

COMPLEX ANALYSIS - A VISUAL AND INTERACTIVE INTRODUCTION



Juan Carlos Ponce Campuzano
University of Queensland

University of Queensland

Complex Analysis - A Visual and Interactive
Introduction

Juan Carlos Ponce Campuzano

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (<https://LibreTexts.org>) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide an accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by [NICE CXone Expert](#) and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptations contact info@LibreTexts.org or visit our main website at <https://LibreTexts.org>.

This text was compiled on 04/01/2026

TABLE OF CONTENTS

Introduction

Licensing

License

Acknowledgements

1: Chapter 1

- 1.1: A Brief History
- 1.2: Terminology and Notation
- 1.3: Geometric Interpretation of the Arithmetic Operations
- 1.4: The Principal Argument
- 1.5: Roots of Complex Numbers
- 1.6: Topology of the Complex Plane

2: Chapter 2

- 2.1: Complex functions
- 2.2: Riemann Sphere
- 2.3: Complex Differentiation
- 2.4: The Logarithmic Function
- 2.5: Riemann Surfaces

3: Chapter 3

- 3.1: Mappings
- 3.2: The Transformation $w=1/z$
- 3.3: Analytic Landscapes
- 3.4: Domain coloring
- 3.5: The Complex Power Function

4: Chapter 4

- 4.1: Curves in the Complex Plane
- 4.2: Complex Integration
- 4.3: Integrals of Functions with Branch Cuts

5: Chapter 5

- 5.1: Series
- 5.2: Taylor Series
- 5.3: Laurent Series
- 5.4: Classification of Singularities
- 5.5: The Mandelbrot Set
- 5.6: The Julia Set

6: Chapter 6

- [6.1: Applications of Conformal Mappings](#)
- [6.2: Complex Potential- Basic examples](#)
- [6.3: Uniform Flow Around a Circle](#)
- [6.4: Joukowski Airfoil](#)

[Index](#)

[Glossary](#)

[Bibliography](#)

[Detailed Licensing](#)

Licensing

A detailed breakdown of this resource's licensing can be found in [Back Matter/Detailed Licensing](#).

License

ISBN: 978-0-6485736-0-9

Published Online 2019

LICENSE

This book's entire contents, including the examples, problems and applets, is released under a [Creative Commons Attribution, Non-Commercial, Share-Alike license](#). This means:

1. You're welcome to use the examples, problems and applets for your personal studies or research. If you are using this book for a project (homework or research), please cite it somewhere in your project.
2. You can only use the book's text, examples, problems and applets for non-commercial projects. That means you can remix or make your own version of the book, and you can fork and create new applets, so long as they are not commercial projects (i.e. publishing a book with a publisher).
3. If you do make a project that forks or remixes this book, it must be released under this same license or a looser one.

If you have any questions about what you can and can't do with these examples or applets, please **get in touch**.

Acknowledgements

Since 2015 I tried many times to integrate my complex analysis notes with applets that I have designed or adapted from other people. Finally, in 2019, I was able to accomplish this task thanks to the development of mathematical, open-source, software capable to run online in (almost) any browser. This book would not have been possible without a ton of people sharing their great work. As much as possible, I've tried to keep track of all the online resources, books and articles that I have used.

A big “thank you” to:

- **Jeffrey Thompson** for allowing me to fork the design of his book **Collision Detection**.
- **Aaron Montag**, **Martin von Gagern**, **Stefan Kranich** and **Michael Strobel** creators, developers and contributors of the **CindyJS** project. Thanks for allowing me to adapt your great applets.
- **Jürgen Richter-Gebert**, one of the authors of the interactive mathematics software **Cinderella**, who now works on the **CindyJS** project.
- The **GeoGebra** developers and community who share their wonderful constructions and applets.
- **Daniel Shiffman**, an amazing teacher who shares his knowledge with the world. I have learned all the basics about programming from his **books** and video tutorials **The Coding Train**.
- **Lauren McCarthy**, the creator of **p5.js**, and to the great community of creative people who use this programming language.
- **Paul Masson**, the creator of **MathCell**. This is JavaScript library for including interactive mathematics in a web browser.
- The equations in the book are displayed thanks to **MathJax**. However, in some applets **KaTeX** is used instead.

Many thanks also to all the people who have supported this project with donations, suggestions and recommendations:

1. Kelly E. Matthews, Alex Iktan, Codi Quetzal, Feya, **Nicolás Guarín-Zapata**, Carla Luciane Klôa Schöninger, Ken Thele, **Sophia Wood**, **Mates Mike**, Christopher Lee, **Alvy**, Pablo Padilla, Jason Cunliffe.

Finally, about the applets:

- The applets “Analytic Landscapes” and “Taylor Series” were adapted for the purposes of this book. The original versions can be found in the **CindyGL-Gallery**.
- I designed all the GeoGebra applets. They can be downloaded from this **GeoGebra book**.
- I also designed all the p5.js applets and the source code can be found at **GitHub**.

CHAPTER OVERVIEW

1: Chapter 1

- 1.1: A Brief History
- 1.2: Terminology and Notation
- 1.3: Geometric Interpretation of the Arithmetic Operations
- 1.4: The Principal Argument
- 1.5: Roots of Complex Numbers
- 1.6: Topology of the Complex Plane

This page titled [1: Chapter 1](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

1.1: A Brief History

Geometric interpretation of quadratic and cubic equations

Consider a quadratic equation

$$x^2 = mx + n. \quad (1.1.1)$$

From elementary school we have learned how to find its solutions, that is, all the values of x that satisfy equation (1). To do so we just need to use the widely known *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (1.1.2)$$

of the *general quadratic equation*

$$ax^2 + bx + c = 0. \quad (1.1.3)$$

By rewriting equation (1) as $x^2 - mx - n = 0$, we can use (2) to obtain

$$x = \frac{m \pm \sqrt{m^2 + 4n}}{2} = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} + n} \quad (1.1.4)$$

If we plot equation (1) we can observe geometrically that it represents the intersection of the parabola $y = x^2$ with the line $y = mx + n$. This can be appreciated in the following applet. Drag the sliders below and observe what happens with the intersection points x_0 and x_1 .

[interactive graph](#)

As you already noticed, we have three cases:

1. There are two points of intersection, i. e. two solutions.
2. There is only one intersection point, i. e. one solution.
3. There are no intersection points, i. e. there are no solutions.

Surprisingly this has been known since ancient times, even without the use of mathematical symbols or computers. We know from clay tablets dated from around 2000 BCE that the Babylonian civilization possessed the quadratic formula, enabling them (in verbal form) to solve quadratic equations. Because the concept of negative numbers had to wait until the sixteenth century to appear, the Babylonians did not consider negative solutions [9, pp. 29-30]. We can also find implicitly equations in the geometry developed by the ancient Greeks, as we would expect when circles, parabolas, and the like are being investigated, but we do not demand that every geometric problem have a solution [9, Ch. 4].

Now let's go back to the quadratic equation $x^2 = mx + n$. Consider the values $m = 0$ and $n = -1$. If we use formula (4), we obtain that

$$x = \pm\sqrt{-1}.$$

What is going on here? Most of you have learned from calculus, or elementary school, that we cannot take the square root of a negative number. Then, how do interpret this value? Geometrically speaking, we can relate the solution $x = \pm\sqrt{-1}$ to the fact that the parabola $y = x^2$ and the line $y = -1$ do not intersect each other, see Figure 1. In other words, there is no solution. In general, if $\frac{m^2}{4} + n < 0$, then equation $x^2 = mx + n$ has no solutions.

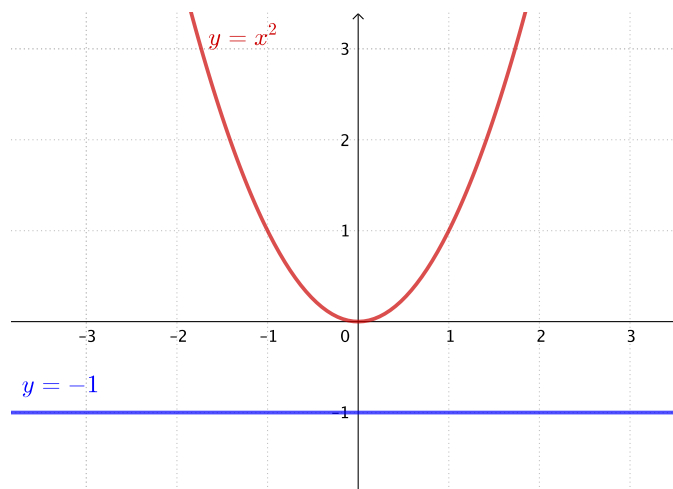


Figure 1: $\setminus(x^2=-1 \setminus)$.

It is argued that this led to the invention of a new number denoted by i , which is equal to $\sqrt{-1}$, and it was given the name “imaginary”. If we use i as a number such that $i^2 = -1$, then this value is the solution of the equation $x^2 = -1$.

It is also a common practice to point out in courses of mathematics that *complex numbers*, denoted as $a + b\sqrt{-1}$, are needed to solve certain quadratic equations, such as $x^2 + 1 = 0$. However, complex numbers emerged, in fact, from the need to solve cubic equations. Furthermore, when quadratic and cubic equations first appeared, at that time, there was no need to have solutions for all equations.

So where did complex numbers really come into importance? To answer this let’s consider the cubic equation

$$x^3 = px + q. \quad (1.1.5)$$

Geometrically, this equation represents the intersection of the cubic $y = x^3$ with the line $y = px + q$, as shown in the following applet. Drag sliders below and observe what happens.

interactive graph

As you can observe no matter what line is defined by the parameters p and q , it will always intersect the cubic somewhere, even when the line $px + q$ is perpendicular to the x-axis and far from the origin (i. e. when p and q are both very large positive/negative numbers). This is because the cubic goes all the way from $-\infty$ to $+\infty$. Thus, there is no line you can draw that won’t intersect this cubic. This example is very different from the quadratic case which was a parabola x^2 and you could define a line $mx + n$ such that it won’t intersect the parabola.

Solution to cubic equations

It is well known that the solution of the cubic $x^3 = px + y$ was developed in the Renaissance (15th and 16th centuries) by Italian mathematicians. Scipione del Ferro (1465-1526) and Niccolò Tartaglia (1500-1557), followed by Girolamo Cardano (1501-1576), showed that $x^3 = px + y$ has a solution given by

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} \quad (1.1.6)$$

This is known as Cardano’s formula and here we are using modern notation.



Figure 2: Tartaglia.

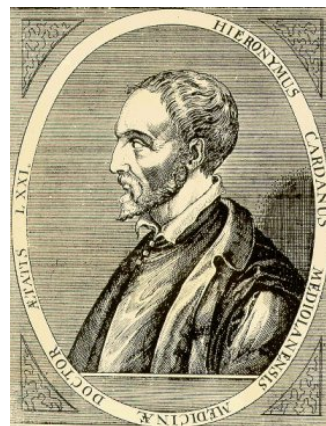


Figure 3: Cardano.

To see how this works consider the equation $x^3 = -6x + 20$. In this case $p = -6$ and $q = 20$. If we plug these numbers into (6), we obtain the solution

$$x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

Simplifying we obtain $x = 2$, a solution of the given equation since

$$8 = (2)^3 = -6(2) + 20 = -12 + 20 = 8$$

Thus, this formula seems to work very well, at least for this case.

Exercise 1.1.1

Try to solve $x^3 = 6x + 6$ using Cardano's formula.

The genesis of imaginary numbers

A few years later after the discovery of Cardano's formula, the Italian engineer-architect Rafael Bombelli (1526-1572) recognized that there was something strange and paradoxical about this formula. He considered the equation

$$x^3 = 15x + 4 \tag{1.1.7}$$

and, with perhaps just a little pondering, you can see that $x = 4$ is a solution. This can also be seen in Figure 2. In fact there are three solutions, but Bombelli did not consider negative values, so we won't either.

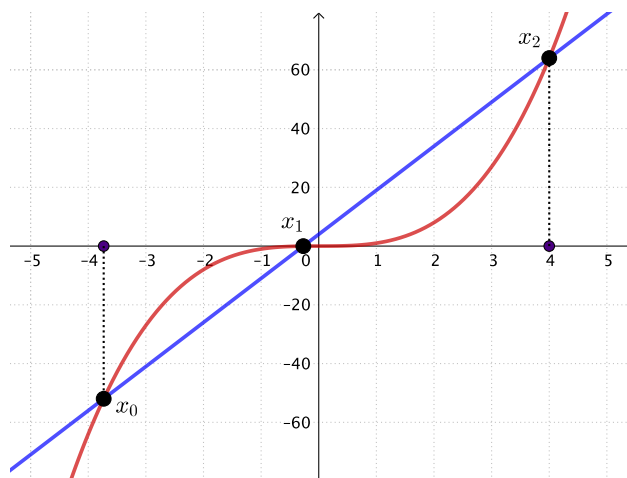


Figure 4: $x^3 = 15 + 4$

Then Bombelli used Cardano's formula to solve $(x^3 = 15 + 4)$. Thus, considering $p = 15$ and $q = 4$, he obtained

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}. \quad (1.1.8)$$

Here he encountered a very unusual value. If Cardano's formula is correct, this number must be equal to 4. Yet this must be nonsense and the value cannot be real, because inside the cube root we are taking the square root of a negative number, an absolute impossibility at that time (and also nowadays). Cardano also encountered this difficulty but he did not face it up. He did once mention imaginary numbers, but in connection with a quadratic equation and accompanied by the comment that these numbers were "as subtle as they are useless" [2, Ch. 37, Rule II]

However, Bombelli overcome this difficulty by seeing that the weird expression (8) that Cardano's formula gives for x is actually real, but expressed in a very unfamiliar manner. This insight did not come easily. As Bombelli wrote in his book *L'Algebra*:

And although to many this will appear an extravagant thing, because even I held this opinion some time ago, since it appeared to me more sophistic than true, nevertheless I searched hard and found the demonstration, which will be noted below. ... But let the reader apply all his strength of mind, for [otherwise] even he will find himself deceived. [1, pp. 293-294; 10]



Figure 5: *L'Algebra* by Rafael Bombelli: frontispiece of the Bologna edition of 1579.

Bombelli's great insight was simply to treat $\sqrt{-1}$ as a number and to operate with it following some specific arithmetic rules (the same kind of rules that we use nowadays). Thus he discovered that

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1} \text{ and } \sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-1}.$$

By substituting these values in (8) Bombelli obtained

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1}.$$

Then he showed that the squared roots of negative numbers cancel each other [1, p. 169; 7, p. 164]. That is

$$x = 2 + 2.$$

And concluded that $x = 4$ is actually the solution of $x^3 = 15x + 4$ obtained from Cardano's formula [1, p. 294], see Figure 6. This trick works in only a few cases, however, it helped to understand imaginary numbers and to take their cube roots or manipulate them when they appear aside real numbers.

 Bombelli's solution

Figure 6: Bombelli's original solution.

Of course, as you can see in Figure 6, Bombelli did not have at his disposal the power of today's algebraic notation (neither computers) and his computations were limited to numbers in "real domain". In fact, most of Italian mathematicians at that time tended to think of cubes or squares as geometric objects rather than algebraic quantities. However, he is credited for proving the reality of the roots of the cubic $x^3 = 15x + 4$, since he demonstrated the extraordinary fact that real numbers could be engendered by imaginary numbers.

Cardano's formula forced mathematicians to confront square roots of negative numbers. This historical incident is another example that negates the widespread view that mathematics is "made up" by mathematicians. As is often the case, it is the mathematics itself that speaks to us. From this time on, imaginary numbers lost some of their mystical character, although their full acceptance as *bona fide* numbers came only in the 1800s.

Exercise 1.1.1

Verify that $\sqrt[3]{2 \pm \sqrt{-121}} = 2 \pm \sqrt{-1}$

The maturing of complex numbers

Many mathematicians after Cardano and Bombelli made important contributions to imaginary (or complex) numbers. For example René Descartes (1596-1650) coined the term "imaginary" in his book in *La Géométrie* of 1637 as follows:

Neither the true nor the false [negative] roots are always real; but sometimes only imaginary. [4, p. 380]

John Wallis (1616-1703) showed how to represent geometrically complex roots of a quadratic equation with real coefficients [8, p. 594]. Carpar Wessel (1745-1818) and Jean-Robert Argan (1768-1822) provided geometric representations of complex numbers as vectors [7, pp. 185-190].



Figure 7: Euler.

Leonard Euler (1707-1783) standardized the notation " $i = \sqrt{-1}$ " [6, p. 184] and used imaginary numbers to solve quadratic and cubic equations, in spite of the fact that he was still suspicious of these numbers. In his *Algebra*, for example, he mentioned:

[...] since all numbers which it is possible to conceive are either greater or less than 0, or are 0 itself, it is evident that we cannot rank the square root of a negative number amongst possible numbers, and we must therefore say that it is an impossible quantity. In this manner we are led to the idea of numbers, which from their nature are impossible; and therefore they are usually called imaginary quantities, because they exist merely in the imagination. [5, p. 43]

Lest anyone takes this as a condemnation, he continued:

notwithstanding this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; [...] nothing prevents us from making use of these imaginary numbers, and employing them in calculation. [5, p. 43]

Later Carl Friedrich Gauss (1777-1855) introduced the term “complex number” referring to numbers of the form $a + bi$ [7, p. 191]. He also gave four proofs of the *fundamental theorem of algebra* over the course of his long career. This theorem tells us that any n th-degree polynomial has n roots, some or all of which may be imaginary. The first proof that Gauss gave was in his doctoral dissertation of 1799. The last (and perhaps the most elegant) allows the use of complex numbers not only for the variable but for the coefficients as well. Since this necessarily depended on the recognition of complex numbers, Gauss helped to solidify the position of these numbers [8, Vol. 2, p. 595].

The first rigorous definition of complex numbers was given by William Rowan Hamilton (1805-1865). In 1833 he proposed to the Irish Academy that a complex number $a+ib$ can be considered as a *couple* (a, b) , with a, b real numbers [7, pp. 192-193]. Then he defined addition and multiplication of *couples* as follows:

$$(a, b) + (c, d) = (a + c, b + d)$$

and

$$(a, b)(c, d) = (ac - bd, bc + ad).$$

This is in fact an algebraic definition of complex numbers. From the didactic and heuristic points of view it is preferable to have complex numbers introduced through a geometrical interpretation. But from the logical point of view the theory of *couples* is much more satisfying, since it shows the consistency of the theory of complex numbers starting from the consistency of the real numbers [3, pág. 175].



Figure 8: Hamilton.



Figure 9: Cauchy.

Among the many mathematicians and scientists who contributed, there are three who stand out as having influenced decisively the course of development of complex analysis [8, Vol.2, Ch. 27]. The first is Augustin-Louis Cauchy (1789-1857), who developed the theory of the complex integral calculus. By using imaginary numbers Cauchy was able to evaluate “real integrals” that heretofore could not be evaluated, obtaining stunning results such as

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

and

$$\int_0^{\pi} \log \sin x dx = -\pi \log 2.$$

The evaluation of real integrals, in addition to solution of the cubic equation and the fundamental theorem of algebra, proved how valuable it was to consider imaginary numbers.



