

MATH 300: MY MATH IDEAS TEXTBOOK



Rory Kinoshita
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Math 300: My Math Ideas Textbook (Kinoshita)

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1.1.1: Introduction

In previous math courses, you've no doubt run into the infamous "word problems." Unfortunately, these problems rarely resemble the type of problems we actually encounter in everyday life. In math books, you usually are told exactly which formula or procedure to use, and are given exactly the information you need to answer the question. In real life, problem solving requires identifying an appropriate formula or procedure, and determining what information you will need (and won't need) to answer the question.

In this chapter, we will review several basic but powerful algebraic ideas: percents, rates, and proportions. We will then focus on the problem solving process, and explore how to use these ideas to solve problems where we don't have perfect information.

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1.1.2: Percents

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties[1]. Who is correct? How can we make sense of these numbers?

Percent literally means "per 100," or "parts per hundred." When we write 40%, this is equivalent to the fraction $\frac{40}{100}$ or the decimal 0.40. Notice that 80 out of 200 and 10 out of 25 are also 40%, since $\frac{80}{200} = \frac{10}{25} = \frac{40}{100}$.

Example 1

243 people out of 400 state that they like dogs. What percent is this?

Solution

$$\frac{243}{400} = 0.6075 = \frac{60.75}{100}. \text{ This is } 60.75\%.$$

Notice that the percent can be found from the equivalent decimal by moving the decimal point two places to the right.

Example 2

Write each as a percent: a) $\frac{1}{4}$ b) 0.02 c) 2.35

Solution

$$\text{a) } \frac{1}{4} = 0.25 = 25\% \quad \text{b) } 0.02 = 2\% \quad \text{c) } 2.35 = 235\%$$

Percents

If we have a *part* that is some *percent* of a *whole*, then

$$\text{percent} = \frac{\text{part}}{\text{whole}}, \text{ or equivalently, } \text{part} = \text{percent} \cdot \text{whole}$$

To do the calculations, we write the percent as a decimal.

Example 3

The sales tax in a town is 9.4%. How much tax will you pay on a \$140 purchase?

Solution

Here, \$140 is the whole, and we want to find 9.4% of \$140. We start by writing the percent as a decimal by moving the decimal point two places to the left (which is equivalent to dividing by 100). We can then compute:

$$\text{tax} = 0.094(140) = \$13.16 \text{ in tax.}$$

Example 4

In the news, you hear "tuition is expected to increase by 7% next year." If tuition this year was \$1200 per quarter, what will it be next year?

Solution

The tuition next year will be the current tuition plus an additional 7%, so it will be 107% of this year's tuition:

$$\$1200(1.07) = \$1284.$$

Alternatively, we could have first calculated 7% of ~~\$1200~~: $\$1200(0.07) = \$84$.

Notice this is *not* the expected tuition for next year (we could only wish). Instead, this is the expected *increase*, so to calculate the expected tuition, we'll need to add this change to the previous year's tuition:

$$\$1200 + \$84 = \$1284.$$

Try it Now 1

A TV originally priced at \$799 is on sale for 30% off. There is then a 9.2% sales tax. Find the price after including the discount and sales tax.

Answer available at the end of this section.

Example 5

The value of a car dropped from ~~\$7400~~ to ~~\$6800~~ over the last year. What percent decrease is this?

Solution

To compute the percent change, we first need to find the dollar value change: $\$6800 - \$7400 = -\$600$. Often we will take the absolute value of this amount, which is called the **absolute change**: $|-600| = 600$.

Since we are computing the decrease relative to the starting value, we compute this percent out of \$7400:

$$\frac{600}{7400} = 0.081 = 8.1\% \text{ decrease. This is called a } \mathbf{relative \ change}.$$

Absolute and Relative Change

Given two quantities,

$$\text{Absolute change} = |\text{ending quantity} - \text{starting quantity}|$$

$$\text{Relative change: } \frac{\text{absolute change}}{\text{starting quantity}}$$

Absolute change has the same units as the original quantity.

Relative change gives a percent change.

The starting quantity is called the **base** of the percent change.

The base of a percent is very important. For example, while Nixon was president, it was argued that marijuana was a “gateway” drug, claiming that 80% of marijuana smokers went on to use harder drugs like cocaine. The problem is, this isn't true. The true claim is that 80% of harder drug users first smoked marijuana. The difference is one of base: 80% of marijuana smokers using hard drugs, vs. 80% of hard drug users having smoked marijuana. These numbers are not equivalent. As it turns out, only one in 2,400 marijuana users actually go on to use harder drugs[2].

Example 6

There are about 75 QFC supermarkets in the U.S. Albertsons has about 215 stores. Compare the size of the two companies.

Solution

When we make comparisons, we must ask first whether an absolute or relative comparison. The absolute difference is $215 - 75 = 140$. From this, we could say “Albertsons has 140 more stores than QFC.” However, if you wrote this in an article or paper, that number does not mean much. The relative difference may be more meaningful. There are two different relative changes we could calculate, depending on which store we use as the base:

Using QFC as the base, $\frac{140}{75} = 1.867$.

This tells us Albertsons is 186.7% larger than QFC.

Using Albertsons as the base, $\frac{140}{215} = 0.651$.

This tells us QFC is 65.1% smaller than Albertsons.

Notice both of these are showing percent *differences*. We could also calculate the size of Albertsons relative to QFC: , which tells us Albertsons is 2.867 times the size of QFC. Likewise, we could calculate the size of QFC relative to Albertsons: , which tells us that QFC is 34.9% of the size of Albertsons.

Example 7

Suppose a stock drops in value by 60% one week, then increases in value the next week by 75%. Is the value higher or lower than where it started?

Solution

To answer this question, suppose the value started at \$100. After one week, the value dropped by 60%:

$$\$100 - \$100(0.60) = \$100 - \$60 = \$40.$$

In the next week, notice that base of the percent has changed to the new value, \$40. Computing the 75% increase:

$$\$40 + \$40(0.75) = \$40 + \$30 = \$70.$$

In the end, the stock is still \$30 lower, or $\frac{\$30}{\$100} = 30\%$ lower, valued than it started.

Try it Now 2

The U.S. federal debt at the end of 2001 was \$5.77 trillion, and grew to \$6.20 trillion by the end of 2002. At the end of 2005 it was \$7.91 trillion, and grew to \$8.45 trillion by the end of 2006[3]. Calculate the absolute and relative increase for 2001-2002 and 2005-2006. Which year saw a larger increase in federal debt?

Answer available at the end of this section.

Example 8

A Seattle Times article on high school graduation rates reported “The number of schools graduating 60 percent or fewer students in four years – sometimes referred to as “dropout factories” – decreased by 17 during that time period. The number of kids attending schools with such low graduation rates was cut in half.”

a) Is the “decrease by 17” number a useful comparison?

b) Considering the last sentence, can we conclude that the number of “dropout factories” was originally 34?

Solution

a) This number is hard to evaluate, since we have no basis for judging whether this is a larger or small change. If the number of “dropout factories” dropped from 20 to 3, that’d be a very significant change, but if the number dropped from 217 to 200, that’d be less of an improvement.

b) The last sentence provides relative change which helps put the first sentence in perspective. We can estimate that the number of “dropout factories” was probably previously around 34. However, it’s possible that students simply moved schools rather than the school improving, so that estimate might not be fully accurate.

Example 9

In the 2004 vice-presidential debates, Edwards’s claimed that US forces have suffered “90% of the coalition casualties” in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies “have taken almost 50 percent” of the casualties. Who is correct?

Solution

Without more information, it is hard for us to judge who is correct, but we can easily conclude that these two percents are talking about different things, so one does not necessarily contradict the other. Edward’s claim was a percent with coalition forces as the base of the percent, while Cheney’s claim was a percent with both coalition and Iraqi security forces as the base of the percent. It turns out both statistics are in fact fairly accurate.

Try it Now 3

In the 2012 presidential elections, one candidate argued that “the president’s plan will cut \$716 billion from Medicare, leading to fewer services for seniors,” while the other candidate rebuts that “our plan does not cut current spending and actually expands benefits for seniors, while implementing cost saving measures.” Are these claims in conflict, in agreement, or not comparable because they’re talking about different things?

Answer available at the end of this section.

We’ll wrap up our review of percents with a couple cautions. First, when talking about a change of quantities that are already measured in percents, we have to be careful in how we describe the change.

Example 10

A politician’s support increases from 40% of voters to 50% of voters. Describe the change.

Solution

We could describe this using an absolute change: $|50\% - 40\%| = 10\%$. Notice that since the original quantities were percents, this change also has the units of percent. In this case, it is best to describe this as an increase of **10 percentage points**.

In contrast, we could compute the percent change: $\frac{10\%}{40\%} = 0.25 = 25\%$ increase. This is the relative change, and we’d say the politician’s support has increased by 25%.

Lastly, a caution against averaging percents.

Example 11

A basketball player scores on 40% of 2-point field goal attempts, and on 30% of 3-point of field goal attempts. Find the player's overall field goal percentage.

Solution

It is very tempting to average these values, and claim the overall average is 35%, but this is likely not correct, since most players make many more 2-point attempts than 3-point attempts. We don't actually have enough information to answer the question. Suppose the player attempted 200 2-point field goals and 100 3-point field goals. Then they made $200(0.40) = 80$ 2-point shots and $100(0.30) = 30$ 3-point shots. Overall, they made 110 shots out of 300, for a $\frac{110}{300} = 0.367 = 36.7\%$ overall field goal percentage.

[1] www.factcheck.org/cheneys_edwards_mangle_facts.html

[2] <http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics>

[3] www.whitehouse.gov/sites/default/files/hist07z1.xls

Answers

Try it Now 1

The sale price is $\$799(0.70) = \$559.30$. After tax, the price is $\$559.30(1.092) = \$610.76$.

Try it Now 2

2001-2002: Absolute change: \$0.43 trillion. Relative change: 7.45%

2005-2006: Absolute change: \$0.54 trillion. Relative change: 6.83%

2005-2006 saw a larger absolute increase, but a smaller relative increase.

Try it Now 3

Without more information, it is hard to judge these arguments. This is compounded by the complexity of Medicare. As it turns out, the \$716 billion is not a cut in current spending, but a cut in future increases in spending, largely reducing future growth in health care payments. In this case, at least the numerical claims in both statements could be considered at least partially true. Here is one source of more information if you're interested: <http://factcheck.org/2012/08/a-campaign-full-of-mediscare/>

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1.1.3: Proportions and Rates

If you wanted to power the city of Seattle using wind power, how many windmills would you need to install? Questions like these can be answered using rates and proportions.

Rates

A rate is the ratio (fraction) of two quantities.

A **unit rate** is a rate with a denominator of one.

Example 12

Your car can drive 300 miles on a tank of 15 gallons. Express this as a rate.

Solution

Expressed as a rate, $\frac{300 \text{ miles}}{15 \text{ gallons}}$. We can divide to find a unit rate: $\frac{20 \text{ miles}}{1 \text{ gallon}}$, which we could also write as $20 \frac{\text{miles}}{\text{gallon}}$, or just 20 miles per gallon.

Proportion Equation

A proportion equation is an equation showing the equivalence of two rates or ratios.

Example 13

Solve the proportion $\frac{5}{3} = \frac{x}{6}$ for the unknown value x .

Solution

This proportion is asking us to find a fraction with denominator 6 that is equivalent to the fraction $\frac{5}{3}$. We can solve this by multiplying both sides of the equation by 6, giving $x = \frac{5}{3} \cdot 6 = 10$.

Example 14

A map scale indicates that $\frac{1}{2}$ inch on the map corresponds with 3 real miles. How many miles apart are two cities that are $2\frac{1}{4}$ inches apart on the map?

Solution

We can set up a proportion by setting equal two $\frac{\text{map inches}}{\text{real miles}}$ rates, and introducing a variable, x , to represent the unknown quantity – the mile distance between the cities.

$$\frac{\frac{1}{2} \text{ map inch}}{3 \text{ miles}} = \frac{2\frac{1}{4} \text{ map inches}}{x \text{ miles}} \quad \text{Multiply both sides by } x \text{ and rewriting the mixed number}$$

$$\frac{1}{2} \cdot x = \frac{9}{4} \quad \text{Multiply both sides by } 3$$

$$\frac{1}{2}x = \frac{27}{4} \quad \text{Multiply both sides by } 2 \text{ (or divide by } \frac{1}{2})$$

$$x = \frac{27}{2} = 13\frac{1}{2} \text{ miles}$$

Many proportion problems can also be solved using **dimensional analysis**, the process of multiplying a quantity by rates to change the units.

Example 15

Your car can drive 300 miles on a tank of 15 gallons. How far can it drive on 40 gallons?

Solution

We could certainly answer this question using a proportion: $\frac{300 \text{ miles}}{15 \text{ gallons}} = \frac{x \text{ miles}}{40 \text{ gallons}}$.

However, we earlier found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit “cancels” and we’re left with a number of miles:

$$40 \text{ gallons} \cdot \frac{20 \text{ miles}}{1 \text{ gallon}} = \frac{40 \text{ gallons}}{1} \cdot \frac{20 \text{ miles}}{\text{gallon}} = 800 \text{ miles}$$

Notice if instead we were asked “how many gallons are needed to drive 50 miles?” we could answer this question by inverting the 20 mile per gallon rate so that the *miles* unit cancels and we’re left with gallons:

$$50 \text{ miles} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ miles}}{1} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ gallons}}{20} = 2.5 \text{ gallons}$$

Dimensional analysis can also be used to do unit conversions. Here are some unit conversions for reference.

Unit Conversions

Length

1 foot (ft) = 12 inches (in) 1 yard (yd) = 3 feet (ft)
 1 mile = 5,280 feet
 1000 millimeters (mm) = 1 meter (m) 100 centimeters (cm) = 1 meter
 1000 meters (m) = 1 kilometer (km) 2.54 centimeters (cm) = 1 inch

Weight and Mass

1 pound (lb) = 16 ounces (oz) 1 ton = 2000 pounds
 1000 milligrams (mg) = 1 gram (g) 1000 grams = 1 kilogram (kg)
 1 kilogram = 2.2 pounds (on earth)

Capacity

1 cup = 8 fluid ounces (fl oz)* 1 pint = 2 cups
 1 quart = 2 pints = 4 cups 1 gallon = 4 quarts = 16 cups
 1000 milliliters (ml) = 1 liter (L)

*Fluid ounces are a capacity measurement for liquids. 1 fluid ounce $\approx$ 1 ounce (weight) for water only.

Example 16

A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

Solution

To answer this question, we need to convert 20 seconds into feet. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don't, we will need to do additional unit conversions. We will need to know that 5280 ft = 1 mile. We might start by converting the 20 seconds into hours:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{180} \text{ hour}$$

Now we can multiply by the 15 miles/hr

$$\frac{1}{180} \text{ hour} \cdot \frac{15 \text{ miles}}{1 \text{ hour}} = \frac{1}{12} \text{ mile}$$

Now we can convert to feet

$$\frac{1}{12} \text{ mile} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

We could have also done this entire calculation in one long set of products:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{15 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

Try it Now 4

A 1000 foot spool of bare 12-gauge copper wire weighs 19.8 pounds. How much will 18 inches of the wire weigh, in ounces?

Answer available at the end of this section.

Notice that with the miles per gallon example, if we double the miles driven, we double the gas used. Likewise, with the map distance example, if the map distance doubles, the real-life distance doubles. This is a key feature of proportional relationships, and one we must confirm before assuming two things are related proportionally.

Example 17

Suppose you're tiling the floor of a 10 ft by 10 ft room, and find that 100 tiles will be needed. How many tiles will be needed to tile the floor of a 20 ft by 20 ft room?

Solution

In this case, while the width the room has doubled, the area has quadrupled. Since the number of tiles needed corresponds with the area of the floor, not the width, 400 tiles will be needed. We could find this using a proportion based on the areas of the rooms:

$$\frac{100 \text{ tiles}}{100\text{ft}^2} = \frac{n \text{ tiles}}{400\text{ft}^2}$$

Other quantities just don't scale proportionally at all.

Example 18

Suppose a small company spends \$1000 on an advertising campaign, and gains 100 new customers from it. How many new customers should they expect if they spend \$10,000?

Solution

While it is tempting to say that they will gain 1000 new customers, it is likely that additional advertising will be less effective than the initial advertising. For example, if the company is a hot tub store, there are likely only a fixed number of people interested in buying a hot tub, so there might not even be 1000 people in the town who would be potential customers.

Sometimes when working with rates, proportions, and percents, the process can be made more challenging by the magnitude of the numbers involved. Sometimes, large numbers are just difficult to comprehend.

Example 19

Compare the 2010 U.S. military budget of \$683.7 billion to other quantities.

Here we have a very large number, about \$683,700,000,000 written out. Of course, imagining a billion dollars is very difficult, so it can help to compare it to other quantities.

If that amount of money was used to pay the salaries of the 1.4 million Walmart employees in the U.S., each would earn over \$488,000.

There are about 300 million people in the U.S. The military budget is about \$2,200 per person.

If you were to put \$683.7 billion in \$100 bills, and count out 1 per second, it would take 216 years to finish counting it.

Example 20

Compare the electricity consumption per capita in China to the rate in Japan.

To address this question, we will first need data. From the CIA[1] website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank[2], we can find the population of China is 1,344,130,000, or 1.344 billion, and the population of Japan is 127,817,277, or 127.8 million.

Solution

Computing the consumption per capita for each country:

China: $\frac{4,693,000,000,000 \text{ KWH}}{1,344,130,000 \text{ people}} \approx 3491.5 \text{ KWH per person}$

Japan: $\frac{859,700,000,000 \text{ KWH}}{127,817,277 \text{ people}} \approx 6726 \text{ KWH per person}$

While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

[1] www.cia.gov/library/publicat.../2042rank.html

[2] <http://data.worldbank.org/indicator/SP.POP.TOT>

Answers

Try it Now 4

$$18 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{19.8 \text{ pounds}}{1000 \text{ feet}} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} \approx 0.475 \text{ ounces}$$

1.1.4: Geometry

Geometric shapes, as well as area and volumes, can often be important in problem solving.

Example 21

You are curious how tall a tree is, but don't have any way to climb it. Describe a method for determining the height.

There are several approaches we could take. We'll use one based on triangles, which requires that it's a sunny day. Suppose the tree is casting a shadow, say 15 ft long. I can then have a friend help me measure my own shadow. Suppose I am 6 ft tall, and cast a 1.5 ft shadow. Since the triangle formed by the tree and its shadow has the same angles as the triangle formed by me and my shadow, these triangles are called **similar triangles** and their sides will scale proportionally. In other words, the ratio of height to width will be the same in both triangles. Using this, we can find the height of the tree, which we'll denote by h :

Solution

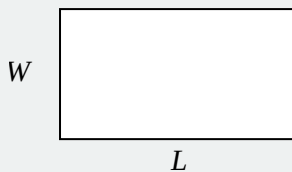
$$\frac{6\text{ft tall}}{1.5\text{ft shadow}} = \frac{h\text{ft tall}}{15\text{ft shadow}}$$

Multiplying both sides by 15, we get $h = 60$. The tree is about 60 ft tall.

It may be helpful to recall some formulas for areas and volumes of a few basic shapes.

Areas

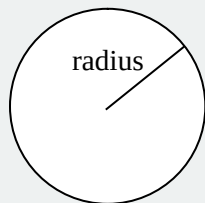
Rectangle



Area: $L \cdot W$

Perimeter: $2L + 2W$

Circle, radius r

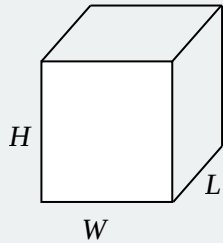


Area: πr^2

Circumference = $2\pi r$

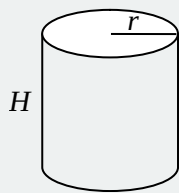
Volumes

Rectangular Box



Volume: $L \cdot W \cdot H$

Cylinder



Volume: $\pi r^2 H$

Example 22

If a 12 inch diameter pizza requires 10 ounces of dough, how much dough is needed for a 16 inch pizza?

Solution

To answer this question, we need to consider how the weight of the dough will scale. The weight will be based on the volume of the dough. However, since both pizzas will be about the same thickness, the weight will scale with the area of the top of the pizza. We can find the area of each pizza using the formula for area of a circle, $A = \pi r^2$:

A 12" pizza has radius 6 inches, so the area will be $\pi 6^2 =$ about 113 square inches.

A 16" pizza has radius 8 inches, so the area will be $\pi 8^2 =$ about 201 square inches.

Notice that if both pizzas were 1 inch thick, the volumes would be 113 in^3 and 201 in^3 respectively, which are at the same ratio as the areas. As mentioned earlier, since the thickness is the same for both pizzas, we can safely ignore it.

We can now set up a proportion to find the weight of the dough for a 16" pizza:

$$\frac{10 \text{ ounces}}{113 \text{ in}^2} = \frac{x \text{ ounces}}{201 \text{ in}^2} \quad \text{Multiply both sides by } 201$$

$$x = 201 \cdot \frac{10}{113} = \text{about } 17.8 \text{ ounces of dough for a 16" pizza.}$$

It is interesting to note that while the diameter is $\frac{16}{12} = 1.33$ times larger, the dough required, which scales with area, is $1.33^2 = 1.78$ times larger.

Example 23



A company makes regular and jumbo marshmallows. The regular marshmallow has 25 calories. How many calories will the jumbo marshmallow have?

We would expect the calories to scale with volume. Since the marshmallows have cylindrical shapes, we can use that formula to find the volume. From the grid in the image, we can estimate the radius and height of each marshmallow.

Solution

The regular marshmallow appears to have a diameter of about 3.5 units, giving a radius of 1.75 units, and a height of about 3.5 units. The volume is about $\pi(1.75)^2(3.5) = 33.7 \text{ units}^3$.

The jumbo marshmallow appears to have a diameter of about 5.5 units, giving a radius of 2.75 units, and a height of about 5 units. The volume is about $\pi(2.75)^2(5) = 118.8 \text{ units}^3$.

We could now set up a proportion, or use rates. The regular marshmallow has 25 calories for 33.7 cubic units of volume. The jumbo marshmallow will have:

$$118.8 \text{ units}^3 \cdot \frac{25 \text{ calories}}{33.7 \text{ units}^3} = 88.1 \text{ calories}$$

It is interesting to note that while the diameter and height are about 1.5 times larger for the jumbo marshmallow, the volume and calories are about $1.5^3 = 3.375$ times larger.

Try it Now 5

A website says that you'll need 48 fifty-pound bags of sand to fill a sandbox that measure 8ft by 8ft by 1ft. How many bags would you need for a sandbox 6ft by 4ft by 1ft?

Answer available at the end of this section.

Answers

Try it Now 5

The original sandbox has volume 64ft^3 . The smaller sandbox has volume 24ft^3 .

$$\frac{48\text{bags}}{64\text{ft}^3} = \frac{x\text{bags}}{24\text{ft}^3} \text{ results in } x = 18 \text{ bags.}$$

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1.1.5: Problem Solving and Estimating

Finally, we will bring together the mathematical tools we've reviewed, and use them to approach more complex problems. In many problems, it is tempting to take the given information, plug it into whatever formulas you have handy, and hope that the result is what you were supposed to find. Chances are, this approach has served you well in other math classes.

This approach does not work well with real life problems. Instead, problem solving is best approached by first starting at the end: identifying exactly what you are looking for. From there, you then work backwards, asking "what information and procedures will I need to find this?" Very few interesting questions can be answered in one mathematical step; often times you will need to chain together a solution pathway, a series of steps that will allow you to answer the question.

Problem Solving Process

1. Identify the question you're trying to answer.
2. Work backwards, identifying the information you will need and the relationships you will use to answer that question.
3. Continue working backwards, creating a solution pathway.
4. If you are missing necessary information, look it up or estimate it. If you have unnecessary information, ignore it.
5. Solve the problem, following your solution pathway.

In most problems we work, we will be approximating a solution, because we will not have perfect information. We will begin with a few examples where we will be able to approximate the solution using basic knowledge from our lives.

Example 24

How many times does your heart beat in a year?

Solution

This question is asking for the rate of heart beats per year. Since a year is a long time to measure heart beats for, if we knew the rate of heart beats per minute, we could scale that quantity up to a year. So the information we need to answer this question is heart beats per minute. This is something you can easily measure by counting your pulse while watching a clock for a minute.

Suppose you count 80 beats in a minute. To convert this beats per year:

$$\frac{80 \text{ beats}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{365 \text{ days}}{1 \text{ year}} = 42,048,000 \text{ beats per year}$$

Example 25

How thick is a single sheet of paper? How much does it weigh?

Solution

While you might have a sheet of paper handy, trying to measure it would be tricky. Instead we might imagine a stack of paper, and then scale the thickness and weight to a single sheet. If you've ever bought paper for a printer or copier, you probably bought a ream, which contains 500 sheets. We could estimate that a ream of paper is about 2 inches thick and weighs about 5 pounds. Scaling these down,

$$\frac{2 \text{ inches}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.004 \text{ inches per sheet}$$

$$\frac{5 \text{ pounds}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.01 \text{ pounds per sheet, or } 0.16 \text{ ounces per sheet.}$$

Example 26

A recipe for zucchini muffins states that it yields 12 muffins, with 250 calories per muffin. You instead decide to make mini-muffins, and the recipe yields 20 muffins. If you eat 4, how many calories will you consume?

Solution

There are several possible solution pathways to answer this question. We will explore one.

To answer the question of how many calories 4 mini-muffins will contain, we would want to know the number of calories in each mini-muffin. To find the calories in each mini-muffin, we could first find the total calories for the entire recipe, then divide it by the number of mini-muffins produced. To find the total calories for the recipe, we could multiply the calories per standard muffin by the number per muffin. Notice that this produces a multi-step solution pathway. It is often easier to solve a problem in small steps, rather than trying to find a way to jump directly from the given information to the solution.

We can now execute our plan:

$$12 \text{ muffins} \cdot \frac{250 \text{ calories}}{\text{muffin}} = 3000 \text{ calories for the whole recipe}$$

$$\frac{3000 \text{ calories}}{20 \text{ mini - muffins}} \text{ gives } 150 \text{ calories per mini-muffin}$$

$$4 \text{ mini muffins} \cdot \frac{150 \text{ calories}}{\text{mini - muffin}} \text{ totals } 600 \text{ calories consumed.}$$

Example 27

You need to replace the boards on your deck. About how much will the materials cost?

Solution

There are two approaches we could take to this problem: 1) estimate the number of boards we will need and find the cost per board, or 2) estimate the area of the deck and find the approximate cost per square foot for deck boards. We will take the latter approach.

For this solution pathway, we will be able to answer the question if we know the cost per square foot for decking boards and the square footage of the deck. To find the cost per square foot for decking boards, we could compute the area of a single board, and divide it into the cost for that board. We can compute the square footage of the deck using geometric formulas. So first we need information: the dimensions of the deck, and the cost and dimensions of a single deck board.

Suppose that measuring the deck, it is rectangular, measuring 16 ft by 24 ft, for a total area of 384ft^2 .

From a visit to the local home store, you find that an 8 foot by 4 inch cedar deck board costs about \$7.50. The area of this board, doing the necessary conversion from inches to feet, is:

$$8 \text{ feet} \cdot 4 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = 2.667\text{ft}^2. \text{ The cost per square foot is then}$$

$$\frac{\$7.50}{2.667\text{ft}^2} = \$2.8125 \text{ per ft}^2.$$

This will allow us to estimate the material cost for the whole 384ft^2 deck

$$\$384\text{ft}^2 \cdot \frac{\$2.8125}{\text{ft}^2} = \$1080 \text{ total cost.}$$

Of course, this cost estimate assumes that there is no waste, which is rarely the case. It is common to add at least 10% to the cost estimate to account for waste.

Example 28

Is it worth buying a Hyundai Sonata hybrid instead the regular Hyundai Sonata?

Solution

To make this decision, we must first decide what our basis for comparison will be. For the purposes of this example, we'll focus on fuel and purchase costs, but environmental impacts and maintenance costs are other factors a buyer might consider.

It might be interesting to compare the cost of gas to run both cars for a year. To determine this, we will need to know the miles per gallon both cars get, as well as the number of miles we expect to drive in a year. From that information, we can find the number of gallons required from a year. Using the price of gas per gallon, we can find the running cost.

From Hyundai's website, the 2013 Sonata will get 24 miles per gallon (mpg) in the city, and 35 mpg on the highway. The hybrid will get 35 mpg in the city, and 40 mpg on the highway.

An average driver drives about 12,000 miles a year. Suppose that you expect to drive about 75% of that in the city, so 9,000 city miles a year, and 3,000 highway miles a year.

We can then find the number of gallons each car would require for the year.

Sonata:

$$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{24 \text{ city miles}} + 3000 \text{ hightway miles} \cdot \frac{1 \text{ gallon}}{35 \text{ highway miles}} = 460.7 \text{ gallons}$$

Hybrid:

$$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{35 \text{ city miles}} + 3000 \text{ hightway miles} \cdot \frac{1 \text{ gallon}}{40 \text{ highway miles}} = 332.1 \text{ gallons}$$

If gas in your area averages about \$3.50 per gallon, we can use that to find the running cost:

$$\text{Sonata: } 460.7 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1612.45$$

$$\text{Hybrid: } 332.1 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1162.35$$

The hybrid will save \$450.10 a year. The gas costs for the hybrid are about $\frac{\$450.10}{\$1612.45} = 0.279 = 27.9\%$ lower than the costs for the standard Sonata.

While both the absolute and relative comparisons are useful here, they still make it hard to answer the original question, since "is it worth it" implies there is some tradeoff for the gas savings. Indeed, the hybrid Sonata costs about \$25,850, compared to the base model for the regular Sonata, at \$20,895.

To better answer the "is it worth it" question, we might explore how long it will take the gas savings to make up for the additional initial cost. The hybrid costs \$4965 more. With gas savings of \$451.10 a year, it will take about 11 years for the gas savings to make up for the higher initial costs.

We can conclude that if you expect to own the car 11 years, the hybrid is indeed worth it. If you plan to own the car for less than 11 years, it may still be worth it, since the resale value of the hybrid may be higher, or for other non-monetary reasons. This is a case where math can help guide your decision, but it can't make it for you.

Try it Now 6

If traveling from Seattle, WA to Spokane WA for a three-day conference, does it make more sense to drive or fly?

Answer available at the end of this section.

Answers

Try it Now 6

There is not enough information provided to answer the question, so we will have to make some assumptions, and look up some values.

Assumptions:

- a) We own a car. Suppose it gets 24 miles to the gallon. We will only consider gas cost.
- b) We will not need to rent a car in Spokane, but will need to get a taxi from the airport to the conference hotel downtown and back.
- c) We can get someone to drop us off at the airport, so we don't need to consider airport parking.
- d) We will not consider whether we will lose money by having to take time off work to drive.

Values looked up (your values may be different)

- a) Flight cost: \$184
- b) Taxi cost: \$25 each way (estimate, according to hotel website)
- c) Driving distance: 280 miles each way
- d) Gas cost: \$3.79 a gallon

Cost for flying: \$184 flight cost + \$50 in taxi fares = \$234.

Cost for driving: 560 miles round trip will require 23.3 gallons of gas, costing \$88.31.

