

MATH 300: MATHEMATICAL IDEAS



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**Math 300: Mathematical Ideas Textbook
(Muranaka)**

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1.0.0: Introduction

Apportionment is the problem of dividing up a fixed number of things among groups of different sizes. In politics, this takes the form of allocating a limited number of representatives amongst voters. This problem, presumably, is older than the United States, but the best-known ways to solve it have their origins in the problem of assigning each state an appropriate number of representatives in the new Congress when the country was formed. States also face this apportionment problem in defining how to draw districts for state representatives. The apportionment problem comes up in a variety of non-political areas too, though. We face several restrictions in this process:

Apportionment Rules

1. The things being divided up can exist only in whole numbers.
2. We must use all of the things being divided up, and we cannot use any more.
3. Each group must get at least one of the things being divided up.
4. The number of things assigned to each group should be at least approximately proportional to the population of the group. (Exact proportionality isn't possible because of the whole number requirement, but we should try to be close, and in any case, if Group A is larger than Group B, then Group B shouldn't get more of the things than Group A does.)

In terms of the apportionment of the United States House of Representatives, these rules imply:

1. We can only have whole representatives (a state can't have 3.4 representatives)
2. We can only use the (currently) 435 representatives available. If one state gets another representative, another state has to lose one.
3. Every state gets at least one representative
4. The number of representatives each state gets should be approximately proportional to the state population. This way, the number of constituents each representative has should be approximately equal.

We will look at four ways of solving the apportionment problem. Three of them (Lowndes's method is the exception) have been used at various times to apportion the U.S. Congress, although the method currently in use (the Huntington-Hill method) is significantly more complicated.

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1.0.1: Hamilton's Method

Alexander Hamilton proposed the method that now bears his name. His method was approved by Congress in 1791, but was vetoed by President Washington. It was later adopted in 1852 and used through 1911. He begins by determining, to several decimal places, how many things each group should get. Since he was interested in the question of Congressional representation, we'll use the language of states and representatives, so he determines how many representatives each state should get. He follows these steps:

Hamilton's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**. Since we can only allocate whole representatives, Hamilton resolves the whole number problem, as follows:
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). These are called the **lower quotas**. Add up the remaining whole numbers. This answer will always be less than or equal to the total number of representatives (and the "or equal to" part happens only in very specific circumstances that are incredibly unlikely to turn up).
4. Assuming that the total from Step 3 was less than the total number of representatives, assign the remaining representatives, one each, to the states whose decimal parts of the quota were largest, until the desired total is reached.

Make sure that each state ends up with at least one representative!

Note on rounding: Today we have technological advantages that Hamilton (and the others) couldn't even have imagined. Take advantage of them, and keep several decimal places.

Example 1

The state of Delaware has three counties: Kent, New Castle, and Sussex. The Delaware state House of Representatives has 41 members. If Delaware wants to divide this representation along county lines (which is *not* required, but let's pretend they do), let's use Hamilton's method to apportion them. The populations of the counties are as follows (from the 2010 Census):

County	Population
Kent	162,310
New Castle	538,479
Sussex	197,145
Total	897,934

Solution

1. First, we determine the divisor: $897,934/41 = 21,900.82927$
2. Now we determine each county's quota by dividing the county's population by the divisor:

County	Population	Quota
Kent	162,310	7.4111
New Castle	538,479	24.5872
Sussex	197,145	9.0017
Total	897,934	

3. Removing the decimal parts of the quotas gives:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	897,934		40

4. We need 41 representatives and this only gives 40. The remaining one goes to the county with the largest decimal part, which is New Castle:

County	Population	Quota	Initial	Final
Kent	162,310	7.4111	7	7
New Castle	538,479	24.5872	24	25
Sussex	197,145	9.0017	9	9
Total	897,934		40	41

Example 2

Use Hamilton's method to apportion the 75 seats of Rhode Island's House of Representatives among its five counties.

County	Population
Bristol	49,875
Kent	166,158
Newport	82,888
Providence	626,667
Washington	126,979
Total	1,052,567

Solution

1. The divisor is $1,052,567/75 = 14,034.22667$

2. Determine each county's quota by dividing its population by the divisor:

County	Population	Quota
Bristol	49,875	3.5538
Kent	166,158	11.8395
Newport	82,888	5.9061
Providence	626,667	44.6528
Washington	126,979	9.0478
Total	1,052,567	

3. Remove the decimal part of each quota:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	1,052,567		72

4. We need 75 representatives and we only have 72, so we assign the remaining three, one each, to the three counties with the largest decimal parts, which are Newport, Kent, and Providence:

County	Population	Quota	Initial	Final
Bristol	49,875	3.5538	3	3
Kent	166,158	11.8395	11	12
Newport	82,888	5.9061	5	6
Providence	626,667	44.6528	44	45
Washington	126,979	9.0478	9	9
Total	1,052,567		72	75

Note that even though Bristol County's decimal part is greater than .5, it isn't big enough to get an additional representative, because three other counties have greater decimal parts.

Hamilton's method obeys something called the Quota Rule. The Quota Rule isn't a law of any sort, but just an idea that some people, including Hamilton, think is a good one.

Quota Rule

The Quota Rule says that the final number of representatives a state gets should be within one of that state's quota. Since we're dealing with whole numbers for our final answers, that means that each state should either go up to the next whole number above its quota, or down to the next whole number below its quota.

Controversy

After seeing Hamilton's method, many people find that it makes sense, it's not that difficult to use (or, at least, the difficulty comes from the numbers that are involved and the amount of computation that's needed, not from the method), and they wonder why anyone would want another method. The problem is that Hamilton's method is subject to several paradoxes. Three of them happened, on separate occasions, when Hamilton's method was used to apportion the United States House of Representatives.

The **Alabama Paradox** is named for an incident that happened during the apportionment that took place after the 1880 census. (A similar incident happened ten years earlier involving the state of Rhode Island, but the paradox is named after Alabama.) The post-1880 apportionment had been completed, using Hamilton's method and the new population numbers from the census. Then it was decided that because of the country's growing population, the House of Representatives should be made larger. That meant that the apportionment would need to be done again, still using Hamilton's method and the same 1880 census numbers, but with more representatives. The assumption was that some states would gain another representative and others would stay with the same number they already had (since there weren't enough new representatives being added to give one more to every state). The paradox is that Alabama ended up *losing* a representative in the process, even though no populations were changed and the total number of representatives increased.

The **New States Paradox** happened when Oklahoma became a state in 1907. Oklahoma had enough population to qualify for five representatives in Congress, so they raised the total number of representatives by five. When the apportionment was recalculated, unsurprisingly Oklahoma received five representatives. However, despite their populations not changing, Maine gained a representative and New York lost one.

The **Population Paradox** happened between the apportionments after the census of 1900 and of 1910. In those ten years, Virginia's population grew at an average annual rate of 1.07%, while Maine's grew at an average annual rate of 0.67%. Virginia started with more people, grew at a faster rate, grew by more people, and ended up with more people than Maine. By itself, that doesn't mean that Virginia should gain representatives or Maine shouldn't, because there are lots of other states involved. But Virginia ended up losing a representative to Maine.

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1.0.2: Jefferson's Method

Thomas Jefferson proposed a different method for apportionment. After Washington vetoed Hamilton's method, Jefferson's method was adopted, and used in Congress from 1791 through 1842. Jefferson, of course, had political reasons for wanting his method to be used rather than Hamilton's. Primarily, his method favors larger states, and his own home state of Virginia was the largest in the country at the time. He would also argue that it's the ratio of people to representatives that is the critical thing, and apportionment methods should be based on that. But the paradoxes we saw also provide mathematical reasons for concluding that Hamilton's method isn't so good, and while Jefferson's method might or might not be the best one to replace it, at least we should look for other possibilities.

The first steps of Jefferson's method are the same as Hamilton's method. He finds the same divisor and the same quota, and cuts off the decimal parts in the same way, giving a total number of representatives that is less than the required total. The difference is in how Jefferson resolves that difference. He says that since we ended up with an answer that is too small, our divisor must have been too big. He changes the divisor by making it smaller, finding new quotas with the new divisor, cutting off the decimal parts, and looking at the new total, until we find a divisor that produces the required total.

Jefferson's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). These are the **lower quotas**. Add up the remaining whole numbers. This answer will always be less than or equal to the total number of representatives.
4. If the total from Step 3 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 3 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 3

We'll return to Delaware and apply Jefferson's method. We begin, as we did with Hamilton's method, by finding the quotas with the original divisor, 21,900.82927:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
Newport	538,479	24.5872	24
Providence	197,145	9.0017	9
Total	897,934		40

Solution

We need 41 representatives, and this divisor gives only 40. We must reduce the divisor until we get 41 representatives. Let's try 21,500 as the divisor:

County	Population	Quota	Initial
Kent	162,310	7.5493	7
Newport	538,479	25.0455	25
Providence	197,145	9.1695	9
Total	897,934		41

This gives us the required 41 representatives, so we're done. If we had fewer than 41, we'd need to reduce the divisor more. If we had more than 41, we'd need to choose a divisor less than the original but greater than the second choice.

Notice that with the new, lower divisor, the quota for New Castle County (the largest county in the state) increased by much more than those of Kent County or Sussex County.

Example 4

We'll apply Jefferson's method for Rhode Island. The original divisor of 14,034.22667 gave these results:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	1,052,567		72

Solution

We need 75 representatives and we only have 72, so we need to use a smaller divisor. Let's try 13,500:

County	Population	Quota	Initial
Bristol	49,875	3.6944	3
Kent	166,158	12.3080	12
Newport	82,888	6.1399	6
Providence	626,667	46.4198	46
Washington	126,979	9.4059	9
Total	1,052,567		76

We've gone too far. We need a divisor that's greater than 13,500 but less than 14,034.22667. Let's try 13,700:

County	Population	Quota	Initial
Bristol	49,875	3.6405	3
Kent	166,158	12.1283	12
Newport	82,888	6.0502	6
Providence	626,667	45.7421	45
Washington	126,979	9.2685	9
Total	1,052,567		75

This works.

Notice, in comparison to Hamilton's method, that although the results were the same, they came about in a different way, and the outcome was almost different. Providence County (the largest) almost went up to 46 representatives before Kent (which is much smaller) got to 12. Although that didn't happen here, it can. Divisor-adjusting methods like Jefferson's are not guaranteed to follow the quota rule!

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1.0.3: Webster's Method

Daniel Webster (1782-1852) proposed a method similar to Jefferson's in 1832. It was adopted by Congress in 1842, but replaced by Hamilton's method in 1852. It was then adopted again in 1901. The difference is that Webster rounds the quotas to the nearest whole number rather than dropping the decimal parts. If that doesn't produce the desired results at the beginning, he says, like Jefferson, to adjust the divisor until it does. (In Jefferson's case, at least the first adjustment will always be to make the divisor smaller. That is not always the case with Webster's method.)

Webster's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Round all the quotas to the nearest whole number (but don't forget what the decimals were). Add up the remaining whole numbers.
4. If the total from Step 3 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. If the total from step 3 was larger than the total number of representatives, increase the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 3 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 5

Again, Delaware, with an initial divisor of 21,900.82927:

County	Population	Quota	Initial
Kent	166,310	7.4111	7
New Castle	538,479	24.5872	25
Sussex	197,145	9.0017	9
Total	897,934		41

Solution

This gives the required total, so we're done.

Example 6

Again, Rhode Island, with an initial divisor of 14,034.22667:

County	Population	Quota	Initial
Bristol	49,875	3.5538	4
Kent	166,158	11.8395	12
Newport	82,888	5.9061	6
Providence	626,667	44.6528	45
Washington	126,979	9.0478	9
Total	1,052,567		76

Solution

This is too many, so we need to increase the divisor. Let's try 14,100:

County	Population	Quota	Initial
Bristol	49,875	3.5372	4
Kent	166,158	11.7843	12
Newport	82,888	5.8786	6
Providence	626,667	5.8786	44
Washington	126,979	9.0056	9
Total	1,052,567		75

This works, so we're done.

Like Jefferson's method, Webster's method carries a bias in favor of states with large populations, but rounding the quotas to the nearest whole number greatly reduces this bias. (Notice that Providence County, the largest, is the one that gets a representative trimmed because of the increased quota.) Also like Jefferson's method, Webster's method does not always follow the quota rule, but it follows the quota rule much more often than Jefferson's method does. (In fact, if Webster's method had been applied to every apportionment of Congress in all of American history, it would have followed the quota rule every single time.)

In 1980, two mathematicians, Peyton Young and Mike Balinski, proved what we now call the Balinski-Young Impossibility Theorem.

Balinski-Young Impossibility Theorem

The Balinski-Young Impossibility Theorem shows that any apportionment method which always follows the quota rule will be subject to the possibility of paradoxes like the Alabama, New States, or Population paradoxes. In other words, we can choose a method that avoids those paradoxes, but only if we are willing to give up the guarantee of following the quota rule.

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1.0.4: Huntington-Hill Method

In 1920, no new apportionment was done, because Congress couldn't agree on the method to be used. They appointed a committee of mathematicians to investigate, and they recommended the Huntington-Hill Method. They continued to use Webster's method in 1931, but after a second report recommending Huntington-Hill, it was adopted in 1941 and is the current method of apportionment used in Congress.

The Huntington-Hill Method is similar to Webster's method, but attempts to minimize the percent differences of how many people each representative will represent.

Huntington-Hill Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off the decimal part of the quota to obtain the lower quota, which we'll call n . Compute $\sqrt{n(n+1)}$, which is the **geometric mean** of the lower quota and one value higher.
4. If the quota is larger than the geometric mean, round up the quota; if the quota is smaller than the geometric mean, round down the quota. Add up the resulting whole numbers to get the **initial allocation**.
5. If the total from Step 4 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. If the total from step 4 was larger than the total number of representatives, increase the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 4 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 7

Again, Delaware, with an initial divisor of 21,900.82927:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Kent	162,310	7.4111	7	7.48	7
New Castle	538,479	24.5872	24	24.49	25
Sussex	197,145	9.0017	9	9.49	9
Total	897,934				41

Solution

This gives the required total, so we're done.

Example 8

Again, Rhode Island, with an initial divisor of 14,034.22667:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Bristol	49,875	3.5538	3	3.46	4
Kent	166,158	11.8395	11	11.49	12
Newport	82,888	5.9061	5	5.48	6
Providence	626,667	44.6528	44	44.50	45
Washington	126,979	9.0478	9	9.49	9
Total	1,052,567				76

Solution

This is too many, so we need to increase the divisor. Let's try 14,100:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Bristol	49,875	3.5372	3	3.46	4
Kent	166,158	11.7843	11	11.49	12
Newport	82,888	5.8786	5	5.48	6
Providence	626,667	44.4445	44	44.50	44
Washington	126,979	9.0056	9	9.49	9
Total	1,052,567				75

This works, so we're done.

In both these cases, the apportionment produced by the Huntington-Hill method was the same as those from Webster's method.

Example 9

Consider a small country with 5 states, two of which are much larger than the others. We need to apportion 70 representatives. We will apportion using both Webster's method and the Huntington-Hill method.

State	Population
A	300,500
B	200,000
C	50,000
D	38,000
E	21,500

Solution

1. The total population is 610,000. Dividing this by the 70 representatives gives the divisor: 8714.286.
2. Dividing each state's population by the divisor gives the quotas.

State	Population	Quota
A	300,500	34.48361
B	200,000	22.95082
C	50,000	5.737705
D	38,000	4.360656
E	21,500	2.467213

Webster's Method

3. Using Webster's method, we round each quota to the nearest whole number

State	Population	Quota	Initial
A	300,500	34.48361	34
B	200,000	22.95082	23
C	50,000	5.737705	6
D	38,000	4.360656	4
E	21,500	2.467213	2

4. Adding these up, they only total 69 representatives, so we adjust the divisor down. Adjusting the divisor down to 8700 gives an updated allocation totaling 70 representatives

State	Population	Quota	Initial
A	300,500	34.54023	35
B	200,000	22.98851	23
C	50,000	5.747126	6
D	38,000	4.367816	4
E	21,500	2.471264	2

Huntington-Hill Method

3. Using the Huntington-Hill method, we round down to find the lower quota, then calculate the geometric mean based on each lower quota. If the quota is less than the geometric mean, we round down; if the quota is more than the geometric mean, we round up.

State	Population	Quota	Lower Quota	Geometric Mean	Initial
A	300,500	34.48361	34	34.49638	34
B	200,000	22.95082	22	22.49444	23
C	50,000	5.737705	5	5.477226	6
D	38,000	4.360656	4	4.472136	4
E	21,500	2.467213	2	2.44949	3

These allocations add up to 70, so we're done.

Notice that this allocation is different than that produced by Webster's method. In this case, state E got the extra seat instead of state A.

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1.0.5: Lowndes' Method

William Lowndes (1782-1822) was a Congressman from South Carolina (a small state) who proposed a method of apportionment that was more favorable to smaller states. Unlike the methods of Hamilton, Jefferson, and Webster, Lowndes's method has never been used to apportion Congress.

Lowndes believed that an additional representative was much more valuable to a small state than to a large one. If a state already has 20 or 30 representatives, getting one more doesn't matter very much. But if it only has 2 or 3, one more is a big deal, and he felt that the additional representatives should go where they could make the most difference.

Like Hamilton's method, Lowndes's method follows the quota rule. In fact, it arrives at the same quotas as Hamilton and the rest, and like Hamilton and Jefferson, it drops the decimal parts. But in deciding where the remaining representatives should go, we divide the decimal part of each state's quota by the whole number part (so that the same decimal part with a smaller whole number is worth more, because it matters more to that state).

Lowndes's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). Add up the remaining whole numbers.
4. Assuming that the total from Step 3 was less than the total number of representatives, divide the decimal part of each state's quota by the whole number part. Assign the remaining representatives, one each, to the states whose **ratio** of decimal part to whole part were largest, until the desired total is reached.

Example 10

We'll do Delaware again. We begin in the same way as with Hamilton's method:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	897,934		40

Solution

We need one more representative. To find out which county should get it, Lowndes says to divide each county's decimal part by its whole number part, with the largest result getting the extra representative:

Kent: $0.4111/7 \approx 0.0587$
 New Castle: $0.5872/24 \approx 0.0245$
 Sussex: $0.0017/9 \approx 0.0002$

The largest of these is Kent's, so Kent gets the 41th representative:

County	Population	Quota	Initial	Ratio	Final
Kent	162,310	7.4111	7	0.0587	8
New Castle	538,479	24.5872	24	0.0245	24
Sussex	197,145	9.0017	9	0.0002	9
Total	897,934		40		41

Example 11

Rhode Island, again beginning in the same way as Hamilton:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	1,052,567		72

Solution

We divide each county's quota's decimal part by its whole number part to determine which three should get the remaining representatives:

Bristol:	$0.5538/3 \approx 0.1846$
Kent:	$0.8395/11 \approx 0.0763$
Newport:	$0.9061/5 \approx 0.1812$
Providence:	$0.6528/44 \approx 0.0148$
Washington:	$0.0478/9 \approx 0.0053$

The three largest of these are Bristol, Newport, and Kent, so they get the remaining three representatives:

County	Population	Quota	Initial	Ratio	Final
Bristol	49,875	3.5538	3	0.1846	4
Kent	166,158	11.8395	11	0.0763	12
Newport	82,888	5.9061	5	0.1812	6
Providence	626,667	44.6528	44	0.0148	44
Washington	126,979	9.0478	9	0.0053	9
Total	1,052,567		72		75

As you can see, there is no "right answer" when it comes to choosing a method for apportionment. Each method has its virtues, and favors different sized states.

1.0.6: Apportionment of Legislative Districts

In most states, there are a fixed number of representatives to the state legislature. Rather than apportioning each county a number of representatives, legislative districts are drawn so that each legislator represents a district. The apportionment process, then, comes in the drawing of the legislative districts, with the goal of having each district include approximately the same number of constituents. Because of this goal, a geographically small city may have several representatives, while a large rural region may be represented by one legislator.

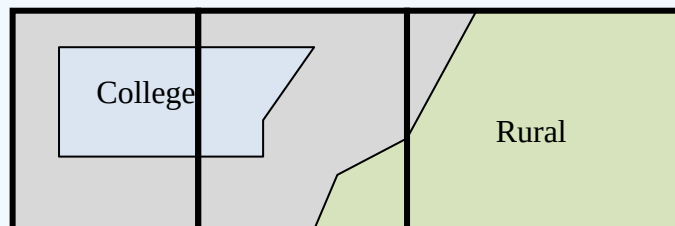
When populations change, it becomes necessary to redistrict the regions each legislator represents (Incidentally, this also occurs for the regions that federal legislators represent). The process of redistricting is typically done by the legislature itself, so not surprisingly it is common to see **gerrymandering**.

Gerrymandering

Gerrymandering is when districts are drawn based on the political affiliation of the constituents to the advantage of those drawing the boundary.

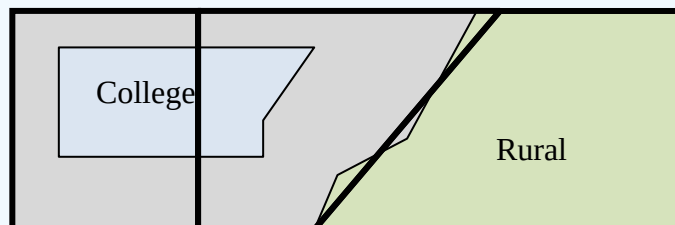
Example 12

Consider three districts, simplified to the three boxes below. On the left there is a college area that typically votes Democratic. On the right is a rural area that typically votes Republican. The rest of the people are more evenly split. The middle district has been voting 50% Democratic and 50% Republican.



Solution

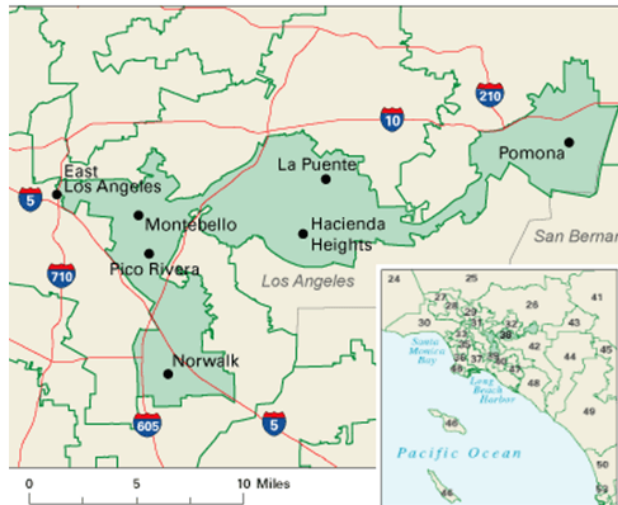
As part of a redistricting, a Democratic led committee could redraw the boundaries so that the middle district includes less of the typically Republican voters, thereby making it more likely that their party will win in that district, while increasing the Republican majority in the third district.



Example 13

The map to the right shows the 38th congressional district in California in 2004[1]. This district was created through a bi-partisan committee of incumbent legislators. This gerrymandering leads to districts that are not competitive; the prevailing party almost always wins with a large margin.

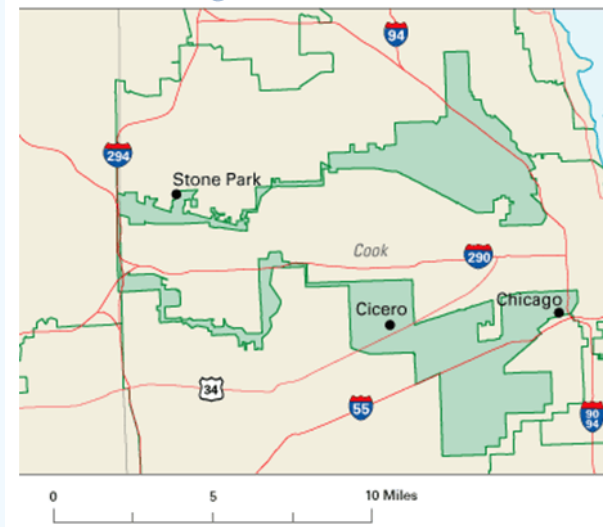
Congressional District 38



Solution

The map to the right shows the 4th congressional district in Illinois in 2004.[2] This district was drawn to contain the two predominantly Hispanic areas of Chicago. The largely Puerto Rican area to the north and the southern Mexican areas are only connected in this districting by a piece of the highway to the west.

Congressional District 4



[1] http://en.Wikipedia.org/wiki/File:Ca...ct_38_2004.png

[2] en.Wikipedia.org/wiki/File:IL...ict_4_2004.png

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1.0.7: Exercises

In exercises 1-8, determine the apportionment using

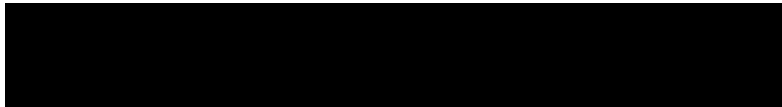
- a. Hamilton's Method
- b. Jefferson's Method
- c. Webster's Method
- d. Huntington-Hill Method
- e. Lowndes' method

1. A college offers tutoring in Math, English, Chemistry, and Biology. The number of students enrolled in each subject is listed below. If the college can only afford to hire 15 tutors, determine how many tutors should be assigned to each subject.

Math: 330 English: 265 Chemistry: 130 Biology: 70

2. Reapportion the previous problem if the college can hire 20 tutors.

3. The number of salespeople assigned to work during a shift is apportioned based on the average number of customers during that shift. Apportion 20 salespeople given the information below.



4. Reapportion the previous problem if the store has 25 salespeople.

5. Three people invest in a treasure dive, each investing the amount listed below. The dive results in 36 gold coins. Apportion those coins to the investors.

Alice: \$7,600 Ben: \$5,900 Carlos: \$1,400

6. Reapportion the previous problem if 37 gold coins are recovered.

7. A small country consists of five states, whose populations are listed below. If the legislature has 119 seats, apportion the seats.

A: 810,000 B: 473,000 C: 292,000 D: 594,000 E: 211,000

8. A small country consists of six states, whose populations are listed below. If the legislature has 200 seats, apportion the seats.

A: 3,411 B: 2,421 C: 11,586 D: 4,494 E: 3,126 F: 4,962

9. A small country consists of three states, whose populations are listed below.

A: 6,000 B: 6,000 C: 2,000

- a. If the legislature has 10 seats, use Hamilton's method to apportion the seats.
- b. If the legislature grows to 11 seats, use Hamilton's method to apportion the seats.
- c. Which apportionment paradox does this illustrate?

10. A state with five counties has 50 seats in their legislature. Using Hamilton's method, apportion the seats based on the 2000 census, then again using the 2010 census. Which apportionment paradox does this illustrate?



11. A school district has two high schools: Lowell, serving 1715 students, and Fairview, serving 7364. The district could only afford to hire 13 guidance counselors.

- a. Determine how many counselors should be assigned to each school using Hamilton's method.
- b. The following year, the district expands to include a third school, serving 2989 students. Based on the divisor from above, how many additional counselors should be hired for the new school?
- c. After hiring that many new counselors, the district recalculates the reapportion using Hamilton's method. Determine the outcome.
- d. Does this situation illustrate any apportionment issues?

12. A small country consists of four states, whose populations are listed below. If the legislature has 116 seats, apportion the seats using Hamilton's method. Does this illustrate any apportionment issues?

A: 33,700 B: 559,500 C: 141,300 D: 89,100

Exploration

13. Explore and describe the similarities, differences, and interplay between weighted voting, fair division (if you've studied it yet), and apportionment.

14. In the methods discussed in the text, it was assumed that the number of seats being apportioned was fixed. Suppose instead that the number of seats could be adjusted slightly, perhaps 10% up or down. Create a method for apportioning that incorporates this additional freedom, and describe why you feel it is the best approach. Apply your method to the apportionment in Exercise 7.

15. Lowndes felt that small states deserved additional seats more than larger states. Suppose you were a legislator from a larger state, and write an argument refuting Lowndes.

16. Research how apportionment of legislative seats is done in other countries around the world. What are the similarities and differences compared to how the United States apportions congress?

17. Adams's method is similar to Jefferson's method, but rounds quotas *up* rather than *down*. This means we usually need a modified divisor that is smaller than the standard divisor. Rework problems 1-8 using Adam's method. Which other method are the results most similar to?

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CHAPTER 2: VOTING THEORY

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2.0.0: Introduction

In many decision making situations, it is necessary to gather the group consensus. This happens when a group of friends decides which movie to watch, when a company decides which product design to manufacture, and when a democratic country elects its leaders.

While the basic idea of voting is fairly universal, the method by which those votes are used to determine a winner can vary. Amongst a group of friends, you may decide upon a movie by voting for all the movies you're willing to watch, with the winner being the one with the greatest approval. A company might eliminate unpopular designs then revote on the remaining. A country might look for the candidate with the most votes.

In deciding upon a winner, there is always one main goal: to reflect the preferences of the people in the most fair way possible.

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2.0.1: Preference Schedules

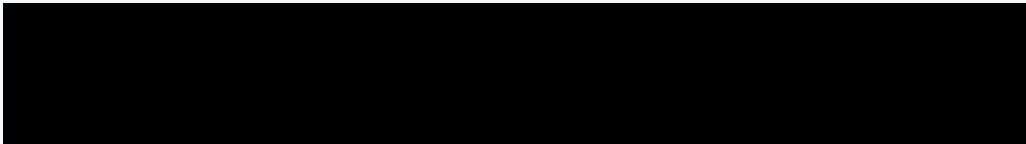
To begin, we're going to want more information than a traditional ballot normally provides. A traditional ballot usually asks you to pick your favorite from a list of choices. This ballot fails to provide any information on how a voter would rank the alternatives if their first choice was unsuccessful.

Preference ballot

A **preference ballot** is a ballot in which the voter ranks the choices in order of preference.

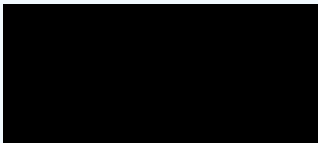
Example 1

A vacation club is trying to decide which destination to visit this year: Hawaii (H), Orlando (O), or Anaheim (A). Their votes are shown below:



Solution

These individual ballots are typically combined into one **preference schedule**, which shows the number of voters in the top row that voted for each option:



Notice that by totaling the vote counts across the top of the preference schedule we can recover the total number of votes cast: $1 + 3 + 3 + 3 = 10$ total votes.

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2.0.2: Plurality

The voting method we're most familiar with in the United States is the **plurality method**.

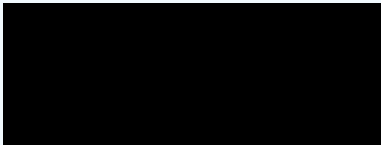
Plurality Method

In this method, the choice with the most first-preference votes is declared the winner. Ties are possible, and would have to be settled through some sort of run-off vote.

This method is sometimes mistakenly called the majority method, or “majority rules”, but it is not necessary for a choice to have gained a majority of votes to win. A majority is over 50%; it is possible for a winner to have a **plurality** without having a majority.

Example 2

In our election from previous pages, we had the preference table:



Solution

For the plurality method, we only care about the first choice options. Totaling them up:

Anaheim: $1+3 = 4$ first-choice votes

Orlando: 3 first-choice votes

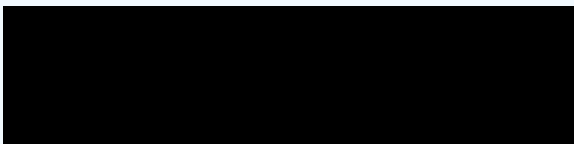
Hawaii: 3 first-choice votes

Anaheim is the winner using the plurality voting method.

Notice that Anaheim won with 4 out of 10 votes, 40% of the votes, which is a plurality of the votes, but not a majority.

Try it Now 1

Three candidates are running in an election for County Executive: Goings (G), McCarthy (M), and Bunney (B)[1]. The voting schedule is shown below. Which candidate wins under the plurality method?



Note: In the third column and last column, those voters only recorded a first-place vote, so we don't know who their second and third choices would have been.

Answer available at the end of this section.

[1] This data is loosely based on the 2008 County Executive election in Pierce County, Washington. See www.co.pierce.wa.us/xml/abtus...ec/summary.pdf

Answers

Try it Now 1

Using plurality method:

G gets $44 + 14 + 20 = 78$ first-choice votes

M gets $70 + 22 = 92$ first-choice votes

B gets $80 + 39 = 119$ first-choice votes

Bunney (B) wins under plurality method.

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2.0.3: What's Wrong with Plurality?

The election from Example 2 may seem totally clean, but there is a problem lurking that arises whenever there are three or more choices. Looking back at our preference table, how would our members vote if they only had two choices?

Anaheim vs Orlando: 7 out of the 10 would prefer Anaheim over Orlando

	1	3	3	3
1 st choice	A	A	O	H
2 nd choice	O	H	H	A
3 rd choice	H	O	A	O

Anaheim vs Hawaii: 6 out of 10 would prefer Hawaii over Anaheim

	1	3	3	3
1 st choice	A	A	O	H
2 nd choice	O	H	H	A
3 rd choice	H	O	A	O

This doesn't seem right, does it? Anaheim just won the election, yet 6 out of 10 voters, 60% of them, would have preferred Hawaii! That hardly seems fair. Marquis de Condorcet, a French philosopher, mathematician, and political scientist wrote about how this could happen in 1785, and for him we name our first **fairness criterion**.

Fairness Criteria

The fairness criteria are statements that seem like they should be true in a fair election.

Condorcet Criterion

If there is a choice that is preferred in every one-to-one comparison with the other choices, that choice should be the winner. We call this winner the **Condorcet Winner**, or Condorcet Candidate.

Example 3

In the election from Example 2, what choice is the Condorcet Winner?

Solution

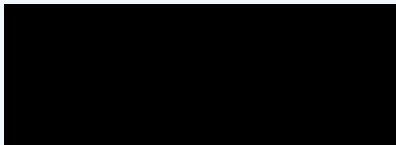
We see above that Hawaii is preferred over Anaheim. Comparing Hawaii to Orlando, we can see 6 out of 10 would prefer Hawaii to Orlando.

	1	3	3	3
1 st choice	A	A	O	H
2 nd choice	O	H	H	A
3 rd choice	H	O	A	O

Since Hawaii is preferred in a one-to-one comparison to both other choices, Hawaii is the Condorcet Winner.

Example 4

Consider a city council election in a district that is historically 60% Democratic voters and 40% Republican voters. Even though city council is technically a nonpartisan office, people generally know the affiliations of the candidates. In this election there are three candidates: Don and Key, both Democrats, and Elle, a Republican. A preference schedule for the votes looks as follows:



Solution

We can see a total of $342 + 214 + 298 = 854$ voters participated in this election. Computing percentage of first place votes:

Don: $214/854 = 25.1\%$

Key: $298/854 = 34.9\%$

Elle: $342/854 = 40.0\%$

So in this election, the Democratic voters split their vote over the two Democratic candidates, allowing the Republican candidate Elle to win under the plurality method with 40% of the vote.

Analyzing this election closer, we see that it violates the Condorcet Criterion. Analyzing the one-to-one comparisons:

Elle vs Don: 342 prefer Elle; 512 prefer Don: Don is preferred

Elle vs Key: 342 prefer Elle; 512 prefer Key: Key is preferred

Don vs Key: 556 prefer Don; 298 prefer Key: Don is preferred

So even though Don had the smallest number of first-place votes in the election, he is the Condorcet winner, being preferred in every one-to-one comparison with the other candidates.

Try it Now 2

Consider the election from Try it Now 1. Is there a Condorcet winner in this election?

Answer available at the end of this section.

Answers

Try it Now 2

Determining the Condorcet Winner:

G vs M: $44 + 14 + 20 = 78$ prefer G, $70 + 22 + 80 = 172$ prefer M: M preferred

G vs B: $44 + 14 + 20 + 70 = 148$ prefer G, $22 + 80 + 39 = 141$ prefer B: G preferred

M vs B: $44 + 70 + 22 = 136$ prefer M, $14 + 80 + 39 = 133$ prefer B: M preferred

M is the Condorcet winner, based on the information we have.

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2.0.4: Insincere Voting

Situations like the one in Example 4 above, when there are more than one candidate that share somewhat similar points of view, can lead to **insincere voting**. Insincere voting is when a person casts a ballot counter to their actual preference for strategic purposes. In the case above, the democratic leadership might realize that Don and Key will split the vote, and encourage voters to vote for Key by officially endorsing him. Not wanting to see their party lose the election, as happened in the scenario above, Don's supporters might insincerely vote for Key, effectively voting against Elle.

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2.0.5: Instant Runoff Voting

Instant Runoff Voting (IRV), also called Plurality with Elimination, is a modification of the plurality method that attempts to address the issue of insincere voting.

Instant Runoff Voting (IRV)

In IRV, voting is done with preference ballots, and a preference schedule is generated. The choice with the *least* first-place votes is then eliminated from the election, and any votes for that candidate are redistributed to the voters' next choice. This continues until a choice has a majority (over 50%).

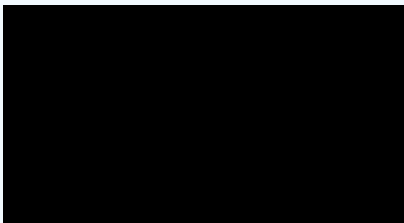
This is similar to the idea of holding runoff elections, but since every voter's order of preference is recorded on the ballot, the runoff can be computed without requiring a second costly election.

This voting method is used in several political elections around the world, including election of members of the Australian House of Representatives, and was used for county positions in Pierce County, Washington until it was eliminated by voters in 2009. A version of IRV is used by the International Olympic Committee to select host nations.

Example 5

Consider the preference schedule below, in which a company's advertising team is voting on five different advertising slogans, called A, B, C, D, and E here for simplicity.

Initial votes

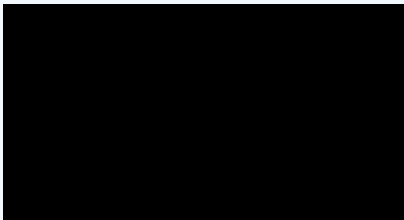


If this was a plurality election, note that B would be the winner with 9 first-choice votes, compared to 6 for D, 4 for C, and 1 for E.

There are total of $3+4+4+6+2+1 = 20$ votes. A majority would be 11 votes. No one yet has a majority, so we proceed to elimination rounds.

Solution

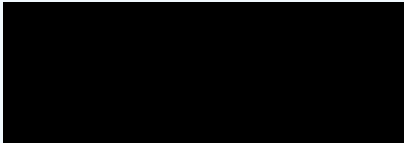
Round 1: We make our first elimination. Choice A has the fewest first-place votes, so we remove that choice



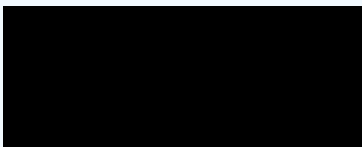
We then shift everyone's choices up to fill the gaps. There is still no choice with a majority, so we eliminate again.



Round 2: We make our second elimination. Choice E has the fewest first-place votes, so we remove that choice, shifting everyone's options to fill the gaps.



Notice that the first and fifth columns have the same preferences now, we can condense those down to one column.



Now B has 9 first-choice votes, C has 4 votes, and D has 7 votes. Still no majority, so we eliminate again.

Round 3: We make our third elimination. C has the fewest votes.



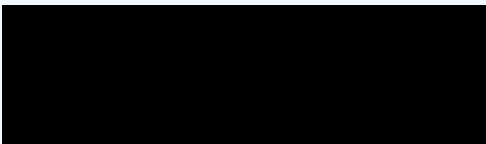
Condensing this down:



D has now gained a majority, and is declared the winner under IRV.

Try it Now 3

Consider again the election from Try it Now 1. Find the winner using IRV.



Answer available at the end of this section.

Answers

Try it Now 3

G has the fewest first-choice votes, so is eliminated first. The 20 voters who did not list a second choice do not get transferred - they simply get eliminated



McCarthy (M) now has a majority, and is declared the winner.

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2.0.6: What's Wrong with IRV?

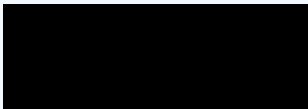
Example 6

Let's return to our City Council Election



Solution

In this election, Don has the smallest number of first place votes, so Don is eliminated in the first round. The 214 people who voted for Don have their votes transferred to their second choice, Key.

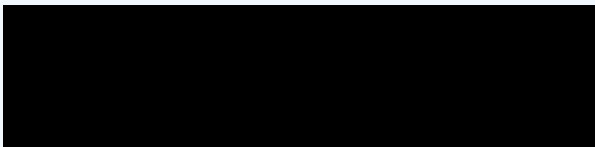


So Key is the winner under the IRV method.

We can immediately notice that in this election, IRV violates the Condorcet Criterion, since we determined earlier that Don was the Condorcet winner. On the other hand, the temptation has been removed for Don's supporters to vote for Key; they now know their vote will be transferred to Key, not simply discarded.

Example 7

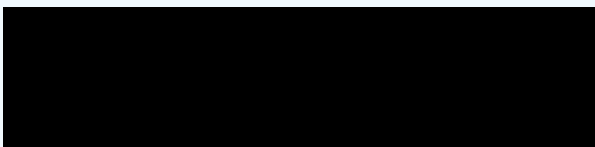
Consider the voting system below.



Solution

In this election, Carter would be eliminated in the first round, and Adams would be the winner with 66 votes to 34 for Brown.

Now suppose that the results were announced, but election officials accidentally destroyed the ballots before they could be certified, and the votes had to be recast. Wanting to "jump on the bandwagon", 10 of the voters who had originally voted in the order Brown, Adams, Carter change their vote to favor the presumed winner, changing those votes to Adams, Brown, Carter.



In this re-vote, Brown will be eliminated in the first round, having the fewest first-place votes. After transferring votes, we find that Carter will win this election with 51 votes to Adams' 49 votes! Even though the only vote changes made *avored* Adams, the change ended up costing Adams the election. This doesn't seem right, and introduces our second fairness criterion:

Monotonicity Criterion

If voters change their votes to increase the preference for a candidate, it should not harm that candidate's chances of winning.

This criterion is violated by this election. Note that even though the criterion is violated in this particular election, it does not mean that IRV always violates the criterion; just that IRV has the potential to violate the criterion in certain elections.

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2.0.7: Borda Count

Borda Count is another voting method, named for Jean-Charles de Borda, who developed the system in 1770.

Borda Count

In this method, points are assigned to candidates based on their ranking; 1 point for last choice, 2 points for second-to-last choice, and so on. The point values for all ballots are totaled, and the candidate with the largest point total is the winner.

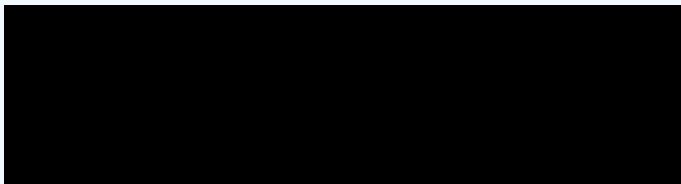
Example 8



A group of mathematicians are getting together for a conference.

The members are coming from four cities: Seattle, Tacoma, Puyallup, and Olympia. Their approximate locations on a map are shown to the right.

The votes for where to hold the conference were:



Solution

In each of the 51 ballots ranking Seattle first, Puyallup will be given 1 point, Olympia 2 points, Tacoma 3 points, and Seattle 4 points. Multiplying the points per vote times the number of votes allows us to calculate points awarded:



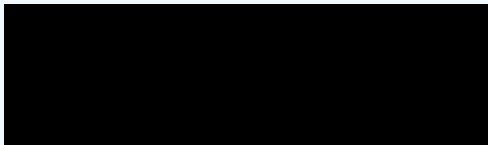
Adding up the points:

- Seattle: $204 + 25 + 10 + 14 = 253$ points
- Tacoma: $153 + 100 + 30 + 42 = 325$ points
- Puyallup: $51 + 75 + 40 + 28 = 194$ points
- Olympia: $102 + 50 + 20 + 56 = 228$ points

Under the Borda Count method, Tacoma is the winner of this vote.

Try it Now 4

Consider again the election from Try it Now 1. Find the winner using Borda Count. Since we have some incomplete preference ballots, for simplicity, give every unranked candidate 1 point, the points they would normally get for last place.



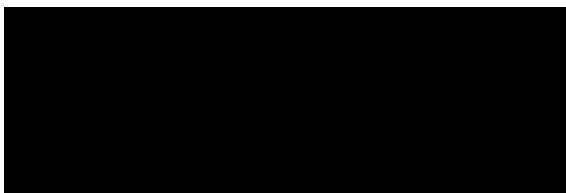
Answer available at the end of this section.

Answers

Try it Now 4

Using Borda Count:

We give 1 point for 3rd place, 2 points for 2nd place, and 3 points for 1st place.



G: $132 + 42 + 60 + 140 + 22 + 80 + 39 = 515$ pts

M: $88 + 14 + 20 + 210 + 66 + 160 + 39 = 597$ pts

B: $44 + 28 + 20 + 70 + 44 + 240 + 117 = 563$ pts

McCarthy (M) would be the winner using Borda Count.

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2.0.8: What's Wrong with Borda Count?

You might have already noticed one potential flaw of the Borda Count from the previous example. In that example, Seattle had a majority of first-choice votes, yet lost the election! This seems odd, and prompts our next fairness criterion:

Majority Criterion

If a choice has a majority of first-place votes, that choice should be the winner.

The election from Example 8 using the Borda Count violates the Majority Criterion. Notice also that this automatically means that the Condorcet Criterion will also be violated, as Seattle would have been preferred by 51% of voters in any head-to-head comparison.

Borda count is sometimes described as a consensus-based voting system, since it can sometimes choose a more broadly acceptable option over the one with majority support. In the example above, Tacoma is probably the best compromise location. This is a different approach than plurality and instant runoff voting that focus on first-choice votes; Borda Count considers every voter's entire ranking to determine the outcome.

Because of this consensus behavior, Borda Count, or some variation of it, is commonly used in awarding sports awards. Variations are used to determine the Most Valuable Player in baseball, to rank teams in NCAA sports, and to award the Heisman trophy.

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2.0.9: Copeland's Method (Pairwise Comparisons)

So far none of our voting methods have satisfied the Condorcet Criterion. The Copeland Method specifically attempts to satisfy the Condorcet Criterion by looking at pairwise (one-to-one) comparisons.

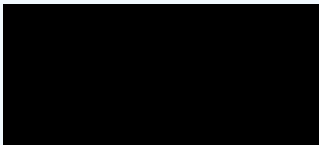
Copeland's Method

In this method, each pair of candidates is compared, using all preferences to determine which of the two is more preferred. The more preferred candidate is awarded 1 point. If there is a tie, each candidate is awarded $\frac{1}{2}$ point. After all pairwise comparisons are made, the candidate with the most points, and hence the most pairwise wins, is declared the winner.

Variations of Copeland's Method are used in many professional organizations, including election of the Board of Trustees for the Wikimedia Foundation that runs Wikipedia.

Example 9

Consider our vacation group example from the beginning of the chapter. Determine the winner using Copeland's Method.



Solution

We need to look at each pair of choices, and see which choice would win in a one-to-one comparison. You may recall we did this earlier when determining the Condorcet Winner. For example, comparing Hawaii vs Orlando, we see that 6 voters, those shaded below in the first table below, would prefer Hawaii to Orlando. Note that Hawaii doesn't have to be the voter's first choice – we're imagining that Anaheim wasn't an option. If it helps, you can imagine removing Anaheim, as in the second table below.

	1	3	3	3
1 st choice	A	A	O	H
2 nd choice	O	H	H	A
3 rd choice	H	O	A	O

	1	3	3	3
1 st choice			O	H
2 nd choice	O	H	H	
3 rd choice	H	O		O

Based on this, in the comparison of Hawaii vs Orlando, Hawaii wins, and receives 1 point.

Comparing Anaheim to Orlando, the 1 voter in the first column clearly prefers Anaheim, as do the 3 voters in the second column. The 3 voters in the third column clearly prefer Orlando. The 3 voters in the last column prefer Hawaii as their first choice, but if they had to choose between Anaheim and Orlando, they'd choose Anaheim, their second choice overall. So, altogether 1+3+3=7 voters prefer Anaheim over Orlando, and 3 prefer Orlando over Anaheim. So, comparing Anaheim vs Orlando: 7 votes to 3 votes: Anaheim gets 1 point.

All together,

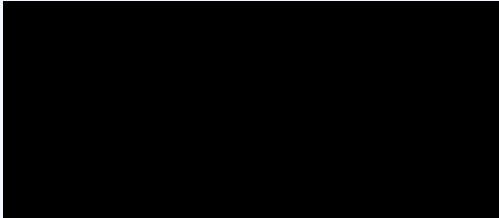
- Hawaii vs Orlando: 6 votes to 4 votes: Hawaii gets 1 point
- Anaheim vs Orlando: 7 votes to 3 votes: Anaheim gets 1 point
- Hawaii vs Anaheim: 6 votes to 4 votes: Hawaii gets 1 point

Hawaii is the winner under Copeland's Method, having earned the most points.

Notice this process is consistent with our determination of a Condorcet Winner.

Example 10

Consider the advertising group's vote we explored earlier. Determine the winner using Copeland's method.



Solution

With 5 candidates, there are 10 comparisons to make:

A vs B: 11 votes to 9 votes	A gets 1 point
A vs C: 3 votes to 17 votes	C gets 1 point
A vs D: 10 votes to 10 votes	A gets $\frac{1}{2}$ point, D gets $\frac{1}{2}$ point
A vs E: 17 votes to 3 votes	A gets 1 point
B vs C: 10 votes to 10 votes	B gets $\frac{1}{2}$ point, C gets $\frac{1}{2}$ point
B vs D: 9 votes to 11 votes	D gets 1 point
B vs E: 13 votes to 7 votes	B gets 1 point
C vs D: 9 votes to 11 votes	D gets 1 point
C vs E: 17 votes to 3 votes	C gets 1 point
D vs E: 17 votes to 3 votes	D gets 1 point

Totaling these up:

A gets $2\frac{1}{2}$ points

B gets $1\frac{1}{2}$ points

C gets $2\frac{1}{2}$ points

D gets $3\frac{1}{2}$ points

E gets 0 points

Using Copeland's Method, we declare D as the winner.

Notice that in this case, D is not a Condorcet Winner. While Copeland's method will also select a Condorcet Candidate as the winner, the method still works in cases where there is no Condorcet Winner.

Try it Now 5

Consider again the election from Try it Now 1. Find the winner using Copeland's method. Since we have some incomplete preference ballots, we'll have to adjust. For example, when comparing M to B, we'll ignore the 20 votes in the third column which do not rank either candidate.



Answer available at the end of this section.

Answers

Try it Now 5

Using Copeland's Method:

Looking back at our work from Try it Now #2, we see

G vs M: $44 + 14 + 20 = 78$ prefer G, $70 + 22 + 80 = 172$ prefer M: M preferred – 1 point

G vs B: $44 + 14 + 20 + 70 = 148$ prefer G, $22 + 80 + 39 = 141$ prefer B: G preferred – 1 point

M vs B: $44 + 70 + 22 = 136$ prefer M, $14 + 80 + 39 = 133$ prefer B: M preferred – 1 point

M earns 2 points; G earns 1 point. M wins under Copeland's method.

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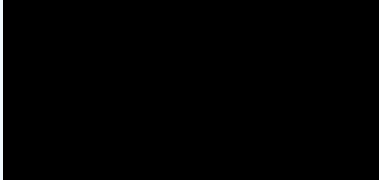
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2.0.10: What's Wrong with Copeland's Method?

As already noted, Copeland's Method does satisfy the Condorcet Criterion. It also satisfies the Majority Criterion and the Monotonicity Criterion. So is this the perfect method? Well, in a word, no.

Example 11

A committee is trying to award a scholarship to one of four students, Anna (A), Brian (B), Carlos (C), and Dimitry (D). The votes are shown below:



Solution

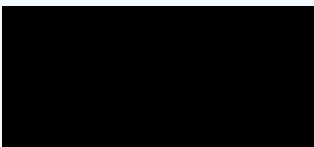
Making the comparisons:

A vs B: 10 votes to 10 votes A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
 A vs C: 14 votes to 6 votes: A gets 1 point
 A vs D: 5 votes to 15 votes: D gets 1 point
 B vs C: 4 votes to 16 votes: C gets 1 point
 B vs D: 15 votes to 5 votes: B gets 1 point
 C vs D: 11 votes to 9 votes: C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points B has $1\frac{1}{2}$ points
 C has 2 points D has 1 point

So Carlos is awarded the scholarship. However, the committee then discovers that Dimitry was not eligible for the scholarship (he failed his last math class). Even though this seems like it shouldn't affect the outcome, the committee decides to recount the vote, removing Dimitry from consideration. This reduces the preference schedule to:



A vs B: 10 votes to 10 votes A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
 A vs C: 14 votes to 6 votes A gets 1 point
 B vs C: 4 votes to 16 votes C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points B has $\frac{1}{2}$ point
 C has 1 point

Suddenly Anna is the winner! This leads us to another fairness criterion.

The Independence of Irrelevant Alternatives (IIA) Criterion

If a non-winning choice is removed from the ballot, it should not change the winner of the election.

Equivalently, if choice A is preferred over choice B, introducing or removing a choice C should not cause B to be preferred over A.

In the election from Example 11, the IIA Criterion was violated.

This anecdote illustrating the IIA issue is attributed to Sidney Morgenbesser:

After finishing dinner, Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point Morgenbesser says "In that case I'll have the blueberry pie."

Another disadvantage of Copeland's Method is that it is fairly easy for the election to end in a tie. For this reason, Copeland's method is usually the first part of a more advanced method that uses more sophisticated methods for breaking ties and determining the winner when there is not a Condorcet Candidate.

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2.0.11: So Where's the Fair Method?

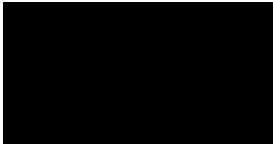
At this point, you're probably asking why we keep looking at method after method just to point out that they are not fully fair. We must be holding out on the perfect method, right?

Unfortunately, no. A mathematical economist, Kenneth Arrow, was able to prove in 1949 that there is no voting method that will satisfy all the fairness criteria we have discussed.

Arrow's Impossibility Theorem

Arrow's Impossibility Theorem states, roughly, that it is not possible for a voting method to satisfy every fairness criteria that we've discussed.

To see a very simple example of how difficult voting can be, consider the election below:



Notice that in this election:

- 10 people prefer A to B
- 10 people prefer B to C
- 10 people prefer C to A

No matter whom we choose as the winner, 2/3 of voters would prefer someone else! This scenario is dubbed **Condorcet's Voting Paradox**, and demonstrates how voting preferences are not transitive (just because A is preferred over B, and B over C, does not mean A is preferred over C). In this election, there is no fair resolution.

It is because of this impossibility of a totally fair method that Plurality, IRV, Borda Count, Copeland's Method, and dozens of variants are all still used. Usually the decision of which method to use is based on what seems most fair for the situation in which it is being applied.

