficient**. A linear function of the form $f(x) = mx + b$ is a polynomial of degree 1 if $m \neq 0$ and degree 0 if $m = 0$. A polynomial of degree 0 is also called a **constant function**. A polynomial function of degree 2 is called a **quadratic function**. In particular, a quadratic function has the form

$$f(x) = ax^2 + bx + c,$$

where $a \neq 0$. A polynomial function of degree **3** is called a **cubic function**.

Power Functions

Some polynomial functions are power functions. A **power function** is any function of the form $f(x) = ax^b$, where a and b are any real numbers. The exponent in a power function can be any real number, but here we consider the case when the exponent is a positive integer. (We consider other cases later.) If the exponent is a positive integer, then $f(x) = ax^n$ is a polynomial. If n is even, then $f(x) = ax^n$ is an even function because $f(-x) = a(-x)^n = ax^n$ if n is even. If n is odd, then $f(x) = ax^n$ is an odd function because $f(-x) = a(-x)^n = -ax^n$ if n is odd (Figure **1.3.4**).

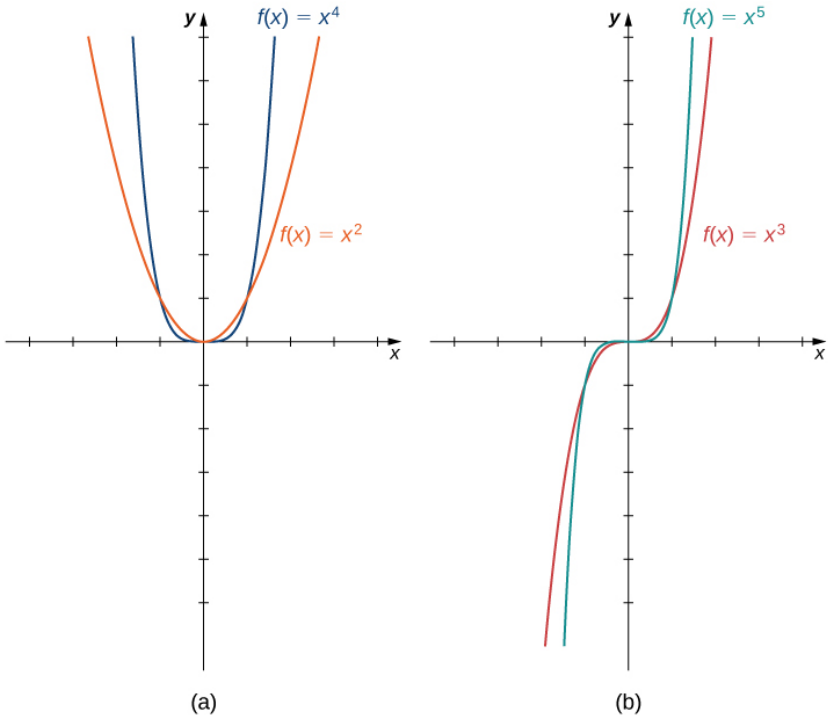


Figure 1.3.4: (a) For any even integer n, $f(x) = ax^n$ is an even function. (b) For any odd integer n, $f(x) = ax^n$ is an odd function.

Behavior at Infinity

To determine the behavior of a function f as the inputs approach infinity, we look at the values $f(x)$ as the inputs, x, become larger. For some functions, the values of $f(x)$ approach a finite number. For example, for the function $f(x) = 2 + 1/x$, the values $1/x$ become closer and closer to zero for all values of x as they get larger and larger. For this function, we say "$f(x)$ approaches two as x goes to infinity," and we write $f(x) \to 2$ as $x \to \infty$. The line $y = 2$ is a horizontal asymptote for the function $f(x) = 2 + 1/x$ because the graph of the function gets closer to the line as x gets larger.

For other functions, the values $f(x)$ may not approach a finite number but instead may become larger for all values of x as they get larger. In that case, we say "$f(x)$ approaches infinity as x approaches infinity," and we write $f(x) \to \infty$ as $x \to \infty$. For example, for the function $f(x) = 3x^2$, the outputs $f(x)$ become larger as the inputs x get larger. We can conclude that the function $f(x) = 3x^2$ approaches infinity as x approaches infinity, and we write $3x^2 \to \infty$ as $x \to \infty$. The behavior as $x \to -\infty$ and the meaning of $f(x) \to -\infty$ as $x \to \infty$ or $x \to -\infty$ can be defined similarly. We can describe what happens to the values of $f(x)$ as $x \to \infty$ and as $x \to -\infty$ as the **end behavior** of the function.

To understand the end behavior for polynomial functions, we can focus on quadratic and cubic functions. The behavior for higher-degree polynomials can be analyzed similarly. Consider a quadratic function $f(x) = ax^2 + bx + c$. If $a > 0$, the values $f(x) \to \infty$ as $x \to \pm\infty$. If $a < 0$, the values $f(x) \to -\infty$ as $x \to \pm\infty$. Since the graph of a quadratic function is a parabola, the parabola opens upward if $a > 0$.; the parabola opens downward if $a < 0$ (Figure 1.3.5a).

Now consider a cubic function $f(x) = ax^3 + bx^2 + cx + d$. If $a > 0$, then $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. If $a < 0$, then $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$. As we can see from both of these graphs, the leading term of the polynomial determines the end *behavior* (Figure 1.3.5b).

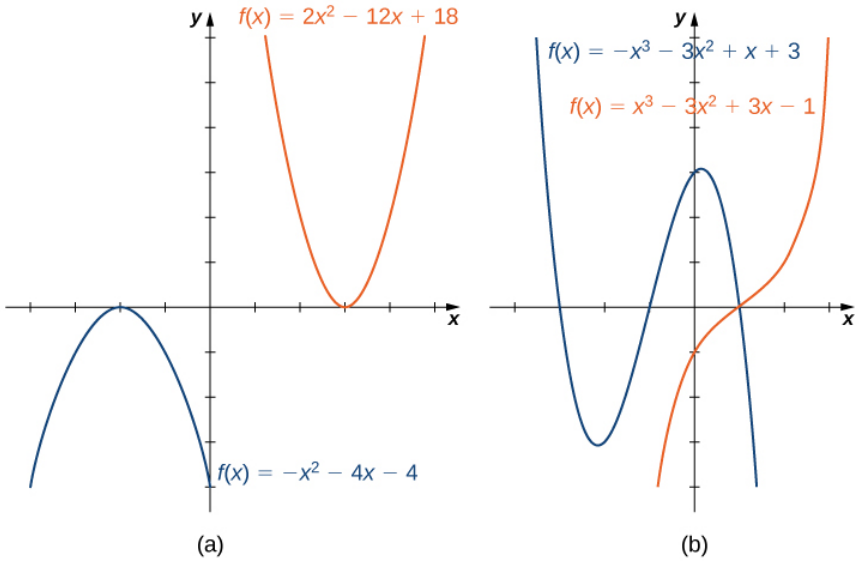


Figure 1.3.5: (a) For a quadratic function, if the leading coefficient $a > 0$,the parabola opens upward. If $a < 0$, the parabola opens downward. (b) For a cubic function f, if the leading coefficient $a > 0$, the values $f(x) \to \infty$ as $x \to \infty$ and the values $f(x) \to -\infty$ as $x \to -\infty$. If the leading coefficient $a < 0$, the opposite is true.

Zeros of Polynomial Functions

Another characteristic of the graph of a polynomial function is where it intersects the x-axis. To determine where a function f intersects the x-axis, we need to solve the equation $f(x) = 0$ for x. In the case of the linear function $f(x) = mx + b$, the x-intercept is given by solving the equation $mx + b = 0$. In this case, we see that the x-intercept is given by $(-b/m, 0)$. In the case of a quadratic function, finding the x-intercept(s) requires finding the zeros of a quadratic equation: $ax^2 + bx + c = 0$. In some cases, it is easy to factor the polynomial $ax^2 + bx + c$ to find the zeros. If not, we make use of the quadratic formula.

The Quadratic Formula

Consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. The solutions of this equation are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.3.1}$$

If the discriminant $b^2 - 4ac > 0$, Equation 1.3.1 tells us there are two real numbers that satisfy the quadratic equation. If $b^2 - 4ac = 0$, this formula tells us there is only one solution, and it is a real number. If $b^2 - 4ac < 0$, no real numbers satisfy the quadratic equation.

In the case of higher-degree polynomials, it may be more complicated to determine where the graph intersects the x-axis. In some instances, it is possible to find the x-intercepts by factoring the polynomial to find its zeros. In other cases, it is impossible to calculate the exact values of the x-intercepts. However, as we see later in the text, in cases such as this, we can use analytical tools to approximate (to a very high degree) where the x-intercepts are located. Here we focus on the graphs of polynomials for which we can calculate their zeros explicitly.

Example 1.3.3: Graphing Polynomial Functions

For the following functions,

a. $f(x) = -2x^2 + 4x - 1$
b. $f(x) = x^3 - 3x^2 - 4x$

i. describe the behavior of $f(x)$ as $x \to \pm\infty$,

ii. find all zeros of f, and

iii. sketch a graph of f.

Solution

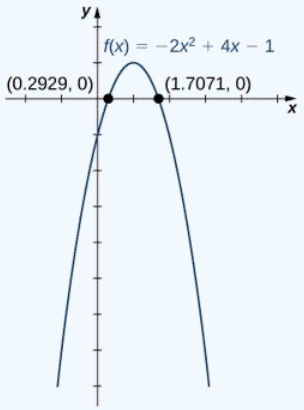
1. The function $f(x) = -2x^2 + 4x - 1$ is a quadratic function.

    1. Because $a = -2 < 0$, as $x \to \pm\infty, f(x) \to -\infty$.

    2. To find the zeros of f, use the quadratic formula. The zeros are

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(-1)}}{2(-2)} = \frac{-4 \pm \sqrt{8}}{-4} = \frac{-4 \pm 2\sqrt{2}}{-4} = \frac{2 \pm \sqrt{2}}{2}.$$

    3. To sketch the graph of f,use the information from your previous answers and combine it with the fact that the graph is a parabola opening downward.



2. The function $f(x) = x^3 - 3x^2 - 4x$ is a cubic function.

    1. Because $a = 1 > 0$, as $x \to \infty$, $f(x) \to \infty$. As $x \to -\infty$, $f(x) \to -\infty$.

    2. To find the zeros of f, we need to factor the polynomial. First, when we factor x out of all the terms, we find

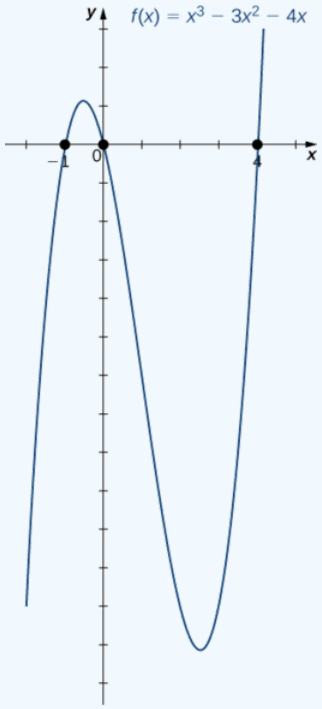
$$f(x) = x(x^2 - 3x - 4).$$

    Then, when we factor the quadratic function $x^2 - 3x - 4$, we find

$$f(x) = x(x - 4)(x + 1).$$

    Therefore, the zeros of f are $x = 0, 4, -1$.

3. Combining the results from parts i. and ii., draw a rough sketch of f.



? Exercise 1.3.2

Consider the quadratic function $f(x) = 3x^2 - 6x + 2$. Find the zeros of f. Does the parabola open upward or downward?

Hint

Use the quadratic formula.

Answer

The zeros are $x = 1 \pm \sqrt{3}/3$. The parabola opens upward.

Mathematical Models

A large variety of real-world situations can be described using **mathematical models**. A mathematical model is a method of simulating real-life situations with mathematical equations. Physicists, engineers, economists, and other researchers develop models by combining observation with quantitative data to develop equations, functions, graphs, and other mathematical tools to describe the behavior of various systems accurately. Models are useful because they help predict future outcomes. Examples of mathematical models include the study of population dynamics, investigations of weather patterns, and predictions of product sales.

As an example, let's consider a mathematical model that a company could use to describe its revenue for the sale of a particular item. The amount of revenue R a company receives for the sale of n items sold at a price of p dollars per item is described by the equation $R = p \cdot n$. The company is interested in how the sales change as the price of the item changes. Suppose the data in Table 1.3.1 show the number of units a company sells as a function of the price per item.

Table 1.3.1: Number of Units Sold n (in Thousands) as a Function of Price per Unit p (in Dollars)

p	6	8	10	12	14
n	19.4	18.5	16.2	13.8	12.2

In Figure 1.3.6, we see the graph the number of units sold (in thousands) as a function of price (in dollars). We note from the shape of the graph that the number of units sold is likely a linear function of price per item, and the data can be closely approximated by

the linear function $n = -1.04p + 26$ for $0 \le p \le 25$, where n predicts the number of units sold in thousands. Using this linear function, the revenue (in thousands of dollars) can be estimated by the quadratic function

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p \text{ for } 0 \le p \le 25.$$

In Example 1.3.4, we use this quadratic function to predict the amount of revenue the company receives depending on the price the company charges per item. Note that we cannot conclude definitively the actual number of units sold for values of p, for which no data are collected. However, given the other data values and the graph shown, it seems reasonable that the number of units sold (in thousands) if the price charged is p dollars may be close to the values predicted by the linear function $n = -1.04p + 26$.

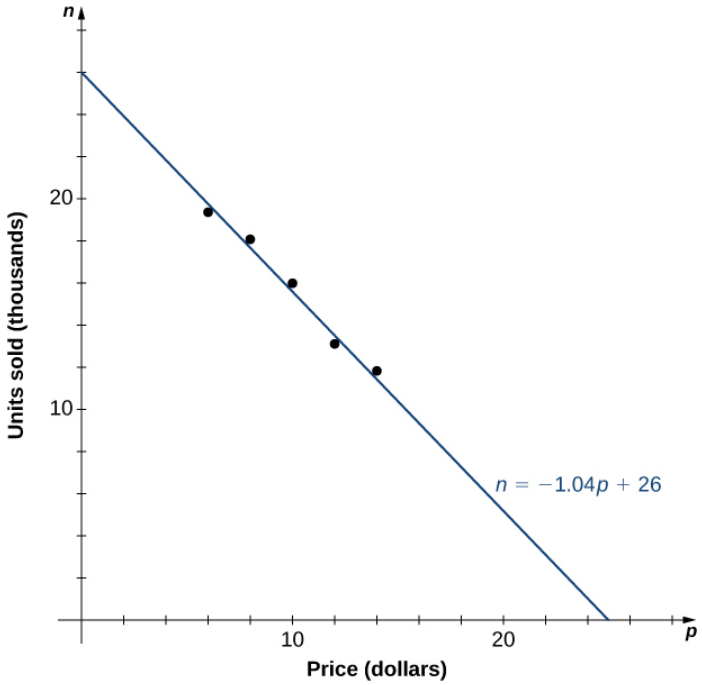


Figure 1.3.6: The data collected for the number of items sold as a function of price is roughly linear. We use the linear function $n = -1.04p + 26$ to estimate this function.

Example 1.3.4: Maximizing Revenue

A company is interested in predicting the amount of revenue it will receive depending on the price it charges for a particular item. Using the data from Table 1.3.1, the company arrives at the following quadratic function to model revenue R as a function of price per item p:

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p$$

for $0 \le p \le 25$.

a. Predict the revenue if the company sells the item at a price of $p = \$5$ and $p = \$17$.
b. Find the zeros of this function and interpret the meaning of the zeros.
c. Sketch a graph of R.
d. Use the graph to determine the value of p that maximizes revenue. Find the maximum revenue.

Solution

a. Evaluating the revenue function at $p = 5$ and $p = 17$, we can conclude that

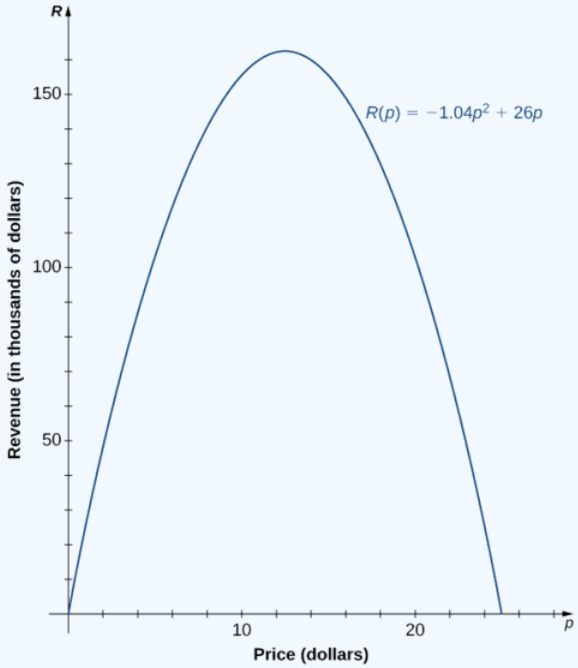
$$R(5) = -1.04(5)^2 + 26(5) = 104, \text{ so revenue} = \$104,000;$$

$$R(17) = -1.04(17)^2 + 26(17) = 141.44, \text{ so revenue} = \$141,440.$$

b. The zeros of this function can be found by solving the equation $-1.04p^2 + 26p = 0$. When we factor the quadratic expression, we get $p(-1.04p + 26) = 0$. The solutions to this equation are given by $p = 0, 25$. For these values of p, the

revenue is zero. When $p = \$0$, the revenue is zero because the company is giving away its merchandise for free. When $p = \$25$,the revenue is zero because the price is too high, and no one will buy any items.

c. Knowing the fact that the function is quadratic, we also know the graph is a parabola. Since the leading coefficient is negative, the parabola opens downward. One property of parabolas is that they are symmetric about the axis of symmetry, so since the zeros are at $p = 0$ and $p = 25$, the parabola must be symmetric about the line halfway between them, or $p = 12.5$.



d. The function is a parabola with zeros at $p = 0$ and $p = 25$, and it is symmetric about the line $p = 12.5$, so the maximum revenue occurs at a price of $p = \$12.50$ per item. At that price, the revenue is $R(p) = -1.04(12.5)^2 + 26(12.5) = \$162,500$.

Algebraic Functions

By allowing for quotients and fractional powers in polynomial functions, we create a larger class of functions. An **algebraic function** is one that involves addition, subtraction, multiplication, division, rational powers, and roots. Two types of algebraic functions are rational functions and root functions.

Just as rational numbers are quotients of integers, rational functions are quotients of polynomials. In particular, a **rational function** is any function of the form $f(x) = p(x)/q(x)$,where $p(x)$ and $q(x)$ are polynomials. For example,

$$f(x) = \frac{3x-1}{5x+2} \text{ and } g(x) = \frac{4}{x^2+1}$$

are rational functions. A **root function** is a power function of the form $f(x) = x^{1/n}$, where n is a positive integer greater than one. For example, $f(x) = x^{1/2} = \sqrt{x}$ is the square-root function and $g(x) = x^{1/3} = \sqrt[3]{x}$ is the cube-root function. By allowing for compositions of root functions and rational functions, we can create other algebraic functions. For example, $f(x) = \sqrt{4-x^2}$ is an algebraic function.

Example 1.3.5: Finding Domain and Range for Algebraic Functions

For each of the following functions, find the domain and range.

a. $f(x) = \dfrac{3x-1}{5x+2}$
b. $f(x) = \sqrt{4-x^2}$

Solution

1. It is not possible to divide by zero, so the domain is the set of real numbers x such that $x \neq -2/5$. To find the range, we need to find the values y for which there exists a real number x such that

$$y = \frac{3x-1}{5x+2}$$

When we multiply both sides of this equation by $5x+2$, we see that x must satisfy the equation

$$5xy + 2y = 3x - 1.$$

From this equation, we can see that x must satisfy

$$2y + 1 = x(3 - 5y).$$

If y=$3/5$, this equation has no solution. On the other hand, as long as $y \neq 3/5$,

$$x = \frac{2y+1}{3-5y}$$

satisfies this equation. We can conclude that the range of f is $\{y \mid y \neq 3/5\}$.

2. To find the domain of f, we need $4 - x^2 \geq 0$. When we factor, we write $4 - x^2 = (2-x)(2+x) \geq 0$. This inequality holds if and only if both terms are positive or both terms are negative. For both terms to be positive, we need to find x such that

$$2 - x \geq 0 \text{ and } 2 + x \geq 0.$$

These two inequalities reduce to $2 \geq x$ and $x \geq -2$. Therefore, the set $\{x \mid -2 \leq x \leq 2\}$ must be part of the domain. For both terms to be negative, we need

$$2 - x \leq 0 \text{ and } 2 + x \leq 0.$$

These two inequalities also reduce to $2 \leq x$ and $x \leq -2$. There are no values of x that satisfy both of these inequalities. Thus, we can conclude the domain of this function is $\{x \mid -2 \leq x \leq 2\}$.

If $-2 \leq x \leq 2$, then $0 \leq 4 - x^2 \leq 4$. Therefore, $0 \leq \sqrt{4-x2} \leq 2$, and the range of f is $\{y \mid 0 \leq y \leq 2\}$.

? Exercise 1.3.3

Find the domain and range for the function $f(x) = (5x+2)/(2x-1)$.

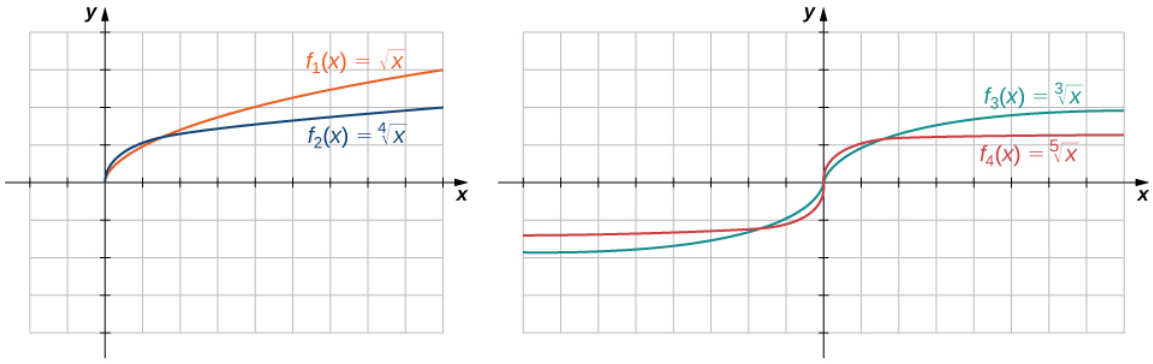
Hint

The denominator cannot be zero. Solve the equation $y = (5x+2)/(2x-1)$ for x to find the range.

Answer

The domain is the set of real numbers x such that $x \neq 1/2$. The range is the set $\{y \mid y \neq 5/2\}$.

The root functions $f(x) = x^{1/n}$ have defining characteristics depending on whether n is odd or even. For all even integers $n \geq 2$, the domain of $f(x) = x^{1/n}$ is the interval $[0, \infty)$. For all odd integers $n \geq 1$, the domain of $f(x) = x^{1/n}$ is the set of all real numbers. Since $x^{1/n} = (-x)^{1/n}$ for odd integers $n, f(x) = x^{1/n}$ is an odd function if n is odd. See the graphs of root functions for different values of n in Figure 1.3.7.



(a) (b)

Figure 1.3.7: (a) If n is even, the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$. (b) If n is odd, the domain of $f(x) = \sqrt[n]{x}$ is $(-\infty, \infty)$ and the function $f(x) = \sqrt[n]{x}$ is an odd function.

Example 1.3.6: Finding Domains for Algebraic Functions

For each of the following functions, determine the domain of the function.

a. $f(x) = \dfrac{3}{x^2 - 1}$
b. $f(x) = \dfrac{2x + 5}{3x^2 + 4}$
c. $f(x) = \sqrt{4 - 3x}$
d. $f(x) = \sqrt[3]{2x - 1}$

Solution

a. You cannot divide by zero, so the domain is the set of values x such that $x^2 - 1 \neq 0$. Therefore, the domain is $\{x \mid x \neq \pm 1\}$.
b. You need to determine the values of x for which the denominator is zero. Since $3x^2 + 4 \geq 4$ for all real numbers x, the denominator is never zero. Therefore, the domain is $(-\infty, \infty)$.
c. Since the square root of a negative number is not a real number, the domain is the set of values x for which $4 - 3x \geq 0$. Therefore, the domain is $\{x \mid x \leq 4/3\}$.
d. The cube root is defined for all real numbers, so the domain is the interval $(-\infty, \infty)$.

Exercise 1.3.4

Find the domain for each of the following functions: $f(x) = (5 - 2x)/(x^2 + 2)$ and $g(x) = \sqrt{5x - 1}$.

Hint

Determine the values of x when the expression in the denominator of f is nonzero, and find the values of x when the expression inside the radical of g is nonnegative.

Answer

The domain of f is $(-\infty, \infty)$. The domain of g is $\{x \mid x \geq 1/5\}$.

Transcendental Functions

Thus far, we have discussed algebraic functions. Some functions, however, cannot be described by basic algebraic operations. These functions are known as **transcendental functions** because they are said to "transcend," or go beyond, algebra. The most common transcendental functions are trigonometric, exponential, and logarithmic functions. A trigonometric function relates the ratios of two sides of a right triangle. They are $\sin x, \cos x, \tan x, \cot x, \sec x,$ and $\csc x$. (We discuss trigonometric functions later in the chapter.) An exponential function is a function of the form $f(x) = b^x$, where the base $b > 0$, $b \neq 1$. A logarithmic function is a function of the form $f(x) = \log_b(x)$ for some constant $b > 0$, $b \neq 1$, where $\log_b(x) = y$ if and only if $b^y = x$. (We also discuss exponential and logarithmic functions later in the chapter.)

Example 1.3.7: Classifying Algebraic and Transcendental Functions

Classify each of the following functions, a. through c., as algebraic or transcendental.

a. $f(x) = \dfrac{\sqrt{x^3+1}}{4x+2}$
b. $f(x) = 2^{x^2}$
c. $f(x) = \sin(2x)$

Solution

a. Since this function involves basic algebraic operations only, it is an algebraic function.
b. This function cannot be written as a formula that involves only basic algebraic operations, so it is transcendental. (Note that algebraic functions can only have powers that are rational numbers.)
c. As in part b, this function cannot be written using a formula involving basic algebraic operations only; therefore, this function is transcendental.

Exercise 1.3.5:

Is $f(x) = x/2$ an algebraic or a transcendental function?

Answer

Algebraic

Piecewise-Defined Functions

Sometimes a function is defined by different formulas on different parts of its domain. A function with this property is known as a piecewise-defined function. The absolute value function is an example of a piecewise-defined function because the formula changes with the sign of x:

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}.$$

Other piecewise-defined functions may be represented by completely different formulas, depending on the part of the domain in which a point falls. To graph a piecewise-defined function, we graph each part of the function in its respective domain, on the same coordinate system. If the formula for a function is different for $x < a$ and $x > a$, we need to pay special attention to what happens at $x = a$ when we graph the function. Sometimes the graph needs to include an open or closed circle to indicate the value of the function at $x = a$. We examine this in the next example.

Example 1.3.8: Graphing a Piecewise-Defined Function

Sketch a graph of the following piecewise-defined function:

$$f(x) = \begin{cases} x+3, & \text{if } x < 1 \\ (x-2)^2, & \text{if } x \geq 1 \end{cases}$$

Solution

Graph the linear function $y = x + 3$ on the interval $(-\infty, 1)$ and graph the quadratic function $y = (x-2)^2$ on the interval $[1, \infty)$. Since the value of the function at $x = 1$ is given by the formula $f(x) = (x-2)^2$, we see that $f(1) = 1$. To indicate this on the graph, we draw a closed circle at the point $(1, 1)$. The value of the function is given by $f(x) = x + 3$ for all $x < 1$, but not at $x = 1$. To indicate this on the graph, we draw an open circle at $(1, 4)$.

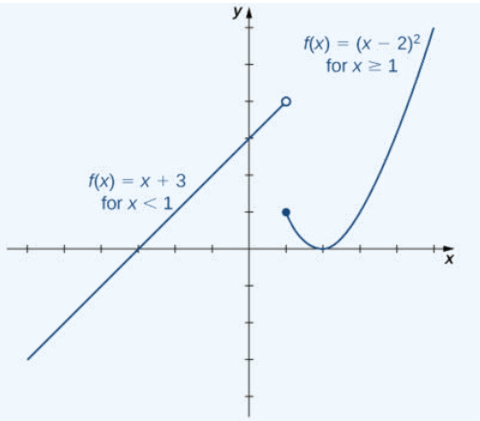
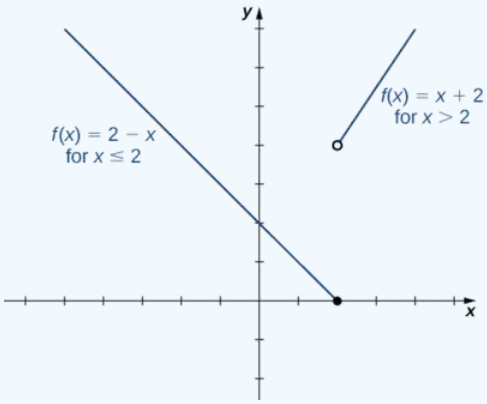


Figure 1.3.8: This piecewise-defined function is linear for $x < 1$ and quadratic for $x \geq 1$.

2) Sketch a graph of the function

$$f(x) = \begin{cases} 2 - x, & \text{if } x \leq 2 \\ x + 2, & \text{if } x > 2 \end{cases}.$$

Solution:



Example 1.3.9: Parking Fees Described by a Piecewise-Defined Function

In a big city, drivers are charged variable rates for parking in a parking garage. They are charged $10 for the first hour or any part of the first hour and an additional $2 for each hour or part thereof up to a maximum of $30 for the day. The parking garage is open from 6 a.m. to 12 midnight.

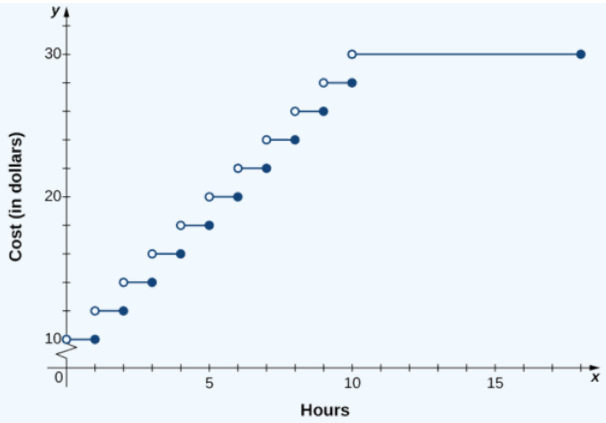
a. Write a piecewise-defined function that describes the cost C to park in the parking garage as a function of hours parked x.
b. Sketch a graph of this function $C(x)$.

Solution

1.Since the parking garage is open 18 hours each day, the domain for this function is $\{x \mid 0 < x \leq 18\}$. The cost to park a car at this parking garage can be described piecewise by the function

$$C(x) = \begin{cases} 10, & \text{for } 0 < x \leq 1 \\ 12, & \text{for } 1 < x \leq 2 \\ 14, & \text{for } 2 < x \leq 3 \\ 16, & \text{for } 3 < x \leq 4 \\ \vdots & \\ 30, & \text{for } 10 < x \leq 18 \end{cases}.$$

2.The graph of the function consists of several horizontal line segments.



? Exercise 1.3.6

The cost of mailing a letter is a function of the weight of the letter. Suppose the cost of mailing a letter is $49¢$ for the first ounce and $21¢$ for each additional ounce. Write a piecewise-defined function describing the cost C as a function of the weight x for $0 < x \leq 3$, where C is measured in cents and x is measured in ounces.

Hint

The piecewise-defined function is constant on the intervals $(0, 1]$, $(1, 2]$, ...

Answer

$$C(x) = \begin{cases} 49, 0 < x \leq 1 \\ 70, 1 < x \leq 2 \\ 91, 2 < x \leq 3 \end{cases}$$

Transformations of Functions

We have seen several cases in which we have added, subtracted, or multiplied constants to form variations of simple functions. In the previous example, for instance, we subtracted 2 from the argument of the function $y = x^2$ to get the function $f(x) = (x - 2)^2$. This subtraction represents a shift of the function $y = x^2$ two units to the right. A shift, horizontally or vertically, is a type of **transformation of a function**. Other transformations include horizontal and vertical scalings, and reflections about the axes.

A vertical shift of a function occurs if we add or subtract the same constant to each output y. For $c > 0$, the graph of $f(x) + c$ is a shift of the graph of $f(x)$ up c units, whereas the graph of $f(x) - c$ is a shift of the graph of $f(x)$ down c units. For example, the graph of the function $f(x) = x^3 + 4$ is the graph of $y = x^3$ shifted up 4 units; the graph of the function $f(x) = x^3 - 4$ is the graph of $y = x^3$ shifted down 4 units (Figure 1.3.9).

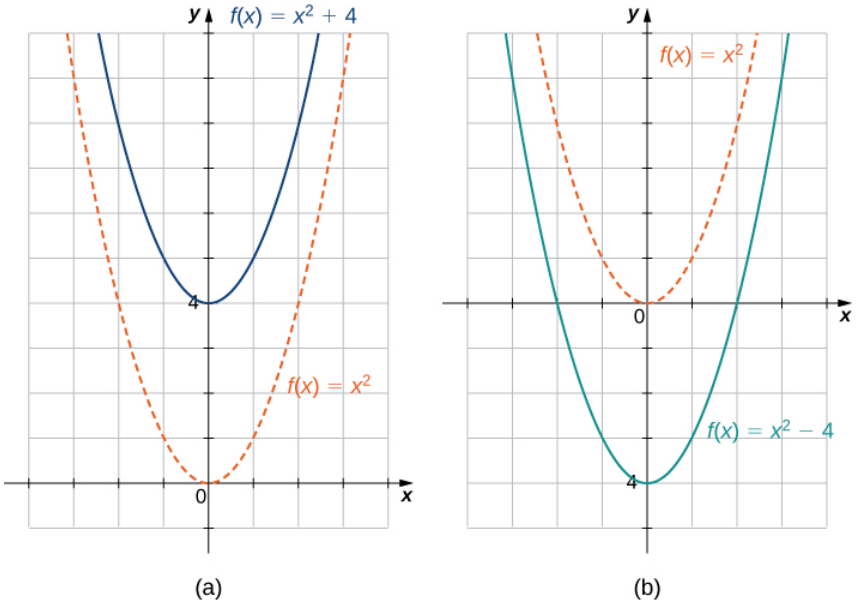


Figure 1.3.9: (a) For $c > 0$, the graph of $y = f(x) + c$ is a vertical shift up c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x) - c$ is a vertical shift down c units of the graph of $y = f(x)$.

A horizontal shift of a function occurs if we add or subtract the same constant to each input x. For $c > 0$, the graph of $f(x + c)$ is a shift of the graph of $f(x)$ to the left c units; the graph of $f(x - c)$ is a shift of the graph of $f(x)$ to the right c units. Why does the graph shift left when adding a constant and shift right when subtracting a constant? To answer this question, let's look at an example.

Consider the function $f(x) = |x + 3|$ and evaluate this function at $x - 3$. Since $f(x - 3) = |x|$ and $x - 3 < x$, the graph of $f(x) = |x + 3|$ is the graph of $y = |x|$ shifted left 3 units. Similarly, the graph of $f(x) = |x - 3|$ is the graph of $y = |x|$ shifted right 3 units (Figure 1.3.10).

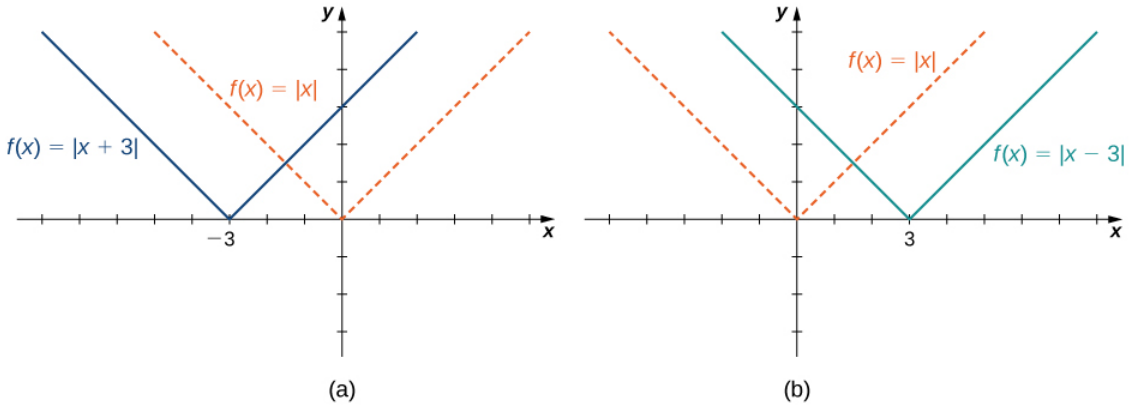


Figure 1.3.10: (a) For $c > 0$, the graph of $y = f(x + c)$ is a horizontal shift left c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x - c)$ is a horizontal shift right c units of the graph of $y = f(x)$.

A vertical scaling of a graph occurs if we multiply all outputs y of a function by the same positive constant. For $c > 0$, the graph of the function $cf(x)$ is the graph of $f(x)$ scaled vertically by a factor of c. If $c > 1$, the values of the outputs for the function $cf(x)$ are larger than the values of the outputs for the function $f(x)$; therefore, the graph has been stretched vertically. If $0 < c < 1$, then the outputs of the function $cf(x)$ are smaller, so the graph has been compressed. For example, the graph of the function $f(x) = 3x^2$ is the graph of $y = x^2$ stretched vertically by a factor of 3, whereas the graph of $f(x) = x^2/3$ is the graph of $y = x^2$ compressed vertically by a factor of 3 (Figure 1.3.11*b*).

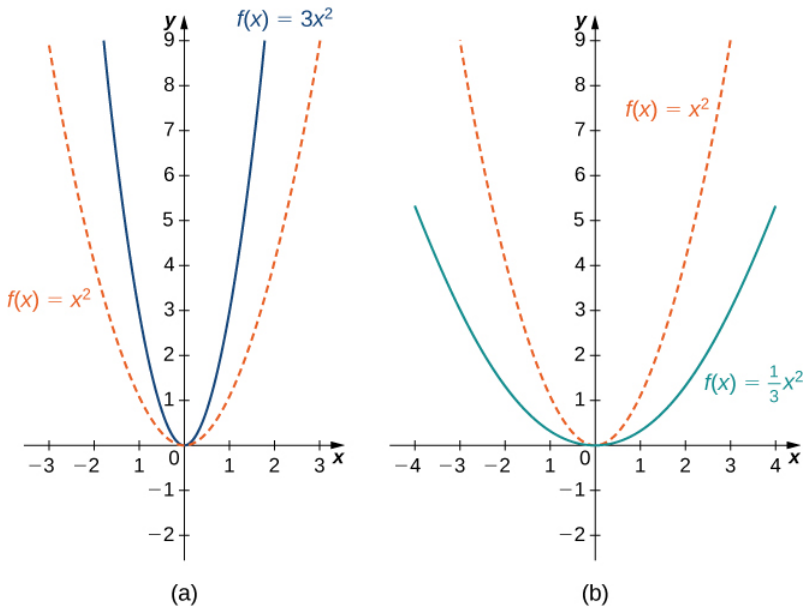


Figure 1.3.11: (a) If $c > 1$, the graph of $y = cf(x)$ is a vertical stretch of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = cf(x)$ is a vertical compression of the graph of $y = f(x)$.

The horizontal scaling of a function occurs if we multiply the inputs x by the same positive constant. For $c > 0$, the graph of the function $f(cx)$ is the graph of $f(x)$ scaled horizontally by a factor of c. If $c > 1$, the graph of $f(cx)$ is the graph of $f(x)$ compressed horizontally. If $0 < c < 1$, the graph of $f(cx)$ is the graph of $f(x)$ stretched horizontally. For example, consider the function $f(x) = \sqrt{2x}$ and evaluate f at $x/2$. Since $f(x/2) = \sqrt{x}$, the graph of $f(x) = \sqrt{2x}$ is the graph of $y = \sqrt{x}$ compressed horizontally. The graph of $y = \sqrt{x/2}$ is a horizontal stretch of the graph of $y = \sqrt{x}$ (Figure 1.3.12).

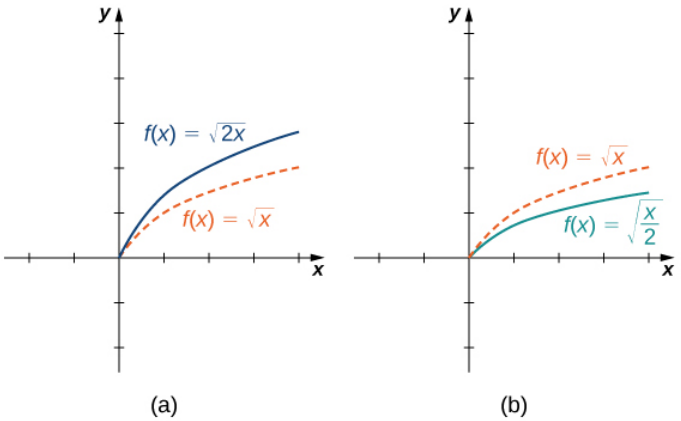


Figure 1.3.12: (a) If $c > 1$, the graph of $y = f(cx)$ is a horizontal compression of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = f(cx)$ is a horizontal stretch of the graph of $y = f(x)$.

We have explored what happens to the graph of a function f when we multiply f by a constant $c > 0$ to get a new function $cf(x)$. We have also discussed what happens to the graph of a function $f$when we multiply the independent variable x by $c > 0$ to get a new function $f(cx)$. However, we have not addressed what happens to the graph of the function if the constant c is negative. If we have a constant $c < 0$, we can write c as a positive number multiplied by -1; but, what kind of transformation do we get when we multiply the function or its argument by -1? When we multiply all the outputs by -1, we get a reflection about the x-axis. When we multiply all inputs by -1, we get a reflection about the y-axis. For example, the graph of $f(x) = -(x^3 + 1)$ is the graph of $y = (x^3 + 1)$ reflected about the x-axis. The graph of $f(x) = (-x)^3 + 1$ is the graph of $y = x^3 + 1$ reflected about the y-axis (Figure 1.3.13).

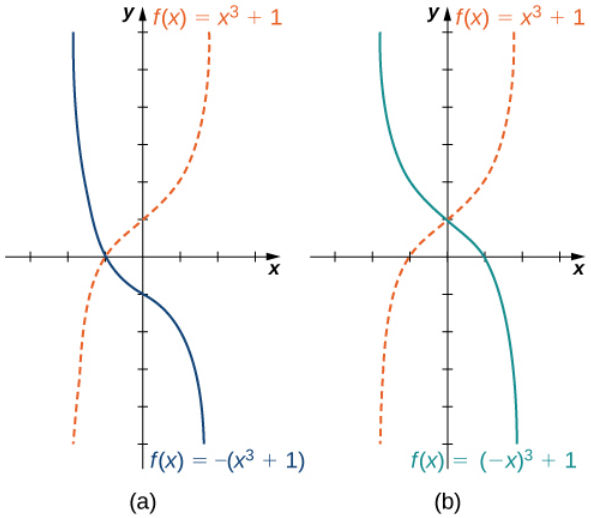


Figure **1.3.13**: (a) The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis. (b) The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y-axis.

If the graph of a function consists of more than one transformation of another graph, it is important to transform the graph in the correct order. Given a function $f(x)$, the graph of the related function $y = cf(a(x + b)) + d$ can be obtained from the graph of $y = f(x)$by performing the transformations in the following order.

- Horizontal shift of the graph of $y = f(x)$. If $b > 0$, shift left. If $b < 0$ shift right.
- Horizontal scaling of the graph of $y = f(x + b)$ by a factor of $|a|$. If $a < 0$, reflect the graph about the y-axis.
- Vertical scaling of the graph of $y = f(a(x + b))$ by a factor of $|c|$. If $c < 0$, reflect the graph about the x -axis.
- Vertical shift of the graph of $y = cf(a(x + b))$. If $d > 0$, shift up. If $d < 0$, shift down.

We can summarize the different transformations and their related effects on the graph of a function in the following table.

Transformation of $f(c > 0)$	Effect of the graph of f
$f(x) + c$	Vertical shift up c units
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if $0 < c < 1$
$f(cx)$	Horizontal stretch if $0 < c < 1$; horizontal compression if $c > 1$
$-f(x)$	Reflection about the x-axis
$f(-x)$	Reflection about the y-axis

Example 1.3.10: Transforming a Function

For each of the following functions, a. and b., sketch a graph by using a sequence of transformations of a well-known function.

a. $f(x) = -|x + 2| - 3$
b. $f(x) = 3\sqrt{-x} + 1$

Solution

1.Starting with the graph of $y = |x|$, shift 2 units to the left, reflect about the x-axis, and then shift down 3 units.

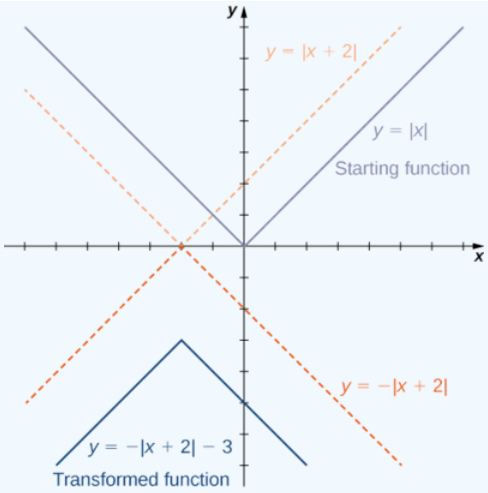


Figure 1.3.14: The function $f(x) = -|x+2| - 3$ can be viewed as a sequence of three transformations of the function $y = |x|$.

2. Starting with the graph of $y = \sqrt{x}$, reflect about the y-axis, stretch the graph vertically by a factor of 3, and move up 1 unit.

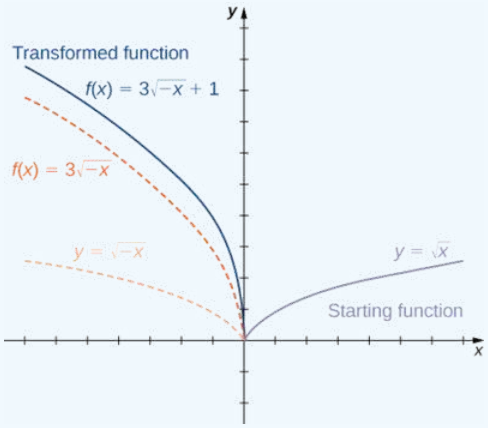


Figure 1.3.15: The function $f(x) = 3\sqrt{-x} + 1$ can be viewed as a sequence of three transformations of the function $y = \sqrt{x}$.

? Exercise 1.3.7

Describe how the function $f(x) = -(x+1)^2 - 4$ can be graphed using the graph of $y = x^2$ and a sequence of transformations

Answer

Shift the graph $y = x^2$ to the left 1 unit, reflect about the x-axis, then shift down 4 units.

Key Concepts

- The power function $f(x) = x^n$ is an even function if n is even and $n \neq 0$, and it is an odd function if n is odd.
- The root function $f(x) = x^{1/n}$ has the domain $[0, \infty)$ if n is even and the domain $(-\infty, \infty)$ if n is odd. If n is odd, then $f(x) = x^{1/n}$ is an odd function.
- The domain of the rational function $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomial functions, is the set of x such that $q(x) \neq 0$.
- Functions that involve the basic operations of addition, subtraction, multiplication, division, and powers are algebraic functions. All other functions are transcendental. Trigonometric, exponential, and logarithmic functions are examples of transcendental functions.
- A polynomial function f with degree $n \geq 1$ satisfies $f(x) \to \pm\infty$ as $x \to \pm\infty$. The sign of the output as $x \to \infty$ depends on the sign of the leading coefficient only and on whether n is even or odd.
- Vertical and horizontal shifts, vertical and horizontal scalings, and reflections about the x- and y-axes are examples of transformations of functions.

Key Equations

- **Point-slope equation of a line**

$$y - y_1 = m(x - x_1)$$

- **Slope-intercept form of a line**

$$y = mx + b$$

- **Standard form of a line**

$$ax + by = c$$

- **Polynomial function**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Glossary

algebraic function
a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable x

cubic function
a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

degree
for a polynomial function, the value of the largest exponent of any term

linear function
a function that can be written in the form $f(x) = mx + b$

logarithmic function
a function of the form $f(x) = \log_b(x)$ for some base $b > 0$, $b \neq 1$ such that $y = \log_b(x)$ if and only if $b^y = x$

mathematical model
A method of simulating real-life situations with mathematical equations

piecewise-defined function
a function that is defined differently on different parts of its domain

point-slope equation
equation of a linear function indicating its slope and a point on the graph of the function

polynomial function
a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

power function
a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

quadratic function
a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$

rational function
a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials

root function

a function of the form $f(x) = x^{1/n}$ for any integer $n \geq 2$

slope

the change in y for each unit change in x

slope-intercept form

equation of a linear function indicating its slope and y-intercept

transcendental function

a function that cannot be expressed by a combination of basic arithmetic operations

transformation of a function

a shift, scaling, or reflection of a function

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1.3E: Exercises

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1.4: Trigonometric Functions

Learning Objectives

- Convert angle measures between degrees and radians.
- Recognize the triangular and circular definitions of the basic trigonometric functions.
- Write the basic trigonometric identities.
- Identify the graphs and periods of the trigonometric functions.
- Describe the shift of a sine or cosine graph from the equation of the function.

Trigonometric functions are used to model many phenomena, including sound waves, vibrations of strings, alternating electrical current, and the motion of pendulums. In fact, almost any repetitive, or cyclical, motion can be modeled by some combination of trigonometric functions. In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

Radian Measure

To use trigonometric functions, we first must understand how to measure the angles. Although we can use both radians and degrees, **radians** are a more natural measurement because they are related directly to the unit circle, a circle with radius 1. The radian measure of an angle is defined as follows. Given an angle θ, let s be the length of the corresponding arc on the unit circle (Figure 1.4.1). We say the angle corresponding to the arc of length 1 has radian measure 1.

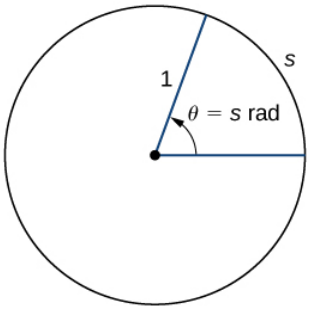


Figure 1.4.1: The radian measure of an angle θ is the arc length s of the associated arc on the unit circle.

Since an angle of $360°$ corresponds to the circumference of a circle, or an arc of length 2π, we conclude that an angle with a degree measure of $360°$ has a radian measure of 2π. Similarly, we see that $180°$ is equivalent to π radians. Table 1.4.1 shows the relationship between common degree and radian values.

Table 1.4.1: Common Angles Expressed in Degrees and Radians

Degrees	Radians	Degrees	Radians
0	0	120	$2\pi/3$
30	$\pi/6$	135	$3\pi/4$
45	$\pi/4$	150	$5\pi/6$
60	$\pi/3$	180	π
90	$\pi/2$		

Converting between Radians and Degrees

a. Express $225°$ using radians.
b. Express $5\pi/3$ rad using degrees.

Solution

Use the fact that $180°$ is equivalent to π radians as a **conversion factor** (Table 1.4.1):

$$1 = \frac{\pi \text{ rad}}{180°} = \frac{180°}{\pi \text{ rad}}.$$

a. $225° = 225° \cdot \left(\frac{\pi}{180°}\right) = \left(\frac{5\pi}{4}\right)$ rad

b. $\frac{5\pi}{3}$ rad $= \frac{5\pi}{3} \cdot \frac{180°}{\pi} = 300°$

Exercise 1.4.1

a. Express $210°$ using radians.

b. Express $11\pi/6$ rad using degrees.

Hint

π radians is equal to 180°

Answer

a. $7\pi/6$

b. 330°

The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angle measures, in radians or degrees, to find the coordinates of a point on any circle—not only on a unit circle—or to find an angle given a point on a circle. They also define the relationship between the sides and angles of a triangle.

To define the trigonometric functions, first consider the unit circle centered at the origin and a point $P = (x, y)$ on the unit circle. Let θ be an angle with an initial side that lies along the positive x-axis and with a terminal side that is the line segment OP. An angle in this position is said to be in standard position (Figure 1.4.2). We can then define the values of the six trigonometric functions for θ in terms of the coordinates x and y.

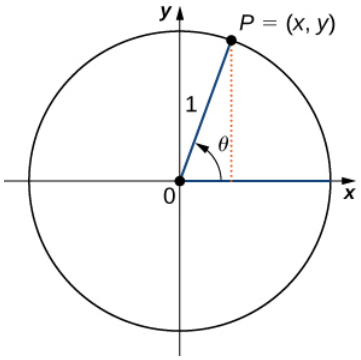


Figure 1.4.2: The angle θ is in standard position. The values of the trigonometric functions for θ are defined in terms of the coordinates x and y.

Definition: Trigonometric functions

Let $P = (x, y)$ be a point on the unit circle centered at the origin O. Let θ be an angle with an initial side along the positive x-axis and a terminal side given by the line segment OP. The trigonometric functions are then defined as

$\sin\theta = y$	$\csc\theta = \frac{1}{y}$
$\cos\theta = x$	$\sec\theta = \frac{1}{x}$
$\tan\theta = \frac{y}{x}$	$\cot\theta = \frac{x}{y}$

If $x = 0, \sec\theta$ and $\tan\theta$ are undefined. If $y = 0$, then $\cot\theta$ and $\csc\theta$ are undefined.

We can see that for a point $P = (x, y)$ on a circle of radius r with a corresponding angle θ, the coordinates x and y satisfy

$$\cos\theta = \frac{x}{r} \tag{1.4.1}$$
$$x = r\cos\theta \tag{1.4.2}$$

and

$$\sin\theta = \frac{y}{r} \tag{1.4.3}$$
$$y = r\sin\theta. \tag{1.4.4}$$

The values of the other trigonometric functions can be expressed in terms of x, y, and r (Figure 1.4.3).

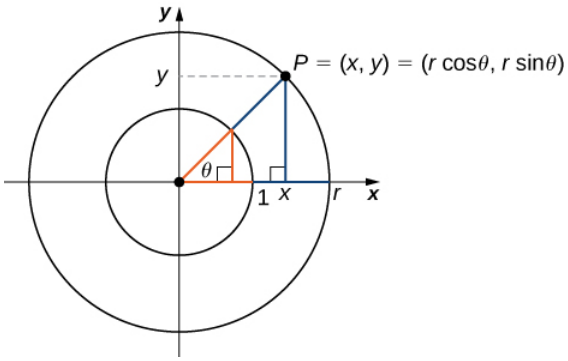


Figure 1.4.3: For a point $P = (x, y)$ on a circle of radius r, the coordinates x and y satisfy $x = r\cos\theta$ and $y = r\sin\theta$.

Table 1.4.2 shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of $\sin\theta$ and $\cos\theta$.

Table 1.4.2: Values of $\sin\theta$ and $\cos\theta$ at Major Angles θ in the First Quadrant

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Example 1.4.2: Evaluating Trigonometric Functions

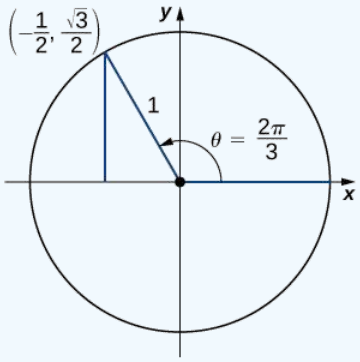
Evaluate each of the following expressions.

a. $\sin\left(\frac{2\pi}{3}\right)$
b. $\cos\left(-\frac{5\pi}{6}\right)$
c. $\tan\left(\frac{15\pi}{4}\right)$

Solution:

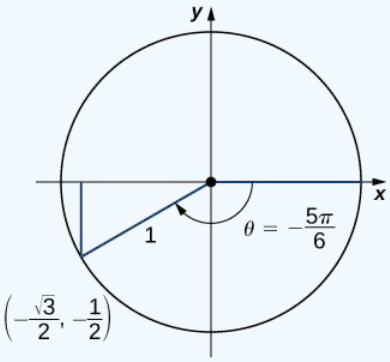
a) On the unit circle, the angle $\theta = \frac{2\pi}{3}$ corresponds to the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore,

$$\sin\left(\frac{2\pi}{3}\right) = y = \left(\frac{\sqrt{3}}{2}\right).$$



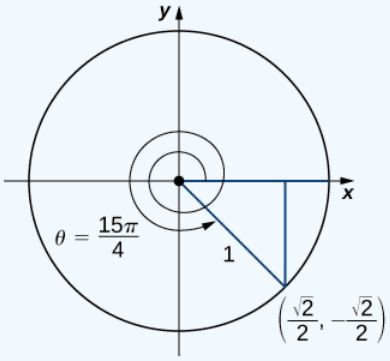
b) An angle $\theta = -\frac{5\pi}{6}$ corresponds to a revolution in the negative direction, as shown. Therefore,

$$\cos\left(-\frac{5\pi}{6}\right) = x = -\frac{\sqrt{3}}{2}.$$



c) An angle $\theta = \frac{15\pi}{4} = 2\pi + \frac{7\pi}{4}$. Therefore, this angle corresponds to more than one revolution, as shown. Knowing the fact that an angle of $\frac{7\pi}{4}$ corresponds to the point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, we can conclude that

$$\tan\left(\frac{15\pi}{4}\right) = \frac{y}{x} = -1.$$



Exercise 1.4.2

Evaluate $\cos(3\pi/4)$ and $\sin(-\pi/6)$.

Hint

Look at angles on the unit circle.

Answer

$$\cos(3\pi/4) = -\sqrt{2}/2$$

$$\sin(-\pi/6) = -1/2$$

As mentioned earlier, the ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle. Let θ be one of the acute angles. Let A be the length of the adjacent leg, O be the length of the opposite leg, and H be the length of the hypotenuse. By inscribing the triangle into a circle of radius H, as shown in Figure 1.4.4, we see that A, H, and O satisfy the following relationships with θ:

$\sin\theta = \frac{O}{H}$	$\csc\theta = \frac{H}{O}$
$\cos\theta = \frac{A}{H}$	$\sec\theta = \frac{H}{A}$
$\tan\theta = \frac{O}{A}$	$\cot\theta = \frac{A}{O}$

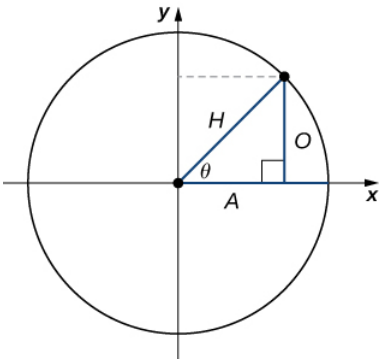


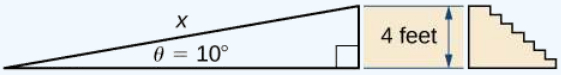
Figure 1.4.4: By inscribing a right triangle in a circle, we can express the ratios of the side lengths in terms of the trigonometric functions evaluated at θ.

Example 1.4.3: Constructing a Wooden Ramp

A wooden ramp is to be built with one end on the ground and the other end at the top of a short staircase. If the top of the staircase is 4 ft from the ground and the angle between the ground and the ramp is to be 10°, how long does the ramp need to be?

Solution

Let x denote the length of the ramp. In the following image, we see that x needs to satisfy the equation $\sin(10°) = 4/x$. Solving this equation for x, we see that $x = 4/\sin(10°) \approx 23.035$ ft.



Exercise 1.4.3

A house painter wants to lean a 20-ft ladder against a house. If the angle between the base of the ladder and the ground is to be 60°, how far from the house should she place the base of the ladder?

Hint

Draw a right triangle with hypotenuse 20.

Answer

10 ft

Trigonometric Identities

A **trigonometric identity** is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined. We can use the identities to help us solve or simplify equations. The main trigonometric identities are listed next.

Trigonometric Identities

Reciprocal identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.4.5}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{1.4.6}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{1.4.7}$$

Addition and subtraction formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double-angle formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{1.4.8}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 \tag{1.4.9}$$

$$= 1 - 2\sin^2\theta \tag{1.4.10}$$

$$= \cos^2\theta - \sin^2\theta \tag{1.4.11}$$

Example 1.4.4: Solving Trigonometric Equations

For each of the following equations, use a trigonometric identity to find all solutions.

a. $1 + \cos(2\theta) = \cos\theta$
b. $\sin(2\theta) = \tan\theta$

Solution

a) Using the double-angle formula for $\cos(2\theta)$, we see that θ is a solution of

$$1 + \cos(2\theta) = \cos\theta$$

if and only if

$$1 + 2\cos^2\theta - 1 = \cos\theta,$$

which is true if and only if

$$2\cos^2\theta - \cos\theta = 0.$$

To solve this equation, it is important to note that we need to factor the left-hand side and not divide both sides of the equation by $\cos\theta$. The problem with dividing by $\cos\theta$ is that it is possible that $\cos\theta$ is zero. In fact, if we did divide both sides of the equation by $\cos\theta$, we would miss some of the solutions of the original equation. Factoring the left-hand side of the equation, we see that θ is a solution of this equation if and only if

$$\cos\theta(2\cos\theta - 1) = 0.$$

Since $\cos\theta = 0$ when

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \ldots,$$

and $\cos\theta = 1/2$ when

$$\theta = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi, \ldots \text{ or } \theta = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi, \ldots,$$

we conclude that the set of solutions to this equation is

$$\theta = \frac{\pi}{2} + n\pi,\ \theta = \frac{\pi}{3} + 2n\pi$$

and

$$\theta = -\frac{\pi}{3} + 2n\pi,\ n = 0, \pm 1, \pm 2, \ldots.$$

b) Using the double-angle formula for $\sin(2\theta)$ and the reciprocal identity for $\tan(\theta)$, the equation can be written as

$$2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}.$$

To solve this equation, we multiply both sides by $\cos\theta$ to eliminate the denominator, and say that if θ satisfies this equation, then θ satisfies the equation

$$2\sin\theta\cos^2\theta - \sin\theta = 0.$$

However, we need to be a little careful here. Even if θ satisfies this new equation, it may not satisfy the original equation because, to satisfy the original equation, we would need to be able to divide both sides of the equation by $\cos\theta$. However, if $\cos\theta = 0$, we cannot divide both sides of the equation by $\cos\theta$. Therefore, it is possible that we may arrive at extraneous solutions. So, at the end, it is important to check for extraneous solutions. Returning to the equation, it is important that we factor $\sin\theta$ out of both terms on the left-hand side instead of dividing both sides of the equation by $\sin\theta$. Factoring the left-hand side of the equation, we can rewrite this equation as

$$\sin\theta(2\cos^2\theta - 1) = 0.$$

Therefore, the solutions are given by the angles θ such that $\sin\theta = 0$ or $\cos^2\theta = 1/2$. The solutions of the first equation are $\theta = 0, \pm\pi, \pm 2\pi, \ldots$. The solutions of the second equation are $\theta = \pi/4, (\pi/4) \pm (\pi/2), (\pi/4) \pm \pi, \ldots$. After checking for extraneous solutions, the set of solutions to the equation is

$$\theta = n\pi$$

and

$$\theta = \frac{\pi}{4} + \frac{n\pi}{2}$$

with $n = 0, \pm 1, \pm 2, \ldots$.

? Exercise 1.4.4

Find all solutions to the equation $\cos(2\theta) = \sin\theta$.

Hint

Use the double-angle formula for cosine (Equation 1.4.8).

Answer

$\theta = \frac{3\pi}{2} + 2n\pi, \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$

for $n = 0, \pm 1, \pm 2, \ldots$.

Example 1.4.5: Proving a Trigonometric Identity

Prove the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$.

Solution:

We start with the Pythagorean identity (Equation 1.4.5)

$$\sin^2\theta + \cos^2\theta = 1.$$

Dividing both sides of this equation by $\cos^2\theta$, we obtain

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}.$$

Since $\sin\theta/\cos\theta = \tan\theta$ and $1/\cos\theta = \sec\theta$, we conclude that

$$\tan^2\theta + 1 = \sec^2\theta.$$

Exercise 1.4.5

Prove the trigonometric identity $1 + \cot^2\theta = \csc^2\theta$.

Answer

Divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

Graphs and Periods of the Trigonometric Functions

We have seen that as we travel around the unit circle, the values of the trigonometric functions repeat. We can see this pattern in the graphs of the functions. Let $P = (x, y)$ be a point on the unit circle and let θ be the corresponding angle . Since the angle θ and $\theta + 2\pi$ correspond to the same point P, the values of the trigonometric functions at θ and at $\theta + 2\pi$ are the same. Consequently, the trigonometric functions are **periodic functions**. The period of a function f is defined to be the smallest positive value p such that $f(x + p) = f(x)$ for all values x in the domain of f. The sine, cosine, secant, and cosecant functions have a period of 2π. Since the tangent and cotangent functions repeat on an interval of length π, their period is π (Figure 1.4.5).

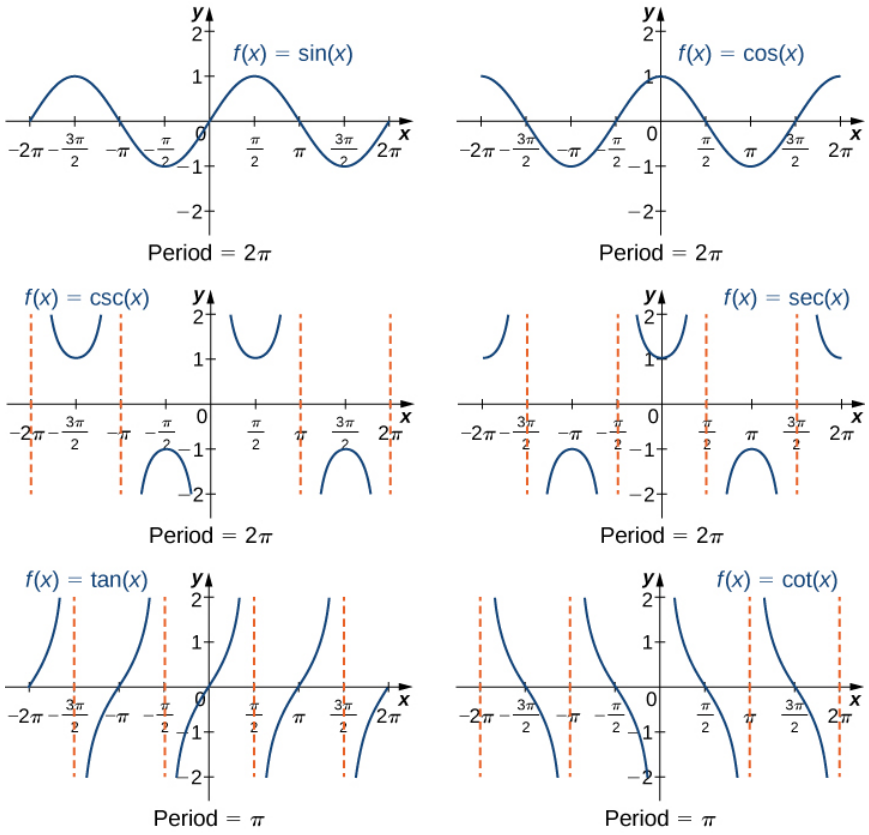


Figure 1.4.5: The six trigonometric functions are periodic.

Just as with algebraic functions, we can apply transformations to trigonometric functions. In particular, consider the following function:

$$f(x) = A\sin(B(x-\alpha)) + C.$$

In Figure 1.4.6, the constant α causes a horizontal or phase shift. The factor B changes the period. This transformed sine function will have a period $2\pi/|B|$. The factor A results in a vertical stretch by a factor of $|A|$. We say $|A|$ is the "amplitude of f." The constant C causes a vertical shift.

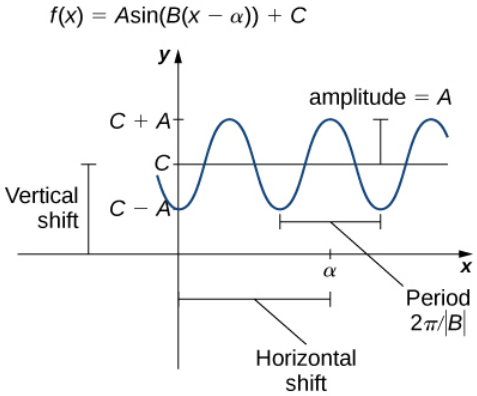


Figure 1.4.6: A graph of a general sine function.

Notice in Figure 1.4.6 that the graph of $y = \cos x$ is the graph of $y = \sin x$ shifted to the left $\pi/2$ units. Therefore, we can write

$$\cos x = \sin(x + \pi/2).$$

Similarly, we can view the graph of $y = \sin x$ as the graph of $y = \cos x$ shifted right $\pi/2$ units, and state that $\sin x = \cos(x - \pi/2)$.

A shifted sine curve arises naturally when graphing the number of hours of daylight in a given location as a function of the day of the year. For example, suppose a city reports that June 21 is the longest day of the year with 15.7 hours and December 21 is the shortest day of the year with 8.3 hours. It can be shown that the function

$$h(t) = 3.7 \sin\left(\frac{2\pi}{365}(x - 80.5)\right) + 12$$

is a model for the number of hours of daylight h as a function of day of the year t (Figure 1.4.7).

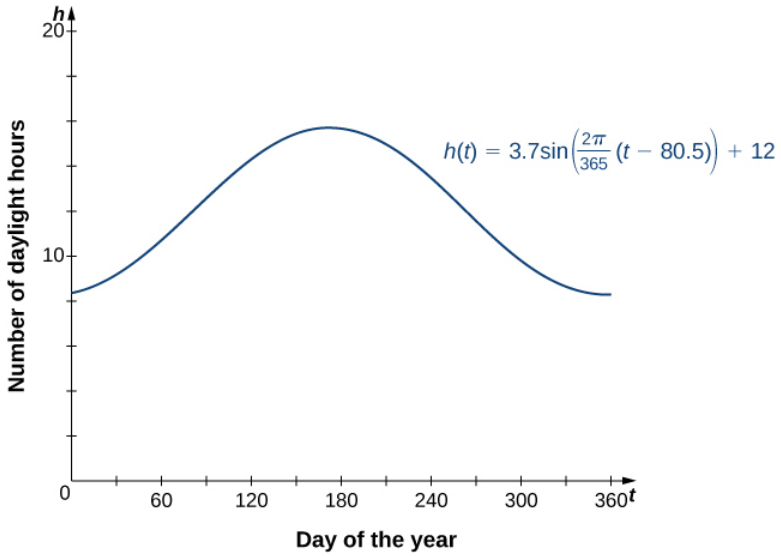


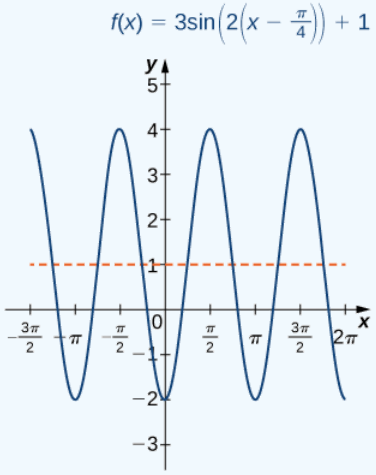
Figure 1.4.7: The hours of daylight as a function of day of the year can be modeled by a shifted sine curve.

Example 1.4.6: Sketching the Graph of a Transformed Sine Curve

Sketch a graph of $f(x) = 3\sin(2(x - \frac{\pi}{4})) + 1$.

Solution

This graph is a phase shift of $y = \sin(x)$ to the right by $\pi/4$ units, followed by a horizontal compression by a factor of 2, a vertical stretch by a factor of 3, and then a vertical shift by 1 unit. The period of f is π.



Exercise 1.4.6

Describe the relationship between the graph of $f(x) = 3\sin(4x) - 5$ and the graph of $y = \sin(x)$.

Hint

The graph of f can be sketched using the graph of $y = \sin(x)$ and a sequence of three transformations.

Answer

To graph $f(x) = 3\sin(4x) - 5$, the graph of $y = \sin(x)$ needs to be compressed horizontally by a factor of 4, then stretched vertically by a factor of 3, then shifted down 5 units. The function f will have a period of $\pi/2$ and an amplitude of 3.

Key Concepts

- Radian measure is defined such that the angle associated with the arc of length 1 on the unit circle has radian measure 1. An angle with a degree measure of $180°$ has a radian measure of π rad.
- For acute angles θ,the values of the trigonometric functions are defined as ratios of two sides of a right triangle in which one of the acute angles is θ.
- For a general angle θ, let (x, y) be a point on a circle of radius r corresponding to this angle θ. The trigonometric functions can be written as ratios involving x, y, and r.
- The trigonometric functions are periodic. The sine, cosine, secant, and cosecant functions have period 2π. The tangent and cotangent functions have period π.

Key Equations

- **Generalized sine function**

$$f(x) = A\sin(B(x - \alpha)) + C$$

Glossary

periodic function

a function is periodic if it has a repeating pattern as the values of x move from left to right

radians

for a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s

trigonometric functions

functions of an angle defined as ratios of the lengths of the sides of a right triangle

trigonometric identity

an equation involving trigonometric functions that is true for all angles θ for which the functions in the equation are defined

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1.5: Inverse Functions

Learning Objectives

- Determine the conditions for when a function has an inverse.
- Use the horizontal line test to recognize when a function is one-to-one.
- Find the inverse of a given function.
- Draw the graph of an inverse function.
- Evaluate inverse trigonometric functions.

An inverse function reverses the operation done by a particular function. In other words, whatever a function does, the inverse function undoes it. In this section, we define an inverse function formally and state the necessary conditions for an inverse function to exist. We examine how to find an inverse function and study the relationship between the graph of a function and the graph of its inverse. Then we apply these ideas to define and discuss properties of the inverse trigonometric functions.

Existence of an Inverse Function

We begin with an example. Given a function f and an output $y = f(x)$, we are often interested in finding what value or values x were mapped to y by f. For example, consider the function $f(x) = x^3 + 4$. Since any output $y = x^3 + 4$, we can solve this equation for x to find that the input is $x = \sqrt[3]{y-4}$. This equation defines x as a function of y. Denoting this function as f^{-1}, and writing $x = f^{-1}(y) = \sqrt[3]{y-4}$, we see that for any x in the domain of f, $f^{-1}f(x)) = f^{-1}(x^3 + 4) = x$. Thus, this new function, f^{-1}, "undid" what the original function f did. A function with this property is called the inverse function of the original function.

Definition: Inverse Functions

Given a function f with domain D and range R, its *inverse function* (if it exists) is the function f^{-1} with domain R and range D such that $f^{-1}(y) = x$ if and only if $f(x) = y$. In other words, for a function f and its inverse f^{-1},

$$f^{-1}(f(x)) = x$$

for all x in D and

$$f(f^{-1}(y)) = y$$

for all y in R.

Note that f^{-1} is read as "f inverse." Here, the -1 is not used as an exponent so

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

Figure 1.5.1shows the relationship between the domain and range of f and the domain and range of f^{-1}.

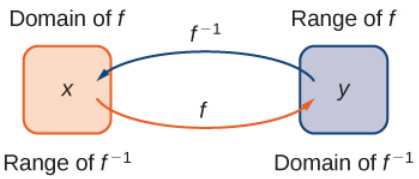


Figure 1.5.1: Given a function f and its inverse f^{-1}, $f^{-1}(y) = x$ if and only if $f(x) = y$. The range of f becomes the domain of f^{-1} and the domain of f becomes the range of f^{-1}.

Recall that a function has exactly one output for each input. Therefore, to define an inverse function, we need to map each input to exactly one output. For example, let's try to find the inverse function for $f(x) = x^2$. Solving the equation $y = x^2$ for x, we arrive at the equation $x = \pm\sqrt{y}$. This equation does not describe x as a function of y because there are two solutions to this equation for every $y > 0$. The problem with trying to find an inverse function for $f(x) = x^2$ is that two inputs are sent to the same output for each

output $y > 0$. The function $f(x) = x^3 + 4$ discussed earlier did not have this problem. For that function, each input was sent to a different output. A function that sends each input to a different output is called a one-to-one function.

Definition: One-to-One functions

We say a function f is a one-to-one function if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

One way to determine whether a function is one-to-one is by looking at its graph. If a function is one-to-one, then no two inputs can be sent to the same output. Therefore, if we draw a horizontal line anywhere in the xy-plane, according to the **horizontal line test**, it cannot intersect the graph more than once. We note that the horizontal line test is different from the vertical line test. The vertical line test determines whether a graph is the graph of a function. The horizontal line test determines whether a function is one-to-one (Figure 1.5.2).

Horizontal Line Test

A function f is one-to-one if and only if every horizontal line intersects the graph of f no more than once.

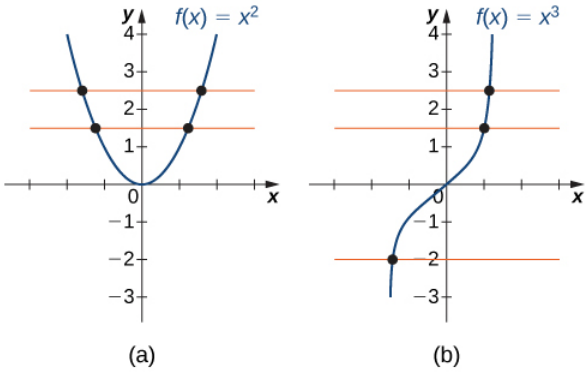
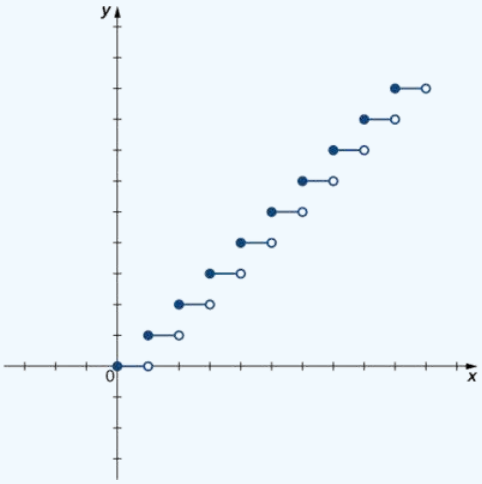


Figure 1.5.2: (a) The function $f(x) = x^2$ is not one-to-one because it fails the horizontal line test. (b) The function $f(x) = x^3$ is one-to-one because it passes the horizontal line test.

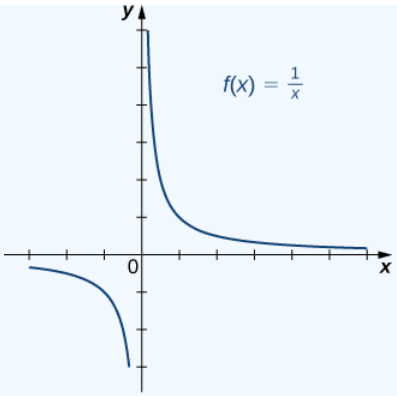
Example 1.5.1: Determining Whether a Function Is One-to-One

For each of the following functions, use the horizontal line test to determine whether it is one-to-one.

a)

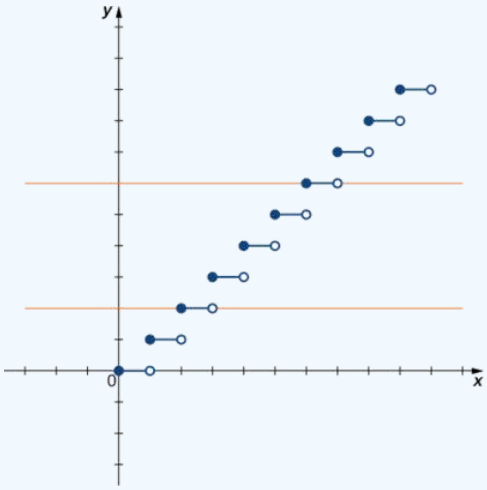


b)

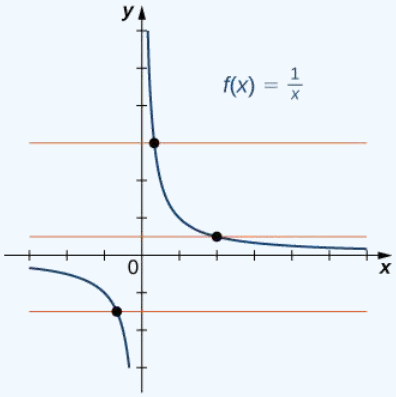


Solution

a) Since the horizontal line $y = n$ for any integer $n \geq 0$ intersects the graph more than once, this function is not one-to-one.

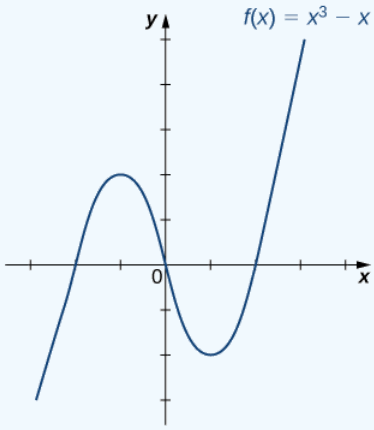


b) Since every horizontal line intersects the graph once (at most), this function is one-to-one.



? Exercise 1.5.1

Is the function f graphed in the following image one-to-one?



Solution

Use the horizontal line test.

Answer

No

Finding a Function's Inverse

We can now consider one-to-one functions and show how to find their inverses. Recall that a function maps elements in the domain of f to elements in the range of f. The inverse function maps each element from the range of f back to its corresponding element from the domain of f. Therefore, to find the inverse function of a one-to-one function f, given any y in the range of f, we need to determine which x in the domain of f satisfies $f(x) = y$. Since f is one-to-one, there is exactly one such value x. We can find that value x by solving the equation $f(x) = y$ for x. Doing so, we are able to write x as a function of y where the domain of this function is the range of f and the range of this new function is the domain of f. Consequently, this function is the inverse of f, and we write $x = f^{-1}(y)$. Since we typically use the variable x to denote the independent variable and y to denote the dependent variable, we often interchange the roles of x and y, and write $y = f^{-1}(x)$. Representing the inverse function in this way is also helpful later when we graph a function f and its inverse f^{-1} on the same axes.

Problem-Solving Strategy: Finding an Inverse Function

1. Solve the equation $y = f(x)$ for x.
2. Interchange the variables x and y and write $y = f^{-1}(x)$.

Example 1.5.2: Finding an Inverse Function

Find the inverse for the function $f(x) = 3x - 4$. State the domain and range of the inverse function. Verify that $f^{-1}(f(x)) = x$.

Solution

Follow the steps outlined in the strategy.

Step 1. If $y = 3x - 4$, then $3x = y + 4$ and $x = \frac{1}{3}y + \frac{4}{3}$.

Step 2. Rewrite as $y = \frac{1}{3}x + \frac{4}{3}$ and let $y = f^{-1}(x)$.Therefore, $f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$.

Since the domain of f is $(-\infty, \infty)$, the range of f^{-1} is $(-\infty, \infty)$. Since the range of f is $(-\infty, \infty)$, the domain of f^{-1} is $(-\infty, \infty)$.

You can verify that $f^{-1}(f(x)) = x$ by writing

$$f^{-1}(f(x)) = f^{-1}(3x - 4) = \tfrac{1}{3}(3x - 4) + \tfrac{4}{3} = x - \tfrac{4}{3} + \tfrac{4}{3} = x.$$

Note that for $f^{-1}(x)$ to be the inverse of $f(x)$, both $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for all x in the domain of the inside function.

? Exercise 1.5.2

Find the inverse of the function $f(x) = 3x/(x-2)$. State the domain and range of the inverse function.

Hint

Use the Problem-Solving Strategy for finding inverse functions.

Answer

$f^{-1}(x) = \dfrac{2x}{x-3}$. The domain of f^{-1} is $\{x \mid x \neq 3\}$. The range of f^{-1} is $\{y \mid y \neq 2\}$.

Graphing Inverse Functions

Let's consider the relationship between the graph of a function f and the graph of its inverse. Consider the graph of f shown in Figure 1.5.3 and a point (a, b) on the graph. Since $b = f(a)$, then $f^{-1}(b) = a$. Therefore, when we graph f^{-1}, the point (b, a) is on the graph. As a result, the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

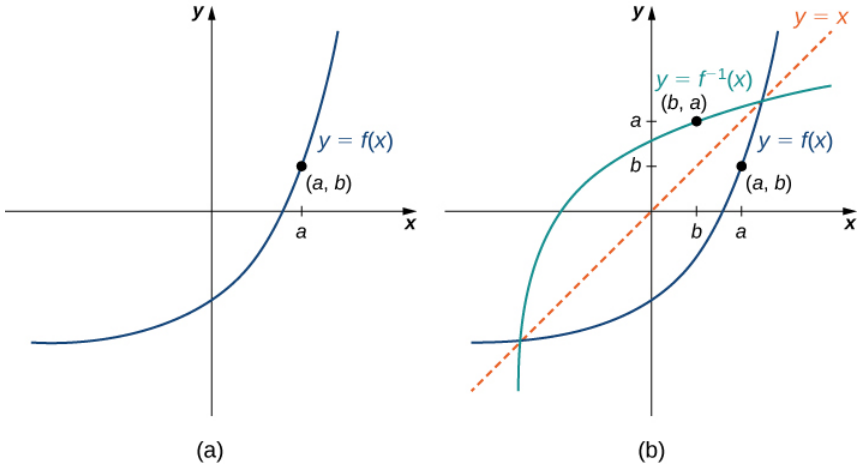
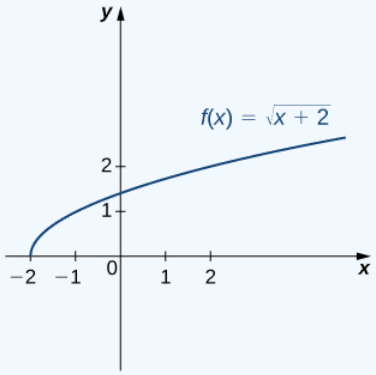


Figure 1.5.3: (a) The graph of this function f shows point (a, b) on the graph of f. (b) Since (a, b) is on the graph of f, the point (b, a) is on the graph of f^{-1}. The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

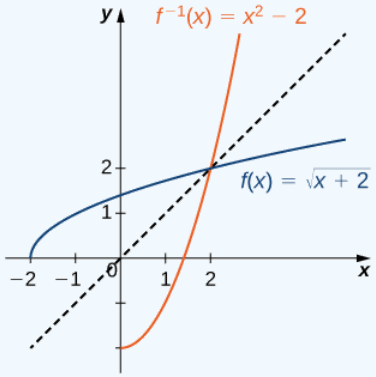
✓ Example 1.5.3: Sketching Graphs of Inverse Functions

For the graph of f in the following image, sketch a graph of f^{-1} by sketching the line $y = x$ and using symmetry. Identify the domain and range of f^{-1}.



Solution

Reflect the graph about the line $y = x$. The domain of f^{-1} is $[0, \infty)$. The range of f^{-1} is $[-2, \infty)$. By using the preceding strategy for finding inverse functions, we can verify that the inverse function is $f^{-1}(x) = x^2 - 2$, as shown in the graph.



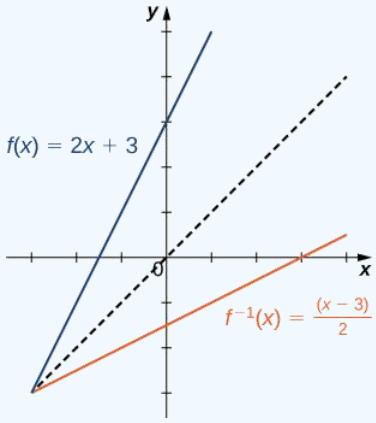
Exercise 1.5.3

Sketch the graph of $f(x) = 2x + 3$ and the graph of its inverse using the symmetry property of inverse functions.

Hint

The graphs are symmetric about the line $y = x$

Answer



Restricting Domains

As we have seen, $f(x) = x^2$ does not have an inverse function because it is not one-to-one. However, we can choose a subset of the domain of f such that the function is one-to-one. This subset is called a **restricted domain**. By restricting the domain of f, we can define a new function g such that the domain of g is the restricted domain of f and $g(x) = f(x)$ for all x in the domain of g. Then we can define an inverse function for g on that domain. For example, since $f(x) = x^2$ is one-to-one on the interval $[0, \infty)$, we can define a new function g such that the domain of g is $[0, \infty)$ and $g(x) = x^2$ for all x in its domain. Since g is a one-to-one function, it has an inverse function, given by the formula $g^{-1}(x) = \sqrt{x}$. On the other hand, the function $f(x) = x^2$ is also one-to-one on the domain $(-\infty, 0]$. Therefore, we could also define a new function h such that the domain of h is $(-\infty, 0]$ and $h(x) = x^2$ for all x in the domain of h. Then h is a one-to-one function and must also have an inverse. Its inverse is given by the formula $h^{-1}(x) = -\sqrt{x}$ (Figure 1.5.4).

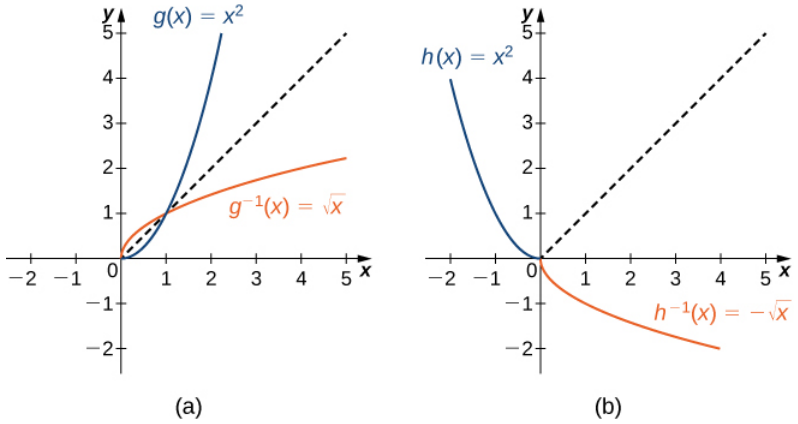


Figure 1.5.4: (a) For $g(x) = x^2$ restricted to $[0, \infty)$, $g^{-1}(x) = \sqrt{x}$. (b) For $h(x) = x^2$ restricted to $(-\infty, 0]$, $h^{-1}(x) = -\sqrt{x}$.

Example 1.5.4: Restricting the Domain

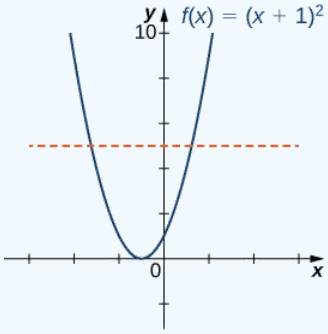
Consider the function $f(x) = (x + 1)^2$.

a. Sketch the graph of f and use the horizontal line test to show that f is not one-to-one.

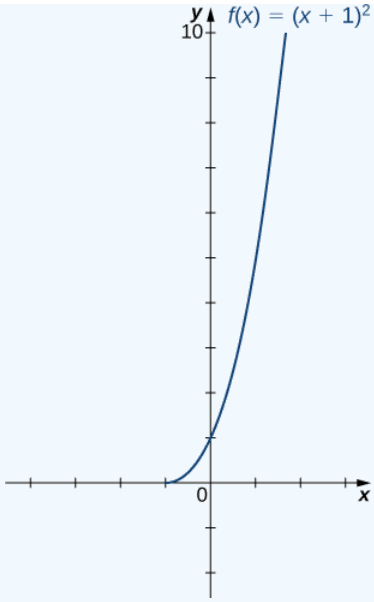
b. Show that f is one-to-one on the restricted domain $[-1, \infty)$. Determine the domain and range for the inverse of f on this restricted domain and find a formula for f^{-1}.

Solution

a) The graph of f is the graph of $y = x^2$ shifted left 1 unit. Since there exists a horizontal line intersecting the graph more than once, f is not one-to-one.



b) On the interval $[-1, \infty)$, f is one-to-one.



The domain and range of f^{-1} are given by the range and domain of f, respectively. Therefore, the domain of f^{-1} is $[0, \infty)$ and the range of f^{-1} is $[-1, \infty)$. To find a formula for f^{-1}, solve the equation $y = (x+1)^2$ for x. If $y = (x+1)^2$, then $x = -1 \pm \sqrt{y}$. Since we are restricting the domain to the interval where $x \geq -1$, we need $\pm\sqrt{y} \geq 0$. Therefore, $x = -1 + \sqrt{y}$. Interchanging x and y, we write $y = -1 + \sqrt{x}$ and conclude that $f^{-1}(x) = -1 + \sqrt{x}$.

? Exercise 1.5.4

Consider $f(x) = 1/x^2$ restricted to the domain $(-\infty, 0)$. Verify that f is one-to-one on this domain. Determine the domain and range of the inverse of f and find a formula for f^{-1}.

Hint

The domain and range of f^{-1} is given by the range and domain of f, respectively. To find f^{-1}, solve $y = 1/x^2$ for x.

Answer

The domain of f^{-1} is $(0, \infty)$. The range of f^{-1} is $(-\infty, 0)$. The inverse function is given by the formula $f^{-1}(x) = -1/\sqrt{x}$.

Inverse Trigonometric Functions

The six basic trigonometric functions are periodic, and therefore they are not one-to-one. However, if we restrict the domain of a trigonometric function to an interval where it is one-to-one, we can define its inverse. Consider the sine function. The sine function is one-to-one on an infinite number of intervals, but the standard convention is to restrict the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. By doing so, we define the inverse sine function on the domain $[-1, 1]$ such that for any x in the interval $[-1, 1]$, the inverse sine function tells us which angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ satisfies $\sin\theta = x$. Similarly, we can restrict the domains of the other trigonometric functions to define **inverse trigonometric functions**, which are functions that tell us which angle in a certain interval has a specified trigonometric value.

Definition: inverse trigonometric functions

The inverse sine function, denoted $\sin^{-1}$ or $\arcsin$, and the inverse cosine function, denoted $\cos^{-1}$ or $\arccos$, are defined on the domain $D = \{x | -1 \leq x \leq 1\}$ as follows:

$$\sin^{-1}(x) = y$$

- if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$;

$$\cos^{-1}(x) = y$$

- if and only if $\cos(y) = x$ and $0 \le y \le \pi$.