Based on these assumptions, driving is cheaper. However, our assumption that we only include gas cost may not be a good one. Tax law allows you deduct \$0.55 (in 2012) for each mile driven, a value that accounts for gas as well as a portion of the car cost, insurance, maintenance, etc. Based on this number, the cost of driving would be \$319.

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1.1.6: Exercises

1. Out of 230 racers who started the marathon, 212 completed the race, 14 gave up, and 4 were disqualified. What percentage did not complete the marathon?
2. Patrick left an \$8 tip on a \$50 restaurant bill. What percent tip is that?
3. Ireland has a 23% VAT (value-added tax, similar to a sales tax). How much will the VAT be on a purchase of a €250 item?
4. Employees in 2012 paid 4.2% of their gross wages towards social security (FICA tax), while employers paid another 6.2%. How much will someone earning \$45,000 a year pay towards social security out of their gross wages?
5. A project on Kickstarter.com was aiming to raise \$15,000 for a precision coffee press. They ended up with 714 supporters, raising 557% of their goal. How much did they raise?
6. Another project on Kickstarter for an iPad stylus raised 1,253% of their goal, raising a total of \$313,490 from 7,511 supporters. What was their original goal?
7. The population of a town increased from 3,250 in 2008 to 4,300 in 2010. Find the absolute and relative (percent) increase.
8. The number of CDs sold in 2010 was 114 million, down from 147 million the previous year[1]. Find the absolute and relative (percent) decrease.
9. A company wants to decrease their energy use by 15%.
 - a. If their electric bill is currently \$2,200 a month, what will their bill be if they're successful?
 - b. If their next bill is \$1,700 a month, were they successful? Why or why not?
10. A store is hoping an advertising campaign will increase their number of customers by 30%. They currently have about 80 customers a day.
 - a. How many customers will they have if their campaign is successful?
 - b. If they increase to 120 customers a day, were they successful? Why or why not?
11. An article reports "attendance dropped 6% this year, to 300." What was the attendance before the drop?
12. An article reports "sales have grown by 30% this year, to \$200 million." What were sales before the growth?
13. The Walden University had 47,456 students in 2010, while Kaplan University had 77,966 students. Complete the following statements:
 - a. Kaplan's enrollment was ___% larger than Walden's.
 - b. Walden's enrollment was ___% smaller than Kaplan's.
 - c. Walden's enrollment was ___% of Kaplan's.
14. In the 2012 Olympics, Usain Bolt ran the 100m dash in 9.63 seconds. Jim Hines won the 1968 Olympic gold with a time of 9.95 seconds.
 - a. Bolt's time was ___% faster than Hines'.
 - b. Hine' time was ___% slower than Bolt's.
 - c. Hine' time was ___% of Bolt's.

15. A store has clearance items that have been marked down by 60%. They are having a sale, advertising an additional 30% off clearance items. What percent of the original price do you end up paying?
16. Which is better: having a stock that goes up 30% on Monday than drops 30% on Tuesday, or a stock that drops 30% on Monday and goes up 30% on Tuesday? In each case, what is the net percent gain or loss?
17. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
- "16.3% of Americans are without health insurance"[2]
 - "only 55.9% of adults receive employer provided health insurance"[3]
18. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
- "We mark up the wholesale price by 33% to come up with the retail price"
 - "The store has a 25% profit margin"
19. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
- "Every year since 1950, the number of American children gunned down has doubled."
 - "The number of child gunshot deaths has doubled from 1950 to 1994."
20. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?[4]
- "75 percent of the federal health care law's taxes would be paid by those earning less than \$120,000 a year"
 - "76 percent of those who would pay the penalty [health care law's taxes] for not having insurance in 2016 would earn under \$120,000"
21. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
- "The school levy is only a 0.1% increase of the property tax rate."
 - "This new levy is a 12% tax hike, raising our total rate to \$9.33 per \$1000 of value."
22. Are the values compared in this statement comparable or not comparable? "Guns have murdered more Americans here at home in recent years than have died on the battlefields of Iraq and Afghanistan. In support of the two wars, more than 6,500 American soldiers have lost their lives. During the same period, however, guns have been used to murder about 100,000 people on American soil"[5]
23. A high school currently has a 30% dropout rate. They've been tasked to decrease that rate by 20%. Find the equivalent percentage point drop.
24. A politician's support grew from 42% by 3 percentage points to 45%. What percent (relative) change is this?
25. Marcy has a 70% average in her class going into the final exam. She says "I need to get a 100% on this final so I can raise my score to 85%." Is she correct?
26. Suppose you have one quart of water/juice mix that is 50% juice, and you add 2 quarts of juice. What percent juice is the final mix?
27. Find a unit rate: You bought 10 pounds of potatoes for \$4.
28. Find a unit rate: Joel ran 1500 meters in 4 minutes, 45 seconds.
29. Solve: $\frac{2}{5} = \frac{6}{x}$.
30. Solve: $\frac{n}{5} = \frac{16}{20}$.

31. A crepe recipe calls for 2 eggs, 1 cup of flour, and 1 cup of milk. How much flour would you need if you use 5 eggs?
32. An 8ft length of 4 inch wide crown molding costs \$14. How much will it cost to buy 40ft of crown molding?
33. Four 3-megawatt wind turbines can supply enough electricity to power 3000 homes. How many turbines would be required to power 55,000 homes?
34. A highway had a landslide, where 3,000 cubic yards of material fell on the road, requiring 200 dump truck loads to clear. On another highway, a slide left 40,000 cubic yards on the road. How many dump truck loads would be needed to clear this slide?
35. Convert 8 feet to inches.
36. Convert 6 kilograms to grams.
37. A wire costs \$2 per meter. How much will 3 kilometers of wire cost?
38. Sugar contains 15 calories per teaspoon. How many calories are in 1 cup of sugar?
39. A car is driving at 100 kilometers per hour. How far does it travel in 2 seconds?
40. A chain weighs 10 pounds per foot. How many ounces will 4 inches weigh?
41. The table below gives data on three movies. Gross earnings is the amount of money the movie brings in. Compare the net earnings (money made after expenses) for the three movies.[6]

Movie Earnings

Movie	Release Date	Budget	Gross Earnings
Saw	10/29/2004	\$1,200,000	\$103,096,345
Titanic	10/29/2004	\$200,000,000	\$1,842,879,955
Jurassic Park	6/11/1993	\$63,000,000	\$923,863,984

42. For the movies in the previous problem, which provided the best return on investment?
43. The population of the U.S. is about 309,975,000, covering a land area of 3,717,000 square miles. The population of India is about 1,184,639,000, covering a land area of 1,269,000 square miles. Compare the population densities of the two countries.
44. The GDP (Gross Domestic Product) of China was \$5,739 billion in 2010, and the GDP of Sweden was \$435 billion. The population of China is about 1,347 million, while the population of Sweden is about 9.5 million. Compare the GDP per capita of the two countries.
45. In June 2012, Twitter was reporting 400 million tweets per day. Each tweet can consist of up to 140 characters (letter, numbers, etc.). Create a comparison to help understand the amount of tweets in a year by imagining each character was a drop of water and comparing to filling something up.
46. The photo sharing site Flickr had 2.7 billion photos in June 2012. Create a comparison to understand this number by assuming each picture is about 2 megabytes in size, and comparing to the data stored on other media like DVDs, iPods, or flash drives.
47. Your chocolate milk mix says to use 4 scoops of mix for 2 cups of milk. After pouring in the milk, you start adding the mix, but get distracted and accidentally put in 5 scoops of mix. How can you adjust the mix if:
 - a. There is still room in the cup?
 - b. The cup is already full?
48. A recipe for sabayon calls for 2 egg yolks, 3 tablespoons of sugar, and $\frac{1}{4}$ cup of white wine. After cracking the eggs, you start measuring the sugar, but accidentally put in 4 tablespoons of sugar. How can you compensate?
49. The Deepwater Horizon oil spill resulted in 4.9 million barrels of oil spilling into the Gulf of Mexico. Each barrel of oil can be

processed into about 19 gallons of gasoline. How many cars could this have fueled for a year? Assume an average car gets 20 miles to the gallon, and drives about 12,000 miles in a year.

50. The store is selling lemons at 2 for \$1. Each yields about 2 tablespoons of juice. How much will it cost to buy enough lemons to make a 9-inch lemon pie requiring $\frac{1}{2}$ cup of lemon juice?

51. A piece of paper can be made into a cylinder in two ways: by joining the short sides together, or by joining the long sides together[7]. Which cylinder would hold more? How much more?

52. Which of these glasses contains more liquid? How much more?



In the next 4 questions, estimate the values by making reasonable approximations for unknown values, or by doing some research to find reasonable values.

53. Estimate how many gallons of water you drink in a year.

54. Estimate how many times you blink in a day.

55. How much does the water in a 6-person hot tub weigh?

56. How many gallons of paint would be needed to paint a two-story house 40 ft long and 30 ft wide?

57. During the landing of the Mars Science Laboratory *Curiosity*, it was reported that the signal from the rover would take 14 minutes to reach earth. Radio signals travel at the speed of light, about 186,000 miles per second. How far was Mars from Earth when *Curiosity* landed?

58. It is estimated that a driver takes, on average, 1.5 seconds from seeing an obstacle to reacting by applying the brake or swerving. How far will a car traveling at 60 miles per hour travel (in feet) before the driver reacts to an obstacle?

59. The flash of lightning travels at the speed of light, which is about 186,000 miles per second. The sound of lightning (thunder) travels at the speed of sound, which is about 750 miles per hour.

a. If you see a flash of lightning, then hear the thunder 4 seconds later, how far away is the lightning?

b. Now let's generalize that result. Suppose it takes n seconds to hear the thunder after a flash of lightning. How far away is the lightning, in terms of n ?

60. Sound travels about 750 miles per hour. If you stand in a parking lot near a building and sound a horn, you will hear an echo.

a. Suppose it takes about $\frac{1}{2}$ a second to hear the echo. How far away is the building[8]?

b. Now let's generalize that result. Suppose it takes n seconds to hear the echo. How far away is the building, in terms of n ?

61. It takes an air pump 5 minutes to fill a twin sized air mattress (39 by 8.75 by 75 inches). How long will it take to fill a queen sized mattress (60 by 8.75 by 80 inches)?
62. It takes your garden hose 20 seconds to fill your 2-gallon watering can. How long will it take to fill
- a. An inflatable pool measuring 3 feet wide, 8 feet long, and 1 foot deep.[9]
 - b. A circular inflatable pool 13 feet in diameter and 3 feet deep.[10]
63. You want to put a 2" thick layer of topsoil for a new 20'x30' garden. The dirt store sells by the cubic yards. How many cubic yards will you need to order?
64. A box of Jell-O costs \$0.50, and makes 2 cups. How much would it cost to fill a swimming pool 4 feet deep, 8 feet wide, and 12 feet long with Jell-O? (1 cubic foot is about 7.5 gallons)
65. You read online that a 15 ft by 20 ft brick patio would cost about \$2,275 to have professionally installed. Estimate the cost of having a 18 by 22 ft brick patio installed.
66. I was at the store, and saw two sizes of avocados being sold. The regular size sold for \$0.88 each, while the jumbo ones sold for \$1.68 each. Which is the better deal?



67. The grocery store has bulk pecans on sale, which is great since you're planning on making 10 pecan pies for a wedding. Your recipe calls for $1\frac{3}{4}$ cups pecans per pie. However, in the bulk section there's only a scale available, not a measuring cup. You run over to the baking aisle and find a bag of pecans, and look at the nutrition label to gather some info. How many pounds of pecans should you buy?

Nutrition Facts	
Serving Size: 1 cup, halves (99 g)	
Servings per Container: about 2	
Amount Per Serving	
Calories 684	Calories from Fat 596
% Daily Value*	
Total Fat 71g	110%
Saturated Fat 6g	31%
Trans Fat	
Cholesterol 0mg	0%

68. Soda is often sold in 20 ounce bottles. The nutrition label for one of these bottles is shown to the right. A packet of sugar (the kind they have at restaurants for your coffee or tea) typically contain 4 grams of sugar in the U.S. Drinking a 20 oz soda is equivalent to eating how many packets of sugar?[11]

Nutrition Facts	
Serving Size: 8 fl oz (240 mL)	
Servings Per Container: about 2.5	
Amount Per Serving	
Calories 110	
% Daily Value*	
Total Fat 0g	0%
Sodium 70mg	3%
Total Carbohydrate 31g	10%
Sugars 30g	
Protein 0g	

For the next set of questions, *first* identify the information you need to answer the question, and *then* turn to the end of the section to find that information. The details may be imprecise; answer the question the best you can with the provided information. Be sure to justify your decision.

69. You're planning on making 6 meatloafs for a party. You go to the store to buy breadcrumbs, and see they are sold by the canister. How many canisters do you need to buy?

70. Your friend wants to cover their car in bottle caps, like in this picture.[12] How many bottle caps are you going to need?



71. You need to buy some chicken for dinner tonight. You found an ad showing that the store across town has it on sale for \$2.99 a pound, which is cheaper than your usual neighborhood store, which sells it for \$3.79 a pound. Is it worth the extra drive?

72. I have an old gas furnace, and am considering replacing it with a new, high efficiency model. Is upgrading worth it?

73. Janine is considering buying a water filter and a reusable water bottle rather than buying bottled water. Will doing so save her money?

74. Marcus is considering going car-free to save money and be more environmentally friendly. Is this financially a good decision?

For the next set of problems, research or make educated estimates for any unknown quantities needed to answer the question.

75. You want to travel from Tacoma, WA to Chico, CA for a wedding. Compare the costs and time involved with driving, flying, and taking a train. Assume that if you fly or take the train you'll need to rent a car while you're there. Which option is best?

76. You want to paint the walls of a 6ft by 9ft storage room that has one door and one window. You want to put on two coats of paint. How many gallons and/or quarts of paint should you buy to paint the room as cheaply as possible?

77. A restaurant in New York tiled their floor with pennies[13]. Just for the materials, is this more expensive than using a more traditional material like ceramic tiles? If each penny has to be laid by hand, estimate how long it would take to lay the pennies for a 12ft by 10ft room. Considering material and labor costs, are pennies a cost-effective replacement for ceramic tiles?

78. You are considering taking up part of your back yard and turning it into a vegetable garden, to grow broccoli, tomatoes, and zucchini. Will doing so save you money, or cost you more than buying vegetables from the store?

79. Barry is trying to decide whether to keep his 1993 Honda Civic with 140,000 miles, or trade it in for a used 2008 Honda Civic. Consider gas, maintenance, and insurance costs in helping him make a decision.

80. Some people claim it costs more to eat vegetarian, while some claim it costs less. Examine your own grocery habits, and compare your current costs to the costs of switching your diet (from omnivore to vegetarian or vice versa as appropriate). Which diet is more cost effective based on your eating habits?

Info for the breadcrumbs question

How much breadcrumbs does the recipe call for?

It calls for $1\frac{1}{2}$ cups of breadcrumbs.

How many meatloafs does the recipe make?

It makes 1 meatloaf.

How many servings does that recipe make?

It says it serves 8.

How big is the canister?

It is cylindrical, 3.5 inches across and 7 inches tall.

What is the net weight of the contents of 1 canister?

15 ounces.

How much does a cup of breadcrumbs weigh?

I'm not sure, but maybe something from the nutritional label will help.

Nutrition Facts	
Serving Size: 1/3 cup (30g)	
Servings per Container: about 14	
Amount Per Serving	
Calories 110	Calories from Fat 15
% Daily Value*	
Total Fat 1.5g	2%

How much does a canister cost?

\$2.39

Info for bottle cap car

What kind of car is that?

A 1993 Honda Accord.

How big is that car / what are the dimensions? Here is some details from MSN autos:

Weight: 2800lb Length: 185.2 in Width: 67.1 in Height: 55.2 in

How much of the car was covered with caps?

Everything but the windows and the underside.

How big is a bottle cap?

Caps are 1 inch in diameter.

Info for chicken problem

How much chicken will you be buying?

Four pounds

How far are the two stores?

My neighborhood store is 2.2 miles away, and takes about 7 minutes. The store across town is 8.9 miles away, and takes about 25 minutes.

What kind of mileage does your car get?

It averages about 24 miles per gallon in the city.

How many gallons does your car hold?

About 14 gallons

How much is gas?

About \$3.69/gallon right now.

Info for furnace problem

How efficient is the current furnace?

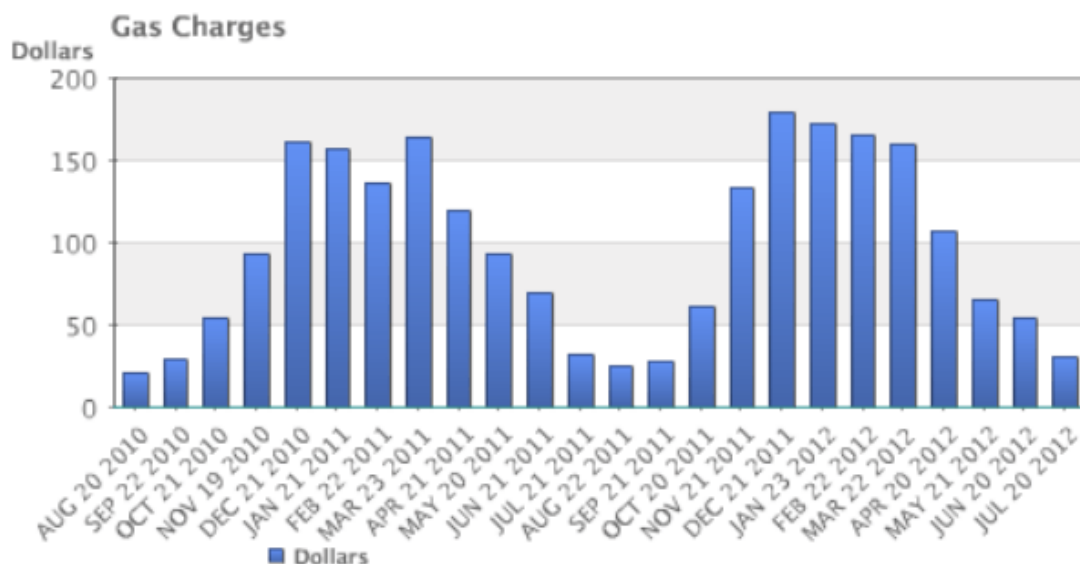
It is a 60% efficient furnace.

How efficient is the new furnace?

It is 94% efficient.

What is your gas bill?

Here is the history for 2 years:

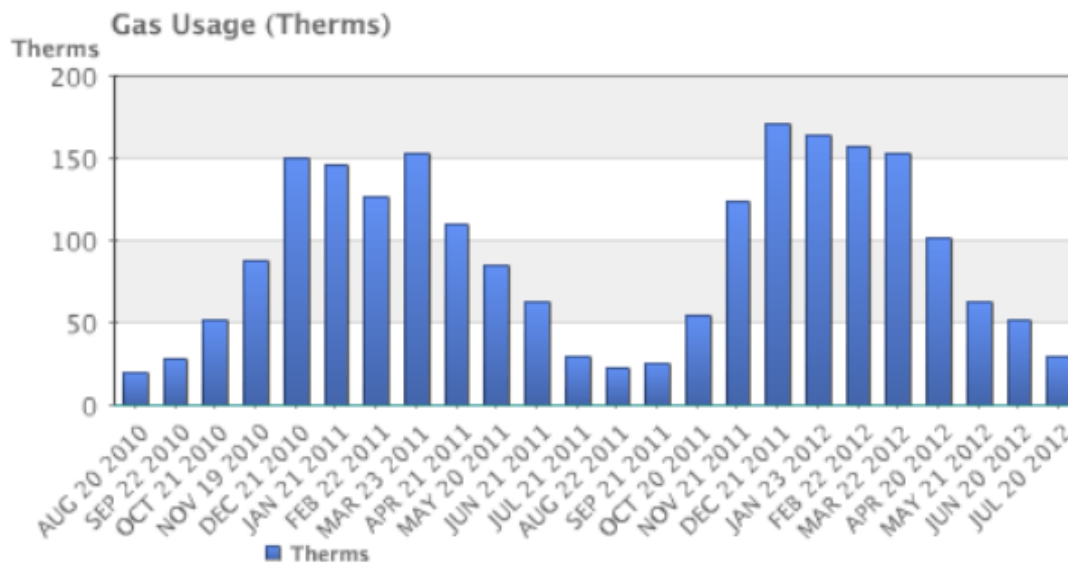


How much do you pay for gas?

There is \$10.34 base charge, plus \$0.39097 per Therm for a delivery charge, and \$0.65195 per Therm for cost of gas.

How much gas do you use?

Here is the history for 2 years:



How much does the new furnace cost?

It will cost \$7,450.

How long do you plan to live in the house?

Probably at least 15 years.

Info for water filter problem

How much water does Janine drink in a day?

She normally drinks 3 bottles a day, each 16.9 ounces.

How much does a bottle of water cost?

She buys 24-packs of 16.9 ounce bottles for \$3.99.

How much does a reusable water bottle cost?

About \$10.

How long does a reusable water bottle last?

Basically forever (or until you lose it).

How much does a water filter cost? How much water will they filter?

- A faucet-mounted filter costs about \$28. Refill filters cost about \$33 for a 3-pack. The box says each filter will filter up to 100 gallons (378 liters)
- A water filter pitcher costs about \$22. Refill filters cost about \$20 for a 4-pack. The box says each filter lasts for 40 gallons or 2 months
- An under-sink filter costs \$130. Refill filters cost about \$60 each. The filter lasts for 500 gallons.

Info for car-free problem

Where does Marcus currently drive? He:

- Drives to work 5 days a week, located 4 miles from his house.
- Drives to the store twice a week, located 7 miles from his house.

- Drives to other locations on average 5 days a week, with locations ranging from 1 mile to 20 miles.
- Drives to his parent's house 80 miles away once a month.

How will he get to these locations without a car?

- For work, he can walk when it's sunny and he gets up early enough. Otherwise he can take a bus, which takes about 20 minutes
- For the store, he can take a bus, which takes about 35 minutes.
- Some of the other locations he can bus to. Sometimes he'll be able to get a friend to pick him up. A few locations he is able to walk to. A couple locations are hard to get to by bus, but there is a ZipCar (short term car rental) location within a few blocks.
- He'll need to get a ZipCar to visit his parents.

How much does gas cost?

About \$3.69/gallon.

How much does he pay for insurance and maintenance?

- He pays \$95/month for insurance.
- He pays \$30 every 3 months for an oil change, and has averaged about \$300/year for other maintenance costs.

How much is he paying for the car?

- He's paying \$220/month on his car loan right now, and has 3 years left on the loan.
- If he sold the car, he'd be able to make enough to pay off the loan.
- If he keeps the car, he's planning on trading the car in for a newer model in a couple years.

What mileage does his car get?

About 26 miles per gallon on average.

How much does a bus ride cost?

\$2.50 per trip, or \$90 for an unlimited monthly pass.

How much does a ZipCar rental cost?

- The "occasional driving plan": \$25 application fee and \$60 annual fee, with no monthly commitment. Monday-Thursday the cost is \$8/hour, or \$72 per day. Friday-Sunday the cost is \$8/hour or \$78/day. Gas, insurance, and 180 miles are included in the cost. Additional miles are \$0.45/mile.
- The "extra value plan": Same as above, but with a \$50 monthly commitment, getting you a 10% discount on the usage costs.

[1] <http://www.cnn.com/2010/SHOWBIZ/Musi...les/index.html>

[2] <http://www.cnn.com/2012/06/27/politi...are/index.html>

[3] <http://www.politico.com/news/stories/0712/78134.html>

[4] <http://factcheck.org/2012/07/twistin...th-care-taxes/>

[5] www.northjersey.com/news/opin...tml?c=y&page=2

[6] <http://www.the-numbers.com/movies/records/budgets.php>

[7] vimeo.com/42501010

[8] vimeo.com/40377128

[9] <http://www.youtube.com/watch?v=DIkweReHZc>

[10] <http://www.youtube.com/watch?v=p9SABH7Yg9M>

[11] <http://www.youtube.com/watch?v=62JMfv0tf3Q>

[12] Photo credit: <http://www.flickr.com/photos/swayze/>, CC-BY

[13] <http://www.notcot.com/archives/2009/...-of-pennie.php>

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2: VOTING THEORY AND APPORTIONMENT

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- 2.1.2: Preference Schedules
- 2.1.3: Plurality
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2.1: VOTING THEORY

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2.1.1: Introduction

In many decision making situations, it is necessary to gather the group consensus. This happens when a group of friends decides which movie to watch, when a company decides which product design to manufacture, and when a democratic country elects its leaders.

While the basic idea of voting is fairly universal, the method by which those votes are used to determine a winner can vary. Amongst a group of friends, you may decide upon a movie by voting for all the movies you're willing to watch, with the winner being the one with the greatest approval. A company might eliminate unpopular designs then revote on the remaining. A country might look for the candidate with the most votes.

In deciding upon a winner, there is always one main goal: to reflect the preferences of the people in the most fair way possible.

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2.1.2: Preference Schedules

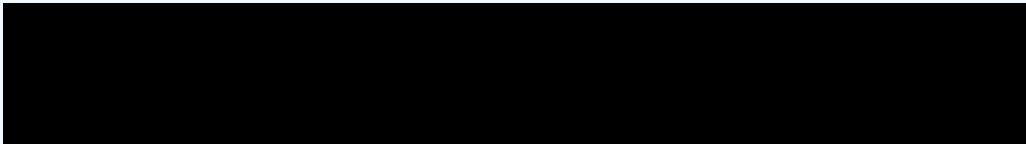
To begin, we're going to want more information than a traditional ballot normally provides. A traditional ballot usually asks you to pick your favorite from a list of choices. This ballot fails to provide any information on how a voter would rank the alternatives if their first choice was unsuccessful.

Preference ballot

A **preference ballot** is a ballot in which the voter ranks the choices in order of preference.

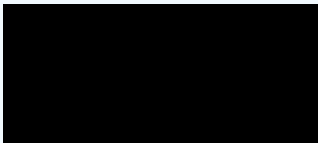
Example 1

A vacation club is trying to decide which destination to visit this year: Hawaii (H), Orlando (O), or Anaheim (A). Their votes are shown below:



Solution

These individual ballots are typically combined into one **preference schedule**, which shows the number of voters in the top row that voted for each option:



Notice that by totaling the vote counts across the top of the preference schedule we can recover the total number of votes cast: $1 + 3 + 3 + 3 = 10$ total votes.

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2.1.3: Plurality

The voting method we're most familiar with in the United States is the **plurality method**.

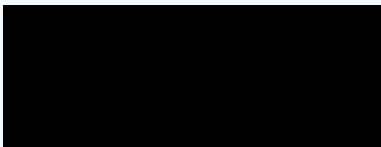
Plurality Method

In this method, the choice with the most first-preference votes is declared the winner. Ties are possible, and would have to be settled through some sort of run-off vote.

This method is sometimes mistakenly called the majority method, or “majority rules”, but it is not necessary for a choice to have gained a majority of votes to win. A majority is over 50%; it is possible for a winner to have a **plurality** without having a majority.

Example 2

In our election from previous pages, we had the preference table:



Solution

For the plurality method, we only care about the first choice options. Totaling them up:

Anaheim: $1+3 = 4$ first-choice votes

Orlando: 3 first-choice votes

Hawaii: 3 first-choice votes

Anaheim is the winner using the plurality voting method.

Notice that Anaheim won with 4 out of 10 votes, 40% of the votes, which is a plurality of the votes, but not a majority.

Try it Now 1

Three candidates are running in an election for County Executive: Goings (G), McCarthy (M), and Bunney (B)[1]. The voting schedule is shown below. Which candidate wins under the plurality method?



Note: In the third column and last column, those voters only recorded a first-place vote, so we don't know who their second and third choices would have been.

Answer available at the end of this section.

[1] This data is loosely based on the 2008 County Executive election in Pierce County, Washington. See www.co.pierce.wa.us/xml/abtus...ec/summary.pdf

Answers

Try it Now 1

Using plurality method:

G gets $44 + 14 + 20 = 78$ first-choice votes

M gets $70 + 22 = 92$ first-choice votes

B gets $80 + 39 = 119$ first-choice votes

Bunney (B) wins under plurality method.

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2.1.4: What's Wrong with Plurality?

The election from Example 2 may seem totally clean, but there is a problem lurking that arises whenever there are three or more choices. Looking back at our preference table, how would our members vote if they only had two choices?

Anaheim vs Orlando: 7 out of the 10 would prefer Anaheim over Orlando

	1	3	3	3
1 st choice	A	A	O	H
2 nd choice	O	H	H	A
3 rd choice	H	O	A	O

Anaheim vs Hawaii: 6 out of 10 would prefer Hawaii over Anaheim

	1	3	3	3
1 st choice	A	A	O	H
2 nd choice	O	H	H	A
3 rd choice	H	O	A	O

This doesn't seem right, does it? Anaheim just won the election, yet 6 out of 10 voters, 60% of them, would have preferred Hawaii! That hardly seems fair. Marquis de Condorcet, a French philosopher, mathematician, and political scientist wrote about how this could happen in 1785, and for him we name our first **fairness criterion**.

Fairness Criteria

The fairness criteria are statements that seem like they should be true in a fair election.

Condorcet Criterion

If there is a choice that is preferred in every one-to-one comparison with the other choices, that choice should be the winner. We call this winner the **Condorcet Winner**, or Condorcet Candidate.

Example 3

In the election from Example 2, what choice is the Condorcet Winner?

Solution

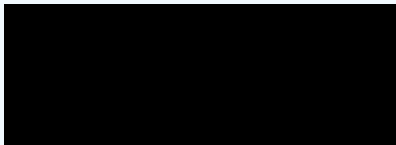
We see above that Hawaii is preferred over Anaheim. Comparing Hawaii to Orlando, we can see 6 out of 10 would prefer Hawaii to Orlando.

	1	3	3	3
1 st choice	A	A	O	H
2 nd choice	O	H	H	A
3 rd choice	H	O	A	O

Since Hawaii is preferred in a one-to-one comparison to both other choices, Hawaii is the Condorcet Winner.

Example 4

Consider a city council election in a district that is historically 60% Democratic voters and 40% Republican voters. Even though city council is technically a nonpartisan office, people generally know the affiliations of the candidates. In this election there are three candidates: Don and Key, both Democrats, and Elle, a Republican. A preference schedule for the votes looks as follows:



Solution

We can see a total of $342 + 214 + 298 = 854$ voters participated in this election. Computing percentage of first place votes:

Don: $214/854 = 25.1\%$

Key: $298/854 = 34.9\%$

Elle: $342/854 = 40.0\%$

So in this election, the Democratic voters split their vote over the two Democratic candidates, allowing the Republican candidate Elle to win under the plurality method with 40% of the vote.

Analyzing this election closer, we see that it violates the Condorcet Criterion. Analyzing the one-to-one comparisons:

Elle vs Don: 342 prefer Elle; 512 prefer Don: Don is preferred

Elle vs Key: 342 prefer Elle; 512 prefer Key: Key is preferred

Don vs Key: 556 prefer Don; 298 prefer Key: Don is preferred

So even though Don had the smallest number of first-place votes in the election, he is the Condorcet winner, being preferred in every one-to-one comparison with the other candidates.