Weierstrass

Figure 10: Weierstrass.



Figure 11: Riemann.

Finally, the other two important mathematicians are Karl Weierstrass (1815-1897) and Bernhard Riemann (1826-1866), who appeared on the mathematical scene about the middle of the nineteenth century. Weierstrass developed the theory from a starting point of convergent power series, and this approach led towards more formal algebraic developments. Riemann contributed a more geometric point of view on the study of complex functions. His ideas had a tremendous impact not only on complex analysis but upon mathematics as a whole, though his views took hold only gradually.

Final comments

The foregoing descriptions of complex numbers are not the end of the story. Various developments in the nineteenth and twentieth centuries enabled us to gain a deeper insight into the role of complex numbers not only in mathematics, but also in engineering and physics. The history of complex numbers is fascinating and I highly encourage the reader to consult the primary sources cited here which could serve as a starting point to delve into the historical details that allowed the emergence and development of complex numbers.

References

1. Bombelli, R. (1579). *L'Algebra*. Bologna.
2. Cardano, H. (1545). *Artis magna, sive de regulis algebraicis, liber unus*. (n.p.): Joh. Petreius.
3. Carrucio, E. (2009). *Mathematics and Logic in History and Contemporary Thought*. (I. Quigly, Trans.) USA: Aldine Transaction. (Original work published 1964).
4. Descartes, R. (1637). *La Géométrie*. Paris: A. Herman, Librairie Scientifique.
5. Euler, L. (1972). *Elements of Algebra*. (Rev. John Hewlett, B.D. F.A.S. &c, Trans.). Springer-Verlag. (Original work published 1770).
6. Euleri, L. (1845). *Institutionum Calculi Integralis*. Volumen Quartum. Petropoli. Impensis Academia Imperialis Scientiarum.
7. González-Velasco, A. E. (2011) *Journey through Mathematics*. Springer Science+Business Media.
8. Kline, M. (1972). *Mathematical Thought from Ancient to Modern Times*. Vols. 1-3. New York: Oxford University Press.
9. Merzbach, U. C & Boyer, C. B. (2011). *A history of mathematics*. 3rd ed. John Wiley & Sons, Inc., Hoboken, New Jersey.
10. O'Connor, J. J. & Robertson, E. F. (2000). *Rafael Bombelli*.

Further reading

- Bagni, G. T. (2009). Bombelli's Algebra (1572) and a New Mathematical Object. *For the Learning of Mathematics*. Vol. 29, No. 2, pp. 29-31.
- Buehler, D. (2014). Incomplete understanding of complex numbers Girolamo Cardano: a case study in the acquisition of mathematical concepts. *Synthese* 191:4231-4252.
- Burton, D. M. (1995). *The history of mathematics: An introduction* (6th ed.) (2005). New York: McGraw-Hill.

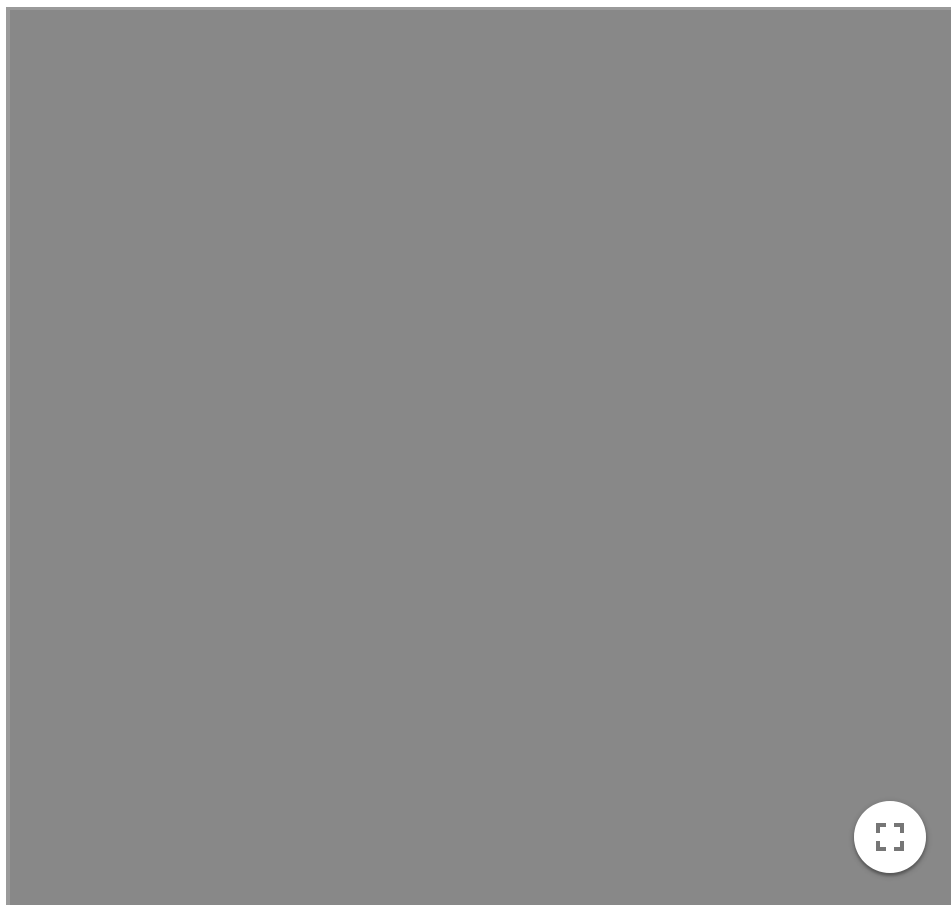
- Gindikin, S. (2007). *Tales of Mathematicians and Physicists*. Second English edition Springer Science+ Business Media, LLC.
- Nahim, P. J. (1998). *An imaginary tale: The story of $\sqrt{-1}$* . USA: Princeton University Press.
- Huffman, C. J. (2019). Mathematical Treasure: Raphael Bombelli's L'algebra. *Convergence*.
- Marsden, J. E. & Tromba, A. J. (2003). *Vector Calculus*. USA: W. H. Freeman and Company.
- Merino, O. (2006). *A Short History of Complex Numbers*.
- Stillwell, J. (2010). *Mathematics and Its History*. Springer Science+Business Media.

This page titled [1.1: A Brief History](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

1.2: Terminology and Notation

A *complex number* z is a number that can be expressed in the form $x + iy$, where x and y are real numbers and i is the imaginary unit, that is, $i^2 = -1$. In this expression, x is the *real part* and y is the *imaginary part* of the complex number.

The complex numbers, denoted by $\mathbb{C}$, extend the concept of the one-dimensional number line to the two-dimensional complex plane (also known as Argand plane) by using the horizontal axis for the real part and the vertical axis for the imaginary part. The analogy with two-dimensional vectors is immediate. The complex number $x + iy$ can be identified with the point (x, y) in the complex plane but also it can be interpreted as a two-dimensional vector.



It is useful to introduce another representation of complex numbers, namely polar coordinates (r, θ) :

$$x = r \cos \theta, \quad y = r \sin \theta \quad (r \geq 0) \quad (1.2.1)$$

Hence the complex number can be written in the alternative polar form:

$$z = x + iy = r(\cos \theta + i \sin \theta). \quad (1.2.2)$$

The radius r is denoted by

$$r = \sqrt{x^2 + y^2} = |z|$$

and naturally gives us a notion of the *absolute value* of z , denoted by $|z|$, that is, it is the length of the vector associated with z . The value $|z|$ is often referred to as the *modulus* of z . The angle θ is called the *argument* (or *phase*) of z and is denoted by $\arg(z)$. When $z \neq 0$, the values of θ can be found from (1) via standard trigonometry:

$$\tan \theta = \frac{y}{x}$$

where the quadrant in which z, y lie is understood as given.

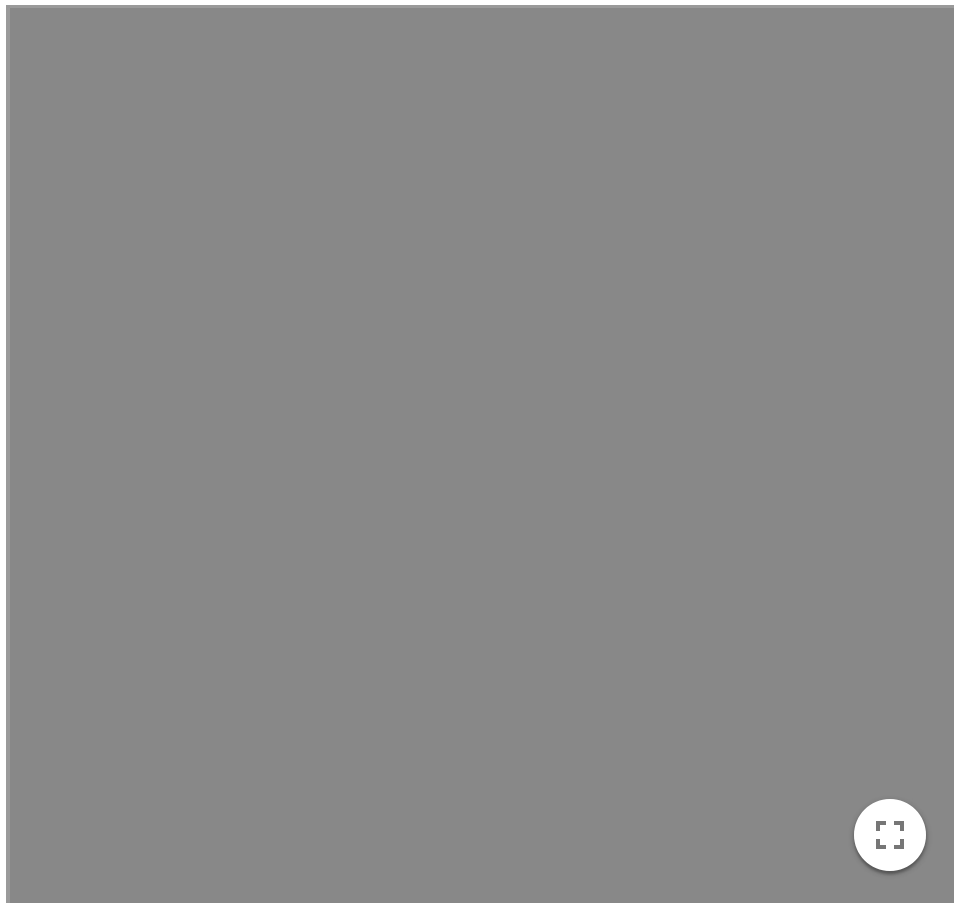
At this point it is convenient to introduce a special exponential function. The polar exponential is defined by

$$\cos\theta + i\sin\theta = e^{i\theta}$$

Hence equation (2) implies that z can be written in the form

$$z = re^{i\theta}.$$

This exponential function has all of the standard properties we are familiar within elementary calculus and is a special case of the complex exponential function.



Finally, the *complex conjugate* of z is defined as

$$\bar{z} = x - iy$$

Addition, subtraction, multiplication, and division of complex numbers follow from the rules governing real numbers. Thus, noting $i^2 = -1$, we have

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

and

$$z_1 \cdot z_2 = (x_1 \pm iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Now, we note that

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

This fact is useful for division of complex numbers,

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

It is easily shown that the commutative, associative, and distributive laws of addition and multiplication hold. Geometrically speaking, addition of two complex numbers is equivalent to that of the parallelogram law of vectors.

Some of the terminology and notation used to describe complex numbers is summarized in Figure 1.

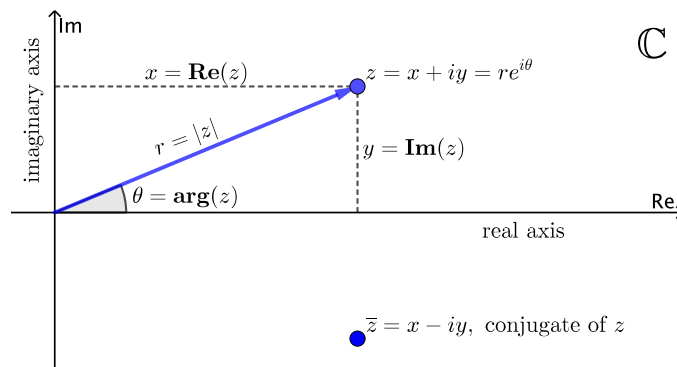


Figure 1: Summarized information.

I suggest you to make yourself comfortable with the concepts, terminology, and notation introduced thus far. To do so, try to convince yourself geometrically (and/or algebraically) of each of the following facts:

$$\begin{aligned}\operatorname{Re}(z) &= \frac{1}{2}(z + \bar{z}) & \operatorname{Im}(z) &= \frac{1}{2i}(z - \bar{z}) & |z| &= \sqrt{x^2 + y^2} \\ \tan(\arg(z)) &= \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} & re^{i\theta} &= r(\cos \theta + i \sin \theta) \\ \bar{\bar{z}} &= z & |z_1 z_2| &= |z_1| |z_2| & \left| \frac{z_1}{z_2} \right| &= \frac{|z_1|}{|z_2|}, (z_2 \neq 0) \\ \overline{z_1 \pm z_2} &= \bar{z}_1 \pm \bar{z}_2 & \overline{z_1 z_2} &= \bar{z}_1 \cdot \bar{z}_2 & \overline{\left(\frac{z_1}{z_2} \right)} &= \frac{\bar{z}_1}{\bar{z}_2}, (z_2 \neq 0) \\ |z_1 \pm z_2| &\leq |z_1| + |z_2| & ||z_1| - |z_2|| &\leq |z_1 \pm z_2|\end{aligned}$$

The following is called the *generalized triangle inequality*:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

When does equality hold?

This page titled [1.2: Terminology and Notation](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

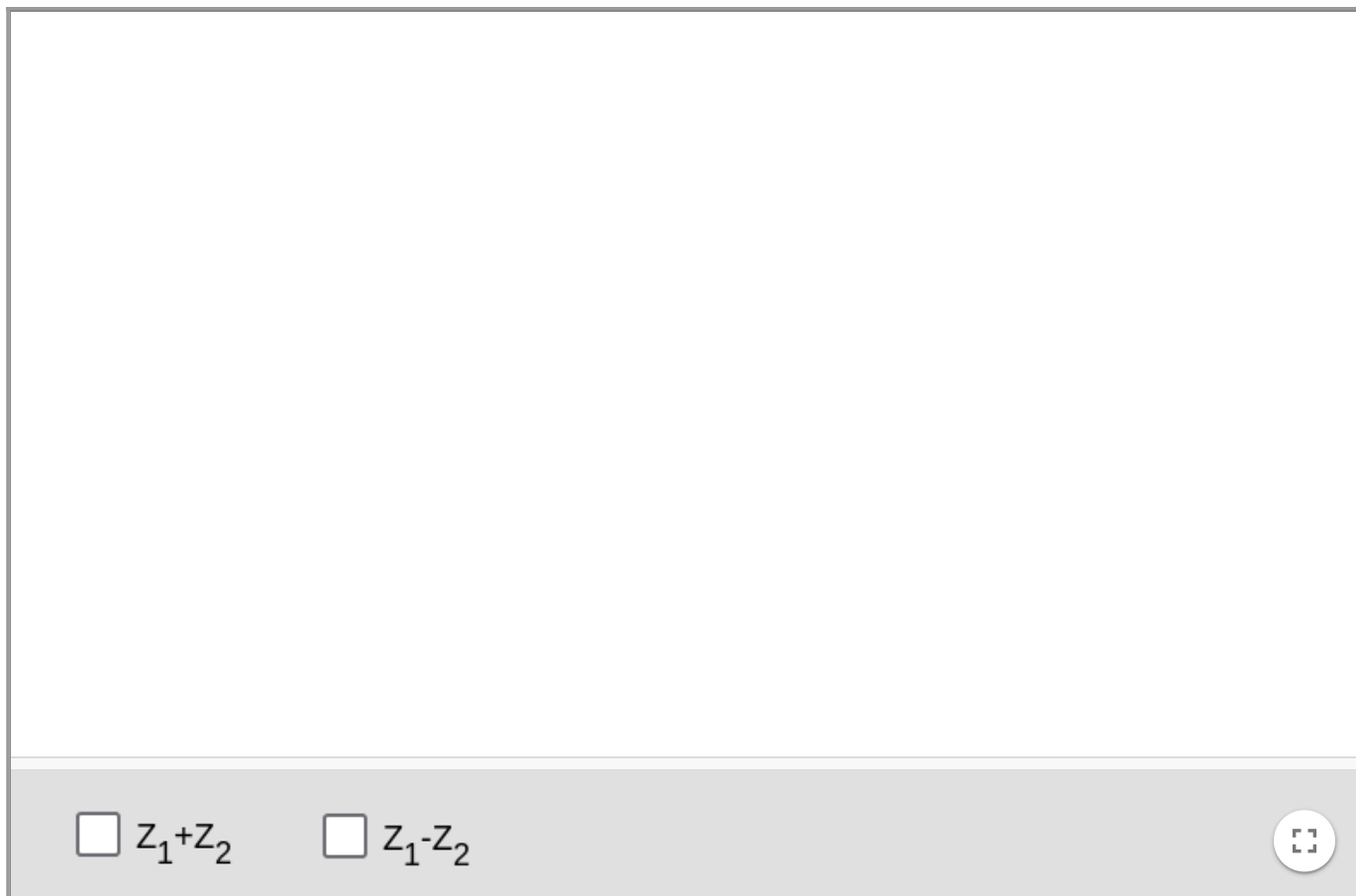
1.3: Geometric Interpretation of the Arithmetic Operations

Addition and Subtraction

Geometrically, addition of two complex numbers Z_1 and Z_2 can be visualized as addition of the vectors by using the *parallelogram law*. The vector sum $Z_1 + Z_2$ is represented by the diagonal of the parallelogram formed by the two original vectors.

The easiest way to represent the difference $Z_1 - Z_2$ is to think in terms of adding a negative vector $Z_1 + (-Z_2)$. The negative vector is the same vector as its positive counterpart, only pointing in the opposite direction.

Use the following applet to explore this geometric interpretation. Activate the boxes below to show the addition or subtraction. You can also drag the points Z_1 and Z_2 around.



Exercise 1.3.1

Can you think about a geometric interpretation of the addition of three complex numbers? In general, what would be a geometric interpretation of the addition of n complex numbers?

Multiplication

In the previous section we defined the multiplication of two complex numbers Z_1 and Z_2 as

$$Z_1 Z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

In this case, to appreciate what happens geometrically we need to consider the polar form of and . That is

$$Z_1 = r_1(\cos\phi_1 + i\sin\phi_1)$$

$$Z_2 = r_2(\cos\phi_2 + i\sin\phi_2)$$

Then the product can be written in the form

$$Z_1 Z_2 = r_1 r_2 [(\cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2) + i(\sin\phi_1 \cos\phi_2 + \cos\phi_1 \sin\phi_2)]$$

Now by means of the addition theorems of the sine and cosine this expression can be simplified to

$$Z_1 Z_2 = r_1 r_2 [\cos(\phi_1 + \phi_2) + i \sin(\phi_1 + \phi_2)].$$

Thus the product $Z_1 Z_2$ has the modulus $r_1 r_2$ and the argument $\phi_1 + \phi_2$.