The inverse tangent function, denoted $\tan^{-1}$ or $\arctan$, and inverse cotangent function, denoted $\cot^{-1}$ or arccot, are defined on the domain $D = \{x | -\infty < x < \infty\}$ as follows:

$$\tan^{-1}(x) = y$$

- if and only if $\tan(y) = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$;

$$\cot^{-1}(x) = y$$

- if and only if $\cot(y) = x$ and $0 < y < \pi$.

The inverse cosecant function, denoted $\csc^{-1}$ or arccsc, and inverse secant function, denoted $\sec^{-1}$ or arcsec, are defined on the domain $D = \{x \,|\, |x| \ge 1\}$ as follows:

$$\csc^{-1}(x) = y$$

- if and only if $\csc(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$;

$$\sec^{-1}(x) = y$$

- if and only if $\sec(y) = x$ and$0 \le y \le \pi$, $y \ne \pi/2$.

To graph the inverse trigonometric functions, we use the graphs of the trigonometric functions restricted to the domains defined earlier and reflect the graphs about the line $y = x$ (Figure 1.5.5).

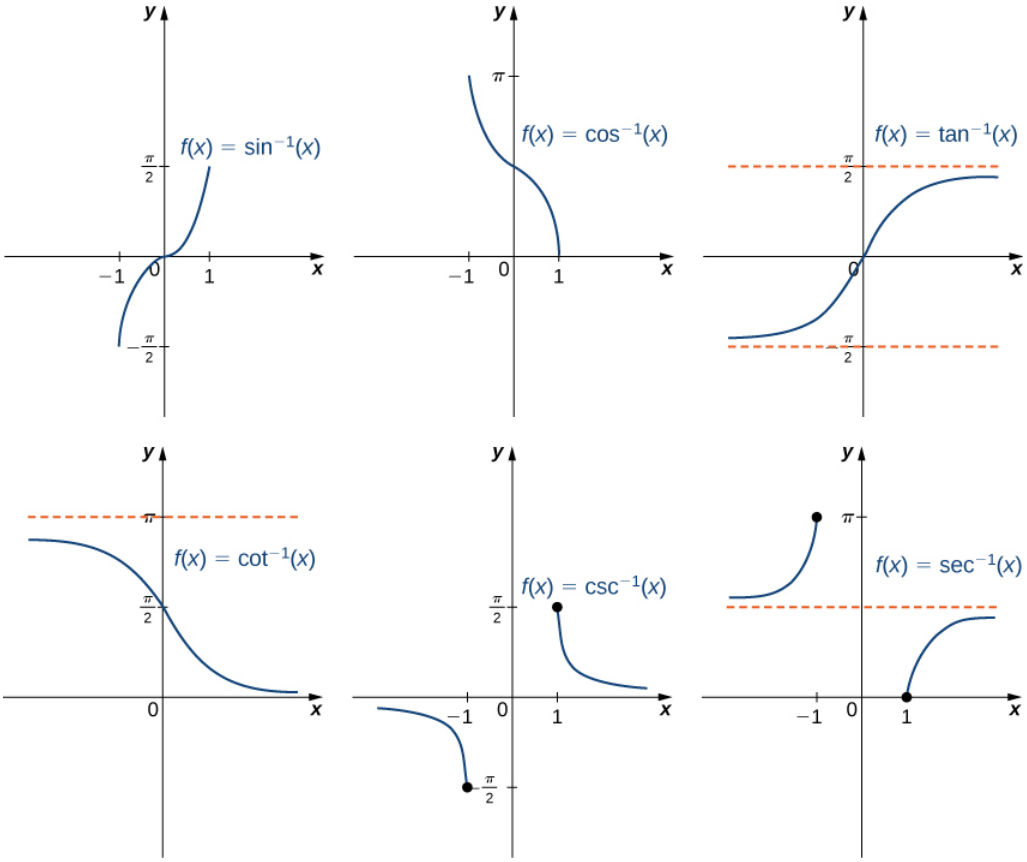


Figure 1.5.5: The graph of each of the inverse trigonometric functions is a reflection about the line $y = x$ of the corresponding restricted trigonometric function.

When evaluating an inverse trigonometric function, the output is an angle. For example, to evaluate $\cos^{-1}\left(\frac{1}{2}\right)$, we need to find an angle θ such that $\cos\theta = \frac{1}{2}$. Clearly, many angles have this property. However, given the definition of $\cos^{-1}$, we need the angle θ that not only solves this equation, but also lies in the interval $[0, \pi]$. We conclude that $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

We now consider a composition of a trigonometric function and its inverse. For example, consider the two expressions $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $\sin^{-1}(\sin(\pi))$.

For the first one, we simplify as follows:

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

For the second one, we have

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0.$$

The inverse function is supposed to "undo" the original function, so why isn't $\sin^{-1}(\sin(\pi)) = \pi$? Recalling our definition of inverse functions, a function f and its inverse f^{-1} satisfy the conditions $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f, so what happened here? The issue is that the inverse sine function, $\sin^{-1}$, is the inverse of the restricted sine function defined on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, for x in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is true that $\sin^{-1}(\sin x) = x$. However, for values of x outside this interval, the equation does not hold, even though $\sin^{-1}(\sin x)$ is defined for all real numbers x.

What about $\sin(\sin^{-1} y)$? Does that have a similar issue? The answer is no. Since the domain of $\sin^{-1}$ is the interval $[-1, 1]$, we conclude that $\sin\left(\sin^{-1} y\right) = y$ if $-1 \le y \le 1$ and the expression is not defined for other values of y. To summarize,

$$\sin(\sin^{-1} y) = y \text{ if } -1 \le y \le 1$$

and

$$\sin^{-1}(\sin x) = x \text{ if } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}.$$

Similarly, for the cosine function,

$$\cos(\cos^{-1} y) = y \text{ if } -1 \le y \le 1$$

and

$$\cos^{-1}(\cos x) = x \text{ if } 0 \le x \le \pi.$$

Similar properties hold for the other trigonometric f*unctions and their inverses.*

Example 1.5.5: Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluate each of the following expressions.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\tan\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$
c. $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$
d. $\sin^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

Solution

a. Evaluating $\sin^{-1}(-\sqrt{3}/2)$ is equivalent to finding the angle θ such that $\sin\theta = -\sqrt{3}/2$ and $-\pi/2 \le \theta \le \pi/2$. The angle $\theta = -\pi/3$ satisfies these two conditions. Therefore, $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$.
b. First we use the fact that $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Then $\tan(-\pi/6) = -1/\sqrt{3}$. Therefore, $\tan(\tan^{-1}(-1/\sqrt{3})) = -1/\sqrt{3}$.
c. To evaluate $\cos^{-1}(\cos(5\pi/4))$,first use the fact that $\cos(5\pi/4) = -\sqrt{2}/2$. Then we need to find the angle θ such that $\cos(\theta) = -\sqrt{2}/2$ and $0 \le \theta \le \pi$. Since $3\pi/4$ satisfies both these conditions, we have $\cos^{-1}(\cos(5\pi/4)) = \cos^{-1}(-\sqrt{2}/2)) = 3\pi/4$.
d. Since $\cos(2\pi/3) = -1/2$, we need to evaluate $\sin^{-1}(-1/2)$. That is, we need to find the angle θ such that $\sin(\theta) = -1/2$ and $-\pi/2 \le \theta \le \pi/2$. Since $-\pi/6$ satisfies both these conditions, we can conclude that $\sin^{-1}(\cos(2\pi/3)) = \sin^{-1}(-1/2) = -\pi/6$.

The Maximum Value of a Function

In many areas of science, engineering, and mathematics, it is useful to know the maximum value a function can obtain, even if we don't know its exact value at a given instant. For instance, if we have a function describing the strength of a roof beam, we would want to know the maximum weight the beam can support without breaking. If we have a function that describes the

speed of a train, we would want to know its maximum speed before it jumps off the rails. Safe design often depends on knowing maximum values.

This project describes a simple example of a function with a maximum value that depends on two equation coefficients. We will see that maximum values can depend on several factors other than the independent variable x.

1. Consider the graph in Figure 1.5.6 of the function $y = \sin x + \cos x$. Describe its overall shape. Is it periodic? How do you know?

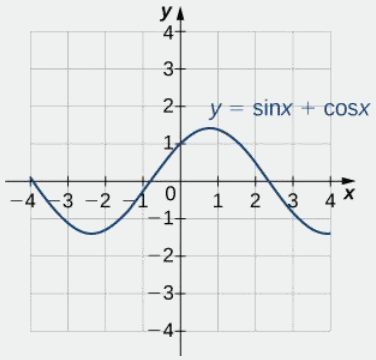


Figure 1.5.6: The graph of $y = \sin x + \cos x$.

Using a graphing calculator or other graphing device, estimate the x- and y-values of the maximum point for the graph (the first such point where $x > 0$). It may be helpful to express the x-value as a multiple of π.

2. Now consider other graphs of the form $y = A\sin x + B\cos x$ for various values of A and B. Sketch the graph when $A = 2$ and $B = 1$, and find the x- and y-values for the maximum point. (Remember to express the x-value as a multiple of π, if possible.) Has it moved?

3. Repeat for $A = 1,\ B = 2$. Is there any relationship to what you found in part (2)?

4. Complete the following table, adding a few choices of your own for A and B:

A	B	x	y	A	B	x	y
0	1			3	4		
1	0			4	3		
1	1			$\sqrt{3}$	1		
1	2			1	$\sqrt{3}$		
2	1			12	5		
2	2			5	12		

5. Try to figure out the formula for the y-values.

6. The formula for the x-values is a little harder. The most helpful points from the table are $(1,1)$, $(1,\sqrt{3})$, $(\sqrt{3},1)$. (Hint: Consider inverse trigonometric functions.)

7. If you found formulas for parts (5) and (6), show that they work together. That is, substitute the x-value formula you found into $y = A\sin x + B\cos x$ and simplify it to arrive at the y-value formula you found.

Key Concepts

- For a function to have an inverse, the function must be one-to-one. Given the graph of a function, we can determine whether the function is one-to-one by using the horizontal line test.
- If a function is not one-to-one, we can restrict the domain to a smaller domain where the function is one-to-one and then define the inverse of the function on the smaller domain.

- For a function f and its inverse f^{-1}, $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f.
- Since the trigonometric functions are periodic, we need to restrict their domains to define the inverse trigonometric functions.
- The graph of a function f and its inverse f^{-1} are symmetric about the line $y = x$.

Key Equations

- **Inverse function**

  $f^{-1}(f(x)) = x$ for all x in D, and $f(f^{-1}(y)) = y$ for all y in R.

Glossary

horizontal line test
a function f is one-to-one if and only if every horizontal line intersects the graph of f, at most, once

inverse function
for a function f, the inverse function f^{-1} satisfies $f^{-1}(y) = x$ if $f(x) = y$

inverse trigonometric functions
the inverses of the trigonometric functions are defined on restricted domains where they are one-to-one functions

one-to-one function
a function f is one-to-one if $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

restricted domain
a subset of the domain of a function f

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1.5E: Exercises

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1.6: Exponential and Logarithmic Functions

Learning Objectives

- Identify the form of an exponential function.
- Explain the difference between the graphs of x^b and b^x.
- Recognize the significance of the number e.
- Identify the form of a logarithmic function.
- Explain the relationship between exponential and logarithmic functions.
- Describe how to calculate a logarithm to a different base.
- Identify the hyperbolic functions, their graphs, and basic identities.

In this section we examine exponential and logarithmic functions. We use the properties of these functions to solve equations involving exponential or logarithmic terms, and we study the meaning and importance of the number e. We also define hyperbolic and inverse hyperbolic functions, which involve combinations of exponential and logarithmic functions. (Note that we present alternative definitions of exponential and logarithmic functions in the chapter Applications of Integrations, and prove that the functions have the same properties with either definition.)

Exponential Functions

Exponential functions arise in many applications. One common example is **population growth.** For example, if a population starts with P_0 individuals and then grows at an annual rate of 2%, its population after 1 year is

$$P(1) = P_0 + 0.02P_0 = P_0(1 + 0.02) = P_0(1.02).$$

Its population after 2 years is

$$P(2) = P(1) + 0.02P(1) = P(1)(1.02) = P_0(1.02)^2.$$

In general, its population after t years is

$$P(t) = P_0(1.02)^t,$$

which is an exponential function. More generally, any function of the form $f(x) = b^x$, where $b > 0$, $b \neq 1$, is an exponential function with **base** b and **exponent** x. Exponential functions have constant bases and variable exponents. Note that a function of the form $f(x) = x^b$ for some constant b is not an exponential function but a power function.

To see the difference between an exponential function and a power function, we compare the functions $y = x^2$ and $y = 2^x$. In Table **1.6.1**, we see that both 2^x and x^2 approach infinity as $x \to \infty$. Eventually, however, 2^x becomes larger than x^2 and grows more rapidly as $x \to \infty$. In the opposite direction, as $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$. The line $y = 0$ is a horizontal asymptote for $y = 2^x$.

Table **1.6.1**

x	-3	-2	-1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
2^x	1/8	1/4	1/2	1	2	4	8	16	32	64

In Figure **1.6.1**, we graph both $y = x^2$ and $y = 2^x$ to show how the graphs differ.

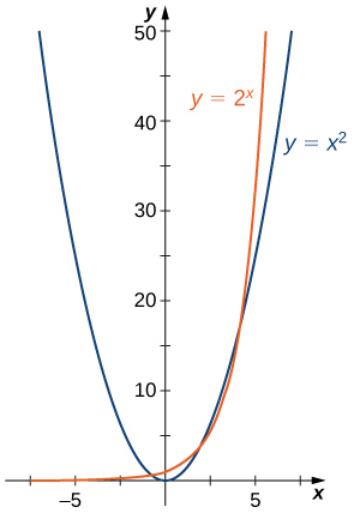


Figure 1.6.1: Both 2^x and x^2 approach infinity as $x \to \infty$, but 2^x grows more rapidly than x^2. As $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$.

Evaluating Exponential Functions

Recall the properties of exponents: If x is a positive integer, then we define $b^x = b \cdot b \cdots b$ (with x factors of b). If x is a negative integer, then $x = -y$ for some positive integer y, and we define $b^x = b^{-y} = 1/b^y$. Also, b^0 is defined to be 1. If x is a rational number, then $x = p/q$, where p and q are integers and $b^x = b^{p/q} = \sqrt[q]{b^p}$. For example, $9^{3/2} = \sqrt{9^3} = \left(\sqrt{9}\right)^3 = 27$. However, how is b^x defined if x is an irrational number? For example, what do we mean by $2^{\sqrt{2}}$? This is too complex a question for us to answer fully right now; however, we can make an approximation.

Table 1.6.2: Values of 2^x for a List of Rational Numbers Approximating $\sqrt{2}$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
2^x	2.639	2.65737	2.66475	2.665119	2.665138	2.665143

In Table 1.6.2, we list some rational numbers approaching $\sqrt{2}$, and the values of 2^x for each rational number x are presented as well. We claim that if we choose rational numbers x getting closer and closer to $\sqrt{2}$, the values of 2^x get closer and closer to some number L. We define that number L to be $2^{\sqrt{2}}$.

Example 1.6.1: Bacterial Growth

Suppose a particular population of bacteria is known to double in size every 4 hours. If a culture starts with 1000 bacteria, the number of bacteria after 4 hours is $n(4) = 1000 \cdot 2$. The number of bacteria after 8 hours is $n(8) = n(4) \cdot 2 = 1000 \cdot 2^2$. In general, the number of bacteria after $4m$ hours is $n(4m) = 1000 \cdot 2^m$. Letting $t = 4m$, we see that the number of bacteria after t hours is $n(t) = 1000 \cdot 2^{t/4}$. Find the number of bacteria after 6 hours, 10 hours, and 24 hours.

Solution

The number of bacteria after 6 hours is given by

$$n(6) = 1000 \cdot 2^{6/4} \approx 2828 \text{ bacteria.}$$

The number of bacteria after 10 hours is given by

$$n(10) = 1000 \cdot 2^{10/4} \approx 5657 \text{ bacteria.}$$

The number of bacteria after 24 hours is given by $n(24) = 1000 \cdot 2^6 = 64,000$ bacteria.

? Exercise 1.6.1

Given the exponential function $f(x) = 100 \cdot 3^{x/2}$, evaluate $f(4)$ and $f(10)$.

Answer

$f(4) = 900$

$f(10) = 24,300.$

Graphing Exponential Functions

For any base $b > 0$, $b \neq 1$, the exponential function $f(x) = b^x$ is defined for all real numbers x and $b^x > 0$. Therefore, the domain of $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. To graph b^x, we note that for $b > 1$, b^x is increasing on $(-\infty, \infty)$ and $b^x \to \infty$ as $x \to \infty$, whereas $b^x \to 0$ as $x \to -\infty$. On the other hand, if $0 < b < 1$, $f(x) = b^x$ is decreasing on $(-\infty, \infty)$ and $b^x \to 0$ as $x \to \infty$ whereas $b^x \to \infty$ as $x \to -\infty$ (Figure 1.6.2).

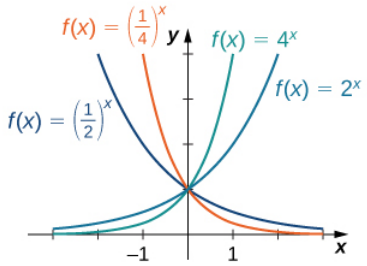


Figure 1.6.2: If $b > 1$, then b^x is increasing on $(-\infty, \infty)$. If $0 < b < 1$, then b^x is decreasing on $(-\infty, \infty)$.

Note that exponential functions satisfy the general laws of exponents. To remind you of these laws, we state them as rules.

Laws of Exponents

For any constants $a > 0$, $b > 0$, and for all x and y,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Example 1.6.2: Using the Laws of Exponents

Use the laws of exponents to simplify each of the following expressions.

a. $\dfrac{(2x^{2/3})^3}{(4x^{-1/3})^2}$

b. $\dfrac{(x^3y^{-1})^2}{(xy^2)^{-2}}$

Soution

a. We can simplify as follows:

$$\frac{(2x^{2/3})^3}{(4x^{-1/3})^2} = \frac{2^3(x^{2/3})^3}{4^2(x^{-1/3})^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

b. We can simplify as follows:

$$\frac{(x^3y^{-1})^2}{(xy^2)^{-2}} = \frac{(x^3)^2(y^{-1})^2}{x^{-2}(y^2)^{-2}} = \frac{x^6y^{-2}}{x^{-2}y^{-4}} = x^6x^2y^{-2}y^4 = x^8y^2.$$

? Exercise 1.6.2

Use the laws of exponents to simplify $\frac{6x^{-3}y^2}{12x^{-4}y^5}$.

Hint

$x^a/x^b = x^{a-b}$

Answer

$x/(2y^3)$

The Number e

A special type of exponential function appears frequently in real-world applications. To describe it, consider the following example of exponential growth, which arises from **compounding interest** in a savings account. Suppose a person invests P dollars in a savings account with an annual interest rate r, compounded annually. The amount of money after 1 year is

$$A(1) = P + rP = P(1+r).$$

The amount of money after 2 years is

$$A(2) = A(1) + rA(1) = P(1+r) + rP(1+r) = P(1+r)^2.$$

More generally, the amount after t years is

$$A(t) = P(1+r)^t.$$

If the money is compounded 2 times per year, the amount of money after half a year is

$$A\left(\frac{1}{2}\right) = P + \left(\frac{r}{2}\right)P = P\left(1 + \left(\frac{r}{2}\right)\right).$$

The amount of money after 1 year is

$$A(1) = A\left(\frac{1}{2}\right) + \left(\frac{r}{2}\right)A\left(\frac{1}{2}\right) = P\left(1+\frac{r}{2}\right) + \frac{r}{2}\left(\left(P(1+\frac{r}{2})\right)\right) = P\left(1+\frac{r}{2}\right)^2.$$

After t years, the amount of money in the account is

$$A(t) = P\left(1+\frac{r}{2}\right)^{2t}.$$

More generally, if the money is compounded n times per year, the amount of money in the account after t years is given by the function

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}.$$

What happens as $n \to \infty$? To answer this question, we let $m = n/r$ and write

$$\left(1+\frac{r}{n}\right)^{nt} = \left(1+\frac{1}{m}\right)^{mrt},$$

and examine the behavior of $(1+1/m)^m$ as $m \to \infty$, using a table of values (Table 1.6.3).

Table 1.6.3: Values of $\left(1+\frac{1}{m}\right)^m$ as $m \to \infty$

m	10	100	1000	10,000	100,000	1,000,000
$\left(1+\frac{1}{m}\right)^m$	2.5937	2.7048	2.71692	2.71815	2.718268	2.718280

Looking at this table, it appears that $(1+1/m)^m$ is approaching a number between 2.7 and 2.8 as $m \to \infty$. In fact, $(1+1/m)^m$ does approach some number as $m \to \infty$. We call this number e. To six decimal places of accuracy,

$$e \approx 2.718282.$$

Leonhard Euler

The letter e was first used to represent this number by the Swiss mathematician Leonhard Euler during the 1720s. Although Euler did not discover the number, he showed many important connections between e and logarithmic functions. We still use the notation e today to honor Euler's work because it appears in many areas of mathematics and because we can use it in many practical applications.

Returning to our savings account example, we can conclude that if a person puts P dollars in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. This function may be familiar. Since functions involving base e arise often in applications, we call the function $f(x) = e^x$ the **natural exponential function**. Not only is this function interesting because of the definition of the number e, but also, as discussed next, its graph has an important property.

Since $e > 1$, we know $f(x) = e^x$ is increasing on $(-\infty, \infty)$. In Figure 1.6.3, we show a graph of $f(x) = e^x$ along with a tangent line to the graph of f at $x = 0$. We give a precise definition of tangent line in the next chapter; but, informally, we say a tangent line to a graph of f at $x = a$ is a line that passes through the point $(a, f(a))$ and has the same "slope" as f at that point . The function $f(x) = e^x$ is the only exponential function b^x with tangent line at $x = 0$ that has a slope of 1. As we see later in the text, having this property makes the natural exponential function the most simple exponential function to use in many instances.

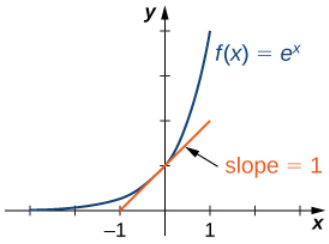


Figure 1.6.3: The graph of $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$.

Example 1.6.3: Compounding Interest

Suppose \$500 is invested in an account at an annual interest rate of $r = 5.5\%$, compounded continuously.

a. Let t denote the number of years after the initial investment and $A(t)$ denote the amount of money in the account at time t. Find a formula for $A(t)$.

b. Find the amount of money in the account after 10 years and after 20 years.

Solution

a. If P dollars are invested in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. Here $P = \$500$ and $r = 0.055$. Therefore, $A(t) = 500e^{0.055t}$.

b. After 10 years, the amount of money in the account is

$$A(10) = 500e^{0.055 \cdot 10} = 500e^{0.55} \approx \$866.63.$$

After 20 years, the amount of money in the account is

$$A(20) = 500e^{0.055 \cdot 20} = 500e^{1.1} \approx \$1,502.08.$$

Exercise 1.6.3

If \$750 is invested in an account at an annual interest rate of 4%, compounded continuously, find a formula for the amount of money in the account after t years. Find the amount of money after 30 years.

Hint

$A(t) = Pe^{rt}$

Answer

$A(t) = 750e^{0.04t}$. After 30 years, there will be approximately $\$2,490.09$.

Logarithmic Functions

Using our understanding of exponential functions, we can discuss their inverses, which are the logarithmic functions. These come in handy when we need to consider any phenomenon that varies over a wide range of values, such as the pH scale in chemistry or decibels in sound levels.

The exponential function $f(x) = b^x$ is *one-to-one*, with domain $(-\infty, \infty)$ and range $(0, \infty)$. Therefore, it has an inverse function, called the logarithmic function with base b. For any $b > 0$, $b \neq 1$, the logarithmic function with base b, denoted $\log_b$, has domain $(0, \infty)$ and range $(-\infty, \infty)$,and satisfies

$$\log_b(x) = y$$

if and only if $b^y = x$.

For example,

$$\log_2(8) = 3$$

since $2^3 = 8$,

$$\log_{10}\left(\frac{1}{100}\right) = -2$$

since $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$,

$$\log_b(1) = 0$$

since $b^0 = 1$ for any base $b > 0$.

Furthermore, since $y = \log_b(x)$ and $y = b^x$ are inverse functions,

$$\log_b(b^x) = x$$

and

$$b^{\log_b(x)} = x.$$

The most commonly used logarithmic function is the function $\log_e$. Since this function uses natural e as its base, it is called the **natural logarithm**. Here we use the notation $\ln(x)$ or $\ln x$ to mean $\log_e(x)$. For example,

$$\ln(e) = \log_e(e) = 1$$
$$\ln(e^3) = \log_e(e^3) = 3$$
$$\ln(1) = \log_e(1) = 0.$$

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

and their graphs are symmetric about the line $y = x$ (Figure 1.6.4).

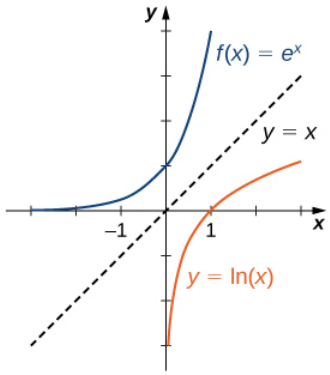


Figure 1.6.4: The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$.

In general, for any base $b > 0$, $b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of the exponential functions, we graph functions $\log_b$ for several values of $b > 1$ (Figure 1.6.5).

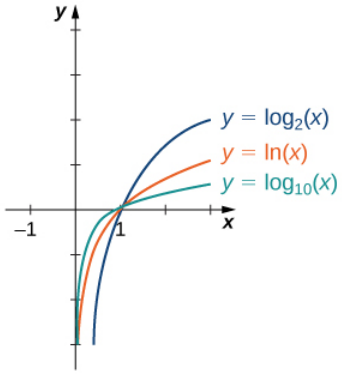


Figure 1.6.5: Graphs of $y = \log_b(x)$ are depicted for $b = 2, e, 10$.

Before solving some equations involving exponential and logarithmic functions, let's review the basic properties of logarithms.

Properties of Logarithms

If $a, b, c > 0$, $b \neq 1$, and r is any real number, then

- **Product property**

$$\log_b(ac) = \log_b(a) + \log_b(c) \tag{1.6.1}$$

- **Quotient property**

$$\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c) \tag{1.6.2}$$

- **Power property**

$$\log_b(a^r) = r\log_b(a) \tag{1.6.3}$$

Example 1.6.4: Solving Equations Involving Exponential Functions

Solve each of the following equations for x.

a. $5^x = 2$

b. $e^x + 6e^{-x} = 5$

Solution

a. Applying the natural logarithm function to both sides of the equation, we have

$$\ln 5^x = \ln 2.$$

Using the power property of logarithms,

$$x \ln 5 = \ln 2.$$

Therefore,

$$x = \frac{\ln 2}{\ln 5}.$$

b. Multiplying both sides of the equation by e^x,we arrive at the equation

$$e^{2x} + 6 = 5e^x.$$

Rewriting this equation as

$$e^{2x} - 5e^x + 6 = 0,$$

we can then rewrite it as a quadratic equation in e^x:

$$(e^x)^2 - 5(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

$$(e^x - 3)(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

Exercise 1.6.4

Solve

$$e^{2x}/(3 + e^{2x}) = 1/2.$$

Hint

First solve the equation for e^{2x}

Answer

$x = \frac{\ln 3}{2}$.

Example 1.6.5: Solving Equations Involving Logarithmic Functions

Solve each of the following equations for x.

a. $\ln\left(\frac{1}{x}\right) = 4$
b. $\log_{10}\sqrt{x} + \log_{10} x = 2$
c. $\ln(2x) - 3\ln(x^2) = 0$

Solution

a. By the definition of the natural logarithm function,

$$\ln\left(\frac{1}{x}\right) = 4$$

- if and only if $e^4 = \frac{1}{x}$.

Therefore, the solution is $x = 1/e^4$.

b. Using the product (Equation 1.6.1) and power (Equation 1.6.3) properties of logarithmic functions, rewrite the left-hand side of the equation as

$$\begin{aligned}\log_{10}\sqrt{x}+\log_{10}x &= \log_{10}x\sqrt{x}\\ &= \log_{10}x^{3/2}\\ &= \frac{3}{2}\log_{10}x.\end{aligned}$$

Therefore, the equation can be rewritten as

$$\frac{3}{2}\log_{10}x = 2$$

or

$$\log_{10}x = \frac{4}{3}.$$

The solution is $x = 10^{4/3} = 10\sqrt[3]{10}$.

c. Using the power property (Equation 1.6.3) of logarithmic functions, we can rewrite the equation as $\ln(2x) - \ln(x^6) = 0$.

Using the quotient property (Equation 1.6.2), this becomes

$$\ln\left(\frac{2}{x^5}\right) = 0$$

Therefore, $2/x^5 = 1$, which implies $x = \sqrt[5]{2}$. We should then check for any extraneous solutions.

? Exercise 1.6.5

Solve $\ln(x^3) - 4\ln(x) = 1$.

Hint

First use the power property, then use the product property of logarithms.

Answer

$x = \frac{1}{e}$

When evaluating a logarithmic function with a calculator, you may have noticed that the only options are $\log_{10}$ or $\log$, called the **common logarithm**, or $\ln$, which is the natural logarithm. However, exponential functions and logarithm functions can be expressed in terms of any desired base b. If you need to use a calculator to evaluate an expression with a different base, you can apply the change-of-base formulas first. Using this change of base, we typically write a given exponential or logarithmic function in terms of the natural exponential and natural logarithmic functions.

Rule: Change-of-Base Formulas

Let $a > 0$, $b > 0$, and $a \neq 1$, $b \neq 1$.

1. $a^x = b^{x\log_b a}$ for any real number x.

   If $b = e$, this equation reduces to $a^x = e^{x\log_e a} = e^{x\ln a}$.

2. $\log_a x = \dfrac{\log_b x}{\log_b a}$ for any real number $x > 0$.

   If $b = e$, this equation reduces to $\log_a x = \dfrac{\ln x}{\ln a}$.

Proof

For the first change-of-base formula, we begin by making use of the power property of logarithmic functions. We know that for any base $b > 0$, $b \neq 1$, $\log_b(a^x) = x\log_b a$. Therefore,

$$b^{\log_b(a^x)} = b^{x\log_b a}.$$

In addition, we know that b^x and $\log_b(x)$ are inverse functions. Therefore,

$$b^{\log_b(a^x)} = a^x.$$

Combining these last two equalities, we conclude that $a^x = b^{x \log_b a}$.

To prove the second property, we show that

$$(\log_b a) \cdot (\log_a x) = \log_b x.$$

Let $u = \log_b a, v = \log_a x$, and $w = \log_b x$. We will show that $u \cdot v = w$. By the definition of logarithmic functions, we know that $b^u = a$, $a^v = x$, and $b^w = x$. From the previous equations, we see that

$$b^{uv} = (b^u)^v = a^v = x = b^w.$$

Therefore, $b^{uv} = b^w$. Since exponential functions are one-to-one, we can conclude that $u \cdot v = w$.

□

Example 1.6.6: Changing Bases

Use a calculating utility to evaluate $\log_3 7$ with the change-of-base formula presented earlier.

Solution

Use the second equation with $a = 3$ and $b = e$: $\log_3 7 = \dfrac{\ln 7}{\ln 3} \approx 1.77124$.

Exercise 1.6.6

Use the change-of-base formula and a calculating utility to evaluate $\log_4 6$.

Hint

Use the change of base to rewrite this expression in terms of expressions involving the natural logarithm function.

Answer

$$\log_4 6 = \frac{\ln 6}{\ln 4} \approx 1.29248$$

Example 1.6.7: The Richter Scale for Earthquakes

In 1935, Charles Richter developed a scale (now known as the Richter scale) to measure the magnitude of an **earthquake**. The scale is a base-10 logarithmic scale, and it can be described as follows: Consider one earthquake with magnitude R_1 on the Richter scale and a second earthquake with magnitude R_2 on the Richter scale. Suppose $R_1 > R_2$, which means the earthquake of magnitude R_1 is stronger, but how much stronger is it than the other earthquake?



Figure 1.6.6: (credit: modification of work by Robb Hannawacker, NPS)

A way of measuring the intensity of an earthquake is by using a seismograph to measure the amplitude of the earthquake waves. If A_1 is the amplitude measured for the first earthquake and A_2 is the amplitude measured for the second earthquake, then the amplitudes and magnitudes of the two earthquakes satisfy the following equation:

$$R_1 - R_2 = \log_{10}\left(\frac{A1}{A2}\right).$$

Consider an earthquake that measures 8 on the Richter scale and an earthquake that measures 7 on the Richter scale. Then,

$$8 - 7 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore,

$$\log_{10}\left(\frac{A1}{A2}\right) = 1,$$

which implies $A_1/A_2 = 10$ or $A_1 = 10A_2$. Since A_1 is 10 times the size of A_2, we say that the first earthquake is 10 times as intense as the second earthquake. On the other hand, if one earthquake measures 8 on the Richter scale and another measures 6, then the relative intensity of the two earthquakes satisfies the equation

$$\log_{10}\left(\frac{A1}{A2}\right) = 8 - 6 = 2.$$

Therefore, $A_1 = 100A_2$.That is, the first earthquake is 100 times more intense than the second earthquake.

How can we use logarithmic functions to compare the relative severity of the magnitude 9 earthquake in Japan in 2011 with the magnitude 7.3 earthquake in Haiti in 2010?

Solution

To compare the Japan and Haiti earthquakes, we can use an equation presented earlier:

$$9 - 7.3 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore, $A_1/A_2 = 10^{1.7}$, and we conclude that the earthquake in Japan was approximately 50 times more intense than the earthquake in Haiti.

? Exercise 1.6.7

Compare the relative severity of a magnitude **8.4** earthquake with a magnitude **7.4** earthquake.

Hint

$R_1 - R_2 = \log_{10}(A1/A2)$.

Answer

The magnitude **8.4** earthquake is roughly **10** times as severe as the magnitude **7.4** earthquake.

Hyperbolic Functions

The hyperbolic functions are defined in terms of certain combinations of e^x and e^{-x}. These functions arise naturally in various engineering and physics applications, including the study of water waves and vibrations of elastic membranes. Another common use for a hyperbolic function is the representation of a hanging chain or cable, also known as a *catenary* (Figure **1.6.7**). If we introduce a coordinate system so that the low point of the chain lies along the y-axis, we can describe the height of the chain in terms of a hyperbolic function. First, we define the **hyperbolic functions.**



Figure **1.6.7**:The shape of a strand of silk in a spider's web can be described in terms of a hyperbolic function. The same shape applies to a chain or cable hanging from two supports with only its own weight. (credit: "Mtpaley", Wikimedia Commons)

Definitions: hyperbolic functions

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The name **cosh** rhymes with "gosh," whereas the name **sinh** is pronounced "cinch." **Tanh, sech, csch,** and **coth** are pronounced "tanch," "seech," "coseech," and "cotanch," respectively.

Using the definition of $\cosh(x)$ and principles of physics, it can be shown that the height of a hanging chain, such as the one in Figure 1.6.8, can be described by the function $h(x) = a\cosh(x/a) + c$ for certain constants a and c.

But why are these functions called hyperbolic functions? To answer this question, consider the quantity $\cosh^2 t - \sinh^2 t$. Using the definition of **cosh** and **sinh**, we see that

$$\cosh^2 t - \sinh^2 t = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = 1.$$

This identity is the analog of the trigonometric identity $\cos^2 t + \sin^2 t = 1$. Here, given a value t, the point $(x, y) = (\cosh t, \sinh t)$ lies on the unit hyperbola $x^2 - y^2 = 1$ (Figure 1.6.8).

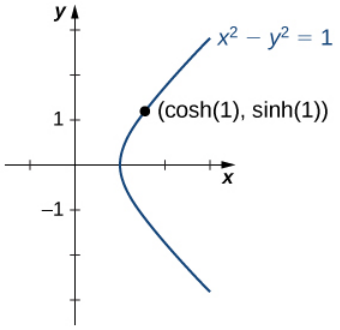


Figure 1.6.8: The unit hyperbola $\cosh^2 t - \sinh^2 t = 1$.

Graphs of Hyperbolic Functions

To graph $\cosh x$ and $\sinh x$, we make use of the fact that both functions approach $(1/2)e^x$ as $x \to \infty$, since $e^{-x} \to 0$ as $x \to \infty$. As $x \to -\infty$, $\cosh x$ approaches $1/2e^{-x}$, whereas $\sinh x$ approaches $-1/2e^{-x}$. Therefore, using the graphs of $1/2e^x, 1/2e^{-x}$, and $-1/2e^{-x}$ as guides, we graph $\cosh x$ and $\sinh x$. To graph $\tanh x$, we use the fact that $\tanh(0) = 0$, $-1 < \tanh(x) < 1$ for all x, $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$ as $x \to -\infty$. The graphs of the other three hyperbolic functions can be sketched using the graphs of $\cosh x$, $\sinh x$, and $\tanh x$ (Figure 1.6.9).

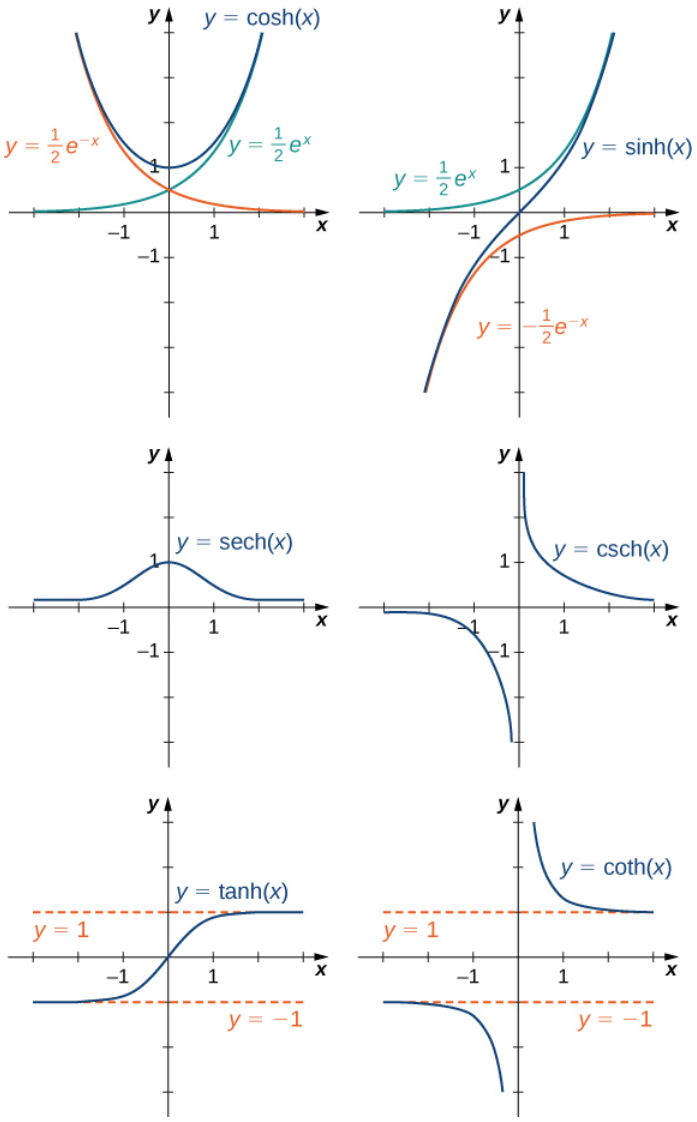


Figure **1.6.9**: The hyperbolic functions involve combinations of e^x and e^{-x}.

Identities Involving Hyperbolic Functions

The identity $\cosh^2 t - \sinh^2 t = 1$, shown in Figure **1.6.8**, is one of several identities involving the hyperbolic functions, some of which are listed next. The first four properties follow easily from the definitions of hyperbolic sine and hyperbolic cosine. Except for some differences in signs, most of these properties are analogous to identities for trigonometric functions.

Identities Involving Hyperbolic Functions

1. $\cosh(-x) = \cosh x$
2. $\sinh(-x) = -\sinh x$
3. $\cosh x + \sinh x = e^x$
4. $\cosh x - \sinh x = e^{-x}$
5. $\cosh^2 x - \sinh^2 x = 1$
6. $1 - \tanh^2 x = \operatorname{sech}^2 x$
7. $\coth^2 x - 1 = \operatorname{csch}^2 x$
8. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
9. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

Example 1.6.8: Evaluating Hyperbolic Functions

a. Simplify $\sinh(5\ln x)$.

b. If $\sinh x = 3/4$, find the values of the remaining five hyperbolic functions.

Solution:

a. Using the definition of the $\sinh$ function, we write

$$\sinh(5\ln x) = \frac{e^{5\ln x} - e^{-5\ln x}}{2} = \frac{e^{\ln(x^5)} - e^{\ln(x^{-5})}}{2} = \frac{x^5 - x^{-5}}{2}.$$

b. Using the identity $\cosh^2 x - \sinh^2 x = 1$,we see that

$$\cosh^2 x = 1 + \left(\tfrac{3}{4}\right)^2 = \frac{25}{16}.$$

Since $\cosh x \geq 1$ for all x, we must have $\cosh x = 5/4$. Then, using the definitions for the other hyperbolic functions, we conclude that $\tanh x = 3/5, \operatorname{csch} x = 4/3, \operatorname{sech} x = 4/5$, and $\coth x = 5/3$.

Exercise 1.6.8

Simplify $\cosh(2\ln x)$.

Hint

Use the definition of the $\cosh$ function and the power property of logarithm functions.

Answer

$(x^2 + x^{-2})/2$

Inverse Hyperbolic Functions

From the graphs of the hyperbolic functions, we see that all of them are one-to-one except $\cosh x$ and $\operatorname{sech} x$. If we restrict the domains of these two functions to the interval $[0, \infty)$, then all the hyperbolic functions are one-to-one, and we can define the **inverse hyperbolic functions**. Since the hyperbolic functions themselves involve exponential functions, the inverse hyperbolic functions involve logarithmic functions.

Definitions: Inverse Hyperbolic Functions

$$\sinh^{-1} x = \operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad \cosh^{-1} x = \operatorname{arccosh} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \operatorname{arccot} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1} x = \operatorname{arcsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \operatorname{arccsch} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

Let's look at how to derive the first equation. The others follow similarly. Suppose $y = \sinh^{-1} x$. Then, $x = \sinh y$ and, by the definition of the hyperbolic sine function, $x = \dfrac{e^y - e^{-y}}{2}$. Therefore,

$$e^y - 2x - e^{-y} = 0.$$

Multiplying this equation by e^y, we obtain

$$e^{2y} - 2xe^y - 1 = 0.$$

This can be solved like a quadratic equation, with the solution

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

Since $e^y > 0$,the only solution is the one with the positive sign. Applying the natural logarithm to both sides of the equation, we conclude that

$$y = \ln(x + \sqrt{x^2 + 1}).$$

Example 1.6.9: Evaluating Inverse Hyperbolic Functions

Evaluate each of the following expressions.

$$\sinh^{-1}(2)$$

$$\tanh^{-1}(1/4)$$

Solution:

$$\sinh^{-1}(2) = \ln(2 + \sqrt{2^2 + 1}) = \ln(2 + \sqrt{5}) \approx 1.4436$$

$$\tanh^{-1}(1/4) = \frac{1}{2}\ln\left(\frac{1 + 1/4}{1 - 1/4}\right) = \frac{1}{2}\ln\left(\frac{5/4}{3/4}\right) = \frac{1}{2}\ln\left(\frac{5}{3}\right) \approx 0.2554$$

Exercise 1.6.9

Evaluate $\tanh^{-1}(1/2)$.

Hint

Use the definition of $\tanh^{-1} x$ and simplify.

Answer

$\frac{1}{2}\ln(3) \approx 0.5493$.

Key Concepts

- The exponential function $y = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$. Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- The logarithmic function $y = \log_b(x)$ is the inverse of $y = b^x$. Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- The natural exponential function is $y = e^x$ and the natural logarithmic function is $y = \ln x = \log_e x$.
- Given an exponential function or logarithmic function in base a, we can make a change of base to convert this function to any base $b > 0$, $b \neq 1$. We typically convert to base e.
- The hyperbolic functions involve combinations of the exponential functions e^x and e^{-x}. As a result, the inverse hyperbolic functions involve the natural logarithm.

Glossary

base

the number b in the exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$

exponent

the value x in the expression b^x

hyperbolic functions

the functions denoted $\sinh$, $\cosh$, $\tanh$, csch, sech, and $\coth$, which involve certain combinations of e^x and e^{-x}

inverse hyperbolic functions

the inverses of the hyperbolic functions where $\cosh$ and sech are restricted to the domain $[0, \infty)$;each of these functions can be expressed in terms of a composition of the natural logarithm function and an algebraic function

natural exponential function

the function $f(x) = e^x$

natural logarithm

the function $\ln x = \log_e x$

number e

as m gets larger, the quantity $(1 + (1/m)^m$ gets closer to some real number; we define that real number to be e; the value of e is approximately 2.718282

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1.6E: Exercises

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1.7: Chapter 1 Review Exercises

True or False? Justify your answer with a proof or a counterexample.

1) A function is always one-to-one.

2) $f \circ g = g \circ f$, assuming f and g are functions.

Answer

False

3) A relation that passes the horizontal and vertical line tests is a one-to-one function.

4) A relation passing the horizontal line test is a function.

Answer

False

State the domain and range of the given functions:

$f = x^2 + 2x - 3$, $g = \ln(x - 5)$, $h = \dfrac{1}{x + 4}$

5) h

6) g

Answer

Domain: $x > 5$, Range: all real numbers

7) $h \circ f$

8) $g \circ f$

Answer

Domain: $x > 2$ and $x < -4$, Range: all real numbers

Find the degree, y-intercept, and zeros for the following polynomial functions.

9) $f(x) = 2x^2 + 9x - 5$

10) $f(x) = x^3 + 2x^2 - 2x$

Answer

Degree of 3, y-intercept: $(0, 0)$, Zeros: 0, $\sqrt{3} - 1$, $-1 - \sqrt{3}$

Simplify the following trigonometric expressions.

11) $\dfrac{\tan^2 x}{\sec^2 x} + \cos^2 x$

12) $\cos^2 x - \sin^2 x$

Answer

$\cos(2x)$

Solve the following trigonometric equations on the interval $\theta = [-2\pi, 2\pi]$ exactly.

13) $6 \cos 2x - 3 = 0$

14) $\sec^2 x - 2 \sec x + 1 = 0$

Answer

$0, \pm 2\pi$

Solve the following logarithmic equations.

15) $5^x = 16$

16) $\log_2(x+4) = 3$

Answer

4

Are the following functions one-to-one over their domain of existence? Does the function have an inverse? If so, find the inverse $f^{-1}(x)$ of the function. Justify your answer.

17) $f(x) = x^2 + 2x + 1$

18) $f(x) = \dfrac{1}{x}$

Answer

One-to-one; yes, the function has an inverse; inverse: $f^{-1}(x) = \dfrac{1}{x}$

For the following problems, determine the largest domain on which the function is one-to-one and find the inverse on that domain.

19) $f(x) = \sqrt{9 - x}$

20) $f(x) = x^2 + 3x + 4$

Answer

$x \geq -\frac{3}{2}, \quad f^{-1}(x) = -\frac{3}{2} + \frac{1}{2}\sqrt{4x - 7}$

21) A car is racing along a circular track with diameter of 1 mi. A trainer standing in the center of the circle marks his progress every 5 sec. After 5 sec, the trainer has to turn 55° to keep up with the car. How fast is the car traveling?

For the following problems, consider a restaurant owner who wants to sell T-shirts advertising his brand. He recalls that there is a fixed cost and variable cost, although he does not remember the values. He does know that the T-shirt printing company charges $440 for 20 shirts and $1000 for 100 shirts.

22) a. Find the equation $C = f(x)$ that describes the total cost as a function of number of shirts and

b. determine how many shirts he must sell to break even if he sells the shirts for $10 each.

Answer

a. $C(x) = 300 + 7x$

b. 100 shirts

23) a. Find the inverse function $x = f^{-1}(C)$ and describe the meaning of this function.

b. Determine how many shirts the owner can buy if he has $8000 to spend.

For the following problems, consider the population of Ocean City, New Jersey, which is cyclical by season.

24) The population can be modeled by $P(t) = 82.5 - 67.5\cos[(\pi/6)t]$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). During a year, in what intervals is the population less than 20,000? During what intervals is the population more than 140,000?

Answer

The population is less than 20,000 from December 8 through January 23 and more than 140,000 from May 29 through August 2

25) In reality, the overall population is most likely increasing or decreasing throughout each year. Let's reformulate the model as $P(t) = 82.5 - 67.5\cos[(\pi/6)t] + t$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). When is the first time the population reaches 200,000?

For the following problems, consider radioactive dating. A human skeleton is found in an archeological dig. Carbon dating is implemented to determine how old the skeleton is by using the equation $y = e^{rt}$, where y is the percentage of radiocarbon still present in the material, t is the number of years passed, and $r = -0.0001210$ is the decay rate of radiocarbon.

26) If the skeleton is expected to be 2000 years old, what percentage of radiocarbon should be present?

Answer

78.51%

27) Find the inverse of the carbon-dating equation. What does it mean? If there is 25% radiocarbon, how old is the skeleton?

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CHAPTER OVERVIEW

2: Limits

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

2.1: Prelude to Limits

2.2: A Preview of Calculus

2.2E: Exercises

2.3: The Limit of a Function

2.3E: Exercises

2.4: The Limit Laws

2.4E: Exercises

2.5: Continuity

2.5E: Exercises

2.6: The Precise Definition of a Limit

2.6E: Exercises

2.7: Chapter 2 Review Exercises

Thumbnail: The function $f(x) = 1/(x-a)^n$ *has infinite limits at* a*. (CC BY; OpenStax)*

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2.1: Prelude to Limits

Science fiction writers often imagine spaceships that can travel to far-off planets in distant galaxies. However, back in 1905, Albert Einstein showed that a limit exists to how fast any object can travel. The problem is that the faster an object moves, the more mass it attains (in the form of energy), according to the equation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. What is this speed limit? (We explore this problem further in the chapter)

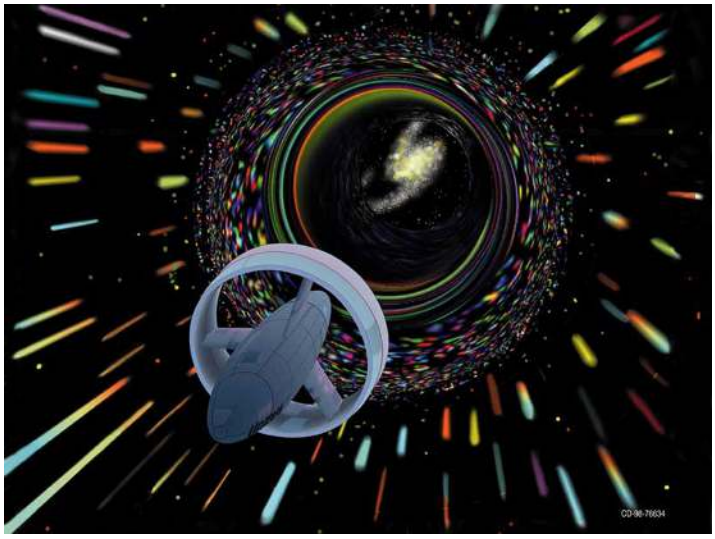


Figure **2.1.1**:The vision of human exploration by the National Aeronautics and Space Administration (NASA) to distant parts of the universe illustrates the idea of space travel at high speeds. But, is there a limit to how fast a spacecraft can go? (credit: NASA)

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

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2.2: A Preview of Calculus

Learning Objectives

- Describe the tangent problem and how it led to the idea of a derivative.
- Explain how the idea of a limit is involved in solving the tangent problem.
- Recognize a tangent to a curve at a point as the limit of secant lines.
- Identify instantaneous velocity as the limit of average velocity over a small time interval.
- Describe the area problem and how it was solved by the integral.
- Explain how the idea of a limit is involved in solving the area problem.
- Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3$, $g(x) = \frac{1}{2}x + 1$, and $h(x) = 2$, shown in Figure 2.2.1.

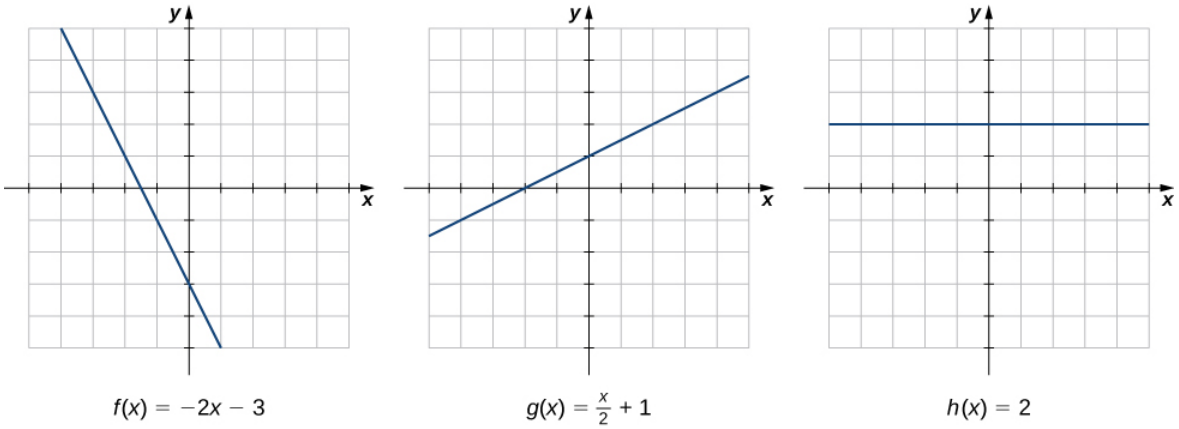


Figure 2.2.1: The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3$, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x-axis, the y-coordinate decreases by 2 units. This **rate of change** is determined by the slope (-2) of the line. Similarly, the slope of $1/2$ in the function $g(x)$ tells us that for every change in x of 1 unit there is a corresponding change in y of $1/2$ unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (Figure 2.2.2). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?

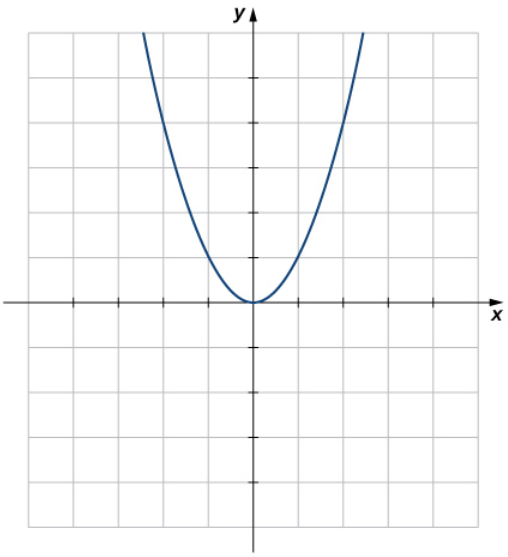


Figure 2.2.2: The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a secant line. Figure 2.2.3 shows a *secant line* to a function $f(x)$ at a point $(a, f(a))$.

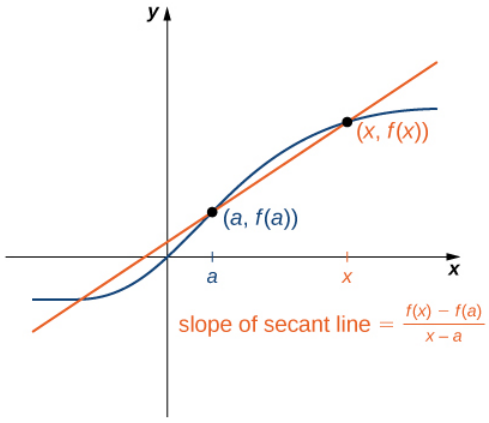


Figure 2.2.3: The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Definition: Secant Line

The *secant* to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{sec} = \frac{f(x) - f(a)}{x - a}. \tag{2.2.1}$$

The accuracy of approximating the rate of change of the function with a secant line depends on how close x is to a. As we see in Figure 2.2.4, if x is closer to a, the slope of the secant line is a better measure of the rate of change of $f(x)$ at a.

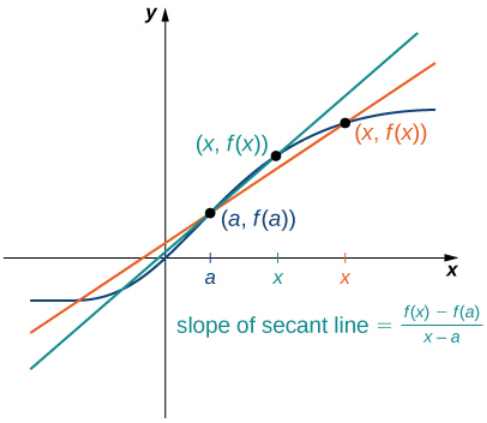


Figure 2.2.4: As x gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at a (Figure 2.2.5). The slope of the tangent line to the graph at a measures the rate of change of the function at a. This value also represents the derivative of the function $f(x)$ at a, or the rate of change of the function at a. This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

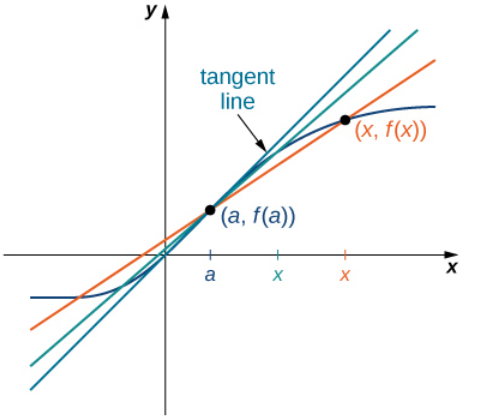


Figure 2.2.5: Solving the Tangent Problem: As x approaches a, the secant lines approach the tangent line.

Example 2.2.1 illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

Example 2.2.1: Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2, 4)$
b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

Solution:

Use the formula for the slope of a secant line (Equation 2.2.1).

a. $m_{sec} = \frac{4-1}{2-1} = 3$

b. $m_{sec} = \frac{\frac{9}{4}-1}{\frac{3}{2}-1} = \frac{5}{2} = 2.5$

The point in part b. is closer to the point $(1, 1)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 (Figure 2.2.6).

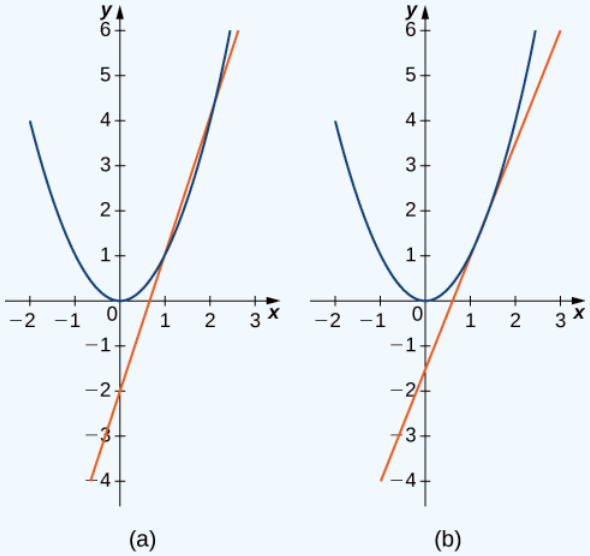


Figure 2.2.6: The secant lines to $f(x) = x^2$ at $(1, 1)$ through (a) $(2, 4)$ and (b) $(\frac{3}{2}, \frac{9}{4})$ provide successively closer approximations to the tangent line to $f(x) = x^2$ at $(1, 1)$.

? Exercise 2.2.1

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and the point $(\frac{5}{4}, \frac{25}{16})$ on the graph of $f(x) = x^2$.

Answer

2.25

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time t. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t = a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the *average velocity* of an object over a time period to be the change in its position divided by the length of the time period.

Definition: Average Velocity

Let $s(t)$ be the position of an object moving along a coordinate axis at time t. The *average velocity* of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$v_{ave} = \frac{s(t) - s(a)}{t - a}. \tag{2.2.2}$$

As t is chosen closer to a, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point a, we let the x-values approach a in the slope of the secant line. Similarly, to find the instantaneous velocity at time a, we let the t-values approach a in the average velocity. This process of letting x or t approach a in an expression is called taking a **limit**. Thus, we may define the *instantaneous velocity* as follows.

Definition: Instantaneous Velocity

For a position function $s(t)$, the *instantaneous velocity* at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of t become closer to a, provided such a value exists.

Example 2.2.2 illustrates this concept of limits and average velocity.

Example 2.2.2: Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground t seconds later (for $0 \le t \le 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$
b. $[0.5, 0.51]$

Solution

Substitute the data into Equation 2.2.2 for the definition of average velocity.

a. $$v_{ave} = \frac{s(0.49) - s(0.5)}{0.49 - 0.5} = -15.84$$

b. $$v_{ave} = \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.016$$

The instantaneous velocity is somewhere between −15.84 and −16.16 ft/sec. A good guess might be −16 ft/sec.

Exercise 2.2.2

An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

Hint

Use Equation 2.2.2 with $v_{ave} = \frac{s(2.001) - s(2)}{2.001 - 2}$.

Answer

12.006001

The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may be interpreted as the area under a curve. This leads us to ask the question: How can we find the area between the graph of a function and the x-axis over an interval (Figure 2.2.7)?

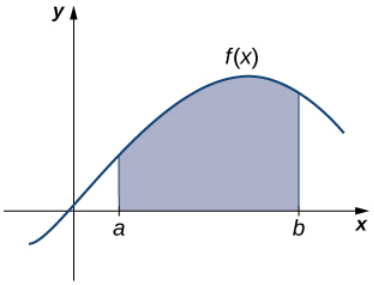


Figure 2.2.7: The Area Problem: How do we find the area of the shaded region?

As in the answer to our previous questions on velocity, we first try to approximate the solution. We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up

the areas of these rectangles (Figure 2.2.8).

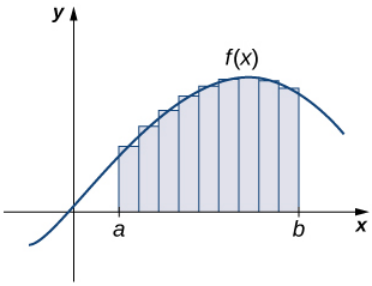


Figure 2.2.8: The area of the region under the curve is approximated by summing the areas of thin rectangles.

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the x-axis over the interval $[a, b]$. Once again, we find ourselves taking a limit. Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

Example 2.2.3: Estimation Using Rectangles

Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in Figure 2.2.9.

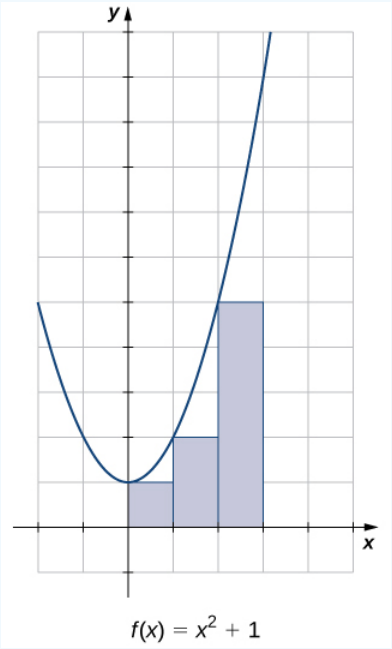


Figure 2.2.9: The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

Solution

The areas of the three rectangles are 1 unit^2, 2 unit^2, and 5 unit^2. Using these rectangles, our area estimate is 8 unit^2.

Exercise 2.2.3

Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in Figure 2.2.10.

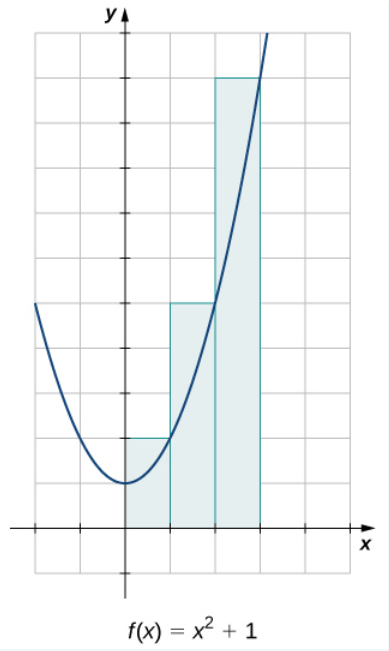


Figure **2.2.10**: The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

Hint

Use Example **2.2.3** as a guide

Answer

17 unit^2

Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions. Suppose, for example, that instead of determining the velocity of an object moving along a coordinate axis, we want to determine the velocity of a rock fired from a catapult at a given time, or of an airplane moving in three dimensions. We might want to graph real-value functions of two variables or determine volumes of solids of the type shown in Figure **2.2.11**. These are only a few of the types of questions that can be asked and answered using *multivariable calculus*. Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables. However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.

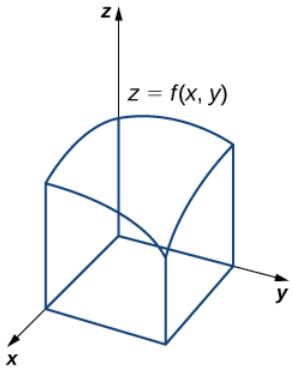


Figure **2.2.11**: We can use multivariable calculus to find the volume between a surface defined by a function of two variables and a plane.

Key Concepts

- Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the rate of change of the function, also called the *derivative*. Calculating a derivative requires finding a limit.
- Integral calculus arose from trying to solve the problem of finding the area of a region between the graph of a function and the x-axis. We can approximate the area by dividing it into thin rectangles and summing the areas of these rectangles. This summation leads to the value of a function called the *integral*. The integral is also calculated by finding a limit and, in fact, is related to the derivative of a function.
- Multivariable calculus enables us to solve problems in three-dimensional space, including determining motion in space and finding volumes of solids.

Key Equations

- **Slope of a Secant Line**

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

- **Average Velocity over Interval [a,t]**

$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$

Glossary

average velocity
the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval $[t, a]$ (if $t < a$ or $[a, t]$ if $t > a$), with a position given by $s(t)$, that is $v_{ave} = \frac{s(t) - s(a)}{t - a}$

differential calculus
the field of calculus concerned with the study of derivatives and their applications

instantaneous velocity
The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t, a]$ and $[a, t]$ approach as the values of t move closer to a, provided such a value exists

integral calculus
the study of integrals and their applications

limit
the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x approaches a

multivariable calculus
the study of the calculus of functions of two or more variables

secant
A secant line to a function $f(x)$ at a is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{sec} = \frac{f(x) - f(a)}{x - a}$

tangent
A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with x-values that approach a; the slope of the tangent line to a graph at a measures the rate of change of the function at a

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2.2E: Exercises

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2.3: The Limit of a Function

Learning Objectives

- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- Define one-sided limits and provide examples.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

- $f(x) = \dfrac{x^2 - 4}{x - 2}$,
- $g(x) = \dfrac{|x - 2|}{x - 2}$, and
- $h(x) = \dfrac{1}{(x - 2)^2}$,

which are shown in Figure **2.3.1**. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

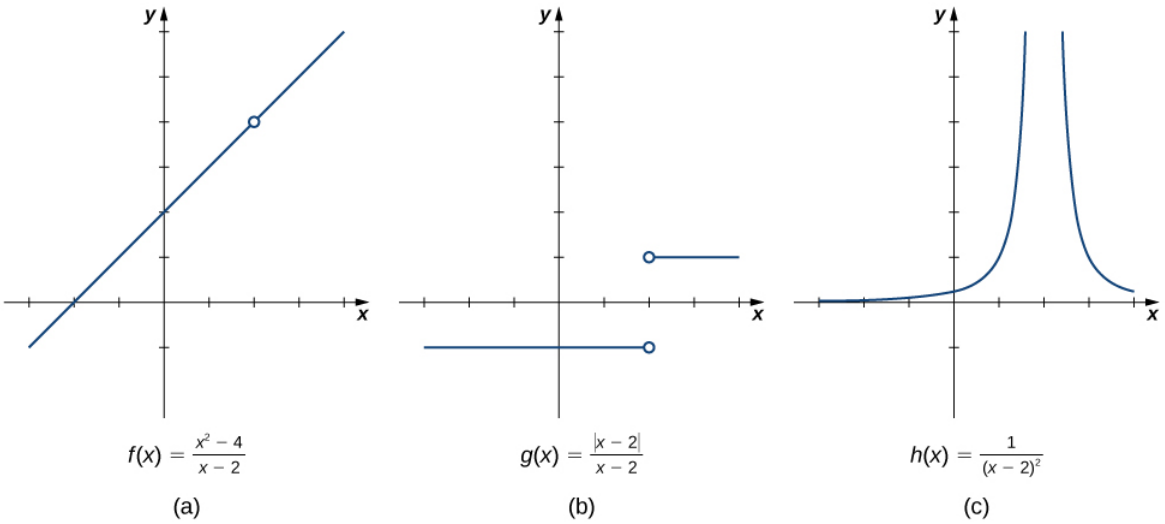


Figure **2.3.1**: These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in Figure **2.3.1**. As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as x approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4$$

From this very brief informal look at one limit, let's start to develop an *intuitive definition of the limit*. We can think of the limit of a function at a number a as being the one real number L that the functional values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following definition:

Definition (Intuitive): Limit

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If all values of the function $f(x)$ approach the real number L as the values of $x (\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. (More succinctly, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L. \tag{2.3.1}$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x-values—one set of values approaching a and less than a, and another set of values approaching a and greater than a. Table 2.3.1 demonstrates what your tables might look like.

Table 2.3.1

x	$f(x)$	x	$f(x)$
$a-0.1$	$f(a-0.1)$	$a+0.1$	$f(a+0.1)$
$a-0.01$	$f(a-0.01)$	$a+0.01$	$f(a+0.01)$
$a-0.001$	$f(a-0.001)$	$a+0.001$	$f(a+0.001)$
$a-0.0001$	$f(a-0.0001)$	$a+0.0001$	$f(a+0.0001)$
Use additional values as necessary.		Use additional values as necessary.	

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a-0.1)$, $f(a-0.01)$, $f(a-0.001)$, $f(a-0.0001)$, and so on, and $f(a+0.1)$, $f(a+0.01)$, $f(a+0.001)$, $f(a+0.0001)$, and so on. (Note: Although we have chosen the x-values $a \pm 0.1$, $a \pm 0.01$, $a \pm 0.001$, $a \pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)

3. If both columns approach a common y-value L, we state $\lim_{x \to a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for x-values near a are in our window. We can use the trace feature to move along the graph of the function and watch the y-value readout as the x-values approach a. If the y-values approach L as our x-values approach a from both directions, then $\lim_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in Examples 2.3.1*A* and 2.3.1*B*.

Example 2.3.1A: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$ using a table of functional values.

Solution

We have calculated the values of $f(x) = \frac{\sin x}{x}$ for the values of x listed in Table 2.3.2.

Table 2.3.2

x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$
-0.1	0.998334166468	0.1	0.998334166468
-0.01	0.999983333417	0.01	0.999983333417
-0.001	0.999999833333	0.001	0.999999833333
-0.0001	0.999999998333	0.0001	0.999999998333

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\frac{\sin x}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. A calculator-or computer-generated graph of $f(x) = \frac{\sin x}{x}$ would be similar to that shown in Figure 2.3.2, and it confirms our estimate.

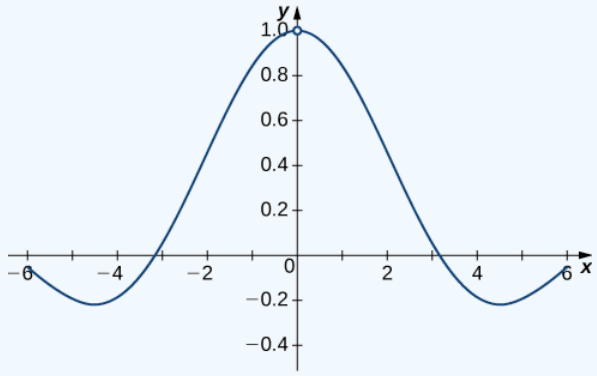


Figure 2.3.2: The graph of $f(x) = (\sin x)/x$ confirms the estimate from Table 2.3.2.

Example 2.3.1B: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$ using a table of functional values.

Solution

As before, we use a table—in this case, Table 2.3.3—to list the values of the function for the given values of x.

Table 2.3.3

x	$\frac{\sqrt{x}-2}{x-4}$	x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869	4.1	0.248456731317
3.99	0.25015644562	4.01	0.24984394501
3.999	0.250015627	4.001	0.249984377
3.9999	0.250001563	4.0001	0.249998438
3.99999	0.25000016	4.00001	0.24999984

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}=0.25$. We confirm this estimate using the graph of $f(x)=\dfrac{\sqrt{x}-2}{x-4}$ shown in Figure **2.3.3**.

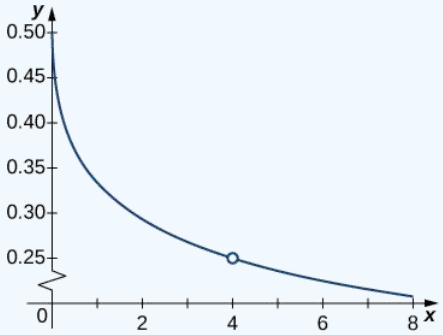


Figure **2.3.3**: The graph of $\frac{\sqrt{x}-2}{x-4}$ confirms the estimate from Table **2.3.3**.

? Exercise 2.3.1

Estimate $\lim\limits_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1}$ using a table of functional values. Use a graph to confirm your estimate.

Hint

Use 0.9, 0.99, 0.999, 0.9999, 0.99999 and 1.1, 1.01, 1.001, 1.0001, 1.00001 as your table values.

Answer

$$\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}=-1$$

At this point, we see from Examples **2.3.1***A* and **2.3.1***B* that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In Example **2.3.2**, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

✔ Example 2.3.2: Evaluating a Limit Using a Graph

For $g(x)$ shown in Figure **2.3.4**, evaluate $\lim\limits_{x\to -1} g(x)$.

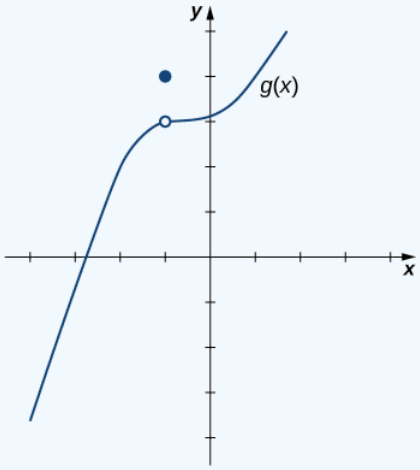


Figure **2.3.4**: The graph of $g(x)$ includes one value not on a smooth curve.

Solution:

Despite the fact that $g(-1) = 4$, as the x-values approach -1 from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x \to -1} g(x) = 3$. Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on Example 2.3.2, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

? Exercise 2.3.2

Use the graph of $h(x)$ in Figure 2.3.5 to evaluate $\lim_{x \to 2} h(x)$, if possible.

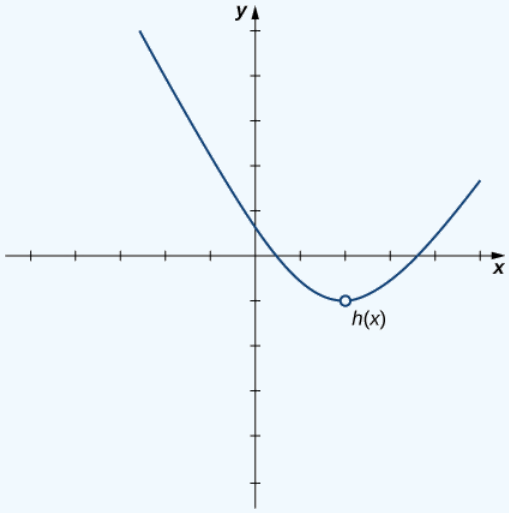


Figure 2.3.5:

Hint

What y-value does the function approach as the x-values approach 2?

Solution

$\lim_{x \to 2} h(x) = -1$.

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Two Important Limits

Let a be a real number and c be a constant.

i. $\lim_{x \to a} x = a$
ii. $\lim_{x \to a} c = c$

We can make the following observations about these two limits.

i. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x \to a} x = a$.
ii. For the second limit, consider Table 2.3.4.

Table 2.3.4

x	$f(x)=c$	x	$f(x)=c$
$a-0.1$	c	$a+0.1$	c
$a-0.01$	c	$a+0.01$	c
$a-0.001$	c	$a+0.001$	c
$a-0.0001$	c	$a+0.0001$	c

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim\limits_{x\to a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example 2.3.3: Evaluating a Limit That Fails to Exist

Evaluate $\lim\limits_{x\to 0} \sin(1/x)$ using a table of values.

Solution

Table 2.3.5 lists values for the function $\sin(1/x)$ for the given values of x.

Table 2.3.5

x	$\sin(1/x)$	x	$\sin(1/x)$
-0.1	0.544021110889	0.1	−0.544021110889
-0.01	0.50636564111	0.01	−0.50636564111
-0.001	−0.8268795405312	0.001	0.8268795405312
-0.0001	0.305614388888	0.0001	−0.305614388888
-0.00001	−0.035748797987	0.00001	0.035748797987
-0.000001	0.349993504187	0.000001	−0.349993504187

After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of x-values approaching 0:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding y-values are

$$1, -1, 1, -1, 1, -1, \ldots.$$

At this point we can indeed conclude that $\lim\limits_{x\to 0} \sin(1/x)$ does not exist. (Mathematicians frequently abbreviate "does not exist" as DNE. Thus, we would write $\lim\limits_{x\to 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in Figure 2.3.6 and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between -1 and 1 as x approaches 0.

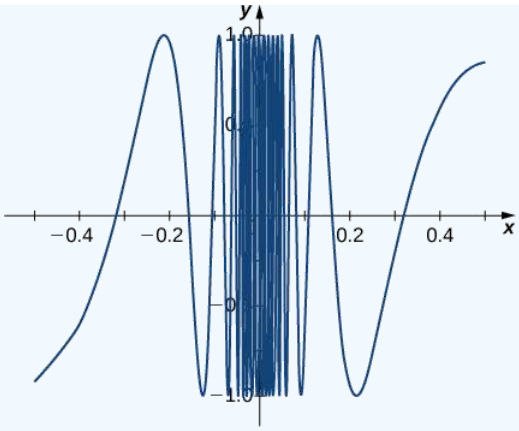


Figure 2.3.6: The graph of $f(x) = \sin(1/x)$ oscillates rapidly between -1 and 1 as x approaches 0.

? Exercise 2.3.3

Use a table of functional values to evaluate $\lim\limits_{x \to 2} \dfrac{|\, x^2 - 4 \,|}{x - 2}$, if possible.

Hint

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 and 2.1, 2.01, 2.001, 2.0001, 2.00001 in your table.

Answer

$\lim\limits_{x \to 2} \dfrac{|\, x^2 - 4 \,|}{x - 2}$ does not exist.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x) = |x - 2|/(x - 2)$ introduced at the beginning of the section (see Figure 2.3.1(b)). As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist—that is, $\lim\limits_{x \to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the x-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or the negative side of 2), $g(x) = -1$. Thus, as x approaches 2 from the left, $g(x)$ approaches -1. Mathematically, we say that the limit as x approaches 2 from the left is -1. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x) = -1.$$

Similarly, as x approaches 2 from the right (or from the positive side), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

Definition: One-sided Limits

We define two types of one-sided limits.

Limit from the left:

Let $f(x)$ be a function defined at all values in an open interval of the form (z, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L.$$

Limit from the right:

Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L.$$

Example 2.3.4: Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$

Solution

We can use tables of functional values again. Observe in Table 2.3.6 that for values of x less than 2, we use $f(x) = x + 1$ and for values of x greater than 2, we use $f(x) = x^2 - 4$.

Table 2.3.6

x	$f(x) = x + 1$	x	$f(x) = x^2 - 4$
1.9	2.9	2.1	0.41
1.99	2.99	2.01	0.0401
1.999	2.999	2.001	0.004001
1.9999	2.9999	2.0001	0.00040001
1.99999	2.99999	2.00001	0.0000400001

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$. Figure 2.3.7 shows a graph of $f(x)$ and reinforces our conclusion about these limits.

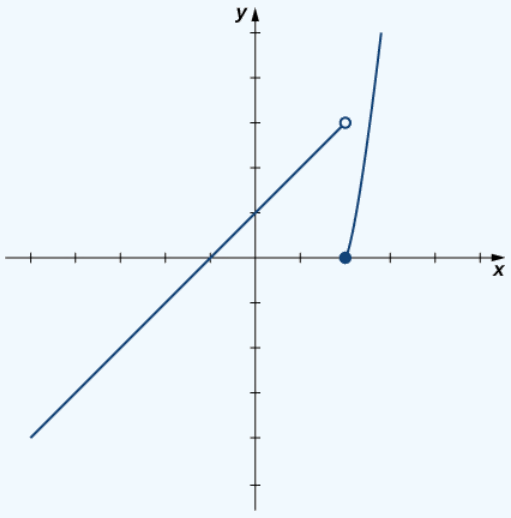


Figure 2.3.7: The graph of $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$ has a break at $x = 2$.

Exercise 2.3.4

Use a table of functional values to estimate the following limits, if possible.

a. $\lim_{x \to 2^-} \dfrac{|x^2 - 4|}{x - 2}$

b. $\lim_{x \to 2^+} \frac{|x^2 - 4|}{x - 2}$

Hint

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 to estimate $\lim_{x \to 2^-} \frac{|x^2 - 4|}{x - 2}$.

Use x-values 2.1, 2.01, 2.001, 2.0001, 2.00001 to estimate $\lim_{x \to 2^+} \frac{|x^2 - 4|}{x - 2}$.

(These tables are available from a previous Checkpoint problem.)

Solution a

a. $\lim_{x \to 2^-} \frac{|x^2 - 4|}{x - 2} = -4$

Solution b

$\lim_{x \to 2^+} \frac{|x^2 - 4|}{x - 2} = 4$

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in Note.

Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

$$\lim_{x \to a} f(x) = L$$

if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see Figure **2.3.1**(c)). From its graph we see that as the values of x approach **2**, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches **2** is positive infinity. Symbolically, we express this idea as

$$\lim_{x \to 2} h(x) = +\infty.$$

More generally, we define *infinite limits* as follows:

Definitions: Infinite Limits

We define three types of *infinite limits*.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty.$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty.$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty.$$

It is important to understand that when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim_{x \to a} f(x) = +\infty$, we always write $\lim_{x \to a} f(x) = +\infty$ rather than $\lim_{x \to a} f(x)$ DNE.

Example 2.3.5: Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \frac{1}{x}$
b. $\lim_{x \to 0^+} \frac{1}{x}$
c. $\lim_{x \to 0} \frac{1}{x}$

Solution

Begin by constructing a table of functional values.

Table 2.3.7

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10,000	0.0001	10,000
-0.00001	-100,000	0.00001	100,000
-0.000001	-1,000,000	0.000001	1,000,000

a. The values of $1/x$ decrease without bound as x approaches 0 from the left. We conclude that

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as x approaches 0 from the right. We conclude that

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x\to 0} \frac{1}{x} \quad \text{DNE.}$$

The graph of $f(x) = 1/x$ in Figure 2.3.8 confirms these conclusions.

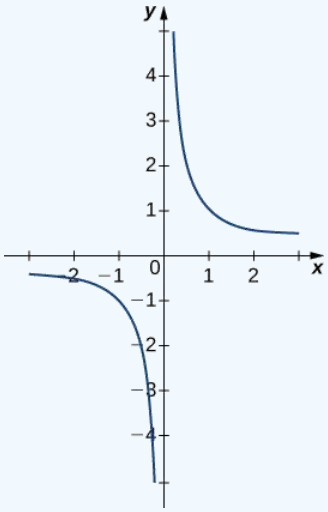


Figure 2.3.8: The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.

Exercise 2.3.5

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim_{x\to 0^-} \frac{1}{x^2}$
b. $\lim_{x\to 0^+} \frac{1}{x^2}$
c. $\lim_{x\to 0} \frac{1}{x^2}$

Hint

Follow the procedures from Example **2.3.5**.

Answer

a. $\lim\limits_{x\to 0^-} \dfrac{1}{x^2} = +\infty$;

b. $\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = +\infty$;

c. $\lim\limits_{x\to 0} \dfrac{1}{x^2} = +\infty$

It is useful to point out that functions of the form $f(x) = 1/(x-a)^n$, where n is a positive integer, have infinite limits as x approaches a from either the left or right (Figure **2.3.9**). These limits are summarized in the above definitions.

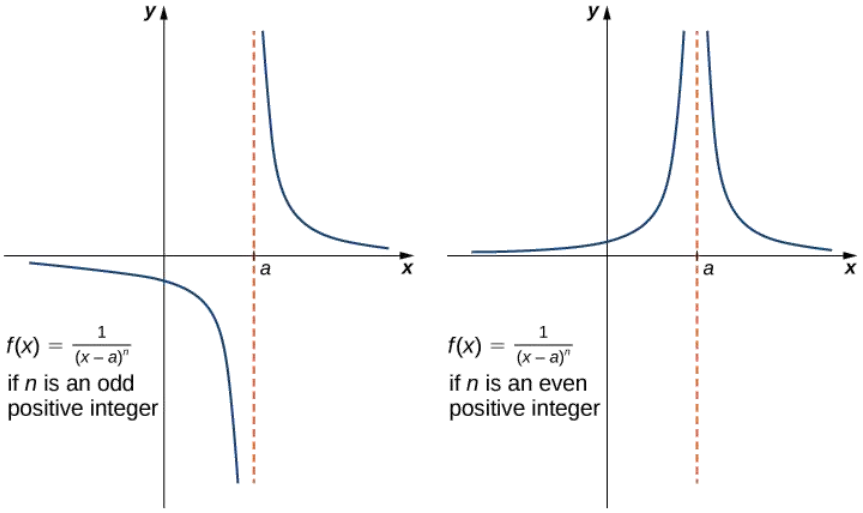


Figure **2.3.9**: The function $f(x) = 1/(x-a)^n$ has infinite limits at a.

Infinite Limits from Positive Integers

If n is a positive even integer, then

$$\lim_{x\to a} \frac{1}{(x-a)^n} = +\infty. \tag{2.3.2}$$

If n is a positive odd integer, then

$$\lim_{x\to a^+} \frac{1}{(x-a)^n} = +\infty \tag{2.3.3}$$

and

$$\lim_{x\to a^-} \frac{1}{(x-a)^n} = -\infty. \tag{2.3.4}$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having x-coordinates very near to a are very close to the vertical line $x = a$. That is, as x approaches a, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a vertical asymptote of the graph. We formally define a *vertical asymptote* as follows:

Definition: Vertical Asymptotes

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a vertical asymptote of $f(x)$.

$$\lim_{x\to a^-} f(x) = +\infty$$

$$\lim_{x\to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = +\infty$$

$$\lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a} f(x) = +\infty$$

$$\lim_{x \to a} f(x) = -\infty$$

Example 2.3.6: Finding a Vertical Asymptote

Evaluate each of the following limits using Equations 2.3.2, 2.3.3, and 2.3.4 above. Identify any vertical asymptotes of the function $f(x) = 1/(x+3)^4$.

a. $\lim_{x \to -3^-} \frac{1}{(x+3)^4}$
b. $\lim_{x \to -3^+} \frac{1}{(x+3)^4}$
c. $\lim_{x \to -3} \frac{1}{(x+3)^4}$

Solution

We can use the above equations directly.

a. $\lim_{x \to -3^-} \frac{1}{(x+3)^4} = +\infty$
b. $\lim_{x \to -3^+} \frac{1}{(x+3)^4} = +\infty$
c. $\lim_{x \to -3} \frac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

Exercise 2.3.6

Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \frac{1}{(x-2)^3}$.

a. $\lim_{x \to 2^-} \frac{1}{(x-2)^3}$
b. $\lim_{x \to 2^+} \frac{1}{(x-2)^3}$
c. $\lim_{x \to 2} \frac{1}{(x-2)^3}$

Answer a

$$\lim_{x \to 2^-} \frac{1}{(x-2)^3} = -\infty$$

Answer b

$$\lim_{x \to 2^+} \frac{1}{(x-2)^3} = +\infty$$

Answer c

$\lim_{x \to 2} \frac{1}{(x-2)^3}$ DNE. The line $x = 2$ is the vertical asymptote of $f(x) = 1/(x-2)^3$.

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example 2.3.7: Behavior of a Function at Different Points

Use the graph of $f(x)$ in Figure **2.3.10** to determine each of the following values:

a. $\lim\limits_{x\to -4^-} f(x)$; $\lim\limits_{x\to -4^+} f(x)$; $\lim\limits_{x\to -4} f(x)$; $f(-4)$
b. $\lim\limits_{x\to -2^-} f(x)$; $\lim\limits_{x\to -2^+} f(x)$; $\lim\limits_{x\to -2} f(x)$; $f(-2)$
c. $\lim\limits_{x\to 1^-} f(x)$; $\lim\limits_{x\to 1^+} f(x)$; $\lim\limits_{x\to 1} f(x)$; $f(1)$
d. $\lim\limits_{x\to 3^-} f(x)$; $\lim\limits_{x\to 3^+} f(x)$; $\lim\limits_{x\to 3} f(x)$; $f(3)$

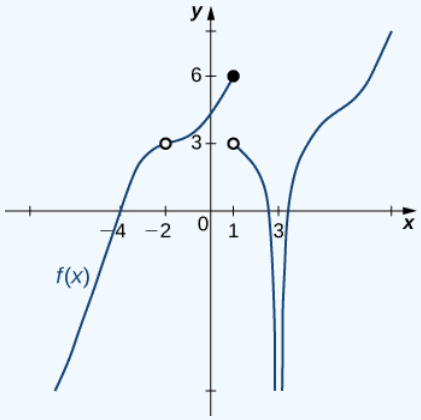


Figure **2.3.10**: The graph shows $f(x)$.

Solution

Using the definitions above and the graph for reference, we arrive at the following values:

a. $\lim\limits_{x\to -4^-} f(x) = 0$; $\lim\limits_{x\to -4^+} f(x) = 0$; $\lim\limits_{x\to -4} f(x) = 0$; $f(-4) = 0$
b. $\lim\limits_{x\to -2^-} f(x) = 3$; $\lim\limits_{x\to -2^+} f(x) = 3$; $\lim\limits_{x\to -2} f(x) = 3$; $f(-2)$ is undefined
c. $\lim\limits_{x\to 1^-} f(x) = 6$; $\lim\limits_{x\to 1^+} f(x) = 3$; $\lim\limits_{x\to 1} f(x)$ DNE; $f(1) = 6$
d. $\lim\limits_{x\to 3^-} f(x) = -\infty$; $\lim\limits_{x\to 3^+} f(x) = -\infty$; $\lim\limits_{x\to 3} f(x) = -\infty$; $f(3)$ is undefined

Exercise 2.3.7

Evaluate $\lim\limits_{x\to 1} f(x)$ for $f(x)$ shown here:

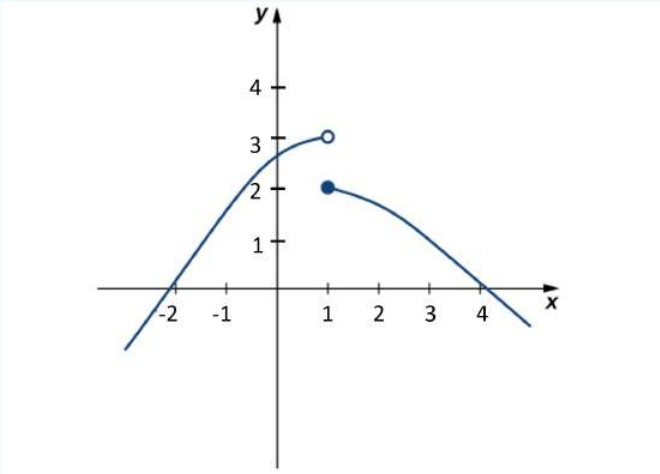


Figure **2.3.11**. The graph of a piecewise function f.

Hint

Compare the limit from the right with the limit from the left.

Answer

$\lim_{x \to 1} f(x)$ does not exist

Example 2.3.8: Einstein's Equation

In the Chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

what is the value of this bound?

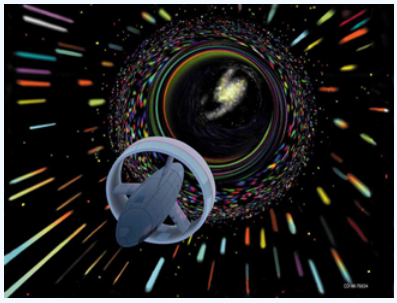


Figure 2.3.12. (Credit:NASA)

Solution

Our starting point is Einstein's equation for the mass of a moving object,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we can graph the ratio of masses m/m_0 as a function of the ratio of speeds, v/c (Figure 2.3.13).

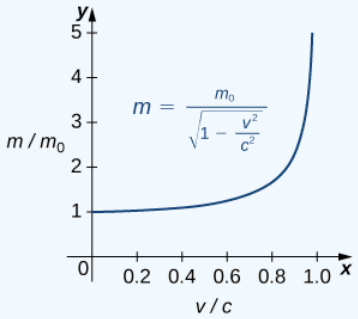


Figure 2.3.13: This graph shows the ratio of masses as a function of the ratio of speeds in Einstein's equation for the mass of a moving object.

We can see that as the ratio of speeds approaches 1—that is, as the speed of the object approaches the speed of light—the ratio of masses increases without bound. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

Table 2.3.8

v/c	$\sqrt{1 - \frac{v^2}{c^2}}$	m/m_o
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.7

Thus, according to Table **2.3.8**:, if an object with mass 100 kg is traveling at 0.9999c, its mass becomes 7071 kg. Since no object can have an infinite mass, we conclude that no object can travel at or more than the speed of light.