Try it Now 2

Consider the election from Try it Now 1. Is there a Condorcet winner in this election?

Answer available at the end of this section.

Answers

Try it Now 2

Determining the Condorcet Winner:

G vs M: $44 + 14 + 20 = 78$ prefer G, $70 + 22 + 80 = 172$ prefer M: M preferred

G vs B: $44 + 14 + 20 + 70 = 148$ prefer G, $22 + 80 + 39 = 141$ prefer B: G preferred

M vs B: $44 + 70 + 22 = 136$ prefer M, $14 + 80 + 39 = 133$ prefer B: M preferred

M is the Condorcet winner, based on the information we have.

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2.1.5: Insincere Voting

Situations like the one in Example 4 above, when there are more than one candidate that share somewhat similar points of view, can lead to **insincere voting**. Insincere voting is when a person casts a ballot counter to their actual preference for strategic purposes. In the case above, the democratic leadership might realize that Don and Key will split the vote, and encourage voters to vote for Key by officially endorsing him. Not wanting to see their party lose the election, as happened in the scenario above, Don's supporters might insincerely vote for Key, effectively voting against Elle.

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2.1.6: Instant Runoff Voting

Instant Runoff Voting (IRV), also called Plurality with Elimination, is a modification of the plurality method that attempts to address the issue of insincere voting.

Instant Runoff Voting (IRV)

In IRV, voting is done with preference ballots, and a preference schedule is generated. The choice with the *least* first-place votes is then eliminated from the election, and any votes for that candidate are redistributed to the voters' next choice. This continues until a choice has a majority (over 50%).

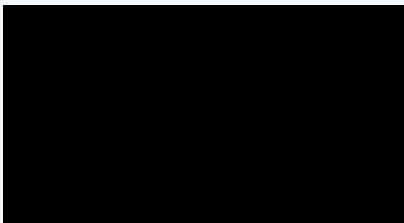
This is similar to the idea of holding runoff elections, but since every voter's order of preference is recorded on the ballot, the runoff can be computed without requiring a second costly election.

This voting method is used in several political elections around the world, including election of members of the Australian House of Representatives, and was used for county positions in Pierce County, Washington until it was eliminated by voters in 2009. A version of IRV is used by the International Olympic Committee to select host nations.

Example 5

Consider the preference schedule below, in which a company's advertising team is voting on five different advertising slogans, called A, B, C, D, and E here for simplicity.

Initial votes

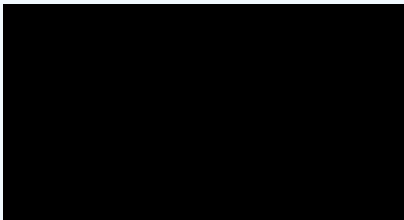


If this was a plurality election, note that B would be the winner with 9 first-choice votes, compared to 6 for D, 4 for C, and 1 for E.

There are total of $3+4+4+6+2+1 = 20$ votes. A majority would be 11 votes. No one yet has a majority, so we proceed to elimination rounds.

Solution

Round 1: We make our first elimination. Choice A has the fewest first-place votes, so we remove that choice



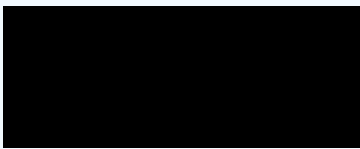
We then shift everyone's choices up to fill the gaps. There is still no choice with a majority, so we eliminate again.



Round 2: We make our second elimination. Choice E has the fewest first-place votes, so we remove that choice, shifting everyone's options to fill the gaps.



Notice that the first and fifth columns have the same preferences now, we can condense those down to one column.



Now B has 9 first-choice votes, C has 4 votes, and D has 7 votes. Still no majority, so we eliminate again.

Round 3: We make our third elimination. C has the fewest votes.



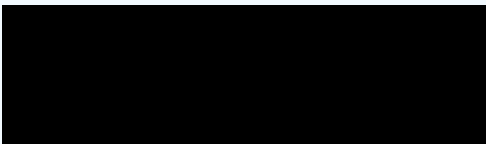
Condensing this down:



D has now gained a majority, and is declared the winner under IRV.

Try it Now 3

Consider again the election from Try it Now 1. Find the winner using IRV.



Answer available at the end of this section.

Answers

Try it Now 3

G has the fewest first-choice votes, so is eliminated first. The 20 voters who did not list a second choice do not get transferred - they simply get eliminated



McCarthy (M) now has a majority, and is declared the winner.

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2.1.7: What's Wrong with IRV?

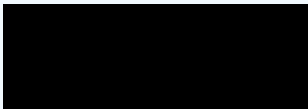
Example 6

Let's return to our City Council Election



Solution

In this election, Don has the smallest number of first place votes, so Don is eliminated in the first round. The 214 people who voted for Don have their votes transferred to their second choice, Key.

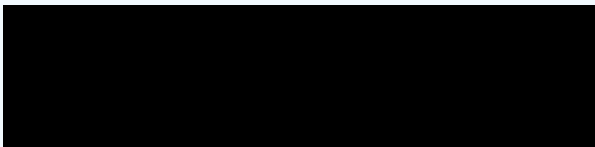


So Key is the winner under the IRV method.

We can immediately notice that in this election, IRV violates the Condorcet Criterion, since we determined earlier that Don was the Condorcet winner. On the other hand, the temptation has been removed for Don's supporters to vote for Key; they now know their vote will be transferred to Key, not simply discarded.

Example 7

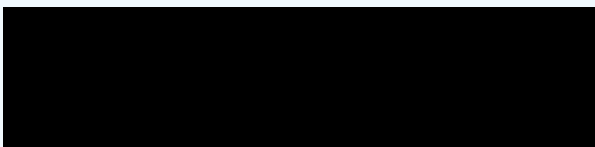
Consider the voting system below.



Solution

In this election, Carter would be eliminated in the first round, and Adams would be the winner with 66 votes to 34 for Brown.

Now suppose that the results were announced, but election officials accidentally destroyed the ballots before they could be certified, and the votes had to be recast. Wanting to "jump on the bandwagon", 10 of the voters who had originally voted in the order Brown, Adams, Carter change their vote to favor the presumed winner, changing those votes to Adams, Brown, Carter.



In this re-vote, Brown will be eliminated in the first round, having the fewest first-place votes. After transferring votes, we find that Carter will win this election with 51 votes to Adams' 49 votes! Even though the only vote changes made *avored* Adams, the change ended up costing Adams the election. This doesn't seem right, and introduces our second fairness criterion:

Monotonicity Criterion

If voters change their votes to increase the preference for a candidate, it should not harm that candidate's chances of winning.

This criterion is violated by this election. Note that even though the criterion is violated in this particular election, it does not mean that IRV always violates the criterion; just that IRV has the potential to violate the criterion in certain elections.

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2.1.8: Borda Count

Borda Count is another voting method, named for Jean-Charles de Borda, who developed the system in 1770.

Borda Count

In this method, points are assigned to candidates based on their ranking; 1 point for last choice, 2 points for second-to-last choice, and so on. The point values for all ballots are totaled, and the candidate with the largest point total is the winner.

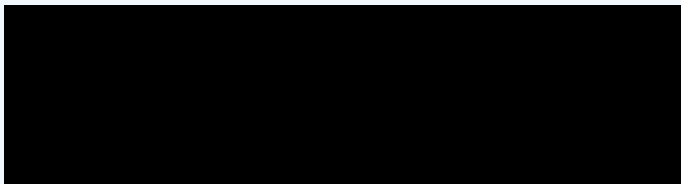
Example 8



A group of mathematicians are getting together for a conference.

The members are coming from four cities: Seattle, Tacoma, Puyallup, and Olympia. Their approximate locations on a map are shown to the right.

The votes for where to hold the conference were:



Solution

In each of the 51 ballots ranking Seattle first, Puyallup will be given 1 point, Olympia 2 points, Tacoma 3 points, and Seattle 4 points. Multiplying the points per vote times the number of votes allows us to calculate points awarded:



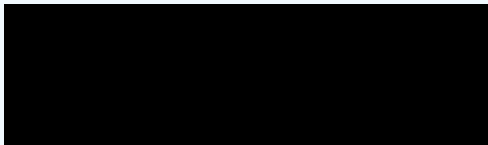
Adding up the points:

- Seattle: $204 + 25 + 10 + 14 = 253$ points
- Tacoma: $153 + 100 + 30 + 42 = 325$ points
- Puyallup: $51 + 75 + 40 + 28 = 194$ points
- Olympia: $102 + 50 + 20 + 56 = 228$ points

Under the Borda Count method, Tacoma is the winner of this vote.

Try it Now 4

Consider again the election from Try it Now 1. Find the winner using Borda Count. Since we have some incomplete preference ballots, for simplicity, give every unranked candidate 1 point, the points they would normally get for last place.



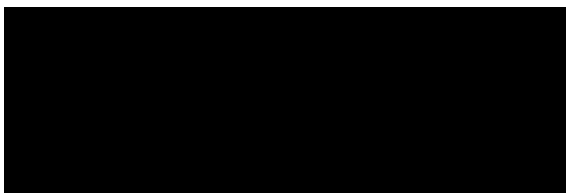
Answer available at the end of this section.

Answers

Try it Now 4

Using Borda Count:

We give 1 point for 3rd place, 2 points for 2nd place, and 3 points for 1st place.



G: $132 + 42 + 60 + 140 + 22 + 80 + 39 = 515$ pts

M: $88 + 14 + 20 + 210 + 66 + 160 + 39 = 597$ pts

B: $44 + 28 + 20 + 70 + 44 + 240 + 117 = 563$ pts

McCarthy (M) would be the winner using Borda Count.

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2.1.9: What's Wrong with Borda Count?

You might have already noticed one potential flaw of the Borda Count from the previous example. In that example, Seattle had a majority of first-choice votes, yet lost the election! This seems odd, and prompts our next fairness criterion:

Majority Criterion

If a choice has a majority of first-place votes, that choice should be the winner.

The election from Example 8 using the Borda Count violates the Majority Criterion. Notice also that this automatically means that the Condorcet Criterion will also be violated, as Seattle would have been preferred by 51% of voters in any head-to-head comparison.

Borda count is sometimes described as a consensus-based voting system, since it can sometimes choose a more broadly acceptable option over the one with majority support. In the example above, Tacoma is probably the best compromise location. This is a different approach than plurality and instant runoff voting that focus on first-choice votes; Borda Count considers every voter's entire ranking to determine the outcome.

Because of this consensus behavior, Borda Count, or some variation of it, is commonly used in awarding sports awards. Variations are used to determine the Most Valuable Player in baseball, to rank teams in NCAA sports, and to award the Heisman trophy.

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2.1.10: Copeland's Method (Pairwise Comparisons)

So far none of our voting methods have satisfied the Condorcet Criterion. The Copeland Method specifically attempts to satisfy the Condorcet Criterion by looking at pairwise (one-to-one) comparisons.

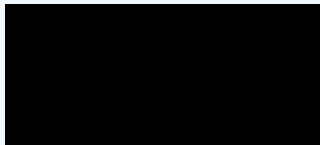
Copeland's Method

In this method, each pair of candidates is compared, using all preferences to determine which of the two is more preferred. The more preferred candidate is awarded 1 point. If there is a tie, each candidate is awarded $\frac{1}{2}$ point. After all pairwise comparisons are made, the candidate with the most points, and hence the most pairwise wins, is declared the winner.

Variations of Copeland's Method are used in many professional organizations, including election of the Board of Trustees for the Wikimedia Foundation that runs Wikipedia.

Example 9

Consider our vacation group example from the beginning of the chapter. Determine the winner using Copeland's Method.



Solution

We need to look at each pair of choices, and see which choice would win in a one-to-one comparison. You may recall we did this earlier when determining the Condorcet Winner. For example, comparing Hawaii vs Orlando, we see that 6 voters, those shaded below in the first table below, would prefer Hawaii to Orlando. Note that Hawaii doesn't have to be the voter's first choice – we're imagining that Anaheim wasn't an option. If it helps, you can imagine removing Anaheim, as in the second table below.

	1	3	3	3
1 st choice	A	A	O	H
2 nd choice	O	H	H	A
3 rd choice	H	O	A	O

	1	3	3	3
1 st choice			O	H
2 nd choice	O	H	H	
3 rd choice	H	O		O

Based on this, in the comparison of Hawaii vs Orlando, Hawaii wins, and receives 1 point.

Comparing Anaheim to Orlando, the 1 voter in the first column clearly prefers Anaheim, as do the 3 voters in the second column. The 3 voters in the third column clearly prefer Orlando. The 3 voters in the last column prefer Hawaii as their first choice, but if they had to choose between Anaheim and Orlando, they'd choose Anaheim, their second choice overall. So, altogether $1+3+3=7$ voters prefer Anaheim over Orlando, and 3 prefer Orlando over Anaheim. So, comparing Anaheim vs Orlando: 7 votes to 3 votes: Anaheim gets 1 point.

All together,

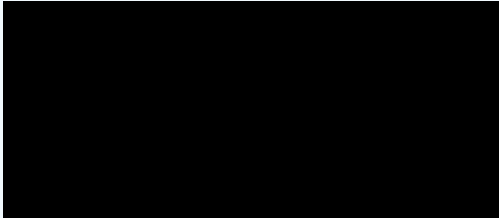
Hawaii vs Orlando: 6 votes to 4 votes: Hawaii gets 1 point
 Anaheim vs Orlando: 7 votes to 3 votes: Anaheim gets 1 point
 Hawaii vs Anaheim: 6 votes to 4 votes: Hawaii gets 1 point

Hawaii is the winner under Copeland's Method, having earned the most points.

Notice this process is consistent with our determination of a Condorcet Winner.

Example 10

Consider the advertising group's vote we explored earlier. Determine the winner using Copeland's method.



Solution

With 5 candidates, there are 10 comparisons to make:

A vs B: 11 votes to 9 votes	A gets 1 point
A vs C: 3 votes to 17 votes	C gets 1 point
A vs D: 10 votes to 10 votes	A gets $\frac{1}{2}$ point, D gets $\frac{1}{2}$ point
A vs E: 17 votes to 3 votes	A gets 1 point
B vs C: 10 votes to 10 votes	B gets $\frac{1}{2}$ point, C gets $\frac{1}{2}$ point
B vs D: 9 votes to 11 votes	D gets 1 point
B vs E: 13 votes to 7 votes	B gets 1 point
C vs D: 9 votes to 11 votes	D gets 1 point
C vs E: 17 votes to 3 votes	C gets 1 point
D vs E: 17 votes to 3 votes	D gets 1 point

Totaling these up:

A gets $2\frac{1}{2}$ points

B gets $1\frac{1}{2}$ points

C gets $2\frac{1}{2}$ points

D gets $3\frac{1}{2}$ points

E gets 0 points

Using Copeland's Method, we declare D as the winner.

Notice that in this case, D is not a Condorcet Winner. While Copeland's method will also select a Condorcet Candidate as the winner, the method still works in cases where there is no Condorcet Winner.

Try it Now 5

Consider again the election from Try it Now 1. Find the winner using Copeland's method. Since we have some incomplete preference ballots, we'll have to adjust. For example, when comparing M to B, we'll ignore the 20 votes in the third column which do not rank either candidate.



Answer available at the end of this section.

Answers

Try it Now 5

Using Copeland's Method:

Looking back at our work from Try it Now #2, we see

G vs M: $44 + 14 + 20 = 78$ prefer G, $70 + 22 + 80 = 172$ prefer M: M preferred – 1 point

G vs B: $44 + 14 + 20 + 70 = 148$ prefer G, $22 + 80 + 39 = 141$ prefer B: G preferred – 1 point

M vs B: $44 + 70 + 22 = 136$ prefer M, $14 + 80 + 39 = 133$ prefer B: M preferred – 1 point

M earns 2 points; G earns 1 point. M wins under Copeland's method.

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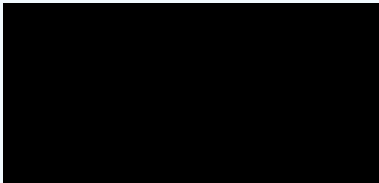
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2.1.11: What's Wrong with Copeland's Method?

As already noted, Copeland's Method does satisfy the Condorcet Criterion. It also satisfies the Majority Criterion and the Monotonicity Criterion. So is this the perfect method? Well, in a word, no.

Example 11

A committee is trying to award a scholarship to one of four students, Anna (A), Brian (B), Carlos (C), and Dimitry (D). The votes are shown below:



Solution

Making the comparisons:

A vs B: 10 votes to 10 votes A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
 A vs C: 14 votes to 6 votes: A gets 1 point
 A vs D: 5 votes to 15 votes: D gets 1 point
 B vs C: 4 votes to 16 votes: C gets 1 point
 B vs D: 15 votes to 5 votes: B gets 1 point
 C vs D: 11 votes to 9 votes: C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points B has $1\frac{1}{2}$ points
 C has 2 points D has 1 point

So Carlos is awarded the scholarship. However, the committee then discovers that Dimitry was not eligible for the scholarship (he failed his last math class). Even though this seems like it shouldn't affect the outcome, the committee decides to recount the vote, removing Dimitry from consideration. This reduces the preference schedule to:



A vs B: 10 votes to 10 votes A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
 A vs C: 14 votes to 6 votes: A gets 1 point
 B vs C: 4 votes to 16 votes: C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points B has $\frac{1}{2}$ point
 C has 1 point

Suddenly Anna is the winner! This leads us to another fairness criterion.

The Independence of Irrelevant Alternatives (IIA) Criterion

If a non-winning choice is removed from the ballot, it should not change the winner of the election.

Equivalently, if choice A is preferred over choice B, introducing or removing a choice C should not cause B to be preferred over A.

In the election from Example 11, the IIA Criterion was violated.

This anecdote illustrating the IIA issue is attributed to Sidney Morgenbesser:

After finishing dinner, Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point Morgenbesser says "In that case I'll have the blueberry pie."

Another disadvantage of Copeland's Method is that it is fairly easy for the election to end in a tie. For this reason, Copeland's method is usually the first part of a more advanced method that uses more sophisticated methods for breaking ties and determining the winner when there is not a Condorcet Candidate.

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2.1.12: So Where's the Fair Method?

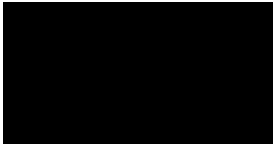
At this point, you're probably asking why we keep looking at method after method just to point out that they are not fully fair. We must be holding out on the perfect method, right?

Unfortunately, no. A mathematical economist, Kenneth Arrow, was able to prove in 1949 that there is no voting method that will satisfy all the fairness criteria we have discussed.

Arrow's Impossibility Theorem

Arrow's Impossibility Theorem states, roughly, that it is not possible for a voting method to satisfy every fairness criteria that we've discussed.

To see a very simple example of how difficult voting can be, consider the election below:



Notice that in this election:

- 10 people prefer A to B
- 10 people prefer B to C
- 10 people prefer C to A

No matter whom we choose as the winner, 2/3 of voters would prefer someone else! This scenario is dubbed **Condorcet's Voting Paradox**, and demonstrates how voting preferences are not transitive (just because A is preferred over B, and B over C, does not mean A is preferred over C). In this election, there is no fair resolution.

It is because of this impossibility of a totally fair method that Plurality, IRV, Borda Count, Copeland's Method, and dozens of variants are all still used. Usually the decision of which method to use is based on what seems most fair for the situation in which it is being applied.

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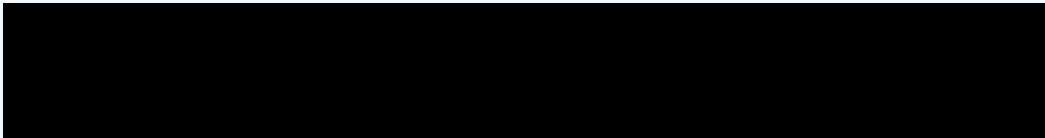
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2.1.13: Approval Voting

Up until now, we've been considering voting methods that require ranking of candidates on a preference ballot. There is another method of voting that can be more appropriate in some decision making scenarios. With Approval Voting, the ballot asks you to mark all choices that you find acceptable. The results are tallied, and the option with the most approval is the winner.

Example 12

A group of friends is trying to decide upon a movie to watch. Three choices are provided, and each person is asked to mark with an "X" which movies they are willing to watch. The results are:



Solution

Totaling the results, we find

Titanic received 5 approvals

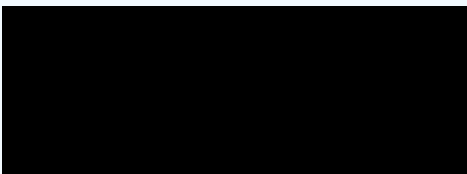
Scream received 6 approvals

The Matrix received 7 approvals.

In this vote, The Matrix would be the winner.

Try it Now 6

Our mathematicians deciding on a conference location from earlier decide to use Approval voting. Their votes are tallied below. Find the winner using Approval voting.



Answer available at the end of this section.

Answers

Try it Now 6

Using Approval voting:

Seattle has $30 + 10 + 15 + 5 = 60$ approval votes

Tacoma has $30 + 15 + 20 + 15 + 5 = 85$ approval votes

Puyallup has $10 + 20 + 25 + 5 = 50$ approval votes

Olympia has $15 + 15 + 5 = 35$ approval votes

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2.1.14: What's Wrong with Approval Voting?

Approval voting can very easily violate the Majority Criterion.

Example 13

Consider the voting schedule:



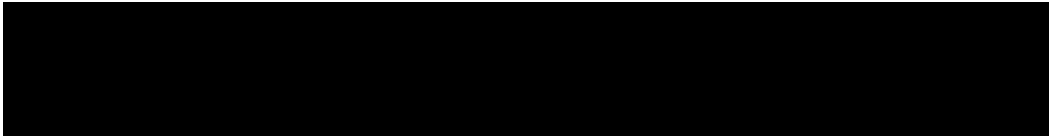
Solution

Clearly A is the majority winner. Now suppose that this election was held using Approval Voting, and every voter marked approval of their top two candidates.

- A would receive approval from 80 voters
- B would receive approval from 100 voters
- C would receive approval from 20 voters

B would be the winner. Some argue that Approval Voting tends to vote the least disliked choice, rather than the most liked candidate.

Additionally, Approval Voting is susceptible to strategic insincere voting, in which a voter does not vote their true preference to try to increase the chances of their choice winning. For example, in the movie example above, suppose Bob and Alice would much rather watch Scream. They remove The Matrix from their approval list, resulting in a different result.



Totaling the results, we find Titanic received 5 approvals, Scream received 6 approvals, and The Matrix received 5 approvals. By voting insincerely, Bob and Alice were able to sway the result in favor of their preference.

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2.1.15: Voting in America

In American politics, there is a lot more to selecting our representatives than simply casting and counting ballots. The process of selecting the president is even more complicated, so we'll save that for the next chapter. Instead, let's look at the process by which state congressional representatives and local politicians get elected.

For most offices, a sequence of two public votes is held: a primary election and the general election. For non-partisan offices like sheriff and judge, in which political party affiliation is not declared, the primary election is usually used to narrow the field of candidates.

Typically, the two candidates receiving the most votes in the primary will then move forward to the general election. While somewhat similar to instant runoff voting, this is actually an example of **sequential voting** - a process in which voters cast totally new ballots after each round of eliminations. Sequential voting has become quite common in television, where it is used in reality competition shows like American Idol.

Congressional, county, and city representatives are partisan offices, in which candidates usually declare themselves a member of a political party, like the Democrats, Republicans, the Green Party, or one of the many other smaller parties. As with non-partisan offices, a primary election is usually held to narrow down the field prior to the general election. Prior to the primary election, the candidate would have met with the political party leaders and gotten their approval to run under that party's affiliation.

In some states a **closed primary** is used, in which only voters who are members of the Democrat party can vote on the Democratic candidates, and similar for Republican voters. In other states, an **open primary** is used, in which any voter can pick the party whose primary they want to vote in. In other states, **caucuses** are used, which are basically meetings of the political parties, only open to party members. Closed primaries are often disliked by independent voters, who like the flexibility to change which party they are voting in. Open primaries do have the disadvantage that they allow raiding, in which a voter will vote in their non-preferred party's primary with the intent of selecting a weaker opponent for their preferred party's candidate.

Washington State currently uses a different method, called a **top 2 primary**, in which voters select from the candidates from all political parties on the primary, and the top two candidates, regardless of party affiliation, move on to the general election. While this method is liked by independent voters, it gives the political parties incentive to select a top candidate internally before the primary, so that two candidates will not split the party's vote.

Regardless of the primary type, the general election is the main election, open to all voters. Except in the case of the top 2 primary, the top candidate from each major political party would be included in the general election. While rules vary state-to-state, for an independent or minor party candidate to get listed on the ballot, they typically have to gather a certain number of signatures to petition for inclusion.

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2.1.16: Exercises

1. To decide on a new website design, the designer asks people to rank three designs that have been created (labeled A, B, and C). The individual ballots are shown below. Create a preference table.

ABC, ABC, ACB, BAC, BCA, BCA, ACB, CAB, CAB, BCA, ACB, ABC

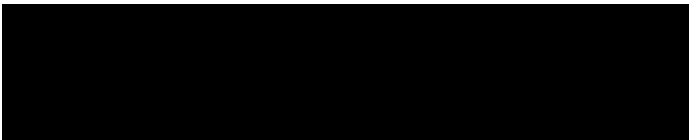
2. To decide on a movie to watch, a group of friends all vote for one of the choices (labeled A, B, and C). The individual ballots are shown below. Create a preference table.

CAB, CBA, BAC, BCA, CBA, ABC, ABC, CBA, BCA, CAB, CAB, BAC

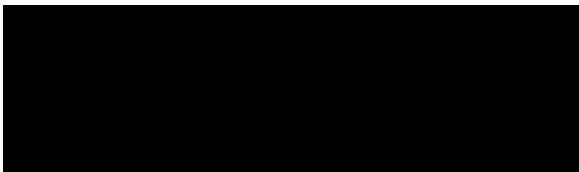
3. The planning committee for a renewable energy trade show is trying to decide what city to hold their next show in. The votes are shown below.



- a. How many voters voted in this election?
 - b. How many votes are needed for a majority? A plurality?
 - c. Find the winner under the plurality method.
 - d. Find the winner under the Borda Count Method.
 - e. Find the winner under the Instant Runoff Voting method.
 - f. Find the winner under Copeland's method.
4. A non-profit agency is electing a new chair of the board. The votes are shown below.



- a. How many voters voted in this election?
 - b. How many votes are needed for a majority? A plurality?
 - c. Find the winner under the plurality method.
 - d. Find the winner under the Borda Count Method.
 - e. Find the winner under the Instant Runoff Voting method.
 - f. Find the winner under Copeland's method.
5. The student government is holding elections for president. There are four candidates (labeled A, B, C, and D for convenience). The preference schedule for the election is:

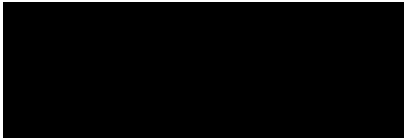


- a. How many voters voted in this election?
- b. How many votes are needed for a majority? A plurality?
- c. Find the winner under the plurality method.
- d. Find the winner under the Borda Count Method.
- e. Find the winner under the Instant Runoff Voting method.

- f. Find the winner under Copeland's method.
6. The homeowners association is deciding a new set of neighborhood standards for architecture, yard maintenance, etc. Four options have been proposed. The votes are:



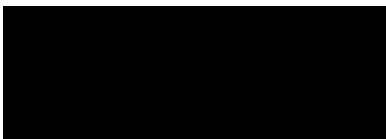
- How many voters voted in this election?
 - How many votes are needed for a majority? A plurality?
 - Find the winner under the plurality method.
 - Find the winner under the Borda Count Method.
 - Find the winner under the Instant Runoff Voting method.
 - Find the winner under Copeland's method.
7. Consider an election with 129 votes.
- If there are 4 candidates, what is the smallest number of votes that a plurality candidate could have?
 - If there are 8 candidates, what is the smallest number of votes that a plurality candidate could have?
8. Consider an election with 953 votes.
- If there are 7 candidates, what is the smallest number of votes that a plurality candidate could have?
 - If there are 8 candidates, what is the smallest number of votes that a plurality candidate could have?
9. Does this voting system having a Condorcet Candidate? If so, find it.



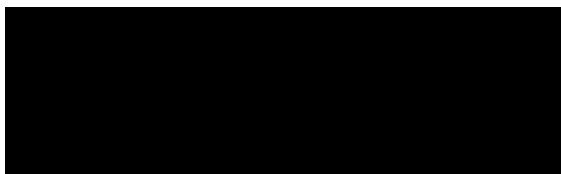
10. Does this voting system having a Condorcet Candidate? If so, find it.

Number of voters	8	7	6
1st choice	A	C	B
2nd choice	B	B	C
3rd choice	C	A	A

11. The marketing committee at a company decides to vote on a new company logo. They decide to use approval voting. Their results are tallied below. Each column shows the number of voters with the particular approval vote. Which logo wins under approval voting?



12. The downtown business association is electing a new chairperson, and decides to use approval voting. The tally is below, where each column shows the number of voters with the particular approval vote. Which candidate wins under approval voting?



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2.1.17: Concepts

1. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If for some reason the election had to be held again and C decided to drop out of the election, which caused B to become the winner, which is the primary fairness criterion violated in this election?
2. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If for some reason the election had to be held again and many people who had voted for C switched their preferences to favor A, which caused B to become the winner, which is the primary fairness criterion violated in this election?
3. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If in a head-to-head comparison a majority of people prefer B to A or C, which is the primary fairness criterion violated in this election?
4. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If B had received a majority of first place votes, which is the primary fairness criterion violated in this election?

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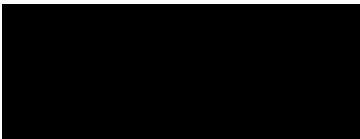
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2.1.18: Exploration

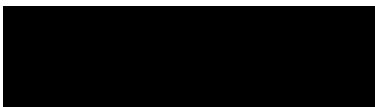
1. In the election shown below under the Plurality method, explain why voters in the third column might be inclined to vote insincerely. How could it affect the outcome of the election?



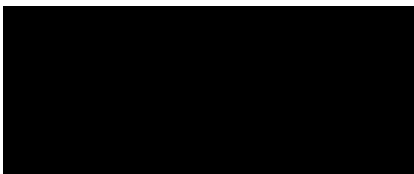
2. In the election shown below under the Borda Count method, explain why voters in the second column might be inclined to vote insincerely. How could it affect the outcome of the election?



3. Compare and contrast the motives of the insincere voters in the two questions above.
4. Consider a two party election with preferences shown below. Suppose a third candidate, C, entered the race, and a segment of voters sincerely voted for that third candidate, producing the preference schedule from #17 above. Explain how other voters might perceive candidate C.