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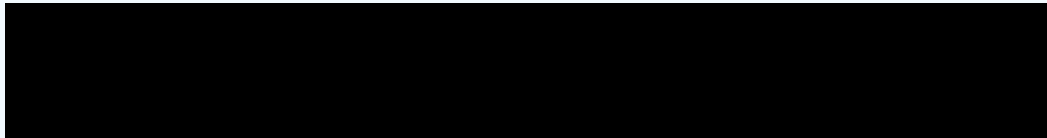
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2.0.12: Approval Voting

Up until now, we've been considering voting methods that require ranking of candidates on a preference ballot. There is another method of voting that can be more appropriate in some decision making scenarios. With Approval Voting, the ballot asks you to mark all choices that you find acceptable. The results are tallied, and the option with the most approval is the winner.

Example 12

A group of friends is trying to decide upon a movie to watch. Three choices are provided, and each person is asked to mark with an "X" which movies they are willing to watch. The results are:



Solution

Totaling the results, we find

Titanic received 5 approvals

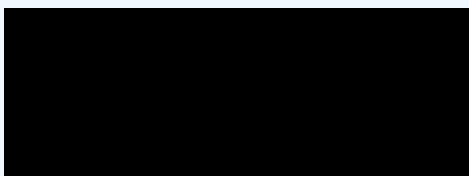
Scream received 6 approvals

The Matrix received 7 approvals.

In this vote, The Matrix would be the winner.

Try it Now 6

Our mathematicians deciding on a conference location from earlier decide to use Approval voting. Their votes are tallied below. Find the winner using Approval voting.



Answer available at the end of this section.

Answers

Try it Now 6

Using Approval voting:

Seattle has $30 + 10 + 15 + 5 = 60$ approval votes

Tacoma has $30 + 15 + 20 + 15 + 5 = 85$ approval votes

Puyallup has $10 + 20 + 25 + 5 = 50$ approval votes

Olympia has $15 + 15 + 5 = 35$ approval votes

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2.0.13: What's Wrong with Approval Voting?

Approval voting can very easily violate the Majority Criterion.

Example 13

Consider the voting schedule:



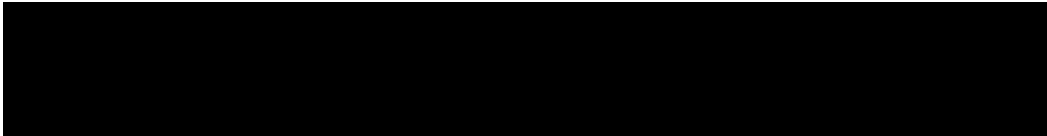
Solution

Clearly A is the majority winner. Now suppose that this election was held using Approval Voting, and every voter marked approval of their top two candidates.

- A would receive approval from 80 voters
- B would receive approval from 100 voters
- C would receive approval from 20 voters

B would be the winner. Some argue that Approval Voting tends to vote the least disliked choice, rather than the most liked candidate.

Additionally, Approval Voting is susceptible to strategic insincere voting, in which a voter does not vote their true preference to try to increase the chances of their choice winning. For example, in the movie example above, suppose Bob and Alice would much rather watch Scream. They remove The Matrix from their approval list, resulting in a different result.



Totaling the results, we find Titanic received 5 approvals, Scream received 6 approvals, and The Matrix received 5 approvals. By voting insincerely, Bob and Alice were able to sway the result in favor of their preference.

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2.0.14: Voting in America

In American politics, there is a lot more to selecting our representatives than simply casting and counting ballots. The process of selecting the president is even more complicated, so we'll save that for the next chapter. Instead, let's look at the process by which state congressional representatives and local politicians get elected.

For most offices, a sequence of two public votes is held: a primary election and the general election. For non-partisan offices like sheriff and judge, in which political party affiliation is not declared, the primary election is usually used to narrow the field of candidates.

Typically, the two candidates receiving the most votes in the primary will then move forward to the general election. While somewhat similar to instant runoff voting, this is actually an example of **sequential voting** - a process in which voters cast totally new ballots after each round of eliminations. Sequential voting has become quite common in television, where it is used in reality competition shows like American Idol.

Congressional, county, and city representatives are partisan offices, in which candidates usually declare themselves a member of a political party, like the Democrats, Republicans, the Green Party, or one of the many other smaller parties. As with non-partisan offices, a primary election is usually held to narrow down the field prior to the general election. Prior to the primary election, the candidate would have met with the political party leaders and gotten their approval to run under that party's affiliation.

In some states a **closed primary** is used, in which only voters who are members of the Democrat party can vote on the Democratic candidates, and similar for Republican voters. In other states, an **open primary** is used, in which any voter can pick the party whose primary they want to vote in. In other states, **caucuses** are used, which are basically meetings of the political parties, only open to party members. Closed primaries are often disliked by independent voters, who like the flexibility to change which party they are voting in. Open primaries do have the disadvantage that they allow raiding, in which a voter will vote in their non-preferred party's primary with the intent of selecting a weaker opponent for their preferred party's candidate.

Washington State currently uses a different method, called a **top 2 primary**, in which voters select from the candidates from all political parties on the primary, and the top two candidates, regardless of party affiliation, move on to the general election. While this method is liked by independent voters, it gives the political parties incentive to select a top candidate internally before the primary, so that two candidates will not split the party's vote.

Regardless of the primary type, the general election is the main election, open to all voters. Except in the case of the top 2 primary, the top candidate from each major political party would be included in the general election. While rules vary state-to-state, for an independent or minor party candidate to get listed on the ballot, they typically have to gather a certain number of signatures to petition for inclusion.

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2.0.15: Exercises

1. To decide on a new website design, the designer asks people to rank three designs that have been created (labeled A, B, and C). The individual ballots are shown below. Create a preference table.

ABC, ABC, ACB, BAC, BCA, BCA, ACB, CAB, CAB, BCA, ACB, ABC

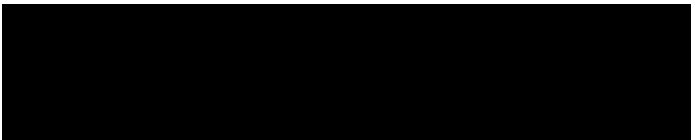
2. To decide on a movie to watch, a group of friends all vote for one of the choices (labeled A, B, and C). The individual ballots are shown below. Create a preference table.

CAB, CBA, BAC, BCA, CBA, ABC, ABC, CBA, BCA, CAB, CAB, BAC

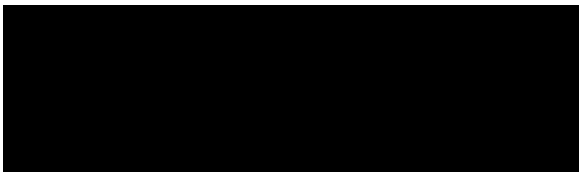
3. The planning committee for a renewable energy trade show is trying to decide what city to hold their next show in. The votes are shown below.



- a. How many voters voted in this election?
 - b. How many votes are needed for a majority? A plurality?
 - c. Find the winner under the plurality method.
 - d. Find the winner under the Borda Count Method.
 - e. Find the winner under the Instant Runoff Voting method.
 - f. Find the winner under Copeland's method.
4. A non-profit agency is electing a new chair of the board. The votes are shown below.



- a. How many voters voted in this election?
 - b. How many votes are needed for a majority? A plurality?
 - c. Find the winner under the plurality method.
 - d. Find the winner under the Borda Count Method.
 - e. Find the winner under the Instant Runoff Voting method.
 - f. Find the winner under Copeland's method.
5. The student government is holding elections for president. There are four candidates (labeled A, B, C, and D for convenience). The preference schedule for the election is:

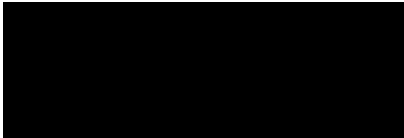


- a. How many voters voted in this election?
- b. How many votes are needed for a majority? A plurality?
- c. Find the winner under the plurality method.
- d. Find the winner under the Borda Count Method.
- e. Find the winner under the Instant Runoff Voting method.

- f. Find the winner under Copeland's method.
6. The homeowners association is deciding a new set of neighborhood standards for architecture, yard maintenance, etc. Four options have been proposed. The votes are:



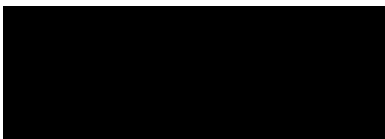
- a. How many voters voted in this election?
- b. How many votes are needed for a majority? A plurality?
- c. Find the winner under the plurality method.
- d. Find the winner under the Borda Count Method.
- e. Find the winner under the Instant Runoff Voting method.
- f. Find the winner under Copeland's method.
7. Consider an election with 129 votes.
 - a. If there are 4 candidates, what is the smallest number of votes that a plurality candidate could have?
 - b. If there are 8 candidates, what is the smallest number of votes that a plurality candidate could have?
8. Consider an election with 953 votes.
 - a. If there are 7 candidates, what is the smallest number of votes that a plurality candidate could have?
 - b. If there are 8 candidates, what is the smallest number of votes that a plurality candidate could have?
9. Does this voting system having a Condorcet Candidate? If so, find it.



10. Does this voting system having a Condorcet Candidate? If so, find it.

Number of voters	8	7	6
1st choice	A	C	B
2nd choice	B	B	C
3rd choice	C	A	A

11. The marketing committee at a company decides to vote on a new company logo. They decide to use approval voting. Their results are tallied below. Each column shows the number of voters with the particular approval vote. Which logo wins under approval voting?



12. The downtown business association is electing a new chairperson, and decides to use approval voting. The tally is below, where each column shows the number of voters with the particular approval vote. Which candidate wins under approval voting?



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2.0.16: Concepts

1. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If for some reason the election had to be held again and C decided to drop out of the election, which caused B to become the winner, which is the primary fairness criterion violated in this election?
2. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If for some reason the election had to be held again and many people who had voted for C switched their preferences to favor A, which caused B to become the winner, which is the primary fairness criterion violated in this election?
3. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If in a head-to-head comparison a majority of people prefer B to A or C, which is the primary fairness criterion violated in this election?
4. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If B had received a majority of first place votes, which is the primary fairness criterion violated in this election?

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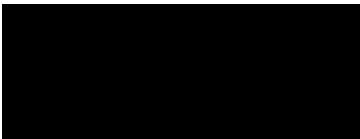
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2.0.17: Exploration

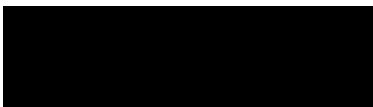
1. In the election shown below under the Plurality method, explain why voters in the third column might be inclined to vote insincerely. How could it affect the outcome of the election?



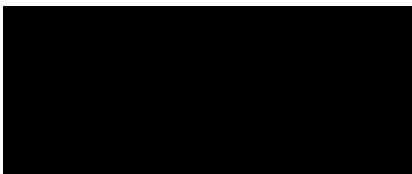
2. In the election shown below under the Borda Count method, explain why voters in the second column might be inclined to vote insincerely. How could it affect the outcome of the election?



3. Compare and contrast the motives of the insincere voters in the two questions above.
4. Consider a two party election with preferences shown below. Suppose a third candidate, C, entered the race, and a segment of voters sincerely voted for that third candidate, producing the preference schedule from #17 above. Explain how other voters might perceive candidate C.



5. In question 18, we showed that the outcome of Borda Count can be manipulated if a group of individuals change their vote.
 - a. Show that it is possible for a single voter to change the outcome under Borda Count if there are four candidates.
 - b. Show that it is not possible for a single voter to change the outcome under Borda Count if there are three candidates.
6. Show that when there is a Condorcet winner in an election, it is impossible for a single voter to manipulate the vote to help a different candidate become a Condorcet winner.
7. The Pareto criterion is another fairness criterion that states: *If every voter prefers choice A to choice B, then B should not be the winner.* Explain why plurality, instant runoff, Borda count, and Copeland's method all satisfy the Pareto condition.
8. Sequential Pairwise voting is a method not commonly used for political elections, but sometimes used for shopping and games of pool. In this method, the choices are assigned an order of comparison, called an agenda. The first two choices are compared. The winner is then compared to the next choice on the agenda, and this continues until all choices have been compared against the winner of the previous comparison.
 - a. Using the preference schedule below, apply Sequential Pairwise voting to determine the winner, using the agenda: A, B, C, D.



- b. Show that Sequential Pairwise voting can violate the Pareto criterion.
 - c. Show that Sequential Pairwise voting can violate the Majority criterion.
9. The Coombs method is a variation of instant runoff voting. In Coombs method, the choice with the most last place votes is

eliminated. Apply Coombs method to the preference schedules from questions 5 and 6.

10. Copeland's Method is designed to identify a Condorcet Candidate if there is one, and is considered a Condorcet Method. There are many Condorcet Methods, which vary primarily in how they deal with ties, which are very common when a Condorcet winner does not exist. Copeland's method does not have a tie-breaking procedure built-in. Research the Schulze method, another Condorcet method that is used by the Wikimedia foundation that runs Wikipedia, and give some examples of how it works.
11. The plurality method is used in most U.S. elections. Some people feel that Ross Perot in 1992 and Ralph Nader in 2000 changed what the outcome of the election would have been if they had not run. Research the outcomes of these elections and explain how each candidate could have affected the outcome of the elections (for the 2000 election, you may wish to focus on the count in Florida). Describe how an alternative voting method could have avoided this issue.
12. Instant Runoff Voting and Approval voting have supporters advocating that they be adopted in the United States and elsewhere to decide elections. Research comparisons between the two methods describing the advantages and disadvantages of each in practice. Summarize the comparisons, and form your own opinion about whether either method should be adopted.
13. In a primary system, a first vote is held with multiple candidates. In some states, each political party has its own primary. In Washington State, there is a "top two" primary, where all candidates are on the ballot and the top two candidates advance to the general election, regardless of party. Compare and contrast the top two primary with general election system to instant runoff voting, considering both differences in the methods, and practical differences like cost, campaigning, fairness, etc.
14. In a primary system, a first vote is held with multiple candidates. In some many states, where voters must declare a party to vote in the primary election, and they are only able to choose between candidates for their declared party. The top candidate from each party then advances to the general election. Compare and contrast this primary with general election system to instant runoff voting, considering both differences in the methods, and practical differences like cost, campaigning, fairness, etc.
15. Sometimes in a voting scenario it is desirable to rank the candidates, either to establish preference order between a set of choices, or because the election requires multiple winners. For example, a hiring committee may have 30 candidates apply, and need to select 6 to interview, so the voting by the committee would need to produce the top 6 candidates. Describe how Plurality, Instant Runoff Voting, Borda Count, and Copeland's Method could be extended to produce a ranked list of candidates.

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CHAPTER 3: SET THEORY

3.0: Sets

3.0.0: Basics of Sets

3.0.1: Union, Intersection, and Complement

3.0.2: Venn Diagrams

3.0.3: Cardinality

3.0.4: Exercises

3.0: SETS

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

3.0.0: Basics of Sets

3.0.1: Union, Intersection, and Complement

3.0.2: Venn Diagrams

3.0.3: Cardinality

3.0.4: Exercises

via Wikipedia).

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3.0.0: Basics of Sets

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:

- The set of all even numbers
- The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:

- $\{1, 3, 9, 12\}$
- $\{\text{red, orange, yellow, green, blue, indigo, purple}\}$

A set simply specifies the contents; order is not important. The set represented by $\{1, 2, 3\}$ is equivalent to the set $\{3, 1, 2\}$.

Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol $\in$ means “is an element of”.

A set that contains no elements, $\{\}$, is called the **empty set** and is notated $\emptyset$

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A , but may not contain all the elements of A .

If B is a subset of A , we write $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.

If B is a proper subset of A , we write $B \subset A$

Example 3

Consider these three sets

A = the set of all even numbers $B = \{2, 4, 6\}$ $C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so is an element of A .

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$

It is also true that $B \subset C$.

C is not a subset of A , since C contains an element, 3, that is not contained in A

Example 4

Suppose a set contains the plays “Much Ado About Nothing”, “MacBeth”, and “A Midsummer’s Night Dream”. What is a larger set this might be a subset of?

Solution

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try it Now 1

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Answer available at the end of this section.

Answers

Try it Now 1

There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

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3.0.1: Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets).

The union is notated $A \cup B$

More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets.

The intersection is notated $A \cap B$

More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$

The **complement** of a set A contains everything that is *not* in the set A .

The complement is notated A , or A^c , or sometimes $\sim A$.

Example 5

Consider the sets:

$$A = \{ \text{red, green, blue} \} \quad B = \{ \text{red, yellow, orange} \} \quad C = \{ \text{red, orange, yellow, green, blue, purple} \}$$

- Find $A \cup B$
- Find $A \cap B$
- Find $A^c \cap C$

Solution

a) The union contains all the elements in either set: $A \cup B = \{ \text{red, green, blue, yellow, orange} \}$

Notice we only list red once.

b) The intersection contains all the elements in both sets: $A \cap B = \{ \text{red} \}$

c) Here we're looking for all the elements that are not in set A and are also in C .

$$A^c \cap C = \{ \text{orange, yellow, purple} \}$$

Try it Now 2

Using the sets from the previous example, find $A \cup C$ and $B^c \cap A$

Answer available at the end of this section.

Notice that in the example above, it would be hard to just ask for A^c , since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections,

or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^C contains all the elements in the universal set that are not in A .

Example 6

- If we were discussing searching for books, the universal set might be all the books in the library.
- If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
- If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7

Suppose the universal set is $U =$ all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then

$$A^c = \{3, 5, 6, 7, 8, 9\}$$

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic - to force an order of operations.

Example 8

Suppose

$$H = \{ \text{cat, dog, rabbit, mouse} \}, F = \{ \text{dog, cow, duck, pig, rabbit} \} \quad W = \{ \text{duck, rabbit, deer, frog, mouse} \}$$

- Find $(H \cap P) \cup W$
- Find $H \cap (F \cup W)$
- Find $(H \cap P) \cap W$

Solution

a) We start with the intersection: $H \cap F = \{ \text{dog, rabbit} \}$

Now we union that result with W : $(H \cap F) \cup W = \{ \text{dog, duck, rabbit, deer, frog, mouse} \}$

b) We start with the union: $F \cup W = \{ \text{dog, cow, rabbit, duck, pig, deer, frog, mouse} \}$

Now we intersect that result with H : $H \cap (F \cup W) = \{ \text{dog, rabbit, mouse} \}$

c) We start with the intersection: $H \cap F = \{ \text{dog, rabbit} \}$

Now we want to find the elements of W that are not in $H \cap F$

$$(H \cap P)^c \cap W = \{ \text{duck, deer, frog, mouse} \}$$

Answers

Try it Now 2

$$A \cup C = \{ \text{red, orange, yellow, green, blue purple} \}$$

$$B^c \cap A = \{ \text{green, blue} \}$$

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3.0.2: Venn Diagrams

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

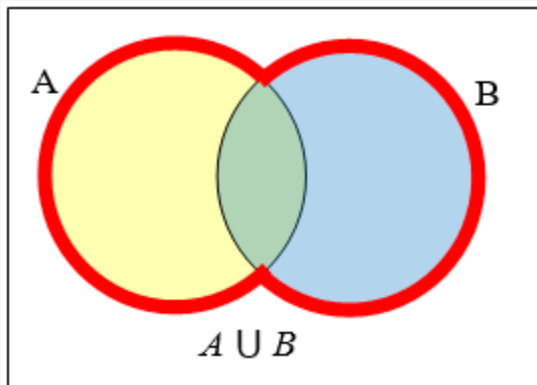
Venn Diagram

A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

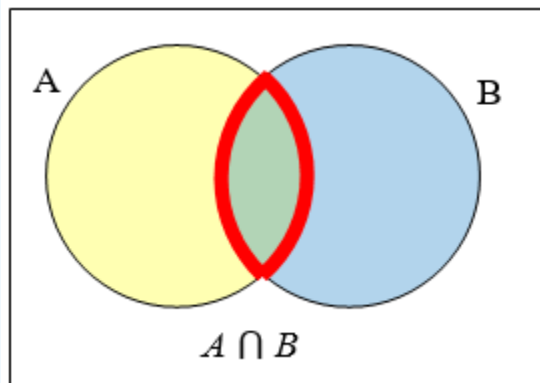
Example 9

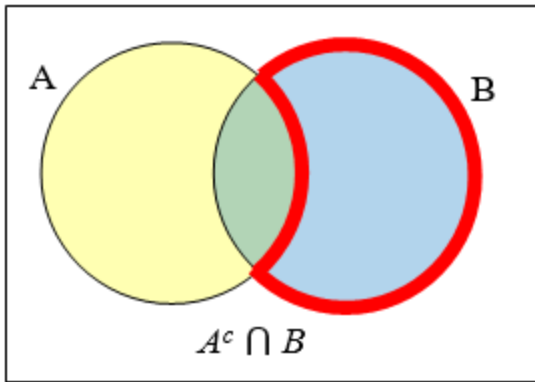
Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$



$A \cup B$ contains all elements in either set.

$A \cup B$ contains all elements in either set.



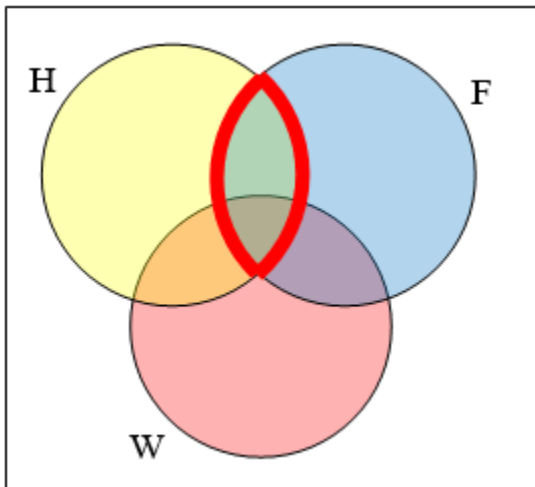


$A \cap B$ contains only those elements in both sets - in the overlap of the circles.

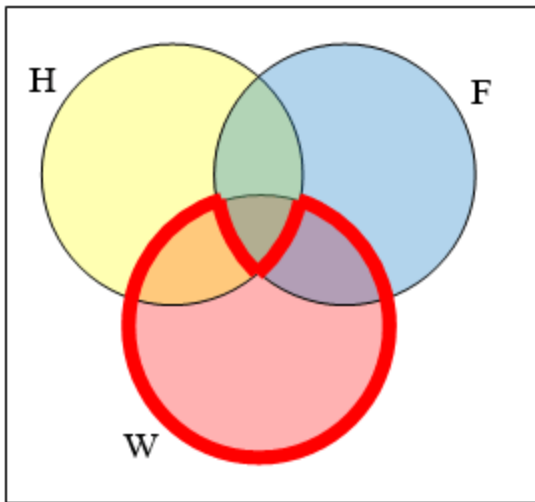
Example 10

Use a Venn diagram to illustrate $(H \cap P)^c \cap W$

We'll start by identifying everything in the set $H \cap P$

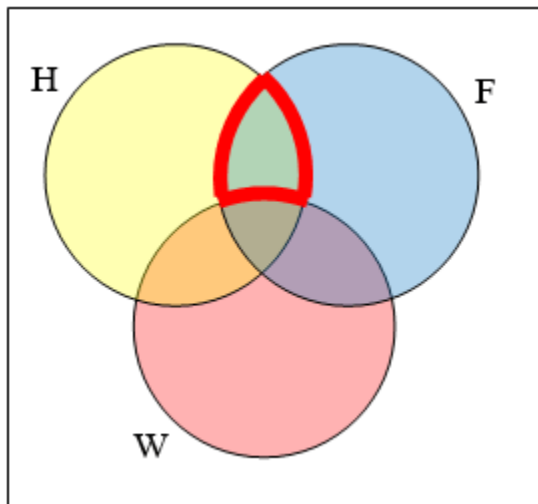


Now, $(H \cap P)^c \cap W$ will contain everything not in the set identified above that is also in set W



Example 11

Create an expression to represent the outlined part of the Venn diagram shown.

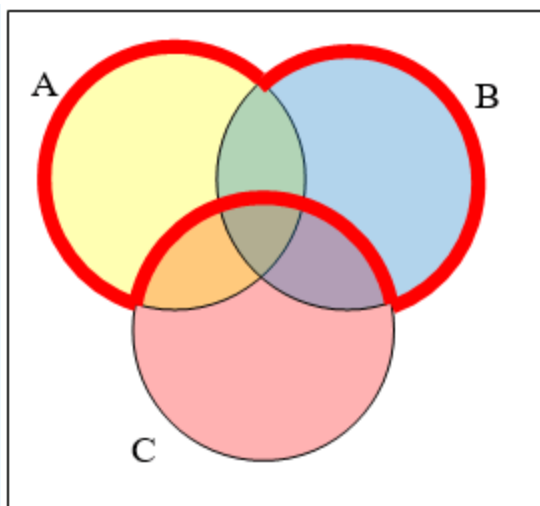


The elements in the outlined set are in sets H and F , but are not in set W

. So we could represent this set as $H \cap F \cap W$

Try it Now 3

Create an expression to represent the outlined portion of the Venn diagram shown



Answer available at the end of this section.

Answers

Try it Now 3

$$A \cup B \cap C^c$$

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3.0.3: Cardinality

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.

The cardinality of the set A is often notated as $|A|$ or $n(A)$

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$

What is the cardinality of B ? $A \cup B$, $A \cap B$?

Solution

The cardinality of B is 4, since there are 4 elements in the set.

The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.

The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

Example 13

What is the cardinality of $P =$ the set of English names for the months of the year?

Solution

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

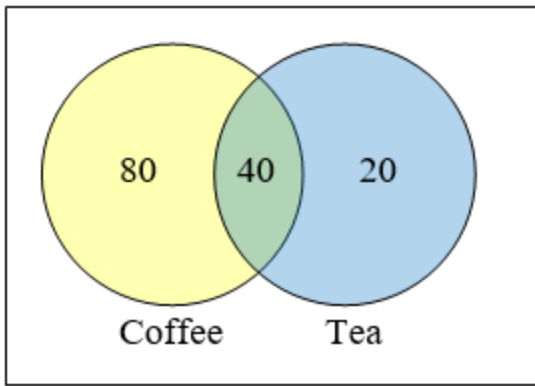
Example 14

A survey asks 200 people “What beverage do you drink in the morning”, and offers choices:

- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

Solution



This question can most easily be answered by creating a Venn diagram.

We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

$$200 - 20 - 80 - 40 = 60 \text{ people who drink neither.}$$

Example 15

A survey asks: Which online services have you used in the last month:

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Solution

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply 70% + 40%, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T , then subtract those in intersection that we've counted twice. In symbols,

$$n(F \cup T) = n(F) + n(T) - n(F \cap T)$$

$$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$. since the universal set contains 100% of people and the cardinality of $F \cup T = 90\%$, the cardinality of $(F \cup T)^c$ must be the other 10%

The previous example illustrated two important properties

Cardinality properties

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A^c) = n(U) - n(A)$$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

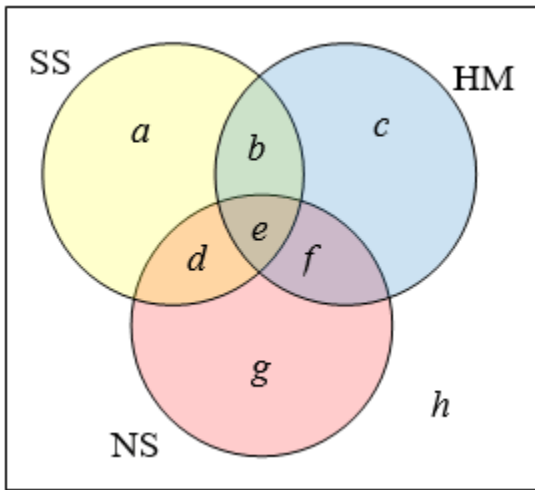
Example 16

Add text here. Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course	26 were taking a HM course
19 were taking a NS course	9 were taking SS and HM
7 were taking SS and NS	10 were taking HM and NS
3 were taking all three	7 were taking none

How many students are only taking a SS course?

Solution



It might help to look at a Venn diagram.

From the given data, we know that there are 3 students in region e and 7 students in region h

since 7 students were taking a SS and NS course, we know that $n(d) + n(e) = 7$. since we know there are 3 students in region 3, there must be

$$7 - 3 = 4 \text{ students in region } d$$

Similarly, since there are 10 students taking HM and NS, which includes regions e and f , there must be

$$10 - 3 = 7 \text{ students in region } f$$

Since 9 students were taking SS and HM, there must be $9 - 3 = 6$ students in region b

Now, we know that 21 students were taking a SS course. This includes students from regions a, b, d , and e . since we know the number of students in all but region a , we can determine that $21 - 6 - 4 - 3 = 8$ students are in region a

8 students are taking only a SS course.

Try it Now 4

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

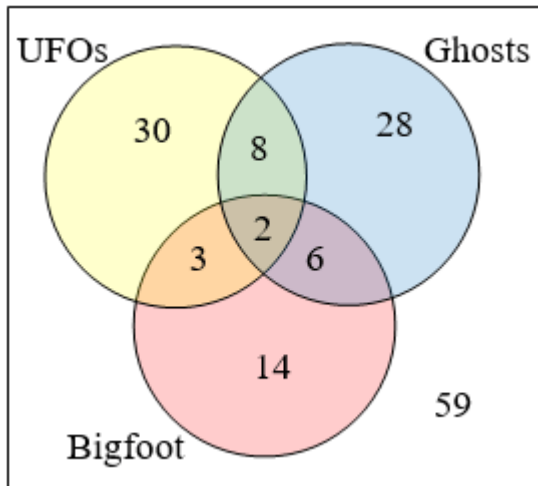
43 believed in UFOs	44 believed in ghosts
25 believed in Bigfoot	10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot	5 believed in UFOs and Bigfoot
2 believed in all three	

How many people surveyed believed in at least one of these things?

Answer available at the end of this section.

Answers

Try it Now 4



Starting with the intersection of all three circles, we work our way out. since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. We work our way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves $150 - 91 = 59$ who believe in none.

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3.0.4: Exercises

1. List out the elements of the set "The letters of the word Mississippi"
2. List out the elements of the set "Months of the year"
3. Write a verbal description of the set $\{3, 6, 9\}$
4. Write a verbal description of the set $\{a, i, e, o, u\}$
5. Is $\{1, 3, 5\}$ a subset of the set of odd integers?
6. Is $\{A, B, C\}$ a subset of the set of letters of the alphabet?

For problems 7-12, consider the sets below, and indicate if each statement is true or false.

$$A = \{1, 2, 3, 4, 5\} \quad B = \{1, 3, 5\} \quad C = \{4, 6\} \quad U = \{\text{numbers from 0 to 10}\}$$

7. $3 \in B$
8. $5 \in C$
9. $B \subset A$
10. $C \subset A$
11. $C \subset B$
12. $C \subset U$

Using the sets from above, and treating U as the Universal set, find each of the following:

13. $A \cup B$
14. $A \cup C$
15. $A \cap C$
16. $B \cap C$
17. A^c
18. B^c

Let $D = \{b, a, c, k\}$, $E = \{t, a, s, k\}$, $F = \{b, a, t, h\}$. Using these sets, find the following:

19. $D^c \cap E$
20. $F^c \cap D$
21. $(D \cap E) \cup F$
22. $D \cap (E \cup P)$
23. $(F \cap E)^c \cap D$
24. $(D \cup E)^c \cap F$

Create a Venn diagram to illustrate each of the following:

25. $(F \cap E) \cup D$

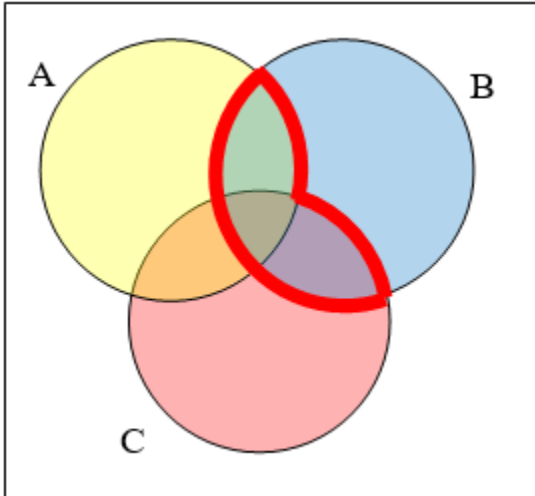
26. $(D \cup E)^c \cap F$

27. $(F^c \cap E') \cap D$

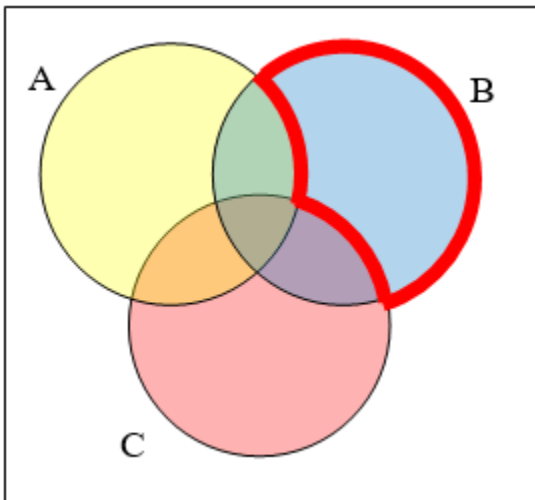
28. $(D \cup E) \cup F$

Write an expression for the shaded region.

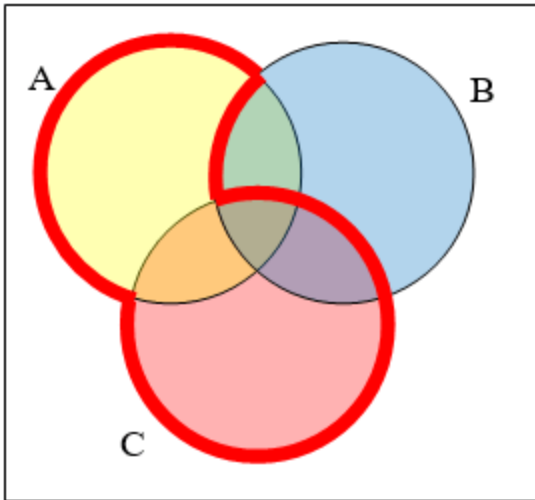
29.



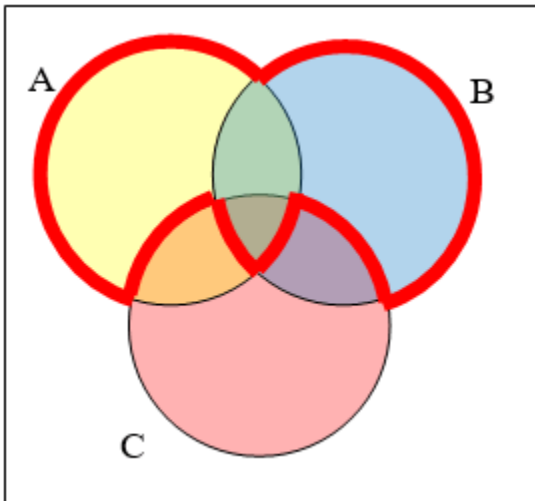
30.



31.



32.



Let $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 3, 5\}$ $C = \{4, 6\}$. Find the cardinality of the given set.

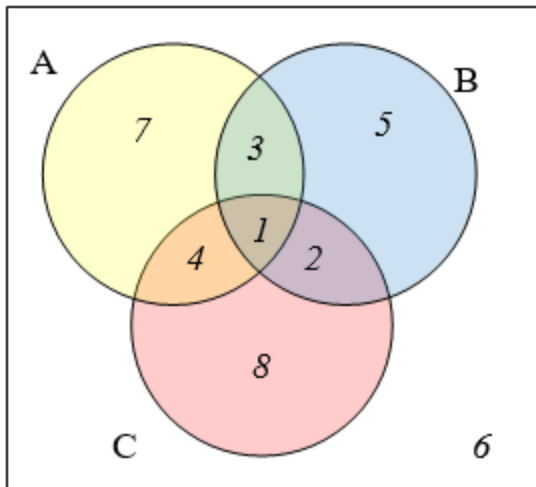
33. $n(A)$

34. $n(B)$

35. $n(A \cup C)$

36. $n(A \cap C)$

The Venn diagram here shows the cardinality of each set. Use this in 37-40 to find the cardinality of given set.



37. $n(A \cap C)$

38. $n(B \cup C)$

39. $n(A \cap B \cap C^c)$

40. $n(A \cap B^c \cap C)$

41. If $n(G) = 20$, $n(H) = 30$, $n(G \cap H) = 5$, find $n(G \cup H)$

42. If $n(G) = 5$, $n(H) = 8$, $n(G \cap H) = 4$, find $n(G \cup H)$

43. A survey was given asking whether they watch movies at home from Netflix, Redbox, or a video store. Use the results to determine how many people use Redbox.

52 only use Netflix	62 only use Redbox
24 only use a video store	16 use only a video store and Redbox
48 use only Netflix and Redbox	30 use only a video store and Netflix
10 use all three	25 use none of these

44. A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?

5 only said color	8 only said size
16 only said brand	20 said only color and size
42 said only color and brand	53 said only size and brand
102 said all three	20 said none of these

45. Use the given information to complete a Venn diagram, then determine: a) how many students have seen exactly one of these movies, and b) how many had seen only *Star Wars*.

18 had seen The Matrix (M)	24 had seen Star Wars (SW)
20 had seen Lord of the Rings (LotR)	10 had seen M and SW
14 had seen LotR and SW	12 had seen M and LotR
6 had seen all three	

46. A survey asked people what alternative transportation modes they use. Using the data to complete a Venn diagram, then determine: a) what percent of people only ride the bus, and b) how many people don't use any alternate transportation.

30 20
25 5
10 12
2

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CHAPTER 4: COUNTING THEORY

4.0: Probability

4.0.0: Counting

4.0: PROBABILITY

4.0.0: Counting

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4.0.0: Counting

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting *efficiently*. When we get to the probability situations a bit later in this chapter we will need to count some *very* large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

Basic Counting

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example 21

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

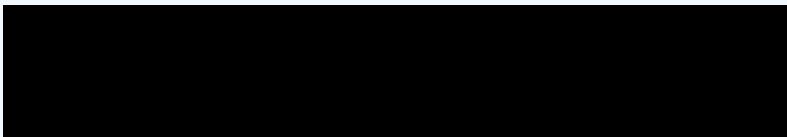
Solution

Solution 1: One way to solve this problem would be to systematically list each possible meal:

soup + hamburger	soup + sandwich	soup + quiche
soup + fajita	soup + pizza	salad + hamburger
salad + sandwich	salad + quiche	salad + fajita
salad + pizza	breadsticks + hamburger	breadsticks + sandwich
breadsticks + quiche	breadsticks + fajita	breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

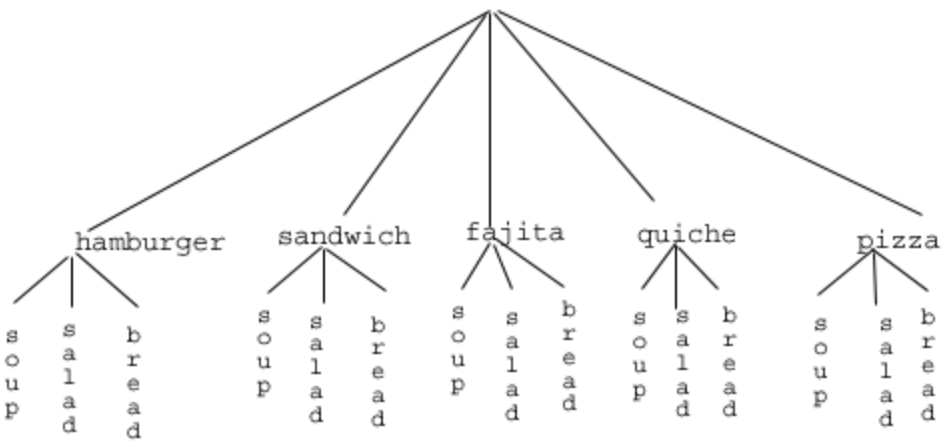
Solution 2: Another way to solve this problem would be to list all the possibilities in a table:



In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care *what* the possible meals are, only *how many* possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

Solution 3: We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose from instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a **tree diagram**:



This is called a "tree" diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5) to get 15. Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the *basic counting rule*:

Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is $m \cdot n$.

This is sometimes called the multiplication rule for probabilities.

Example 22

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

Solution

There are 21 choices from the first category and 18 for the second, so there are $21 \cdot 18 = 378$ possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example 23

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution

There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are $3 \cdot 5 \cdot 2 = 30$ possibilities.

Example 24

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

Solution

There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in $2 \cdot 2 \cdot 2 = 8$ different ways. Recall that another way to write $2 \cdot 2 \cdot 2$ is 2^3 which is much more compact.

Try it Now 6

Suppose at a particular restaurant you have eight choices for an appetizer, eleven choices for a main course and five choices for dessert. If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Answer available at the end of this section.

Permutations

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

Example 25

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

Solution

This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is: the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to spell a code word with the letters MATH.

In this example, we needed to calculate $n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$. This calculation shows up often in mathematics, and is called the **factorial**, and is notated $n!$

Factorial

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$$

Example 26

How many ways can five different door prizes be distributed among five people?

Solution

There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

Now we will consider some slightly different examples.

Example 27

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of \$100, the second is worth \$25 and the third is worth \$10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Solution

Using the Basic Counting Rule, there are 25 choices for the person who receives the \$100 certificate, 24 remaining choices for the \$25 certificate and 23 choices for the \$10 certificate, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways in which the prizes can be awarded.

Example 28

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Solution

Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are $8 \cdot 7 \cdot 6 = 336$ ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the \$100 prize, or a different gold medalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a \$10 gift certificate; in this case the order of selection is *not* important since each of the three people receive the same prize. Situations where the order is *not* important will be discussed in the next section.)

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order r items out of n possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by

$$n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$

If you don't see why $(n - r + 1)$ is the right number to use for the last factor, just think back to the first example in this section, where we calculated $25 \cdot 24 \cdot 23$ to get 13,800. In this case $n = 25$ and $r = 3$, so $n - r + 1 = 25 - 3 + 1 = 23$, which is exactly the right number for the final factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often, we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* and where the *order of selection is important*. In this situation we write:

Permutations

$${}_nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

We say that there are ${}_nP_r$ **permutations** of size r that may be selected from among n choices *without replacement* when *order matters*.

It turns out that we can express this result more simply using factorials.

$${}_nP_r = \frac{n!}{(n-r)!}$$

In practicality, we usually use technology rather than factorials or repeated multiplication to compute permutations.

Example 29

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Solution

Since we are choosing 4 paintings out of 9 *without replacement* where the *order of selection is important* there are ${}_9P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024$ permutations.

Example 30

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

Solution

We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is ${}_{16}P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43,680$.

Try it Now 7

How many 5 character passwords can be made using the letters A through Z

- if repeats are allowed
- if no repeats are allowed

Answer available at the end of this section.

Combinations

In the previous section we considered the situation where we chose r items out of n possibilities *without replacement* and where the *order of selection was important*. We now consider a similar situation in which the order of selection is *not* important.

Example 31

A charity benefit is attended by 25 people at which three \$50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Solution

Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get \$50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

ABC ACB BAC BCA CAB CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are $3 \cdot 2 \cdot 1 = 6$ ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be $\frac{13,800}{6} = 2300$.

We can generalize the situation in this example above to any problem of choosing a collection of items *without replacement* where the *order of selection is not important*. If we are choosing r items out of n possibilities (instead of 3 out of 25 as in the previous examples), the number of possible choices will be given by , and we could use this formula for computation. However this situation arises so frequently that we attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* where the *order of selection is not important*.

Combinations

The ideal gas law is easy to remember and apply in solving problems, as long as you get the **proper values a**

$${}_nC_r = \frac{{}_nP_r}{r!}$$

We say that there are ${}_nC_r$ **combinations** of size r that may be selected from among n choices *without replacement* where *order doesn't matter*.

We can also write the combinations formula in terms of factorials:

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Example 32

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Solution

Since we are choosing 4 people out of 35 *without replacement* where the *order of selection is not important* there are $= {}_{35}C_4$
 $= \frac{35 \cdot 34 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1} = 52,360$ combinations.

Try it Now 8

The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

Answer available at the end of this section.

In the preceding Try it Now problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same of course would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form:

Example 33

The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

Solution

In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are ${}_{15}C_{10} = 3003$ ways to choose the 10 Republicans and ${}_{14}C_9 = 2002$ ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be $3003 \cdot 2002 = 6,012,006$ possible ways of selecting the members of the Defense Subcommittee.

Answers

Try it Now 6

$8 \cdot 11 \cdot 5 = 440$ menu combinations

Try it Now 7

There are 26 characters.

- $26^5 = 11,881,376$.
- ${}_{26}P_5 = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7,893,600$

Try it Now 8

Order does not matter. ${}_{29}C_{19} = 20,030,010$ possible subcommittees

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CHAPTER 5: PROBABILITY

5.0: Probability

5.0.0: Introduction to Probability

5.0.1: Basic Concepts

5.0.2: Working with Events

5.0.3: Bayes Theorem

5.0.4: Counting

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5.0: PROBABILITY

5.0.0: Introduction to Probability

5.0.1: Basic Concepts

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5.0.0: Introduction to Probability

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says “there is a 10% chance of rain tomorrow,” she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose there is a situation with n equally likely possible outcomes and that m of those n outcomes correspond to a particular event; then the **probability** of that event is defined as $\frac{m}{n}$.

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5.0.1: Basic Concepts

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further

The **sample space** is the set of all possible simple events.

Example 1

If we roll a standard 6-sided die, describe the sample space and some simple events.

The sample space is the set of all possible simple events: $\{1, 2, 3, 4, 5, 6\}$



Some examples of simple events:

We roll a 1

We roll a 5

Some compound events:

We roll a number bigger than 4

We roll an even number

Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event E using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event } E}{\text{Total number of equally - likely outcomes}}$$

Example 2

If we roll a 6-sided die, calculate

- P(rolling a 1)
- P(rolling a number bigger than 4)

Solution

Recall that the sample space is $\{1, 2, 3, 4, 5, 6\}$

- There is one outcome corresponding to “rolling a 1”, so the probability is $\frac{1}{6}$
- There are two outcomes bigger than a 4, so the probability is $\frac{2}{6} = \frac{1}{3}$

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

Example 3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

Solution

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20} = \frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Try it Now 1

At some random moment, you look at your clock and note the minutes reading.

- What is probability the minutes reading is 15?
- What is the probability the minutes reading is 15 or less?

Answer available at the end of this section.

Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

Solution

There are 52 cards in the deck and 4 Aces so

$$P(Ace) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – if there are no outcomes that correspond with the event. The largest possible probability is 1 – if all possible outcomes correspond with the event.

Certain and Impossible events

An **impossible event** has a probability of 0.

A **certain event** has a probability of 1.

The probability of any event must be $0 \leq P(E) \leq 1$.

In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

Answers

Try it Now 1

There are 60 possible readings, from 00 to 59.

- $\frac{1}{60}$
- $\frac{16}{60}$ (counting 00 through 15)

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5.0.2: Working with Events

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six}) = \frac{1}{6}$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = \frac{5}{6}$. Notice that

$$P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event E that corresponds to m of these outcomes. Then the remaining $n - m$ outcomes correspond to E not happening, thus

$$P(\text{not } E) = \frac{n - m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

Complement of an Event

The **complement** of an event is the event “ E doesn’t happen”

The notation $\bar{E}$ is used for the complement of event E .

We can compute the probability of the complement using $P(\bar{E}) = 1 - P(E)$

Notice also that $P(E) = 1 - P(\bar{E})$

Example 5

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

Solution

There are 13 hearts in the deck, so

$$P(\text{heart}) = \frac{13}{52} = \frac{1}{4}.$$

The probability of *not* drawing a heart is the complement:

$$P(\text{ not heart }) = 1 - P(\text{ heart }) = 1 - \frac{1}{4} = \frac{3}{4}$$

Probability of two independent events

Example 6

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

Solution

We could list all possible outcomes: $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$.

Notice there are $2 \cdot 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example was looking at two independent events.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 7

Are these events independent?

- A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.
- The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).
- You draw a card from a deck, then draw a second card without replacing the first.

Solution

- The probability that a head comes up on the second toss is $\frac{1}{2}$ regardless of whether or not a head came up on the first toss, so these events are independent.
- These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.
- The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

$P(A \text{ and } B)$ for Independent Events

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 8

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

Solution

The probability of choosing a white pair of socks is $\frac{6}{10}$.

The probability of choosing a white tee shirt is $\frac{3}{7}$.

The probability of both being white is $\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}$

Try it Now 2

A card is pulled a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Answer available at the end of this section.

Example 9

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Solution

Here, there are still 12 possible outcomes:

$$\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are $H1, H2, H3, H4, H5, H6, T6$), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12} - \frac{1}{12} = \frac{7}{12}$.

$P(A \text{ or } B)$

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 10

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

Solution

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{King or Queen}) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen}) = 0$. Using our probability rule, we could have said:

$$P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) - P(\text{King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52}$$

In the last example, the events were **mutually exclusive**, so $P(A \text{ or } B) = P(A) + P(B)$.

Example 11

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Solution

- Half the cards are red, so $P(\text{red}) = \frac{26}{52}$
- There are four kings, so $P(\text{King}) = \frac{4}{52}$
- There are two red kings, so $P(\text{Red and King}) = \frac{2}{52}$

We can then calculate

$$P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$$

Try it Now 3

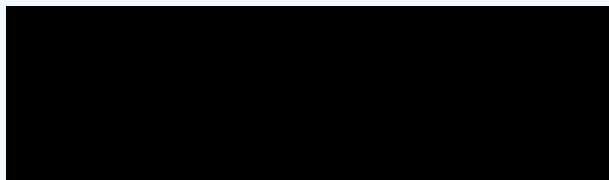
In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

Answer available at the end of this section.

Example 12

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

- Has a red car *and* got a speeding ticket
- Has a red car *or* got a speeding ticket.



Solution

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $\frac{15}{665} \approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665} \approx 0.2932$.

We also could have found this probability by:

$$P(\text{had a red car}) + P(\text{got a speeding ticket}) - P(\text{had a red car and got a speeding ticket}) = \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665}$$

Conditional Probability

Often it is required to compute the probability of an event given that another event has occurred.

Example 13

What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

Solution

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case the "condition" is that the first card is an ace. Symbolically, we write this as:

$P(\text{ace on second draw} \mid \text{an ace on the first draw})$.

The vertical bar "|" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw." What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $\frac{3}{51} = \frac{1}{17}$.

Thus, the probability of both cards being aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$.

Conditional Probability

The probability the event B occurs, given that event A has happened, is represented as

$$P(B|A)$$

This is read as "the probability of B given A "

Example 14

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

Solution

These are two independent events, so the probability of the die rolling a 6 is $\frac{1}{6}$, regardless of the result of the coin flip.

Example 15

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a) Has a speeding ticket *given* they have a red car

b) Has a red car *given* they have a speeding ticket



Solution

a) Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

$$P(\text{ticket} \mid \text{red car}) = \frac{15}{150} = \frac{1}{10} = 0.1$$

b) Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

$$P(\text{red car} \mid \text{ticket}) = \frac{15}{60} = \frac{1}{4} = 0.25.$$

Notice from the last example that $P(B|A)$ is **not** equal to $P(A|B)$.

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

Conditional Probability Formula

If Events A and B are not independent, then

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Example 16

If you pull 2 cards out of a deck, what is the probability that both are spades?

Solution

The probability that the first card is a spade is $\frac{13}{52}$.

The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck, and one less total cards.

The probability that both cards are spades is $\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx 0.0588$

Example 17

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

Solution

You can satisfy this condition by having Case A or Case B, as follows:

Case A) you can get the Ace of Diamonds first and then a black card or

Case B) you can get a black card first and then the Ace of Diamonds.

Let's calculate the probability of Case A. The probability that the first card is the Ace of Diamonds is $\frac{1}{52}$. The probability that the second card is black given that the first card is the Ace of Diamonds is $\frac{26}{51}$ because 26 of the remaining 51 cards are black. The probability is therefore $\frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$.

Now for Case B: the probability that the first card is black is $\frac{26}{52} = \frac{1}{2}$. The probability that the second card is the Ace of Diamonds given that the first card is black is $\frac{1}{51}$. The probability of Case B is therefore $\frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102}$, the same as the probability of Case 1.

Recall that the probability of A or B is $P(A) + P(B) - P(A \text{ and } B)$. In this problem, $P(A \text{ and } B) = 0$ since the first card cannot be the Ace of Diamonds and be a black card. Therefore, the probability of Case A or Case B is $\frac{1}{102} + \frac{1}{102} = \frac{2}{102} = \frac{1}{51}$. The probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is $\frac{1}{51}$.

Try it Now 4

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

Answer available at the end of this section.

Example 18

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

- $P(\text{not pregnant} \mid \text{positive test result})$
- $P(\text{positive test result} \mid \text{not pregnant})$

Solution

a) Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant. $P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} \approx 0.067$.

b) Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test. $P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263$.

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

Answers

Try it Now 2

since the second draw is made after replacing the first card, these events are independent The probability of an ace on each draw is $\frac{4}{52} = \frac{1}{13}$, so the probability of an Ace on both draws is $\frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$

Try it Now 3

$$P(\text{white sock and white tee}) = \frac{6}{10} \cdot \frac{3}{7} = \frac{9}{35}$$

$$P(\text{white sock or white tee}) = \frac{6}{10} + \frac{3}{7} - \frac{9}{35} = \frac{27}{35}$$

Try it Now 4

$$\frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$$

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5.0.3: Bayes Theorem

In this section we concentrate on the more complex conditional probability problems we began looking at in the last section.

Example 19

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

Solution

There are two ways to approach the solution to this problem. One involves an important result in probability theory called Bayes' theorem. We will discuss this theorem a bit later, but for now we will use an alternative and, we hope, much more intuitive approach.

Let's break down the information in the problem piece by piece.

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). The percentage 0.1% can be converted to a decimal number by moving the decimal place two places to the left, to get 0.001. In turn, 0.001 can be rewritten as a fraction: $1/1000$. This tells us that about 1 in every 1000 people has the disease. (If we wanted we could write $P(\text{disease})=0.001$.)

A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it). This part is fairly straightforward: everyone who has the disease will test positive, or alternatively everyone who tests negative does not have the disease. (We could also say $P(\text{positive} | \text{disease})=1$.)

The false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). This is even more straightforward. Another way of looking at it is that of every 100 people who are tested and do not have the disease, 5 will test positive even though they do not have the disease. (We could also say that $P(\text{positive} | \text{no disease})=0.05$.)

Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease? Here we want to compute $P(\text{disease}|\text{positive})$. We already know that $P(\text{positive}|\text{disease})=1$, but remember that conditional probabilities are not equal if the conditions are switched.

Rather than thinking in terms of all these probabilities we have developed, let's create a hypothetical situation and apply the facts as set out above. First, suppose we randomly select 1000 people and administer the test. How many do we expect to have the disease? Since about $1/1000$ of all people are afflicted with the disease, $\frac{1}{1000}$ of 1000 people is 1. (Now you know why we chose 1000.) Only 1 of 1000 test subjects actually has the disease; the other 999 do not.

We also know that 5% of all people who do not have the disease will test positive. There are 999 disease-free people, so we would expect $(0.05)(999) = 49.95$ (so, about 50) people to test positive who do not have the disease.

Now back to the original question, computing $P(\text{disease}|\text{positive})$. There are 51 people who test positive in our example (the one unfortunate person who actually has the disease, plus the 50 people who tested positive but don't). Only one of these people has the disease, so

$$P(\text{disease} | \text{positive}) \approx \frac{1}{51} \approx 0.0196$$

or less than 2%. Does this surprise you? This means that of all people who test positive, over 98% *do not have the disease*.

The answer we got was slightly approximate, since we rounded 49.95 to 50. We could redo the problem with 100,000 test subjects, 100 of whom would have the disease and $(0.05)(99,900) = 4995$ test positive but do not have the disease, so the exact probability of having the disease if you test positive is

$$P(\text{disease} \mid \text{positive}) \approx \frac{100}{5095} \approx 0.0196$$

which is pretty much the same answer.

But back to the surprising result. *Of all people who test positive, over 98% do not have the disease.* If your guess for the probability a person who tests positive has the disease was wildly different from the right answer (2%), don't feel bad. The exact same problem was posed to doctors and medical students at the Harvard Medical School 25 years ago and the results revealed in a 1978 *New England Journal of Medicine* article. Only about 18% of the participants got the right answer. Most of the rest thought the answer was closer to 95% (perhaps they were misled by the false positive rate of 5%).

So at least you should feel a little better that a bunch of doctors didn't get the right answer either (assuming you thought the answer was much higher). But the significance of this finding and similar results from other studies in the intervening years lies not in making math students feel better but in the possibly catastrophic consequences it might have for patient care. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. Or worse, as in the early days of the AIDS crisis when being HIV-positive was often equated with a death sentence, the patient might take a drastic action and commit suicide.

As we have seen in this hypothetical example, the most responsible course of action for treating a patient who tests positive would be to counsel the patient that they most likely do *not* have the disease and to order further, more reliable, tests to verify the diagnosis.

One of the reasons that the doctors and medical students in the study did so poorly is that such problems, when presented in the types of statistics courses that medical students often take, are solved by use of Bayes' theorem, which is stated as follows:

Bayes' Theorem

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\bar{A})P(B|\bar{A})}$$

In our earlier example, this translates to

$$P(\text{disease} \mid \text{positive}) = \frac{P(\text{disease})P(\text{positive} \mid \text{disease})}{P(\text{disease})P(\text{positive} \mid \text{disease}) + P(\text{no disease})P(\text{positive} \mid \text{no disease})}$$

Plugging in the numbers gives

$$P(\text{disease} \mid \text{positive}) = \frac{(0.001)(1)}{(0.001)(1) + (0.999)(0.05)} \approx 0.0196$$

which is exactly the same answer as our original solution.

The problem is that you (or the typical medical student, or even the typical math professor) are much more likely to be able to remember the original solution than to remember Bayes' theorem. Psychologists, such as Gerd Gigerenzer, author of *Calculated Risks: How to Know When Numbers Deceive You*, have advocated that the method involved in the original solution (which Gigerenzer calls the method of "natural frequencies") be employed in place of Bayes' Theorem. Gigerenzer performed a study and found that those educated in the natural frequency method were able to recall it far longer than those who were taught Bayes' theorem. When one considers the possible life-and-death consequences associated with such calculations it seems wise to heed his advice.

Example 20

A certain disease has an incidence rate of 2%. If the false negative rate is 10% and the false positive rate is 1%, compute the probability that a person who tests positive actually has the disease.

Solution

Imagine 10,000 people who are tested. Of these 10,000, 200 will have the disease; 10% of them, or 20, will test negative and the remaining 180 will test positive. Of the 9800 who do not have the disease, 98 will test positive. So of the 278 total people who test positive, 180 will have the disease. Thus

$$P(\text{disease} \mid \text{positive}) = \frac{180}{278} \approx 0.647$$

so about 65% of the people who test positive will have the disease.

Using Bayes theorem directly would give the same result:

$$P(\text{disease} \mid \text{positive}) = \frac{(0.02)(0.90)}{(0.02)(0.90) + (0.98)(0.01)} = \frac{0.018}{0.0278} \approx 0.647$$

Try it Now 5

A certain disease has an incidence rate of 0.5%. If there are no false negatives and if the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

Answer available at the end of this section.

Answers

Try it Now 5

Out of 100,000 people, 500 would have the disease. Of those, all 500 would test positive. Of the 99,500 without the disease, 2,985 would falsely test positive and the other 96,515 would test negative.

$$P(\text{disease} \mid \text{positive}) = \frac{500}{500 + 2985} = \frac{500}{3485} \approx 14.3\%$$

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5.0.4: Counting

Probability using Permutations and Combinations

We can use permutations and combinations to help us answer more complex probability questions

Example 34

A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

Solution

There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \cdot 9 \cdot 8 \cdot 7$, or notice that this is the same as the permutation ${}_{10}P_4 = 5040$.

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is $\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504$

Example 35

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

Solution

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$$

Example 36

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of \$1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

Solution

As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is

given by ${}_6C_5 = 6$ and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1 = 42$. Thus the number of favorable outcomes is then given by the Basic Counting Rule: ${}_6C_5 \cdot {}_{42}C_1 = 6 \cdot 42 = 252$. So the probability of winning the second prize is.

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$$

Try it Now 9

A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

Answer available at the end of this section.

Example 37

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

Solution

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be ${}_{48}C_4$ ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be ${}_4C_1 \cdot {}_{48}C_4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$$P(\text{one Ace}) = \frac{({}_4C_1)({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$$

Example 38

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

Solution

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{two Aces}) = \frac{({}_4C_2)({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now 10

Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Answer available at the end of this section.

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer ($\frac{30}{365}$, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example 39

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

Solution

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves "What is the alternative to having at least one shared birthday?" In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to "at least one" is having **none**. In other words, since this is a complementary event,

$$P(\text{at least one}) = 1 - P(\text{none})$$

We will start, then, by computing the probability that there is no shared birthday. Let's imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let's ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is $\frac{364}{365}$. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is $\frac{363}{365}$.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{no shared birthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday}) = 1 - 0.9918 = 0.0082.$$

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

Example 40

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Solution

Continuing the pattern of the previous example, the answer should be

$$P(\text{shared birthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

Example 41

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Solution

Here we can calculate

$$P(\text{shared birthday}) = 1 - \frac{365 P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page: <http://statweb.stanford.edu/~susan/surprise/Birthday.html>, and once the applet has loaded, select 30 birthdays and then keep clicking Start and Reset. If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

Try it Now 11

Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

Answer available at the end of this section.

Answers

Try it Now 9

There are $5^{10} = 9,765,625$ different ways the exam can be answered. There are 10 possible locations for the one missed question, and in each of those locations there are 4 wrong answers, so there are 40 ways the test could be answered with one wrong answer.

$$P(9 \text{ answers correct}) = \frac{40}{5^{10}} \approx 0.0000041 \text{ chance}$$

Try it Now 10

$$P(\text{three Aces and two Kings}) = \frac{{}_4C_3 \cdot {}_4C_2 \cdot {}_{52}C_5}{{}_{52}C_5} = \frac{24}{2598960} \approx 0.0000092$$

Try it Now 11

$$P(\text{shared birthday}) = 1 - \frac{365 P_{10}}{365^{10}} \approx 0.117$$

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5.0.5: Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 42



[1]In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

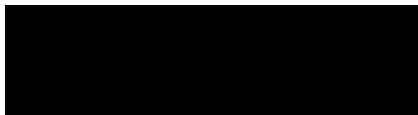
Solution

Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won" $-\$2/\$38 \approx -\$0.053$. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:



Notice that if we multiply each outcome by its corresponding probability we get $\$35 \cdot \frac{1}{38} = 0.9211$ and $-\$1 \cdot \frac{37}{38} = -0.9737$, and if we add these numbers we get $0.9211 + (-0.9737) \approx -0.053$, which is the expected value we computed above.

Expected Value

Expected Value is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

Try it Now 12

You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

Answer available at the end of this section.

Example 43

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. If they match 5 numbers, then win \$1,000. It costs \$1 to buy a ticket. Find the expected value.

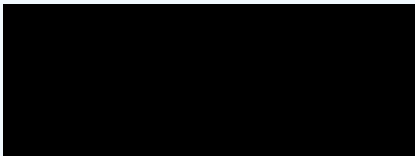
Solution

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815 \text{ for all 6 numbers,}$$

$$\frac{{}_6C_5({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205 \text{ for 5 numbers.}$$

Our probabilities and outcome values are:



The expected value, then is:

$$(\$999,999) \cdot \frac{1}{12271512} + (\$999) \cdot \frac{252}{12271512} + (-\$1) \cdot \frac{12271259}{12271512} \approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose \$1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

Try it Now 13

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him \$1. If any two dice match values, you get \$2. What is the expected value of this game? Would you play?

Answer available at the end of this section.

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example 44

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year[2]. An insurance company charges \$275 for a life-insurance policy that pays a \$100,000 death benefit. What is the expected value for the person buying the insurance?

Solution

The probabilities and outcomes are



The expected value is $(\$99,725)(0.00242) + (-\$275)(0.99758) = -\$33$.

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

[1] Photo CC-BY-SA <http://www.flickr.com/photos/stoneflower/>

[2] According to the estimator at www.numericalexample.com/index.html?article&id=91

Answers

Try it Now 12

$$(\$3,995) \cdot \frac{1}{2000} + (-\$5) \cdot \frac{1999}{2000} \approx -\$3.00$$

Try it Now 13

Suppose you roll the first die. The probability the second will be different is $\frac{5}{6}$. The probability that the third roll is different than the previous two is $\frac{4}{6}$, so the probability that the three dice are different is $\frac{5}{6} \cdot \frac{4}{6} = \frac{20}{36}$. The probability that two dice will match is the complement, $1 - \frac{20}{36} = \frac{16}{36}$.

The expected value is: $(\$2) \cdot \frac{16}{36} + (-\$1) \cdot \frac{20}{36} = \frac{12}{36} \approx \$0.33$. Yes, it is in your advantage to play. On average, you'd win \$0.33 per play.

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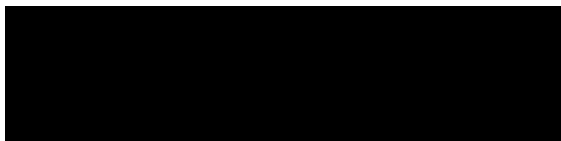
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5.0.6: Exercises

1. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of the given event.
 - a. A red ball is drawn
 - b. A white ball is drawn
2. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. What is the probability of drawing one slip of paper from the hat at random and getting:
 - a. A consonant
 - b. A vowel
3. A group of people were asked if they had run a red light in the last year. 150 responded "yes", and 185 responded "no". Find the probability that if a person is chosen at random, they have run a red light in the last year.
4. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dogs, and 40 indicated they don't enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.
5. Compute the probability of tossing a six-sided die (with sides numbered 1 through 6) and getting a 5.
6. Compute the probability of tossing a six-sided die and getting a 7.
7. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female.



8. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person had no credit cards.



9. Compute the probability of tossing a six-sided die and getting an even number.
10. Compute the probability of tossing a six-sided die and getting a number less than 3.
11. If you pick one card at random from a standard deck of cards, what is the probability it will be a King?
12. If you pick one card at random from a standard deck of cards, what is the probability it will be a Diamond?
13. Compute the probability of rolling a 12-sided die and getting a number other than 8.
14. If you pick one card at random from a standard deck of cards, what is the probability it is not the Ace of Spades?
15. Referring to the grade table from question #7, what is the probability that a student chosen at random did NOT earn a C?
16. Referring to the credit card table from question #8, what is the probability that a person chosen at random has at least one credit card?

17. A six-sided die is rolled twice. What is the probability of showing a 6 on both rolls?
18. A fair coin is flipped twice. What is the probability of showing heads on both flips?
19. A die is rolled twice. What is the probability of showing a 5 on the first roll and an even number on the second roll?
20. Suppose that 21% of people own dogs. If you pick two people at random, what is the probability that they both own a dog?
21. Suppose a jar contains 17 red marbles and 32 blue marbles. If you reach in the jar and pull out 2 marbles at random, find the probability that both are red.
22. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. If you pull out two slips at random, find the probability that both are vowels.
23. Bert and Ernie each have a well-shuffled standard deck of 52 cards. They each draw one card from their own deck. Compute the probability that:
- Bert and Ernie both draw an Ace.
 - Bert draws an Ace but Ernie does not.
 - neither Bert nor Ernie draws an Ace.
 - Bert and Ernie both draw a heart.
 - Bert gets a card that is not a Jack and Ernie draws a card that is not a heart.
24. Bert has a well-shuffled standard deck of 52 cards, from which he draws one card; Ernie has a 12-sided die, which he rolls at the same time Bert draws a card. Compute the probability that:
- Bert gets a Jack and Ernie rolls a five.
 - Bert gets a heart and Ernie rolls a number less than six.
 - Bert gets a face card (Jack, Queen or King) and Ernie rolls an even number.
 - Bert gets a red card and Ernie rolls a fifteen.
 - Bert gets a card that is not a Jack and Ernie rolls a number that is not twelve.
25. Compute the probability of drawing a King from a deck of cards and then drawing a Queen.
26. Compute the probability of drawing two spades from a deck of cards.
27. A math class consists of 25 students, 14 female and 11 male. Two students are selected at random to participate in a probability experiment. Compute the probability that
- a male is selected, then a female.
 - a female is selected, then a male.
 - two males are selected.
 - two females are selected.
 - no males are selected.
28. A math class consists of 25 students, 14 female and 11 male. Three students are selected at random to participate in a probability experiment. Compute the probability that
- a male is selected, then two females.
 - a female is selected, then two males.
 - two females are selected, then one male.
 - three males are selected.
 - three females are selected.
29. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female and earned an A.



30. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person was male and had two or more credit cards.



31. A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd-numbered.

32. A jar contains 4 red marbles numbered 1 to 4 and 10 blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability the marble is blue or even-numbered.

33. Referring to the table from #29, find the probability that a student chosen at random is female or earned a B.

34. Referring to the table from #30, find the probability that a person chosen at random is male or has no credit cards.

35. Compute the probability of drawing the King of hearts or a Queen from a deck of cards.

36. Compute the probability of drawing a King or a heart from a deck of cards.

37. A jar contains 5 red marbles numbered 1 to 5 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is

- a. Even-numbered given that the marble is red.
- b. Red given that the marble is even-numbered.

38. A jar contains 4 red marbles numbered 1 to 4 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is

- a. Odd-numbered given that the marble is blue.
- b. Blue given that the marble is odd-numbered.

39. Compute the probability of flipping a coin and getting heads, given that the previous flip was tails.

40. Find the probability of rolling a "1" on a fair die, given that the last 3 rolls were all ones.

41. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student speaks French, given that the student is female.

42. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student is male, given that the student speaks French.

43. A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 90% of the time if the person has the virus and 10% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".

- a. Find the probability that a person has the virus given that they have tested positive, i.e. find $P(A|B)$.
- b. Find the probability that a person does not have the virus given that they test negative, i.e. find $P(\text{not } A | \text{not } B)$.

44. A certain virus infects one in every 2000 people. A test used to detect the virus in a person is positive 96% of the time if the person has the virus and 4% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".

- a. Find the probability that a person has the virus given that they have tested positive, i.e. find $P(A|B)$.
- b. Find the probability that a person does not have the virus given that they test negative, i.e. find $P(\text{not } A | \text{not } B)$.

45. A certain disease has an incidence rate of 0.3%. If the false negative rate is 6% and the false positive rate is 4%, compute the probability that a person who tests positive actually has the disease.

46. A certain disease has an incidence rate of 0.1%. If the false negative rate is 8% and the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

47. A certain group of symptom-free women between the ages of 40 and 50 are randomly selected to participate in mammography screening. The incidence rate of breast cancer among such women is 0.8%. The false negative rate for the mammogram is 10%. The false positive rate is 7%. If a the mammogram results for a particular woman are positive (indicating that she has breast cancer), what is the probability that she actually has breast cancer?

48. About 0.01% of men with no known risk behavior are infected with HIV. The false negative rate for the standard HIV test 0.01% and the false positive rate is also 0.01%. If a randomly selected man with no known risk behavior tests positive for HIV, what is the probability that he is actually infected with HIV?

49. A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?

50. At a restaurant you can choose from 3 appetizers, 8 entrees, and 2 desserts. How many different three-course meals can you have?

51. How many three-letter "words" can be made from 4 letters "FGHI" if

- a. repetition of letters is allowed
- b. repetition of letters is not allowed

52. How many four-letter "words" can be made from 6 letters "AEBWDP" if

- a. repetition of letters is allowed
- b. repetition of letters is not allowed

53. All of the license plates in a particular state feature three letters followed by three digits (e.g. ABC 123). How many different license plate numbers are available to the state's Department of Motor Vehicles?

54. A computer password must be eight characters long. How many passwords are possible if only the 26 letters of the alphabet are allowed?

55. A pianist plans to play 4 pieces at a recital. In how many ways can she arrange these pieces in the program?

56. In how many ways can first, second, and third prizes be awarded in a contest with 210 contestants?

57. Seven Olympic sprinters are eligible to compete in the 4 x 100 m relay race for the USA Olympic team. How many four-person relay teams can be selected from among the seven athletes?

58. A computer user has downloaded 25 songs using an online file-sharing program and wants to create a CD-R with ten songs to use in his portable CD player. If the order that the songs are placed on the CD-R is important to him, how many different CD-Rs could he make from the 25 songs available to him?

59. In western music, an octave is divided into 12 pitches. For the film *Close Encounters of the Third Kind*, director Steven Spielberg asked composer John Williams to write a five-note theme, which aliens would use to communicate with people on Earth. Disregarding rhythm and octave changes, how many five-note themes are possible if no note is repeated?

60. In the early twentieth century, proponents of the Second Viennese School of musical composition (including Arnold Schönberg, Anton Webern and Alban Berg) devised the twelve-tone technique, which utilized a tone row consisting of all 12 pitches from the chromatic scale in any order, but with not pitches repeated in the row. Disregarding rhythm and octave changes, how many tone rows are possible?

61. In how many ways can 4 pizza toppings be chosen from 12 available toppings?
62. At a baby shower 17 guests are in attendance and 5 of them are randomly selected to receive a door prize. If all 5 prizes are identical, in how many ways can the prizes be awarded?
63. In the 6/50 lottery game, a player picks six numbers from 1 to 50. How many different choices does the player have if order doesn't matter?
64. In a lottery daily game, a player picks three numbers from 0 to 9. How many different choices does the player have if order doesn't matter?
65. A jury pool consists of 27 people. How many different ways can 11 people be chosen to serve on a jury and one additional person be chosen to serve as the jury foreman?
66. The United States Senate Committee on Commerce, Science, and Transportation consists of 23 members, 12 Republicans and 11 Democrats. The Surface Transportation and Merchant Marine Subcommittee consists of 8 Republicans and 7 Democrats. How many ways can members of the Subcommittee be chosen from the Committee?
67. You own 16 CDs. You want to randomly arrange 5 of them in a CD rack. What is the probability that the rack ends up in alphabetical order?
68. A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.
69. In a lottery game, a player picks six numbers from 1 to 48. If 5 of the 6 numbers match those drawn, they player wins second prize. What is the probability of winning this prize?
70. In a lottery game, a player picks six numbers from 1 to 48. If 4 of the 6 numbers match those drawn, they player wins third prize. What is the probability of winning this prize?
71. Compute the probability that a 5-card poker hand is dealt to you that contains all hearts.
72. Compute the probability that a 5-card poker hand is dealt to you that contains four Aces.
73. A bag contains 3 gold marbles, 6 silver marbles, and 28 black marbles. Someone offers to play this game: You randomly select on marble from the bag. If it is gold, you win \$3. If it is silver, you win \$2. If it is black, you lose \$1. What is your expected value if you play this game?
74. A friend devises a game that is played by rolling a single six-sided die once. If you roll a 6, he pays you \$3; if you roll a 5, he pays you nothing; if you roll a number less than 5, you pay him \$1. Compute the expected value for this game. Should you play this game?
75. In a lottery game, a player picks six numbers from 1 to 23. If the player matches all six numbers, they win 30,000 dollars. Otherwise, they lose \$1. Find the expected value of this game.
76. A game is played by picking two cards from a deck. If they are the same value, then you win \$5, otherwise you lose \$1. What is the expected value of this game?
77. A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of \$350. If they offer a 2 year extended warranty for \$48, what is the company's expected value of each warranty sold?
78. An insurance company estimates the probability of an earthquake in the next year to be 0.0013. The average damage done by an earthquake it estimates to be \$60,000. If the company offers earthquake insurance for \$100, what is their expected value of the policy?

Exploration

Some of these questions were adapted from puzzles at mindyourdecisions.com.

79. A small college has been accused of gender bias in its admissions to graduate programs.

- a. Out of 500 men who applied, 255 were accepted. Out of 700 women who applied, 240 were accepted. Find the acceptance rate for each gender. Does this suggest bias?
- b. The college then looked at each of the two departments with graduate programs, and found the data below. Compute the acceptance rate within each department by gender. Does this suggest bias?
- c. Looking at our results from Parts a and b, what can you conclude? Is there gender bias in this college's admissions? If so, in which direction?

80. A bet on "black" in Roulette has a probability of $\frac{18}{38}$ of winning. If you win, you double your money. You can bet anywhere from \$1 to \$100 on each spin.

- a. Suppose you have \$10, and are going to play until you go broke or have \$20. What is your best strategy for playing?
- b. Suppose you have \$10, and are going to play until you go broke or have \$30. What is your best strategy for playing?

81. Your friend proposes a game: You flip a coin. If it's heads, you win \$1. If it's tails, you lose \$1. However, you are worried the coin might not be fair coin. How could you change the game to make the game fair, without replacing the coin?

82. Fifty people are in a line. The first person in the line to have a birthday matching someone in front of them will win a prize. Of course, this means the first person in the line has no chance of winning. Which person has the highest likelihood of winning?

83. Three people put their names in a hat, then each draw a name, as part of a randomized gift exchange. What is the probability that no one draws their own name? What about with four people?

84. How many different "words" can be formed by using all the letters of each of the following words exactly once?

- a. "ALICE"
- b. "APPLE"

85. How many different "words" can be formed by using all the letters of each of the following words exactly once?

- a. "TRUMPS"
- b. "TEETER"

86. The *Monty Hall problem* is named for the host of the game show *Let's make a Deal*. In this game, there would be three doors, behind one of which there was a prize. The contestant was asked to choose one of the doors. Monty Hall would then open one of the other doors to show there was no prize there. The contestant was then asked if they wanted to stay with their original door, or switch to the other unopened door. Is it better to stay or switch, or does it matter?

87. Suppose you have two coins, where one is a fair coin, and the other coin comes up heads 70% of the time. What is the probability you have the fair coin given each of the following outcomes from a series of flips?

- a. 5 Heads and 0 Tails
- b. 8 Heads and 3 Tails
- c. 10 Heads and 10 Tails
- d. 3 Heads and 8 Tails

88. Suppose you have six coins, where five are fair coins, and one coin comes up heads 80% of the time. What is the probability you have a fair coin given each of the following outcomes from a series of flips?

- a. 5 Heads and 0 Tails
- b. 8 Heads and 3 Tails
- c. 10 Heads and 10 Tails
- d. 3 Heads and 8 Tails

89. In this problem, we will explore probabilities from a series of events.

- a. If you flip 20 coins, how many would you expect to come up "heads", on average? Would you expect every flip of 20 coins to come up with exactly that many heads?
- b. If you were to flip 20 coins, what would you consider a "usual" result? An "unusual" result?

- c. Flip 20 coins (or one coin 20 times) and record how many come up “heads”. Repeat this experiment 9 more times. Collect the data from the entire class.
- d. When flipping 20 coins, what is the theoretic probability of flipping 20 heads?
- e. Based on the class’s experimental data, what appears to be the probability of flipping 10 heads out of 20 coins?
- f. The formula ${}_nC_x p^x (1-p)^{n-x}$ will compute the probability of an event with probability p occurring x times out of n , such as flipping x heads out of n coins where the probability of heads is $p = \frac{1}{2}$. Use this to compute the theoretic probability of flipping 10 heads out of 20 coins.
- g. If you were to flip 20 coins, based on the class’s experimental data, what range of values would you consider a “usual” result? What is the combined probability of these results? What would you consider an “unusual” result? What is the combined probability of these results?
- h. We’ll now consider a simplification of a case from the 1960s. In the area, about 26% of the jury eligible population was black. In the court case, there were 100 men on the juror panel, of which 8 were black. Does this provide evidence of racial bias in jury selection?

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CHAPTER 6: MISCELLANEOUS EXTRA TOPICS

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6.0: HISTORICAL COUNTING SYSTEMS

In the following chapter, we will try to focus on two main ideas. The first will be an examination of basic number and counting systems and the symbols that we use for numbers. We will look at our own modern (Western) number system as well those of a couple of selected civilizations to see the differences and diversity that is possible when humans start counting. The second idea we will look at will be base systems. By comparing our own base-ten (decimal) system with other bases, we will quickly become aware that the system that we are so used to, when slightly changed, will challenge our notions about numbers and what symbols for those numbers actually mean.

6.0.0: Introduction and Basic Number and Counting Systems

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6.0.0: Introduction and Basic Number and Counting Systems

Introduction

As we begin our journey through the history of mathematics, one question to be asked is “Where do we start?” Depending on how you view mathematics or numbers, you could choose any of a number of launching points from which to begin. Howard Eves suggests the following list of possibilities.[i]

Where to start the study of the history of mathematics...

- At the first logical geometric “proofs” traditionally credited to Thales of Miletus (600 BCE).
- With the formulation of methods of measurement made by the Egyptians and Mesopotamians/Babylonians.
- Where prehistoric peoples made efforts to organize the concepts of size, shape, and number.
- In pre-human times in the very simple number sense and pattern recognition that can be displayed by certain animals, birds, etc.
- Even before that in the amazing relationships of numbers and shapes found in plants.
- With the spiral nebulae, the natural course of planets, and other universe phenomena.

We can choose no starting point at all and instead agree that mathematics has *always* existed and has simply been waiting in the wings for humans to discover. Each of these positions can be defended to some degree and which one you adopt (if any) largely depends on your philosophical ideas about mathematics and numbers.

Nevertheless, we need a starting point. Without passing judgment on the validity of any of these particular possibilities, we will choose as our starting point the emergence of the idea of number and the process of counting as our launching pad. This is done primarily as a practical matter given the nature of this course. In the following chapter, we will try to focus on two main ideas. The first will be an examination of basic number and counting systems and the symbols that we use for numbers. We will look at our own modern (Western) number system as well those of a couple of selected civilizations to see the differences and diversity that is possible when humans start counting. The second idea we will look at will be base systems. By comparing our own base-ten (decimal) system with other bases, we will quickly become aware that the system that we are so used to, when slightly changed, will challenge our notions about numbers and what symbols for those numbers actually mean.

Recognition of More vs. Less

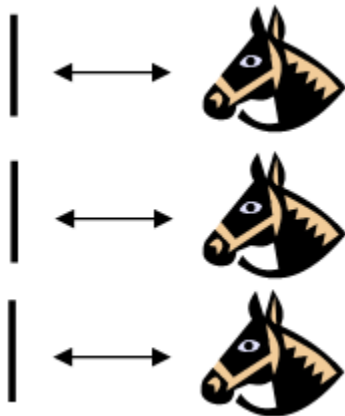
The idea of numbers and the process of counting goes back far beyond when history began to be recorded. There is some archeological evidence that suggests that humans were counting as far back as 50,000 years ago.[ii] However, we do not really know how this process started or developed over time. The best we can do is to make a good guess as to how things progressed. It is probably not hard to believe that even the earliest humans had some sense of *more* and *less*. Even some small animals have been shown to have such a sense. For example, one naturalist tells of how he would secretly remove one egg each day from a plover’s nest. The mother was diligent in laying an extra egg every day to make up for the missing egg. Some research has shown that hens can be trained to distinguish between even and odd numbers of pieces of food.[iii] With these sorts of findings in mind, it is not hard to conceive that early humans had (at least) a similar sense of more and less. However, our conjectures about how and when these ideas emerged among humans are simply that; educated guesses based on our own assumptions of what might or could have been.

The Need for Simple Counting

As societies and humankind evolved, simply having a sense of more or less, even or odd, etc., would prove to be insufficient to meet the needs of everyday living. As tribes and groups formed, it became important to be able to know how many members were in the group, and perhaps how many were in the enemy’s camp. Certainly it was important for them to know if the flock of sheep or other possessed animals were increasing or decreasing in size. “Just how many of them do we have, anyway?” is a question that we do not have a hard time imagining them asking themselves (or each other).

In order to count items such as animals, it is often conjectured that one of the earliest methods of doing so would be with “tally

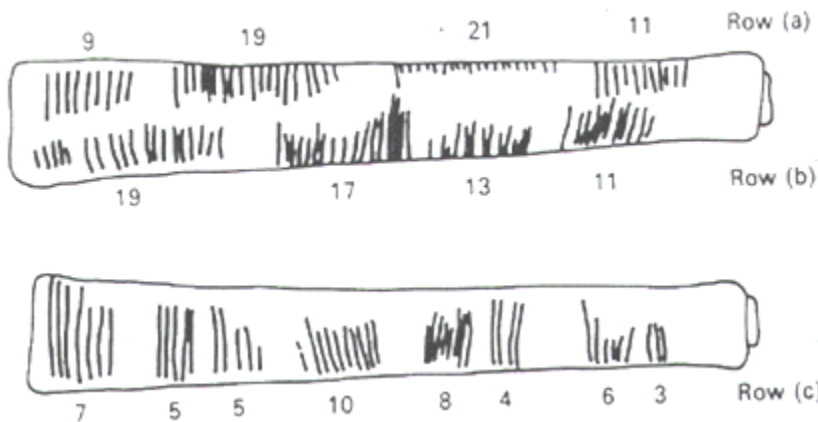
sticks.” These are objects used to track the numbers of items to be counted. With this method, each “stick” (or pebble, or whatever counting device being used) represents one animal or object. This method uses the idea of **one to one correspondence**. In a one to one correspondence, items that are being counted are uniquely linked with some counting tool.



In the picture to the right, you see each stick corresponding to one horse. By examining the collection of sticks in hand one knows how many animals should be present. You can imagine the usefulness of such a system, at least for smaller numbers of items to keep track of. If a herder wanted to “count off” his animals to make sure they were all present, he could mentally (or methodically) assign each stick to one animal and continue to do so until he was satisfied that all were accounted for.

Of course, in our modern system, we have replaced the sticks with more abstract objects. In particular, the top stick is replaced with our symbol “1,” the second stick gets replaced by a “2” and the third stick is represented by the symbol “3,” but we are getting ahead of ourselves here. These modern symbols took many centuries to emerge.

Another possible way of employing the “tally stick” counting method is by making marks or cutting notches into pieces of wood, or even tying knots in string (as we shall see later). In 1937, Karl Absolon discovered a wolf bone that goes back possibly 30,000 years. It is believed to be a counting device.[iv] Another example of this kind of tool is the Ishango Bone, discovered in 1960 at Ishango, and shown below.[v] It is reported to be between six and nine thousand years old and shows what appear to be markings used to do counting of some sort.



The markings on rows (a) and (b) each add up to 60. Row (b) contains the prime numbers between 10 and 20. Row (c) seems to illustrate for the method of doubling and multiplication used by the Egyptians. It is believed that this may also represent a lunar phase counter.

Spoken Words

As methods for counting developed, and as language progressed as well, it is natural to expect that spoken words for numbers would appear. Unfortunately, the developments of these words, especially those corresponding to the numbers from one through ten, are not easy to trace. Past ten, however, we do see some patterns:

Eleven comes from “ein lifon,” meaning “one left over.”

Twelve comes from “twe lif,” meaning “two left over.”

Thirteen comes from “Three and ten” as do fourteen through nineteen.

Twenty appears to come from “twe-tig” which means “two tens.”

Hundred probably comes from a term meaning “ten times.”

Written Numbers

When we speak of “written” numbers, we have to be careful because this could mean a variety of things. It is important to keep in mind that modern paper is relatively new in the overall timeline of human existence, so “writing” in times past often took on forms that might look quite unfamiliar to us today.

As we saw earlier, some might consider wooden sticks with notches carved in them as writing as these are means of recording information on a medium that can be “read” by others. Of course, the symbols used (simple notches) certainly did not leave a lot of flexibility for communicating a wide variety of ideas or information.

Other mediums on which “writing” may have taken place include carvings in stone or clay tablets, rag paper made by hand (12th century in Europe, but earlier in China), papyrus (invented by the Egyptians and used up until the Greeks), and parchments from animal skins. And these are just a few of the many possibilities.

These are just a few examples of early methods of counting and simple symbols for representing numbers. Extensive books, articles and research have been done on this topic and could provide enough information to fill this entire course if we allowed it to. The range and diversity of creative thought that has been used in the past to describe numbers and to count objects and people is staggering. Unfortunately, we don’t have time to examine them all, but it is fun and interesting to look at one system in more detail to see just how ingenious people have been.

[i] Eves, Howard; An Introduction to the History of Mathematics, p. 9.

[ii] Eves, p. 9.

[iii] McLeish, John; The Story of Numbers - How Mathematics Has Shaped Civilization, p. 7.

[iv] Bunt, Lucas; Jones, Phillip; Bedient, Jack; The Historical Roots of Elementary Mathematics, p. 2.

[v] http://www.math.buffalo.edu/mad/Ancient-Africa/mad_zaire-uganda.html

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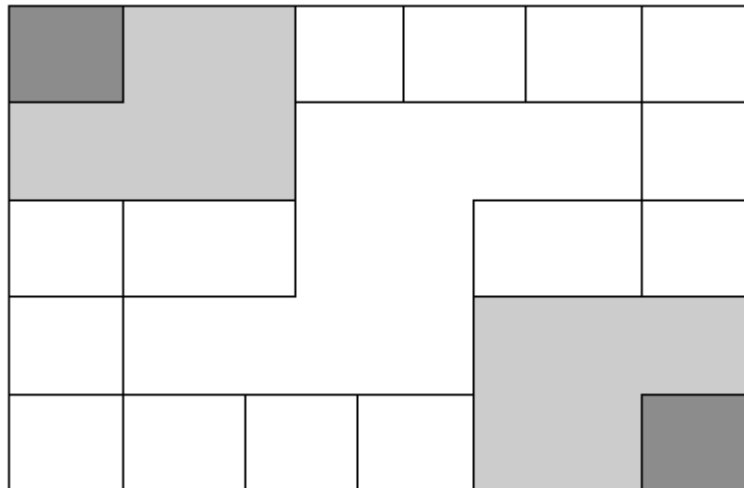
6.0.1: The Number and Counting System of the Inca Civilization

Background

There is generally a lack of books and research material concerning the historical foundations of the Americas. Most of the “important” information available concentrates on the eastern hemisphere, with Europe as the central focus. The reasons for this may be twofold: first, it is thought that there was a lack of specialized mathematics in the American regions; second, many of the secrets of ancient mathematics in the Americas have been closely guarded.[i] The Peruvian system does not seem to be an exception here. Two researchers, Leland Locke and Erland Nordenskiöld, have carried out research that has attempted to discover what mathematical knowledge was known by the Incas and how they used the Peruvian quipu, a counting system using cords and knots, in their mathematics. These researchers have come to certain beliefs about the quipu that we will summarize here.

Counting Boards

It should be noted that the Incas did not have a complicated system of computation. Where other peoples in the regions, such as the Mayans, were doing computations related to their rituals and calendars, the Incas seem to have been more concerned with the simpler task of record-keeping. To do this, they used what are called the “quipu” to record quantities of items. (We will describe them in more detail in a moment.) However, they first often needed to do computations whose results would be

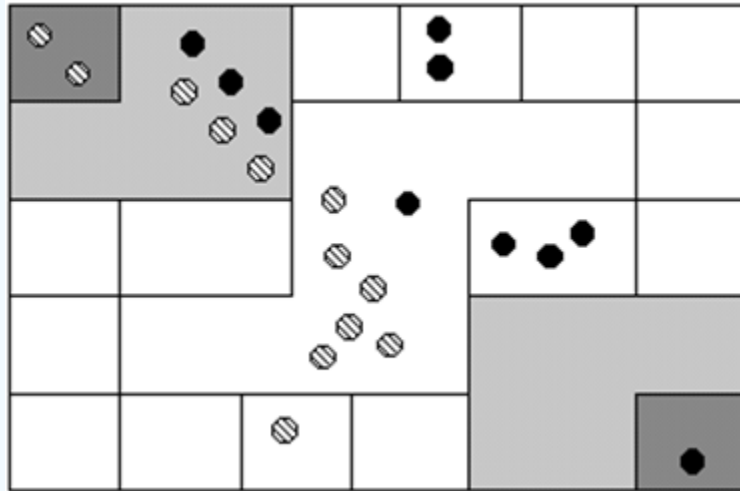


recorded on quipu. To do these computations, they would sometimes use a counting board constructed with a slab of stone. In the slab were cut rectangular and square compartments so that an octagonal (eight-sided) region was left in the middle. Two opposite corner rectangles were raised. Another two sections were mounted on the original surface of the slab so that there were actually three levels available. In the figure shown, the darkest shaded corner regions represent the highest, third level. The lighter shaded regions surrounding the corners are the second highest levels, while the clear white rectangles are the compartments cut into the stone slab.

Pebbles were used to keep accounts and their positions within the various levels and compartments gave totals. For example, a pebble in a smaller (white) compartment represented one unit. Note that there are 12 such squares around the outer edge of the figure. If a pebble was put into one of the two (white) larger, rectangular compartments, its value was doubled. When a pebble was put in the octagonal region in the middle of the slab, its value was tripled. If a pebble was placed on the second (shaded) level, its value was multiplied by six. And finally, if a pebble was found on one of the two highest corner levels, its value was multiplied by twelve. Different objects could be counted at the same time by representing different objects by different colored pebbles.

Example 1

Suppose you have the following counting board with two different kind of pebbles places as illustrated. Let the solid black pebble represent a dog and the striped pebble represent a cat. How many dogs are being represented?



Solution

- There are two black pebbles in the outer square regions...these represent 2 dogs.
- There are three black pebbles in the larger (white) rectangular compartments. These represent 6 dogs.
- There is one black pebble in the middle region...this represents 3 dogs.
- There are three black pebbles on the second level...these represent 18 dogs.
- Finally, there is one black pebble on the highest corner level...this represents 12 dogs.

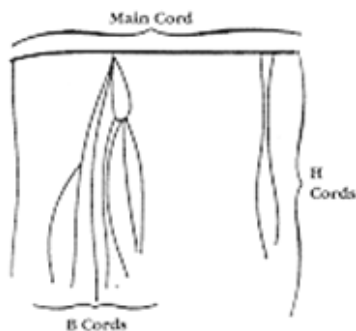
We then have a total of $2 + 6 + 3 + 18 + 12 = 41$ dogs.

Try it Now 1

How many cats are represented on this board?

Answer available at the end of this section.

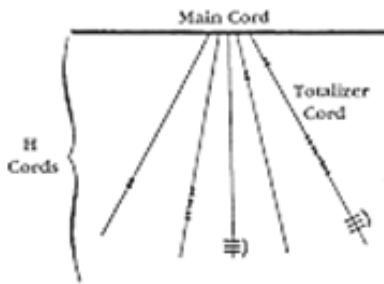
The Quipu



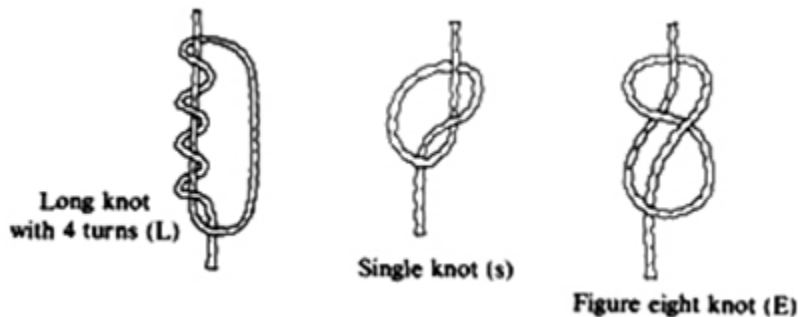
This kind of board was good for doing quick computations, but it did not provide a good way to keep a permanent recording of quantities or computations. For this purpose, they used the quipu. The quipu is a collection of cords with knots in them. These cords and knots are carefully arranged so that the position and type of cord or knot gives specific

information on how to decipher the cord.

A quipu is made up of a main cord which has other cords (branches) tied to it. See pictures to the right.[ii]



Locke called the branches H cords. They are attached to the main cord. B cords, in turn, were attached to the H cords. Most of these cords would have knots on them. Rarely are knots found on the main cord, however, and tend to be mainly on the H and B cords. A quipu might also have a “totalizer” cord that summarizes all of the information on the cord group in one place.



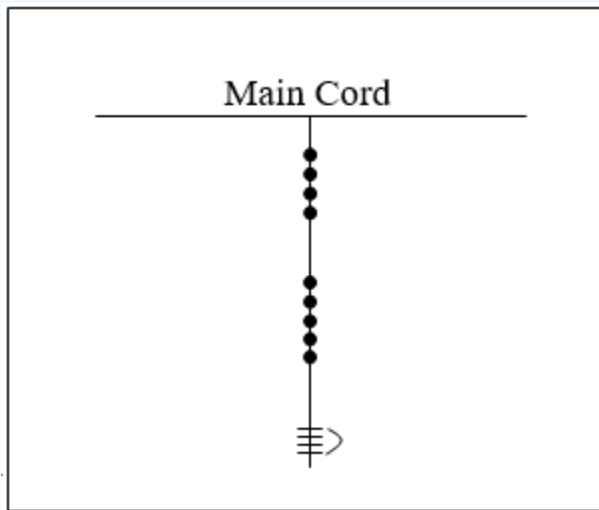
Locke points out that there are three types of knots, each representing a different value, depending on the kind of knot used and its position on the cord. The Incas, like us, had a decimal (base-ten) system, so each kind of knot had a specific decimal value. The Single knot, pictured in the middle of the diagram[iii] was used to denote tens, hundreds, thousands, and ten thousands. They would be on the upper levels of the H cords. The figure-eight knot on the end was used to denote the integer “one.” Every other integer from 2 to 9 was represented with a long knot, shown on the left of the figure. (Sometimes long knots were used to represent tens and hundreds.) Note that the long knot has several turns in it...the number of turns indicates which integer is being represented. The units (ones) were placed closest to the bottom of the cord, then tens right above them, then the hundreds, and so on.

In order to make reading these pictures easier, we will adopt a convention that is consistent. For the long knot with turns in it (representing the numbers 2 through 9), we will use the following notation: $\equiv >$

The four horizontal bars represent four turns and the curved arc on the right links the four turns together. This would represent the number 4.

We will represent the single knot with a large dot (·) and we will represent the figure eight knot with a sideways eight (∞).

Example 2

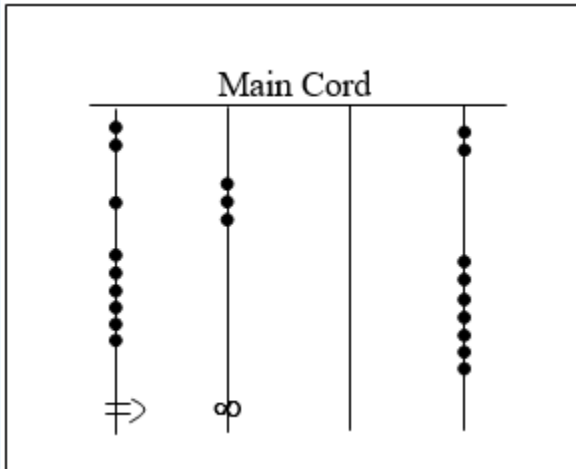


What number is represented on the cord shown?

Solution

On the cord, we see a long knot with four turns in it...this represents four in the ones place. Then 5 single knots appear in the tens position immediately above that, which represents 5 tens, or 50. Finally, 4 single knots are tied in the hundreds, representing four 4 hundreds, or 400. Thus, the total shown on this cord is 454.

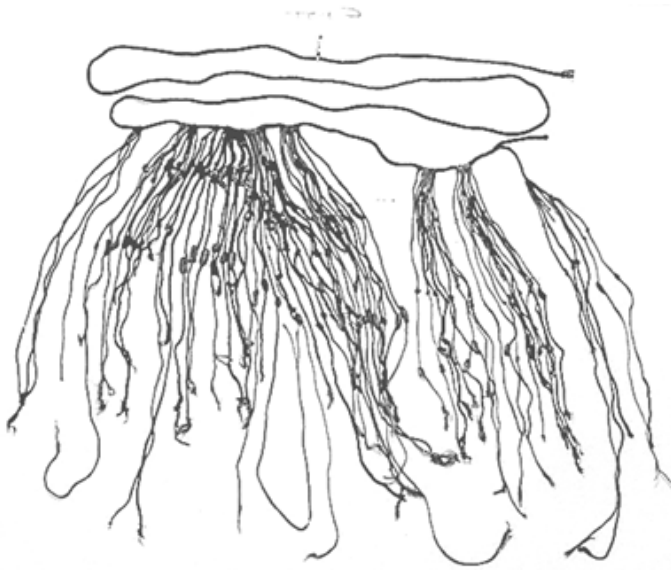
Try it Now 2



What numbers are represented on each of the four cords hanging from the main cord?

Answer available at the end of this section.

The colors of the cords had meaning and could distinguish one object from another. One color could represent llamas, while a different color might represent sheep, for example. When all the colors available were exhausted, they would have to be re-used. Because of this, the ability to read the quipu became a complicated task and specially trained individuals did this job. They were called Quipucamayoc, which means keeper of the quipus. They would build, guard, and decipher quipus.



As you can see from this photograph of an actual quipu, they could get quite complex.

There were various purposes for the quipu. Some believe that they were used to keep an account of their traditions and history, using knots to record history rather than some other formal system of writing. One writer has even suggested that the quipu replaced writing as it formed a role in the Incan postal system.[iv] Another proposed use of the quipu is as a translation tool. After the conquest of the Incas by the Spaniards and subsequent “conversion” to Catholicism, an Inca supposedly could use the quipu to confess their sins to a priest. Yet another proposed use of the quipu was to record numbers related to magic and astronomy, although this is not a widely accepted interpretation.

The mysteries of the quipu have not been fully explored yet. Recently, Ascher and Ascher have published a book, *The Code of the Quipu: A Study in Media, Mathematics, and Culture*, which is “an extensive elaboration of the logical-numerical system of the quipu.”[v] For more information on the quipu, you may want to check out the following Internet link:

www.anthropology.wisc.edu/salomon/Chaysimire/khipus.php

We are so used to seeing the symbols 1, 2, 3, 4, etc. that it may be somewhat surprising to see such a creative and innovative way to compute and record numbers. Unfortunately, as we proceed through our mathematical education in grade and high school, we receive very little information about the wide range of number systems that have existed and which still exist all over the world. That’s not to say our own system is not important or efficient. The fact that it has survived for hundreds of years and shows no sign of going away any time soon suggests that we may have finally found a system that works well and may not need further improvement, but only time will tell that whether or not that conjecture is valid or not. We now turn to a brief historical look at how our current system developed over history.

[i] Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher*, Issue 60 (Oct., 1967), p. 623-28.

[ii] Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher*, Issue 60 (Oct., 1967), p. 623-28.

[iii] wiscinfo.doit.wisc.edu/chaysimire/titulo2/khipus/what.htm

[iv] Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher*, Issue 60 (Oct., 1967), p. 623-28.

[v] <http://www.cs.uidaho.edu/~casey931/seminar/quipu.html>

Answers

Try it Now 1

$$1 + 6 \times 3 + 3 \times 6 + 2 \times 12 = 61 \text{ cats.}$$

Try it Now 2

From left to right:

Cord 1 = 2,162

Cord 2 = 301

Cord 3 = 0

Cord 4 = 2,070

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6.0.2: The Hindu-Arabic Number System

The Evolution of a System

Our own number system, composed of the ten symbols {0,1,2,3,4,5,6,7,8,9} is called the *Hindu-Arabic system*. This is a base-ten (decimal) system since place values increase by powers of ten. Furthermore, this system is positional, which means that the position of a symbol has bearing on the value of that symbol within the number. For example, the position of the symbol 3 in the number 435,681 gives it a value much greater than the value of the symbol 8 in that same number. We'll explore base systems more thoroughly later. The development of these ten symbols and their use in a positional system comes to us primarily from India.[i]



It was not until the 15th century that the symbols that we are familiar with today first took form in Europe. However, the history of these numbers and their development goes back hundreds of years. One important source of information on this topic is the writer al-Biruni, whose picture is shown here.[ii] Al-Biruni, who was born in modern day Uzbekistan, had visited India on several occasions and made comments on the Indian number system. When we look at the origins of the numbers that al-Biruni encountered, we have to go back to the third century B.C.E. to explore their origins. It is then that the Brahmi numerals were being used.

The Brahmi numerals were more complicated than those used in our own modern system. They had separate symbols for the numbers 1 through 9, as well as distinct symbols for 10, 100, 1000, ..., also for 20, 30, 40, ..., and others for 200, 300, 400, ..., 900. The Brahmi symbols for 1, 2, and 3 are shown below.[iii]

1	2	3
—	=	≡

Brahmi one, two, three

These numerals were used all the way up to the 4th century C.E., with variations through time and geographic location. For example, in the first century C.E., one particular set of Brahmi numerals took on the following form[iv]:

1	2	3	4	5	6	7	8	9
—	=	≡	+	h	4	7	5	?

From the 4th century on, you can actually trace several different paths that the Brahmi numerals took to get to different points and incarnations. One of those paths led to our current numeral system, and went through what are called the Gupta numerals. The Gupta numerals were prominent during a time ruled by the Gupta dynasty and were spread throughout that empire as they conquered lands during the 4th through 6th centuries. They have the following form[v]:

1	2	3	4	5	6	7	8	9
—	=	≡	५	८	६	७	८	३

How the numbers got to their Gupta form is open to considerable debate. Many possible hypotheses have been offered, most of which boil down to two basic types[vi]. The first type of hypothesis states that the numerals came from the initial letters of the names of the numbers. This is not uncommon...the Greek numerals developed in this manner. The second type of hypothesis states that they were derived from some earlier number system. However, there are other hypotheses that are offered, one of which is by the researcher Ifrah. His theory is that there were originally nine numerals, each represented by a corresponding number of vertical lines. One possibility is this:[vii]

1	2	3	4	5	6	7	8	9
I	II	III	II II	III II	III III	III III	III III	III III

Because these symbols would have taken a lot of time to write, they eventually evolved into cursive symbols that could be written more quickly. If we compare these to the Gupta numerals above, we can try to see how that evolutionary process might have taken place, but our imagination would be just about all we would have to depend upon since we do not know exactly how the process unfolded.

The Gupta numerals eventually evolved into another form of numerals called the Nagari numerals, and these continued to evolve until the 11th century, at which time they looked like this:[viii]

1	2	3	4	5	6	7	8	9	0
१	२	३	४	५	६	७	८	९	०

Note that by this time, the symbol for 0 has appeared! The Mayans in the Americas had a symbol for zero long before this, however, as we shall see later in the chapter.

These numerals were adopted by the Arabs, most likely in the eighth century during Islamic incursions into the northern part of India.[ix] It is believed that the Arabs were instrumental in spreading them to other parts of the world, including Spain (see below).

Other examples of variations up to the eleventh century include:

Devangari, eighth century[x]:

१ २ ३ ४ ५ ६ ७ ८ ९

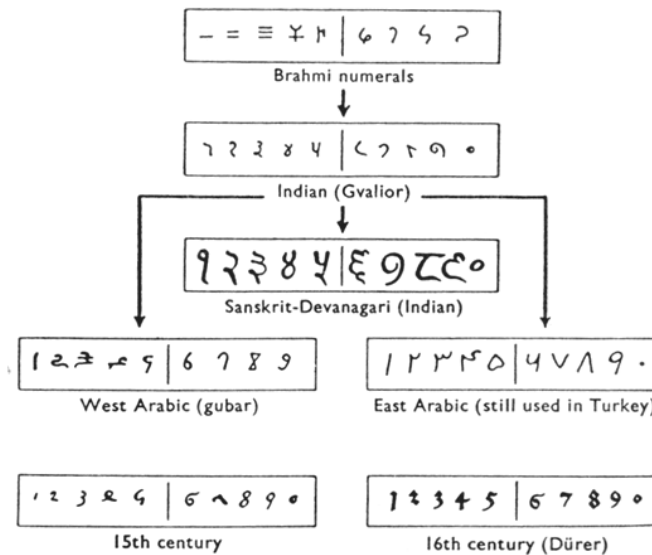
West Arab Gobar, tenth century[xi]:

1 2 3 4 5 6 7 8 9

Spain, 976 C.E.[xii]:

1 2 3 4 5 6 7 8 9

Finally, one more graphic[xiii] shows various forms of these numerals as they developed and eventually converged to the 15th century in Europe.



The Positional System

More important than the form of the number symbols is the development of the place value system. Although it is in slight dispute, the earliest known document in which the Indian system displays a positional system dates back to 346 C.E. However, some evidence suggests that they may have actually developed a positional system as far back as the first century C.E.

The Indians were not the first to use a positional system. The Babylonians used a positional system with 60 as their base. However, there is not much evidence that the Babylonian system had much impact on later numeral systems, except with the Greeks. Also, the Chinese had a base-10 system, probably derived from the use of a counting board[xiv]. Some believe that the positional system used in India was derived from the Chinese system.

Wherever it may have originated, it appears that around 600 C.E., the Indians abandoned the use of symbols for numbers higher than nine and began to use our familiar system where the position of the symbol determines its overall value.[xv] Numerous documents from the seventh century demonstrate the use of this positional system.