In the following applet, you can appreciate what happens to the argument of the product. Drag the points Z_1 and Z_2 around and observe the behaviour the angles. Then drag the slider below.



Exercise 1.3.2

Consider now

$$Z_1 = r_1(\cos\phi_1 + i\sin\phi_1)$$

$$Z_2 = r_2(\cos\phi_2 + i\sin\phi_2)$$

such that $Z_2 \neq 0$. Find the polar representation of $\frac{Z_1}{Z_2}$. What is the geometric interpretation of this expression?

Multiplication of complex numbers as stretching (squeezing) and rotation

In the applet below a set of points are defined randomly on the complex plane. Then each point is multiplied by a given complex number z . On the right-side screen, drag around the point z and analyze the behaviour of the points (●) multiplied by z and try to answer the following questions:

- What happens when z is inside, or outside, the unit circle?
- What happens if z moves only around the unit circle?

Note: You can also study the behaviour of the points (●) multiplied by $\frac{1}{z}$ by activating the box `Multiply by 1/z`.

$$z = 1 + 0i$$

☒ Multiply by z

☐ Multiply by $1/z$



As you already have noticed the geometric interpretation of multiplication of complex numbers is stretching (or squeezing) and rotation of vectors in the plane.

In the previous applet, with the option `Multiply by  $z$` , set $n = 1$ by dragging the slider to the left side. In this case, the applet shows the three complex numbers

$$z_0, z \text{ and } z_1 = z_0 \cdot z,$$

represented as vectors. When z_0 and z are non zero, then

- the modulus of z_1 is equal to $|z_0 \cdot z|$, and
- the argument of z_1 is equal to $\text{Arg}(z_0 + z)$.

If $|z| > 1$, we deal with stretching. If $|z| < 1$, it is a case of squeezing.

Exercise 1.3.3

Use the same applet, with the option `Multiply by  $1/z$` , to investigate what happens when we multiply by $\frac{1}{z}$. Set $n = 1$ by dragging the slider to the left side to show the three complex numbers

$$z_0, z \text{ and } z_2 = z_0 \cdot \frac{1}{z}.$$

What happens to the modulus and argument of z_2 ?

This page titled [1.3: Geometric Interpretation of the Arithmetic Operations](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

1.4: The Principal Argument

The Argument

In this text the notation $\arg(z)$ is used to designate an arbitrary argument of z , which means that $\arg(z)$ is a set rather than a number. In particular, the relation

$$\arg(z_1) = \arg(z_2)$$

is not an equation, but expresses equality of two sets.

As a consequence, two non-zero complex numbers $r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $r_2(\cos\varphi_2 + i\sin\varphi_2)$ are equal if and only if

$$r_1 = r_2, \text{ and } \varphi_1 = \varphi_2 + 2k\pi,$$

where $k \in \mathbb{Z}$.

In order to make the argument of z a well-defined number, it is sometimes restricted to the interval $(-\pi, \pi]$. This special choice is called the *principal value* or the *main branch* of the argument and is written as $\text{Arg}(z)$.

Note that there is no general convention about the definition of the principal value, sometimes its values are supposed to be in the interval $[0, 2\pi)$. This ambiguity is a perpetual source of misunderstandings and errors.

The Principal Argument

The principal value $\text{Arg}(z)$ of a complex number $z = x + iy$ is normally given by

$$\Theta = \arctan\left(\frac{y}{x}\right),$$

where y/x is the slope, and $\arctan$ converts slope to angle. But this is correct only when $x > 0$, so the quotient is defined and the angle lies between $-\pi/2$ and $\pi/2$. We need to extend this definition to cases where x is not positive, considering the principal value of the argument separately on the four quadrants.

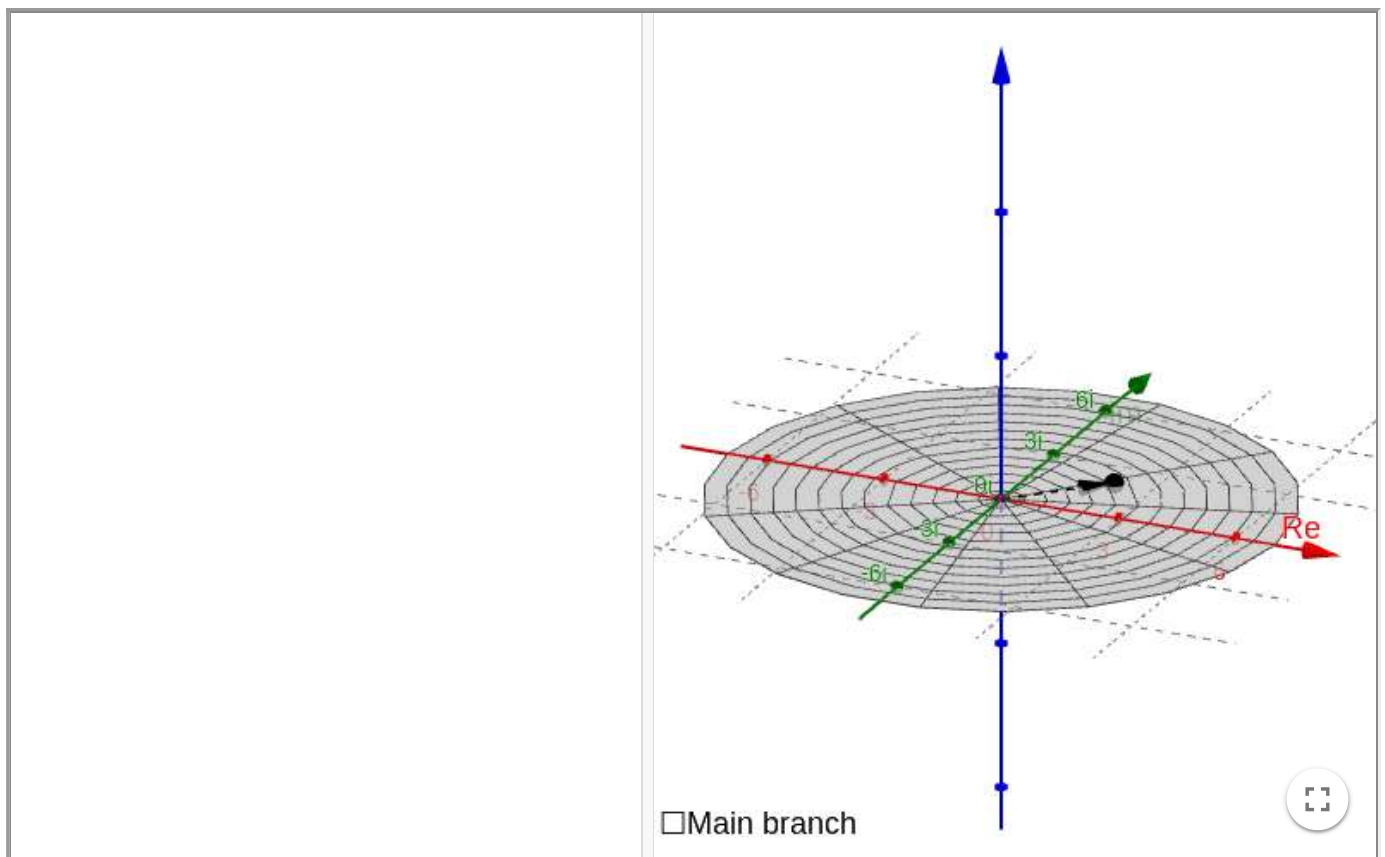
The function $\text{Arg}(z) : \mathbb{C} \setminus \{0\} \rightarrow (-\pi, \pi]$ is defined as follows:

$$\text{Arg}(z) = \begin{cases} \arctan\frac{y}{x} & \text{if } x > 0, y \in \mathbb{R} \\ \arctan\frac{y}{x} + \pi & \text{if } x < 0, y \geq 0 \\ \arctan\frac{y}{x} - \pi & \text{if } x < 0, y < 0 \\ \frac{\pi}{2} & \text{if } x = 0, y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0 \\ \text{undefined} & \text{if } x = 0, y = 0 \end{cases}$$

Thus, if $z = r(\cos\Theta + i\sin\Theta)$, with $r > 0$ and $-\pi < \Theta < \pi$, then

$$\arg(z) = \text{Arg}(z) + 2n\pi, \quad n \in \mathbb{Z}.$$

We can visualize the multiple-valued nature of $\arg$ by using [Riemann surfaces](#). The following interactive shows some of the infinite values of $\arg$. Each branch is identified with a different color.



This page titled [1.4: The Principal Argument](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

1.5: Roots of Complex Numbers

Recall that if $z = x + iy$ is a nonzero complex number, then it can be written in polar form as

$$z = r(\cos\theta + i\sin\theta)$$

where $r = \sqrt{x^2 + y^2}$ and θ is the angle, in radians, from the positive x-axis to the ray connecting the origin to the point z .

Now, de Moivre's formula establishes that if $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Let w be a complex number. Using de Moivre's formula will help us to solve the equation

$$z^n = w$$

for z when w is given.

Suppose that $w = r(\cos\theta + i\sin\theta)$ and $z = \rho(\cos\psi + i\sin\psi)$. Then de Moivre's formula gives

$$z^n = \rho^n(\cos n\psi + i\sin n\psi)$$

It follows that

$$\rho^n = r = |w|$$

by uniqueness of the polar representation and

$$n\psi = \theta + k(2\pi),$$

where k is some integer. Thus

$$z = \sqrt[n]{r}[\cos(\frac{\theta}{n} + \frac{2k\pi}{n}) + i\sin(\frac{\theta}{n} + \frac{2k\pi}{n})].$$

Each value of $k = 0, 1, 2, \dots, n-1$ gives a different value of z . Any other value of k merely repeats one of the values of z corresponding to $k = 0, 1, 2, \dots, n-1$. Thus there are exactly n th roots of a nonzero complex number.

Using Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

the complex number $z = r(\cos\theta + i\sin\theta)$ can also be written in exponential form as

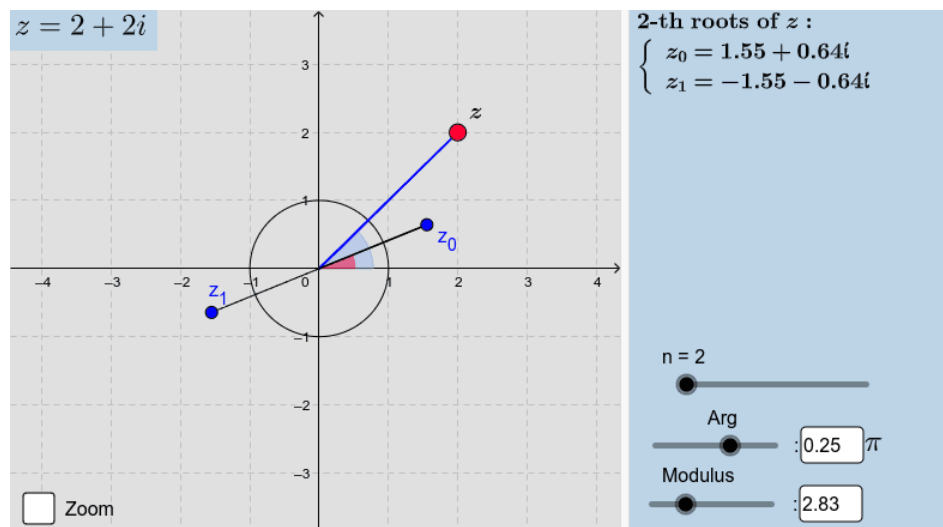
$$z = re^{i\theta}$$

Thus, the n th roots of a nonzero complex number $z \neq 0$ can also be expressed as

$$z = \sqrt[n]{r} \exp \left[i \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) \right] \quad (1.5.1)$$

where $k = 0, 1, 2, \dots, n-1$.

The applet below shows a geometrical representation of the n th roots of a complex number, up to $n = 10$. Drag the red point around to change the value of z or drag the sliders.



Code

Enter the following script in [GeoGebra](#) to explore it yourself and make your own version. The symbol # indicates comments.

```
#Complex number

Z = 1 + i

#Modulus of Z

r = abs(Z)

#Angle of Z

theta = atan2(y(Z), x(Z))

#Number of roots

n = Slider(2, 10, 1, 1, 150, false, true, false, false)

#Plot n-roots

nRoots = Sequence(r^(1 / n) * exp( i * ( theta / n + 2 * pi * k / n ) ), k, 0, n-1)
```



Exercise 1.5.1

From the exponential form (1) of the roots, show that all the n th roots lie on the circle $|z| = \sqrt[n]{r}$ about the origin and are equally spaced every $\frac{2\pi}{n}$ radians, starting with argument $\frac{\theta}{n}$.

This page titled [1.5: Roots of Complex Numbers](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

1.6: Topology of the Complex Plane

Neighbourhoods

An ε neighbourhood, also called *open ball* or *open disk*, of a complex number z_0 consists of all points z lying inside but not on a circle centred at z_0 and with radius $\varepsilon > 0$ and is expressed by

$$B_\varepsilon(z_0) = \{z : |z - z_0| < \varepsilon\} \quad (1.6.1)$$

The *closed* ε neighbourhood of z_0 is expressed by

$$\overline{B}_\varepsilon(z_0) = \{z : |z - z_0| \leq \varepsilon\} \quad (1.6.2)$$

And finally, a *deleted* ε neighbourhood of z_0 , also called *punctured balls or disks*, is expressed by

$$B_\varepsilon(z_0) \setminus \{z_0\} = \{z : 0 < |z - z_0| < \varepsilon\} \quad (1.6.3)$$

Figure 1 shows the geometrical representation of the following examples:

1. $B_1(0) = \{z : |z| < 1\}$
2. $\overline{B}_{\frac{7}{8}}(-1 - \sqrt{2}i) = \{z : |z - (-1 - \sqrt{2}i)| \leq \frac{7}{8}\}$
3. $\overline{B}_{\frac{1}{2}}(2 + \sqrt{3}i) \setminus \{2 + \sqrt{3}i\} = \{z : 0 < |z - (2 + \sqrt{3}i)| \leq \frac{1}{2}\}$

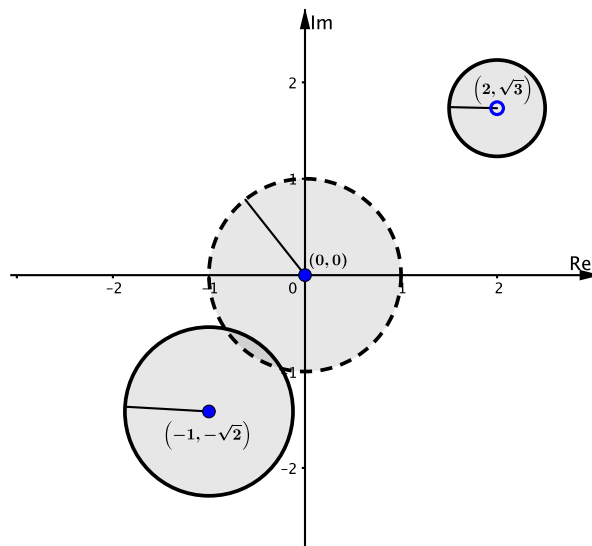


Figure 1: Geometrical representation of neighbourhoods.

Classification of points

A point z_0 is said to be an *interior point* of a set $S \subset \mathbb{C}$ whenever there is some neighbourhood of z_0 that contains only points of S ; it is called an *exterior point* of S when there exists a neighbourhood of it containing no points of S . If z_0 is neither of these, it is a *boundary point* of S . A boundary point is, therefore, a point all of whose neighbourhoods contain at least one point in S and at least one point not in S . The totality of all boundary points is called the *boundary* of S .

In this text we will use the following notation:

1. $\text{Int } S = \{z : z \text{ is an interior point of } S\}$
2. $\text{Ext } S = \{z : z \text{ is an exterior point of } S\}$
3. $\partial S = \{z : z \text{ is a boundary point of } S\}$

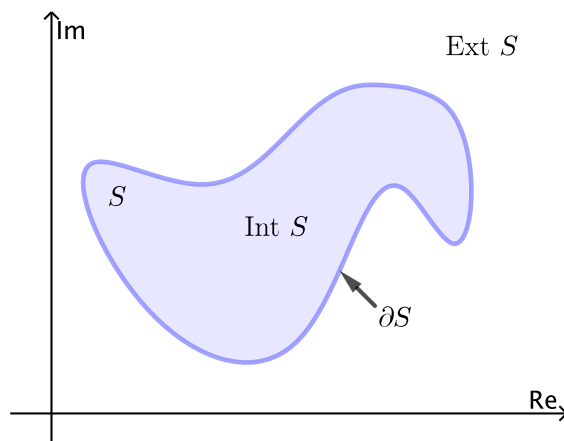


Figure 2: Interior, boundary, and exterior of set S .

Considering the previous examples of neighbourhoods, let

$$S_1 = B_1(0), S_2 = \bar{B}_{\frac{7}{8}}(-1 - \sqrt{2}i) \text{ and } S_3 = \bar{B}_{\frac{1}{2}}(2 + \sqrt{3}i) \setminus \{2 + \sqrt{3}i\}.$$

Thus, for S_1 we have:

- $\text{Int } S_1 = B_1(0)$
- $\text{Ext } S_1 = \{z : |z| > 1\}$
- $\partial S_1 = \{z : |z| = 1\}$

For S_2 we have:

- $\text{Int } S_2 = B_{\frac{7}{8}}(-1 - \sqrt{2}i)$
- $\text{Ext } S_2 = \{z : |z - (-1 - \sqrt{2}i)| > \frac{7}{8}\}$
- $\partial S_2 = \{z : |z - (-1 - \sqrt{2}i)| = \frac{7}{8}\}$

And finally, for S_3 we have:

- $\text{Int } S_3 = B_{\frac{1}{2}}(2 + \sqrt{3}i)$
- $\text{Ext } S_3 = \{z : |z - (2 + \sqrt{3}i)| > \frac{1}{2}\}$
- $\partial S_3 = \{z : |z - (2 + \sqrt{3}i)| = \frac{1}{2}\} \cup \{0\}$

Topological space

A set S is *open* if for every $z \in S$, exists $\epsilon > 0$ such that

$$B_\epsilon(z) \subset S.$$

That is, $\text{Int } S = S$. A set S is *closed* if it contains all of its boundary points, that is

$$\partial S \subseteq S.$$

The set $\mathbb{C}$ is both open and closed since it has no boundary points.

The set $\mathbb{C}$, together with the collection $\tau = \{S \subseteq \mathbb{C} : S \text{ is open}\}$ is a **topological space**, and this is expressed by the pair $(\mathbb{C}, \tau)$. The topological space $(\mathbb{C}, \tau)$ satisfies the following:

1. $\emptyset$ and $\mathbb{C}$ are open.
2. Whenever two or more sets are open, then so is their union.
3. Whenever sets S_1 and S_2 are open, then so is $S_1 \cap S_2$.