5. In question 18, we showed that the outcome of Borda Count can be manipulated if a group of individuals change their vote.
 - a. Show that it is possible for a single voter to change the outcome under Borda Count if there are four candidates.
 - b. Show that it is not possible for a single voter to change the outcome under Borda Count if there are three candidates.
6. Show that when there is a Condorcet winner in an election, it is impossible for a single voter to manipulate the vote to help a different candidate become a Condorcet winner.
7. The Pareto criterion is another fairness criterion that states: *If every voter prefers choice A to choice B, then B should not be the winner.* Explain why plurality, instant runoff, Borda count, and Copeland's method all satisfy the Pareto condition.
8. Sequential Pairwise voting is a method not commonly used for political elections, but sometimes used for shopping and games of pool. In this method, the choices are assigned an order of comparison, called an agenda. The first two choices are compared. The winner is then compared to the next choice on the agenda, and this continues until all choices have been compared against the winner of the previous comparison.
 - a. Using the preference schedule below, apply Sequential Pairwise voting to determine the winner, using the agenda: A, B, C, D.



- b. Show that Sequential Pairwise voting can violate the Pareto criterion.
 - c. Show that Sequential Pairwise voting can violate the Majority criterion.
9. The Coombs method is a variation of instant runoff voting. In Coombs method, the choice with the most last place votes is

eliminated. Apply Coombs method to the preference schedules from questions 5 and 6.

10. Copeland's Method is designed to identify a Condorcet Candidate if there is one, and is considered a Condorcet Method. There are many Condorcet Methods, which vary primarily in how they deal with ties, which are very common when a Condorcet winner does not exist. Copeland's method does not have a tie-breaking procedure built-in. Research the Schulze method, another Condorcet method that is used by the Wikimedia foundation that runs Wikipedia, and give some examples of how it works.
11. The plurality method is used in most U.S. elections. Some people feel that Ross Perot in 1992 and Ralph Nader in 2000 changed what the outcome of the election would have been if they had not run. Research the outcomes of these elections and explain how each candidate could have affected the outcome of the elections (for the 2000 election, you may wish to focus on the count in Florida). Describe how an alternative voting method could have avoided this issue.
12. Instant Runoff Voting and Approval voting have supporters advocating that they be adopted in the United States and elsewhere to decide elections. Research comparisons between the two methods describing the advantages and disadvantages of each in practice. Summarize the comparisons, and form your own opinion about whether either method should be adopted.
13. In a primary system, a first vote is held with multiple candidates. In some states, each political party has its own primary. In Washington State, there is a "top two" primary, where all candidates are on the ballot and the top two candidates advance to the general election, regardless of party. Compare and contrast the top two primary with general election system to instant runoff voting, considering both differences in the methods, and practical differences like cost, campaigning, fairness, etc.
14. In a primary system, a first vote is held with multiple candidates. In some many states, where voters must declare a party to vote in the primary election, and they are only able to choose between candidates for their declared party. The top candidate from each party then advances to the general election. Compare and contrast this primary with general election system to instant runoff voting, considering both differences in the methods, and practical differences like cost, campaigning, fairness, etc.
15. Sometimes in a voting scenario it is desirable to rank the candidates, either to establish preference order between a set of choices, or because the election requires multiple winners. For example, a hiring committee may have 30 candidates apply, and need to select 6 to interview, so the voting by the committee would need to produce the top 6 candidates. Describe how Plurality, Instant Runoff Voting, Borda Count, and Copeland's Method could be extended to produce a ranked list of candidates.

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2.2: APPORTIONMENT

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2.2.1: Introduction

Apportionment is the problem of dividing up a fixed number of things among groups of different sizes. In politics, this takes the form of allocating a limited number of representatives amongst voters. This problem, presumably, is older than the United States, but the best-known ways to solve it have their origins in the problem of assigning each state an appropriate number of representatives in the new Congress when the country was formed. States also face this apportionment problem in defining how to draw districts for state representatives. The apportionment problem comes up in a variety of non-political areas too, though. We face several restrictions in this process:

Apportionment Rules

1. The things being divided up can exist only in whole numbers.
2. We must use all of the things being divided up, and we cannot use any more.
3. Each group must get at least one of the things being divided up.
4. The number of things assigned to each group should be at least approximately proportional to the population of the group. (Exact proportionality isn't possible because of the whole number requirement, but we should try to be close, and in any case, if Group A is larger than Group B, then Group B shouldn't get more of the things than Group A does.)

In terms of the apportionment of the United States House of Representatives, these rules imply:

1. We can only have whole representatives (a state can't have 3.4 representatives)
2. We can only use the (currently) 435 representatives available. If one state gets another representative, another state has to lose one.
3. Every state gets at least one representative
4. The number of representatives each state gets should be approximately proportional to the state population. This way, the number of constituents each representative has should be approximately equal.

We will look at four ways of solving the apportionment problem. Three of them (Lowndes's method is the exception) have been used at various times to apportion the U.S. Congress, although the method currently in use (the Huntington-Hill method) is significantly more complicated.

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2.2.2: Hamilton's Method

Alexander Hamilton proposed the method that now bears his name. His method was approved by Congress in 1791, but was vetoed by President Washington. It was later adopted in 1852 and used through 1911. He begins by determining, to several decimal places, how many things each group should get. Since he was interested in the question of Congressional representation, we'll use the language of states and representatives, so he determines how many representatives each state should get. He follows these steps:

Hamilton's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**. Since we can only allocate whole representatives, Hamilton resolves the whole number problem, as follows:
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). These are called the **lower quotas**. Add up the remaining whole numbers. This answer will always be less than or equal to the total number of representatives (and the "or equal to" part happens only in very specific circumstances that are incredibly unlikely to turn up).
4. Assuming that the total from Step 3 was less than the total number of representatives, assign the remaining representatives, one each, to the states whose decimal parts of the quota were largest, until the desired total is reached.

Make sure that each state ends up with at least one representative!

Note on rounding: Today we have technological advantages that Hamilton (and the others) couldn't even have imagined. Take advantage of them, and keep several decimal places.

Example 1

The state of Delaware has three counties: Kent, New Castle, and Sussex. The Delaware state House of Representatives has 41 members. If Delaware wants to divide this representation along county lines (which is *not* required, but let's pretend they do), let's use Hamilton's method to apportion them. The populations of the counties are as follows (from the 2010 Census):

County	Population
Kent	162,310
New Castle	538,479
Sussex	197,145
Total	897,934

Solution

1. First, we determine the divisor: $897,934/41 = 21,900.82927$
2. Now we determine each county's quota by dividing the county's population by the divisor:

County	Population	Quota
Kent	162,310	7.4111
New Castle	538,479	24.5872
Sussex	197,145	9.0017
Total	897,934	

3. Removing the decimal parts of the quotas gives:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	897,934		40

4. We need 41 representatives and this only gives 40. The remaining one goes to the county with the largest decimal part, which is New Castle:

County	Population	Quota	Initial	Final
Kent	162,310	7.4111	7	7
New Castle	538,479	24.5872	24	25
Sussex	197,145	9.0017	9	9
Total	897,934		40	41

Example 2

Use Hamilton's method to apportion the 75 seats of Rhode Island's House of Representatives among its five counties.

County	Population
Bristol	49,875
Kent	166,158
Newport	82,888
Providence	626,667
Washington	126,979
Total	1,052,567

Solution

1. The divisor is $1,052,567/75 = 14,034.22667$

2. Determine each county's quota by dividing its population by the divisor:

County	Population	Quota
Bristol	49,875	3.5538
Kent	166,158	11.8395
Newport	82,888	5.9061
Providence	626,667	44.6528
Washington	126,979	9.0478
Total	1,052,567	

3. Remove the decimal part of each quota:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	1,052,567		72

4. We need 75 representatives and we only have 72, so we assign the remaining three, one each, to the three counties with the largest decimal parts, which are Newport, Kent, and Providence:

County	Population	Quota	Initial	Final
Bristol	49,875	3.5538	3	3
Kent	166,158	11.8395	11	12
Newport	82,888	5.9061	5	6
Providence	626,667	44.6528	44	45
Washington	126,979	9.0478	9	9
Total	1,052,567		72	75

Note that even though Bristol County's decimal part is greater than .5, it isn't big enough to get an additional representative, because three other counties have greater decimal parts.

Hamilton's method obeys something called the Quota Rule. The Quota Rule isn't a law of any sort, but just an idea that some people, including Hamilton, think is a good one.

Quota Rule

The Quota Rule says that the final number of representatives a state gets should be within one of that state's quota. Since we're dealing with whole numbers for our final answers, that means that each state should either go up to the next whole number above its quota, or down to the next whole number below its quota.

Controversy

After seeing Hamilton's method, many people find that it makes sense, it's not that difficult to use (or, at least, the difficulty comes from the numbers that are involved and the amount of computation that's needed, not from the method), and they wonder why anyone would want another method. The problem is that Hamilton's method is subject to several paradoxes. Three of them happened, on separate occasions, when Hamilton's method was used to apportion the United States House of Representatives.

The **Alabama Paradox** is named for an incident that happened during the apportionment that took place after the 1880 census. (A similar incident happened ten years earlier involving the state of Rhode Island, but the paradox is named after Alabama.) The post-1880 apportionment had been completed, using Hamilton's method and the new population numbers from the census. Then it was decided that because of the country's growing population, the House of Representatives should be made larger. That meant that the apportionment would need to be done again, still using Hamilton's method and the same 1880 census numbers, but with more representatives. The assumption was that some states would gain another representative and others would stay with the same number they already had (since there weren't enough new representatives being added to give one more to every state). The paradox is that Alabama ended up *losing* a representative in the process, even though no populations were changed and the total number of representatives increased.

The **New States Paradox** happened when Oklahoma became a state in 1907. Oklahoma had enough population to qualify for five representatives in Congress, so they raised the total number of representatives by five. When the apportionment was recalculated, unsurprisingly Oklahoma received five representatives. However, despite their populations not changing, Maine gained a representative and New York lost one.

The **Population Paradox** happened between the apportionments after the census of 1900 and of 1910. In those ten years, Virginia's population grew at an average annual rate of 1.07%, while Maine's grew at an average annual rate of 0.67%. Virginia started with more people, grew at a faster rate, grew by more people, and ended up with more people than Maine. By itself, that doesn't mean that Virginia should gain representatives or Maine shouldn't, because there are lots of other states involved. But Virginia ended up losing a representative to Maine.

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2.2.3: Jefferson's Method

Thomas Jefferson proposed a different method for apportionment. After Washington vetoed Hamilton's method, Jefferson's method was adopted, and used in Congress from 1791 through 1842. Jefferson, of course, had political reasons for wanting his method to be used rather than Hamilton's. Primarily, his method favors larger states, and his own home state of Virginia was the largest in the country at the time. He would also argue that it's the ratio of people to representatives that is the critical thing, and apportionment methods should be based on that. But the paradoxes we saw also provide mathematical reasons for concluding that Hamilton's method isn't so good, and while Jefferson's method might or might not be the best one to replace it, at least we should look for other possibilities.

The first steps of Jefferson's method are the same as Hamilton's method. He finds the same divisor and the same quota, and cuts off the decimal parts in the same way, giving a total number of representatives that is less than the required total. The difference is in how Jefferson resolves that difference. He says that since we ended up with an answer that is too small, our divisor must have been too big. He changes the divisor by making it smaller, finding new quotas with the new divisor, cutting off the decimal parts, and looking at the new total, until we find a divisor that produces the required total.

Jefferson's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). These are the **lower quotas**. Add up the remaining whole numbers. This answer will always be less than or equal to the total number of representatives.
4. If the total from Step 3 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 3 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 3

We'll return to Delaware and apply Jefferson's method. We begin, as we did with Hamilton's method, by finding the quotas with the original divisor, 21,900.82927:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
Newport	538,479	24.5872	24
Providence	197,145	9.0017	9
Total	897,934		40

Solution

We need 41 representatives, and this divisor gives only 40. We must reduce the divisor until we get 41 representatives. Let's try 21,500 as the divisor:

County	Population	Quota	Initial
Kent	162,310	7.5493	7
Newport	538,479	25.0455	25
Providence	197,145	9.1695	9
Total	897,934		41

This gives us the required 41 representatives, so we're done. If we had fewer than 41, we'd need to reduce the divisor more. If we had more than 41, we'd need to choose a divisor less than the original but greater than the second choice.

Notice that with the new, lower divisor, the quota for New Castle County (the largest county in the state) increased by much more than those of Kent County or Sussex County.

Example 4

We'll apply Jefferson's method for Rhode Island. The original divisor of 14,034.22667 gave these results:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	1,052,567		72

Solution

We need 75 representatives and we only have 72, so we need to use a smaller divisor. Let's try 13,500:

County	Population	Quota	Initial
Bristol	49,875	3.6944	3
Kent	166,158	12.3080	12
Newport	82,888	6.1399	6
Providence	626,667	46.4198	46
Washington	126,979	9.4059	9
Total	1,052,567		76

We've gone too far. We need a divisor that's greater than 13,500 but less than 14,034.22667. Let's try 13,700:

County	Population	Quota	Initial
Bristol	49,875	3.6405	3
Kent	166,158	12.1283	12
Newport	82,888	6.0502	6
Providence	626,667	45.7421	45
Washington	126,979	9.2685	9
Total	1,052,567		75

This works.

Notice, in comparison to Hamilton's method, that although the results were the same, they came about in a different way, and the outcome was almost different. Providence County (the largest) almost went up to 46 representatives before Kent (which is much smaller) got to 12. Although that didn't happen here, it can. Divisor-adjusting methods like Jefferson's are not guaranteed to follow the quota rule!

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2.2.4: Webster's Method

Daniel Webster (1782-1852) proposed a method similar to Jefferson's in 1832. It was adopted by Congress in 1842, but replaced by Hamilton's method in 1852. It was then adopted again in 1901. The difference is that Webster rounds the quotas to the nearest whole number rather than dropping the decimal parts. If that doesn't produce the desired results at the beginning, he says, like Jefferson, to adjust the divisor until it does. (In Jefferson's case, at least the first adjustment will always be to make the divisor smaller. That is not always the case with Webster's method.)

Webster's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Round all the quotas to the nearest whole number (but don't forget what the decimals were). Add up the remaining whole numbers.
4. If the total from Step 3 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. If the total from step 3 was larger than the total number of representatives, increase the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 3 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 5

Again, Delaware, with an initial divisor of 21,900.82927:

County	Population	Quota	Initial
Kent	166,310	7.4111	7
New Castle	538,479	24.5872	25
Sussex	197,145	9.0017	9
Total	897,934		41

Solution

This gives the required total, so we're done.

Example 6

Again, Rhode Island, with an initial divisor of 14,034.22667:

County	Population	Quota	Initial
Bristol	49,875	3.5538	4
Kent	166,158	11.8395	12
Newport	82,888	5.9061	6
Providence	626,667	44.6528	45
Washington	126,979	9.0478	9
Total	1,052,567		76

Solution

This is too many, so we need to increase the divisor. Let's try 14,100:

County	Population	Quota	Initial
Bristol	49,875	3.5372	4
Kent	166,158	11.7843	12
Newport	82,888	5.8786	6
Providence	626,667	5.8786	44
Washington	126,979	9.0056	9
Total	1,052,567		75

This works, so we're done.

Like Jefferson's method, Webster's method carries a bias in favor of states with large populations, but rounding the quotas to the nearest whole number greatly reduces this bias. (Notice that Providence County, the largest, is the one that gets a representative trimmed because of the increased quota.) Also like Jefferson's method, Webster's method does not always follow the quota rule, but it follows the quota rule much more often than Jefferson's method does. (In fact, if Webster's method had been applied to every apportionment of Congress in all of American history, it would have followed the quota rule every single time.)

In 1980, two mathematicians, Peyton Young and Mike Balinski, proved what we now call the Balinski-Young Impossibility Theorem.

Balinski-Young Impossibility Theorem

The Balinski-Young Impossibility Theorem shows that any apportionment method which always follows the quota rule will be subject to the possibility of paradoxes like the Alabama, New States, or Population paradoxes. In other words, we can choose a method that avoids those paradoxes, but only if we are willing to give up the guarantee of following the quota rule.

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2.2.5: Huntington-Hill Method

In 1920, no new apportionment was done, because Congress couldn't agree on the method to be used. They appointed a committee of mathematicians to investigate, and they recommended the Huntington-Hill Method. They continued to use Webster's method in 1931, but after a second report recommending Huntington-Hill, it was adopted in 1941 and is the current method of apportionment used in Congress.

The Huntington-Hill Method is similar to Webster's method, but attempts to minimize the percent differences of how many people each representative will represent.

Huntington-Hill Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off the decimal part of the quota to obtain the lower quota, which we'll call n . Compute $\sqrt{n(n+1)}$, which is the **geometric mean** of the lower quota and one value higher.
4. If the quota is larger than the geometric mean, round up the quota; if the quota is smaller than the geometric mean, round down the quota. Add up the resulting whole numbers to get the **initial allocation**.
5. If the total from Step 4 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. If the total from step 4 was larger than the total number of representatives, increase the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 4 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 7

Again, Delaware, with an initial divisor of 21,900.82927:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Kent	162,310	7.4111	7	7.48	7
New Castle	538,479	24.5872	24	24.49	25
Sussex	197,145	9.0017	9	9.49	9
Total	897,934				41

Solution

This gives the required total, so we're done.

Example 8

Again, Rhode Island, with an initial divisor of 14,034.22667:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Bristol	49,875	3.5538	3	3.46	4
Kent	166,158	11.8395	11	11.49	12
Newport	82,888	5.9061	5	5.48	6
Providence	626,667	44.6528	44	44.50	45
Washington	126,979	9.0478	9	9.49	9
Total	1,052,567				76

Solution

This is too many, so we need to increase the divisor. Let's try 14,100:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Bristol	49,875	3.5372	3	3.46	4
Kent	166,158	11.7843	11	11.49	12
Newport	82,888	5.8786	5	5.48	6
Providence	626,667	44.4445	44	44.50	44
Washington	126,979	9.0056	9	9.49	9
Total	1,052,567				75

This works, so we're done.

In both these cases, the apportionment produced by the Huntington-Hill method was the same as those from Webster's method.

Example 9

Consider a small country with 5 states, two of which are much larger than the others. We need to apportion 70 representatives. We will apportion using both Webster's method and the Huntington-Hill method.

State	Population
A	300,500
B	200,000
C	50,000
D	38,000
E	21,500

Solution

1. The total population is 610,000. Dividing this by the 70 representatives gives the divisor: 8714.286.
2. Dividing each state's population by the divisor gives the quotas.

State	Population	Quota
A	300,500	34.48361
B	200,000	22.95082
C	50,000	5.737705
D	38,000	4.360656
E	21,500	2.467213

Webster's Method

3. Using Webster's method, we round each quota to the nearest whole number

State	Population	Quota	Initial
A	300,500	34.48361	34
B	200,000	22.95082	23
C	50,000	5.737705	6
D	38,000	4.360656	4
E	21,500	2.467213	2

4. Adding these up, they only total 69 representatives, so we adjust the divisor down. Adjusting the divisor down to 8700 gives an updated allocation totaling 70 representatives

State	Population	Quota	Initial
A	300,500	34.54023	35
B	200,000	22.98851	23
C	50,000	5.747126	6
D	38,000	4.367816	4
E	21,500	2.471264	2

Huntington-Hill Method

3. Using the Huntington-Hill method, we round down to find the lower quota, then calculate the geometric mean based on each lower quota. If the quota is less than the geometric mean, we round down; if the quota is more than the geometric mean, we round up.

State	Population	Quota	Lower Quota	Geometric Mean	Initial
A	300,500	34.48361	34	34.49638	34
B	200,000	22.95082	22	22.49444	23
C	50,000	5.737705	5	5.477226	6
D	38,000	4.360656	4	4.472136	4
E	21,500	2.467213	2	2.44949	3

These allocations add up to 70, so we're done.

Notice that this allocation is different than that produced by Webster's method. In this case, state E got the extra seat instead of state A.

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2.2.6: Lowndes' Method

William Lowndes (1782-1822) was a Congressman from South Carolina (a small state) who proposed a method of apportionment that was more favorable to smaller states. Unlike the methods of Hamilton, Jefferson, and Webster, Lowndes's method has never been used to apportion Congress.

Lowndes believed that an additional representative was much more valuable to a small state than to a large one. If a state already has 20 or 30 representatives, getting one more doesn't matter very much. But if it only has 2 or 3, one more is a big deal, and he felt that the additional representatives should go where they could make the most difference.

Like Hamilton's method, Lowndes's method follows the quota rule. In fact, it arrives at the same quotas as Hamilton and the rest, and like Hamilton and Jefferson, it drops the decimal parts. But in deciding where the remaining representatives should go, we divide the decimal part of each state's quota by the whole number part (so that the same decimal part with a smaller whole number is worth more, because it matters more to that state).

Lowndes's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). Add up the remaining whole numbers.
4. Assuming that the total from Step 3 was less than the total number of representatives, divide the decimal part of each state's quota by the whole number part. Assign the remaining representatives, one each, to the states whose **ratio** of decimal part to whole part were largest, until the desired total is reached.

Example 10

We'll do Delaware again. We begin in the same way as with Hamilton's method:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	897,934		40

Solution

We need one more representative. To find out which county should get it, Lowndes says to divide each county's decimal part by its whole number part, with the largest result getting the extra representative:

Kent: $0.4111/7 \approx 0.0587$
 New Castle: $0.5872/24 \approx 0.0245$
 Sussex: $0.0017/9 \approx 0.0002$

The largest of these is Kent's, so Kent gets the 41th representative:

County	Population	Quota	Initial	Ratio	Final
Kent	162,310	7.4111	7	0.0587	8
New Castle	538,479	24.5872	24	0.0245	24
Sussex	197,145	9.0017	9	0.0002	9
Total	897,934		40		41

Example 11

Rhode Island, again beginning in the same way as Hamilton:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	1,052,567		72

Solution

We divide each county's quota's decimal part by its whole number part to determine which three should get the remaining representatives:

Bristol:	$0.5538/3 \approx 0.1846$
Kent:	$0.8395/11 \approx 0.0763$
Newport:	$0.9061/5 \approx 0.1812$
Providence:	$0.6528/44 \approx 0.0148$
Washington:	$0.0478/9 \approx 0.0053$

The three largest of these are Bristol, Newport, and Kent, so they get the remaining three representatives:

County	Population	Quota	Initial	Ratio	Final
Bristol	49,875	3.5538	3	0.1846	4
Kent	166,158	11.8395	11	0.0763	12
Newport	82,888	5.9061	5	0.1812	6
Providence	626,667	44.6528	44	0.0148	44
Washington	126,979	9.0478	9	0.0053	9
Total	1,052,567		72		75

As you can see, there is no "right answer" when it comes to choosing a method for apportionment. Each method has its virtues, and favors different sized states.

2.2.7: Apportionment of Legislative Districts

In most states, there are a fixed number of representatives to the state legislature. Rather than apportioning each county a number of representatives, legislative districts are drawn so that each legislator represents a district. The apportionment process, then, comes in the drawing of the legislative districts, with the goal of having each district include approximately the same number of constituents. Because of this goal, a geographically small city may have several representatives, while a large rural region may be represented by one legislator.

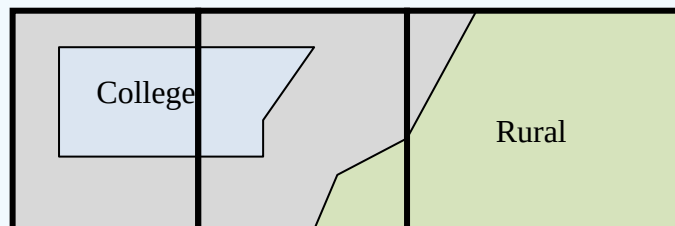
When populations change, it becomes necessary to redistrict the regions each legislator represents (Incidentally, this also occurs for the regions that federal legislators represent). The process of redistricting is typically done by the legislature itself, so not surprisingly it is common to see **gerrymandering**.

Gerrymandering

Gerrymandering is when districts are drawn based on the political affiliation of the constituents to the advantage of those drawing the boundary.

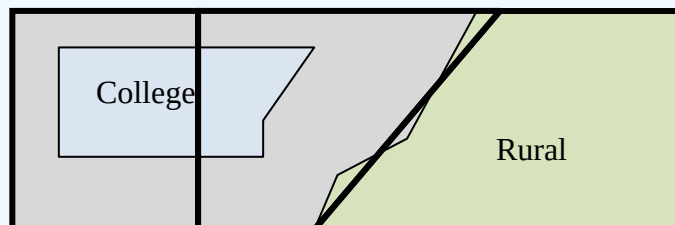
Example 12

Consider three districts, simplified to the three boxes below. On the left there is a college area that typically votes Democratic. On the right is a rural area that typically votes Republican. The rest of the people are more evenly split. The middle district has been voting 50% Democratic and 50% Republican.



Solution

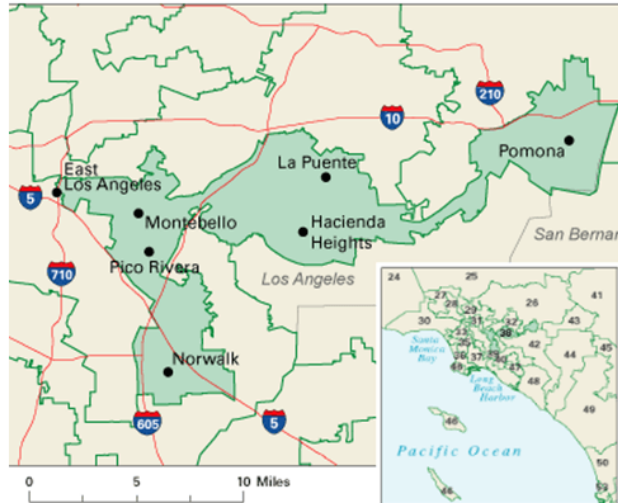
As part of a redistricting, a Democratic led committee could redraw the boundaries so that the middle district includes less of the typically Republican voters, thereby making it more likely that their party will win in that district, while increasing the Republican majority in the third district.



Example 13

The map to the right shows the 38th congressional district in California in 2004[1]. This district was created through a bi-partisan committee of incumbent legislators. This gerrymandering leads to districts that are not competitive; the prevailing party almost always wins with a large margin.

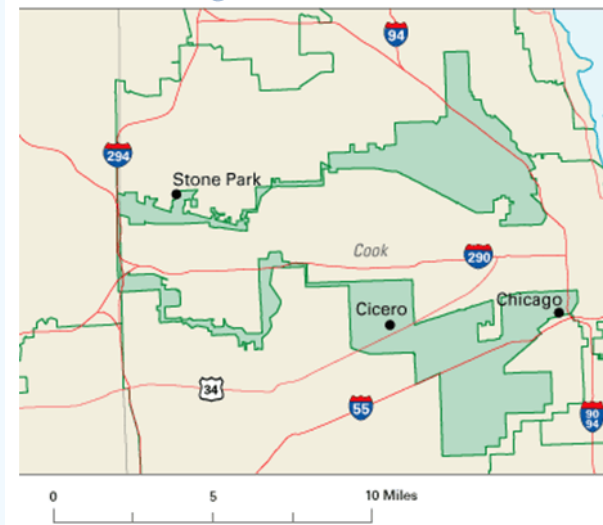
Congressional District 38



Solution

The map to the right shows the 4th congressional district in Illinois in 2004.[2] This district was drawn to contain the two predominantly Hispanic areas of Chicago. The largely Puerto Rican area to the north and the southern Mexican areas are only connected in this districting by a piece of the highway to the west.

Congressional District 4



[1] http://en.Wikipedia.org/wiki/File:Ca...ct_38_2004.png

[2] en.Wikipedia.org/wiki/File:IL...ict_4_2004.png

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2.2.8: Exercises

In exercises 1-8, determine the apportionment using

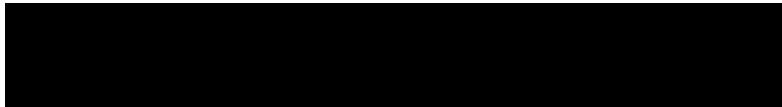
- a. Hamilton's Method
- b. Jefferson's Method
- c. Webster's Method
- d. Huntington-Hill Method
- e. Lowndes' method

1. A college offers tutoring in Math, English, Chemistry, and Biology. The number of students enrolled in each subject is listed below. If the college can only afford to hire 15 tutors, determine how many tutors should be assigned to each subject.

Math: 330 English: 265 Chemistry: 130 Biology: 70

2. Reapportion the previous problem if the college can hire 20 tutors.

3. The number of salespeople assigned to work during a shift is apportioned based on the average number of customers during that shift. Apportion 20 salespeople given the information below.



4. Reapportion the previous problem if the store has 25 salespeople.

5. Three people invest in a treasure dive, each investing the amount listed below. The dive results in 36 gold coins. Apportion those coins to the investors.

Alice: \$7,600 Ben: \$5,900 Carlos: \$1,400

6. Reapportion the previous problem if 37 gold coins are recovered.

7. A small country consists of five states, whose populations are listed below. If the legislature has 119 seats, apportion the seats.

A: 810,000 B: 473,000 C: 292,000 D: 594,000 E: 211,000

8. A small country consists of six states, whose populations are listed below. If the legislature has 200 seats, apportion the seats.

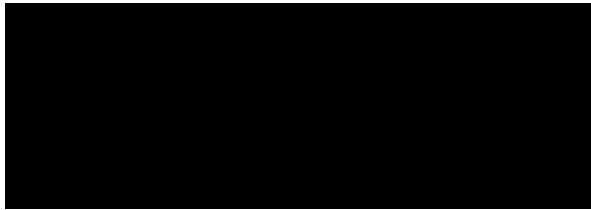
A: 3,411 B: 2,421 C: 11,586 D: 4,494 E: 3,126 F: 4,962

9. A small country consists of three states, whose populations are listed below.

A: 6,000 B: 6,000 C: 2,000

- a. If the legislature has 10 seats, use Hamilton's method to apportion the seats.
- b. If the legislature grows to 11 seats, use Hamilton's method to apportion the seats.
- c. Which apportionment paradox does this illustrate?

10. A state with five counties has 50 seats in their legislature. Using Hamilton's method, apportion the seats based on the 2000 census, then again using the 2010 census. Which apportionment paradox does this illustrate?



11. A school district has two high schools: Lowell, serving 1715 students, and Fairview, serving 7364. The district could only afford to hire 13 guidance counselors.

- a. Determine how many counselors should be assigned to each school using Hamilton's method.
- b. The following year, the district expands to include a third school, serving 2989 students. Based on the divisor from above, how many additional counselors should be hired for the new school?
- c. After hiring that many new counselors, the district recalculates the reapportion using Hamilton's method. Determine the outcome.
- d. Does this situation illustrate any apportionment issues?

12. A small country consists of four states, whose populations are listed below. If the legislature has 116 seats, apportion the seats using Hamilton's method. Does this illustrate any apportionment issues?

A: 33,700 B: 559,500 C: 141,300 D: 89,100

Exploration

13. Explore and describe the similarities, differences, and interplay between weighted voting, fair division (if you've studied it yet), and apportionment.

14. In the methods discussed in the text, it was assumed that the number of seats being apportioned was fixed. Suppose instead that the number of seats could be adjusted slightly, perhaps 10% up or down. Create a method for apportioning that incorporates this additional freedom, and describe why you feel it is the best approach. Apply your method to the apportionment in Exercise 7.

15. Lowndes felt that small states deserved additional seats more than larger states. Suppose you were a legislator from a larger state, and write an argument refuting Lowndes.