Interestingly, the earliest dated inscriptions using the system with a symbol for zero come from Cambodia. In 683, the 605th year of the Saka era is written with three digits and a dot in the middle. The 608th year uses three digits with a modern 0 in the middle.[xvi] The dot as a symbol for zero also appears in a Chinese work (*Chiu-chih li*). The author of this document gives a strikingly clear description of how the Indian system works:

Using the [Indian] numerals, multiplication and division are carried out. Each numeral is written in one stroke. When a number is counted to ten, it is advanced into the higher place. In each vacant place a dot is always put. Thus the numeral is always denoted in each place. Accordingly there can be no error in determining the place. With the numerals, calculations is easy..."[xvii]

Transmission to Europe

It is not completely known how the system got transmitted to Europe. Traders and travelers of the Mediterranean coast may have carried it there. It is found in a tenth-century Spanish manuscript and may have been introduced to Spain by the Arabs, who invaded the region in 711 C.E. and were there until 1492.

In many societies, a division formed between those who used numbers and calculation for practical, every day business and those who used them for ritualistic purposes or for state business.[xviii] The former might often use older systems while the latter were inclined to use the newer, more elite written numbers. Competition between the two groups arose and continued for quite some time.



In a 14th century manuscript of Boethius' *The Consolations of Philosophy*, there appears a well-known drawing of two mathematicians. One is a merchant and is using an abacus (the "abacist"). The other is a Pythagorean philosopher (the "algorist") using his "sacred" numbers. They are in a competition that is being judged by the goddess of number. By 1500 C.E., however, the newer symbols and system had won out and has persevered until today. The Seattle Times recently reported that the Hindu-Arabic numeral system has been included in the book *The Greatest Inventions of the Past 2000 Years*. [xix]

One question to answer is *why* the Indians would develop such a positional notation. Unfortunately, an answer to that question is not currently known. Some suggest that the system has its origins with the Chinese counting boards. These boards were portable and it is thought that Chinese travelers who passed through India took their boards with them and ignited an idea in Indian mathematics.[xx] Others, such as G. G. Joseph propose that it is the Indian fascination with very large numbers that drove them to develop a system whereby these kinds of big numbers could easily be written down. In this theory, the system developed entirely within the Indian mathematical framework without considerable influence from other civilizations.

[i] www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

[ii] www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Al-Biruni.html

[iii] www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

[iv] www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

[v] Ibid

[vi] Ibid

[vii] Ibid

[viii] Ibid

[ix] Katz, page 230

[x] Burton, David M., *History of Mathematics, An Introduction*, p. 254-255

[xi] Ibid

[xii] Ibid

[xiii] Katz, page 231.

[xiv] Ibid, page 230

[xv] Ibid, page 231.

[xvi] Ibid, page 232.

[xvii] Ibid, page 232.

[xviii] McLeish, p. 18

[xix] seattletimes.nwsources.com/news/health-science/html98/invs_20000201.html, Seattle Times, Feb. 1, 2000

[xx] Ibid, page 232.

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6.0.3: The Development and Use of Different Number Bases

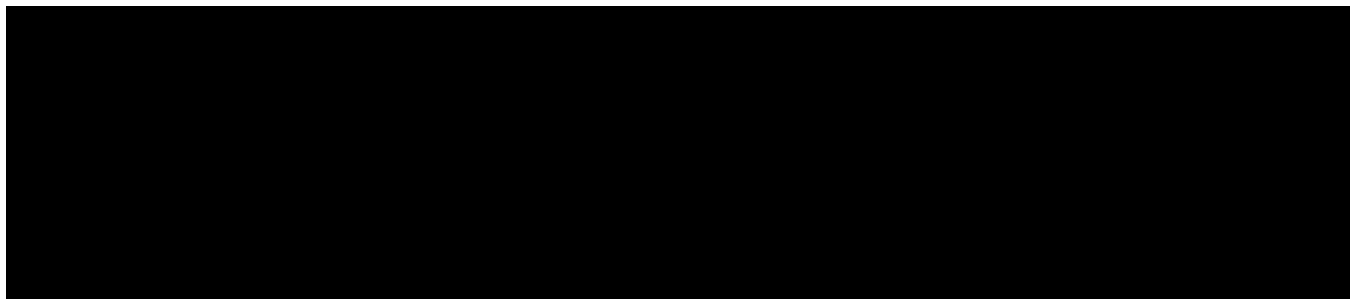
Introduction and Basics

During the previous discussions, we have been referring to positional base systems. In this section of the chapter, we will explore exactly what a base system is and what it means if a system is “positional.” We will do so by first looking at our own familiar, base-ten system and then deepen our exploration by looking at other possible base systems. In the next part of this section, we will journey back to [Mayan civilization](#) and look at their unique base system, which is based on the number 20 rather than the number 10.

A base system is a structure within which we count. The easiest way to describe a base system is to think about our own base-ten system. The base-ten system, which we call the “decimal” system, requires a total of ten different symbols/digits to write any number. They are, of course, 0, 1, 2, ..., 9.

The decimal system is also an example of a *positional* base system, which simply means that the position of a digit gives its place value. Not all civilizations had a positional system even though they did have a base with which they worked.

In our base-ten system, a number like 5,783,216 has meaning to us because we are familiar with the system and its places. As we know, there are six ones, since there is a 6 in the ones place. Likewise, there are seven “hundred thousands,” since the 7 resides in that place. Each digit has a value that is explicitly determined by its position within the number. We make a distinction between digit, which is just a symbol such as 5, and a number, which is made up of one or more digits. We can take this number and assign each of its digits a value. One way to do this is with a table, which follows:



From the third column in the table we can see that each place is simply a multiple of ten. Of course, this makes sense given that our base is ten. The digits that are multiplying each place simply tell us how many of that place we have. We are restricted to having at most 9 in any one place before we have to “carry” over to the next place. We cannot, for example, have 11 in the hundreds place. Instead, we would carry 1 to the thousands place and retain 1 in the hundreds place. This comes as no surprise to us since we readily see that 11 hundreds is the same as one thousand, one hundred. Carrying is a pretty typical occurrence in a base system.

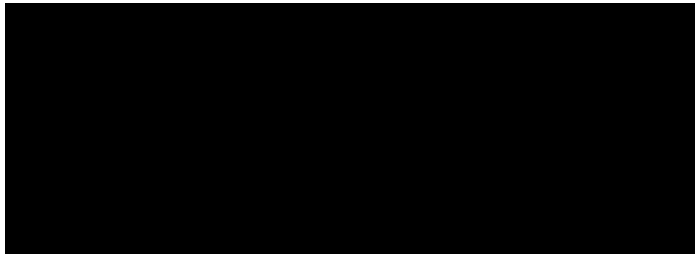
However, base-ten is not the only option we have. Practically any positive integer greater than or equal to 2 can be used as a base for a number system. Such systems can work just like the decimal system except the number of symbols will be different and each position will depend on the base itself.

Other Bases

For example, let’s suppose we adopt a base-five system. The only modern digits we would need for this system are 0,1,2,3 and 4. What are the place values in such a system? To answer that, we start with the ones place, as most base systems do. However, if we were to count in this system, we could only get to four (4) before we had to jump up to the next place. Our base is 5, after all! What is that next place that we would jump to? It would not be tens, since we are no longer in base-ten. We’re in a different numerical world. As the base-ten system progresses from 10^0 to 10^1 , so the base-five system moves from 5^0 to $5^1 = 5$. Thus, we move from the ones to the fives.

After the fives, we would move to the 5^2 place, or the twenty fives. Note that in base-ten, we would have gone from the tens to the hundreds, which is, of course, 10^2 .

Let's take an example and build a table. Consider the number 30412_5 in base five. We will write this as 30412_5 , where the subscript 5 is not part of the number but indicates the base we're using. First off, note that this is NOT the number "thirty thousand, four hundred twelve." We must be careful not to impose the base-ten system on this number. Here's what our table might look like. We will use it to convert this number to our more familiar base-ten system.



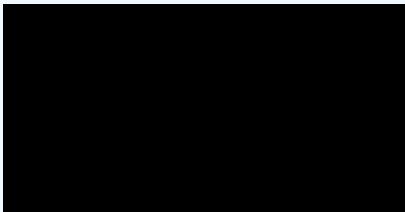
As you can see, the number 30412_5 is equivalent to 1,982 in base-ten. We will say $30412_5 = 1982_{10}$. All of this may seem strange to you, but that's only because you are so used to the only system that you've ever seen.

Example 3

Convert 6234_7 to a base 10 number.

Solution

We first note that we are given a base-7 number that we are to convert. Thus, our places will start at the ones (7^0), and then move up to the 7's, 49's (7^2), etc. Here's the breakdown:



Thus $6234_7 = 2181_{10}$

Try it Now 3

Convert 41065_7 to a base 10 number.

Answer available at the end of this section.

Converting from Base 10 to Other Bases

Converting from an unfamiliar base to the familiar decimal system is not that difficult once you get the hang of it. It's only a matter of identifying each place and then multiplying each digit by the appropriate power. However, going the other direction can be a little trickier. Suppose you have a base-ten number and you want to convert to base-five. Let's start with some simple examples before we get to a more complicated one.

Example 4

Convert twelve to a base-five number.

Solution

We can probably easily see that we can rewrite this number as follows:

$$12 = (2 \times 5) + (2 \times 1)$$

Hence, we have two fives and 2 ones. Hence, in base-five we would write twelve as 22_5 . Thus, $12_{10} = 22_5$

Example 5

Convert sixty-nine to a base-five number.

Solution

We can see now that we have more than 25, so we rewrite sixty-nine as follows:

$$69 = (2 \times 25) + (3 \times 5) + (4 \times 1)$$

Here, we have two twenty-fives, 3 fives, and 4 ones. Hence, in base five we have 234_5 . Thus, $69_{10} = 234_5$

Example 6

Convert the base-seven number 3261_7 to base 10

Solution

The powers of 7 are:

$$7^0 = 1$$

$$7^1 = 7$$

$$7^2 = 49$$

$$7^3 = 343$$

Etc...

$$3261_7 = (3 \times 343) + (2 \times 49) + (6 \times 7) + (1 \times 1) = 1170_{10}$$

Thus $3261_7 = 1170_{10}$

Try it Now 4

Convert 143 to base 5

Answer available at the end of this section.

Try it Now 5

Convert the base-three number 21021_3 to base 10.

Answer available at the end of this section.

In general, when converting from base-ten to some other base, it is often helpful to determine the highest power of the base that will divide into the given number at least once. In the last example, $5^2 = 25$ is the largest power of five that is present in 69, so that was our starting point. If we had moved to $5^3 = 125$, then 125 would not divide into 69 at least once.

Converting from Base 10 to Base b

1. Find the highest power of the base b that will divide into the given number at least once and then divide.
2. Write down the whole number part, then use the remainder from division in the next step.
3. Repeat step two, dividing by the next highest power of the base b , writing down the whole number part (including 0), and using the remainder in the next step.
4. Continue until the remainder is smaller than the base. This last remainder will be in the “ones” place.
5. Collect all your whole number parts to get your number in base b notation.

Example 7

Convert the base-ten number 348 to base-five.

Solution

The powers of five are:

$$5^0 = 1$$

$$5^1 = 5$$

$$5^2 = 25$$

$$5^3 = 125$$

$$5^4 = 625$$

Etc..

since 348 is smaller than 625, but bigger than 125, we see that $5^3 = 125$ is the highest power of five present in 348. So we divide 125 into 348 to see how many of them there are:

$$348 \div 125 = 2 \text{ with remainder } 98$$

We write down the whole part, 2, and continue with the remainder. There are 98 left over, so we see how many 25's (the next smallest power of five) there are in the remainder:

$$98 \div 25 = 3 \text{ with remainder } 23$$

We write down the whole part, 2, and continue with the remainder. There are 23 left over, so we look at the next place, the 5's:

$$23 \div 5 = 4 \text{ with remainder } 3$$

This leaves us with 3, which is less than our base, so this number will be in the “ones” place. We are ready to assemble our base-five number:

$$348 = (2 \times 5^3) + (3 \times 5^2) + (4 \times 5^1) + (3 \times 1)$$

Hence, our base-five number is 2343. We'll say that $348_{10} = 2343_5$

Example 8

Convert the base-ten number 4509 to base-seven.

Solution

The powers of 7 are:

$$7^0 = 1$$

$$7^1 = 7$$

$$7^2 = 49$$

$$7^3 = 343$$

$$7^4 = 2401$$

$$7^5 = 16807$$

Etc...

The highest power of 7 that will divide into 4509 is $7^4 = 2401$

With division, we see that it will go in 1 time with a remainder of 2108. So we have 1 in the 7^4 place.

The next power down is $7^3 = 343$, which goes into 2108 six times with a new remainder of 50. So we have 6 in the 7^3 place.

The next power down is $7^2 = 49$, which goes into 50 once with a new remainder of 1. So there is a 1 in the 7^2 place.

The next power down is 7^1 but there was only a remainder of 1, so that means there is a 0 in the 7's place and we still have 1 as a remainder.

That, of course, means that we have 1 in the ones place.

$$4509 \div 7^4 = 1 \text{ R } 2108$$

$$2108 \div 7^3 = 6 \text{ R } 50$$

$$50 \div 7^2 = 1 \text{ R } 1$$

$$1 \div 7^1 = 0 \text{ R } 1$$

$$1 \div 7^0 = 1$$

$$4509_{10} = 16101_7$$

Putting all of this together means that $4509_{10} = 16101_7$

Try it Now 6

Convert 657_{10} to a base 4 number.

Answer available at the end of this section.

Try it Now 7

Convert 8377_{10} to a base 8 number.

Answer available at the end of this section.

Another Method For Converting From Base 10 to Other Bases

As you read the solution to this last example and attempted the "Try it Now" problems, you may have had to repeatedly stop and think about what was going on. The fact that you are probably struggling to follow the explanation and reproduce the process yourself is mostly due to the fact that the non-decimal systems are so unfamiliar to you. In fact, the only system that you are probably comfortable with is the decimal system.

As budding mathematicians, you should always be asking questions like “How could I simplify this process?” In general, that is one of the main things that mathematicians do...they look for ways to take complicated situations and make them easier or more familiar. In this section we will attempt to do that.

To do so, we will start by looking at our own decimal system. What we do may seem obvious and maybe even intuitive but that's the point. We want to find a process that we readily recognize works and makes sense to us in a familiar system and then use it to extend our results to a different, unfamiliar system.

Let's start with the decimal number, 4863_{10} . We will convert this number to base 10. Yeah, I know it's already in base 10, but if you carefully follow what we're doing, you'll see it makes things work out very nicely with other bases later on. We first note that the highest power of 10 that will divide into 4863 at least once is $10^3 = 1000$. *In general, this is the first step in our new process; we find the highest power that a given base that will divide at least once into our given number.*

We now divide 1000 into 4863:

$$4863 \div 1000 = 4.863$$

This says that there are four thousands in 4863 (obviously). However, it also says that there are 0.863 thousands in 4863. This fractional part is our remainder and will be converted to lower powers of our base (10). If we take that decimal and multiply by 10 (since that's the base we're in) we get the following:

$$0.863 \times 10 = 8.63$$

Why multiply by 10 at this point? We need to recognize here that 0.863 thousands is the same as 8.63 hundreds. Think about that until it sinks in.

$$(0.863)(1000) = 863$$

$$(8.63)(100) = 863$$

These two statements are equivalent. So, what we are really doing here by multiplying by 10 is rephrasing or converting from one place (thousands) to the next place down (hundreds).

$$0.863 \times 10 \Rightarrow 8.63$$

$$(\text{Parts of Thousands}) \times 10 \Rightarrow \text{Hundreds}$$

What we have now is 8 hundreds and a remainder of 0.63 hundreds, which is the same as 6.3 tens. We can do this again with the 0.63 that remains after this first step.

$$0.63 \times 10 \Rightarrow 6.3$$

$$\text{Hundreds} \times 10 \Rightarrow \text{Tens}$$

So we have six tens and 0.3 tens, which is the same as 3 ones, our last place value.

Now here's the punch line. Let's put all of the together in one place:

$$\begin{array}{rcl}
 4863 \div 1000 & = & \textcircled{4}.863 \\
 \downarrow & & \\
 0.863 \times 10 & = & \textcircled{8}.63 \\
 \downarrow & & \\
 0.63 \times 10 & = & \textcircled{6}.3 \\
 \downarrow & & \\
 0.3 \times 10 & = & \textcircled{3}.0
 \end{array}$$

Note that in each step, the remainder is carried down to the next step and multiplied by 10, the base. Also, at each step, the whole

number part, which is circled, gives the digit that belongs in that particular place. What is amazing is that this works for any base! So, to convert from a base 10 number to some other base, b , we have the following steps we can follow:

Converting from Base 10 to Base b : Another method

1. Find the highest power of the base b that will divide into the given number at least once and then divide.
2. Keep the whole number part, and multiply the fractional part by the base b .
3. Repeat step two, keeping the whole number part (including 0), carrying the fractional part to the next step until only a whole number result is obtained.
4. Collect all your whole number parts to get your number in base b notation.

We will illustrate this procedure with some examples.

Example 9

Convert the base 10 number, 348_{10} , to base 5.

Solution

This is actually a conversion that we have done in a previous example. The powers of five are:

$$\begin{aligned} 5^0 &= 1 \\ 5^1 &= 5 \\ 5^2 &= 25 \\ 5^3 &= 125 \\ 5^4 &= 625 \end{aligned}$$

Etc...

The highest power of five that will go into 348 at least once is 5^3

We divide by 125 and then proceed.

$$\begin{aligned} 348 \div 5^3 &= \textcircled{2}.784 \\ &\downarrow \\ 0.784 \times 5 &= \textcircled{3}.92 \\ &\downarrow \\ 0.92 \times 5 &= \textcircled{4}0.6 \\ &\downarrow \\ 0.6 \times 5 &= \textcircled{3}.0 \end{aligned}$$

By keeping all the whole number parts, from top bottom, gives 2343 as our base 5 number. Thus, $2343_5 = 348_{10}$

We can compare our result with what we saw earlier, or simply check with our calculator, and find that these two numbers really are equivalent to each other.

Example 10

Convert the base 10 number, 3007_{10} , to base 5.

Solution

The highest power of 5 that divides at least once into 3007 is $5^4 = 625$. Thus, we have:

$$3007 \div 625 = (4).8112$$

$$0.8112 \times 5 = (4).056$$

$$0.056 \times 5 = (0).28$$

$$0.28 \times 5 = (1)0.4$$

$$0.4 \times 5 = (2)0.0$$

This gives us that $3007_{10} = 44012_5$. Notice that in the third line that multiplying by 5 gave us 0 for our whole number part. We don't discard that! The zero tells us that a zero in that place. That is, there are no 5^2 's in this number

This last example shows the importance of using a calculator in certain situations and taking care to avoid clearing the calculator's memory or display until you get to the very end of the process.

Example 11

Convert the base 10 number, 63201_{10} , to base 7.

Solution

The powers of 7 are:

$$7^0 = 1$$

$$7^1 = 7$$

$$7^2 = 49$$

$$7^3 = 343$$

$$7^4 = 2401$$

$$7^5 = 16807$$

etc...

The highest power of 7 that will divide at least once into 63201 is 7^5 . When we do the initial division on a calculator, we get the following:

$$63201 \div 7^5 = 3.760397453$$

The decimal part actually fills up the calculators display and we don't know if it terminates at some point or perhaps even repeats down the road. So if we clear our calculator at this point, we will introduce error that is likely to keep this process from ever ending. To avoid this problem, we leave the result in the calculator and simply subtract 3 from this to get the fractional part all by itself. DO NOT ROUND OFF! Subtraction and then multiplication by seven gives:

$$63201 \div 7^5 = (3).760397453$$

$$0.760397453 \times 7 = (5).322782174$$

$$0.322782174 \times 7 = (2).259475219$$

$$0.259475219 \times 7 = (1).816326531$$

$$0.816326531 \times 7 = (5).714285714$$

$$0.714285714 \times 7 = (5).000000000$$

Yes, believe it or not, that last product is exactly 5, *as long as you don't clear anything out on your calculator*. This gives us our final result: $63201_{10} = 352155_7$.

If we round, even to two decimal places in each step, clearing our calculator out at each step along the way, we will get a series of numbers that do not terminate, but begin repeating themselves endlessly. (Try it!) We end up with something that doesn't make any sense, at least not in this context. So be careful to use your calculator cautiously on these conversion problems.

Also, remember that if your first division is by 7^5 , then you expect to have 6 digits in the final answer, corresponding to the places for 7^5 , 7^4 and so on down to 7^0 . If you find yourself with more than 6 digits due to rounding errors, you know something went wrong

Try it Now 8

Convert the base 10 number, 9352_{10} , to base 5.

Answer available at the end of this section.

Try it Now 9

Convert the base 10 number, 1500_{10} , to base 3.

Be careful not to clear your calculator on this one. Also, if you're not careful in each step, you may not get all of the digits you're looking for, so move slowly and with caution.

Answer available at the end of this section.

Answers

Try it Now 3

$$41065_7 = 9994_{10}$$

Try it Now 4

$$143_{10} = 1033_5$$

Try it Now 5

$$21021_3 = 196_{10}$$

Try it Now 6

$$657_{10} = 22101_4$$

Try it Now 7

$$8377_{10} = 20271_8$$

Try it Now 8

$$9352_{10} = 244402_5$$

Try it Now 9

$$1500_{10} = 2001120_3$$

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6.0.4: The Mayan Numeral System

Background

As you might imagine, the development of a base system is an important step in making the counting process more efficient. Our own base-ten system probably arose from the fact that we have 10 fingers (including thumbs) on two hands. This is a natural development. However, other civilizations have had a variety of bases other than ten. For example, the Natives of Queensland used a base-two system, counting as follows: “one, two, two and one, two two’s, much.” Some Modern South American Tribes have a base-five system counting in this way: “one, two, three, four, hand, hand and one, hand and two,” and so on. The Babylonians used a base-sixty (sexagesimal) system. In this chapter, we wrap up with a specific example of a civilization that actually used a base system other than 10.



1. Chichen Itza
2. Uxmal
3. Tulum
4. Palenque
5. Bonampak, Yaxchilan
6. Tikal
7. Altun Ha
8. Copan

The [Mayan civilization](#) is generally dated from 1500 B.C.E to 1700 C.E. The Yucatan Peninsula (see map[i]) in Mexico was the scene for the development of one of the most advanced civilizations of the ancient world. The Mayans had a sophisticated ritual system that was overseen by a priestly class. This class of priests developed a philosophy with time as divine and eternal.[ii] The calendar, and calculations related to it, were thus very important to the ritual life of the priestly class, and hence the Mayan people. In fact, much of what we know about this culture comes from their calendar records and astronomy data. Another important source of information on the Mayans is the writings of Father Diego de Landa, who went to Mexico as a missionary in 1549.



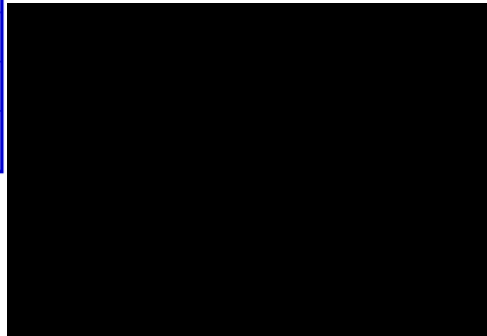
There were two numeral systems developed by the Mayans - one for the common people and one for the priests. Not only did these two systems use different symbols, they also used different base systems. For the priests, the number system was governed by ritual. The days of the year were thought to be gods, so the formal symbols for the days were decorated heads,[iii] like the sample to the left[iv] Since the basic calendar was based on 360 days, the priestly numeral system used a mixed base system employing multiples of 20 and 360. This makes for a confusing system, the details of which we will skip.

The Mayan Number System

Instead, we will focus on the numeration system of the “common” people, which used a more consistent base system. As we

stated earlier, the Mayans used a base-20 system, called the “vigesimal” system. Like our system, it is positional, meaning that the position of a numeric symbol indicates its place value. In the following table you can see the place value in its vertical format.[v]

Number	Vertical Form	Number	Vertical Form
0		10	
1		11	
2		12	
3		13	
4		14	
5		15	
6		16	
7		17	
8		18	
9		19	



In order to write numbers down, there were only three symbols needed in this system. A horizontal bar represented the quantity 5, a dot represented the quantity 1, and a special symbol (thought to be a shell) represented zero. The Mayan system may have been the first to make use of zero as a placeholder/number. The first 20 numbers are shown in the table to the right.[vi]

Unlike our system, where the ones place starts on the right and then moves to the left, the Mayan systems places the ones on the bottom of a vertical orientation and moves up as the place value increases.

When numbers are written in vertical form, there should never be more than four dots in a single place. When writing Mayan numbers, every group of five dots becomes one bar. Also, there should never be more than three bars in a single place...four bars would be converted to one dot in the next place up. It's the same as 10 getting converted to a 1 in the next place up when we carry during addition.

Example 12

What is the value of this number, which is shown in vertical form?




Solution

Starting from the bottom, we have the ones place. There are two bars and three dots in this place. Since each bar is worth 5, we have 13 ones when we count the three dots in the ones place. Looking to the place value above it (the twenties places), we see there are three dots so we have three twenties.



Hence we can write this number in base-ten as:

$$\begin{aligned}(3 \times 20^1) + (13 \times 20^0) &= (3 \times 20) + (13 \times 1) \\ &= 60 + 13 \\ &= 73\end{aligned}$$

Example 13

What is the value of the following Mayan number?



Solution

This number has 11 in the ones place, zero in the 20's place, and 18 in the $20^2=400$'s place. Hence, the value of this number in base-ten is:

$$18 \times 400 + 0 \times 20 + 11 \times 1 = 7211$$

Try it Now 10

Convert the Mayan number below to base 10.



Answer available at the end of this section.

Example 14

Convert the base 10 number 3575_{10} to Mayan numerals.

Solution

This problem is done in two stages. First we need to convert to a base 20 number. We will do so using the method provided in the last section of the text. The second step is to convert that number to Mayan symbols.

The highest power of 20 that will divide into 3575 is $20^2 = 400$, so we start by dividing that and then proceed from there:

$$3575 \div 400 = 8.9375$$

$$0.9375 \times 20 = 18.75$$

$$(0.75 \times 20 = 15.0$$

This means that $3575_{10} = 8, 18, 15_{20}$

The second step is to convert this to Mayan notation. This number indicates that we have 15 in the ones position. That's three bars at the bottom of the number. We also have 18 in the 20's place, so that's three bars and three dots in the second position. Finally, we have 8 in the 400's place, so that's one bar and three dots on the top. We get the following



Note that in the previous example a new notation was used when we wrote $8, 18, 15_{20}$. The commas between the three numbers 8, 18, and 15 are now separating place values for us so that we can keep them separate from each other. This use of the comma is slightly different than how they're used in the decimal system. When we write a number in base 10, such as 7,567,323, the commas are used primarily as an aide to read the number easily but they do not separate single place values from each other. We will need this notation whenever the base we use is larger than 10.

Writing numbers with bases bigger than 10

When the base of a number is larger than 10, separate each "digit" with a comma to make the separation of digits clear.

For example, in base 20, to write the number corresponding to $17 \times 20^2 + 6 \times 20^1 + 13 \times 20^0$, we'd write $17, 6, 13_{20}$.

Try it Now 11

Convert the base 10 number 10553_{10} to Mayan numerals.

Answer available at the end of this section.

Try it Now 12

Convert the base 10 number 5617 to Mayan numerals.

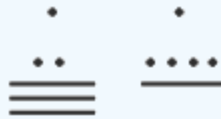
Answer available at the end of this section.

Adding Mayan Numbers

When adding Mayan numbers together, we'll adopt a scheme that the Mayans probably did not use but which will make life a little easier for us.

Example 15

Add, in Mayan, the numbers 37 and 29: [vii]



Solution

First draw a box around each of the vertical places. This will help keep the place values from being mixed up.



Next, put all of the symbols from both numbers into a single set of places (boxes), and to the right of this new number draw a set of empty boxes where you will place the final sum:

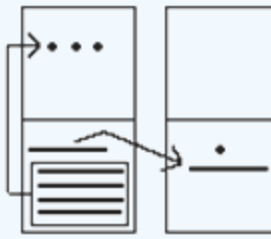


You are now ready to start carrying. Begin with the place that has the lowest value, just as you do with Arabic numbers. Start at the bottom place, where each dot is worth 1. There are six dots, but a maximum of four are allowed in any one place; once you get to five dots, you must convert to a bar. Since five dots make one bar, we draw a bar through five of the dots, leaving us with one dot which is under the four-dot limit. Put this dot into the bottom place of the empty set of boxes you just drew:

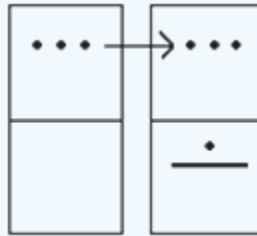


Now look at the bars in the bottom place. There are five, and the maximum number the place can hold is three. *Four bars are equal to one dot in the next highest place.*

Whenever we have four bars in a single place we will automatically convert that to a *dot* in the next place up. We draw a circle around four of the bars and an arrow up to the dots' section of the higher place. At the end of that arrow, draw a new dot. That dot represents 20 just the same as the other dots in that place. Not counting the circled bars in the bottom place, there is one bar left. One bar is under the three-bar limit; put it under the dot in the set of empty places to the right.



Now there are only three dots in the next highest place, so draw them in the corresponding empty box.



We can see here that we have 3 twenties (60), and 6 ones, for a total of 66. We check and note that $37 + 29 = 66$, so we have done this addition correctly. Is it easier to just do it in base-ten? Probably, but that's only because it's more familiar to you. Your task here is to try to learn a new base system and how addition can be done in slightly different ways than what you have seen in the past. Note, however, that the concept of carrying is still used, just as it is in our own addition algorithm.

Try it Now 13

Try adding 174 and 78 in Mayan by first converting to Mayan numbers and then working entirely within that system. Do not add in base-ten (decimal) until the very end when you *check* your work.

Answer available at the end of this section.

Conclusion

In this chapter, we have briefly sketched the development of numbers and our counting system, with the emphasis on the “brief” part. There are numerous sources of information and research that fill many volumes of books on this topic. Unfortunately, we cannot begin to come close to covering all of the information that is out there.

We have only scratched the surface of the wealth of research and information that exists on the development of numbers and counting throughout human history. What is important to note is that the system that we use every day is a product of thousands of years of progress and development. It represents contributions by many civilizations and cultures. It does not come down to us from the sky, a gift from the gods. It is not the creation of a textbook publisher. It is indeed as human as we are, as is the rest of mathematics. Behind every symbol, formula and rule there is a human face to be found, or at least sought.

Furthermore, we hope that you now have a basic appreciation for just how interesting and diverse number systems can get. Also, we're pretty sure that you have also begun to recognize that we take our own number system for granted so much that when we try to adapt to other systems or bases, we find ourselves truly having to concentrate and think about what is going on.

[i] www.gorp.com/gorp/location/latamer/map_maya.htm

[ii] Bidwell, James; Mayan Arithmetic in *Mathematics Teacher*, Issue 74 (Nov., 1967), p. 762-68.

[iii] www.ukans.edu/~lctls/Mayan/numbers.html

[iv] www.ukans.edu/~lctls/Mayan/numbers.html

[v] Bidwell

[vi] www.vpds.wsu.edu/fair_95/gym/UM001.html

[vii] forum.swarthmore.edu/k12/mayan.math/mayan2.html

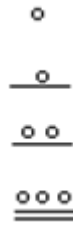
Answers

Try it Now 10

1562

Try it Now 11

$10553_{10} = 1, 6, 7, 13_{20}$

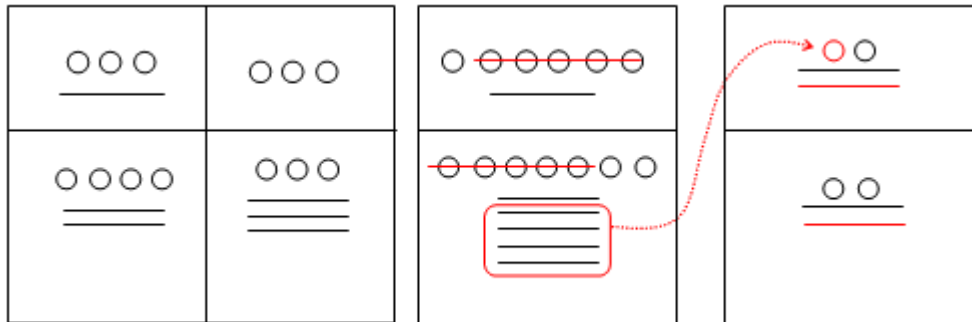


Try it Now 12

$5617_{10} = 14, 0, 17_{20}$. Note that there is a zero in the 20's place, so you'll need to use the appropriate zero symbol in between the ones and 400's places.

Try it Now 13

A sample solution is shown.



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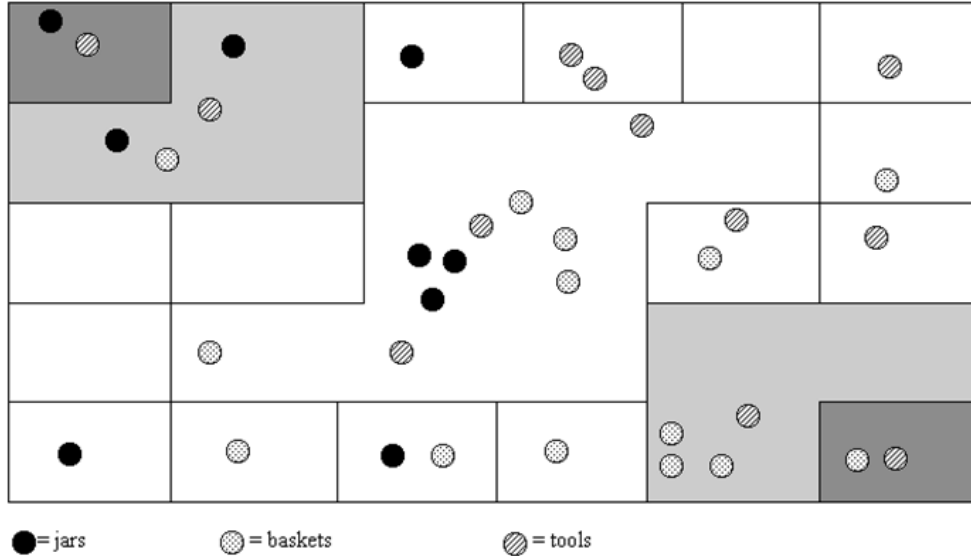
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6.0.5: Exercise

Skills

Counting Board And Quipu

1) In the following Peruvian counting board, determine how many of each item is represented. Please show all of your calculations along with some kind of explanation of how you got your answer. Note the key at the bottom of the drawing.



2) Draw a [quipu](#) with a main cord that has branches (H cords) that show each of the following numbers on them. (You should produce one drawing for this problem with the cord for part **a** on the left and moving to the right for parts **b** through **d**.)

a. 232	b. 5065
c. 23451	d. 3002

Basic Base Conversions

3) 423 in base 5 to base 10	4) 3044 in base 5 to base 10
5) 387 in base 10 to base 5	6) 2546 in base 10 to base 5
7) 110101 in base 2 to base 10	8) 11010001 in base 2 to base 10
9) 100 in base 10 to base 2	10) 2933 in base 10 to base 2
11) Convert 653 in base 7 to base 10	12) Convert 653 in base 10 to base 7
13) 3412 in base 5 to base 2	14) 10011011 in base 2 to base 5

(Hint: convert first to base 10 then to the final desired base)

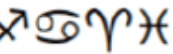
The Caidoz System

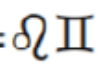
Suppose you were to discover an ancient base-12 positional system made up twelve symbols. Let's call this base system the Caidoz system. Here are the symbols for each of the numbers 0 through 12:

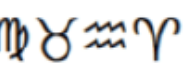
0 = 	6 = 
1 = 	7 = 
2 = 	8 = 
3 = 	9 = 
4 = 	10 = 
5 = 	11 = 

Convert each of the following numbers in Caidoz to base 10

15) 

16) 

17) 

18) 

Convert the following base 10 numbers to Caidoz, using the symbols shown above.

19) 175	20) 3030
21) 10000	22) 5507

Mayan Conversions

Convert the following numbers to Mayan notation. Show your calculations used to get your answers.

23) 135	24) 234
25) 360	26) 1215
27) 10500	28) 1100000

Convert the following Mayan numbers to decimal (base-10) numbers. Show all calculations.

29. 

30. 

31. 

32. 

James Bidwell has suggested that Mayan addition was done by “simply combining bars and dots and carrying to the next higher place.” He goes on to say, “After the combining of dots and bars, the second step is to exchange every five dots for one bar in the same position.” After converting the following base 10 numbers into vertical Maya notation (in base 20, of course), perform the indicated addition:

33) 32 + 11	34) 82 + 15
35) 35 + 148	36) 2412 + 5000

37) $450 + 844$	38) $10000 + 20000$
39) $4500 + 3500$	40) $130000 + 30000$

41) Use the fact that the Mayans had a base-20 number system to complete the following multiplication table. The table entries should be in Mayan notation. Remember: Their zero looked like this... . Xerox and then cut out the table below, fill it in, and paste it onto your homework assignment if you do not want to duplicate the table with a ruler.

(To think about but not write up: Bidwell claims that only these entries are needed for “Mayan multiplication.” What does he mean?)

Binary and Hexadecimal Conversions

Modern computers operate in a world of “on” and “off” electronic switches, so use a **binary** counting system – base 2, consisting of only two digits: 0 and 1.

Convert the following binary numbers to decimal (base-10) numbers.

42) 1001	43) 1101
44) 110010	45) 101110

Convert the following base-10 numbers to binary

46) 7	47) 12
48) 36	49) 27

Four binary digits together can represent any base-10 number from 0 to 15. To create a more human-readable representation of binary-coded numbers, hexadecimal numbers, base 16, are commonly used. Instead of using the $8, 13, 12_{16}$ notation used earlier, the letter A is used to represent the digit 10, B for 11, up to F for 15, so $8, 13, 12_{16}$ would be written as 8DC.

Convert the following hexadecimal numbers to decimal (base-10) numbers.

50) C3	51) 4D
52) 3A6	53) BC2

Convert the following base-10 numbers to hexadecimal

54) 152	55) 176
56) 2034	57) 8263

Exploration

58) What are the advantages and disadvantages of bases other than ten.

59) Supposed you are charged with creating a base-15 number system. What symbols would you use for your system and why? Explain with at least two specific examples how you would convert between your base-15 system and the decimal system.

60) Describe an interesting aspect of Mayan civilization that we did not discuss in class. Your findings must come from some source such as an encyclopedia article, or internet site and you must provide reference(s) of the materials you used (either the publishing information or Internet address).

61) For a Papuan tribe in southeast New Guinea, it was necessary to translate the bible passage John 5:5 “And a certain man was there, which had an infirmity 30 and 8 years” into “A man lay ill one man, both hands, five and three years.” Based on your

own understanding of bases systems (and some common sense), furnish an explanation of the translation. Please use complete sentences to do so. (Hint: To do this problem, I am asking you to think about how base systems work, where they come from, and how they are used. You won't necessarily find an "answer" in readings or such...you'll have to think it through and come up with a reasonable response. Just make sure that you clearly explain why the passage was translated the way that it was.)

62) The Mayan calendar was largely discussed leading up to December 2012. Research how the Mayan calendar works, and how the counts are related to the number based they use.

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6.1: FINANCE

6.1.0: Introduction

6.1.1: Simple Interest

6.1.2: Compound Interest

6.1.3: Annuities

6.1.4: Payout Annuities

6.1.5: Loans

6.1.6: Remaining Loan Balance

6.1.7: Which equation to use?

6.1.8: Solving for time

6.1.9: Exercise

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6.1.0: Introduction

We have to work with money every day. While balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more mathematics.

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6.1.1: Simple Interest

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed \$100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = \$5$. The total amount you would repay would be \$105, the original principal plus the interest.

Simple One-time Interest

$$I = P_0 r$$

$$A = P_0 + I = P_0 + P_0 r = P_0(1 + r)$$

where

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate (in decimal form. Example: 5% = 0.05)

Example 1

A friend asks to borrow \$300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

Solution

$P_0 = \$300$ the principal
 $r = 0.03$ 3% rate
 $I = \$300(0.03) = \$9$. You will earn \$9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a \$1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Solution

Each year, you would earn 5% interest: $1000(0.05) = \$50$ in interest. So over the course of five years, you would earn a total of \$250 in interest. When the bond matures, you would receive back the \$1,000 you originally paid, leaving you with a total of \$1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time

$$I = P_0 r t$$

$$A = P_0 + I = P_0 + P_0rt = P_0(1 + rt)$$

where

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate in decimal form
- t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.
A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3: Treasury Notes

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a \$1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Solution

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$P_0 = \$1000$ the principal
 $r = 0.02$ 2% rate per half-year
 $t = 8$ 4 years = 8 half-years
 $I = \$1000(0.02)(8) = \$160$. You will earn \$160 interest total over the four years.

Try it Now 1

A loan company charges \$30 interest for a one month loan of \$500. Find the annual interest rate they are charging.

Answer available at the end of this section.

Answers

Try it Now 1

$I = \$30$ of interest
 $P_0 = \$500$ principal
 $r = \frac{30}{500} = 0.06$ per month
 $(0.06)(12) = 0.72$ per year
They are charging an annual interest rate of 72%.

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6.1.2: Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit \$1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3\%}{12} = 0.25\%$ per month.

In the first month,

$$P_0 = \$1000$$

$$r = 0.0025(0.25\%)$$

$$I = \$1000(0.0025) = \$2.50$$

$$A = \$1000 + \$2.50 = \$1002.50$$

In the first month, we will earn \$2.50 in interest, raising our account balance to \$1002.50.

In the second month,

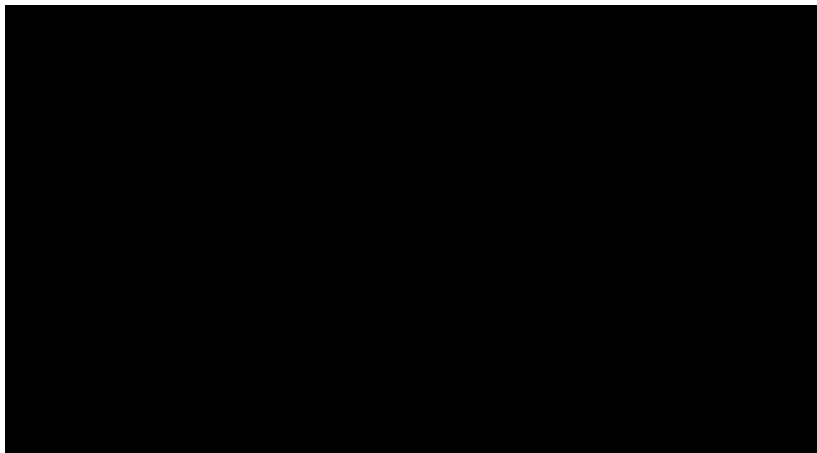
$$P_0 = \$1002.50$$

$$I = \$1002.50(0.0025) = \$2.51 \text{ (rounded)}$$

$$A = \$1000 + \$2.50 = \$1002.50$$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original \$1000 we deposited, but we also earned interest on the \$2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:



To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

$$P_0 = \$1000$$

$$P_m = (1 + 0.0025)P_{m-1}$$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$$P_0 = \$1000$$

$$P_1 = 1.0025P_0 = 1.0025(1000)$$

$$P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = 1.0025^2(1000)$$

$$P_3 = 1.0025P_2 = 1.0025(1.0025^2(1000)) = 1.0025^3(1000)$$

$$P_4 = 1.0025P_3 = 1.0025(1.0025^3(1000)) = 1.0025^4(1000)$$

Observing a pattern, we could conclude

$$P_m = (1.0025)^m(\$1000)$$

Notice that the ~~\$1000~~ in the equation was P_0 , the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

$$P_m = P_0 \left(1 + \frac{r}{k}\right)^m$$

In this formula:

m is the number of compounding periods (months in our example)

r is the annual interest rate

k is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If N is the number of years, then $m = Nk$. Making this change gives us the standard formula for compound interest.

Compound Interest

$$P_N = P_0 \left(1 + \frac{r}{k}\right)^{Nk}$$

P_N is the balance in the account after N years.

P_0 is the starting balance of the account (also called initial deposit, or principal)

r is the annual interest rate in decimal form

k is the number of compounding periods in one year.

If the compounding is done annually (once a year), $k = 1$.

If the compounding is done quarterly, $k = 4$.

If the compounding is done monthly, $k = 12$.

If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let

it sit there earning interest.

Example 4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

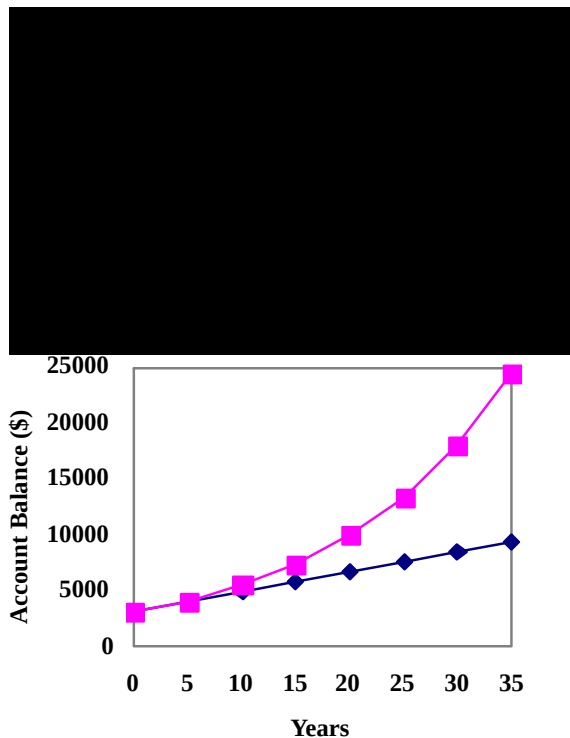
Solution

In this example,

$P_0 = \$3000$ the initial deposit
 $r = 0.06$ 6% annual rate
 $k = 12$ 12 months in 1 year
 $N = 20$ since we're looking for how much we'll have after 20 years

So $P_{20} = 3000 \left(1 + \frac{0.06}{12}\right)^{20 \times 12} = \$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Evaluating exponents on the calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$. But when we need to calculate something like 1.005^{240} , it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

$[\wedge]$, $[y^x]$, or $[x^y]$

To evaluate 1.005^{240} we'd type $1.005 [\wedge] 240$, or $1.005 [y^x] 240$. Try it out - you should get something around 3.3102044758.

Example 5

You know that you will need \$40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Solution

We're looking for P_0 .

$r = 0.04$ 4%
 $k = 4$ 4 quarters in 1 year
 $N = 18$ Since we know the balance in 18 years
 $P_{18} = \$40,000$ The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for P_0 .

$$40000 = P_0 \left(1 + \frac{0.04}{4} \right)^{18 \times 4}$$

$$40000 = P_0 (2.0471)$$

$$P_0 = \frac{40000}{2.0471} = \$19539.84$$

So you would need to deposit \$19,539.84 now to have \$40,000 in 18 years.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better.

Example 6

To see why not over-rounding is so important, suppose you were investing \$1000 at 5% interest compounded monthly for 30 years.

Solution

$P_0 = \$1000$ the initial deposit
 $r = 0.05$ 5%
 $k = 12$ 12 months in 1 year
 $N = 30$ since we're looking for the amount after 30 years

If we first compute $\frac{r}{k}$, we find $\frac{0.05}{12} = 0.0041666666667$

Here is the effect of rounding this to different values:



If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough - \$5 off of \$4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

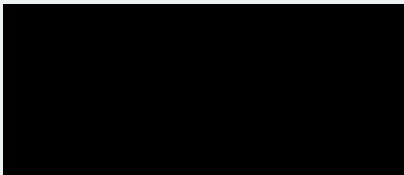
Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$$P_{30} = 1000 \left(1 + \frac{0.05}{12} \right)^{12 \times 30}$$

We can quickly calculate $12 \times 30 = 360$, giving $P_{30} = 1000 \left(1 + \frac{0.05}{12} \right)^{360}$.

Now we can use the calculator.



Using your calculator continued

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 [×] (1 [+] 0.05 [÷] 12) [y^x] 360 [=]

6.1.3: Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1}$$

For a savings annuity, we simply need to add a deposit, d , to the account with each compounding period:

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1} + d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit \$100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$$r = 0.06 \text{ (6\%)}$$

$$k = 12 \text{ (12 compounds/deposits per year)}$$

$$d = \$100 \text{ (our deposit per month)}$$

Writing out the recursive equation gives

$$P_m = \left(1 + \frac{0.06}{12}\right) P_{m-1} + 100 = (1.005)P_{m-1} + 100$$

Assuming we start with an empty account, we can begin using this relationship:

$$P_0 = 0$$

$$P_1 = (1.005)P_0 + 100 = 100$$

$$P_2 = (1.005)P_1 + 100 = (1.005)(100) + 100 = 100(1.005) + 100$$

$$P_3 = (1.005)P_2 + 100 = (1.005)(100(1.005) + 100) + 100 = 100(1.005)^2 + 100(1.005) + 100$$

Continuing this pattern, after m deposits, we'd have saved:

$$P_m = 100(1.005)^{m-1} + 100(1.005)^{m-2} + \dots + 100(1.005) + 100$$

In other words, after m months, the first deposit will have earned compound interest for $m - 1$ months. The second deposit will have earned interest for $m - 2$ months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005P_m = 1.005(100(1.005)^{m-1} + 100(1.005)^{m-2} + \dots + 100(1.005) + 100)$$

Distributing on the right side of the equation gives

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \dots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$\begin{aligned} 1.005P_m &= 100(1.005)^m + 100(1.005)^{m-1} + \dots + 100(1.005) \\ P_m &= 100(1.005)^{m-1} + \dots + 100(1.005) + 100 \end{aligned}$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005P_m - P_m = 100(1.005)^m - 100$$

Solving for P_m

$$0.005P_m = 100((1.005)^m - 1)$$

$$P_m = \frac{100((1.005)^m - 1)}{0.005}$$

Replacing m months with $12N$, where N is measured in years, gives

$$P_N = \frac{100((1.005)^{12N} - 1)}{0.005}$$

Recall 0.005 was $\frac{r}{k}$ and 100 was the deposit d . 12 was k , the number of deposit each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula

$$P_N = \frac{d((1+\frac{r}{k})^{Nk} - 1)}{(\frac{r}{k})}$$

P_N is the balance in the account after N years.

d is the regular deposit (the amount you deposit each year, each month, etc.)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, $k = 12$.

If you make your deposits every year, use yearly compounding, $k = 1$.

If you make your deposits every quarter, use quarterly compounding, $k = 4$.

Etc.

When do you use this

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Example 7

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit \$100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

Solution

In this example,

$d = \$100$ the monthly deposit
 $r = 0.06$ 6% annual rate
 $k = 12$ since we're doing monthly deposits, we'll compound monthly
 $N = 20$ we want the amount after 20 years

Putting this into the equation:

$$P_{20} = \frac{100 \left(\left(1 + \frac{0.06}{12} \right)^{20(12)} - 1 \right)}{\left(\frac{0.06}{12} \right)}$$

$$P_{20} = \frac{100((1.005)^{240} - 1)}{(0.005)}$$

$$P_{20} = \frac{100(3.310 - 1)}{(0.005)}$$

$$P_{20} = \frac{100(2.310)}{(0.005)} = \$46200$$

The account will grow to \$46,200 after 20 years.

Notice that you deposited into the account a total of \$24,000 (\$100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $\$46,200 - \$24,000 = \$22,200$.

Example 8

You want to have \$200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Solution

In this example,

We're looking for d .

$r = 0.08$ 8% annual rate
 $k = 12$ since we're doing monthly deposits, we'll compound monthly
 $N = 30$ 30 years
 $P_{30} = \$200,000$ The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for d .

$$200,000 = \frac{d \left(\left(1 + \frac{0.08}{12} \right)^{30(12)} - 1 \right)}{\left(\frac{0.08}{12} \right)}$$

$$200,000 = \frac{d \left((1.00667)^{360} - 1 \right)}{(0.00667)}$$

$$200,000 = d(1491.57)$$

$$d = \frac{200,000}{1491.57} = \$134.09$$

So you would need to deposit \$134.09 each month to have \$200,000 in 30 years if your account earns 8% interest

Try it Now 2

A more conservative investment account pays 3% interest. If you deposit \$5 a day into this account, how much will you have after 10 years? How much is from interest?

Answer available at the end of this section.

Answers

Try it Now 2

$d = \$5$ the daily deposit

$r = 0.03$ 3% annual rate

$k = 365$ since we're doing daily deposits, we'll compound daily

$N = 10$ we want the amount after 10 years

$$P_{10} = \frac{5 \left(\left(1 + \frac{0.03}{365} \right)^{365 \times 10} - 1 \right)}{\frac{0.03}{365}} = \$21,282.07$$

We would have deposited a total of $\$5 \cdot 365 \cdot 10 = \$18,250$, so \$3,032.07 is from interest

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6.1.4: Payout Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved \$500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$P_0 = \frac{d(1 - (1 + \frac{r}{k})^{-Nk})}{(\frac{r}{k})}$$

P_0 is the balance in the account at the beginning (starting amount, or principal).

d is the regular withdrawal (the amount you take out each year, each month, etc.)

r is the annual interest rate (in decimal form. Example: 5% = 0.05)

k is the number of compounding periods in one year.

N is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Example 9

After retiring, you want to be able to take \$1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Solution

In this example,

$d = \$1000$ the monthly withdrawal
 $r = 0.06$ 6% annual rate
 $k = 12$ since we're doing monthly withdrawals, we'll compound monthly
 $N = 20$ since we're taking withdrawals for 20 years

We're looking for P_0 ; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$P_0 = \frac{1000 \left(1 - \left(1 + \frac{0.06}{12} \right)^{-20(12)} \right)}{\left(\frac{0.06}{12} \right)}$$

$$P_0 = \frac{1000 \times (1 - (1.005)^{-240})}{(0.005)}$$

$$P_0 = \frac{1000 \times (1 - 0.302)}{(0.005)} = \$139,600$$

You will need to have \$139,600 in your account when you retire.

Notice that you withdrew a total of \$240,000 (\$1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is $\$240,000 - \$139,600 = \$100,400$ in interest.

Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this $[(-)]$, some with $[+/-]$. The button is often near the $=$ key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005^{-240} you'd type something like: $1.005[\wedge][(-)]240$

If your calculator only shows one value at a time, then usually you hit the $(-)$ key after a number to negate it, so you'd hit: $1.005[y^x]240[(-)] =$

Give it a try - you should get $1.005^{-240} = 0.302096$

Example 10

You know you will have \$500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

Solution

In this example,

We're looking for d .