Remark: The technical definition of topological space is a bit unintuitive, particularly if you haven't studied topology. In essence, it states that the geometric properties of subsets of $\mathbb{C}$ will be preserved when continuous transformations (functions or mappings) are applied.

The *closure* of a set S is the closed set consisting of all points in S together with the boundary of S . In other words

$$\bar{S} = S \cup \partial S.$$

An open set S is *connected* if each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in S .

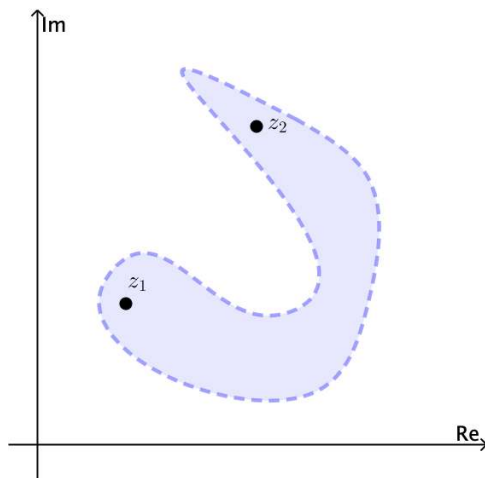


Figure 3: Connected set.

Notice for example that the open set $|z| < 1$ is connected. The *annulus* $1 < |z| < 2$ is open and also connected, see Figures 4 and 5.

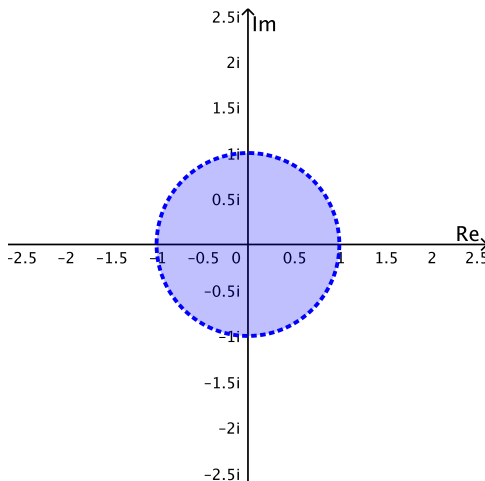


Figure 4: $|z| < 1$

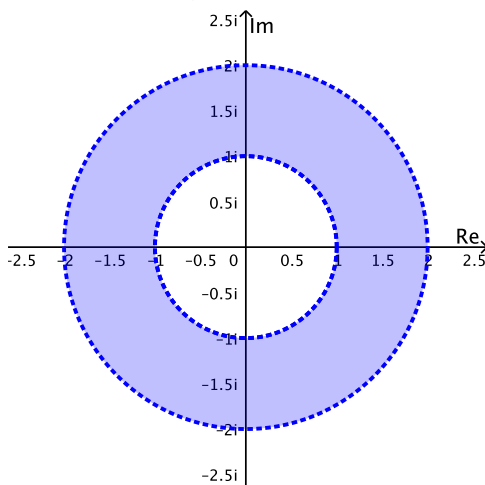


Figure 5: $|z| < 2$

A nonempty open set that is connected is called a *domain*. In this context, any neighbourhood is a domain. A domain together with some, none, or all of its boundary points is referred to as a *region*. In other words, a set whose interior is a domain is called a *region*. A set S is bounded if there is $R > 0$ such that

$$S \subset B_R(0) = \{z \in \mathbb{C} : |z| < R\}.$$

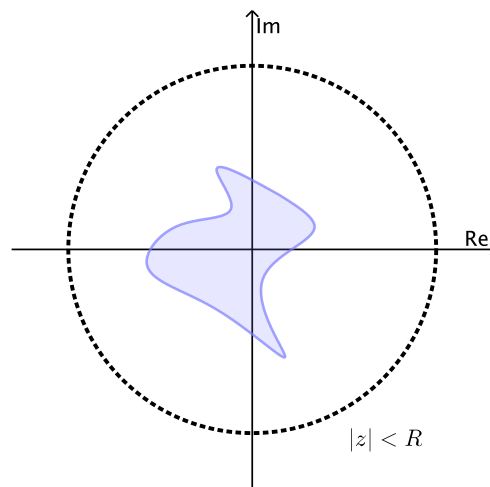


Figure 6: Bounded set.

Exercise 1.6.1

Sketch the set S of points in the complex plane satisfying the given inequality. Determine whether the set is (a) open, (b) closed, (c) a domain, (d) bounded, or (e) connected.

- $\text{Im}(z) < 0$
- $-1 < \text{Re}(z) < 1$
- $|z| > 1$
- $2 \leq |z - 3 + 4i| \leq 5$

This page titled [1.6: Topology of the Complex Plane](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

CHAPTER OVERVIEW

2: Chapter 2

[2.1: Complex functions](#)

[2.2: Riemann Sphere](#)

[2.3: Complex Differentiation](#)

[2.4: The Logarithmic Function](#)

[2.5: Riemann Surfaces](#)

This page titled [2: Chapter 2](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

2.1: Complex functions

 $f(z)=z^2$

Complex functions

Let S be a set of complex numbers. A *function* f defined on S is a rule that assigns to each z in S a complex number w . The number w is called the value of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set S is called the domain of definition of f .

If only one value of w corresponds to each value of z , we say that w is a *single-valued* function of z or that $f(z)$ is single-valued. If more than one value of w corresponds to each value of z , we say that w is a *multiple-valued* or *many-valued* function of z .

A *multiple-valued* function can be considered as a collection of single-valued functions, each member of which is called a *branch* of the function. In general, we consider one particular member as a *principal branch* of the multiple-valued function and the value of the function corresponding to this branch as the *principal value*.

Example 2.1.1

The function $w = z^2$ is a single-valued function of z . On the other hand, if $w = z^{\frac{1}{2}}$, then to each value of z there are two values of w . Hence, the function

$$w = z^{\frac{1}{2}}$$

is a multiple-valued (in this case two-valued) function of z .

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, so that

$$u + iv = f(x + iy)$$

Each of the real numbers u and v depends on the real variables x and y , and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y :

$$f(z) = u(x, y) + iv(x, y). \quad (2.1.1)$$

If the polar coordinates r and θ , instead of x and y , are used, then

$$u + iv = f(re^{i\theta})$$

where $w = u + iv$ and $z = re^{i\theta}$. In this case, we write

$$f(z) = u(r, \theta) + iv(r, \theta). \quad (2.1.2)$$

Example 2.1.2

Example 2: If $f(z) = z^2$ then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i(2xy).$$

Hence

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

When we use polar coordinates, we have

$$u(r, \theta) = r^2 \cos 2\theta \quad \text{and} \quad v(r, \theta) = r^2 \sin 2\theta.$$

Question: What happens when in either of equations (1) and (2) the function v always has a value zero?

Examples of complex functions

Polynomial functions

For $a_n, a_{n-1}, \dots, a_0$ complex constants we define

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

where $a_n \neq 0$ and n is a positive integer called the *degree* of the polynomial $p(z)$.

Rational functions: Ratios

$$\frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials and $q(z) \neq 0$.

Exponential function

Exponential function: If $z = x + iy$, the exponential function e^z is defined by writing

$$e^z = e^x e^{iy}.$$

Because

$$e^{iy} = \cos y + i \sin y,$$

then we have

$$e^z = e^x (\cos y + i \sin y).$$

Logarithmic function

In a similar fashion, the complex logarithm is a complex extension of the usual real natural (i.e., base e) logarithm. In terms of polar coordinates $z = re^{i\theta}$, the complex logarithm has the form

$$\log z = \log(re^{i\theta}) = \log r + \log e^{i\theta} = \log r + i\theta.$$

We will explore in detail this function in the following section.

Trigonometric functions

The sine and cosine of a complex variable z are defined as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The other four trigonometric functions are defined in terms of the sine and cosine functions with the following relations:

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z} & \cot z &= \frac{\cos z}{\sin z} \\ \sec z &= \frac{1}{\cos z} & \csc z &= \frac{1}{\sin z}. \end{aligned}$$

Hyperbolic trigonometric functions

The hyperbolic sine and the hyperbolic cosine of a complex variable are defined as they are with a real variable; that is,

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

The other four hyperbolic functions are defined in terms of the hyperbolic sine and cosine functions with the relations:

$$\begin{aligned} \tanh z &= \frac{\sinh z}{\cosh z} & \coth z &= \frac{\cosh z}{\sinh z} \\ \operatorname{sech} z &= \frac{1}{\cosh z} & \operatorname{csch} z &= \frac{1}{\sinh z}. \end{aligned}$$

Explore the real and imaginary components

Use the following applet to explore the real and imaginary components of some complex functions.

[INTERACTIVE GRAPH](#)

▼ Code

Enter the following scripts in [GeoGebra](#) to explore it yourself. Open the 3D view. The symbol # indicates comments.

```
#Define complex function

f(z) := z + 1/z

#Define components

Re = Surface(u, v, real( f(u + i v) ), u, -5, 5, v, -5, 5)

Im = Surface(u, v, imaginary( f(u + i v) ), u, -5, 5, v, -5, 5)
```

This page titled [2.1: Complex functions](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

2.2: Riemann Sphere

The Point to Infinity

For some purposes it is convenient to introduce the *point to infinity*, denoted by ∞ , in addition to the points $z \in \mathbb{C}$. We must be careful in doing so, because it can lead to confusion an abuse of the symbol ∞ . However, with care it can be useful, if we want to be able to talk about infinite limits and limits at infinity.

In contrast to the real line, to which $+\infty$ and $-\infty$ can be added, we have only one ∞ for $\mathbb{C}$. The reason is that $\mathbb{C}$ has no natural ordering as $\mathbb{R}$ does. Formally we add a symbol ∞ to $\mathbb{C}$ to obtain the *extended complex plane*, denoted by $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, and define operations with ∞ by the rules

$$\begin{aligned} z + \infty &= \infty \\ z \cdot \infty &= \infty \quad \text{provided } z \neq 0 \\ \infty + \infty &= \infty \\ \infty \cdot \infty &= \infty \\ \frac{z}{\infty} &= 0 \end{aligned}$$

for $z \in \mathbb{C}$. Notice that some operations are not defined:

$$\frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty$$

and so forth are for the same reasons that they are in the calculus of real numbers.

The extended complex plane can be mapped onto the surface of a sphere whose south pole corresponds to the origin and whose north pole to the point ∞ . All other points of the complex plane can be mapped in a one-to-one fashion to points on the surface of the sphere by using the following construction. Connect the point z in the plane with the north pole using a straight line. This line intersects the sphere at the point $P(z)$. In this way each point $z = x + iy$ on the complex plane corresponds uniquely to a point $P(z)$ on the surface of the sphere. This construction is called the *stereographic projection* and is illustrated in the following applet.

In the following applet we can observe the unit sphere whose south pole corresponds to the origin of the z plane. Drag the point defined on the z plane, or the sliders, to explore the behaviour of the point $P(z)$ on the sphere.

INTERACTIVE GRAPH

The extended complex plane is sometimes referred to as the *compactified* (closed) complex plane. It is often useful to view the complex plane in this way, and knowledge of the construction of the stereographic projection is valuable in certain advanced treatments.

Now we can introduce the following limit concepts:

1. $\lim_{z \rightarrow \infty} f(z) = z_0$ means: For any $\varepsilon > 0$, there is an $R > 0$ such that $|f(z) - z_0| < \varepsilon$ whenever $|z| > R$.
2. $\lim_{z \rightarrow z_0} f(z) = \infty$ means: For any $R > 0$, there is a $\delta > 0$ such that $|f(z)| > R$ whenever $0 < |z - z_0| < \delta$.
3. $\lim_{z \rightarrow \infty} f(z) = \infty$ means: For any $M > 0$, there is an $R > 0$ such that $|f(z)| > M$ whenever $|z| > R$.

Example 2.2.1

Example 1: If $f(z) = 1/z^2$, for $z \neq 0$, then

$$\lim_{z \rightarrow \infty} f(z) = 0.$$

In fact, given $\varepsilon > 0$ we have

$$\left| \frac{1}{z^2} - 0 \right| = \frac{1}{|z|^2} = \frac{1}{|z|^2} < \varepsilon$$

by taking

$$|z| > \frac{1}{\sqrt{\varepsilon}} = R.$$

Example 2.2.2

Let $f(z) = 1/(z-3)$, for $z \neq 3$. Then

$$\lim_{z \rightarrow 3} f(z) = \infty.$$

In fact, for any given $R > 0$ the inequality

$$\frac{1}{|z-3|} > R$$

holds whenever

$$0 < |z-3| < \frac{1}{R} = \delta.$$

Example 2.2.3

For $f(z) = e^z$, defined on $D = \{z : \operatorname{Re}(z) > 0\}$, we have

$$\lim_{z \rightarrow \infty} f(z) = \infty.$$

In fact, for any given $M > 0$ we have

$$|f(z)| = |e^z| = e^x > M$$

whenever $x > \ln M$. Hence, it suffices to take $R > \max\{0, \ln R\}$.

It is worth to mention that for $f(z) = e^z$ defined on $D_1 = \{z : \operatorname{Re}(z) = 0\}$, i.e., for $f(iy) = e^{iy}$, there is no limit as $z = iy \rightarrow \infty$, because along the imaginary axis, $e^{iy} = \cos y + i \sin y$ is periodic and not constant.

Finally, for $f(z) = e^z$ defined on $D_2 = \{z : \operatorname{Re}(z) < 0\}$ we have that

$$\lim_{z \rightarrow \infty} f(z) = 0.$$

There is an easier way to calculate the limits from Examples 1-3. The following theorem provides a very useful method.

Theorem :

If z_0 and w_0 are points in the z and w planes, respectively, then

$$\lim_{z \rightarrow z_0} f(z) = \infty \quad \text{if and only if} \quad \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0,$$

$$\lim_{z \rightarrow \infty} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0,$$

$$\lim_{z \rightarrow \infty} f(z) = \infty \quad \text{if and only if} \quad \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = \infty.$$

Using this result, we can easily find that

$$\lim_{z \rightarrow -1} \frac{iz+3}{z+1} = \infty \quad \text{since} \quad \lim_{z \rightarrow -1} \frac{z+1}{iz+3} = 0$$

and

$$\lim_{z \rightarrow \infty} \frac{2z+i}{z+1} = 2 \quad \text{since} \quad \lim_{z \rightarrow 0} \frac{(2/z)+i}{(1/z)+1} = \frac{2+iz}{1+z} = 2.$$

Furthermore,

$$\lim_{z \rightarrow \infty} \frac{2z^3-1}{z^2+1} = \infty \quad \text{since} \quad \lim_{z \rightarrow 0} \frac{(1/z^3)+1}{(2/z^3)-1} = \frac{z+z^3}{2-z^3} = 0.$$

This page titled [2.2: Riemann Sphere](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

2.3: Complex Differentiation

The notion of the complex derivative is the basis of complex function theory. The definition of complex derivative is similar to the derivative of a real function. However, despite a superficial similarity, complex differentiation is a deeply different theory.

A complex function $f(z)$ is *differentiable* at a point $z_0 \in \mathbb{C}$ if and only if the following limit difference quotient exists

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}. \quad (2.3.1)$$

Alternatively, letting $\Delta z = z - z_0$, we can write

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \quad (2.3.2)$$

We often drop the subscript on z_0 and introduce the number

$$\Delta w = f(z + \Delta z) - f(z).$$

which denotes the change in the value $w = f(z)$ corresponding to a change Δz in the point at which f is evaluated. Then we can write equation (2) as

$$\frac{dw}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}.$$

Despite the fact that the formula (1) for a derivative is identical in form to that of the derivative of a real-valued function, a significant point to note is that $f'(z_0)$ follows from a two-dimensional limit. Thus for $f'(z_0)$ to exist, the relevant limit must exist independent of the direction from which z approaches the limit point z_0 . For a function of one real variable we only have two directions, that is, $x < x_0$ and $x > x_0$.



Figure 1: There are an infinite variety of directions to approach z_0 .

A remarkable feature of complex differentiation is that the existence of one complex derivative automatically implies the existence of infinitely many! This is in contrast to the case of the function of real variable $g(x)$, in which $g'(x)$ can exist without the existence of $g''(x)$.

Cauchy-Riemann equations

Now let's see a remarkable consequence of definition (1). First we will see what happens when we approach z_0 along the two simplest directions - horizontal and vertical. If we set

$$z = z_0 + h = (x_0 + h) + iy_0, \quad h \in \mathbb{R},$$

then $z \rightarrow z_0$ along a horizontal line as $h \rightarrow 0$. If we write f in terms of its real and imaginary components, that is

$$f(z) = u(x, y) + iv(x, y),$$

then

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h}$$

then

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0+h+iy_0) - f(x_0+iy_0)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} \right] + i \lim_{h \rightarrow 0} \left[\frac{v(x_0+h, y_0) - v(x_0, y_0)}{h} \right] \\ &= u_x(x_0, y_0) + i v_x(x_0, y_0) \end{aligned}$$

where $u_x(x_0, y_0)$ and $v_x(x_0, y_0)$ denote the first-order partial derivatives with respect to x of the function u and v , respectively, at (x_0, y_0) . If now we set

$$z = z_0 + ik = x_0 + i(y_0 + k), \quad k \in \mathbb{R},$$

then $z \rightarrow 0$ along a vertical line as $k \rightarrow 0$. Therefore, we also have

$$\begin{aligned} f'(z_0) &= \lim_{k \rightarrow 0} \frac{f(z_0 + ik) - f(z_0)}{ik} = \lim_{k \rightarrow 0} \left[-i \frac{f(x_0 + i(y_0 + k)) - f(x_0 + iy_0)}{k} \right] \\ &= \lim_{k \rightarrow 0} \left[\frac{v(x_0, y_0 + k) - v(x_0, y_0)}{k} - i \frac{u(x_0, y_0 + k) - u(x_0, y_0)}{k} \right] \\ &= v_y(x_0, y_0) - iu_y(x_0, y_0) \end{aligned}$$

where the partial derivatives of u and v are, this time, with respect to y . By equating the real and imaginary parts of these two formulae for the complex derivative [Extra \left or missing \right](#), we notice that the real and imaginary components of $f(z)$ must satisfy a homogeneous linear system of partial differential equations:

$$u_x = v_y, \quad u_y = -v_x.$$

These are the *Cauchy-Riemann equations* named after the famous nineteenth century mathematicians Augustin-Louis Cauchy and Bernhard Riemann, two of the founders of modern complex analysis.