16. Research how apportionment of legislative seats is done in other countries around the world. What are the similarities and differences compared to how the United States apportions congress?

17. Adams's method is similar to Jefferson's method, but rounds quotas *up* rather than *down*. This means we usually need a modified divisor that is smaller than the standard divisor. Rework problems 1-8 using Adam's method. Which other method are the results most similar to?

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3: FINANCE

3.1: Finance

3.1.1: Introduction

3.1.2: Simple Interest

3.1.3: Compound Interest

3.1.4: Annuities

3.1.5: Payout Annuities

3.1.6: Loans

3.1.7: Remaining Loan Balance

3.1.8: Which equation to use?

3.1.9: Solving for time

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3.1: FINANCE

3.1.1: Introduction

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3.1.4: Annuities

3.1.5: Payout Annuities

3.1.6: Loans

3.1.7: Remaining Loan Balance

3.1.8: Which equation to use?

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3.1.1: Introduction

We have to work with money every day. While balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more mathematics.

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3.1.2: Simple Interest

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed \$100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = \$5$. The total amount you would repay would be \$105, the original principal plus the interest.

Simple One-time Interest

$$I = P_0 r$$

$$A = P_0 + I = P_0 + P_0 r = P_0(1 + r)$$

where

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate (in decimal form. Example: $5\% = 0.05$)

Example 1

A friend asks to borrow \$300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

Solution

$P_0 = \$300$ the principal
 $r = 0.03$ 3% rate
 $I = \$300(0.03) = \$9$. You will earn \$9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a \$1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Solution

Each year, you would earn 5% interest: $1000(0.05) = \$50$ in interest. So over the course of five years, you would earn a total of \$250 in interest. When the bond matures, you would receive back the \$1,000 you originally paid, leaving you with a total of \$1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time

$$I = P_0 r t$$

$$A = P_0 + I = P_0 + P_0rt = P_0(1 + rt)$$

where

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate in decimal form
- t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.
A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3: Treasury Notes

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a \$1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Solution

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$P_0 = \$1000$ the principal
 $r = 0.02$ 2% rate per half-year
 $t = 8$ 4 years = 8 half-years
 $I = \$1000(0.02)(8) = \$160$. You will earn \$160 interest total over the four years.

Try it Now 1

A loan company charges \$30 interest for a one month loan of \$500. Find the annual interest rate they are charging.

Answer available at the end of this section.

Answers

Try it Now 1

$I = \$30$ of interest
 $P_0 = \$500$ principal
 $r = \frac{30}{500} = 0.06$ per month
 $(0.06)(12) = 0.72$ per year
They are charging an annual interest rate of 72%.

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3.1.3: Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit \$1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3\%}{12} = 0.25\%$ per month.

In the first month,

$$P_0 = \$1000$$

$$r = 0.0025(0.25\%)$$

$$I = \$1000(0.0025) = \$2.50$$

$$A = \$1000 + \$2.50 = \$1002.50$$

In the first month, we will earn \$2.50 in interest, raising our account balance to \$1002.50.

In the second month,

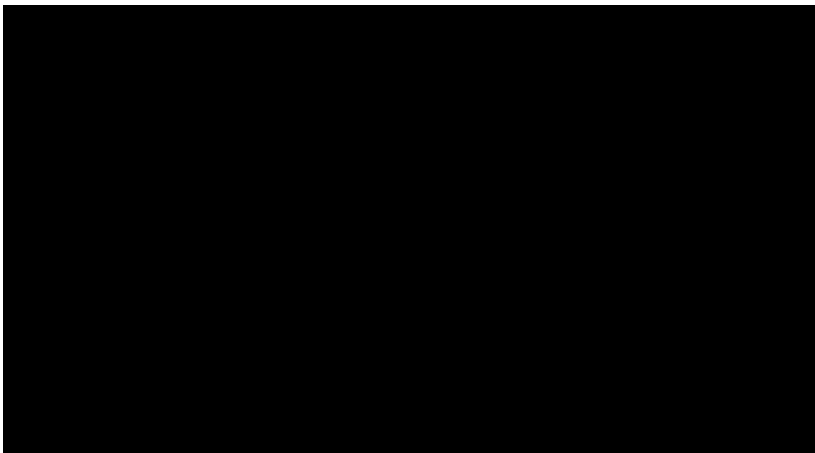
$$P_0 = \$1002.50$$

$$I = \$1002.50(0.0025) = \$2.51 \text{ (rounded)}$$

$$A = \$1000 + \$2.50 = \$1002.50$$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original \$1000 we deposited, but we also earned interest on the \$2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:



To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

$$P_0 = \$1000$$

$$P_m = (1 + 0.0025)P_{m-1}$$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$$P_0 = \$1000$$

$$P_1 = 1.0025P_0 = 1.0025(1000)$$

$$P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = 1.0025^2(1000)$$

$$P_3 = 1.0025P_2 = 1.0025(1.0025^2(1000)) = 1.0025^3(1000)$$

$$P_4 = 1.0025P_3 = 1.0025(1.0025^3(1000)) = 1.0025^4(1000)$$

Observing a pattern, we could conclude

$$P_m = (1.0025)^m(\$1000)$$

Notice that the ~~\$1000~~ in the equation was P_0 , the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

$$P_m = P_0 \left(1 + \frac{r}{k}\right)^m$$

In this formula:

m is the number of compounding periods (months in our example)

r is the annual interest rate

k is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If N is the number of years, then $m = Nk$. Making this change gives us the standard formula for compound interest.

Compound Interest

$$P_N = P_0 \left(1 + \frac{r}{k}\right)^{Nk}$$

P_N is the balance in the account after N years.

P_0 is the starting balance of the account (also called initial deposit, or principal)

r is the annual interest rate in decimal form

k is the number of compounding periods in one year.

If the compounding is done annually (once a year), $k = 1$.

If the compounding is done quarterly, $k = 4$.

If the compounding is done monthly, $k = 12$.

If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let

it sit there earning interest.

Example 4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

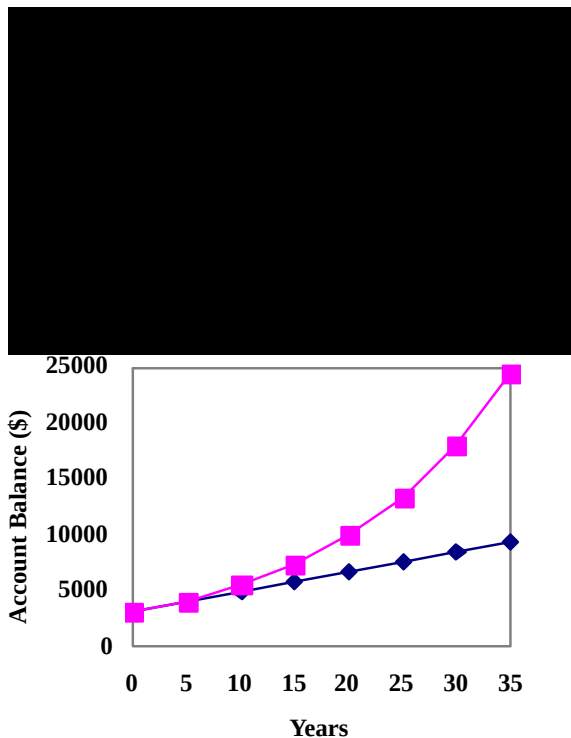
Solution

In this example,

$P_0 = \$3000$ the initial deposit
 $r = 0.06$ 6% annual rate
 $k = 12$ 12 months in 1 year
 $N = 20$ since we're looking for how much we'll have after 20 years

So $P_{20} = 3000 \left(1 + \frac{0.06}{12}\right)^{20 \times 12} = \$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Evaluating exponents on the calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$. But when we need to calculate something like 1.005^{240} , it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

$[\wedge]$, $[y^x]$, or $[x^y]$

To evaluate 1.005^{240} we'd type $1.005 [\wedge] 240$, or $1.005 [y^x] 240$. Try it out - you should get something around 3.3102044758.

Example 5

You know that you will need \$40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Solution

We're looking for P_0 .

$r = 0.04$ 4%
 $k = 4$ 4 quarters in 1 year
 $N = 18$ Since we know the balance in 18 years
 $P_{18} = \$40,000$ The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for P_0 .

$$40000 = P_0 \left(1 + \frac{0.04}{4} \right)^{18 \times 4}$$

$$40000 = P_0(2.0471)$$

$$P_0 = \frac{40000}{2.0471} = \$19539.84$$

So you would need to deposit \$19,539.84 now to have \$40,000 in 18 years.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better.

Example 6

To see why not over-rounding is so important, suppose you were investing \$1000 at 5% interest compounded monthly for 30 years.

Solution

$P_0 = \$1000$ the initial deposit
 $r = 0.05$ 5%
 $k = 12$ 12 months in 1 year
 $N = 30$ since we're looking for the amount after 30 years

If we first compute $\frac{r}{k}$, we find $\frac{0.05}{12} = 0.0041666666667$

Here is the effect of rounding this to different values:



If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough - \$5 off of \$4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

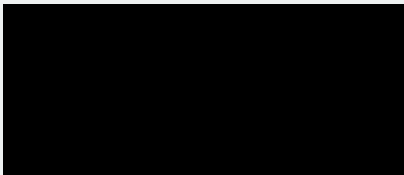
Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$$P_{30} = 1000 \left(1 + \frac{0.05}{12} \right)^{12 \times 30}$$

We can quickly calculate $12 \times 30 = 360$, giving $P_{30} = 1000 \left(1 + \frac{0.05}{12} \right)^{360}$.

Now we can use the calculator.



Using your calculator continued

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 [×] (1 [+] 0.05 [÷] 12) [y^x] 360 [=]

3.1.4: Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1}$$

For a savings annuity, we simply need to add a deposit, d , to the account with each compounding period:

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1} + d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit \$100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$$r = 0.06 \text{ (6\%)}$$

$$k = 12 \text{ (12 compounds/deposits per year)}$$

$$d = \$100 \text{ (our deposit per month)}$$

Writing out the recursive equation gives

$$P_m = \left(1 + \frac{0.06}{12}\right) P_{m-1} + 100 = (1.005)P_{m-1} + 100$$

Assuming we start with an empty account, we can begin using this relationship:

$$P_0 = 0$$

$$P_1 = (1.005)P_0 + 100 = 100$$

$$P_2 = (1.005)P_1 + 100 = (1.005)(100) + 100 = 100(1.005) + 100$$

$$P_3 = (1.005)P_2 + 100 = (1.005)(100(1.005) + 100) + 100 = 100(1.005)^2 + 100(1.005) + 100$$

Continuing this pattern, after m deposits, we'd have saved:

$$P_m = 100(1.005)^{m-1} + 100(1.005)^{m-2} + \dots + 100(1.005) + 100$$

In other words, after m months, the first deposit will have earned compound interest for $m - 1$ months. The second deposit will have earned interest for $m - 2$ months. Last month's deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005P_m = 1.005(100(1.005)^{m-1} + 100(1.005)^{m-2} + \dots + 100(1.005) + 100)$$

Distributing on the right side of the equation gives

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \dots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$\begin{aligned} 1.005P_m &= 100(1.005)^m + 100(1.005)^{m-1} + \dots + 100(1.005) \\ P_m &= 100(1.005)^{m-1} + \dots + 100(1.005) + 100 \end{aligned}$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005P_m - P_m = 100(1.005)^m - 100$$

Solving for P_m

$$0.005P_m = 100((1.005)^m - 1)$$

$$P_m = \frac{100((1.005)^m - 1)}{0.005}$$

Replacing m months with $12N$, where N is measured in years, gives

$$P_N = \frac{100((1.005)^{12N} - 1)}{0.005}$$

Recall 0.005 was $\frac{r}{k}$ and 100 was the deposit d . 12 was k , the number of deposit each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula

$$P_N = \frac{d((1+\frac{r}{k})^{Nk} - 1)}{(\frac{r}{k})}$$

P_N is the balance in the account after N years.

d is the regular deposit (the amount you deposit each year, each month, etc.)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, $k = 12$.

If you make your deposits every year, use yearly compounding, $k = 1$.

If you make your deposits every quarter, use quarterly compounding, $k = 4$.

Etc.

When do you use this

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Example 7

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit \$100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

Solution

In this example,

$d = \$100$ the monthly deposit
 $r = 0.06$ 6% annual rate
 $k = 12$ since we're doing monthly deposits, we'll compound monthly
 $N = 20$ we want the amount after 20 years

Putting this into the equation:

$$P_{20} = \frac{100 \left(\left(1 + \frac{0.06}{12} \right)^{20(12)} - 1 \right)}{\left(\frac{0.06}{12} \right)}$$

$$P_{20} = \frac{100((1.005)^{240} - 1)}{(0.005)}$$

$$P_{20} = \frac{100(3.310 - 1)}{(0.005)}$$

$$P_{20} = \frac{100(2.310)}{(0.005)} = \$46200$$

The account will grow to \$46,200 after 20 years.

Notice that you deposited into the account a total of \$24,000 (\$100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $\$46,200 - \$24,000 = \$22,200$.

Example 8

You want to have \$200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Solution

In this example,

We're looking for d .

$r = 0.08$ 8% annual rate
 $k = 12$ since we're doing monthly deposits, we'll compound monthly
 $N = 30$ 30 years
 $P_{30} = \$200,000$ The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for d .

$$200,000 = \frac{d \left(\left(1 + \frac{0.08}{12} \right)^{30(12)} - 1 \right)}{\left(\frac{0.08}{12} \right)}$$

$$200,000 = \frac{d \left((1.00667)^{360} - 1 \right)}{(0.00667)}$$

$$200,000 = d(1491.57)$$

$$d = \frac{200,000}{1491.57} = \$134.09$$

So you would need to deposit \$134.09 each month to have \$200,000 in 30 years if your account earns 8% interest

Try it Now 2

A more conservative investment account pays 3% interest. If you deposit \$5 a day into this account, how much will you have after 10 years? How much is from interest?

Answer available at the end of this section.

Answers

Try it Now 2

$d = \$5$ the daily deposit

$r = 0.03$ 3% annual rate

$k = 365$ since we're doing daily deposits, we'll compound daily

$N = 10$ we want the amount after 10 years

$$P_{10} = \frac{5 \left(\left(1 + \frac{0.03}{365} \right)^{365 \times 10} - 1 \right)}{\frac{0.03}{365}} = \$21,282.07$$

We would have deposited a total of $\$5 \cdot 365 \cdot 10 = \$18,250$, so \$3,032.07 is from interest

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3.1.5: Payout Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved \$500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$P_0 = \frac{d(1 - (1 + \frac{r}{k})^{-Nk})}{(\frac{r}{k})}$$

P_0 is the balance in the account at the beginning (starting amount, or principal).

d is the regular withdrawal (the amount you take out each year, each month, etc.)

r is the annual interest rate (in decimal form. Example: 5% = 0.05)

k is the number of compounding periods in one year.

N is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Example 9

After retiring, you want to be able to take \$1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Solution

In this example,

$d = \$1000$ the monthly withdrawal
 $r = 0.06$ 6% annual rate
 $k = 12$ since we're doing monthly withdrawals, we'll compound monthly
 $N = 20$ since we're taking withdrawals for 20 years

We're looking for P_0 ; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$P_0 = \frac{1000 \left(1 - \left(1 + \frac{0.06}{12} \right)^{-20(12)} \right)}{\left(\frac{0.06}{12} \right)}$$

$$P_0 = \frac{1000 \times (1 - (1.005)^{-240})}{(0.005)}$$

$$P_0 = \frac{1000 \times (1 - 0.302)}{(0.005)} = \$139,600$$

You will need to have \$139,600 in your account when you retire.

Notice that you withdrew a total of \$240,000 (\$1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is $\$240,000 - \$139,600 = \$100,400$ in interest.

Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this $[(-)]$, some with $[+/-]$. The button is often near the $=$ key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005^{-240} you'd type something like: $1.005[\wedge][(-)]240$

If your calculator only shows one value at a time, then usually you hit the $(-)$ key after a number to negate it, so you'd hit: $1.005[y^x]240[(-)] =$

Give it a try - you should get $1.005^{-240} = 0.302096$

Example 10

You know you will have \$500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

Solution

In this example,

We're looking for d .

$r = 0.08$ 8% annual rate
 $k = 12$ since we're doing monthly withdrawals
 $N = 30$ since we're taking withdrawals for 30 years
 $P_0 = \$500,000$ we are beginning with \$500,000

In this case, we're going to have to set up the equation, and solve for d .

$$500,000 = \frac{d \left(1 - \left(1 + \frac{0.08}{12} \right)^{-30(12)} \right)}{\left(\frac{0.08}{12} \right)}$$

$$500,000 = \frac{d(1-(1.00667)^{-360})}{(0.00667)}$$

$$500,000 = d(136.232)$$

$$d = \frac{500,000}{136.232} = \$3670.21$$

You would be able to withdraw \$3,670.21 each month for 30 years.

Try it Now 3

A donor gives \$100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Answer available at the end of this section.

Answers

Try it Now 3

$d =$ unknown

$r = 0.04$ 4% annual rate

$k = 1$ since we're doing annual scholarships

$N = 20$ since we're taking withdrawals for 20 years

$P_0 = \$100,000$ we are starting with \$100,000

$$100,000 = \frac{d(1-(1+\frac{0.04}{1})^{-20 \times 1})}{\frac{0.04}{1}}$$

Solving for d gives \$7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, $\$100,000(0.04) = \$4,000$ a year would have been available.

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3.1.6: Loans

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had \$10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests \$10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula

$$P_0 = \frac{d(1 - (1 + \frac{r}{k})^{-Nk})}{(\frac{r}{k})}$$

P_0 is the balance in the account at the beginning (the principal, or amount of the loan).

d is your loan payment (your monthly payment, annual payment, etc)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

N is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Loans: Many payments

Example 11

You can afford \$200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with \$200 per month?

Solution

In this example,

$d = \$200$ the monthly loan payment
 $r = 0.03$ 3% annual rate
 $k = 12$ since we're doing monthly payments, we'll compound monthly
 $N = 5$ since we're making monthly payments for 5 years

We're looking for P_0 , the starting amount of the loan.

$$P_0 = \frac{200 \left(1 - \left(1 + \frac{0.03}{12} \right)^{-5(12)} \right)}{\left(\frac{0.03}{12} \right)}$$

$$P_0 = \frac{200(1 - (1.0025)^{-60})}{(0.0025)}$$

$$P_0 = \frac{200(1 - 0.861)}{(0.0025)} = \$11,120$$

You can afford a \$11,120 loan.

You will pay a total of \$12,000 (\$200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying $\$12,000 - \$11,120 = \$880$ interest total.

Example 12

You want to take out a \$140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Solution

In this example,

We're looking for d .

$r = 0.06$ 6% annual rate
 $k = 12$ since we're doing monthly payments, we'll compound monthly
 $N = 30$ since we're making monthly payments for 30 years
 $P_0 = \$140,000$ the starting loan amount

In this case, we're going to have to set up the equation, and solve for d .

$$140,000 = \frac{d \left(1 - \left(1 + \frac{0.06}{12} \right)^{-30(12)} \right)}{\left(\frac{0.06}{12} \right)}$$

$$140,000 = \frac{d(1 - (1.005)^{-360})}{(0.005)}$$

$$140,000 = d(166.792)$$

$$d = \frac{140,000}{166.792} = \$839.37$$

You will make payments of \$839.37 per month for 30 years.

You're paying a total of \$302,173.20 to the loan company: \$839.37 per month for 360 months. You are paying a total of $\$302,173.20 - \$140,000 = \$162,173.20$ in interest over the life of the loan.

Try it Now 4

Janine bought \$3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Answer available at the end of this section.

Answers

Try it Now 4

$d =$ unknown

$r = 0.16$ 16% annual rate

$k = 12$ since we're doing monthly payments, we'll compound monthly

$N = 2$ 2 year to repay

$P_0 = 3,000$ the starting loan amount \$3,000 loan

$$3,000 = \frac{d \left(1 - \left(1 + \frac{0.16}{12} \right)^{-2 \times 12} \right)}{\frac{0.16}{12}}$$

Solving for d gives \$146.89 as monthly payments.

In total, she will pay \$3,525.36 to the store, meaning she will pay \$525.36 in interest over the two years.

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3.1.7: Remaining Loan Balance

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were \$1,000 a month, after a year you will *not* have paid off \$12,000 of the loan balance.

To determine the remaining loan balance, we can think "how much loan will these loan payments be able to pay off in the remaining time on the loan?"

Example 13

If a mortgage at a 6% interest rate has payments of \$1,000 a month, how much will the loan balance be 10 years from the end the loan?

Solution

To determine this, we are looking for the amount of the loan that can be paid off by \$1,000 a month payments in 10 years. In other words, we're looking for P_0 when

$d = \$1,000$ the monthly loan payment
 $r = 0.06$ 6% annual rate
 $k = 12$ since we're doing monthly payments, we'll compound monthly
 $N = 10$ since we're making monthly payments for 10 more years

$$\begin{aligned}
 P_0 &= \frac{1000 \left(1 - \left(1 + \frac{0.06}{12} \right)^{-10(12)} \right)}{\left(\frac{0.06}{12} \right)} \\
 &= \frac{1000 (1 - (1.005)^{-120})}{(0.005)} \\
 &= \frac{1000 (1 - (1.005)^{-120})}{(0.005)} \\
 &= \frac{1000(1 - 0.5496)}{(0.005)} \\
 &= \$90,073.45
 \end{aligned}$$

The loan balance with 10 years remaining on the loan will be \$90,073.45

Often times answering remaining balance questions requires two steps:

1. Calculating the monthly payments on the loan
2. Calculating the remaining loan balance based on the *remaining time* on the loan

Example 14

A couple purchases a home with a \$180,000 mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

Solution

First we will calculate their monthly payments.

We're looking for d .

$r = 0.04$ 4% annual rate
 $k = 12$ since they're paying monthly
 $N = 30$ 30 years
 $P_0 = \$180,000$ the starting loan amount

We set up the equation and solve for d .

$$\begin{aligned}
 180,000 &= \frac{d \left(1 - \left(1 + \frac{0.04}{12} \right)^{-30(12)} \right)}{\left(\frac{0.04}{12} \right)} \\
 180,000 &= \frac{d (1 - (1.00333)^{-360})}{(0.00333)} \\
 180,000 &= d(209.562) \\
 d &= \frac{180,000}{209.562} \\
 &= \$858.93
 \end{aligned}$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

$d = \$858.93$ the monthly loan payment we calculated above
 $r = 0.04$ 4% annual rate
 $k = 12$ since they're paying monthly
 $N = 25$ since they'd be making monthly payments for 25 more years

$$\begin{aligned}
 P_0 &= \frac{858.93 \left(1 - \left(1 + \frac{0.04}{12} \right)^{-25(12)} \right)}{\left(\frac{0.04}{12} \right)} \\
 &= \frac{858.93 (1 - (1.00333)^{-300})}{(0.00333)} \\
 &= \frac{858.93(1 - 0.369)}{(0.00333)} \\
 &\approx \$162,758
 \end{aligned}$$

The loan balance after 5 years, with 25 years remaining on the loan, will be \$162,758

Over that 5 years, the couple has paid off $\$180,000 - \$162,758 = \$17,242$ of the loan balance. They have paid a total of $\$858.93$ a month for 5 years (60 months), for a total of $\$51,535.80$, so $\$51,535.80 - \$17,242 = \$34,292.80$ of what they have paid so far has been interest.

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3.1.8: Which equation to use?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan", "amortize" (the fancy word for loans), "finance (a car)", or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a basic Annuity problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a Payout Annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

Try it Now 5

For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loans problem. Then solve each problem.

- Marcy received an inheritance of \$20,000, and invested it at 6% interest. She is going to use it for college, withdrawing money for tuition and expenses each quarter. How much can she take out each quarter if she has 3 years of school left?
- Paul wants to buy a new car. Rather than take out a loan, he decides to save \$200 a month in an account earning 3% interest compounded monthly. How much will he have saved up after 3 years?
- Keisha is managing investments for a non-profit company. They want to invest some money in an account earning 5% interest compounded annually with the goal to have \$30,000 in the account in 6 years. How much should Keisha deposit into the account?
- Miao is going to finance new office equipment at a 2% rate over a 4 year term. If she can afford monthly payments of \$100, how much new equipment can she buy?
- How much would you need to save every month in an account earning 4% interest to have \$5,000 saved up in two years?

Answer available at the end of this section.

Answers

Try it Now 5

- This is a payout annuity problem. She can pull out \$1833.60 a quarter.
- This is a savings annuity problem. He will have saved up \$7,524.11.
- This is compound interest problem. She would need to deposit \$22,386.46.
- This is a loans problem. She can buy \$4,609.33 of new equipment.
- This is a savings annuity problem. You would need to save \$200.46 each month

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3.1.9: Solving for time

Often we are interested in how long it will take to accumulate money or how long we'd need to extend a loan to bring payments down to a reasonable level.

Note: This section assumes you've covered solving exponential equations using logarithms, either in prior classes or in the growth models chapter.

Example 15

If you invest \$2000 at 6% compounded monthly, how long will it take the account to double in value?

Solution

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

$$\begin{aligned} P_0 &= \$2000 && \text{the initial deposit} \\ r &= 0.06 && \text{6\% annual rate} \\ k &= 12 && \text{12 months in 1 year} \end{aligned}$$

So our general equation is $P_N = 2000 \left(1 + \frac{0.06}{12}\right)^{N \times 12}$. We also know that we want our ending amount to be double of \$2000, which is \$4000, so we're looking for N so that $P_N = 4000$. To solve this, we set our equation for P_N equal to 4000.

$$\begin{aligned} 4000 &= 2000 \left(1 + \frac{0.06}{12}\right)^{N \times 12} && \text{Divide both sides by 2000} \\ 2 &= (1.005)^{12N} && \text{To solve for the exponent, take the log of both sides} \\ \log(2) &= \log((1.005)^{12N}) && \text{Use the exponent property of logs on the right side} \\ \log(2) &= 12N \log(1.005) && \text{Now we can divide both sides by } 12 \log 1.005 \\ \frac{\log(2)}{12 \log(1.005)} &= N && \text{Approximating this to a decimal} \\ N &= 11.581 \end{aligned}$$

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded $\log(2)$ to 0.301 and $\log(1.005)$ to 0.00217, then your final answer would have been about 11.577 years.

Example 16

If you invest \$100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to \$10,000?

Solution

This is a savings annuity problem since we are making regular deposits into the account.

$$\begin{aligned} d &= \$1000 && \text{the monthly deposit} \\ r &= 0.03 && \text{3\% annual rate} \\ k &= 12 && \text{since we're doing monthly deposits, we'll compound monthly} \end{aligned}$$

We don't know N , but we want P_N to be \$10,000

Putting this into the equation:

$$10,000 = \frac{100 \left(\left(1 + \frac{0.03}{12} \right)^{N(12)} - 1 \right)}{\left(\frac{0.03}{12} \right)} \quad \text{Simplifying the fractions a bit}$$

$$10,000 = \frac{100((1.0025)^{12N} - 1)}{0.0025}$$

We want to isolate the exponential term, 1.0025^{12N} , so multiply both sides by 0.0025

$$25 = 100((1.0025)^{12N} - 1) \quad \text{Divide both sides by 100}$$

$$0.25 = (1.0025)^{12N} - 1 \quad \text{Add 1 to both sides}$$

$$1.25 = (1.0025)^{12N} \quad \text{Now take the log of both sides}$$

$$\log(1.25) = \log((1.0025)^{12N}) \quad \text{Use the exponent property of logs}$$

$$\log(1.25) = 12N \log(1.0025) \quad \text{Divide by } 12\log(1.0025)$$

$$\frac{\log(1.25)}{12\log(1.0025)} = N \quad \text{Approximating to a decimal}$$

$$N = 7.447 \text{ years}$$

It will take about 7.447 years to grow the account to \$10,000.

Try it Now 6

Joel is considering putting a \$1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of \$30 a month?

Answer available at the end of this section.

Answers

Try it Now 6

$d = \$30$ The monthly payments
 $r = 0.12$ 12% annual rate
 $k = 12$ since we're doing monthly deposits
 $P_0 = \$1000$ we're starting with a \$1,000 loan

We are solving for N , the time to pay off the loan

$$1,000 = \frac{30 \left(1 - \left(1 + \frac{0.12}{12} \right)^{-N(12)} \right)}{\frac{0.12}{12}}$$

Solving for N gives 3.396. It will take about 3.4 years to pay off the purchase.

3.1.10: Exercise

Skills

1. A friend lends you \$200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?
2. Suppose you obtain a \$3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?
3. A T-bill is a type of bond that is sold at a discount over the face value. For example, suppose you buy a 13-week T-bill with a face value of \$10,000 for \$9,800. This means that in 13 weeks, the government will give you the face value, earning you \$200. What annual interest rate have you earned?
4. Suppose you are looking to buy a \$5000 face value 26-week T-bill. If you want to earn at least 1% annual interest, what is the most you should pay for the T-bill?
5. You deposit \$300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?
6. How much will \$1000 deposited in an account earning 7% interest compounded annually be worth in 20 years?
7. You deposit \$2000 in an account earning 3% interest compounded monthly.
 - a. How much will you have in the account in 20 years?
 - b. How much interest will you earn?
8. You deposit \$10,000 in an account earning 4% interest compounded monthly.
 - a. How much will you have in the account in 25 years?
 - b. How much interest will you earn?
9. How much would you need to deposit in an account now in order to have \$6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.
10. How much would you need to deposit in an account now in order to have \$20,000 in the account in 4 years? Assume the account earns 5% interest.
11. You deposit \$200 each month into an account earning 3% interest compounded monthly.
 - a. How much will you have in the account in 30 years?
 - b. How much total money will you put into the account?
 - c. How much total interest will you earn?
12. You deposit \$1000 each year into an account earning 8% compounded annually.
 - a. How much will you have in the account in 10 years?
 - b. How much total money will you put into the account?
 - c. How much total interest will you earn?
13. Jose has determined he needs to have \$800,000 for retirement in 30 years. His account earns 6% interest.
 - a. How much would he need to deposit in the account each month?
 - b. How much total money will he put into the account?
 - c. How much total interest will he earn?
14. You wish to have \$3000 in 2 years to buy a fancy new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?
15. You want to be able to withdraw \$30,000 each year for 25 years. Your account earns 8% interest.
 - a. How much do you need in your account at the beginning
 - b. How much total money will you pull out of the account?