Key Concepts

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

Key Equations

- **Intuitive Definition of the Limit**

$$\lim_{x \to a} f(x) = L$$

- **Two Important Limits**

$$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c$$

- **One-Sided Limits**

$$\lim_{x \to a^-} f(x) = L \qquad \lim_{x \to a^+} f(x) = L$$

- **Infinite Limits from the Left**

$$\lim_{x \to a^-} f(x) = +\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

- **Infinite Limits from the Right**

$$\lim_{x \to a^+} f(x) = +\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

- **Two-Sided Infinite Limits**

$$\lim_{x \to a} f(x) = +\infty\text{: } \lim_{x \to a^-} f(x) = +\infty \text{ and } \lim_{x \to a^+} f(x) = +\infty$$

$$\lim_{x \to a} f(x) = -\infty\text{: } \lim_{x \to a^-} f(x) = -\infty \text{ and } \lim_{x \to a^+} f(x) = -\infty$$

Glossary

infinite limit

A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

intuitive definition of the limit

If all values of the function $f(x)$ approach the real number L as the values of $x (\neq a)$ approach a, $f(x)$ approaches L

one-sided limit

A one-sided limit of a function is a limit taken from either the left or the right

vertical asymptote

A function has a vertical asymptote at $x = a$ if the limit as x approaches a from the right or left is infinite

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2.3E: Exercises

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2.4: The Limit Laws

Learning Objectives

- Recognize the basic limit laws.
- Use the limit laws to evaluate the limit of a function.
- Evaluate the limit of a function by factoring.
- Use the limit laws to evaluate the limit of a polynomial or rational function.
- Evaluate the limit of a function by factoring or by using conjugates.
- Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated previously and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Theorem 2.4.1: Basic Limit Results

For any real number a and any constant c,

I. $\lim\limits_{x \to a} x = a$
II. $\lim\limits_{x \to a} c = c$

Example 2.4.1: Evaluating a Basic Limit

Evaluate each of the following limits using "Basic Limit Results."

a. $\lim\limits_{x \to 2} x$
b. $\lim\limits_{x \to 2} 5$

Solution

a. The limit of x as x approaches a is a: $\lim\limits_{x \to 2} x = 2$.
b. The limit of a constant is that constant: $\lim\limits_{x \to 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Theorem 2.4.2: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

- **Sum law for limits**:

$$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

- **Difference law for limits**:

$$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

- **Constant multiple law for limits**:

$$\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$$

- **Product law for limits**:

$$\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

- **Quotient law for limits**:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$$

for $M \neq 0$.

- **Power law for limits**:

$$\lim_{x \to a} \big(f(x)\big)^n = \big(\lim_{x \to a} f(x)\big)^n = L^n$$

for every positive integer n.

- **Root law for limits**:

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$$

for all L if n is odd and for $L \geq 0$ if n is even.

We now practice applying these limit laws to evaluate a limit.

Example 2.4.2A: Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate

$$\lim_{x \to -3}(4x + 2).$$

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \to -3}(4x + 2) &= \lim_{x \to -3} 4x + \lim_{x \to -3} 2 && \text{Apply the sum law.} \\ &= 4 \cdot \lim_{x \to -3} x + \lim_{x \to -3} 2 && \text{Apply the constant multiple law.} \\ &= 4 \cdot (-3) + 2 = -10. && \text{Apply the basic limit results and simplify.} \end{aligned}$$

Example 2.4.2B: Using Limit Laws Repeatedly

Use the limit laws to evaluate

$$\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}.$$

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\begin{aligned}
\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4} &= \frac{\lim_{x\to 2}(2x^2-3x+1)}{\lim_{x\to 2}(x^3+4)} && \text{Apply the quotient law, make sure that } (2)^3+4\neq 0.\\
&= \frac{2\cdot\lim_{x\to 2}x^2-3\cdot\lim_{x\to 2}x+\lim_{x\to 2}1}{\lim_{x\to 2}x^3+\lim_{x\to 2}4} && \text{Apply the sum law and constant multiple law.}\\
&= \frac{2\cdot\left(\lim_{x\to 2}x\right)^2-3\cdot\lim_{x\to 2}x+\lim_{x\to 2}1}{\left(\lim_{x\to 2}x\right)^3+\lim_{x\to 2}4} && \text{Apply the power law.}\\
&= \frac{2(4)-3(2)+1}{(2)^3+4}=\frac{1}{4}. && \text{Apply the basic limit laws and simplify.}
\end{aligned}$$

? Exercise 2.4.2

Use the limit laws to evaluate $\lim_{x\to 6}(2x-1)\sqrt{x+4}$. In each step, indicate the limit law applied.

Hint

Begin by applying the product law.

Answer

$11\sqrt{10}$

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x\to a}f(x)=f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Theorem 2.4.3: Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\lim_{x\to a}p(x)=p(a)$$

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}$$

when $q(a)\neq 0$.

To see that this theorem holds, consider the polynomial

$$p(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0.$$

By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned}
\lim_{x\to a}p(x) &= \lim_{x\to a}(c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0)\\
&= c_n\left(\lim_{x\to a}x\right)^n+c_{n-1}\left(\lim_{x\to a}x\right)^{n-1}+\cdots+c_1\left(\lim_{x\to a}x\right)+\lim_{x\to a}c_0\\
&= c_na^n+c_{n-1}a^{n-1}+\cdots+c_1a+c_0\\
&= p(a)
\end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a)\neq 0$,

then

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Example 2.4.3: Evaluating a Limit of a Rational Function

Evaluate the $\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4}$.

Solution

Since 3 is in the domain of the rational function $f(x) = \frac{2x^2 - 3x + 1}{5x + 4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4} = \frac{10}{19}.$$

Exercise 2.4.3

Evaluate $\lim_{x \to -2} (3x^3 - 2x + 7)$.

Hint

Use LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS as reference

Answer

−13

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x \to a} f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x \neq a$, $f(x) = g(x)$ over some open interval containing a, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

To understand this idea better, consider the limit $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

The function

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

and the function $g(x) = x + 1$ are identical for all values of $x \neq 1$. The graphs of these two functions are shown in Figure 2.4.1.

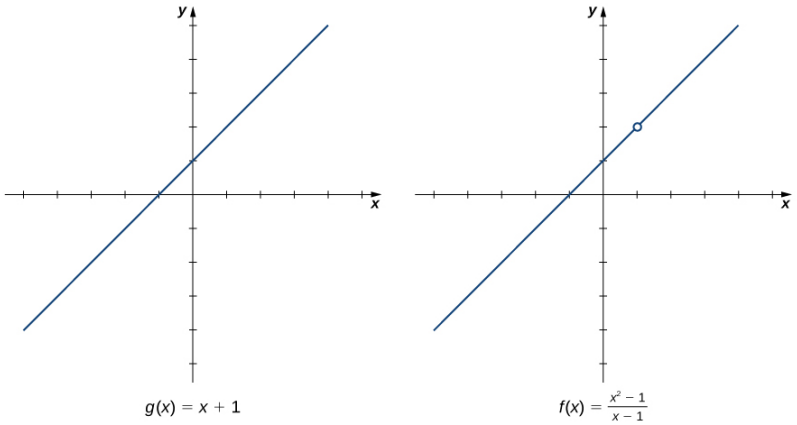


Figure 2.4.1: The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)=2.$$

The limit has the form $\lim_{x\to a} f(x)/g(x)$, where $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form $0/0$

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x)=f(x)/g(x)$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. Example 2.4.4 illustrates the factor-and-cancel technique; Example 2.4.5 shows multiplying by a conjugate. In Example 2.4.6, we look at simplifying a complex fraction.

Example 2.4.4: Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x\to 3}\dfrac{x^2-3x}{2x^2-5x-3}$.

Solution

Step 1. The function $f(x)=\dfrac{x^2-3x}{2x^2-5x-3}$ is undefined for $x=3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}=\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}$$

Step 2. For all $x \neq 3$, $\dfrac{x^2-3x}{2x^2-5x-3}=\dfrac{x}{2x+1}$. Therefore,

$$\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}=\lim_{x\to 3}\frac{x}{2x+1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x\to 3}\frac{x}{2x+1}=\frac{3}{7}.$$

Exercise 2.4.4

Evaluate $\lim_{x\to -3}\dfrac{x^2+4x+3}{x^2-9}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

$\dfrac{1}{3}$

Example 2.4.5: Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim_{x\to -1}\dfrac{\sqrt{x+2}-1}{x+1}$.

Solution

Step 1. $\dfrac{\sqrt{x+2}-1}{x+1}$ has the form $0/0$ at -1. Let's begin by multiplying by $\sqrt{x+2}+1$, the conjugate of $\sqrt{x+2}-1$, on the numerator and denominator:

$$\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}=\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}\cdot\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:

$$=\lim_{x\to -1}\frac{x+1}{(x+1)(\sqrt{x+2}+1)}.$$

Step 3. Then we cancel:

$$=\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}=\frac{1}{2}.$$

Exercise 2.4.5

Evaluate $\lim_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

$\dfrac{1}{4}$

Example 2.4.6: Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim\limits_{x\to 1}\dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$.

Solution

Step 1. $\dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x\to 1}\frac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}=\lim_{x\to 1}\frac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}\cdot\frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

$$=\lim_{x\to 1}\frac{2-(x+1)}{2(x-1)(x+1)}.$$

Step 3. Then, we simplify the numerator:

$$=\lim_{x\to 1}\frac{-x+1}{2(x-1)(x+1)}.$$

Step 4. Now we factor out -1 from the numerator:

$$=\lim_{x\to 1}\frac{-(x-1)}{2(x-1)(x+1)}.$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$=\lim_{x\to 1}\frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x\to 1}\frac{-1}{2(x+1)}=-\frac{1}{4}.$$

Exercise 2.4.6

Evaluate $\lim\limits_{x\to -3}\dfrac{\dfrac{1}{x+2}+1}{x+3}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

-1

Example 2.4.7 does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example 2.4.7: Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim\limits_{x\to 0}\left(\dfrac{1}{x}+\dfrac{5}{x(x-5)}\right)$.

Solution:

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\frac{1}{x}+\frac{5}{x(x-5)}=\frac{x-5+5}{x(x-5)}=\frac{x}{x(x-5)}.$$

Thus,

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=\lim_{x\to 0}\frac{x}{x(x-5)}=\lim_{x\to 0}\frac{1}{x-5}=-\frac{1}{5}.$$

? Exercise 2.4.7

Evaluate $\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{4}{x^2-2x-3}\right)$.

Hint

Use the same technique as Example 2.4.7. Don't forget to factor x^2-2x-3 before getting a common denominator.

Answer

$\frac{1}{4}$

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim_{x\to a^-}h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b,a); for a limit of the form $\lim_{x\to a^+}h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a,c). Example 2.4.8*A* illustrates this point.

✓ Example 2.4.8*A*: Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim_{x\to 3^-}\sqrt{x-3}$
b. $\lim_{x\to 3^+}\sqrt{x-3}$

Solution

Figure 2.4.2 illustrates the function $f(x)=\sqrt{x-3}$ and aids in our understanding of these limits.

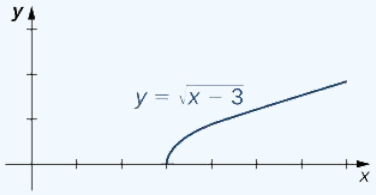


Figure 2.4.2: The graph shows the function $f(x)=\sqrt{x-3}$.

a. The function $f(x)=\sqrt{x-3}$ is defined over the interval $[3,+\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim_{x\to 3^-}\sqrt{x-3}$. In fact, since $f(x)=\sqrt{x-3}$ is undefined to the left of 3, $\lim_{x\to 3^-}\sqrt{x-3}$ does not exist.

b. Since $f(x)=\sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim_{x\to 3^+}\sqrt{x-3}$. By applying these limit laws we obtain $\lim_{x\to 3^+}\sqrt{x-3}=0$.

In Example 2.4.8*B* we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example 2.4.8B: Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x-3, & \text{if } x < 2 \\ (x-3)^2, & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim\limits_{x \to 2^-} f(x)$
b. $\lim\limits_{x \to 2^+} f(x)$
c. $\lim\limits_{x \to 2} f(x)$

Solution

Figure 2.4.3 illustrates the function $f(x)$ and aids in our understanding of these limits.

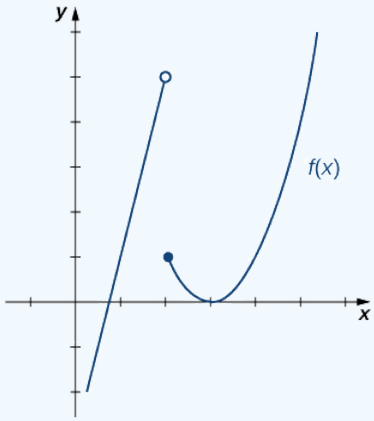


Figure 2.4.3: This graph shows a function $f(x)$.

a. Since $f(x) = 4x - 3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x - 3$ and apply the limit laws:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x - 3) = 5$$

b. Since $f(x) = (x-3)^2$for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x-3)^2$ and apply the limit laws:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (x-3)^2 = 1.$$

c. Since $\lim\limits_{x \to 2^-} f(x) = 5$ and $\lim\limits_{x \to 2^+} f(x) = 1$, we conclude that $\lim\limits_{x \to 2} f(x)$ does not exist.

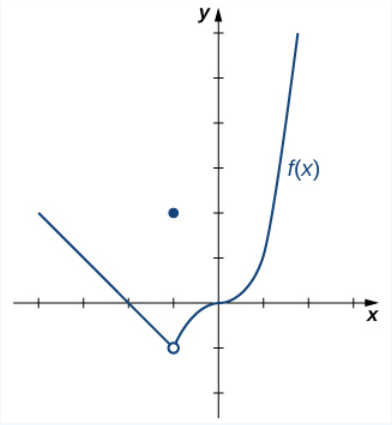
Exercise 2.4.8

Graph $f(x) = \begin{cases} -x-2, & \text{if } x < -1 \\ 2, & \text{if } x = -1 \\ x^3, & \text{if } x > -1 \end{cases}$ and evaluate $\lim\limits_{x \to -1^-} f(x)$.

Hint

Use the method in Example 2.4.8B to evaluate the limit.

Answer



$$\lim_{x \to -1^-} f(x) = -1$$

We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0, K \neq 0$ at a.

Example 2.4.9: Evaluating a Limit of the Form $K/0$, $K \neq 0$ Using the Limit Laws

Evaluate $\lim_{x \to 2^-} \frac{x-3}{x^2-2x}$.

Solution

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as x approaches 2 from the left, the numerator approaches -1; and the denominator approaches 0. Consequently, the magnitude of $\frac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = \lim_{x \to 2^-} \frac{x-3}{x(x-2)}$$

Step 2. Since $x-2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x-2)$ from the rest of the function:

$$= \lim_{x \to 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}$$

Step 3. Using the Limit Laws, we can write:

$$= \left(\lim_{x \to 2^-} \frac{x-3}{x}\right) \cdot \left(\lim_{x \to 2^-} \frac{1}{x-2}\right).$$

Step 4. $\lim_{x \to 2^-} \frac{x-3}{x} = -\frac{1}{2}$ and $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = +\infty.$$

Exercise 2.4.9

Evaluate $\lim_{x \to 1} \frac{x+2}{(x-1)^2}$.

Solution

Use the methods from Example 2.4.9.

Answer

$+\infty$

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem,** proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by "squeezing" a function, with a limit at a point a that is unknown, between two functions having a common known limit at a. Figure 2.4.4 illustrates this idea.

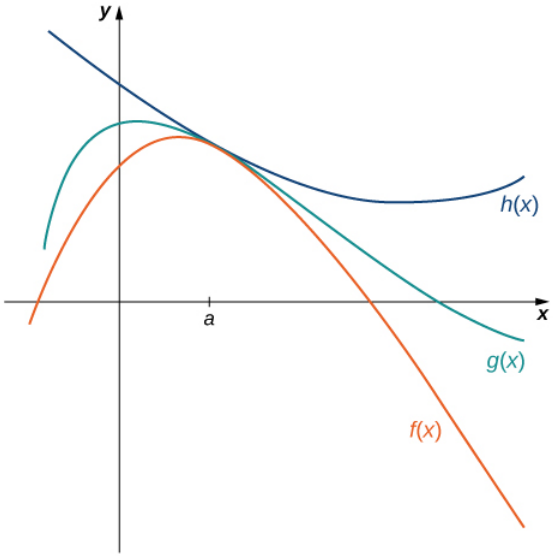


Figure 2.4.4: The Squeeze Theorem applies when $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Theorem 2.4.4: The Squeeze Theorem

Let $f(x), g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \le g(x) \le h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.

Example 2.4.10: Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim_{x \to 0} x \cos x$.

Solution

Because $-1 \le \cos x \le 1$ for all x, we have $-x \le x \cos x \le x$ for $x \ge 0$ and $-x \ge x \cos x \ge x$ for $x \le 0$ (if x is negative the direction of the inequalities changes when we multiply). Since $\lim_{x \to 0}(-x) = 0 = \lim_{x \to 0} x$, from the squeeze theorem, we obtain $\lim_{x \to 0} x \cos x = 0$. The graphs of $f(x) = -x$, $g(x) = x \cos x$, and $h(x) = x$ are shown in Figure 2.4.5.

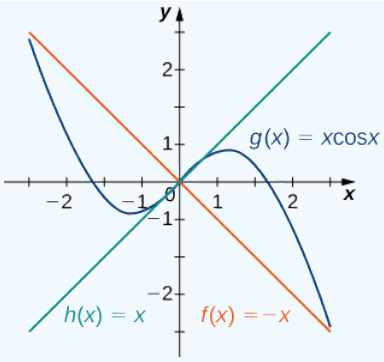


Figure 2.4.5: The graphs of $f(x)$, $g(x)$, and $h(x)$ are shown around the point $x = 0$.

? Exercise 2.4.10

Use the squeeze theorem to evaluate $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.

Hint

Use the fact that $-x^2 \leq x^2 \sin(1/x) \leq x^2$ to help you find two functions such that $x^2 \sin(1/x)$ is squeezed between them.

Answer

0

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim_{\theta \to 0} \sin \theta$. Consider the unit circle shown in Figure 2.4.6. In the figure, we see that $\sin \theta$ is the y-coordinate on the unit circle and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \frac{\pi}{2}$, we have $0 < \sin \theta < \theta$.

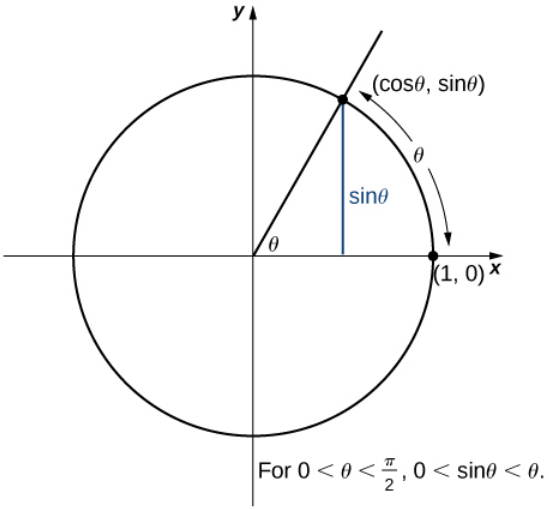


Figure 2.4.6: The sine function is shown as a line on the unit circle.

Because $\lim_{\theta \to 0^+} 0 = 0$ and $\lim_{\theta \to 0^+} \theta = 0$, by using the squeeze theorem we conclude that

$$\lim_{\theta \to 0^+} \sin \theta = 0.$$

To see that $\lim_{\theta \to 0^-} \sin \theta = 0$ as well, observe that for $-\frac{\pi}{2} < \theta < 0, 0 < -\theta < \frac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta$. Consequently, $0 < -\sin \theta < -\theta$. It follows that $0 > \sin \theta > \theta$. An application of the squeeze theorem produces the desired limit. Thus, since $\lim_{\theta \to 0^+} \sin \theta = 0$ and $\lim_{\theta \to 0^-} \sin \theta = 0$,

$$\lim_{\theta \to 0} \sin \theta = 0$$

Next, using the identity $\cos \theta = \sqrt{1 - \sin^2 \theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we see that

$$\lim_{\theta \to 0} \cos \theta = \lim_{\theta \to 0} \sqrt{1 - \sin^2 \theta} = 1.$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$. To evaluate this limit, we use the unit circle in Figure 2.4.7. Notice that this figure adds one additional triangle to Figure 2.4.6. We see that the length of the side opposite angle θ in this new triangle is $\tan \theta$. Thus, we see that for $0 < \theta < \frac{\pi}{2}$, we have $\sin \theta < \theta < \tan \theta$.

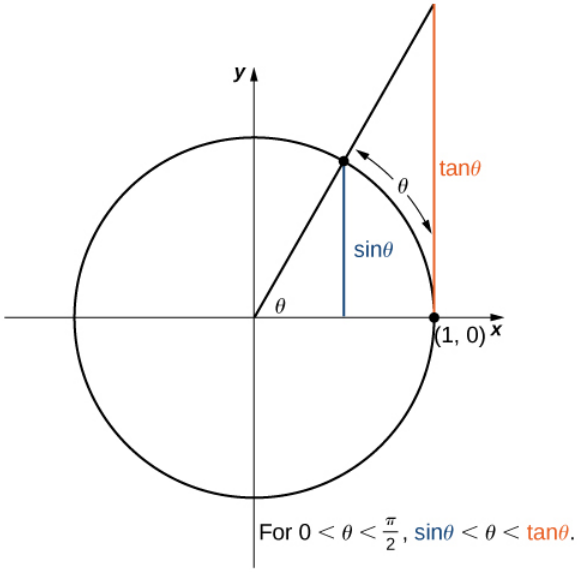


Figure 2.4.7: The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin \theta$ in all parts of the inequality, we obtain

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Equivalently, we have

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

Since $\lim_{\theta \to 0^+} 1 = 1 = \lim_{\theta \to 0^+} \cos \theta$, we conclude that $\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1$, by the squeeze theorem. By applying a manipulation similar to that used in demonstrating that $\lim_{\theta \to 0^-} \sin \theta = 0$, we can show that $\lim_{\theta \to 0^-} \frac{\sin \theta}{\theta} = 1$. Thus,

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

In Example 2.4.11, we use this limit to establish $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$. This limit also proves useful in later chapters.

Example 2.4.11: Evaluating an Important Trigonometric Limit

Evaluate $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta}$.

Solution

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

$$\begin{aligned}\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}\\ &= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}\\ &= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}\\ &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}\\ &= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\cdot\left(\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}\right)\\ &= 1\cdot\frac{0}{2}=0.\end{aligned}$$

Therefore,

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0.$$

? Exercise 2.4.11

Evaluate $\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}$.

Hint

Multiply numerator and denominator by $1+\cos\theta$.

Answer

0

Deriving the Formula for the Area of a Circle

Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician *Archimedes* (ca. 287−212; BCE) was particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit, but we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of n triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1.Express the height h and the base b of the isosceles triangle in Figure 2.4.8 in terms of θ and r.

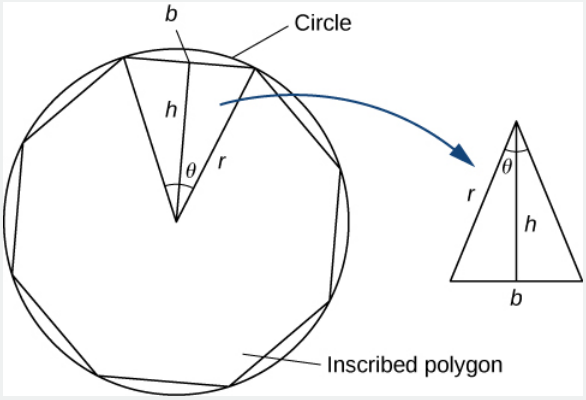


Figure 2.4.8

2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and r.

(Substitute $\frac{1}{2}\sin\theta$ for $\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ in your expression.)

3. If an n-sided regular polygon is inscribed in a circle of radius r, find a relationship between θ and n. Solve this for n. Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)

4. Find an expression for the area of the n-sided polygon in terms of r and θ.

5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero. (Hint: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$).

The technique of estimating areas of regions by using polygons is revisited in Introduction to Integration.

Key Concepts

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions,

$$\lim_{x\to a} f(x) = f(a).$$

- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.
- The squeeze theorem allows you to find the limit of a function if the function is always greater than one function and less than another function with limits that are known.

Key Equations

- **Basic Limit Results**

$$\lim_{x\to a} x = a \qquad \lim_{x\to a} c = c$$

- **Important Limits**

$$\lim_{\theta\to 0}\sin\theta = 0$$

$$\lim_{\theta\to 0}\cos\theta = 1$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$$

Glossary

constant multiple law for limits

the limit law

$$\lim_{x\to a} cf(x) = c\cdot\lim_{x\to a} f(x) = cL$$

difference law for limits

the limit law

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$$

limit laws

the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$; let c be a constant

power law for limits

the limit law

$$\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n = L^n$$

for every positive integer n

product law for limits

the limit law

$$\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

quotient law for limits

the limit law $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ for M≠0

root law for limits

the limit law $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even

squeeze theorem

states that if $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ over an open interval containing a and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ where L is a real number, then $\lim_{x \to a} g(x) = L$

sum law for limits

The limit law $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

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2.4E: Exercises

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2.5: Continuity

Learning Objectives

- Explain the three conditions for continuity at a point.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- State the theorem for limits of composite functions.
- Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity* at a point where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in Figure **2.5.1**. We see that the graph of $f(x)$ has a hole at a. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a, we need the following condition:

i. $f(a)$ is defined

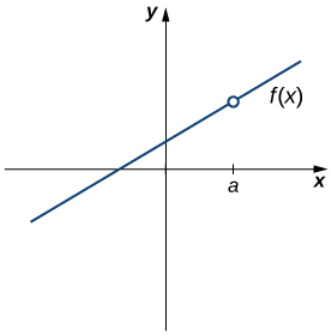


Figure **2.5.1**: The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

However, as we see in Figure **2.5.2**, this condition alone is insufficient to guarantee continuity at the point a. Although $f(a)$ is defined, the function has a gap at a. In this example, the gap exists because $\lim_{x \to a} f(x)$ does not exist. We must add another condition for continuity at a—namely,

ii. $\lim_{x \to a} f(x)$ exists

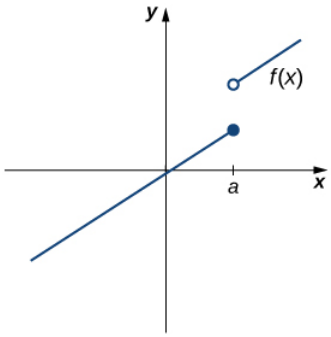


Figure **2.5.2**: The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x)$ does not exist.

However, as we see in Figure **2.5.3**, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a. We must add a third condition to our list:

iii. $\lim_{x \to a} f(x) = f(a)$

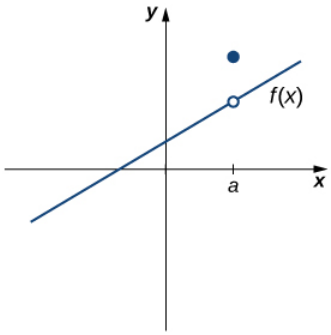


Figure **2.5.3**: The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Definition: Continuous at a Point

A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

i. $f(a)$ is defined
ii. $\lim_{x \to a} f(x)$ exists
iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** a if it fails to be continuous at a.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If $\lim_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x \to a} f(x)$. If $\lim_{x \to a} f(x) \neq f(a)$, then the function is not continuous at a. If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at a.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

Example 2.5.1A: Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(2)$. We can see that $f(2) = 0/0$, which is undefined. Therefore, $f(x) = \dfrac{x^2 - 4}{x - 2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in Figure 2.5.4.

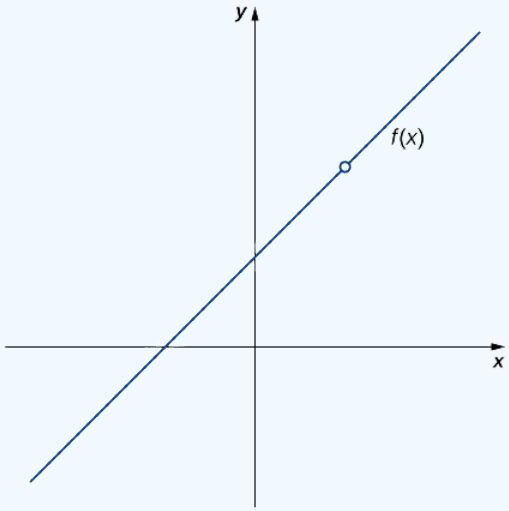


Figure 2.5.4: The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

Example 2.5.1B: Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4, & \text{if } x \leq 3 \\ 4x - 8, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(3)$.

$$f(3) = -(3^2) + 4 = -5.$$

Thus, $f(3)$ is defined. Next, we calculate $\lim\limits_{x \to 3} f(x)$. To do this, we must compute $\lim\limits_{x \to 3^-} f(x)$ and $\lim\limits_{x \to 3^+} f(x)$:

$$\lim_{x \to 3^-} f(x) = -(3^2) + 4 = -5$$

and

$$\lim_{x \to 3^+} f(x) = 4(3) - 8 = 4.$$

Therefore, $\lim\limits_{x \to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in Figure 2.5.5.

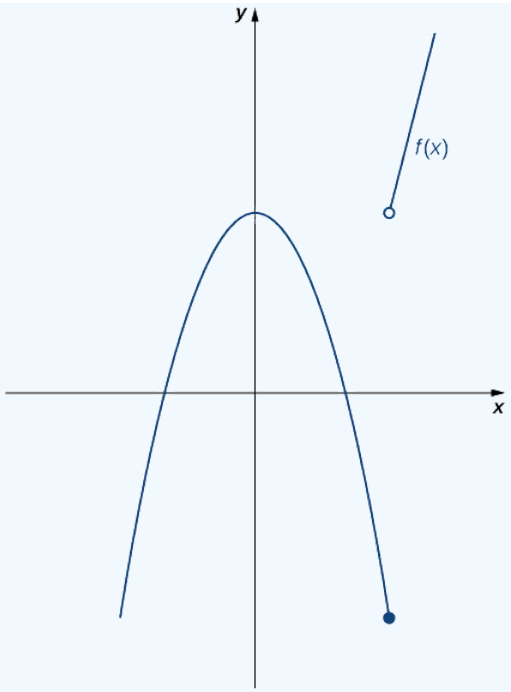


Figure 2.5.5: The function $f(x)$ is not continuous at 3 because $\lim_{x \to 3} f(x)$ does not exist.

Example 2.5.1C: Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution

First, observe that

$$f(0) = 1$$

Next,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Last, compare $f(0)$ and $\lim_{x \to 0} f(x)$. We see that

$$f(0) = 1 = \lim_{x \to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

Exercise 2.5.1

Using the definition, determine whether the function $f(x) = \begin{cases} 2x + 1, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ -x + 4, & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

Hint

Check each condition of the definition.

Answer

f is not continuous at 1 because $f(1) = 2 \neq 3 = \lim_{x \to 1} f(x)$.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Theorem 2.5.1: Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x \to a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply Theorem 2.5.1 to determine the points at which a given rational function is continuous.

Example 2.5.2: Continuity of a Rational Function

For what values of x is $f(x) = \frac{x+1}{x-5}$ continuous?

Solution

The rational function $f(x) = \frac{x+1}{x-5}$ is continuous for every value of x except $x = 5$.

Exercise 2.5.2

For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Hint

Use the Continuity of Polynomials and Rational Functions stated above.

Answer

$f(x)$ is continuous at every real number.

Types of Discontinuities

As we have seen in Example 2.5.1*A* and Example 2.5.1*B*, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. Figure 2.5.6 illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

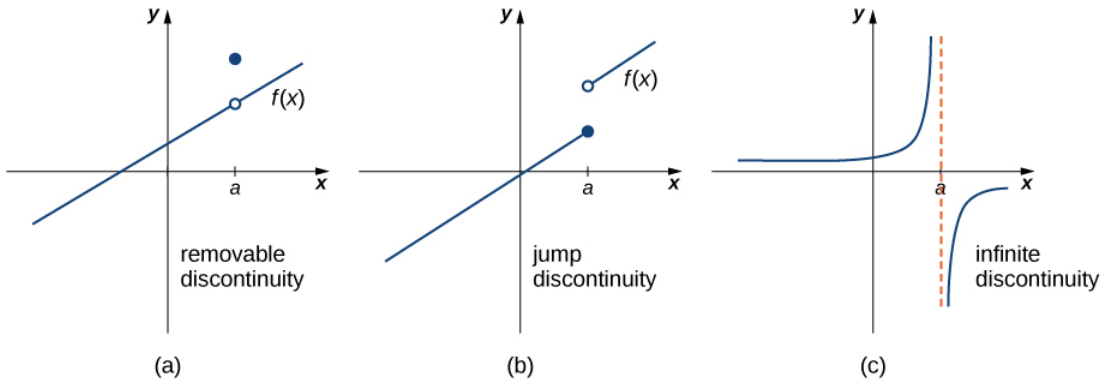


Figure 2.5.6: Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x\to a} f(x)$ exists. (Note: When we state that $\lim_{x\to a} f(x)$ exists, we mean that $\lim_{x\to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. (Note: When we state that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

Example 2.5.3: Classifying a Discontinuity

In Example 2.5.1A, we showed that $f(x) = \dfrac{x^2-4}{x-2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution

To classify the discontinuity at 2 we must evaluate $\lim_{x\to 2} f(x)$:

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2} (x+2) \\ &= 4.\end{aligned}$$

Since f is discontinuous at 2 and $\lim_{x\to 2} f(x)$ exists, f has a removable discontinuity at $x = 2$.

Example 2.5.4: Classifying a Discontinuity

In Example 2.5.1B, we showed that $f(x) = \begin{cases} -x^2+4, & \text{if } x \leq 3 \\ 4x-8, & \text{if } x > 3 \end{cases}$ is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

Solution

Earlier, we showed that f is discontinuous at 3 because $\lim_{x\to 3} f(x)$ does not exist. However, since $\lim_{x\to 3^-} f(x) = -5$ and $\lim_{x\to 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example 2.5.5: Classifying a Discontinuity

Determine whether $f(x) = \dfrac{x+2}{x+1}$ is continuous at -1. If the function is discontinuous at -1, classify the discontinuity as removable, jump, or infinite.

Solution

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at -1. To determine the type of discontinuity, we must determine the limit at -1. We see that $\lim\limits_{x \to -1^-} \dfrac{x+2}{x+1} = -\infty$ and $\lim\limits_{x \to -1^+} \dfrac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at -1.

Exercise 2.5.3

For $f(x) = \begin{cases} x^2, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases}$, decide whether f is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Hint

Consider the definitions of the various kinds of discontinuity stated above. If the function is discontinuous at 1, look at $\lim\limits_{x \to 1} f(x)$

Answer

Discontinuous at 1; removable

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Definition: Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if $\lim\limits_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at a if $\lim\limits_{x \to a^-} f(x) = f(a)$

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b. Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim\limits_{x \to a^+} f(x) = f(a)$ and $\lim\limits_{x \to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim\limits_{x \to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

Example 2.5.6: Continuity on an Interval

State the interval(s) over which the function $f(x) = \dfrac{x-1}{x^2+2x}$ is continuous.

Solution

Since $f(x) = \dfrac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$. Thus, $f(x)$ is continuous over each of the intervals $(-\infty, -2), (-2, 0)$, and $(0, +\infty)$.

Example 2.5.7: Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4-x^2}$ is continuous.

Solution

From the limit laws, we know that $\lim\limits_{x \to a} \sqrt{4-x^2} = \sqrt{4-a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim\limits_{x \to -2^+} \sqrt{4-x^2} = 0$ exists and $\lim\limits_{x \to 2^-} \sqrt{4-x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

Exercise 2.5.4

State the interval(s) over which the function $f(x) = \sqrt{x+3}$ is continuous.

Hint

Use Example 2.5.7 as a guide.

Answer

$[-3, +\infty)$

Theorem 2.5.2 allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Theorem 2.5.2: Composite Function Theorem

If $f(x)$ is continuous at L and $\lim\limits_{x \to a} g(x) = L$, then

$$\lim_{x \to a} f\big(g(x)\big) = f\Big(\lim_{x \to a} g(x)\Big) = f(L). \tag{2.5.1}$$

Before we move on to Example 2.5.8, recall that earlier, in the section on limit laws, we showed $\lim\limits_{x \to 0} \cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In Example 2.5.8, we see how to combine this result with the composite function theorem.

Example 2.5.8: Limit of a Composite Cosine Function

Evaluate $\lim\limits_{x \to \pi/2} \cos\left(x - \dfrac{\pi}{2}\right)$.

Solution

The given function is a composite of $\cos x$ and $x - \frac{\pi}{2}$. Since $\lim\limits_{x \to \pi/2} \left(x - \dfrac{\pi}{2}\right) = 0$ and $\cos x$ is continuous at 0, we may apply the composite function theorem. Thus,

$$\lim_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\lim_{x \to \pi/2} \left(x - \frac{\pi}{2}\right)\right) = \cos(0) = 1.$$

Exercise 2.5.4:

Evaluate $\lim\limits_{x \to \pi} \sin(x - \pi)$.

Hint

$f(x) = \sin x$ is continuous at 0. Use Example 2.5.8 as a guide.

Answer

0

The proof of the next theorem uses the composite function theorem as well as the continuity of $f(x) = \sin x$ and $g(x) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Theorem 2.5.3: Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim_{x \to a} \cos x = \cos a$ for all values of a.

$$\begin{aligned}
\lim_{x \to a} \cos x &= \lim_{x \to a} \cos((x - a) + a) && \text{Rewrite } x = x - a + a. \\
&= \lim_{x \to a} (\cos(x - a) \cos a - \sin(x - a) \sin a) && \text{Apply the identity for the cosine of the sum of two angles.} \\
&= \cos(\lim_{x \to a}(x - a)) \cos a - \sin(\lim_{x \to a}(x - a)) \sin a && \text{Since } \lim_{x \to a}(x - a) = 0, \text{ and } \sin x \text{ and } \cos x \text{ are continuous at } 0. \\
&= \cos(0) \cos a - \sin(0) \sin a && \text{Evaluate } \cos(0) \text{ and } \sin(0) \text{ and simplify.} \\
&= 1 \cdot \cos a - 0 \cdot \sin a = \cos a.
\end{aligned}$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

□

As you can see, the composite function theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem.**

The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$ in Figure 2.5.7.

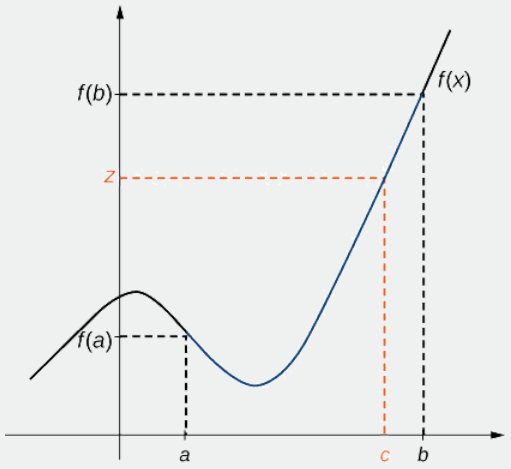


Figure 2.5.7: There is a number $c \in [a, b]$ that satisfies $f(c) = z$.

Example 2.5.9: Application of the Intermediate Value Theorem

Show that $f(x) = x - \cos x$ has at least one zero.

Solution

Since $f(x) = x - \cos x$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a, b) that satisfies $f(c) = 0$. Note that

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

$$f(\tfrac{\pi}{2}) = \tfrac{\pi}{2} - \cos\tfrac{\pi}{2} = \tfrac{\pi}{2} > 0.$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0, \pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

Example 2.5.10: When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$ and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

Solution

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x) = (x - 1)^2$. It satisfies $f(0) = 1 > 0, f(2) = 1 > 0$, and $f(1) = 0$.

Example 2.5.11: When Can You Apply the Intermediate Value Theorem?

For $f(x) = 1/x, f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

Solution

No. The function is not continuous over $[-1, 1]$. The Intermediate Value Theorem does not apply here.

Exercise 2.5.5

Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Hint

Find $f(0)$ and $f(1)$. Apply the Intermediate Value Theorem.

Answer

$f(0) = 1 > 0,\ f(1) = -2 < 0;\ f(x)$ is continuous over $[0, 1]$. It must have a zero on this interval.

Key Concepts

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The composite function theorem states: If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} f\big(g(x)\big) = f\big(\lim_{x \to a} g(x)\big) = f(L)$.

- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

Glossary

continuity at a point

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim_{x \to a} f(x)$ exists, and (3) $\lim x \to a f(x) = f(a)$

continuity from the left

A function is continuous from the left at b if $\lim_{x \to b^-} f(x) = f(b)$

continuity from the right

A function is continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$

continuity over an interval

a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b), and it is continuous from the right at a and from the left at b

discontinuity at a point

A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

infinite discontinuity

An infinite discontinuity occurs at a point a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

Intermediate Value Theorem

Let f be continuous over a closed bounded interval $[a, b]$ if z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

jump discontinuity

A jump discontinuity occurs at a point a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

removable discontinuity

A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a, but $\lim_{x \to a} f(x)$ exists

---

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2.5E: Exercises

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2.6: The Precise Definition of a Limit

Learning Objectives

- Describe the epsilon-delta definition of a limit.
- Apply the epsilon-delta definition to find the limit of a function.
- Describe the epsilon-delta definitions of one-sided limits and infinite limits.
- Use the epsilon-delta definition to prove the limit laws.

By now you have progressed from the very informal definition of a limit in the introduction of this chapter to the intuitive understanding of a limit. At this point, you should have a very strong intuitive sense of what the limit of a function means and how you can find it. In this section, we convert this intuitive idea of a limit into a formal definition using precise mathematical language. The formal definition of a limit is quite possibly one of the most challenging definitions you will encounter early in your study of calculus; however, it is well worth any effort you make to reconcile it with your intuitive notion of a limit. Understanding this definition is the key that opens the door to a better understanding of calculus.

Quantifying Closeness

Before stating the formal definition of a limit, we must introduce a few preliminary ideas. Recall that the distance between two points a and b on a number line is given by $|a-b|$.

- The statement $|f(x)-L|<\varepsilon$ may be interpreted as: The distance between $f(x)$ and L is less than ε.
- The statement $0<|x-a|<\delta$ may be interpreted as: $x\neq a$ and the distance between x and a is less than δ.

It is also important to look at the following equivalences for absolute value:

- The statement $|f(x)-L|<\varepsilon$ is equivalent to the statement $L-\varepsilon<f(x)<L+\varepsilon$.
- The statement $0<|x-a|<\delta$ is equivalent to the statement $a-\delta<x<a+\delta$ and $x\neq a$.

With these clarifications, we can state the formal **epsilon-delta definition of the limit**.

Definition: Finite Limits (Formal)

Let $f(x)$ be defined for all $x\neq a$ over an open interval containing a. Let L be a real number. Then

$$\lim_{x\to a} f(x)=L$$

if, for every $\varepsilon>0$, there exists a $\delta>0$, such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$.

This definition may seem rather complex from a mathematical point of view, but it becomes easier to understand if we break it down phrase by phrase. The statement itself involves something called a **universal quantifier** (for every $\varepsilon>0$), an **existential quantifier** (there exists a $\delta>0$), and, last, a **conditional statement** (if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$). Let's take a look at Table **2.6.1**, which breaks down the definition and translates each part.

Table **2.6.1**

Definition	Translation
1. For every $\varepsilon>0$,	1. For every positive distance ε from L,
2. there exists a $\delta>0$,	2. There is a positive distance δ from a,
3. such that	3. such that
4. if $0<\|x-a\|<\delta$, then $\|f(x)-L\|<\varepsilon$.	4. if x is closer than δ to a and $x\neq a$, then $f(x)$ is closer than ε to L.

We can get a better handle on this definition by looking at the definition geometrically. Figure **2.6.1** shows possible values of δ for various choices of $\varepsilon>0$ for a given function $f(x)$, a number a, and a limit L at a. Notice that as we choose smaller values of ε (the

distance between the function and the limit), we can always find a δ small enough so that if we have chosen an x value within δ of a, then the value of $f(x)$ is within ε of the limit L.

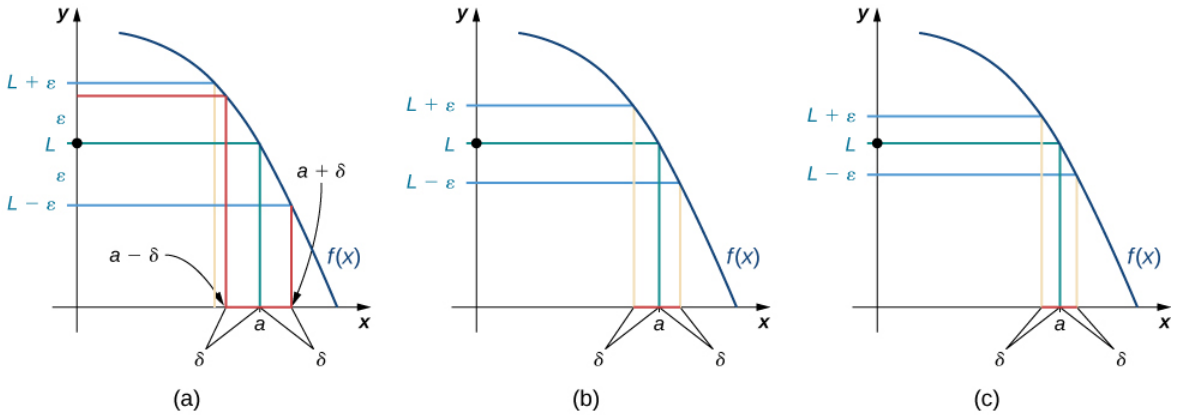


Figure **2.6.1**: These graphs show possible values of δ, given successively smaller choices of ε.

Note

Visit the following applet to experiment with finding values of δ for selected values of ε:

- The epsilon-delta definition of limit

Example **2.6.1** shows how you can use this definition to prove a statement about the limit of a specific function at a specified value.

Example 2.6.1: Proving a Statement about the Limit of a Specific Function

Prove that $\lim\limits_{x \to 1} (2x + 1) = 3$.

Solution

Let $\varepsilon > 0$.

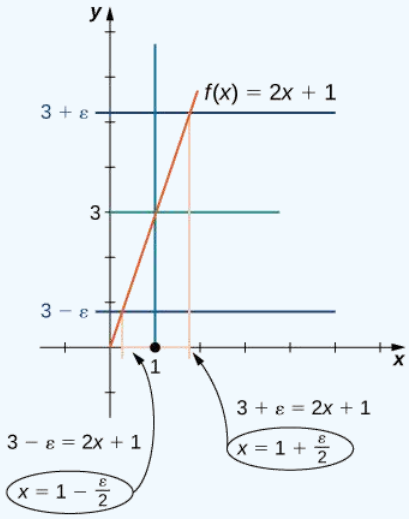
The first part of the definition begins "For every $\varepsilon > 0$."This means we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon > 0$," we signal our intent to do so.

Choose $\delta = \frac{\varepsilon}{2}$.

The definition continues with "there exists a $\delta > 0$. " The phrase "there exists" in a mathematical statement is always a signal for a scavenger hunt. In other words, we must go and find δ. So, where exactly did $\delta = \varepsilon/2$ come from? There are two basic approaches to tracking down δ. One method is purely algebraic and the other is geometric.

We begin by tackling the problem from an algebraic point of view. Since ultimately we want $|(2x + 1) - 3| < \varepsilon$, we begin by manipulating this expression: $|(2x + 1) - 3| < \varepsilon$ is equivalent to $|2x - 2| < \varepsilon$, which in turn is equivalent to $|2||x - 1| < \varepsilon$. Last, this is equivalent to $|x - 1| < \varepsilon/2$. Thus, it would seem that $\delta = \varepsilon/2$ is appropriate.

We may also find δ through geometric methods. Figure **2.6.2** demonstrates how this is done.



δ is the length of the smaller of the two distances marked in brown.

$$\delta = \min\{1 + \tfrac{\varepsilon}{2} - 1,\ 1 - (1 - \tfrac{\varepsilon}{2})\}$$
$$= \min\{\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\}$$
$$= \tfrac{\varepsilon}{2}$$

Figure **2.6.2**: This graph shows how we find δ geometrically.

Assume $0 < |x-1| < \delta$. When δ has been chosen, our goal is to show that if $0 < |x-1| < \delta$, then $|(2x+1)-3| < \varepsilon$. To prove any statement of the form "If this, then that," we begin by assuming "this" and trying to get "that."

Thus,

$|(2x+1)-3| = |2x-2|$ **property of absolute value**

$= |2(x-1)|$

$= |2||x-1|\ |2| = 2$

$= 2|x-1|$

$< 2 \cdot \delta$ **here's where we use the assumption that** $0 < |x-1| < \delta$

$= 2 \cdot \frac{\varepsilon}{2} = \varepsilon$ **here's where we use our choice of** $\delta = \varepsilon/2$

Analysis

In this part of the proof, we started with $|(2x+1)-3|$ and used our assumption $0 < |x-1| < \delta$ in a key part of the chain of inequalities to get $|(2x+1)-3|$ to be less than ε. We could just as easily have manipulated the assumed inequality $0 < |x-1| < \delta$ to arrive at $|(2x+1)-3| < \varepsilon$ as follows:

$0 < |x-1| < \delta \Rightarrow |x-1| < \delta$

$\Rightarrow -\delta < x-1 < \delta$

$\Rightarrow -\frac{\varepsilon}{2} < x-1 < \frac{\varepsilon}{2}$

$\Rightarrow -\varepsilon < 2x-2 < \varepsilon$

$\Rightarrow |2x-2| < \varepsilon$

$\Rightarrow |(2x+1)-3| < \varepsilon.$

Therefore, $\lim\limits_{x \to 1} (2x+1) = 3$. (Having completed the proof, we state what we have accomplished.)

After removing all the remarks, here is a final version of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon/2$.

Assume $0 < |x-1| < \delta$.

Thus,

$$
\begin{aligned}
|(2x+1)-3| &= |2x-2| \\
&= |2(x-1)| \\
&= |2||x-1| \\
&= 2|x-1| \\
&< 2 \cdot \delta \\
&= 2 \cdot \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Therefore, $\lim\limits_{x \to 1} (2x+1) = 3$.

The following Problem-Solving Strategy summarizes the type of proof we worked out in Example **2.6.1**.

Problem-Solving Strategy: Proving That $\lim\limits_{x \to a} f(x) = L$ for a Specific Function $f(x)$

1. Let's begin the proof with the following statement: Let $\varepsilon > 0$.
2. Next, we need to obtain a value for δ. After we have obtained this value, we make the following statement, filling in the blank with our choice of δ: Choose $\delta =$_____.
3. The next statement in the proof should be (at this point, we fill in our given value for a): Assume $0 < |x-a| < \delta$.
4. Next, based on this assumption, we need to show that $|f(x) - L| < \varepsilon$, where $f(x)$ and L are our function $f(x)$ and our limit L. At some point, we need to use $0 < |x-a| < \delta$.
5. We conclude our proof with the statement: Therefore, $\lim\limits_{x \to a} f(x) = L$.

Example 2.6.2: Proving a Statement about a Limit

Complete the proof that $\lim\limits_{x \to -1} (4x+1) = -3$ by filling in the blanks.

Let _____.

Choose $\delta =$_____.

Assume $0 < |x-$_____$| < \delta$.

Thus, |_____−_____|=_____ε.

Solution

We begin by filling in the blanks where the choices are specified by the definition. Thus, we have

Let $\varepsilon > 0$.

Choose δ=_____.

Assume $0 < |x-(-1)| < \delta$. (or equivalently, $0 < |x+1| < \delta$.)

Thus, $|(4x+1)-(-3)| = |4x+4| = |4||x+1| < 4\delta$_____$\varepsilon$.

Focusing on the final line of the proof, we see that we should choose $\delta = \frac{\varepsilon}{4}$.

We now complete the final write-up of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \frac{\varepsilon}{4}$.

Assume $0 < |x-(-1)| < \delta$ (or equivalently, $0 < |x+1| < \delta$.)

Thus, $|(4x+1)-(-3)| = |4x+4| = |4||x+1| < 4\delta = 4(\varepsilon/4) = \varepsilon$.

Exercise 2.6.1

Complete the proof that $\lim_{x\to 2}(3x-2)=4$ by filling in the blanks.

Let _____.

Choose δ =_____.

Assume $0<|x-$____$|<$____.

Thus,

|_____−____|=_____ε.

Therefore, $\lim_{x\to 2}(3x-2)=4$.

Hint

Follow the outline in the Problem-Solving Strategy that we worked out in full in Example **2.6.2**.

Answer

Let $\varepsilon>0$; choose $\delta=\frac{\varepsilon}{3}$; assume $0<|x-2|<\delta$.

Thus, $|(3x-2)-4|=|3x-6|=|3|\cdot|x-2|<3\cdot\delta=3\cdot(\varepsilon/3)=\varepsilon$.

Therefore, $\lim_{x\to 2}(3x-2)=4$.

In Examples **2.6.1** and **2.6.2**, the proofs were fairly straightforward, since the functions with which we were working were linear. In Example **2.6.3**, we see how to modify the proof to accommodate a nonlinear function.

Example 2.6.3: Proving a Statement about the Limit of a Specific Function (Geometric Approach)

Prove that $\lim_{x\to 2}x^2=4$.

Solution

1. Let $\varepsilon>0$. The first part of the definition begins "For every $\varepsilon>0$," so we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon>0$," we signal our intent to do so.

2. Without loss of generality, assume $\varepsilon\le 4$. Two questions present themselves: Why do we want $\varepsilon\le 4$ and why is it okay to make this assumption? In answer to the first question: Later on, in the process of solving for δ, we will discover that δ involves the quantity $\sqrt{4-\varepsilon}$. Consequently, we need $\varepsilon\le 4$. In answer to the second question: If we can find $\delta>0$ that "works" for $\varepsilon\le 4$, then it will "work" for any $\varepsilon>4$ as well. Keep in mind that, although it is always okay to put an upper bound on ε, it is never okay to put a lower bound (other than zero) on ε.

3. Choose $\delta=\min\{2-\sqrt{4-\varepsilon},\sqrt{4+\varepsilon}-2\}$. Figure **2.6.3** shows how we made this choice of δ.

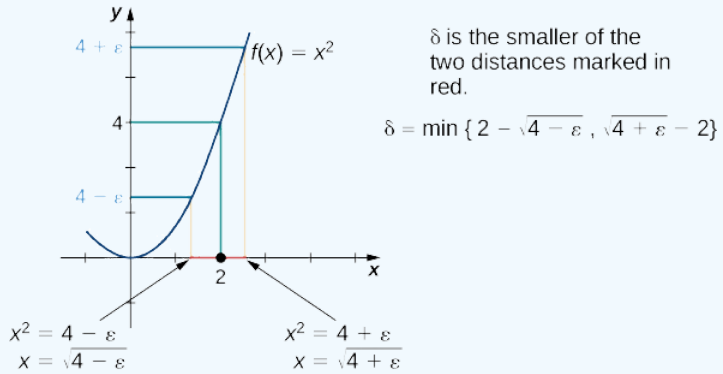


Figure **2.6.3**: This graph shows how we find δ geometrically for a given ε for the proof in Example **2.6.3**.

4. We must show: If $0<|x-2|<\delta$, then $|x^2-4|<\varepsilon$, so we must begin by assuming

$$0 < |x-2| < \delta.$$

We don't really need $0 < |x-2|$ (in other words, $x \neq 2$) for this proof. Since $0 < |x-2| < \delta \Rightarrow |x-2| < \delta$, it is okay to drop $0 < |x-2|$.

$$|x-2| < \delta.$$

Hence,

$$-\delta < x-2 < \delta.$$

Recall that $\delta = \min\{2-\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2\}$. Thus, $\delta \geq 2-\sqrt{4-\varepsilon}$ and consequently $-(2-\sqrt{4-\varepsilon}) \leq -\delta$. We also use $\delta \leq \sqrt{4+\varepsilon}-2$ here. We might ask at this point: Why did we substitute $2-\sqrt{4-\varepsilon}$ for δ on the left-hand side of the inequality and $\sqrt{4+\varepsilon}-2$ on the right-hand side of the inequality? If we look at Figure 2.6.3, we see that $2-\sqrt{4-\varepsilon}$ corresponds to the distance on the left of 2 on the x-axis and $\sqrt{4+\varepsilon}-2$ corresponds to the distance on the right. Thus,

$$-(2-\sqrt{4-\varepsilon}) \leq -\delta < x-2 < \delta \leq \sqrt{4+\varepsilon}-2.$$

We simplify the expression on the left:

$$-2+\sqrt{4-\varepsilon} < x-2 < \sqrt{4+\varepsilon}-2.$$

Then, we add 2 to all parts of the inequality:

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}.$$

We square all parts of the inequality. It is okay to do so, since all parts of the inequality are positive:

$$4-\varepsilon < x^2 < 4+\varepsilon.$$

We subtract 4 from all parts of the inequality:

$$-\varepsilon < x^2-4 < \varepsilon.$$

Last,

$$|x^2-4| < \varepsilon.$$

5. Therefore,

$$\lim_{x\to 2} x^2 = 4.$$

? Exercise 2.6.2

Find δ corresponding to $\varepsilon > 0$ for a proof that $\lim_{x\to 9} \sqrt{x} = 3$.

Hint

Draw a graph similar to the one in Example 2.6.3.

Answer

Choose $\delta = \min\{9-(3-\varepsilon)^2,\ (3+\varepsilon)^2-9\}$.

The geometric approach to proving that the limit of a function takes on a specific value works quite well for some functions. Also, the insight into the formal definition of the limit that this method provides is invaluable. However, we may also approach limit proofs from a purely algebraic point of view. In many cases, an algebraic approach may not only provide us with additional insight into the definition, it may prove to be simpler as well. Furthermore, an algebraic approach is the primary tool used in proofs of statements about limits. For Example 2.6.4, we take on a purely algebraic approach.

Example 2.6.4: Proving a Statement about the Limit of a Specific Function (Algebraic Approach)

Prove that $\lim_{x \to -1} (x^2 - 2x + 3) = 6$.

Solution

Let's use our outline from the Problem-Solving Strategy:

1. Let $\varepsilon > 0$.

2. Choose $\delta = \min\{1, \varepsilon/5\}$. This choice of δ may appear odd at first glance, but it was obtained by taking a look at our ultimate desired inequality: $|\,(x^2 - 2x + 3) - 6\,| < \varepsilon$. This inequality is equivalent to $|x + 1| \cdot |x - 3| < \varepsilon$. At this point, the temptation simply to choose $\delta = \frac{\varepsilon}{x-3}$ is very strong. Unfortunately, our choice of δ must depend on ε only and no other variable. If we can replace $|x - 3|$ by a numerical value, our problem can be resolved. This is the place where assuming $\delta \leq 1$ comes into play. The choice of $\delta \leq 1$ here is arbitrary. We could have just as easily used any other positive number. In some proofs, greater care in this choice may be necessary. Now, since $\delta \leq 1$ and $|x + 1| < \delta \leq 1$, we are able to show that $|x - 3| < 5$. Consequently, $|x + 1| \cdot |x - 3| < |x + 1| \cdot 5$. At this point we realize that we also need $\delta \leq \varepsilon/5$. Thus, we choose $\delta = \min\{1, \varepsilon/5\}$.

3. Assume $0 < |x + 1| < \delta$. Thus,

$$|x + 1| < 1 \text{ and } |x + 1| < \frac{\varepsilon}{5}.$$

Since $|x + 1| < 1$, we may conclude that $-1 < x + 1 < 1$. Thus, by subtracting 4 from all parts of the inequality, we obtain $-5 < x - 3 < -1$. Consequently, $|x - 3| < 5$. This gives us

$$|(x^2 - 2x + 3) - 6| = |x + 1| \cdot |x - 3| < \frac{\varepsilon}{5} \cdot 5 = \varepsilon.$$

Therefore,

$$\lim_{x \to -1} (x^2 - 2x + 3) = 6.$$

Exercise 2.6.3

Complete the proof that $\lim_{x \to 1} x^2 = 1$.

Let $\varepsilon > 0$; choose $\delta = \min\{1, \varepsilon/3\}$; assume $0 < |x - 1| < \delta$.

Since $|x - 1| < 1$, we may conclude that $-1 < x - 1 < 1$. Thus, $1 < x + 1 < 3$. Hence, $|x + 1| < 3$.

Hint

Use Example 2.6.4 as a guide.

Answer

$|\,x^2 - 1\,| = |x - 1| \cdot |x + 1| < \varepsilon/3 \cdot 3 = \varepsilon$

You will find that, in general, the more complex a function, the more likely it is that the algebraic approach is the easiest to apply. The algebraic approach is also more useful in proving statements about limits.

Proving Limit Laws

We now demonstrate how to use the epsilon-delta definition of a limit to construct a rigorous proof of one of the limit laws. The **triangle inequality** is used at a key point of the proof, so we first review this key property of absolute value.

Definition: The Triangle Inequality

The **triangle inequality** states that if a and b are any real numbers, then $|a + b| \leq |a| + |b|$.

Proof

We prove the following limit law: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} (f(x) + g(x)) = L + M$.

Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ so that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \varepsilon/2$.

Choose $\delta_2 > 0$ so that if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \varepsilon/2$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.

Assume $0 < |x - a| < \delta$.

Thus,

$$0 < |x - a| < \delta_1 \text{ and } 0 < |x - a| < \delta_2.$$

Hence,

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

□

We now explore what it means for a limit not to exist. The limit $\lim_{x \to a} f(x)$ does not exist if there is no real number L for which $\lim_{x \to a} f(x) = L$. Thus, for all real numbers L, $\lim_{x \to a} f(x) \neq L$. To understand what this means, we look at each part of the definition of $\lim_{x \to a} f(x) = L$ together with its opposite. A translation of the definition is given in Table 2.6.2.

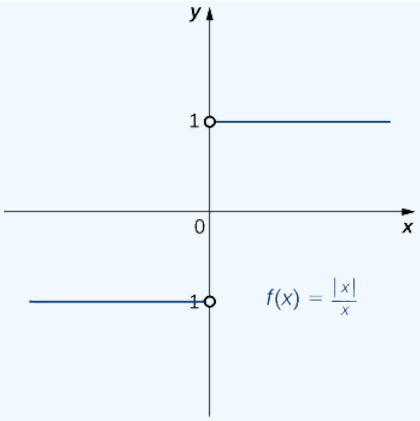
Table 2.6.2. Translation of the Definition of $\lim_{x \to a} f(x) = L$ and its Opposite

Definition	Opposite
1. For every $\varepsilon > 0$,	1. There exists $\varepsilon > 0$ so that
2. there exists a $\delta > 0$, so that	2. for every $\delta > 0$,
3. if $0 < \lvert x - a \rvert < \delta$, then $\lvert f(x) - L \rvert < \varepsilon$.	3. There is an x satisfying $0 < \lvert x - a \rvert < \delta$ so that $\lvert f(x) - L \rvert \geq \varepsilon$.

Finally, we may state what it means for a limit not to exist. The limit $\lim_{x \to a} f(x)$ does not exist if for every real number L, there exists a real number $\varepsilon > 0$ so that for all $\delta > 0$, there is an x satisfying $0 < |x - a| < \delta$, so that $|f(x) - L| \geq \varepsilon$. Let's apply this in Example 2.6.5 to show that a limit does not exist.

Example 2.6.5: Showing That a Limit Does Not Exist

Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. The graph of $f(x) = |x|/x$ is shown here:



Solution

Suppose that L is a candidate for a limit. Choose $\varepsilon = 1/2$.

Let $\delta > 0$. Either $L \geq 0$ or $L < 0$. If $L \geq 0$, then let $x = -\delta/2$.

Thus,

$$|x - 0| = |-\tfrac{\delta}{2} - 0| = \tfrac{\delta}{2} < \delta$$

and

$$\left|\frac{|-\frac{\delta}{2}|}{-\frac{\delta}{2}} - L\right| = |-1 - L| = L + 1 \geq 1 > \tfrac{1}{2} = \varepsilon.$$

On the other hand, if $L < 0$, then let $x = \delta/2$. Thus,

$$|x - 0| = |\tfrac{\delta}{2} - 0| = \tfrac{\delta}{2} < \delta$$

and

$$\left|\frac{|\frac{\delta}{2}|}{\frac{\delta}{2}} - L\right| = |1 - L| = |L| + 1 \geq 1 > \tfrac{1}{2} = \varepsilon.$$

Thus, for any value of L, $\lim\limits_{x \to 0} \dfrac{|x|}{x} \neq L$.

One-Sided Limits

Just as we first gained an intuitive understanding of limits and then moved on to a more rigorous definition of a limit, we now revisit one-sided limits. To do this, we modify the epsilon-delta definition of a limit to give formal epsilon-delta definitions for limits from the right and left at a point. These definitions only require slight modifications from the definition of the limit. In the definition of the limit from the right, the inequality $0 < x - a < \delta$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are greater than (to the right of) a. Similarly, in the definition of the limit from the left, the inequality $-\delta < x - a < 0$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are less than (to the left of) a.

Definition: One-Sided Limits (Formal)

Limit from the Right: Let $f(x)$ be defined over an open interval of the form (a, b) where $a < b$. Then

$$\lim_{x \to a^+} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$.

Limit from the Left: Let $f(x)$ be defined over an open interval of the form (b, c) where $b < c$. Then,

$$\lim_{x \to c^-} f(x) = L$$

if for every $\varepsilon > 0$,there exists a $\delta > 0$ such that if $-\delta < x - c < 0$, then $|f(x) - L| < \varepsilon$.

Example 2.6.6: Proving a Statement about a Limit From the Right

Prove that

$$\lim_{x \to 4^+} \sqrt{x-4} = 0.$$

Solution

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon^2$. Since we ultimately want $|\sqrt{x-4} - 0| < \varepsilon$, we manipulate this inequality to get $\sqrt{x-4} < \varepsilon$ or, equivalently, $0 < x - 4 < \varepsilon^2$, making $\delta = \varepsilon^2$ a clear choice. We may also determine δ geometrically, as shown in Figure 2.6.4.

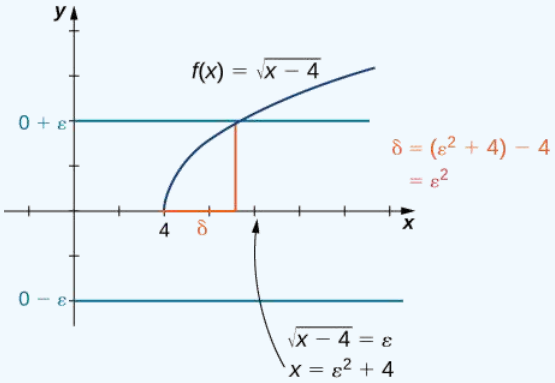


Figure 2.6.4: This graph shows how we find δ for the proof in Example 2.6.6.

Assume $0 < x - 4 < \delta$. Thus, $0 < x - 4 < \varepsilon^2$. Hence, $0 < \sqrt{x-4} < \varepsilon$. Finally, $\left|\sqrt{x-4} - 0\right| < \varepsilon$. Therefore, $\lim_{x \to 4^+} \sqrt{x-4} = 0$.

Exercise 2.6.4

Find δ corresponding to ε for a proof that $\lim_{x \to 1^-} \sqrt{1-x} = 0$.

Hint

Sketch the graph and use Example 2.6.6 as a solving guide.

Answer

$\delta = \varepsilon^2$

Infinite Limits

We conclude the process of converting our intuitive ideas of various types of limits to rigorous formal definitions by pursuing a formal definition of infinite limits. To have $\lim_{x \to a} f(x) = +\infty$, we want the values of the function $f(x)$ to get larger and larger as x approaches a. Instead of the requirement that $|f(x) - L| < \varepsilon$ for arbitrarily small ε when $0 < |x - a| < \delta$ for small enough δ, we want $f(x) > M$ for arbitrarily large positive M when $0 < |x - a| < \delta$ for small enough δ. Figure 2.6.5 illustrates this idea by showing the value of δ for successively larger values of M.

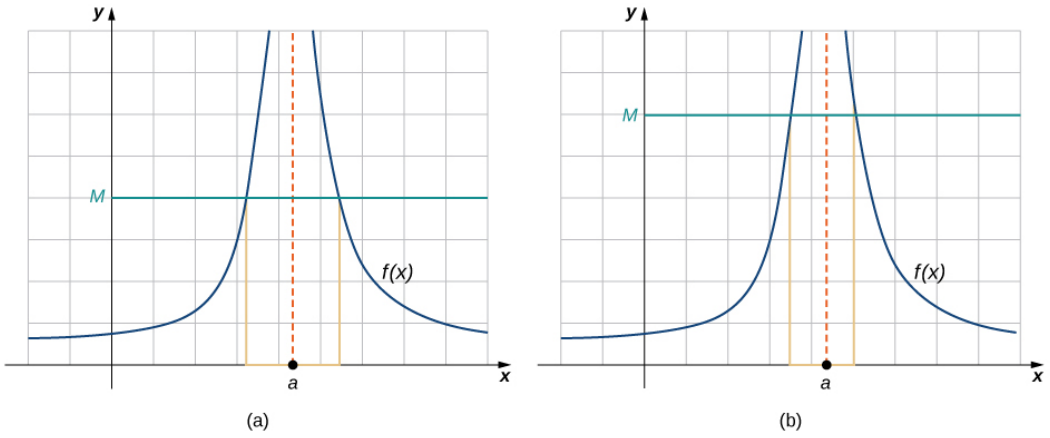


Figure 2.6.5: These graphs plot values of δ for M to show that $\lim\limits_{x\to a} f(x) = +\infty$.

Definition: Infinite Limits (Formal)

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have an infinite limit

$$\lim_{x\to a} f(x) = +\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$.

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have a negative infinite limit

$$\lim_{x\to a} f(x) = -\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) < -M$.

Example 2.6.7: Proving a Statement about an Infinite Limit

Prove that $\lim\limits_{x\to 3} \dfrac{1}{(x-3)^2} = \infty$.

Solution

We use a very similar approach to our previous Problem-Solving Strategy. We first find an appropriate $\delta > 0$. Then we write our proof.

Step 1: First we find an appropriate $\delta > 0$.

1. Let M be any real number such that $M > 0$.

2. Let $f(x) = \dfrac{1}{(x-3)^2} > M$. Then we solve for the expression $x - 3$.

Multiplying both sides of the inequality by the positive quantity $(x-3)^2$ and dividing both sides by the positive quantity M gives us:

$$\frac{1}{M} > (x-3)^2$$

Taking the square root of both sides, we have,

$$\sqrt{\frac{1}{M}} > |x-3|. \qquad \left(\text{Remember that } \sqrt{x^2} = |x|.\right)$$

Rewriting this statement gives us, $0 < |x-3| < \sqrt{\dfrac{1}{M}}$. From this we choose $\delta = \sqrt{\dfrac{1}{M}}$.

Step 2: Now we write a proof.

3. Let $\delta = \sqrt{\frac{1}{M}}$ and assume $0 < |x-3| < \delta = \sqrt{\frac{1}{M}}$.

Thus,

$$|x-3| < \sqrt{\frac{1}{M}}.$$

Squaring both sides gives us,

$$(x-3)^2 < \frac{1}{M}.$$

Taking the reciprocal of both sides (and remembering that this will reverse the direction of the inequality),

$$\frac{1}{(x-3)^2} > M.$$

Therefore, we have proven that

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty.$$

A very similar proof will be needed for a limit that is equal to $-\infty$.

Note that a one-sided limit approach will often need to be taken with this type of limit. For example, to prove: $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

Key Concepts

- The intuitive notion of a limit may be converted into a rigorous mathematical definition known as the epsilon-delta definition of the limit.
- The epsilon-delta definition may be used to prove statements about limits.
- The epsilon-delta definition of a limit may be modified to define one-sided limits.
- A similar definition of an infinite limit can be used to prove statements about infinite limits.

Glossary

epsilon-delta definition of the limit

$\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

triangle inequality

If a and b are any real numbers, then $|a+b| \leq |a| + |b|$

formal definition of an infinite limit

$\lim_{x \to a} f(x) = \infty$ if for every $M > 0$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $f(x) > M$
$\lim_{x \to a} f(x) = -\infty$ if for every $M > 0$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $f(x) < -M$

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- Paul Seeburger (Monroe Community College), added Example **2.6.7** and entries for infinite limits under Key Concepts and the Glossary.

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2.6E: Exercises

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2.7: Chapter 2 Review Exercises

***True or False*. In exercises 1 - 4, justify your answer with a proof or a counterexample.**

1) A function has to be continuous at $x = a$ if the $\lim_{x \to a} f(x)$ exists.

2) You can use the quotient rule to evaluate $\lim_{x \to 0} \frac{\sin x}{x}$.

Answer

False, since we cannot have $\lim_{x \to 0} x = 0$ in the denominator.

3) If there is a vertical asymptote at $x = a$ for the function $f(x)$, then f is undefined at the point $x = a$.

4) If $\lim_{x \to a} f(x)$ does not exist, then f is undefined at the point $x = a$.

Answer

False. A jump discontinuity is possible.

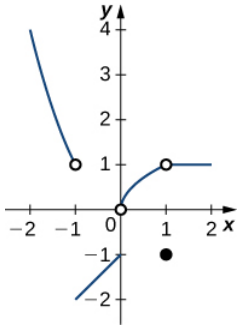
5) Using the graph, find each limit or explain why the limit does not exist.

a. $\lim_{x \to -1} f(x)$

b. $\lim_{x \to 1} f(x)$

c. $\lim_{x \to 0^+} f(x)$

d. $\lim_{x \to 2} f(x)$



In exercises 6 - 15, evaluate the limit algebraically or explain why the limit does not exist.

6) $\lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2}$

Answer

5

7) $\lim_{x \to 0} 3x^2 - 2x + 4$

8) $\lim_{x \to 3} \frac{x^3 - 2x^2 - 1}{3x - 2}$

Answer

8/7

9) $\lim_{x \to \pi/2} \frac{\cot x}{\cos x}$

10) $\lim_{x \to -5} \frac{x^2 + 25}{x + 5}$

Answer

DNE

11) $\lim_{x\to 2} \dfrac{3x^2-2x-8}{x^2-4}$

12) $\lim_{x\to 1} \dfrac{x^2-1}{x^3-1}$

Answer

$2/3$

13) $\lim_{x\to 1} \dfrac{x^2-1}{\sqrt{x}-1}$

14) $\lim_{x\to 4} \dfrac{4-x}{\sqrt{x}-2}$

Answer

-4

15) $\lim_{x\to 4} \dfrac{1}{\sqrt{x}-2}$

In exercises 16 - 17, use the squeeze theorem to prove the limit.

16) $\lim_{x\to 0} x^2\cos(2\pi x) = 0$

Answer

Since $-1 \le \cos(2\pi x) \le 1$, then $-x^2 \le x^2\cos(2\pi x) \le x^2$. Since $\lim_{x\to 0} x^2 = 0 = \lim_{x\to 0} -x^2$, it follows that $\lim_{x\to 0} x^2\cos(2\pi x) = 0$.

17) $\lim_{x\to 0} x^3 \sin\left(\dfrac{\pi}{x}\right) = 0$

18) Determine the domain such that the function $f(x) = \sqrt{x-2} + xe^x$ is continuous over its domain.

Answer

$[2, \infty]$

In exercises 19 - 20, determine the value of c such that the function remains continuous. Draw your resulting function to ensure it is continuous.

19) $f(x) = \begin{cases} x^2+1, & \text{if } x > c \\ 2^x, & \text{if } x \le c \end{cases}$

20) $f(x) = \begin{cases} \sqrt{x+1}, & \text{if } x > -1 \\ x^2+c, & \text{if } x \le -1 \end{cases}$

In exercises 21 - 22, use the precise definition of limit to prove the limit.

21) $\lim_{x\to 1} (8x+16) = 24$

22) $\lim_{x\to 0} x^3 = 0$

Answer

$\delta = \sqrt[3]{\varepsilon}$

23) A ball is thrown into the air and the vertical position is given by $x(t) = -4.9t^2 + 25t + 5$. Use the Intermediate Value Theorem to show that the ball must land on the ground sometime between 5 sec and 6 sec after the throw.

24) A particle moving along a line has a displacement according to the function $x(t) = t^2 - 2t + 4$, where x is measured in meters and t is measured in seconds. Find the average velocity over the time period $t = [0, 2]$.

Answer

0 m/sec

25) From the previous exercises, estimate the instantaneous velocity at $t = 2$ by checking the average velocity within $t = 0.01$ sec.

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CHAPTER OVERVIEW

3: Derivatives

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

3.1: Prelude to Derivatives
3.2: Defining the Derivative
3.2E: Exercises
3.3: The Derivative as a Function
3.3E: Exercises
3.4: Differentiation Rules
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3.6: Derivatives of Trigonometric Functions
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Thumbnail: Derivatives (CC BY; OpenStax)

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3.1: Prelude to Derivatives

The Hennessey Venom GT is one of the fastest cars in the world. In 2014, it reached a record-setting speed of 270.49 mph. It can go from 0 to 200 mph in 14.51 seconds. The techniques in this chapter can be used to calculate the acceleration the Venom achieves in this feat.)



Figure **3.1.1**: The Hennessey Venom GT can go from 0 to 200 mph in 14.51 seconds. (credit: modification of work by Codex41, Flickr)

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

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3.2: Defining the Derivative

Learning Objectives

- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope of a tangent line.
- Identify the derivative as the limit of a difference quotient.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.
- Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac **Newton (1643–1727)** and the German Gottfried **Leibniz (1646–1716)**, whose images appear in Figure **3.2.1**. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.



Figure **3.2.1**: Newton and Leibniz are credited with developing calculus independently.

Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in Figure **3.2.2**. The slope of this line is given by an equation in the form of a difference quotient:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with $a + h$, where h is a value close to a. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a + h, f(a + h))$. In this case, we find the secant line has a slope given by the following difference quotient with increment h:

$$m_{sec} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

Definition: Difference Quotient

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a}$$

is a *difference quotient*.

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a+h) - f(a)}{h}$$

is a difference quotient with increment h.

These two expressions for calculating the slope of a secant line are illustrated in Figure **3.2.2**. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.

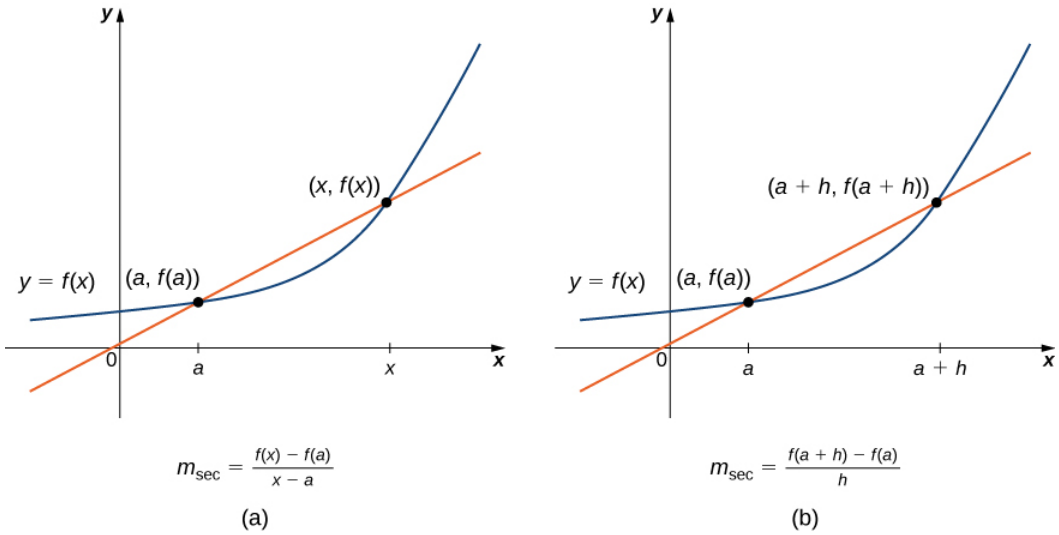


Figure **3.2.2**: We can calculate the slope of a secant line in either of two ways.

In Figure **3.2.3***a* we see that, as the values of x approach a, the slopes of the secant lines provide better estimates of the rate of change of the function at a. Furthermore, the secant lines themselves approach the tangent line to the function at a, which represents the limit of the secant lines. Similarly, Figure **3.2.3***b* shows that as the values of h get closer to 0, the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at a, as shown in Figure **3.2.3***c*.

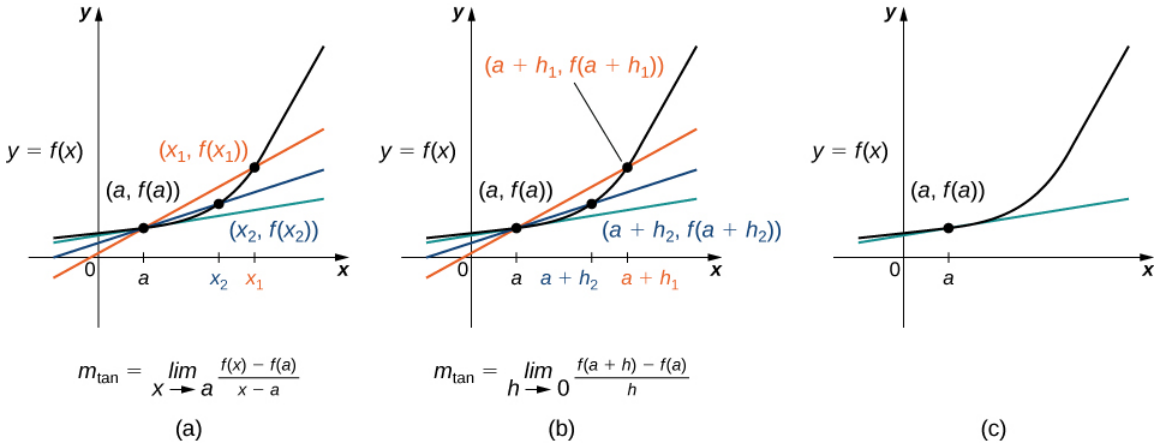


Figure **3.2.3**: The secant lines approach the tangent line (shown in green) as the second point approaches the first.

In Figure 3.2.4 we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to 1. In fact, the graph of $f(x)$ itself appears to be **locally linear** in the immediate vicinity of $x = 1$.

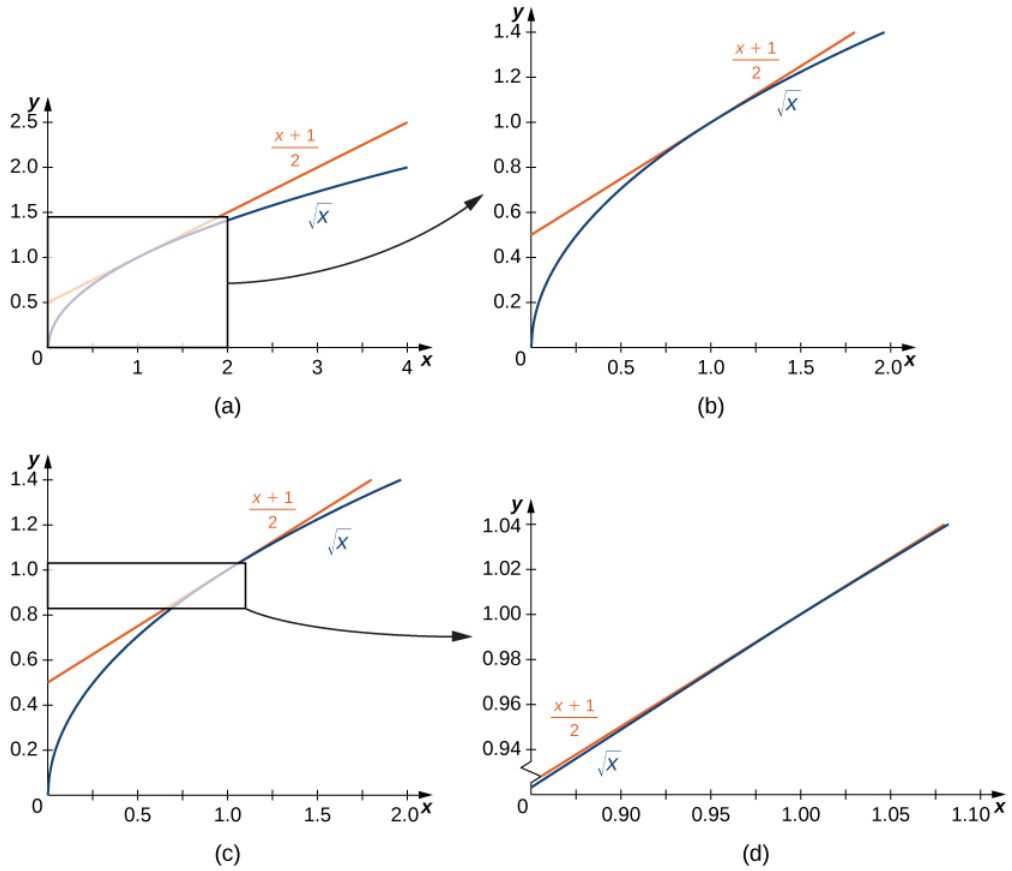


Figure 3.2.4: For values of x close to 1, the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

Definition: Tangent Line

Let $f(x)$ be a function defined in an open interval containing a. The tangent line to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.2.1}$$

provided this limit exists.

Equivalently, we may define the *tangent line* to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{3.2.2}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

Example 3.2.1: Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

First find the slope of the tangent line. In this example, use Equation 3.2.1.

$$m_{tan} = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \qquad \text{Apply the definition.}$$

$$= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} \qquad \text{Substitute } f(x) = x^2 \text{ and } f(3) = 9$$

$$= \lim_{x \to 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \to 3}(x + 3) = 6 \qquad \text{Factor the numerator to evaluate the limit.}$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x = 3$, it passes through the point $(3, f(3))$. We have $f(3) = 9$, so the tangent line passes through the point $(3, 9)$.

Using the point-slope equation of the line with the slope $m = 6$ and the point $(3, 9)$, we obtain the line $y - 9 = 6(x - 3)$. Simplifying, we have $y = 6x - 9$. The graph of $f(x) = x^2$ and its tangent line at 3 are shown in Figure 3.2.5.

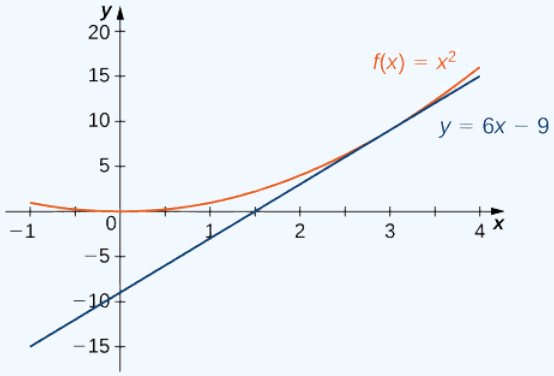


Figure 3.2.5: The tangent line to $f(x)$ at $x = 3$.

Example 3.2.2: The Slope of a Tangent Line Revisited

Use Equation 3.2.2 to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

The steps are very similar to Example 3.2.1. See Equation 3.2.2 for the definition.

$$m_{tan} = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} \qquad \text{Apply the definition.}$$

$$= \lim_{h \to 0} \frac{(3 + h)^2 - 9}{h} \qquad \text{Substitute } f(3 + h) = (3 + h)^2 \text{ and } f(3) = 9$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} \qquad \text{Expand and simplify to evaluate the limit.}$$

$$= \lim_{h \to 0} \frac{h(6 + h)}{h} = \lim_{h \to 0}(6 + h) = 6$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

Example 3.2.3: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

Solution

We can use Equation 3.2.1, but as we have seen, the results are the same if we use Equation 3.2.2.

Misplaced &

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f(2) = \frac{1}{2}$. Since the tangent line passes through the point $(2, \frac{1}{2})$ we can use the point-slope equation of a line to find the equation of the tangent line. Thus the tangent line has the equation $y = -\frac{1}{4}x + 1$. The graphs of $f(x) = \frac{1}{x}$ and $y = -\frac{1}{4}x + 1$ are shown in Figure 3.2.6.

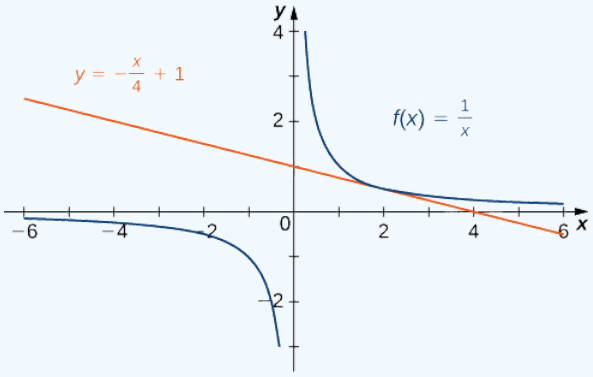


Figure 3.2.6:The line is tangent to $f(x)$ at $x = 2$.

? Exercise 3.2.1

Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

Hint

Use either Equation 3.2.1 or Equation 3.2.2. Multiply the numerator and the denominator by a conjugate.

Answer

$\frac{1}{4}$

The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative.** The process of finding a derivative is called **differentiation**.

Definition: Derivative

Let $f(x)$ be a function defined in an open interval containing a. The **derivative** of the function $f(x)$ at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.2.3}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{3.2.4}$$

Example 3.2.4: Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f'(3)$ using Equation 3.2.3.

Solution

Create a table using values of x just below 3 and just above 3.

x	$\frac{x^2-9}{x-3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

After examining the table, we see that a good estimate is $f'(3) = 6$.

Exercise 3.2.2

For $f(x) = x^2$, use a table to estimate $f'(3)$ using Equation 3.2.4.

Hint

Evaluate $\frac{(x+h)^2 - x^2}{h}$ at $h = -0.1, -0.01, -0.001, 0.001, 0.01, 0.1$

Answer

6

Example 3.2.6: Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 3.2.3.

Solution

Substitute the given function and value directly into the equation.

$$
\begin{aligned}
f'(x) &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} && \text{Apply the definition.} \\
&= \lim_{x \to 2} \frac{(3x^2 - 4x + 1) - 5}{x - 2} && \text{Substitute } f(x) = 3x^2 - 4x + 1 \text{ and } f(2) = 5. \\
&= \lim_{x \to 2} \frac{(x-2)(3x+2)}{x - 2} && \text{Simplify and factor the numerator.} \\
&= \lim_{x \to 2} (3x + 2) && \text{Cancel the common factor.} \\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

Example 3.2.7: Revisiting the Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 3.2.4.

Solution

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in Example 3.2.6.

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} && \text{Apply the definition.}\\
&= \lim_{h\to 0}\frac{(3(2+h)^2-4(2+h)+1)-5}{h} && \text{Substitute } f(2)=5 \text{ and } f(2+h)=3(2+h)^2-4(2+h)+1.\\
&= \lim_{h\to 0}\frac{3(4+4h+h^2)-8-4h+1-5}{h} && \text{Expand the numerator.}\\
&= \lim_{h\to 0}\frac{12+12h+3h^2-12-4h}{h} && \text{Distribute and begin simplifying the numerator.}\\
&= \lim_{h\to 0}\frac{3h^2+8h}{h} && \text{Finish simplifying the numerator.}\\
&= \lim_{h\to 0}\frac{h(3h+8)}{h} && \text{Factor the numerator.}\\
&= \lim_{h\to 0}(3h+8) && \text{Cancel the common factor.}\\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

The results are the same whether we use Equation 3.2.3 or Equation 3.2.4.

? Exercise 3.2.4

For $f(x)=x^2+3x+2$, find $f'(1)$.

Hint

Use either Equation 3.2.3, Equation 3.2.4, or try both.

Answer

$f'(1)=5$

Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the **average velocity** of the object over a time interval $[a,t]$ if $t>a$ or $[t,a]$ if $t<a$ is given by the difference quotient

$$v_{ave}=\frac{s(t)-s(a)}{t-a}. \tag{3.2.5}$$

As the values of t approach a, the values of v_{ave} approach the value we call the **instantaneous velocity** at a. That is, instantaneous velocity at a, denoted $v(a)$, is given by

$$v(a)=s'(a)=\lim_{t\to a}\frac{s(t)-s(a)}{t-a}. \tag{3.2.6}$$

To better understand the relationship between average velocity and instantaneous velocity, see Figure 3.2.7. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t=a$ whose position at time t is given by the function $s(t)$. The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a,t]$.

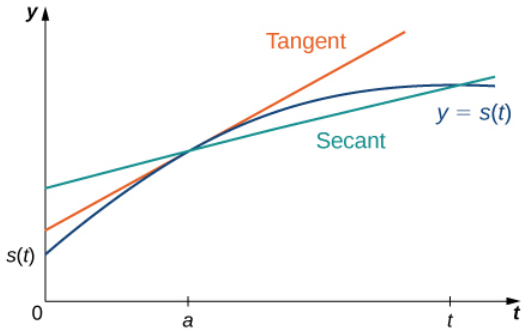


Figure 3.2.7: The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

We can use Equation 3.2.6 to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using Equation 3.2.5.

Example 3.2.8: Estimating Velocity

A lead weight on a spring is oscillating up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin t$ (Figure 3.2.8). Use a table of values to estimate $v(0)$. Check the estimate by using Equation 3.2.6.

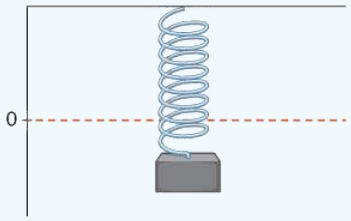


Figure 3.2.8: A lead weight suspended from a spring in vertical oscillatory motion.

Solution

We can estimate the instantaneous velocity at $t = 0$ by computing a table of average velocities using values of t approaching 0, as shown in Table 3.2.2.

Table 3.2.2: Average velocities using values of t approaching 0

t	$\frac{\sin t - \sin 0}{t-0} = \frac{\sin t}{t}$
−0.1	0.998334166
−0.01	0.9999833333
−0.001	0.999999833
0.001	0.999999833
0.01	0.9999833333
0.1	0.998334166

From the table we see that the average velocity over the time interval $[-0.1, 0]$ is 0.998334166, the average velocity over the time interval $[-0.01, 0]$ is 0.9999833333, and so forth. Using this table of values, it appears that a good estimate is $v(0) = 1$.

By using Equation 3.2.6, we can see that

$$v(0) = s'(0) = \lim_{t \to 0} \frac{\sin t - \sin 0}{t - 0} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$$

Thus, in fact, $v(0) = 1$.