Theorem 2.3.1

A complex function $f(z) = u(x, y) + iv(x, y)$ has a complex derivative $f'(z)$ if and only if its real and imaginary part are continuously differentiable and satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

In this case, the complex derivative of $f(z)$ is equal to any of the following expressions:

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Example 2.3.1

Consider the function $f(z) = z^2$, which can be written as

$$z^2 = (x^2 - y^2) + i(2xy).$$

Its real part $u^2 = x^2 - y^2$ and imaginary part $v = 2xy$ satisfy the Cauchy-Riemann equations, since

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x.$$

Theorem 1 implies that $f(z) = z^2$ is differentiable. Its derivative turns out to be

$$f'(z) = u_x + iv_x = v_y - iu_y = 2x + i2y = 2(x + iy) = 2z.$$

Fortunately, the complex derivative has all of the usual rules that we have learned in real-variable calculus. For example,

$$\frac{d}{dz} z^n = nz^{n-1}, \quad \frac{d}{dz} e^{cz} = ce^{cz}, \quad \frac{d}{dz} \log z = \frac{1}{z}$$

and so on. In this case, the power n can be a real number (or even complex in view of the identity $z^n = e^{n \log z}$), while c is any complex constant. The [exponential formulae](#) for the complex trigonometric and hyperbolic functions implies that they also satisfy the standard rules

$$\begin{aligned} \frac{d}{dz} \sin z &= \cos z, & \frac{d}{dz} \cos z &= -\sin z \\ \frac{d}{dz} \sinh z &= \cosh z, & \frac{d}{dz} \cosh z &= \sinh z \end{aligned}$$

The formulae for differentiating sums, products, ratios, inverses, and compositions of complex functions are all identical to their real counterparts, with similar proofs. This means that you don't need to learn any new rules for performing complex differentiation!

Analytic functions

Let $f : A \rightarrow \mathbb{C}$ where $A \subset \mathbb{C}$ is an open set. The function is said to be *analytic* on A if f is differentiable at each $z_0 \in A$. The word “holomorphic”, which is sometimes used, is synonymous with the word “analytic”. The phrase “*analytic at z_0* ” means f is analytic on a neighborhood of z_0 .

This page titled [2.3: Complex Differentiation](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

2.4: The Logarithmic Function

Consider z any nonzero complex number. We would like to solve for w , the equation

$$e^w = z. \quad (2.4.1)$$

If $\Theta = \text{Arg}(z)$ with $-\pi < \Theta < \pi$, then z and w can be written as follows

$$z = re^{i\Theta} \quad \text{and} \quad w = u + iv.$$

Then equation (1) becomes

$$e^u e^{iv} = re^{i\Theta}.$$

Thus, we have

$$e^u = r \quad \text{and} \quad v = \Theta + 2n\pi$$

where $n \in \mathbb{Z}$. Since $e^u = r$ is the same as $u = \ln r$, it follows that equation (1) is satisfied if and only if w has one of the values

$$w = \ln r + i(\Theta + 2n\pi) \quad (n \in \mathbb{Z}).$$

Therefore, the (multiple-valued) logarithmic function of a nonzero complex variable $z = re^{i\Theta}$ is defined by the formula

$$\log z = \ln r + i(\Theta + 2n\pi) \quad (n \in \mathbb{Z}). \quad (2.4.2)$$

Example 2.4.1

Example 1: Calculate $\log z$ for $z = -1 - \sqrt{3}i$.

Solution: If $z = -1 - \sqrt{3}i$, then $r = 2$ and $\Theta = -\frac{2\pi}{3}$. Hence

$$\log(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2n\pi\right) = \ln 2 + 2\left(n - \frac{1}{3}\right)\pi i$$

with $n \in \mathbb{Z}$.

The principal value of $\log z$ is the value obtained from equation (2) when $n = 0$ and is denoted by $\text{Log } z$. Thus

$$\text{Log } z = \ln r + i\Theta.$$

The function $\text{Log } z$ is well defined and single-valued when $z \neq 0$ and that

$$\log z = \text{Log } z + 2n\pi i \quad (n \in \mathbb{Z})$$

This is reduced to the usual logarithm in calculus when z is a positive real number.

Example 2.4.2

Example 2: Calculate $\log(1)$ and $\log(-1)$.

Solution: From expression (2)

$$\log(1) = \ln 1 + i(0 + 2n\pi) = 2n\pi i \quad (n \in \mathbb{Z})$$

and

$$\log(-1) = \ln 1 + i(\pi + 2n\pi) = (2n + 1)\pi i \quad (n \in \mathbb{Z})$$

Notice that $\text{Log}(1) = 0$ and $\log(-1) = \pi i$.

Expression (2) is also equivalent to the following:

$$\begin{aligned} \log z &= \ln |z| + i \arg(z) \\ &= \ln |z| + i \text{Arg}(z) + 2ni\pi \quad (n \in \mathbb{Z}) \end{aligned}$$

Some basic properties of the function $\log z$ are the following:

1. $\log(z_1 z_2) = \log z_1 + \log z_2$
2. $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$
3. There may hold $\text{Log}(z_1 z_2) \neq \text{Log} z_1 + \text{Log} z_2$

Branches of Logarithms

From definition (2) let $\theta = \Theta + 2n\pi (n \in \mathbb{Z})$, so we can write

$$\log z = \ln r + i\theta. \quad (2.4.3)$$

Now, let α be any real number. If we restrict the value of θ so that $\alpha < \theta < \alpha + 2\pi$, then the function

$$\log z = \ln r + i\theta \quad (r > 0, \alpha < \theta < \alpha + 2\pi), \quad (2.4.4)$$

with components

$$u(r, \theta) = \ln r, \quad v(r, \theta) = \theta,$$

is a *single-value* and continuous function in the stated domain.

A *branch* of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f . The requirement of analyticity, of course, prevents F from taking on a random selection of the values of f .

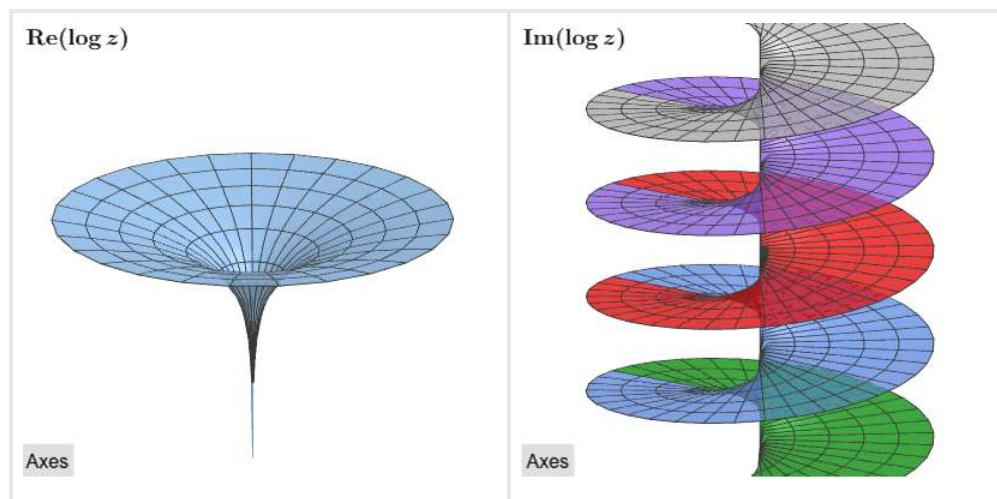
Observe that for each fixed α , the single-valued function (4) is a branch of the multiple-valued function (3). The function

$$\text{Log} z = \ln r + i\Theta \quad (r > 0, -\pi < \theta < \pi), \quad (2.4.5)$$

is called the principal branch.

A *branch cut* is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f . Points on the branch cut for F are *singular points* of F , and any point that is common to all branch cuts of f is called a *branch point*. The origin and the ray $\theta = \alpha$ make up the branch cut for the branch (4) of the logarithmic function. The branch cut for the principal branch (5) consists of the origin and the ray $\theta = \pi$. The origin is evidently a branch point for branches of the multiple-valued logarithmic function.

We can visualize the multiple-valued nature of $\log z$ by using [Riemann surfaces](#). The following interactive images show the real and imaginary components of $\log z$. Each branch of the imaginary part is identified with a different color.



Special care must be taken in using branches of the logarithmic function, especially since expected identities involving logarithms do not always carry over from calculus.

Final Remark

Notice that for $z \neq 0$, we have

$$e^{\log z} = z \quad \text{and} \quad \log(e^z) = z + 2n\pi i \quad (2.4.6)$$

with $n \in \mathbb{Z}$.

Example 2.4.3

Calculate $e^{\log z}$, and $\log(e^z)$ for $z = 4i$.

Solution

If $z = 4i$, then $e^z = e^{4i}$. Hence

$$\log(e^{4i}) = 4i + 2n\pi i$$

with $n \in \mathbb{Z}$. On the other hand, we have

$$e^{\log(4i)} = 4i.$$

This page titled [2.4: The Logarithmic Function](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

2.5: Riemann Surfaces

This page titled [2.5: Riemann Surfaces](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

CHAPTER OVERVIEW

3: Chapter 3

[3.1: Mappings](#)

[3.2: The Transformation \$w=1/z\$](#)

[3.3: Analytic Landscapes](#)

[3.4: Domain coloring](#)

[3.5: The Complex Power Function](#)

This page titled [3: Chapter 3](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

3.1: Mappings

A complex function $w = f(z)$ can be regarded as a *mapping* or *transformation* of the points in the $z = x + iy$ plane to the points of the $w = u + iv$ plane. In real variables in one dimension, this notion amounts to understanding the graph $y = f(x)$, that is, the mapping of the points x to $y = f(x)$.

In complex variables, the situation is more difficult due to the fact that we have four dimensions. Thus a graphical depiction such as in the real one-dimensional case is not feasible. Rather, one considers the two complex planes, z and w , separately and asks how a region in the z plane transforms or maps to a corresponding region or image in the w plane.

The applet below visualizes the action of a complex function as a mapping from a subset of the z -plane to the w -plane. For example, the light purple regions are the domain set and the range of the function, respectively. Any point z of the domain set is mapped to the corresponding point $f(z) = w$ in the range. Of course, we can also choose a different domain (i.e. a triangle or square) to apply the mapping. In this manner the function maps (transforms) the colored objects from the domain to the range. Drag the triangle and square (or points) defined on the z -plane to observe the effect of the transformation in the w -plane.

INTERACTIVE GRAPH

Remark: In complex analysis the notion of domain has two different meanings. The first one alludes to the domain set of a function, while the second pertains to any open and connected subset of the complex plane or the Riemann sphere. Most domain sets of complex functions we shall encounter in this book will indeed be domains in the topological sense.

This page titled [3.1: Mappings](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

3.2: The Transformation $w=1/z$

Consider the equation

$$w = \frac{1}{z}$$

which establishes a one-to-one correspondence between the nonzero points of the z and w planes. Since $z\bar{z} = |z|^2$, the mapping can be described by means of the successive transformations

$$g(z) = \frac{z}{|z|^2}.$$

The first transformation $g(z)$ is an inversion with respect to the unit circle $|z| = 1$. That is, the image of a nonzero point z is the point $g(z)$ with the properties

$$|g(z)| = \frac{1}{|z|} \quad \text{and} \quad \arg g(z) = \arg z.$$

Thus the points exterior to the circle $|z| = 1$ are mapped onto the nonzero points interior to it, and conversely. Any point on the circle is mapped onto itself. The second transformation $f(z) = \overline{g(z)}$ is simply a reflection in the real axis.



If we consider the function

$$T(z) = \frac{1}{z}, \quad z \neq 0$$

we can define T at the origin and at the point at infinity so as to be continuous on the *extended* complex plane. In order to make T continuous on the extended plane, then, we write

$$T(0) = \infty, \quad T(\infty) = 0, \quad \text{and} \quad T(z) = \frac{1}{z}$$

for the remaining values of z .

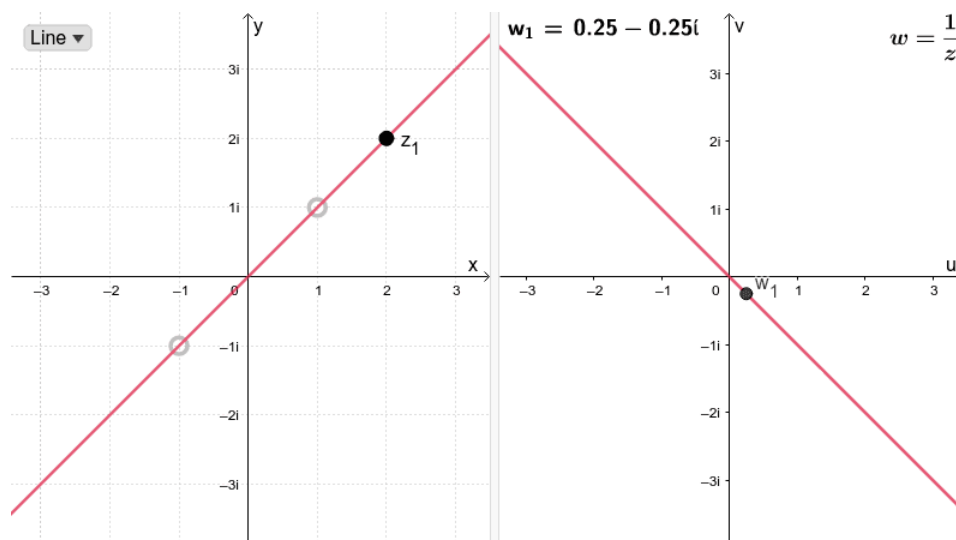
Mappings by $(1/z)$

An interesting property of the mapping $w = 1/z$ is that it transforms circles and lines into circles and lines.

You can observe this intuitively in the following applet. Things to try:

- Select between a **Line** or **Circle**.
- Drag points around on the left-side window. You can also change the position of the line or circle by dragging the grey points.

INTERACTIVE GRAPH



Observe carefully what happens to the points w_1, w_2 (the image of z_1 and z_2 , respectively) on the w -plane, shown on the right-side window.

- What do you notice when the line on the xy -plane crosses the origin?
- What happens when the circle on the xy -plane crosses the origin?

When A, B, C and D are all real numbers satisfying the condition $B^2 + C^2 > 4AD$, the equation

$$A(x^2 + y^2) + Bx + Cy + D = 0 \quad (1)$$

represents an arbitrary circle or line, where $A \neq 0$ for a circle and $A = 0$ for a line.

By using the method of completing the squares, we can rewrite equation (1) as follows

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2 + C^2 - 4AD}}{2A}\right)^2$$

This makes evident the need for condition $B^2 + C^2 > 4AD$ when $A \neq 0$. When $A = 0$, the condition becomes $B^2 + C^2 > 0$, which means that B and C are not both zero.

Now, using the relations

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i} \quad (2)$$

we can rewrite equation (1) in the form

$$2Az\bar{z} + (B - iC)z + (B + iC)\bar{z} + 2D = 0. \quad (3)$$

Since $w = 1/z$, equation (3) becomes

$$2Dw\bar{w} + (B + iC)w + (B - iC)\bar{w} + 2A = 0$$

and using the relations

$$u = \frac{w+\bar{w}}{2}, \quad v = \frac{w-\bar{w}}{2i}, \quad (4)$$

we obtain

$$D(u^2 + v^2) + Bu - Cv + A = 0 \quad (5)$$

which also represents a circle or line.

It is now clear from equations (1) and (5) that

1. a circle ($A \neq 0$) not passing through the origin ($D \neq 0$) in the z plane is transformed into a circle not passing through the origin in the w plane;
2. a circle ($A \neq 0$) through the origin ($D = 0$) in the z plane is transformed into a line that does not pass through the origin in the w plane;
3. a line ($A = 0$) not passing through the origin ($D \neq 0$) in the z plane is transformed into a circle through the origin in the w plane;
4. a line ($A = 0$) through the origin ($D = 0$) in the z plane is transformed into a line through the origin in the w plane.

Exercise 3.2.1

Verify that the expression $D(u^2 + v^2) + Bu - Cv + A = 0$ can be obtained from (1) using the relations (2) and (4).

This page titled 3.2: The Transformation $w=1/z$ is shared under a CC BY-NC-SA license and was authored, remixed, and/or curated by Juan Carlos Ponce Campuzano.

SECTION OVERVIEW

3.3: Analytic Landscapes

A brief history

A traditional concept for visualizing complex functions is the so called *analytic landscape*. Probably introduced by [Edmond Maillet](#) in 1903, it depicts the graph of the absolute value of a function.

In the first half of the preceding century analytic landscapes became rather popular. The Figure 1 reproduces a historical illustration from the book *Funktionentafeln mit Formeln und Kurven* by Jahnke and Emde of 1909. It shows the analytic landscape of the complex [Gamma function](#) and reached an almost iconic status. Today it is hard to believe that this detailed hand-drawn picture could be created without the help of computers!

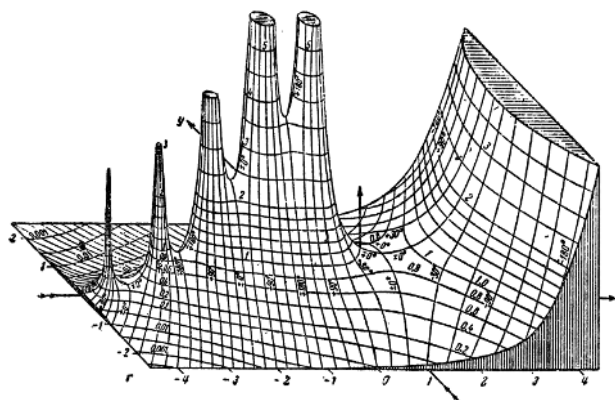


Figure 3.3.1: A historical analytic landscape of $\Gamma(z)$.

In the era of black-and-white illustrations this shortcoming was often compensated by endowing the analytic landscape with lines of constant argument as in the previous figure, where the argument is explicitly indicated by its numerical value. Modern computer technology allows us to achieve the same effect much better using colors, which yields the *colored analytic landscape* shown in Figure 2.