$r = 0.08$ 8% annual rate
 $k = 12$ since we're doing monthly withdrawals
 $N = 30$ since we're taking withdrawals for 30 years
 $P_0 = \$500,000$ we are beginning with \$500,000

In this case, we're going to have to set up the equation, and solve for d .

$$500,000 = \frac{d \left(1 - \left(1 + \frac{0.08}{12} \right)^{-30(12)} \right)}{\left(\frac{0.08}{12} \right)}$$

$$500,000 = \frac{d(1-(1.00667)^{-360})}{(0.00667)}$$

$$500,000 = d(136.232)$$

$$d = \frac{500,000}{136.232} = \$3670.21$$

You would be able to withdraw \$3,670.21 each month for 30 years.

Try it Now 3

A donor gives \$100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Answer available at the end of this section.

Answers

Try it Now 3

$d =$ unknown

$r = 0.04$ 4% annual rate

$k = 1$ since we're doing annual scholarships

$N = 20$ since we're taking withdrawals for 20 years

$P_0 = \$100,000$ we are starting with \$100,000

$$100,000 = \frac{d(1-(1+\frac{0.04}{1})^{-20 \times 1})}{\frac{0.04}{1}}$$

Solving for d gives \$7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, $\$100,000(0.04) = \$4,000$ a year would have been available.

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6.1.5: Loans

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had \$10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests \$10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula

$$P_0 = \frac{d(1 - (1 + \frac{r}{k})^{-Nk})}{(\frac{r}{k})}$$

P_0 is the balance in the account at the beginning (the principal, or amount of the loan).

d is your loan payment (your monthly payment, annual payment, etc)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

N is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Loans: Many payments

Example 11

You can afford \$200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with \$200 per month?

Solution

In this example,

$d = \$200$ the monthly loan payment
 $r = 0.03$ 3% annual rate
 $k = 12$ since we're doing monthly payments, we'll compound monthly
 $N = 5$ since we're making monthly payments for 5 years

We're looking for P_0 , the starting amount of the loan.

$$P_0 = \frac{200 \left(1 - \left(1 + \frac{0.03}{12} \right)^{-5(12)} \right)}{\left(\frac{0.03}{12} \right)}$$

$$P_0 = \frac{200(1 - (1.0025)^{-60})}{(0.0025)}$$

$$P_0 = \frac{200(1 - 0.861)}{(0.0025)} = \$11,120$$

You can afford a \$11,120 loan.

You will pay a total of \$12,000 (\$200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying $\$12,000 - \$11,120 = \$880$ interest total.

Example 12

You want to take out a \$140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Solution

In this example,

We're looking for d .

$r = 0.06$ 6% annual rate
 $k = 12$ since we're doing monthly payments, we'll compound monthly
 $N = 30$ since we're making monthly payments for 30 years
 $P_0 = \$140,000$ the starting loan amount

In this case, we're going to have to set up the equation, and solve for d .

$$140,000 = \frac{d \left(1 - \left(1 + \frac{0.06}{12} \right)^{-30(12)} \right)}{\left(\frac{0.06}{12} \right)}$$

$$140,000 = \frac{d(1 - (1.005)^{-360})}{(0.005)}$$

$$140,000 = d(166.792)$$

$$d = \frac{140,000}{166.792} = \$839.37$$

You will make payments of \$839.37 per month for 30 years.

You're paying a total of \$302,173.20 to the loan company: \$839.37 per month for 360 months. You are paying a total of $\$302,173.20 - \$140,000 = \$162,173.20$ in interest over the life of the loan.

Try it Now 4

Janine bought \$3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Answer available at the end of this section.

Answers

Try it Now 4

$d =$ unknown

$r = 0.16$ 16% annual rate

$k = 12$ since we're doing monthly payments, we'll compound monthly

$N = 2$ 2 year to repay

$P_0 = 3,000$ the starting loan amount \$3,000 loan

$$3,000 = \frac{d \left(1 - \left(1 + \frac{0.16}{12} \right)^{-2 \times 12} \right)}{\frac{0.16}{12}}$$

Solving for d gives \$146.89 as monthly payments.

In total, she will pay \$3,525.36 to the store, meaning she will pay \$525.36 in interest over the two years.

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6.1.6: Remaining Loan Balance

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were \$1,000 a month, after a year you will *not* have paid off \$12,000 of the loan balance.

To determine the remaining loan balance, we can think "how much loan will these loan payments be able to pay off in the remaining time on the loan?"

Example 13

If a mortgage at a 6% interest rate has payments of \$1,000 a month, how much will the loan balance be 10 years from the end the loan?

Solution

To determine this, we are looking for the amount of the loan that can be paid off by \$1,000 a month payments in 10 years. In other words, we're looking for P_0 when

$d = \$1,000$ the monthly loan payment
 $r = 0.06$ 6% annual rate
 $k = 12$ since we're doing monthly payments, we'll compound monthly
 $N = 10$ since we're making monthly payments for 10 more years

$$\begin{aligned}
 P_0 &= \frac{1000 \left(1 - \left(1 + \frac{0.06}{12} \right)^{-10(12)} \right)}{\left(\frac{0.06}{12} \right)} \\
 &= \frac{1000 (1 - (1.005)^{-120})}{(0.005)} \\
 &= \frac{1000 (1 - (1.005)^{-120})}{(0.005)} \\
 &= \frac{1000(1 - 0.5496)}{(0.005)} \\
 &= \$90,073.45
 \end{aligned}$$

The loan balance with 10 years remaining on the loan will be \$90,073.45

Often times answering remaining balance questions requires two steps:

1. Calculating the monthly payments on the loan
2. Calculating the remaining loan balance based on the *remaining time* on the loan

Example 14

A couple purchases a home with a \$180,000 mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

Solution

First we will calculate their monthly payments.

We're looking for d .

$r = 0.04$ 4% annual rate
 $k = 12$ since they're paying monthly
 $N = 30$ 30 years
 $P_0 = \$180,000$ the starting loan amount

We set up the equation and solve for d .

$$\begin{aligned}
 180,000 &= \frac{d \left(1 - \left(1 + \frac{0.04}{12} \right)^{-30(12)} \right)}{\left(\frac{0.04}{12} \right)} \\
 180,000 &= \frac{d (1 - (1.00333)^{-360})}{(0.00333)} \\
 180,000 &= d(209.562) \\
 d &= \frac{180,000}{209.562} \\
 &= \$858.93
 \end{aligned}$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

$d = \$858.93$ the monthly loan payment we calculated above
 $r = 0.04$ 4% annual rate
 $k = 12$ since they're paying monthly
 $N = 25$ since they'd be making monthly payments for 25 more years

$$\begin{aligned}
 P_0 &= \frac{858.93 \left(1 - \left(1 + \frac{0.04}{12} \right)^{-25(12)} \right)}{\left(\frac{0.04}{12} \right)} \\
 &= \frac{858.93 (1 - (1.00333)^{-300})}{(0.00333)} \\
 &= \frac{858.93(1 - 0.369)}{(0.00333)} \\
 &\approx \$162,758
 \end{aligned}$$

The loan balance after 5 years, with 25 years remaining on the loan, will be \$162,758

Over that 5 years, the couple has paid off $\$180,000 - \$162,758 = \$17,242$ of the loan balance. They have paid a total of $\$858.93$ a month for 5 years (60 months), for a total of $\$51,535.80$, so $\$51,535.80 - \$17,242 = \$34,292.80$ of what they have paid so far has been interest.

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6.1.7: Which equation to use?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan", "amortize" (the fancy word for loans), "finance (a car)", or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a basic Annuity problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a Payout Annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

Try it Now 5

For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loans problem. Then solve each problem.

- Marcy received an inheritance of \$20,000, and invested it at 6% interest. She is going to use it for college, withdrawing money for tuition and expenses each quarter. How much can she take out each quarter if she has 3 years of school left?
- Paul wants to buy a new car. Rather than take out a loan, he decides to save \$200 a month in an account earning 3% interest compounded monthly. How much will he have saved up after 3 years?
- Keisha is managing investments for a non-profit company. They want to invest some money in an account earning 5% interest compounded annually with the goal to have \$30,000 in the account in 6 years. How much should Keisha deposit into the account?
- Miao is going to finance new office equipment at a 2% rate over a 4 year term. If she can afford monthly payments of \$100, how much new equipment can she buy?
- How much would you need to save every month in an account earning 4% interest to have \$5,000 saved up in two years?

Answer available at the end of this section.

Answers

Try it Now 5

- This is a payout annuity problem. She can pull out \$1833.60 a quarter.
- This is a savings annuity problem. He will have saved up \$7,524.11.
- This is compound interest problem. She would need to deposit \$22,386.46.
- This is a loans problem. She can buy \$4,609.33 of new equipment.
- This is a savings annuity problem. You would need to save \$200.46 each month

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6.1.8: Solving for time

Often we are interested in how long it will take to accumulate money or how long we'd need to extend a loan to bring payments down to a reasonable level.

Note: This section assumes you've covered solving exponential equations using logarithms, either in prior classes or in the growth models chapter.

Example 15

If you invest \$2000 at 6% compounded monthly, how long will it take the account to double in value?

Solution

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

$$\begin{aligned} P_0 &= \$2000 && \text{the initial deposit} \\ r &= 0.06 && \text{6\% annual rate} \\ k &= 12 && \text{12 months in 1 year} \end{aligned}$$

So our general equation is $P_N = 2000 \left(1 + \frac{0.06}{12}\right)^{N \times 12}$. We also know that we want our ending amount to be double of \$2000, which is \$4000, so we're looking for N so that $P_N = 4000$. To solve this, we set our equation for P_N equal to 4000.

$$\begin{aligned} 4000 &= 2000 \left(1 + \frac{0.06}{12}\right)^{N \times 12} && \text{Divide both sides by 2000} \\ 2 &= (1.005)^{12N} && \text{To solve for the exponent, take the log of both sides} \\ \log(2) &= \log((1.005)^{12N}) && \text{Use the exponent property of logs on the right side} \\ \log(2) &= 12N \log(1.005) && \text{Now we can divide both sides by } 12 \log 1.005 \\ \frac{\log(2)}{12 \log(1.005)} &= N && \text{Approximating this to a decimal} \\ N &= 11.581 \end{aligned}$$

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded $\log(2)$ to 0.301 and $\log(1.005)$ to 0.00217, then your final answer would have been about 11.577 years.

Example 16

If you invest \$100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to \$10,000?

Solution

This is a savings annuity problem since we are making regular deposits into the account.

$$\begin{aligned} d &= \$1000 && \text{the monthly deposit} \\ r &= 0.03 && \text{3\% annual rate} \\ k &= 12 && \text{since we're doing monthly deposits, we'll compound monthly} \end{aligned}$$

We don't know N , but we want P_N to be \$10,000

Putting this into the equation:

$$10,000 = \frac{100 \left(\left(1 + \frac{0.03}{12} \right)^{N(12)} - 1 \right)}{\left(\frac{0.03}{12} \right)} \quad \text{Simplifying the fractions a bit}$$

$$10,000 = \frac{100((1.0025)^{12N} - 1)}{0.0025}$$

We want to isolate the exponential term, 1.0025^{12N} , so multiply both sides by 0.0025

$$25 = 100((1.0025)^{12N} - 1) \quad \text{Divide both sides by 100}$$

$$0.25 = (1.0025)^{12N} - 1 \quad \text{Add 1 to both sides}$$

$$1.25 = (1.0025)^{12N} \quad \text{Now take the log of both sides}$$

$$\log(1.25) = \log((1.0025)^{12N}) \quad \text{Use the exponent property of logs}$$

$$\log(1.25) = 12N \log(1.0025) \quad \text{Divide by } 12\log(1.0025)$$

$$\frac{\log(1.25)}{12\log(1.0025)} = N \quad \text{Approximating to a decimal}$$

$$N = 7.447 \text{ years}$$

It will take about 7.447 years to grow the account to \$10,000.

Try it Now 6

Joel is considering putting a \$1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of \$30 a month?

Answer available at the end of this section.

Answers

Try it Now 6

$d = \$30$ The monthly payments
 $r = 0.12$ 12% annual rate
 $k = 12$ since we're doing monthly deposits
 $P_0 = \$1000$ we're starting with a \$1,000 loan

We are solving for N , the time to pay off the loan

$$1,000 = \frac{30 \left(1 - \left(1 + \frac{0.12}{12} \right)^{-N(12)} \right)}{\frac{0.12}{12}}$$

Solving for N gives 3.396. It will take about 3.4 years to pay off the purchase.

6.1.9: Exercise

Skills

1. A friend lends you \$200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?
2. Suppose you obtain a \$3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?
3. A T-bill is a type of bond that is sold at a discount over the face value. For example, suppose you buy a 13-week T-bill with a face value of \$10,000 for \$9,800. This means that in 13 weeks, the government will give you the face value, earning you \$200. What annual interest rate have you earned?
4. Suppose you are looking to buy a \$5000 face value 26-week T-bill. If you want to earn at least 1% annual interest, what is the most you should pay for the T-bill?
5. You deposit \$300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?
6. How much will \$1000 deposited in an account earning 7% interest compounded annually be worth in 20 years?
7. You deposit \$2000 in an account earning 3% interest compounded monthly.
 - a. How much will you have in the account in 20 years?
 - b. How much interest will you earn?
8. You deposit \$10,000 in an account earning 4% interest compounded monthly.
 - a. How much will you have in the account in 25 years?
 - b. How much interest will you earn?
9. How much would you need to deposit in an account now in order to have \$6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.
10. How much would you need to deposit in an account now in order to have \$20,000 in the account in 4 years? Assume the account earns 5% interest.
11. You deposit \$200 each month into an account earning 3% interest compounded monthly.
 - a. How much will you have in the account in 30 years?
 - b. How much total money will you put into the account?
 - c. How much total interest will you earn?
12. You deposit \$1000 each year into an account earning 8% compounded annually.
 - a. How much will you have in the account in 10 years?
 - b. How much total money will you put into the account?
 - c. How much total interest will you earn?
13. Jose has determined he needs to have \$800,000 for retirement in 30 years. His account earns 6% interest.
 - a. How much would he need to deposit in the account each month?
 - b. How much total money will he put into the account?
 - c. How much total interest will he earn?
14. You wish to have \$3000 in 2 years to buy a fancy new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?
15. You want to be able to withdraw \$30,000 each year for 25 years. Your account earns 8% interest.
 - a. How much do you need in your account at the beginning
 - b. How much total money will you pull out of the account?

- c. How much of that money is interest?
16. How much money will I need to have at retirement so I can withdraw \$60,000 a year for 20 years from an account earning 8% compounded annually?
- How much do you need in your account at the beginning
 - How much total money will you pull out of the account?
 - How much of that money is interest?
17. You have \$500,000 saved for retirement. Your account earns 6% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 20 years?
18. Loren already knows that he will have \$500,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 10% interest, how much could the annuity provide each month?
19. You can afford a \$700 per month mortgage payment. You've found a 30 year loan at 5% interest.
- How big of a loan can you afford?
 - How much total money will you pay the loan company?
 - How much of that money is interest?
20. Marie can afford a \$250 per month car payment. She's found a 5 year loan at 7% interest.
- How expensive of a car can she afford?
 - How much total money will she pay the loan company?
 - How much of that money is interest?
21. You want to buy a \$25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?
22. You decide finance a \$12,000 car at 3% compounded monthly for 4 years. What will your monthly payments be? How much interest will you pay over the life of the loan?
23. You want to buy a \$200,000 home. You plan to pay 10% as a down payment, and take out a 30 year loan for the rest.
- How much is the loan amount going to be?
 - What will your monthly payments be if the interest rate is 5%?
 - What will your monthly payments be if the interest rate is 6%?
24. Lynn bought a \$300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.
- Find her monthly payments.
 - How much interest will she pay over the life of the loan?
25. Emile bought a car for \$24,000 three years ago. The loan had a 5 year term at 3% interest rate, making monthly payments. How much does he still owe on the car?
26. A friend bought a house 15 years ago, taking out a \$120,000 mortgage at 6% for 30 years, making monthly payments. How much does she still owe on the mortgage?
27. Pat deposits \$6,000 into an account earning 4% compounded monthly. How long will it take the account to grow to \$10,000?
28. Kay is saving \$200 a month into an account earning 5% interest. How long will it take her to save \$20,000?
29. James has \$3,000 in credit card debt, which charges 14% interest. How long will it take to pay off the card if he makes the minimum payment of \$60 a month?
30. Chris has saved \$200,000 for retirement, and it is in an account earning 6% interest. If she withdraws \$3,000 a month, how long will the money last?

Concepts

31. Suppose you invest \$50 a month for 5 years into an account earning 8% compounded monthly. After 5 years, you leave the money, without making additional deposits, in the account for another 25 years. How much will you have in the end?

32. Suppose you put off making investments for the first 5 years, and instead made deposits of \$50 a month for 25 years into an account earning 8% compounded monthly. How much will you have in the end?
33. Mike plans to make contributions to his retirement account for 15 years. After the last contribution, he will start withdrawing \$10,000 a quarter for 10 years. Assuming Mike's account earns 8% compounded quarterly, how large must his quarterly contributions be during the first 15 years, in order to accomplish his goal?
34. Kendra wants to be able to make withdrawals of \$60,000 a year for 30 years after retiring in 35 years. How much will she have to save each year up until retirement if her account earns 7% interest?
35. You have \$2,000 to invest, and want it to grow to \$3,000 in two years. What interest rate would you need to find to make this possible?
36. You have \$5,000 to invest, and want it to grow to \$20,000 in ten years. What interest rate would you need to find to make this possible?
37. You plan to save \$600 a month for the next 30 years for retirement. What interest rate would you need to have \$1,000,000 at retirement?
38. You really want to buy a used car for \$11,000, but can only afford \$200 a month. What interest rate would you need to find to be able to afford the car, assuming the loan is for 60 months?

Exploration

39. Pay day loans are short term loans that you take out against future paychecks: The company advances you money against a future paycheck. Either visit a pay day loan company, or look one up online. Be forewarned that many companies do not make their fees obvious, so you might need to do some digging or look at several companies.
 - a. Explain the general method by which the loan works.
 - b. We will assume that we need to borrow \$500 and that we will pay back the loan in 14 days. Determine the total amount that you would need to pay back and the effective loan rate. The effective loan rate is the percentage of the original loan amount that you pay back. It is not the same as the APR (annual rate) that is probably published.
 - c. If you cannot pay back the loan after 14 days, you will need to get an extension for another 14 days. Determine the fees for an extension, determine the total amount you will be paying for the now 28 day loan, and compute the effective loan rate.
10. Suppose that 10 years ago you bought a home for \$110,000, paying 10% as a down payment, and financing the rest at 9% interest for 30 years.
 - a. Let's consider your existing mortgage:
 - i. How much money did you pay as your down payment?
 - ii. How much money was your mortgage (loan) for?
 - iii. What is your current monthly payment?
 - iv. How much total interest will you pay over the life of the loan?
 - b. This year, you check your loan balance. Only part of your payments have been going to pay down the loan; the rest has been going towards interest. You see that you still have \$88,536 left to pay on your loan. Your house is now valued at \$150,000.

How much of the loan have you paid off? (i.e., how much have you reduced the loan balance by? Keep in mind that interest is charged each month - it's not part of the loan balance.)

- i. How much money have you paid to the loan company so far?
 - ii. How much interest have you paid so far?
 - iii. How much equity do you have in your home (equity is value minus remaining debt)
 - c. Since interest rates have dropped, you consider refinancing your mortgage at a lower 6% rate.
 - i. If you took out a new 30 year mortgage at 6% for your remaining loan balance, what would your new monthly payments be?
 - ii. How much interest will you pay over the life of the new loan?

10. d. Notice that if you refinance, you are going to be making payments on your home for another 30 years. In addition to the 10 years you've already been paying, that's 40 years total.
- How much will you save each month because of the lower monthly payment?
 - How much total interest will you be paying (you need to consider the amount from 2c and 3b)
 - Does it make sense to refinance? (there isn't a correct answer to this question. Just give your opinion and your reason)

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6.2: LOGIC

6.2.0: Logic

6.2.1: Boolean Logic

6.2.2: Conditional Statements

6.2.3: Quantified Statements

6.2.4: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

6.2.5: Truth Tables- Conditional, Biconditional

6.2.6: Evaluating Deductive Arguments with Truth Tables

6.2.7: Forms of Valid Arguments

6.2.8: Logical Fallacies in Common Language

6.2.9: Exercises

6.2.10: De Morgan's Laws

6.2.11: Arguments

6.2.12: Evaluating Deductive Arguments with Euler Diagrams

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6.2.0: Logic

Logic is, basically, the study of valid reasoning. When searching the internet, we use Boolean logic – terms like “and” and “or” – to help us find specific web pages that fit in the sets we are interested in. After exploring this form of logic, we will look at logical arguments and how we can determine the validity of a claim.

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6.2.1: Boolean Logic

We can often classify items as belonging to sets. If you went the library to search for a book and they asked you to express your search using unions, intersections, and complements of sets, that would feel a little strange. Instead, we typically using words like “and”, “or”, and “not” to connect our keywords together to form a search. These words, which form the basis of **Boolean logic**, are directly related to our set operations. (Boolean logic was developed by the 19th-century English mathematician George Boole.)

Boolean logic

Boolean logic combines multiple statements that are either true or false into an expression that is either true or false.

In connection to sets, a search is true if the element is part of the set.

Suppose M is the set of all mystery books, and C is the set of all comedy books. If we search for “mystery”, we are looking for all the books that are an element of the set M ; the search is true for books that are in the set.

When we search for “mystery *and* comedy”, we are looking for a book that is an element of both sets, in the intersection. If we were to search for “mystery *or* comedy”, we are looking for a book that is a mystery, a comedy, or both, which is the union of the sets. If we searched for “*not* comedy”, we are looking for any book in the library that is not a comedy, the complement of the set C .

Connection to Set Operations

A and B elements in the intersection $A \cap B$

A or B elements in the union $A \cup B$

not A elements in the complement A^c

Notice here that *or* is not exclusive. This is a difference between the Boolean logic use of the word and common everyday use. When your significant other asks “do you want to go to the park *or* the movies?” they usually are proposing an exclusive choice – one option or the other, but not both. In Boolean logic, the *or* is not exclusive – more like being asked at a restaurant “would you like fries or a drink with that?” Answering “both, please” is an acceptable answer.

Example 1

Suppose we are searching a library database for Mexican universities. Express a reasonable search using Boolean logic.**Solution**

We could start with the search “Mexico *and* university”, but would be likely to find results for the U.S. state New Mexico. To account for this, we could revise our search to read:

Mexico *and* university *not* “New Mexico”

In most internet search engines, it is not necessary to include the word *and*; the search engine assumes that if you provide two keywords you are looking for both. In Google’s search, the keyword *or* has be capitalized as OR, and a negative sign in front of a word is used to indicate *not*. Quotes around a phrase indicate that the entire phrase should be looked for. The search from the previous example on Google could be written:

Mexico university -“New Mexico”

Example 2

Describe the numbers that meet the condition:

even and less than 10 and greater than 0

Solution

The numbers that satisfy all three requirements are $\{2, 4, 6, 8\}$

Sometimes statements made in English can be ambiguous. For this reason, Boolean logic uses parentheses to show precedent, just like in algebraic order of operations.

Example 3

Describe the numbers that meet the condition:

odd number and less than 20 and greater than 0 and (multiple of 3 or multiple of 5)

Solution

The first three conditions limit us to the set $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

The last grouped conditions tell us to find elements of this set that are also either a multiple of 3 or a multiple of 5. This leaves us with the set $\{3, 5, 9, 15\}$

Notice that we would have gotten a very different result if we had written

(odd number and less than 20 and greater than 0 and multiple of 3) or multiple of 5

The first grouped set of conditions would give $\{3, 9, 15\}$. When combined with the last condition, though, this set expands without limits:

$\{3, 5, 9, 15, 20, 25, 30, 35, 40, 45, \dots\}$

Example 4

The English phrase “Go to the store and buy me eggs and bagels or cereal” is ambiguous; it is not clear whether the requestor is asking for eggs always along with either bagels or cereal, or whether they’re asking for either the combination of eggs and bagels, or just cereal.

For this reason, using parentheses clarifies the intent:

Eggs and (bagels or cereal) means Option 1: Eggs and bagels, Option 2: Eggs and cereal

(Eggs and bagels) or cereal means Option 1: Eggs and bagels, Option 2: Cereal

Be aware that when a string of conditions is written without grouping symbols, it is often interpreted from the left to right, resulting in the latter interpretation.

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6.2.2: Conditional Statements

Beyond searching, Boolean logic is commonly used in spreadsheet applications like Excel to do conditional calculations. A **statement** is something that is either true or false. A statement like $3 < 5$ is true; a statement like “a rat is a fish” is false. A statement like “ $x < 5$ ” is true for some values of x and false for others. When an action is taken or not depending on the value of a statement, it forms a **conditional**.

Statements and Conditionals

A **statement** is either true or false.

A **conditional** is a compound statement of the form

“if p then q ” or “if p then q , else s ”

Example 5

In common language, an example of a conditional statement would be “If it is raining, then we’ll go to the mall. Otherwise we’ll go for a hike.”

The statement “If it is raining” is the condition – this may be true or false for any given day. If the condition is true, then we will follow the first course of action, and go to the mall. If the condition is false, then we will use the alternative, and go for a hike.

Example 6.2.2.1

As mentioned earlier, conditional statements are commonly used in spreadsheet applications like Excel or Google Sheets. In Excel, you can enter an expression like

$= \text{IF} (A1 < 2000, A1 + 1, A1 \times 2)$

Notice that after the IF, there are three parts. The first part is the condition, and the second two are calculations. Excel will look at the value in cell A1 and compare it to 2000. If that condition is true, then the first calculation is used, and 1 is added to the value of A1 and the result is stored. If the condition is false, then the second calculation is used, and A1 is multiplied by 2 and the result is stored.

In other words, this statement is equivalent to saying “If the value of A1 is less than 2000, then add 1 to the value in A1. Otherwise, multiply A1 by 2”.

Example 7

The expression $= \text{IF} (A1 > 5, 2 \times A1, 3 \times A1)$ is used. Find the result if A1 is 3, and the result if A1 is 8.

Solution

If $A1 > 5$, then calculate $2 \times A1$. Otherwise, calculate $3 \times A1$.

If A1 is 3, then the condition is false, since $3 > 5$ is not true, so we do the alternate action, and multiply by 3, giving $3 \times 3 = 9$

If A1 is 8, then the condition is true, since $8 > 5$, so we multiply the value by 2, giving $2 \times 8 = 16$

Example 8

An accountant needs to withhold 15% of income for taxes if the income is below \$30,000, and 20% of income if the income is \$30,000 or more. Write an expression that would calculate the amount to withhold.

Our conditional needs to compare the value to 30,000. If the income is less than 30,000, we need to calculate 15% of the income: $0.15 \times \text{income}$. If the income is more than 30,000, we need to calculate 20% of the income: $0.20 \times \text{income}$.

In words we could write "If income < 30,000, then multiply by 0.15, otherwise multiply by 0.20". In Excel, we would write:

= IF (A1 < 30000, 0.15 × A1, 0.20 × A1)

As we did earlier, we can create more complex conditions by using the operators *and*, *or*, and *not* to join simpler conditions together.

Example 9

A parent might say to their child "if you clean your room and take out the garbage, then you can have ice cream."

Here, there are two simpler conditions:

- 1) The child cleaning her room
- 2) The child taking out the garbage

Since these conditions were joined with *and*, the combined conditional will be true only if both simpler conditions are true; if either chore is not completed, then the parent's condition is not met.

Notice that if the parent had said "if you clean your room *or* take out the garbage, then you can have ice cream", then the child would need to complete only one chore to meet the condition.

Suppose you wanted to have something happen when a certain value is between 100 and 300. To create the condition " $A1 < 300$ and $A1 > 100$ " in Excel, you would need to enter " $\text{AND}(A1 < 300, A1 > 100)$ ". Likewise, for the condition " $A1 = 4$ or $A1 = 6$ " you would enter " $\text{OR}(A1 = 4, A1 = 6)$ ".

Example 10

In a spreadsheet, cell A1 contains annual income, and A2 contains number of dependents.

A certain tax credit applies if someone with no dependents earns less than \$10,000, or if someone with dependents earns less than \$20,000. Write a rule that describes this.

There are two ways the rule is met:

income is less than 10,000 *and* dependents is 0, *or*

income is less than 20,000 *and* dependents is not 0.

Informally, we could write these as

$(A1 < 10000 \text{ and } A2 = 0)$ or $(A1 < 20000 \text{ and } A2 > 0)$

In Excel's format, we'd write

= IF (OR(AND(A1 < 10000, A2 = 0), AND(A1 < 20000, A2 > 0)), "you qualify", "you don't qualify")

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6.2.3: Quantified Statements

Words that describe an entire set, such as “all”, “every”, or “none”, are called **universal quantifiers** because that set could be considered a universal set. In contrast, words or phrases such as “some”, “one”, or “at least one” are called **existential quantifiers** because they describe the existence of at least one element in a set.

Quantifiers

A **universal quantifier** states that an entire set of things share a characteristic.

An **existential quantifier** states that a set contains at least one element.

Something interesting happens when we **negate** – or state the opposite of – a quantified statement.

Example 11

Suppose your friend says “Everybody cheats on their taxes.” What is the minimum amount of evidence you would need to prove your friend wrong?

To show that it is not true that everybody cheats on their taxes, all you need is one person who does not cheat on their taxes. It would be perfectly fine to produce more people who do not cheat, but one counterexample is all you need.

It is important to note that you do not need to show that absolutely nobody cheats on their taxes.

Example 12

Suppose your friend says “One of these six cartons of milk is leaking.” What is the minimum amount of evidence you would need to prove your friend wrong?

Solution

In this case, you would need to check all six cartons and show that none of them is leaking. You cannot disprove your friend’s statement by checking only one of the cartons.

When we negate a statement with a universal quantifier, we get a statement with an existential quantifier, and vice-versa.

Negating a quantified statement

The negation of “all A are B” is “at least one A is not B”.

The negation of “no A are B” is “at least one A is B”.

The negation of “at least one A is B” is “no A are B”.

The negation of “at least one A is not B” is “all A are B”.

Example 13

“Somebody brought a flashlight.” Write the negation of this statement.

The negation is “Nobody brought a flashlight.”

Example 14

“There are no prime numbers that are even.” Write the negation of this statement.

The negation is “At least one prime number is even.”

Try it Now 1

Write the negation of “All Icelandic children learn English in school.”

Answer available at the end of this section.

Answers

Try it Now 1

At least one Icelandic child did not learn English in school.

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6.2.4: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

Before we focus on truth tables, we're going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

Symbols

The symbol $\wedge$ is used for and: A and B is notated $A \wedge B$

The symbol $\vee$ is used for or: A or B is notated $A \vee B$

The symbol $\sim$ is used for not: not A is notated $\sim A$

You can remember the first two symbols by relating them to the shapes for the union and intersection. $A \wedge B$ would be the elements that exist in both sets, in $A \cap B$. Likewise, $A \vee B$ would be the elements that exist in either set, in $A \cup B$. When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

Example 15

Translate each statement into symbolic notation. Let P represent "I like Pepsi" and let C represent "I like Coke".

- I like Pepsi or I like Coke.
- I like Pepsi and I like Coke.
- I do not like Pepsi.
- It is not the case that I like Pepsi or Coke.
- I like Pepsi and I do not like Coke.

Solution

- $P \vee C$
- $P \wedge C$
- $\sim P$
- $\sim (P \vee C)$
- $P \wedge \sim C$

As you can see, we can use parentheses to organize more complicated statements.

Try it Now 2

Translate "We have carrots or we will not make soup" into symbols. Let C represent "we have carrots" and let S represent "we will make soup".

Answer available at the end of this section.

Because complex Boolean statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true and false.

Truth Table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 16

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise".

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use *S* to designate "is a sectional", and *C* to designate "has a chaise".

A truth table for this situation would look like this:

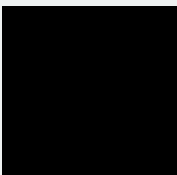


In the table, T is used for true, and F for false. In the first row, if *S* is true and *C* is also true, then the complex statement "*S or C*" is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that *or* in logic is not exclusive; if the couch has both features, it meets the condition.)

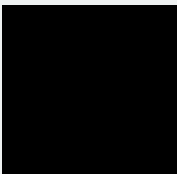
In the previous example about the couch, the truth table was really just summarizing what we already know about how the *or* statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

Basic truth tables

Conjunction



Disjunction



Negation



Truth tables really become useful when we analyze more complex Boolean statements.

Example 17

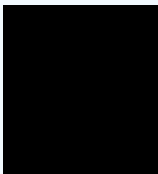
Create a truth table for the statement $A \vee \sim B$

Solution

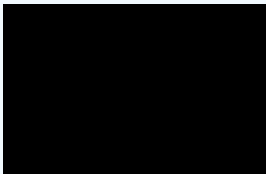
When we create the truth table, we need to list all the possible truth value combinations for A and B . Notice how the first column contains 2 Ts followed by 2 Fs, and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.



After creating columns with those initial values, we create a third column for the expression $\sim B$. Now we will temporarily ignore the column for A and write the truth values for $\sim B$



Next we can find the truth values of $A \vee \sim B$, using the first and third columns.



The truth table shows that $A \vee \sim B$ is true in three cases and false in one case. If you're wondering what the point of this is, suppose it is the last day of the baseball season and two teams, who are not playing each other, are competing for the final playoff spot. Anaheim will make the playoffs if it wins its game or if Boston does not win its game. (Anaheim owns the tie-breaker; if both teams win, or if both teams lose, then Anaheim gets the playoff spot.) If A = Anaheim wins its game and B = Boston wins its game, then $A \vee \sim B$ represents the situation "Anaheim wins its game or Boston does not win its game". The truth table shows us the different scenarios related to Anaheim making the playoffs. In the first row, Anaheim wins its game and Boston wins its game, so it is true that Anaheim makes the playoffs. In the second row, Anaheim wins and Boston does not win, so it is true that Anaheim makes the playoffs. In the third row, Anaheim does not win its game and Boston wins its game, so it is false that Anaheim makes the playoffs. In the fourth row, Anaheim does not win and Boston does not win, so it is true that Anaheim makes the playoffs.

Try it Now 3

Create a truth table for this statement: $\sim A \wedge B$

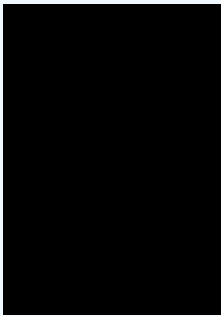
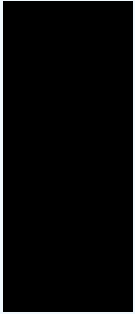
Answer available at the end of this section.

Example 18

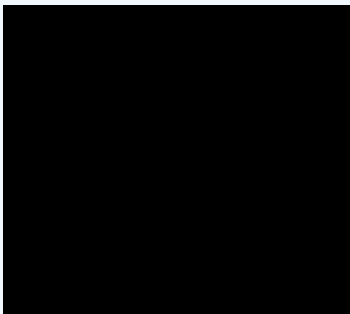
Create a truth table for the statement $A \wedge \sim (B \vee C)$

Solution

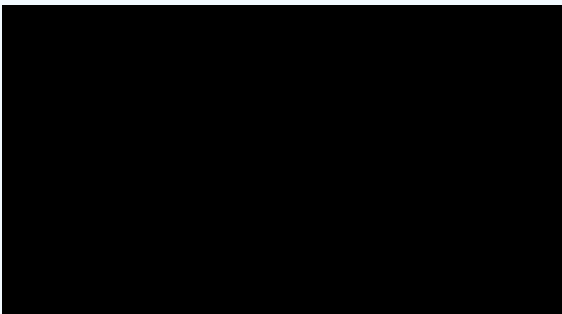
It helps to work from the inside out when creating a truth table, and to create columns in the table for intermediate operations. We start by listing all the possible truth value combinations for A , B , and C . Notice how the first column contains 4 Ts followed by 4Fs, the second column contains 2Ts, 2Fs, then repeats, and the last column alternates T, F, T, F ... This pattern ensures that all 8 combinations are considered. After creating columns with those initial values, we create a fourth column for the innermost expression, $B \vee C$. Now we will temporarily ignore the column for A and focus on B and C , writing the truth values for $B \vee C$



Next we can find the negation of $B \vee C$, working off the $B \vee C$ column we just created. (Ignore the first three columns and simply negate the values in the $B \vee C$ column.)



Finally, we find the values of A and $\sim (B \vee C)$. (Ignore the second, third, and fourth columns.)



It turns out that this complex expression is true in only one case: when A is true, B is false, and C is false. To illustrate this situation, suppose that Anaheim will make the playoffs if: (1) Anaheim wins, and (2) neither Boston nor Cleveland wins. IFF is the only scenario in which Anaheim will make the playoffs.

Try it Now 4

Create a truth table for this statement: $(\sim A \wedge B) \vee \sim B$

Answer available at the end of this section.

Answers

Try it Now 2

$$C \vee \sim S$$

Try it Now 3



Try it Now 4



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6.2.5: Truth Tables- Conditional, Biconditional

We discussed conditional statements earlier, in which we take an action based on the value of the condition. We are now going to look at another version of a conditional, sometimes called an implication, which states that the second part must logically follow from the first.

Conditional

A conditional is a logical compound statement in which a statement p , called the antecedent, implies a statement q , called the consequent.

A conditional is written as $p \rightarrow q$ and is translated as "if p , then q ".

Example 19

The English statement "If it is raining, then there are clouds in the sky" is a conditional statement. It makes sense because if the antecedent "it is raining" is true, then the consequent "there are clouds in the sky" must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consequent becomes irrelevant.

Example 20

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday's game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

There are four possible outcomes:

- 1) You pay for expedited shipping and receive the jersey by Friday
- 2) You pay for expedited shipping and don't receive the jersey by Friday
- 3) You don't pay for expedited shipping and receive the jersey by Friday
- 4) You don't pay for expedited shipping and don't receive the jersey by Friday

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don't receive the jersey by Friday. The first outcome is exactly what was promised, so there's no problem with that. The third outcome is not a lie because the website never said what would happen if you didn't pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn't make any promises about when the jersey would arrive if you didn't pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the antecedent is false, we cannot make any judgment about the consequent. The website never said that paying for expedited shipping was the *only* way to receive the jersey by Friday.

Example 21

A friend tells you “If you upload that picture to Facebook, you’ll lose your job.” Under what conditions can you say that your friend was wrong?

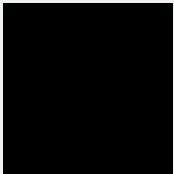
There are four possible outcomes:

- 1) You upload the picture and lose your job
- 2) You upload the picture and don’t lose your job
- 3) You don’t upload the picture and lose your job
- 4) You don’t upload the picture and don’t lose your job

There is only one possible case in which you can say your friend was wrong: the second outcome in which you upload the picture but still keep your job. In the last two cases, your friend didn’t say anything about what would happen if you didn’t upload the picture, so you can’t say that their statement was wrong. Even if you didn’t upload the picture and lost your job anyway, your friend never said that you were guaranteed to keep your job if you didn’t upload the picture; you might lose your job for missing a shift or punching your boss instead.

In traditional logic, a conditional is considered true as long as there are no cases in which the antecedent is true and the consequent is false.

Truth table for the conditional



Again, if the antecedent p is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true.

Example 22

Construct a truth table for the statement $(m \wedge \sim p) \rightarrow r$

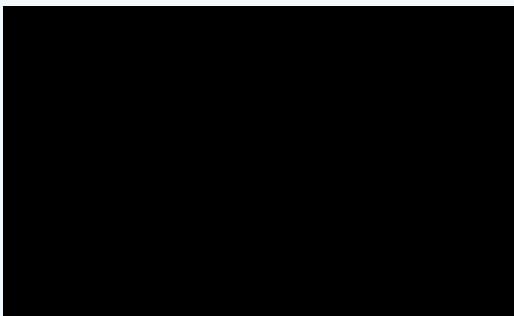
Solution

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the antecedent, $m \wedge \sim p$.





Now we can create a column for the conditional. Because it can be confusing to keep track of all the Ts and Fs, why don't we copy the column for r to the right of the column for $m \wedge \sim p$? This makes it a lot easier to read the conditional from left to right.



When m is true, p is false, and r is false- the fourth row of the table-then the antecedent $m \wedge \sim p$ will be true but the consequent false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by $(m \wedge \sim p) \rightarrow r$, consider this: let m = we order meatballs, p = we order pasta, and r = Rob is happy. The statement $(m \wedge \sim p) \rightarrow r$ is "if we order meatballs and don't order pasta, then Rob is happy". If m is true (we order meatballs), p is false (we don't order pasta), and r is false (Rob is not happy), then the statement is false, because we satisfied the antecedent but Rob did not satisfy the consequent.

For any conditional, there are three related statements, the converse, the inverse, and the contrapositive.

Related Statments

The original conditional is "if p , then q " $p \rightarrow q$

The converse is "if q , then p " $q \rightarrow p$

The inverse is "if not p , then not q " $\sim p \rightarrow \sim q$

The contrapositive is "if not q , then not p " $\sim q \rightarrow \sim p$

Example 23

Consider again the conditional “If it is raining, then there are clouds in the sky.” It seems reasonable to assume that this is true.

The converse would be “If there are clouds in the sky, then it is raining.” This is not always true.

The inverse would be “If it is not raining, then there are not clouds in the sky.” Likewise, this is not always true.

The contrapositive would be “If there are not clouds in the sky, then it is not raining.” This statement is true, and is equivalent to the original conditional.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Conditional	Converse	Inverse	Contrapositive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalent

Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a conditional statement are logically equivalent.

In other words, the original statement and the contrapositive must agree with each other; they must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other; they must both be true, or they must both be false.

Be aware that symbolic logic cannot represent the English language perfectly. For example, we may need to change the verb tense to show that one thing occurred before another.

Example 24

Suppose this statement is true: “If I eat this giant cookie, then I will feel sick.” Which of the following statements must also be true?

- If I feel sick, then I ate that giant cookie.
- If I don't eat this giant cookie, then I won't feel sick.
- If I don't feel sick, then I didn't eat that giant cookie.

Solution

- a. This is the converse, which is not necessarily true. I could feel sick for some other reason, such as drinking sour milk.
- b. This is the inverse, which is not necessarily true. Again, I could feel sick for some other reason; avoiding the cookie doesn't guarantee that I won't feel sick.
- c. This is the contrapositive, which is true, but we have to think somewhat backwards to explain it. If I ate the cookie, I would feel sick, but since I don't feel sick, I must not have eaten the cookie.

Notice again that the original statement and the contrapositive have the same truth value (both are true), and the converse and the inverse have the same truth value (both are false).

Try it Now 5

"If you microwave salmon in the staff kitchen, then I will be mad at you." If this statement is true, which of the following statements must also be true?

- a. If you don't microwave salmon in the staff kitchen, then I won't be mad at you.
- b. If I am not mad at you, then you didn't microwave salmon in the staff kitchen.
- c. If I am mad at you, then you microwaved salmon in the staff kitchen.

Answer available at the end of this section.

Consider the statement "If you park here, then you will get a ticket." What set of conditions would prove this statement false?

- a. You don't park here and you get a ticket.
- b. You don't park here and you don't get a ticket.
- c. You park here and you don't get a ticket.

The first two statements are irrelevant because we don't know what will happen if you park somewhere else. The third statement, however contradicts the conditional statement "If you park here, then you will get a ticket" because you parked here but didn't get a ticket. This example demonstrates a general rule; the negation of a conditional can be written as a conjunction: "It is not the case that if you park here, then you will get a ticket" is equivalent to "You park here *and* you do not get a ticket."

The Negation of a Conditional

The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

$$\sim (p \rightarrow q) \text{ is equivalent to } p \wedge \sim q$$

Example 25

Which of the following statements is equivalent to the negation of "If you don't grease the pan, then the food will stick to it"?

- a. I didn't grease the pan and the food didn't stick to it.
- b. I didn't grease the pan and the food stuck to it.
- c. I greased the pan and the food didn't stick to it.

Solution

- a. This is correct; it is the conjunction of the antecedent and the negation of the consequent. To disprove that not greasing the pan will cause the food to stick, I have to not grease the pan and have the food not stick.
- b. This is essentially the original statement with no negation; the "if...then" has been replaced by "and".
- c. This essentially agrees with the original statement and cannot disprove it.

Try it Now 6

"If you go swimming less than an hour after eating lunch, then you will get cramps." Which of the following statements is equivalent to the negation of this statement?

- I went swimming more than an hour after eating lunch and I got cramps.
- I went swimming less than an hour after eating lunch and I didn't get cramps.
- I went swimming more than an hour after eating lunch and I didn't get cramps.

Answer available at the end of this section.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider "If you submit your hours today, then you will be paid next Friday." What the payroll rep really means is "If you submit your hours today, then you will be paid next Friday, and if you don't submit your hours today, then you won't be paid next Friday." The conditional statement if t , then p also includes the inverse of the statement: if not t , then not p . A more compact way to express this statement is "You will be paid next Friday *if and only if* you submit your timesheet today." A statement of this form is called a **biconditional**.

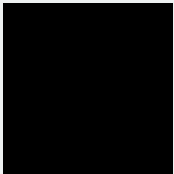
Biconditional

A biconditional is a logical conditional statement in which the antecedent and consequent are interchangeable.

A biconditional is written as $p \leftrightarrow q$ and is translated as " p if and only if q ".

Because a biconditional statement $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$, we may think of it as a conditional statement combined with its converse: if p , then q and if q , then p . The double-headed arrow shows that the conditional statement goes from left to right and from right to left. A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

Truth table for the biconditional



Notice that the fourth row, where both components are false, is true; if you don't submit your timesheet and you don't get paid, the person from payroll told you the truth.

Example 26

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?

- It is noon on Thursday and the garbage truck did not come down my street this morning.
- It is Monday and the garbage truck is coming down my street.
- It is Wednesday at 11:59PM and the garbage truck did not come down my street today.

Solution

- This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.

- b. This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.
- c. This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

Try it Now 7

Suppose this statement is true: “I wear my running shoes if and only if I am exercising.” Determine whether each of the following statements must be true or false.

- a. I am exercising and I am not wearing my running shoes.
- b. I am wearing my running shoes and I am not exercising.
- c. I am not exercising and I am not wearing my running shoes.

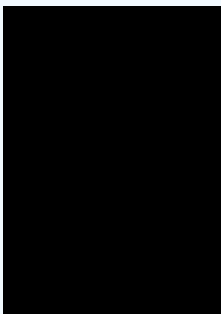
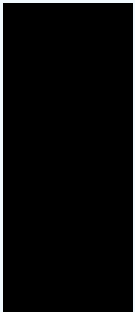
Answer available at the end of this section.

Example 27

Create a truth table for the statement $(A \vee B) \leftrightarrow \sim C$

Solution

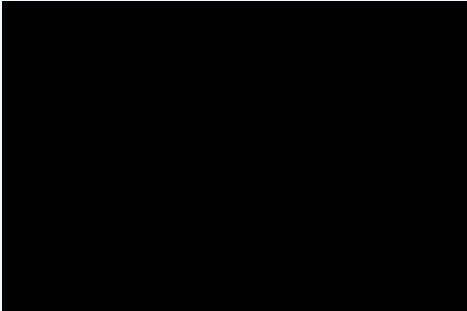
Whenever we have three component statements, we start by listing all the possible truth value combinations for A , B , and C . After creating those three columns, we can create a fourth column for the antecedent, $A \vee B$. Now we will temporarily ignore the column for C and focus on A and B , writing the truth values for $A \vee B$.



Next we can create a column for the negation of C . (Ignore the $A \vee B$ column and simply negate the values in the C column.)



Finally, we find the truth values of $(A \vee B) \leftrightarrow \sim C$. Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.



To illustrate this situation, suppose your boss needs you to do either project A or project B (or both, if you have the time). If you do one of the projects, you will not get a crummy review (C is for crummy). So $(A \vee B) \leftrightarrow \sim C$ means "You will not get a crummy review if and only if you do project A or project B ." Looking at a few of the rows of the truth table, we can see how this works out. In the first row, A , B , and C are all true: you did both projects and got a crummy review, which is not what your boss told you would happen! That is why the final result of the first row is false. In the fourth row, A is true, B is false, and C is false: you did project A and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row, A , B , and C are all false: you didn't do either project and did not get a crummy review. This is not what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)

Answers

Try it Now 5

Choice b is correct because it is the contrapositive of the original statement.

Try it Now 6

Choice b is equivalent to the negation; it keeps the first part the same and negates the second part.

Try it Now 7

Choices a & b are false; c is true.

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6.2.6: Evaluating Deductive Arguments with Truth Tables

Arguments can also be analyzed using truth tables, although this can be a lot of work.

Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for the statement. If it is always true, then the argument is valid.

Example 34

Consider the argument

Premise: If you bought bread, then you went to the store.

Premise: You bought bread.

Conclusion: You went to the store.

Solution

While this example is fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then form a conditional statement showing that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

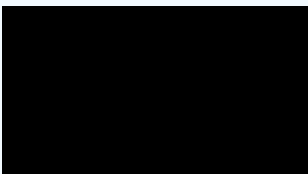
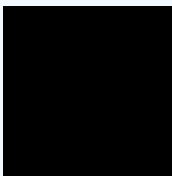
We'll let b represent "you bought bread" and s represent "you went to the store". Then the argument becomes:

Premise: $b \rightarrow s$

Premise: b

Conclusion: s

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(b \rightarrow s) \wedge b] \rightarrow s$?





Since the truth table for $[(b \rightarrow s) \wedge b] \rightarrow s$ is always true, this is a valid argument.

Try it Now 13

Determine whether the argument is valid:

Premise: If I have a shovel, I can dig a hole.

Premise: I dug a hole.

Conclusion: Therefore, I had a shovel.

Answer available at the end of this section.

Example 35

Premise: If I go to the mall, then I'll buy new jeans.

Premise: If I buy new jeans, I'll buy a shirt to go with it.

Conclusion: If I go to the mall, I'll buy a shirt.

Solution

Let m = I go to the mall, j = I buy jeans, and s = I buy a shirt.

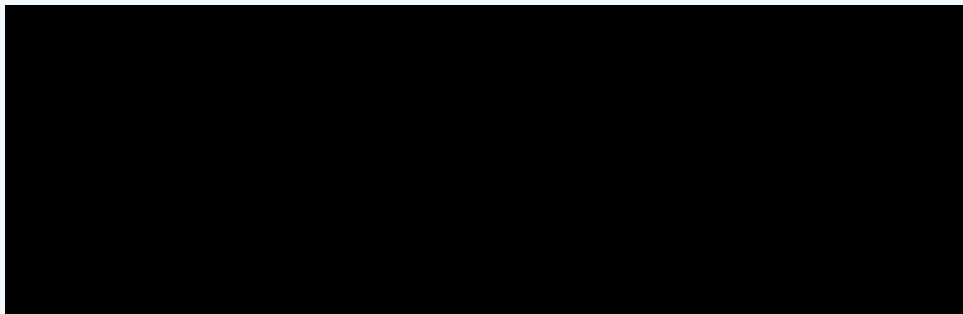
The premises and conclusion can be stated as:

Premise: $m \rightarrow j$

Premise: $j \rightarrow s$

Conclusion: $m \rightarrow s$

We can construct a truth table for $[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$. Try to recreate each step and see how the truth table was constructed.

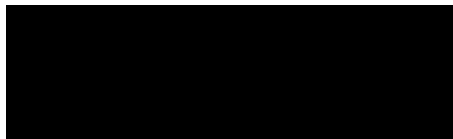


From the final column of the truth table, we can see this is a valid argument.

Answers

Try it Now 13

Let S = have a shovel, D = dig a hole. The first premise is equivalent to $S \rightarrow D$. The second premise is D . The conclusion is S . We are testing $[(S \rightarrow D) \wedge D] \rightarrow S$



This is not a tautology, so this is an invalid argument.

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6.2.7: Forms of Valid Arguments

Rather than making a truth table for every argument, we may be able to recognize certain common forms of arguments that are valid (or invalid). If we can determine that an argument fits one of the common forms, we can immediately state whether it is valid or invalid.

The Law of Detachment (*Modus Ponens*)

The law of detachment applies when a conditional and its antecedent are given as premises, and the consequent is the conclusion. The general form is:

Premise: $p \rightarrow q$
Premise: p
Conclusion: q

The Latin name, *modus ponens*, translates to “mode that affirms”.

Example 36

Recall this argument from an earlier example:

Premise: If you bought bread, then you went to the store.
Premise: You bought bread.
Conclusion: You went to the store.

In symbolic form:

Premise: $b \rightarrow s$
Premise: b
Conclusion: s

This argument has the structure described by the law of detachment. (The second premise and the conclusion are simply the two parts of the first premise detached from each other.) Instead of making a truth table, we can say that this argument is valid by stating that it satisfies the law of detachment.

The Law of Contraposition (*Modus Tollens*)

The law of contraposition applies when a conditional and the negation of its consequent are given as premises, and the negation of its antecedent is the conclusion. The general form is:

Premise: $p \rightarrow q$
Premise: $\sim q$
Conclusion: $\sim p$

The Latin name, *modus tollens*, translates to “mode that denies”.

Notice that the second premise and the conclusion look like the contrapositive of the first premise, $\sim q \rightarrow \sim p$, but they have been detached. You can think of the law of contraposition as a combination of the law of detachment and the fact that the contrapositive is logically equivalent to the original statement.

Example 37

Premise: If I drop my phone into the swimming pool, my phone will be ruined.
Premise: My phone isn't ruined.
Conclusion: I didn't drop my phone into the swimming pool.

If we let d = I drop the phone in the pool and r = the phone is ruined, then we can represent the argument this way:

Premise: $d \rightarrow r$
Premise: $\sim r$
Conclusion: $\sim d$

The form of this argument matches what we need to invoke the law of contraposition, so it is a valid argument.

Try it Now 14

Is this argument valid?

Premise: If you brushed your teeth before bed, then your toothbrush will be wet.
Premise: Your toothbrush is dry.
Conclusion: You didn't brush your teeth before bed.

Answer available at the end of this section.

The Transitive Property (Hypothetical Syllogism)

The transitive property has as its premises a series of conditionals, where the consequent of one is the antecedent of the next. The conclusion is a conditional with the same antecedent as the first premise and the same consequent as the final premise. The general form is:

Premise: $p \rightarrow q$
Premise: $q \rightarrow r$
Conclusion: $p \rightarrow r$

The earlier example about buying a shirt at the mall is an example illustrating the transitive property. It describes a chain reaction: if the first thing happens, then the second thing happens, and if the second thing happens, then the third thing happens. Therefore, if we want to ignore the second thing, we can say that if the first thing happens, then we know the third thing will happen. We don't have to mention the part about buying jeans; we can simply say that the first event leads to the final event. We could even have more than two premises; as long as they form a chain reaction, the transitive property will give us a valid argument.

Example 38

Premise: If a soccer player commits a reckless foul, she will receive a yellow card.
Premise: If Hayley receives a yellow card, she will be suspended for the next match.
Conclusion: If Hayley commits a reckless foul, she will be suspended for the next match.