Exercise 3.2.5

A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64,\ 0 \le t \le 2$. Find its instantaneous velocity 1 second after it is dropped, using Equation 3.2.6.

Hint

$v(t) = s'(t)$. Follow the earlier examples of the derivative using Equation 3.2.6.

Answer

−32 ft/s

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

Definition: Instantaneous Rate of Change

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Example 3.2.9: Chapter Opener: Estimating Rate of Change of Velocity

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its **acceleration**) as it approaches 229.9 mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?



Figure 3.2.9: (credit: modification of work by Codex41, Flickr)

Solution: First observe that 60 mph = 88 ft/s, 100 mph ≈146.67 ft/s, 200 mph ≈293.33 ft/s, and 229.9 mph ≈337.19 ft/s. We can summarize the information in a table.

Table 3.2.3: $v(t)$ at different values of t

t	$v(t)$
0	0
3.05	88
5.88	147.67
14.51	293.33
19.96	337.19

Now compute the average acceleration of the car in feet per second on intervals of the form $[t, 19.96]$ as t approaches 19.96, as shown in the following table.

Average acceleration

t	$\frac{v(t)-v(19.96)}{t-19.96}=\frac{v(t)-337.19}{t-19.96}$
0.0	16.89
3.05	14.74
5.88	13.46
14.51	8.05

The rate at which the car is accelerating is decreasing as its velocity approaches 229.9 mph (337.19 ft/s).

Example 3.2.10: Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70°F$ at 9 p.m., reaches a low of $60°$ during the night, and rises back to $70°$ by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10$, where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution

Since midnight is 3 hours past 9 p.m., we want to compute $T'(3)$. Refer to Equation 3.2.3.

$$\begin{aligned}
T'(3) &= \lim_{t\to 3}\frac{T(t)-T(3)}{t-3} && \text{Apply the definition.}\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+70-61.6}{t-3} && \text{Substitute } T(t)=0.4t^2-4t+70 \text{ and } T(3)=61.6.\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+8.4}{t-3} && \text{Simplify.}\\
&= \lim_{t\to 3}\frac{0.4(t-3)(t-7)}{t-3} &&\\
&= \lim_{t\to 3}0.4(t-7) && \text{Cancel.}\\
&= -1.6 && \text{Evaluate the limit.}
\end{aligned}$$

The instantaneous rate of change of the temperature at midnight is $-1.6°F$ per hour.

Example 3.2.11: Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10,000$ dollars. Find the rate of change of profit when $10,000$ games are produced. Should the toy company increase or decrease production?

Solution

The profit $P(x)$ earned by producing x gaming systems is $R(x) - C(x)$, where $R(x)$ is the revenue obtained from the sale of x games. Since the company can sell x games at $p = -0.01x + 400$ per game,

$$R(x) = xp = x(-0.01x + 400) = -0.01x^2 + 400x.$$

Consequently,

$$P(x) = -0.01x^2 + 300x - 10,000.$$

Therefore, evaluating the rate of change of profit gives

$$
\begin{aligned}
P'(10000) &= \lim_{x \to 10000} \frac{P(x) - P(10000)}{x - 10000} \\
&= \lim_{x \to 10000} \frac{-0.01x^2 + 300x - 10000 - 1990000}{x - 10000} \\
&= \lim_{x \to 10000} \frac{-0.01x^2 + 300x - 2000000}{x - 10000} \\
&= 100
\end{aligned}
$$

Since the rate of change of profit $P'(10,000) > 0$ and $P(10,000) > 0$, the company should increase production.

? Exercise 3.2.6

A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10$. The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25)$, the rate of change of profit when the price is $\$3.25$ and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

Hint

Use Example 3.2.11 for a guide.

Answer

$P'(3.25) = 20 > 0$; raise prices

Key Concepts

- The slope of the tangent line to a curve measures the instantaneous rate of change of a curve. We can calculate it by finding the limit of the difference quotient or the difference quotient with increment h.
- The derivative of a function $f(x)$ at a value a is found using either of the definitions for the slope of the tangent line.
- Velocity is the rate of change of position. As such, the velocity $v(t)$ at time t is the derivative of the position $s(t)$ at time t. Average velocity is given by

$$v_{ave} = \frac{s(t) - s(a)}{t - a}.$$

Instantaneous velocity is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}.$$

- We may estimate a derivative by using a table of values.

Key Equations

- **Difference quotient**

$$Q = \frac{f(x) - f(a)}{x - a}$$

- **Difference quotient with increment h**

$$Q = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

- **Slope of tangent line**

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- **Derivative of f(x) at a**

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- **Average velocity**

$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$

- **Instantaneous velocity**

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

Glossary

derivative

the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

difference quotient

of a function $f(x)$ at a is given by

$$\frac{f(a+h) - f(a)}{h} \text{ or } \frac{f(x) - f(a)}{x - a}$$

differentiation

the process of taking a derivative

instantaneous rate of change

the rate of change of a function at any point along the function a, also called $f'(a)$, or the derivative of the function at a

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3.3: The Derivative as a Function

Learning Objectives

- Define the derivative function of a given function.
- Graph a derivative function from the graph of a given function.
- State the connection between derivatives and continuity.
- Describe three conditions for when a function does not have a derivative.
- Explain the meaning of a higher-order derivative.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. If we differentiate a position function at a given time, we obtain the velocity at that time. It seems reasonable to conclude that knowing the derivative of the function at every point would produce valuable information about the behavior of the function. However, the process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section we define the derivative function and learn a process for finding it.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Definition: Derivative Function

Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.3.1}$$

A function $f(x)$ is said to be **differentiable at** a if $f'(a)$ exists. More generally, a function is said to be **differentiable on** S if it is differentiable at every point in an open set S, and a differentiable function is one in which $f'(x)$ exists on its domain.

In the next few examples we use Equation 3.3.1 to find the derivative of a function.

Example 3.3.1: Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.

Solution

Start directly with the definition of the derivative function.

Substitute $f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$	
$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$	Multiply numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ without distributing in the denominator.
$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$	Multiply the numerators and simplify.
$= \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)}$	Cancel the h.
$= \frac{1}{2\sqrt{x}}$	Evaluate the limit

Example 3.3.2: Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

Solution

Follow the same procedure here, but without having to multiply by the conjugate.

Substitute $f(x+h) = (x+h)^2 - 2(x+h)$ and $f(x) = x^2 - 2x$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim_{h \to 0} \frac{((x+h)^2 - 2(x+h)) - (x^2 - 2x)}{h}$	
$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$	Expand $(x+h)^2 - 2(x+h)$.
$= \lim_{h \to 0} \frac{2xh - 2h + h^2}{h}$	Simplify
$= \lim_{h \to 0} \frac{h(2x - 2 + h)}{h}$	Factor out h from the numerator
$= \lim_{h \to 0} (2x - 2 + h)$	Cancel the common factor of h
$= 2x - 2$	Evaluate the limit

Exercise 3.3.1

Find the derivative of $f(x) = x^2$.

Hint

Use Equation 3.3.1 and follow the example.

Answer

$f'(x) = 2x$

We use a variety of different notations to express the derivative of a function. In Example 3.3.2 we showed that if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$. If we had expressed this function in the form $y = x^2 - 2x$, we could have expressed the derivative as $y' = 2x - 2$ or $\frac{dy}{dx} = 2x - 2$. We could have conveyed the same information by writing $\frac{d}{dx}(x^2 - 2x) = 2x - 2$. Thus, for the function $y = f(x)$, each of the following notations represents the derivative of $f(x)$:

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}\big(f(x)\big).$$

In place of $f'(a)$ we may also use $\left.\frac{dy}{dx}\right|_{x=a}$. Use of the $\frac{dy}{dx}$ notation (called Leibniz notation) is quite common in engineering and physics. To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\frac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx (Figure 3.3.1). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to x, is expressed as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

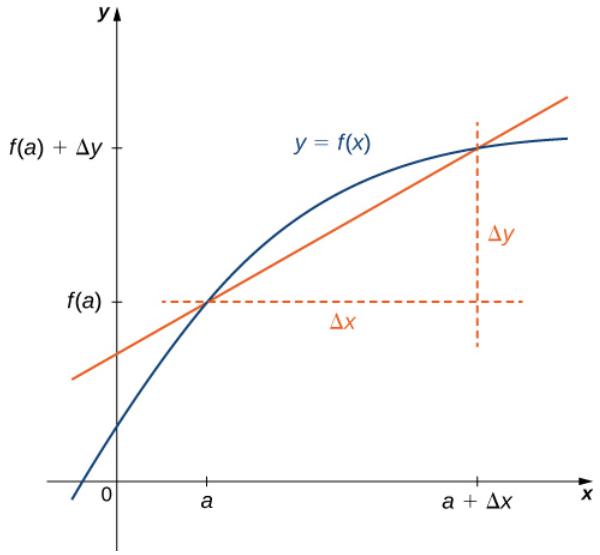


Figure **3.3.1**: The derivative is expressed as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Graphing a Derivative

We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In Example **3.3.1**, we found that for $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$. If we graph these functions on the same axes, as in Figure **3.3.2**, we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \to 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.

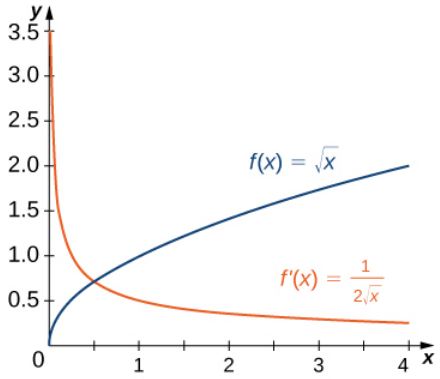


Figure **3.3.2**: The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.

In Example **3.3.2**, we found that for $f(x) = x^2 - 2x$, $f'(x) = 2x - 2$. The graphs of these functions are shown in Figure **3.3.3**. Observe that $f(x)$ is decreasing for $x < 1$. For these same values of x, $f'(x) < 0$. For values of $x > 1$, $f(x)$ is increasing and $f'(x) > 0$. Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.

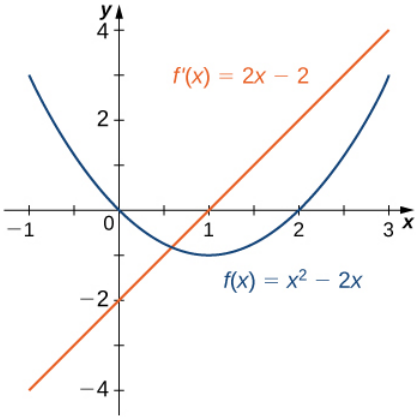
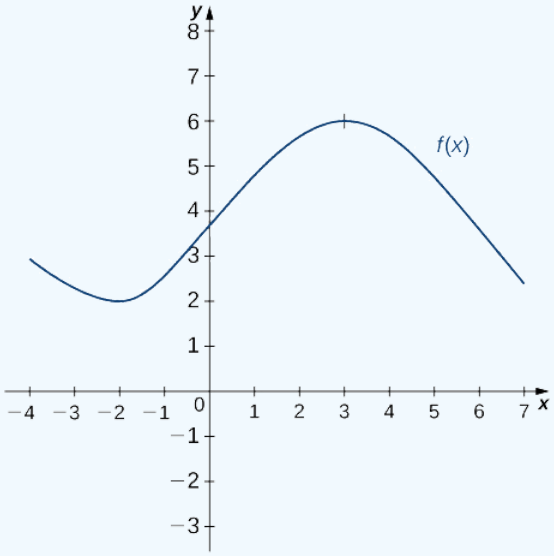


Figure **3.3.3**: The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent

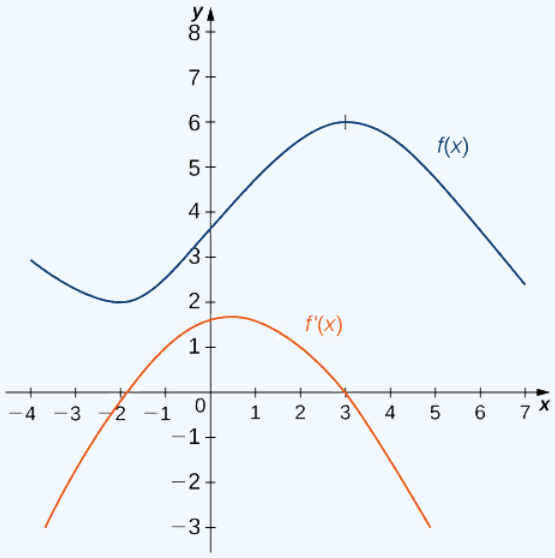
Example 3.3.3: Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



Solution

The solution is shown in the following graph. Observe that $f(x)$ is increasing and $f'(x) > 0$ on $(-2, 3)$. Also, $f(x)$ is decreasing and $f'(x) < 0$ on $(-\infty, -2)$ and on $(3, +\infty)$. Also note that $f(x)$ has horizontal tangents at -2 and 3, and $f'(-2) = 0$ and $f'(3) = 0$.



Exercise 3.3.2

Sketch the graph of $f(x) = x^2 - 4$. On what interval is the graph of $f'(x)$ above the x-axis?

Hint

The graph of $f'(x)$ is positive where $f(x)$ is increasing.

Answer

$(0, +\infty)$

Derivatives and Continuity

Now that we can graph a derivative, let's examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there; however, a function that is continuous at a point need not be differentiable at that point. In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

Proof

If $f(x)$ is differentiable at a, then $f'(a)$ exists and, if we let $h = x - a$, we have $x = a + h$, and as $h = x - a \to 0$, we can see that $x \to a$.

Then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

can be rewritten as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim_{x \to a} f(x) = f(a)$. Thus,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\big(f(x)-f(a)+f(a)\big)\\ &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)+f(a)\right) \qquad \text{Multiply and divide } (f(x)-f(a)) \text{ by } x-a.\\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\cdot\left(\lim_{x\to a}(x-a)\right)+\lim_{x\to a} f(a)\\ &= f'(a)\cdot 0+f(a)\\ &= f(a).\end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{x\to a} f(x) = f(a)$, we conclude that f is continuous at a.

□

We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To determine an answer to this question, we examine the function $f(x) = |x|$. This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f(x) = |x|$,

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{|x|-|0|}{x-0} = \lim_{x\to 0}\frac{|x|}{x}.$$

This limit does not exist because

$$\lim_{x\to 0^-}\frac{|x|}{x} = -1 \text{ and } \lim_{x\to 0^+}\frac{|x|}{x} = 1.$$

See Figure 3.3.4.

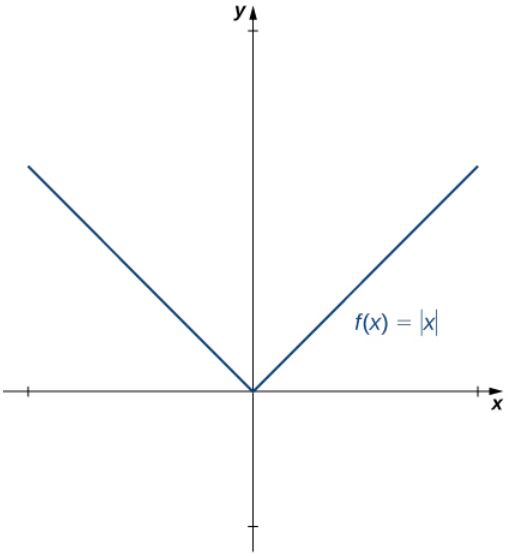


Figure 3.3.4: The function $f(x) = |x|$ is continuous at 0 but is not differentiable at 0.

Let's consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f(x) = \sqrt[3]{x}$:

$$f'(0) = \lim_{x\to 0}\frac{\sqrt[3]{x}-0}{x-0} = \lim_{x\to 0}\frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Thus $f'(0)$ does not exist. A quick look at the graph of $f(x) = \sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at 0 (Figure 3.3.5).

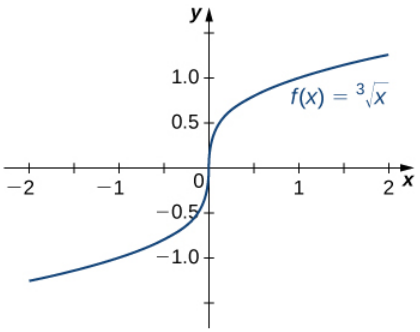


Figure 3.3.5: The function $f(x) = \sqrt[3]{x}$ has a vertical tangent at $x = 0$. It is continuous at 0 but is not differentiable at 0.

The function $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ also has a derivative that exhibits interesting behavior at 0.

We see that

$$f'(0) = \lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right).$$

This limit does not exist, essentially because the slopes of the secant lines continuously change direction as they approach zero (Figure 3.3.6).

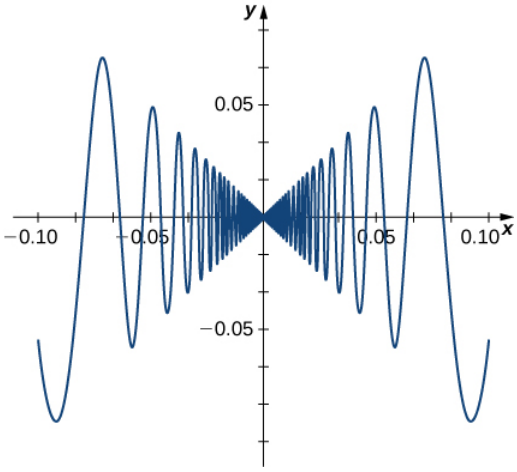


Figure 3.3.6: The function $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is not differentiable at 0.

In summary:

1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f(x) = |x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be "smooth" at that point.
3. As we saw in the example of $f(x) = \sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.
4. As we saw with $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

Example 3.3.4: A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line (Figure 3.3.7). The function that describes the track is to have the form $f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c, & \text{if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2}, & \text{if } x \geq -10 \end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.

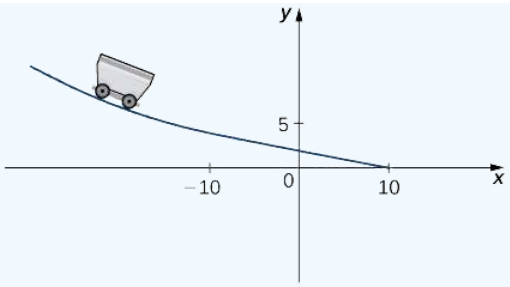


Figure **3.3.7**: For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution

For the function to be continuous at $x = -10$, $\lim_{x \to 10^-} f(x) = f(-10)$. Thus, since

$$\lim_{x \to -10^-} f(x) = \frac{1}{10}(-10)^2 - 10b + c = 10 - 10b + c$$

and $f(-10) = 5$, we must have $10 - 10b + c = 5$. Equivalently, we have $c = 10b - 5$.

For the function to be differentiable at -10,

$$f'(10) = \lim_{x \to -10} \frac{f(x) - f(-10)}{x + 10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

$$\begin{aligned}
\lim_{x \to -10^-} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x \to -10^-} \frac{\frac{1}{10}x^2 + bx + c - 5}{x + 10} \\
&= \lim_{x \to -10^-} \frac{\frac{1}{10}x^2 + bx + (10b - 5) - 5}{x + 10} && \text{Substitute } c = 10b - 5. \\
&= \lim_{x \to -10^-} \frac{x^2 - 100 + 10bx + 100b}{10(x + 10)} \\
&= \lim_{x \to -10^-} \frac{(x + 10)(x - 10 + 10b)}{10(x + 10)} && \text{Factor by grouping} \\
&= b - 2
\end{aligned}$$

We also have

$$\begin{aligned}
\lim_{x \to -10^+} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x \to -10^+} \frac{-\frac{1}{4}x + \frac{5}{2} - 5}{x + 10} \\
&= \lim_{x \to -10^+} \frac{-(x + 10)}{4(x + 10)} \\
&= -\frac{1}{4}
\end{aligned}$$

This gives us $b - 2 = -\frac{1}{4}$. Thus $b = \frac{7}{4}$ and $c = 10(\frac{7}{4}) - 5 = \frac{25}{2}$.

Exercise 3.3.3

Find values of a and b that make $f(x) = \begin{cases} ax + b, & \text{if } x < 3 \\ x^2, & \text{if } x \geq 3 \end{cases}$ both continuous and differentiable at 3.

Hint

Use Example **3.3.4** as a guide.

Answer

$a = 6$ and $b = -9$

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y = f(x)$ can be expressed in any of the following forms:

$$f''(x),\ f'''(x),\ f^{(4)}(x),\ \ldots,\ f^{(n)}(x)$$

$$y''(x),\ y'''(x),\ y^{(4)}(x),\ \ldots,\ y^{(n)}(x)$$

$$\frac{d^2y}{dx^2},\ \frac{d^3y}{dy^3},\ \frac{d^4y}{dy^4},\ \ldots,\ \frac{d^ny}{dy^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly.

Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$.

Example 3.3.5: Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution

First find $f'(x)$.

Substitute $f(x) = 2x^2 - 3x + 1$ and $f(x+h) = 2(x+h)^2 - 3(x+h) + 1$ into $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

$f'(x) = \lim_{h\to 0}\frac{(2(x+h)^2 - 3(x+h) + 1) - (2x^2 - 3x + 1)}{h}$	
$= \lim_{h\to 0}\frac{4xh + 2h^2 - 3h}{h}$	Simplify the numerator.
$= \lim_{h\to 0}(4x + 2h - 3)$	Factor out the h in the numerator and cancel with the h in the denominator.
$= 4x - 3$	Take the limit.

Next, find $f''(x)$ by taking the derivative of $f'(x) = 4x - 3$.

$f''(x) = \lim_{h\to 0}\frac{f'(x+h) - f'(x)}{h}$	Use $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ with $f'(x)$ in place of $f(x)$.
$= \lim_{h\to 0}\frac{(4(x+h) - 3) - (4x - 3)}{h}$	Substitute $f'(x+h) = 4(x+h) - 3$ and $f'(x) = 4x - 3$.
$= \lim_{h\to 0} 4$	Simplify.
$= 4$	Take the limit.

Exercise 3.3.4

Find $f''(x)$ for $f(x) = x^2$.

Hint

We found $f'(x) = 2x$ in a previous checkpoint. Use Equation 3.3.1 to find the derivative of $f'(x)$

Answer

$f''(x) = 2$

Example 3.3.6: Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t.

Solution

Since $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$, we begin by finding the derivative of $s(t)$:

$$\begin{aligned} s'(t) &= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h \to 0} \frac{3(t+h)^2 - 4(t+h) + 1 - (3t^2 - 4t + 1)}{h} \\ &= 6t - 4. \end{aligned}$$

Next,

$$\begin{aligned} s''(t) &= \lim_{h \to 0} \frac{s'(t+h) - s'(t)}{h} \\ &= \lim_{h \to 0} \frac{6(t+h) - 4 - (6t - 4)}{h} \\ &= 6. \end{aligned}$$

Thus, $a = 6 \text{ m/s}^2$.

Exercise 3.3.5

For $s(t) = t^3$, find $a(t)$.

Hint

Use Example **3.3.6** as a guide.

Answer

$a(t) = 6t$

Key Concepts

- The derivative of a function $f(x)$ is the function whose value at x is $f'(x)$.
- The graph of a derivative of a function $f(x)$ is related to the graph of $f(x)$. Where $f(x)$ has a tangent line with positive slope, $f'(x) > 0$. Where $f(x)$ has a tangent line with negative slope, $f'(x) < 0$. Where $f(x)$ has a horizontal tangent line, $f'(x) = 0$.
- If a function is differentiable at a point, then it is continuous at that point. A function is not differentiable at a point if it is not continuous at the point, if it has a vertical tangent line at the point, or if the graph has a sharp corner or cusp.
- Higher-order derivatives are derivatives of derivatives, from the second derivative to the nth derivative.

Key Equations

- **The derivative function**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Glossary

derivative function

gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

differentiable at a

a function for which $f'(a)$ exists is differentiable at a

differentiable on S

a function for which $f'(x)$ exists for each x in the open set S is differentiable on S

differentiable function

a function for which $f'(x)$ exists is a differentiable function

higher-order derivative

a derivative of a derivative, from the second derivative to the nth derivative, is called a higher-order derivative

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) added explanation of the alternative definition of the derivative used in the proof of that differentiability implies continuity.

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3.3E: Exercises

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3.4: Differentiation Rules

Learning Objectives

- State the constant, constant multiple, and power rules.
- Apply the sum and difference rules to combine derivatives.
- Use the product rule for finding the derivative of a product of functions.
- Use the quotient rule for finding the derivative of a quotient of functions.
- Extend the power rule to functions with negative exponents.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit.

The process that we could use to evaluate $\frac{d}{dx}\left(\sqrt[3]{x}\right)$ using the definition, while similar, is more complicated.

In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c$. For this function, both $f(x) = c$ and $f(x+h) = c$, so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{c - c}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 = 0. \end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

The Constant Rule

Let c be a constant. If $f(x) = c$, then $f'(x) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example 3.4.1: Applying the Constant Rule

Find the derivative of $f(x) = 8$.

Solution

This is just a one-step application of the rule: $f'(8) = 0$.

Exercise 3.4.1

Find the derivative of $g(x) = -3$.

Hint

Use the preceding example as a guide

Answer

0

The Power Rule

We have shown that

$$\frac{d}{dx}\left(x^2\right) = 2x \quad \text{and} \quad \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}\left(x^n\right)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x) = x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$. As we go through this derivation, pay special attention to the portion of the expression in boldface, as the technique used in this case is essentially the same as the technique used to prove the general case.

Example 3.4.2: Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

Solution:

$\frac{d}{dx}\left(x^3\right) = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$	
$= \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$	Notice that the first term in the expansion of $(x+h)^3$ is x^3 and the second term is $3x^2h$. All other terms contain powers of h that are two or greater
$= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h}$	In this step the x^3 terms have been cancelled, leaving only terms containing h.
$= \lim_{h\to 0}\frac{h(3x^2 + 3xh + h^2)}{h}$	Factor out the common factor of h.
$= \lim_{h\to 0}(3x^2 + 3xh + h^2)$	After cancelling the common factor of h,the only term not containing h is $3x^2$.
$= 3x^2$	Let h go to 0.

? Exercise 3.4.2

Find $\frac{d}{dx}(x^4)$.

Hint

Use $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$ and follow the procedure outlined in the preceding example.

Answer

$\frac{d}{dx}(x^4) = 4x^3$

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x) = 3^x$.

The Power Rule

Let n be a positive integer. If $f(x) = x^n$,then

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof

For $f(x) = x^n$ where n is a positive integer, we have

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

Since

$$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n,$$

we see that

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n.$$

Next, divide both sides by h:

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n}{h}.$$

Thus,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}.$$

Finally,

$$f'(x) = \lim_{h\to 0}(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1})$$

$$= nx^{n-1}.$$

□

Example 3.4.3: Applying the Power Rule

Find the derivative of the function $f(x) = x^{10}$ by applying the power rule.

Solution

Using the power rule with $n = 10$, we obtain

$$f'(x) = 10x^{10-1} = 10x^9.$$

Exercise 3.4.3

Find the derivative of $f(x) = x^7$.

Hint

Use the power rule with $n = 7$.

Answer

$f'(x) = 7x^6$

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big);$$

that is,

$$\text{for } s(x) = f(x) + g(x), \quad s'(x) = f'(x) + g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g :

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

that is,

$$\text{for } d(x) = f(x) - g(x), \quad d'(x) = f'(x) - g'(x).$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}\big(kf(x)\big) = k\frac{d}{dx}\big(f(x)\big);$$

that is,

$$\text{for } m(x) = kf(x), \quad m'(x) = kf'(x).$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $s(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

$$s'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}.$$

By substituting $s(x+h) = f(x+h) + g(x+h)$ and $s(x) = f(x) + g(x)$, we obtain

$$s'(x) = \lim_{h \to 0} \frac{\big(f(x+h) + g(x+h)\big) - \big(f(x) + g(x)\big)}{h}.$$

Rearranging and regrouping the terms, we have

$$s'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right).$$

We now apply the sum law for limits and the definition of the derivative to obtain

$$s'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).$$

□

Example 3.4.4: Applying the Constant Multiple Rule

Find the derivative of $g(x) = 3x^2$ and compare it to the derivative of $f(x) = x^2$.

Solution

We use the power rule directly:

$$g'(x) = \frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3(2x) = 6x.$$

Since $f(x) = x^2$ has derivative $f'(x) = 2x$, we see that the derivative of $g(x)$ is 3 times the derivative of $f(x)$. This relationship is illustrated in Figure 3.4.1.

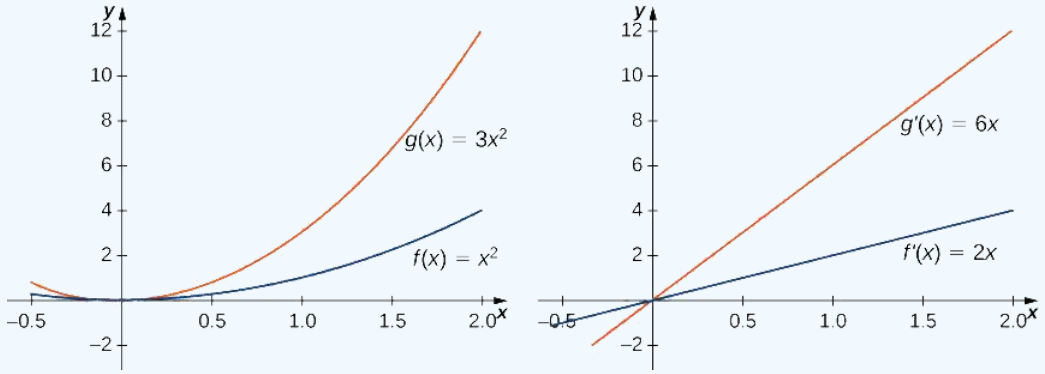


Figure 3.4.1: The derivative of $g(x)$ is 3 times the derivative of $f(x)$.

Example 3.4.5: Applying Basic Derivative Rules

Find the derivative of $f(x) = 2x^5 + 7$.

Solution

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(2x^5 + 7\right) \\
&= \frac{d}{dx}(2x^5) + \frac{d}{dx}(7) && \text{Apply the sum rule.} \\
&= 2\frac{d}{dx}(x^5) + \frac{d}{dx}(7) && \text{Apply the constant multiple rule.} \\
&= 2(5x^4) + 0 && \text{Apply the power rule and the constant rule.} \\
&= 10x^4 && \text{Simplify.}
\end{aligned}
$$

? Exercise 3.4.4

Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$.

Hint

Use the preceding example as a guide.

Answer

$f'(x) = 6x^2 - 12x$.

✓ Example 3.4.6: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$

Solution

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

$$f(1) = 1^2 - 4(1) + 6 = 3.$$

This gives us the point $(1, 3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have

$$f'(x) = 2x - 4$$

so the slope of the tangent line is $f'(1) = -2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y - 3 = -2(x - 1).$$

Putting the equation of the line in slope-intercept form, we obtain

$$y = -2x + 5.$$

? Exercise 3.4.5

Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

Hint

Use the preceding example as a guide.

Answer

$y = 12x - 23$

The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product

of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).$$

That is,

$$\text{if } p(x) = f(x)g(x), \quad \text{then } p'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim_{h \to 0} g(x+h) = g(x)$.

By applying the limit definition of the derivative to $p(x) = f(x)g(x)$, we obtain

$$p'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$p'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

$$p'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}.$$

Rearranging, we obtain

$$\begin{aligned} p'(x) &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) \right) + \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \cdot f(x) \right) \\ &= \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right) \cdot \left(\lim_{h \to 0} g(x+h) \right) + \left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \right) \cdot f(x) \end{aligned}$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

$$p'(x) = f'(x)g(x) + g'(x)f(x).$$

□

Example 3.4.7: Applying the Product Rule to Constant Functions

For $p(x) = f(x)g(x)$, use the product rule to find $p'(2)$ if $f(2) = 3,\ f'(2) = -4,\ g(2) = 1$, and $g'(2) = 6$.

Solution

Since $p(x) = f(x)g(x)$, $p'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

$p'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14.$

Example 3.4.8: Applying the Product Rule to Binomials

For $p(x) = (x^2 + 2)(3x^3 - 5x)$, find $p'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$. Thus,

$$p'(x) = f'(x)g(x) + g'(x)f(x) = (2x)(3x^3 - 5x) + (9x^2 - 5)(x^2 + 2).$$

Simplifying, we have

$$p'(x) = 15x^4 + 3x^2 - 10.$$

To check, we see that $p(x) = 3x^5 + x^3 - 10x$ and, consequently, $p'(x) = 15x^4 + 3x^2 - 10$.

Exercise 3.4.6

Use the product rule to obtain the derivative of $p(x) = 2x^5(4x^2 + x)$.

Hint

Set $f(x) = 2x^5$ and $g(x) = 4x^2 + x$ and use the preceding example as a guide.

Answer

$p'(x) = 10x^4(4x^2 + x) + (8x + 1)(2x^5) = 56x^6 + 12x^5$.

The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}(x^2) = 2x, \text{ not } \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{\big(g(x)\big)^2}.$$

That is, if

$$q(x) = \frac{f(x)}{g(x)}$$

then

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2}.$$

The proof of the quotient rule is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example 3.4.9: Applying the Quotient Rule

Use the quotient rule to find the derivative of $q(x) = \dfrac{5x^2}{4x+3}$.

Solution

Let $f(x) = 5x^2$ and $g(x) = 4x + 3$. Thus, $f'(x) = 10x$ and $g'(x) = 4$.

Substituting into the quotient rule, we have

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x+3) - 4(5x^2)}{(4x+3)^2}.$$

Simplifying, we obtain

$$q'(x) = \frac{20x^2 + 30x}{(4x+3)^2}$$

Exercise 3.4.7

Find the derivative of $h(x) = \dfrac{3x+1}{4x-3}$.

Hint

Apply the quotient rule with $f(x) = 3x + 1$ and $g(x) = 4x - 3$.

Answer

$h'(x) = -\dfrac{13}{(4x-3)^2}$.

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

Extended Power Rule

If k is a negative integer, then

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

Proof

If k is a negative integer, we may set $n = -k$, so that n is a positive integer with $k = -n$. Since for each positive integer n,$x^{-n} = \dfrac{1}{x^n}$, we may now apply the quotient rule by setting $f(x) = 1$ and $g(x) = x^n$. In this case, $f'(x) = 0$ and $g'(x) = nx^{n-1}$. Thus,

$$\frac{d}{dx}(x^{-n}) = \frac{0(x^n) - 1(nx^{n-1})}{(x^n)^2}.$$

Simplifying, we see that

$$\begin{aligned}\frac{d}{dx}(x^{-n}) &= \frac{-nx^{n-1}}{x^{2n}} \\ &= -nx^{(n-1)-2n} \\ &= -nx^{-n-1}.\end{aligned}$$

Finally, observe that since $k = -n$, by substituting we have

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

□

Example 3.4.10: Using the Extended Power Rule

Find $\frac{d}{dx}(x^{-4})$.

Solution

By applying the extended power rule with $k = -4$, we obtain

$$\frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5}.$$

Example 3.4.11: Using the Extended Power Rule and the Constant Multiple Rule

Use the extended power rule and the constant multiple rule to find $f(x) = \frac{6}{x^2}$.

Solution

It may seem tempting to use the quotient rule to find this derivative, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this function by first rewriting it as $f(x) = 6x^{-2}$.

$$\begin{aligned} f'(x) = \frac{d}{dx}\left(\frac{6}{x^2}\right) = \frac{d}{dx}\left(6x^{-2}\right) & \quad \text{Rewrite } \frac{6}{x^2} \text{ as } 6x^{-2}. \\ = 6\frac{d}{dx}\left(x^{-2}\right) & \quad \text{Apply the constant multiple rule.} \\ = 6(-2x^{-3}) & \quad \text{Use the extended power rule to differentiate } x^{-2}. \\ = -12x^{-3} & \quad \text{Simplify.} \end{aligned}$$

Exercise 3.4.8

Find the derivative of $g(x) = \frac{1}{x^7}$ using the extended power rule.

Hint

Rewrite $g(x) = \frac{1}{x^7} = x^{-7}$. Use the extended power rule with $k = -7$.

Answer

$g'(x) = -7x^{-8}$.

Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example 3.4.12: Combining Differentiation Rules

For $k(x) = 3h(x) + x^2 g(x)$, find $k'(x)$.

Solution: Finding this derivative requires the sum rule, the constant multiple rule, and the product rule.

$k'(x)=\frac{d}{dx}\left(3h(x)+x^2g(x)\right)=\frac{d}{dx}\left(3h(x)\right)+\frac{d}{dx}\left(x^2g(x)\right)$	Apply the sum rule.
$=3\frac{d}{dx}\left(h(x)\right)+\left(\frac{d}{dx}(x^2)g(x)+\frac{d}{dx}(g(x))x^2\right)$	Apply the constant multiple rule to differentiate $3h(x)$ and the product rule to differentiate $x^2g(x)$.
$=3h'(x)+2xg(x)+g'(x)x^2$	

Example 3.4.13: Extending the Product Rule

For $k(x)=f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x), g(x), h(x)$, and their derivatives.

Solution

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x)=(f(x)g(x))\cdot h(x)$. Thus,

$$k'(x)=\frac{d}{dx}\left(f(x)g(x)\right)\cdot h(x)+\frac{d}{dx}\left(h(x)\right)\cdot\left(f(x)g(x)\right) \quad \text{Apply the product rule to the product of } f(x)g(x) \text{ and } h(x).$$

$$=\left(f'(x)g(x)+g'(x)f(x)\right)h(x)+h'(x)f(x)g(x) \quad \text{Apply the product rule to } f(x)g(x)$$

$$=f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x) \quad \text{Simplify.}$$

Example 3.4.14: Combining the Quotient Rule and the Product Rule

For $h(x)=\frac{2x^3k(x)}{3x+2}$, find $h'(x)$.

Solution

This procedure is typical for finding the derivative of a rational function.

$$h'(x)=\frac{\frac{d}{dx}(2x^3k(x))\cdot(3x+2)-\frac{d}{dx}(3x+2)\cdot(2x^3k(x))}{(3x+2)^2} \quad \text{Apply the quotient rule.}$$

$$=\frac{(6x^2k(x)+k'(x)\cdot 2x^3)(3x+2)-3(2x^3k(x))}{(3x+2)^2} \quad \text{Apply the product rule to find } \frac{d}{dx}(2x^3k(x)). \text{ Use } \frac{d}{dx}(3x+2)=3.$$

$$=\frac{-6x^3k(x)+18x^3k(x)+12x^2k(x)+6x^4k'(x)+4x^3k'(x)}{(3x+2)^2} \quad \text{Simplify}$$

Exercise 3.4.9

Find $\frac{d}{dx}(3f(x)-2g(x))$.

Hint

Apply the difference rule and the constant multiple rule.

Answer

$3f'(x)-2g'(x)$.

Example 3.4.15: Determining Where a Function Has a Horizontal Tangent

Determine the values of x for which $f(x)=x^3-7x^2+8x+1$ has a horizontal tangent line.

Solution

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'(x)=0$.

Since $f'(x)=3x^2-14x+8=(3x-2)(x-4)$,

we must solve $(3x-2)(x-4)=0$. Thus we see that the function has horizontal tangent lines at $x=\frac{2}{3}$ and $x=4$ as shown in the following graph.

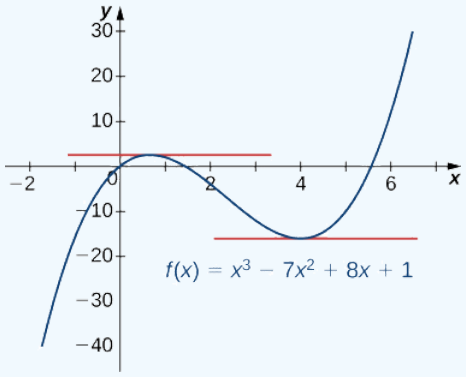


Figure 3.4.2: This function has horizontal tangent lines at $x=2/3$ and $x=4$.

Example 3.4.16: Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s(t)=\frac{t}{t^2+1}$. What is the initial velocity of the object?

Solution

Since the initial velocity is $v(0)=s'(0)$, begin by finding $s'(t)$ by applying the quotient rule:

$$s'(t)=\frac{1(t^2+1)-2t(t)}{(t^2+1)^2}=\frac{1-t^2}{(t^2+1)^2}.$$

After evaluating, we see that $v(0)=1$.

Exercise 3.4.10

Find the values of x for which the line tangent to the graph of $f(x)=4x^2-3x+2$ has a tangent line parallel to the line $y=2x+3$.

Hint

Solve $f'(x)=2$.

Answer

$\frac{5}{8}$

Formula One Grandstands

Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (Figure 3.4.3).



Figure **3.4.3**: The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ (Figure **3.4.4**). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9, 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

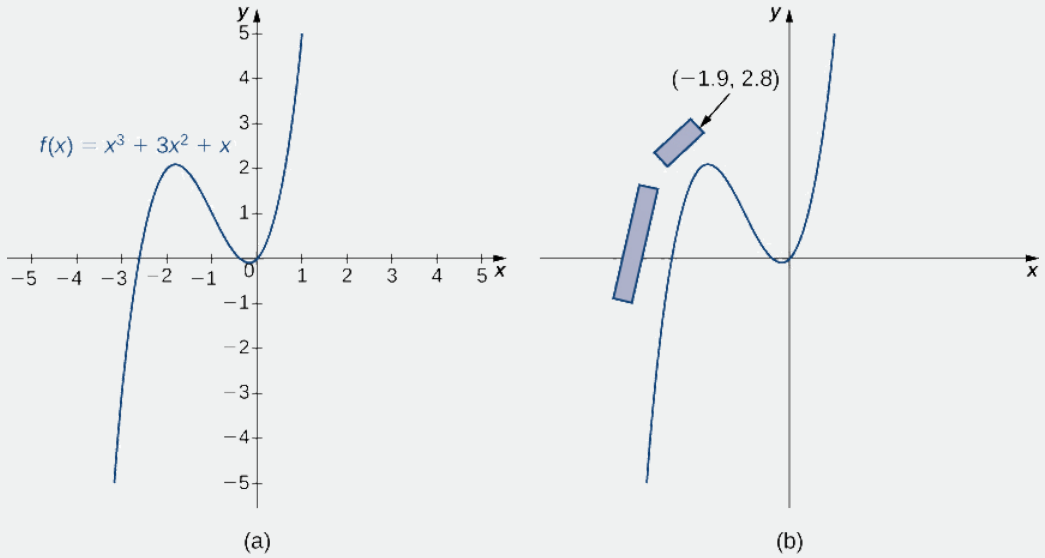


Figure **3.4.4**: (a) One section of the racetrack can be modeled by the function $f(x) = x^3 + 3x^2 + x$. (b) The front corner of the grandstand is located at $(-1.9, 2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x, y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the x-coordinate of the point where the tangent line crosses the line $y = 2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?
4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

Key Concepts

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.
- The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g.
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function.
- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

Glossary

constant multiple rule

the derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative: $\frac{d}{dx}\big(cf(x)\big) = cf'(x)$

constant rule

the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0$, where c is a constant

difference rule

the derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x)$

power rule

the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

product rule

the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + g'(x)f(x)$

quotient rule

the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2}$

sum rule

the derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$

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3.4E: Exercises

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3.5: Derivatives as Rates of Change

Learning Objectives

- Determine a new value of a quantity from the old value and the amount of change.
- Calculate the average rate of change and explain how it differs from the instantaneous rate of change.
- Apply rates of change to displacement, velocity, and acceleration of an object moving along a straight line.
- Predict the future population from the present value and the population growth rate.
- Use derivatives to calculate marginal cost and revenue in a business situation.

In this section we look at some applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include **acceleration** and velocity in physics, **population growth rates** in biology, and marginal functions in economics.

Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the given point. If $f(x)$ is a function defined on an interval $[a, a+h]$, then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by

$$f(a+h) - f(a).$$

The **average rate of change** of the function f over that same interval is the ratio of the amount of change over that interval to the corresponding change in the x values. It is given by

$$\frac{f(a+h) - f(a)}{h}.$$

As we already know, the instantaneous rate of change of $f(x)$ at a is its derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For small enough values of h, $f'(a) \approx \frac{f(a+h)-f(a)}{h}$. We can then solve for $f(a+h)$ to get the amount of change formula:

$$f(a+h) \approx f(a) + f'(a)h. \tag{3.5.1}$$

We can use this formula if we know only $f(a)$and $f'(a)$ and wish to estimate the value of $f(a+h)$. For example, we may use the current population of a city and the rate at which it is growing to estimate its population in the near future. As we can see in Figure **3.5.1**, we are approximating $f(a+h)$ by the y coordinate at a+h on the line tangent to $f(x)$ at $x = a$. Observe that the accuracy of this estimate depends on the value of h as well as the value of $f'(a)$.

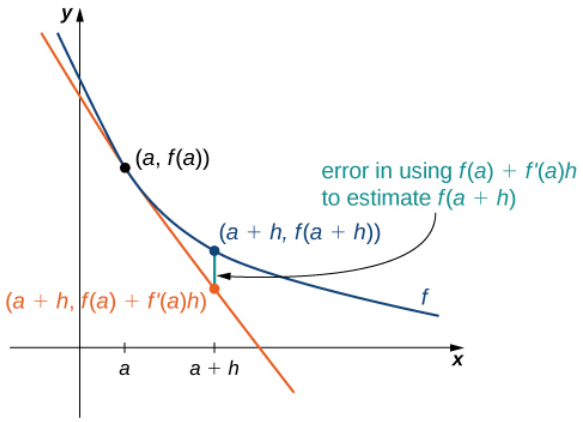


Figure **3.5.1**: The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f(a+h) \approx f(a) + f'(a)h$.

Example 3.5.1: Estimating the Value of a Function

If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$.

Solution

Begin by finding h. We have $h = 3.2 - 3 = 0.2$. Thus,

$$f(3.2) = f(3 + 0.2) \approx f(3) + (0.2)f'(3) = 2 + 0.2(5) = 3.$$

Exercise 3.5.1

Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

Hint

Use the same process as in the preceding example.

Answer

-4.4

Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

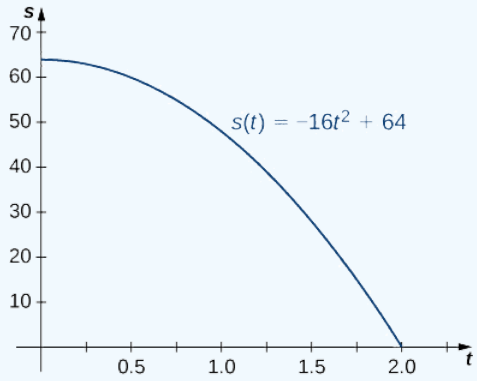
Definition

Let $s(t)$ be a function giving the position of an object at time t.

- The velocity of the object at time t is given by $v(t) = s'(t)$.
- The speed of the object at time t is given by $|v(t)|$.
- The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

Example 3.5.2: Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.



a. What is the instantaneous velocity of the ball when it hits the ground?
b. What is the average velocity during its fall?

Solution

The first thing to do is determine how long it takes the ball to reach the ground. To do this, set $s(t) = 0$. Solving $-16t^2 + 64 = 0$, we get $t = 2$, so it takes 2 seconds for the ball to reach the ground.

a. The instantaneous velocity of the ball as it strikes the ground is $v(2)$. Since $v(t) = s'(t) = -32t$, we obtain $v(t) = -64$ ft/s.
b. The average velocity of the ball during its fall is

$$v_{ave} = \frac{s(2)-s(0)}{2-0} = \frac{0-64}{2} = -32 \text{ ft/s}.$$

Example 3.5.3: Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

a. Is the particle moving from left to right or from right to left at time $t = 1$?
b. Is the particle speeding up or slowing down at time $t = 1$?

Solution

Begin by finding $v(t)$ and $a(t)$.

$v(t) = s'(t) = 3t^2 - 4$ and $a(t) = v'(t) = s''(t) = 6t$.

Evaluating these functions at $t = 1$, we obtain $v(1) = -1$ and $a(1) = 6$.

a. Because $v(1) < 0$, the particle is moving from right to left.
b. Because $v(1) < 0$ and $a(1) > 0$, velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing $|v(t)|$ to decrease. The particle is slowing down.

Example 3.5.4: Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4,\ t \geq 0$.

a. Find $v(t)$.
b. At what time(s) is the particle at rest?
c. On what time intervals is the particle moving from left to right? From right to left?
d. Use the information obtained to sketch the path of the particle along a coordinate axis.

Solution

a. The velocity is the derivative of the position function:

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

b. The particle is at rest when $v(t) = 0$, so set $3t^2 - 18t + 24 = 0$. Factoring the left-hand side of the equation produces $3(t-2)(t-4) = 0$. Solving, we find that the particle is at rest at $t = 2$ and $t = 4$.

c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0$. Figure 3.5.2 gives the analysis of the sign of $v(t)$ for $t \geq 0$, but it does not represent the axis along which the particle is moving.



Figure 3.5.2: The sign of $v(t)$ determines the direction of the particle.

- Since $3t^2 - 18t + 24 > 0$ on $[0, 2) \cup (4, +\infty)$, the particle is moving from left to right on these intervals.
- Since $3t^2 - 18t + 24 < 0$ on $(2, 4)$, the particle is moving from right to left on this interval.

d. Before we can sketch the graph of the particle, we need to know its position at the time it starts moving $(t = 0)$ and at the times that it changes direction $(t = 2, 4)$. We have $s(0) = 4$, $s(2) = 24$, and $s(4) = 20$. This means that the particle begins on the coordinate axis at 4 and changes direction at 24 and 20 on the coordinate axis. The path of the particle is shown on a coordinate axis in Figure 3.5.3.

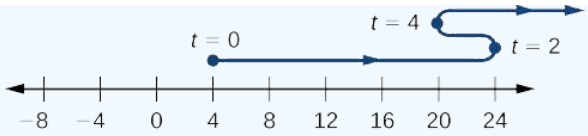


Figure 3.5.3: The path of the particle can be determined by analyzing $v(t)$.

? Exercise 3.5.2

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1$. Is the particle moving from right to left or from left to right at time $t = 3$?

Hint

Find $v(3)$ and look at the sign.

Answer

left to right

Population Change

In addition to analyzing velocity, speed, acceleration, and position, we can use derivatives to analyze various types of populations, including those as diverse as bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and consequently can be represented by the derivative of the size of the population.

Definition

If $P(t)$ is the number of entities present in a population, then the population growth rate of $P(t)$ is defined to be $P'(t)$.

✓ Example 3.5.5: Estimating a Population

The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

Solution

Let $P(t)$ be the population (in thousands) t years from now. Thus, we know that $P(0) = 10$ and based on the information, we anticipate $P(5) = 30$. Now estimate $P'(0)$, the current growth rate, using

$$P'(0) \approx \frac{P(5) - P(0)}{5 - 0} = \frac{30 - 10}{5} = 4.$$

By applying Equation 3.5.1 to $P(t)$, we can estimate the population 2 years from now by writing

$$P(2) \approx P(0) + (2)P'(0) \approx 10 + 2(4) = 18;$$

thus, in 2 years the population will be 18,000.

? Exercise 3.5.3

The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3,000$. If $P'(0) = 100$, estimate the size of the population in 3 days, where t is measured in days.

Hint

Use $P(3) \approx P(0) + 3P'(0)$

Answer

3,300

Changes in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

Definition

- If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.
- If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

We can roughly approximate

$$MC(x) = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

by choosing an appropriate value for h. Since x represents objects, a reasonable and small value for h is 1. Thus, by substituting $h = 1$, we get the approximation $MC(x) = C'(x) \approx C(x+1) - C(x)$. Consequently, $C'(x)$ for a given value of x can be thought of as the change in cost associated with producing one additional item. In a similar way, $MR(x) = R'(x)$ approximates the revenue obtained by selling one additional item, and $MP(x) = P'(x)$ approximates the profit obtained by producing and selling one additional item.

Example 3.5.6: Applying Marginal Revenue

Assume that the number of barbeque dinners that can be sold, x, can be related to the price charged, p, by the equation $p(x) = 9 - 0.03x, 0 \le x \le 300$.

In this case, the revenue in dollars obtained by selling x barbeque dinners is given by

$$R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x \text{ for } 0 \le x \le 300.$$

Use the marginal revenue function to estimate the revenue obtained from selling the 101st barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

Solution

First, find the marginal revenue function: $MR(x) = R'(x) = -0.06x + 9$.

Next, use $R'(100)$ to approximate $R(101) - R(100)$, the revenue obtained from the sale of the 101st dinner. Since $R'(100) = 3$, the revenue obtained from the sale of the 101st dinner is approximately \$3.

The actual revenue obtained from the sale of the 101st dinner is

$$R(101) - R(100) = 602.97 - 600 = 2.97, \text{ or } \$2.97.$$

The marginal revenue is a fairly good estimate in this case and has the advantage of being easy to compute.

Exercise 3.5.4

Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

Hint

Use $P'(100)$ to approximate $P(101) - P(100)$.

Answer

$2

Key Concepts

- Using $f(a+h) \approx f(a) + f'(a)h$, it is possible to estimate $f(a+h)$ given $f'(a)$ and $f(a)$.
- The rate of change of position is velocity, and the rate of change of velocity is acceleration. Speed is the absolute value, or magnitude, of velocity.
- The population growth rate and the present population can be used to predict the size of a future population.
- Marginal cost, marginal revenue, and marginal profit functions can be used to predict, respectively, the cost of producing one more item, the revenue obtained by selling one more item, and the profit obtained by producing and selling one more item.

Glossary

acceleration
is the rate of change of the velocity, that is, the derivative of velocity

amount of change
the amount of a function $f(x)$ over an interval $[x, x+h] is f(x+h) - f(x)$

average rate of change
is a function $f(x)$ over an interval $[x, x+h]$ is $\frac{f(x+h)-f(a)}{b-a}$

marginal cost
is the derivative of the cost function, or the approximate cost of producing one more item

marginal revenue
is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

marginal profit
is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

population growth rate
is the derivative of the population with respect to time

speed
is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time t whose velocity is given by $v(t)$

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3.6: Derivatives of Trigonometric Functions

Learning Objectives

- Find the derivatives of the sine and cosine function.
- Find the derivatives of the standard trigonometric functions.
- Calculate the higher-order derivatives of the sine and cosine.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Being able to calculate the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, for values of h very close to 0,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

We see that by using $h = 0.01$,

$$\frac{d}{dx}(\sin x) \approx \frac{\sin(x + 0.01) - \sin x}{0.01}$$

By setting

$$D(x) = \frac{\sin(x + 0.01) - \sin x}{0.01}$$

and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ (Figure 3.6.1).

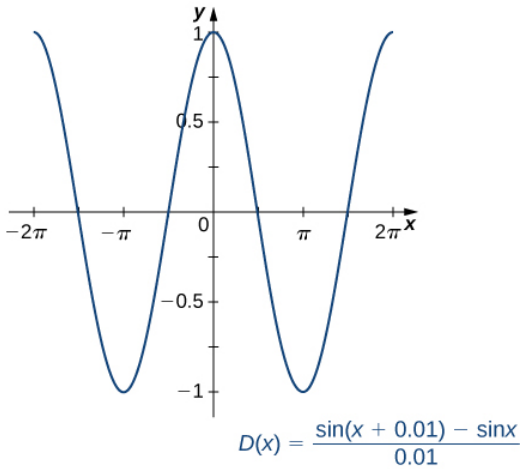


Figure 3.6.1: The graph of the function $D(x)$ looks a lot like a cosine curve.

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

The Derivatives of $\sin x$ and $\cos x$

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

$$\frac{d}{dx}(\sin x) = \cos x$$

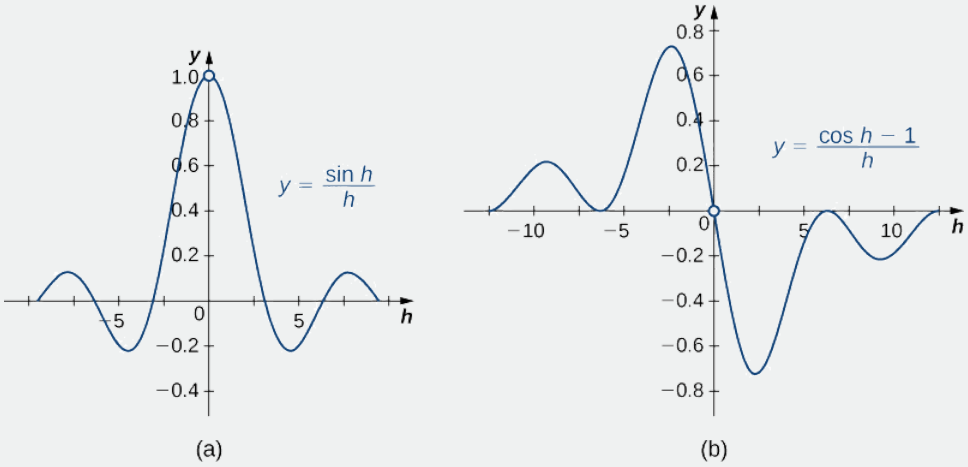
$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof

Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

The graphs of $y = \frac{\sin h}{h}$ and $y = \frac{\cos h - 1}{h}$ are shown in Figure 3.6.2.



(a) (b)

Figure 3.6.2: These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.

We also recall the following trigonometric identity for the sine of the sum of two angles:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.

$$
\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} && \text{Apply the definition of the derivative.}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} && \text{Use trig identity for the sine of the sum of two angles.}\\
&= \lim_{h\to 0}\left(\frac{\sin x\cos h-\sin x}{h}+\frac{\cos x\sin h}{h}\right) && \text{Regroup.}\\
&= \lim_{h\to 0}\left(\sin x\left(\frac{\cos h-1}{h}\right)+(\cos x)\left(\frac{\sin h}{h}\right)\right) && \text{Factor out } \sin x \text{ and } \cos x\\
&= (\sin x)\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)+(\cos x)\lim_{h\to 0}\left(\frac{\sin h}{h}\right) && \text{Factor } \sin x \text{ and } \cos x \text{ out of limits.}\\
&= (\sin x)(0)+(\cos x)(1) && \text{Apply trig limit formulas.}\\
&= \cos x && \text{Simplify.}
\end{aligned}
$$

□

Figure 3.6.3 shows the relationship between the graph of $f(x)=\sin x$ and its derivative $f'(x)=\cos x$. Notice that at the points where $f(x)=\sin x$ has a horizontal tangent, its derivative $f'(x)=\cos x$ takes on the value zero. We also see that where f$(x)=\sin x$ is increasing, $f'(x)=\cos x>0$ and where $f(x)=\sin x$ is decreasing, $f'(x)=\cos x<0$.

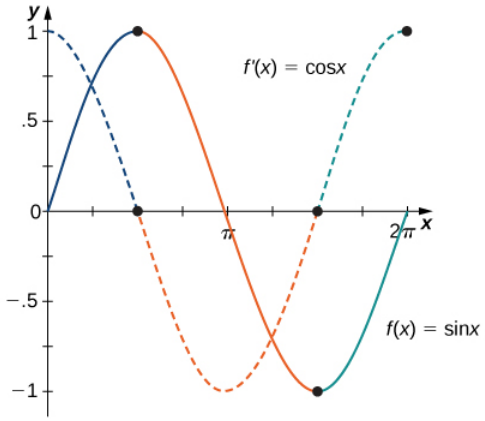


Figure 3.6.3: Where $f(x)$ has a maximum or a minimum, $f'(x)=0$ that is, $f'(x)=0$ where $f(x)$ has a horizontal tangent. These points are noted with dots on the graphs

Example 3.6.1: Differentiating a Function Containing $\sin x$

Find the derivative of $f(x)=5x^3\sin x$.

Solution

Using the product rule, we have

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(5x^3)\cdot\sin x+\frac{d}{dx}(\sin x)\cdot 5x^3\\
&= 15x^2\cdot\sin x+\cos x\cdot 5x^3.
\end{aligned}
$$

After simplifying, we obtain

$$f'(x)=15x^2\sin x+5x^3\cos x.$$

? Exercise 3.6.1

Find the derivative of $f(x) = \sin x \cos x$.

Hint

Don't forget to use the product rule.

Answer

$$f'(x) = \cos^2 x - \sin^2 x$$

✓ Example 3.6.2: Finding the Derivative of a Function Containing cos x

Find the derivative of $g(x) = \dfrac{\cos x}{4x^2}$.

Solution

By applying the quotient rule, we have

$$g'(x) = \frac{(-\sin x)4x^2 - 8x(\cos x)}{(4x^2)^2}.$$

Simplifying, we obtain

$$g'(x) = \frac{-4x^2 \sin x - 8x \cos x}{16x^4} = \frac{-x \sin x - 2 \cos x}{4x^3}.$$

? Exercise 3.6.2

Find the derivative of $f(x) = \dfrac{x}{\cos x}$.

Hint

Use the quotient rule.

Answer

$$f'(x) = \frac{\cos x + x \sin x}{\cos^2 x}$$

✓ Example 3.6.3: An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 \sin t - t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Solution

To determine when the particle is at rest, set $s'(t) = v(t) = 0$. Begin by finding $s'(t)$. We obtain

$$s'(t) = 2 \cos t - 1,$$

so we must solve

$$2 \cos t - 1 = 0 \text{ for } 0 \le t \le 2\pi.$$

The solutions to this equation are $t = \dfrac{\pi}{3}$ and $t = \dfrac{5\pi}{3}$. Thus the particle is at rest at times $t = \dfrac{\pi}{3}$ and $t = \dfrac{5\pi}{3}$.

? Exercise 3.6.3

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Hint

Use the previous example as a guide.

Answer

$t = \frac{\pi}{3}, \quad t = \frac{2\pi}{3}$

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the quotient rule to find formulas for their derivatives.

✓ Example 3.6.4: The Derivative of the Tangent Function

Find the derivative of $f(x) = \tan x$.

Solution

Start by expressing $\tan x$ as the quotient of $\sin x$ and $\cos x$:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Now apply the quotient rule to obtain

$$f'(x) = \frac{\cos x \cos x - (-\sin x)\sin x}{(\cos x)^2}.$$

Simplifying, we obtain

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Recognizing that $\cos^2 x + \sin^2 x = 1$, by the Pythagorean theorem, we now have

$$f'(x) = \frac{1}{\cos^2 x}$$

Finally, use the identity $\sec x = \frac{1}{\cos x}$ to obtain

$$f'(x) = \sec^2 x.$$

? Exercise 3.6.4

Find the derivative of $f(x) = \cot x$.

Hint

Rewrite $\cot x$ as $\frac{\cos x}{\sin x}$ and use the quotient rule.

Answer

$f'(x) = -\csc^2 x$

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The derivatives of the remaining trigonometric functions are as follows:

$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{3.6.1}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \tag{3.6.2}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \tag{3.6.3}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x. \tag{3.6.4}$$

Example 3.6.5: Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution

To find the equation of the tangent line, we need a point and a slope at that point. To find the point, compute

$$f\left(\tfrac{\pi}{4}\right) = \cot \tfrac{\pi}{4} = 1.$$

Thus the tangent line passes through the point $\left(\frac{\pi}{4}, 1\right)$. Next, find the slope by finding the derivative of $f(x) = \cot x$ and evaluating it at $\frac{\pi}{4}$:

$$f'(x) = -\csc^2 x \text{ and } f'\left(\tfrac{\pi}{4}\right) = -\csc^2\left(\tfrac{\pi}{4}\right) = -2.$$

Using the point-slope equation of the line, we obtain

$$y - 1 = -2\left(x - \tfrac{\pi}{4}\right)$$

or equivalently,

$$y = -2x + 1 + \tfrac{\pi}{2}.$$

Example 3.6.6: Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x \tan x$.

Solution

To find this derivative, we must use both the sum rule and the product rule. Using the sum rule, we find

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x \tan x).$$

In the first term, $\frac{d}{dx}(\csc x) = -\csc x \cot x$, and by applying the product rule to the second term we obtain

$$\frac{d}{dx}(x \tan x) = (1)(\tan x) + (\sec^2 x)(x).$$

Therefore, we have

$$f'(x) = -\csc x \cot x + \tan x + x \sec^2 x.$$

Exercise 3.6.5

Find the derivative of $f(x) = 2 \tan x - 3 \cot x$.

Hint

Use the rule for differentiating a constant multiple and the rule for differentiating a difference of two functions.

Answer

$f'(x) = 2\sec^2 x + 3\csc^2 x$

? Exercise 3.6.6

Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Hint

Evaluate the derivative at $x = \frac{\pi}{6}$.

Answer

$\frac{4}{3}$

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

✓ Example 3.6.7: Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

Solution

Each step in the chain is straightforward:

$$\begin{aligned} y &= \sin x \\ \frac{dy}{dx} &= \cos x \\ \frac{d^2y}{dx^2} &= -\sin x \\ \frac{d^3y}{dx^3} &= -\cos x \\ \frac{d^4y}{dx^4} &= \sin x \end{aligned}$$

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of $\sin x$ equals $\sin x$, so

$$\frac{d^4}{dx^4}(\sin x) = \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \ldots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x$$

$$\frac{d^5}{dx^5}(\sin x) = \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \ldots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x.$$

? Exercise 3.6.7

For $y = \cos x$, find $\frac{d^4y}{dx^4}$.

Hint

See the previous example.

Answer

$\cos x$

Example 3.6.8: Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution

We can see right away that for the 74th derivative of $\sin x$, $74 = 4(18) + 2$, so

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

Exercise 3.6.8

For $y = \sin x$, find $\frac{d^{59}}{dx^{59}}(\sin x)$.

Hint

$$\frac{d^{59}}{dx^{59}}(\sin x) = \frac{d^{4 \cdot 14+3}}{dx^{4 \cdot 14+3}}(\sin x)$$

Answer

$-\cos x$

Example 3.6.9: An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

Solution

First find $v(t) = s'(t)$

$$v(t) = s'(t) = -\cos t.$$

Thus,

$$v\left(\tfrac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Next, find $a(t) = v'(t)$. Thus, $a(t) = v'(t) = \sin t$ and we have

$$a\left(\tfrac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Since $v\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} < 0$ and $a\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} > 0$, we see that velocity and acceleration are acting in opposite directions; that is, the object is being accelerated in the direction opposite to the direction in which it is traveling. Consequently, the particle is slowing down.

Exercise 3.6.9

A block attached to a spring is moving vertically. Its position at time t is given by $s(t) = 2 \sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

Hint

Use Example **3.6.9** as a guide.

Answer

$v\left(\frac{5\pi}{6}\right) = -\sqrt{3} < 0$ and $a\left(\frac{5\pi}{6}\right) = -1 < 0$. The block is speeding up.

Key Concepts

- We can find the derivatives of $\sin x$ and $\cos x$ by using the definition of derivative and the limit formulas found earlier. The results are

  $$\frac{d}{dx}\big(\sin x\big) = \cos x \quad \text{and} \quad \frac{d}{dx}\big(\cos x\big) = -\sin x.$$

- With these two formulas, we can determine the derivatives of all six basic trigonometric functions.

Key Equations

- **Derivative of sine function**

  $$\frac{d}{dx}(\sin x) = \cos x$$

- **Derivative of cosine function**

  $$\frac{d}{dx}(\cos x) = -\sin x$$

- **Derivative of tangent function**

  $$\frac{d}{dx}(\tan x) = \sec^2 x$$

- **Derivative of cotangent function**

  $$\frac{d}{dx}(\cot x) = -\csc^2 x$$

- **Derivative of secant function**

  $$\frac{d}{dx}(\sec x) = \sec x \tan x$$

- **Derivative of cosecant function**

  $$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

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3.6E: Exercises

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3.7: The Chain Rule

Learning Objectives

- State the chain rule for the composition of two functions.
- Apply the chain rule together with the power rule.
- Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
- Recognize the chain rule for a composition of three or more functions.
- Describe the proof of the chain rule.

We have seen the techniques for differentiating basic functions ($x^n, \sin x, \cos x$, etc.) as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x) = \sin(x^3)$ or $k(x) = \sqrt{3x^2 + 1}$. In this section, we study the rule for finding the derivative of the composition of two or more functions.

Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the *chain rule*, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x) = \sin(x^3)$. We can think of the derivative of this function with respect to x as the rate of change of $\sin(x^3)$ relative to the change in x. Consequently, we want to know how $\sin(x^3)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin(x^3)$. This chain reaction gives us hints as to what is involved in computing the derivative of $\sin(x^3)$. First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin(x^3)$ suggests that the derivative of $\sin(u)$ with respect to u, where $u = x^3$, is also part of the final derivative.

We can take a more formal look at the derivative of $h(x) = \sin(x^3)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x) = \sin(x^3)$.

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x - a}$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression $x^3 - a^3$ to obtain

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} \cdot \frac{x^3 - a^3}{x - a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x = a$. That is,

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \frac{d}{dx}(x^3)\Big|_{x=a} = 3a^2.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u = x^3$ and observing that as $x \to a, u \to a^3$:

$$\begin{aligned} \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} &= \lim_{u \to a^3} \frac{\sin u - \sin(a^3)}{u - a^3} \\ &= \frac{d}{du}(\sin u)\Big|_{u=a^3} \\ &= \cos(a^3) \end{aligned}.$$

Thus, $h'(a) = \cos(a^3) \cdot 3a^2$.

In other words, if $h(x) = \sin(x^3)$, then $h'(x) = \cos(x^3) \cdot 3x^2$. Thus, if we think of $h(x) = \sin(x^3)$ as the composition $(f \circ g)(x) = f\big(g(x)\big)$ where $f(x) = \sin x$ and $g(x) = x^3$, then the derivative of $h(x) = \sin(x^3)$ is the product of the derivative of $g(x) = x^3$ and the derivative of the function $f(x) = \sin x$ evaluated at the function $g(x) = x^3$. At this point, we anticipate that for $h(x) = \sin\big(g(x)\big)$, it is quite likely that $h'(x) = \cos\big(g(x)\big)g'(x)$. As we determined above, this is the case for $h(x) = \sin(x^3)$.

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f\big(g(x)\big)$$

is given by

$$h'(x) = f'\big(g(x)\big) \cdot g'(x).$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h(x) = f\big(g(x)\big)$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'\big(g(x)\big)$.
3. Find $g'(x)$.
4. Write $h'(x) = f'\big(g(x)\big) \cdot g'(x)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = \big(g(x)\big)^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x) = \big(g(x)\big)^n$ as $f\big(g(x)\big)$ where $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Thus, $f'\big(g(x)\big) = n\big(g(x)\big)^{n-1}$. This leads us to the derivative of a power function using the chain rule,

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x)$$

Rule: Power Rule for Composition of Functions (General Power Rule)

For all values of x for which the derivative is defined, if

$$h(x) = \big(g(x)\big)^n,$$

Then

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x). \tag{3.7.1}$$

Example 3.7.1: Using the Chain and Power Rules

Find the derivative of $h(x) = \dfrac{1}{(3x^2+1)^2}$.

Solution

First, rewrite $h(x) = \dfrac{1}{(3x^2+1)^2} = (3x^2+1)^{-2}$.

Applying the power rule with $g(x) = 3x^2+1$, we have

$$h'(x) = -2(3x^2+1)^{-3} \cdot 6x.$$

Rewriting back to the original form gives us

$$h'(x) = \frac{-12x}{(3x^2+1)^3}$$

Exercise 3.7.1

Find the derivative of $h(x) = (2x^3+2x-1)^4$.

Hint

Use the General Power Rule (Equation 3.7.1) with $g(x) = 2x^3+2x-1$.

Answer

$h'(x) = 4(2x^3+2x-1)^3(6x^2+2) = 8(3x^2+1)(2x^3+2x-1)^3$

Example 3.7.2: Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

Solution

First recall that $\sin^3 x = (\sin x)^3$, so we can rewrite $h(x) = \sin^3 x$ as $h(x) = (\sin x)^3$.

Applying the power rule with $g(x) = \sin x$, we obtain

$$h'(x) = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x.$$

Example 3.7.3: Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \dfrac{1}{(3x-5)^2}$ at $x = 2$.

Solution

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2) = \dfrac{1}{(3(2)-5)^2} = 1$, the point is $(2, 1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x) = (3x-5)^{-2}$ and apply the power rule to obtain

$$h'(x) = -2(3x-5)^{-3}(3) = -6(3x-5)^{-3}.$$

By substituting, we have $h'(2) = -6(3(2)-5)^{-3} = -6$.

Therefore, the line has equation $y - 1 = -6(x-2)$. Rewriting, the equation of the line is $y = -6x + 13$.

? Exercise 3.7.2

Find the equation of the line tangent to the graph of $f(x) = (x^2 - 2)^3$ at $x = -2$.

Hint

Use the preceding example as a guide.

Answer

$y = -48x - 88$

Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the product rule.

✓ Example 3.7.4: Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos\big(g(x)\big)$.

Solution

Think of $h(x) = \cos\big(g(x)\big)$ as $f\big(g(x)\big)$ where $f(x) = \cos x$. Since $f'(x) = -\sin x$, we have $f'\big(g(x)\big) = -\sin\big(g(x)\big)$. Then we do the following calculation.

$$\begin{aligned} h'(x) &= f'\big(g(x)\big) \cdot g'(x) && \text{Apply the chain rule.} \\ &= -\sin\big(g(x)\big) \cdot g'(x) && \text{Substitute } f'\big(g(x)\big) = -\sin\big(g(x)\big). \end{aligned}$$

Thus, the derivative of $h(x) = \cos\big(g(x)\big)$ is given by $h'(x) = -\sin\big(g(x)\big) \cdot g'(x)$.

In the following example we apply the rule that we have just derived.

✓ Example 3.7.5: Using the Chain Rule on a Cosine Function

Find the derivative of $h(x) = \cos(5x^2)$.

Solution

Let $g(x) = 5x^2$. Then $g'(x) = 10x$. Using the result from the previous example,

$$h'(x) = -\sin(5x^2) \cdot 10x = -10x \sin(5x^2)$$

✓ Example 3.7.6: Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x) = \sec(4x^5 + 2x)$.

Solution

Apply the chain rule to $h(x) = \sec\big(g(x)\big)$ to obtain

$$h'(x) = \sec(g(x)) \tan\big(g(x)\big) \cdot g'(x).$$

In this problem, $g(x) = 4x^5 + 2x$, so we have $g'(x) = 20x^4 + 2$. Therefore, we obtain

$$h'(x) = \sec(4x^5 + 2x) \tan(4x^5 + 2x)(20x^4 + 2) = (20x^4 + 2)\sec(4x^5 + 2x) \tan(4x^5 + 2x).$$

? Exercise 3.7.3

Find the derivative of $h(x) = \sin(7x + 2)$.

Hint

Apply the chain rule to $h(x) = \sin\big(g(x)\big)$ first and then use $g(x) = 7x + 2$.

Answer

$h'(x) = 7\cos(7x + 2)$

At this point we provide a list of derivative formulas that may be obtained by applying the chain rule in conjunction with the formulas for derivatives of trigonometric functions. Their derivations are similar to those used in the examples above. For convenience, formulas are also given in Leibniz's notation, which some students find easier to remember. (We discuss the chain rule using Leibniz's notation at the end of this section.) It is not absolutely necessary to memorize these as separate formulas as they are all applications of the chain rule to previously learned formulas.

Using the Chain Rule with Trigonometric Functions

For all values of x for which the derivative is defined,

$\frac{d}{dx}\Big(\sin(g(x))\Big) = \cos(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\sin u\Big) = \cos u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\cos(g(x))\Big) = -\sin(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\cos u\Big) = -\sin u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\tan(g(x))\Big) = \sec^2(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\tan u\Big) = \sec^2 u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\cot(g(x))\Big) = -\csc^2(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\cot u\Big) = -\csc^2 u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\sec(g(x))\Big) = \sec(g(x))\tan(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\sec u\Big) = \sec u \tan u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\csc(g(x))\Big) = -\csc(g(x))\cot(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\csc u\Big) = -\csc u \cot u \cdot \frac{du}{dx}.$

Example 3.7.7: Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x + 1)^5(3x - 2)^7$.

Solution

First apply the product rule, then apply the chain rule to each term of the product.

$$
\begin{aligned}
h'(x) &= \frac{d}{dx}\big((2x+1)^5\big) \cdot (3x-2)^7 + \frac{d}{dx}\big((3x-2)^7\big) \cdot (2x+1)^5 && \text{Apply the product rule.} \\
&= 5(2x+1)^4 \cdot 2 \cdot (3x-2)^7 + 7(3x-2)^6 \cdot 3 \cdot (2x+1)^5 && \text{Apply the chain rule.} \\
&= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{Simplify.} \\
&= (2x+1)^4(3x-2)^6(10(3x-2) + 21(2x+1)) && \text{Factor out } (2x+1)^4(3x-2)^6 \\
&= (2x+1)^4(3x-2)^6(72x+1) && \text{Simplify.}
\end{aligned}
$$

Exercise 3.7.4

Find the derivative of $h(x) = \dfrac{x}{(2x+3)^3}$.

Hint

Start out by applying the quotient rule. Remember to use the chain rule to differentiate the denominator.

Answer

$h'(x) = \dfrac{3 - 4x}{(2x+3)^4}$

Composites of Three or More Functions

We can now combine the chain rule with other rules for differentiating functions, but when we are differentiating the composition of three or more functions, we need to apply the chain rule more than once. If we look at this situation in general terms, we can generate a formula, but we do not need to remember it, as we can simply apply the chain rule multiple times.

In general terms, first we let

$$k(x) = h\Big(f\big(g(x)\big)\Big).$$

Then, applying the chain rule once we obtain

$$k'(x) = \frac{d}{dx}\Big(h\big(f\big(g(x)\big)\big)\Big) = h'\Big(f\big(g(x)\big)\Big) \cdot \frac{d}{dx}\Big(f\big(g(x)\big)\Big).$$

Applying the chain rule again, we obtain

$$k'(x) = h'\Big(f\big(g(x)\big)\Big) \cdot f'\big(g(x)\big) \cdot g'(x).$$

Rule: Chain Rule for a Composition of Three Functions

For all values of x for which the function is differentiable, if

$$k(x) = h\Big(f\big(g(x)\big)\Big),$$

then

$$k'(x) = h'\Big(f\big(g(x)\big)\Big) \cdot f'\big(g(x)\big) \cdot g'(x).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

Example 3.7.8: Differentiating a Composite of Three Functions

Find the derivative of $k(x) = \cos^4(7x^2 + 1)$.

Solution

First, rewrite $k(x)$ as

$$k(x) = \big(\cos(7x^2 + 1)\big)^4.$$

Then apply the chain rule several times.

$$\begin{aligned} k'(x) &= 4(\cos(7x^2+1))^3 \cdot \frac{d}{dx}\big(\cos(7x^2+1)\big) && \text{Apply the chain rule.} \\ &= 4(\cos(7x^2+1))^3(-\sin(7x^2+1)) \cdot \frac{d}{dx}\big(7x^2+1\big) && \text{Apply the chain rule.} \\ &= 4(\cos(7x^2+1))^3(-\sin(7x^2+1))(14x) && \text{Apply the chain rule.} \\ &= -56x\sin(7x^2+1)\cos^3(7x^2+1) && \text{Simplify} \end{aligned}$$

Exercise 3.7.5

Find the derivative of $h(x) = \sin^6(x^3)$.

Hint

Rewrite $h(x) = \sin^6(x^3) = \big(\sin(x^3)\big)^6$ and use Example 3.7.8 as a guide.

Answer

$h'(x) = 18x^2 \sin^5(x^3)\cos(x^3)$

Example 3.7.9: Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(2t) + \cos(3t)$. What is the velocity of the particle at time $t = \frac{\pi}{6}$?

Solution

To find $v(t)$, the velocity of the particle at time t, we must differentiate $s(t)$. Thus,

$$v(t) = s'(t) = 2\cos(2t) - 3\sin(3t).$$

Proof of Chain Rule

At this point, we present a very informal proof of the chain rule. For simplicity's sake we ignore certain issues: For example, we assume that $g(x) \neq g(a)$ for $x \neq a$ in some open interval containing a. We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:

$$h'(a) = \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{x - a}.$$

Rewriting, we obtain

$$h'(a) = \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}.$$

Although it is clear that

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a),$$

it is not obvious that

$$\lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} = f'\big(g(a)\big).$$

To see that this is true, first recall that since g is differentiable at a, g is also continuous at a. Thus,

$$\lim_{x \to a} g(x) = g(a).$$

Next, make the substitution $y = g(x)$and $b = g(a)$ and use change of variables in the limit to obtain

$$\lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'\big(g(a)\big).$$

Finally,

$$h'(a) = \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} = f'\big(g(a)\big) \cdot g'(a).$$

□

Example 3.7.10: Using the Chain Rule with Functional Values

Let $h(x) = f\big(g(x)\big)$. If $g(1) = 4, g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

Solution

Use the chain rule, then substitute.

$$\begin{aligned} h'(1) &= f'\big(g(1)\big) \cdot g'(1) && \text{Apply the chain rule.} \\ &= f'(4) \cdot 3 && \text{Substitute } g(1) = 4 \text{ and } g'(1) = 3. \\ &= 7 \cdot 3 && \text{Substitute } f'(4) = 7. \\ &= 21 && \text{Simplify.} \end{aligned}$$

? Exercise 3.7.6

Given $h(x) = f(g(x))$. If $g(2) = -3, g'(2) = 4$, and $f'(-3) = 7$, find $h'(2)$.

Hint

Follow Example 3.7.10.

Answer

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The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This notation for the chain rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Thus,

$$h'(x) = \frac{dy}{dx}$$

$$f'(g(x)) = f'(u) = \frac{dy}{du}$$

and

$$g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'\big(g(x)\big) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

✓ Example 3.7.11: Taking a Derivative Using Leibniz's Notation I

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the quotient rule,

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and

$$\frac{dy}{du} = 5u^4.$$

Finally, we put it all together.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} && \text{Apply the chain rule.}\\
&= 5u^4\cdot\frac{2}{(3x+2)^2} && \text{Substitute } \frac{dy}{du}=5u^4 \text{ and } \frac{du}{dx}=\frac{2}{(3x+2)^2}.\\
&= 5\left(\frac{x}{3x+2}\right)^4\cdot\frac{2}{(3x+2)^2} && \text{Substitute } u=\frac{x}{3x+2}.\\
&= \frac{10x^4}{(3x+2)^6} && \text{Simplify.}
\end{aligned}$$

It is important to remember that, when using the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Example 3.7.12: Taking a Derivative Using Leibniz's Notation II

Find the derivative of $y=\tan(4x^2-3x+1)$.

Solution

First, let $u=4x^2-3x+1$. Then $y=\tan u$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$:

$$\frac{du}{dx}=8x-3 \text{ and } \frac{dy}{du}=\sec^2 u.$$

Finally, we put it all together.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} && \text{Apply the chain rule.}\\
&= \sec^2 u\cdot(8x-3) && \text{Use } \frac{du}{dx}=8x-3 \text{ and } \frac{dy}{du}=\sec^2 u.\\
&= \sec^2(4x^2-3x+1)\cdot(8x-3) && \text{Substitute } u=4x^2-3x+1.
\end{aligned}$$

Exercise 3.7.7

Use Leibniz's notation to find the derivative of $y=\cos(x^3)$. Make sure that the final answer is expressed entirely in terms of the variable x.

Hint

Let $u=x^3$.

Answer

$\frac{dy}{dx}=-3x^2\sin(x^3)$.

Key Concepts

- The chain rule allows us to differentiate compositions of two or more functions. It states that for $h(x)=f\big(g(x)\big)$,

$$h'(x)=f'\big(g(x)\big)\cdot g'(x).$$

  In Leibniz's notation this rule takes the form

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

- We can use the chain rule with other rules that we have learned, and we can derive formulas for some of them.
- The chain rule combines with the power rule to form a new rule:

$$\text{If } h(x)=\big(g(x)\big)^n, \text{ then } h'(x)=n\big(g(x)\big)^{n-1}\cdot g'(x).$$

- When applied to the composition of three functions, the chain rule can be expressed as follows: If $h(x)=f\big(g\big(k(x)\big)\big)$, then $h'(x)=f'\big(g\big(k(x)\big)\big)\cdot g'\big(k(x)\big)\cdot k'(x)$.

Key Equations

- **The chain rule**

$$h'(x) = f'\big(g(x)\big) \cdot g'(x)$$

- **The power rule for functions**

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x)$$

Glossary

chain rule

the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

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3.8: Derivatives of Inverse Functions

Learning Objectives

- Calculate the derivative of an inverse function.
- Recognize the derivatives of the standard inverse trigonometric functions.

In this section we explore the relationship between the derivative of a function and the derivative of its inverse. For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without having to use the limit definition of the derivative. In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions. This formula may also be used to extend the power rule to rational exponents.

The Derivative of an Inverse Function

We begin by considering a function and its inverse. If $f(x)$ is both invertible and differentiable, it seems reasonable that the inverse of $f(x)$ is also differentiable. Figure **3.8.1** shows the relationship between a function $f(x)$ and its inverse $f^{-1}(x)$. Look at the point $(a,\ f^{-1}(a))$ on the graph of $f^{-1}(x)$ having a tangent line with a slope of

$$\left(f^{-1}\right)'(a) = \frac{p}{q}.$$

This point corresponds to a point $(f^{-1}(a),\ a)$ on the graph of $f(x)$ having a tangent line with a slope of

$$f'\left(f^{-1}(a)\right) = \frac{q}{p}.$$

Thus, if $f^{-1}(x)$ is differentiable at a, then it must be the case that

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

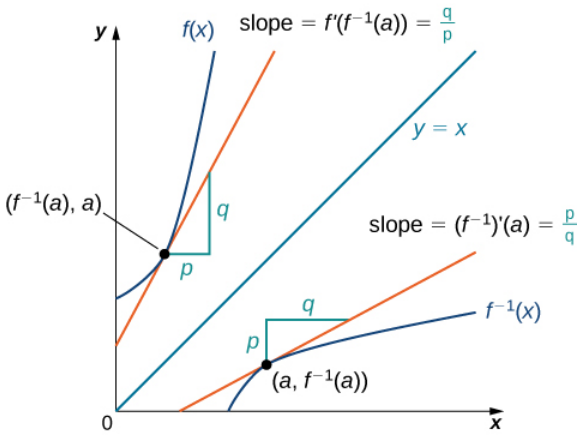


Figure **3.8.1**:The tangent lines of a function and its inverse are related; so, too, are the derivatives of these functions.

We may also derive the formula for the derivative of the inverse by first recalling that $x = f\left(f^{-1}(x)\right)$. Then by differentiating both sides of this equation (using the chain rule on the right), we obtain

$$1 = f'\left(f^{-1}(x)\right)\left(f^{-1}\right)'(x)).$$

Solving for $\left(f^{-1}\right)'(x)$, we obtain

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

We summarize this result in the following theorem.

Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\left(f^{-1}(x)\right) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}. \tag{3.8.1}$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'\left(g(x)\right)}. \tag{3.8.2}$$

Example 3.8.1: Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \dfrac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Solution

The inverse of $g(x) = \dfrac{x+2}{x}$ is $f(x) = \dfrac{2}{x-1}$.

We will use Equation 3.8.2 and begin by finding $f'(x)$. Thus,

$$f'(x) = \frac{-2}{(x-1)^2}$$

and

$$f'\left(g(x)\right) = \frac{-2}{(g(x)-1)^2} = \frac{-2}{\left(\frac{x+2}{x}-1\right)^2} = -\frac{x^2}{2}.$$

Finally,

$$g'(x) = \frac{1}{f'\left(g(x)\right)} = -\frac{2}{x^2}.$$

We can verify that this is the correct derivative by applying the quotient rule to $g(x)$ to obtain

$$g'(x) = -\frac{2}{x^2}.$$

Exercise 3.8.1

Use the inverse function theorem to find the derivative of $g(x) = \dfrac{1}{x+2}$. Compare the result obtained by differentiating $g(x)$ directly.

Hint

Use the preceding example as a guide.

Answer

$$g'(x) = -\frac{1}{(x+2)^2}$$

Example 3.8.2: Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \sqrt[3]{x}$.

Solution

The function $g(x) = \sqrt[3]{x}$ is the inverse of the function $f(x) = x^3$. Since $g'(x) = \dfrac{1}{f'\big(g(x)\big)}$, begin by finding $f'(x)$. Thus,

$$f'(x) = 3x^2$$

and

$$f'\big(g(x)\big) = 3\big(\sqrt[3]{x}\big)^2 = 3x^{2/3}$$

Finally,

$$g'(x) = \frac{1}{3x^{2/3}}.$$

If we were to differentiate $g(x)$ directly, using the power rule, we would first rewrite $g(x) = \sqrt[3]{x}$ as a power of x to get,

$$g(x) = x^{1/3}$$

Then we would differentiate using the power rule to obtain

$$g'(x) = \tfrac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}.$$

Exercise 3.8.2

Find the derivative of $g(x) = \sqrt[5]{x}$ by applying the inverse function theorem.

Hint

$g(x)$ is the inverse of $f(x) = x^5$.

Answer

$g(x) = \tfrac{1}{5}x^{-4/5}$

From the previous example, we see that we can use the inverse function theorem to extend the power rule to exponents of the form $\dfrac{1}{n}$, where n is a positive integer. This extension will ultimately allow us to differentiate x^q, where q is any rational number.

Extending the Power Rule to Rational Exponents

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n)-1}.$$

Also, if n is a positive integer and m is an arbitrary integer, then

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}.$$

Proof

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$. Since $g'(x) = \dfrac{1}{f'\big(g(x)\big)}$, begin by finding $f'(x)$. Thus,

$$f'(x) = nx^{n-1} \text{ and } f'\big(g(x)\big) = n\left(x^{1/n}\right)^{n-1} = nx^{(n-1)/n}.$$

Finally,

$$g'(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{(1-n)/n} = \frac{1}{n}x^{(1/n)-1}.$$

To differentiate $x^{m/n}$ we must rewrite it as $(x^{1/n})^m$ and apply the chain rule. Thus,

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{d}{dx}\left(\left(x^{1/n}\right)^m\right) = m\left(x^{1/n}\right)^{m-1} \cdot \frac{1}{n}x^{(1/n)-1} = \frac{m}{n}x^{(m/n)-1}.$$

□

Example 3.8.3: Applying the Power Rule to a Rational Power

Find the equation of the line tangent to the graph of $y = x^{2/3}$ at $x = 8$.

Solution

First find $\frac{dy}{dx}$ and evaluate it at $x = 8$. Since

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3}$$

and

$$\left.\frac{dy}{dx}\right|_{x=8} = \frac{1}{3}$$

the slope of the tangent line to the graph at $x = 8$ is $\frac{1}{3}$.

Substituting $x = 8$ into the original function, we obtain $y = 4$. Thus, the tangent line passes through the point $(8, 4)$. Substituting into the point-slope formula for a line, we obtain the tangent line

$$y = \tfrac{1}{3}x + \tfrac{4}{3}.$$

Exercise 3.8.3

Find the derivative of $s(t) = \sqrt{2t+1}$.

Hint

Use the chain rule.

Answer

$s'(t) = (2t+1)^{-1/2}$

Derivatives of Inverse Trigonometric Functions

We now turn our attention to finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text. The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually algebraic functions. Previously, derivatives of algebraic functions have proven to be algebraic functions and derivatives of trigonometric functions have been shown to be trigonometric functions. Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

Example 3.8.4A: Derivative of the Inverse Sine Function

Use the inverse function theorem to find the derivative of $g(x) = \sin^{-1} x$.

Solution

Since for x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin x$ is the inverse of $g(x) = \sin^{-1} x$, begin by finding $f'(x)$. Since

$$f'(x) = \cos x$$

and

$$f'\big(g(x)\big) = \cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$$

we see that

$$g'(x) = \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{f'\big(g(x)\big)} = \frac{1}{\sqrt{1-x^2}}$$

Analysis

To see that $\cos(\sin^{-1} x) = \sqrt{1-x^2}$, consider the following argument. Set $\sin^{-1} x = \theta$. In this case, $\sin\theta = x$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. We begin by considering the case where $0 < \theta < \frac{\pi}{2}$. Since θ is an acute angle, we may construct a right triangle having acute angle θ, a hypotenuse of length 1 and the side opposite angle θ having length x. From the Pythagorean theorem, the side adjacent to angle θ has length $\sqrt{1-x^2}$. This triangle is shown in Figure 3.8.2 Using the triangle, we see that $\cos(\sin^{-1} x) = \cos\theta = \sqrt{1-x^2}$.

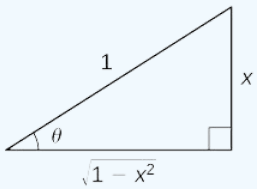


Figure 3.8.2: Using a right triangle having acute angle θ, a hypotenuse of length 1, and the side opposite angle θ having length x, we can see that $\cos(\sin^{-1} x) = \cos\theta = \sqrt{1-x^2}$.

In the case where $-\frac{\pi}{2} < \theta < 0$, we make the observation that $0 < -\theta < \frac{\pi}{2}$ and hence

$$\cos\left(\sin^{-1} x\right) = \cos\theta = \cos(-\theta) = \sqrt{1-x^2}.$$

Now if $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}, x = 1$ or $x = -1$, and since in either case $\cos\theta = 0$ and $\sqrt{1-x^2} = 0$, we have

$$\cos\left(\sin^{-1} x\right) = \cos\theta = \sqrt{1-x^2}.$$

Consequently, in all cases,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}.$$

✔ Example 3.8.4B: Applying the Chain Rule to the Inverse Sine Function

Apply the chain rule to the formula derived in Example 3.8.4A to find the derivative of $h(x) = \sin^{-1}\big(g(x)\big)$ and use this result to find the derivative of $h(x) = \sin^{-1}(2x^3)$.

Solution

Applying the chain rule to $h(x) = \sin^{-1}\big(g(x)\big)$, we have

$$h'(x) = \frac{1}{\sqrt{1-\big(g(x)\big)^2}} g'(x).$$

Now let $g(x) = 2x^3$, so $g'(x) = 6x^2$. Substituting into the previous result, we obtain

$$\begin{aligned} h'(x) &= \frac{1}{\sqrt{1-4x^6}} \cdot 6x^2 \\ &= \frac{6x^2}{\sqrt{1-4x^6}} \end{aligned}$$

? Exercise 3.8.4

Use the inverse function theorem to find the derivative of $g(x) = \tan^{-1} x$.

Hint

The inverse of $g(x)$ is $f(x) = \tan x$. Use Example 3.8.4A as a guide.