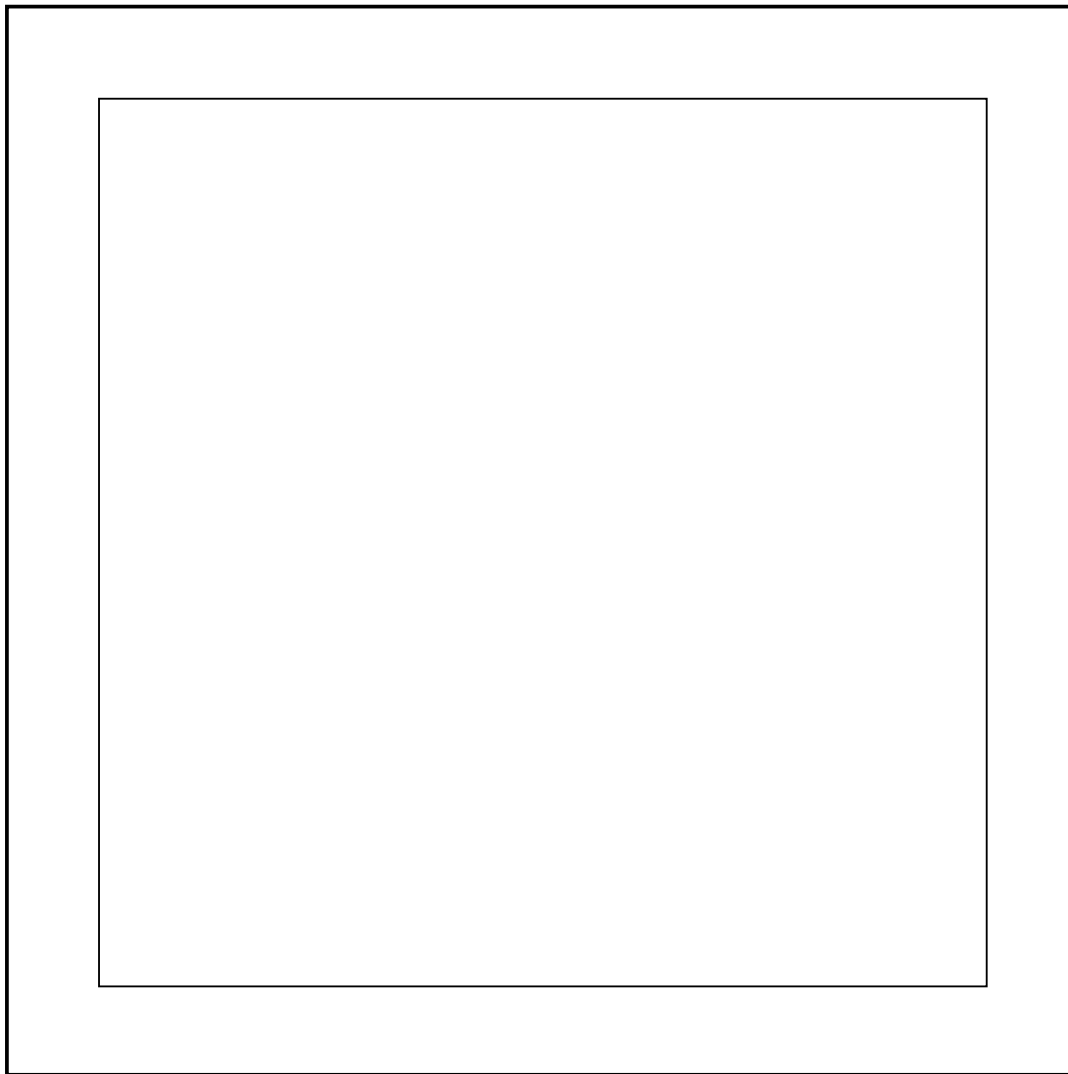


Figure 3.3.2: Colored analytic landscape of $\Gamma(z)$.

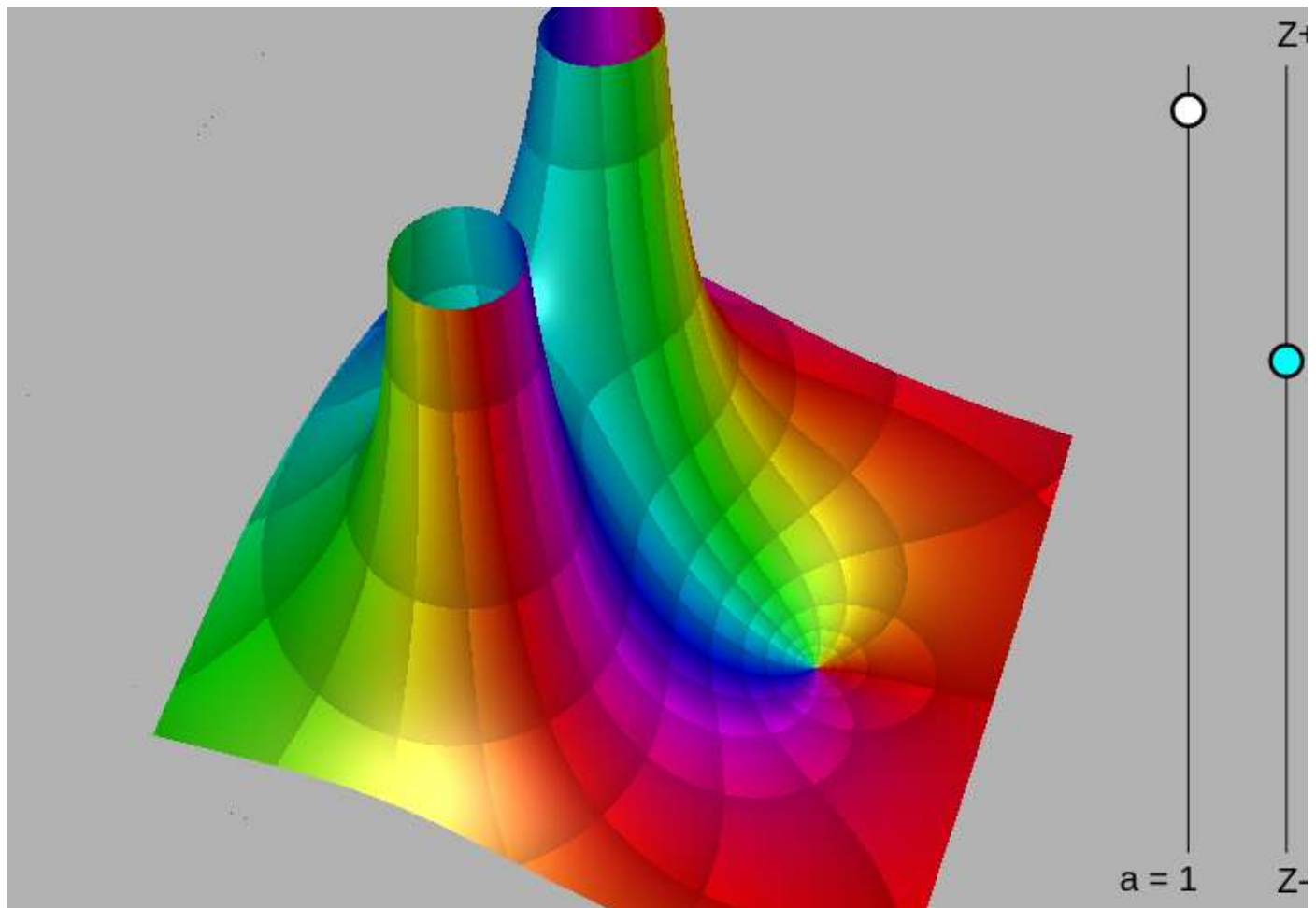
Dynamic Exploration

Complex functions $f : \mathbb{C} \rightarrow \mathbb{C}$ can be visualized by plotting the function $g : \mathbb{R} \rightarrow \mathbb{R}$ with

$$g(x, y) = |f(x + iy)|.$$

The color of each point $(x, y, g(x, y))$ indicates **the phase (or argument)** of the complex number $f(x + iy)$.

In the applet below you can explore colored analytic landscapes. Use the blue slider on the right side for zooming out/in. The black slider defines a real scalar $a \in [-0.14, 1.14]$.



Choose one function $f(z)$ from the following list: ▼

Figure 3.3.3: Interactive Applet

In practice, it is often difficult to generate analytic landscapes which allow us to read off properties of the complex function easily and precisely. An alternative approach is not only simpler but even more general: Instead of drawing a graph in $\mathbb{R}^3$, we can depict a function directly on its domain by color coding its values completely. This method is called *domain coloring*.

Note: The last applet was written by [Aaron Montag](#) using [CindyJS](#). The source code can be found at [GitHub](#).

This page titled [3.3: Analytic Landscapes](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

3.4: Domain coloring

Complex phase portraits

A way to visualize complex functions $f : \mathbb{C} \rightarrow \mathbb{C}$ is using phase-portraits. A complex number can be assigned a color according to its argument/phase. Positive numbers are colored red; negative numbers are colored in cyan and numbers with a non-zero imaginary part are colored as in Figure 1, which shows a phase portrait for the function $f(z) = z$.

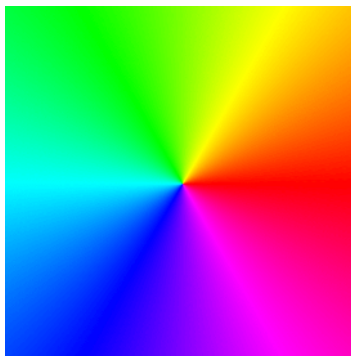


Figure 1: Phase portrait of $f(z) = z$.

In his book *Visual Complex Functions*, Elias Wegert employs phase portraits with contour lines of phase and modulus (*enhanced phase portraits*) for the study of the theory of complex functions. See for example Figures 2 and 3 for the function $f(z) = z$.

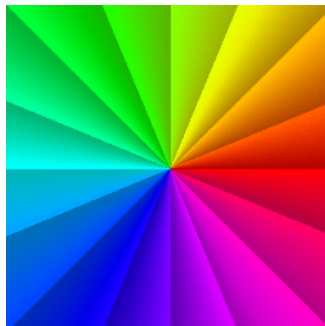


Figure 2: Contour lines of phase.

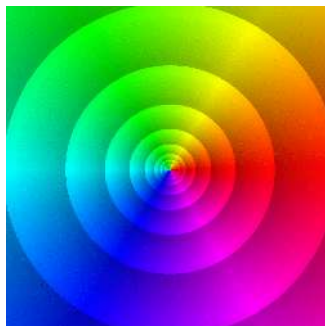


Figure 3: Contour lines of modulus.

We say that a complex function f has a root (or a zero) at z_0 , if $f(z_0) = 0$. We say that z_0 is a pole when $f(z_0)$ is undefined. With the use of enhanced phase portraits, roots and poles of a complex function $f(z)$ can be easily spotted at the points where all colors meet. Figures 4 and 5 show the enhanced phase portrait of the functions

$$f(z) = z \quad \text{and} \quad g(z) = 1/z,$$

respectively. Observe the contrast between the level curves of modulus in each case.

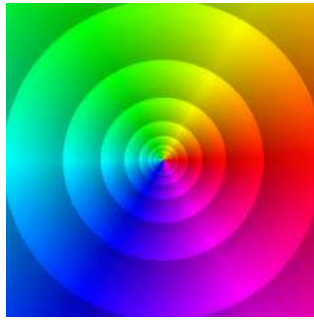


Figure 4: Root at $z = 0$.

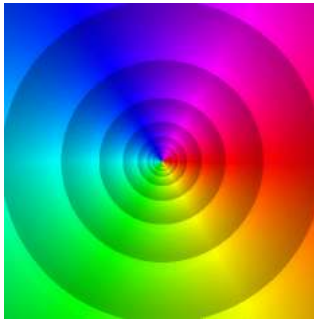


Figure 5: Pole at $z = 0$.

Consider now the function

$$f(z) = \frac{z-1}{z^2+z+1} \quad (1)$$

which has a root at $z_0 = 1$ and two poles at

$$z_1 = \frac{-1+\sqrt{3}i}{2} \quad \text{and} \quad z_2 = \frac{-1-\sqrt{3}i}{2}.$$

Figure 6 shows the enhanced portrait of (1) with level curves of the modulus. Notice the behaviour of the level curves of the modulus around the root (right side) and the poles (left side). Can you see the difference?

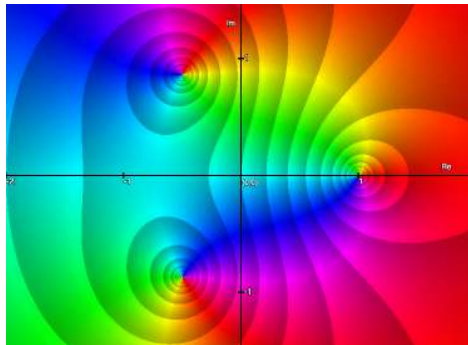
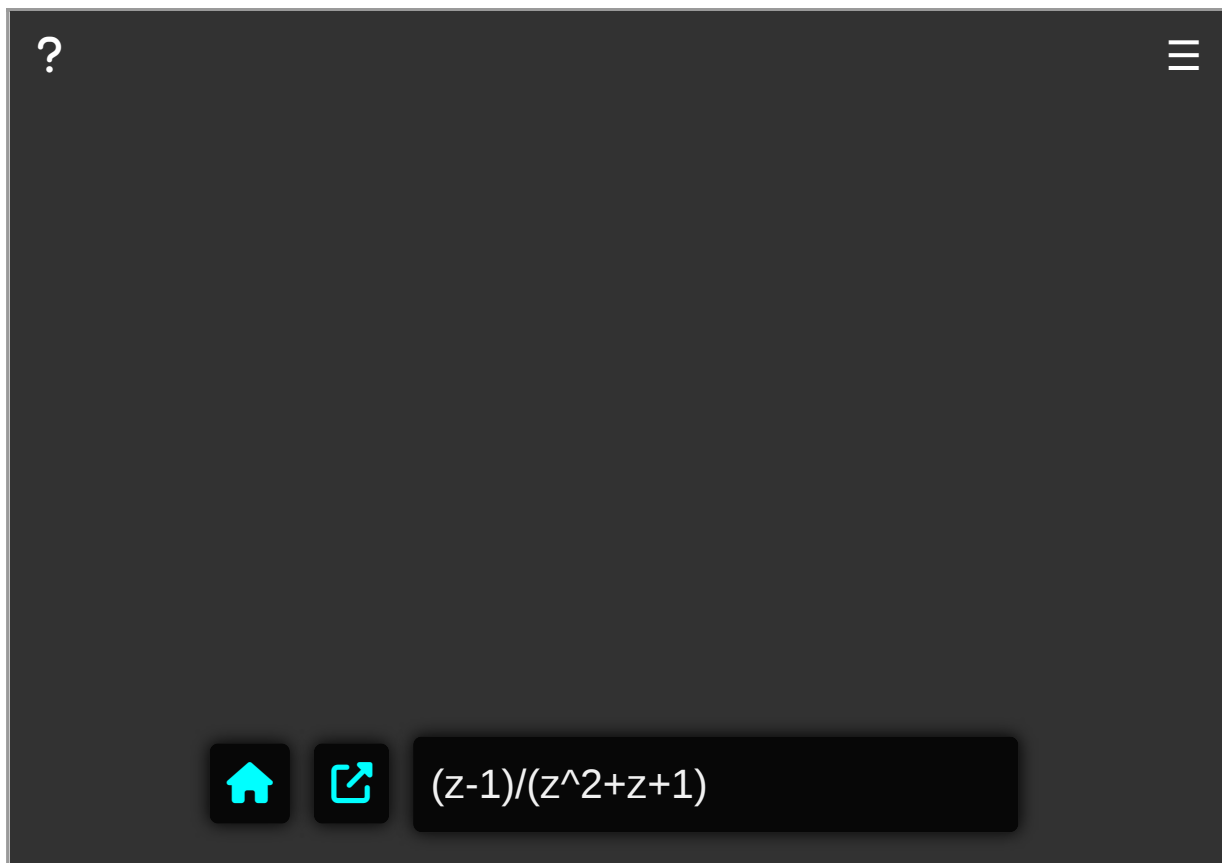


Figure 6: Plot of $f(z) = (z-1)/(z^2+z+1)$ with level curves of the modulus.

Explore complex functions

Use the applet below to explore enhanced phase portraits of complex functions.



INTERACTIVE GRAPH

Warning!

If we do not impose additional restrictions, like continuity or differentiability, the isochromatic sets of complex functions can be arbitrary - but this is not so for *analytic* functions, which are the objects of prime interest in this text.

In fact, analytic functions are (almost) uniquely determined by their (pure) phase portraits, but this is not so for general functions. For example, the functions f (analytic) and g (not analytic) defined by

$$f(z) = \frac{z-1}{z^2+z+1}, \quad g(z) = (z-1)(\bar{z}^2 + \bar{z} + 1) \quad (2)$$

have the same phase (except at their zeros and poles) though they are completely different.

Since pure phase portraits do not always display enough information for exploring general complex functions, I recommend the use of their enhanced versions with contour lines of modulus and phase in such cases. Figure 7 shows two such portraits of the functions f (left) and g (right) defined in (2).

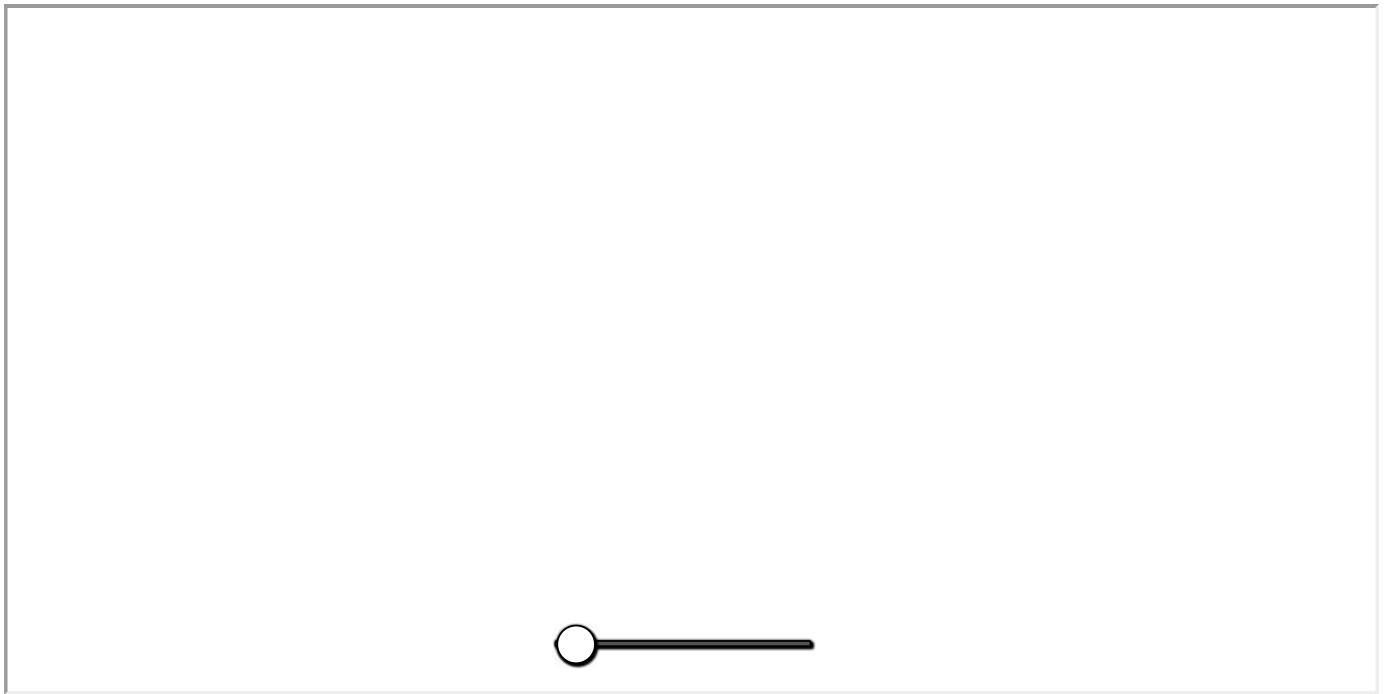


Figure 7: Enhanced phase portraits of f (left) and g (right). Drag slider.

A notable distinction between the two portraits is the shape of the tiles. In the left picture most of them are almost squares and have right-angled corners. In contrast, many tiles in the portrait of g are prolate and their angles differ significantly from $\pi/2$ - at some points the contour lines of modulus and phase are even mutually tangent.