- c. How much of that money is interest?
16. How much money will I need to have at retirement so I can withdraw \$60,000 a year for 20 years from an account earning 8% compounded annually?
- How much do you need in your account at the beginning
 - How much total money will you pull out of the account?
 - How much of that money is interest?
17. You have \$500,000 saved for retirement. Your account earns 6% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 20 years?
18. Loren already knows that he will have \$500,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 10% interest, how much could the annuity provide each month?
19. You can afford a \$700 per month mortgage payment. You've found a 30 year loan at 5% interest.
- How big of a loan can you afford?
 - How much total money will you pay the loan company?
 - How much of that money is interest?
20. Marie can afford a \$250 per month car payment. She's found a 5 year loan at 7% interest.
- How expensive of a car can she afford?
 - How much total money will she pay the loan company?
 - How much of that money is interest?
21. You want to buy a \$25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?
22. You decide finance a \$12,000 car at 3% compounded monthly for 4 years. What will your monthly payments be? How much interest will you pay over the life of the loan?
23. You want to buy a \$200,000 home. You plan to pay 10% as a down payment, and take out a 30 year loan for the rest.
- How much is the loan amount going to be?
 - What will your monthly payments be if the interest rate is 5%?
 - What will your monthly payments be if the interest rate is 6%?
24. Lynn bought a \$300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.
- Find her monthly payments.
 - How much interest will she pay over the life of the loan?
25. Emile bought a car for \$24,000 three years ago. The loan had a 5 year term at 3% interest rate, making monthly payments. How much does he still owe on the car?
26. A friend bought a house 15 years ago, taking out a \$120,000 mortgage at 6% for 30 years, making monthly payments. How much does she still owe on the mortgage?
27. Pat deposits \$6,000 into an account earning 4% compounded monthly. How long will it take the account to grow to \$10,000?
28. Kay is saving \$200 a month into an account earning 5% interest. How long will it take her to save \$20,000?
29. James has \$3,000 in credit card debt, which charges 14% interest. How long will it take to pay off the card if he makes the minimum payment of \$60 a month?
30. Chris has saved \$200,000 for retirement, and it is in an account earning 6% interest. If she withdraws \$3,000 a month, how long will the money last?

Concepts

31. Suppose you invest \$50 a month for 5 years into an account earning 8% compounded monthly. After 5 years, you leave the money, without making additional deposits, in the account for another 25 years. How much will you have in the end?

32. Suppose you put off making investments for the first 5 years, and instead made deposits of \$50 a month for 25 years into an account earning 8% compounded monthly. How much will you have in the end?
33. Mike plans to make contributions to his retirement account for 15 years. After the last contribution, he will start withdrawing \$10,000 a quarter for 10 years. Assuming Mike's account earns 8% compounded quarterly, how large must his quarterly contributions be during the first 15 years, in order to accomplish his goal?
34. Kendra wants to be able to make withdrawals of \$60,000 a year for 30 years after retiring in 35 years. How much will she have to save each year up until retirement if her account earns 7% interest?
35. You have \$2,000 to invest, and want it to grow to \$3,000 in two years. What interest rate would you need to find to make this possible?
36. You have \$5,000 to invest, and want it to grow to \$20,000 in ten years. What interest rate would you need to find to make this possible?
37. You plan to save \$600 a month for the next 30 years for retirement. What interest rate would you need to have \$1,000,000 at retirement?
38. You really want to buy a used car for \$11,000, but can only afford \$200 a month. What interest rate would you need to find to be able to afford the car, assuming the loan is for 60 months?

Exploration

39. Pay day loans are short term loans that you take out against future paychecks: The company advances you money against a future paycheck. Either visit a pay day loan company, or look one up online. Be forewarned that many companies do not make their fees obvious, so you might need to do some digging or look at several companies.
 - a. Explain the general method by which the loan works.
 - b. We will assume that we need to borrow \$500 and that we will pay back the loan in 14 days. Determine the total amount that you would need to pay back and the effective loan rate. The effective loan rate is the percentage of the original loan amount that you pay back. It is not the same as the APR (annual rate) that is probably published.
 - c. If you cannot pay back the loan after 14 days, you will need to get an extension for another 14 days. Determine the fees for an extension, determine the total amount you will be paying for the now 28 day loan, and compute the effective loan rate.
10. Suppose that 10 years ago you bought a home for \$110,000, paying 10% as a down payment, and financing the rest at 9% interest for 30 years.
 - a. Let's consider your existing mortgage:
 - i. How much money did you pay as your down payment?
 - ii. How much money was your mortgage (loan) for?
 - iii. What is your current monthly payment?
 - iv. How much total interest will you pay over the life of the loan?
 - b. This year, you check your loan balance. Only part of your payments have been going to pay down the loan; the rest has been going towards interest. You see that you still have \$88,536 left to pay on your loan. Your house is now valued at \$150,000.

How much of the loan have you paid off? (i.e., how much have you reduced the loan balance by? Keep in mind that interest is charged each month - it's not part of the loan balance.)

- i. How much money have you paid to the loan company so far?
 - ii. How much interest have you paid so far?
 - iii. How much equity do you have in your home (equity is value minus remaining debt)
 - c. Since interest rates have dropped, you consider refinancing your mortgage at a lower 6% rate.
 - i. If you took out a new 30 year mortgage at 6% for your remaining loan balance, what would your new monthly payments be?
 - ii. How much interest will you pay over the life of the new loan?

10. d. Notice that if you refinance, you are going to be making payments on your home for another 30 years. In addition to the 10 years you've already been paying, that's 40 years total.
- i. How much will you save each month because of the lower monthly payment?
 - ii. How much total interest will you be paying (you need to consider the amount from 2c and 3b)
 - iii. Does it make sense to refinance? (there isn't a correct answer to this question. Just give your opinion and your reason)

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4: SETS

4.1: Sets

4.1.1: Basics of Sets

4.1.2: Union, Intersection, and Complement

4.1.3: Venn Diagrams

4.1.4: Cardinality

4.1.5: Exercises

4.1: SETS

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

4.1.1: Basics of Sets

4.1.2: Union, Intersection, and Complement

4.1.3: Venn Diagrams

4.1.4: Cardinality

4.1.5: Exercises

via Wikipedia).

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4.1.1: Basics of Sets

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:

- The set of all even numbers
- The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:

- $\{1, 3, 9, 12\}$
- $\{\text{red, orange, yellow, green, blue, indigo, purple}\}$

A set simply specifies the contents; order is not important. The set represented by $\{1, 2, 3\}$ is equivalent to the set $\{3, 1, 2\}$.

Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol $\in$ means “is an element of”.

A set that contains no elements, $\{\}$, is called the **empty set** and is notated $\emptyset$

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A , but may not contain all the elements of A .

If B is a subset of A , we write $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.

If B is a proper subset of A , we write $B \subset A$

Example 3

Consider these three sets

A = the set of all even numbers $B = \{2, 4, 6\}$ $C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so is an element of A .

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$

It is also true that $B \subset C$.

C is not a subset of A , since C contains an element, 3, that is not contained in A

Example 4

Suppose a set contains the plays “Much Ado About Nothing”, “MacBeth”, and “A Midsummer’s Night Dream”. What is a larger set this might be a subset of?

Solution

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try it Now 1

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Answer available at the end of this section.

Answers

Try it Now 1

There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

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4.1.2: Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets).

The union is notated $A \cup B$

More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets.

The intersection is notated $A \cap B$

More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$

The **complement** of a set A contains everything that is *not* in the set A .

The complement is notated A , or A^c , or sometimes $\sim A$.

Example 5

Consider the sets:

$$A = \{ \text{red, green, blue} \} \quad B = \{ \text{red, yellow, orange} \} \quad C = \{ \text{red, orange, yellow, green, blue, purple} \}$$

- Find $A \cup B$
- Find $A \cap B$
- Find $A^c \cap C$

Solution

a) The union contains all the elements in either set: $A \cup B = \{ \text{red, green, blue, yellow, orange} \}$

Notice we only list red once.

b) The intersection contains all the elements in both sets: $A \cap B = \{ \text{red} \}$

c) Here we're looking for all the elements that are not in set A and are also in C .

$$A^c \cap C = \{ \text{orange, yellow, purple} \}$$

Try it Now 2

Using the sets from the previous example, find $A \cup C$ and $B^c \cap A$

Answer available at the end of this section.

Notice that in the example above, it would be hard to just ask for A^c , since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections,

or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in A .

Example 6

- If we were discussing searching for books, the universal set might be all the books in the library.
- If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
- If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7

Suppose the universal set is $U =$ all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then

$$A^c = \{3, 5, 6, 7, 8, 9\}$$

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic - to force an order of operations.

Example 8

Suppose

$$H = \{ \text{cat, dog, rabbit, mouse} \}, F = \{ \text{dog, cow, duck, pig, rabbit} \} \quad W = \{ \text{duck, rabbit, deer, frog, mouse} \}$$

- Find $(H \cap P) \cup W$
- Find $H \cap (F \cup W)$
- Find $(H \cap P) \cap W$

Solution

a) We start with the intersection: $H \cap F = \{ \text{dog, rabbit} \}$

Now we union that result with W : $(H \cap F) \cup W = \{ \text{dog, duck, rabbit, deer, frog, mouse} \}$

b) We start with the union: $F \cup W = \{ \text{dog, cow, rabbit, duck, pig, deer, frog, mouse} \}$

Now we intersect that result with H : $H \cap (F \cup W) = \{ \text{dog, rabbit, mouse} \}$

c) We start with the intersection: $H \cap F = \{ \text{dog, rabbit} \}$

Now we want to find the elements of W that are not in $H \cap F$

$$(H \cap P)^c \cap W = \{ \text{duck, deer, frog, mouse} \}$$

Answers

Try it Now 2

$$A \cup C = \{ \text{red, orange, yellow, green, blue, purple} \}$$

$$B^c \cap A = \{ \text{green, blue} \}$$

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4.1.3: Venn Diagrams

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

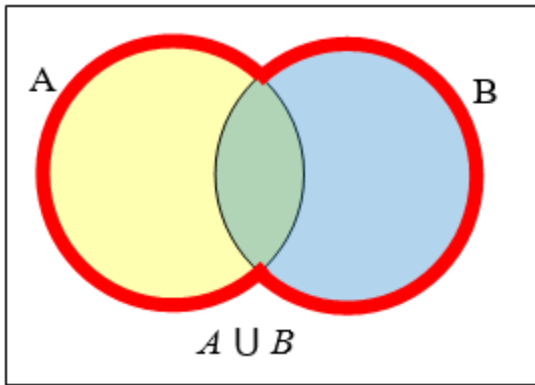
Venn Diagram

A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

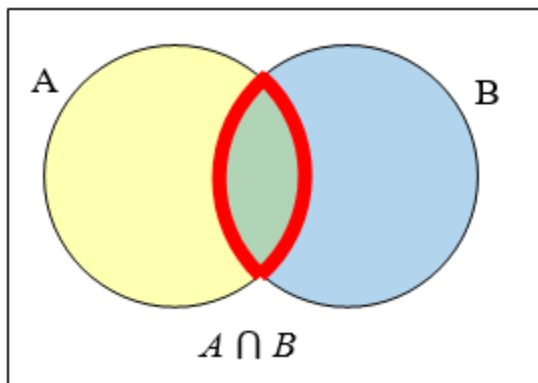
Example 9

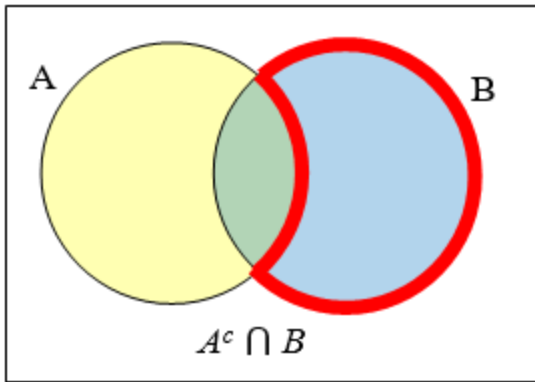
Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$



$A \cup B$ contains all elements in either set.

$A \cup B$ contains all elements in either set.



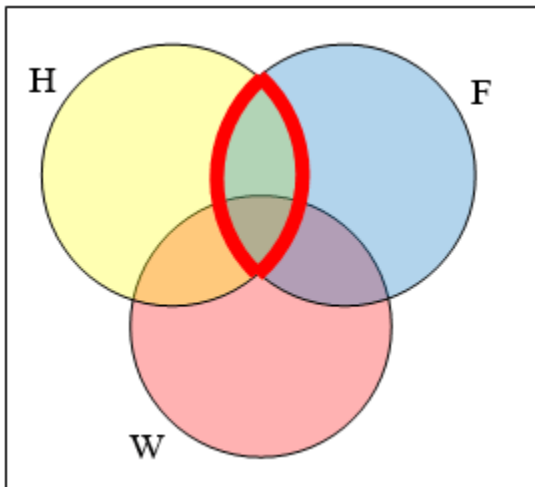


$A \cap B$ contains only those elements in both sets - in the overlap of the circles.

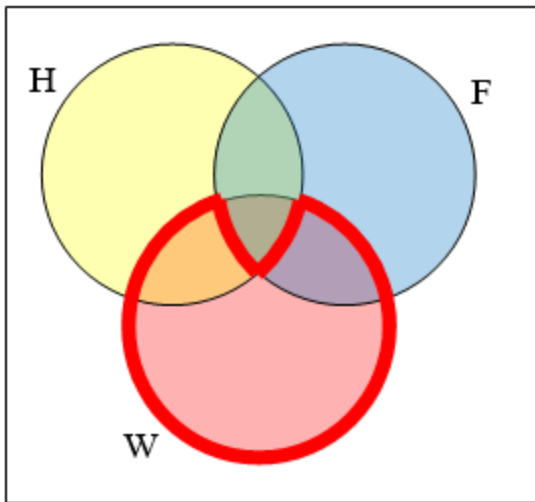
Example 10

Use a Venn diagram to illustrate $(H \cap P)^c \cap W$

We'll start by identifying everything in the set $H \cap P$

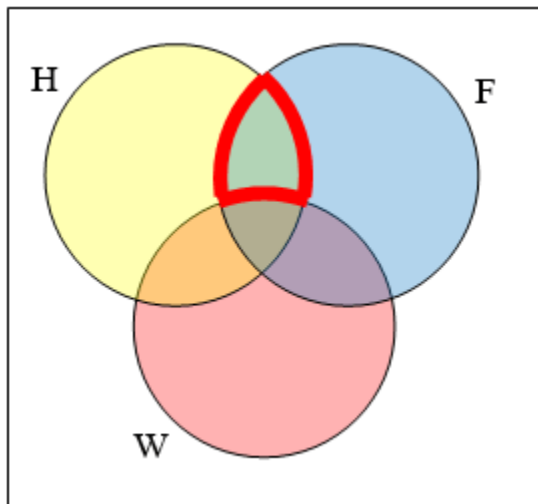


Now, $(H \cap P)^c \cap W$ will contain everything not in the set identified above that is also in set W



Example 11

Create an expression to represent the outlined part of the Venn diagram shown.

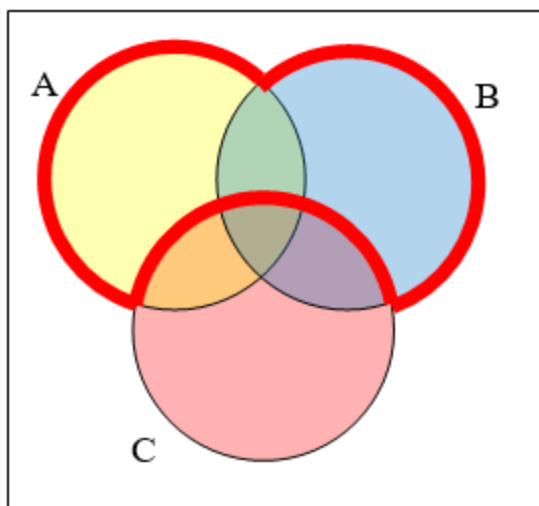


The elements in the outlined set are in sets H and F , but are not in set W

. So we could represent this set as $H \cap F \cap W$

Try it Now 3

Create an expression to represent the outlined portion of the Venn diagram shown



Answer available at the end of this section.

Answers

Try it Now 3

$$A \cup B \cap C^c$$

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4.1.4: Cardinality

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.

The cardinality of the set A is often notated as $|A|$ or $n(A)$

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$

What is the cardinality of B ? $A \cup B$, $A \cap B$?

Solution

The cardinality of B is 4, since there are 4 elements in the set.

The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.

The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

Example 13

What is the cardinality of P = the set of English names for the months of the year?

Solution

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

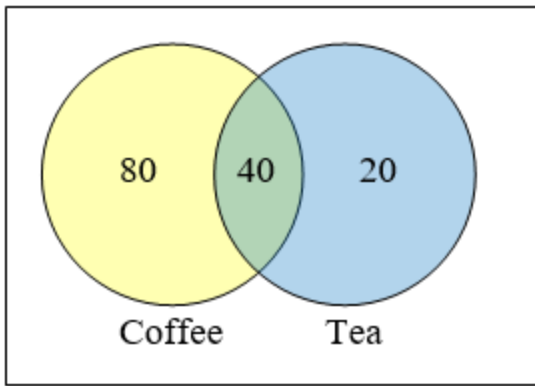
Example 14

A survey asks 200 people “What beverage do you drink in the morning”, and offers choices:

- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

Solution



This question can most easily be answered by creating a Venn diagram.

We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

$$200 - 20 - 80 - 40 = 60 \text{ people who drink neither.}$$

Example 15

A survey asks: Which online services have you used in the last month:

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Solution

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply 70% + 40%, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T , then subtract those in intersection that we've counted twice. In symbols,

$$n(F \cup T) = n(F) + n(T) - n(F \cap T)$$

$$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$. since the universal set contains 100% of people and the cardinality of $F \cup T = 90\%$, the cardinality of $(F \cup T)^c$ must be the other 10%

The previous example illustrated two important properties

Cardinality properties

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A^c) = n(U) - n(A)$$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

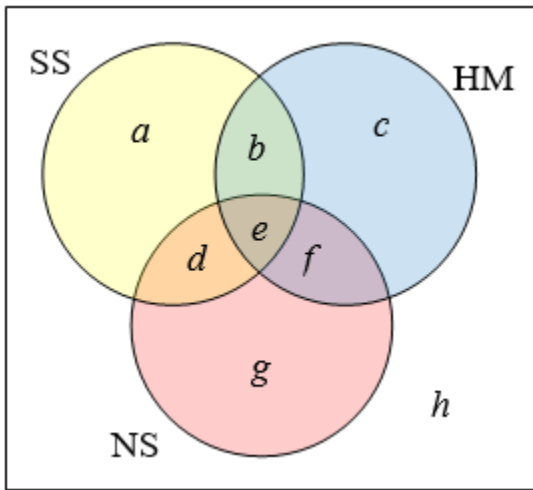
Example 16

Add text here. Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course	26 were taking a HM course
19 were taking a NS course	9 were taking SS and HM
7 were taking SS and NS	10 were taking HM and NS
3 were taking all three	7 were taking none

How many students are only taking a SS course?

Solution



It might help to look at a Venn diagram.

From the given data, we know that there are 3 students in region e and 7 students in region h

since 7 students were taking a SS and NS course, we know that $n(d) + n(e) = 7$. since we know there are 3 students in region 3, there must be

$7 - 3 = 4$ students in region d

Similarly, since there are 10 students taking HM and NS, which includes regions e and f , there must be

$10 - 3 = 7$ students in region f

Since 9 students were taking SS and HM, there must be $9 - 3 = 6$ students in region b

Now, we know that 21 students were taking a SS course. This includes students from regions a, b, d , and e . since we know the number of students in all but region a , we can determine that $21 - 6 - 4 - 3 = 8$ students are in region a

8 students are taking only a SS course.

Try it Now 4

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

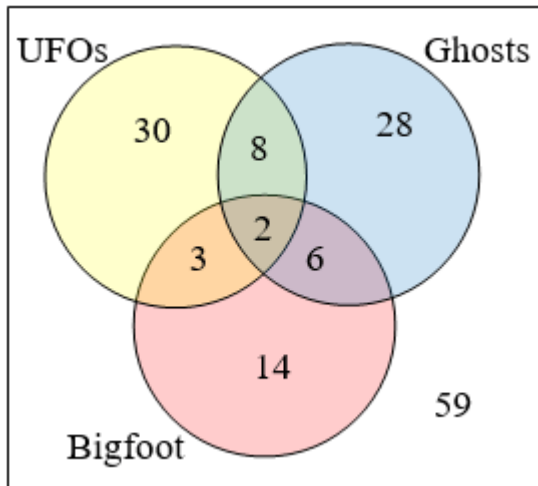
43 believed in UFOs	44 believed in ghosts
25 believed in Bigfoot	10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot	5 believed in UFOs and Bigfoot
2 believed in all three	

How many people surveyed believed in at least one of these things?

Answer available at the end of this section.

Answers

Try it Now 4



Starting with the intersection of all three circles, we work our way out. since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. We work our way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves $150 - 91 = 59$ who believe in none.

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4.1.5: Exercises

1. List out the elements of the set "The letters of the word Mississippi"
2. List out the elements of the set "Months of the year"
3. Write a verbal description of the set $\{3, 6, 9\}$
4. Write a verbal description of the set $\{a, i, e, o, u\}$
5. Is $\{1, 3, 5\}$ a subset of the set of odd integers?
6. Is $\{A, B, C\}$ a subset of the set of letters of the alphabet?

For problems 7-12, consider the sets below, and indicate if each statement is true or false.

$$A = \{1, 2, 3, 4, 5\} \quad B = \{1, 3, 5\} \quad C = \{4, 6\} \quad U = \{\text{numbers from 0 to 10}\}$$

7. $3 \in B$
8. $5 \in C$
9. $B \subset A$
10. $C \subset A$
11. $C \subset B$
12. $C \subset U$

Using the sets from above, and treating U as the Universal set, find each of the following:

13. $A \cup B$
14. $A \cup C$
15. $A \cap C$
16. $B \cap C$
17. A^c
18. B^c

Let $D = \{b, a, c, k\}$, $E = \{t, a, s, k\}$, $F = \{b, a, t, h\}$. Using these sets, find the following:

19. $D^c \cap E$
20. $F^c \cap D$
21. $(D \cap E) \cup F$
22. $D \cap (E \cup P)$
23. $(F \cap E)^c \cap D$
24. $(D \cup E)^c \cap F$

Create a Venn diagram to illustrate each of the following:

25. $(F \cap E) \cup D$

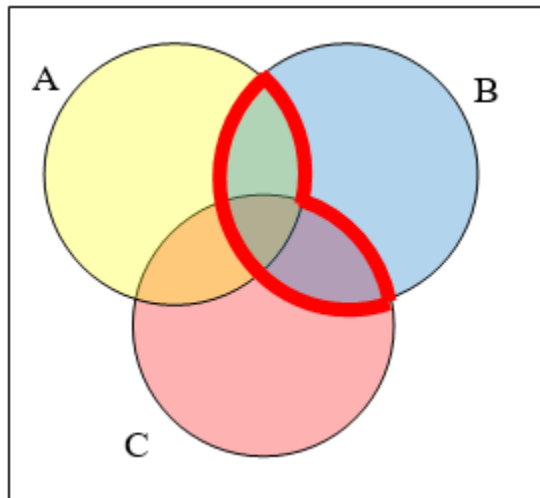
26. $(D \cup E)^c \cap F$

27. $(F^c \cap E') \cap D$

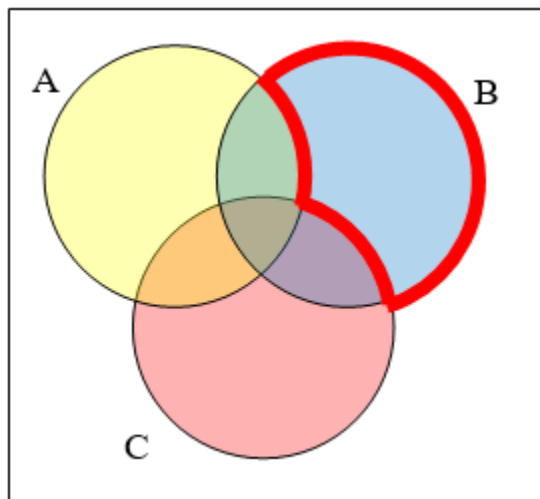
28. $(D \cup E) \cup F$

Write an expression for the shaded region.

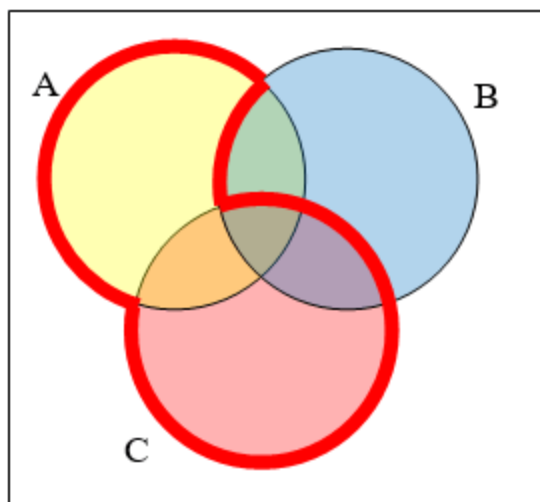
29.



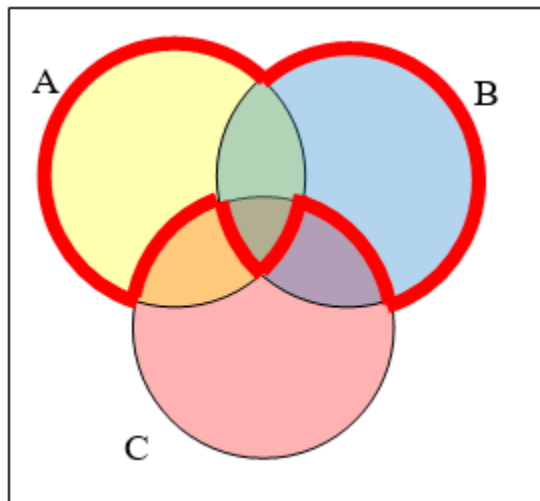
30.



31.



32.



Let $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 3, 5\}$ $C = \{4, 6\}$. Find the cardinality of the given set.

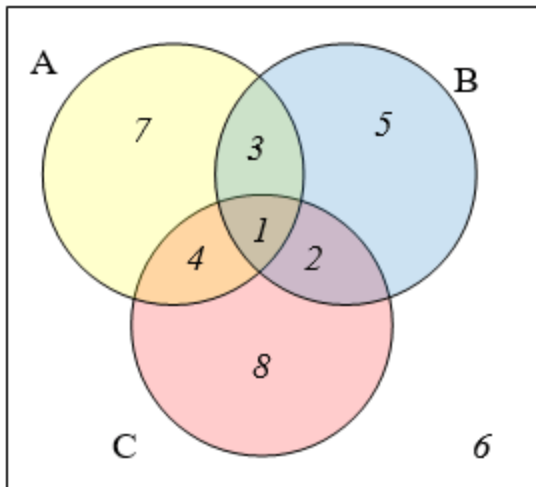
33. $n(A)$

34. $n(B)$

35. $n(A \cup C)$

36. $n(A \cap C)$

The Venn diagram here shows the cardinality of each set. Use this in 37-40 to find the cardinality of given set.



37. $n(A \cap C)$

38. $n(B \cup C)$

39. $n(A \cap B \cap C^c)$

40. $n(A \cap B^c \cap C)$

41. If $n(G) = 20$, $n(H) = 30$, $n(G \cap H) = 5$, find $n(G \cup H)$

42. If $n(G) = 5$, $n(H) = 8$, $n(G \cap H) = 4$, find $n(G \cup H)$

43. A survey was given asking whether they watch movies at home from Netflix, Redbox, or a video store. Use the results to determine how many people use Redbox.

52 only use Netflix	62 only use Redbox
24 only use a video store	16 use only a video store and Redbox
48 use only Netflix and Redbox	30 use only a video store and Netflix
10 use all three	25 use none of these

44. A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?

5 only said color	8 only said size
16 only said brand	20 said only color and size
42 said only color and brand	53 said only size and brand
102 said all three	20 said none of these

45. Use the given information to complete a Venn diagram, then determine: a) how many students have seen exactly one of these movies, and b) how many had seen only *Star Wars*.

18 had seen The Matrix (M)	24 had seen Star Wars (SW)
20 had seen Lord of the Rings (LotR)	10 had seen M and SW
14 had seen LotR and SW	12 had seen M and LotR
6 had seen all three	

46. A survey asked people what alternative transportation modes they use. Using the data to complete a Venn diagram, then determine: a) what percent of people only ride the bus, and b) how many people don't use any alternate transportation.

30 20
25 5
10 12
2

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5: LOGIC

5.1: Logic

5.1.1: Logic

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5.1.5: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

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5.1: LOGIC

5.1.1: Logic

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5.1.9: Logical Fallacies in Common Language

5.1.10: Exercises

5.1.11: De Morgan's Laws

5.1.12: Arguments

5.1.13: Evaluating Deductive Arguments with Euler Diagrams

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5.1.1: Logic

Logic is, basically, the study of valid reasoning. When searching the internet, we use Boolean logic – terms like “and” and “or” – to help us find specific web pages that fit in the sets we are interested in. After exploring this form of logic, we will look at logical arguments and how we can determine the validity of a claim.

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5.1.2: Boolean Logic

We can often classify items as belonging to sets. If you went the library to search for a book and they asked you to express your search using unions, intersections, and complements of sets, that would feel a little strange. Instead, we typically using words like “and”, “or”, and “not” to connect our keywords together to form a search. These words, which form the basis of **Boolean logic**, are directly related to our set operations. (Boolean logic was developed by the 19th-century English mathematician George Boole.)

Boolean logic

Boolean logic combines multiple statements that are either true or false into an expression that is either true or false.

In connection to sets, a search is true if the element is part of the set.

Suppose M is the set of all mystery books, and C is the set of all comedy books. If we search for “mystery”, we are looking for all the books that are an element of the set M ; the search is true for books that are in the set.

When we search for “mystery *and* comedy”, we are looking for a book that is an element of both sets, in the intersection. If we were to search for “mystery *or* comedy”, we are looking for a book that is a mystery, a comedy, or both, which is the union of the sets. If we searched for “*not* comedy”, we are looking for any book in the library that is not a comedy, the complement of the set C .

Connection to Set Operations

A and B elements in the intersection $A \cap B$

A or B elements in the union $A \cup B$

not A elements in the complement A^c

Notice here that *or* is not exclusive. This is a difference between the Boolean logic use of the word and common everyday use. When your significant other asks “do you want to go to the park *or* the movies?” they usually are proposing an exclusive choice – one option or the other, but not both. In Boolean logic, the *or* is not exclusive – more like being asked at a restaurant “would you like fries or a drink with that?” Answering “both, please” is an acceptable answer.

Example 1

Suppose we are searching a library database for Mexican universities. Express a reasonable search using Boolean logic.**Solution**

We could start with the search “Mexico *and* university”, but would be likely to find results for the U.S. state New Mexico. To account for this, we could revise our search to read:

Mexico *and* university *not* “New Mexico”

In most internet search engines, it is not necessary to include the word *and*; the search engine assumes that if you provide two keywords you are looking for both. In Google’s search, the keyword *or* has be capitalized as OR, and a negative sign in front of a word is used to indicate *not*. Quotes around a phrase indicate that the entire phrase should be looked for. The search from the previous example on Google could be written:

Mexico university -“New Mexico”

Example 2

Describe the numbers that meet the condition:

even and less than 10 and greater than 0

Solution

The numbers that satisfy all three requirements are $\{2, 4, 6, 8\}$

Sometimes statements made in English can be ambiguous. For this reason, Boolean logic uses parentheses to show precedent, just like in algebraic order of operations.