If we let r = committing a reckless foul, y = receiving a yellow card, and s = being suspended, then our argument looks like this:

Premise: $r \rightarrow y$
Premise: $y \rightarrow s$
Conclusion: $r \rightarrow s$

This argument has the exact structure required to use the transitive property, so it is a valid argument.

Try it Now 15

Is this argument valid?

Premise: If the old lady swallows a fly, she will swallow a spider.
Premise: If the old lady swallows a spider, she will swallow a bird.
Premise: If you brushed your teeth before bed, then your toothbrush will be wet.
Premise: If you brushed your teeth before bed, then your toothbrush will be wet.
Premise: If the old lady swallows a bird, she will swallow a cat.
Premise: If the old lady swallows a cat, she will swallow a dog.
Premise: If the old lady swallows a dog, she will swallow a goat.
Premise: If the old lady swallows a goat, she will swallow a cow.
Premise: If the old lady swallows a cow, she will swallow a horse.
Premise: If the old lady swallows a horse, she will die, of course.
Conclusion: If the old lady swallows a fly, she will die, of course.

Answer available at the end of this section.

Disjunctive Syllogism

In a disjunctive syllogism, the premises consist of an *or* statement and the negation of one of the options. The conclusion is the other option. The general form is:

Premise: $p \vee q$
Premise: $\sim p$
Conclusion: q

The order of the two parts of the disjunction isn't important. In other words, we could have the premises $p \vee q$ and $\sim q$, and the conclusion p

Example 39

Premise: I can either drive or take the train.
Premise: I refuse to drive.
Conclusion: I will take the train.

If we let $d = I$ drive and $t = I$ take the train, then the symbolic representation of the argument is:

Premise: $d \vee t$
Premise: $\sim d$
Conclusion: t

This argument is valid because it has the form of a disjunctive syllogism. I have two choices, and one of them is not going to happen, so the other one must happen.

Try it Now 16

Is this argument valid?

Premise: Alison was required to write a 10-page paper or give a 5-minute speech.

Premise: Alison did not give a 5-minute speech.

Conclusion: Alison wrote a 10-page paper.

Answer available at the end of this section.

Keep in mind that, when you are determining the validity of an argument, you must assume that the premises are true. If you don't agree with one of the premises, you need to keep your personal opinion out of it. Your job is to pretend that the premises are true and then determine whether they force you to accept the conclusion. You may attack the premises in a court of law or a political discussion, of course, but here we are focusing on the structure of the arguments, not the truth of what they actually say.

We have just looked at four forms of valid arguments; there are two common forms that represent *invalid* arguments, which are also called *fallacies*.

The Fallacy of the Converse

The fallacy of the converse arises when a conditional and its consequent are given as premises, and the antecedent is the conclusion. The general form is:

Premise: $p \rightarrow q$

Premise: q

Conclusion: p

Notice that the second premise and the conclusion look like the converse of the first premise, $q \rightarrow p$, but they have been detached. The fallacy of the converse incorrectly tries to assert that the converse of a statement is equivalent to that statement.

Example 40

Premise: If I drink coffee after noon, then I have a hard time falling asleep that night.

Premise: I had a hard time falling asleep last night.

Conclusion: I drank coffee after noon yesterday.

If we let c = I drink coffee after noon and h = I have a hard time falling asleep, then our argument looks like this:

Premise: $c \rightarrow h$

Premise: h

Conclusion: c

This argument uses converse reasoning, so it is an invalid argument. There could be plenty of other reasons why I couldn't fall asleep: I could be worried about money, my neighbors might have been setting off fireworks, ...

Try it Now 17

Is this argument valid?

Premise: If you pull that fire alarm, you will get in big trouble.

Premise: You got in big trouble.

Conclusion: You must have pulled the fire alarm.

Answer available at the end of this section.

The Fallacy of the Inverse

The fallacy of the inverse occurs when a conditional and the negation of its antecedent are given as premises, and the negation of the consequent is the conclusion. The general form is:

Premise: $p \rightarrow q$
Premise: $\sim p$
Conclusion: $\sim q$

Again, notice that the second premise and the conclusion look like the inverse of the first premise, $\sim p \rightarrow \sim q$, but they have been detached. The fallacy of the inverse incorrectly tries to assert that the inverse of a statement is equivalent to that statement.

Example 41

Premise: If you listen to the Grateful Dead, then you are a hippie.
Premise: Sky doesn't listen to the Grateful Dead.
Conclusion: Sky is not a hippie.

If we let g = listen to the Grateful Dead and h = is a hippie, then this is the argument:

Premise: $g \rightarrow h$
Premise: $\sim g$
Conclusion: $\sim h$

This argument is invalid because it uses inverse reasoning. The first premise does not imply that all hippies listen to the Grateful Dead; there could be some hippies who listen to Phish instead.

Try it Now 18

Is this argument valid?

Premise: If a hockey player trips an opponent, he will be assessed a 2-minute penalty.
Premise: Alexei did not trip an opponent.
Conclusion: Alexei will not be assessed a 2-minute penalty.

Answer available at the end of this section.

Of course, arguments are not limited to these six basic forms; some arguments have more premises, or premises that need to be rearranged before you can see what is really happening. There are plenty of other forms of arguments that are invalid. If an argument doesn't seem to fit the pattern of any of these common forms, though, you may want to use a Venn diagram or a truth table instead.

Lewis Carroll, author of *Alice's Adventures in Wonderland*, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected using syllogisms. The following example is one such puzzle.

Example 42

Solve the puzzle. In other words, find a logical conclusion from these premises.

All babies are illogical.

Nobody is despised who can manage a crocodile.

Illogical persons are despised.

Let b = is a baby, d = is despised, i = is illogical, and m = can manage a crocodile.

Then we can write the premises as:

$$b \rightarrow i$$

$$m \rightarrow \sim d$$

$$i \rightarrow d$$

Writing the second premise correctly can be a challenge; it can be rephrased as “If you can manage a crocodile, then you are not despised.”

Using the transitive property with the first and third premises, we can conclude that $b \rightarrow d$, that all babies are despised. Using the contrapositive of the second premise, $d \rightarrow \sim m$, we can then use the transitive property with $b \rightarrow d$ to conclude that $b \rightarrow \sim m$, that babies cannot manage crocodiles. While it is silly, this is a logical conclusion from the given premises.

Example 43

Premise: If I work hard, I'll get a raise.

Premise: If I get a raise, I'll buy a boat.

Conclusion: If I don't buy a boat, I must not have worked hard.

If we let h = working hard, r = getting a raise, and b = buying a boat, then we can represent our argument symbolically:

Premise: $h \rightarrow r$

Premise: $r \rightarrow b$

Conclusion: $\sim b \rightarrow \sim h$

Using the transitive property with the two premises, we can conclude that $h \rightarrow b$, if I work hard, then I will buy a boat. When we learned about the contrapositive, we saw that the conditional statement $h \rightarrow b$ is equivalent to $\sim b \rightarrow \sim h$. Therefore, the conclusion is indeed a logical syllogism derived from the premises.

Try it Now 19

Is this argument valid?

Premise: If I go to the party, I'll be really tired tomorrow.

Premise: If I go to the party, I'll get to see friends.

Conclusion: If I don't see friends, I won't be tired tomorrow.

Answer available at the end of this section.

Answers

Try it Now 14

Let b = brushed teeth and w = toothbrush is wet.

Premise: $b \rightarrow w$

Premise: $\sim w$

Conclusion: $\sim b$

This argument is valid by the Law of Contraposition.

Try it Now 15

This argument is valid by the Transitive Property, which can involve more than two premises, as long as they continue the chain reaction. The premises $f \rightarrow s, s \rightarrow b, b \rightarrow c, c \rightarrow d, d \rightarrow g, g \rightarrow w, w \rightarrow h, h \rightarrow x$ can be reduced to $f \rightarrow x$. (Because we had already used c and d we decided to use w for cow and x for death. If the old lady swallows the fly, she will eventually eat a horse and die.

Try it Now 16

Let p = wrote a paper and s = gave a speech.

Premise: $p \vee s$

Premise: $\neg s$

Conclusion: p

This argument is valid by Disjunctive Syllogism. Alison had to do one or the other; she didn't choose the speech, so she must have chosen the paper.

Try it Now 17

Let f = pulled fire alarm and t = got in big trouble.

Premise: $f \rightarrow t$

Premise: t

Conclusion: f

Try it Now 18

Let t = tripped and p = got a penalty.

Premise: $t \rightarrow p$

Premise: $\sim t$

Conclusion: $\sim p$

This argument is invalid because it has the form of the Fallacy of the Inverse. Alexei may have gotten a penalty for an infraction other than tripping.

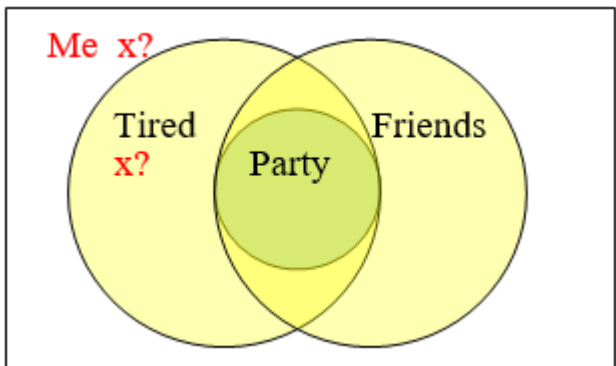
Try it Now 19

Let p = go to party, t = be tired, and f = see friends.

Premise: $p \rightarrow t$

Premise: $p \rightarrow f$

Conclusion: $\neg f \rightarrow \neg t$



We could try to rewrite the second premise using the contrapositive to state $\sim f \rightarrow \sim p$, but that does not allow us to form a syllogism. If I don't see friends, then I didn't go the party, but that is not sufficient to claim I won't be tired tomorrow. Maybe I stayed up all night watching movies.

A Venn diagram can help, if we set it up correctly. The "party" circle must be completely contained within the intersection of the other circles. We know that I am somewhere outside the "friends" circle, but we cannot determine whether I am in the "tired" circle. All we really know for sure is that I didn't go to the party.

6.2.8: Logical Fallacies in Common Language

In the previous discussion, we saw that logical arguments can be invalid when the premises are not true, when the premises are not sufficient to guarantee the conclusion, or when there are invalid chains in logic. There are a number of other ways in which arguments can be invalid, a sampling of which are given here.

Ad hominem

An ad hominem argument attacks the person making the argument, ignoring the argument itself.

Example 44

“Jane says that whales aren’t fish, but she’s only in the second grade, so she can’t be right.”

Here the argument is attacking Jane, not the validity of her claim, so this is an ad hominem argument.

Example 45

“Jane says that whales aren’t fish, but everyone knows that they’re really mammals. She’s so stupid.”

This certainly isn’t very nice, but it is *not* ad hominem since a valid counterargument is made along with the personal insult.

Appeal to ignorance

This type of argument assumes something is true because it hasn’t been proven false.

Example 46

“Nobody has proven that photo isn’t of Bigfoot, so it must be Bigfoot.”

Appeal to authority

These arguments attempt to use the authority of a person to prove a claim. While often authority can provide strength to an argument, problems can occur when the person’s opinion is not shared by other experts, or when the authority is irrelevant to the claim.

Example 47

“A diet high in bacon can be healthy; Doctor Atkins said so.”

Here, an appeal to the authority of a doctor is used for the argument. This generally would provide strength to the argument, except that the opinion that eating a diet high in saturated fat runs counter to general medical opinion. More supporting evidence would be needed to justify this claim.

Example 48

“Jennifer Hudson lost weight with Weight Watchers, so their program must work.”

Here, there is an appeal to the authority of a celebrity. While her experience does provide evidence, it provides no more than any other person’s experience would.

Appeal to consequence

An appeal to consequence concludes that a premise is true or false based on whether the consequences are desirable or not.

Example 49

“Humans will travel faster than light: faster-than-light travel would be beneficial for space travel.”

False dilemma

A false dilemma argument falsely frames an argument as an “either or” choice, without allowing for additional options.

Example 50

“Either those lights in the sky were an airplane or aliens. There are no airplanes scheduled for tonight, so it must be aliens.”

This argument ignores the possibility that the lights could be something other than an airplane or aliens.

Circular reasoning

Circular reasoning is an argument that relies on the conclusion being true for the premise to be true.

Example 51

“I shouldn’t have gotten a C in that class; I’m an A student!”

In this argument, the student is claiming that because they’re an A student, though shouldn’t have gotten a C. But because they got a C, they’re not an A student.

Post hoc (post hoc ergo propter hoc)

A post hoc argument claims that because two things happened sequentially, then the first must have caused the second.

Example 52

“Today I wore a red shirt, and my football team won! I need to wear a red shirt every time they play to make sure they keep winning.”

Straw man

A straw man argument involves misrepresenting the argument in a less favorable way to make it easier to attack.

Example 53

“Senator Jones has proposed reducing military funding by 10%. Apparently he wants to leave us defenseless against attacks by terrorists”

Here the arguer has represented a 10% funding cut as equivalent to leaving us defenseless, making it easier to attack Senator Jones’ position.

Correlation implies causation

Similar to post hoc, but without the requirement of sequence, this fallacy assumes that just because two things are related one must have caused the other. Often there is a third variable not considered.

Example 54

“Months with high ice cream sales also have a high rate of deaths by drowning. Therefore, ice cream must be causing people to drown.”

This argument is implying a causal relation, when really both are more likely dependent on the weather; that ice cream and drowning are both more likely during warm summer months.

Try it Now 20

Identify the logical fallacy in each of the arguments

- Only an untrustworthy person would run for office. The fact that politicians are untrustworthy is proof of this.
- Since the 1950s, both the atmospheric carbon dioxide level and [obesity](#) levels have increased sharply. Hence, atmospheric carbon dioxide causes obesity.
- The oven was working fine until you started using it, so you must have broken it.
- You can’t give me a D in the class because I can’t afford to retake it.
- The senator wants to increase support for food stamps. He wants to take the taxpayers’ hard-earned money and give it away to lazy people. This isn’t fair, so we shouldn’t do it.

Answer available at the end of this section.

It may be difficult to identify one particular fallacy for an argument. Consider this argument: “Emma Watson says she’s a feminist, but she posed for these racy pictures. I’m a feminist, and no self-respecting feminist would do that.” Could this be ad hominem, saying that Emma Watson has no self-respect? Could it be appealing to authority because the person making the argument claims to be a feminist? Could it be a false dilemma because the argument assumes that a woman is either a feminist or not, with no gray area in between?

Answers

Try it Now 20

- Circular
- Correlation does not imply causation
- Post hoc
- Appeal to consequence

e. Straw man

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6.2.9: Exercises

Boolean Logic

For questions 1-2, list the set of integers that satisfy the given conditions.

1. A positive multiple of 5 and not a multiple of 2
2. Greater than 12 and less than or equal to 18

Quantified Statements

For questions 3-4, write the negation of each quantified statement.

3. Everyone failed the quiz today.
4. Someone in the car needs to use the restroom.

Truth Tables

5. Translate each statement from symbolic notation into English sentences. Let A represent "Elvis is alive" and let G represent "Elvis gained weight".

- a. $A \vee G$
- b. $\sim (A \wedge G)$
- c. $G \rightarrow \sim A$
- d. $A \leftrightarrow \sim G$

For questions 6-9, create a truth table for each statement.

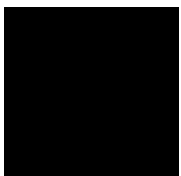
6. $A \wedge \sim B$
7. $\sim (\sim A \vee B)$
8. $(A \wedge B) \rightarrow C$
9. $(A \vee B) \rightarrow \sim C$

Questions 10 – 13 : In this lesson, we have been studying the inclusive or, which allows both A and B to be true. The exclusive or does not allow both to be true; it translates to "either A or B , but not both."

10. For each situation, decide whether the "or" is most likely exclusive or inclusive.

- a. An entrée at a restaurant includes soup or a salad.
- b. You should bring an umbrella or a raincoat with you.
- c. We can keep driving on I-5 or get on I-405 at the next exit.
- d. You should save this document on your computer or a flash drive.

11. Complete the truth table for the exclusive or.



12. Complete the truth table for $(A \vee B) \wedge \sim (A \wedge B)$.



13. Compare your answers for questions 11 and 12. Can you explain the similarities?

Conditional Statements

14. Consider the statement “If you are under age 17, then you cannot attend this movie.”

- a. Write the converse.
- b. Write the inverse.
- c. Write the contrapositive.

15. Assume that the statement “If you swear, then you will get your mouth washed out with soap” is true. Which of the following statements must also be true?

- a. If you don’t swear, then you won’t get your mouth washed out with soap.
- b. If you don’t get your mouth washed out with soap, then you didn’t swear.
- c. If you get your mouth washed out with soap, then you swore.

For questions 16-18, write the negation of each conditional statement.

16. If you don’t look both ways before crossing the street, then you will get hit by a car.

17. If Luke faces Vader, then Obi-Wan cannot interfere.

18. If you weren’t talking, then you wouldn’t have missed the instructions.

19. Assume that the biconditional statement “You will play in the game if and only if you attend all practices this week” is true. Which of the following situations could happen?

- a. You attended all practices this week and didn’t play in the game.
- b. You didn’t attend all practices this week and played in the game.
- c. You didn’t attend all practices this week and didn’t play in the game.

De Morgan’s Laws

For questions 20-21, use De Morgan’s Laws to rewrite each conjunction as a disjunction, or each disjunction as a conjunction.

20. It is not true that Tina likes Sprite or 7-Up.

21. It is not the case that you need a dated receipt and your credit card to return this item.

22. Go back and look at the truth tables in Exercises 6 & 7. Explain why the results are identical.

Deductive Arguments

For questions 23-28, use a Venn diagram or truth table or common form of an argument to decide whether each argument is valid or invalid.

23. If a person is on this reality show, they must be self-absorbed. Laura is not self-absorbed. Therefore, Laura cannot be on this reality show.

24. If you are a triathlete, then you have outstanding endurance. LeBron James is not a triathlete. Therefore, LeBron does not have outstanding endurance.

25. Jamie must scrub the toilets or hose down the garbage cans. Jamie refuses to scrub the toilets. Therefore, Jamie will hose

down the garbage cans.

26. Some of these kids are rude. Jimmy is one of these kids. Therefore, Jimmy is rude!

27. Every student brought a pencil or a pen. Marcie brought a pencil. Therefore, Marcie did not bring a pen.

28. If a creature is a chimpanzee, then it is a primate. If a creature is a primate, then it is a mammal. Bobo is a mammal. Therefore, Bobo is a chimpanzee.

Logical Fallacies

For questions 28-30, name the type of logical fallacy being used.

29. If you don't want to drive from Boston to New York, then you will have to take the train.

30. New England Patriots quarterback Tom Brady likes his footballs slightly underinflated. The "Cheatriots" have a history of bending or breaking the rules, so Brady must have told the equipment manager to make sure that the footballs were underinflated.

31. Whenever our smoke detector beeps, my kids eat cereal for dinner. The loud beeping sound must make them want to eat cereal for some reason.

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6.2.10: De Morgan's Laws

A contemporary of Boole's, Augustus De Morgan, formalized two rules of logic that had previously been known informally. They allow us to rewrite the negation of a conjunction as a disjunction, and vice-versa.

For example, suppose you want to schedule a meeting with two colleagues at 4 : 30PM on Friday, and you need both of them to be available at that time. What situation would make it impossible to have the meeting? It is NOT the case that colleague a is available AND colleague b is available: $\sim (a \wedge b)$. This situation is equivalent to either colleague a NOT being available OR colleague b NOT being available: $\sim a \vee \sim b$

De Morgan's Laws

The negation of a conjunction is equivalent to the disjunction of the negation of the statements making up the conjunction. To negate an "and" statement, negate each part and change the "and" to "or".

$$\sim (p \wedge q) \text{ is equivalent to } \sim p \vee \sim q$$

The negation of a disjunction is equivalent to the conjunction of the negation of the statements making up the disjunction. To negate an "or" statement, negate each part and change the "or" to "and".

$$\sim (p \vee q) \text{ is equivalent to } \sim p \wedge \sim q$$

Example 28

For Valentine's Day, you did not get your sweetie flowers or candy: Which of the following statements is logically equivalent?

- a. You did not get them flowers or did not get them candy.
- b. You did not get them flowers and did not get them candy.
- c. You got them flowers or got them candy.

Solution

- a. This statement does not go far enough; it leaves open the possibility that you got them one of the two things.
- b. This statement is equivalent to the original; $\sim (f \vee c)$ is equivalent to $\sim f \wedge \sim c$
- c. This statement says that you got them something, but we know that you did not.

Try it Now 8

To serve as the President of the US, a person must have been born in the US, must be at least 35 years old, and must have lived in the US for at least 14 years. What minimum set of conditions would disqualify someone from serving as President?

Answer available at the end of this section.

Answers

Try it Now 8

Failing to meet just one of the three conditions is all it takes to be disqualified. A person is disqualified if they were not born in the US, or are not at least 35 years old, or have not lived in the US for at least 14 years. The key word here is "or" instead of "and".

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6.2.11: Arguments

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

Argument types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

Example 29

The argument “when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store” is an inductive argument.

The premises are:

I forgot my purse last week

I forgot my purse today

The conclusion is:

I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

Example 30

The argument “every day for the past year, a plane flies over my house at 2:00 P.M. A plane will fly over my house every day at 2:00 P.M.” is a stronger inductive argument, since it is based on a larger set of evidence. While it is not necessarily true—the airline may have cancelled its afternoon flight—it is probably a safe bet.

Evaluating inductive arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest that it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. Common scientific theories, like Newton’s theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence, such as when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes it easier to evaluate.

Evaluating deductive arguments

A deductive argument is considered valid if, assuming that all the premises are true, the conclusion follows logically from those premises. In other words, when the premises are all true, the conclusion *must* be true.

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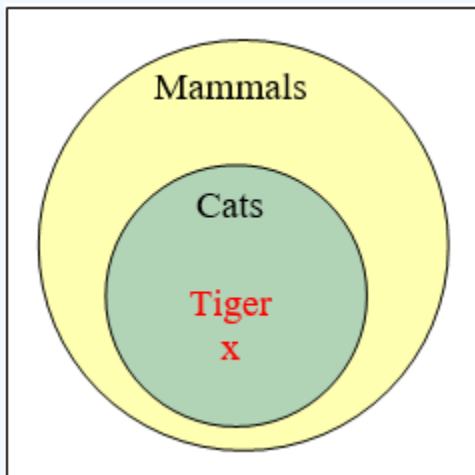
6.2.12: Evaluating Deductive Arguments with Euler Diagrams

We can interpret a deductive argument visually with an Euler diagram, which is essentially the same thing as a Venn diagram. This can make it easier to determine whether the argument is valid or invalid.

Example 31

Consider the deductive argument “All cats are mammals and a tiger is a cat, so a tiger is a mammal.” Is this argument valid?

Solution



The premises are:

All cats are mammals.

A tiger is a cat.

The conclusion is:

A tiger is a mammal.

Both the premises are true. To see that the premises must logically lead to the conclusion, we can use a Venn diagram. From the first premise, we draw the set of cats as a subset of the set of mammals. From the second premise, we are told that a tiger is contained within the set of cats. From that, we can see in the Venn diagram that the tiger must also be inside the set of mammals, so the conclusion is valid.

Analyzing arguments with Euler diagrams

To analyze an argument with an Euler diagram:

- 1) Draw an Euler diagram based on the premises of the argument
- 2) The argument is invalid if there is a way to draw the diagram that makes the conclusion false
- 3) The argument is valid if the diagram cannot be drawn to make the conclusion false
- 4) If the premises are insufficient to determine the location of an element or a set mentioned in the conclusion, then the argument is invalid.

Try it Now 9

Determine the validity of this argument:

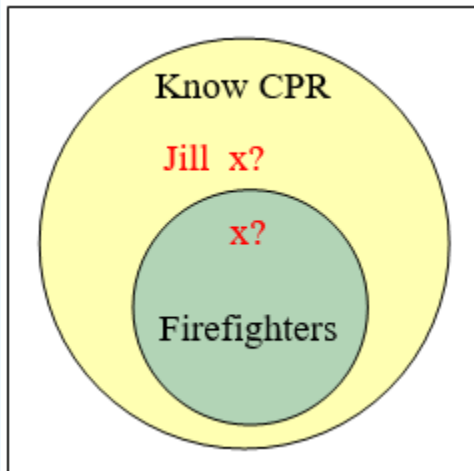
Premise: All cats are scared of vacuum cleaners.

Premise: Max is a cat.

Conclusion: Max is scared of vacuum cleaners.

Answer available at the end of this section.

Example 32



Premise: All firefighters know CPR.

Premise: Jill knows CPR.

Conclusion: Jill is a firefighter.

Solution

From the first premise, we know that firefighters all lie inside the set of those who know CPR. (Firefighters are a subset of people who know CPR.) From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know whether she also is a member of the smaller subset that is firefighters.

Since the conclusion does not necessarily follow from the premises, this is an invalid argument. It's possible that Jill is a firefighter, but the structure of the argument doesn't allow us to conclude that she definitely is.

It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are concerned with whether the premises are enough to prove the conclusion.

Try it Now 10

Determine the validity of this argument:

Premise: All bicycles have two wheels.

Premise: This Harley-Davidson has two wheels.

Conclusion: This Harley-Davidson is a bicycle.

Answer available at the end of this section.

Try it Now 11

Determine the validity of this argument:

Premise: No cows are purple.

Premise: Fido is not a cow.

Conclusion: Fido is purple.

Answer available at the end of this section.

In addition to these categorical style premises of the form “all ___”, “some ___”, and “no ___”, it is also common to see premises that are conditionals.

Example 33

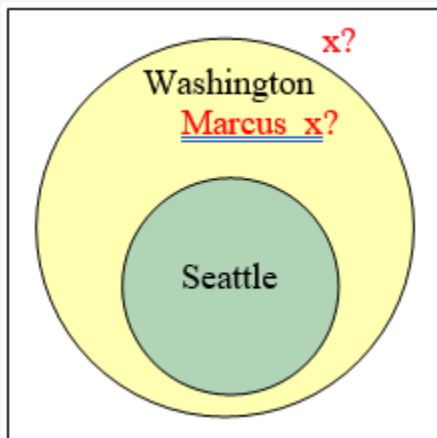
Premise: If you live in Seattle, you live in Washington.

Premise: Marcus does not live in Seattle.

Conclusion: Marcus does not live in Washington.

Solution

From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether Marcus lives in Washington or not. This is an invalid argument.



Try it Now 12

Determine the validity of this argument:

Premise: If you have lipstick on your collar, then you are cheating on me.

Premise: If you are cheating on me, then I will divorce you.

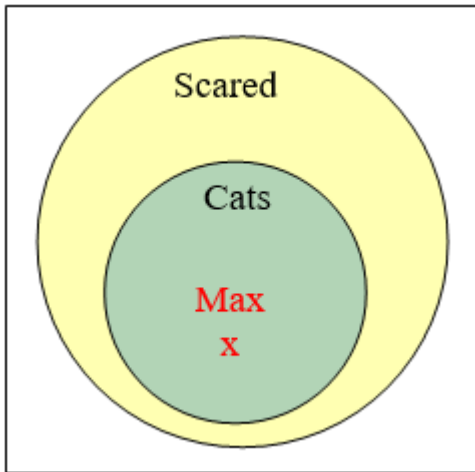
Premise: You do not have lipstick on your collar.

Conclusion: I will not divorce you.

Answer available at the end of this section.

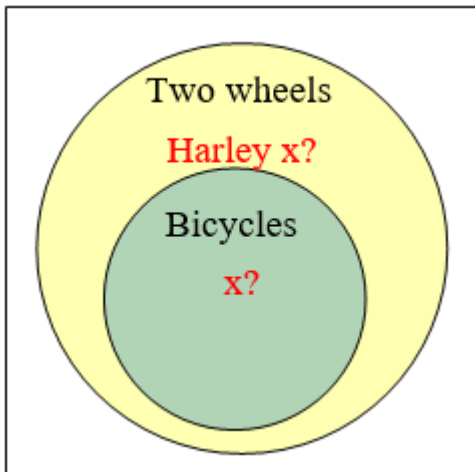
Answers

Try it Now 9



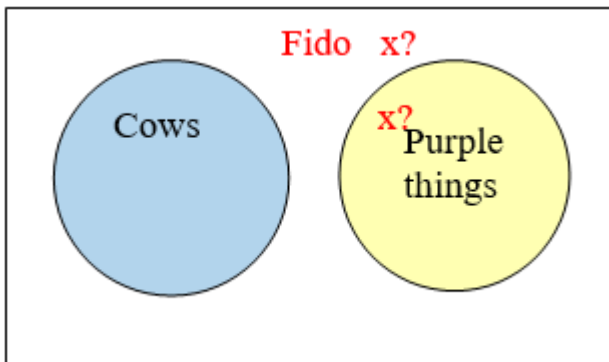
Valid. Cats are a subset of creatures that are scared by vacuum cleaners. Max is in the set of cats, so he must also be in the set of creatures that are scared by vacuum cleaners.

Try it Now 10



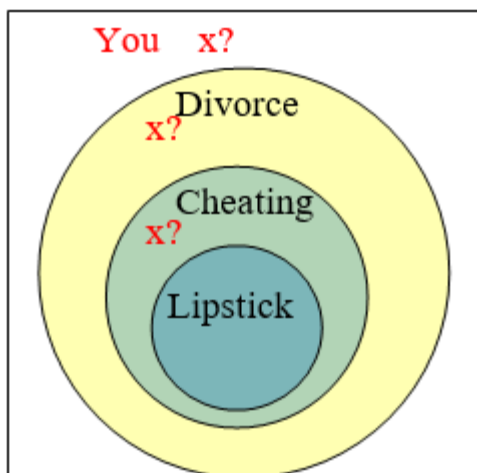
Invalid. The set of bicycles is a subset of the set of vehicles with two wheels; the Harley-Davidson is in the set of two-wheeled vehicles but not necessarily in the smaller circle.

Try it Now 11



Invalid. Since no cows are purple, we know there is no overlap between the set of cows and the set of purple things. We know Fido is not in the cow set, but that is not enough to conclude that Fido is in the purple things set.

Tryt it Now 12



Invalid. Lipstick on your collar is a subset of scenarios in which you are cheating, and cheating is a subset of the scenarios in which I will divorce you. Although it is wonderful that you don't have lipstick on your collar, you could still be cheating on me, and I will divorce you. In fact, even if you aren't cheating on me, I might divorce you for another reason. You'd better shape up.

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6.3: GROWTH

Population growth is a current topic in the media today. The world population is growing by over 70 million people every year. Predicting populations in the future can have an impact on how countries plan to manage resources for more people. The tools needed to help make predictions about future populations are growth models like the exponential function. This chapter will discuss real world phenomena, like population growth and radioactive decay, using three different growth models.

The growth functions to be examined are linear, exponential, and logistic growth models. Each type of model will be used when data behaves in a specific way and for different types of scenarios. Data that grows by the same amount in each iteration uses a different model than data that increases by a percentage.

6.3.0: Linear Growth

6.3.1: Exponential Growth

6.3.2: Special Cases- Doubling Time and Half-Life

6.3.3: Natural Growth and Logistic Growth

6.3.4: Exercises

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6.3.0: Linear Growth

Starting at the age of 25, imagine if you could save \$20 per week, every week, until you retire, how much money would you have stuffed under your mattress at age 65? To solve this problem, we could use a linear growth model. Linear growth has the characteristic of growing by the same amount in each unit of time. In this example, there is an increase of \$20 per week; a constant amount is placed under the mattress in the same unit of time.

If we start with \$0 under the mattress, then at the end of the first year we would have $\$20 \cdot 52 = \$1040$. So, this means you could add \$1040 under your mattress every year. At the end of 40 years, you would have $\$1040 \cdot 40 = \$41,600$ for retirement. This is not the best way to save money, but we can see that it is calculated in a systematic way.

Definition: Linear Growth

A quantity grows linearly if it grows by a constant amount for each unit of time.

Example 6.3.0.1: City Growth

Suppose in Flagstaff Arizona, the number of residents increased by 1000 people per year. If the initial population was 46,080 in 1990, can you predict the population in 2013? This is an example of linear growth because the population grows by a constant amount. We list the population in future years below by adding 1000 people for each passing year.

	1990	1991	1992	1993	1994	1995	1996
Year	0	1	2	3	4	5	6
Population	46,080	47,080	48,080	49,080	50,080	51,080	52,080

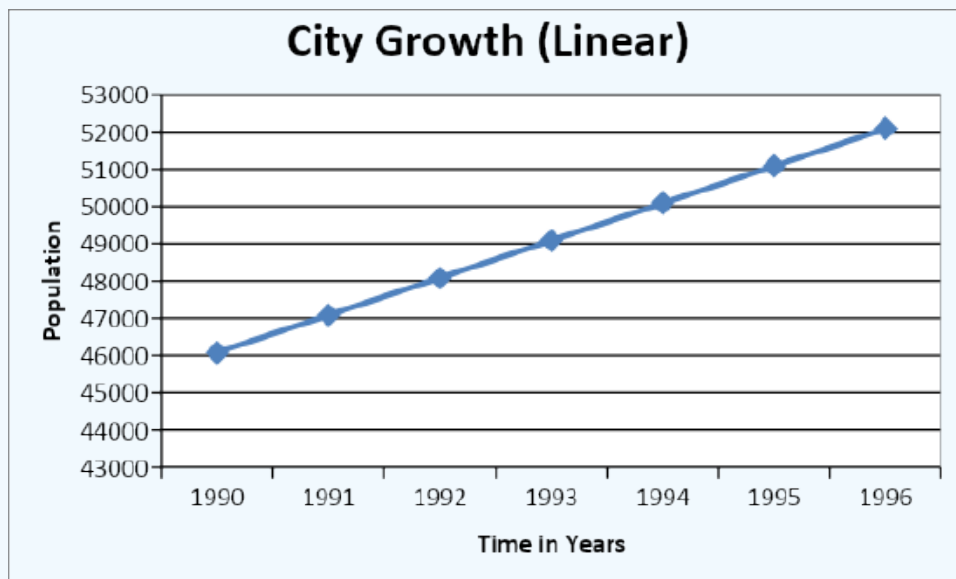


Figure 6.3.0.1: Graph of Linear Population Growth

The population growth can be modeled with a linear equation. The initial population P_0 is 46,080. The future population depends on the number of years, t , after the initial year. The model is $P(t) = 46,080 + 1000t$

To predict the population in 2013, we identify how many years it has been from 1990, which is year zero. So $n = 23$ for the year 2013.

$$P(23) = 46,080 + 1000(23) = 69,080$$

The population of Flagstaff in 2013 would be 69,080 people.

Linear Growth Model: Linear growth begins with an initial population called P_0 . In each time period or generation t , the population changes by a constant amount called the common difference d . The basic model is:

$$P(t) = P_0 + td$$

Example 6.3.0.2: Antique Frog Collection

Dora has inherited a collection of 30 antique frogs. Each year she vows to buy two frogs a month to grow the collection. This is an additional 24 frogs per year. How many frogs will she have in six years? How long will it take her to reach 510 frogs?

Solution

The initial population is $P_0 = 30$ and the common difference is $d = 24$. The linear growth model for this problem is:

$$P(t) = 30 + 24t$$

The first question asks how many frogs Dora will have in six years so, $t = 6$.

$$P(6) = 30 + 24(6) = 30 + 144 = 174$$

frogs.

The second question asks for the time it will take for Dora to collect 510 frogs. So, $P(t) = 510$ and we will solve for t .

$$510 = 30 + 24t$$

$$480 = 24t$$

$$20 = t$$

It will take 20 years to collect 510 antique frogs.

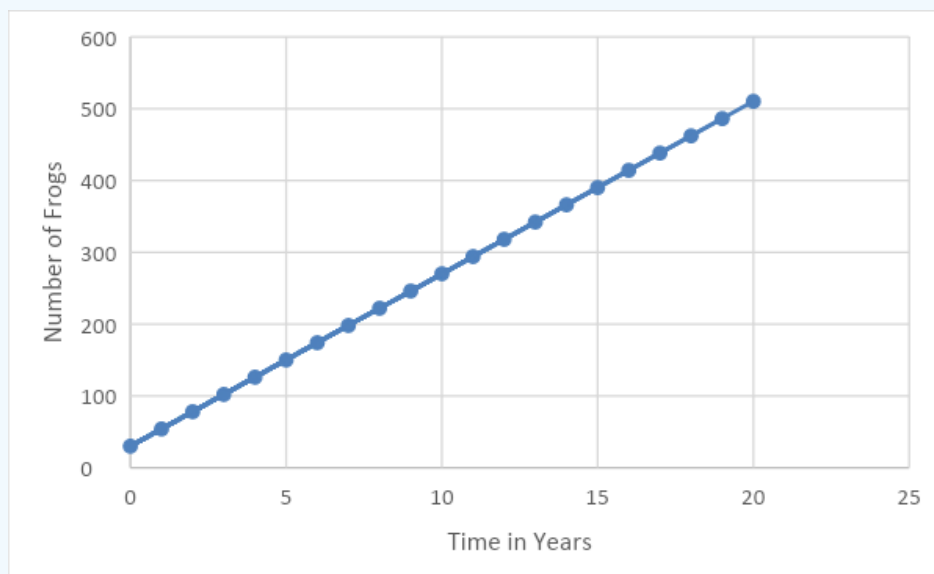


Figure 6.3.0.2: Graph of Antique Frog Collection

Note: The graph of the number of antique frogs Dora accumulates over time follows a straight line.

Example 6.3.0.3: Car Depreciation

Assume a car depreciates by the same amount each year. Joe purchased a car in 2010 for \$16,800. In 2014 it is worth \$12,000. Find the linear growth model. Predict how much the car will be worth in 2020.

$$P_0 = 16,800 \text{ and } P_4 = 12,000$$

To find the linear growth model for this problem, we need to find the common difference d .

$$\begin{aligned} P(t) &= P_0 + td \\ 12,000 &= 16,800 + 4d \\ -4800 &= 4d \\ -1200 &= d \end{aligned}$$

The common difference of depreciation each year is $d = \$ -1200$. Thus, the linear growth model for this problem is: $P(t) = 16,800 - 1200t$

Now, to find out how much the car will be worth in 2020, we need to know how many years that is from the purchase year. Since it is ten years later, $t = 10$.

$$P(10) = 16,800 - 1200(10) = 16,800 - 12,000 = 4,800$$

The car is worth \$4800 in 2020.

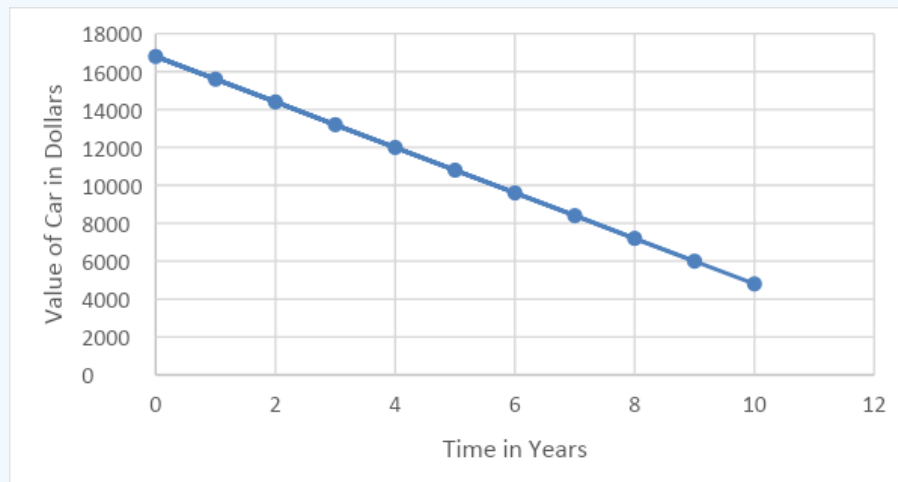


Figure 6.3.0.3: Graph of Car Value Depreciation

Note: The value of the car over time follows a decreasing straight line.

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6.3.1: Exponential Growth

The next growth we will examine is exponential growth. Linear growth occurs by adding the same amount in each unit of time. Exponential growth happens when an initial population increases by the same percentage or factor over equal time increments or generations. This is known as relative growth and is usually expressed as percentage. For example, let's say a population is growing by 1.6% each year. For every 1000 people in the population, there will be $1000 \cdot 0.016 = 16$ more people added per year.

Definition: Exponential Growth

A quantity grows exponentially if it grows by a constant factor or rate for each unit of time.

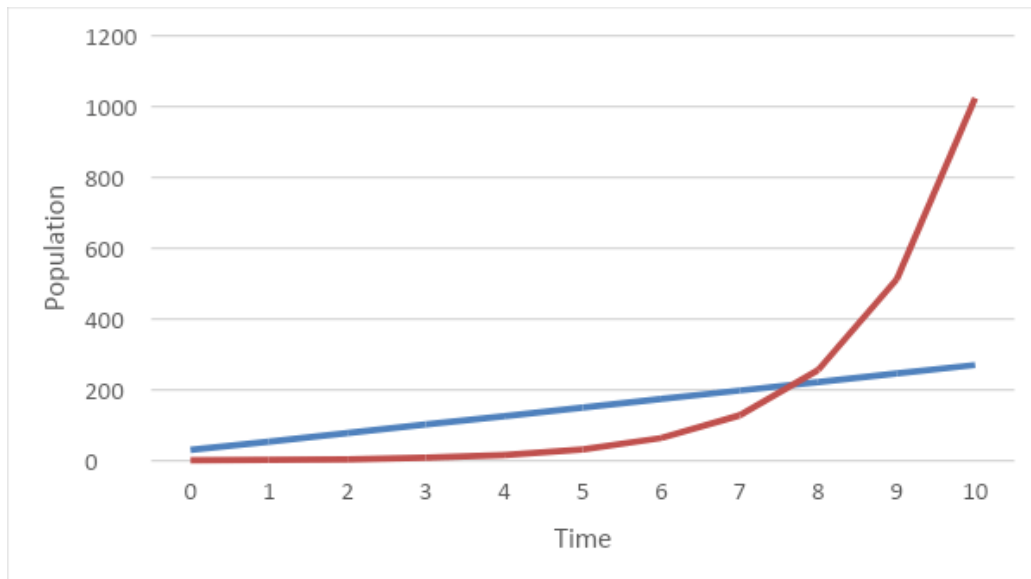


Figure 6.3.1.1: Graphical Comparison of Linear and Exponential Growth

In this graph, the blue straight line represents linear growth and the red curved line represents exponential growth.

Example 6.3.1.1: City Growth

A city is growing at a rate of 1.6% per year. The initial population in 2010 is $P_0 = 125,000$. Calculate the city's population over the next few years.

Solution

The relative growth rate is 1.6%. This means an additional 1.6% is added on to 100% of the population that already exists each year. This is a factor of 1.016.

- Population in 2011

$$125,000(1.016)^1 = 127,000$$

- Population in 2012

$$127,000(1.016) = 125,000(1.016)^2 = 129,032$$

- Population in 2013

$$129,032(1.016) = 125,000(1.016)^3 = 131,097$$

We can create an equation for the city's growth. Each year the population is 1.6% more than the previous year.

$$P(t) = 125,000(1 + 0.016)^t$$



Figure 6.3.1.2: Graph of City Growth

Note: The graph of the city growth follows an exponential growth model

Example 6.3.1.2: A Shrinking Population

St. Louis, Missouri has declined in population at a rate of 1.6 % per year over the last 60 years. The population in 1950 was 857,000. Find the population in 2014. (Wikipedia, n.d.)

Solution

$$P_0 = 857,000$$

The relative growth rate is 1.6%. This means 1.6% of the population is subtracted from 100% of the population that already exists each year. This is a factor of 98.4%.

- Population in 1951

$$857,000(0.984)^1 = 843,288$$

- Population in 1952

$$843,288(0.984) = 857,000(0.984)^2 = 829,795$$

- Population in 1953

$$829,795(0.984) = 857,000(0.984)^3 = 816,519$$

We can create an equation for the city's growth. Each year the population is 1.6% less than the previous year.

$$P(t) = 857,000(1 - 0.016)^t$$

So the population of St. Louis Missouri in 2014, when $t = 64$, is:

$$\begin{aligned} P(64) &= 857,000(1 - 0.016)^{64} \\ &= 857,000(0.984)^{64} \\ &= 305,258 \end{aligned}$$

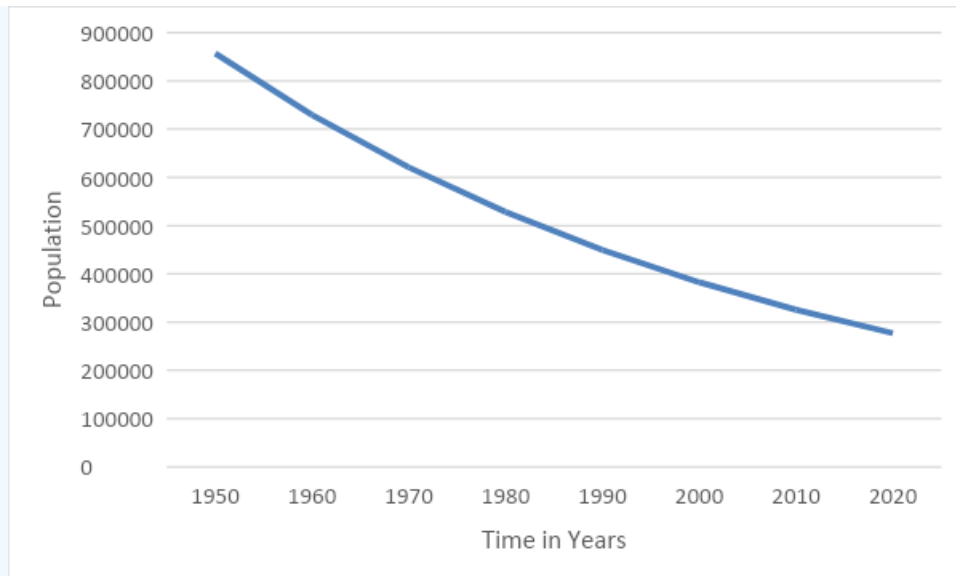


Figure 6.3.1.3: Graph of St. Louis, Missouri Population Decline

Note: The graph of the population of St. Louis, Missouri over time follows a declining exponential growth model.

Definition: Exponential Growth Model $P(t) = P_0(1 + r)^t$

P_0 is the initial population.

r is the relative growth rate.

t is the time unit.

r is the positive if the population is increasing and negative if the population is decreasing.

Example 6.3.1.3: Inflation

The average inflation rate of the U.S. dollar over the last five years is 1.7% per year. If a new car cost \$18,000 five years ago, how much would it cost today? (U.S. Inflation Calculator, n.d.)

Solution

To solve this problem, we use the exponential growth model with $r = 1.7\%$.

$P_0 = 18,000$ and $t = 5$

$$P(t) = 18,000(1 + 0.017)^t$$

$$P(5) = 18,000(1 + 0.017)^5 = 19,582.91$$

This car would cost \$19,582.91 today.

Example 6.3.1.4: Ebola Epidemic in Sierra Leone

In May of 2014 there were 15 cases of Ebola in Sierra Leone. By August, there were 850 cases. If the virus is spreading at the same rate (exponential growth), how many cases will there be in February of 2015? (McKenna, 2014)

Solution

To solve this problem, we have to find three things; the growth rate per month, the exponential growth model, and the number of cases of Ebola in February 2015. First calculate the growth rate per month. To do this, use the initial population $P_0 = 15$, in May 2014. Also, in August, three months later, the number of cases was 850 so, $P(3) = 850$.

Use these values and the exponential growth model to solve for r .

$$\begin{aligned} P(t) &= P_0(1+r)^t \\ 850 &= 15(1+r)^3 \\ 56.67 &= (1+r)^3 \\ \sqrt[3]{56.67} &= \sqrt[3]{(1+r)^3} \\ 3.84 &= 1+r \\ 2.84 &= r \end{aligned}$$

The growth rate is 284% per month. Thus, the exponential growth model is:

$$P(t) = 15(1 + 2.84)^t = 15(3.84)^t$$

Now, we use this to calculate the number of cases of Ebola in Sierra Leone in February 2015, which is 9 months after the initial outbreak so, $t = 9$

$$P(9) = 15(3.84)^9 = 2,725,250$$

If this same exponential growth rate continues, the number of Ebola cases in Sierra Leone in February 2015 would be 2,725,250.

This is a bleak prediction for the community of Sierra Leone. Fortunately, the growth rate of this deadly virus should be reduced by the world community and World Health Organization by providing the needed means to fight the initial spread.

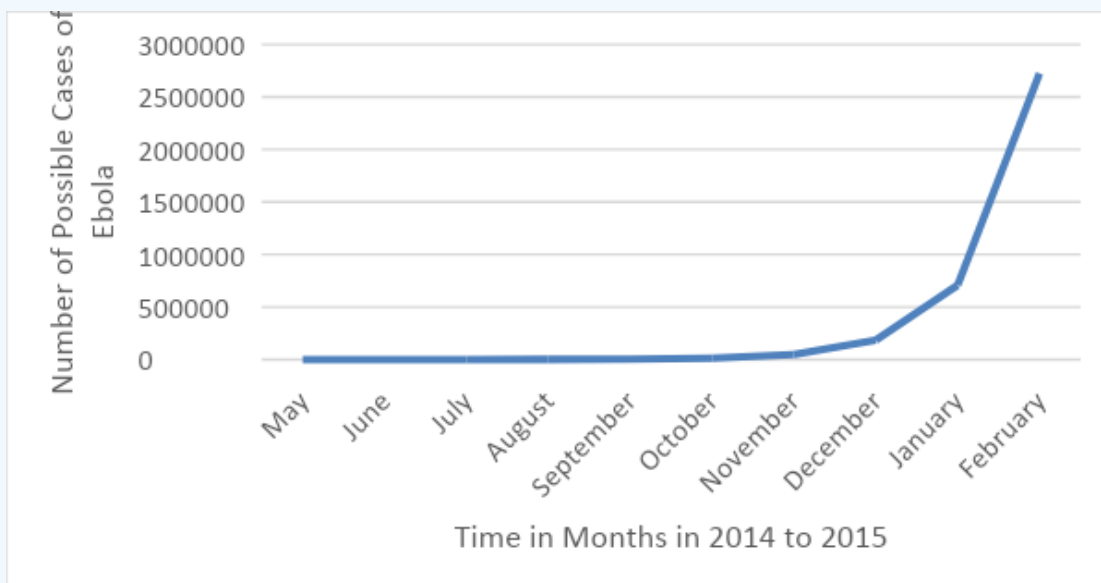


Figure 6.3.1.4: Graph of Ebola Virus in Sierra Leone

Note: The graph of the number of possible Ebola cases in Sierra Leone over time follows an increasing exponential growth model.

Example 6.3.1.5: Population Decline in Puerto Rico

According to a new forecast, the population of Puerto Rico is in decline. If the population in 2010 is 3,978,000 and the prediction for the population in 2050 is 3,697,000, find the annual percent decrease rate. (Bloomberg Businessweek, n.d.)

Solution

To solve this problem we use the exponential growth model. We need to solve for r .

$$P(t) = P_0(1 + r)^t \text{ where } t = 40 \text{ years}$$

$$P(40) = 3,697,000 \text{ and } P_0 = 3,978,000$$

$$3,697,000 = 3,978,000(1 + r)^{40}$$

$$0.92936 = (1 + r)^{40}$$

$$\sqrt[40]{0.92936} = \sqrt[40]{(1 + r)^{40}}$$

$$0.99817 = 1 + r$$

$$-0.0018 = r$$

The annual percent decrease is 0.18%.

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6.3.2: Special Cases- Doubling Time and Half-Life

Example 6.3.2.1: April Fool's Joke

Let's say that on April 1st I say I will give you a penny, on April 2nd two pennies, four pennies on April 3rd, and that I will double the amount each day until the end of the month. How much money would I have agreed to give you on April 30? With $P_0 = \$0.01$, we get the following table:

Table 6.3.2.1: April Fool's Joke

Day	Dollar Amount
April 1 = P_0	0.01
April 2 = P_1	0.02
April 3 = P_2	0.04
April 4 = P_3	0.08
April 5 = P_4	0.16
April 6 = P_5	0.32
....	
April 30 = P_{29}	?

In this example, the money received each day is 100% more than the previous day. If we use the exponential growth model $P(t) = P_0(1 + r)^t$ with $r = 1$, we get the doubling time model.

$$P(t) = P_0(1 + 1)^t = P_0(2)^t$$

We use it to find the dollar amount when $t = 29$ which represents April 30

$$P(29) = 0.01(2)^{29} = \$5,368,709.12$$

Surprised? That is a lot of pennies.

Doubling Time Model

Example 6.3.2.2: E. coli Bacteria

A water tank up on the San Francisco Peaks is contaminated with a colony of 80,000 *E. coli* bacteria. The population doubles every five days. We want to find a model for the population of bacteria present after t days. The amount of time it takes the population to double is five days, so this is our time unit. After t days have passed, then $t/5$ is the number of time units that have passed. Starting with the initial amount of 80,000 bacteria, our doubling model becomes:

$$P(t) = 80,000(2)^{\frac{t}{5}}$$

Using this model, how large is the colony in two weeks' time? We have to be careful that the units on the times are the same; 2 weeks = 14 days.

Solution

$$P(14) = 80,000(2)^{\frac{14}{5}} = 557,152$$

The colony is now 557,152 bacteria.

Definition: Doubling Time Model

If D is the doubling time of a quantity (the amount of time it takes the quantity to double) and P_0 is the initial amount of the quantity then the amount of the quantity present after t units of time is $P(t) = P_0(2)^{\frac{t}{D}}$

Example 6.3.2.3: Files

The doubling time of a population of flies is eight days. If there are initially 100 flies, how many flies will there be in 17 days?

Solution

To solve this problem, use the doubling time model with $D = 8$ and $P_0 = 100$ so the doubling time model for this problem is:

$$P(t) = 100(2)^{t/8}$$

When $t = 17$ days,

$$P(17) = 100(2)^{\frac{17}{8}} = 436$$

There are 436 flies after 17 days.