Answer

$$g'(x) = \frac{1}{1+x^2}$$

The derivatives of the remaining inverse trigonometric functions may also be found by using the inverse function theorem. These formulas are provided in the following theorem.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \tag{3.8.3}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}} \tag{3.8.4}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \tag{3.8.5}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2} \tag{3.8.6}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}} \tag{3.8.7}$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}} \tag{3.8.8}$$

Example 3.8.5A: Applying Differentiation Formulas to an Inverse Tangent Function

Find the derivative of $f(x) = \tan^{-1}(x^2)$.

Solution

Let $g(x) = x^2$, so $g'(x) = 2x$. Substituting into Equation 3.8.5, we obtain

$$f'(x) = \frac{1}{1+(x^2)^2} \cdot (2x).$$

Simplifying, we have

$$f'(x) = \frac{2x}{1+x^4}.$$

Example 3.8.5B: Applying Differentiation Formulas to an Inverse Sine Function

Find the derivative of $h(x) = x^2 \sin^{-1} x$.

Solution

By applying the product rule, we have

$$h'(x) = 2x \sin^{-1} x + \frac{1}{\sqrt{1-x^2}} \cdot x^2$$

Exercise 3.8.5

Find the derivative of $h(x) = \cos^{-1}(3x-1)$.

Hint

Use Equation 3.8.4. with $g(x) = 3x - 1$

Answer

$$h'(x) = \frac{-3}{\sqrt{6x - 9x^2}}$$

Example 3.8.6: Applying the Inverse Tangent Function

The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

Solution

Begin by differentiating $s(t)$ in order to find $v(t)$.Thus,

$$v(t) = s'(t) = \frac{1}{1 + \left(\frac{1}{t}\right)^2} \cdot \frac{-1}{t^2}.$$

Simplifying, we have

$$v(t) = -\frac{1}{t^2 + 1}.$$

Thus, $v(1) = -\frac{1}{2}$.

Exercise 3.8.6

Find the equation of the line tangent to the graph of $f(x) = \sin^{-1} x$ at $x = 0$.

Hint

$f'(0)$ is the slope of the tangent line.

Answer

$y = x$

Key Concepts

- The inverse function theorem allows us to compute derivatives of inverse functions without using the limit definition of the derivative.
- We can use the inverse function theorem to develop differentiation formulas for the inverse trigonometric functions.

Key Equations

- **Inverse function theorem**

  $(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$ whenever $f'\left(f^{-1}(x)\right) \neq 0$ and $f(x)$ is differentiable.

- **Power rule with rational exponents**

  $\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}$.

- **Derivative of inverse sine function**

  $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$

- **Derivative of inverse cosine function**

  $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1 - x^2}}$

Derivative of inverse tangent function

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2}$$

Derivative of inverse cotangent function

$$\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$$

Derivative of inverse secant function

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}}$$

Derivative of inverse cosecant function

$$\frac{d}{dx}\left(\csc^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}$$

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) added the second half of Example **3.8.2**.

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3.11: Chapter 3 Review Exercises

Chapter Review Exercises

True or False? Justify the answer with a proof or a counterexample.

1) Every function has a derivative.

Answer

False

2) A continuous function has a continuous derivative.

3) A continuous function has a derivative.

Answer

False

4) If a function is differentiable, it is continuous.

In exercises 5 and 6, use the limit definition of the derivative to exactly evaluate the derivative.

5) $f(x) = \sqrt{x+4}$

Answer

$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

6) $f(x) = \frac{3}{x}$

In exercises 7 - 15, find the derivatives of the given functions.

7) $f(x) = 3x^3 - \frac{4}{x^2}$

Answer

$f'(x) = 9x^2 + \frac{8}{x^3}$

9) $f(x) = (4 - x^2)^3$

10) $f(x) = e^{\sin x}$

Answer

$f'(x) = e^{\sin x} \cos x$

11) $f(x) = \ln(x+2)$

12) $f(x) = x^2 \cos x + x \tan x$

Answer

$f'(x) = x \sec^2 x + 2x \cos x + \tan x - x^2 \sin x$

13) $f(x) = \sqrt{3x^2+2}$

14) $f(x) = \frac{x}{4} \sin^{-1}(x)$

Answer

$f'(x) = \frac{1}{4}\left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1} x\right)$

15) $x^2 y = (y+2) + xy \sin x$

In exercises 16 - 18, find the indicated derivatives of various orders.

16) First derivative of $y = x(\ln x)\cos x$

Answer

$$\frac{dy}{dx} = \cos x \cdot (\ln x + 1) - x(\ln x)\sin x$$

17) Third derivative of $y = (3x + 2)^2$

18) Second derivative of $y = 4^x + x^2 \sin x$

Answer

$$\frac{d^2y}{dx^2} = 4^x(\ln 4)^2 + 2\sin x + 4x\cos x - x^2 \sin x$$

In exercises 19 and 20, find the equation of the tangent line to the following equations at the specified point.

19) $y = \cos^{-1}(x) + x$ at $x = 0$

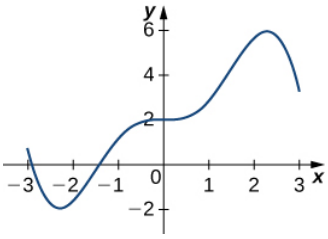
20) $y = x + e^x - \frac{1}{x}$ at $x = 1$

Answer

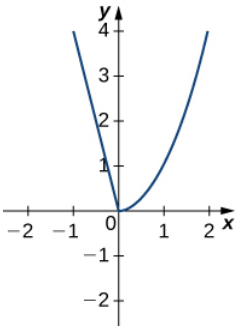
$$y = (2 + e)x - 2$$

In exercises 21 and 22, draw the derivative of the functions with the given graphs.

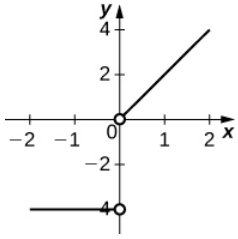
21)



22)



Answer



Questions 22 and 23 concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t) = 1.9 + 2.9\cos(\frac{\pi}{6}t)$, where t is measured in hours after midnight, and the height is measured in feet.

22) Find and graph the derivative. What is the physical meaning?

23) Find $w'(3)$. What is the physical meaning of this value?

Answer

$w'(3) = -\frac{2.9\pi}{6}$. At 3 a.m. the tide is decreasing at a rate of 1.514 ft/hr.

Questions 24 and 25 consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

Wind Speeds of Hurricane KatrinaSource: news.nationalgeographic.com/n..._timeline.html.

24) Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

25) Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Answer

-7.5. The wind speed is decreasing at a rate of 7.5 mph/hr

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4: Applications of Derivatives

Page ID

77087

4.1: Prelude to Applications of Derivatives

A rocket is being launched from the ground and cameras are recording the event. A video camera is located on the ground a certain distance from the launch pad. At what rate should the angle of inclination (the angle the camera makes with the ground) change to allow the camera to record the flight of the rocket as it heads upward?



Figure **4.1.1**: As a rocket is being launched, at what rate should the angle of a video camera change to continue viewing the rocket? (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

A rocket launch involves two related quantities that change over time. Being able to solve this type of problem is just one application of derivatives introduced in this chapter. We also look at how derivatives are used to find maximum and minimum values of functions. As a result, we will be able to solve applied optimization problems, such as maximizing revenue and minimizing surface area. In addition, we examine how derivatives are used to evaluate complicated limits, to approximate roots of functions, and to provide accurate graphs of functions.

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4.2: Related Rates

Learning Objectives

- Express changing quantities in terms of derivatives.
- Find relationships among the derivatives in a given problem.
- Use the chain rule to find the rate of change of one quantity that depends on the rate of change of other quantities.

We have seen that for quantities that are changing over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related. For example, if a balloon is being filled with air, both the radius of the balloon and the volume of the balloon are increasing. In this section, we consider several problems in which two or more related quantities are changing and we study how to determine the relationship between the rates of change of these quantities.

Setting up Related-Rates Problems

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, V, is related to the rate of change in the radius, r. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r. Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

Example 4.2.1: Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\ \text{cm}^3/\text{sec}$ (Figure 4.2.1). How fast is the radius increasing when the radius is 3 cm?

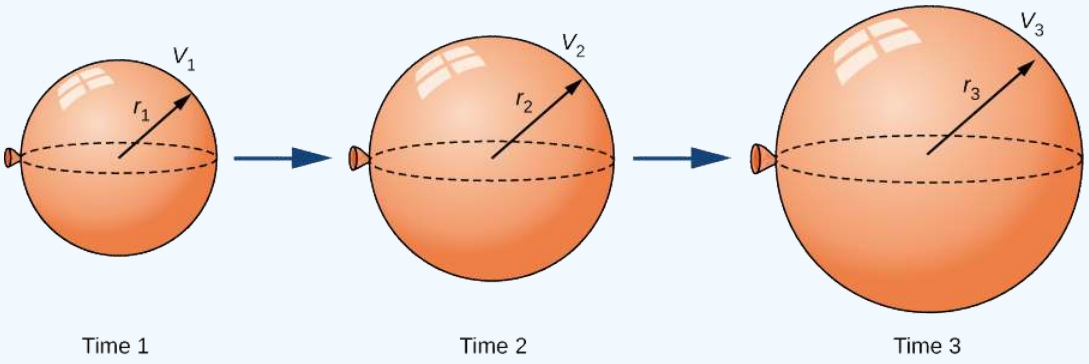


Figure 4.2.1: As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Solution

The volume of a sphere of radius r centimeters is

$$V = \tfrac{4}{3}\pi r^3\ \text{cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

$$V(t) = \tfrac{4}{3}\pi\big[r(t)\big]^3\text{cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

$$V'(t) = 4\pi\big[r(t)\big]^2 r'(t).$$

The balloon is being filled with air at the constant rate of $2\,\text{cm}^3/\text{sec}$, so $V'(t) = 2\,\text{cm}^3/\text{sec}$. Therefore,

$$2\,\text{cm}^3/\text{sec} = \Big(4\pi\big[r(t)\big]^2\ \text{cm}^2\Big)\cdot\Big(r'(t)\ \text{cm/s}\Big),$$

which implies

$$r'(t) = \frac{1}{2\pi[r(t)]^2} \text{ cm/sec.}$$

When the radius $r = 3$ cm,

$$r'(t) = \frac{1}{18\pi} \text{ cm/sec.}$$

? Exercise 4.2.1

What is the instantaneous rate of change of the radius when $r = 6$ cm?

Hint

$\frac{dr}{dt} = \frac{1}{2\pi r^2}$

Answer

$\frac{1}{72\pi}$ cm/sec, or approximately 0.0044 cm/sec

Before looking at other examples, let's outline the problem-solving strategy we will be using to solve related-rates problems.

Problem-Solving Strategy: Solving a Related-Rates Problem

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1.
4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change

Note that when solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4. We examine this potential error in the following example.

Examples of the Process

Let's now implement the strategy just described to solve several related-rates problems. The first example involves a plane flying overhead. The relationship we are studying is between the speed of the plane and the rate at which the distance between the plane and a person on the ground is changing.

✓ Example 4.2.2: An Airplane Flying at a Constant Elevation

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

Solution

Step 1. Draw a picture, introducing variables to represent the different quantities involved.

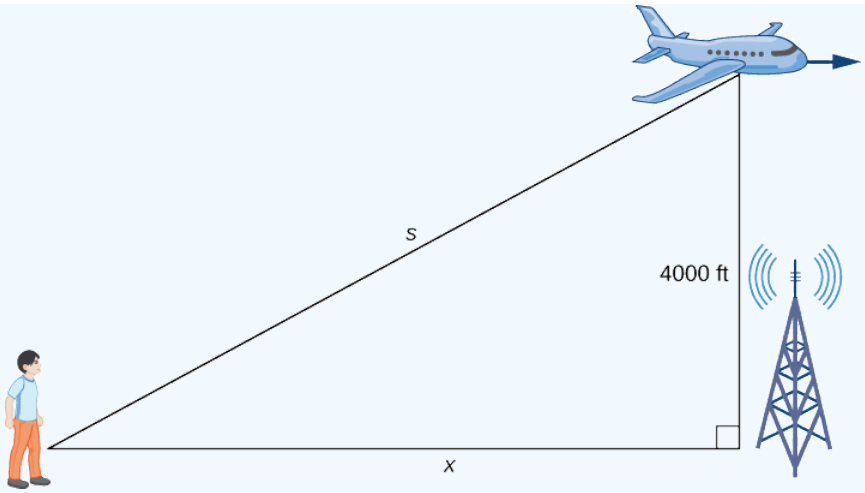


Figure 4.2.2: An airplane is flying at a constant height of 4000 ft. The distance between the person and the airplane and the person and the place on the ground directly below the airplane are changing. We denote those quantities with the variables s and x, respectively.

As shown, x denotes the distance between the man and the position on the ground directly below the airplane. The variable s denotes the distance between the man and the plane. Note that both x and s are functions of time. We do not introduce a variable for the height of the plane because it remains at a constant elevation of 4000 ft. Since an object's height above the ground is measured as the shortest distance between the object and the ground, the line segment of length 4000 ft is perpendicular to the line segment of length x feet, creating a right triangle.

Step 2. Since x denotes the horizontal distance between the man and the point on the ground below the plane, dx/dt represents the speed of the plane. We are told the speed of the plane is 600 ft/sec. Therefore, $\frac{dx}{dt} = 600$ ft/sec. Since we are asked to find the rate of change in the distance between the man and the plane when the plane is directly above the radio tower, we need to find ds/dt when $x = 3000$ ft.

Step 3. From the figure, we can use the **Pythagorean theorem** to write an equation relating x and s:

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Step 4. Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation

$$x\frac{dx}{dt} = s\frac{ds}{dt}.$$

Step 5. Find the rate at which the distance between the man and the plane is increasing when the plane is directly over the radio tower. That is, find $\frac{ds}{dt}$ when $x = 3000$ ft. Since the speed of the plane is 600 ft/sec, we know that $\frac{dx}{dt} = 600$ ft/sec. We are not given an explicit value for s; however, since we are trying to find $\frac{ds}{dt}$ when $x = 3000$ ft, we can use the Pythagorean theorem to determine the distance s when $x = 3000$ ft and the height is 4000 ft. Solving the equation

$$3000^2 + 4000^2 = s^2$$

for s, we have $s = 5000$ ft at the time of interest. Using these values, we conclude that ds/dt

is a solution of the equation

$$(3000)(600) = (5000) \cdot \frac{ds}{dt}.$$

Therefore,

$$\frac{ds}{dt} = \frac{3000 \cdot 600}{5000} = 360 \text{ ft/sec}.$$

Note: When solving related-rates problems, it is important not to substitute values for the variables too soon. For example, in step 3, we related the variable quantities $x(t)$ and $s(t)$ by the equation

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Since the plane remains at a constant height, it is not necessary to introduce a variable for the height, and we are allowed to use the constant 4000 to denote that quantity. However, the other two quantities are changing. If we mistakenly substituted $x(t) = 3000$ into the equation before differentiating, our equation would have been

$$3000^2 + 4000^2 = [s(t)]^2.$$

After differentiating, our equation would become

$$0 = s(t)\frac{ds}{dt}.$$

As a result, we would incorrectly conclude that $\frac{ds}{dt} = 0$.

? Exercise 4.2.2

What is the speed of the plane if the distance between the person and the plane is increasing at the rate of 300 ft/sec?

Hint

$\frac{ds}{dt} = 300$ ft/sec

Answer

500 ft/sec

We now return to the problem involving the rocket launch from the beginning of the chapter.

✓ Example 4.2.3: Chapter Opener - A Rocket Launch

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec.



Figure 4.2.3: (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

Solution

Step 1. Draw a picture introducing the variables.

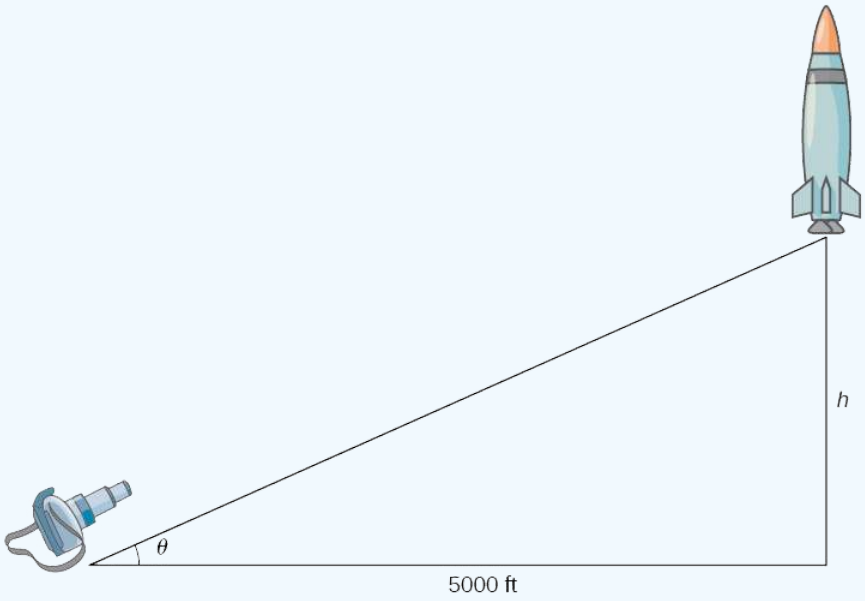


Figure 4.2.4: A camera is positioned 5000 ft from the launch pad of the rocket. The height of the rocket and the angle of the camera are changing with respect to time. We denote those quantities with the variables h and θ,respectively.

Let h denote the height of the rocket above the launch pad and θ be the angle between the camera lens and the ground.

Step 2. We are trying to find the rate of change in the angle of the camera with respect to time when the rocket is 1000 ft off the ground. That is, we need to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At that time, we know the velocity of the rocket is $\frac{dh}{dt} = 600$ ft/sec.

Step 3. Now we need to find an equation relating the two quantities that are changing with respect to time: h and θ. How can we create such an equation? Using the fact that we have drawn a right triangle, it is natural to think about trigonometric functions. Recall that $\tan\theta$ is the ratio of the length of the opposite side of the triangle to the length of the adjacent side. Thus, we have

$$\tan\theta = \frac{h}{5000}.$$

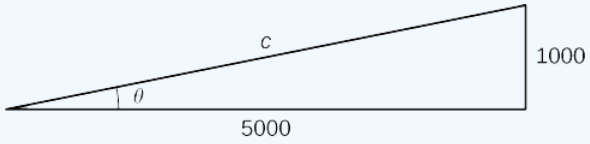
This gives us the equation

$$h = 5000\tan\theta.$$

Step 4. Differentiating this equation with respect to time t, we obtain

$$\frac{dh}{dt} = 5000\sec^2\theta\frac{d\theta}{dt}.$$

Step 5. We want to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At this time, we know that $\frac{dh}{dt} = 600$ ft/sec. We need to determine $\sec^2\theta$. Recall that $\sec\theta$ is the ratio of the length of the hypotenuse to the length of the adjacent side. We know the length of the adjacent side is 5000 ft. To determine the length of the hypotenuse, we use the Pythagorean theorem, where the length of one leg is 5000 ft, the length of the other leg is $h = 1000$ ft, and the length of the hypotenuse is c feet as shown in the following figure.



We see that

$$1000^2 + 5000^2 = c^2$$

and we conclude that the hypotenuse is

$$c = 1000\sqrt{26}\text{ ft}.$$

Therefore, when $h = 1000$, we have

$$\sec^2\theta = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

Recall from step 4 that the equation relating $\frac{d\theta}{dt}$ to our known values is

$$\frac{dh}{dt} = 5000\sec^2\theta\frac{d\theta}{dt}.$$

When $h = 1000$ ft, we know that $\frac{dh}{dt} = 600$ ft/sec and $\sec^2\theta = \frac{26}{25}$. Substituting these values into the previous equation, we arrive at the equation

$$600 = 5000\left(\frac{26}{25}\right)\frac{d\theta}{dt}.$$

Therefore, $\frac{d\theta}{dt} = \frac{3}{26}$ rad/sec.

Exercise 4.2.3

What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

Hint

Find $\frac{d\theta}{dt}$ when $h = 2000$ ft. At that time, $\frac{dh}{dt} = 500$ ft/sec.

Answer

$\frac{1}{10}$ rad/sec

In the next example, we consider water draining from a cone-shaped funnel. We compare the rate at which the level of water in the cone is decreasing with the rate at which the volume of water is decreasing.

Example 4.2.4: Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\ \text{ft}^3/\text{sec}$. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}$ ft?

Solution

Step 1: Draw a picture introducing the variables.

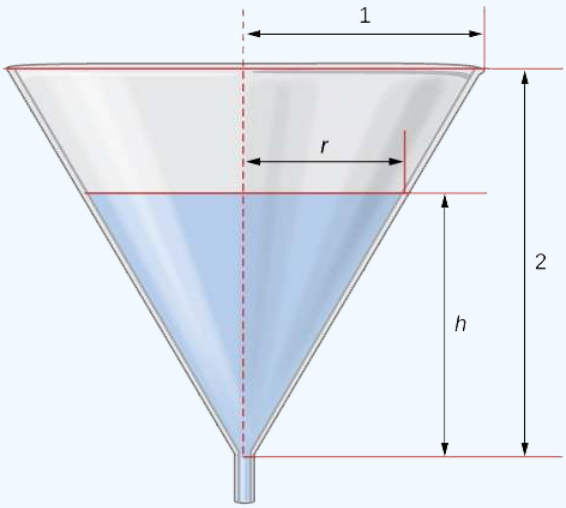


Figure 4.2.5: Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of water are changing over time. We denote these quantities with the variables h and r, respectively.

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that $\frac{dV}{dt} = -0.03$ ft/sec.

Step 3: The volume of water in the cone is

$$V = \tfrac{1}{3}\pi r^2 h.$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

$$V = \tfrac{1}{3}\pi\left(\tfrac{h}{2}\right)^2 h = \tfrac{\pi}{12}h^3.$$

Step 4: Applying the chain rule while differentiating both sides of this equation with respect to time t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03\,\text{ft}^3/\text{sec}$, we know that $\frac{dV}{dt} = -0.03\,\text{ft}^3/\text{sec}$. Therefore,

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2\frac{dh}{dt},$$

which implies

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

It follows that

$$\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153\,\text{ft/sec}.$$

? Exercise 4.2.4

At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

Hint

We need to find $\frac{dh}{dt}$ when $h = \frac{1}{4}$.

Answer

-0.61 ft/sec

Key Concepts

- To solve a related rates problem, first draw a picture that illustrates the relationship between the two or more related quantities that are changing with respect to time.
- In terms of the quantities, state the information given and the rate to be found.
- Find an equation relating the quantities.
- Use differentiation, applying the chain rule as necessary, to find an equation that relates the rates.
- Be sure not to substitute a variable quantity for one of the variables until after finding an equation relating the rates.

Glossary

related rates

are rates of change associated with two or more related quantities that are changing over time

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4.2E: Exercises

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4.3: Linear Approximations and Differentials

Learning Objectives

- Describe the linear approximation to a function at a point.
- Write the linearization of a given function.
- Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
- Calculate the relative error and percentage error in using a differential approximation.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials Introduction to Power Series and Functions.

Linear Approximation of a Function at a Point

Consider a function f that is differentiable at a point $x = a$. Recall that the tangent line to the graph of f at a is given by the equation

$$y = f(a) + f'(a)(x - a).$$

For example, consider the function $f(x) = \frac{1}{x}$ at $a = 2$. Since f is differentiable at $x = 2$ and $f'(x) = -\frac{1}{x^2}$, we see that $f'(2) = -\frac{1}{4}$. Therefore, the tangent line to the graph of f at $a = 2$ is given by the equation

$$y = \frac{1}{2} - \frac{1}{4}(x - 2).$$

Figure 4.3.1*a* shows a graph of $f(x) = \frac{1}{x}$ along with the tangent line to f at $x = 2$. Note that for x near 2, the graph of the tangent line is close to the graph of f. As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near 2. For example, if $x = 2.1$, the y value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.475.$$

The actual value of $f(2.1)$ is given by

$$f(2.1) = \frac{1}{2.1} \approx 0.47619.$$

Therefore, the tangent line gives us a fairly good approximation of $f(2.1)$ (Figure 4.3.1*b*). However, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if $x = 10$, the y-value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(10 - 2) = \frac{1}{2} - 2 = -1.5,$$

whereas the value of the function at $x = 10$ is $f(10) = 0.1$.

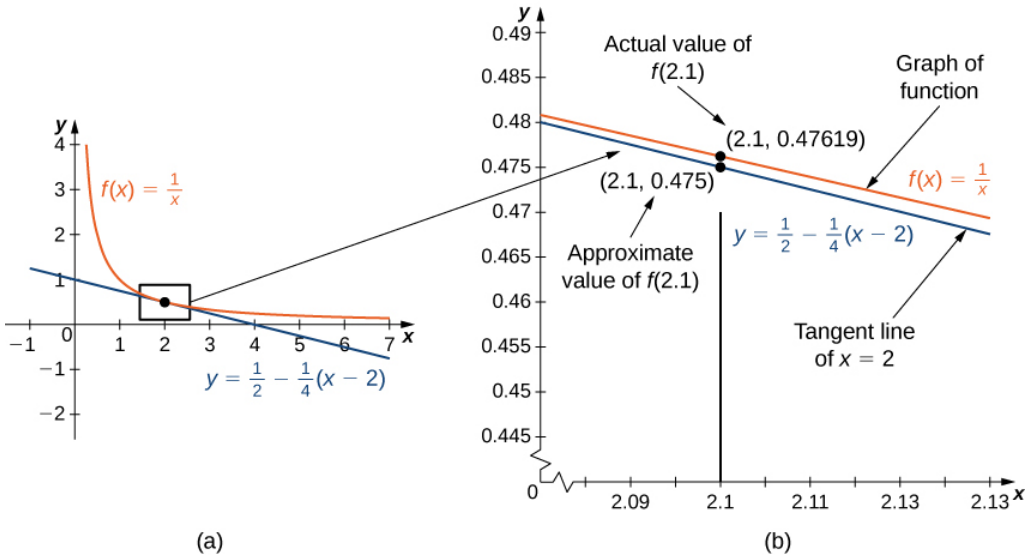


Figure 4.3.1: (a) The tangent line to $f(x) = 1/x$ at $x = 2$ provides a good approximation to f for x near 2. (b) At $x = 2.1$, the value of y on the tangent line to $f(x) = 1/x$ is 0.475. The actual value of $f(2.1)$ is $1/2.1$, which is approximately 0.47619.

In general, for a differentiable function f, the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near a. Therefore, we can write

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

We call the linear function

$$L(x) = f(a) + f'(a)(x - a) \tag{4.3.1}$$

the **linear approximation,** or **tangent line approximation**, of f at $x = a$. This function L is also known as the **linearization** of f at $x = a$.

To show how useful the linear approximation can be, we look at how to find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9$.

Example 4.3.1: Linear Approximation of $\sqrt{x}$

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}$.

Solution

Since we are looking for the linear approximation at $x = 9$, using Equation 4.3.1 we know the linear approximation is given by

$$L(x) = f(9) + f'(9)(x - 9).$$

We need to find $f(9)$ and $f'(9)$.

$$f(x) = \sqrt{x} \Rightarrow f(9) = \sqrt{9} = 3$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Therefore, the linear approximation is given by Figure 4.3.2.

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$

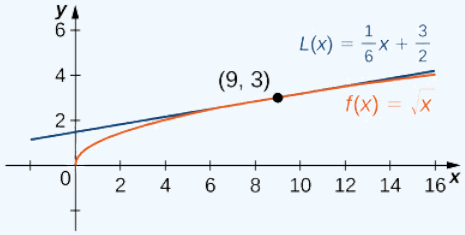


Figure 4.3.2: The local linear approximation to $f(x) = \sqrt{x}$ at $x = 9$ provides an approximation to f for x near 9.

Analysis

Using a calculator, the value of $\sqrt{9.1}$ to four decimal places is 3.0166. The value given by the linear approximation, 3.0167, is very close to the value obtained with a calculator, so it appears that using this linear approximation is a good way to estimate $\sqrt{x}$, at least for x near 9. At the same time, it may seem odd to use a linear approximation when we can just push a few buttons on a calculator to evaluate $\sqrt{9.1}$. However, how does the calculator evaluate $\sqrt{9.1}$? The calculator uses an approximation! In fact, calculators and computers use approximations all the time to evaluate mathematical expressions; they just use higher-degree approximations.

? Exercise 4.3.1

Find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$. Use it to approximate $\sqrt[3]{8.1}$ to five decimal places.

Hint

$L(x) = f(a) + f'(a)(x - a)$

Answer

$L(x) = 2 + \frac{1}{12}(x - 8)$; 2.00833

✓ Example 4.3.2: Linear Approximation of $\sin x$

Find the linear approximation of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ and use it to approximate $\sin(62°)$.

Solution

First we note that since $\frac{\pi}{3}$ rad is equivalent to 60°, using the linear approximation at $x = \pi/3$ seems reasonable. The linear approximation is given by

$$L(x) = f(\tfrac{\pi}{3}) + f'(\tfrac{\pi}{3})(x - \tfrac{\pi}{3}).$$

We see that

$$f(x) = \sin x \Rightarrow f(\tfrac{\pi}{3}) = \sin(\tfrac{\pi}{3}) = \tfrac{\sqrt{3}}{2}$$

$$f'(x) = \cos x \Rightarrow f'(\tfrac{\pi}{3}) = \cos(\tfrac{\pi}{3}) = \tfrac{1}{2}$$

Therefore, the linear approximation of f at $x = \pi/3$ is given by Figure 4.3.3.

$$L(x) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(x - \tfrac{\pi}{3})$$

To estimate $\sin(62°)$ using L, we must first convert 62° to radians. We have $62° = \frac{62\pi}{180}$ radians, so the estimate for $\sin(62°)$ is given by

$$\sin(62°) = f(\tfrac{62\pi}{180}) \approx L(\tfrac{62\pi}{180}) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(\tfrac{62\pi}{180} - \tfrac{\pi}{3}) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(\tfrac{2\pi}{180}) = \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{180} \approx 0.88348.$$

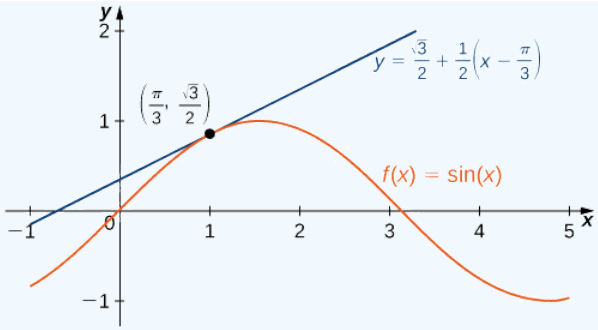


Figure 4.3.3: The linear approximation to $f(x) = \sin x$ at $x = \pi/3$ provides an approximation to $\sin x$ for x near $\pi/3$.

? Exercise 4.3.2

Find the linear approximation for $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

Hint

$L(x) = f(a) + f'(a)(x - a)$

Answer

$L(x) = -x + \frac{\pi}{2}$

Linear approximations may be used in estimating roots and powers. In the next example, we find the linear approximation for $f(x) = (1 + x)^n$ at $x = 0$, which can be used to estimate roots and powers for real numbers near 1. The same idea can be extended to a function of the form $f(x) = (m + x)^n$ to estimate roots and powers near a different number m.

✓ Example 4.3.3: Approximating Roots and Powers

Find the linear approximation of $f(x) = (1 + x)^n$ at $x = 0$. Use this approximation to estimate $(1.01)^3$.

Solution

The linear approximation at $x = 0$ is given by

$$L(x) = f(0) + f'(0)(x - 0).$$

Because

$$f(x) = (1 + x)^n \Rightarrow f(0) = 1$$

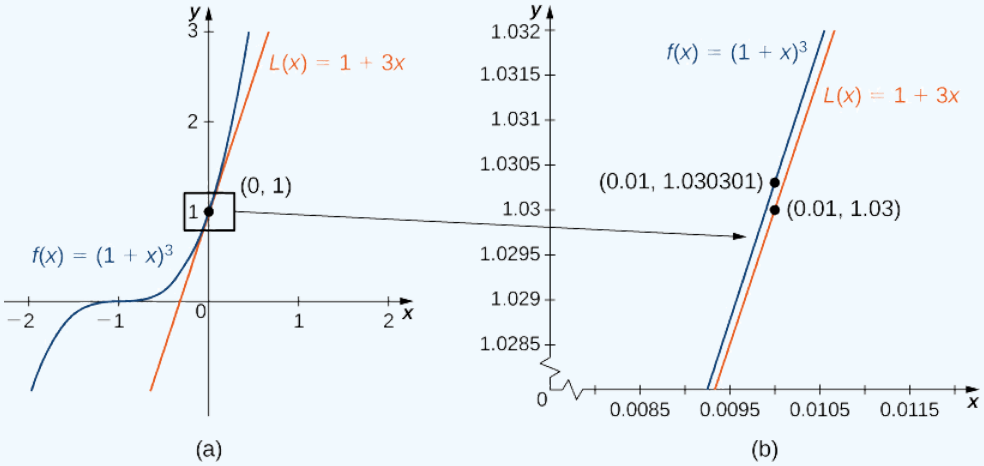
$$f'(x) = n(1 + x)^{n-1} \Rightarrow f'(0) = n,$$

the linear approximation is given by Figure 4.3.4a.

$$L(x) = 1 + n(x - 0) = 1 + nx$$

We can approximate $(1.01)^3$ by evaluating $L(0.01)$ when $n = 3$. We conclude that

$$(1.01)^3 = f(1.01) \approx L(1.01) = 1 + 3(0.01) = 1.03.$$



(a) (b)

Figure 4.3.4: (a) The linear approximation of $f(x)$ at $x = 0$ is $L(x)$. (b) The actual value of 1.01^3 is 1.030301. The linear approximation of $f(x)$ at $x = 0$ estimates 1.01^3 to be 1.03.

? Exercise 4.3.3

Find the linear approximation of $f(x) = (1 + x)^4$ at $x = 0$ without using the result from the preceding example.

Hint

$f'(x) = 4(1 + x)^3$

Answer

$L(x) = 1 + 4x$

Differentials

We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input. To discuss this more formally, we define a related concept: **differentials**. Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.

When we first looked at derivatives, we used the Leibniz notation dy/dx to represent the derivative of y with respect to x. Although we used the expressions dy and dx in this notation, they did not have meaning on their own. Here we see a meaning to the expressions dy and dx. Suppose $y = f(x)$ is a differentiable function. Let dx be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$dy = f'(x)\,dx. \tag{4.3.2}$$

It is important to notice that dy is a function of both x and dx. The expressions dy and dx are called differentials. We can divide both sides of Equation 4.3.2 by dx, which yields

$$\frac{dy}{dx} = f'(x). \tag{4.3.3}$$

This is the familiar expression we have used to denote a derivative. Equation 4.3.3 is known as the **differential form** of Equation 4.3.2.

✓ Example 4.3.4: Computing Differentials

For each of the following functions, find dy and evaluate when $x = 3$ and $dx = 0.1$.

a. $y = x^2 + 2x$
b. $y = \cos x$

Solution

The key step is calculating the derivative. When we have that, we can obtain dy directly.

a. Since $f(x) = x^2 + 2x$, we know $f'(x) = 2x + 2$, and therefore

$$dy = (2x + 2)\,dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = (2 \cdot 3 + 2)(0.1) = 0.8.$$

b. Since $f(x) = \cos x$, $f'(x) = -\sin(x)$. This gives us

$$dy = -\sin x\,dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = -\sin(3)(0.1) = -0.1\sin(3).$$

? Exercise 4.3.4

For $y = e^{x^2}$, find dy.

Hint

$dy = f'(x)\,dx$

Answer

$dy = 2xe^{x^2}\,dx$

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point a. Suppose the input x changes by a small amount. We are interested in how much the output y changes. If x changes from a to $a + dx$, then the change in x is dx (also denoted Δx), and the change in y is given by

$$\Delta y = f(a + dx) - f(a).$$

Instead of calculating the exact change in y, however, it is often easier to approximate the change in y by using a linear approximation. For x near a, $f(x)$ can be approximated by the linear approximation (Equation 4.3.1)

$$L(x) = f(a) + f'(a)(x - a).$$

Therefore, if dx is small,

$$f(a + dx) \approx L(a + dx) = f(a) + f'(a)(a + dx - a).$$

That is,

$$f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)\,dx.$$

In other words, the actual change in the function f if x increases from a to $a + dx$ is approximately the difference between $L(a + dx)$ and $f(a)$, where $L(x)$ is the linear approximation of f at a. By definition of $L(x)$, this difference is equal to $f'(a)\,dx$. In summary,

$$\Delta y = f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)\,dx = dy.$$

Therefore, we can use the differential $dy = f'(a)\,dx$ to approximate the change in y if x increases from $x = a$ to $x = a + dx$. We can see this in the following graph.

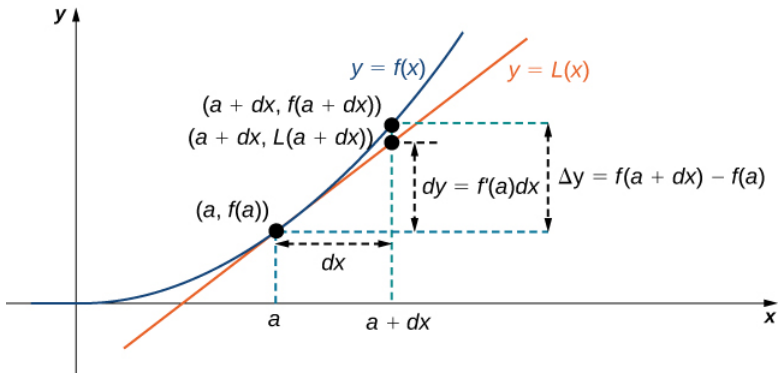


Figure 4.3.5: The differential $dy = f'(a)\,dx$ is used to approximate the actual change in y if x increases from a to $a + dx$.

We now take a look at how to use differentials to approximate the change in the value of the function that results from a small change in the value of the input. Note the calculation with differentials is much simpler than calculating actual values of functions and the result is very close to what we would obtain with the more exact calculation.

Example 4.3.5: Approximating Change with Differentials

Let $y = x^2 + 2x$. Compute Δy and dy at $x = 3$ if $dx = 0.1$.

Solution

The actual change in y if x changes from $x = 3$ to $x = 3.1$ is given by

$$\Delta y = f(3.1) - f(3) = [(3.1)^2 + 2(3.1)] - [3^2 + 2(3)] = 0.81.$$

The approximate change in y is given by $dy = f'(3)\,dx$. Since $f'(x) = 2x + 2$, we have

$$dy = f'(3)\,dx = (2(3) + 2)(0.1) = 0.8.$$

Exercise 4.3.5

For $y = x^2 + 2x$, find Δy and dy at $x = 3$ if $dx = 0.2$.

Hint

$dy = f'(3)\,dx,\ \Delta y = f(3.2) - f(3)$

Answer

$dy = 1.6,\ \Delta y = 1.64$

Calculating the Amount of Error

Any type of measurement is prone to a certain amount of error. In many applications, certain quantities are calculated based on measurements. For example, the area of a circle is calculated by measuring the radius of the circle. An error in the measurement of the radius leads to an error in the computed value of the area. Here we examine this type of error and study how differentials can be used to estimate the error.

Consider a function f with an input that is a measured quantity. Suppose the exact value of the measured quantity is a, but the measured value is $a + dx$. We say the measurement error is dx (or Δx). As a result, an error occurs in the calculated quantity $f(x)$. This type of error is known as a **propagated error** and is given by

$$\Delta y = f(a + dx) - f(a).$$

Since all measurements are prone to some degree of error, we do not know the exact value of a measured quantity, so we cannot calculate the propagated error exactly. However, given an estimate of the accuracy of a measurement, we can use differentials to approximate the propagated error Δy. Specifically, if f is a differentiable function at a,the propagated error is

$$\Delta y \approx dy = f'(a)\,dx.$$

Unfortunately, we do not know the exact value a. However, we can use the measured value $a + dx$, and estimate

$$\Delta y \approx dy \approx f'(a + dx)\,dx.$$

In the next example, we look at how differentials can be used to estimate the error in calculating the volume of a box if we assume the measurement of the side length is made with a certain amount of accuracy.

Example 4.3.6: Volume of a Cube

Suppose the side length of a cube is measured to be 5 cm with an accuracy of 0.1 cm.

a. Use differentials to estimate the error in the computed volume of the cube.
b. Compute the volume of the cube if the side length is (i) 4.9 cm and (ii) 5.1 cm to compare the estimated error with the actual potential error.

Solution

a. The measurement of the side length is accurate to within ± 0.1 cm. Therefore,

$$-0.1 \le dx \le 0.1.$$

The volume of a cube is given by $V = x^3$, which leads to

$$dV = 3x^2 dx.$$

Using the measured side length of 5 cm, we can estimate that

$$-3(5)^2(0.1) \le dV \le 3(5)^2(0.1).$$

Therefore,

$$-7.5 \le dV \le 7.5.$$

b. If the side length is actually 4.9 cm, then the volume of the cube is

$$V(4.9) = (4.9)^3 = 117.649 \text{cm}^3.$$

If the side length is actually 5.1 cm, then the volume of the cube is

$$V(5.1) = (5.1)^3 = 132.651 \text{cm}^3.$$

Therefore, the actual volume of the cube is between 117.649 and 132.651. Since the side length is measured to be 5 cm, the computed volume is $V(5) = 5^3 = 125$. Therefore, the error in the computed volume is

$$117.649 - 125 \le \Delta V \le 132.651 - 125.$$

That is,

$$-7.351 \le \Delta V \le 7.651.$$

We see the estimated error dV is relatively close to the actual potential error in the computed volume.

Exercise 4.3.6

Estimate the error in the computed volume of a cube if the side length is measured to be 6 cm with an accuracy of 0.2 cm.

Hint

$dV = 3x^2 dx$

Answer

The volume measurement is accurate to within $21.6\,\text{cm}^3$.

The measurement error dx $(= \Delta x)$ and the propagated error Δy are absolute errors. We are typically interested in the size of an error relative to the size of the quantity being measured or calculated. Given an absolute error Δq for a particular quantity, we define the **relative error** as $\frac{\Delta q}{q}$, where q is the actual value of the quantity. The **percentage error** is the relative error expressed as a percentage. For example, if we measure the height of a ladder to be 63 in. when the actual height is 62 in., the absolute error is 1 in. but the relative error is $\frac{1}{62} = 0.016$, or 1.6%. By comparison, if we measure the width of a piece of cardboard to be 8.25 in. when the actual width is 8 in., our absolute error is $\frac{1}{4}$ in., whereas the relative error is $\frac{0.25}{8} = \frac{1}{32}$, or 3.1%. Therefore, the percentage error in the measurement of the cardboard is larger, even though 0.25 in. is less than 1 in.

Example 4.3.7: Relative and Percentage Error

An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ± 80 mi. Let's use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

Solution: If the measurement of the radius is accurate to within ± 80, we have

$$-80 \leq dr \leq 80.$$

Since the volume of a sphere is given by $V = (\frac{4}{3})\pi r^3$, we have

$$dV = 4\pi r^2 dr.$$

Using the measured radius of 4000 mi, we can estimate

$$-4\pi(4000)^2(80) \leq dV \leq 4\pi(4000)^2(80).$$

To estimate the relative error, consider $\frac{dV}{V}$. Since we do not know the exact value of the volume V, use the measured radius $r = 4000$ mi to estimate V. We obtain $V \approx (\frac{4}{3})\pi(4000)^3$. Therefore the relative error satisfies

$$\frac{-4\pi(4000)^2(80)}{4\pi(4000)^3/3} \leq \frac{dV}{V} \leq \frac{4\pi(4000)^2(80)}{4\pi(4000)^3/3},$$

which simplifies to

$$-0.06 \leq \frac{dV}{V} \leq 0.06.$$

The relative error is 0.06 and the percentage error is 6%.

Exercise 4.3.7

Determine the percentage error if the radius of Earth is measured to be 3950 mi with an error of ± 100 mi.

Hint

Use the fact that $dV = 4\pi r^2 dr$ to find dV/V.

Answer

7.6%

Key Concepts

- A differentiable function $y = f(x)$ can be approximated at a by the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

- For a function $y = f(x)$, if x changes from a to $a + dx$, then

$$dy = f'(x)\,dx$$

is an approximation for the change in y. The actual change in y is

$$\Delta y = f(a + dx) - f(a).$$

- A measurement error dx can lead to an error in a calculated quantity $f(x)$. The error in the calculated quantity is known as the propagated error. The propagated error can be estimated by

$$dy \approx f'(x)\,dx.$$

- To estimate the relative error of a particular quantity q, we estimate $\frac{\Delta q}{q}$.

Key Equations

- **Linear approximation**

$L(x) = f(a) + f'(a)(x - a)$

- **A differential**

$$dy = f'(x)\,dx$$

Glossary

differential

the differential dx is an independent variable that can be assigned any nonzero real number; the differential dy is defined to be $dy = f'(x)\,dx$

differential form

given a differentiable function $y = f'(x)$, the equation $dy = f'(x)\,dx$ is the differential form of the derivative of y with respect to x

linear approximation

the linear function $L(x) = f(a) + f'(a)(x - a)$ is the linear approximation of f at $x = a$

percentage error

the relative error expressed as a percentage

propagated error

the error that results in a calculated quantity $f(x)$ resulting from a measurement error dx

relative error

given an absolute error Δq for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

tangent line approximation (linearization)

since the linear approximation of f at $x = a$ is defined using the equation of the tangent line, the linear approximation of f at $x = a$ is also known as the tangent line approximation to f at $x = a$

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4.3E: Exercises

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4.4: Maxima and Minima

Learning Objectives

- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical points of a function over a closed interval.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance, because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty, f(x) \to \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (Figure 4.4.1).

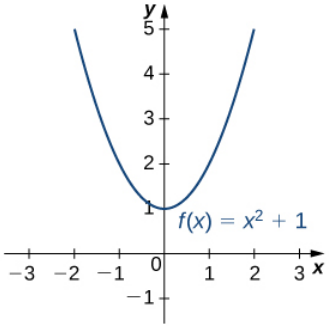


Figure 4.4.1: The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Definition: Absolute Extrema

Let f be a function defined over an interval I and let $c \in I$. We say f has an *absolute maximum* on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an *absolute minimum* on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has *an absolute extremum* on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term absolute here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. It occurs at $x = 0$, as shown in Figure 4.4.2(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. Figure 4.4.2 shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.

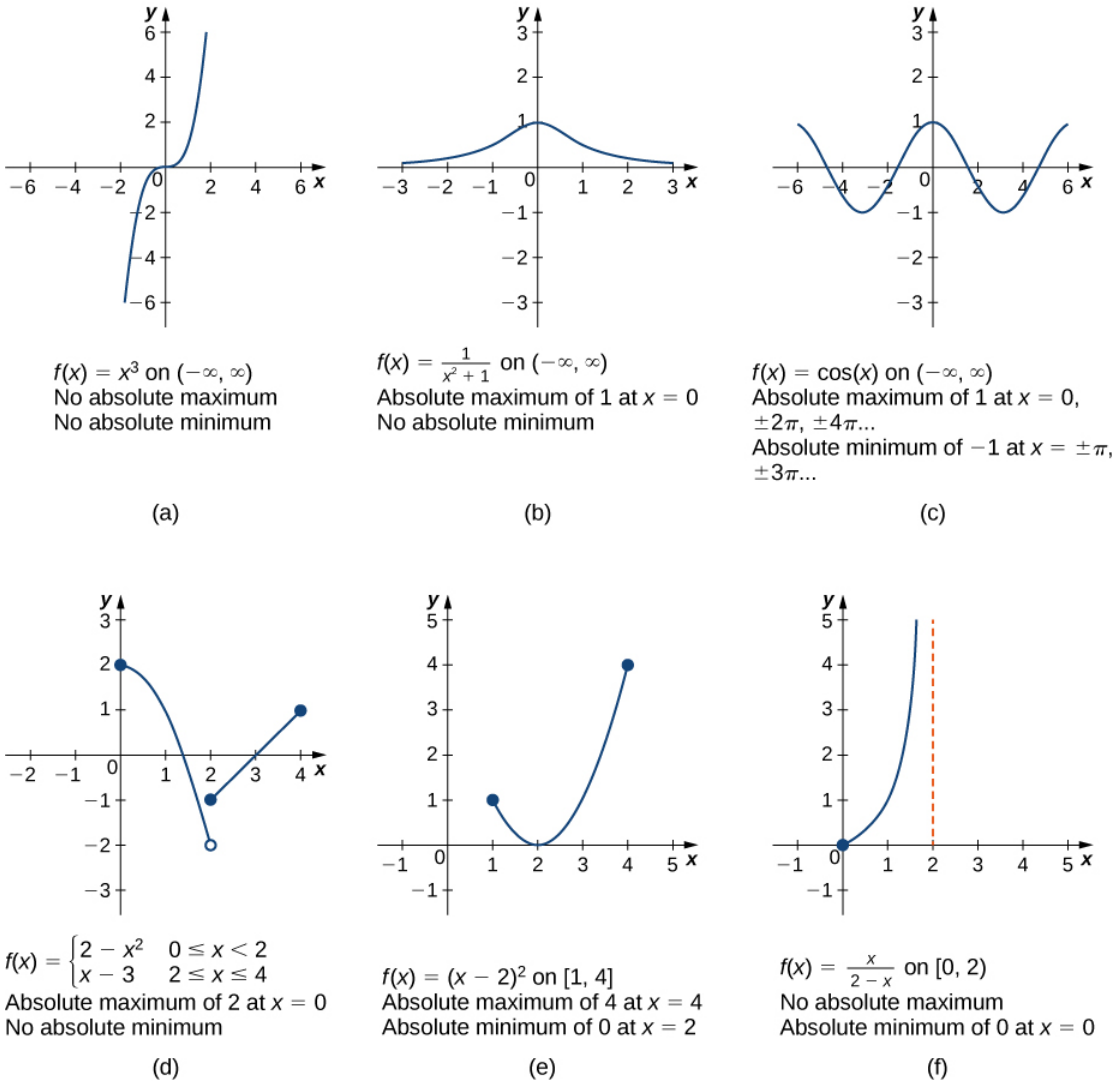


Figure 4.4.2: Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Theorem 4.4.1: Extreme Value Theorem

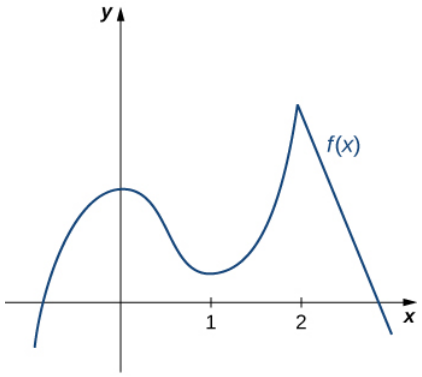
If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I. For example, consider the functions shown in Figure 4.4.2 (d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let's examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

Local Extrema and Critical Points

Consider the function f shown in Figure **4.4.3**. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.



$f(x)$ defined on $(-\infty, \infty)$
Local maxima at $x = 0$ and $x = 2$
Local minimum at $x = 1$

Figure **4.4.3**: This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Definition: Local Extrema

A function f has a *local maximum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a *local minimum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a *local extremum* at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at Figure **4.4.3** again. The local extrema occur at $x = 0, x = 1$, and $x = 2$. Notice that at $x = 0$ and $x = 1$, the derivative $f'(x) = 0$. At $x = 2$, the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c$, the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Definition: Critical Points

Let c be an interior point in the domain of f. We say that c is a *critical point* of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c$, then c must be a critical point of f. This fact is known as **Fermat's theorem.**

Theorem 4.4.2: Fermat's Theorem

If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

Proof

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f'(c) = 0$. To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since f has a local extremum at c, f has a local maximum or local minimum at c. Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I$. Since f is differentiable at c, from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c)$. Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c,} \tag{4.4.1}$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near c, but $x > c$, we have $\frac{f(x)-f(c)}{x-c} \leq 0$. From Equation 4.4.1 we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

□

From Fermat's theorem, we conclude that if f has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$. In Figure 4.4.4, we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

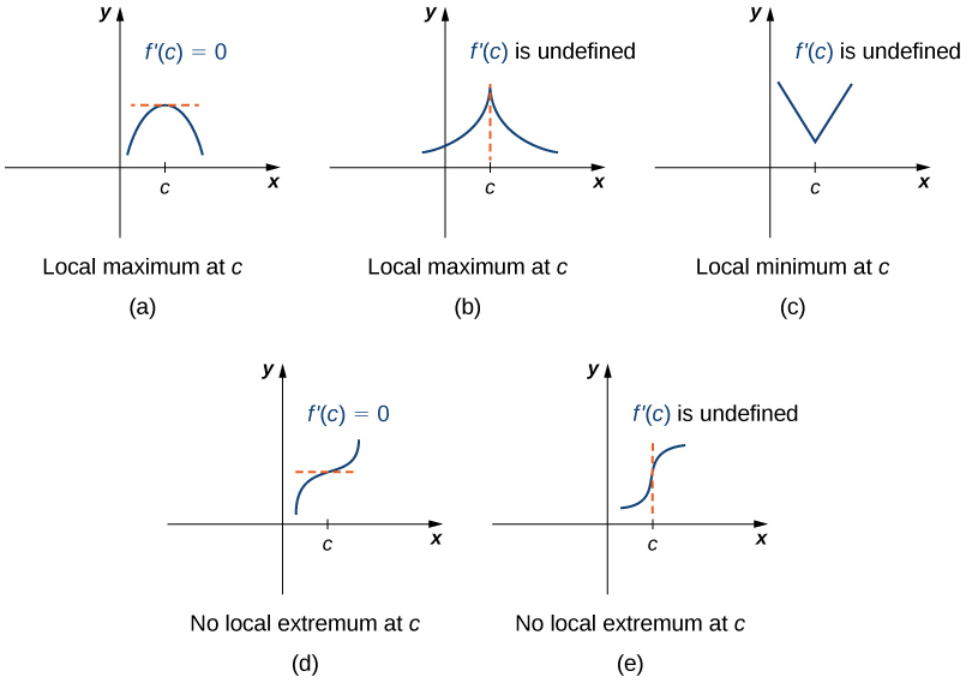


Figure 4.4.4: (a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 4.4.1: Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$
b. $f(x) = (x^2 - 1)^3$
c. $f(x) = \frac{4x}{1+x^2}$

Solution

a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x-4)(x-1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in Figure 4.4.5, we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

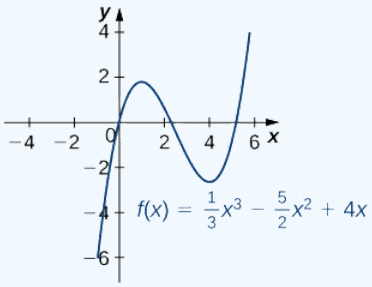


Figure 4.4.5: This function has a local maximum and a local minimum.

b. Using the chain rule, we see the derivative is

$$f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in Figure 4.4.6, we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum at $x = 1$ or $x = -1$.

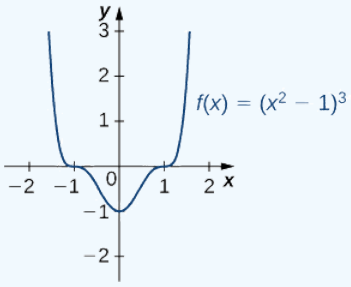


Figure 4.4.6: This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

c. By the quotient rule, we see that the derivative is

$$f'(x) = \frac{4(1+x^2) - 4x(2x)}{(1+x^2)^2} = \frac{4 - 4x^2}{(1+x^2)^2}.$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in Figure 4.4.7, we see that f has an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I, then f has a local extremum at c.)

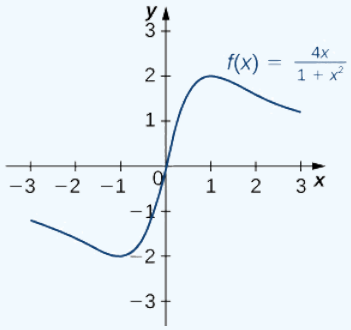


Figure 4.4.7: This function has an absolute maximum and an absolute minimum.

? Exercise 4.4.1

Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Hint

Calculate $f'(x)$.

Answer

$x = \frac{-2}{3}, x = 1$

Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in Figure 4.4.2, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by Fermat's Theorem, the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem 4.4.3: Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let's examine a procedure for locating absolute extrema.

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). From "Location of Absolute Extrema," the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example 4.4.2: Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.
b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3$, f' is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f(\frac{3}{2})$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f(\frac{3}{2})$ and find

$$f\left(\tfrac{3}{2}\right) = \tfrac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	Conclusion
1	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$. The absolute minimum of f over the interval $[1, 3]$ is -2, and it occurs at $x = 3$ as shown in Figure 4.4.8.

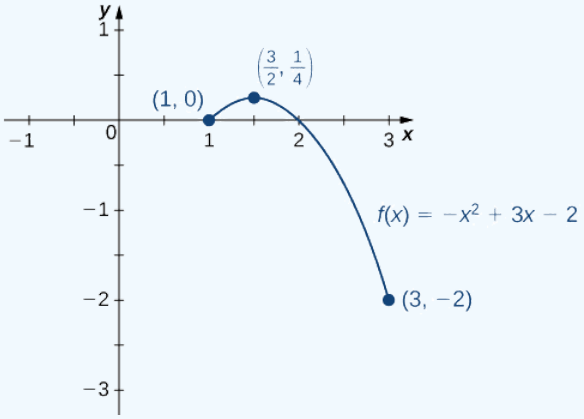


Figure 4.4.8: This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

$$f(0) = 0 \text{ and } f(2) = 4 - 3\,(2)^{2/3} \approx -0.762$$

Step 2. The derivative of f is given by

$$f'(x) = 2x - \tfrac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	-2	Absolute minimum
2	-0.762	

We conclude that the absolute maximum of f over the interval $[0, 2]$ is zero, and it occurs at $x = 0$. The absolute minimum is -2, and it occurs at $x = 1$ as shown in Figure 4.4.9.

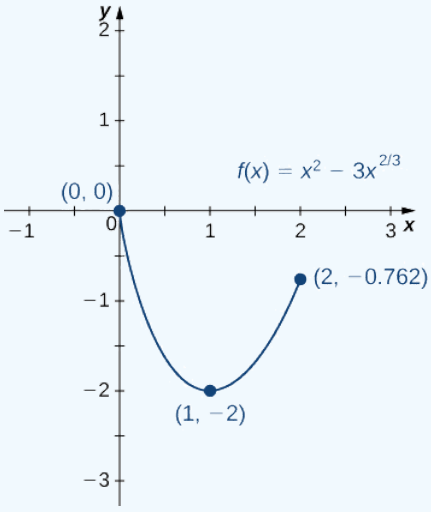


Figure 4.4.9: This function has an absolute maximum at an endpoint of the interval.

? Exercise 4.4.2

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

Hint

Look for critical points. Evaluate f at all critical points and at the endpoints.

Answer

The absolute maximum is 3 and it occurs at $x = 4$. The absolute minimum is -1 and it occurs at $x = 2$.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c, then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

Key Concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

Glossary

absolute extremum
if f has an absolute maximum or absolute minimum at c, we say f has an absolute extremum at c

absolute maximum
if $f(c) \geq f(x)$ for all x in the domain of f, we say f has an absolute maximum at c

absolute minimum
if $f(c) \leq f(x)$ for all x in the domain of f, we say f has an absolute minimum at c

critical point
if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

extreme value theorem
if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem
if f has a local extremum at c, then c is a critical point of f

local extremum
if f has a local maximum or local minimum at c, we say f has a local extremum at c

local maximum
if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I$, we say f has a local maximum at c

local minimum
if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I$, we say f has a local minimum at c

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4.4E: Exercises

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4.5: The Mean Value Theorem

Learning Objectives

- Explain the meaning of Rolle's theorem.
- Describe the significance of the Mean Value Theorem.
- State three important consequences of the Mean Value Theorem.

The *Mean Value Theorem* is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. Figure 4.5.1 illustrates this theorem.

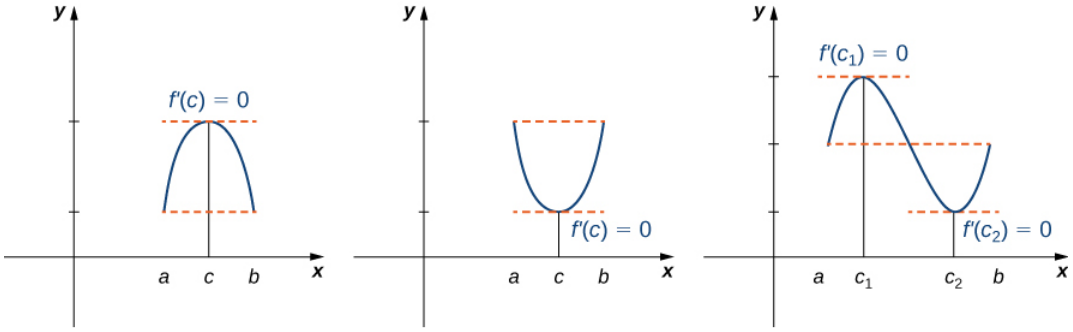


Figure 4.5.1: If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b.

Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof

Let $k = f(a) = f(b)$. We consider three cases:

1. $f(x) = k$ for all $x \in (a, b)$.
2. There exists $x \in (a, b)$ such that $f(x) > k$.
3. There exists $x \in (a, b)$ such that $f(x) < k$.

Case 1: If $f(x) = k$ for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: Since f is a continuous function over the closed, bounded interval $[a, b]$, by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k$, the absolute maximum is greater than k. Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because f has a maximum at an interior point c, and f is differentiable at c, by Fermat's theorem, $f'(c) = 0$.

Case 3: The case when there exists a point $x \in (a, b)$ such that $f(x) < k$ is analogous to case 2, with maximum replaced by minimum.

□

An important point about Rolle's theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over $[-1, 1]$ and $f(-1) = 0 = f(1)$, but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown in the following figure.

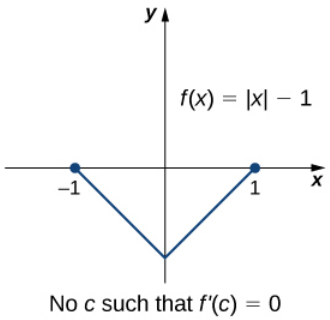


Figure 4.5.2: Since $f(x) = |x| - 1$ is not differentiable at $x = 0$, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points c where $f'(c) = 0$.

Example 4.5.1: Using Rolle's Theorem

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

a. $f(x) = x^2 + 2x$ over $[-2, 0]$
b. $f(x) = x^3 - 4x$ over $[-2, 2]$

Solution

a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0)$. Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0$. Since $f'(x) = 2x + 2 = 2(x + 1)$, we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.

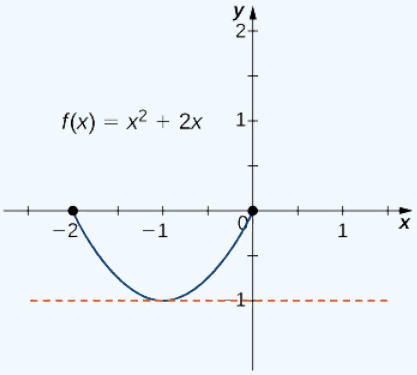


Figure 4.5.3: This function is continuous and differentiable over $[-2,0]$, $f'(c) = 0$ when $c = -1$.

b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.

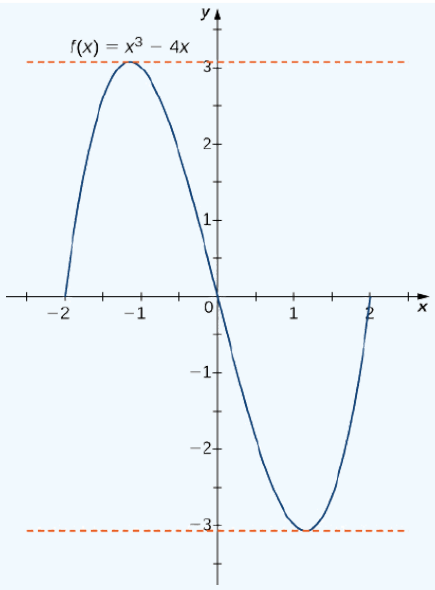


Figure 4.5.4: For this polynomial over $[-2,2]$, $f'(c)=0$ at $x=\pm 2/\sqrt{3}$.

Exercise 4.5.1

Verify that the function $f(x)=2x^2-8x+6$ defined over the interval $[1,3]$ satisfies the conditions of Rolle's theorem. Find all points c guaranteed by Rolle's theorem.

Hint

Find all values c, where $f'(c)=0$.

Answer

$c=2$

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f that are zero at the endpoints. The Mean Value Theorem generalizes Rolle's theorem by considering functions that are not necessarily zero at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem (Figure 4.5.5). The Mean Value Theorem states that if f is continuous over the closed interval $[a,b]$ and differentiable over the open interval (a,b), then there exists a point $c\in(a,b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a,f(a))$ and $(b,f(b))$.

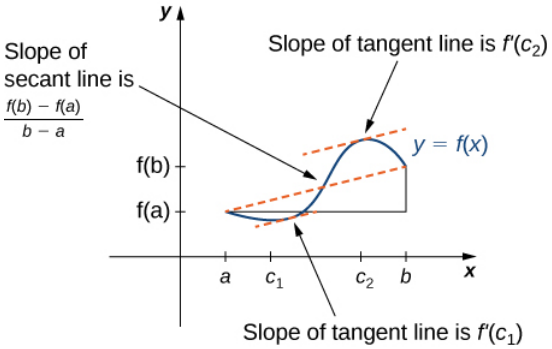


Figure 4.5.5: The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point (x, y) on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

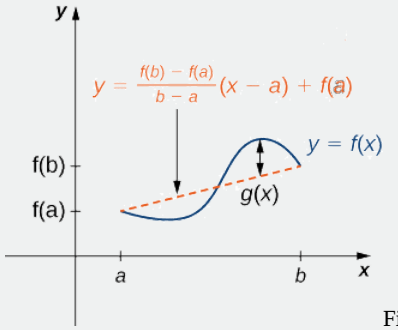


Figure 4.5.6: The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point (x, y) on the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over $[a, b]$, g is also continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

In the next example, we show how the Mean Value Theorem can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

Example 4.5.2: Verifying that the Mean Value Theorem Applies

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

Solution

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$ (Figure 4.5.7). To determine which value(s) of c are guaranteed, first calculate the derivative of f. The derivative $f'(x) = \frac{1}{(2\sqrt{x})}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}.$$

We want to find c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Solving this equation for c, we obtain $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

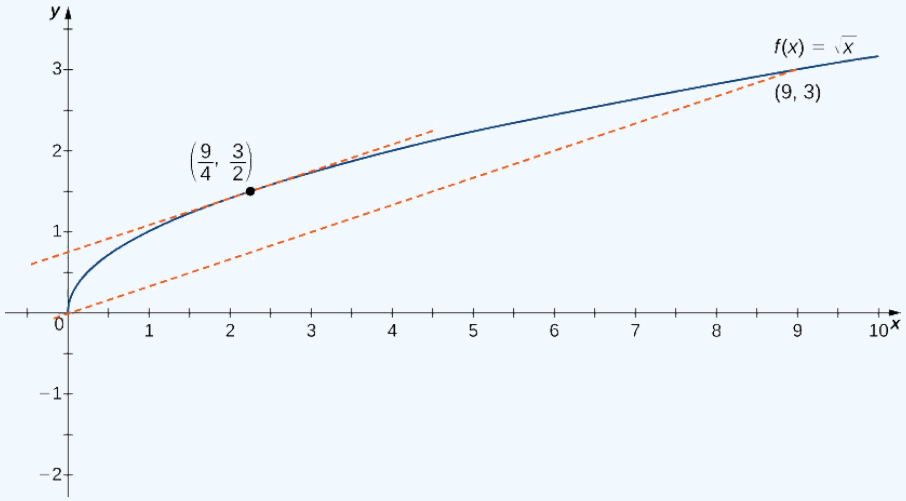


Figure 4.5.7: The slope of the tangent line at $c = 9/4$ is the same as the slope of the line segment connecting (0,0) and (9,3).

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for 1 h down a straight road with an average velocity of 45 mph. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0 \le t \le 1$ h. Assuming that the position function $s(t)$ is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c \in (0, 1)$, the speed of the car was exactly

$$v(c) = s'(c) = \frac{s(1) - s(0)}{1 - 0} = 45 \text{ mph}.$$

Example 4.5.3: Mean Value Theorem and Velocity

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100$.

a. Determine how long it takes before the rock hits the ground.
b. Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.
c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

Solution

a. When the rock hits the ground, its position is $s(t) = 0$. Solving the equation $-16t^2 + 100 = 0$ for t, we find that $t = \pm\frac{5}{2} sec$. Since we are only considering $t \geq 0$, the ball will hit the ground $\frac{5}{2}$ sec after it is dropped.

b. The average velocity is given by

$$v_{avg} = \frac{s(5/2) - s(0)}{5/2 - 0} = \frac{0 - 100}{5/2} = -40 \text{ ft/sec}.$$

c. The instantaneous velocity is given by the derivative of the position function. Therefore, we need to find a time t such that $v(t) = s'(t) = v_{avg} = -40$ ft/sec. Since $s(t)$ is continuous over the interval $[0, 5/2]$ and differentiable over the interval $(0, 5/2)$, by the Mean Value Theorem, there is guaranteed to be a point $c \in (0, 5/2)$ such that

$$s'(c) = \frac{s(5/2) - s(0)}{5/2 - 0} = -40.$$

Taking the derivative of the position function $s(t)$, we find that $s'(t) = -32t$. Therefore, the equation reduces to $s'(c) = -32c = -40$. Solving this equation for c, we have $c = \frac{5}{4}$. Therefore, $\frac{5}{4}$ sec after the rock is dropped, the instantaneous velocity equals the average velocity of the rock during its free fall: -40 ft/sec.

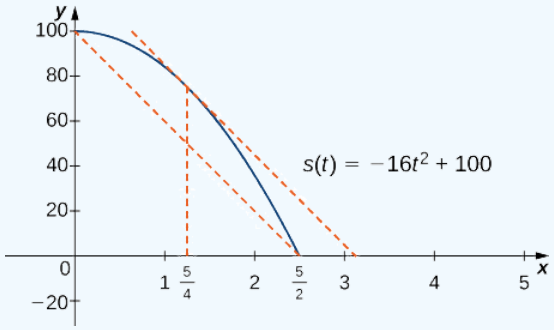


Figure 4.5.8: At time $t = 5/4$ sec, the velocity of the rock is equal to its average velocity from the time it is dropped until it hits the ground.

Exercise 4.5.2

Suppose a ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity of the ball equals its average velocity.

Hint

First, determine how long it takes for the ball to hit the ground. Then, find the average velocity of the ball from the time it is dropped until it hits the ground.

Answer

$\frac{5}{2\sqrt{2}}$ sec

Corollaries of the Mean Value Theorem

Let's now look at three corollaries of the Mean Value Theorem. These results have important consequences, which we use in upcoming sections.

At this point, we know the derivative of any constant function is zero. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f'(x) = 0$ for all x in some interval I, then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

Corollary 1: Functions with a Derivative of Zero

Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x) =$ constant for all $x \in I$.

Proof

Since f is differentiable over I, f must be continuous over I. Suppose $f(x)$ is not constant for all x in I. Then there exist $a, b \in I$, where $a \neq b$ and $f(a) \neq f(b)$. Choose the notation so that $a < b$. Therefore,

$$\frac{f(b) - f(a)}{b - a} \neq 0.$$

Since f is a differentiable function, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in I$ such that $f'(c) \neq 0$, which contradicts the assumption that $f'(x) = 0$ for all $x \in I$.

□

From "Corollary 1: Functions with a Derivative of Zero," it follows that if two functions have the same derivative, they differ by, at most, a constant.

Corollary 2: Constant Difference Theorem

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C.

Proof

Let $h(x) = f(x) - g(x)$. Then, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By Corollary 1, there is a constant C such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

□

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Recall that a function f is increasing over I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas f is decreasing over I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive, then the function is increasing; if the derivative is negative, then the function is decreasing (Figure **4.5.9**). We make use of this fact in the next section, where we show how to use the derivative of a function to locate local maximum and minimum values of the function, and how to determine the shape of the graph.

This fact is important because it means that for a given function f, if there exists a function F such that $F'(x) = f(x)$; then, the only other functions that have a derivative equal to f are $F(x) + C$ for some constant C. We discuss this result in more detail later in the chapter.

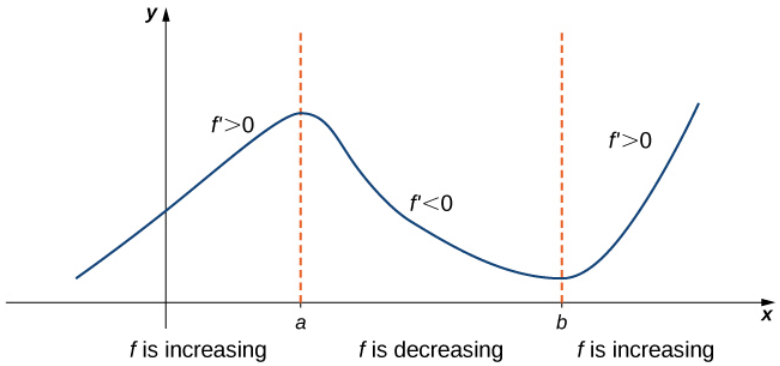


Figure **4.5.9**: If a function has a positive derivative over some interval I, then the function increases over that interval I; if the derivative is negative over some interval I, then the function decreases over that interval I.

Corollary 3: Increasing and Decreasing Functions

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.
ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

Proof

We will prove i.; the proof of ii. is similar. Suppose f is not an increasing function on I. Then there exist a and b in I such that $a < b$, but $f(a) \geq f(b)$. Since f is a differentiable function over I, by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, $a < b$ tells us that $b - a > 0$. We conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction, and therefore f must be an increasing function over I.

□

Key Concepts

- If f is continuous over $[a, b]$ and differentiable over (a, b) and $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$. This is Rolle's theorem.
- If f is continuous over $[a, b]$ and differentiable over (a, b), then there exists a point $c \in (a, b)$ such that

  $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

  This is the Mean Value Theorem.
- If $f'(x) = 0$ over an interval I, then f is constant over I.
- If two differentiable functions f and g satisfy $f'(x) = g'(x)$ over I, then $f(x) = g(x) + C$ for some constant C.
- If $f'(x) > 0$ over an interval I, then f is increasing over I. If $f'(x) < 0$ over I, then f is decreasing over I.

Glossary

mean value theorem

if f is continuous over $[a, b]$ and differentiable over (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

rolle's theorem

if f is continuous over $[a, b]$ and differentiable over (a, b), and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$

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4.5E: Exercises

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4.6: Derivatives and the Shape of a Graph

Learning Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function f has a local extremum at a point c, then c must be a critical point of f. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0$, but f does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over I. On the other hand, if the derivative of the function is negative over an interval I, then the function is decreasing over I as shown in the following figure.

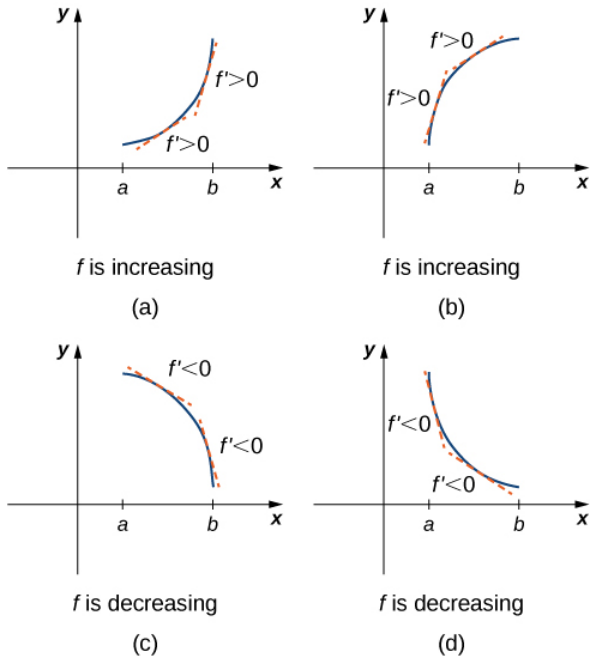


Figure 4.6.1: Both functions are increasing over the interval (a, b). At each point x, the derivative $f'(x) > 0$. Both functions are decreasing over the interval (a, b). At each point x, the derivative $f'(x) < 0$.

A continuous function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c. Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c. If f is a continuous function over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c. If f is differentiable at c, the only way that f' can change sign as x increases through c is if $f'(c) = 0$. Therefore, for a function f that is continuous over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is

undefined. Consequently, to locate local extrema for a function f, we look for points c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such points are called critical points of f.

Note that f need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In Figure 4.6.2, we show that if a continuous function f has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if f has a local extremum at a critical point, then the sign of f' switches as x increases through that point.

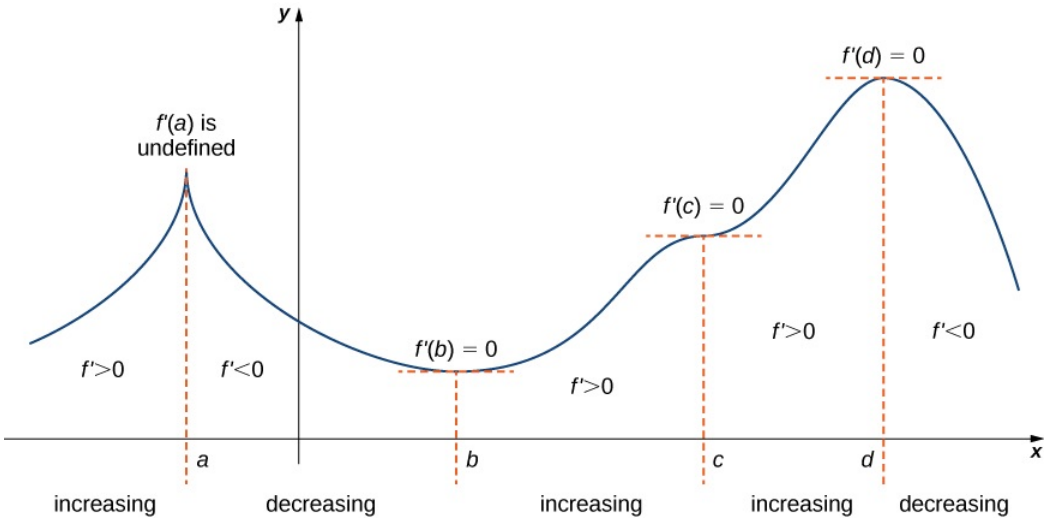


Figure 4.6.2: The function f has four critical points: a, b, c,and d. The function f has local maxima at a and d, and a local minimum at b. The function f does not have a local extremum at c. The sign of f' changes at all local extrema.

Using Figure 4.6.2, we summarize the main results regarding local extrema.

- If a continuous function f has a local extremum, it must occur at a critical point c.
- The function has a local extremum at the critical point c if and only if the derivative f' switches sign as x increases through c.
- Therefore, to test whether a function has a local extremum at a critical point c, we must determine the sign of $f'(x)$ to the left and right of c.

This result is known as the **first derivative test.**

First Derivative Test

Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possibly at point c, then $f(c)$ satisfies one of the following descriptions:

i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.
ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.
iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example 4.6.1: Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is $f'(x) = 3x^2 - 6x - 9$. To find the critical points, we need to find where $f'(x) = 0$. Factoring the polynomial, we conclude that the critical points must satisfy

$$3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0.$$

Therefore, the critical points are $x = 3, -1$. Now divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$.

Step 2. Since f' is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$ and determine the sign of f' at each of these points. For example, let's choose $x = -2$, $x = 0$, and $x = 4$ as test points.

Table: 4.6.1: First Derivative Test for $f(x) = x^3 - 3x^2 - 9x - 1$.

Interval	Test Point	Sign of $f'(x) = 3(x-3)(x+1)$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	(+)(−)(−)=+	f is increasing.
$(-1, 3)$	$x = 0$	(+)(−)(+)=-	f is decreasing.
$(3, \infty)$	$x = 4$	(+)(+)(+)=+	f is increasing.

Step 3. Since f' switches sign from positive to negative as x increases through -1, f has a local maximum at $x = -1$. Since f' switches sign from negative to positive as x increases through 3, f has a local minimum at $x = 3$. These analytical results agree with the following graph.

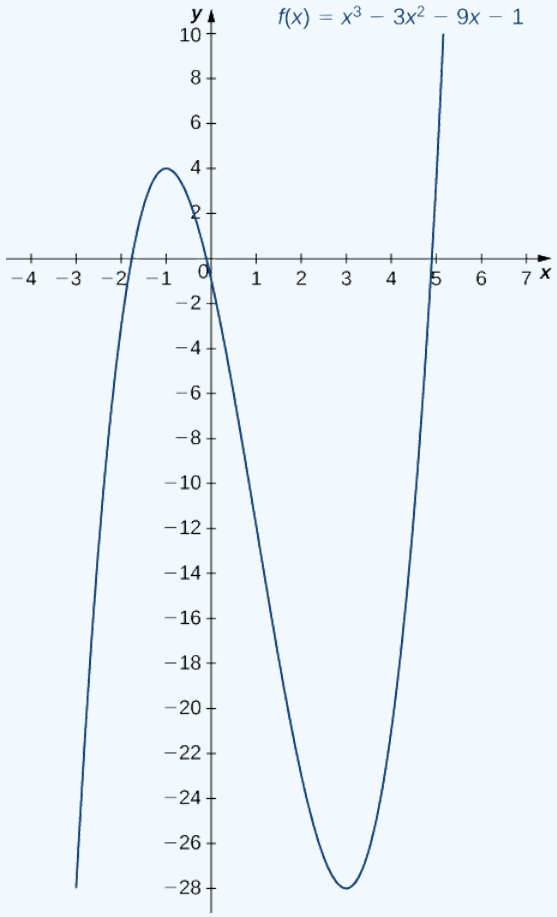


Figure 4.6.3: The function f has a maximum at $x = -1$ and a minimum at $x = 3$

? Exercise 4.6.1

Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

Hint

Find all critical points of f and determine the signs of $f'(x)$ over particular intervals determined by the critical points.

Answer

f has a local minimum at -2 and a local maximum at 3.

Example 4.6.2: Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3x^{2/3}} - \frac{5x^{2/3}}{3} = \frac{5 - 5x^{4/3}}{3x^{2/3}} = \frac{5(1 - x^{4/3})}{3x^{2/3}}.$$

The derivative $f'(x) = 0$ when $1 - x^{4/3} = 0$. Therefore, $f'(x) = 0$ at $x = \pm 1$. The derivative $f'(x)$ is undefined at $x = 0$. Therefore, we have three critical points: $x = 0$, $x = 1$, and $x = -1$. Consequently, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

Step 2: Since f' is continuous over each subinterval, it suffices to choose a test point x in each of the intervals from step 1 and determine the sign of f' at each of these points. The points $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are test points for these intervals.

Table: 4.6.2: First Derivative Test for $f(x) = 5x^{1/3} - x^{5/3}$.

Interval	Test Point	Sign of $f'(x) = \frac{5(1-x^{4/3})}{3x^{2/3}}$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	$\frac{(+)(-)}{+} = -$	f is decreasing.
$(-1, 0)$	$x = -\frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(1, \infty)$	$x = 2$	$\frac{(+)(-)}{+} = -$	f is decreasing.

Step 3: Since f is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0)$, f has a local minimum at $x = -1$. Since f is increasing over the interval $(-1, 0)$ and the interval $(0, 1)$, f does not have a local extremum at $x = 0$. Since f is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty)$, f has a local maximum at $x = 1$. The analytical results agree with the following graph.

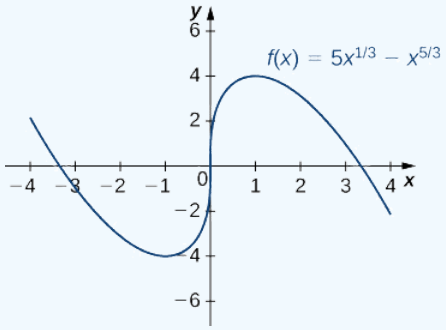


Figure 4.6.4: The function f has a local minimum at $x = -1$ and a local maximum at $x = 1$

Exercise 4.6.2

Use the first derivative test to find all local extrema for $f(x) = \dfrac{3}{x - 1}$.

Hint

The only critical point of f is $x = 1$.

Answer

f has no local extrema because f' does not change sign at $x = 1$.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure 4.6.5*a* shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. Figure 4.6.5*b* shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Definition: concavity test

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

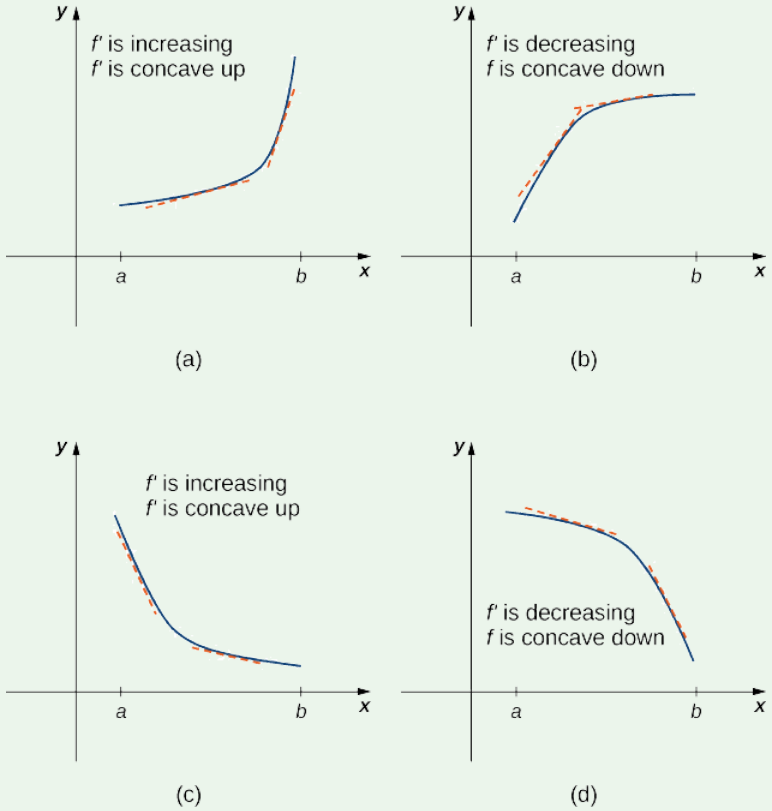


Figure 4.6.5: (a), (c) Since f' is increasing over the interval (a, b), we say f is concave up over $(a, b).(b), (d)$ Since f' is decreasing over the interval (a, b), we say f is concave down over (a, b).

In general, without having the graph of a function f, how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary 3, we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0$. Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the *concavity test*.

Test for Concavity

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I
ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (Figure 4.6.6). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a, f(a))$ is an **inflection point** of f.

Definition: inflection point

If f is continuous at a and f changes concavity at a, the point $(a,\ f(a))$ is an *inflection point* of f.

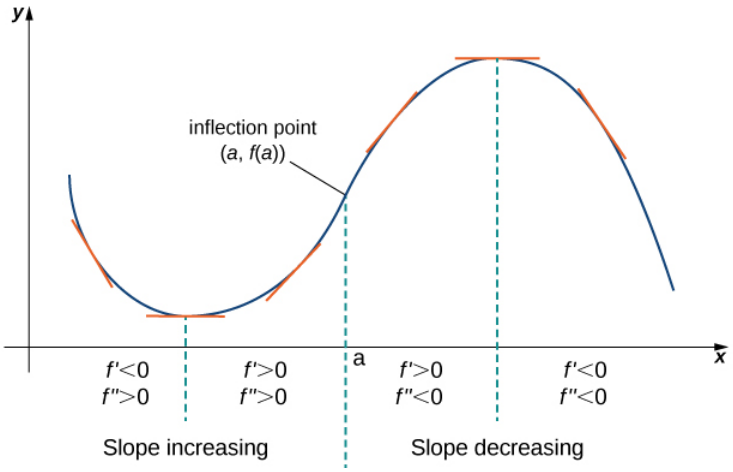


Figure 4.6.6: Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example 4.6.3: Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9$, so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers x, we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0$, we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Table: 4.6.3: Test for Concavity for $f(x) = x^3 - 6x^2 + 9x + 30$.

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	$-$	f is concave down
$(2, \infty)$	$x = 3$	$+$	f is concave up

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. Figure 4.6.7 confirms the analytical results.

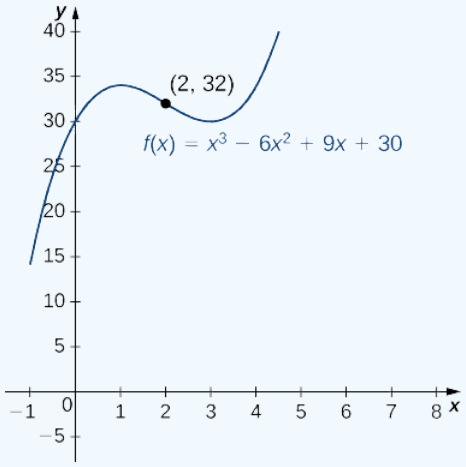


Figure 4.6.7: The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

Exercise 4.6.3

For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where f is concave up and all intervals where f is concave down.

Hint

Find where $f''(x) = 0$

Answer

f is concave up over the interval $(-\infty, \frac{1}{2})$ and concave down over the interval $(\frac{1}{2}, \infty)$

We now summarize, in Table 4.6.4, the information that the first and second derivatives of a function f provide about the graph of f, and illustrate this information in Figure 4.6.8.

Table: 4.6.4: What Derivatives Tell Us about Graphs

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

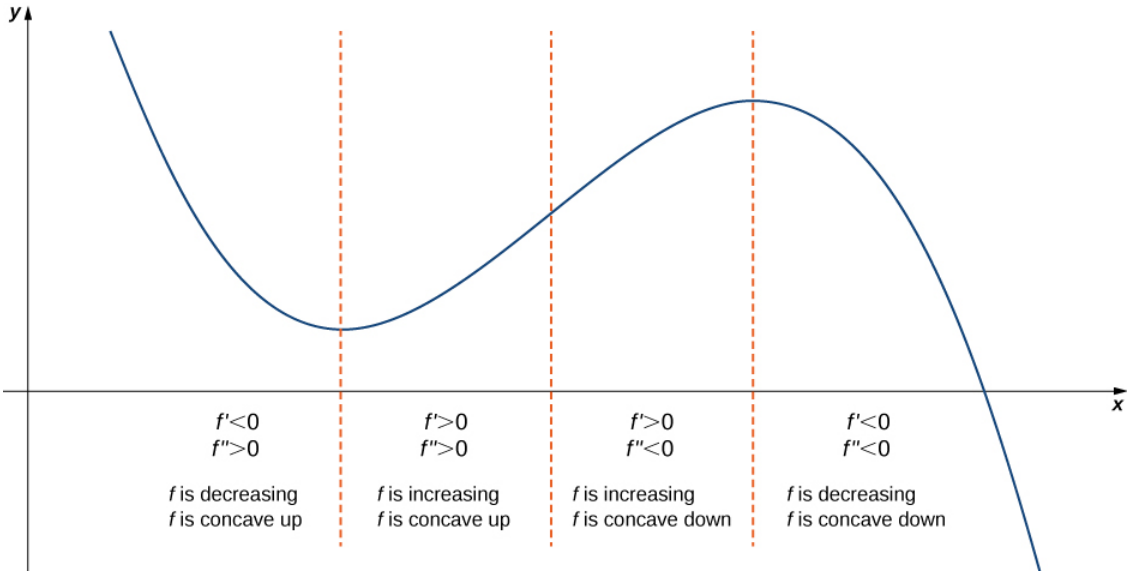


Figure 4.6.8:Consider a twice-differentiable function f over an open interval I. If $f'(x) > 0$ for all $x \in I$, the function is increasing over I. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I.

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extremum, it must occur at a critical point. However, a function need not have a local extremum at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a. Suppose $f''(a) < 0$. Since f'' is continuous over $I, f''(x) < 0$ for all $x \in I$ (Figure 4.6.9). Then, by Corollary 3, f' is a decreasing function over I. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$.

On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, then $f''(x) > 0$ for all $x \in I$ (Figure 4.6.9). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.

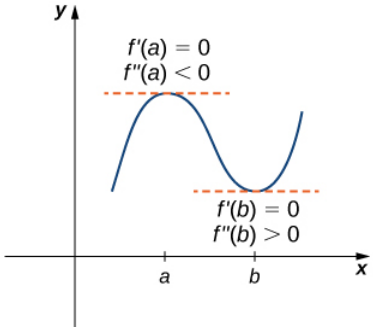


Figure 4.6.9: Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval I containing a such that for all x in I, f is increasing if $x < a$ and f is decreasing if $x > a$. As a result, f has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval I containing b such that for all x in I, f is decreasing if $x < b$ and f is increasing if $x > b$. As a result, f has a local minimum at $x = b$.

Second Derivative Test

Suppose $f'(c) = 0$ and f'' is continuous over an interval containing c.

i. If $f''(c) > 0$, then f has a local minimum at c.
ii. If $f''(c) < 0$, then f has a local maximum at c.
iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then f may have a local maximum, local minimum, or neither at c. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at $x = 0$, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0$.

Example 4.6.4: Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0$. The derivative is $f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extremum at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

Table: 4.6.5: Second Derivative Test for $f(x) = x^5 - 5x^3$.

x	$f''(x)$	Conclusion
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2(x^2 - 3)$ for $x \in (-\sqrt{3}, 0)$ and $x \in (0, \sqrt{3})$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.