This page titled [3.4: Domain coloring](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

3.5: The Complex Power Function

The *generalized complex power function* is defined as:

$$f(z) = z^c = \exp(c \log z), \quad \text{with } z \neq 0. \quad (1)$$

Due to the multi-valued nature of $\log z$, it follows that (1) is also multi-valued for any non-integer value of c , with a branch point at $z = 0$. In other words

$$f(z) = z^c = \exp(c \log z) = \exp[c(\text{Log } z + 2n\pi i)], \quad \text{with } n \in \mathbb{Z}.$$

On the other hand, we have that the *generalized exponential function*, for $c \neq 0$, is defined as:

$$f(z) = c^z = \exp(z \log c) = \exp[z(\text{Log } c + 2n\pi i)], \quad (2)$$

with $n \in \mathbb{Z}$.

Notice that (2) possesses no branch point (or any other type of singularity) in the infinite complex z -plane. Thus, we can regard the equation (2) as defining a set of independent single-value functions for each value of n .

This is reason why the multi-valued *nature* of the function $f(z) = z^c$ differs from the multi-valued function $f(z) = c^z$.

Typically, the $n = 0$ case is the most useful, in which case, we would simply define:

$$w = c^z = \exp(z \log c) = \exp(z \text{Log } c),$$

with $c \neq 0$.

This conforms with the definition of exponential function

$$e^z = e^x(\cos y + i \sin y)$$

where $c = e$ (the Euler constant).

Use the following applet to explore functions (1) and (2) defined on the region $[-3, 3] \times [-3, 3]$. The enhanced phase portrait is used with contour lines of modulus and phase. Drag the points to change the value of c in each case. You can also deactivate the contour lines, if you want.

INTERACTIVE GRAPH

Final remark: In practice, many textbooks treat the *generalized exponential function* as a single-valued function, $c^z = \exp(z \text{Log } c)$, only when c is a positive real number. For any other value of c , the multi-valued function $c^z = \exp(z \text{Log } c)$ is preferred.

This page titled [3.5: The Complex Power Function](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

CHAPTER OVERVIEW

4: Chapter 4

[4.1: Curves in the Complex Plane](#)

[4.2: Complex Integration](#)

[4.3: Integrals of Functions with Branch Cuts](#)

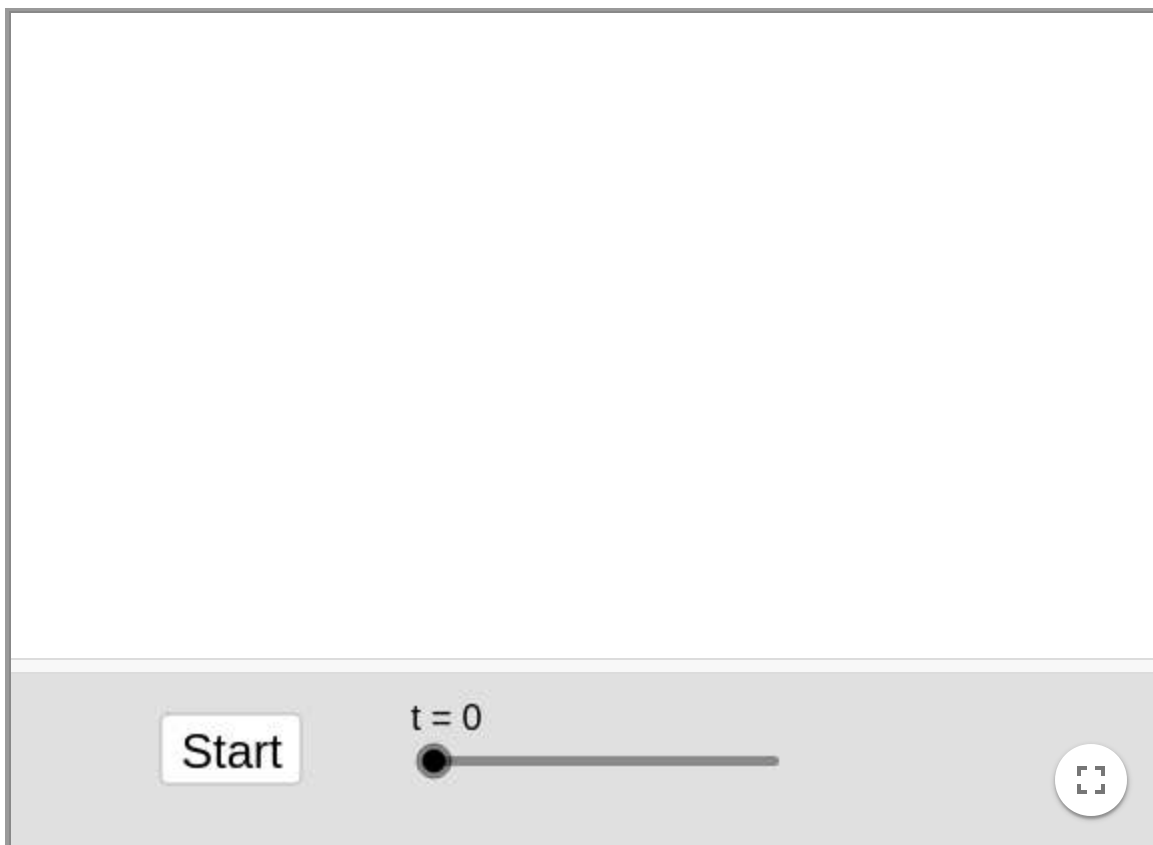
This page titled [4: Chapter 4](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

4.1: Curves in the Complex Plane

Suppose the continuous real-valued functions $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, are parametric equations of a curve C in the complex plane. If we use these equations as the real and imaginary parts in $z = x + iy$, we can describe the points z on C by means of a complex-valued function of a real variable t called a *parametrization* of C :

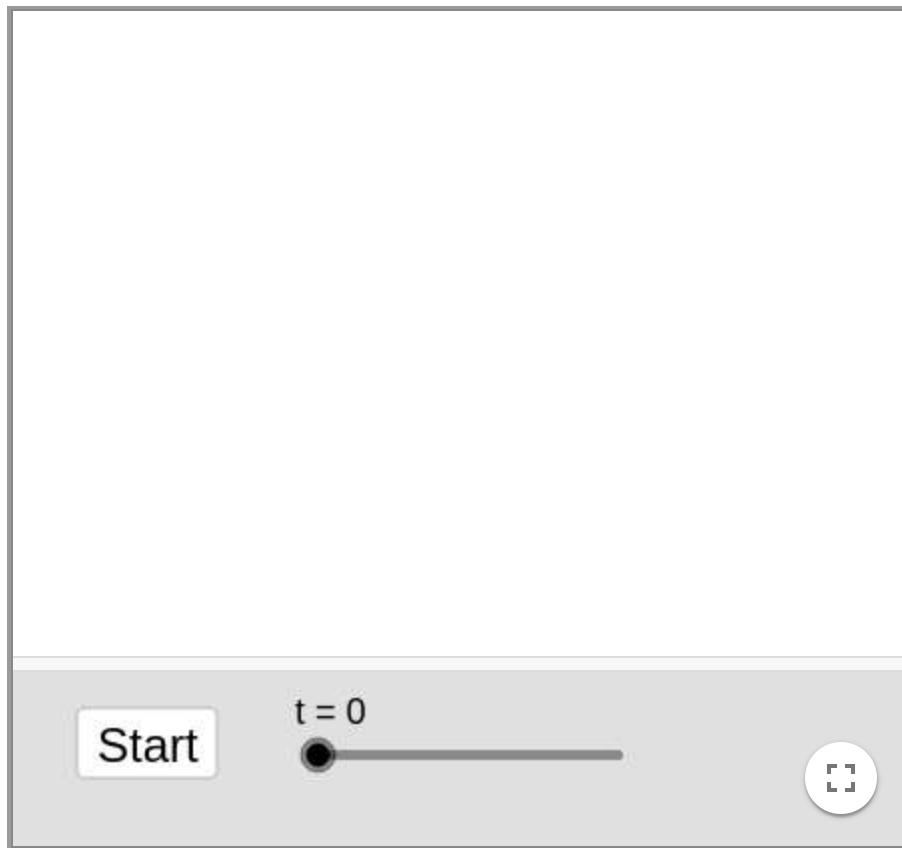
$$z(t) = x(t) + iy(t), \quad a \leq t \leq b. \quad (1)$$

The point $z(a) = x(a) + iy(a)$ or $z_0 = (x(a), y(a))$ is called the *initial point* of C and $z(b) = x(b) + iy(b)$ or $z_1 = (x(b), y(b))$ is its *terminal point*. The expression $z(t) = x(t) + iy(t)$ could also be interpreted as a two-dimensional vector function. Consequently, $z(a)$ and $z(b)$ can be interpreted as position vectors. As t varies from $t = a$ to $t = b$ we can envision the curve C being traced out by the moving arrowhead of $z(t)$. This can be appreciated in the following applet with $0 \leq t \leq 1$.



Press **Start** to animate. You can move the points to change the curve.

For example, the parametric equations $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$, describe a unit circle centered at the origin. A parametrization of this circle is $z(t) = \cos t + i \sin t$, or $z(t) = e^{it}$, $0 \leq t \leq 2\pi$.



Press Start to animate.

Contours

The notions of curves in the complex plane that are smooth, piecewise smooth, simple, closed, and simple closed are easily formulated in terms of the vector function (1). Suppose the derivative of (1) is $z'(t) = x'(t) + iy'(t)$. We say a curve C in the complex plane is *smooth* if $z'(t)$ is continuous and never zero in the interval $a \leq t \leq b$. As shown in Figure 2, since the vector $z'(t)$ is not zero at any point P on C , the vector $z'(t)$ is tangent to C at P . Thus, a smooth curve has a continuously turning tangent; or in other words, a smooth curve can have no sharp corners or cusps. See Figure 2.

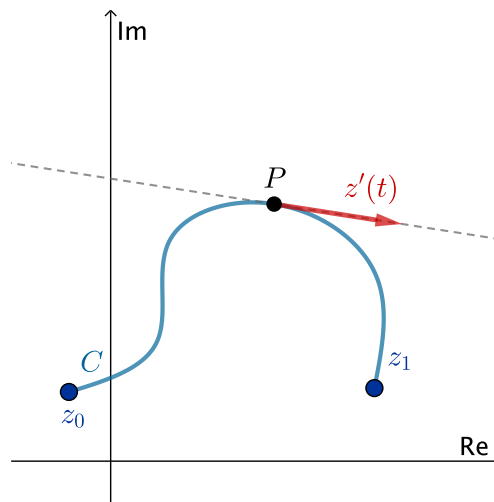


Figure 1: $z'(t) = x'(t) + iy(t)$ as a tangent vector.

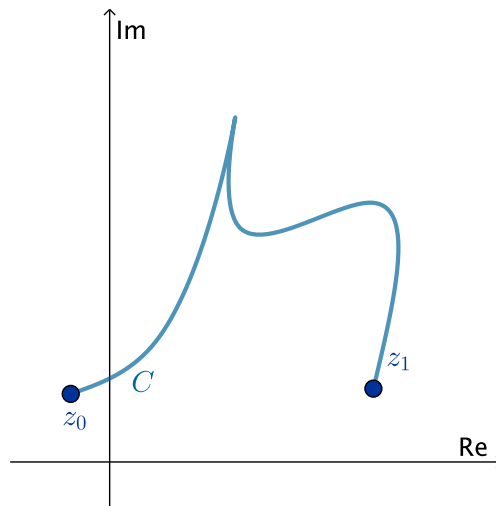


Figure 2: Curve C is not smooth since it has a cusp.

A *piecewise smooth curve* C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \dots, C_n$ are joined together.

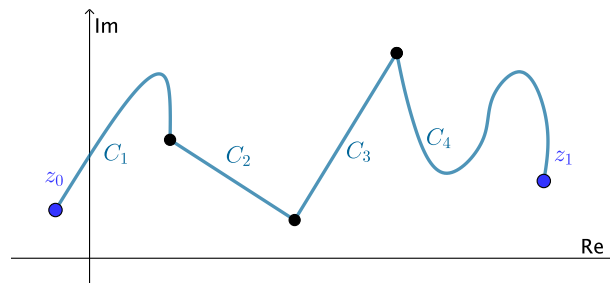


Figure 3: A piecewise smooth curve.

A curve C in the complex plane is said to be a *simple* if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, except possibly for $t = a$ and $t = b$. C is a *closed curve* if $z(a) = z(b)$.

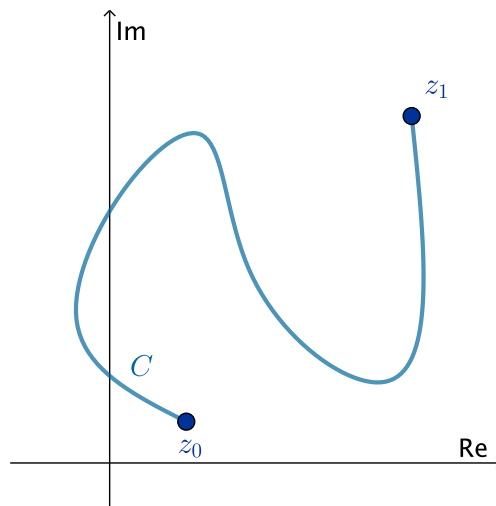


Figure 4: Simple curve.

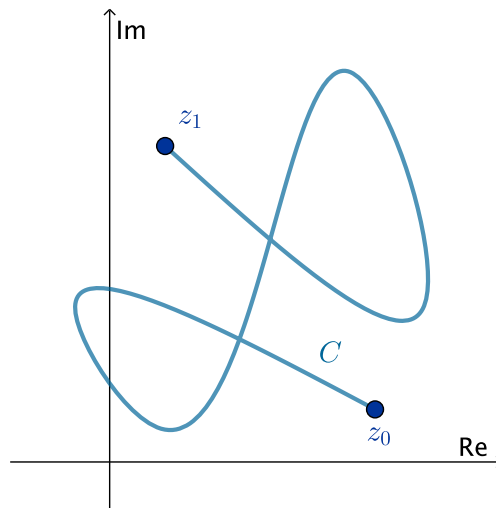


Figure 5: Non simple curve.

C is a simple closed curve if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$ and $z(a) = z(b)$.

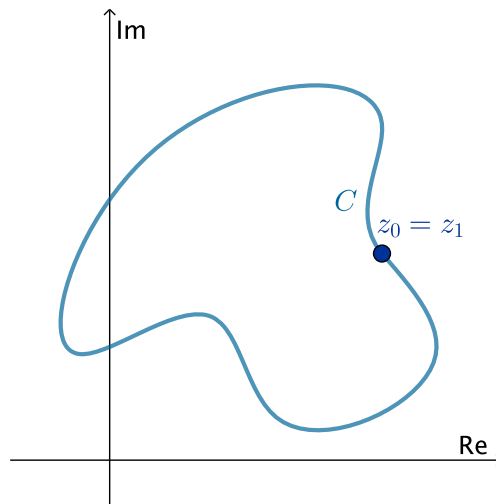


Figure 4: Simple closed curve.

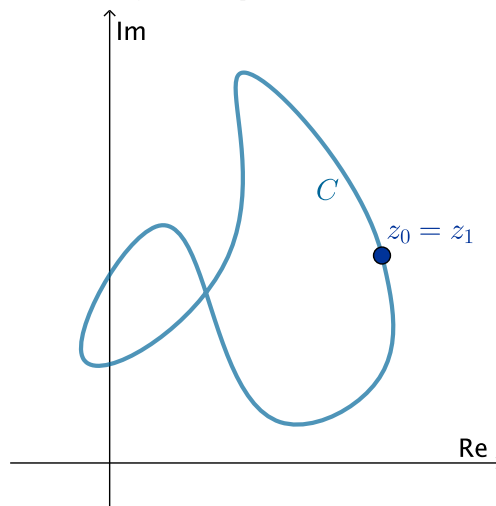


Figure 5: Non simple closed curve.

In complex analysis, a piecewise smooth curve C is called a *contour* or *path*. We define the *positive direction* on a contour C to be the direction on the curve corresponding to increasing values of the parameter t . It is also said that the curve C has *positive orientation*. In the case of a simple closed countour C , the *positive direction* corresponds to the *counter-clockwise* direction. For

example, the circle $z(t) = e^{it}$, $0 \leq t \leq 2\pi$, has positive orientation. The *negative direction* on a contour C is the direction opposite the positive direction. If C has an orientation, the *opposite curve*, that is, a curve with opposite orientation, is denoted by $-C$. On a simple closed curve, the negative direction corresponds to the clockwise direction. For instance, the circle $z(t) = e^{-it}$, $0 \leq t \leq 2\pi$, has negative orientation.

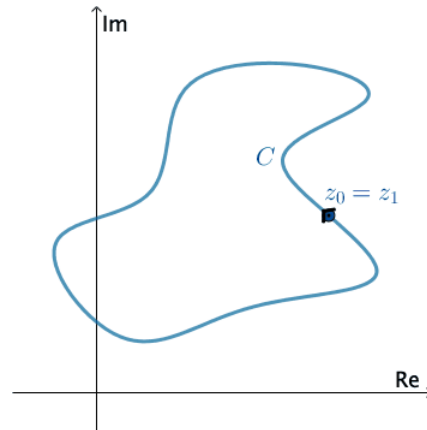


Figure 6: Positively oriented.

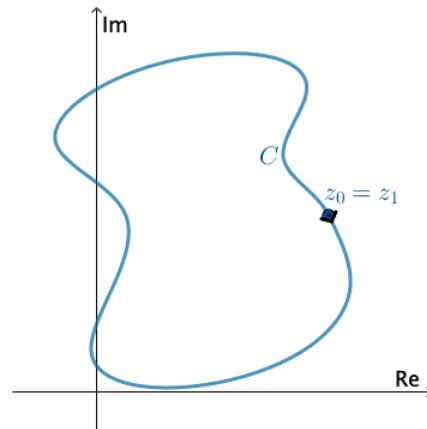


Figure 7: Negatively oriented.



Press **Start** to animate. You can change the direction of C .

Exercise 4.1.1

Exercise: There is no unique parametrization for a contour C . You should verify that

$$z(t) = e^{it} = \cos t + i \sin t, 0 \leq t \leq 2\pi$$

$$z(t) = e^{2\pi it} = \cos(2\pi t) + i \sin(2\pi t), 0 \leq t \leq 1$$

$$z(t) = e^{\pi/2 it} = \cos\left(\frac{\pi}{2}t\right) + i \sin\left(\frac{\pi}{2}t\right), 0 \leq t \leq 4$$

are all parametrizations, oriented in the positive direction, for the unit circle $|z| = 1$.

This page titled [4.1: Curves in the Complex Plane](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

4.2: Complex Integration

The magic and power of calculus ultimately rests on the amazing fact that differentiation and integration are mutually inverse operations. And, just as complex functions enjoy remarkable differentiability properties not shared by their real counterparts, so the sublime beauty of complex integration goes far beyond its real progenitor.