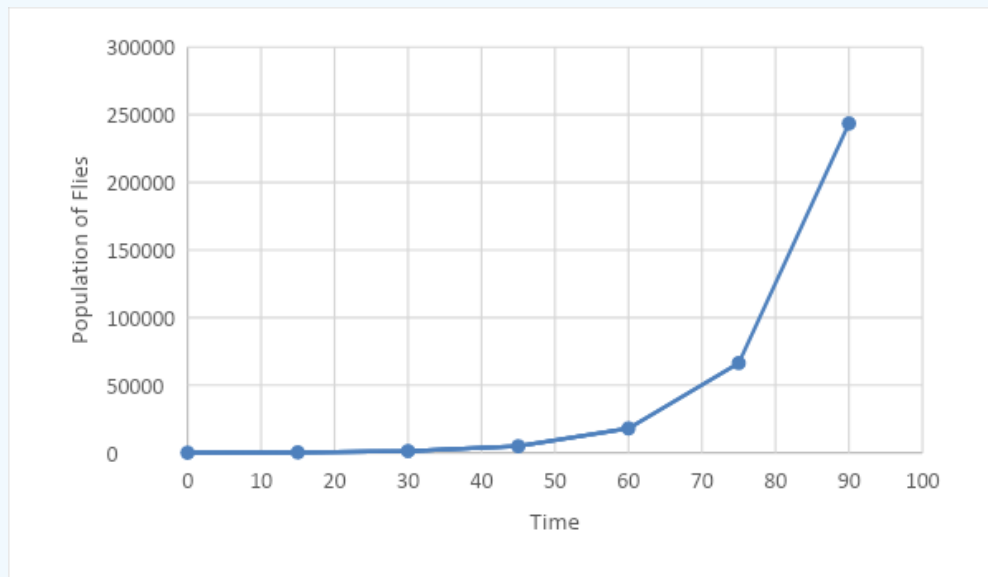


Figure 6.3.2.2: Graph of Fly Population in Three Months

Note: The population of flies follows an exponential growth model.

Sometimes we want to solve for the length of time it takes for a certain population to grow given their doubling time. To solve for the exponent, we use the $\log$ button on the calculator.

Example 6.3.2.4: Bacteria Growth

Suppose that a bacteria population doubles every six hours. If the initial population is 4000 individuals, how many hours would it take the population to increase to 25,000?

Solution

$P_0 = 4000$ and $D = 6$, so the doubling time model for this problem is:

$$P(t) = 4000(2)^{\frac{t}{6}}$$

Now, find t when $P(t) = 25,000$

$$25,000 = 4000(2)^{\frac{t}{6}}$$

$$\frac{25,000}{4000} = \frac{4000(2)^{\frac{t}{6}}}{4000}$$

$$6.25 = (2)^{\frac{t}{6}}$$

Now, take the log of both sides of the equation.

$$\log 6.25 = \log(2)^{\frac{t}{6}}$$

The exponent comes down using rules of logarithms.

$$\log 6.25 = \left(\frac{t}{6}\right)\log(2)$$

Now, calculate $\log 6.25$ and $\log 2$ with your calculator.

$$\begin{aligned} 0.7959 &= \left(\frac{t}{6}\right) \cdot 0.3010 \\ \frac{0.7959}{0.3010} &= \frac{t}{6} \\ 2.644 &= \frac{t}{6} \\ t &= 15.9 \end{aligned}$$

The population would increase to 25,000 bacteria in approximately 15.9 hours.

Rule of 70

There is a simple formula for approximating the doubling time of a population. It is called the rule of 70 and it is an approximation for growth rates less than 15%. Do not use this formula if the growth rate is 15% or greater.

Rule of 70

For a quantity growing at a constant percentage rate (not written as a decimal), R , per time period, the doubling time is approximately given by

$$D \approx \frac{70}{R}$$

Example 6.3.2.5: Bird Population

A bird population on a certain island has an annual growth rate of 2.5% per year. Approximate the number of years it will take the population to double. If the initial population is 20 birds, use it to find the bird population of the island in 17 years.

Solution

To solve this problem, first approximate the population doubling time.

Doubling time $D \approx \frac{70}{2.5} = 28$ years.

With the bird population doubling in 28 years, we use the doubling time model to find the population is 17 years.

$$P(t) = 20(2)^{\frac{t}{28}}$$

When $t = 17$ years

$$P(17) = 20(2)^{\frac{17}{28}} = 30.46$$

There will be 30 birds on the island in 17 years.

Example 6.3.2.6: Cancer Growth Rate

A certain cancerous tumor doubles in size every six months. If the initial size of the tumor is four cells, how many cells will there in three years? In seven years?

Solution

To calculate the number of cells in the tumor, we use the doubling time model. Change the time units to be the same. The doubling time is six months = 0.5 years.

$$P(t) = 4(2)^{\frac{t}{0.5}}$$

When $t = 3$ years

$$P(3) = 4(2)^{\frac{3}{0.5}} = 256 \text{ cells}$$

When $t = 7$ years

$$P(7) = 4(2)^{\frac{7}{0.5}} = 65,536 \text{ cells}$$

Example 6.3.2.7: Approximating Annual Growth Rate

Suppose that a certain city's population doubles every 12 years. What is the approximate annual growth rate of the city?

Solution

By solving the doubling time model for the growth rate, we can solve this problem.

$$\begin{aligned} D &\approx \frac{70}{R} \\ R \cdot D &\approx \frac{70}{\cancel{R}} \cdot \cancel{R} \\ RD &\approx 70 \\ \frac{\cancel{R}}{D} &\approx \frac{70}{D} \\ \text{Annual growth rate } R &\approx \frac{70}{D} \\ R &= \frac{70}{12} = 5.83\% \end{aligned}$$

The annual growth rate of the city is approximately 5.83%

Exponential Decay and Half-Life Model

The half-life of a material is the time it takes for a quantity of material to be cut in half. This term is commonly used when describing radioactive metals like uranium or plutonium. For example, the half-life of carbon-14 is 5730 years.

If a substance has a half-life, this means that half of the substance will be gone in a unit of time. In other words, the amount decreases by 50% per unit of time. Using the exponential growth model with a decrease of 50%, we have

$$P(t) = P_0(1 - 0.5)^t = P_0\left(\frac{1}{2}\right)^t$$

Example 6.3.2.8: Half-Life

Let's say a substance has a half-life of eight days. If there are 40 grams present now, how much is left after three days?

Solution

We want to find a model for the quantity of the substance that remains after t days. The amount of time it takes the quantity to be reduced by half is eight days, so this is our time unit. After t days have passed, then $t/8$ is the number of time units that have passed. Starting with the initial amount of 40, our half-life model becomes:

$$P(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{8}}$$

With $t = 3$

$$P(3) = 40\left(\frac{1}{2}\right)^{\frac{3}{8}} = 30.8$$

There are 30.8 grams of the substance remaining after three days.

Definition: Half-Life Model

If H is the half-life of a quantity (the amount of time it takes the quantity be cut in half) and P_0 is the initial amount of the quantity then the amount of the quantity present after t units of time is

$$P(t) = P_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$$

Example 6.3.2.9

Lead-209 is a radioactive isotope. It has a half-life of 3.3 hours. Suppose that 40 milligrams of this isotope is created in an experiment, how much is left after 14 hours?

Solution

Use the half-life model to solve this problem.

$P_0 = 40$ and $H = 3.3$, so the half-life model for this problem is:

$$P(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{3.3}}$$

With $t = 14$ hours,

$$P(14) = 40\left(\frac{1}{2}\right)^{\frac{14}{3.3}} = 2.1$$

There are 2.1 milligrams of Lead-209 remaining after 14 hours.

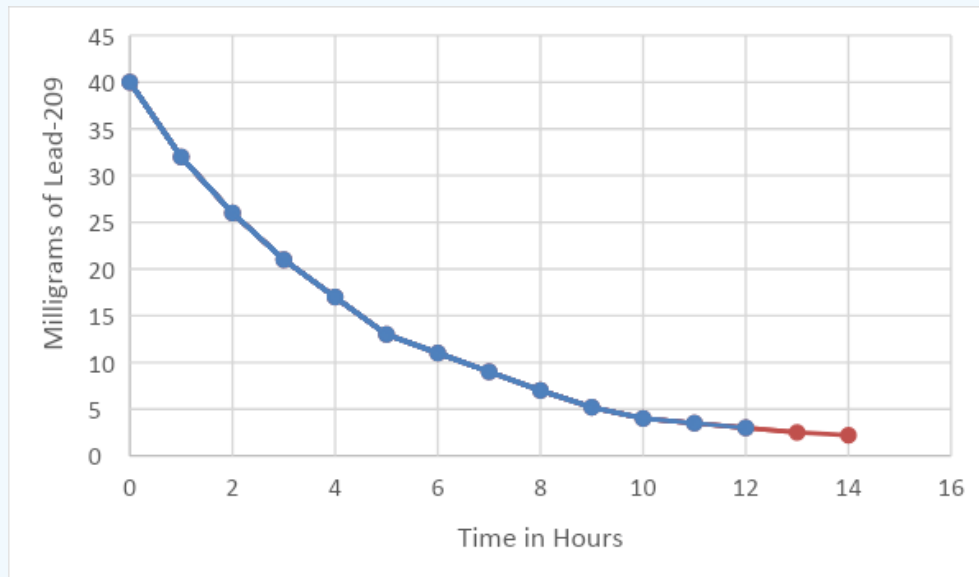


Figure 6.3.2.3: Lead-209 Decay Graph

Note: The milligrams of Lead-209 remaining follows a decreasing exponential growth model.

Example 6.3.2.10: Nobelium-259

Nobelium-259 has a half-life of 58 minutes. If you have 1000 grams, how much will be left in two hours?

Solution

We solve this problem using the half-life model. Before we begin, it is important to note the time units. The half-life is given in minutes and we want to know how much is left in two hours. Convert hours to minutes when using the model: two hours = 120 minutes.

$P_0 = 1000$ and $H = 58$ minutes, so the half-life model for this problem is:

$$P(t) = 1000\left(\frac{1}{2}\right)^{\frac{t}{58}}$$

With $t = 120$ minutes,

$$P(120) = 1000\left(\frac{1}{2}\right)^{\frac{120}{58}} = 238.33$$

There are 238 grams of Nobelium-259 is remaining after two hours.

Example 6.3.2.11: Carbon-14

Radioactive carbon-14 is used to determine the age of artifacts because it concentrates in organisms only when they are alive. It has a half-life of 5730 years. In 1947, earthenware jars containing what are known as the Dead Sea Scrolls were found. Analysis indicated that the scroll wrappings contained 76% of their original carbon-14. Estimate the age of the Dead Sea Scrolls.

Solution

In this problem, we want to estimate the age of the scrolls. In 1947, 76% of the carbon-14 remained. This means that the amount remaining at time t divided by the original amount of carbon-14, P_0 , is equal to 76%. So, $\frac{P(t)}{P_0} = 0.76$ we use this fact to solve for t .

$$\begin{aligned} P(t) &= P_0 \left(\frac{1}{2}\right)^{\frac{t}{5730}} \\ \frac{P(t)}{P_0} &= \left(\frac{1}{2}\right)^{\frac{t}{5730}} \\ 0.76 &= \left(\frac{1}{2}\right)^{\frac{t}{5730}} \end{aligned}$$

Now, take the log of both sides of the equation.

$$\log 0.76 = \log \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

The exponent comes down using rules of logarithms.

$$\log 0.76 = \left(\frac{t}{5730}\right) \log \frac{1}{2}$$

Now, calculate $\log 0.76$ and $\log \frac{1}{2}$ with your calculator.

$$\begin{aligned} -0.1192 &= \left(\frac{t}{5730}\right) \cdot (-0.3010) \\ \frac{-0.1192}{-0.3010} &= \frac{t}{5730} \\ 0.3960 &= \frac{t}{5730} \\ t &= 2269.08 \end{aligned}$$

The Dead Sea Scrolls are well over 2000 years old.

Example 6.3.2.12: Plutonium

Plutonium has a half-life of 24,000 years. Suppose that 50 pounds of it was dumped at a nuclear waste site. How long would it take for it to decay into 10 lbs?

Solution

$P_0 = 50$ and $H = 24,000$ minutes, so the half-life model for this problem is:

$$P(t) = 50 \left(\frac{1}{2}\right)^{\frac{t}{24,000}}$$

Now, find t when $P_t = 10$.

$$\begin{aligned} 10 &= 50 \left(\frac{1}{2}\right)^{\frac{t}{24,000}} \\ \frac{10}{50} &= \left(\frac{1}{2}\right)^{\frac{t}{24,000}} \\ 0.2 &= \left(\frac{1}{2}\right)^{\frac{t}{24,000}} \end{aligned}$$

Now, take the log of both sides of the equation.

$$\log 0.2 = \log \left(\frac{1}{2}\right)^{\frac{t}{24,000}}$$

The exponent comes down using rules of logarithms.

$$\log 0.2 = \left(\frac{t}{24,000}\right) \log \frac{1}{2}$$

Now, calculate $\log 0.2$ and $\log \frac{1}{2}$ with your calculator.

$$\begin{aligned} -0.6990 &= \left(\frac{t}{24,000}\right) \cdot (-0.3010) \\ \frac{-0.6990}{-0.3010} &= \frac{t}{24,000} \\ 2.322 &= \frac{t}{24,000} \\ t &= 55,728 \end{aligned}$$

The quantity of plutonium would decrease to 10 pounds in approximately 55,728 years.

Rule of 70 for Half-Life:

There is simple formula for approximating the half-life of a population. It is called the rule of 70 and is an approximation for decay rates less than 15%. Do not use this formula if the decay rate is 15% or greater.

Rule of 70

For a quantity decreasing at a constant percentage (not written as a decimal), R , per time period, the half-life is approximately given by:

$$\text{Half-life } H \approx \frac{70}{R}$$

Example 6.3.2.13: Elephant Population

The population of wild elephants is decreasing by 7% per year. Approximate the half -life for this population. If there are currently 8000 elephants left in the wild, how many will remain in 25 years?

Solution

To solve this problem, use the half-life approximation formula.

$$\text{Half-Life } H \approx \frac{70}{7} = 10 \text{ years}$$

$P_0 = 7000$ and $H = 10$ years, so the half-life model for this problem is:

$$P(t) = 7000\left(\frac{1}{2}\right)^{\frac{t}{10}}$$

When $t = 25$,

$$P(25) = 7000\left(\frac{1}{2}\right)^{\frac{25}{10}} = 1237.4$$

There will be approximately 1237 wild elephants left in 25 years.

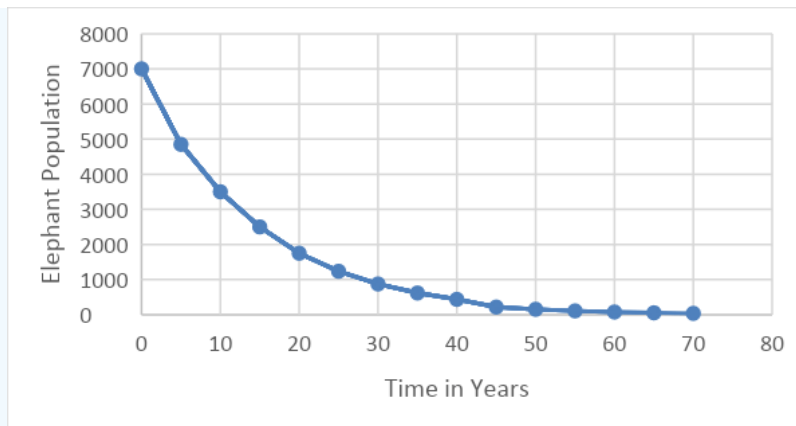


Figure 6.3.2.4: Elephant Population over a 70 Year Span.

Note: The population of elephants follows a decreasing exponential growth model.

Review of Exponent Rules and Logarithm Rules

Rules of Exponents	Rules of Logarithm for the Common Logarithm (Base 10)
Definition of an Exponent $a^n = a \cdot a \cdot a \cdot a \cdot \dots \cdot a$ (n a's multiplied together)	Definition of a Logarithm $10^y = x$ if and only if $\log x = y$
Zero Rule $a^0 = 1$	
Product Rule $a^m \cdot a^n = a^{m+n}$	Product Rule $\log(xy) = \log(x) + \log(y)$
Quotient Rule $\frac{a^m}{a^n} = a^{m-n}$	Quotient Rule $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$
Power Rule $(a^n)^m = a^{n \cdot m}$	Power Rule $\log x^r = r \log(x) (x > 0)$
Distributive Rules $(ab)^n = a^n \cdot b^n, \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\log 10^x = x \log(10) = x$
Negative Exponent Rules $a^{-n} = \frac{1}{a^n}, \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$10^{\log x} = x (x > 0)$

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6.3.3: Natural Growth and Logistic Growth

In this chapter, we have been looking at linear and exponential growth. Another very useful tool for modeling population growth is the natural growth model. This model uses base e , an irrational number, as the base of the exponent instead of $(1 + r)$. You may remember learning about e in a previous class, as an exponential function and the base of the natural logarithm.

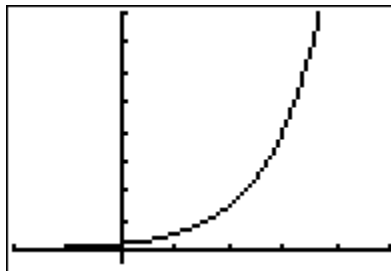
Definition: The Natural Growth Model

The Natural Growth Model is

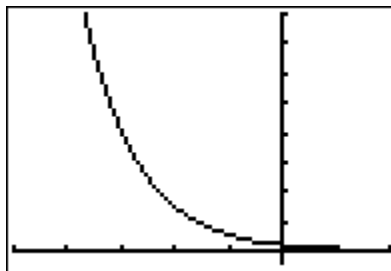
$$P(t) = P_0 e^{kt}$$

where P_0 is the initial population, k is the growth rate per unit of time, and t is the number of time periods.

Given $P_0 > 0$, if $k > 0$, this is an exponential growth model, if $k < 0$, this is an exponential decay model.



a. Natural growth function
 $P(t) = e^t$



b. Natural decay function
 $P(t) = e^{-t}$

Figure 6.3.3.1: Natural Growth and Decay Graphs

Example 6.3.3.1: Drugs in the Bloodstream

When a certain drug is administered to a patient, the number of milligrams remaining in the bloodstream after t hours is given by the model

$$P(t) = 40e^{-.25t}$$

How many milligrams are in the blood after two hours?

Solution

To solve this problem, we use the given equation with $t = 2$

$$P(2) = 40e^{-.25(2)}$$

$$P(2) = 24.26$$

There are approximately 24.6 milligrams of the drug in the patient's bloodstream after two hours.

In the next example, we can see that the exponential growth model does not reflect an accurate picture of population growth for natural populations.

Example 6.3.3.2: Ants in the Yard

Bob has an ant problem. On the first day of May, Bob discovers he has a small red ant hill in his back yard, with a population of about 100 ants. If conditions are just right red ant colonies have a growth rate of 240% per year during the first four years. If Bob does nothing, how many ants will he have next May? How many in five years?

Solution

We solve this problem using the natural growth model.

$$P(t) = 100e^{2.4t}$$

In one year, $t = 1$, we have

$$P(1) = 100e^{2.4(1)} = 1102 \text{ ants}$$

In one year, $t = 5$, we have

$$P(5) = 100e^{2.4(5)} = 16,275,479 \text{ ants}$$

That is a lot of ants! Bob will not let this happen in his back yard!

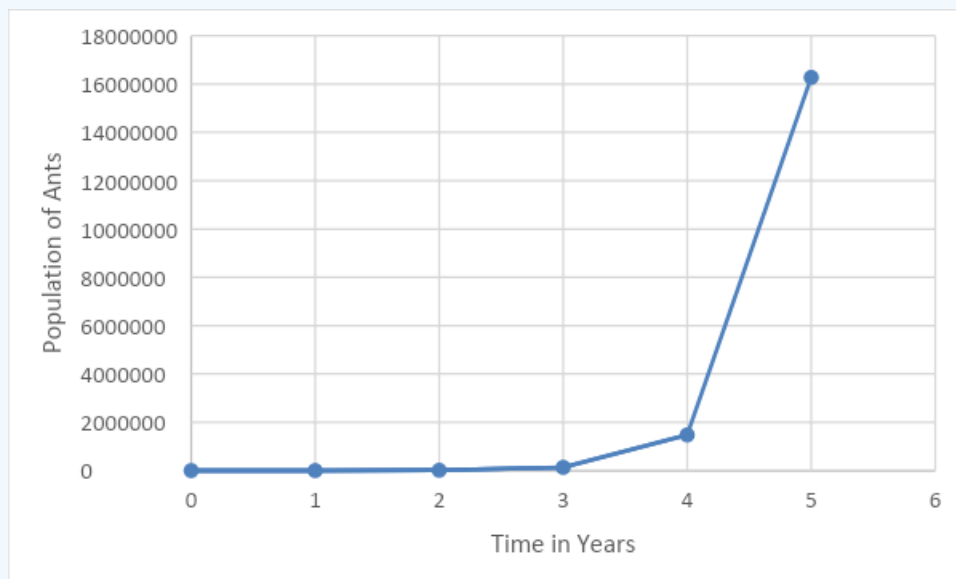


Figure 6.3.3.2: Graph of Ant Population Growth in Bob's Yard.

Note: The population of ants in Bob's back yard follows an exponential (or natural) growth model.

The problem with exponential growth is that the population grows without bound and, at some point, the model will no longer predict what is actually happening since the amount of resources available is limited. Populations cannot continue to grow on a purely physical level, eventually death occurs and a limiting population is reached.

Another growth model for living organisms in the logistic growth model. The logistic growth model has a maximum population called the carrying capacity. As the population grows, the number of individuals in the population grows to the carrying capacity and stays there. This is the maximum population the environment can sustain.

Logistic Growth Model

$$P(t) = \frac{M}{1 + ke^{-ct}}$$

where M , c , and k are positive constants and t is the number of time periods.

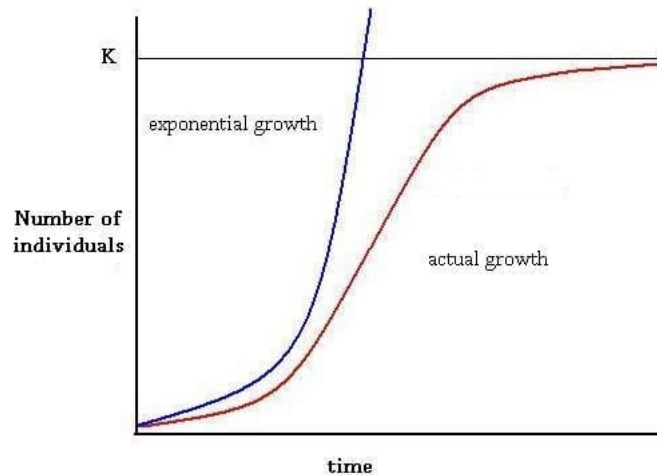


Figure 6.3.3.1: Comparison of Exponential Growth and Logistic Growth

The horizontal line K on this graph illustrates the carrying capacity. However, this book uses M to represent the carrying capacity rather than K .

(Logistic Growth Image 1, n.d.)

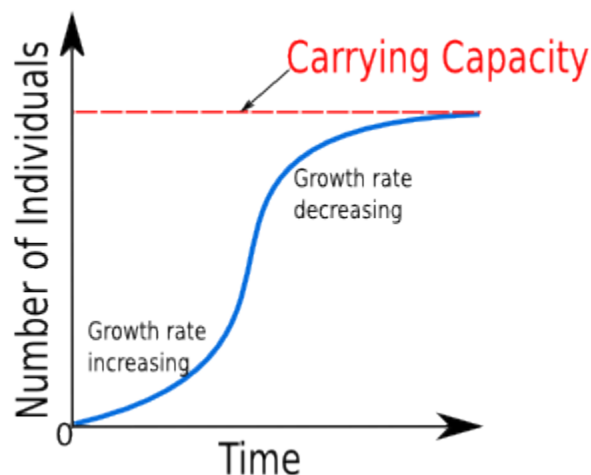


Figure 6.3.3.4: Logistic Growth Model

(Logistic Growth Image 2, n.d.)

The graph for logistic growth starts with a small population. When the population is small, the growth is fast because there is more elbow room in the environment. As the population approaches the carrying capacity, the growth slows.

Example 6.3.3.3: Bird Population

The population of an endangered bird species on an island grows according to the logistic growth model.

$$P(t) = \frac{3640}{1 + 25e^{-0.04t}}$$

Identify the initial population. What will be the bird population in five years? What will be the population in 150 years? What will be the population in 500 years?

Solution

We know the initial population, P_0 , occurs when $t = 0$.

$$P_0 = P(0) = \frac{3640}{1 + 25e^{-0.04(0)}} = 140$$

Calculate the population in five years, when $t = 5$.

$$P(5) = \frac{3640}{1 + 25e^{-0.04(5)}} = 169.6$$

The island will be home to approximately 170 birds in five years

Calculate the population in 150 years, when $t = 150$.

$$P(150) = \frac{3640}{1 + 25e^{-0.04(150)}} = 3427.6$$

The island will be home to approximately 3428 birds in 150 years.

Calculate the population in 500 years, when $t = 500$.

$$P(500) = \frac{3640}{1 + 25e^{-0.04(500)}} = 3640.0$$

The island will be home to approximately 3640 birds in 500 years.

This example shows that the population grows quickly between five years and 150 years, with an overall increase of over 3000 birds; but, slows dramatically between 150 years and 500 years (a longer span of time) with an increase of just over 200 birds.

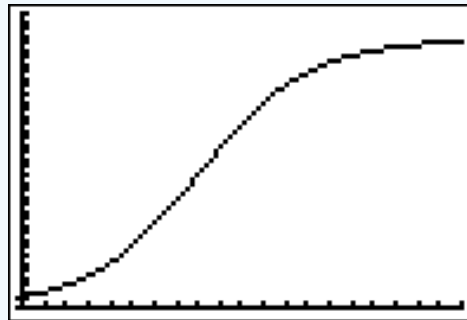


Figure 6.3.3.5: Bird Population over a 200-Year Span

Example 6.3.3.4: Student Population at Northern Arizona University

The student population at NAU can be modeled by the logistic growth model below, with initial population taken from the early 1960's. We will use 1960 as the initial population date.

$$P(t) = \frac{30,000}{1 + 5e^{-0.06t}}$$

Determine the initial population and find the population of NAU in 2014. What will be NAU's population in 2050? From this model, what do you think is the carrying capacity of NAU?

Solution

We solve this problem by substituting in different values of time.

When $t = 0$, we get the initial population P_0 .

$$P_0 = P(0) = \frac{30,000}{1 + 5e^{-0.06(0)}} = \frac{30,000}{6} = 5000$$

The initial population of NAU in 1960 was 5000 students.

In the year 2014, 54 years have elapsed so, $t = 54$.

$$P(54) = \frac{30,000}{1 + 5e^{-0.06(54)}} = \frac{30,000}{1 + 5e^{-3.24}} = \frac{30,000}{1.19582} = 25,087$$

There are 25,087 NAU students in 2014.

In 2050, 90 years have elapsed so, $t = 90$.

$$P(90) = \frac{30,000}{1 + 5e^{-0.06(90)}} = \frac{30,000}{1 + 5e^{-5.4}} = 29,337$$

There are 29,337 NAU students in 2050.

Finally, to predict the carrying capacity, look at the population 200 years from 1960, when $t = 200$.

$$P(200) = \frac{30,000}{1 + 5e^{-0.06(200)}} = \frac{30,000}{1 + 5e^{-12}} = \frac{30,000}{1.00003} = 29,999$$

Thus, the carrying capacity of NAU is 30,000 students.

It appears that the numerator of the logistic growth model, M , is the carrying capacity.

Carrying Capacity

Given the logistic growth model $P(t) = \frac{M}{1 + ke^{-ct}}$, the carrying capacity of the population is M . M , the carrying capacity, is the maximum population possible within a certain habitat.

Example 6.3.3.5

Suppose that in a certain fish hatchery, the fish population is modeled by the logistic growth model where t is measured in years.

$$P(t) = \frac{12,000}{1 + 11e^{-0.2t}}$$

What is the carrying capacity of the fish hatchery? How long will it take for the population to reach 6000 fish?

Solution

The carrying capacity of the fish hatchery is $M = 12,000$ fish.

Now, we need to find the number of years it takes for the hatchery to reach a population of 6000 fish. We must solve for t when $P(t) = 6000$.

$$\begin{aligned}
 6000 &= \frac{12,000}{1 + 11e^{-0.2t}} \\
 (1 + 11e^{-0.2t}) \cdot 6000 &= \frac{12,000}{1 + 11e^{-0.2t}} \cdot (1 + 11e^{-0.2t}) \\
 (1 + 11e^{-0.2t}) \cdot 6000 &= 12,000 \\
 \frac{(1 + 11e^{-0.2t}) \cdot \cancel{6000}}{\cancel{6000}} &= \frac{12,000}{6000} \\
 1 + 11e^{-0.2t} &= 2 \\
 11e^{-0.2t} &= 1 \\
 e^{-0.2t} &= \frac{1}{11} = 0.090909
 \end{aligned}$$

Take the natural logarithm (ln on the calculator) of both sides of the equation.

$$\begin{aligned}
 \ln e^{-0.2t} &= \ln 0.090909 \\
 \ln e^{-0.2t} &= -0.2t \text{ by the rules of logarithms.} \\
 -0.2t &= \ln 0.090909 \\
 t &= \frac{\ln 0.090909}{-0.2} \\
 t &= 11.999
 \end{aligned}$$

It will take approximately 12 years for the hatchery to reach 6000 fish.

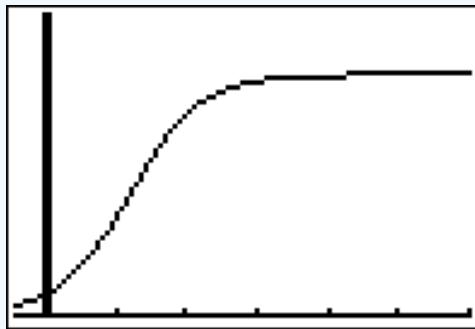


Figure 6.3.3.6: Fish Population over a 30-Year Period.

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6.3.4: Exercises

1. Goodyear AZ is one of the fastest growing cities in the nation according to the census bureau. In 2012, the population was about 72,800. The city's population grew by 3800 people from 2012 to 2013. If the growth keeps up in a linear fashion, create a population model for Goodyear. How many people will live there in 10 years? How many people will live there in 50 years? (U.S. Census, 2014)
2. Gilbert AZ is one of the fastest growing cities in the nation according to the census bureau. In 2012, the population was about 229,800. The city's population grew by 9200 people from 2012 to 2013. If the growth keeps up in a linear fashion, create a population model for Gilbert. How many people will live there in 10 years? How many people will live there in 50 years? (U.S. Census, 2014)
3. Apache County AZ, is shrinking according to the census bureau. In 2012, the population was about 71,700. The county's population decreased by 1147 people from 2012 to 2013. If the decline is linear, create a population model for Apache County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)
4. Cochise County AZ, is shrinking according to the census bureau. In 2012, the population was about 129,472. The county's population decreased by 2600 people from 2012 to 2013. If the decline is linear, create a population model for Cochise County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)
5. Mohave County AZ, is shrinking according to the census bureau. In 2012, the population was about 203,030. The county's population decreased by 1200 people from 2012 to 2013. If the decline is linear, create a population model for Mohave County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)
6. The 2012 Kia Sedona LX has one of the largest depreciation values of any car. Suppose a 2012 Kia Sedona sold for \$24,900, and its value depreciates by \$3400 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)
7. The 2013 Chevy Impala has one of the largest depreciation values of any car. Suppose a 2013 Chevy Impala sold for \$27,800 and its value depreciates by \$3600 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)
8. The 2013 Jaguar XJ AWD has one of the largest depreciation values of any car. Suppose a 2013 Jaguar XJ AWD sold for \$74,500 and its value depreciates by \$10,400 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)
9. The 2012 Jeep Liberty Limited Sport 2WD has one of the largest depreciation values of any car. Suppose a 2012 Jeep Liberty Limited Sport 2WD sold for \$23,400 and its value depreciates by \$3,040 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)
10. Suppose that in January, the maximum water depth at Lake Powell, Arizona was 528 feet. The water evaporates at an average rate of 1.2 feet per month. Find a model for the rate at which the water evaporates. If it does not rain at all, what will be the depth of Lake Powell in May and in September?
11. Suppose that the maximum water depth at Lake Tahoe, California in 2014 was 1644 feet. Because of the drought, the water level has been decreasing at an average rate of 6.2 feet per year. Find a model for the rate at which the water level decreases. If it there is no precipitation at all, what will be the depth of Lake Tahoe be in two years, and in five years?
12. Suppose the homes in Arizona have appreciated an average of 8% per year in the last five years. If the average home in a suburb sold for \$225,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?
13. Suppose the homes in Massachusetts have appreciated an average of 13% per year in the last five years. If the average home in a suburb sold for \$205,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?

14. Suppose the homes in Michigan have depreciated an average of 17% per year in the last five years. If the average home in a suburb sold for \$215,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?
15. Suppose the homes in Nevada have depreciated an average of 15% per year in the last five years. If the average home in a suburb sold for \$318,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?
16. The cost of a home in Flagstaff AZ was \$89,000 in 1992. In 2007, the same home appraised for \$349,000. Assuming the home value grew according to the exponential growth model, find the annual growth rate of this home over this 15-year period. If the growth continued at this rate, what would the home be worth in 2020?
17. The cost of a home in Bullhead City AZ was \$109,000 in 1992. In 2007, the same home appraised for \$352,000. Assuming the home value grew according to the exponential growth model, find the annual growth rate of this home over this 15-year period. If the growth continued at this rate, what would the home be worth in 2020?
18. The population of West Virginia is in decline. The population in 2014 was 1,850,326 and the population had decreased by 0.14% from 2010. How many people were living in West Virginia in 2010? Create a model for this population. If the decline continues at this rate, how many people will reside in West Virginia in 2020? (Wikipedia, n.d.)
19. Assume that the population of Arizona grew by 2.4% per year between the years 2000 to 2010. The number of Native American living in Arizona was 257,426 in 2010. How many Native Americans were living in Arizona in 2000? Create a model for this population. If the growth continues at this rate, how many Native Americans will reside in Arizona in 2020?
20. Assume the population of the U.S. grew by 0.96% per year between the years 2000 and 2010. The number of Hispanic Americans was 55,740,000 in 2010. How many Hispanic Americans were living in the U.S. in 2000? Create a model for this population. If the growth continues at this rate, how many Hispanic Americans will reside in The U.S. in 2020?
21. Assume the population of Michigan decreased by 0.6% per year between the years 2000 to 2010. The population of Michigan was 9,970,000 in 2010. How many people were living in Michigan in 2000? Create a model for this population. If the growth continues at this rate, how many people will reside in Michigan in 2020?
22. The doubling time of a population of aphids is 12 days. If there are initially 200 aphids, how many aphids will there be in 17 days?
23. The doubling time of a population of rabbits is six months. If there are initially 26 rabbits, how many rabbits will there be in 17 months?
24. The doubling time of a population of shrews is three months. If there are initially 32 shrews, how many shrews will there be in 21 months?
25. The doubling time of a population of hamsters is 1.2 years. If there are initially 43 hamsters, how many hamsters will there be in 7 years?
26. A certain cancerous tumor doubles in size in three months. If the initial size of the tumor is two cells, how many months will it take for the tumor to grow to 60,000 cells? How many cells will there be in 1.5 years? In three years?
27. A certain cancerous tumor doubles in size in 1.5 months. If the initial size of the tumor is eight cells, how many months will it take for the tumor to grow to 40,000 cells? How many cells will there be in six months? In 2.5 years?
28. A bird population on a certain island has an annual growth rate of 1.5% per year. Approximate the number of years it will take the population to double. If the initial population is 130 birds, use it to find the bird population of the island in 14 years.
29. The beaver population on Kodiak Island has an annual growth rate of 1.2% per year. Approximate the number of years it will take the population to double. If the initial population is 32 beavers, use it to find the population of beavers on the island in 20 years.
30. The black-footed ferret population in Arizona has an annual growth rate of 0.5% per year. Approximate the number of years it will take the population to double. If the initial population is 230 ferrets, use it to find the ferret population in AZ in 12 years.

31. The Mexican gray wolf population in southern Arizona increased from 72 individuals to 92 individuals from 2012 to 2013. What is the annual growth rate? Approximate the number of years it will take the population to double. Create the doubling time model and use it to find the population of Mexican gray wolves in 10 years.
32. There is a small population of Sonoran pronghorn antelope in a captive breeding program on the Cabeza Prieta National Wildlife Refuge in southwest Arizona. Recently, the population increased from 122 individuals to 135 individuals from 2012 to 2013. Approximate the number of years it will take the population to double. Create the doubling time model and use it to find the population of Sonoran pronghorn antelope in 10 years.
33. Lead-209 is a radioactive isotope. It has a half-life of 3.3 hours. Suppose that 56 milligrams of this isotope is created in an experiment, how much is left after 12 hours?
34. Titanium-44 has a half-life of 63 years. If there is 560 grams of this isotope, how much is left after 1200 years?
35. Uranium-232 has a half-life of 68.9 years. If there is 160 grams of this isotope, how much is left after 1000 years?
36. Nickel-63 has a half-life of 100 years. If there is 16 grams of this isotope, how much is left after 145 years?
37. Radium-226 has a half-life of 1600 years. If there is 56 grams of this isotope, how much is left after 145,000 years?
38. The population of wild desert tortoises is decreasing by 72% per year. Approximate the half-life for this population. If there are currently 100,000 tortoises left in the wild, how many will remain in 20 years?
39. The population of pygmy owls is decreasing by 4.4% per year. Approximate the half-life for this population. If there are currently 41 owls left in the wild, how many will remain in five years?
40. Radioactive carbon-14 is used to determine the age of artifacts because it concentrates in organisms only when they are alive. It has a half-life of 5730 years. In 2004, carbonized plant remains were found where human artifacts were unearthed along the Savannah River in Allendale County. Analysis indicated that the plant remains contained 0.2% of their original carbon-14. Estimate the age of the plant remains. (Wikipedia, n.d.)

41. The population of a bird species on an island grows according to the logistic model below.

$$P(t) = \frac{43,240}{1 + 12e^{-0.05t}}$$

Identify the initial population. What will be the bird population in five years? In 150 years? In 500 years? What is the carrying capacity of the bird population on the island?

42. Suppose the population of students at Arizona State University grows according to the logistic model below with the initial population taken from 2009.

$$P(t) = \frac{85,240}{1 + 0.6e^{-0.15t}}$$

Identify the initial population. What will be the campus population in five years? What will be the population in 20 years? What is the carrying capacity of students at ASU?

43. Suppose the population of students at Ohio State University grows according to the logistic model below with the initial population taken from 1970.

$$P(t) = \frac{90,000}{1 + 4e^{-0.06t}}$$

Identify the initial population. What will be the campus population in five years? What will be the population in 20 years? What is the carrying capacity of students at OSU?

44. Suppose that in a certain shrimp farm, the shrimp population is modeled by the logistic model below where t is measured in years.

$$P(t) = \frac{120,000}{1 + 25e^{-0.12t}}$$

Find the initial population. How long will it take for the population to reach 6000 shrimp? What is the carrying capacity of shrimp on the farm?

45. Suppose that in a certain oyster farm, the oyster population is modeled by the logistic model below where t is measured in years.

$$P(t) = \frac{18,000}{1 + 19e^{-0.36t}}$$

Find the initial population. How long will it take for the population to reach 2000 oysters? What is the carrying capacity of oysters on the farm?

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6.4: GRAPH THEORY

Graph theory deals with routing and network problems and if it is possible to find a “best” route, whether that means the least expensive, least amount of time or the least distance. Some examples of routing problems are routes covered by postal workers, UPS drivers, police officers, garbage disposal personnel, water meter readers, census takers, tour buses, etc. Some examples of network problems are telephone networks, railway systems, canals, roads, pipelines, and computer chips.

6.4.0: Graph Theory

6.4.1: Networks

6.4.2: Euler Circuits

6.4.3: Hamiltonian Circuits

6.4.4: Exercises

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6.4.0: Graph Theory

There are several definitions that are important to understand before delving into Graph Theory. They are:

- A graph is a picture of dots called **vertices** and lines called **edges**.
- An edge that starts and ends at the same vertex is called a **loop**.
- If there are two or more edges directly connecting the same two vertices, then these edges are called **multiple edges**.
- If there is a way to get from one vertex of a graph to all the other vertices of the graph, then the graph is **connected**.
- If there is even one vertex of a graph that cannot be reached from every other vertex, then the graph is **disconnected**.

Example 6.4.0.1: Graph Example

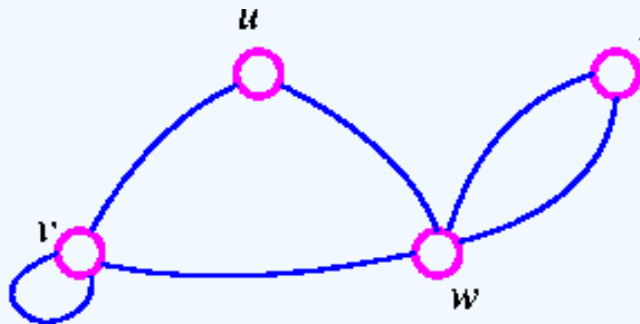


Figure 6.4.0.1: Graph 1

In the above graph, the **vertices** are U, V, W, and Z and the **edges** are UV, VV, VW, UW, WZ1, and WZ2.

This is a **connected** graph. VV is a loop. WZ1, and WZ2 are **multiple edges**.

Example 6.4.0.2: Graph Example



Figure 6.4.0.2: Graph 2 Figure 6.4.0.3: Graph 3

The graph in Figure 6.4.0.2 is connected while the graph in Figure 6.4.0.3 is disconnected.

Definition: Graph Concepts and Terminology

- **Order of a Network:** the number of vertices in the entire network or graph
- **Adjacent Vertices:** two vertices that are connected by an edge
- **Adjacent Edges:** two edges that share a common vertex
- **Degree of a Vertex:** the number of edges at that vertex
- **Path:** a sequence of vertices with each vertex adjacent to the next one that starts and ends at different vertices and travels over any edge only once
- **Circuit:** a path that starts and ends at the same vertex
- **Bridge:** an edge such that if it were removed from a connected graph, the graph would become disconnected

Example 6.4.0.3: Graph Terminology

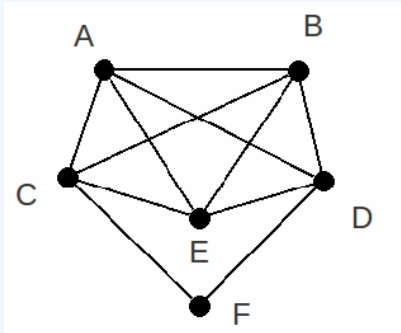


Figure 6.4.0.4: Graph 4

In the above graph the following is true:

- Vertex A is adjacent to vertex B, vertex C, vertex D, and vertex E.
- Vertex F is adjacent to vertex C, and vertex D.
- Edge DF is adjacent to edge BD, edge AD, edge CF, and edge DE.

The degrees of the vertices:

A	4
B	4
C	4
D	4
E	4
F	2

Here are some paths in the above graph: (there are many more than listed)

A,B,D

A,B,C,E

F,D,E,B,C

Here are some circuits in the above graph: (there are many more than listed)

B,A,D,B

B,C,F,D,B

F,C, E, D, F

The above graph does not have any bridges.

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6.4.1: Networks

A network is a connection of vertices through edges. The internet is an example of a network with computers as the vertices and the connections between these computers as edges.

Definition: Spanning Subgraph

A **spanning subgraph** is a graph that joins all of the vertices of a more complex graph, but does not create a circuit

Example 6.4.1.1: Spanning Subgraph

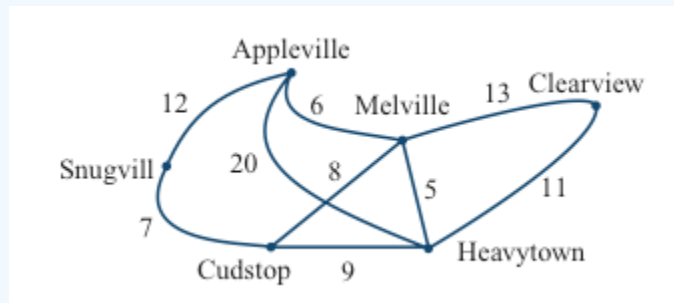


Figure 6.4.1.1: Map of Connecting Towns

This is a graph showing how six cities are linked by roads. This graph has many spanning subgraphs but two examples are shown below.

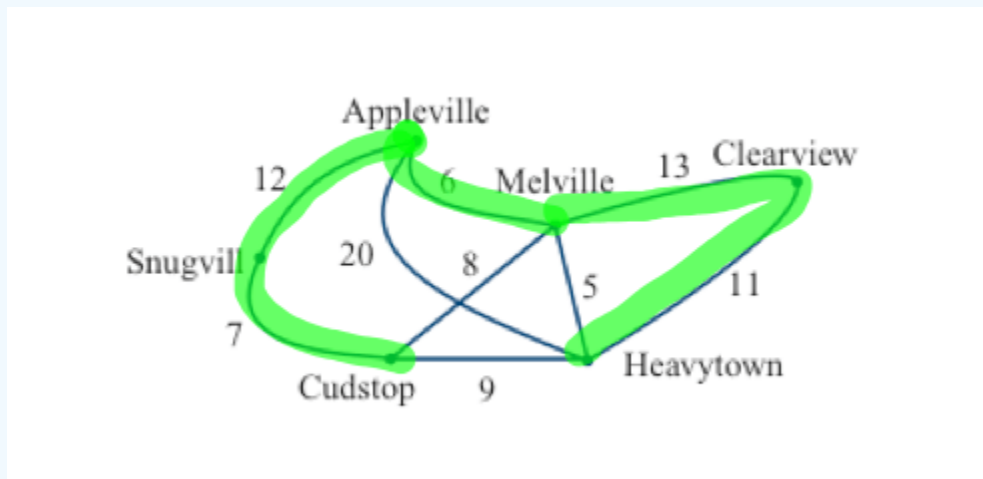


Figure 6.4.1.2: Spanning Subgraph 1

This graph spans all of the cities (vertices) of the original graph, but does not contain any circuits.

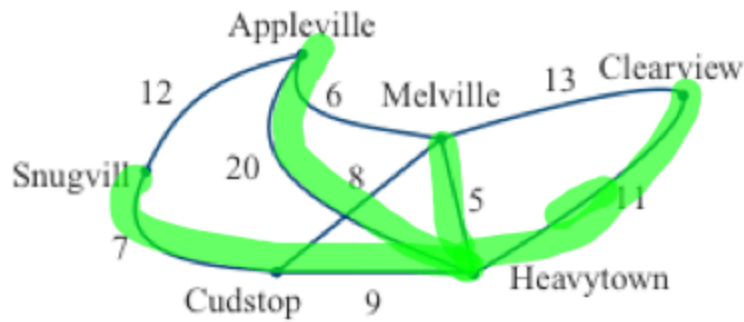


Figure 6.4.1.3: Spanning Subgraph 2

This graph spans all of the cities (vertices) of the original graph, but does not contain any circuits.

Definition: Tree

A **tree** is a graph that is connected and has no circuits. Therefore, a spanning subgraph is a tree and the examples of spanning subgraphs in Example 6.4.1.1 above are also trees.

Properties of Trees

1. If a graph is a tree, there is one and only one path joining any two vertices. Conversely, if there is one and only one path joining any two vertices of a graph, the graph must be a tree.
2. In a tree, every edge is a bridge. Conversely, if every edge of a connected graph is a bridge, then the graph must be a tree.
3. A tree with N vertices must have $N-1$ edges.
4. A connected graph with N vertices and $N-1$ edges must be a tree.

Example 6.4.1.2: Tree Properties

Consider the spanning subgraph highlighted in green shown in Figure 6.4.1.2.

1. Tree Property 1

Look at the vertices Appleville and Heavytown. Since the graph is a tree, there is only one path joining these two cities. Also, since there is only one path between any two cities on the whole graph, then the graph must be a tree.

2. Tree Property 2

Since the graph is a tree, notice that every edge of the graph is a bridge, which is an edge such that if it were removed the graph would become disconnected.

3. Tree Property 3

Since the graph is a tree and it has six vertices, it must have $N - 1$ or six - 1 = five edges.

4. Tree Property 4

Since the graph is connected and has six vertices and five edges, it must be a tree.

Example 6.4.1.3: More Examples of Trees

All of the graphs shown below are trees and they all satisfy the tree properties.

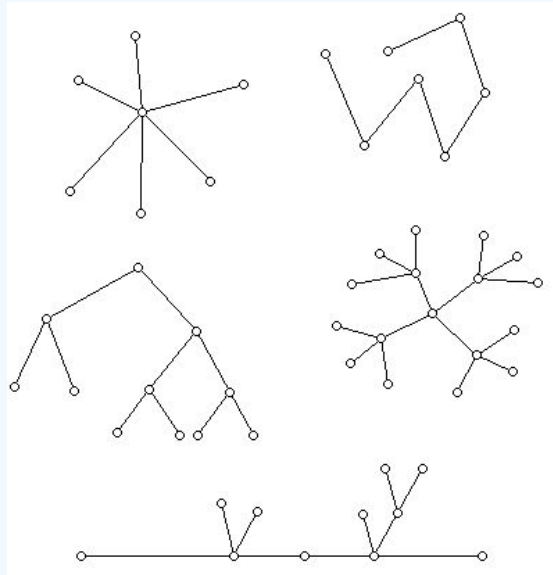


Figure 6.4.1.4: More Examples of Trees

Definition: Minimum Spanning Tree

A **minimum spanning tree** is the tree that spans all of the vertices in a problem with the least cost (or time, or distance).

Example 6.4.1.4: Minimum Spanning Tree

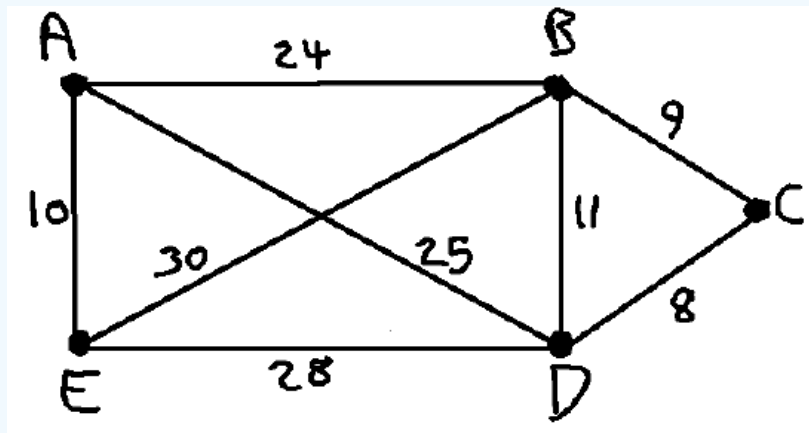


Figure 6.4.1.5: Weighted Graph 1

The above is a weighted graph where the numbers on each edge represent the cost of each edge. We want to find the minimum spanning tree of this graph so that we can find a network that will reach all vertices for the least total cost.

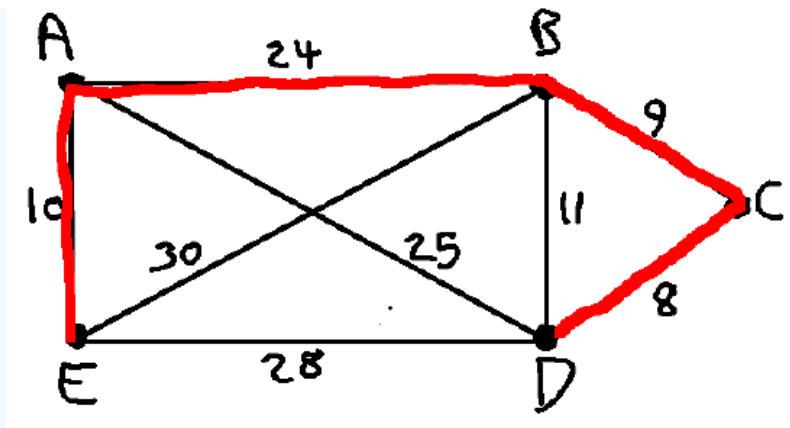


Figure 6.4.1.6: Minimum Spanning Tree for Weighted Graph 1

This is the minimum spanning tree for the graph with a total cost of 51.

Kruskal's Algorithm

Since some graphs are much more complicated than the previous example, we can use Kruskal's Algorithm to always be able to find the minimum spanning tree for any graph.

1. Find the cheapest link in the graph. If there is more than one, pick one at random. Mark it in red.
2. Find the next cheapest link in the graph. If there is more than one, pick one at random. Mark it in red.
3. Continue doing this as long as the next cheapest link does not create a red circuit.
4. You are done when the red edges span every vertex of the graph without any circuits. The red edges are the MST (minimum spanning tree).

Example 6.4.1.5: Using Kruskal's Algorithm

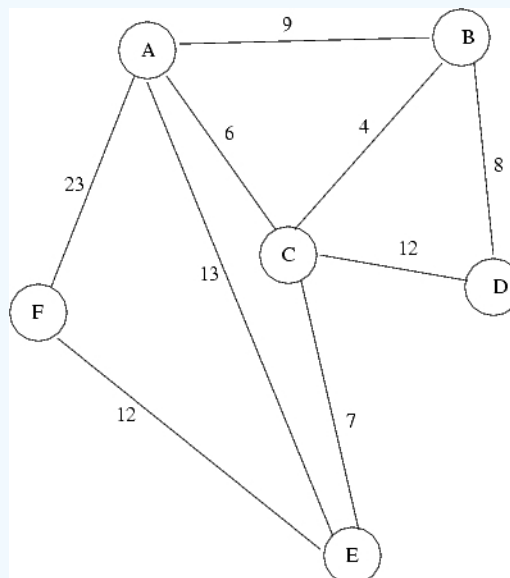


Figure 6.4.1.7: Weighted Graph 2

Suppose that it is desired to install a new fiber optic cable network between the six cities (A, B, C, D, E, and F) shown above for the least total cost. Also, suppose that the fiber optic cable can only be installed along the roadways shown above. The weighted graph above shows the cost (in millions of dollars) of installing the fiber optic cable along each roadway. We want to find the minimum spanning tree for this graph using Kruskal's Algorithm.

Step 1: Find the cheapest link of the whole graph and mark it in red. The cheapest link is between B and C with a cost of four million dollars.

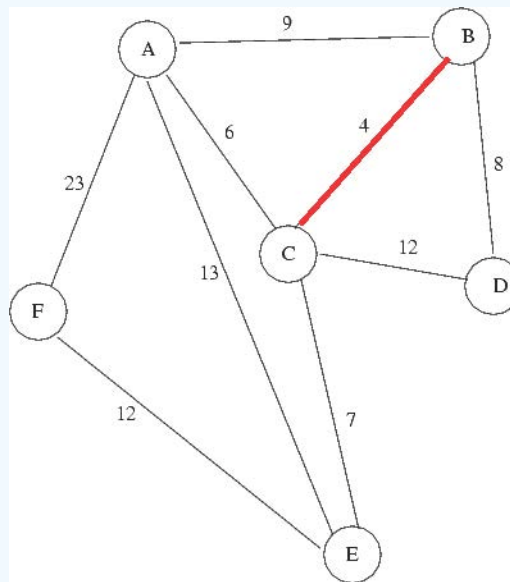


Figure 6.4.1.8: Kruskal's Algorithm Step 1

Step 2: Find the next cheapest link of the whole graph and mark it in red. The next cheapest link is between A and C with a cost of six million dollars.