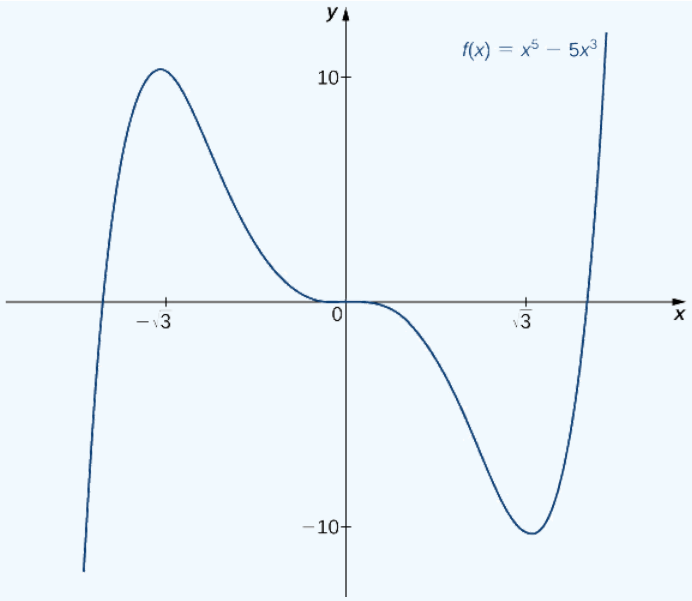


Figure 4.6.10:The function f has a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$

? Exercise 4.6.4

Consider the function $f(x) = x^3 - (\frac{3}{2})x^2 - 18x$. The points $c = 3, -2$ satisfy $f'(c) = 0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

Hint

$f''(x) = 6x - 3$

Answer

f has a local maximum at -2 and a local minimum at 3.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x \to \pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

Key Concepts

- If c is a critical point of f and $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at c.
- If c is a critical point of f and $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local minimum at c.
- If $f''(x) > 0$ over an interval I, then f is concave up over I.
- If $f''(x) < 0$ over an interval I, then f is concave down over I.
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

Glossary

concave down

if f is differentiable over an interval I and f' is decreasing over I, then f is concave down over I

concave up

if f is differentiable over an interval I and f' is increasing over I, then f is concave up over I

concavity

the upward or downward curve of the graph of a function

concavity test

suppose f is twice differentiable over an interval I; if $f'' > 0$ over I, then f is concave up over I; if $f'' <$ over I, then f is concave down over I

first derivative test

let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through c, then f has a local maximum at c; if f' changes sign from negative to positive as x increases through c, then f has a local minimum at c; if f' does not change sign as x increases through c, then f does not have a local extremum at c

inflection point

if f is continuous at c and f changes concavity at c, the point $(c, f(c))$ is an inflection point of f

second derivative test

suppose $f'(c) = 0$ and f'' is continuous over an interval containing c; if $f''(c) > 0$, then f has a local minimum at c; if $f''(c) < 0$, then f has a local maximum at c; if $f''(c) = 0$, then the test is inconclusive

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4.6E: Exercises

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4.7: Limits at Infinity and Asymptotes

Learning Objectives

- Calculate the limit of a function as x increases or decreases without bound.
- Recognize a horizontal asymptote on the graph of a function.
- Estimate the end behavior of a function as x increases or decreases without bound.
- Recognize an oblique asymptote on the graph of a function.
- Analyze a function and its derivatives to draw its graph.

We have shown how to use the first and second derivatives of a function to describe the shape of a graph. To graph a function f defined on an unbounded domain, we also need to know the behavior of f as $x \to \pm\infty$. In this section, we define limits at infinity and show how these limits affect the graph of a function. At the end of this section, we outline a strategy for graphing an arbitrary function f.

We begin by examining what it means for a function to have a finite limit at infinity. Then we study the idea of a function with an infinite limit at infinity. Back in Introduction to Functions and Graphs, we looked at vertical asymptotes; in this section we deal with horizontal and oblique asymptotes.

Limits at Infinity and Horizontal Asymptotes

Recall that $\lim\limits_{x \to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as long as x is sufficiently close to a. We can extend this idea to limits at infinity. For example, consider the function $f(x) = 2 + \frac{1}{x}$. As can be seen graphically in Figure 4.7.1 and numerically in Table 4.7.1, as the values of x get larger, the values of $f(x)$ approach 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim\limits_{x \to \infty} f(x) = 2$. Similarly, for $x < 0$, as the values $|x|$ get larger, the values of $f(x)$ approach 2. We say the limit as x approaches $-\infty$ of $f(x)$ is 2 and write $\lim\limits_{x \to -\infty} f(x) = 2$.

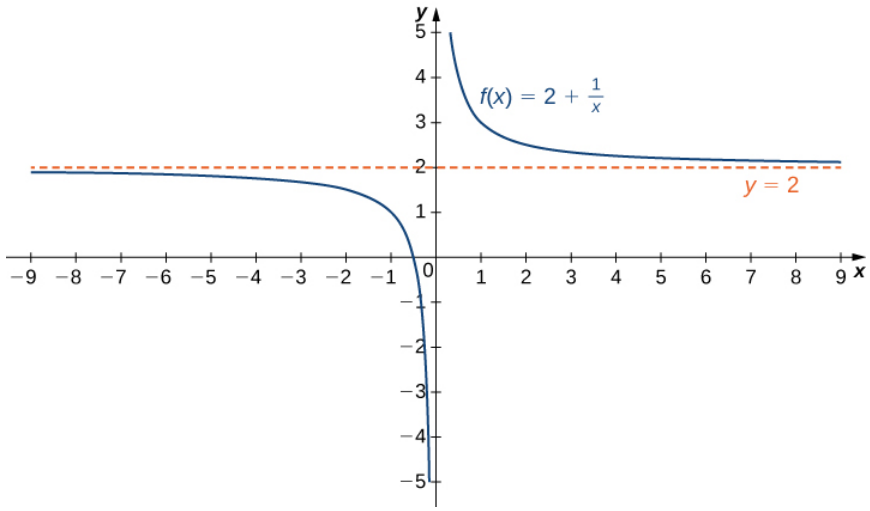


Figure 4.7.1:The function approaches the asymptote $y = 2$ as x approaches $\pm\infty$.

Table 4.7.1: Values of a function f as $x \to \pm\infty$

x	10	100	1,000	10,000
$2+\frac{1}{x}$	2.1	2.01	2.001	2.0001
x	−10	−100	−1000	−10,000
$2+\frac{1}{x}$	1.9	1.99	1.999	1.9999

More generally, for any function f, we say the limit as $x \to \infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as x is sufficiently large. In that case, we write $\lim_{x\to\infty} f(x) = L$. Similarly, we say the limit as $x \to -\infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as $x < 0$ and $|x|$ is sufficiently large. In that case, we write $\lim_{x\to-\infty} f(x) = L$. We now look at the definition of a function having a limit at infinity.

Definition: Limit at Infinity (Informal)

If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, we say the function f has a **limit at infinity** and write

$$\lim_{x\to\infty} f(x) = L.$$

If the values of $f(x)$ becomes arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function f has a limit at negative infinity and write

$$\lim_{x\to-\infty} f(x) = L.$$

If the values $f(x)$ are getting arbitrarily close to some finite value L as $x \to \infty$ or $x \to -\infty$, the graph of f approaches the line $y = L$. In that case, the line $y = L$ is a horizontal asymptote of f (Figure 4.7.2). For example, for the function $f(x) = \frac{1}{x}$, since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of $f(x) = \frac{1}{x}$.

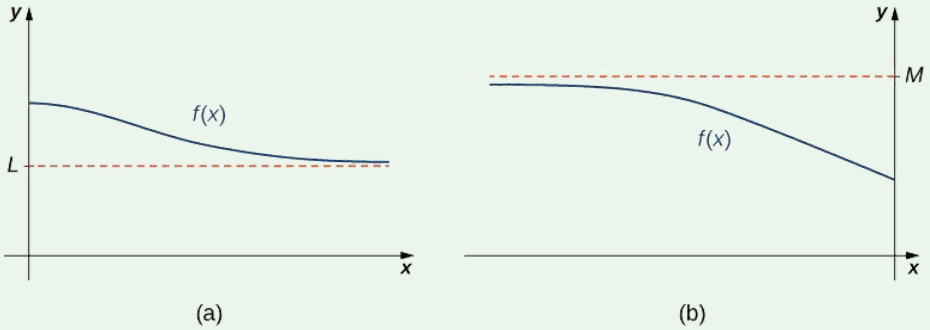


Figure 4.7.2: (a) As $x \to \infty$, the values of f are getting arbitrarily close to L. The line $y = L$ is a horizontal asymptote of f. (b) As $x \to -\infty$, the values of f are getting arbitrarily close to M. The line $y = M$ is a horizontal asymptote of f.

Definition: Horizontal Asymptote

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a *horizontal asymptote* of f.

A function cannot cross a vertical asymptote because the graph must approach infinity (or $-\infty$) from at least one direction as x approaches the vertical asymptote. However, a function may cross a horizontal asymptote. In fact, a function may cross a horizontal asymptote an unlimited number of times. For example, the function $f(x) = \frac{\cos x}{x} + 1$ shown in Figure 4.7.3 intersects the horizontal asymptote $y = 1$ an infinite number of times as it oscillates around the asymptote with ever-decreasing amplitude.

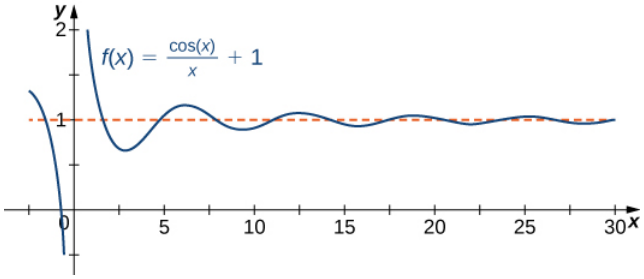


Figure 4.7.3: The graph of $f(x) = (\cos x)/x + 1$ crosses its horizontal asymptote $y = 1$ an infinite number of times.

The algebraic limit laws and squeeze theorem we introduced in Introduction to Limits also apply to limits at infinity. We illustrate how to use these laws to compute several limits at infinity.

Example 4.7.1: Computing Limits at Infinity

For each of the following functions f, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Determine the horizontal asymptote(s) for f.

a. $f(x) = 5 - \dfrac{2}{x^2}$
b. $f(x) = \dfrac{\sin x}{x}$
c. $f(x) = \tan^{-1}(x)$

Solution

a. Using the algebraic limit laws, we have

$$\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right) = \lim_{x\to\infty} 5 - 2\left(\lim_{x\to\infty}\frac{1}{x}\right)\cdot\left(\lim_{x\to\infty}\frac{1}{x}\right) = 5 - 2\cdot 0 = 5.$$

Similarly, $\lim_{x\to-\infty} f(x) = 5$. Therefore, $f(x) = 5 - \dfrac{2}{x^2}$ has a horizontal asymptote of $y = 5$ and f approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

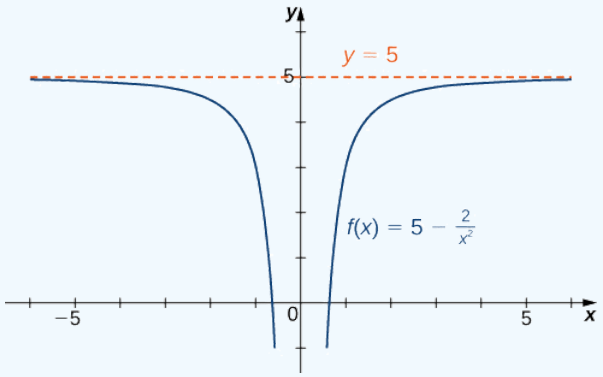


Figure 4.7.4: This function approaches a horizontal asymptote as $x \to \pm\infty$.

b. Since $-1 \le \sin x \le 1$ for all x, we have

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

for all $x \neq 0$. Also, since

$$\lim_{x\to\infty}\frac{-1}{x} = 0 = \lim_{x\to\infty}\frac{1}{x},$$

we can apply the squeeze theorem to conclude that

$$\lim_{x\to\infty}\frac{\sin x}{x} = 0.$$

Similarly,

$$\lim_{x\to-\infty}\frac{\sin x}{x} = 0.$$

Thus, $f(x) = \dfrac{\sin x}{x}$ has a horizontal asymptote of $y = 0$ and $f(x)$ approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

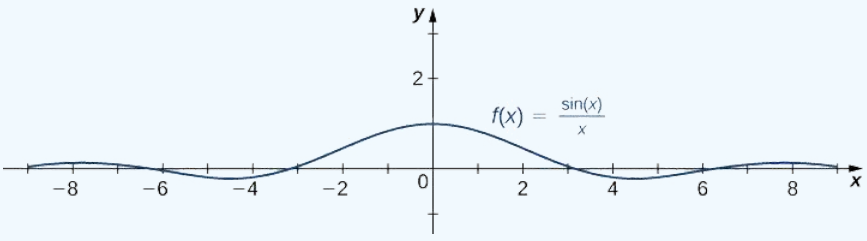


Figure 4.7.5: This function crosses its horizontal asymptote multiple times.

c. To evaluate $\lim_{x\to\infty} \tan^{-1}(x)$ and $\lim_{x\to-\infty} \tan^{-1}(x)$, we first consider the graph of $y = \tan(x)$ over the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ as shown in the following graph.

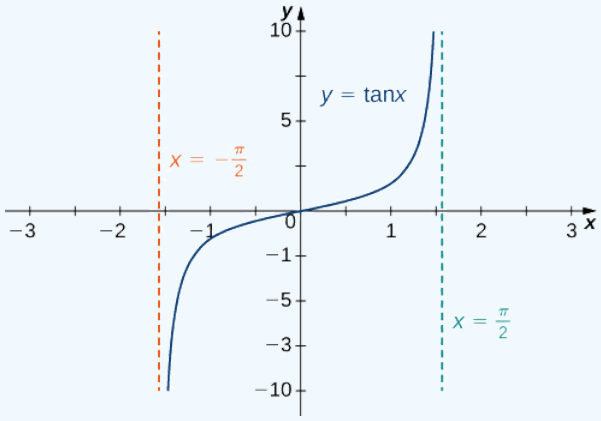


Figure 4.7.6: The graph of $y = \tan x$ has vertical asymptotes at $x = \pm\frac{\pi}{2}$

Since

$$\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty,$$

it follows that

$$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}.$$

Similarly, since

$$\lim_{x\to-\frac{\pi}{2}^+} \tan x = -\infty,$$

it follows that

$$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}.$$

As a result, $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes of $f(x) = \tan^{-1}(x)$ as shown in the following graph.

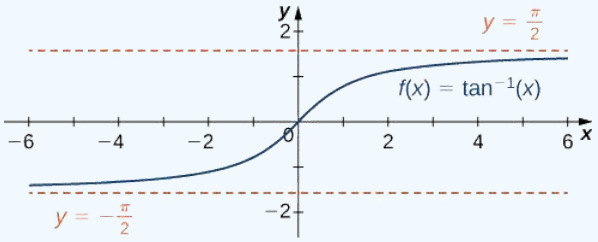


Figure 4.7.7: This function has two horizontal asymptotes.

? Exercise 4.7.1

Evaluate $\lim_{x\to-\infty} \left(3 + \frac{4}{x}\right)$ and $\lim_{x\to\infty} \left(3 + \frac{4}{x}\right)$. Determine the horizontal asymptotes of $f(x) = 3 + \frac{4}{x}$, if any.

Hint

$\lim_{x\to\pm\infty} \frac{1}{x} = 0$

Answer

Both limits are 3. The line $y = 3$ is a horizontal asymptote.

Infinite Limits at Infinity

Sometimes the values of a function f become arbitrarily large as $x \to \infty$(or as $x \to -\infty$). In this case, we write $\lim\limits_{x\to\infty} f(x) = \infty$ (or $\lim\limits_{x\to-\infty} f(x) = \infty$). On the other hand, if the values of f are negative but become arbitrarily large in magnitude as $x \to \infty$ (or as $x \to -\infty$), we write $\lim\limits_{x\to\infty} f(x) = -\infty$ (or $\lim\limits_{x\to-\infty} f(x) = -\infty$).

For example, consider the function $f(x) = x^3$. As seen in Table **4.7.2** and Figure **4.7.8**, as $x \to \infty$ the values $f(x)$ become arbitrarily large. Therefore, $\lim\limits_{x\to\infty} x^3 = \infty$. On the other hand, as $x \to -\infty$, the values of $f(x) = x^3$ are negative but become arbitrarily large in magnitude. Consequently, $\lim\limits_{x\to-\infty} x^3 = -\infty$.

Table **4.7.2**: Values of a power function as $x \to \pm\infty$

x	10	20	50	100	1000
x^3	1000	8000	125,000	1,000,000	1,000,000,000
x	−10	−20	−50	−100	−1000
x^3	−1000	−8000	−125,000	−1,000,000	−1,000,000,000

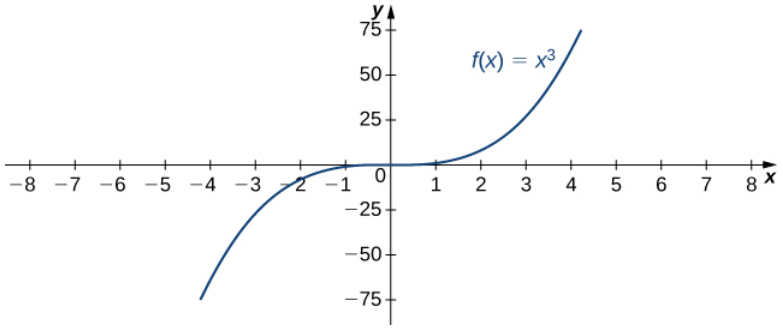


Figure **4.7.8**: For this function, the functional values approach ±infinity as $x \to \pm\infty$.

Definition: Infinite Limit at Infinity (Informal)

We say a function f has an infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = \infty.$$

if $f(x)$ becomes arbitrarily large for x sufficiently large. We say a function has a negative infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = -\infty.$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. Similarly, we can define infinite limits as $x \to -\infty$.

Formal Definitions

Earlier, we used the terms arbitrarily close, arbitrarily large, and sufficiently large to define limits at infinity informally. Although these terms provide accurate descriptions of limits at infinity, they are not precise mathematically. Here are more formal definitions of limits at infinity. We then look at how to use these definitions to prove results involving limits at infinity.

Definition: Limit at Infinity (Formal)

We say a function f has a *limit at infinity*, if there exists a real number L such that for all $\varepsilon > 0$, there exists $N > 0$ such that

$$|f(x) - L| < \varepsilon$$

for all $x > N$. in that case, we write

$$\lim_{x\to\infty} f(x) = L$$

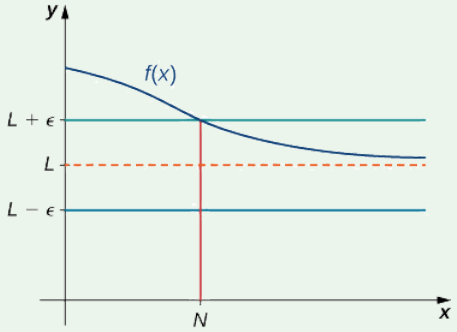


Figure 4.7.9: For a function with a limit at infinity, for all $x > N, |f(x) - L| < \varepsilon$.

Earlier in this section, we used graphical evidence in Figure 4.7.1 and numerical evidence in Table 4.7.1 to conclude that $\lim_{x\to\infty}\left(2+\frac{1}{x}\right)=2$. Here we use the formal definition of limit at infinity to prove this result rigorously.

Example 4.7.2:

Use the formal definition of limit at infinity to prove that $\lim_{x\to\infty}\left(2+\frac{1}{x}\right)=2$.

Solution

Let $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon}$. Therefore, for all $x > N$, we have

$$\left|2+\frac{1}{x}-2\right| = \left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{N} = \varepsilon$$

Exercise 4.7.2

Use the formal definition of limit at infinity to prove that $\lim_{x\to\infty}\left(3-\frac{1}{x^2}\right)=3$.

Hint

Let $N = \frac{1}{\sqrt{\varepsilon}}$.

Answer

Let $\varepsilon > 0$. Let $N = \frac{1}{\sqrt{\varepsilon}}$. Therefore, for all $x > N$, we have

$$\left|3-\frac{1}{x^2}-3\right| = \frac{1}{x^2} < \frac{1}{N^2} = \varepsilon$$

Therefore, $\lim_{x\to\infty}(3-1/x^2)=3$.

We now turn our attention to a more precise definition for an infinite limit at infinity.

Definition: Infinite Limit at Infinity (Formal)

We say a function f has an *infinite limit at infinity* and write

$$\lim_{x\to\infty} f(x) = \infty$$

if for all $M > 0$, there exists an $N > 0$ such that

$$f(x) > M$$

for all $x > N$ (see Figure 4.7.10).

We say a function has a negative infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = -\infty$$

if for all $M < 0$, there exists an $N > 0$ such that

$$f(x) < M$$

for all $x > N$.

Similarly we can define limits as $x \to -\infty$.

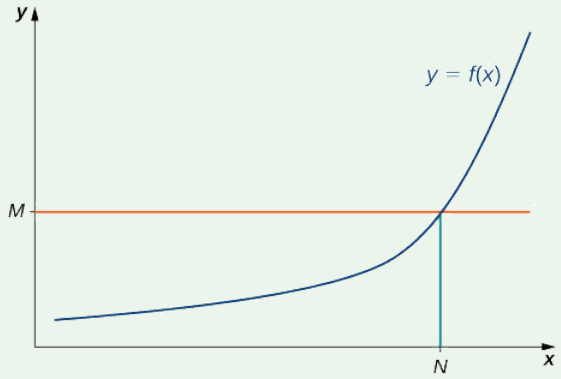


Figure 4.7.10: For a function with an infinite limit at infinity, for all $x > N$, $f(x) > M$.

Earlier, we used graphical evidence (Figure 4.7.8) and numerical evidence (Table 4.7.2) to conclude that $\lim_{x\to\infty} x^3 = \infty$. Here we use the formal definition of infinite limit at infinity to prove that result.

Example 4.7.3

Use the formal definition of infinite limit at infinity to prove that $\lim_{x\to\infty} x^3 = \infty$.

Solution

Let $M > 0$. Let $N = \sqrt[3]{M}$. Then, for all $x > N$, we have

$$x^3 > N^3 = (\sqrt[3]{M})^3 = M.$$

Therefore, $\lim_{x\to\infty} x^3 = \infty$.

Exercise 4.7.3

Use the formal definition of infinite limit at infinity to prove that $\lim_{x\to\infty} 3x^2 = \infty$.

Hint

Let $N = \sqrt{\frac{M}{3}}$.

Answer

Let $M > 0$. Let $N = \sqrt{\frac{M}{3}}$. Then, for all $x > N$, we have

$$3x^2 > 3N^2 = 3\left(\sqrt{\frac{M}{3}}\right)^2 = \frac{3M}{3} = M$$

End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

1. The function $f(x)$ approaches a horizontal asymptote $y = L$.
2. The function $f(x) \to \infty$ or $f(x) \to -\infty$.
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior.

Let's consider several classes of functions here and look at the different types of end behaviors for these functions.

End Behavior for Polynomial Functions

Consider the power function $f(x) = x^n$ where n is a positive integer. From Figure 4.7.11 and Figure 4.7.12, we see that

$$\lim_{x \to \infty} x^n = \infty; \; n = 1, 2, 3, \ldots$$

and

$$\lim_{x \to -\infty} x^n = \begin{cases} \infty, & n = 2, 4, 6, \ldots \\ -\infty, & n = 1, 3, 5, \ldots . \end{cases}$$

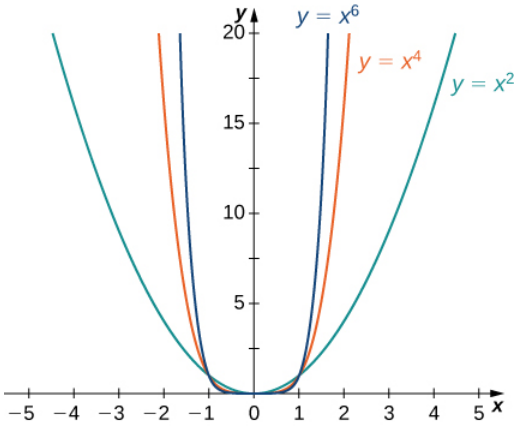


Figure 4.7.11: For power functions with an even power n, $\lim_{x \to \infty} x^n = \infty = \lim_{x \to -\infty} x^n$.

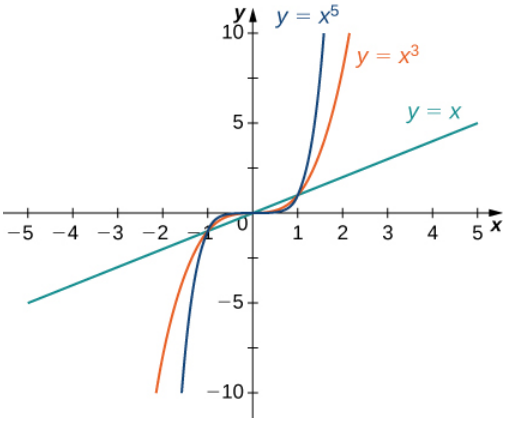


Figure 4.7.12: For power functions with an odd power n, $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$.

Using these facts, it is not difficult to evaluate $\lim_{x \to \infty} cx^n$ and $\lim_{x \to -\infty} cx^n$, where c is any constant and n is a positive integer. If $c > 0$, the graph of $y = cx^n$ is a vertical stretch or compression of $y = x^n$, and therefore

$$\lim_{x \to \infty} cx^n = \lim_{x \to \infty} x^n \text{ and } \lim_{x \to -\infty} cx^n = \lim_{x \to -\infty} x^n \text{ if } c > 0.$$

If $c < 0$, the graph of $y = cx^n$ is a vertical stretch or compression combined with a reflection about the x-axis, and therefore

$$\lim_{x \to \infty} cx^n = -\lim_{x \to \infty} x^n \text{ and } \lim_{x \to -\infty} cx^n = -\lim_{x \to -\infty} x^n \text{ if } c < 0.$$

If $c = 0, y = cx^n = 0$, in which case $\lim_{x \to \infty} cx^n = 0 = \lim_{x \to -\infty} cx^n$.

Example 4.7.4: Limits at Infinity for Power Functions

For each function f, evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

a. $f(x) = -5x^3$

b. $f(x) = 2x^4$

Solution

a. Since the coefficient of x^3 is -5, the graph of $f(x) = -5x^3$ involves a vertical stretch and reflection of the graph of $y = x^3$ about the x-axis. Therefore, $\lim_{x\to\infty}(-5x^3) = -\infty$ and $\lim_{x\to-\infty}(-5x^3) = \infty$.

b. Since the coefficient of x^4 is 2, the graph of $f(x) = 2x^4$ is a vertical stretch of the graph of $y = x^4$. Therefore, $\lim_{x\to\infty} 2x^4 = \infty$ and $\lim_{x\to-\infty} 2x^4 = \infty$.

? Exercise 4.7.4

Let $f(x) = -3x^4$. Find $\lim_{x\to\infty} f(x)$.

Hint

The coefficient -3 is negative.

Answer

$-\infty$

We now look at how the limits at infinity for power functions can be used to determine $\lim_{x\to\pm\infty} f(x)$ for any polynomial function f. Consider a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a^1 x + a^0$$

of degree $n \geq 1$ so that $a_n \neq 0$.

Factoring, we see that

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \ldots + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}\right).$$

As $x \to \pm\infty$, all the terms inside the parentheses approach zero except the first term. We conclude that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n.$$

For example, the function $f(x) = 5x^3 - 3x^2 + 4$ behaves like $g(x) = 5x^3$ as $x \to \pm\infty$ as shown in Figure 4.7.13 and Table 4.7.3.

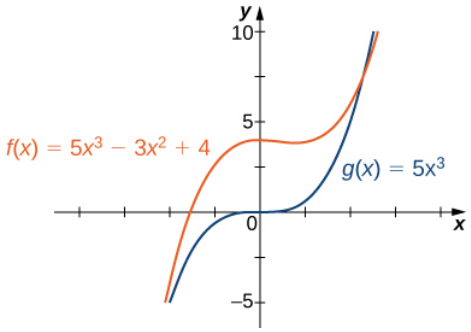


Figure 4.7.13: The end behavior of a polynomial is determined by the behavior of the term with the largest exponent.

Table 4.7.3: A polynomial's end behavior is determined by the term with the largest exponent

x	10	100	1000
$f(x) = 5x^3 - 3x^2 + 4$	4704	4,970,004	4,997,000,004
$g(x) = 5x^3$	5000	5,000,000	5,000,000,000
x	−10	−100	−000
$f(x) = 5x^3 - 3x^2 + 4$	−5296	−5,029,996	−5,002,999,996
$g(x) = 5x^3$	−5000	−5,000,000	−5,000,000,000

End Behavior for Algebraic Functions

The end behavior for rational functions and functions involving radicals is a little more complicated than for polynomials. In Example 4.7.5, we show that the limits at infinity of a rational function $f(x) = \frac{p(x)}{q(x)}$ depend on the relationship between the degree of the numerator and the degree of the denominator. To evaluate the limits at infinity for a rational function, we divide the numerator and denominator by the highest power of x appearing in the denominator. This determines which term in the overall expression dominates the behavior of the function at large values of x.

Example 4.7.5: Determining End Behavior for Rational Functions

For each of the following functions, determine the limits as $x \to \infty$ and $x \to -\infty$. Then, use this information to describe the end behavior of the function.

a. $f(x) = \frac{3x-1}{2x+5}$ (Note: The degree of the numerator and the denominator are the same.)

b. $f(x) = \frac{3x^2+2x}{4x^3-5x+7}$ (Note: The degree of numerator is less than the degree of the denominator.)

c. $f(x) = \frac{3x^2+4x}{x+2}$ (Note: The degree of the numerator is greater than the degree of the denominator.)

Solution

a. The highest power of x in the denominator is x. Therefore, dividing the numerator and denominator by x and applying the algebraic limit laws, we see that

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{3x-1}{2x+5} &= \lim_{x\to\pm\infty} \frac{3-1/x}{2+5/x} \\ &= \frac{\lim_{x\to\pm\infty}(3-1/x)}{\lim_{x\to\pm\infty}(2+5/x)} \\ &= \frac{\lim_{x\to\pm\infty} 3 - \lim_{x\to\pm\infty} 1/x}{\lim_{x\to\pm\infty} 2 + \lim_{x\to\pm\infty} 5/x} \\ &= \frac{3-0}{2+0} = \frac{3}{2}.\end{aligned}$$

Since $\lim_{x\to\pm\infty} f(x) = \frac{3}{2}$, we know that $y = \frac{3}{2}$ is a horizontal asymptote for this function as shown in the following graph.

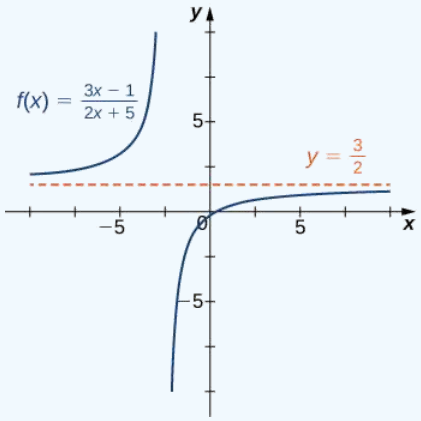


Figure **4.7.14**: The graph of this rational function approaches a horizontal asymptote as $x \to \pm\infty$.

b. Since the largest power of x appearing in the denominator is x^3, divide the numerator and denominator by x^3. After doing so and applying algebraic limit laws, we obtain

$$\lim_{x\to\pm\infty} \frac{3x^2+2x}{4x^3-5x+7} = \lim_{x\to\pm\infty} \frac{3/x+2/x^2}{4-5/x^2+7/x^3} = \frac{3\cdot 0+2\cdot 0}{4-5\cdot 0+7\cdot 0} = \frac{0}{4} = 0.$$

Therefore f has a horizontal asymptote of $y = 0$ as shown in the following graph.

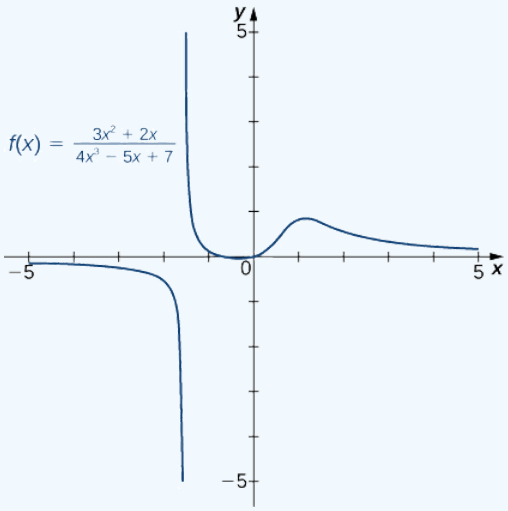


Figure **4.7.15**: The graph of this rational function approaches the horizontal asymptote $y = 0$ as $x \to \pm\infty$.

c. Dividing the numerator and denominator by x, we have

$$\lim_{x\to\pm\infty} \frac{3x^2+4x}{x+2} = \lim_{x\to\pm\infty} \frac{3x+4}{1+2/x}.$$

As $x \to \pm\infty$, the denominator approaches 1. As $x \to \infty$, the numerator approaches $+\infty$. As $x \to -\infty$, the numerator approaches $-\infty$. Therefore $\lim_{x\to\infty} f(x) = \infty$, whereas $\lim_{x\to-\infty} f(x) = -\infty$ as shown in the following figure.

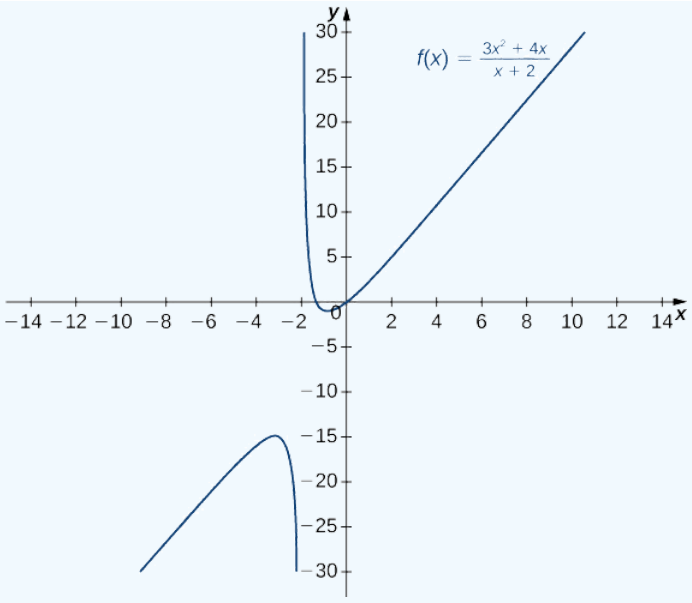


Figure 4.7.16: As $x \to \infty$, the values $f(x) \to \infty$. As $x \to -\infty$, the values $f(x) \to -\infty$.

? Exercise 4.7.5

Evaluate $\lim_{x \to \pm\infty} \dfrac{3x^2 + 2x - 1}{5x^2 - 4x + 7}$ and use these limits to determine the end behavior of $f(x) = \dfrac{3x^2 + 2x - 1}{5x^2 - 4x + 7}$.

Hint

Divide the numerator and denominator by x^2.

Answer

$\frac{3}{5}$

Before proceeding, consider the graph of $f(x) = \dfrac{3x^2 + 4x}{x + 2}$ shown in Figure 4.7.16. As $x \to \infty$ and $x \to -\infty$, the graph of f appears almost linear. Although f is certainly not a linear function, we now investigate why the graph of f seems to be approaching a linear function. First, using long division of polynomials, we can write

$$f(x) = \frac{3x^2 + 4x}{x + 2} = 3x - 2 + \frac{4}{x + 2}.$$

Since $\dfrac{4}{x + 2} \to 0$ as $x \to \pm\infty$, we conclude that

$$\lim_{x \to \pm\infty} (f(x) - (3x - 2)) = \lim_{x \to \pm\infty} \frac{4}{x + 2} = 0.$$

Therefore, the graph of f approaches the line $y = 3x - 2$ as $x \to \pm\infty$. This line is known as an **oblique asymptote** for f (Figure 4.7.17).

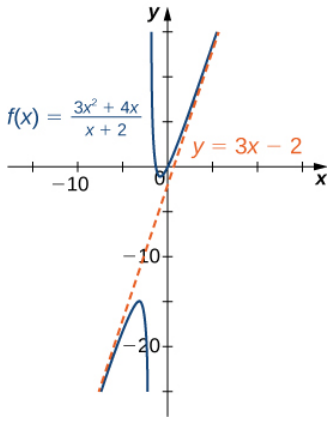


Figure 4.7.17: The graph of the rational function $f(x) = (3x^2 + 4x)/(x + 2)$ approaches the oblique asymptote $y = 3x - 2$ as $x \to \pm\infty$.

We can summarize the results of Example 4.7.5 to make the following conclusion regarding end behavior for rational functions. Consider a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0},$$

where $a_n \neq 0$ and $b_m \neq 0$.

1. If the degree of the numerator is the same as the degree of the denominator ($n = m$), then f has a horizontal asymptote of $y = a_n/b_m$ as $x \to \pm\infty$.
2. If the degree of the numerator is less than the degree of the denominator ($n < m$), then f has a horizontal asymptote of $y = 0$ as $x \to \pm\infty$.
3. If the degree of the numerator is greater than the degree of the denominator ($n > m$), then f does not have a horizontal asymptote. The limits at infinity are either positive or negative infinity, depending on the signs of the leading terms. In addition, using long division, the function can be rewritten as

   $$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

   where the degree of $r(x)$ is less than the degree of $q(x)$. As a result, $\lim_{x\to\pm\infty} r(x)/q(x) = 0$. Therefore, the values of $[f(x) - g(x)]$ approach zero as $x \to \pm\infty$. If the degree of $p(x)$ is exactly one more than the degree of $q(x)$ (i.e., $n = m + 1$), the function $g(x)$ is a linear function. In this case, we call $g(x)$ an oblique asymptote.

Now let's consider the end behavior for functions involving a radical.

Example 4.7.6: Determining End Behavior for a Function Involving a Radical

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{3x - 2}{\sqrt{4x^2 + 5}}$ and describe the end behavior of f.

Solution

Let's use the same strategy as we did for rational functions: divide the numerator and denominator by a power of x. To determine the appropriate power of x, consider the expression $\sqrt{4x^2 + 5}$ in the denominator. Since

$$\sqrt{4x^2 + 5} \approx \sqrt{4x^2} = 2|x|$$

for large values of x in effect x appears just to the first power in the denominator. Therefore, we divide the numerator and denominator by $|x|$. Then, using the fact that $|x| = x$ for $x > 0, |x| = -x$ for $x < 0$, and $|x| = \sqrt{x^2}$ for all x, we calculate the limits as follows:

$$\lim_{x\to\infty}\frac{3x-2}{\sqrt{4x^2+5}}=\lim_{x\to\infty}\frac{(1/|x|)(3x-2)}{(1/|x|)\sqrt{4x^2+5}}$$
$$=\lim_{x\to\infty}\frac{(1/x)(3x-2)}{\sqrt{(1/x^2)(4x^2+5)}}$$
$$=\lim_{x\to\infty}\frac{3-2/x}{\sqrt{4+5/x^2}}=\frac{3}{\sqrt{4}}=\frac{3}{2}$$

$$\lim_{x\to-\infty}\frac{3x-2}{\sqrt{4x^2+5}}=\lim_{x\to-\infty}\frac{(1/|x|)(3x-2)}{(1/|x|)\sqrt{4x^2+5}}$$
$$=\lim_{x\to-\infty}\frac{(-1/x)(3x-2)}{\sqrt{(1/x^2)(4x^2+5)}}$$
$$=\lim_{x\to-\infty}\frac{-3+2/x}{\sqrt{4+5/x^2}}=\frac{-3}{\sqrt{4}}=\frac{-3}{2}.$$

Therefore, $f(x)$ approaches the horizontal asymptote $y=\frac{3}{2}$ as $x\to\infty$ and the horizontal asymptote $y=-\frac{3}{2}$ as $x\to-\infty$ as shown in the following graph.

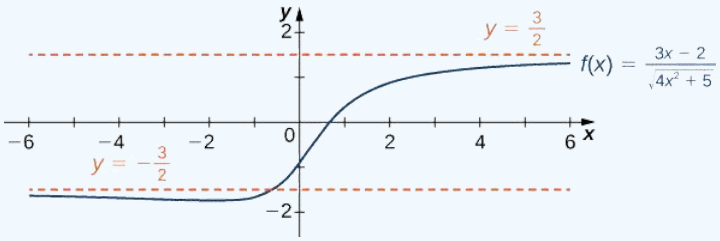


Figure 4.7.18: This function has two horizontal asymptotes and it crosses one of the asymptotes.

? Exercise 4.7.6

Evaluate $\lim_{x\to\infty}\frac{\sqrt{3x^2+4}}{x+6}$.

Hint

Divide the numerator and denominator by x.

Answer

$\sqrt{3}$

Determining End Behavior for Transcendental Functions

The six basic trigonometric functions are periodic and do not approach a finite limit as $x\to\pm\infty$. For example, $\sin x$ oscillates between 1 and −1 (Figure 4.7.19). The tangent function x has an infinite number of vertical asymptotes as $x\to\pm\infty$; therefore, it does not approach a finite limit nor does it approach $\pm\infty$ as $x\to\pm\infty$ as shown in Figure 4.7.20.

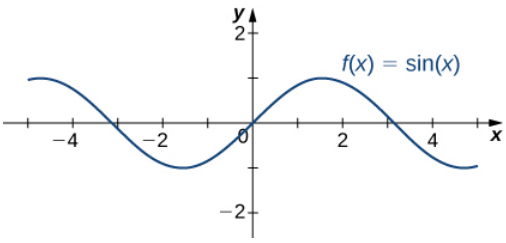


Figure 4.7.19: The function $f(x)=\sin x$ oscillates between 1 and −1 as $x\to\pm\infty$

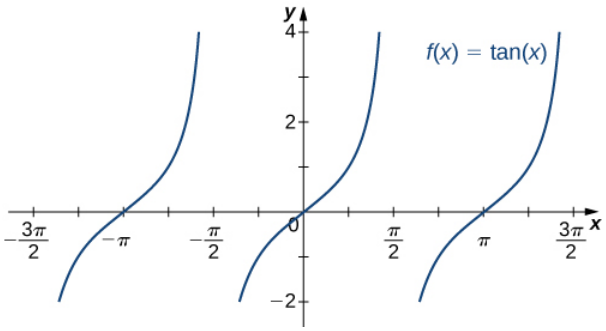


Figure 4.7.20: The function $f(x) = \tan x$ does not approach a limit and does not approach $\pm\infty$ as $x \to \pm\infty$

Recall that for any base $b > 0,\ b \neq 1$, the function $y = b^x$ is an exponential function with domain $(-\infty, \infty)$ and range $(0, \infty)$. If $b > 1,\ y = b^x$ is increasing over $(-\infty, \infty)$. If $0 < b < 1,\ y = b^x$ is decreasing over $(-\infty, \infty)$. For the natural exponential function $f(x) = e^x$, $e \approx 2.718 > 1$. Therefore, $f(x) = e^x$ is increasing on $(-\infty, \infty)$ and the range is $(0, \infty)$. The exponential function $f(x) = e^x$ approaches ∞ as $x \to \infty$ and approaches 0 as $x \to -\infty$ as shown in Table 4.7.4 and Figure 4.7.21.

Table 4.7.4: End behavior of the natural exponential function

x	−5	−2	0	2	5
e^x	0.00674	0.135	1	7.389	148.413

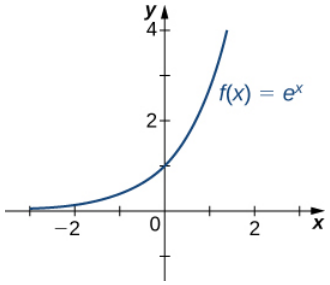


Figure 4.7.21: The exponential function approaches zero as $x \to -\infty$ and approaches ∞ as $x \to \infty$.

Recall that the natural logarithm function $f(x) = \ln(x)$ is the inverse of the natural exponential function $y = e^x$. Therefore, the domain of $f(x) = \ln(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of $f(x) = \ln(x)$ is the reflection of the graph of $y = e^x$ about the line $y = x$. Therefore, $\ln(x) \to -\infty$ as $x \to 0^+$ and $\ln(x) \to \infty$ as $x \to \infty$ as shown in Figure 4.7.22 and Table 4.7.5.

Table 4.7.5: End behavior of the natural logarithm function

x	0.01	0.1	1	10	100
$\ln(x)$	−4.605	−2.303	0	2.303	4.605

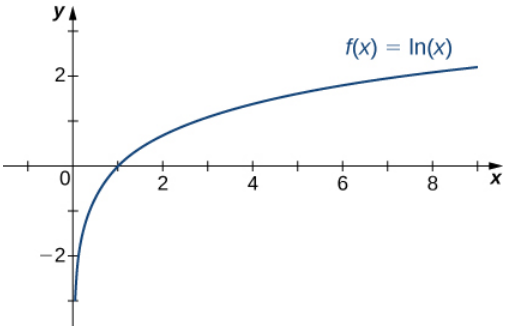


Figure 4.7.22: The natural logarithm function approaches ∞ as $x \to \infty$.

Example 4.7.7: Determining End Behavior for a Transcendental Function

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{2+3e^x}{7-5e^x}$ and describe the end behavior of f.

Solution

To find the limit as $x \to \infty$, divide the numerator and denominator by e^x:

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2+3e^x}{7-5e^x} \\ &= \lim_{x\to\infty} \frac{(2/e^x)+3}{(7/e^x)-5.}\end{aligned}$$

As shown in Figure 4.7.21, $e^x \to \infty$ as $x \to \infty$. Therefore,

$$\lim_{x\to\infty} \frac{2}{e^x} = 0 = \lim_{x\to\infty} \frac{7}{e^x}.$$

We conclude that $\lim_{x\to\infty} f(x) = -\dfrac{3}{5}$, and the graph of f approaches the horizontal asymptote $y = -\frac{3}{5}$ as $x \to \infty$. To find the limit as $x \to -\infty$, use the fact that $e^x \to 0$ as $x \to -\infty$ to conclude that $\lim_{x\to-\infty} f(x) = \dfrac{2}{7}$, and therefore the graph of $f(x)$ approaches the horizontal asymptote $y = \frac{2}{7}$ as $x \to -\infty$.

Exercise 4.7.7

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{3e^x - 4}{5e^x + 2}$.

Hint

$\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.

Answer

$\lim_{x\to\infty} f(x) = \dfrac{3}{5}, \quad \lim_{x\to-\infty} f(x) = -2$

Guidelines for Drawing the Graph of a Function

We now have enough analytical tools to draw graphs of a wide variety of algebraic and transcendental functions. Before showing how to graph specific functions, let's look at a general strategy to use when graphing any function.

Problem-Solving Strategy: Drawing the Graph of a Function

Given a function f, use the following steps to sketch a graph of f:

1. Determine the domain of the function.
2. Locate the x- and y-intercepts.
3. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to determine the end behavior. If either of these limits is a finite number L, then $y = L$ is a horizontal asymptote. If either of these limits is ∞ or $-\infty$, determine whether f has an oblique asymptote. If f is a rational function such that $f(x) = \dfrac{p(x)}{q(x)}$, where the degree of the numerator is greater than the degree of the denominator, then f can be written as

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x),}$$

where the degree of $r(x)$ is less than the degree of $q(x)$. The values of $f(x)$ approach the values of $g(x)$ as $x \to \pm\infty$. If $g(x)$ is a linear function, it is known as an oblique asymptote.
4. Determine whether f has any vertical asymptotes.
5. Calculate f'. Find all critical points and determine the intervals where f is increasing and where f is decreasing. Determine whether f has any local extrema.

6. Calculate f''. Determine the intervals where f is concave up and where f is concave down. Use this information to determine whether f has any inflection points. The second derivative can also be used as an alternate means to determine or verify that f has a local extremum at a critical point.

Now let's use this strategy to graph several different functions. We start by graphing a polynomial function.

Example 4.7.8: Sketching a Graph of a Polynomial

Sketch a graph of $f(x) = (x-1)^2(x+2)$.

Solution

Step 1: Since f is a polynomial, the domain is the set of all real numbers.

Step 2: When $x = 0$, $f(x) = 2$. Therefore, the y-intercept is $(0, 2)$. To find the x-intercepts, we need to solve the equation $(x-1)^2(x+2) = 0$, which gives us the x-intercepts $(1, 0)$ and $(-2, 0)$

Step 3: We need to evaluate the end behavior of f. As $x \to \infty$, $(x-1)^2 \to \infty$ and $(x+2) \to \infty$. Therefore, $\lim\limits_{x \to \infty} f(x) = \infty$.

As $x \to -\infty$, $(x-1)^2 \to \infty$ and $(x+2) \to -\infty$. Therefore, $\lim\limits_{x \to -\infty} f(x) = -\infty$.

To get even more information about the end behavior of f, we can multiply the factors of f. When doing so, we see that

$$f(x) = (x-1)^2(x+2) = x^3 - 3x + 2.$$

Since the leading term of f is x^3, we conclude that f behaves like $y = x^3$ as $x \to \pm\infty$.

Step 4: Since f is a polynomial function, it does not have any vertical asymptotes.

Step 5: The first derivative of f is

$$f'(x) = 3x^2 - 3.$$

Therefore, f has two critical points: $x = 1, -1$. Divide the interval $(-\infty, \infty)$ into the three smaller intervals: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. Then, choose test points $x = -2, x = 0$, and $x = 2$ from these intervals and evaluate the sign of $f'(x)$ at each of these test points, as shown in the following table.

Interval	Test point	Sign of Derivative $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$	Conclusion
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing
$(-1, 1)$	$x = 0$	$(+)(-)(+) = -$	f decreasing
$(1, \infty)$	$x = 2$	$(+)(+)(+) = +$	f is increasing

From the table, we see that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. Evaluating $f(x)$ at those two points, we find that the local maximum value is $f(-1) = 4$ and the local minimum value is $f(1) = 0$.

Step 6: The second derivative of f is

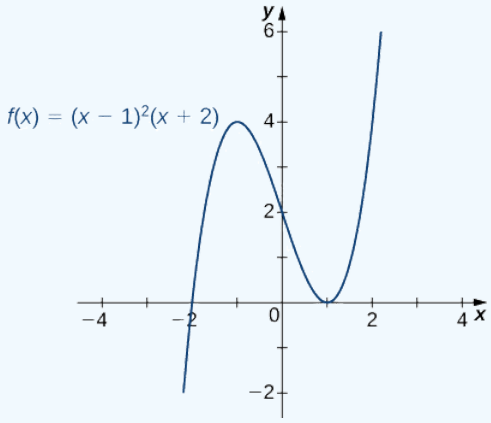
$$f''(x) = 6x.$$

The second derivative is zero at $x = 0$. Therefore, to determine the concavity of f, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$ and $(0, \infty)$, and choose test points $x = -1$ and $x = 1$ to determine the concavity of f on each of these smaller intervals as shown in the following table.

Interval	Test Point	Sign of $f''(x) = 6x$	Conclusion
$(-\infty, 0)$	$x = -1$	$-$	f is concave down.
$(0, \infty)$	$x = 1$	$+$	f is concave up.

We note that the information in the preceding table confirms the fact, found in step 5, that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. In addition, the information found in step 5—namely, f has a local maximum at $x = -1$ and a local minimum at $x = 1$, and $f'(x) = 0$ at those points—combined with the fact that f'' changes sign only at $x = 0$ confirms the results found in step 6 on the concavity of f.

Combining this information, we arrive at the graph of $f(x) = (x-1)^2(x+2)$ shown in the following graph.



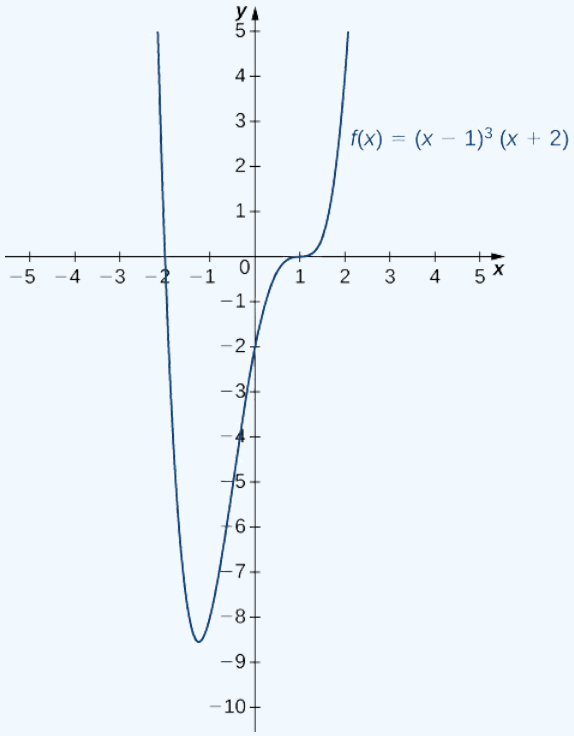
Exercise 4.7.8

Sketch a graph of $f(x) = (x-1)^3(x+2)$.

Hint

f is a fourth-degree polynomial.

Answer



Example 4.7.9: Sketching a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2}{1-x^2}$.

Solution

Step 1: The function f is defined as long as the denominator is not zero. Therefore, the domain is the set of all real numbers x except $x = \pm 1$.

Step 2: Find the intercepts. If $x = 0$, then $f(x) = 0$, so 0 is an intercept. If $y = 0$, then $\dfrac{x^2}{1-x^2} = 0$, which implies $x = 0$. Therefore, $(0, 0)$ is the only intercept.

Step 3: Evaluate the limits at infinity. Since $f$is a rational function, divide the numerator and denominator by the highest power in the denominator: x^2.We obtain

$$\lim_{x \to \pm\infty} \frac{x^2}{1-x^2} = \lim_{x \to \pm\infty} \frac{1}{\frac{1}{x^2} - 1} = -1.$$

Therefore, f has a horizontal asymptote of $y = -1$ as $x \to \infty$ and $x \to -\infty$.

Step 4: To determine whether f has any vertical asymptotes, first check to see whether the denominator has any zeroes. We find the denominator is zero when $x = \pm 1$. To determine whether the lines $x = 1$ or $x = -1$ are vertical asymptotes of f, evaluate $\lim\limits_{x \to 1} f(x)$ and $\lim\limits_{x \to -1} f(x)$. By looking at each one-sided limit as $x \to 1$, we see that

$$\lim_{x \to 1^+} \frac{x^2}{1-x^2} = -\infty \text{ and } \lim_{x \to 1^-} \frac{x^2}{1-x^2} = \infty.$$

In addition, by looking at each one-sided limit as $x \to -1$, we find that

$$\lim_{x \to -1^+} \frac{x^2}{1-x^2} = \infty \text{ and } \lim_{x \to -1^-} \frac{x^2}{1-x^2} = -\infty.$$

Step 5: Calculate the first derivative:

$$f'(x) = \frac{(1-x^2)(2x) - x^2(-2x)}{\left(1-x^2\right)^2} = \frac{2x}{\left(1-x^2\right)^2}.$$

Critical points occur at points x where $f'(x) = 0$ or $f'(x)$ is undefined. We see that $f'(x) = 0$ when $x = 0$. The derivative f' is not undefined at any point in the domain of f. However, $x = \pm 1$ are not in the domain of f. Therefore, to determine where f is increasing and where f is decreasing, divide the interval $(-\infty, \infty)$ into four smaller intervals: $(-\infty, -1), (-1, 0), (0, 1)$, and $(1, \infty)$, and choose a test point in each interval to determine the sign of $f'(x)$ in each of these intervals. The values $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are good choices for test points as shown in the following table.

Interval	Test point	Sign of $f'(x) = \frac{2x}{(1-x^2)^2}$	Conclusion
$(-\infty, -1)$	$x = -2$	$-/+ = -$	f is decreasing.
$(-1, 0)$	$x = -1/2$	$-/+ = -$	f is decreasing.
$(0, 1)$	$x = 1/2$	$+/+ = +$	f is increasing.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing.

From this analysis, we conclude that f has a local minimum at $x = 0$ but no local maximum.

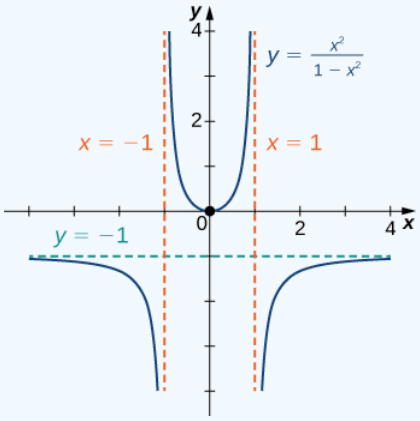
Step 6: Calculate the second derivative:

$$f''(x) = \frac{(1-x^2)^2(2) - 2x(2(1-x^2)(-2x))}{(1-x^2)^4}$$
$$= \frac{(1-x^2)[2(1-x^2)+8x^2]}{\left(1-x^2\right)^4}$$
$$= \frac{2(1-x^2)+8x^2}{\left(1-x^2\right)^3}$$
$$= \frac{6x^2+2}{\left(1-x^2\right)^3}.$$

To determine the intervals where f is concave up and where f is concave down, we first need to find all points x where $f''(x) = 0$ or $f''(x)$ is undefined. Since the numerator $6x^2 + 2 \neq 0$ for any x, $f''(x)$ is never zero. Furthermore, f'' is not undefined for any x in the domain of f. However, as discussed earlier, $x = \pm 1$ are not in the domain of f. Therefore, to determine the concavity of f, we divide the interval $(-\infty, \infty)$ into the three smaller intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, and choose a test point in each of these intervals to evaluate the sign of $f''(x)$. The values $x = -2$, $x = 0$, and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{6x^2+2}{(1-x^2)^3}$	Conclusion
$(-\infty, -1)$	$x = -2$	$+/- = -$	f is concave down.
$(-1, 1)$	$x = 0$	$+/+ = +$	f is concave up
$(1, \infty)$	$x = 2$	$+/- = -$	f is concave down.

Combining all this information, we arrive at the graph of f shown below. Note that, although f changes concavity at $x = -1$ and $x = 1$, there are no inflection points at either of these places because f is not continuous at $x = -1$ or $x = 1$.



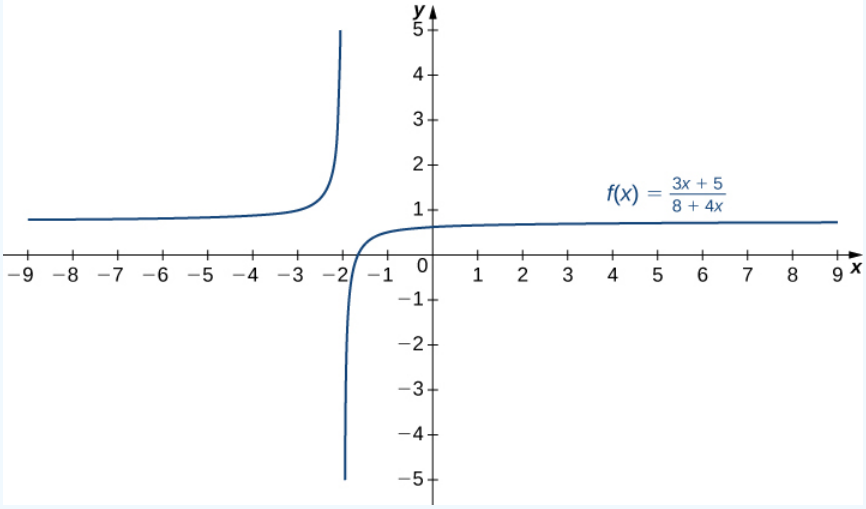
Exercise 4.7.9

Sketch a graph of $f(x) = \dfrac{3x+5}{8+4x}$.

Hint

A line $y = L$ is a horizontal asymptote of f if the limit as $x \to \infty$ or the limit as $x \to -\infty$ of $f(x)$ is L. A line $x = a$ is a vertical asymptote if at least one of the one-sided limits of f as $x \to a$ is ∞ or $-\infty$.

Answer



Example 4.7.10: Sketching a Rational Function with an Oblique Asymptote

Sketch the graph of $f(x) = \dfrac{x^2}{x-1}$

Solution

Step 1: The domain of f is the set of all real numbers x except $x = 1$.

Step 2: Find the intercepts. We can see that when $x = 0$, $f(x) = 0$, so $(0, 0)$ is the only intercept.

Step 3: Evaluate the limits at infinity. Since the degree of the numerator is one more than the degree of the denominator, f must have an oblique asymptote. To find the oblique asymptote, use long division of polynomials to write

$$f(x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

Since $\dfrac{1}{x-1} \to 0$ as $x \to \pm\infty$, $f(x)$ approaches the line $y = x + 1$ as $x \to \pm\infty$. The line $y = x + 1$ is an oblique asymptote for f.

Step 4: To check for vertical asymptotes, look at where the denominator is zero. Here the denominator is zero at $x = 1$. Looking at both one-sided limits as $x \to 1$, we find

$$\lim_{x \to 1^+} \frac{x^2}{x-1} = \infty \text{ and } \lim_{x \to 1^-} \frac{x^2}{x-1} = -\infty.$$

Therefore, $x = 1$ is a vertical asymptote, and we have determined the behavior of f as x approaches 1 from the right and the left.

Step 5: Calculate the first derivative:

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

We have $f'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$. Therefore, $x = 0$ and $x = 2$ are critical points. Since f is undefined at $x = 1$, we need to divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0), (0, 1), (1, 2)$, and $(2, \infty)$, and choose a test point from each interval to evaluate the sign of $f'(x)$ in each of these smaller intervals. For example, let $x = -1, x = \frac{1}{2}, x = \frac{3}{2}$, and $x = 3$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f'(x) = \frac{x^2 - 2x}{(x-1)^2}$	Conclusion
$(-\infty, 0)$	$x = -1$	$(-)(-)/+=+$	f is increasing.
$(0, 1)$	$x = 1/2$	$(+)(-)/+=-$	f is decreasing.
$(1, 2)$	$x = 3/2$	$(+)(-)/+=-$	f is decreasing.
$(2, \infty)$	$x = 3$	$(+)(+)/+=+$	f is increasing.

From this table, we see that f has a local maximum at $x = 0$ and a local minimum at $x = 2$. The value of f at the local maximum is $f(0) = 0$ and the value of f at the local minimum is $f(2) = 4$. Therefore, $(0, 0)$ and $(2, 4)$ are important points on the graph.

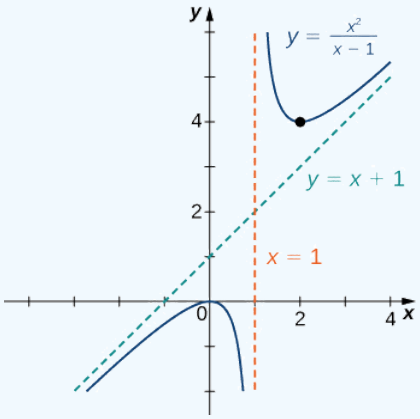
Step 6. Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{(x-1)^2(2x-2) - 2(x-1)(x^2-2x)}{(x-1)^4} \\ &= \frac{2(x-1)[(x-1)^2 - (x^2-2x)]}{(x-1)^4} \\ &= \frac{2[x^2 - 2x + 1 - x^2 + 2x]}{(x-1)^3} \\ &= \frac{2}{(x-1)^3}. \end{aligned}$$

We see that $f''(x)$ is never zero or undefined for x in the domain of f. Since f is undefined at $x = 1$, to check concavity we just divide the interval $(-\infty, \infty)$ into the two smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose a test point from each interval to evaluate the sign of $f''(x)$ in each of these intervals. The values $x = 0$ and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{2}{(x-1)^3}$	Conclusion
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is concave down.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is concave up

From the information gathered, we arrive at the following graph for f.



? Exercise 4.7.10

Find the oblique asymptote for $f(x) = \dfrac{3x^3 - 2x + 1}{2x^2 - 4}$.

Hint

Use long division of polynomials.

Answer

$y = \frac{3}{2}x$

✓ Example 4.7.11: Sketching the Graph of a Function with a Cusp

Sketch a graph of $f(x) = (x-1)^{2/3}$

Solution

Step 1: Since the cube-root function is defined for all real numbers x and $(x-1)^{2/3} = (\sqrt[3]{x-1})^2$, the domain of f is all real numbers.

Step 2: To find the y-intercept, evaluate $f(0)$. Since $f(0) = 1$, the y-intercept is $(0, 1)$. To find the x-intercept, solve $(x-1)^{2/3} = 0$. The solution of this equation is $x = 1$, so the x-intercept is $(1, 0)$.

Step 3: Since $\lim\limits_{x \to \pm\infty} (x-1)^{2/3} = \infty$, the function continues to grow without bound as $x \to \infty$ and $x \to -\infty$.

Step 4: The function has no vertical asymptotes.

Step 5: To determine where f is increasing or decreasing, calculate f'. We find

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

This function is not zero anywhere, but it is undefined when $x = 1$. Therefore, the only critical point is $x = 1$. Divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points in each of these intervals to determine the sign of $f'(x)$ in each of these smaller intervals. Let $x = 0$ and $x = 2$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f'(x) = \frac{2}{3(x-1)^{1/3}}$	Conclusion
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is decreasing
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing

We conclude that f has a local minimum at $x = 1$. Evaluating f at $x = 1$, we find that the value of f at the local minimum is zero. Note that $f'(1)$ is undefined, so to determine the behavior of the function at this critical point, we need to examine $\lim\limits_{x \to 1} f'(x)$. Looking at the one-sided limits, we have

$$\lim_{x \to 1^+} \frac{2}{3(x-1)^{1/3}} = \infty \text{ and } \lim_{x \to 1^-} \frac{2}{3(x-1)^{1/3}} = -\infty.$$

Therefore, f has a cusp at $x = 1$.

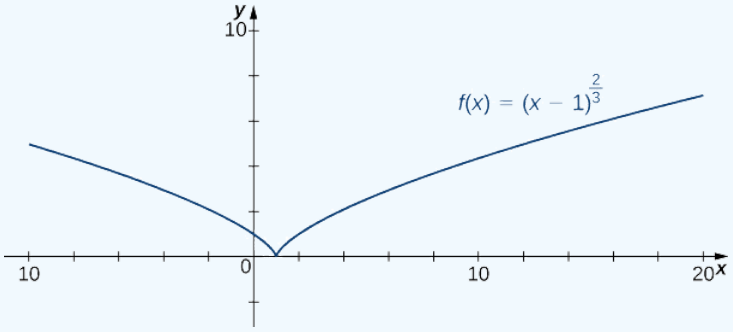
Step 6: To determine concavity, we calculate the second derivative of f:

$$f''(x) = -\frac{2}{9}(x-1)^{-4/3} = \frac{-2}{9(x-1)^{4/3}}.$$

We find that $f''(x)$ is defined for all x, but is undefined when $x = 1$. Therefore, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points to evaluate the sign of $f''(x)$ in each of these intervals. As we did earlier, let $x = 0$ and $x = 2$ be test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{-2}{9(x-1)^{4/3}}$	Conclusion
$(-\infty, 1)$	$x = 0$	$-/+ = -$	f is concave down
$(1, \infty)$	$x = 2$	$-/+ = -$	f is concave down

From this table, we conclude that f is concave down everywhere. Combining all of this information, we arrive at the following graph for f.



Exercise 4.7.11

Consider the function $f(x) = 5 - x^{2/3}$. Determine the point on the graph where a cusp is located. Determine the end behavior of f.

Hint

A function f has a cusp at a point a if $f(a)$ exists, $f'(a)$ is undefined, one of the one-sided limits as $x \to a$ of $f'(x)$ is $+\infty$, and the other one-sided limit is $-\infty$.

Answer

The function f has a cusp at $(0, 5)$, since $\lim_{x \to 0^-} f'(x) = \infty$ and $\lim_{x \to 0^+} f'(x) = -\infty$. For end behavior, $\lim_{x \to \pm\infty} f(x) = -\infty$.

Key Concepts

- The limit of $f(x)$ is L as $x \to \infty$ (or as $x \to -\infty$) if the values $f(x)$ become arbitrarily close to L as x becomes sufficiently large.
- The limit of $f(x)$ is ∞ as $x \to \infty$ if $f(x)$ becomes arbitrarily large as x becomes sufficiently large. The limit of $f(x)$ is $-\infty$ as $x \to \infty$ if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large as x becomes sufficiently large. We can define the limit of $f(x)$ as x approaches $-\infty$ similarly.
- For a polynomial function $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where $a_n \neq 0$, the end behavior is determined by the leading term $a_n x^n$. If $n \neq 0, p(x)$ approaches ∞ or $-\infty$at each end.
- For a rational function $f(x) = \frac{p(x)}{q(x),}$ the end behavior is determined by the relationship between the degree of p and the degree of q. If the degree of p is less than the degree of q, the line $y = 0$ is a horizontal asymptote for f. If the degree of p is equal to the degree of q, then the line $y = \frac{a_n}{b_n}$ is a horizontal asymptote, where a_n and b_n are the leading coefficients of p and q, respectively. If the degree of p is greater than the degree of q, then f approaches ∞ or $-\infty$ at each end.

Glossary

end behavior

the behavior of a function as $x \to \infty$ and $x \to -\infty$

horizontal asymptote

if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of f

infinite limit at infinity

a function that becomes arbitrarily large as x becomes large

limit at infinity

a function that approaches a limit value L as x becomes large

oblique asymptote

the line $y = mx + b$ if $f(x)$ approaches it as $x \to \infty$ or$x \to -\infty$

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4.7E: Exercises

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4.8: Applied Optimization Problems

Learning Objectives

- Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the optimization problems that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in Example 4.8.1, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example 4.8.1: Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (Figure 4.8.1). Given $100\,\text{ft}$ of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

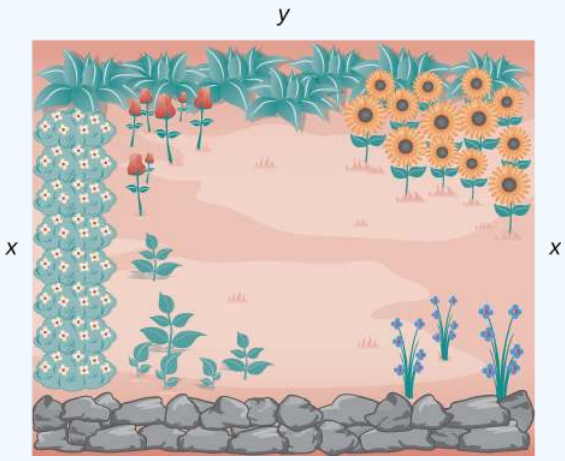


Figure 4.8.1: We want to determine the measurements x and y that will create a garden with a maximum area using $100\,\text{ft}$ of fencing.

Solution

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is $100\,\text{ft}$. From Figure 4.8.1, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden.

Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ (Figure 4.8.2). We conclude that the maximum area must occur when $x = 25$.

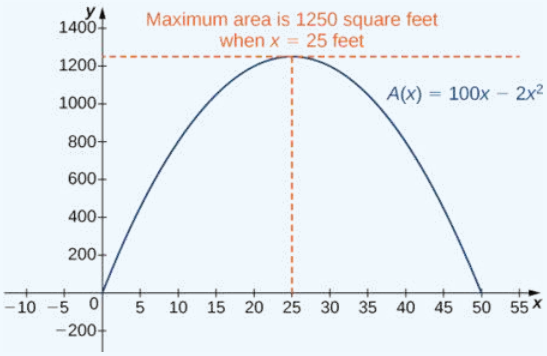


Figure 4.8.2: To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$.

Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50$ ft. The area of this garden is $1250\,\text{ft}^2$.

Exercise 4.8.1

Determine the maximum area if we want to make the same rectangular garden as in Figure 4.8.2, but we have 200 ft of fencing.

Hint

We need to maximize the function $A(x) = 200x - 2x^2$ over the interval $[0, 100]$.

Answer

The maximum area is $5000\,\text{ft}^2$.

Now let's look at a general strategy for solving optimization problems similar to Example 4.8.1.

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.

4. Write any equations relating the independent variables in the formula from step **3**. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step **4** based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step **4**. This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example 4.8.2: Maximizing the Volume of a Box

An open-top box is to be made from a **24 in.** by **36 in.** piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution

Step 1: Let x be the side length of the square to be removed from each corner (Figure **4.8.3**). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.

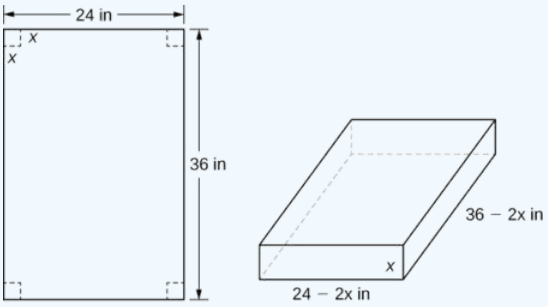


Figure **4.8.3**: A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is

$$V = L \cdot W \cdot H,$$

where L, W,and H are the length, width, and height, respectively.

Step 4: From Figure **4.8.3**, we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$\begin{aligned} V(x) &= (36 - 2x)(24 - 2x)x \\ &= 4x^3 - 120x^2 + 864x \end{aligned}.$$

Step 5: To determine the domain of consideration, let's examine Figure **4.8.3**. Certainly, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, **24 in.**; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12)$. Since V is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12$, the maximum must occur at a critical point. The derivative is

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by 12, the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are

$$\begin{aligned} x &= \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} \\ &= \frac{20 \pm \sqrt{112}}{2} \\ &= \frac{20 \pm 4\sqrt{7}}{2} \\ &= 10 \pm 2\sqrt{7} \end{aligned}.$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}$ in. The maximum volume is

$$V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825 \text{ in}^3.$$

as shown in the following graph.

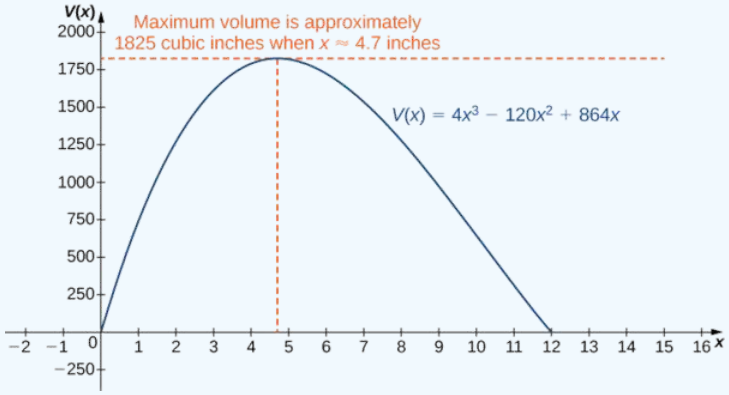


Figure 4.8.4: Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

? Exercise 4.8.2

Suppose the dimensions of the cardboard in Example 4.8.2 are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Hint

The volume of the box is $L \cdot W \cdot H$.

Answer

$V(x) = x(20 - 2x)(30 - 2x)$. The domain is $[0, 10]$.

✓ Example 4.8.3: Minimizing Travel Time

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution

Step 1: Let x be the distance running and let y be the distance swimming (Figure 4.8.5). Let T be the time it takes to get from the cabin to the island.

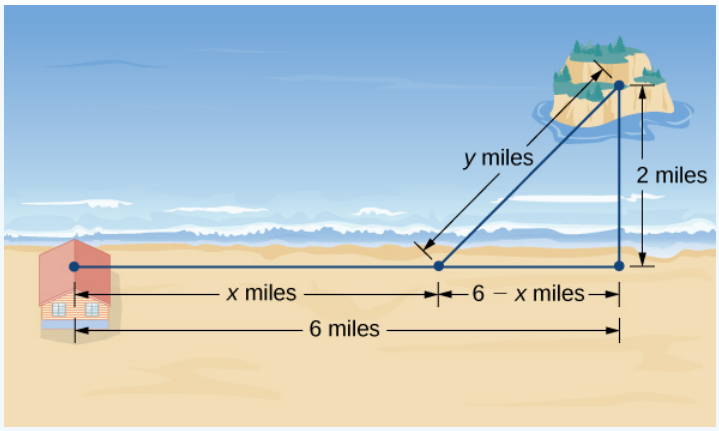


Figure 4.8.5: How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate × Time ($D = R \times T$), the time spent running is

$$T_{running} = \frac{D_{running}}{R_{running}} = \frac{x}{8},$$

and the time spent swimming is

$$T_{swimming} = \frac{D_{swimming}}{R_{swimming}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From Figure 4.8.5, the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x$ mi. Therefore, by the Pythagorean theorem, $2^2 + (6-x)^2 = y^2$, and we obtain $y = \sqrt{(6-x)^2 + 4}$. Thus, the total time spent traveling is given by the function

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6-x)^2 + 4}}{3}.$$

Step 5: From Figure 4.8.5, we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

$$\begin{aligned} T'(x) &= \frac{1}{8} - \frac{1}{2}\frac{[(6-x)^2 + 4]^{-1/2}}{3} \cdot 2(6-x) \\ &= \frac{1}{8} - \frac{(6-x)}{3\sqrt{(6-x)^2 + 4}} \end{aligned}$$

If $T'(x) = 0$,, then

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2 + 4}} \tag{4.8.1}$$

Therefore,

$$3\sqrt{(6-x)^2 + 4} = 8(6-x). \tag{4.8.2}$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

$$9[(6-x)^2 + 4] = 64(6-x)^2,$$

which implies

$$55(6-x)^2 = 36.$$

We conclude that if x is a critical point, then x satisfies

$$(x-6)^2 = \frac{36}{55}.$$

[Note that since we are squaring, $(x-6)^2 = (6-x)^2$.]

Therefore, the possibilities for critical points are

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + 6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - 6/\sqrt{55}$ is in the domain. Since we squared both sides of Equation 4.8.2 to arrive at the possible critical points, it remains to verify that $x = 6 - 6/\sqrt{55}$ satisfies Equation 4.8.1. Since $x = 6 - 6/\sqrt{55}$ does satisfy that equation, we conclude that $x = 6 - 6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of x, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}$. We find that $T(0) \approx 2.108\,\text{h}$ and $T(6) \approx 1.417\,\text{h}$, whereas

$$T(6 - 6/\sqrt{55}) \approx 1.368\,\text{h}.$$

Therefore, we conclude that T has a local minimum at $x \approx 5.19$ mi.

? Exercise 4.8.3

Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the rate of 2.5 mph and runs at a rate of 6 mph. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

Hint

The time $T = T_{running} + T_{swimming}$.

Answer

$$T(x) = \frac{x}{6} + \frac{\sqrt{(15-x)^2 + 1}}{2.5}$$

In business, companies are interested in **maximizing revenue.** In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

✓ Example 4.8.4: Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \le p \le 200$, the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge \$50 per day or less, they will rent all their cars. If they charge \$200 per day or more, they will not rent any cars. Assuming the owners plan to charge customers between \$50 per day and \$200 per day to rent a car, how much should they charge to maximize their revenue?

Solution

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize R.

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \times p$.

Step 4: Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p$, the revenue R can be represented by the function

$$\begin{aligned} R(p) &= n \times p \\ &= (1000 - 5p)p \\ &= -5p^2 + 1000p. \end{aligned}$$

Step 5: Since the owners plan to charge between $\$50$ per car per day and $\$200$ per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200]$.

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000$. Therefore, the critical point is $p = 100$. When $p = 100, R(100) = \$50,000$. When $p = 50, R(p) = \$37,500$. When $p = 200, R(p) = \$0$.

Therefore, the absolute maximum occurs at $p = \$100$. The car rental company should charge $\$100$ per day per car to maximize revenue as shown in the following figure.

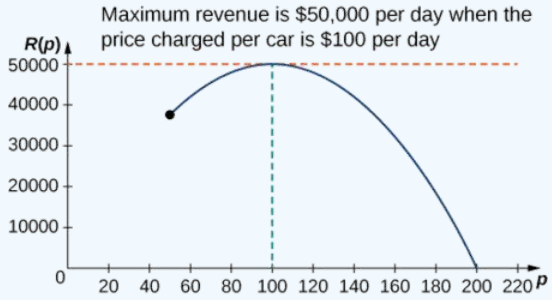


Figure 4.8.6: To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

? Exercise 4.8.4

A car rental company charges its customers p dollars per day, where $60 \le p \le 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Hint

$R(p) = n \times p$, where n is the number of cars rented and p is the price charged per car.

Answer

The company should charge $\$75$ per car per day.

✓ Example 4.8.5: Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

Solution

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.

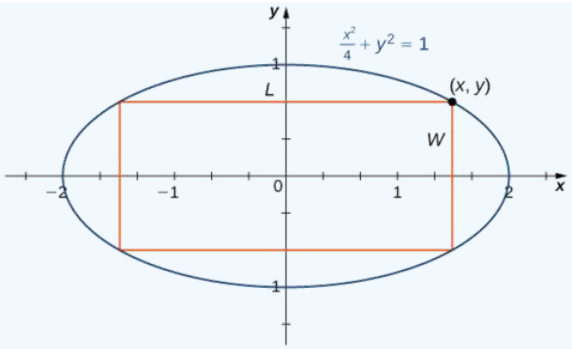


Figure 4.8.7: We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A.

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in Figure 4.8.7. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4} + y^2 = 1$ and $y > 0$, we have $y = \sqrt{1 - \frac{x^2}{4}}$. Therefore, the area is

$$A = LW = (2x)(2y) = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}$$

Step 5: From Figure 4.8.7, we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy $0 < x < 2$. Therefore, the problem reduces to looking for the maximum value of $A(x)$ over the open interval $(0, 2)$. Since $A(x)$ will have an absolute maximum (and absolute minimum) over the closed interval $[0, 2]$, we consider $A(x) = 2x\sqrt{4 - x^2}$ over the interval $[0, 2]$. If the absolute maximum occurs at an interior point, then we have found an absolute maximum in the open interval.

Step 6: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A(x) = 0$. For $0 < x < 2$, $A(x) > 0$.

Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

$$\begin{aligned} A'(x) &= 2\sqrt{4 - x^2} + 2x \cdot \frac{1}{2\sqrt{4 - x^2}}(-2x) \\ &= 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}} \\ &= \frac{8 - 4x^2}{\sqrt{4 - x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0$. We can see that if x is a solution of

$$\frac{8 - 4x^2}{\sqrt{4 - x^2}} = 0, \tag{4.8.3}$$

then x must satisfy

$$8 - 4x^2 = 0.$$

Therefore, $x^2 = 2$. Thus, $x = \pm\sqrt{2}$ are the possible solutions of Equation 4.8.3. Since we are considering x over the interval $[0, 2]$, $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of Equation 4.8.3. Since $x = \sqrt{2}$ is a solution of Equation 4.8.3, we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0, 2]$.

Therefore, $A(x)$ must have an absolute maximum at the critical point $x = \sqrt{2}$. To determine the dimensions of the rectangle, we need to find the length L and the width W. If $x = \sqrt{2}$ then

$$y = \sqrt{1 - \frac{(\sqrt{2})^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \dfrac{2}{\sqrt{2}} = \sqrt{2}$. The area of this rectangle is $A = LW = (2\sqrt{2})(\sqrt{2}) = 4$.

Exercise 4.8.5

Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

Hint

If (x, y) is the vertex of the square that lies in the first quadrant, then the area of the square is $A = (2x)(2y) = 4xy$.

Answer

$A(x) = 4x\sqrt{1 - x^2}$. The domain of consideration is $[0, 1]$.

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

Example 4.8.6: Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of $216\,\text{in}^3$ is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.

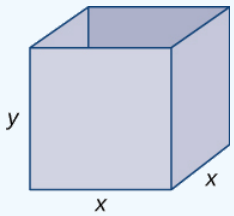


Figure 4.8.8: We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

$$S = 4xy + x^2.$$

Step 4: Since the volume of this box is x^2y and the volume is given as $216\,\text{in}^3$, the constraint equation is

$$x^2y = 216.$$

Solving the constraint equation for y, we have $y = \dfrac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2.$$

Therefore, $S(x) = \dfrac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \to 0^+$, $S(x) \to \infty$. Also, as $x \to \infty$, $S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S'(x) = 0$ when $2x = \dfrac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see Figure 4.8.9).

When $x = 6\sqrt[3]{2}$, $y = \dfrac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$ in. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}$ in. and $y = 3\sqrt[3]{2}$ in. With these dimensions, the surface area is

$$S(6\sqrt[3]{2}) = \frac{864}{6\sqrt[3]{2}} + (6\sqrt[3]{2})^2 = 108\sqrt[3]{4}\,\text{in}^2$$

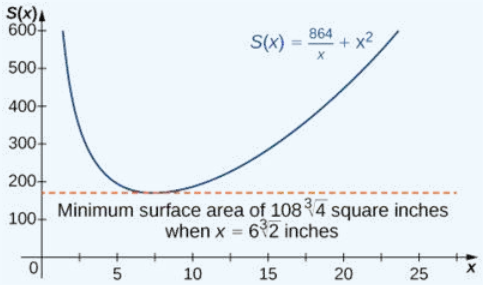


Figure 4.8.9: We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

? Exercise 4.8.6

Consider the same open-top box, which is to have volume $216\,\text{in}^3$. Suppose the cost of the material for the base is $20¢/\text{in}^2$ and the cost of the material for the sides is $30¢/\text{in}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

Hint

If the cost of one of the sides is $30¢/\text{in}^2$, the cost of that side is $0.30xy$ dollars.

Answer

$c(x) = \dfrac{259.2}{x} + 0.2x^2$ dollars

Key Concepts

- To solve an optimization problem, begin by drawing a picture and introducing variables.
- Find an equation relating the variables.
- Find a function of one variable to describe the quantity that is to be minimized or maximized.
- Look for critical points to locate local extrema.

Glossary

optimization problems

problems that are solved by finding the maximum or minimum value of a function

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4.8E: Exercises

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4.9: L'Hôpital's Rule

Learning Objectives

- Recognize when to apply L'Hôpital's rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
- Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L'Hôpital's rule,** uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L'Hôpital's Rule

L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

However, what happens if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? We call this one of the **indeterminate forms,** of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

and

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4.$$

For $\lim_{x\to 0}\frac{\sin x}{x}$ we were able to show, using a geometric argument, that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more importantly, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and such that $g'(a) \neq 0$ For x near a,we can write

$$f(x) \approx f(a) + f'(a)(x-a)$$

and

$$g(x) \approx g(a) + g'(a)(x-a).$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}.$$

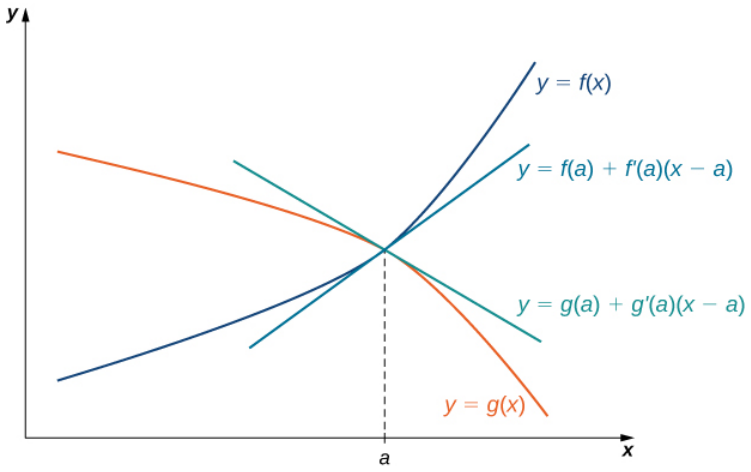


Figure 4.9.1: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at a, then f is continuous at a, and therefore $f(a) = \lim_{x\to a} f(x) = 0$. Similarly, $g(a) = \lim_{x\to a} g(x) = 0$. If we also assume that f' and g' are continuous at $x = a$, then $f'(a) = \lim_{x\to a} f'(x)$ and $g'(a) = \lim_{x\to a} g'(x)$. Using these ideas, we conclude that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}$. Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

L'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ or $a = -\infty$.

Proof

We provide a proof of this theorem in the special case when f, g, f', and g' are all continuous over an open interval containing a. In that case, since $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and f and g are continuous at a, it follows that $f(a) = 0 = g(a)$. Therefore,

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && \text{Since } f(a)=0=g(a)\\
&= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} && \text{Multiply numerator and denominator by } \frac{1}{x-a}\\
&= \frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} && \text{The limit of a quotient is the quotient of the limits.}\\
&= \frac{f'(a)}{g'(a)} && \text{By the definition of the derivative}\\
&= \frac{\lim\limits_{x\to a}f'(x)}{\lim\limits_{x\to a}g'(x)} && \text{By the continuity of } f' \text{ and } g'\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)}. && \text{The limit of a quotient}
\end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example 4.9.1: Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$
b. $\lim\limits_{x\to 1}\dfrac{\sin(\pi x)}{\ln x}$
c. $\lim\limits_{x\to\infty}\dfrac{e^{1/x}-1}{1/x}$
d. $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^2}$

Solution

a. Since the numerator $1-\cos x\to 0$ and the denominator $x\to 0$, we can apply L'Hôpital's rule to evaluate this limit. We have

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x)}=\lim_{x\to 0}\frac{\sin x}{1}=\frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0}1}=\frac{0}{1}=0.$$

b. As $x\to 1$, the numerator $\sin(\pi x)\to 0$ and the denominator $\ln(x)\to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\begin{aligned}
\lim_{x\to 1}\frac{\sin(\pi x)}{\ln x} &= \lim_{x\to 1}\frac{\pi\cos(\pi x)}{1/x}\\
&= \lim_{x\to 1}(\pi x)\cos(\pi x)\\
&= (\pi\cdot 1)(-1)=-\pi.
\end{aligned}$$

c. As $x\to\infty$, the numerator $e^{1/x}-1\to 0$ and the denominator $\frac{1}{x}\to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to\infty}\frac{e^{1/x}-1}{\frac{1}{x}}=\lim_{x\to\infty}\frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to\infty}e^{1/x}=e^0=1.$$

d. As $x\to 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to 0}\frac{\sin x - x}{x^2}=\lim_{x\to 0}\frac{\cos x-1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x \to 0$, we apply L'Hôpital's rule again. In doing so, we see that

$$\lim_{x \to 0} \frac{\cos x - 1}{2x} = \lim_{x \to 0} \frac{-\sin x}{2} = 0.$$

Therefore, we conclude that

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} = 0.$$

? Exercise 4.9.1

Evaluate $\lim_{x \to 0} \frac{x}{\tan x}$.

Hint

$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$

Answer

1

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$. Limits of this form are classified as indeterminate forms of type ∞/∞. Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, ∞/∞ is used to represent a quotient of limits, each of which is ∞ or $-\infty$.

L'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x \to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$). Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or $-\infty$, or the limit is one-sided.

✓ Example 4.9.2: Applying L'Hôpital's Rule (∞/∞) Case

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x \to \infty} \frac{3x + 5}{2x + 1}$

b. $\lim_{x \to 0^+} \frac{\ln x}{\cot x}$

Solution

a. Since $3x + 5$ and $2x + 1$ are first-degree polynomials with positive leading coefficients, $\lim_{x \to \infty} (3x + 5) = \infty$ and $\lim_{x \to \infty} (2x + 1) = \infty$. Therefore, we apply L'Hôpital's rule and obtain

$$\lim_{x \to \infty} \frac{3x + 5}{2x + 1} = \lim_{x \to \infty} \frac{3}{2} = \frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x \to \infty} \frac{3x + 5}{2x + 1} = \lim_{x \to \infty} \frac{3 + 5/x}{2 + 1/x} = \frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to 0^+} \cot x = \infty$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x\to 0^+} \frac{1}{-x\csc^2 x}.$$

Now as $x \to 0^+, \csc^2 x \to \infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

$$\lim_{x\to 0^+} \frac{1}{-x\csc^2 x} = \lim_{x\to 0^+} \frac{\sin^2 x}{-x}.$$

Now $\lim_{x\to 0^+} \sin^2 x = 0$ and $\lim_{x\to 0^+} -x = 0$, so we apply L'Hôpital's rule again. We find

$$\lim_{x\to 0^+} \frac{\sin^2 x}{-x} = \lim_{x\to 0^+} \frac{2\sin x\cos x}{-1} = \frac{0}{-1} = 0.$$

We conclude that

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = 0.$$

? Exercise 4.9.2

Evaluate $\lim_{x\to\infty} \frac{\ln x}{5x}$.

Hint

$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$

Answer

0

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or ∞/∞. Consider the following example.

✓ Example 4.9.3: When L'Hôpital's Rule Does Not Apply

Consider $\lim_{x\to 1} \frac{x^2+5}{3x+4}$.

Show that the limit cannot be evaluated by applying L'Hôpital's rule.

Solution

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

$$\frac{d}{dx}(x^2+5) = 2x$$

and

$$\frac{d}{dx}(3x+4) = 3.$$

At which point we would conclude erroneously that

$$\lim_{x\to 1} \frac{x^2+5}{3x+4} = \lim_{x\to 1} \frac{2x}{3} = \frac{2}{3}.$$

However, since $\lim_{x\to 1}(x^2+5)=6$ and $\lim_{x\to 1}(3x+4)=7$, we actually have

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\frac{6}{7}.$$

We can conclude that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}\neq\lim_{x\to 1}\frac{\frac{d}{dx}(x^2+5)}{\frac{d}{dx}(3x+4)}.$$

? Exercise 4.9.3

Explain why we cannot apply L'Hôpital's rule to evaluate $\lim_{x\to 0^+}\frac{\cos x}{x}$. Evaluate $\lim_{x\to 0^+}\frac{\cos x}{x}$ by other means.

Hint

Determine the limits of the numerator and denominator separately.

Answer

$\lim_{x\to 0^+}\cos x=1$. Therefore, we cannot apply L'Hôpital's rule. The limit of the quotient is ∞.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and ∞/∞. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0\cdot\infty, \infty-\infty, 1^\infty, \infty^0$, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\frac{0}{0}$ or ∞/∞.

Indeterminate Form of Type $0\cdot\infty$

Suppose we want to evaluate $\lim_{x\to a}(f(x)\cdot g(x))$, where $f(x)\to 0$ and $g(x)\to\infty$ (or $-\infty$) as $x\to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0\cdot\infty$ to denote the form that arises in this situation. The expression $0\cdot\infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x\to\infty$. For example, let n be a positive integer and consider

$$f(x)=\frac{1}{(x^n+1)}\text{ and }g(x)=3x^2.$$

As $x\to\infty, f(x)\to 0$ and $g(x)\to\infty$. However, the limit as $x\to\infty$ of $f(x)g(x)=\frac{3x^2}{(x^n+1)}$ varies, depending on n. If $n=2$, then $\lim_{x\to\infty}f(x)g(x)=3$. If $n=1$, then $\lim_{x\to\infty}f(x)g(x)=\infty$. If $n=3$, then $\lim_{x\to\infty}f(x)g(x)=0$. Here we consider another limit involving the indeterminate form $0\cdot\infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

✓ Example 4.9.4: Indeterminate Form of Type $0\cdot\infty$

Evaluate $\lim_{x\to 0^+}x\ln x$.