Example 3

Describe the numbers that meet the condition:

odd number and less than 20 and greater than 0 and (multiple of 3 or multiple of 5)

Solution

The first three conditions limit us to the set $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

The last grouped conditions tell us to find elements of this set that are also either a multiple of 3 or a multiple of 5. This leaves us with the set $\{3, 5, 9, 15\}$

Notice that we would have gotten a very different result if we had written

(odd number and less than 20 and greater than 0 and multiple of 3) or multiple of 5

The first grouped set of conditions would give $\{3, 9, 15\}$. When combined with the last condition, though, this set expands without limits:

$\{3, 5, 9, 15, 20, 25, 30, 35, 40, 45, \dots\}$

Example 4

The English phrase “Go to the store and buy me eggs and bagels or cereal” is ambiguous; it is not clear whether the requestor is asking for eggs always along with either bagels or cereal, or whether they’re asking for either the combination of eggs and bagels, or just cereal.

For this reason, using parentheses clarifies the intent:

Eggs and (bagels or cereal) means Option 1: Eggs and bagels, Option 2: Eggs and cereal

(Eggs and bagels) or cereal means Option 1: Eggs and bagels, Option 2: Cereal

Be aware that when a string of conditions is written without grouping symbols, it is often interpreted from the left to right, resulting in the latter interpretation.

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5.1.3: Conditional Statements

Beyond searching, Boolean logic is commonly used in spreadsheet applications like Excel to do conditional calculations. A **statement** is something that is either true or false. A statement like $3 < 5$ is true; a statement like “a rat is a fish” is false. A statement like “ $x < 5$ ” is true for some values of x and false for others. When an action is taken or not depending on the value of a statement, it forms a **conditional**.

Statements and Conditionals

A **statement** is either true or false.

A **conditional** is a compound statement of the form

“if p then q ” or “if p then q , else s ”

Example 5

In common language, an example of a conditional statement would be “If it is raining, then we’ll go to the mall. Otherwise we’ll go for a hike.”

The statement “If it is raining” is the condition – this may be true or false for any given day. If the condition is true, then we will follow the first course of action, and go to the mall. If the condition is false, then we will use the alternative, and go for a hike.

Example 5.1.3.1

As mentioned earlier, conditional statements are commonly used in spreadsheet applications like Excel or Google Sheets. In Excel, you can enter an expression like

= IF (A1 < 2000, A1 + 1, A1 × 2)

Notice that after the IF, there are three parts. The first part is the condition, and the second two are calculations. Excel will look at the value in cell A1 and compare it to 2000. If that condition is true, then the first calculation is used, and 1 is added to the value of A1 and the result is stored. If the condition is false, then the second calculation is used, and A1 is multiplied by 2 and the result is stored.

In other words, this statement is equivalent to saying “If the value of A1 is less than 2000, then add 1 to the value in A1. Otherwise, multiply A1 by 2”.

Example 7

The expression = IF (A1 > 5, 2 × A1, 3 × A1) is used. Find the result if A1 is 3, and the result if A1 is 8.

Solution

If $A1 > 5$, then calculate $2 \times A1$. Otherwise, calculate $3 \times A1$.

If A1 is 3, then the condition is false, since $3 > 5$ is not true, so we do the alternate action, and multiply by 3, giving $3 \times 3 = 9$

If A1 is 8, then the condition is true, since $8 > 5$, so we multiply the value by 2, giving $2 \times 8 = 16$

Example 8

An accountant needs to withhold 15% of income for taxes if the income is below \$30,000, and 20% of income if the income is \$30,000 or more. Write an expression that would calculate the amount to withhold.

Our conditional needs to compare the value to 30,000. If the income is less than 30,000, we need to calculate 15% of the income: $0.15 \times \text{income}$. If the income is more than 30,000, we need to calculate 20% of the income: $0.20 \times \text{income}$.

In words we could write "If income < 30,000, then multiply by 0.15, otherwise multiply by 0.20". In Excel, we would write:

= IF (A1 < 30000, 0.15 × A1, 0.20 × A1)

As we did earlier, we can create more complex conditions by using the operators *and*, *or*, and *not* to join simpler conditions together.

Example 9

A parent might say to their child "if you clean your room and take out the garbage, then you can have ice cream."

Here, there are two simpler conditions:

- 1) The child cleaning her room
- 2) The child taking out the garbage

Since these conditions were joined with *and*, the combined conditional will be true only if both simpler conditions are true; if either chore is not completed, then the parent's condition is not met.

Notice that if the parent had said "if you clean your room *or* take out the garbage, then you can have ice cream", then the child would need to complete only one chore to meet the condition.

Suppose you wanted to have something happen when a certain value is between 100 and 300. To create the condition " $A1 < 300$ and $A1 > 100$ " in Excel, you would need to enter " $\text{AND}(A1 < 300, A1 > 100)$ ". Likewise, for the condition " $A1 = 4$ or $A1 = 6$ " you would enter " $\text{OR}(A1 = 4, A1 = 6)$ ".

Example 10

In a spreadsheet, cell A1 contains annual income, and A2 contains number of dependents.

A certain tax credit applies if someone with no dependents earns less than \$10,000, or if someone with dependents earns less than \$20,000. Write a rule that describes this.

There are two ways the rule is met:

income is less than 10,000 *and* dependents is 0, *or*

income is less than 20,000 *and* dependents is not 0.

Informally, we could write these as

$(A1 < 10000 \text{ and } A2 = 0) \text{ or } (A1 < 20000 \text{ and } A2 > 0)$

In Excel's format, we'd write

IF (OR(AND(A1 < 10000, A2 = 0), AND(A1 < 20000, A2 > 0)), "you qualify", "you don't qualify")

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5.1.4: Quantified Statements

Words that describe an entire set, such as “all”, “every”, or “none”, are called **universal quantifiers** because that set could be considered a universal set. In contrast, words or phrases such as “some”, “one”, or “at least one” are called **existential quantifiers** because they describe the existence of at least one element in a set.

Quantifiers

A **universal quantifier** states that an entire set of things share a characteristic.

An **existential quantifier** states that a set contains at least one element.

Something interesting happens when we **negate** – or state the opposite of – a quantified statement.

Example 11

Suppose your friend says “Everybody cheats on their taxes.” What is the minimum amount of evidence you would need to prove your friend wrong?

To show that it is not true that everybody cheats on their taxes, all you need is one person who does not cheat on their taxes. It would be perfectly fine to produce more people who do not cheat, but one counterexample is all you need.

It is important to note that you do not need to show that absolutely nobody cheats on their taxes.

Example 12

Suppose your friend says “One of these six cartons of milk is leaking.” What is the minimum amount of evidence you would need to prove your friend wrong?

Solution

In this case, you would need to check all six cartons and show that none of them is leaking. You cannot disprove your friend’s statement by checking only one of the cartons.

When we negate a statement with a universal quantifier, we get a statement with an existential quantifier, and vice-versa.

Negating a quantified statement

The negation of “all A are B” is “at least one A is not B”.

The negation of “no A are B” is “at least one A is B”.

The negation of “at least one A is B” is “no A are B”.

The negation of “at least one A is not B” is “all A are B”.

Example 13

“Somebody brought a flashlight.” Write the negation of this statement.

The negation is “Nobody brought a flashlight.”

Example 14

“There are no prime numbers that are even.” Write the negation of this statement.

The negation is “At least one prime number is even.”

Try it Now 1

Write the negation of “All Icelandic children learn English in school.”

Answer available at the end of this section.

Answers

Try it Now 1

At least one Icelandic child did not learn English in school.

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5.1.5: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

Before we focus on truth tables, we're going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

Symbols

The symbol $\wedge$ is used for and: A and B is notated $A \wedge B$

The symbol $\vee$ is used for or: A or B is notated $A \vee B$

The symbol $\sim$ is used for not: not A is notated $\sim A$

You can remember the first two symbols by relating them to the shapes for the union and intersection. $A \wedge B$ would be the elements that exist in both sets, in $A \cap B$. Likewise, $A \vee B$ would be the elements that exist in either set, in $A \cup B$. When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

Example 15

Translate each statement into symbolic notation. Let P represent "I like Pepsi" and let C represent "I like Coke".

- I like Pepsi or I like Coke.
- I like Pepsi and I like Coke.
- I do not like Pepsi.
- It is not the case that I like Pepsi or Coke.
- I like Pepsi and I do not like Coke.

Solution

- $P \vee C$
- $P \wedge C$
- $\sim P$
- $\sim (P \vee C)$
- $P \wedge \sim C$

As you can see, we can use parentheses to organize more complicated statements.

Try it Now 2

Translate "We have carrots or we will not make soup" into symbols. Let C represent "we have carrots" and let S represent "we will make soup".

Answer available at the end of this section.

Because complex Boolean statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true and false.

Truth Table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 16

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise".

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use *S* to designate "is a sectional", and *C* to designate "has a chaise".

A truth table for this situation would look like this:

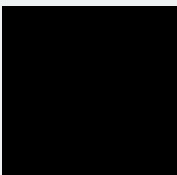


In the table, T is used for true, and F for false. In the first row, if *S* is true and *C* is also true, then the complex statement "*S or C*" is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that *or* in logic is not exclusive; if the couch has both features, it meets the condition.)

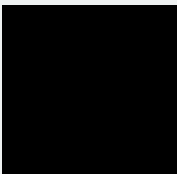
In the previous example about the couch, the truth table was really just summarizing what we already know about how the *or* statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

Basic truth tables

Conjunction



Disjunction



Negation



Truth tables really become useful when we analyze more complex Boolean statements.

Example 17

Create a truth table for the statement $A \vee \sim B$

Solution

When we create the truth table, we need to list all the possible truth value combinations for A and B . Notice how the first column contains 2 Ts followed by 2 Fs, and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.

After creating columns with those initial values, we create a third column for the expression $\sim B$. Now we will temporarily ignore the column for A and write the truth values for $\sim B$

Next we can find the truth values of $A \vee \sim B$, using the first and third columns.

The truth table shows that $A \vee \sim B$ is true in three cases and false in one case. If you're wondering what the point of this is, suppose it is the last day of the baseball season and two teams, who are not playing each other, are competing for the final playoff spot. Anaheim will make the playoffs if it wins its game or if Boston does not win its game. (Anaheim owns the tie-breaker; if both teams win, or if both teams lose, then Anaheim gets the playoff spot.) If A = Anaheim wins its game and B = Boston wins its game, then $A \vee \sim B$ represents the situation "Anaheim wins its game or Boston does not win its game". The truth table shows us the different scenarios related to Anaheim making the playoffs. In the first row, Anaheim wins its game and Boston wins its game, so it is true that Anaheim makes the playoffs. In the second row, Anaheim wins and Boston does not win, so it is true that Anaheim makes the playoffs. In the third row, Anaheim does not win its game and Boston wins its game, so it is false that Anaheim makes the playoffs. In the fourth row, Anaheim does not win and Boston does not win, so it is true that Anaheim makes the playoffs.

Try it Now 3

Create a truth table for this statement: $\sim A \wedge B$

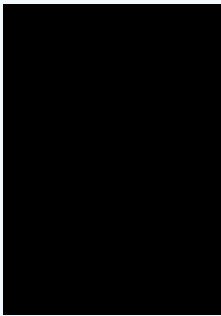
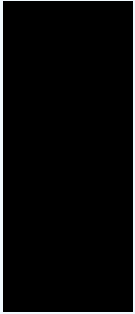
Answer available at the end of this section.

Example 18

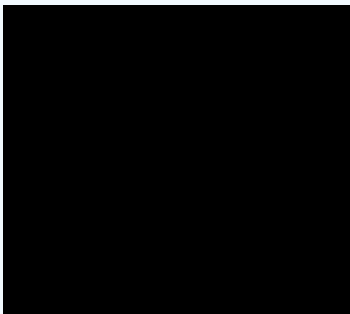
Create a truth table for the statement $A \wedge \sim (B \vee C)$

Solution

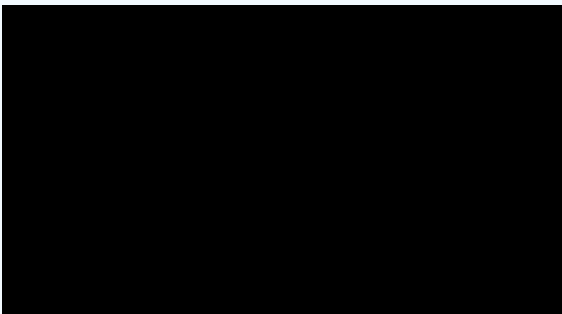
It helps to work from the inside out when creating a truth table, and to create columns in the table for intermediate operations. We start by listing all the possible truth value combinations for A , B , and C . Notice how the first column contains 4 Ts followed by 4Fs, the second column contains 2Ts, 2Fs, then repeats, and the last column alternates T, F, T, F ... This pattern ensures that all 8 combinations are considered. After creating columns with those initial values, we create a fourth column for the innermost expression, $B \vee C$. Now we will temporarily ignore the column for A and focus on B and C , writing the truth values for $B \vee C$



Next we can find the negation of $B \vee C$, working off the $B \vee C$ column we just created. (Ignore the first three columns and simply negate the values in the $B \vee C$ column.)



Finally, we find the values of A and $\sim (B \vee C)$. (Ignore the second, third, and fourth columns.)



It turns out that this complex expression is true in only one case: when A is true, B is false, and C is false. To illustrate this situation, suppose that Anaheim will make the playoffs if: (1) Anaheim wins, and (2) neither Boston nor Cleveland wins. IFF is the only scenario in which Anaheim will make the playoffs.

Try it Now 4

Create a truth table for this statement: $(\sim A \wedge B) \vee \sim B$

Answer available at the end of this section.

Answers

Try it Now 2

$$C \vee \sim S$$

Try it Now 3



Try it Now 4



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5.1.6: Truth Tables- Conditional, Biconditional

We discussed conditional statements earlier, in which we take an action based on the value of the condition. We are now going to look at another version of a conditional, sometimes called an implication, which states that the second part must logically follow from the first.

Conditional

A conditional is a logical compound statement in which a statement p , called the antecedent, implies a statement q , called the consequent.

A conditional is written as $p \rightarrow q$ and is translated as "if p , then q ".

Example 19

The English statement "If it is raining, then there are clouds in the sky" is a conditional statement. It makes sense because if the antecedent "it is raining" is true, then the consequent "there are clouds in the sky" must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consequent becomes irrelevant.

Example 20

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday's game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

There are four possible outcomes:

- 1) You pay for expedited shipping and receive the jersey by Friday
- 2) You pay for expedited shipping and don't receive the jersey by Friday
- 3) You don't pay for expedited shipping and receive the jersey by Friday
- 4) You don't pay for expedited shipping and don't receive the jersey by Friday

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don't receive the jersey by Friday. The first outcome is exactly what was promised, so there's no problem with that. The third outcome is not a lie because the website never said what would happen if you didn't pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn't make any promises about when the jersey would arrive if you didn't pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the antecedent is false, we cannot make any judgment about the consequent. The website never said that paying for expedited shipping was the *only* way to receive the jersey by Friday.

Example 21

A friend tells you “If you upload that picture to Facebook, you’ll lose your job.” Under what conditions can you say that your friend was wrong?

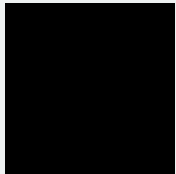
There are four possible outcomes:

- 1) You upload the picture and lose your job
- 2) You upload the picture and don’t lose your job
- 3) You don’t upload the picture and lose your job
- 4) You don’t upload the picture and don’t lose your job

There is only one possible case in which you can say your friend was wrong: the second outcome in which you upload the picture but still keep your job. In the last two cases, your friend didn’t say anything about what would happen if you didn’t upload the picture, so you can’t say that their statement was wrong. Even if you didn’t upload the picture and lost your job anyway, your friend never said that you were guaranteed to keep your job if you didn’t upload the picture; you might lose your job for missing a shift or punching your boss instead.

In traditional logic, a conditional is considered true as long as there are no cases in which the antecedent is true and the consequent is false.

Truth table for the conditional



Again, if the antecedent p is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true.

Example 22

Construct a truth table for the statement $(m \wedge \sim p) \rightarrow r$

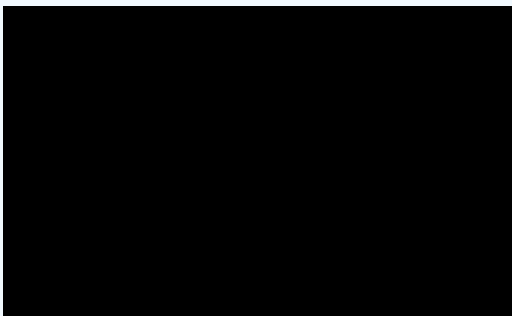
Solution

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the antecedent, $m \wedge \sim p$.





Now we can create a column for the conditional. Because it can be confusing to keep track of all the Ts and Fs, why don't we copy the column for r to the right of the column for $m \wedge \sim p$? This makes it a lot easier to read the conditional from left to right.



When m is true, p is false, and r is false- the fourth row of the table-then the antecedent $m \wedge \sim p$ will be true but the consequent false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by $(m \wedge \sim p) \rightarrow r$, consider this: let m = we order meatballs, p = we order pasta, and r = Rob is happy. The statement $(m \wedge \sim p) \rightarrow r$ is "if we order meatballs and don't order pasta, then Rob is happy". If m is true (we order meatballs), p is false (we don't order pasta), and r is false (Rob is not happy), then the statement is false, because we satisfied the antecedent but Rob did not satisfy the consequent.

For any conditional, there are three related statements, the converse, the inverse, and the contrapositive.

Related Statments

The original conditional is "if p , then q " $p \rightarrow q$

The converse is "if q , then p " $q \rightarrow p$

The inverse is "if not p , then not q " $\sim p \rightarrow \sim q$

The contrapositive is "if not q , then not p " $\sim q \rightarrow \sim p$

Example 23

Consider again the conditional “If it is raining, then there are clouds in the sky.” It seems reasonable to assume that this is true.

The converse would be “If there are clouds in the sky, then it is raining.” This is not always true.

The inverse would be “If it is not raining, then there are not clouds in the sky.” Likewise, this is not always true.

The contrapositive would be “If there are not clouds in the sky, then it is not raining.” This statement is true, and is equivalent to the original conditional.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Conditional	Converse	Inverse	Contrapositive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalent

Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a conditional statement are logically equivalent.

In other words, the original statement and the contrapositive must agree with each other; they must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other; they must both be true, or they must both be false.

Be aware that symbolic logic cannot represent the English language perfectly. For example, we may need to change the verb tense to show that one thing occurred before another.

Example 24

Suppose this statement is true: “If I eat this giant cookie, then I will feel sick.” Which of the following statements must also be true?

- If I feel sick, then I ate that giant cookie.
- If I don't eat this giant cookie, then I won't feel sick.
- If I don't feel sick, then I didn't eat that giant cookie.

Solution

- a. This is the converse, which is not necessarily true. I could feel sick for some other reason, such as drinking sour milk.
- b. This is the inverse, which is not necessarily true. Again, I could feel sick for some other reason; avoiding the cookie doesn't guarantee that I won't feel sick.
- c. This is the contrapositive, which is true, but we have to think somewhat backwards to explain it. If I ate the cookie, I would feel sick, but since I don't feel sick, I must not have eaten the cookie.

Notice again that the original statement and the contrapositive have the same truth value (both are true), and the converse and the inverse have the same truth value (both are false).

Try it Now 5

"If you microwave salmon in the staff kitchen, then I will be mad at you." If this statement is true, which of the following statements must also be true?

- a. If you don't microwave salmon in the staff kitchen, then I won't be mad at you.
- b. If I am not mad at you, then you didn't microwave salmon in the staff kitchen.
- c. If I am mad at you, then you microwaved salmon in the staff kitchen.

Answer available at the end of this section.

Consider the statement "If you park here, then you will get a ticket." What set of conditions would prove this statement false?

- a. You don't park here and you get a ticket.
- b. You don't park here and you don't get a ticket.
- c. You park here and you don't get a ticket.

The first two statements are irrelevant because we don't know what will happen if you park somewhere else. The third statement, however contradicts the conditional statement "If you park here, then you will get a ticket" because you parked here but didn't get a ticket. This example demonstrates a general rule; the negation of a conditional can be written as a conjunction: "It is not the case that if you park here, then you will get a ticket" is equivalent to "You park here *and* you do not get a ticket."

The Negation of a Conditional

The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

$$\sim (p \rightarrow q) \text{ is equivalent to } p \wedge \sim q$$

Example 25

Which of the following statements is equivalent to the negation of "If you don't grease the pan, then the food will stick to it"?

- a. I didn't grease the pan and the food didn't stick to it.
- b. I didn't grease the pan and the food stuck to it.
- c. I greased the pan and the food didn't stick to it.

Solution

- a. This is correct; it is the conjunction of the antecedent and the negation of the consequent. To disprove that not greasing the pan will cause the food to stick, I have to not grease the pan and have the food not stick.
- b. This is essentially the original statement with no negation; the "if...then" has been replaced by "and".
- c. This essentially agrees with the original statement and cannot disprove it.

Try it Now 6

"If you go swimming less than an hour after eating lunch, then you will get cramps." Which of the following statements is equivalent to the negation of this statement?

- I went swimming more than an hour after eating lunch and I got cramps.
- I went swimming less than an hour after eating lunch and I didn't get cramps.
- I went swimming more than an hour after eating lunch and I didn't get cramps.

Answer available at the end of this section.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider "If you submit your hours today, then you will be paid next Friday." What the payroll rep really means is "If you submit your hours today, then you will be paid next Friday, and if you don't submit your hours today, then you won't be paid next Friday." The conditional statement if t , then p also includes the inverse of the statement: if not t , then not p . A more compact way to express this statement is "You will be paid next Friday *if and only if* you submit your timesheet today." A statement of this form is called a **biconditional**.

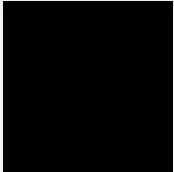
Biconditional

A biconditional is a logical conditional statement in which the antecedent and consequent are interchangeable.

A biconditional is written as $p \leftrightarrow q$ and is translated as " p if and only if q ".

Because a biconditional statement $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$, we may think of it as a conditional statement combined with its converse: if p , then q and if q , then p . The double-headed arrow shows that the conditional statement goes from left to right and from right to left. A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

Truth table for the biconditional



Notice that the fourth row, where both components are false, is true; if you don't submit your timesheet and you don't get paid, the person from payroll told you the truth.

Example 26

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?

- It is noon on Thursday and the garbage truck did not come down my street this morning.
- It is Monday and the garbage truck is coming down my street.
- It is Wednesday at 11:59PM and the garbage truck did not come down my street today.

Solution

- This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.

- b. This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.
- c. This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

Try it Now 7

Suppose this statement is true: “I wear my running shoes if and only if I am exercising.” Determine whether each of the following statements must be true or false.

- a. I am exercising and I am not wearing my running shoes.
- b. I am wearing my running shoes and I am not exercising.
- c. I am not exercising and I am not wearing my running shoes.

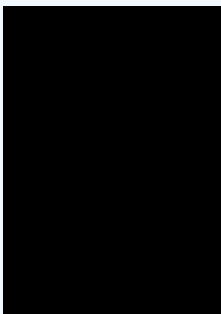
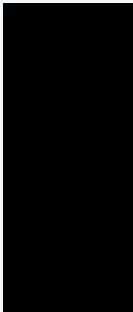
Answer available at the end of this section.

Example 27

Create a truth table for the statement $(A \vee B) \leftrightarrow \sim C$

Solution

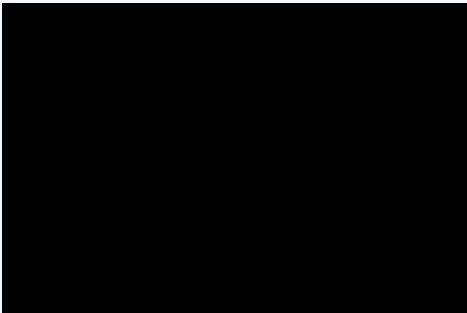
Whenever we have three component statements, we start by listing all the possible truth value combinations for A , B , and C . After creating those three columns, we can create a fourth column for the antecedent, $A \vee B$. Now we will temporarily ignore the column for C and focus on A and B , writing the truth values for $A \vee B$.



Next we can create a column for the negation of C . (Ignore the $A \vee B$ column and simply negate the values in the C column.)



Finally, we find the truth values of $(A \vee B) \leftrightarrow \sim C$. Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.



To illustrate this situation, suppose your boss needs you to do either project A or project B (or both, if you have the time). If you do one of the projects, you will not get a crummy review (C is for crummy). So $(A \vee B) \leftrightarrow \sim C$ means "You will not get a crummy review if and only if you do project A or project B ." Looking at a few of the rows of the truth table, we can see how this works out. In the first row, A , B , and C are all true: you did both projects and got a crummy review, which is not what your boss told you would happen! That is why the final result of the first row is false. In the fourth row, A is true, B is false, and C is false: you did project A and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row, A , B , and C are all false: you didn't do either project and did not get a crummy review. This is not what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)

Answers

Try it Now 5

Choice b is correct because it is the contrapositive of the original statement.

Try it Now 6

Choice b is equivalent to the negation; it keeps the first part the same and negates the second part.

Try it Now 7

Choices a & b are false; c is true.

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5.1.7: Evaluating Deductive Arguments with Truth Tables

Arguments can also be analyzed using truth tables, although this can be a lot of work.

Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for the statement. If it is always true, then the argument is valid.

Example 34

Consider the argument

Premise: If you bought bread, then you went to the store.

Premise: You bought bread.

Conclusion: You went to the store.

Solution

While this example is fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then form a conditional statement showing that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

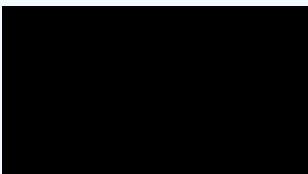
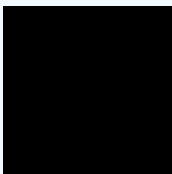
We'll let b represent "you bought bread" and s represent "you went to the store". Then the argument becomes:

Premise: $b \rightarrow s$

Premise: b

Conclusion: s

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(b \rightarrow s) \wedge b] \rightarrow s$?





Since the truth table for $[(b \rightarrow s) \wedge b] \rightarrow s$ is always true, this is a valid argument.

Try it Now 13

Determine whether the argument is valid:

Premise: If I have a shovel, I can dig a hole.

Premise: I dug a hole.

Conclusion: Therefore, I had a shovel.

Answer available at the end of this section.

Example 35

Premise: If I go to the mall, then I'll buy new jeans.

Premise: If I buy new jeans, I'll buy a shirt to go with it.

Conclusion: If I go to the mall, I'll buy a shirt.

Solution

Let m = I go to the mall, j = I buy jeans, and s = I buy a shirt.

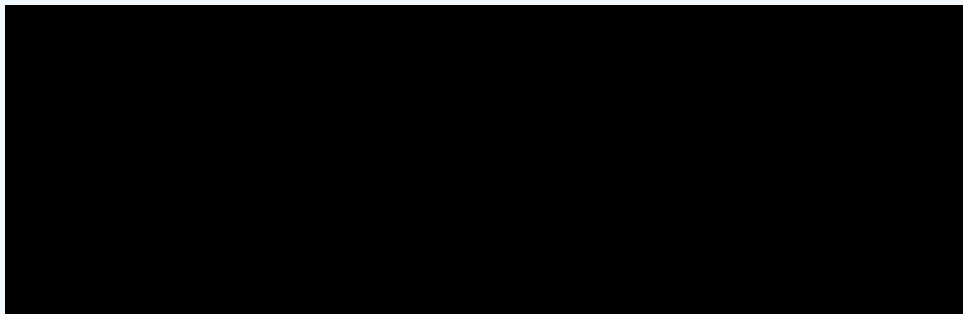
The premises and conclusion can be stated as:

Premise: $m \rightarrow j$

Premise: $j \rightarrow s$

Conclusion: $m \rightarrow s$

We can construct a truth table for $[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$. Try to recreate each step and see how the truth table was constructed.

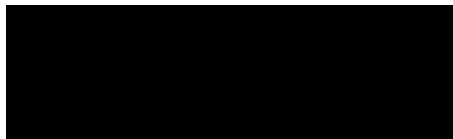


From the final column of the truth table, we can see this is a valid argument.

Answers

Try it Now 13

Let S = have a shovel, D = dig a hole. The first premise is equivalent to $S \rightarrow D$. The second premise is D . The conclusion is S . We are testing $[(S \rightarrow D) \wedge D] \rightarrow S$



This is not a tautology, so this is an invalid argument.

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5.1.8: Forms of Valid Arguments

Rather than making a truth table for every argument, we may be able to recognize certain common forms of arguments that are valid (or invalid). If we can determine that an argument fits one of the common forms, we can immediately state whether it is valid or invalid.

The Law of Detachment (*Modus Ponens*)

The law of detachment applies when a conditional and its antecedent are given as premises, and the consequent is the conclusion. The general form is:

Premise: $p \rightarrow q$
Premise: p
Conclusion: q

The Latin name, *modus ponens*, translates to “mode that affirms”.

Example 36

Recall this argument from an earlier example:

Premise: If you bought bread, then you went to the store.
Premise: You bought bread.
Conclusion: You went to the store.

In symbolic form:

Premise: $b \rightarrow s$
Premise: b
Conclusion: s

This argument has the structure described by the law of detachment. (The second premise and the conclusion are simply the two parts of the first premise detached from each other.) Instead of making a truth table, we can say that this argument is valid by stating that it satisfies the law of detachment.

The Law of Contraposition (*Modus Tollens*)

The law of contraposition applies when a conditional and the negation of its consequent are given as premises, and the negation of its antecedent is the conclusion. The general form is:

Premise: $p \rightarrow q$
Premise: $\sim q$
Conclusion: $\sim p$

The Latin name, *modus tollens*, translates to “mode that denies”.

Notice that the second premise and the conclusion look like the contrapositive of the first premise, $\sim q \rightarrow \sim p$, but they have been detached. You can think of the law of contraposition as a combination of the law of detachment and the fact that the contrapositive is logically equivalent to the original statement.

Example 37

Premise: If I drop my phone into the swimming pool, my phone will be ruined.
Premise: My phone isn't ruined.
Conclusion: I didn't drop my phone into the swimming pool.

If we let d = I drop the phone in the pool and r = the phone is ruined, then we can represent the argument this way:

Premise: $d \rightarrow r$
Premise: $\sim r$
Conclusion: $\sim d$

The form of this argument matches what we need to invoke the law of contraposition, so it is a valid argument.

Try it Now 14

Is this argument valid?

Premise: If you brushed your teeth before bed, then your toothbrush will be wet.
Premise: Your toothbrush is dry.
Conclusion: You didn't brush your teeth before bed.

Answer available at the end of this section.