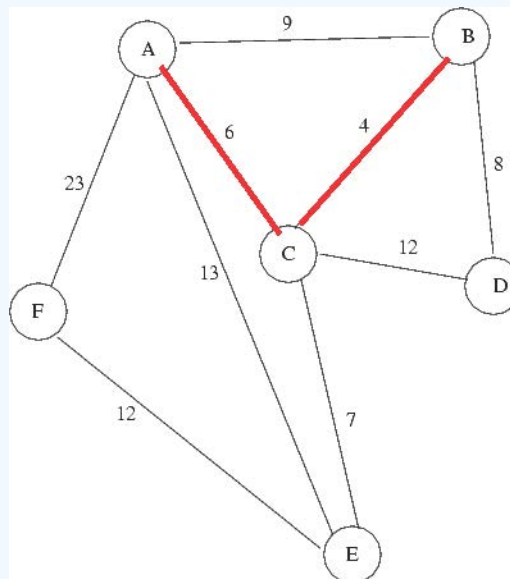


Figure 6.4.1.9: Kruskal's Algorithm Step 2

Step 3: Find the next cheapest link of the whole graph and mark it in red as long as it does not create a red circuit. The next cheapest link is between C and E with a cost of seven million dollars.

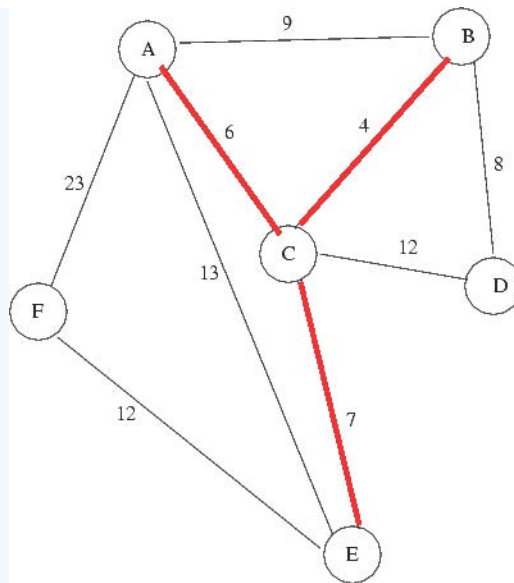


Figure 6.4.1.10: Kruskal's Algorithm Step 3

Step 4: Find the next cheapest link of the whole graph and mark it in red as long as it does not create a red circuit. The next cheapest link is between B and D with a cost of eight million dollars.

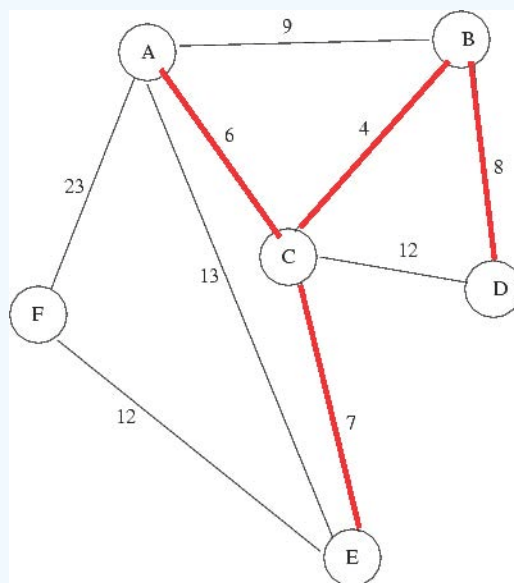


Figure 6.4.1.11: Kruskal's Algorithm Step 4

Step 5: Find the next cheapest link of the whole graph and mark it in red as long as it does not create a red circuit. The next cheapest link is between A and B with a cost of nine million dollars, but that would create a red circuit so we cannot use it. Therefore, the next cheapest link after that is between E and F with a cost of 12 million dollars, which we are able to use. We cannot use the link between C and D which also has a cost of 12 million dollars because it would create a red circuit.

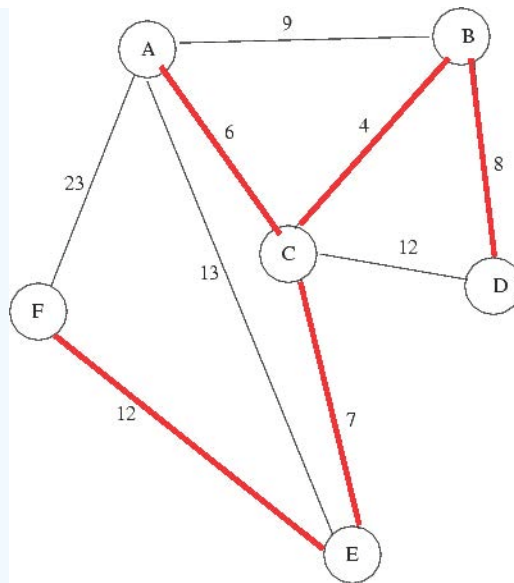


Figure 6.4.1.12: Kruskal's Algorithm Step 5

This was the last step and we now have the minimum spanning tree for the weighted graph with a total cost of \$37,000,000.

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6.4.2: Euler Circuits

Leonhard Euler first discussed and used Euler paths and circuits in 1736. Rather than finding a minimum spanning tree that visits every vertex of a graph, an Euler path or circuit can be used to find a way to visit every edge of a graph once and only once. This would be useful for checking parking meters along the streets of a city, patrolling the streets of a city, or delivering mail.

Definition: Euler Path

A path that travels through every edge of a connected graph once and only once and starts and ends at different vertices

Example 6.4.2.1: Euler Path

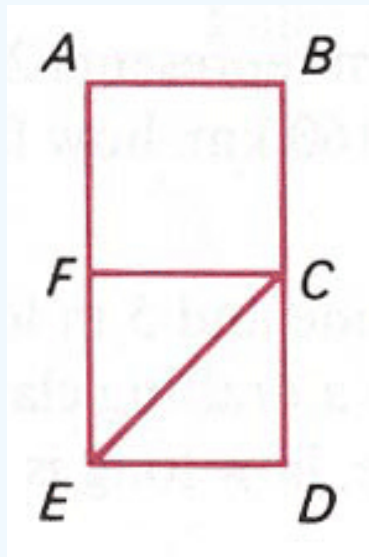


Figure 6.4.2.1: Euler Path Example

One Euler path for the above graph is F, A, B, C, F, E, C, D, E as shown below.

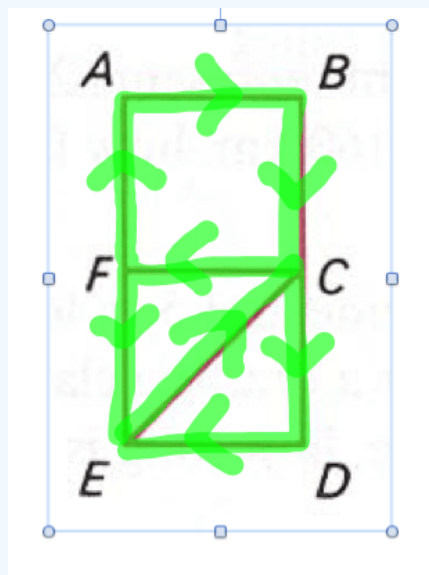


Figure 6.4.2.2: Euler Path

This Euler path travels every edge once and only once and starts and ends at different vertices. This graph cannot have an Euler circuit since no Euler path can start and end at the same vertex without crossing over at least one edge more than once.

Definition: Euler Circuit

An Euler path that starts and ends at the same vertex

Example 6.4.2.2: Euler Circuit

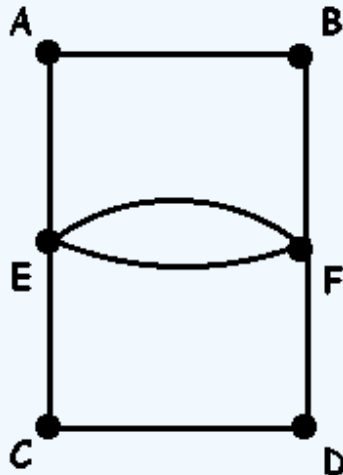


Figure 6.4.2.3: Euler Circuit Example

One Euler circuit for the above graph is E, A, B, F, E, F, D, C, E as shown below.

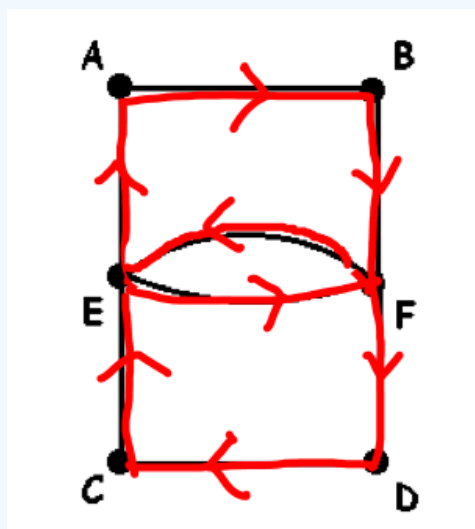


Figure 6.4.2.4: Euler Circuit

This Euler path travels every edge once and only once and starts and ends at the same vertex. Therefore, it is also an Euler circuit.

Euler's Theorem 6.4.2.1: If a graph has any vertices of odd degree, then it cannot have an Euler circuit.

If a graph is connected and every vertex has an even degree, then it has at least one Euler circuit (usually more).

Euler's Theorem 6.4.2.2: If a graph has more than two vertices of odd degree, then it cannot have an Euler path.

If a graph is connected and has exactly two vertices of odd degree, then it has at least one Euler path (usually more). Any such path must start at one of the odd-degree vertices and end at the other one.

Euler's Theorem 6.4.2.3: The sum of the degrees of all the vertices of a graph equals twice the number of edges (and therefore must be an even number).

Therefore, the number of vertices of odd degree must be even.

Finding Euler Circuits

1. Be sure that every vertex in the network has even degree.
2. Begin the Euler circuit at any vertex in the network.
3. As you choose edges, never use an edge that is the only connection to a part of the network that you have not already visited.
4. Label the edges in the order that you travel them and continue this until you have travelled along every edge exactly once and you end up at the starting vertex.

Example 6.4.2.3: Finding an Euler Circuit

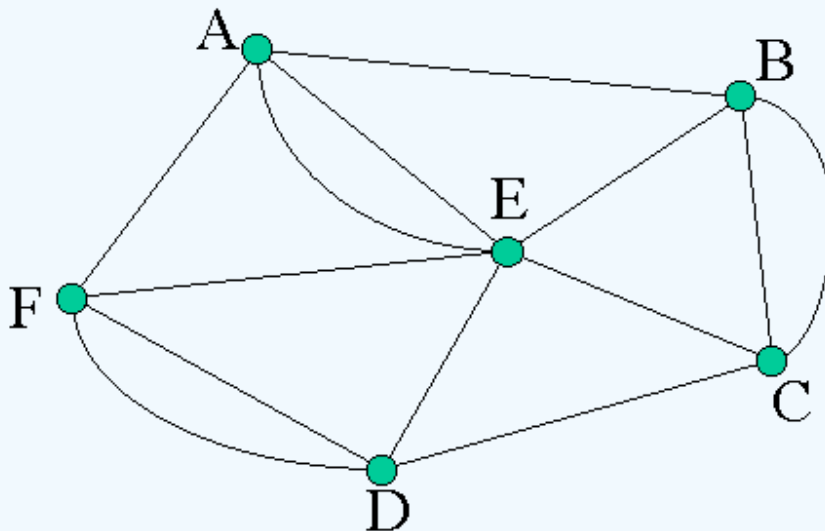


Figure 6.4.2.5: Graph for Finding an Euler Circuit

The graph shown above has an Euler circuit since each vertex in the entire graph is even degree. Thus, start at one even vertex, travel over each vertex once and only once, and end at the starting point. One example of an Euler circuit for this graph is A, E, A, B, C, B, E, C, D, E, F, D, F, A. This is a circuit that travels over every edge once and only once and starts and ends in the same place. There are other Euler circuits for this graph. This is just one example.

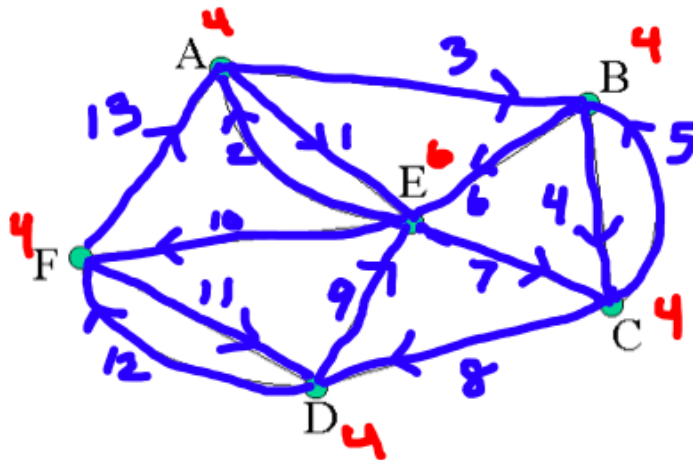


Figure 6.4.2.6: Euler Circuit

The degree of each vertex is labeled in red. The ordering of the edges of the circuit is labeled in blue and the direction of the circuit is shown with the blue arrows.

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6.4.3: Hamiltonian Circuits

The Traveling Salesman Problem (TSP) is any problem where you must visit every vertex of a weighted graph once and only once, and then end up back at the starting vertex. Examples of TSP situations are package deliveries, fabricating circuit boards, scheduling jobs on a machine and running errands around town.

Hamilton Circuit: a circuit that must pass through each vertex of a graph once and only once

Hamilton Path: a path that must pass through each vertex of a graph once and only once

Example 6.4.3.1: Hamilton Path:

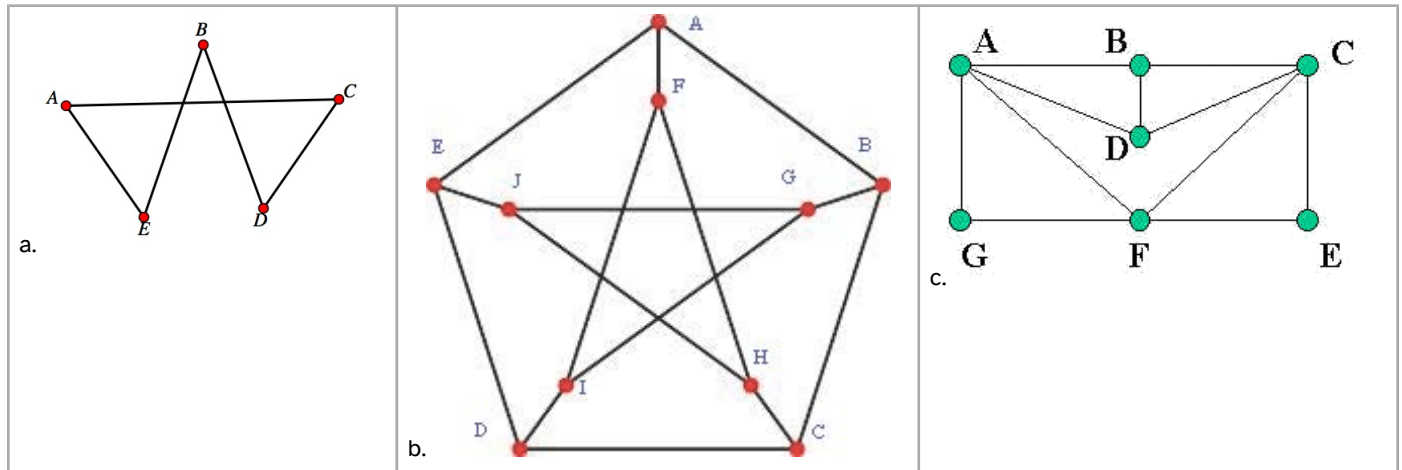


Figure 6.4.3.1: Examples of Hamilton Paths

Not all graphs have a Hamilton circuit or path. There is no way to tell just by looking at a graph if it has a Hamilton circuit or path like you can with an Euler circuit or path. You must do trial and error to determine this. By the way if a graph has a Hamilton circuit then it has a Hamilton path. Just do not go back to home.

Graph a. has a Hamilton circuit (one example is ACDBEA)

Graph b. has no Hamilton circuits, though it has a Hamilton path (one example is ABCDEJGIFH)

Graph c. has a Hamilton circuit (one example is AGFECDBA)

Complete Graph: A complete graph is a graph with N vertices in which every pair of vertices is joined by exactly one edge. The symbol used to denote a complete graph is K_N .

Example 6.4.3.2: Complete Graphs

a. K_2	b. K_3	c. K_4	d. K_5
----------	----------	----------	----------

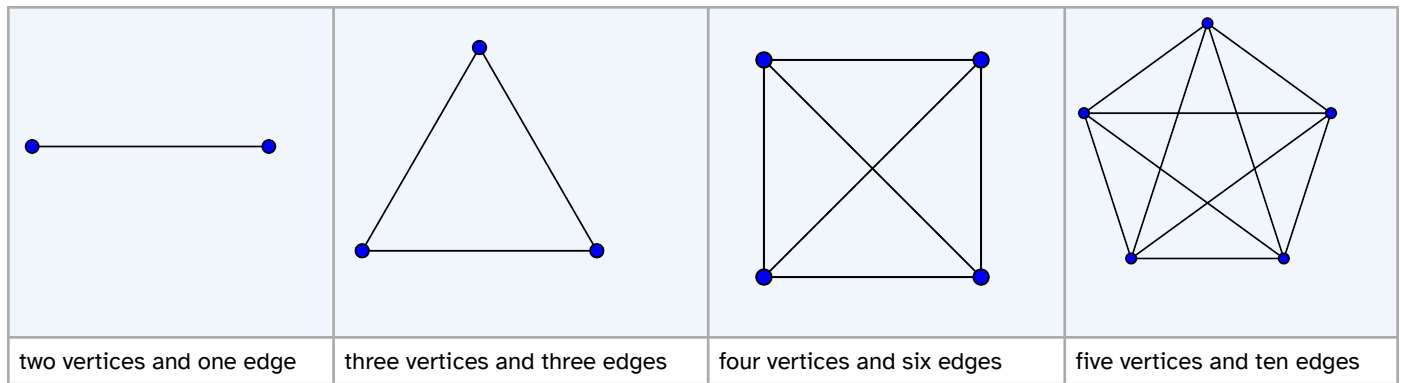


Figure 6.4.3.2: Complete Graphs for $N = 2, 3, 4$, and 5

In each complete graph shown above, there is exactly one edge connecting each pair of vertices. There are no loops or multiple edges in complete graphs. Complete graphs do have Hamilton circuits.

Reference Point: the starting point of a Hamilton circuit

Example 6.4.3.3: Reference Point in a Complete Graph

Many Hamilton circuits in a complete graph are the same circuit with different starting points. For example, in the graph K_3 , shown below in Figure 6.4.3.3, ABCA is the same circuit as BCAB, just with a different starting point (reference point). We will typically assume that the reference point is A.

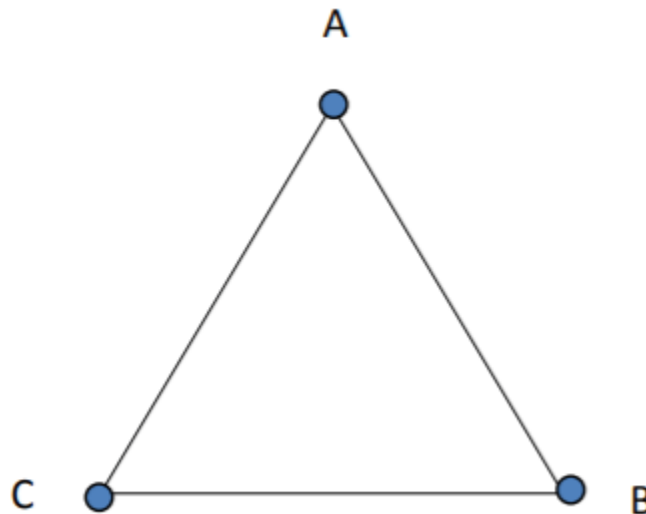


Figure
6.4.3.3: K_3
3

Number of Hamilton Circuits: A complete graph with N vertices is $(N-1)!$ Hamilton circuits. Since half of the circuits are mirror images of the other half, there are actually only half this many unique circuits.

Example 6.4.3.4: Number of Hamilton Circuits

How many Hamilton circuits does a graph with five vertices have?

$(N - 1)! = (5 - 1)! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ Hamilton circuits.

How to solve a Traveling Salesman Problem (TSP):

A traveling salesman problem is a problem where you imagine that a traveling salesman goes on a business trip. He starts in his home city (A) and then needs to travel to several different cities to sell his wares (the other cities are B, C, D, etc.). To solve a TSP, you need to find the cheapest way for the traveling salesman to start at home, A, travel to the other cities, and then return home to A at the end of the trip. This is simply finding the Hamilton circuit in a complete graph that has the smallest overall weight. There are several different algorithms that can be used to solve this type of problem.

A. Brute Force Algorithm

1. List all possible Hamilton circuits of the graph.
2. For each circuit find its total weight.
3. The circuit with the least total weight is the optimal Hamilton circuit.

Example 6.4.3.5: Brute Force Algorithm:

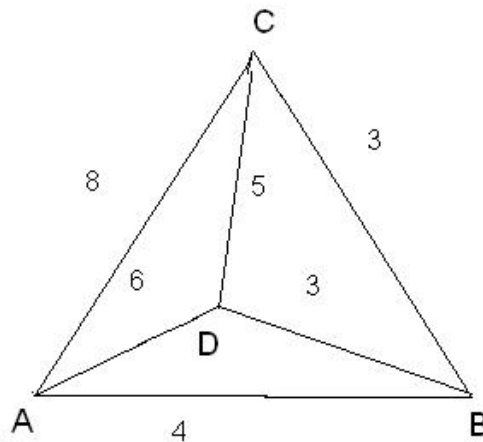


Figure 6.4.3.4: Complete Graph for Brute Force Algorithm

Suppose a delivery person needs to deliver packages to three locations and return to the home office A. Using the graph shown above in Figure 6.4.3.4, find the shortest route if the weights on the graph represent distance in miles.

Recall the way to find out how many Hamilton circuits this complete graph has. The complete graph above has four vertices, so the number of Hamilton circuits is:

$(N - 1)! = (4 - 1)! = 3! = 3 \cdot 2 \cdot 1 = 6$ Hamilton circuits.

However, three of those Hamilton circuits are the same circuit going the opposite direction (the mirror image).

Hamilton Circuit	Mirror Image	Total Weight (Miles)
ABCD	ADCBA	18
ABDCA	ACDBA	20
ACBDA	ADBCA	20

The solution is ABCD (or ADCBA) with total weight of 18 mi. This is the optimal solution.

B. Nearest-Neighbor Algorithm:

1. Pick a vertex as the starting point.

2. From the starting point go to the vertex with an edge with the smallest weight. If there is more than one choice, choose at random.
3. Continue building the circuit, one vertex at a time from among the vertices that have not been visited yet.
4. From the last vertex, return to the starting point.

Example 6.4.3.6: Nearest-Neighbor Algorithm

A delivery person needs to deliver packages to four locations and return to the home office A as shown in Figure 6.4.3.5 below. Find the shortest route if the weights represent distances in miles.

Starting at A, E is the nearest neighbor since it has the least weight, so go to E. From E, B is the nearest neighbor so go to B. From B, C is the nearest neighbor so go to C. From C, the first nearest neighbor is B, but you just came from there. The next nearest neighbor is A, but you do not want to go there yet because that is the starting point. The next nearest neighbor is E, but you already went there. So go to D. From D, go to A since all other vertices have been visited.

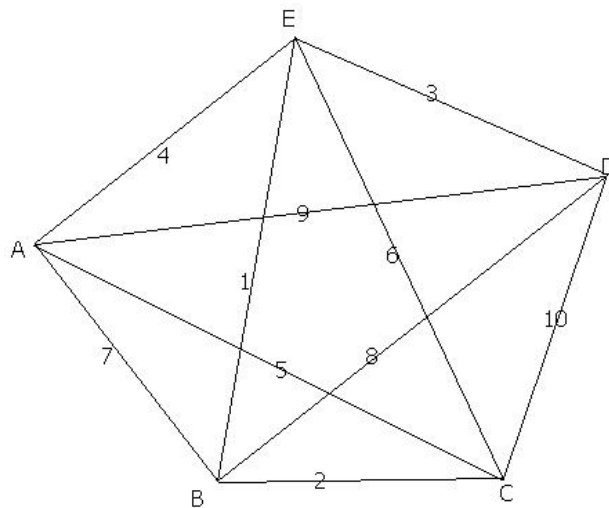


Figure 6.4.3.5: Complete Graph for Nearest-Neighbor Algorithm

The solution is AEB CDA with a total weight of 26 miles. This is not the optimal solution, but it is close and it was a very efficient method.

C. Repetitive Nearest-Neighbor Algorithm:

1. Let X be any vertex. Apply the Nearest-Neighbor Algorithm using X as the starting vertex and calculate the total cost of the circuit obtained.
2. Repeat the process using each of the other vertices of the graph as the starting vertex.
3. Of the Hamilton circuits obtained, keep the best one. If there is a designated starting vertex, rewrite this circuit with that vertex as the reference point.

Example 6.4.3.7: Repetitive Nearest-Neighbor Algorithm

Suppose a delivery person needs to deliver packages to four locations and return to the home office A. Find the shortest route if the weights on the graph represent distances in kilometers.

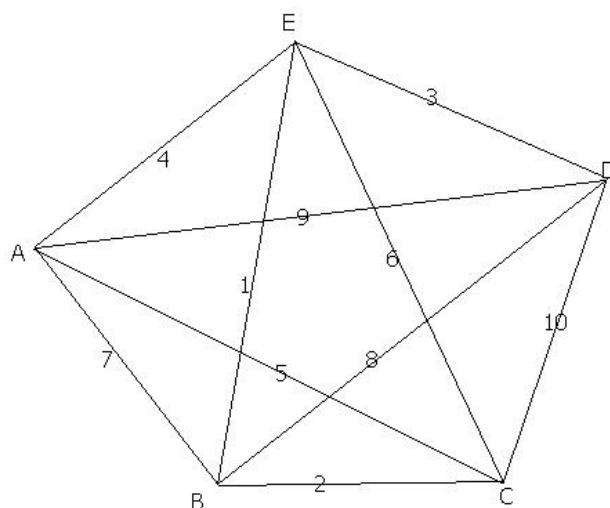


Figure 6.4.3.6: Complete Graph for Repetitive Nearest-Neighbor Algorithm

Starting at A, the solution is AEBCDA with total weight of 26 miles as we found in Example 6.4.3.6. See this solution below in Figure 6.4.3.7.

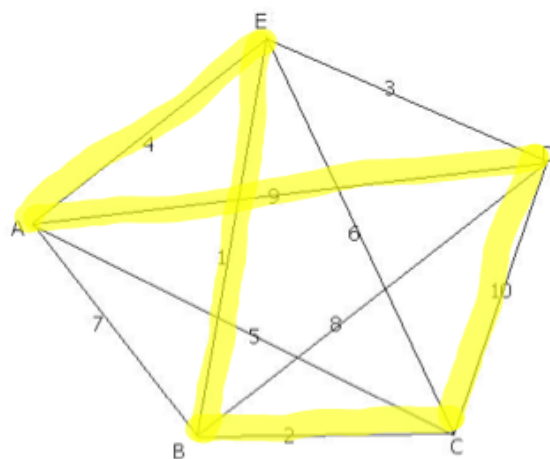


Figure 6.4.3.7: Starting at A

Starting at B, the solution is BEDACB with total weight of 20 miles.

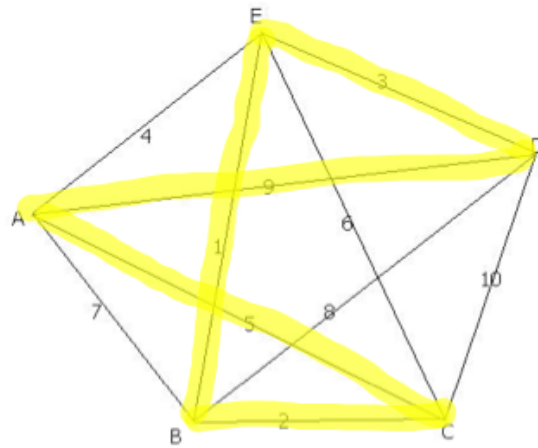


Figure 6.4.3.8: Starting at B

Starting at C, the solution is CBEDAC with total weight of 20 miles.

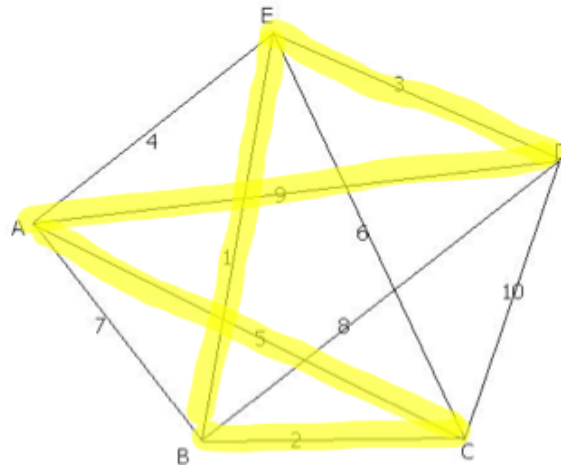


Figure 6.4.3.9: Starting at C

Starting at D, the solution is DEBCAD with total weight of 20 miles.

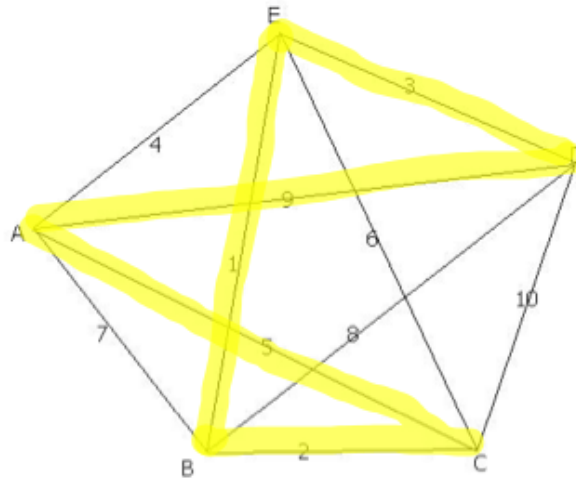


Figure 6.4.3.10: Starting at D

Starting at E, solution is EBCADE with total weight of 20 miles.

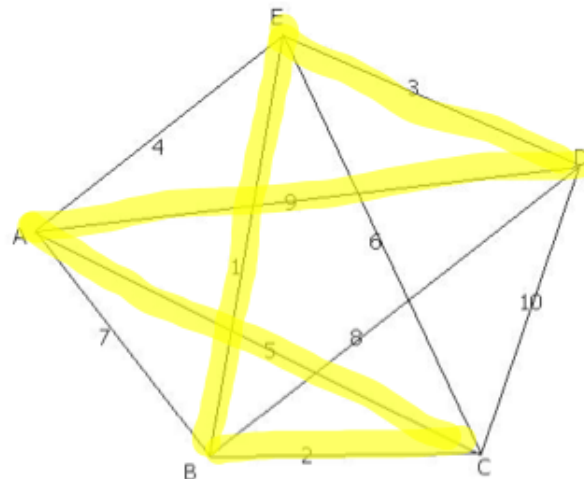


Figure 6.4.3.11: Starting at E

Now, you can compare all of the solutions to see which one has the lowest overall weight. The solution is any of the circuits starting at B, C, D, or E since they all have the same weight of 20 miles. Now that you know the best solution using this method, you can rewrite the circuit starting with any vertex. Since the home office in this example is A, let's rewrite the solutions starting with A. Thus, the solution is ACBEDA or ADEBCA.

D. Cheapest-Link Algorithm

1. Pick the link with the smallest weight first (if there is a tie, randomly pick one). Mark the corresponding edge in red.
2. Pick the next cheapest link and mark the corresponding edge in red.
3. Continue picking the cheapest link available. Mark the corresponding edge in red except when a) it closes a circuit or b) it results in three edges coming out of a single vertex.
4. When there are no more vertices to link, close the red circuit.

Example 6.4.3.8: Cheapest-Link Algorithm

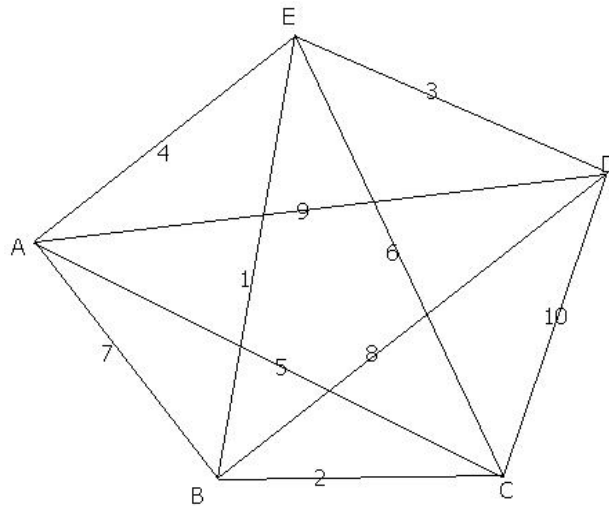


Figure 6.4.3.12: Complete Graph for Cheapest-Link Algorithm

Suppose a delivery person needs to deliver packages to four locations and return to the home office A. Find the shortest route if the weights represent distances in miles.

Step 1: Find the cheapest link of the graph and mark it in blue. The cheapest link is between B and E with a weight of one mile.

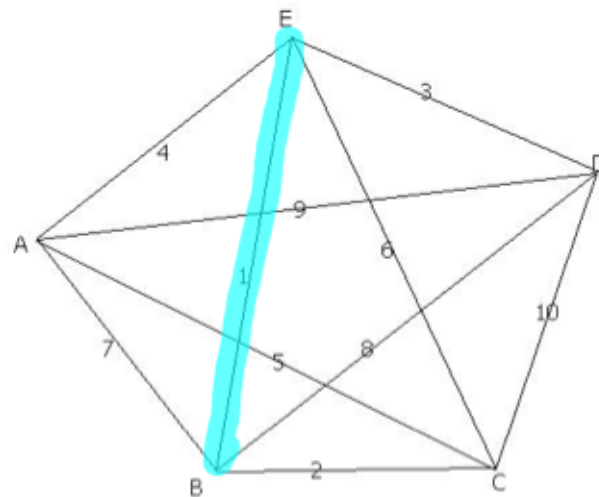


Figure 6.4.3.13: Step 1

Step 2: Find the next cheapest link of the graph and mark it in blue. The next cheapest link is between B and C with a weight of two miles.

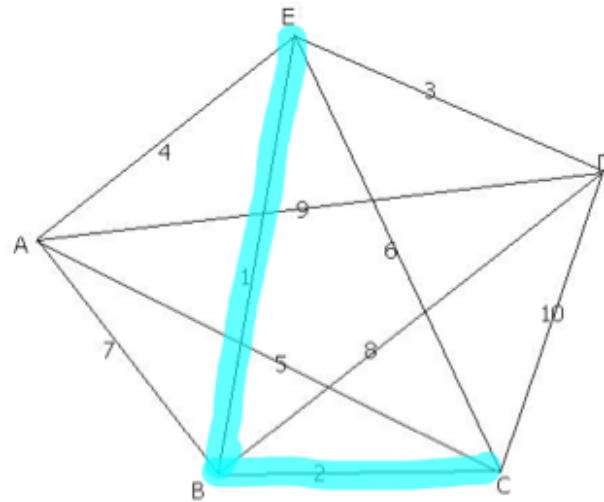


Figure 6.4.3.14: Step 2

Step 3: Find the next cheapest link of the graph and mark it in blue provided it does not make a circuit or it is not a third edge coming out of a single vertex. The next cheapest link is between D and E with a weight of three miles.

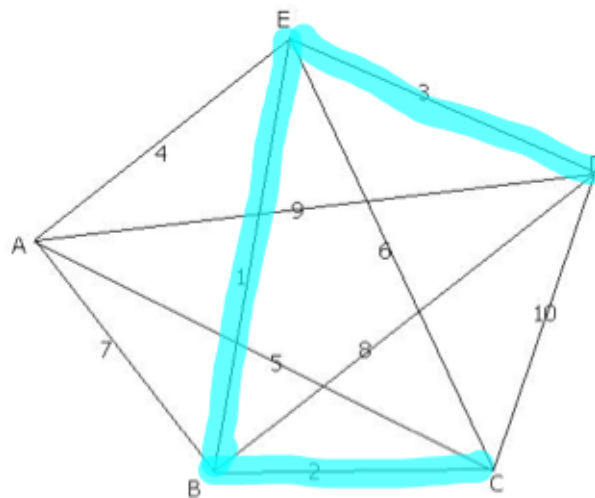


Figure 6.4.3.15: Step 3

Step 4: Find the next cheapest link of the graph and mark it in blue provided it does not make a circuit or it is not a third edge coming out of a single vertex. The next cheapest link is between A and E with a weight of four miles, but it would be a third edge coming out of a single vertex. The next cheapest link is between A and C with a weight of five miles. Mark it in blue.

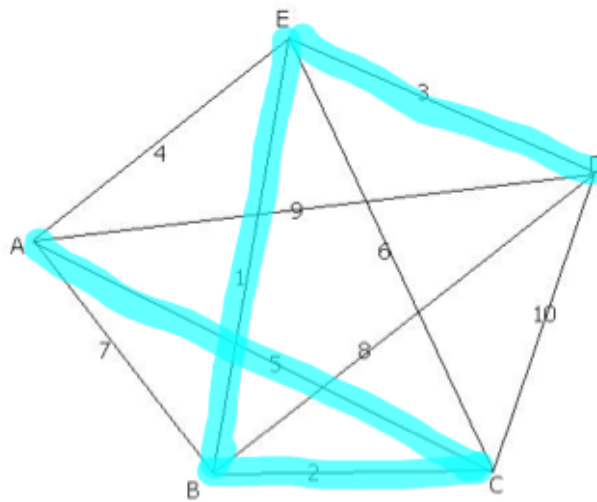


Figure 6.4.3.16: Step 4

Step 5: Since all vertices have been visited, close the circuit with edge DA to get back to the home office, A. This is the only edge we could close the circuit with because AB creates three edges coming out of vertex B and BD also created three edges coming out of vertex B.

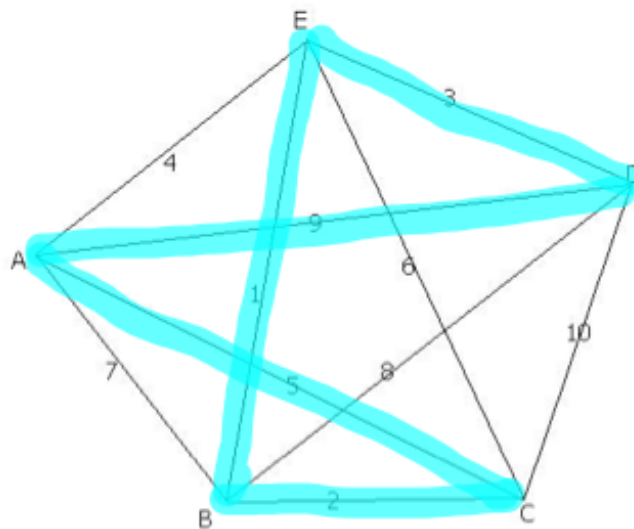


Figure 6.4.3.17: Step 5

The solution is ACBEDA or ADEBCA with total weight of 20 miles.

Efficient Algorithm: an algorithm for which the number of steps needed to carry it out grows in proportion to the size of the input to the problem.

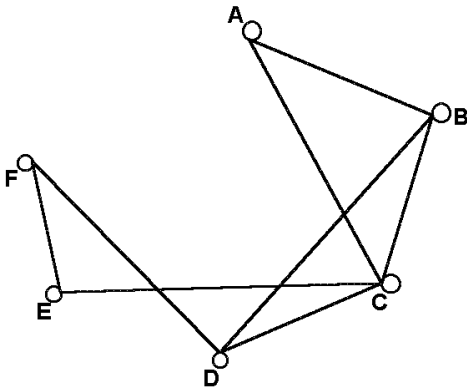
Approximate Algorithm: any algorithm for which the number of steps needed to carry it out grows in proportion to the size of the input to the problem. There is no known algorithm that is efficient and produces the optimal solution.

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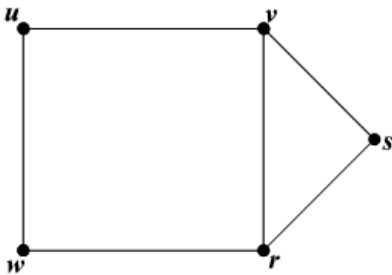
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6.4.4: Exercises

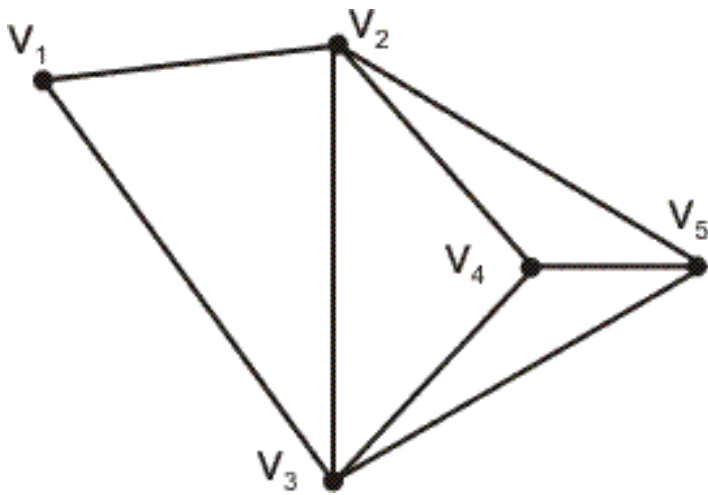
1. Answer the following questions based on the graph below.



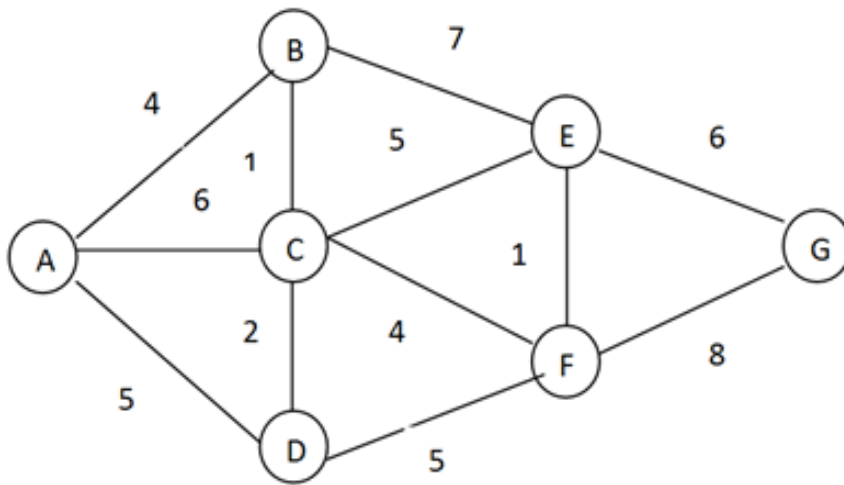
1. What are the vertices?
 2. Is this graph connected?
 3. What is the degree of vertex C?
 4. Edge FE is adjacent to which edges?
 5. Does this graph have any bridges?
2. Answer the following questions based on the graph below.



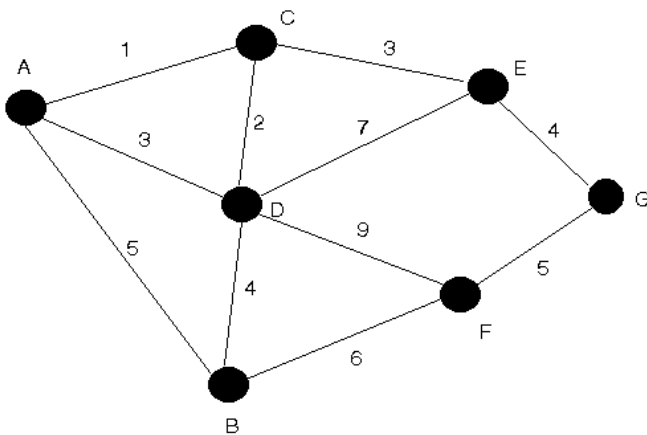
1. What are the vertices?
 2. What is the degree of vertex u ?
 3. What is the degree of vertex s ?
 4. What is one circuit in the graph?
3. Draw a spanning subgraph in the graph below.



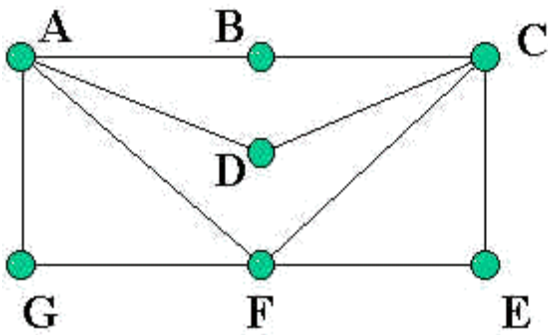
4. Find the minimum spanning tree in the graph below using Kruskal's Algorithm.



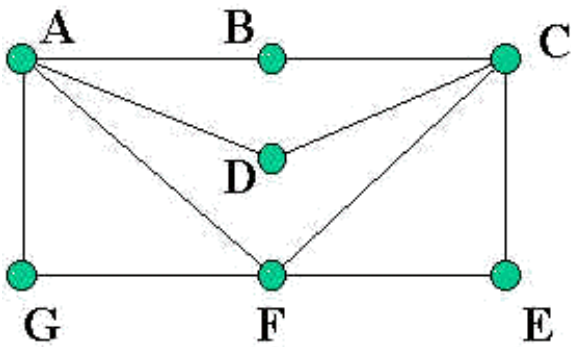
5. Find the minimum spanning tree in the graph below using Kruskal's Algorithm.



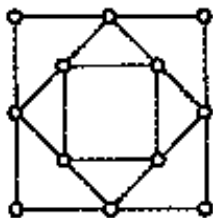
6. Find an Euler Path in the graph below.



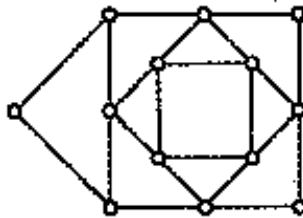
7. Find an Euler circuit in the graph below.



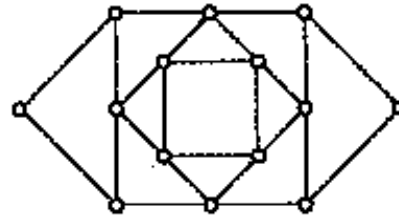
8. Which graphs below have Euler Paths? Which graphs have Euler circuits?



(a)

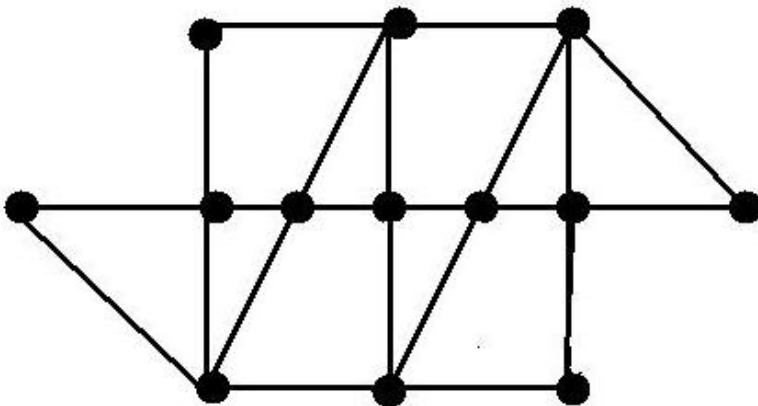


(b)

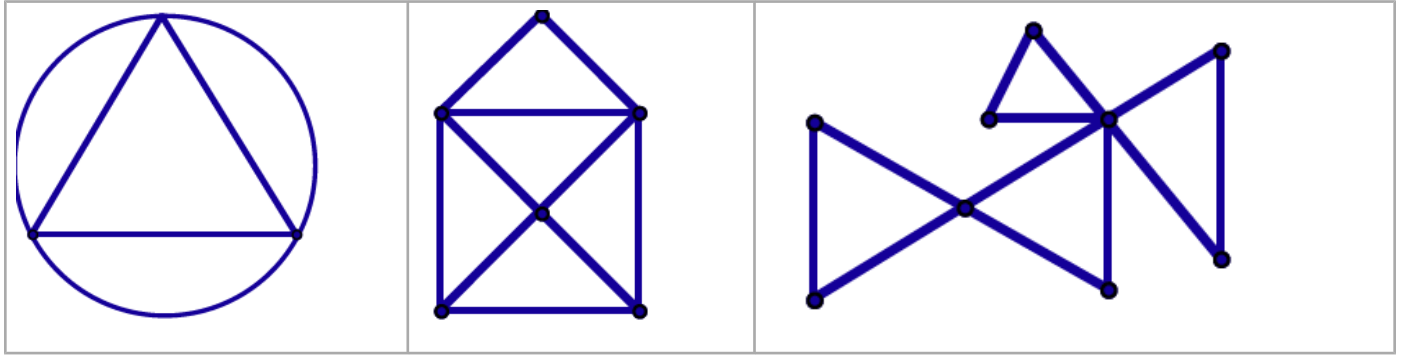


(c)

9. Highlight an Euler circuit in the graph below.



10. For each of the graphs below, write the degree of each vertex next to each vertex.

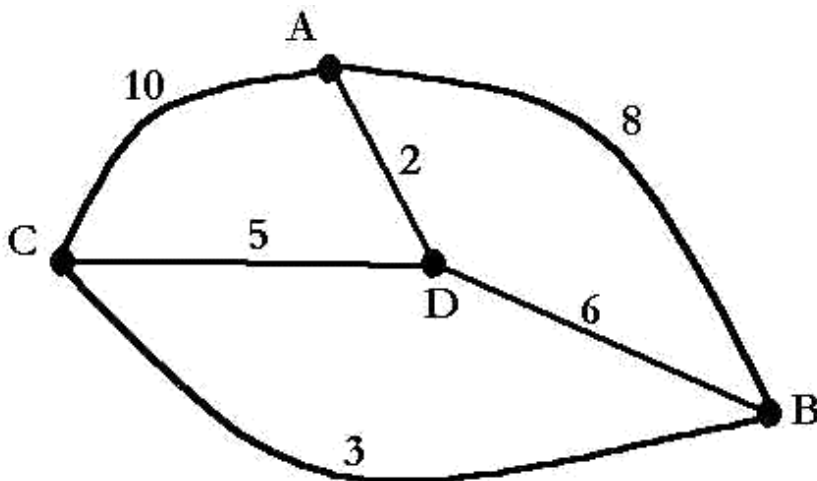


11. Circle whether each of the graphs below has an Euler circuit, an Euler path, or neither.

1. Euler circuit 2. Euler path 3. Neither	1. Euler circuit 2. Euler path 3. Neither	1. Euler circuit 2. Euler path 3. Neither

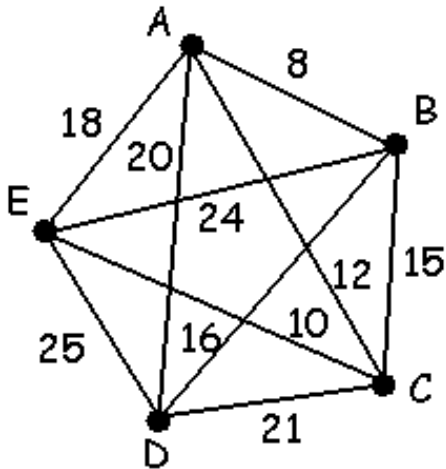
12. How many Hamilton circuits does a complete graph with 6 vertices have?

13. Suppose you need to start at A, visit all three vertices, and return to the starting point A. Using the Brute Force Algorithm, find the shortest route if the weights represent distances in miles.

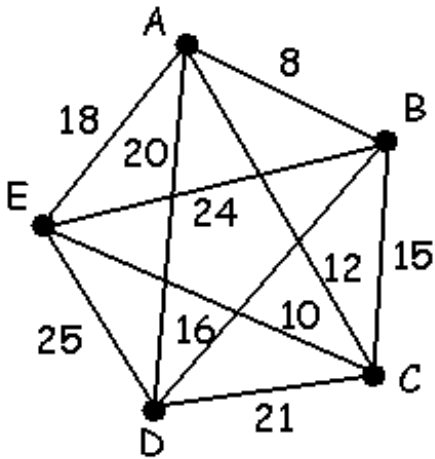


14. If a graph is connected and _____, the graph will have an Euler circuit.

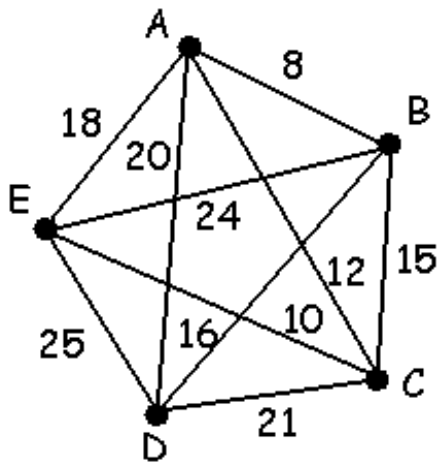
1. the graph has an even number of vertices
 2. the graph has an even number of edges
 3. the graph has all vertices of even degree
 4. the graph has only two odd vertices
15. Starting at vertex A, use the Nearest-Neighbor Algorithm to find the shortest route if the weights represent distances in miles.



16. Find a Hamilton circuit using the Repetitive Nearest-Neighbor Algorithm.



17. Find a Hamilton circuit using the Cheapest-Link Algorithm.



18. Which is a circuit that traverses each edge of the graph exactly once?

A. Euler circuit b. Hamilton circuit c. Minimum Spanning Tree

19. Which is a circuit that traverses each vertex of the graph exactly once?

A. Euler circuit b. Hamilton circuit c. Minimum Spanning Tree

20. For each situation, would you find an Euler circuit or a Hamilton Circuit?

1. The department of Public Works must inspect all streets in the city to remove dangerous debris.
2. Relief food supplies must be delivered to eight emergency shelters located at different sites in a large city.
3. The Department of Public Works must inspect traffic lights at intersections in the city to determine which are still working.
4. An insurance claims adjuster must visit 11 homes in various neighborhoods to write reports.

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S

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