Solution

First, rewrite the function $x\ln x$ as a quotient to apply L'Hôpital's rule. If we write

$$x\ln x=\frac{\ln x}{1/x}$$

we see that $\ln x \to -\infty$ as $x \to 0^+$ and $\frac{1}{x} \to \infty$ as $x \to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

We conclude that

$$\lim_{x\to 0^+} x \ln x = 0.$$

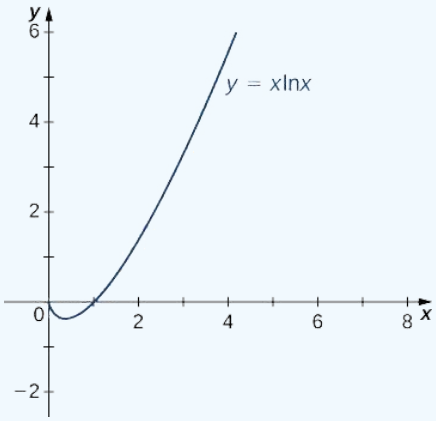


Figure 4.9.2: Finding the limit at $x = 0$ of the function $f(x) = x \ln x$.

? Exercise 4.9.4

Evaluate $\lim_{x\to 0} x \cot x$.

Hint

Write $x \cot x = \frac{x \cos x}{\sin x}$

Answer

1

Indeterminate Form of Type ∞-∞

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim_{x\to\infty} (f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x\to\infty} (f(x) - g(x)) = \lim_{x\to\infty} (3x^3 - 3x^2 - 5) = \infty.$$

On the other hand, if $n = 2$, then

$$\lim_{x\to\infty} (f(x) - g(x)) = \lim_{x\to\infty} (3x^2 - 3x^2 - 5) = -5.$$

However, if $n = 1$, then

$$\lim_{x\to\infty} (f(x) - g(x)) = \lim_{x\to\infty} (3x - 3x^2 - 5) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example 4.9.5: Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x^2} - \dfrac{1}{\tan x} \right)$.

Solution

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0$. Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} = \lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \to 0^+$, $(\sec^2 x) - 2x \to 1$ and $x^2 \sec^2 x + 2x \tan x \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right) = \infty.$$

Exercise 4.9.5

Evaluate $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x} \right)$.

Hint

Rewrite the difference of fractions as a single fraction.

Answer

0

Indeterminate Forms Involving Exponents

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions $0^0, \infty^0$, and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim\limits_{x \to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

$$y = f(x)^{g(x)}.$$

Then,

$$\ln y = \ln(f(x)^{g(x)}) = g(x) \ln(f(x)).$$

Therefore,

$$\lim_{x\to a}[\ln(y)] = \lim_{x\to a}[g(x)\ln(f(x))].$$

Since $\lim_{x\to a} f(x) = \infty$, we know that $\lim_{x\to a}\ln(f(x)) = \infty$. Therefore, $\lim_{x\to a} g(x)\ln(f(x))$ is of the indeterminate form $0\cdot\infty$, and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x\to a} g(x)\ln(f(x)) = L$, where L may be ∞ or $-\infty$. Then

$$\lim_{x\to a}[\ln(y)] = L.$$

Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to a} y\right) = L,$$

which gives us

$$\lim_{x\to a} y = \lim_{x\to a} f(x)^{g(x)} = e^L.$$

Example 4.9.6: Indeterminate Form of Type ∞^0

Evaluate $\lim_{x\to\infty} x^{1/x}$.

Solution

Let $y = x^{1/x}$.Then,

$$\ln(x^{1/x}) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x\to\infty}\frac{\ln x}{x}$. Applying L'Hôpital's rule, we obtain

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0.$$

Therefore, $\lim_{x\to\infty}\ln y = 0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to\infty} y\right) = 0,$$

which leads to

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} y = e^{\ln\left(\lim_{x\to\infty} y\right)} = e^0 = 1.$$

Hence,

$$\lim_{x\to\infty} x^{1/x} = 1.$$

Exercise 4.9.6

Evaluate $\lim_{x\to\infty} x^{1/\ln(x)}$.

Hint

Let $y = x^{1/\ln(x)}$ and apply the natural logarithm to both sides of the equation.

Answer

e

Example 4.9.7: Indeterminate Form of Type 0^0

Evaluate $\lim\limits_{x\to 0^+} x^{\sin x}$.

Solution

Let

$$y = x^{\sin x}.$$

Therefore,

$$\ln y = \ln(x^{\sin x}) = \sin x \ln x.$$

We now evaluate $\lim\limits_{x\to 0^+} \sin x \ln x$. Since $\lim\limits_{x\to 0^+} \sin x = 0$ and $\lim\limits_{x\to 0^+} \ln x = -\infty$, we have the indeterminate form $0\cdot\infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x \ln x$ as a fraction. We could write

$$\sin x \ln x = \frac{\sin x}{1/\ln x}$$

or

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x}.$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\sin x}{1/\ln x} = \lim_{x\to 0^+} \frac{\cos x}{-1/(x(\ln x)^2)} = \lim_{x\to 0^+} (-x(\ln x)^2 \cos x).$$

Unfortunately, we not only have another expression involving the indeterminate form $0\cdot\infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x,}$$

and applying L'Hôpital's rule, we obtain

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $\csc x = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as

$$\lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x} = \lim_{x\to 0^+} \left[\frac{\sin x}{x}\cdot(-\tan x)\right] = \left(\lim_{x\to 0^+} \frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0^+}(-\tan x)\right) = 1\cdot 0 = 0.$$

We conclude that $\lim\limits_{x\to 0^+} \ln y = 0$. Therefore, $\ln\left(\lim\limits_{x\to 0^+} y\right) = 0$ and we have

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^{\sin x} = e^0 = 1.$$

Hence,

$$\lim_{x\to 0^+} x^{\sin x} = 1.$$

Exercise 4.9.7

Evaluate $\lim\limits_{x\to 0^+} x^x$.

Hint

Let $y = x^x$ and take the natural logarithm of both sides of the equation.

Answer

1

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as Table 4.9.1 shows, the values of x^3 are growing much faster than the values of x^2.

Table 4.9.1: Comparing the Growth Rates of x^2 and x^3

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

In fact,

$$\lim_{x \to \infty} \frac{x^3}{x^2} = \lim_{x \to \infty} x = \infty.$$

or, equivalently

$$\lim_{x \to \infty} \frac{x^2}{x^3} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in Table 4.9.2. In fact,

$$\lim_{x \to \infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

Table 4.9.2: Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = \infty \quad \text{or, equivalently,} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 4.9.8: Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}.$$

a. $f(x)=x^2$ and $g(x)=e^x$
b. $f(x)=\ln(x)$ and $g(x)=x^2$

Solution

a. Since $\lim_{x\to\infty}x^2=\infty$ and $\lim_{x\to\infty}e^x=\infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty}\left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}.$$

Since $\lim_{x\to\infty}2x=\infty$ and $\lim_{x\to\infty}e^x=\infty$, we can apply L'Hôpital's rule again. Since

$$\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0,$$

we conclude that

$$\lim_{x\to\infty}\frac{x^2}{e^x}=0.$$

Therefore, e^x grows more rapidly than x^2 as $x\to\infty$ (See Figure 4.9.3 and Table 4.9.3)

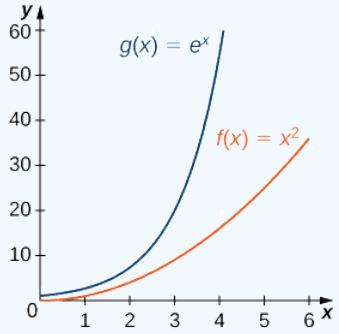


Figure 4.9.3: An exponential function grows at a faster rate than a power function.

Table 4.9.3: Growth rates of a power function and an exponential function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22,026	3,269,017	485,165,195

b. Since $\lim_{x\to\infty}\ln x=\infty$ and $\lim_{x\to\infty}x^2=\infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty}\frac{\ln x}{x^2}$. We obtain

$$\lim_{x\to\infty}\frac{\ln x}{x^2}=\lim_{x\to\infty}\frac{1/x}{2x}=\lim_{x\to\infty}\frac{1}{2x^2}=0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x\to\infty$ (see Figure 4.9.4 and Table 4.9.4).

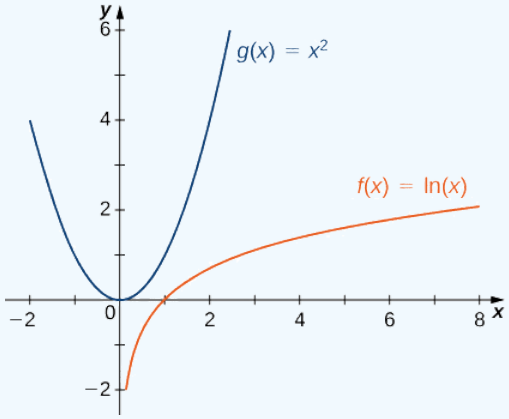


Figure 4.9.4: A power function grows at a faster rate than a logarithmic function.

Table 4.9.4: Growth rates of a power function and a logarithmic function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

? Exercise 4.9.8

Compare the growth rates of x^{100} and 2^x.

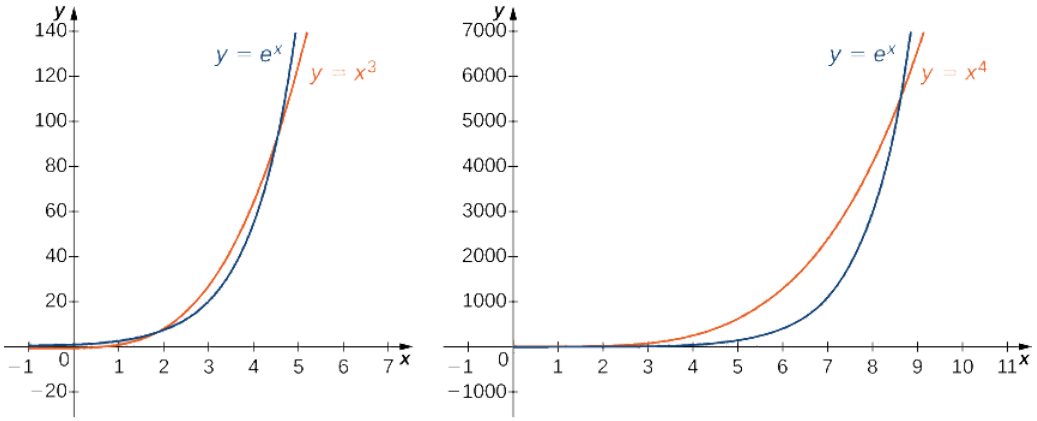
Hint

Apply L'Hôpital's rule to $x^{100}/2^x$.

Answer

The function 2^x grows faster than x^{100}.

Using the same ideas as in Example 4.9.8a. it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In Figure 4.9.5 and Table 4.9.5, we compare e^x with x^3 and x^4 as $x \to \infty$.



(a) (b)

Figure 4.9.5: The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

Table 4.9.5: An exponential function grows at a faster rate than any power function

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In Figure 4.9.6 and Table 4.9.6, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

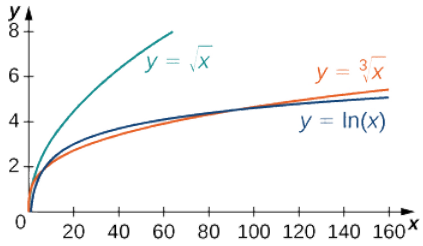


Figure 4.9.6: The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

Table 4.9.6: A logarithmic function grows at a slower rate than any root function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Key Concepts

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞.
- The exponential function e^x grows faster than any power function $x^p, p > 0$.
- The logarithmic function $\ln x$ grows more slowly than any power function $x^p, p > 0$.

Glossary

indeterminate forms

When evaluating a limit, the forms $\frac{0}{0}, \infty/\infty, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0$, and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is.

L'Hôpital's rule

If f and g are differentiable functions over an interval a, except possibly at a, and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is ∞ or $-\infty$.

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4.9E: Exercises

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4.10: Newton's Method

Learning Objectives

- Describe the steps of Newton's method.
- Explain what an iterative process means.
- Recognize when Newton's method does not work.
- Apply iterative processes to various situations.

In many areas of pure and applied mathematics, we are interested in finding solutions to an equation of the form $f(x) = 0$. For most functions, however, it is difficult—if not impossible—to calculate their zeroes explicitly. In this section, we take a look at a technique that provides a very efficient way of approximating the **zeroes of functions**. This technique makes use of tangent line approximations and is behind the method used often by calculators and computers to find zeroes.

Describing Newton's Method

Consider the task of finding the solutions of $f(x) = 0$. If f is the first-degree polynomial $f(x) = ax + b$, then the solution of $f(x) = 0$ is given by the formula $x = -\frac{b}{a}$. If f is the second-degree polynomial $f(x) = ax^2 + bx + c$, the solutions of $f(x) = 0$ can be found by using the quadratic formula. However, for polynomials of degree 3 or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree 5 or greater, it is known that no such formulas exist. For example, consider the function

$$f(x) = x^5 + 8x^4 + 4x^3 - 2x - 7.$$

No formula exists that allows us to find the solutions of $f(x) = 0$. Similar difficulties exist for nonpolynomial functions. For example, consider the task of finding solutions of $\tan(x) - x = 0$. No simple formula exists for the solutions of this equation. In cases such as these, we can use Newton's method to approximate the roots.

Newton's method makes use of the following idea to approximate the solutions of $f(x) = 0$. By sketching a graph of f, we can estimate a root of $f(x) = 0$. Let's call this estimate x_0. We then draw the tangent line to f at x_0. If $f'(x_0) \neq 0$, this tangent line intersects the x-axis at some point $(x_1, 0)$. Now let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next we draw the tangent line to f at x_1. If $f'(x_1) \neq 0$, this tangent line also intersects the x-axis, producing another approximation, x_2. We continue in this way, deriving a list of approximations: $x_0, x_1, x_2, \ldots$. Typically, the numbers $x_0, x_1, x_2, \ldots$ quickly approach an actual root x^*, as shown in the following figure.

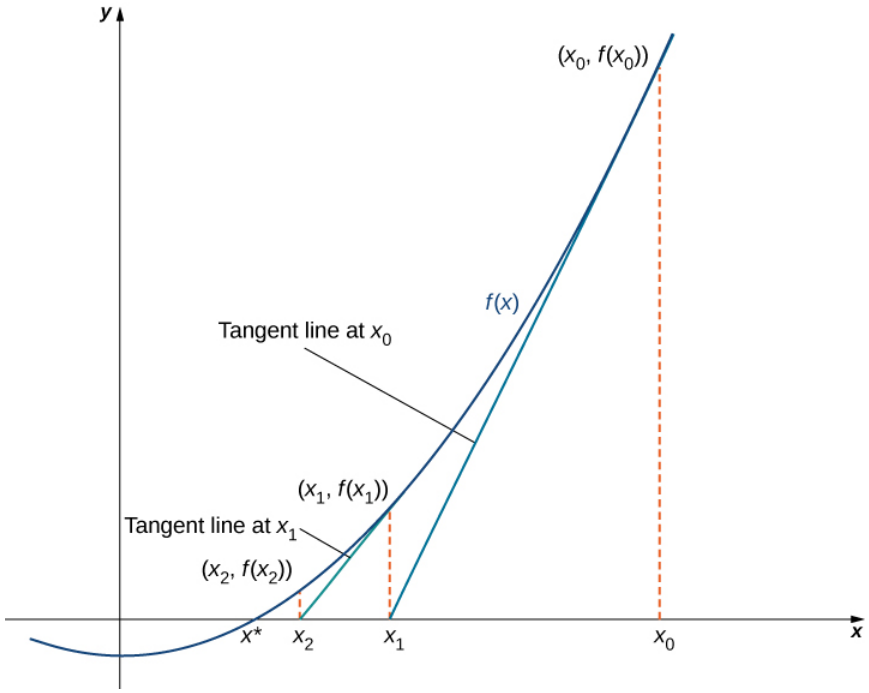


Figure 4.10.1:The approximations $x_0, x_1, x_2, \ldots$ approach the actual root x^*. The approximations are derived by looking at tangent lines to the graph of f.

Now let's look at how to calculate the approximations $x_0, x_1, x_2, \ldots$. If x_0 is our first approximation, the approximation x_1 is defined by letting $(x_1, 0)$ be the x-intercept of the tangent line to f at x_0. The equation of this tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

Solving this equation for x_1, we conclude that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point $(x_2, 0)$ is the x-intercept of the tangent line to f at x_1. Therefore, x_2 satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for $n > 0, x_n$ satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{4.10.1}$$

Next we see how to make use of this technique to approximate the root of the polynomial $f(x) = x^3 - 3x + 1$.

✔ Example 4.10.1: Finding a Root of a Polynomial

Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find x_1, x_2, x_3, x_4, and x_5.

Solution

From Figure 4.10.2, we see that f has one root over the interval $[1, 2]$. Therefore $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use Equation 4.10.1. Since $f(x) = x^3 - 3x + 1$, the derivative is $f'(x) = 3x^2 - 3$. Using Equation 4.10.1 with $n = 1$ (and a calculator that displays 10 digits), we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, x_2, we use Equation 4.10.1 with $n = 2$ and the value of x_1 stored on the calculator. We find that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Continuing in this way, we obtain the following results:

- $x_1 \approx 1.666666667$
- $x_2 \approx 1.548611111$
- $x_3 \approx 1.532390162$
- $x_4 \approx 1.532088989$
- $x_5 \approx 1.532088886$
- $x_6 \approx 1.532088886.$

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.

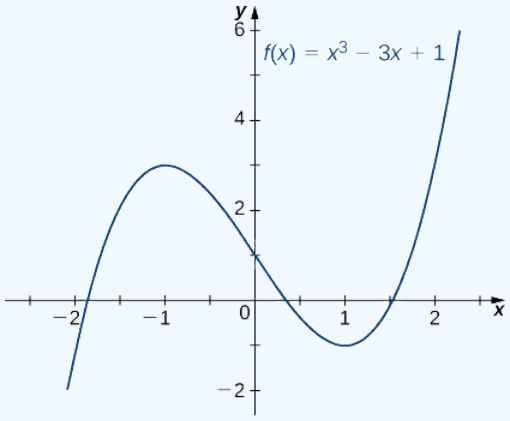


Figure 4.10.2: The function $f(x) = x^3 - 3x + 1$ has one root over the interval $[1, 2]$.

? Exercise 4.10.1

Letting $x_0 = 0$, let's use Newton's method to approximate the root of $f(x) = x^3 - 3x + 1$ over the interval $[0, 1]$ by calculating x_1 and x_2.

Hint

Use Equation 4.10.1.

Answer

$x_1 \approx 0.33333333$
$x_2 \approx 0.347222222$

Newton's method can also be used to approximate square roots. Here we show how to approximate $\sqrt{2}$. This method can be modified to approximate the square root of any positive number.

✓ Example 4.10.2: Finding a Square Root

Use Newton's method to approximate $\sqrt{2}$ (Figure 4.10.3). Let $f(x) = x^2 - 2$, let $x_0 = 2$, and calculate x_1, x_2, x_3, x_4, x_5. (We note that since $f(x) = x^2 - 2$ has a zero at $\sqrt{2}$, the initial value $x_0 = 2$ is a reasonable choice to approximate $\sqrt{2}$).

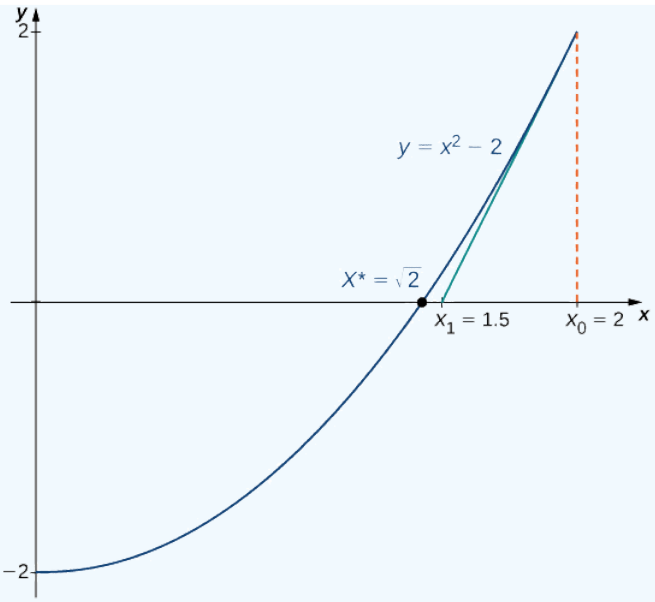


Figure **4.10.3**: We can use Newton's method to find $\sqrt{2}$.

Solution

For $f(x) = x^2 - 2,\ f'(x) = 2x$. From Equation 4.10.1, we know that

$$\begin{aligned} x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} \\ &= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\ &= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right). \end{aligned}$$

Therefore,

$$x_1 = \tfrac{1}{2}\left(x_0 + \tfrac{2}{x_0}\right) = \tfrac{1}{2}\left(2 + \tfrac{2}{2}\right) = 1.5$$

$$x_2 = \tfrac{1}{2}\left(x_1 + \tfrac{2}{x_1}\right) = \tfrac{1}{2}\left(1.5 + \tfrac{2}{1.5}\right) \approx 1.416666667.$$

Continuing in this way, we find that

$$x_1 = 1.5$$

$$x_2 \approx 1.416666667$$

$$x_3 \approx 1.414215686$$

$$x_4 \approx 1.414213562$$

$$x_5 \approx 1.414213562.$$

Since we obtained the same value for x_4 and x_5, it is unlikely that the value x_n will change on any subsequent application of Newton's method. We conclude that $\sqrt{2} \approx 1.414213562$.

? Exercise 4.10.2

Use Newton's method to approximate $\sqrt{3}$ by letting $f(x) = x^2 - 3$ and $x_0 = 3$. Find x_1 and x_2.

Hint

For $f(x) = x^2 - 3$, Equation 4.10.1 reduces to $x_n = \frac{x_{n-1}}{2} + \frac{3}{2x_{n-1}}$.

Answer

$x_1 = 2$
$x_2 = 1.75$

When using Newton's method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$, we can rewrite Equation 4.10.1 as $x_n = F(x_{n-1})$. This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an iterative process. Shortly, we examine other iterative processes. First, let's look at the reasons why Newton's method could fail to find a root.

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.
2. The approximations $x_0, x_1, x_2, \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see Figure 4.10.4). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In Example 4.10.3, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

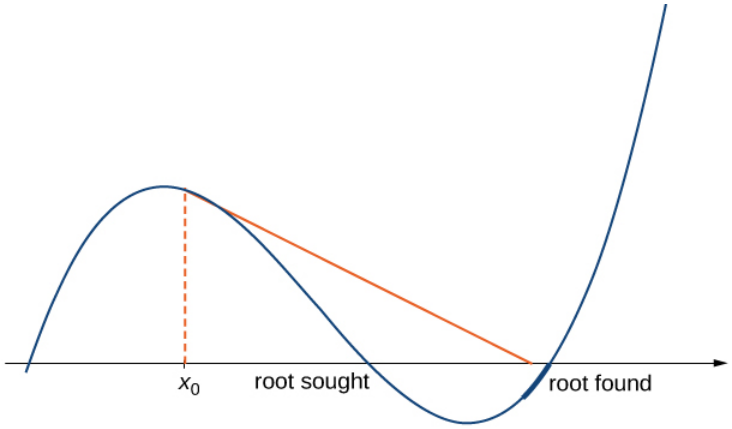


Figure 4.10.4: If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

Example 4.10.3: When Newton's Method Fails

Consider the function $f(x) = x^3 - 2x + 2$. Let $x_0 = 0$. Show that the sequence $x_1, x_2, \ldots$ fails to approach a root of f.

Solution

For $f(x) = x^3 - 2x + 2$, the derivative is $f'(x) = 3x^2 - 2$.Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers $x_0, x_1, x_2, \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$ (Figure 4.10.5). Fortunately, if we choose an initial approximation x_0 closer to the actual

root, we can avoid this situation.

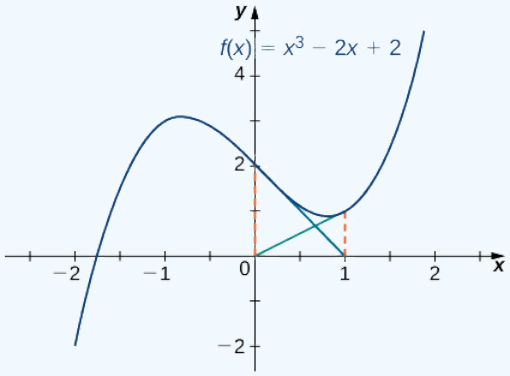


Figure 4.10.5: The approximations continue to alternate between 0 and 1 and never approach the root of f.

Exercise 4.10.3

For $f(x) = x^3 - 2x + 2$, let $x_0 = -1.5$ and find x_1 and x_2.

Hint

Use Equation 4.10.1.

Answer

$x_1 \approx -1.842105263$
$x_2 \approx -1.772826920$

From Example 4.10.3, we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

Other Iterative Processes

As mentioned earlier, Newton's method is a type of iterative process. We now look at an example of a different type of iterative process.

Consider a function F and an initial number x_0. Define the subsequent numbers x_n by the formula $x_n = F(x_{n-1})$. This process is an iterative process that creates a list of numbers $x_0, x_1, x_2, \ldots, x_n, \ldots$. This list of numbers may approach a finite number x^* as n gets larger, or it may not. In Example 4.10.4, we see an example of a function F and an initial guess x_0 such that the resulting list of numbers approaches a finite value.

Example 4.10.4: Finding a Limit for an Iterative Process

Let $F(x) = \frac{1}{2}x + 4$ and let $x_0 = 0$. For all $n \geq 1$, let $x_n = F(x_{n-1})$. Find the values x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to this list of numbers $x_1, x_2, x_3, \ldots, x_n, \ldots$ as $n \to \infty$. If the list of numbers $x_1, x_2, x_3, \ldots$ approaches a finite number x^*, then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F.

Solution

If $x_0 = 0$, then

- $x_1 = \frac{1}{2}(0) + 4 = 4$
- $x_2 = \frac{1}{2}(4) + 4 = 6$
- $x_3 = \frac{1}{2}(6) + 4 = 7$
- $x_4 = \frac{1}{2}(7) + 4 = 7.5$
- $x_5 = \frac{1}{2}(7.5) + 4 = 7.75$

- $x_6 = \frac{1}{2}(7.75) + 4 = 7.875$
- $x_7 = \frac{1}{2}(7.875) + 4 = 7.9375$
- $x_8 = \frac{1}{2}(7.9375) + 4 = 7.96875$
- $x_9 = \frac{1}{2}(7.96875) + 4 = 7.984375.$

From this list, we conjecture that the values x_n approach 8.

Figure 4.10.6 provides a graphical argument that the values approach 8 as $n \to \infty$. Starting at the point (x_0, x_0), we draw a vertical line to the point $(x_0, F(x_0))$. The next number in our list is $x_1 = F(x_0)$. We use x_1 to calculate x_2. Therefore, we draw a horizontal line connecting (x_0, x_1) to the point (x_1, x_1) on the line $y = x$, and then draw a vertical line connecting (x_1, x_1) to the point $(x_1, F(x_1))$. The output $F(x_1)$ becomes x_2. Continuing in this way, we could create an infinite number of line segments. These line segments are trapped between the lines $F(x) = \frac{x}{2} + 4$ and $y = x$. The line segments get closer to the intersection point of these two lines, which occurs when $x = F(x)$. Solving the equation $x = \frac{x}{2} + 4$, we conclude they intersect at $x = 8$. Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers $x_0, x_1, x_2, \ldots$ approaches $x^* = 8$ as $n \to \infty$.

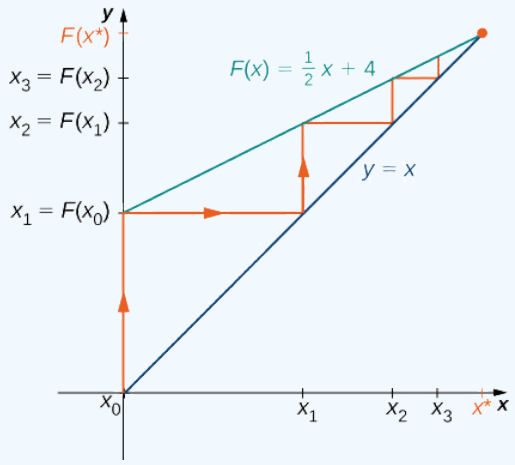


Figure 4.10.6: This iterative process approaches the value $x^* = 8$.

Exercise 4.10.4

Consider the function $F(x) = \frac{1}{3}x + 6$. Let $x_0 = 0$ and let $x_n = F(x_{n-1})$ for $n \geq 2$. Find x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to the list of numbers $x_1, x_2, x_3, \ldots x_n, \ldots$ as $n \to \infty$.

Hint

Consider the point where the lines $y = x$ and $y = F(x)$ intersect.

Answer

$x_1 = 6,\ x_2 = 8,\ x_3 = \frac{26}{3},\ x_4 = \frac{80}{9},\ x_5 = \frac{242}{27};\ x^* = 9$

Iterative Processes and Chaos

Iterative processes can yield some very interesting behavior. In this section, we have seen several examples of iterative processes that converge to a fixed point. We also saw in Example 4.10.3 that the iterative process bounced back and forth between two values. We call this kind of behavior a 2-cycle. Iterative processes can converge to cycles with various periodicities, such as 2−cycles, 4−cycles (where the iterative process repeats a sequence of four values), 8-cycles, and so on.

Some iterative processes yield what mathematicians call chaos. In this case, the iterative process jumps from value to value in a seemingly random fashion and never converges or settles into a cycle. Although a complete exploration of chaos is beyond the scope of this text, in this project we look at one of the key properties of a chaotic iterative process: sensitive dependence on

initial conditions. This property refers to the concept that small changes in initial conditions can generate drastically different behavior in the iterative process.

Probably the best-known example of chaos is the **Mandelbrot set** (see Figure 4.10.7), named after Benoit Mandelbrot (1924–2010), who investigated its properties and helped popularize the field of chaos theory. The Mandelbrot set is usually generated by computer and shows fascinating details on enlargement, including self-replication of the set. Several colorized versions of the set have been shown in museums and can be found online and in popular books on the subject.

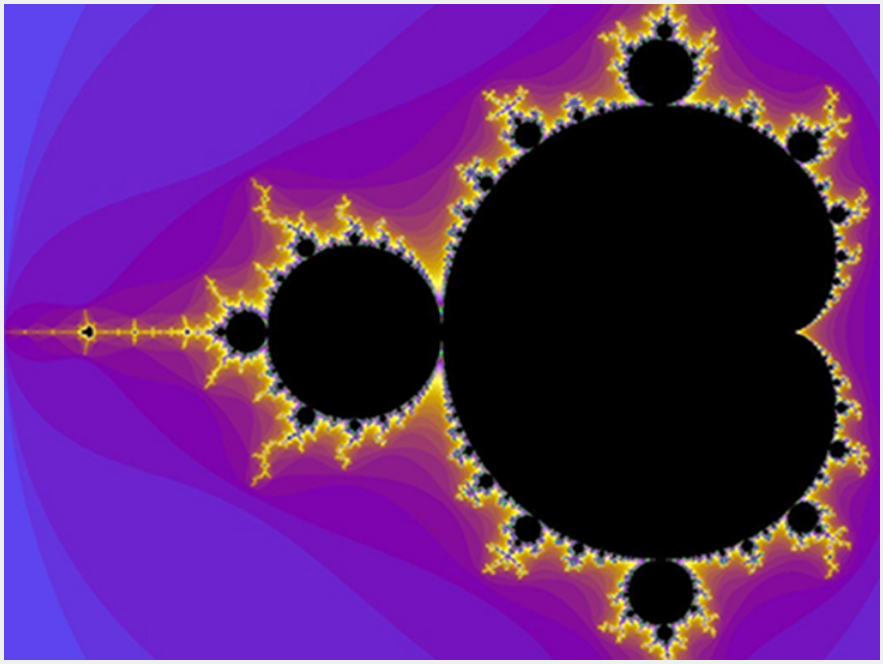


Figure 4.10.7: The Mandelbrot set is a well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.

In this project we use the logistic map

$$f(x) = rx(1 - x)$$

where $x \in [0, 1]$ and $r > 0$

as the function in our iterative process. The logistic map is a deceptively simple function; but, depending on the value of r, the resulting iterative process displays some very interesting behavior. It can lead to fixed points, cycles, and even chaos.

To visualize the long-term behavior of the iterative process associated with the logistic map, we will use a tool called a cobweb diagram. As we did with the iterative process we examined earlier in this section, we first draw a vertical line from the point $(x_0, 0)$ to the point $(x_0, f(x_0)) = (x_0, x_1)$. We then draw a horizontal line from that point to the point (x_1, x_1), then draw a vertical line to $(x_1, f(x_1)) = (x_1, x_2)$, and continue the process until the long-term behavior of the system becomes apparent. Figure 4.10.8 shows the long-term behavior of the logistic map when $r = 3.55$ and $x_0 = 0.2$. (The first 100 iterations are not plotted.) The long-term behavior of this iterative process is an 8-cycle.

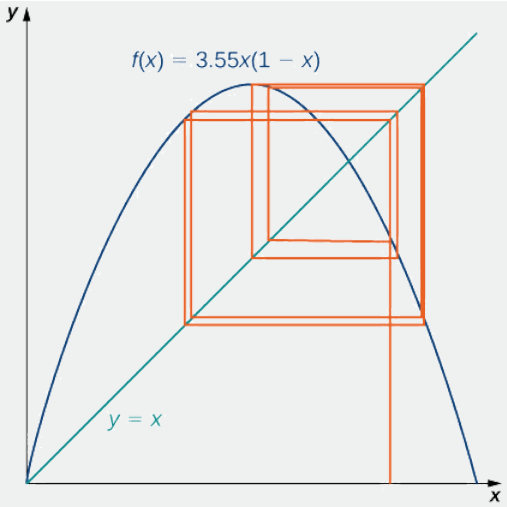


Figure 4.10.8: A cobweb diagram for $f(x) = 3.55x(1 - x)$ is presented here. The sequence of values results in an 8-cycle.

1. Let $r = 0.5$ and choose $x_0 = 0.2$. Either by hand or by using a computer, calculate the first 10 values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, 2−cycle, 4−cycle.)?
2. What happens when $r = 2$?
3. For $r = 3.2$ and $r = 3.5$, calculate the first 100 sequence values. Generate a cobweb diagram for each iterative process. (Several free applets are available online that generate cobweb diagrams for the logistic map.) What is the long-term behavior in each of these cases?
4. Now let $r = 4$. Calculate the first 100 sequence values and generate a cobweb diagram. What is the long-term behavior in this case?
5. Repeat the process for $r = 4$, but let $x_0 = 0.201$. How does this behavior compare with the behavior for $x_0 = 0.2$?

Key Concepts

- Newton's method approximates roots of $f(x) = 0$ by starting with an initial approximation x_0, then uses tangent lines to the graph of f to create a sequence of approximations $x_1, x_2, x_3, \ldots$.
- Typically, Newton's method is an efficient method for finding a particular root. In certain cases, Newton's method fails to work because the list of numbers $x_0, x_1, x_2, \ldots$ does not approach a finite value or it approaches a value other than the root sought.
- Any process in which a list of numbers $x_0, x_1, x_2, \ldots$ is generated by defining an initial number x_0 and defining the subsequent numbers by the equation $x_n = F(x_{n-1})$ for some function F is an iterative process. Newton's method is an example of an iterative process, where the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$ for a given function f.

Glossary

iterative process

process in which a list of numbers $x_0, x_1, x_2, x_3 \ldots$ is generated by starting with a number x_0 and defining $x_n = F(x_{n-1})$ for $n \geq 1$

Newton's method

method for approximating roots of $f(x) = 0$; using an initial guess x_0; each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

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4.11: Antiderivatives

Learning Objectives

- Find the general antiderivative of a given function.
- Explain the terms and notation used for an indefinite integral.
- State the power rule for integrals.
- Use antidifferentiation to solve simple initial-value problems.

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f, how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f. Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in Derivatives of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$—that is, $v(t) = s'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$—that is, $a(t) = v'(t) = s''(t)$. Now suppose we are given an acceleration function a, but not the velocity function v or the position function s. Since $a(t) = v'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text (Introduction to Techniques of Integration).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F with derivative f, we call F an antiderivative of f.

Definition: Antiderivative

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x$.

Are there any other antiderivatives of f?

Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives.

Are there any others that are not of the form $x^2 + C$ for some constant C?

The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x)$, then $F(x) - G(x) = C$ for some constant C. This fact leads to the following important theorem.

Theorem 4.11.1: General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

I. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;

II. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 4.11.1: Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x) = 3x^2$
b. $f(x) = \dfrac{1}{x}$
c. $f(x) = \cos x$
d. $f(x) = e^x$

Solution:

a. Because

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

then $F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C, and every function of the form $x^3 + C$ is an antiderivative of $3x^2$.

b. Let $f(x) = \ln|x|$.

For $x > 0,\ f(x) = \ln|x| = \ln(x)$ and

$$\frac{d}{dx}\Big(\ln x\Big) = \frac{1}{x}.$$

For $x < 0,\ f(x) = \ln|x| = \ln(-x)$ and

$$\frac{d}{dx}\Big(\ln(-x)\Big) = -\frac{1}{-x} = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}\Big(\ln|x|\Big) = \frac{1}{x}.$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\dfrac{1}{x}$. Therefore, every antiderivative of $\dfrac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\ln|x| + C$ is an antiderivative of $\dfrac{1}{x}$.

c. We have

$$\frac{d}{dx}\Big(\sin x\Big) = \cos x,$$

so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x + C$ for some constant C and every function of the form $\sin x + C$ is an antiderivative of $\cos x$.

d. Since

$$\frac{d}{dx}\left(e^x\right) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x.

Exercise 4.11.1

Find all antiderivatives of $f(x) = \sin x$.

Hint

What function has a derivative of $\sin x$?

Answer

$F(x) = -\cos x + C$

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f, we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here we introduce notation for antiderivatives. If F is an antiderivative of f, we say that $F(x) + C$ is the most general antiderivative of f and write

$$\int f(x)\,dx = F(x) + C.$$

The symbol $\int$ is called an integral sign, and $\int f(x)\,dx$ is called the indefinite integral of f.

Definition: Indefinite Integrals

Given a function f, the indefinite integral of f, denoted

$$\int f(x)\,dx,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)\,dx = F(x) + C.$$

The expression $f(x)$ is called the integrand and the variable x is the variable of integration.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as integrating f.

For a function f and an antiderivative F, the functions $F(x) + C$, where C is any real number, is often referred to as the family of antiderivatives of f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form $x^2 + C$, we write

$$\int 2x\,dx = x^2 + C.$$

The collection of all functions of the form $x^2 + C$, where C is any real number, is known as the family of antiderivatives of $2x$. Figure 4.11.1 shows a graph of this family of antiderivatives.

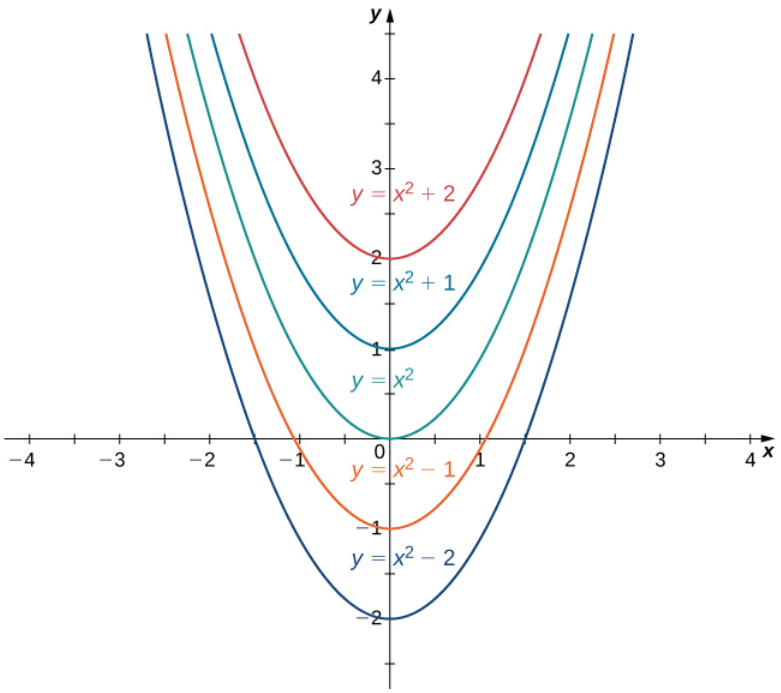


Figure 4.11.1: The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C$, where C is any real number.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n.$$

This fact is known as the power rule for integrals.

Power Rule for Integrals

For $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in Appendix B.

Table 4.11.1: Integration Formulas

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}\left(k\right) = 0$	$\int k\,dx = \int kx^0\,dx = kx + C$
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}\left(\ln\lvert x\rvert\right) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}\left(e^x\right) = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}\left(\sin x\right) = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\left(\cos x\right) = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
$\frac{d}{dx}\left(\sec^{-1}\lvert x\rvert\right) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}\lvert x\rvert + C$

From the definition of indefinite integral of f, we know

$$\int f(x)\,dx = F(x) + C$$

if and only if F is an antiderivative of f.

Therefore, when claiming that

$$\int f(x)\,dx = F(x) + C$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example 4.11.2: Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)\,dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int \left(x + e^x\right)dx = \frac{x^2}{2} + e^x + C$

b. $\int xe^x\,dx = xe^x - e^x + C$

Solution:

a. Since

$$\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x,$$

the statement

$$\int \left(x + e^x\right) dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x, respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

$$\frac{d}{dx}\left(xe^x - e^x + C\right) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

$$\int xe^x\, dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2e^x/2$ is not an antiderivative of xe^x since

$$\frac{d}{dx}\left(\frac{x^2e^x}{2}\right) = xe^x + \frac{x^2e^x}{2} \neq xe^x.$$

In general, the product of antiderivatives is not an antiderivative of a product.

? Exercise 4.11.2

Verify that $\int x\cos x\ dx = x\sin x + \cos x + C$.

Hint

Calculate $\frac{d}{dx}\left(x\sin x + \cos x + C\right)$.

Answer

$\frac{d}{dx}\left(x\sin x + \cos x + C\right) = \sin x + x\cos x - \sin x = x\cos x$

In Table 4.11.1, we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f + g$. In Example 4.11.2a we showed that an antiderivative of the sum $x + e^x$ is given by the sum $\frac{x^2}{2} + e^x$—that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and g, respectively, then

$$\frac{d}{dx}\left(F(x) + G(x)\right) = F'(x) + G'(x) = f(x) + g(x).$$

Therefore, $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and we have

$$\int \left(f(x) + g(x)\right) dx = F(x) + G(x) + C.$$

Similarly,

$$\int \left(f(x) - g(x)\right) dx = F(x) - G(x) + C.$$

In addition, consider the task of finding an antiderivative of $kf(x)$, where k is any real number. Since

$$\frac{d}{dx}\left(kF(x)\right) = k\frac{d}{dx}\left(F(x)\right) = kF'(x)$$

for any real number k, we conclude that

$$\int kf(x)\,dx = kF(x) + C.$$

These properties are summarized next.

Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g, respectively, and let k be any real number.

Sums and Differences

$$\int \big(f(x) \pm g(x)\big)\,dx = F(x) \pm G(x) + C$$

Constant Multiples

$$\int kf(x)\,dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated. (See Example **4.11.2***b* for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in Introduction to Integration. In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example 4.11.3: Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int \big(5x^3 - 7x^2 + 3x + 4\big)\,dx$

b. $\int \dfrac{x^2 + 4\sqrt[3]{x}}{x}\,dx$

c. $\int \dfrac{4}{1+x^2}\,dx$

d. $\int \tan x \cos x\,dx$

Solution:

a. Using Properties of Indefinite Integrals, we can integrate each of the four terms in the integrand separately. We obtain

$$\int \big(5x^3 - 7x^2 + 3x + 4\big)\,dx = \int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx.$$

From the second part of Properties of Indefinite Integrals, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx = 5\int x^3\,dx - 7\int x^2\,dx + 3\int x\,dx + 4\int 1\,dx.$$

Using the power rule for integrals, we conclude that

$$\int \big(5x^3 - 7x^2 + 3x + 4\big)\,dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x}.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

$$\int \left(x + \frac{4}{x^{2/3}}\right) dx = \int x\, dx + 4\int x^{-2/3}\, dx$$
$$= \frac{1}{2}x^2 + 4\frac{1}{\left(\frac{-2}{3}\right)+1}x^{(-2/3)+1} + C$$
$$= \frac{1}{2}x^2 + 12x^{1/3} + C.$$

c. Using Properties of Indefinite Integrals, write the integral as

$$4\int \frac{1}{1+x^2}\, dx.$$

Then, use the fact that $\tan^{-1}(x)$ is an antiderivative of $\frac{1}{1+x^2}$ to conclude that

$$\int \frac{4}{1+x^2}\, dx = 4\tan^{-1}(x) + C.$$

d. Rewrite the integrand as

$$\tan x \cos x = \frac{\sin x}{\cos x} \cdot \cos x = \sin x.$$

Therefore,

$$\int \tan x \cos x\, dx = \int \sin x\, dx = -\cos x + C.$$

? Exercise 4.11.3

Evaluate $\int \left(4x^3 - 5x^2 + x - 7\right) dx$.

Hint

Integrate each term in the integrand separately, making use of the power rule.

Answer

$$\int \left(4x^3 - 5x^2 + x - 7\right) dx = \quad x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 - 7x + C$$

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A differential equation is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x) \tag{4.11.1}$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of Equation 4.11.1 are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2\, dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x)$$

with the additional condition

$$y(x_0) = y_0$$

is an example of an initial-value problem. The condition $y(x_0) = y_0$ is known as an initial condition. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.

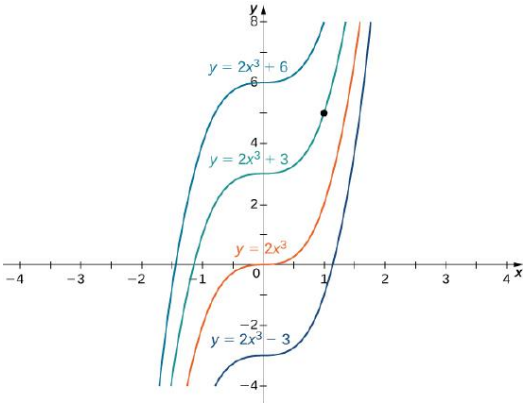


Figure 4.11.2: Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example 4.11.4: Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin x, \quad y(0) = 5.$$

Solution

First we need to solve the differential equation. If $\frac{dy}{dx} = \sin x$, then

$$y = \int \sin(x)\, dx = -\cos x + C.$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition $y(0) = 5$ means we need a constant C such that $-\cos x + C = 5$. Therefore,

$$C = 5 + \cos(0) = 6.$$

The solution of the initial-value problem is $y = -\cos x + 6$.

Exercise 4.11.4

Solve the initial value problem $\frac{dy}{dx} = 3x^{-2}, \quad y(1) = 2.$

Hint

Find all antiderivatives of $f(x) = 3x^{-2}$.

Answer

$$y = -\frac{3}{x} + 5$$

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car. We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 4.11.5:

A car is traveling at the rate of 88 ft/sec (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec^2.

a. How many seconds elapse before the car stops?
b. How far does the car travel during that time?

Solution

a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.

The car is traveling at a rate of 88 ft/sec. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is

$$a(t) = -15 \text{ ft/sec}^2.$$

The acceleration is the derivative of the velocity,

$$v'(t) = -15.$$

Therefore, we have an initial-value problem to solve:

$$v'(t) = -15, \quad v(0) = 88.$$

Integrating, we find that

$$v(t) = -15t + C.$$

Since $v(0) = 88, C = 88$. Thus, the velocity function is

$$v(t) = -15t + 88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t + 88 = 0$, we obtain $t = \frac{88}{15}$ sec.

b. To find how far the car travels during this time, we need to find the position of the car after $\frac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem

$$s'(t) = -15t + 88, \quad s(0) = 0.$$

Integrating, we have

$$s(t) = -\frac{15}{2}t^2 + 88t + C.$$

Since $s(0) = 0$, the constant is $C = 0$. Therefore, the position function is

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

? Exercise 4.11.5

Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop? How far will the car travel?

Hint

$v(t) = -15t + 44$.

Answer

2.93 sec, 64.5 ft

Key Concepts

- If F is an antiderivative of f, then every antiderivative of f is of the form $F(x) + C$ for some constant C.
- Solving the initial-value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0$$

requires us first to find the set of antiderivatives of f and then to look for the particular antiderivative that also satisfies the initial condition.

Glossary

antiderivative

a function F such that $F'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

indefinite integral

the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)\,dx$ to denote the indefinite integral of f

initial value problem

a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

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4.11E: Exercises

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4.12: Chapter 4 Review Exercises

True or False? Justify your answer with a proof or a counterexample. Assume that $f(x)$ is continuous and differentiable unless stated otherwise.

1) If $f(-1) = -6$ and $f(1) = 2$, then there exists at least one point $x \in [-1, 1]$ such that $f'(x) = 4$.

Answer

True, by Mean Value Theorem

2) If $f'(c) = 0$, there is a maximum or minimum at $x = c$.

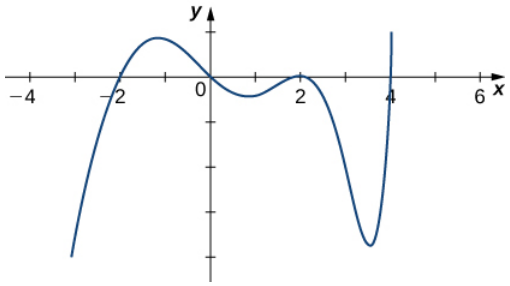
3) There is a function such that $f(x) < 0, f'(x) > 0$, and $f''(x) < 0$. (A graphical "proof" is acceptable for this answer.)

Answer

True

4) There is a function such that there is both an inflection point and a critical point for some value $x = a$.

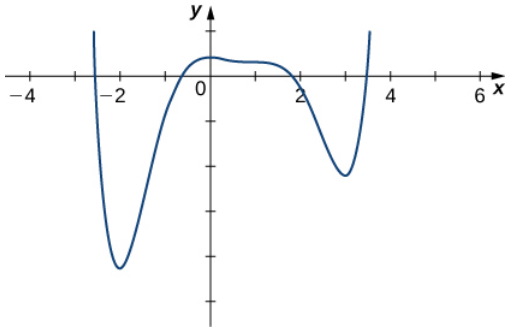
5) Given the graph of f', determine where f is increasing or decreasing.



Answer

Increasing: $(-2, 0) \cup (4, \infty)$, decreasing: $(-\infty, -2) \cup (0, 4)$

6) The graph of f is given below. Draw f'.



7) Find the linear approximation $L(x)$ to $y = x^2 + \tan(\pi x)$ near $x = \frac{1}{4}$.

Answer

$L(x) = \frac{17}{16} + \frac{1}{2}(1 + 4\pi)\left(x - \frac{1}{4}\right)$

8) Find the differential of $y = x^2 - 5x - 6$ and evaluate for $x = 2$ with $dx = 0.1$.

Find the critical points and the local and absolute extrema of the following functions on the given interval.

9) $f(x) = x + \sin^2(x)$ over $[0, \pi]$

Answer

Critical point: $x = \frac{3\pi}{4}$,
Absolute minimum: 0 when $x = 0$,
Absolute maximum: π when $x = \pi$

Solution:

10) $f(x) = 3x^4 - 4x^3 - 12x^2 + 6$ over $[-3, 3]$

Determine over which intervals the following functions are increasing, decreasing, concave up, and concave down.

11) $x(t) = 3t^4 - 8t^3 - 18t^2$

Answer

Increasing: $(-1, 0) \cup (3, \infty)$,
Decreasing: $(-\infty, -1) \cup (0, 3)$,
Concave up: $\left(-\infty, \frac{1}{3}\left(2 - \sqrt{13}\right)\right) \cup \left(\frac{1}{3}\left(2 + \sqrt{13}\right), \infty\right)$,
Concave down: $\left(\frac{1}{3}\left(2 - \sqrt{13}\right), \frac{1}{3}\left(2 + \sqrt{13}\right)\right)$

12) $y = x + \sin(\pi x)$

13) $g(x) = x - \sqrt{x}$

Answer

Increasing: $\left(\frac{1}{4}, \infty\right)$,
Decreasing: $\left(0, \frac{1}{4}\right)$,
Concave up: $(0, \infty)$,
Concave down: nowhere

14) $f(\theta) = \sin(3\theta)$

Evaluate the following limits.

15) $\lim\limits_{x \to \infty} \dfrac{3x\sqrt{x^2 + 1}}{\sqrt{x^4 - 1}}$

Answer

3

16) $\lim\limits_{x \to \infty} \cos\left(\dfrac{1}{x}\right)$

17) $\lim\limits_{x \to 1} \dfrac{x - 1}{\sin(\pi x)}$

Answer

$-\frac{1}{\pi}$

18) $\lim\limits_{x \to \infty} (3x)^{1/x}$

Use Newton's method to find the first two iterations, given the starting point.

19) $y = x^3 + 1, \quad x_0 = 0.5$

Answer

$x_1 = -1,\ x_2 = -1$

20) $\dfrac{1}{x + 1} = \dfrac{1}{2}, \quad x_0 = 0$

Find the antiderivatives $F(x)$ of the following functions.

21) $g(x) = \sqrt{x} - \frac{1}{x^2}$

Answer

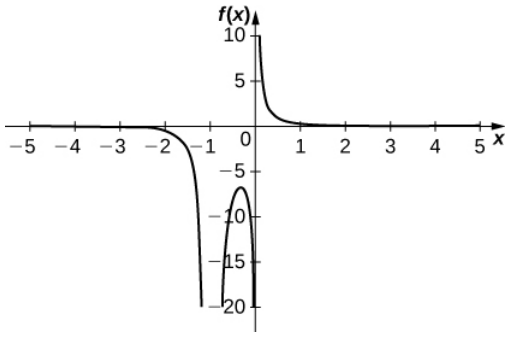
$$F(x) = \frac{2x^{3/2}}{3} + \frac{1}{x} + C$$

22) $f(x) = 2x + 6\cos x, \quad F(\pi) = \pi^2 + 2$

Graph the following functions by hand. Make sure to label the inflection points, critical points, zeros, and asymptotes.

23) $y = \frac{1}{x(x+1)^2}$

Answer



Inflection points: none;
Critical points: $x = -\frac{1}{3}$;
Zeros: none;
Vertical asymptotes: $x = -1,\ x = 0$;
Horizontal asymptote: $y = 0$

24) $y = x - \sqrt{4 - x^2}$

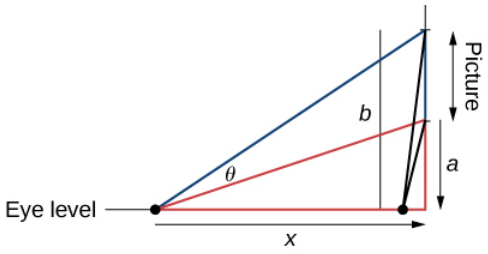
25) A car is being compacted into a rectangular solid. The volume is decreasing at a rate of $2\ \text{m}^3/\text{sec}$. The length and width of the compactor are square, but the height is not the same length as the length and width. If the length and width walls move toward each other at a rate of 0.25 m/sec, find the rate at which the height is changing when the length and width are 2 m and the height is 1.5 m.

Answer

The height is decreasing at a rate of 0.125 m/sec

26) A rocket is launched into space; its kinetic energy is given by $K(t) = \frac{1}{2}m(t)v(t)^2$, where K is the kinetic energy in joules, m is the mass of the rocket in kilograms, and v is the velocity of the rocket in meters/second. Assume the velocity is increasing at a rate of $15\ \text{m/sec}^2$ and the mass is decreasing at a rate of 10 kg/sec because the fuel is being burned. At what rate is the rocket's kinetic energy changing when the mass is 2000 kg and the velocity is 5000 m/sec? Give your answer in mega-Joules (MJ), which is equivalent to 10^6 J.

27) The famous **Regiomontanus' problem** for angle maximization was proposed during the 15th century. A painting hangs on a wall with the bottom of the painting a distance a feet above eye level, and the top b feet above eye level. What distance x (in feet) from the wall should the viewer stand to maximize the angle subtended by the painting, θ?



Answer

$x = \sqrt{ab}$ feet

28) An airline sells tickets from Tokyo to Detroit for $1200. There are 500 seats available and a typical flight books 350 seats. For every $10 decrease in price, the airline observes an additional five seats sold. What should the fare be to maximize profit? How many passengers would be onboard?

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CHAPTER OVERVIEW

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5.1: Approximating Areas

Learning Objectives

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the **method of exhaustion**, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x-axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ, sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20.$$

We could probably skip writing a couple of terms and write

$$1+2+3+4+\cdots+19+20,$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i$$

which is much more compact. Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i$$

where a_i describes the terms to be added, and the i is called the *index*. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n$. For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2 + s_3 + s_4 + s_5 + s_6 + s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a **dummy variable**. We can use any letter we like for the index. Typically, mathematicians use i, j, k, m, and n for indices.

Let's try a couple of examples of using sigma notation.

Example 5.1.1: Using Sigma Notation

a. Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.

b. Write the sum in sigma notation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

Solution

a. Write

$$\sum_{i=1}^{5} 3^i = 3 + 3^2 + 3^3 + 3^4 + 3^5 = 363.$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

Exercise 5.1.1

Write in sigma notation and evaluate the sum of terms 2^i for $i = 3, 4, 5, 6$.

Hint

Use the solving steps in Example 5.1.1 as a guide.

Answer

$$\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$$

The properties associated with the summation process are given in the following rule.

Rule: Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \le m \le n$.

i. $\sum_{i=1}^{n} c = nc$

ii. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

iii. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

iv. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

v. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

Proof

We prove properties (ii.) and (iii.) here, and leave proof of the other properties to the Exercises.

(ii.) We have

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + a_3 + \cdots + a_n) = c\sum_{i=1}^{n} a_i.$$

(iii.) We have

$$\sum_{i=1}^{n}(a_i+b_i)=(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\cdots+(a_n+b_n) \tag{5.1.1}$$

$$=(a_1+a_2+a_3+\cdots+a_n)+(b_1+b_2+b_3+\cdots+b_n) \tag{5.1.2}$$

$$=\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}b_i. \tag{5.1.3}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n}i=1+2+\cdots+n=\frac{n(n+1)}{2}. \tag{5.1.4}$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n}i^2=1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}. \tag{5.1.5}$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n}i^3=1^3+2^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}. \tag{5.1.6}$$

Example 5.1.2: Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

a. The sum of the terms $(i-3)^2$ for $i=1,2,\ldots,200$.
b. The sum of the terms (i^3-i^2) for $i=1,2,3,4,5,6$

Solution

a. Multiplying out $(i-3)^2$, we can break the expression into three terms.

$$\begin{aligned}\sum_{i=1}^{200}(i-3)^2&=\sum_{i=1}^{200}(i^2-6i+9)\\&=\sum_{i=1}^{200}i^2-\sum_{i=1}^{200}6i+\sum_{i=1}^{200}9\\&=\sum_{i=1}^{200}i^2-6\sum_{i=1}^{200}i+\sum_{i=1}^{200}9\\&=\frac{200(200+1)(400+1)}{6}-6\left[\frac{200(200+1)}{2}\right]+9(200)\\&=2,686,700-120,600+1800\\&=2,567,900\end{aligned}$$

b. Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

$$\begin{aligned}\sum_{i=1}^{6}(i^3-i^2) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\ &= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6} \\ &= \frac{1764}{4} - \frac{546}{6} \\ &= 350\end{aligned}$$

? Exercise 5.1.2

Find the sum of the values of $4+3i$ for $i = 1, 2, \ldots, 100$.

Hint

Use the properties of sigma notation to solve the problem.

Answer

$15,550$

✓ Example 5.1.3: Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

Solution

Using Equation 5.1.6, we have

$$\sum_{i=0}^{10} i^3 = \frac{(10)^2(10+1)^2}{4} = \frac{100(121)}{4} = 3025$$

? Exercise 5.1.3

Evaluate the sum indicated by the notation $\sum_{k=1}^{20}(2k+1)$.

Hint

Use the rule on sum and powers of integers (Equations 5.1.4-5.1.6).

Answer

440

Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x-axis below, the line $x = a$ on the left, and the line $x = b$ on the right (Figure 5.1.1).

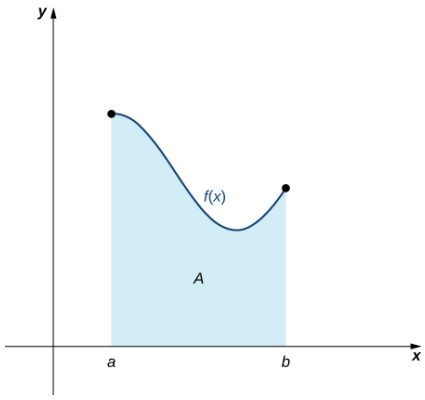


Figure 5.1.1: An area (shaded region) bounded by the curve $f(x)$ at top, the x-axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b$, and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

for $i = 1, 2, 3, \ldots, n$.

We denote the width of each subinterval with the notation Δx, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

for $i = 1, 2, 3, \ldots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

Definition: Partitions

A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** (or uniform partition) of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

Rule: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure 5.1.2). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

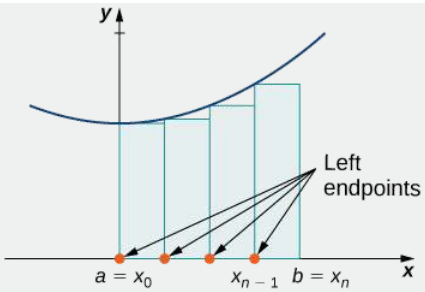


Figure 5.1.2: In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Rule: Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\,\Delta x$ and the approximation for A is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

The notation R_n indicates this is a **right-endpoint approximation** for A (Figure 5.1.3).

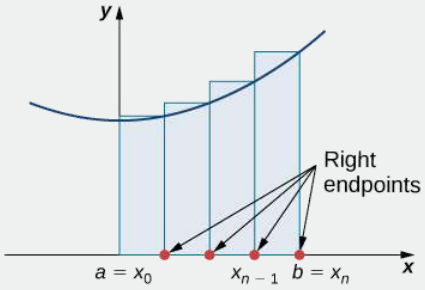


Figure 5.1.3: In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in Figure 5.1.2.

The graphs in Figure 5.1.4 represent the curve $f(x) = \frac{x^2}{2}$. In Figure 5.1.4b we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width 0.5. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1}, the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\ &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375 \text{ units}^2 \end{aligned}$$

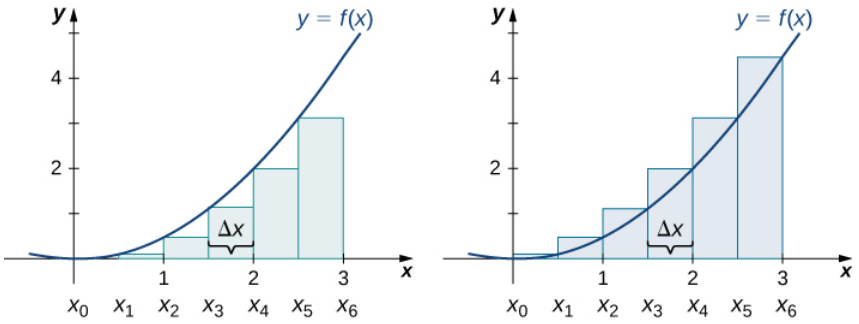


Figure 5.1.4: Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In Figure 5.1.4*b*, we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\ &= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875 \text{ units}^2. \end{aligned}$$

Example 5.1.4: Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \dfrac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$ are shown in Figure 5.1.5. Using a left-endpoint approximation, the heights are $f(0) = 0$, $f(0.5) = 0.25$, $f(1) = 1$, and $f(1.5) = 2.25$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 1.75 \text{ units}^2 \end{aligned}$$

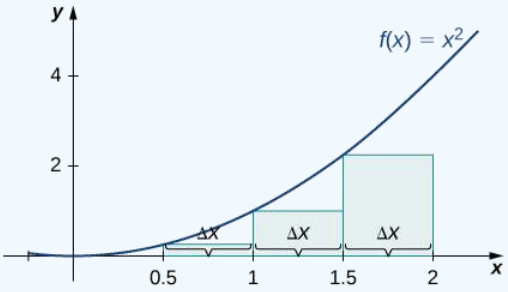


Figure 5.1.5: The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure 5.1.6. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 3.75 \text{ units}^2 \end{aligned}$$

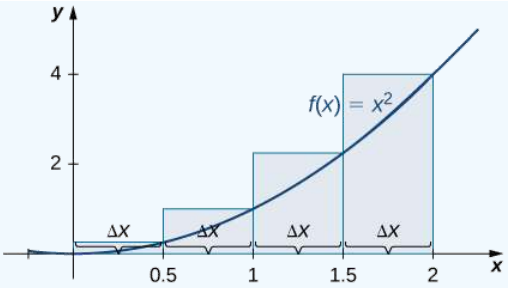


Figure 5.1.6: The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is $1.75\ \text{units}^2$; the right-endpoint approximation is $3.75\ \text{units}^2$.

Exercise 5.1.4

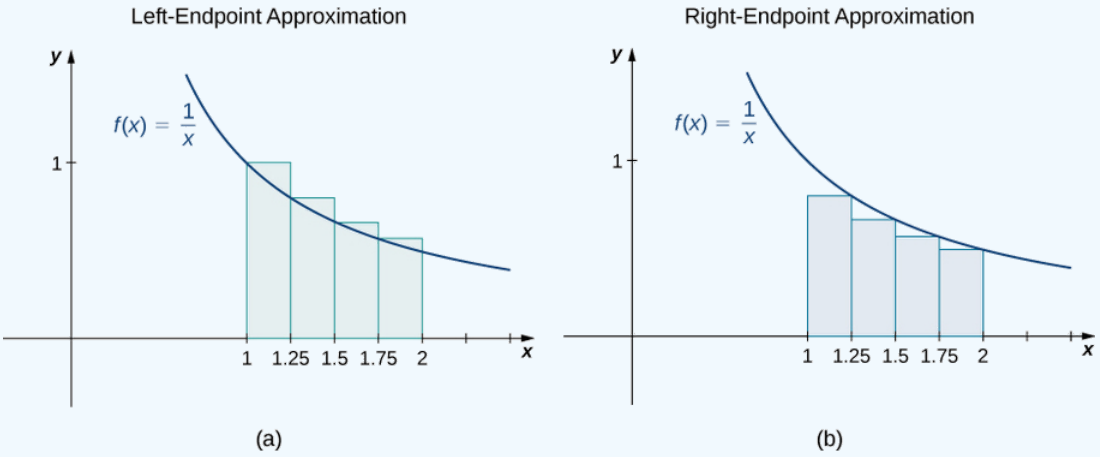
Sketch left-endpoint and right-endpoint approximations for $f(x) = \frac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Hint

Follow the solving strategy in Example 5.1.4 step-by-step.

Answer

The left-endpoint approximation is $0.7595\ \text{units}^2$. The right-endpoint approximation is $0.6345\ \text{units}^2$. See the below Media.



Looking at Figure 5.1.4 and the graphs in Example 5.1.4, we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n, first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. Figure 5.1.7 shows the area of the region under the curve $f(x) = (x-1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$$

The area is approximated by the summed areas of the rectangles, or

$$L_4 = f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)0.5 = 7.5\ \text{units}^2$$

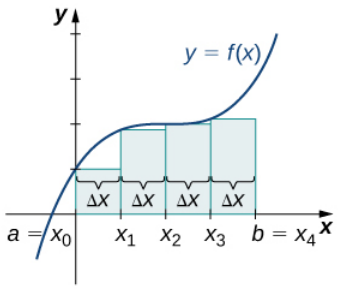


Figure 5.1.7: With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

Figure 5.1.8 shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in Figure 5.1.7 with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

$$L_8 = f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) = 7.75 \text{ units}^2$$

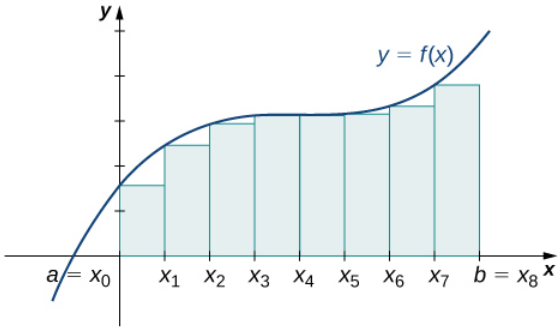


Figure 5.1.8: The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in Figure 5.1.9 shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$L_{32} = f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) = 7.9375 \text{ units}^2.$$

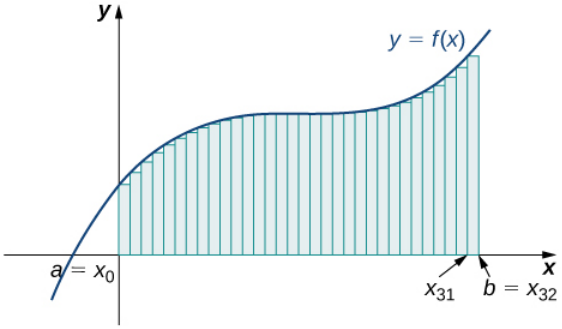


Figure 5.1.9: Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles (Figure 5.1.10), yields an area

$$R_4 = f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) = 8.5 \text{ units}^2.$$

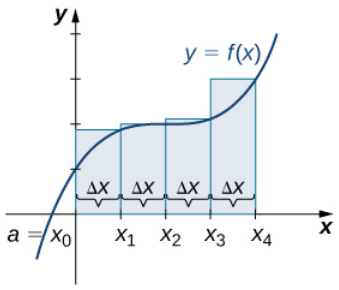


Figure 5.1.10: Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \frac{2-0}{8} = 0.25$. The graph is shown in Figure 5.1.11. The area is

$$R_8 = f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) = 8.25 \text{ units}^2$$

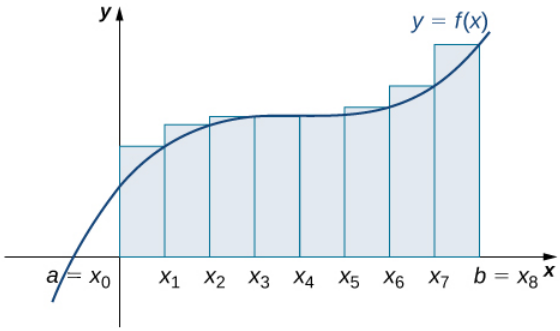


Figure 5.1.11: Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area (Figure 5.1.12). The area is approximately

$$R_{32} = f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) = 8.0625 \text{ units}^2$$

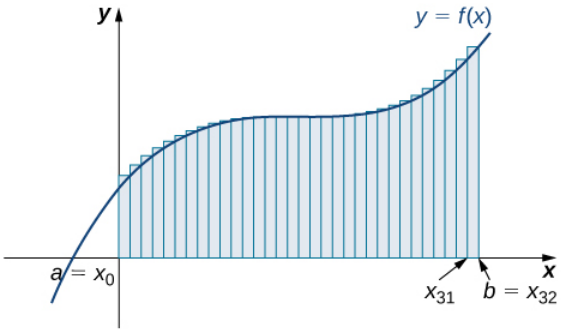


Figure 5.1.12: The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as n gets larger. Furthermore, as n increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. Table 5.1.15 shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as n gets larger and larger is very important, and we now explore this idea in more detail.

Table 5.1.15: Converging Values of Left- and Right-Endpoint Approximations as n Increases

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 4$	7.5	8.5
$n = 8$	7.75	8.25
$n = 32$	7.94	8.06

Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point x_i^* in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition: Riemann sum

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be any partition of $[a, b]$. Let Δx_i be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

At this point, we'll choose a regular partition P, as we have in our examples above. This forces all Δx_i to be equal to $\Delta x = \dfrac{b-a}{n}$ for any natural number of intervals n.

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ be a Riemann sum for $f(x)$ with a regular partition P. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

See a graphical demonstration of the construction of a Riemann sum.

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in detail in the chapter on Sequences and Series; however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of x_i^*. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (in other words, it does not depend on the choice of x_i^*).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for x_i^*. Although any choice for x_i^* gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high (overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for x_i^* to guarantee one result or the other.

If we want an overestimate, for example, we can choose x_i^* such that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*) \geq f(x)$ for all $x \in [x_i - 1, x_i]$. In other words, we choose x_i^* so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select x_i^* in this way, then the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose x_i^* so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Example 5.1.5: Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

Solution

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \frac{1}{4}$. We can list the intervals as $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, and $[1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, Figure shows that a lower sum is obtained by using the right endpoints.

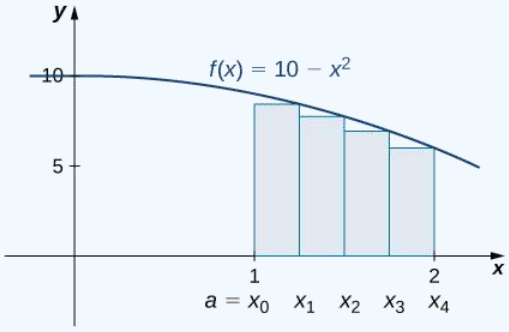


Figure 5.1.13: The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x-axis on $[1, 2]$, and it shows a lower sum.

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4}(10 - x^2)(0.25) &= 0.25[10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2] \\ &= 0.25[8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28 \text{ units}^2.\end{aligned}$$

The area of 7.28 units^2 is a lower sum and an underestimate.

Exercise 5.1.5

a. Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$.
b. Sketch the approximation.

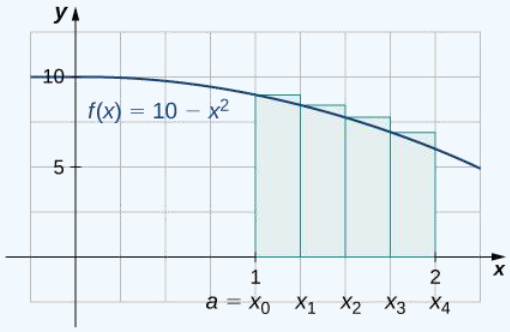
Hint

$f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

Answer

a. Upper sum=8.0313 units^2.

b.



Example 5.1.6: Finding Lower and Upper Sums for $f(x) = \sin x$

Find a lower sum for $f(x) = \sin x$ over the interval $[a, b] = [0, \frac{\pi}{2}]$; let $n = 6$.

Solution

Let's first look at the graph in Figure 5.1.14 to get a better idea of the area of interest.

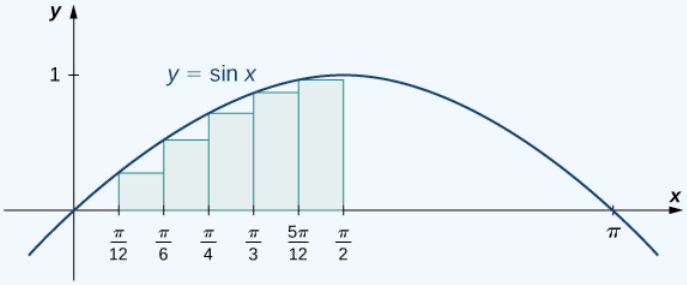


Figure 5.1.14: The graph of $y = \sin x$ is divided into six regions: $\Delta x = \frac{\pi/2}{6} = \frac{\pi}{12}$.

The intervals are $[0, \frac{\pi}{12}]$, $[\frac{\pi}{12}, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{3}]$, $[\frac{\pi}{3}, \frac{5\pi}{12}]$, and $[\frac{5\pi}{12}, \frac{\pi}{2}]$. Note that $f(x) = \sin x$ is increasing on the interval $[0, \frac{\pi}{2}]$, so a left-endpoint approximation gives us the lower sum. A left-endpoint approximation is the Riemann sum $\sum_{i=0}^{5} \sin x_i \left(\frac{\pi}{12}\right)$.We have

$$A \approx \sin(0)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{12}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{6}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{4}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{3}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{5\pi}{12}\right)\left(\tfrac{\pi}{12}\right) \approx 0.863 \text{ units}^2.$$

Exercise 5.1.6

Using the function $f(x) = \sin x$ over the interval $[0, \frac{\pi}{2}]$, find an upper sum; let $n = 6$.

Hint

Follow the steps from Example 5.1.6.

Answer

$A \approx 1.125 \text{ units}^2$

Key Concepts

- The use of sigma (summation) notation of the form $\sum_{i=1}^{n} a_i$ is useful for expressing long sums of values in compact form.
- For a continuous function defined over an interval $[a, b]$, the process of dividing the interval into n equal parts, extending a rectangle to the graph of the function, calculating the areas of the series of rectangles, and then summing the areas yields an approximation of the area of that region.

- When using a regular partition, the width of each rectangle is $\Delta x = \frac{b-a}{n}$.
- Riemann sums are expressions of the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$, and can be used to estimate the area under the curve $y = f(x)$. Left- and right-endpoint approximations are special kinds of Riemann sums where the values of x_i^* are chosen to be the left or right endpoints of the subintervals, respectively.
- Riemann sums allow for much flexibility in choosing the set of points x_i^* at which the function is evaluated, often with an eye to obtaining a lower sum or an upper sum.

Key Equations

- **Properties of Sigma Notation**

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$$

- **Sums and Powers of Integers**

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Left-Endpoint Approximation**

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

- **Right-Endpoint Approximation**

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

Glossary

left-endpoint approximation

an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

lower sum

a sum obtained by using the minimum value of $f(x)$ on each subinterval

partition

a set of points that divides an interval into subintervals

regular partition

a partition in which the subintervals all have the same width

riemann sum

an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

right-endpoint approximation

the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

sigma notation

(also, **summation notation**) the Greek letter sigma (Σ) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

upper sum

a sum obtained by using the maximum value of $f(x)$ on each subinterval

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5.1E: Exercises

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5.2: The Definite Integral

Learning Objectives

- State the definition of the definite integral.
- Explain the terms integrand, limits of integration, and variable of integration.
- Explain when a function is integrable.
- Describe the relationship between the definite integral and net area.
- Use geometry and the properties of definite integrals to evaluate them.
- Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Definition: Definite Integral

If $f(x)$ is a function defined on an interval $[a, b]$, the ***definite integral*** of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an integrable function.

The integral symbol in the previous definition should look familiar. We have seen similar notation in the chapter on Applications of Derivatives, where we used the indefinite integral symbol (without the a and b above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter we examine how these concepts are related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century and is one of the contributions of Gottfried Wilhelm **Leibniz**, who is often considered to be the codiscoverer of calculus, along with Isaac Newton. The integration symbol $\int$ is an elongated S, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers a and b are x-values and are called the **limits of integration**; specifically, a is the lower limit and b is the upper limit. To clarify, we are using the word limit in two different ways in the context of the definite integral. First, we talk about the limit of a sum as $n \to \infty$. Second, the boundaries of the region are called the *limits of integration.*

We call the function $f(x)$ the **integrand**, and the dx indicates that $f(x)$ is a function with respect to x, called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a **dummy variable**, and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a, b]$, then the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Functions that are not continuous on $[a, b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities or removable discontinuities on a closed interval are integrable.

It is also worth noting here that we have retained the use of a regular partition in the Riemann sums. This restriction is not strictly necessary. Any partition can be used to form a Riemann sum. However, if a nonregular partition is used to define the definite integral, it is not sufficient to take the limit as the number of subintervals goes to infinity. Instead, we must take the limit as the width of the largest subinterval goes to zero. This introduces a little more complex notation in our limits and makes the calculations more difficult without really gaining much additional insight, so we stick with regular partitions for the Riemann sums.

Example 5.2.1: Evaluating an Integral Using the Definition

Use the definition of the definite integral to evaluate $\int_0^2 x^2\,dx$. Use a right-endpoint approximation to generate the Riemann sum.

Solution

We first want to set up a Riemann sum. Based on the limits of integration, we have $a = 0$ and $b = 2$. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[0, 2]$. Then

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}.$$

Since we are using a right-endpoint approximation to generate Riemann sums, for each i, we need to calculate the function value at the right endpoint of the interval $[x_{i-1}, x_i]$. The right endpoint of the interval is x_i, and since P is a regular partition,

$$x_i = x_0 + i\Delta x = 0 + i\left[\frac{2}{n}\right] = \frac{2i}{n}.$$

Thus, the function value at the right endpoint of the interval is

$$f(x_i) = x_i^2 = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}.$$

Then the Riemann sum takes the form

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{4i^2}{n^2}\right)\frac{2}{n} = \sum_{i=1}^{n} \frac{8i^2}{n^3} = \frac{8}{n^3}\sum_{i=1}^{n} i^2.$$

Using the summation formula for $\sum_{i=1}^{n} i^2$, we have

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\ &= \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right] \\ &= \frac{16n^3+24n^2+n}{6n^3} \\ &= \frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}.\end{aligned}$$

Now, to calculate the definite integral, we need to take the limit as $n \to \infty$. We get

$$\begin{aligned}\int_0^2 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty} \left(\frac{8}{3} + \frac{4}{n} + \frac{1}{6n^2}\right) \\ &= \lim_{n\to\infty} \left(\frac{8}{3}\right) + \lim_{n\to\infty} \left(\frac{4}{n}\right) + \lim_{n\to\infty} \left(\frac{1}{6n^2}\right) \\ &= \frac{8}{3} + 0 + 0 = \frac{8}{3}.\end{aligned}$$

? Exercise 5.2.1

Use the definition of the definite integral to evaluate $\int_0^3 (2x-1)\,dx$.

Use a right-endpoint approximation to generate the Riemann sum.

Hint

Use the solving strategy from Example **5.2.1**.

Answer

6

Evaluating Definite Integrals

Evaluating definite integrals this way can be quite tedious because of the complexity of the calculations. Later in this chapter we develop techniques for evaluating definite integrals *without* taking limits of Riemann sums. However, for now, we can rely on the fact that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area. We do this to confirm that definite integrals do, indeed, represent areas, so we can then discuss what to do in the case of a curve of a function dropping below the x-axis.

✓ Example 5.2.2: Using Geometric Formulas to Calculate Definite Integrals

Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\,dx$.

Solution

The function describes a semicircle with radius 3. To find

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx$$

we want to find the area under the curve over the interval $[3,6]$. The formula for the area of a circle is $A = \pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A = \frac{1}{2}\pi r^2$. The shaded area in Figure **5.2.1** covers one-half of the semicircle, or $A = \frac{1}{4}\pi r^2$. Thus,

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9}{4}\pi \approx 7.069.$$

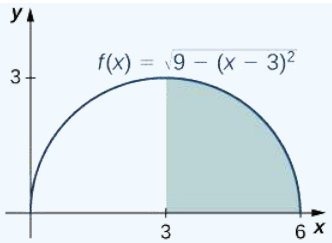


Figure 5.2.1: The value of the integral of the function $f(x)$ over the interval $[3, 6]$ is the area of the shaded region.

? Exercise 5.2.2

Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x + 3)\, dx$.

Hint

Graph the function $f(x)$ and calculate the area under the function on the interval $[2, 4]$.

Answer

18 square units

Area and the Definite Integral

When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret "the area under the curve" when $f(x)$ is negative?

Net Signed Area

Let us return to the Riemann sum. Consider, for example, the function $f(x) = 2 - 2x^2$ (shown in Figure 5.2.2) on the interval $[0, 2]$. Use $n = 8$ and choose $\{x_i^*\}$ as the left endpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width Δx. When $f(x_i^*)$ is positive, the product $f(x_i^*)\Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*)\Delta x$ represents the negative of the area of the rectangle. The Riemann sum then becomes

$$\sum_{i=1}^{8} f(x_i^*)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

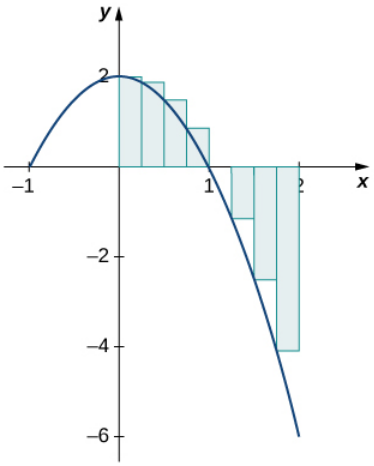


Figure 5.2.2: For a function that is partly negative, the Riemann sum is the area of the rectangles above the x-axis less the area of the rectangles below the x-axis.

Taking the limit as $n \to \infty$, the Riemann sum approaches the area between the curve above the x-axis and the x-axis, less the area between the curve below the x-axis and the x-axis, as shown in Figure 5.2.3. Then,

$$\int_0^2 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = A_1 - A_2.$$

The quantity $A_1 - A_2$ is called the **net signed area.**

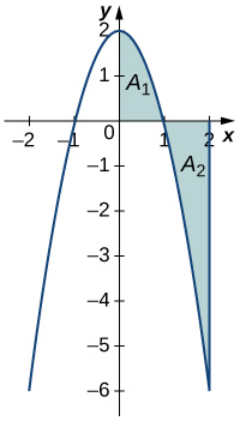


Figure 5.2.3: In the limit, the definite integral equals area A_1 less area A_2, or the net signed area.

Notice that net signed area can be positive, negative, or zero. If the area above the x-axis is larger, the net signed area is positive. If the area below the x-axis is larger, the net signed area is negative. If the areas above and below the x-axis are equal, the net signed area is zero.

Example 5.2.3: Finding the Net Signed Area

Find the net signed area between the curve of the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution

The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$ (Figure 5.2.4). Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle A_1, above the axis, is

$$A_1 = \frac{1}{2}3(6) = 9,$$

where 3 is the base and $2(3) = 6$ is the height. The area of triangle A_2, below the axis, is

$$A_2 = \frac{1}{2}(3)(6) = 9,$$

where 3 is the base and 6 is the height. Thus, the net area is

$$\int_{-3}^{3} 2x\,dx = A_1 - A_2 = 9 - 9 = 0.$$

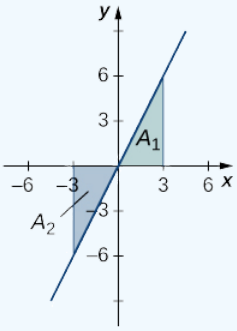


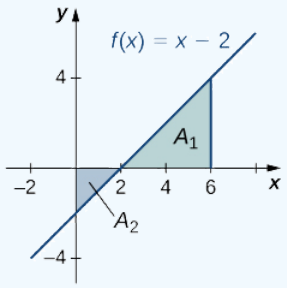
Figure 5.2.4: The area above the curve and below the x-axis equals the area below the curve and above the x-axis.

Analysis

If A_1 is the area above the x-axis and A_2 is the area below the x-axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

Exercise 5.2.3

Find the net signed area of $f(x) = x - 2$ over the interval $[0, 6]$, illustrated in the following image.



Hint

Use the solving method described in Example 5.2.3.

Answer

6

Total Area

One application of the definite integral is finding **displacement** when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. This is a very important application of the definite integral, and we examine it in more detail later in the chapter. For now, we're just going to look at some basics to get a feel for how this works by studying constant velocities.

When velocity is a constant, the area under the curve is just velocity times time. This idea is already very familiar. If a car travels away from its starting position in a straight line at a speed of 70 mph for 2 hours, then it is 140 miles away from its original position (Figure 5.2.5). Using integral notation, we have

$$\int_0^2 70\,dt = 140 \text{ miles.}$$

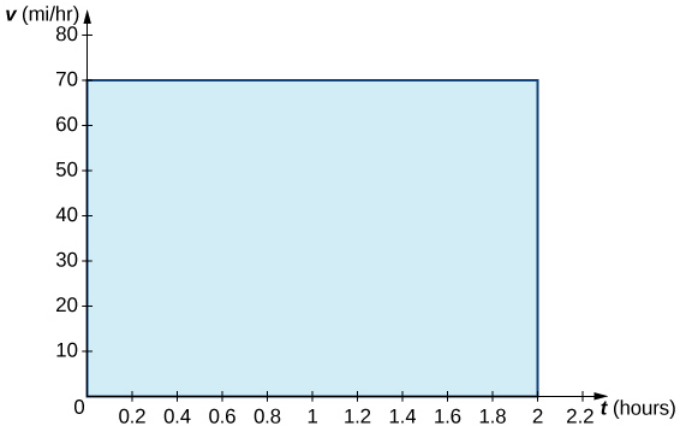


Figure 5.2.5: The area under the curve $v(t) = 70$ tells us how far the car is from its starting point at a given time.

In the context of displacement, net signed area allows us to take direction into account. If a car travels straight north at a speed of 60 mph for 2 hours, it is 120 miles north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at it starting position (Figure 5.2.6). Again, using integral notation, we have

$$\int_0^2 60\,dt + \int_2^5 -40\,dt = 120 - 120 = 0.$$

In this case the displacement is zero.

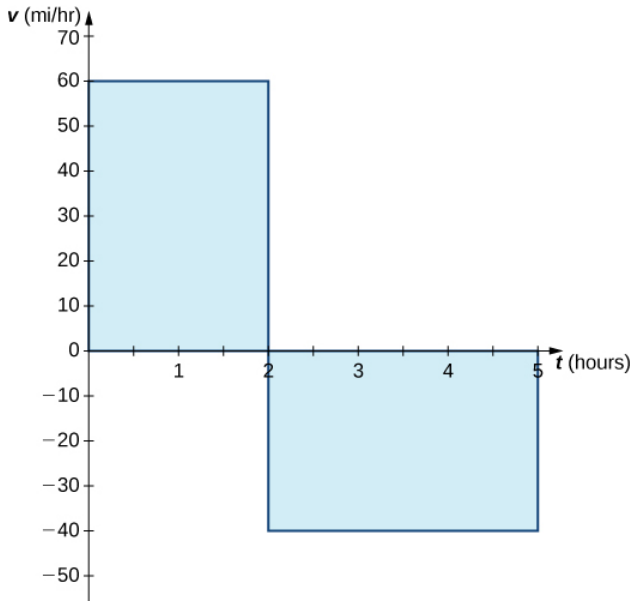


Figure 5.2.6: The area above the axis and the area below the axis are equal, so the net signed area is zero.

Suppose we want to know how far the car travels overall, regardless of direction. In this case, we want to know the area between the curve and the t-axis, regardless of whether that area is above or below the axis. This is called the **total area.**

Graphically, it is easiest to think of calculating total area by adding the areas above the axis and the areas below the axis (rather than subtracting the areas below the axis, as we did with net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is

$$\int_0^2 |60|\,dt + \int_2^5 |-40|\,dt = \int_0^2 60\,dt + \int_2^5 40\,dt = 120 + 120 = 240.$$

Bringing these ideas together formally, we state the following definitions.

Definition: Net Signed Area

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x-axis that lies above the axis and let A_2 represent the area between $f(x)$ and the x-axis that lies below the axis. Then, the *net signed area* between $f(x)$ and the x-axis is given by

$$\int_a^b f(x)\,dx = A_1 - A_2.$$

The total area between $f(x)$ and the x-axis is given by

$$\int_a^b |f(x)|\,dx = A_1 + A_2.$$

Example 5.2.4: Finding the Total Area

Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0, 6]$.

Solution

Calculate the x-intercept as $(2, 0)$ (set $y = 0$, solve for x). To find the total area, take the area below the x-axis over the subinterval $[0, 2]$ and add it to the area above the x-axis on the subinterval $[2, 6]$ (Figure 5.2.7).

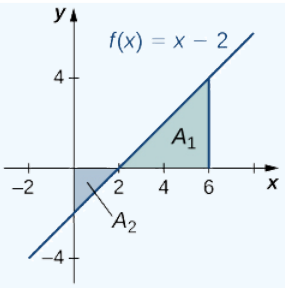


Figure 5.2.7: The total area between the line and the x-axis over $[0, 6]$ is A_2 plus A_1.

We have

$$\int_0^6 |(x-2)|\,dx = A_2 + A_1.$$

Then, using the formula for the area of a triangle, we obtain

$$A_2 = \frac{1}{2}bh = \frac{1}{2}\cdot 2 \cdot 2 = 2$$

$$A_1 = \frac{1}{2}bh = \frac{1}{2}\cdot 4 \cdot 4 = 8.$$

The total area, then, is

$$A_1 + A_2 = 8 + 2 = 10\,\text{units}^2.$$

? Exercise 5.2.4

Find the total area between the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Hint

Review the solving strategy in Example 5.2.4.

Answer

$18\,\text{units}^2$

Properties of the Definite Integral

The properties of indefinite integrals apply to definite integrals as well. Definite integrals also have properties that relate to the limits of integration. These properties, along with the rules of integration that we examine later in this chapter, help us manipulate expressions to evaluate definite integrals.

Rule: Properties of the Definite Integral

1.

$$\int_a^a f(x)\,dx = 0 \tag{5.2.1}$$

If the limits of integration are the same, the integral is just a line and contains no area.

2.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If the limits are reversed, then place a negative sign in front of the integral.

3.

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

The integral of a sum is the sum of the integrals.

4.

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

The integral of a difference is the difference of the integrals.

5.

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

for constant c. The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

6.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

✔ Example 5.2.5: Using the Properties of the Definite Integral

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2, 1]$ as the sum of three definite integrals.

Solution

Using integral notation, we have $\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx$. We apply properties 3. and 5. to get

$$\begin{aligned}\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx &= \int_{-2}^{1} -3x^3\,dx + \int_{-2}^{1} 2x\,dx + \int_{-2}^{1} 2\,dx \\ &= -3\int_{-2}^{1} x^3\,dx + 2\int_{-2}^{1} x\,dx + \int_{-2}^{1} 2\,dx.\end{aligned}$$

? Exercise 5.2.5

Use the properties of the definite integral to express the definite integral of $f(x) = 6x^3 - 4x^2 + 2x - 3$ over the interval $[1, 3]$ as the sum of four definite integrals.

Hint

Use the solving strategy from Example 5.2.5 and the properties of definite integrals.

Answer

$$6\int_1^3 x^3\,dx - 4\int_1^3 x^2\,dx + 2\int_1^3 x\,dx - \int_1^3 3\,dx$$

✔ Example 5.2.6: Using the Properties of the Definite Integral

If it is known that $\int_0^8 f(x)\,dx = 10$ and $\int_0^5 f(x)\,dx = 5$, find the value of $\int_5^8 f(x)\,dx$.