The Transitive Property (Hypothetical Syllogism)

The transitive property has as its premises a series of conditionals, where the consequent of one is the antecedent of the next. The conclusion is a conditional with the same antecedent as the first premise and the same consequent as the final premise. The general form is:

Premise: $p \rightarrow q$
Premise: $q \rightarrow r$
Conclusion: $p \rightarrow r$

The earlier example about buying a shirt at the mall is an example illustrating the transitive property. It describes a chain reaction: if the first thing happens, then the second thing happens, and if the second thing happens, then the third thing happens. Therefore, if we want to ignore the second thing, we can say that if the first thing happens, then we know the third thing will happen. We don't have to mention the part about buying jeans; we can simply say that the first event leads to the final event. We could even have more than two premises; as long as they form a chain reaction, the transitive property will give us a valid argument.

Example 38

Premise: If a soccer player commits a reckless foul, she will receive a yellow card.
Premise: If Hayley receives a yellow card, she will be suspended for the next match.
Conclusion: If Hayley commits a reckless foul, she will be suspended for the next match.

If we let r = committing a reckless foul, y = receiving a yellow card, and s = being suspended, then our argument looks like this:

Premise: $r \rightarrow y$
Premise: $y \rightarrow s$
Conclusion: $r \rightarrow s$

This argument has the exact structure required to use the transitive property, so it is a valid argument.

Try it Now 15

Is this argument valid?

Premise: If the old lady swallows a fly, she will swallow a spider.
Premise: If the old lady swallows a spider, she will swallow a bird.
Premise: If you brushed your teeth before bed, then your toothbrush will be wet.
Premise: If you brushed your teeth before bed, then your toothbrush will be wet.
Premise: If the old lady swallows a bird, she will swallow a cat.
Premise: If the old lady swallows a cat, she will swallow a dog.
Premise: If the old lady swallows a dog, she will swallow a goat.
Premise: If the old lady swallows a goat, she will swallow a cow.
Premise: If the old lady swallows a cow, she will swallow a horse.
Premise: If the old lady swallows a horse, she will die, of course.
Conclusion: If the old lady swallows a fly, she will die, of course.

Answer available at the end of this section.

Disjunctive Syllogism

In a disjunctive syllogism, the premises consist of an *or* statement and the negation of one of the options. The conclusion is the other option. The general form is:

Premise: $p \vee q$
Premise: $\sim p$
Conclusion: q

The order of the two parts of the disjunction isn't important. In other words, we could have the premises $p \vee q$ and $\sim q$, and the conclusion p

Example 39

Premise: I can either drive or take the train.
Premise: I refuse to drive.
Conclusion: I will take the train.

If we let $d = I$ drive and $t = I$ take the train, then the symbolic representation of the argument is:

Premise: $d \vee t$
Premise: $\sim d$
Conclusion: t

This argument is valid because it has the form of a disjunctive syllogism. I have two choices, and one of them is not going to happen, so the other one must happen.

Try it Now 16

Is this argument valid?

Premise: Alison was required to write a 10-page paper or give a 5-minute speech.

Premise: Alison did not give a 5-minute speech.

Conclusion: Alison wrote a 10-page paper.

Answer available at the end of this section.

Keep in mind that, when you are determining the validity of an argument, you must assume that the premises are true. If you don't agree with one of the premises, you need to keep your personal opinion out of it. Your job is to pretend that the premises are true and then determine whether they force you to accept the conclusion. You may attack the premises in a court of law or a political discussion, of course, but here we are focusing on the structure of the arguments, not the truth of what they actually say.

We have just looked at four forms of valid arguments; there are two common forms that represent *invalid* arguments, which are also called *fallacies*.

The Fallacy of the Converse

The fallacy of the converse arises when a conditional and its consequent are given as premises, and the antecedent is the conclusion. The general form is:

Premise: $p \rightarrow q$

Premise: q

Conclusion: p

Notice that the second premise and the conclusion look like the converse of the first premise, $q \rightarrow p$, but they have been detached. The fallacy of the converse incorrectly tries to assert that the converse of a statement is equivalent to that statement.

Example 40

Premise: If I drink coffee after noon, then I have a hard time falling asleep that night.

Premise: I had a hard time falling asleep last night.

Conclusion: I drank coffee after noon yesterday.

If we let c = I drink coffee after noon and h = I have a hard time falling asleep, then our argument looks like this:

Premise: $c \rightarrow h$

Premise: h

Conclusion: c

This argument uses converse reasoning, so it is an invalid argument. There could be plenty of other reasons why I couldn't fall asleep: I could be worried about money, my neighbors might have been setting off fireworks, ...

Try it Now 17

Is this argument valid?

Premise: If you pull that fire alarm, you will get in big trouble.

Premise: You got in big trouble.

Conclusion: You must have pulled the fire alarm.

Answer available at the end of this section.

The Fallacy of the Inverse

The fallacy of the inverse occurs when a conditional and the negation of its antecedent are given as premises, and the negation of the consequent is the conclusion. The general form is:

Premise: $p \rightarrow q$
Premise: $\sim p$
Conclusion: $\sim q$

Again, notice that the second premise and the conclusion look like the inverse of the first premise, $\sim p \rightarrow \sim q$, but they have been detached. The fallacy of the inverse incorrectly tries to assert that the inverse of a statement is equivalent to that statement.

Example 41

Premise: If you listen to the Grateful Dead, then you are a hippie.
Premise: Sky doesn't listen to the Grateful Dead.
Conclusion: Sky is not a hippie.

If we let g = listen to the Grateful Dead and h = is a hippie, then this is the argument:

Premise: $g \rightarrow h$
Premise: $\sim g$
Conclusion: $\sim h$

This argument is invalid because it uses inverse reasoning. The first premise does not imply that all hippies listen to the Grateful Dead; there could be some hippies who listen to Phish instead.

Try it Now 18

Is this argument valid?

Premise: If a hockey player trips an opponent, he will be assessed a 2-minute penalty.
Premise: Alexei did not trip an opponent.
Conclusion: Alexei will not be assessed a 2-minute penalty.

Answer available at the end of this section.

Of course, arguments are not limited to these six basic forms; some arguments have more premises, or premises that need to be rearranged before you can see what is really happening. There are plenty of other forms of arguments that are invalid. If an argument doesn't seem to fit the pattern of any of these common forms, though, you may want to use a Venn diagram or a truth table instead.

Lewis Carroll, author of *Alice's Adventures in Wonderland*, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected using syllogisms. The following example is one such puzzle.

Example 42

Solve the puzzle. In other words, find a logical conclusion from these premises.

All babies are illogical.

Nobody is despised who can manage a crocodile.

Illogical persons are despised.

Let b = is a baby, d = is despised, i = is illogical, and m = can manage a crocodile.

Then we can write the premises as:

$$b \rightarrow i$$

$$m \rightarrow \sim d$$

$$i \rightarrow d$$

Writing the second premise correctly can be a challenge; it can be rephrased as “If you can manage a crocodile, then you are not despised.”

Using the transitive property with the first and third premises, we can conclude that $b \rightarrow d$, that all babies are despised. Using the contrapositive of the second premise, $d \rightarrow \sim m$, we can then use the transitive property with $b \rightarrow d$ to conclude that $b \rightarrow \sim m$, that babies cannot manage crocodiles. While it is silly, this is a logical conclusion from the given premises.

Example 43

Premise: If I work hard, I'll get a raise.

Premise: If I get a raise, I'll buy a boat.

Conclusion: If I don't buy a boat, I must not have worked hard.

If we let h = working hard, r = getting a raise, and b = buying a boat, then we can represent our argument symbolically:

Premise: $h \rightarrow r$

Premise: $r \rightarrow b$

Conclusion: $\sim b \rightarrow \sim h$

Using the transitive property with the two premises, we can conclude that $h \rightarrow b$, if I work hard, then I will buy a boat. When we learned about the contrapositive, we saw that the conditional statement $h \rightarrow b$ is equivalent to $\sim b \rightarrow \sim h$. Therefore, the conclusion is indeed a logical syllogism derived from the premises.

Try it Now 19

Is this argument valid?

Premise: If I go to the party, I'll be really tired tomorrow.

Premise: If I go to the party, I'll get to see friends.

Conclusion: If I don't see friends, I won't be tired tomorrow.

Answer available at the end of this section.

Answers

Try it Now 14

Let b = brushed teeth and w = toothbrush is wet.

Premise: $b \rightarrow w$

Premise: $\sim w$

Conclusion: $\sim b$

This argument is valid by the Law of Contraposition.

Try it Now 15

This argument is valid by the Transitive Property, which can involve more than two premises, as long as they continue the chain reaction. The premises $f \rightarrow s, s \rightarrow b, b \rightarrow c, c \rightarrow d, d \rightarrow g, g \rightarrow w, w \rightarrow h, h \rightarrow x$ can be reduced to $f \rightarrow x$. (Because we had already used c and d we decided to use w for cow and x for death. If the old lady swallows the fly, she will eventually eat a horse and die.

Try it Now 16

Let p = wrote a paper and s = gave a speech.

Premise: $p \vee s$

Premise: $\neg s$

Conclusion: p

This argument is valid by Disjunctive Syllogism. Alison had to do one or the other; she didn't choose the speech, so she must have chosen the paper.

Try it Now 17

Let f = pulled fire alarm and t = got in big trouble.

Premise: $f \rightarrow t$

Premise: t

Conclusion: f

Try it Now 18

Let t = tripped and p = got a penalty.

Premise: $t \rightarrow p$

Premise: $\sim t$

Conclusion: $\sim p$

This argument is invalid because it has the form of the Fallacy of the Inverse. Alexei may have gotten a penalty for an infraction other than tripping.

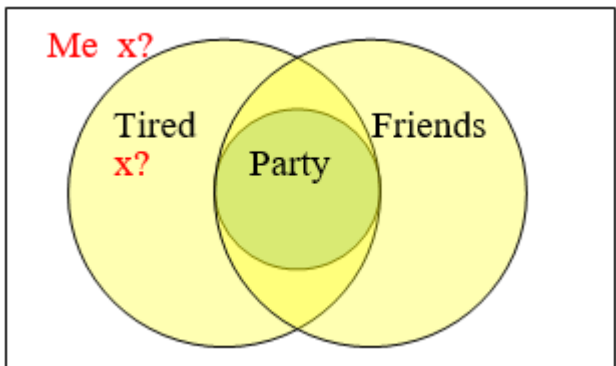
Try it Now 19

Let p = go to party, t = be tired, and f = see friends.

Premise: $p \rightarrow t$

Premise: $p \rightarrow f$

Conclusion: $\neg f \rightarrow \neg t$



We could try to rewrite the second premise using the contrapositive to state $\sim f \rightarrow \sim p$, but that does not allow us to form a syllogism. If I don't see friends, then I didn't go the party, but that is not sufficient to claim I won't be tired tomorrow. Maybe I stayed up all night watching movies.

A Venn diagram can help, if we set it up correctly. The "party" circle must be completely contained within the intersection of the other circles. We know that I am somewhere outside the "friends" circle, but we cannot determine whether I am in the "tired" circle. All we really know for sure is that I didn't go to the party.

5.1.9: Logical Fallacies in Common Language

In the previous discussion, we saw that logical arguments can be invalid when the premises are not true, when the premises are not sufficient to guarantee the conclusion, or when there are invalid chains in logic. There are a number of other ways in which arguments can be invalid, a sampling of which are given here.

Ad hominem

An ad hominem argument attacks the person making the argument, ignoring the argument itself.

Example 44

“Jane says that whales aren’t fish, but she’s only in the second grade, so she can’t be right.”

Here the argument is attacking Jane, not the validity of her claim, so this is an ad hominem argument.

Example 45

“Jane says that whales aren’t fish, but everyone knows that they’re really mammals. She’s so stupid.”

This certainly isn’t very nice, but it is *not* ad hominem since a valid counterargument is made along with the personal insult.

Appeal to ignorance

This type of argument assumes something is true because it hasn’t been proven false.

Example 46

“Nobody has proven that photo isn’t of Bigfoot, so it must be Bigfoot.”

Appeal to authority

These arguments attempt to use the authority of a person to prove a claim. While often authority can provide strength to an argument, problems can occur when the person’s opinion is not shared by other experts, or when the authority is irrelevant to the claim.

Example 47

“A diet high in bacon can be healthy; Doctor Atkins said so.”

Here, an appeal to the authority of a doctor is used for the argument. This generally would provide strength to the argument, except that the opinion that eating a diet high in saturated fat runs counter to general medical opinion. More supporting evidence would be needed to justify this claim.

Example 48

“Jennifer Hudson lost weight with Weight Watchers, so their program must work.”

Here, there is an appeal to the authority of a celebrity. While her experience does provide evidence, it provides no more than any other person’s experience would.

Appeal to consequence

An appeal to consequence concludes that a premise is true or false based on whether the consequences are desirable or not.

Example 49

“Humans will travel faster than light: faster-than-light travel would be beneficial for space travel.”

False dilemma

A false dilemma argument falsely frames an argument as an “either or” choice, without allowing for additional options.

Example 50

“Either those lights in the sky were an airplane or aliens. There are no airplanes scheduled for tonight, so it must be aliens.”

This argument ignores the possibility that the lights could be something other than an airplane or aliens.

Circular reasoning

Circular reasoning is an argument that relies on the conclusion being true for the premise to be true.

Example 51

“I shouldn’t have gotten a C in that class; I’m an A student!”

In this argument, the student is claiming that because they’re an A student, though shouldn’t have gotten a C. But because they got a C, they’re not an A student.

Post hoc (post hoc ergo propter hoc)

A post hoc argument claims that because two things happened sequentially, then the first must have caused the second.

Example 52

“Today I wore a red shirt, and my football team won! I need to wear a red shirt every time they play to make sure they keep winning.”

Straw man

A straw man argument involves misrepresenting the argument in a less favorable way to make it easier to attack.

Example 53

“Senator Jones has proposed reducing military funding by 10%. Apparently he wants to leave us defenseless against attacks by terrorists”

Here the arguer has represented a 10% funding cut as equivalent to leaving us defenseless, making it easier to attack Senator Jones’ position.

Correlation implies causation

Similar to post hoc, but without the requirement of sequence, this fallacy assumes that just because two things are related one must have caused the other. Often there is a third variable not considered.

Example 54

“Months with high ice cream sales also have a high rate of deaths by drowning. Therefore, ice cream must be causing people to drown.”

This argument is implying a causal relation, when really both are more likely dependent on the weather; that ice cream and drowning are both more likely during warm summer months.

Try it Now 20

Identify the logical fallacy in each of the arguments

- Only an untrustworthy person would run for office. The fact that politicians are untrustworthy is proof of this.
- Since the 1950s, both the atmospheric carbon dioxide level and [obesity](#) levels have increased sharply. Hence, atmospheric carbon dioxide causes obesity.
- The oven was working fine until you started using it, so you must have broken it.
- You can’t give me a D in the class because I can’t afford to retake it.
- The senator wants to increase support for food stamps. He wants to take the taxpayers’ hard-earned money and give it away to lazy people. This isn’t fair, so we shouldn’t do it.

Answer available at the end of this section.

It may be difficult to identify one particular fallacy for an argument. Consider this argument: “Emma Watson says she’s a feminist, but she posed for these racy pictures. I’m a feminist, and no self-respecting feminist would do that.” Could this be ad hominem, saying that Emma Watson has no self-respect? Could it be appealing to authority because the person making the argument claims to be a feminist? Could it be a false dilemma because the argument assumes that a woman is either a feminist or not, with no gray area in between?

Answers

Try it Now 20

- Circular
- Correlation does not imply causation
- Post hoc
- Appeal to consequence

e. Straw man

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5.1.10: Exercises

Boolean Logic

For questions 1-2, list the set of integers that satisfy the given conditions.

1. A positive multiple of 5 and not a multiple of 2
2. Greater than 12 and less than or equal to 18

Quantified Statements

For questions 3-4, write the negation of each quantified statement.

3. Everyone failed the quiz today.
4. Someone in the car needs to use the restroom.

Truth Tables

5. Translate each statement from symbolic notation into English sentences. Let A represent "Elvis is alive" and let G represent "Elvis gained weight".

- a. $A \vee G$
- b. $\sim (A \wedge G)$
- c. $G \rightarrow \sim A$
- d. $A \leftrightarrow \sim G$

For questions 6-9, create a truth table for each statement.

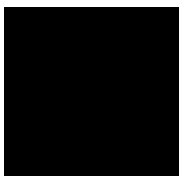
6. $A \wedge \sim B$
7. $\sim (\sim A \vee B)$
8. $(A \wedge B) \rightarrow C$
9. $(A \vee B) \rightarrow \sim C$

Questions 10 – 13 : In this lesson, we have been studying the inclusive or, which allows both A and B to be true. The exclusive or does not allow both to be true; it translates to "either A or B , but not both."

10. For each situation, decide whether the "or" is most likely exclusive or inclusive.

- a. An entrée at a restaurant includes soup or a salad.
- b. You should bring an umbrella or a raincoat with you.
- c. We can keep driving on I-5 or get on I-405 at the next exit.
- d. You should save this document on your computer or a flash drive.

11. Complete the truth table for the exclusive or.



12. Complete the truth table for $(A \vee B) \wedge \sim (A \wedge B)$.



13. Compare your answers for questions 11 and 12. Can you explain the similarities?

Conditional Statements

14. Consider the statement “If you are under age 17, then you cannot attend this movie.”

- a. Write the converse.
- b. Write the inverse.
- c. Write the contrapositive.

15. Assume that the statement “If you swear, then you will get your mouth washed out with soap” is true. Which of the following statements must also be true?

- a. If you don’t swear, then you won’t get your mouth washed out with soap.
- b. If you don’t get your mouth washed out with soap, then you didn’t swear.
- c. If you get your mouth washed out with soap, then you swore.

For questions 16-18, write the negation of each conditional statement.

16. If you don’t look both ways before crossing the street, then you will get hit by a car.

17. If Luke faces Vader, then Obi-Wan cannot interfere.

18. If you weren’t talking, then you wouldn’t have missed the instructions.

19. Assume that the biconditional statement “You will play in the game if and only if you attend all practices this week” is true. Which of the following situations could happen?

- a. You attended all practices this week and didn’t play in the game.
- b. You didn’t attend all practices this week and played in the game.
- c. You didn’t attend all practices this week and didn’t play in the game.

De Morgan’s Laws

For questions 20-21, use De Morgan’s Laws to rewrite each conjunction as a disjunction, or each disjunction as a conjunction.

20. It is not true that Tina likes Sprite or 7-Up.

21. It is not the case that you need a dated receipt and your credit card to return this item.

22. Go back and look at the truth tables in Exercises 6 & 7. Explain why the results are identical.

Deductive Arguments

For questions 23-28, use a Venn diagram or truth table or common form of an argument to decide whether each argument is valid or invalid.

23. If a person is on this reality show, they must be self-absorbed. Laura is not self-absorbed. Therefore, Laura cannot be on this reality show.

24. If you are a triathlete, then you have outstanding endurance. LeBron James is not a triathlete. Therefore, LeBron does not have outstanding endurance.

25. Jamie must scrub the toilets or hose down the garbage cans. Jamie refuses to scrub the toilets. Therefore, Jamie will hose

down the garbage cans.

26. Some of these kids are rude. Jimmy is one of these kids. Therefore, Jimmy is rude!

27. Every student brought a pencil or a pen. Marcie brought a pencil. Therefore, Marcie did not bring a pen.

28. If a creature is a chimpanzee, then it is a primate. If a creature is a primate, then it is a mammal. Bobo is a mammal. Therefore, Bobo is a chimpanzee.

Logical Fallacies

For questions 28-30, name the type of logical fallacy being used.

29. If you don't want to drive from Boston to New York, then you will have to take the train.

30. New England Patriots quarterback Tom Brady likes his footballs slightly underinflated. The "Cheatriots" have a history of bending or breaking the rules, so Brady must have told the equipment manager to make sure that the footballs were underinflated.

31. Whenever our smoke detector beeps, my kids eat cereal for dinner. The loud beeping sound must make them want to eat cereal for some reason.

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5.1.11: De Morgan's Laws

A contemporary of Boole's, Augustus De Morgan, formalized two rules of logic that had previously been known informally. They allow us to rewrite the negation of a conjunction as a disjunction, and vice-versa.

For example, suppose you want to schedule a meeting with two colleagues at 4 : 30PM on Friday, and you need both of them to be available at that time. What situation would make it impossible to have the meeting? It is NOT the case that colleague a is available AND colleague b is available: $\sim (a \wedge b)$. This situation is equivalent to either colleague a NOT being available OR colleague b NOT being available: $\sim a \vee \sim b$

De Morgan's Laws

The negation of a conjunction is equivalent to the disjunction of the negation of the statements making up the conjunction. To negate an "and" statement, negate each part and change the "and" to "or".

$$\sim (p \wedge q) \text{ is equivalent to } \sim p \vee \sim q$$

The negation of a disjunction is equivalent to the conjunction of the negation of the statements making up the disjunction. To negate an "or" statement, negate each part and change the "or" to "and".

$$\sim (p \vee q) \text{ is equivalent to } \sim p \wedge \sim q$$

Example 28

For Valentine's Day, you did not get your sweetie flowers or candy: Which of the following statements is logically equivalent?

- a. You did not get them flowers or did not get them candy.
- b. You did not get them flowers and did not get them candy.
- c. You got them flowers or got them candy.

Solution

- a. This statement does not go far enough; it leaves open the possibility that you got them one of the two things.
- b. This statement is equivalent to the original; $\sim (f \vee c)$ is equivalent to $\sim f \wedge \sim c$
- c. This statement says that you got them something, but we know that you did not.

Try it Now 8

To serve as the President of the US, a person must have been born in the US, must be at least 35 years old, and must have lived in the US for at least 14 years. What minimum set of conditions would disqualify someone from serving as President?

Answer available at the end of this section.

Answers

Try it Now 8

Failing to meet just one of the three conditions is all it takes to be disqualified. A person is disqualified if they were not born in the US, or are not at least 35 years old, or have not lived in the US for at least 14 years. The key word here is "or" instead of "and".

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5.1.12: Arguments

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

Argument types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

Example 29

The argument “when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store” is an inductive argument.

The premises are:

I forgot my purse last week

I forgot my purse today

The conclusion is:

I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

Example 30

The argument “every day for the past year, a plane flies over my house at 2:00 P.M. A plane will fly over my house every day at 2:00 P.M.” is a stronger inductive argument, since it is based on a larger set of evidence. While it is not necessarily true—the airline may have cancelled its afternoon flight—it is probably a safe bet.

Evaluating inductive arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest that it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. Common scientific theories, like Newton’s theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence, such as when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes it easier to evaluate.

Evaluating deductive arguments

A deductive argument is considered valid if, assuming that all the premises are true, the conclusion follows logically from those premises. In other words, when the premises are all true, the conclusion *must* be true.

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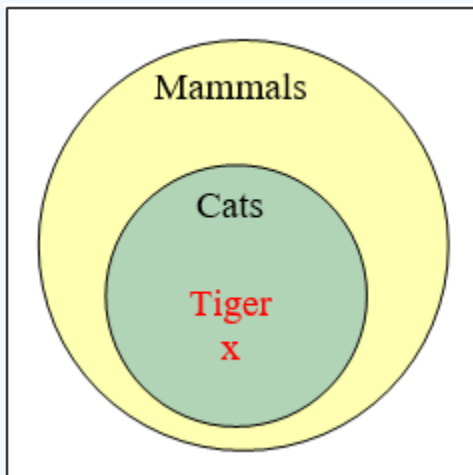
5.1.13: Evaluating Deductive Arguments with Euler Diagrams

We can interpret a deductive argument visually with an Euler diagram, which is essentially the same thing as a Venn diagram. This can make it easier to determine whether the argument is valid or invalid.

Example 31

Consider the deductive argument “All cats are mammals and a tiger is a cat, so a tiger is a mammal.” Is this argument valid?

Solution



The premises are:

All cats are mammals.

A tiger is a cat.

The conclusion is:

A tiger is a mammal.

Both the premises are true. To see that the premises must logically lead to the conclusion, we can use a Venn diagram. From the first premise, we draw the set of cats as a subset of the set of mammals. From the second premise, we are told that a tiger is contained within the set of cats. From that, we can see in the Venn diagram that the tiger must also be inside the set of mammals, so the conclusion is valid.

Analyzing arguments with Euler diagrams

To analyze an argument with an Euler diagram:

- 1) Draw an Euler diagram based on the premises of the argument
- 2) The argument is invalid if there is a way to draw the diagram that makes the conclusion false
- 3) The argument is valid if the diagram cannot be drawn to make the conclusion false
- 4) If the premises are insufficient to determine the location of an element or a set mentioned in the conclusion, then the argument is invalid.

Try it Now 9

Determine the validity of this argument:

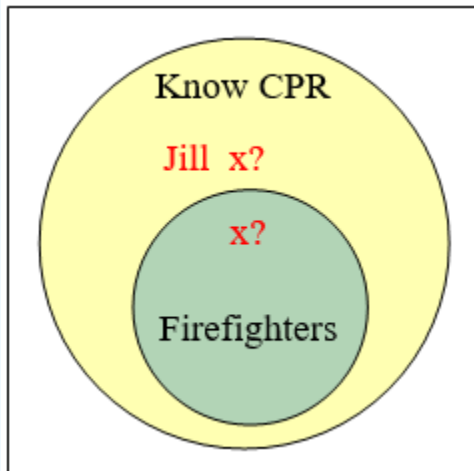
Premise: All cats are scared of vacuum cleaners.

Premise: Max is a cat.

Conclusion: Max is scared of vacuum cleaners.

Answer available at the end of this section.

Example 32



Premise: All firefighters know CPR.

Premise: Jill knows CPR.

Conclusion: Jill is a firefighter.

Solution

From the first premise, we know that firefighters all lie inside the set of those who know CPR. (Firefighters are a subset of people who know CPR.) From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know whether she also is a member of the smaller subset that is firefighters.

Since the conclusion does not necessarily follow from the premises, this is an invalid argument. It's possible that Jill is a firefighter, but the structure of the argument doesn't allow us to conclude that she definitely is.

It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are concerned with whether the premises are enough to prove the conclusion.

Try it Now 10

Determine the validity of this argument:

Premise: All bicycles have two wheels.

Premise: This Harley-Davidson has two wheels.

Conclusion: This Harley-Davidson is a bicycle.

Answer available at the end of this section.

Try it Now 11

Determine the validity of this argument:

Premise: No cows are purple.

Premise: Fido is not a cow.

Conclusion: Fido is purple.

Answer available at the end of this section.

In addition to these categorical style premises of the form “all ___”, “some ___”, and “no ___”, it is also common to see premises that are conditionals.

Example 33

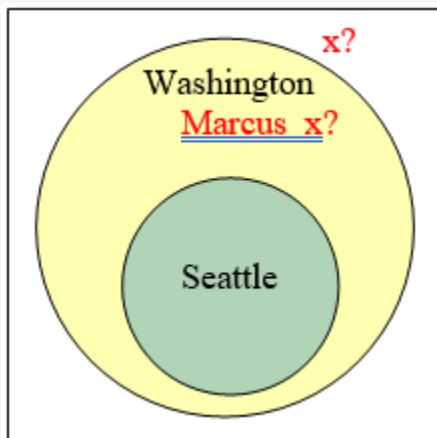
Premise: If you live in Seattle, you live in Washington.

Premise: Marcus does not live in Seattle.

Conclusion: Marcus does not live in Washington.

Solution

From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether Marcus lives in Washington or not. This is an invalid argument.



Try it Now 12

Determine the validity of this argument:

Premise: If you have lipstick on your collar, then you are cheating on me.

Premise: If you are cheating on me, then I will divorce you.

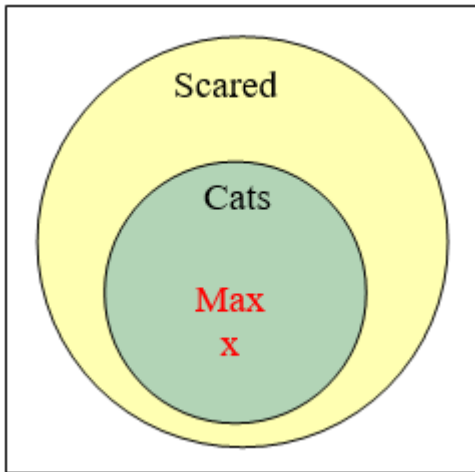
Premise: You do not have lipstick on your collar.

Conclusion: I will not divorce you.

Answer available at the end of this section.

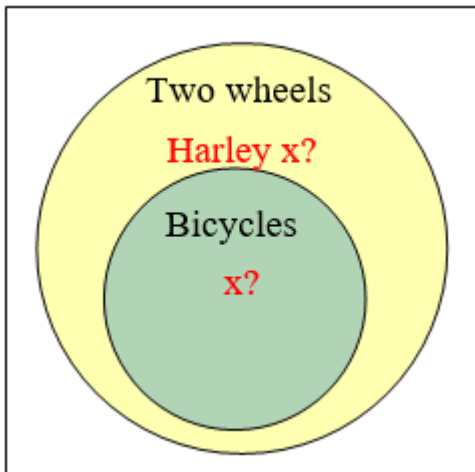
Answers

Try it Now 9



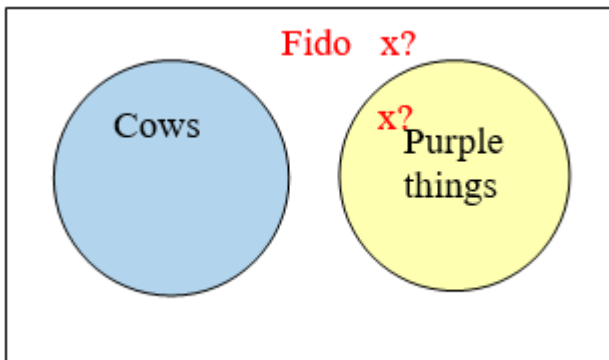
Valid. Cats are a subset of creatures that are scared by vacuum cleaners. Max is in the set of cats, so he must also be in the set of creatures that are scared by vacuum cleaners.

Try it Now 10



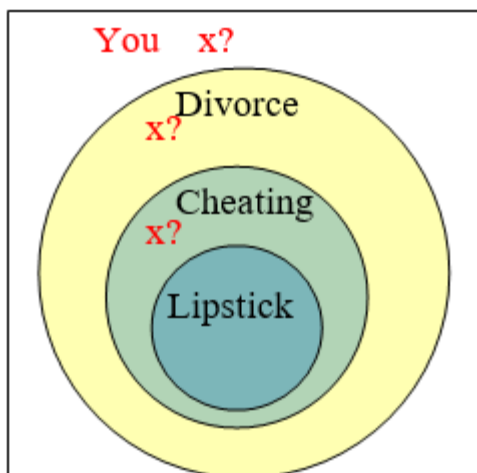
Invalid. The set of bicycles is a subset of the set of vehicles with two wheels; the Harley-Davidson is in the set of two-wheeled vehicles but not necessarily in the smaller circle.

Try it Now 11



Invalid. Since no cows are purple, we know there is no overlap between the set of cows and the set of purple things. We know Fido is not in the cow set, but that is not enough to conclude that Fido is in the purple things set.

Tryt it Now 12



Invalid. Lipstick on your collar is a subset of scenarios in which you are cheating, and cheating is a subset of the scenarios in which I will divorce you. Although it is wonderful that you don't have lipstick on your collar, you could still be cheating on me, and I will divorce you. In fact, even if you aren't cheating on me, I might divorce you for another reason. You'd better shape up.

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