Solution

By property 6,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Thus,

$$\int_0^8 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^8 f(x)\,dx$$

$$10 = 5 + \int_5^8 f(x)\,dx$$

$$5 = \int_5^8 f(x)\,dx.$$

? Exercise 5.2.6

If it is known that $\int_1^5 f(x)\,dx = -3$ and $\int_2^5 f(x)\,dx = 4$, find the value of $\int_1^2 f(x)\,dx$.

Hint

Use the solving strategy from Example **5.2.6** and the rule on properties of definite integrals.

Answer

-7

Comparison Properties of Integrals

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration. Intuitively, we might say that if a function $f(x)$ is above another function $g(x)$, then the area between $f(x)$ and the x-axis is greater than the area between $g(x)$ and the x-axis. This is true depending on the interval over which the comparison is made. The properties of definite integrals are valid whether $a < b$, $a = b$, or $a > b$. The following properties, however, concern only the case $a \le b$, and are used when we want to compare the sizes of integrals.

Comparison Theorem

i. If $f(x) \ge 0$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \ge 0.$$

ii. If $f(x) \ge g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$$

iii. If m and M are constants such that $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Example 5.2.7: Comparing Two Functions over a Given Interval

Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0,1]$.

Solution

Graphing these functions is necessary to understand how they compare over the interval $[0,1]$. Initially, when graphed on a graphing calculator, $f(x)$ appears to be above $g(x)$ everywhere. However, on the interval $[0,1]$, the graphs appear to be on top of each other. We need to zoom in to see that, on the interval $[0,1]$, $g(x)$ is above $f(x)$. The two functions intersect at $x=0$ and $x=1$ (Figure 5.2.8).

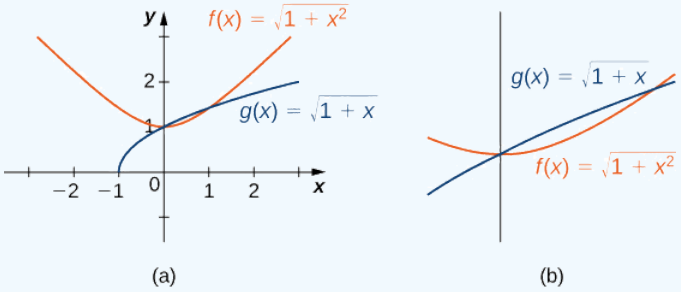


Figure 5.2.8: (a) The function $f(x)$ appears above the function $g(x)$ except over the interval $[0,1]$ (b) Viewing the same graph with a greater zoom shows this more clearly.

We can see from the graph that over the interval $[0,1]$, $g(x) \geq f(x)$. Comparing the integrals over the specified interval $[0,1]$, we also see that $\int_0^1 g(x)\,dx \geq \int_0^1 f(x)\,dx$ (Figure 5.2.9). The thin, red-shaded area shows just how much difference there is between these two integrals over the interval $[0,1]$.

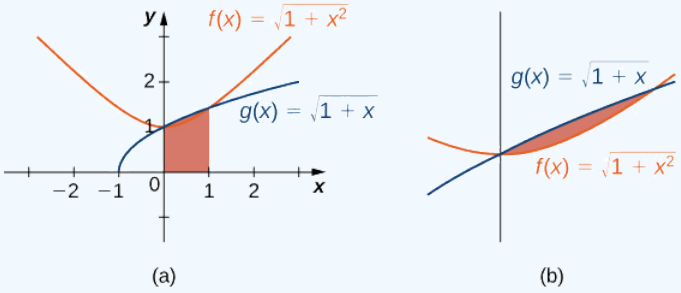


Figure 5.2.9: (a) The graph shows that over the interval $[0,1], g(x) \geq f(x)$, where equality holds only at the endpoints of the interval. (b) Viewing the same graph with a greater zoom shows this more clearly.

Average Value of a Function

We often need to find the average of a set of numbers, such as an average test grade. Suppose you received the following test scores in your algebra class: 89, 90, 56, 78, 100, and 69. Your semester grade is your average of test scores and you want to know what grade to expect. We can find the average by adding all the scores and dividing by the number of scores. In this case, there are six test scores. Thus,

$$\frac{89+90+56+78+100+69}{6} = \frac{482}{6} \approx 80.33.$$

Therefore, your average test grade is approximately 80.33, which translates to a B− at most schools.

Suppose, however, that we have a function $v(t)$ that gives us the speed of an object at any time t, and we want to find the object's average speed. The function $v(t)$ takes on an infinite number of values, so we can't use the process just described. Fortunately, we can use a definite integral to find the average value of a function such as this.

Let $f(x)$ be continuous over the interval $[a,b]$ and let $[a,b]$ be divided into n subintervals of width $\Delta x = (b-a)/n$. Choose a representative x_i^* in each subinterval and calculate $f(x_i^*)$ for $i = 1, 2, \ldots, n$. In other words, consider each $f(x_i^*)$ as a sampling of the function over each subinterval. The average value of the function may then be approximated as

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n},$$

which is basically the same expression used to calculate the average of discrete values.

But we know $\Delta x = \frac{b-a}{n}$, so $n = \frac{b-a}{\Delta x}$, and we get

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\left(\frac{b-a}{\Delta x}\right)}.$$

Following through with the algebra, the numerator is a sum that is represented as $\sum_{i=1}^{n} f(x_i^*)$, and we are dividing by a fraction. To divide by a fraction, invert the denominator and multiply. Thus, an approximate value for the average value of the function is given by

$$\frac{\sum_{i=1}^{n} f(x_i^*)}{\left(\frac{b-a}{\Delta x}\right)} = \left(\frac{\Delta x}{b-a}\right)\sum_{i=1}^{n} f(x_i^*) = \left(\frac{1}{b-a}\right)\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

This is a Riemann sum. Then, to get the exact average value, take the limit as n goes to infinity. Thus, the average value of a function is given by

$$\frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx.$$

Definition: Average Value of a Function

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the *average value of the function* $f(x)$ (or f_{ave}) on $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{5.2.2}$$

Example 5.2.8: Finding the Average Value of a Linear Function

Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.

Solution

First, graph the function on the stated interval, as shown in Figure 5.2.10.

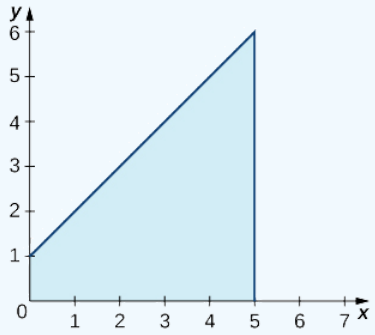


Figure 5.2.10:The graph shows the area under the function $(x) = x + 1$ over $[0, 5]$.

The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a + b)$, where h represents height, and a and b represent the two parallel sides. Then,

$$\int_0^5 x + 1\,dx = \frac{1}{2}h(a + b) = \frac{1}{2}\cdot 5\cdot(1 + 6) = \frac{35}{2}.$$

Thus the average value of the function is

$$\frac{1}{5-0}\int_0^5 x + 1\,dx = \frac{1}{5}\cdot\frac{35}{2} = \frac{7}{2}.$$

? Exercise 5.2.7

Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$.

Hint

Use the average value formula (Equation 5.2.2), and use geometry to evaluate the integral.

Answer

3

Key Concepts

- The definite integral can be used to calculate net signed area, which is the area above the x-axis less the area below the x-axis. Net signed area can be positive, negative, or zero.
- The component parts of the definite integral are the integrand, the variable of integration, and the limits of integration.
- Continuous functions on a closed interval are integrable. Functions that are not continuous may still be integrable, depending on the nature of the discontinuities.
- The properties of definite integrals can be used to evaluate integrals.
- The area under the curve of many functions can be calculated using geometric formulas.
- The average value of a function can be calculated using definite integrals.

Key Equations

- **Definite Integral**

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- **Properties of the Definite Integral**

$$\int_a^a f(x)\,dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ for constant } c$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Glossary

average value of a function

(or f_{ave}) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

definite integral

a primary operation of calculus; the area between the curve and the x-axis over a given interval is a definite integral

integrable function

a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as n goes to infinity exists

integrand

the function to the right of the integration symbol; the integrand includes the function being integrated

limits of integration

these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

net signed area

the area between a function and the x-axis such that the area below the x-axis is subtracted from the area above the x-axis; the result is the same as the definite integral of the function

total area

total area between a function and the x-axis is calculated by adding the area above the x-axis and the area below the x-axis; the result is the same as the definite integral of the absolute value of the function

variable of integration

indicates which variable you are integrating with respect to; if it is x, then the function in the integrand is followed by dx

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5.3: The Fundamental Theorem of Calculus

Learning Objectives

- Describe the meaning of the Mean Value Theorem for Integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 1.
- Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 2.
- Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- Explain the relationship between differentiation and integration.

In the previous two sections, we looked at the definite integral and its relationship to the area under the curve of a function. Unfortunately, so far, the only tools we have available to calculate the value of a definite integral are geometric area formulas and limits of Riemann sums, and both approaches are extremely cumbersome. In this section we look at some more powerful and useful techniques for evaluating definite integrals.

These new techniques rely on the relationship between differentiation and integration. This relationship was discovered and explored by both Sir Isaac Newton and Gottfried Wilhelm Leibniz (among others) during the late 1600s and early 1700s, and it is codified in what we now call the **Fundamental Theorem of Calculus**, which has two parts that we examine in this section. Its very name indicates how central this theorem is to the entire development of calculus.

Isaac Newton's contributions to mathematics and physics changed the way we look at the world. The relationships he discovered, codified as Newton's laws and the law of universal gravitation, are still taught as foundational material in physics today, and his calculus has spawned entire fields of mathematics.

Before we get to this crucial theorem, however, let's examine another important theorem, the Mean Value Theorem for Integrals, which is needed to prove the Fundamental Theorem of Calculus.

The Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals states that a continuous function on a closed interval takes on its average value at some point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point c exists in an interval $[a, b]$ such that the value of the function at c is equal to the average value of $f(x)$ over $[a, b]$. We state this theorem mathematically with the help of the formula for the average value of a function that we presented at the end of the preceding section.

Theorem 5.3.1: The Mean Value Theorem for Integrals

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

This formula can also be stated as

$$\int_a^b f(x)\, dx = f(c)(b-a). \tag{5.3.1}$$

Since $f(x)$ is continuous on $[a, b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—m and M, respectively—on $[a, b]$. Then, for all x in $[a, b]$, we have $m \le f(x) \le M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \le \int_a^b f(x)\, dx \le M(b-a).$$

Proof

Since $f(x)$ is continuous on $[a,b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—m and M, respectively—on $[a,b]$. Then, for all x in $[a,b]$, we have $m \le f(x) \le M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Dividing by $b-a$ gives us

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

Since $\frac{1}{b-a}\int_a^b f(x)\,dx$ is a number between m and M, and since $f(x)$ is continuous and assumes the values m and M over $[a,b]$, by the Intermediate Value Theorem, there is a number c over $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

and the proof is complete.

□

Example 5.3.1: Finding the Average Value of a Function

Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0,4]$ and find c such that $f(c)$ equals the average value of the function over $[0,4]$.

Solution

The formula states the mean value of $f(x)$ is given by

$$\frac{1}{4-0}\int_0^4 (8-2x)\,dx.$$

We can see in Figure 5.3.1 that the function represents a straight line and forms a right triangle bounded by the x- and y-axes. The area of the triangle is $A = \frac{1}{2}(base)(height)$. We have

$$A = \frac{1}{2}(4)(8) = 16.$$

The average value is found by multiplying the area by $1/(4-0)$. Thus, the average value of the function is

$$\frac{1}{4}(16) = 4$$

Set the average value equal to $f(c)$ and solve for c.

$$\begin{aligned} 8 - 2c &= 4 \\ c &= 2 \end{aligned}$$

At $c = 2$, $f(2) = 4$.

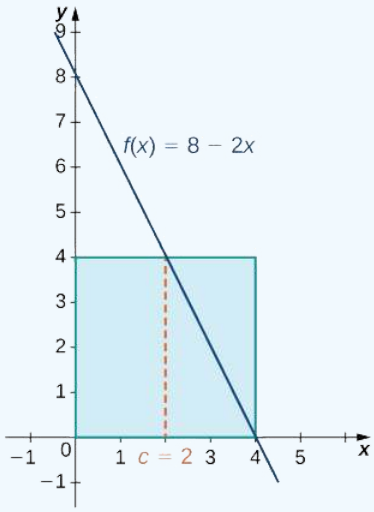


Figure 5.3.1: By the Mean Value Theorem, the continuous function $f(x)$ takes on its average value at c at least once over a closed interval.

? Exercise 5.3.1

Find the average value of the function $f(x) = \frac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

Hint

Use the procedures from Example 5.3.1 to solve the problem

Answer

The average value is 1.5 and $c = 3$.

✓ Example 5.3.2: Finding the Point Where a Function Takes on Its Average Value

Given $\int_0^3 x^2\,dx = 9$, find c such that $f(c)$ equals the average value of $f(x) = x^2$ over $[0, 3]$.

Solution

We are looking for the value of c such that

$$f(c) = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}(9) = 3.$$

Replacing $f(c)$ with c^2, we have

$$c^2 = 3$$
$$c = \pm\sqrt{3}.$$

Since $-\sqrt{3}$ is outside the interval, take only the positive value. Thus, $c = \sqrt{3}$ (Figure 5.3.2).

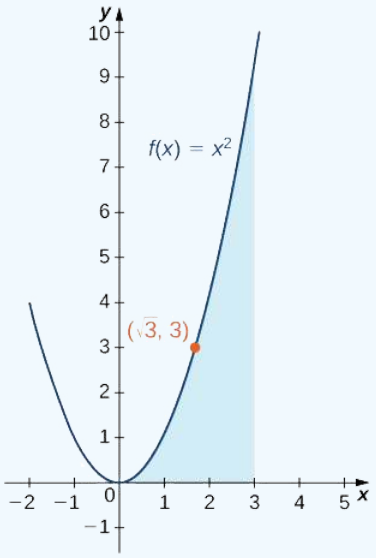


Figure 5.3.2: Over the interval $[0, 3]$, the function $f(x) = x^2$ takes on its average value at $c = \sqrt{3}$.

? Exercise 5.3.2

Given $\int_0^3 (2x^2 - 1)\,dx = 15$, find c such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[0, 3]$.

Hint

Use the procedures from Example 5.3.2 to solve the problem.

Answer

$c = \sqrt{3}$

Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums or calculating areas. The theorem is comprised of two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1,** is stated here. Part 1 establishes the relationship between differentiation and integration.

Theorem 5.3.2: The Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then $F'(x) = f(x)$ over $[a, b]$.

Before we delve into the proof, a couple of subtleties are worth mentioning here. First, a comment on the notation. Note that we have defined a function, $F(x)$, as the definite integral of another function, $f(t)$, from the point a to the point x. At first glance, this is confusing, because we have said several times that a definite integral is a number, and here it looks like it's a function. The key here is to notice that for any particular value of x, the definite integral is a number. So the function $F(x)$ returns a number (the value of the definite integral) for each value of x.

Second, it is worth commenting on some of the key implications of this theorem. There is a reason it is called the *Fundamental Theorem of Calculus*. Not only does it establish a relationship between integration and differentiation, but also it guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative.

Proof: Fundamental Theorem of Calculus, Part 1

Applying the definition of the derivative, we have

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)dt - \int_a^x f(t)\,dt \right] \\ &= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt \right] \\ &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt. \end{aligned}$$

Looking carefully at this last expression, we see $\frac{1}{h} \int_x^{x+h} f(t)\,dt$ is just the average value of the function $f(x)$ over the interval $[x, x+h]$. Therefore, by Equation 5.3.1, there is some number c in $[x, x+h]$ such that

$$\frac{1}{h} \int_x^{x+h} f(t)\,dt = f(c).$$

In addition, since c is between x and h, c approaches x as h approaches zero. Also, since $f(x)$ is continuous, we have

$$\lim_{h \to 0} f(c) = \lim_{c \to x} f(c) = f(x)$$

Putting all these pieces together, we have

$$F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h \to 0} f(c) = f(x),$$

and the proof is complete.

□

Example 5.3.3: Finding a Derivative with the Fundamental Theorem of Calculus

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3 + 1}\,dt.$$

Solution

According to the Fundamental Theorem of Calculus, the derivative is given by

$$g'(x) = \frac{1}{x^3 + 1}.$$

Exercise 5.3.3

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of $g(r) = \int_0^r \sqrt{x^2 + 4}\,dx$.

Hint

Follow the procedures from Example 5.3.3 to solve the problem.

Answer

$g'(r) = \sqrt{r^2 + 4}$

Example 5.3.4: Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives

Let $F(x) = \int_1^{\sqrt{x}} \sin t \, dt$. Find $F'(x)$.

Solution

Letting $u(x) = \sqrt{x}$, we have $F(x) = \int_1^{u(x)} \sin t \, dt$.

Thus, by the Fundamental Theorem of Calculus and the chain rule,

$$F'(x) = \sin(u(x))\frac{du}{dx} = \sin(u(x)) \cdot \left(\frac{1}{2}x^{-1/2}\right) = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

Exercise 5.3.4

Let $F(x) = \int_1^{x^3} \cos t \, dt$. Find $F'(x)$.

Hint

Use the chain rule to solve the problem.

Answer

$F'(x) = 3x^2 \cos x^3$

Example 5.3.5: Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration

Let $F(x) = \int_x^{2x} t^3 \, dt$. Find $F'(x)$.

Solution

We have $F(x) = \int_x^{2x} t^3 \, dt$. Both limits of integration are variable, so we need to split this into two integrals. We get

$$\begin{aligned} F(x) &= \int_x^{2x} t^3 \, dt = \int_x^0 t^3 \, dt + \int_0^{2x} t^3 \, dt \\ &= -\int_0^x t^3 \, dt + \int_0^{2x} t^3 \, dt. \end{aligned}$$

Differentiating the first term, we obtain

$$\frac{d}{dx}\left[-\int_0^x t^3 \, dt\right] = -x^3.$$

Differentiating the second term, we first let $u(x) = 2x$. Then,

$$\begin{aligned} \frac{d}{dx}\left[\int_0^{2x} t^3 \, dt\right] &= \frac{d}{dx}\left[\int_0^{u(x)} t^3 \, dt\right] \\ &= (u(x))^3 \frac{du}{dx} \\ &= (2x)^3 \cdot 2 = 16x^3. \end{aligned}$$

Thus,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left[-\int_0^x t^3 \, dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3 \, dt\right] \\ &= -x^3 + 16x^3 = 15x^3 \end{aligned}$$

? Exercise 5.3.5

Let $F(x) = \int_x^{x^2} \cos t \, dt$. Find $F'(x)$.

Hint

Use the procedures from Example **5.3.5** to solve the problem

Answer

$F'(x) = 2x \cos x^2 - \cos x$

Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

After finding approximate areas by adding the areas of n rectangles, the application of this theorem is straightforward by comparison. It almost seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

Theorem 5.3.3: The Fundamental Theorem of Calculus, Part 2

If $f(x)$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{5.3.2}$$

We often see the notation $F(x)|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the upper limit (in this case, b), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, a).

The **Fundamental Theorem of Calculus, Part 2** (also known as the **evaluation theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Proof

Let $P = x_i, i = 0, 1, \ldots, n$ be a regular partition of $[a, b]$. Then, we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)] \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

Now, we know F is an antiderivative of f over $[a, b]$, so by the Mean Value Theorem for derivatives (see The Mean Value Theorem) for $i = 0, 1, \ldots, n$ we can find c_i in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\,\Delta x.$$

Then, substituting into the previous equation, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x.$$

Taking the limit of both sides as $n \to \infty$, we obtain

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx.$$

□

Example 5.3.6: Evaluating an Integral with the Fundamental Theorem of Calculus

Use Equation 5.3.2 to evaluate

$$\int_{-2}^{2} (t^2 - 4)\,dt.$$

Solution

Recall the power rule for Antiderivatives:

If $y = x^n$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Use this rule to find the antiderivative of the function and then apply the theorem. We have

$$\begin{aligned}
\int_{-2}^{2} (t^2 - 4)dt &= \left(\frac{t^3}{3} - 4t\right)|_{-2}^{2} \\
&= \left[\frac{(2)^3}{3} - 4(2)\right] - \left[\frac{(-2)^3}{3} - 4(-2)\right] \\
&= \left[\frac{8}{3} - 8\right] - \left[-\frac{8}{3} + 8\right] \\
&= \frac{8}{3} - 8 + \frac{8}{3} - 8 \\
&= \frac{16}{3} - 16 = -\frac{32}{3}.
\end{aligned}$$

Analysis

Notice that we did not include the "$+C$" term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2 (Equation 5.3.2), any antiderivative works. So, for convenience, we chose the antiderivative with $C = 0$. If we had chosen another antiderivative, the constant term would have canceled out. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in Figure 5.3.3. Note that the region between the curve and the x-axis is all below the x-axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.

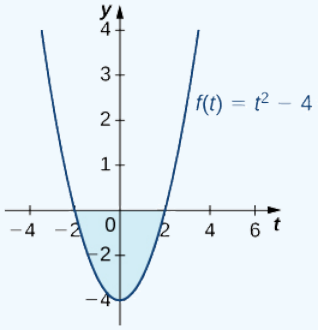


Figure 5.3.3: The evaluation of a definite integral can produce a negative value, even though area is always positive.

Example 5.3.7: Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2 (Equation 5.3.2):

$$\int_1^9 \frac{x-1}{\sqrt{x}}dx.$$

Solution

First, eliminate the radical by rewriting the integral using rational exponents. Then, separate the numerator terms by writing each one over the denominator:

$$\int_1^9 \frac{x-1}{x^{1/2}}\,dx = \int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right)\,dx.$$

Use the properties of exponents to simplify:

$$\int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right)\,dx = \int_1^9 (x^{1/2} - x^{-1/2})\,dx.$$

Now, integrate using the power rule:

$$\begin{aligned}\int_1^9 (x^{1/2} - x^{-1/2})\,dx &= \left(\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}}\right)\Big|_1^9 \\ &= \left[\frac{(9)^{3/2}}{\frac{3}{2}} - \frac{(9)^{1/2}}{\frac{1}{2}}\right] - \left[\frac{(1)^{3/2}}{\frac{3}{2}} - \frac{(1)^{1/2}}{\frac{1}{2}}\right] \\ &= \left[\frac{2}{3}(27) - 2(3)\right] - \left[\frac{2}{3}(1) - 2(1)\right] \\ &= 18 - 6 - \frac{2}{3} + 2 = \frac{40}{3}.\end{aligned}$$

See Figure 5.3.4.

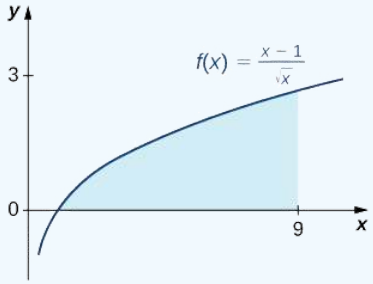


Figure 5.3.4: The area under the curve from $x = 1$ to $x = 9$ can be calculated by evaluating a definite integral.

Exercise 5.3.6

Use Note to evaluate $\int_1^2 x^{-4}\,dx$.

Hint

Use the power rule.

Answer

$\frac{7}{24}$

Example 5.3.8: A Roller-Skating Race

James and Kathy are racing on roller skates. They race along a long, straight track, and whoever has gone the farthest after 5 sec wins a prize. If James can skate at a velocity of $f(t) = 5 + 2t$ ft/sec and Kathy can skate at a velocity of $g(t) = 10 + \cos\left(\frac{\pi}{2}t\right)$ ft/sec, who is going to win the race?

Solution

We need to integrate both functions over the interval $[0, 5]$ and see which value is bigger. For James, we want to calculate

$$\int_0^5 (5 + 2t)\, dt.$$

Using the power rule, we have

$$\begin{aligned}\int_0^5 (5 + 2t)\, dt &= \left(5t + t^2\right)\Big|_0^5 \\ &= (25 + 25) \\ &= 50.\end{aligned}$$

Thus, James has skated 50 ft after 5 sec. Turning now to Kathy, we want to calculate

$$\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)\, dt.$$

We know $\sin t$ is an antiderivative of $\cos t$, so it is reasonable to expect that an antiderivative of $\cos\left(\frac{\pi}{2}t\right)$ would involve $\sin\left(\frac{\pi}{2}t\right)$. However, when we differentiate $\sin\left(\pi^2 t\right)$, we get $\pi^2 \cos\left(\pi^2 t\right)$ as a result of the chain rule, so we have to account for this additional coefficient when we integrate. We obtain

$$\begin{aligned}\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)\, dt &= \left(10t + \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\right)\Big|_0^5 \\ &= \left(50 + \frac{2}{\pi}\right) - \left(0 - \frac{2}{\pi}\sin 0\right) \approx 50.6.\end{aligned}$$

Kathy has skated approximately 50.6 ft after 5 sec. Kathy wins, but not by much!

Exercise 5.3.7

Suppose James and Kathy have a rematch, but this time the official stops the contest after only 3 sec. Does this change the outcome?

Hint

Change the limits of integration from those in Example 5.3.7.

Answer

Kathy still wins, but by a much larger margin: James skates 24 ft in 3 sec, but Kathy skates 29.3634 ft in 3 sec.

A Parachutist in Free Fall

Julie is an avid skydiver with more than 300 jumps under her belt and has mastered the art of making adjustments to her body position in the air to control how fast she falls. If she arches her back and points her belly toward the ground, she reaches a terminal velocity of approximately 120 mph (176 ft/sec). If, instead, she orients her body with her head straight down, she falls faster, reaching a terminal velocity of 150 mph (220 ft/sec).



Figure 5.3.5: Skydivers can adjust the velocity of their dive by changing the position of their body during the free fall. (credit: Jeremy T. Lock)

Since Julie will be moving (falling) in a downward direction, we assume the downward direction is positive to simplify our calculations. Julie executes her jumps from an altitude of 12,500 ft. After she exits the aircraft, she immediately starts falling at a velocity given by $v(t) = 32t$.

She continues to accelerate according to this velocity function until she reaches terminal velocity. After she reaches terminal velocity, her speed remains constant until she pulls her ripcord and slows down to land.

On her first jump of the day, Julie orients herself in the slower "belly down" position (terminal velocity is 176 ft/sec). Using this information, answer the following questions.

1. How long after she exits the aircraft does Julie reach terminal velocity?
2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.
3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?
4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from the time she leaves the airplane until the time her feet touch the ground. On Julie's second jump of the day, she decides she wants to fall a little faster and orients herself in the "head down" position. Her terminal velocity in this position is 220 ft/sec. Answer these questions based on this velocity:
5. How long does it take Julie to reach terminal velocity in this case?
6. Before pulling her ripcord, Julie reorients her body in the "belly down" position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Some jumpers wear "**wingsuits**" (Figure 5.3.6). These suits have fabric panels between the arms and legs and allow the wearer to glide around in a free fall, much like a flying squirrel. (Indeed, the suits are sometimes called "flying squirrel suits.") When wearing these suits, terminal velocity can be reduced to about 30 mph (44 ft/sec), allowing the wearers a much longer time in the air. Wingsuit flyers still use parachutes to land; although the vertical velocities are within the margin of safety, horizontal velocities can exceed 70 mph, much too fast to land safely.



Figure **5.3.6**: The fabric panels on the arms and legs of a wingsuit work to reduce the vertical velocity of a skydiver's fall. (credit: Richard Schneider)

Answer the following question based on the velocity in a wingsuit.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air

Key Concepts

- The Mean Value Theorem for Integrals states that for a continuous function over a closed interval, there is a value c such that $f(c)$ equals the average value of the function.
- The Fundamental Theorem of Calculus, Part 1 shows the relationship between the derivative and the integral.
- The Fundamental Theorem of Calculus, Part 2 is a formula for evaluating a definite integral in terms of an antiderivative of its integrand. The total area under a curve can be found using this formula.

Key Equations

- **Mean Value Theorem for Integrals**

  If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

  $$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

- **Fundamental Theorem of Calculus, Part 1**

  If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

  $$F(x) = \int_a^x f(t)\, dt,$$

  then

  $$F'(x) = f(x).$$

- **Fundamental Theorem of Calculus, Part 2**

  If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

  $$\int_a^b f(x)\, dx = F(b) - F(a).$$

Glossary

fundamental theorem of calculus

the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

fundamental theorem of calculus, part 1

uses a definite integral to define an antiderivative of a function

fundamental theorem of calculus, part 2

(also, **evaluation theorem**) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

mean value theorem for integrals

guarantees that a point c exists such that $f(c)$ is equal to the average value of the function

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5.4: Integration Formulas and the Net Change Theorem

Learning Objectives

- Apply the basic integration formulas.
- Explain the significance of the net change theorem.
- Use the net change theorem to solve applied problems.
- Apply the integrals of odd and even functions.

In this section, we use some basic integration formulas studied previously to solve some key applied problems. It is important to note that these formulas are presented in terms of indefinite integrals. Although definite and indefinite integrals are closely related, there are some key differences to keep in mind. A definite integral is either a number (when the limits of integration are constants) or a single function (when one or both of the limits of integration are variables). An indefinite integral represents a family of functions, all of which differ by a constant. As you become more familiar with integration, you will get a feel for when to use definite integrals and when to use indefinite integrals. You will naturally select the correct approach for a given problem without thinking too much about it. However, until these concepts are cemented in your mind, think carefully about whether you need a definite integral or an indefinite integral and make sure you are using the proper notation based on your choice.

Basic Integration Formulas

Recall the integration formulas given in the section on Antiderivatives and the properties of definite integrals. Let's look at a few examples of how to apply these formulas and properties.

Example 5.4.1: Integrating a Function Using the Power Rule

Use the power rule to integrate the function $\int_1^4 \sqrt{t}(1+t)\,dt$.

Solution

The first step is to rewrite the function and simplify it so we can apply the power rule:

$$\begin{aligned}\int_1^4 \sqrt{t}(1+t)\,dt &= \int_1^4 t^{1/2}(1+t)\,dt \\ &= \int_1^4 (t^{1/2}+t^{3/2})\,dt.\end{aligned}$$

Now apply the power rule:

$$\begin{aligned}\int_1^4 (t^{1/2}+t^{3/2})\,dt &= \left(\frac{2}{3}t^{3/2}+\frac{2}{5}t^{5/2}\right)\Big|_1^4 \\ &= \left[\frac{2}{3}(4)^{3/2}+\frac{2}{5}(4)^{5/2}\right]-\left[\frac{2}{3}(1)^{3/2}+\frac{2}{5}(1)^{5/2}\right] \\ &= \frac{256}{15}.\end{aligned}$$

Exercise 5.4.1

Find the definite integral of $f(x)=x^2-3x$ over the interval $[1,3]$.

Hint

Follow the process from Example 5.4.1 to solve the problem.

Answer

$$\int_1^3 (x^2 - 3x)\, dx = -\frac{10}{3}$$

The Net Change Theorem

The **net change theorem** considers the integral of a **rate of change**. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)dx \tag{5.4.1}$$

or

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{5.4.2}$$

Subtracting $F(a)$ from both sides of the Equation 5.4.1 yields Equation 5.4.2. Since they are equivalent formulas, which one we use depends on the application.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let's apply the net change theorem to a **velocity** function in which the result is **displacement.**

We looked at a simple example of this in The Definite Integral section. Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in Figure 5.4.1.

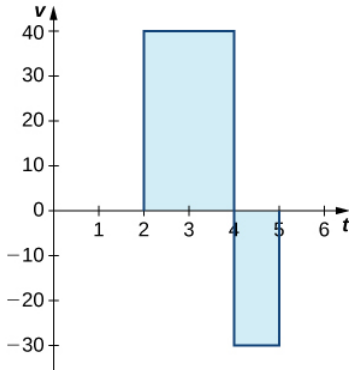


Figure 5.4.1: The graph shows speed versus time for the given motion of a car.

Just as we did before, we can use definite integrals to calculate the net displacement as well as the total distance traveled. The **net displacement** is given by

$$\int_2^5 v(t)\, dt = \int_2^4 40\, dt + \int_4^5 -30\, dt = 80 - 30 = 50.$$

Thus, at 5 p.m. the car is 50 mi north of its starting position. The **total distance** traveled is given by

$$\int_2^5 |v(t)|\, dt = \int_2^4 40\, dt + \int_4^5 30\, dt = 80 + 30 = 110.$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include both positive and negative values. In other words, the velocity function accounts for both forward distance and backward distance. To find net displacement, integrate the velocity function over the interval. Total distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.

Example 5.4.2: Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

Solution

Applying the net change theorem, we have

$$\int_0^3 (3t - 5)\, dt = \left(\frac{3t^2}{2} - 5t\right)\Big|_0^3 = \left[\frac{3(3)^2}{2} - 5(3)\right] - 0 = \frac{27}{2} - 15 = \frac{27}{2} - \frac{30}{2} = -\frac{3}{2}.$$

The net displacement is $-\frac{3}{2}$ m (Figure 5.4.2).

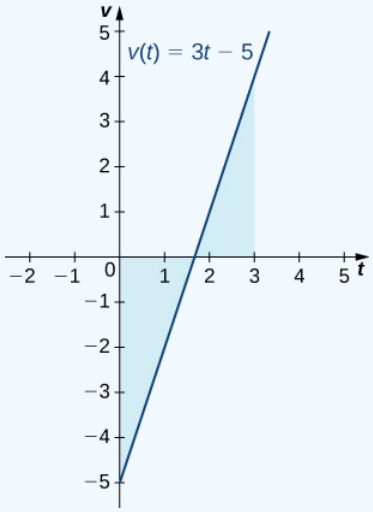


Figure 5.4.2: The graph shows velocity versus time for a particle moving with a linear velocity function.

Example 5.4.3: Finding the Total Distance Traveled

Use Example 5.4.2 to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

Solution

The total distance traveled includes both the positive and the negative values. Therefore, we must integrate the absolute value of the velocity function to find the total distance traveled.

To continue with the example, use two integrals to find the total distance. First, find the t-intercept of the function, since that is where the division of the interval occurs. Set the equation equal to zero and solve for t. Thus,

$$\begin{aligned} 3t - 5 &= 0 \\ 3t &= 5 \\ t &= \frac{5}{3}. \end{aligned}$$

The two subintervals are $[0, \frac{5}{3}]$ and $[\frac{5}{3}, 3]$. To find the total distance traveled, integrate the absolute value of the function. Since the function is negative over the interval $[0, \frac{5}{3}]$, we have $|v(t)| = -v(t)$ over that interval. Over $[\frac{5}{3}, 3]$, the function is positive, so $|v(t)| = v(t)$. Thus, we have

$$\begin{aligned}\int_0^3 |v(t)|\,dt &= \int_0^{5/3} -v(t)\,dt + \int_{5/3}^3 v(t)\,dt \\ &= \int_0^{5/3} 5-3t\,dt + \int_{5/3}^3 3t-5\,dt \\ &= \left(5t-\frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2}-5t\right)\Big|_{5/3}^3 \\ &= \left[5(\frac{5}{3})-\frac{3(5/3)^2}{2}\right]-0+\left[\frac{27}{2}-15\right]-\left[\frac{3(5/3)^2}{2}-\frac{25}{3}\right] \\ &= \frac{25}{3}-\frac{25}{6}+\frac{27}{2}-15-\frac{25}{6}+\frac{25}{3} \\ &= \frac{41}{6}\end{aligned}$$

So, the total distance traveled is $\frac{41}{6}$ m.

? Exercise 5.4.2

Find the net displacement and total distance traveled in meters given the velocity function $f(t)=\frac{1}{2}e^t-2$ over the interval $[0,2]$.

Hint

Follow the procedures from Examples 5.4.2 and 5.4.3. Note that $f(t)\le 0$ for $t\le \ln 4$ and $f(t)\ge 0$ for $t\ge \ln 4$.

Answer

Net displacement: $\frac{e^2-9}{2}\approx -0.8055$ m; total distance traveled: $4\ln 4-7.5+\frac{e^2}{2}\approx 1.740$ m.

Applying the Net Change Theorem

The net change theorem can be applied to the flow and consumption of fluids, as shown in Example 5.4.4.

✓ Example 5.4.4: How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t=0$ and the boat consumes gasoline at the rate of $5-t^3$ gal/hr, how much gasoline is used in the first 2 hours?

Solution

Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval [0,2]. We have

$$\begin{aligned}\int_0^2 (5-t^3)\,dt &= \left(5t-\frac{t^4}{4}\right)|_0^2 \\ &= \left[5(2)-\frac{(2)^4}{4}\right]-0 \\ &= 10-\frac{16}{4} \\ &= 6.\end{aligned}$$

Thus, the motorboat uses 6 gal of gas in 2 hours.

✓ Example 5.4.5: Chapter Opener: Iceboats

As we saw at the beginning of the chapter, top **iceboat** racers can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed.



Figure **5.4.3**: (credit: modification of work by Carter Brown, Flickr)

Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew gets his iceboat set up, though, the wind begins to pick up. During his first half hour of iceboating, the wind speed increases according to the function $v(t) = 20t + 5$. For the second half hour of Andrew's outing, the wind remains steady at 15 mph. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

Solution

To figure out how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\text{Distance} = \int_0^1 2v(t)\,dt.$$

Substituting the expressions we were given for $v(t)$, we get

$$\begin{aligned} \int_0^1 2v(t)\,dt &= \int_0^{1/2} 2v(t)\,dt + \int_{1/2}^1 2v(t)\,dt \\ &= \int_0^{1/2} 2(20t+5)\,dt + \int_{1/3}^1 2(15)\,dt \\ &= \int_0^{1/2} (40t+10)\,dt + \int_{1/2}^1 30\,dt \\ &= \left[20t^2 + 10t\right]\Big|_0^{1/2} + \left[30t\right]\Big|_{1/2}^1 \\ &= \left(\frac{20}{4} + 5\right) - 0 + (30 - 15) \\ &= 25. \end{aligned}$$

Andrew is 25 mi from his starting point after 1 hour.

? Exercise 5.4.3

Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15$. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ -10t + 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}.$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

Hint

Don't forget that Andrew's iceboat moves twice as fast as the wind.

Answer

17.5 mi

Integrating Even and Odd Functions

We saw in Functions and Graphs that an **even function** is a function in which $f(-x) = f(x)$ for all x in the domain—that is, the graph of the curve is unchanged when x is replaced with $-x$. The graphs of even functions are symmetric about the y-axis. An **odd function** is one in which $f(-x) = -f(x)$ for all x in the domain, and the graph of the function is symmetric about the origin.

Integrals of even functions, when the limits of integration are from $-a$ to a, involve two equal areas, because they are symmetric about the y-axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a]$, evaluate to zero because the areas above and below the x-axis are equal.

Integrals of Even and Odd Functions

For continuous even functions such that $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

For continuous odd functions such that $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Example 5.4.6: Integrating an Even Function

Integrate the even function $\int_{-2}^{2}(3x^8 - 2)\,dx$ and verify that the integration formula for even functions holds.

Solution

The symmetry appears in the graphs in Figure 5.4.4. Graph (a) shows the region below the curve and above the x-axis. We have to zoom in to this graph by a huge amount to see the region. Graph (b) shows the region above the curve and below the x-axis. The signed area of this region is negative. Both views illustrate the symmetry about the y-axis of an even function. We have

$$\begin{aligned}\int_{-2}^{2}(3x^8 - 2)\,dx &= \left(\frac{x^9}{3} - 2x\right)\Big|_{-2}^{2} \\ &= \left[\frac{(2)^9}{3} - 2(2)\right] - \left[\frac{(-2)^9}{3} - 2(-2)\right] \\ &= \left(\frac{512}{3} - 4\right) - \left(-\frac{512}{3} + 4\right) \\ &= \frac{1000}{3}.\end{aligned}$$

To verify the integration formula for even functions, we can calculate the integral from 0 to 2 and double it, then check to make sure we get the same answer.

$$\int_{0}^{2}(3x^8 - 2)\,dx = \left(\frac{x^9}{3} - 2x\right)\Big|_{0}^{2} = \frac{512}{3} - 4 = \frac{500}{3}$$

Since $2 \cdot \frac{500}{3} = \frac{1000}{3}$, we have verified the formula for even functions in this particular example.

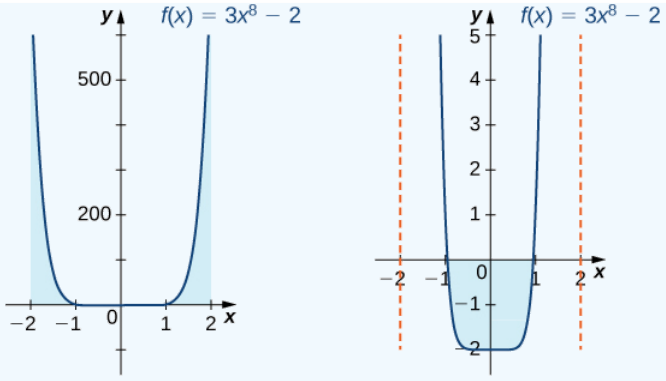


Figure 5.4.4: Graph (a) shows the positive area between the curve and the x-axis, whereas graph (b) shows the negative area between the curve and the x-axis. Both views show the symmetry about the y-axis.

Example 5.4.7: Integrating an Odd Function

Evaluate the definite integral of the odd function $-5\sin x$ over the interval $[-\pi, \pi]$.

Solution

The graph is shown in Figure 5.4.5. We can see the symmetry about the origin by the positive area above the x-axis over $[-\pi, 0]$, and the negative area below the x-axis over $[0, \pi]$. we have

$$\begin{aligned}\int_{-\pi}^{\pi} -5\sin x\, dx &= -5(-\cos x)\Big|_{-\pi}^{\pi} \\ &= 5\cos x\Big|_{-\pi}^{\pi} \\ &= [5\cos \pi] - [5\cos(-\pi)] \\ &= -5 - (-5) = 0.\end{aligned}$$

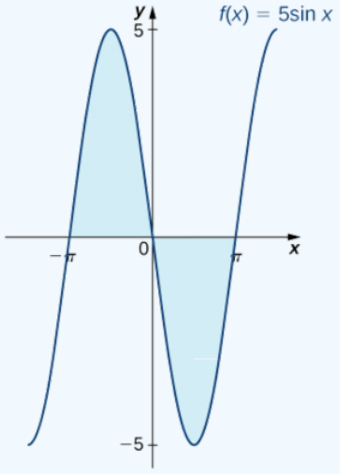


Figure 5.4.5:The graph shows areas between a curve and the x-axis for an odd function.

Exercise 5.4.4

Integrate the function $\int_{-2}^{2} x^4\, dx$.

Hint

Integrate an even function.

Answer

$\frac{64}{5}$

Key Concepts

- The net change theorem states that when a quantity changes, the final value equals the initial value plus the integral of the rate of change. Net change can be a positive number, a negative number, or zero.
- The area under an even function over a symmetric interval can be calculated by doubling the area over the positive x-axis. For an odd function, the integral over a symmetric interval equals zero, because half the area is negative.

Key Equations

- Net Change Theorem

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

or

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Glossary

net change theorem

if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

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5.4E: Exercises

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5.5: Substitution

Learning Objectives

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

The Fundamental Theorem of Calculus gave us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called *integration by substitution*, to help us find antiderivatives. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f\big[g(x)\big]g'(x)\,dx$. For example, in the integral

$$\int (x^2-3)^3\,2x\,dx. \tag{5.5.1}$$

we have

$$f(x)=x^3$$

and

$$g(x)=x^2-3.$$

Then

$$g'(x)=2x.$$

and

$$f[g(x)]g'(x)=(x^2-3)^3(2x),$$

and we see that our integrand is in the correct form. The method is called substitution because we substitute part of the integrand with the variable u and part of the integrand with du. It is also referred to as *change of variables* because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

Substitution with Indefinite Integrals

Let $u=g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f[g(x)]g'(x)\,dx &= \int f(u)\,du\\ &= F(u)+C\\ &= F(g(x))+C\end{aligned}$$

Proof

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{d}{dx}\big[F(g(x))\big]=F'(g(x))g'(x)=f[g(x)]g'(x).$$

Integrating both sides with respect to x, we see that

$$\int f[g(x)]g'(x)\,dx=F(g(x))+C.$$

If we now substitute $u = g(x)$, and $du = g'(x)\,dx$, we get

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C.$$

□

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x\,dx$.

Rewrite the integral (Equation 5.5.1) in terms of u:

$$\int (x^2 - 3)^3 (2x\,dx) = \int u^3\,du.$$

Using the power rule for integrals, we have

$$\int u^3\,du = \frac{u^4}{4} + C.$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C.$$

We can generalize the procedure in the following Problem-Solving Strategy.

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$. such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$. into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Write the result in terms of x and the expression $g(x)$.

Example 5.5.1: Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x(3x^2 + 4)^4\,dx$.

Solution

The first step is to choose an expression for u. We choose $u = 3x^2 + 4$ because then $du = 6x\,dx$ and we already have du in the integrand. Write the integral in terms of u:

$$\int 6x(3x^2 + 4)^4\,dx = \int u^4\,du.$$

Remember that du is the derivative of the expression chosen for u, regardless of what is inside the integrand. Now we can evaluate the integral with respect to u:

$$\int u^4\,du = \frac{u^5}{5} + C = \frac{(3x^2 + 4)^5}{5} + C.$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for C of 1, we let $y = \frac{1}{5}(3x^2 + 4)^5 + 1$. We have

$$y = \frac{1}{5}(3x^2 + 4)^5 + 1,$$

so

$$y' = \left(\frac{1}{5}\right) 5(3x^2+4)^4 6x$$
$$= 6x(3x^2+4)^4.$$

This is exactly the expression we started with inside the integrand.

? Exercise 5.5.1

Use substitution to find the antiderivative of $\int 3x^2(x^3-3)^2\,dx$.

Hint

Let $u = x^3 - 3$.

Answer

$$\int 3x^2(x^3-3)^2\,dx = \frac{1}{3}(x^3-3)^3 + C$$

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

✓ Example 5.5.2: Using Substitution with Alteration

Use substitution to find the antiderivative of

$$\int z\sqrt{z^2-5}\,dz.$$

Solution

Rewrite the integral as $\int z(z^2-5)^{1/2}\,dz$. Let $u = z^2 - 5$ and $du = 2z\,dz$. Now we have a problem because $du = 2z\,dz$ and the original expression has only $z\,dz$. We have to alter our expression for du or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}$. we can solve this problem. Thus,

$$u = z^2 - 5$$
$$du = 2z\,dz$$
$$\frac{1}{2}du = \frac{1}{2}(2z)\,dz = z\,dz.$$

Write the integral in terms of u, but pull the $\frac{1}{2}$ outside the integration symbol:

$$\int z(z^2-5)^{1/2}\,dz = \frac{1}{2}\int u^{1/2}\,du.$$

Integrate the expression in u:

$$\frac{1}{2}\int u^{1/2}\,du = \left(\frac{1}{2}\right)\frac{u^{3/2}}{\frac{3}{2}} + C$$
$$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2} + C$$
$$= \frac{1}{3}u^{3/2} + C$$
$$= \frac{1}{3}(z^2-5)^{3/2} + C$$

? Exercise 5.5.2

Use substitution to find the antiderivative of $\int x^2(x^3+5)^9\,dx$.

Hint

Multiply the du equation by $\frac{1}{3}$.

Answer

$$\int x^2(x^3+5)^9\,dx = \frac{(x^3+5)^{10}}{30} + C$$

✓ Example 5.5.3: Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t}\,dt$.

Solution

We know the derivative of $\cos t$ is $-\sin t$, so we set $u = \cos t$. Then $du = -\sin t\,dt$.

Substituting into the integral, we have

$$\int \frac{\sin t}{\cos^3 t}\,dt = -\int \frac{du}{u^3}.$$

Evaluating the integral, we get

$$-\int \frac{du}{u^3} = -\int u^{-3}\,du = -\left(-\frac{1}{2}\right)u^{-2} + C.$$

Putting the answer back in terms of t, we get

$$\int \frac{\sin t}{\cos^3 t}\,dt = \frac{1}{2u^2} + C = \frac{1}{2\cos^2 t} + C.$$

? Exercise 5.5.3

Use substitution to evaluate the integral $\int \frac{\cos t}{\sin^2 t}\,dt$.

Hint

Use the process from Example **5.5.3** to solve the problem.

Answer

$$\int \frac{\cos t}{\sin^2 t}\,dt = -\frac{1}{\sin t} + C$$

? Exercise 5.5.4

Use substitution to evaluate the indefinite integral $\int \cos^3 t \sin t\,dt$.

Hint

Use the process from Example **5.5.3** to solve the problem.

Answer

$$\int \cos^3 t \sin t\,dt = -\frac{\cos^4 t}{4} + C$$

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When we are done, u should be the only variable in the integrand. In some cases, this means solving for the original variable in terms of u. This technique should become clear in the next example.

Example 5.5.4: Finding an Antiderivative Using u-Substitution

Use substitution to find the antiderivative of

$$\int \frac{x}{\sqrt{x-1}}\,dx.$$

Solution

If we let $u = x - 1$, then $du = dx$. But this does not account for the x in the numerator of the integrand. We need to express x in terms of u. If $u = x - 1$, then $x = u + 1$. Now we can rewrite the integral in terms of u:

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right)\,du = \int \left(u^{1/2} + u^{-1/2}\right)\,du.$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned}
\int (u^{1/2} + u^{-1/2})\,du &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\
&= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C \\
&= (x-1)^{1/2}\left[\frac{2}{3}(x-1) + 2\right] + C \\
&= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + \frac{6}{3}\right) \\
&= (x-1)^{1/2}\left(\frac{2}{3}x + \frac{4}{3}\right) \\
&= \frac{2}{3}(x-1)^{1/2}(x+2) + C.
\end{aligned}$$

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Substitution with Definite Integrals

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Then

$$\begin{aligned}\int_a^b f[g(x)]g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b}\\ &= F(g(b)) - F(g(a))\\ &= F(u)\Big|_{u=g(a)}^{u=g(b)}\\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

and we have the desired result.

Example 5.5.5: Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate

$$\int_0^1 x^2(1+2x^3)^5\,dx.$$

Solution

Let $u = 1 + 2x^3$, so $du = 6x^2\,dx$. Since the original function includes one factor of x^2 and $du = 6x^2\,dx$, multiply both sides of the du equation by $1/6$. Then,

$$\begin{aligned} du &= 6x^2\,dx\\ \text{becomes}\quad \frac{1}{6}du &= x^2\,dx.\end{aligned}$$

To adjust the limits of integration, note that when $x = 0, u = 1 + 2(0) = 1$, and when $x = 1,\ u = 1 + 2(1) = 3$.

Then

$$\int_0^1 x^2(1+2x^3)^5 dx = \frac{1}{6}\int_1^3 u^5\,du.$$

Evaluating this expression, we get

$$\begin{aligned}\frac{1}{6}\int_1^3 u^5\,du &= \left(\frac{1}{6}\right)\left(\frac{u^6}{6}\right)\Big|_1^3\\ &= \frac{1}{36}\left[(3)^6 - (1)^6\right]\\ &= \frac{182}{9}.\end{aligned}$$

Exercise 5.5.5

Use substitution to evaluate the definite integral $\int_{-1}^0 y(2y^2-3)^5\,dy$.

Hint

Use the steps from Example 5.5.5 to solve the problem.

Answer

$$\int_{-1}^0 y(2y^2-3)^5\,dy = \frac{91}{3}$$

? Exercise 5.5.6

Use substitution to evaluate $\int_0^1 x^2 \cos\left(\frac{\pi}{2}x^3\right)\,dx$.

Hint

Use the process from Example 5.5.5 to solve the problem.

Answer

$$\int_0^1 x^2 \cos\left(\frac{\pi}{2}x^3\right)\,dx = \frac{2}{3\pi} \approx 0.2122$$

✓ Example 5.5.6: Using Substitution with an Exponential Function

Use substitution to evaluate

$$\int_0^1 xe^{4x^2+3}\,dx.$$

Solution

Let $u = 4x^3 + 3$. Then, $du = 8x\,dx$. To adjust the limits of integration, we note that when $x = 0$, $u = 3$, and when $x = 1$, $u = 7$. So our substitution gives

$$\begin{aligned}\int_0^1 xe^{4x^2+3}\,dx &= \frac{1}{8}\int_3^7 e^u\,du \\ &= \frac{1}{8}e^u\Big|_3^7 \\ &= \frac{e^7 - e^3}{8} \\ &\approx 134.568\end{aligned}$$

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution. Also, we have the option of replacing the original expression for u after we find the antiderivative, which means that we do not have to change the limits of integration. These two approaches are shown in Example 5.5.7.

✓ Example 5.5.7: Using Substitution to Evaluate a Trigonometric Integral

Use substitution to evaluate

$$\int_0^{\pi/2} \cos^2\theta\,d\theta.$$

Solution

Let us first use a trigonometric identity to rewrite the integral. The trig identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ allows us to rewrite the integral as

$$\int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta.$$

Then,

$$\begin{aligned}\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)\,d\theta &= \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\,d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} d\theta + \int_0^{\pi/2} \cos 2\theta\,d\theta.\end{aligned}$$

We can evaluate the first integral as it is, but we need to make a substitution to evaluate the second integral. Let $u = 2\theta$. Then, $du = 2\,d\theta$, or $\frac{1}{2}\,du = d\theta$. Also, when $\theta = 0$, $u = 0$, and when $\theta = \pi/2$, $u = \pi$. Expressing the second integral in terms of u, we have

$$\begin{aligned}\frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\, d\theta &= \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\left(\frac{1}{2}\right)\int_0^{\pi} \cos u\, du \\ &= \frac{\theta}{2}\Big|_{\theta=0}^{\theta=\pi/2} + \frac{1}{4}\sin u\Big|_{u=0}^{u=\pi} \\ &= \left(\frac{\pi}{4} - 0\right) + (0 - 0) = \frac{\pi}{4}\end{aligned}$$

Key Concepts

- Substitution is a technique that simplifies the integration of functions that are the result of a chain-rule derivative. The term 'substitution' refers to changing variables or substituting the variable u and du for appropriate expressions in the integrand.
- When using substitution for a definite integral, we also have to change the limits of integration.

Key Equations

- **Substitution with Indefinite Integrals**

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

- **Substitution with Definite Integrals**

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Glossary

change of variables

the substitution of a variable, such as u, for an expression in the integrand

integration by substitution

a technique for integration that allows integration of functions that are the result of a chain-rule derivative

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5.5E: Exercises

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5.6: Integrals Involving Exponential and Logarithmic Functions

Learning Objectives

- Integrate functions involving exponential functions.
- Integrate functions involving logarithmic functions.

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

Integrals of Exponential Functions

The exponential function is perhaps the most efficient function in terms of the operations of calculus. The exponential function, $y = e^x$, is its own derivative and its own integral.

Rule: Integrals of Exponential Functions

Exponential functions can be integrated using the following formulas.

$$\int e^x \, dx = e^x + C \tag{5.6.1}$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \tag{5.6.2}$$

Example 5.6.1: Finding an Antiderivative of an Exponential Function

Find the antiderivative of the exponential function e^{-x}.

Solution

Use substitution, setting $u = -x$, and then $du = -1\,dx$. Multiply the du equation by -1, so you now have $-du = dx$. Then,

$$\int e^{-x} \, dx = -\int e^u \, du = -e^u + C = -e^{-x} + C.$$

Exercise 5.6.1

Find the antiderivative of the function using substitution: $x^2 e^{-2x^3}$.

Hint

Let u equal the exponent on e.

Answer

$$\int x^2 e^{-2x^3} \, dx = -\frac{1}{6} e^{-2x^3} + C$$

A common mistake when dealing with exponential expressions is treating the exponent on e the same way we treat exponents in polynomial expressions. We cannot use the power rule for the exponent on e. This can be especially confusing when we have both exponentials and polynomials in the same expression, as in the previous checkpoint. In these cases, we should always double-check to make sure we're using the right rules for the functions we're integrating.

Example 5.6.2: Square Root of an Exponential Function

Find the antiderivative of the exponential function $e^x\sqrt{1+e^x}$.

Solution

First rewrite the problem using a rational exponent:

$$\int e^x\sqrt{1+e^x}\,dx = \int e^x(1+e^x)^{1/2}\,dx.$$

Using substitution, choose $u = 1+e^x$. Then, $du = e^x\,dx$. We have

$$\int e^x(1+e^x)^{1/2}\,dx = \int u^{1/2}\,du.$$

Then

$$\int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$$

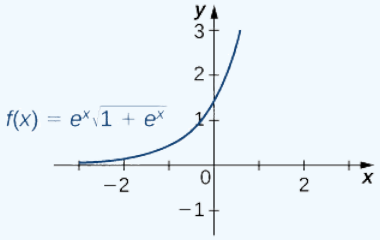


Figure 5.6.1: The graph shows an exponential function times the square root of an exponential function.

Exercise 5.6.2

Find the antiderivative of $e^x(3e^x-2)^2$.

Hint

Let $u = 3e^x - 2$.

Answer

$$\int e^x(3e^x-2)^2\,dx = \frac{1}{9}(3e^x-2)^3 + C$$

Example 5.6.3: Using Substitution with an Exponential Function

Use substitution to evaluate the indefinite integral $\int 3x^2e^{2x^3}\,dx$.

Solution

Here we choose to let u equal the expression in the exponent on e. Let $u = 2x^3$ and $du = 6x^2\,dx$. Again, du is off by a constant multiplier; the original function contains a factor of $3x^2$, not $6x^2$. Multiply both sides of the equation by $\frac{1}{2}$ so that the integrand in u equals the integrand in x. Thus,

$$\int 3x^2e^{2x^3}\,dx = \frac{1}{2}\int e^u\,du.$$

Integrate the expression in u and then substitute the original expression in x back into the u-integral:

$$\frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x^3} + C.$$

Exercise 5.6.3

Evaluate the indefinite integral $\int 2x^3 e^{x^4}\,dx$.

Hint

Let $u = x^4$.

Answer

$$\int 2x^3 e^{x^4}\,dx = \frac{1}{2}e^{x^4} + C$$

As mentioned at the beginning of this section, exponential functions are used in many real-life applications. The number e is often associated with compounded or accelerating growth, as we have seen in earlier sections about the derivative. Although the derivative represents a rate of change or a growth rate, the integral represents the total change or the total growth. Let's look at an example in which integration of an exponential function solves a common business application.

A price–demand function tells us the relationship between the quantity of a product demanded and the price of the product. In general, price decreases as quantity demanded increases. The marginal price–demand function is the derivative of the price–demand function and it tells us how fast the price changes at a given level of production. These functions are used in business to determine the price–elasticity of demand, and to help companies determine whether changing production levels would be profitable.

Example 5.6.4: Finding a Price–Demand Equation

Find the price–demand equation for a particular brand of toothpaste at a supermarket chain when the demand is 50 tubes per week at \$2.35 per tube, given that the marginal price—demand function, $p'(x)$, for x number of tubes per week, is given as

$$p'(x) = -0.015e^{-0.01x}.$$

If the supermarket chain sells 100 tubes per week, what price should it set?

Solution

To find the price–demand equation, integrate the marginal price–demand function. First find the antiderivative, then look at the particulars. Thus,

$$p(x) = \int -0.015e^{-0.01x}\,dx = -0.015\int e^{-0.01x}\,dx.$$

Using substitution, let $u = -0.01x$ and $du = -0.01\,dx$. Then, divide both sides of the du equation by -0.01. This gives

$$\frac{-0.015}{-0.01}\int e^u\,du = 1.5\int e^u\,du = 1.5e^u + C = 1.5e^{-0.01x} + C.$$

The next step is to solve for C. We know that when the price is \$2.35 per tube, the demand is 50 tubes per week. This means

$$p(50) = 1.5e^{-0.01(50)} + C = 2.35.$$

Now, just solve for C:

$$C = 2.35 - 1.5e^{-0.5} = 2.35 - 0.91 = 1.44.$$

Thus,

$$p(x) = 1.5e^{-0.01x} + 1.44.$$

If the supermarket sells 100 tubes of toothpaste per week, the price would be

$$p(100) = 1.5e^{-0.01(100)} + 1.44 = 1.5e^{-1} + 1.44 \approx 1.99.$$

The supermarket should charge \$1.99 per tube if it is selling 100 tubes per week.

Example 5.6.5: Evaluating a Definite Integral Involving an Exponential Function

Evaluate the definite integral $\int_1^2 e^{1-x}\,dx$.

Solution

Again, substitution is the method to use. Let $u = 1 - x$, so $du = -1\,dx$ or $-du = dx$. Then $\int e^{1-x}\,dx = -\int e^u\,du$.

Next, change the limits of integration. Using the equation $u = 1 - x$, we have:

$$\text{When } x = 1, \quad u = 1 - (1) = 0,$$

$$\text{and when } x = 2, \quad u = 1 - (2) = -1.$$

The integral then becomes

$$\begin{aligned}\int_1^2 e^{1-x}\,dx &= -\int_0^{-1} e^u\,du \\ &= \int_{-1}^0 e^u\,du \\ &= e^u\Big|_{-1}^0 = e^0 - (e^{-1}) \\ &= -e^{-1} + 1.\end{aligned}$$

See Figure 5.6.2.

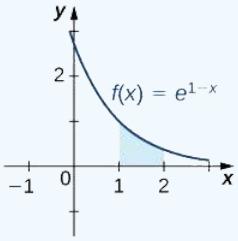


Figure 5.6.2: The indicated area can be calculated by evaluating a definite integral using substitution.

Exercise 5.6.4

Evaluate $\int_0^2 e^{2x}\,dx$.

Hint

Let $u = 2x$.

Answer

$$\frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}(e^4 - 1)$$

Example 5.6.6: Growth of Bacteria in a Culture

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where t is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with $10,000$ bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time t. How many bacteria are in the dish after 2 hours?

Solution

We have

$$Q(t) = \int 3^t dt = \frac{3^t}{\ln 3} + C.$$

Then, at $t = 0$ we have $Q(0) = 10 = \frac{1}{\ln 3} + C$, so $C \approx 9.090$ and we get

$$Q(t) = \frac{3^t}{\ln 3} + 9.090.$$

At time $t = 2$, we have

$$\begin{aligned} Q(2) &= \frac{3^2}{\ln 3} + 9.090 \\ &\approx 17.282. \end{aligned}$$

After 2 hours, there are 17,282 bacteria in the dish.

Exercise 5.6.5

From Example, suppose the bacteria grow at a rate of $q(t) = 2^t$. Assume the culture still starts with $10,000$ bacteria. Find $Q(t)$. How many bacteria are in the dish after 3 hours?

Hint

Use the procedure from Example **5.6.6** to solve the problem

Answer

$$Q(t) = \frac{2^t}{\ln 2} + 8.557.$$
$$Q(3) \approx 20,099$$

So there are $20,099$ bacteria in the dish after 3 hours.

Example 5.6.7: Fruit Fly Population Growth

Suppose a population of **fruit flies** increases at a rate of $g(t) = 2e^{0.02t}$, in flies per day. If the initial population of fruit flies is 100 flies, how many flies are in the population after 10 days?

Solution

Let $G(t)$ represent the number of flies in the population at time t. Applying the net change theorem, we have

$$\begin{aligned} G(10) = G(0) + \int_0^{10} 2e^{0.02t}\, dt \\ &= 100 + \left[\frac{2}{0.02} e^{0.02t}\right] |_0^{10} \\ &= 100 + \left[100e^{0.02t}\right] |_0^{10} \\ &= 100 + 100e^{0.2} - 100 \\ &\approx 122. \end{aligned}$$

There are 122 flies in the population after 10 days.

Exercise 5.6.6

Suppose the rate of growth of the fly population is given by $g(t) = e^{0.01t}$, and the initial fly population is 100 flies. How many flies are in the population after 15 days?

Hint

Use the process from Example **5.6.7** to solve the problem.

Answer

There are **116** flies.

Example 5.6.8: Evaluating a Definite Integral Using Substitution

Evaluate the definite integral using substitution:

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx.$$

Solution

This problem requires some rewriting to simplify applying the properties. First, rewrite the exponent on e as a power of x, then bring the x^2 in the denominator up to the numerator using a negative exponent. We have

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^2 e^{x^{-1}}x^{-2}\,dx.$$

Let $u = x^{-1}$, the exponent on e. Then

$$du = -x^{-2}\,dx$$

$$-du = x^{-2}\,dx.$$

Bringing the negative sign outside the integral sign, the problem now reads

$$-\int e^u\,du.$$

Next, change the limits of integration:

$$u = (1)^{-1} = 1$$

$$u = (2)^{-1} = \frac{1}{2}.$$

Notice that now the limits begin with the larger number, meaning we can multiply by -1 and interchange the limits. Thus,

$$-\int_1^{1/2} e^u\,du = \int_{1/2}^1 e^u\,du = e^u\big|_{1/2}^1 = e - e^{1/2} = e - \sqrt{e}.$$

Exercise 5.6.7

Evaluate the definite integral using substitution:

$$\int_1^2 \frac{1}{x^3}e^{4x^{-2}}\,dx.$$

Hint

Let $u = 4x^{-2}$.

Answer

$$\int_1^2 \frac{1}{x^3}e^{4x^{-2}}\,dx = \frac{1}{8}[e^4 - e].$$

Integrals Involving Logarithmic Functions

Integrating functions of the form $f(x) = x^{-1}$ result in the absolute value of the natural log function, as shown in the following rule. Integral formulas for other logarithmic functions, such as $f(x) = \ln x$ and $f(x) = \log_a x$, are also included in the rule.

Rule: Integration Formulas Involving Logarithmic Functions

The following formulas can be used to evaluate integrals involving logarithmic functions.

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int \ln x\,dx = x\ln x - x + C = x(\ln x - 1) + C$$

$$\int \log_a x\,dx = \frac{x}{\ln a}(\ln x - 1) + C$$

Example 5.6.9: Finding an Antiderivative Involving $\ln x$

Find the antiderivative of the function $\dfrac{3}{x-10}$.

Solution

First factor the **3** outside the integral symbol. Then use the u^{-1} rule. Thus,

$$\int \frac{3}{x-10}\,dx = 3\int \frac{1}{x-10}\,dx = 3\int \frac{du}{u} = 3\ln|u| + C = 3\ln|x-10| + C, \quad x \neq 10.$$

See Figure 5.6.3.

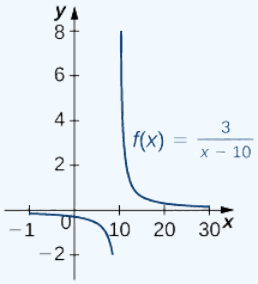


Figure 5.6.3: The domain of this function is $x \neq 10$.

Exercise 5.6.8

Find the antiderivative of $\dfrac{1}{x+2}$.

Hint

Follow the pattern from Example 5.6.9 to solve the problem.

Answer

$$\int \frac{1}{x+2}\,dx = \ln|x+2| + C$$

Example 5.6.10: Finding an Antiderivative of a Rational Function

Find the antiderivative of $\dfrac{2x^3+3x}{x^4+3x^2}$.

Solution

This can be rewritten as $\int (2x^3+3x)(x^4+3x^2)^{-1}\,dx$. Use substitution.

Let $u = x^4+3x^2$, then $du = (4x^3+6x)\,dx$. Alter du by factoring out the **2**. Thus,

$$du = (4x^3+6x)\,dx = 2(2x^3+3x)\,dx$$

$$\frac{1}{2}\,du = (2x^3 + 3x)\,dx.$$

Rewrite the integrand in u:

$$\int (2x^3 + 3x)(x^4 + 3x^2)^{-1}\,dx = \frac{1}{2}\int u^{-1}\,du.$$

Then we have

$$\frac{1}{2}\int u^{-1}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|\,x^4 + 3x^2\,| + C.$$

Example 5.6.11: Finding an Antiderivative of a Logarithmic Function

Find the antiderivative of the log function $\log_2 x$.

Solution

Follow the format in the formula listed in the rule on integration formulas involving logarithmic functions. Based on this format, we have

$$\int \log_2 x\,dx = \frac{x}{\ln 2}(\ln x - 1) + C.$$

Exercise 5.6.9

Find the antiderivative of $\log_3 x$.

Hint

Follow Example 5.6.11 and refer to the rule on integration formulas involving logarithmic functions.

Answer

$$\int \log_3 x\,dx = \frac{x}{\ln 3}(\ln x - 1) + C$$

Example 5.6.12 is a definite integral of a trigonometric function. With trigonometric functions, we often have to apply a trigonometric property or an identity before we can move forward. Finding the right form of the integrand is usually the key to a smooth integration.

Example 5.6.12: Evaluating a Definite Integral

Evaluate the definite integral

$$\int_0^{\pi/2} \frac{\sin x}{1 + \cos x}\,dx.$$

Solution

We need substitution to evaluate this problem. Let $u = 1 + \cos x$ so $du = -\sin x\,dx$.

Rewrite the integral in terms of u, changing the limits of integration as well. Thus,

$$u = 1 + \cos(0) = 2$$
$$u = 1 + \cos\left(\frac{\pi}{2}\right) = 1.$$

Then

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin x}{1+\cos x} &= -\int_2^1 u^{-1}\,du \\ &= \int_1^2 u^{-1}\,du \\ &= \ln|u|\Big|_1^2 \\ &= [\ln 2 - \ln 1] = \ln 2\end{aligned}$$

Key Concepts

- Exponential and logarithmic functions arise in many real-world applications, especially those involving growth and decay.
- Substitution is often used to evaluate integrals involving exponential functions or logarithms.

Key Equations

- **Integrals of Exponential Functions**

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

- **Integration Formulas Involving Logarithmic Functions**

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int \ln x\,dx = x\ln x - x + C = x(\ln x - 1) + C$$

$$\int \log_a x\,dx = \frac{x}{\ln a}(\ln x - 1) + C$$

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5.7: Integrals Resulting in Inverse Trigonometric Functions

Learning Objectives

- Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall, that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also, we previously developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Trigonometric Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \tag{5.7.1}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \tag{5.7.2}$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C \tag{5.7.3}$$

Proof of the first formula

Let $y = \sin^{-1}\frac{x}{a}$. Then $a\sin y = x$. Now using implicit differentiation, we obtain

$$\frac{d}{dx}(a\sin y) = \frac{d}{dx}(x)$$

$$a\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a\cos y}.$$

For $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \cos y \ge 0$. Thus, applying the Pythagorean identity $\sin^2 y + \cos^2 y = 1$, we have $\cos y = \sqrt{1-\sin^2 y}$. This gives

$$\frac{1}{a\cos y} = \frac{1}{a\sqrt{1-\sin^2 y}} \tag{5.7.4}$$

$$= \frac{1}{\sqrt{a^2-a^2\sin^2 y}} \tag{5.7.5}$$

$$= \frac{1}{\sqrt{a^2-x^2}}. \tag{5.7.6}$$

Then for $-a \le x \le a$, we have

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + C.$$

□

Example 5.7.1: Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.$$

Solution

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^{1/2} = \sin^{-1} \tfrac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

Note that since the integrand is simply the derivative of $\sin^{-1} x$, we are really just using this fact to find the antiderivative here.

Exercise 5.7.1

Find the indefinite integral using an inverse trigonometric function and substitution for $\int \frac{dx}{\sqrt{9-x^2}}$.

Hint

Use the formula in the rule on integration formulas resulting in inverse trigonometric functions.

Answer

$$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$

In many integrals that result in inverse trigonometric functions in the antiderivative, we may need to use substitution to see how to use the integration formulas provided above.

Example 5.7.2: Finding an Antiderivative Involving an Inverse Trigonometric Function using substitution

Evaluate the integral

$$\int \frac{dx}{\sqrt{4-9x^2}}.$$

Solution

Substitute $u = 3x$. Then $du = 3\,dx$ and we have

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}.$$

Applying the formula with $a = 2$, we obtain

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C.$$

Exercise 5.7.2

Find the antiderivative of $\int \frac{dx}{\sqrt{1-16x^2}}$.

Hint

Substitute $u = 4x$.

Answer

$$\int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4}\sin^{-1}(4x) + C$$

Example 5.7.3: Evaluating a Definite Integral

Evaluate the definite integral

$$\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}}.$$

Solution

The format of the problem matches the inverse sine formula. Thus,

$$\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u \Big|_0^{\sqrt{3}/2} = [\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)] - [\sin^{-1}(0)] = \frac{\pi}{3}.$$

Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out −1 and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example 5.7.4: Finding an Antiderivative Involving the Inverse Tangent Function

Find the antiderivative of $\int \frac{1}{9+x^2}\,dx$.

Solution

Apply the formula with $a = 3$. Then,

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Exercise 5.7.3

Find the antiderivative of $\int \frac{dx}{16+x^2}$.

Hint

Follow the steps in Example 5.7.4.

Answer

$$\int \frac{dx}{16+x^2} = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$$

Example 5.7.5: Applying the Integration Formulas WITH SUBSTITUTION

Find an antiderivative of $\int \frac{1}{1+4x^2}\,dx$.

Solution

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1} u + C$. So we use substitution, letting $u = 2x$, then $du = 2\,dx$ and $\frac{1}{2}\,du = dx$.Then, we have

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(2x) + C.$$

? Exercise 5.7.4

Use substitution to find the antiderivative of $\int \frac{dx}{25+4x^2}$.

Hint

Use the solving strategy from Example **5.7.5** and the rule on integration formulas resulting in inverse trigonometric functions.

Answer

$$\int \frac{dx}{25+4x^2} = \frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right) + C$$

✓ Example 5.7.6: Evaluating a Definite Integral

Evaluate the definite integral $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2}$.

Solution

Use the formula for the inverse tangent. We have

$$\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1}x\Big|_{\sqrt{3}/3}^{\sqrt{3}} = [\tan^{-1}(\sqrt{3})] - [\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

? Exercise 5.7.5

Evaluate the definite integral $\int_0^2 \frac{dx}{4+x^2}$.

Hint

Follow the procedures from Example **5.7.6** to solve the problem.

Answer

$$\int_0^2 \frac{dx}{4+x^2} = \frac{\pi}{8}$$

Key Concepts

- Formulas for derivatives of inverse trigonometric functions developed in Derivatives of Exponential and Logarithmic Functions lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.

Key Equations

- **Integrals That Produce Inverse Trigonometric Functions**

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$$

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- Includes some added textual clarifications and edits by Paul Seeburger (Monroe Community College)

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5.7E: Exercises

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5.8: Chapter 5 Review Exercises

In exercises 1 - 4, answer *True or False.* Justify your answer with a proof or a counterexample. Assume all functions f and g are continuous over their domains.

1) If $f(x) > 0,\ f'(x) > 0$ for all x, then the right-hand rule underestimates the integral $\int_a^b f(x)\,dx$. Use a graph to justify your answer.

Answer

False

2) $\int_a^b f(x)^2\,dx = \int_a^b f(x)\,dx$

3) If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Answer

True

4) All continuous functions have an antiderivative.

In exercises 5 - 8, evaluate the Riemann sums L_4 and R_4 for the given functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.

5) $y = 3x^2 - 2x + 1)$ over $[-1,1]$

Answer

$L_4 = 5.25,\ R_4 = 3.25,$ exact answer: 4

6) $y = \ln(x^2+1)$ over $[0,e]$

7) $y = x^2 \sin x$ over $[0,\pi]$

Answer

$L_4 = 5.364,\ R_4 = 5.364,$ exact answer: 5.870

8) $y = \sqrt{x} + \frac{1}{x}$ over $[1,4]$

In exercises 9 - 12, evaluate the integrals.

9) $\int_{-1}^{1} (x^3 - 2x^2 + 4x)\,dx$

Answer

$-\frac{4}{3}$

10) $\int_0^4 \frac{3t}{\sqrt{1+6t^2}}\,dt$

11) $\int_{\pi/3}^{\pi/2} 2\sec(2\theta)\tan(2\theta)\,d\theta$

Answer

1

12) $\int_0^{\pi/4} e^{\cos^2 x} \sin x \cos x\,dx$

In exercises 13 - 16, find the antiderivative.

13) $\int \frac{dx}{(x+4)^3}$

Answer

$$-\frac{1}{2(x+4)^2}+C$$

14) $\int x\ln(x^2)\,dx$

15) $\int \frac{4x^2}{\sqrt{1-x^6}}\,dx$

Answer

$$\frac{4}{3}\sin^{-1}(x^3)+C$$

16) $\int \frac{e^{2x}}{1+e^{4x}}\,dx$

In exercises 17 - 20, find the derivative.

17) $\frac{d}{dt}\int_0^t \frac{\sin x}{\sqrt{1+x^2}}\,dx$

Answer

$$\frac{\sin t}{\sqrt{1+t^2}}$$

18) $\frac{d}{dx}\int_1^{x^3} \sqrt{4-t^2}\,dt$

19) $\frac{d}{dx}\int_1^{\ln(x)} (4t+e^t)\,dt$

Answer

$$4\frac{\ln x}{x}+1$$

20) $\frac{d}{dx}\int_0^{\cos x} e^{t^2}\,dt$

In exercises 21 - 23, consider the historic average cost per gigabyte of RAM on a computer.

Year	5-Year Change ($)
1980	0
1985	−5,468,750
1990	−755,495
1995	−73,005
2000	−29,768
2005	−918
2010	−177

21) If the average cost per gigabyte of RAM in 2010 is $12, find the average cost per gigabyte of RAM in 1980.

Answer

$6,328,113

Solution: $6,328,113

22) The average cost per gigabyte of RAM can be approximated by the function $C(t) = 8,500,000(0.65)^t$, where t is measured in years since 1980, and C is cost in US dollars. Find the average cost per gigabyte of RAM for the period from 1980 to 2010.

23) Find the average cost of 1 GB RAM from 2005 to 2010.

Answer

$73.36

24) The velocity of a bullet from a rifle can be approximated by $v(t) = 6400t^2 - 6505t + 2686$, where t is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot: $0 \leq t \leq 0.5$. What is the total distance the bullet travels in 0.5 sec?

25) What is the average velocity of the bullet for the first half-second?

Answer

$\frac{19117}{12}$ ft/sec, or about 1593 ft/sec

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CHAPTER OVERVIEW

6: Applications of Integration

In this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

6.1: Prelude to Applications of Integration

6.2: Areas between Curves

6.2E: Exercises

Thumbnail: A region between two functions.

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6.1: Prelude to Applications of Integration

The Hoover Dam is an engineering marvel. When Lake Mead, the reservoir behind the dam, is full, the dam withstands a great deal of force. However, water levels in the lake vary considerably as a result of droughts and varying water demands. Later in this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.



Figure **6.1.1**: Hoover Dam is one of the United States' iconic landmarks, and provides irrigation and hydroelectric power for millions of people in the southwest United States. (credit: modification of work by Lynn Betts, Wikimedia).

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6.2: Areas between Curves

Learning Objectives

- Determine the area of a region between two curves by integrating with respect to the independent variable.
- Find the area of a compound region.
- Determine the area of a region between two curves by integrating with respect to the dependent variable.

In Introduction to Integration, we developed the concept of the definite integral to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of x, beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of y.

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in Figure **6.2.1**.

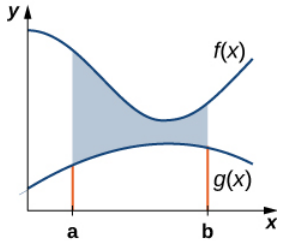


Figure **6.2.1**: The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$

As we did before, we are going to partition the interval on the **x-axis** and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. Figure **6.2.2***a* shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$. Figure **6.2.2***b* shows a representative rectangle in detail.

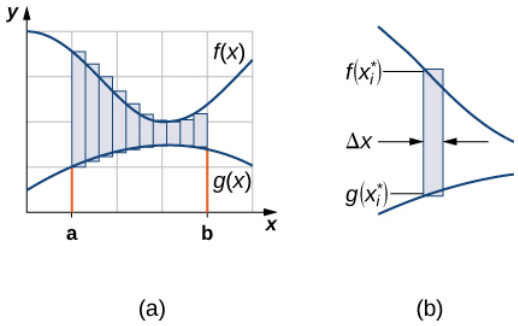


Figure **6.2.2**: (a)We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x = \int_a^b [f(x) - g(x)]dx.$$

These findings are summarized in the following theorem.

Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_a^b [f(x) - g(x)]dx.$$

We apply this theorem in the following example.

Example 6.2.1: Finding the Area of a Region between Two Curves I

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

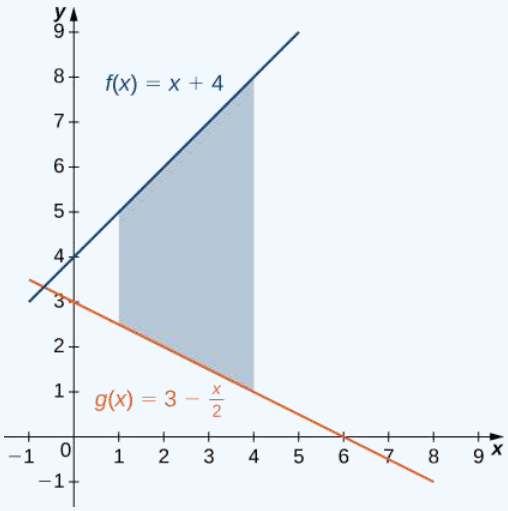


Figure 6.2.3: A region between two curves is shown where one curve is always greater than the other.

We have

$$A = \int_a^b [f(x) - g(x)]\,dx$$

$$= \int_1^4 [(x + 4) - (3 - \frac{x}{2})]\,dx = \int_1^4 \left[\frac{3x}{2} + 1\right]\,dx$$

$$= [\frac{3x^2}{4} + x]\Big|_1^4 = (16 - \frac{7}{4}) = \frac{57}{4}.$$

The area of the region is $\frac{57}{4} units^2$.

Exercise 6.2.1

If $\mathbf{R}$ is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region $\mathbf{R}$.

Hint

Graph the functions to determine which function's graph forms the upper bound and which forms the lower bound, then follow the process used in Example.

Answer

12 units^2

In Example 6.2.1, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

Example 6.2.2: Finding the Area of a Region between Two Curves II

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

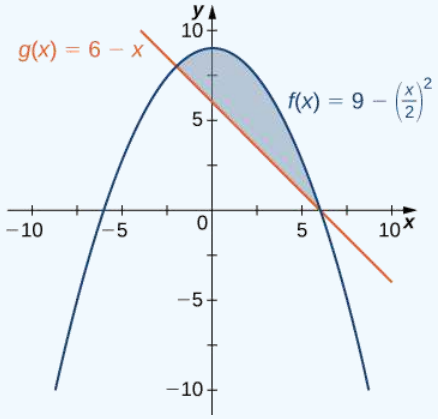


Figure 6.2.4: This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$\begin{aligned} f(x) &= g(x) \\ 9 - (\frac{x}{2})^2 &= 6 - x \\ 9 - \frac{x^2}{4} &= 6 - x \\ 36 - x^2 &= 24 - 4x \\ x^2 - 4x - 12 &= 0 \\ (x - 6)(x + 2) &= 0. \end{aligned}$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain

$$A = \int_a^b [f(x) - g(x)]\,dx$$
$$= \int_{-2}^{6} \left[9 - (\frac{x}{2})^2 - (6 - x)\right]\,dx$$
$$= \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right]\,dx$$
$$= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Big|_{-2}^{6} = \frac{64}{3}.$$

The area of the region is $64/3$ units^2.

? Exercise 6.2.2

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region $\mathbf{R}$.

Hint

Use the process from Example 6.2.2.

Answer

$\frac{3}{10}$ unit^2

Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

Finding the Area of a Region between Curves That Cross

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let $\mathbf{R}$ denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_a^b |f(x) - g(x)|dx.$$

In practice, applying this theorem requires us to break up the interval $[a, b]$ and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

✓ Example 6.2.3: Finding the Area of a Region Bounded by Functions That Cross

If $\mathbf{R}$ is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

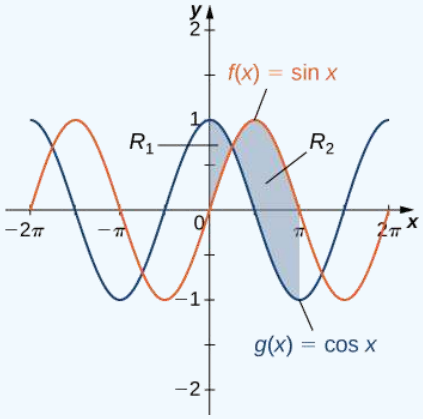


Figure 6.2.5: The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4], \cos x \geq \sin x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x.$$

On the other hand, for $x \in [\pi/4, \pi], \sin x \geq \cos x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then

$$\begin{aligned} A &= \int_a^b |f(x) - g(x)|dx \\ &= \int_0^\pi |\sin x - \cos x|dx = \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^\pi (\sin x - \cos x)dx \\ &= [\sin x + \cos x]|_0^{\pi/4} + [-\cos x - \sin x]|_{\pi/4}^\pi \\ &= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}. \end{aligned}$$

The area of the region is $2\sqrt{2}$ units^2.

? Exercise 6.2.3

If $\mathbf{R}$ is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[\pi/2, 2\pi]$, find the area of region $\mathbf{R}$.

Hint

The two curves intersect at $x = (5\pi)/4$.

Answer

$2 + 2\sqrt{2}$ units^2

✓ Example 6.2.4: Finding the Area of a Complex Region

Consider the region depicted in Figure 6.2.6. Find the area of $\mathbf{R}$.

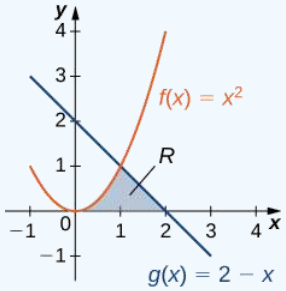


Figure **6.2.6**: Two integrals are required to calculate the area of this region.

Solution

As with Example **6.2.3**, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the x-axis, so we have

$$A_1 = \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the **x-axis**, so we have

$$A_2 = \int_1^2 (2 - x) dx = [2x - \frac{x^2}{2}] \Big|_1^2 = \frac{1}{2}.$$

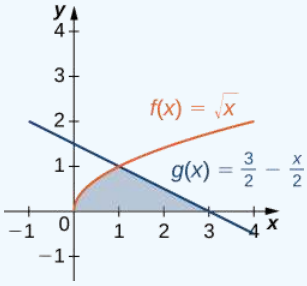
Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $5/6$ units^2.

? Exercise 6.2.4

Consider the region depicted in the following figure. Find the area of **R**.



Hint

The two curves intersect at **x=1**

Answer

$\frac{5}{3}$ units^2

Regions Defined with Respect to y

In Example **6.2.4**, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y, instead of as functions of x? Review Figure. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of y. However, based on the graph, it is clear we are interested in the positive square root.) Similarly, the right graph is represented by

the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of y, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y, we need to evaluate one integral only. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in Figure 6.2.7.

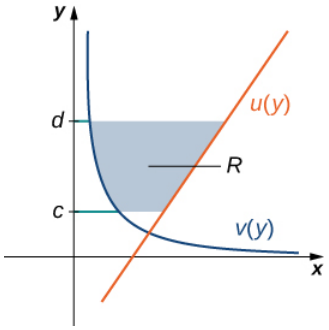


Figure 6.2.7: We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the **y-axis** and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n$, let $Q = y_i$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \ldots, n$, choose a point $y_i^* \in [y_{i-1}, y_i]$, then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y^0 *_i)$ to $u(y_i^*)$. Figure 6.2.8*a* shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10$. Figure 6.2.8*b* shows a representative rectangle in detail.

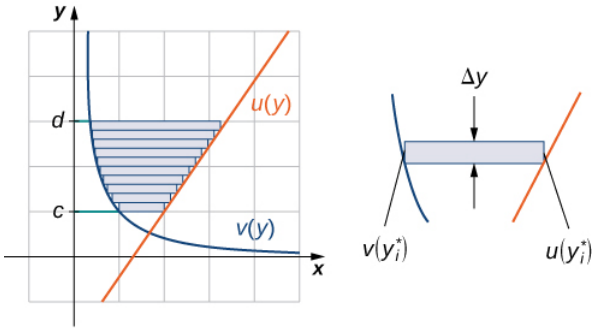


Figure 6.2.8: (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y)$, with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is Δy and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y.$$

This is a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$\begin{aligned} A &= \lim_{n \to \infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y \\ &= \int_c^d [u(y) - v(y)] dy. \end{aligned}$$

These findings are summarized in the following theorem.

Finding the Area between Two Curves, Integrating along the y-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$for all $y \in [c, d]$. Let $\mathbf{R}$ denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_c^d [u(y) - v(y)]dy.$$

Example 6.2.5: Integrating with Respect to y

Let's revisit Example **6.2.4**, only this time let's integrate with respect to y. Let $\mathbf{R}$ be the region depicted in Figure **6.2.9**. Find the area of $\mathbf{R}$ by integrating with respect to y.

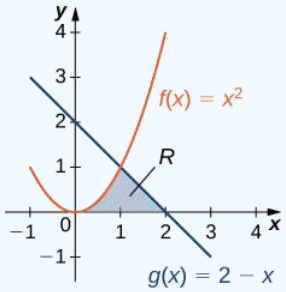


Figure **6.2.9**: The area of region $\mathbf{R}$ can be calculated using one integral only when the curves are treated as functions of y.

Solution

We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

Now we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.

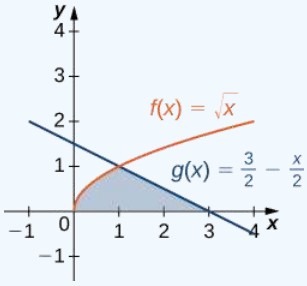
Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y)]dy \\ &= \int_0^1 [(2 - y) - \sqrt{y}]dy \\ &= [2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}] \,|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

The area of the region is $5/6$ units^2.

Exercise 6.2.5

Let's revisit the checkpoint associated with Example **6.2.4**, only this time, let's integrate with respect to y. Let $\mathbf{R}$ be the region depicted in the following figure. Find the area of $\mathbf{R}$ by integrating with respect to y.



Hint

Follow the process from the previous example.

Answer

$\frac{5}{3}$ units^2

Key Concepts

- Just as definite integrals can be used to find the area under a curve, they can also be used to find the area between two curves.
- To find the area between two curves defined by functions, integrate the difference of the functions.
- If the graphs of the functions cross, or if the region is complex, use the absolute value of the difference of the functions. In this case, it may be necessary to evaluate two or more integrals and add the results to find the area of the region.
- Sometimes it can be easier to integrate with respect to y to find the area. The principles are the same regardless of which variable is used as the variable of integration.

Key Equations

- **Area between two curves, integrating on the x-axis**

$$A = \int_a^b [f(x) - g(x)]dx$$

- **Area between two curves, integrating on the y-axis**

$$A = \int_c^d [u(y) - v(y)]dy$$

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6.2E: Exercises

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CHAPTER OVERVIEW

7: Appendices

7.1: A- Table of Derivatives
7.2: B- Table of Integrals

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7.1: A- Table of Derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{for real numbers } n$
5. $\frac{d}{dx}(cf(x)) = cf'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
18. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
19. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
20. $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$
22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
23. $\frac{d}{dx}(b^x) = b^x \ln b$
24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$
26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
28. $\frac{d}{dx}(\cosh x) = \sinh x$
29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
32. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2} \quad (|x| < 1)$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}} \quad (0 < x < 1)$
34. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
35. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2} \quad (|x| > 1)$
36. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}} \quad (x \neq 0)$

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7.2: B- Table of Integrals

Basic Integrals

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

Trigonometric Integrals

18. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
19. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
20. $\int \tan^2 u\,du = \tan u - u + C$
21. $\int \cot^2 u\,du = -\cot u - u + C$
22. $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$
23. $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$
24. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$
25. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\,du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\,du = \frac{-1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

29. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

30. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

31. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

32. $\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

33. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

34. $\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u \sin u\,du = \sin u - u\cos u + C$

38. $\int u \cos u\,du = \cos u + u\sin u + C$

39. $\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$

40. $\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$

41. Erroneous nesting of equation structures

Exponential and Logarithmic Integrals

42. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$

43. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

44. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

45. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

46. $\int \ln u\,du = u\ln u - u + C$

47. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\,du = \cosh u + C$

50. $\int \cosh u\,du = \sinh u + C$

51. $\int \tanh u\,du = \ln\cosh u + C$

52. $\int \coth u\,du = \ln|\sinh u| + C$

53. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

54. $\int \operatorname{csch} u\,du = \ln|\tanh \frac{1}{2}u| + C$

55. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

56. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

57. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

58. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

60. $\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

61. $\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

62. $\int u\sin^{-1} u\,du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u\cos^{-1} u\,du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u\tan^{-1} u\,du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

65. $\int u^n \sin^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n \cos^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

67. $\int u^n \tan^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right) + C$

69. $\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u + \sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u + \sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u + \sqrt{a^2+u^2}\right) + C$

73. $\int \frac{u^2}{\sqrt{a^2+u^2}}\,du = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

76. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

78. $\int u^2\sqrt{u^2 - a^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

79. $\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{|u|} + C$

80. $\int \frac{\sqrt{u^2 - a^2}}{u^2}\, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

81. $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

82. $\int \frac{u^2}{\sqrt{u^2 - a^2}}\, du = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

83. $\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

84. $\int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

86. $\int u^2\sqrt{a^2 - u^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

87. $\int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

88. $\int \frac{\sqrt{a^2 - u^2}}{u^2}\, du = \frac{-1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

89. $\int \frac{u^2}{\sqrt{a^2 - u^2}}\, du = \frac{1}{2}\left(-u\sqrt{a^2 - u^2} + a^2\sin^{-1}\frac{u}{a}\right) + C$

90. $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

91. $\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

92. $\int (a^2 - u^2)^{3/2}\, du = -\frac{u}{8}\left(2u^2 - 5a^2\right)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

93. $\int \frac{du}{(a^2 - u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Integrals Involving $2au - u^2$, $a > 0$

94. $\int \sqrt{2au - u^2}\, du = \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

95. $\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\left(\frac{a - u}{a}\right) + C$

96. $\int u\sqrt{2au - u^2}\, du = \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

97. $\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$

Integrals Involving $a + bu$, $a \neq 0$

98. $\int \frac{u}{a + bu}\, du = \frac{1}{b^2}(a + bu - a\ln|a + bu|) + C$

99. $\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

100. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

101. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

102. $\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

103. $\int \frac{u}{u(a+bu)^2}\,du = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

104. $\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

105. $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

106. $\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

107. $\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$

108. $\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, & \text{if } a > 0 \\ \frac{\sqrt{2}}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, & \text{if } a < 0 \end{cases}$

109. $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

110. $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

111. $\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

112. $\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du$

113. $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

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