Peter J. Oliver

Contour integral

Consider a contour C parametrized by $z(t) = x(t) + iy(t)$ for $a \leq t \leq b$. We define the *integral* of the complex function along C to be the complex number

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt. \quad (1)$$

Here we assume that $f(z(t))$ is piecewise continuous on the interval $a \leq t \leq b$ and refer to the function $f(z)$ as being piecewise continuous on C . Since C is a contour, $z'(t)$ is also piecewise continuous on $a \leq t \leq b$; and so the existence of integral (1) is ensured.

The right hand side of (1) is an ordinary real integral of a complex-valued function; that is, if $w(t) = u(t) + iv(t)$, then

$$\int_a^b w(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt \quad (2)$$

Now let us write the integrand

$$f(z) = u(x, y) + iv(x, y)$$

in terms of its real and imaginary parts, as well as the differential

$$dz = \frac{dz}{dt} dt = \left(\frac{dx}{dt} + i \frac{dy}{dt} \right) dt = dx + i dy$$

Then the complex integral (1) splits up into a pair of real line integrals:

$$\int_C f(z) dz = \int_C (u + iv)(dx + i dy) = \int_C (u dx - v dy) + i \int_C (v dx + u dy). \quad (3)$$

Example 4.2.1

Let's evaluate $\int_C f(\bar{z}) dz$, where C is given by $x = 3t$, $y = t^2$, $y = t^2$, with $-1 \leq t \leq 4$.

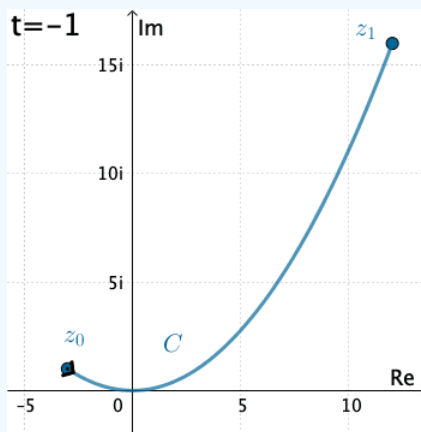


Figure 1: , with $-1 \leq t \leq 4$

Here we have that C is $z(t) = 3t + it^2$. Therefore, with the identification $f(z) = \bar{z}$ we have

$$f(z(t)) = \overline{3t + it^2} = 3t - it^2.$$

Also, $z'(t) = 3 + 2it$, and so the integral is

$$\begin{aligned}\int_C \bar{z} dz &= \int_{-1}^4 (3t - it^2)(3 + 2it) dt \\ &= \int_{-1}^4 (2t^3 + 9t + 3t^2i) dt \\ &= \int_{-1}^4 (2t^3 + 9t) dt + i \int_{-1}^4 3t^2 dt \\ &= \left(\frac{1}{2}t^4 + \frac{9}{2}t^2\right)\Big|_{-1}^4 + it^3\Big|_{-1}^4 \\ &= 195 + 65i.\end{aligned}$$

Example 4.2.2

Now let's evaluate $\int_C \frac{1}{z} dz$, where C is the circle $x = \cos t$, $y = \sin t$, with $0 \leq t \leq 2\pi$.

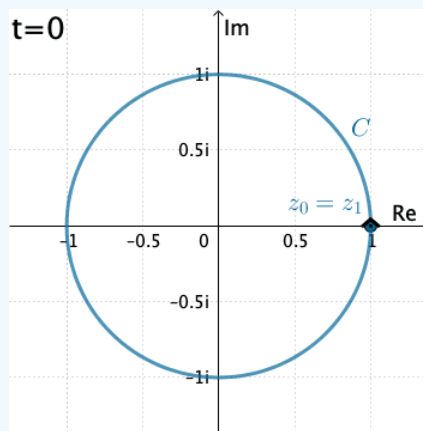


Figure 2: $C : z(t) = \cos t + i \sin t = e^{it}$, with $0 \leq t \leq 2\pi$.

In this case C is $z(t) = \cos t + i \sin t = e^{it}$,

$$f(z(t)) = \frac{1}{e^{it}}$$

and, $z'(t) = ie^{it}$. Thus

$$\int_C \frac{1}{z} dz = \int_0^{2\pi} (e^{-it}) ie^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$

Numerical evaluation of complex integrals

Exploration 1

Use the following applet to explore numerically the integral

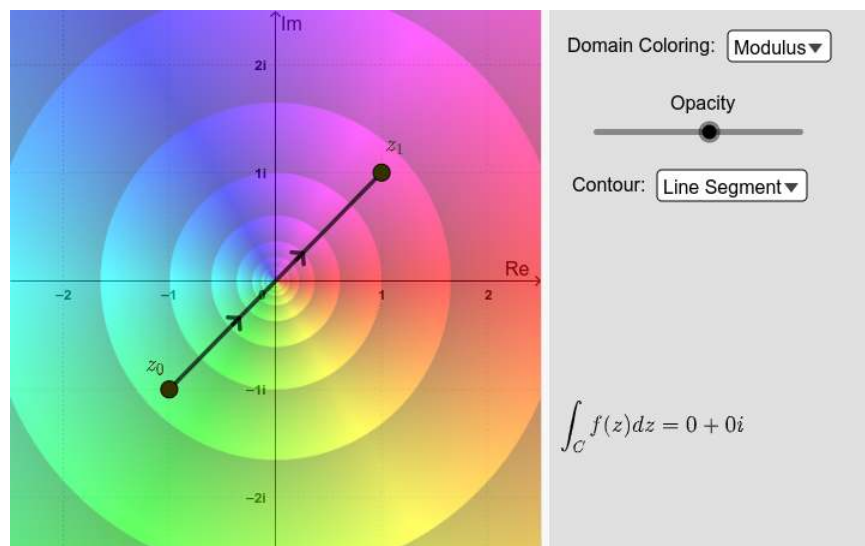
$$\int_C \bar{z} dz$$

with different contours C :

- Line segments.
- Semicircles.
- Circles, positively and negatively oriented.

You can also change the **domain coloring** plotting option. Drag the points around and observe carefully what happens. Then solve Exercise 1 below.

[INTERACTIVE GRAPH](#)



The arrows on the contours indicate direction.

Exercise 4.2.1

Exercise 1: Use definition (1) to evaluate $\int_C \bar{z} dz$, for the following contours C from $z_0 = -2i$ to $z_1 = 2i$:

1. Line segment. That is, $z(t) = -2i(1-t) + 2it$, with $0 \leq t \leq 1$.
2. Right-hand semicircle. That is, $z(\theta) = 2e^{i\theta}$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
3. Left-hand semicircle. That is, $z(\theta) = -2e^{-i\theta}$ with $0 \leq \theta \leq \pi$.

Use the applet to confirm your results.

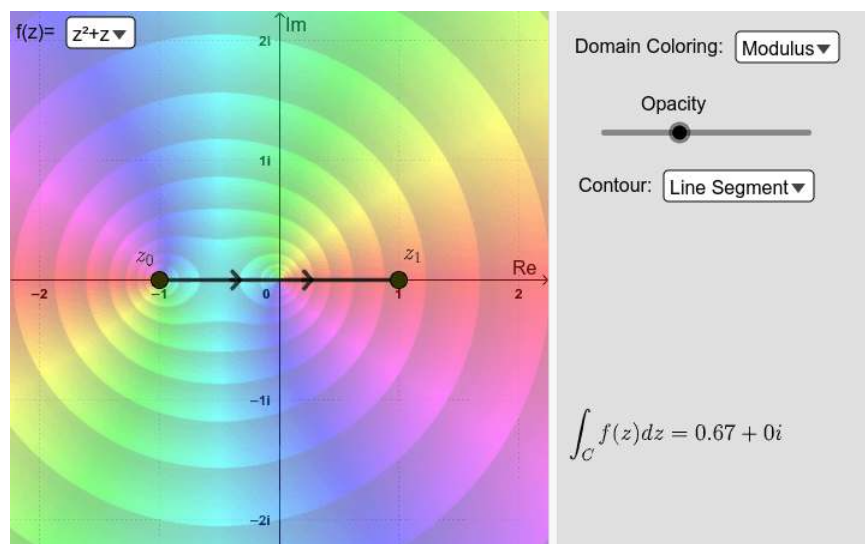
What conclusions (if any) can you draw about the function $\bar{z}$ from this?

Exploration 2

Now use the applet below to explore numerically the integrals

$$\int_C (z^2 + z) dz; \quad \int_C \frac{1}{z^2} dz$$

with different contours C (line segments, semicircles, and circles). Drag the points around and observe carefully what happens. You can select the functions $z^2 + z$ or $1/z^2$ from the list at the left-top corner. Then solve Exercises 2 and 3.



Exercise 4.2.2

Exercise 2: Consider the integral

$$I_1 = \int_C (z^2 + z) dz.$$

Use the applet to analyze the value of I_1 in the following cases:

1. C is any contour from $z_0 = -1 - i$ to $z_1 = 1 + i$.
2. C is the circle with center z_0 and radius $r > 0$, $|z - z_0| = r$; positively or negatively oriented. In this cases select `Circle ↺` or `Circle ↻`.

What conclusions (if any) can you draw about the value of I_1 and the function $z^2 + z$ from this?

Exercise 4.2.3

Exercise 3: Now considering integral

$$I_2 = \int_C \frac{1}{z^2} dz.$$

First, in the applet select the function $f(z) = 1/z^2$. Then analyze the values of I_2 in the following cases:

1. C is any contour from $z_0 = -i$ to $z_1 = i$. What happens when you select `Line Segment` in the applet? What happens when you select `Semicircles`?
2. C is the circle with center z_0 and radius $r > 0$, $|z - z_0| = r$; positively or negatively oriented. In this case select `Circle ↺` or `Circle ↻`. What happens if $z = 0$ is inside or outside the circle? What happens if $z = 0$ lies on the contour, e.g. when $(z_0 = 1)$ and $r = 1$?

What conclusions (if any) can you draw about the value of I_2 and the function $\frac{1}{z^2}$ from this?

Antiderivatives

Although the value of a contour integral of a function $f(z)$ from a fixed point z_0 to a fixed point z_1 depends, in general, on the path that is taken, there are certain functions whose integrals from z_0 to z_1 have values that are independent of path, as you have seen in Exercises 2 and 3. These examples also illustrate the fact that the values of integrals around closed paths are sometimes, but not always, zero. The next theorem is useful in determining when integration is independent of path and, moreover, when an integral around a closed path has value zero. This is known as the complex version of the *Fundamental Theorem of Calculus*.

Theorem 4.2.1

Let $f(z) = F'(z)$ be the derivative of a single-valued complex function $F(z)$ defined on a domain $\Omega \subset \mathbb{C}$. Let C be any countour lying entirely in Ω with initial point z_0 and final point z_1 . Then

$$\int_C f(z) dz = F(z)|_{z_0}^{z_1} = F(z_1) - F(z_0).$$

Proof: This follows from definition (1) and the chain rule. That is

$$\begin{aligned} \int_C f(z) dz &= \int_C F'(z(t)) \frac{dz}{dt} dt \\ &= \int_a^b \frac{d}{dt} F(z(t)) dt \\ &= F(z(b)) - F(z(a)) \\ &= F(z_1) - F(z_0) \end{aligned}$$

where $z_0 = z_a$ and $z_1 = z_b$ are the endpoints of the contour C . ■

This page titled [4.2: Complex Integration](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

4.3: Integrals of Functions with Branch Cuts

When we consider multiple-valued functions, the path in a contour integral can contain a point on a branch cut of the integrand involved. The next two examples illustrate this.

Example 1

Let C be the semicircular path from $z_0 = 3$ to $z_1 = -3$. That is $z(\theta) = 3e^{i\theta}$, with $0 \leq \theta \leq \pi$. Here we would like to evaluate the integral

$$I = \int_C z^{1/2} dz. \quad (4.3.1)$$

To do so, we need to choose a particular branch of the multiple-valued function $z^{1/2}$. For example, we will use the principal branch

$$|z| > 0, \quad -\pi < \text{Arg}(z) < \pi. \quad (4.3.2)$$

In this case, notice that although the principal branch of $z^{1/2} = \exp\left(\frac{1}{2} \text{Log } z\right)$ is not defined at the end point $z_1 = -3$ of the contour C , the integral I nevertheless exists.

Use the following applet to explore the value of I for the given contour C . Just drag the points z_0 and z_1 to the corresponding values. You can also select other contours and explore what happens when they cross the branch cut $\{z : x = 0 \text{ and } y \leq 0\}$.

INTERACTIVE GRAPH

As you already have figured it out, the integral (1) exists because the integrand is **piecewise continuous** on C . To confirm this, observe that when $z(\theta) = 3e^{i\theta}$, then

$$f(z(\theta)) = \exp\left(\frac{1}{2} \ln 3 + i\theta\right) = \sqrt{3}e^{i\theta/2}.$$

The left-hand limits of the real and imaginary components of the function

$$\begin{aligned} f(z(\theta)) z'(\theta) &= \sqrt{3}e^{i\theta/2} \cdot 3ie^{i\theta} \\ &= 3\sqrt{3}i e^{i3\theta/2} \\ &= -3\sqrt{3} \sin \frac{3\theta}{2} + i 3\sqrt{3} \cos \frac{3\theta}{2} \quad (0 \leq \theta < \pi) \end{aligned}$$

at $\theta = \pi$ exist. That is

$$\lim_{\theta \rightarrow \pi^-} -3\sqrt{3} \sin \frac{3\theta}{2} = 3\sqrt{3} \quad \text{and} \quad \lim_{\theta \rightarrow \pi^-} 3\sqrt{3} \cos \frac{3\theta}{2} = 0.$$

This means that $f(z(\theta))z'(\theta)$ is continuous on the closed interval $0 \leq \theta \leq \pi$ when its value at $\theta = \pi$ is defined as $3\sqrt{3}$. Therefore

$$\begin{aligned} I &= \int_C f(z(\theta)) z'(\theta) d\theta = 3\sqrt{3}i \int_0^\pi e^{i\theta/2} d\theta = 3\sqrt{3}i \left[\frac{2}{3i} e^{i3\theta/2} \right]_0^\pi \\ &= 3\sqrt{3}i \left[-\frac{2}{3i} (1 + i) \right] = -2\sqrt{3}(1 + i). \end{aligned}$$

Exercise 4.3.1

Exercise 1: Evaluate $\int_C z^{1/2} dz$ for the contour $C : z(\theta) = e^{i\theta}$, with $-\pi \leq \theta \leq \pi$. You can use the applet to confirm your results.

Remark 1: You can not evaluate the integral for the contour $C : z(t) = e^{it}$, with $0 \leq t \leq 2\pi$. Why?

Remark 2: Notice that $\int_C z^{1/2} dz = 0$ for any circle not intersecting the branch cut $\{z : x = 0 \text{ and } y \leq 0\}$. Why?

Example 2

Consider the principal branch

$$f(z) = z^i = \exp[i \operatorname{Log} z] \quad \text{with } |z| > 0, -\pi < \operatorname{Arg} z < \pi \quad (4.3.3)$$

and C the upper half circle from $z = -1$ to $z = 1$; that is, $z(t) = -e^{-int}$ with $0 \leq t \leq 1$.

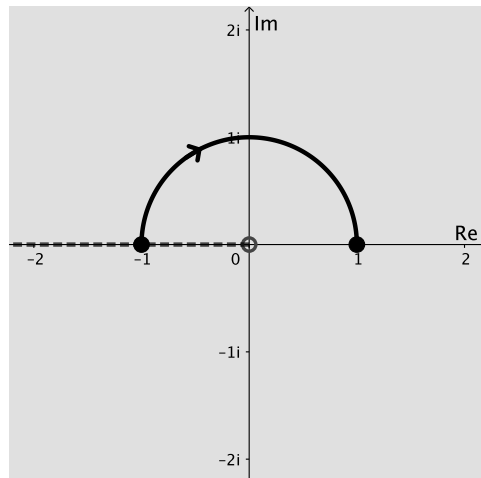


Figure 1: $z(t) = -e^{-int}$, with $0 \leq t \leq 1$.

It is not difficult to verify that

$$I = \int_C z^i dz = \frac{1 + e^{-\pi}}{2} (1 - i). \quad (4.3.4)$$

Use the following applet to confirm this. You can also analyze $\int_C z^i dz$ for other contours.

INTERACTIVE GRAPH

Remark 3: Notice that $\int_C z^i dz = 0$ for any circle not intersecting the branch cut $\{z : x = 0, y \leq 0\}$. Why?

In general, we can calculate $\int$ for any contour C from $z = -1$ to $z = 1$ lying above the real axis. We just need to find an antiderivative of z^i . Unfortunately, we *can not* use the principal branch, defined in (3), since this branch is not even defined at $z = -1$. But the integrand can be replaced by the branch

$$z^i = \exp[i \log z] \quad \text{with } |z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}.$$

since it agrees with the integrand along C .

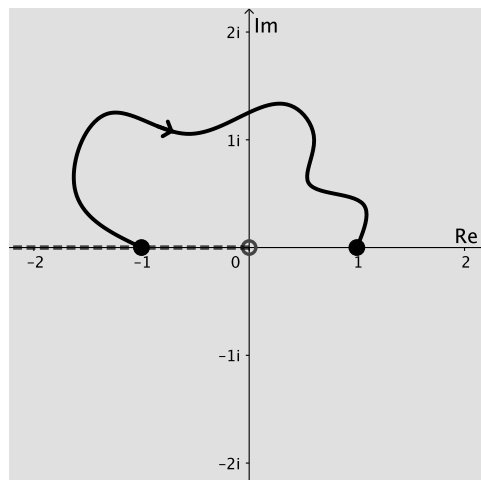


Figure 2: Principal branch.

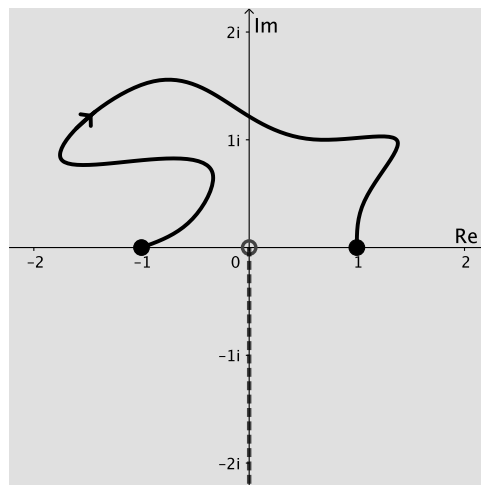


Figure 3: An appropriate branch.

Using an antiderivative of this new branch, we obtain

$$\int_{-1}^1 z^i dz = \left. \frac{z^{i+1}}{i+1} \right|_{-1}^1 = \frac{1 + e^{-\pi}}{2} (1 - i), \quad (4.3.5)$$

which is the same value as (4).

Exercise 4.3.2

Evaluate $\left. \frac{z^{i+1}}{i+1} \right|_{-1}^1$ to confirm the value of (5).

This page titled [4.3: Integrals of Functions with Branch Cuts](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

CHAPTER OVERVIEW

5: Chapter 5

[5.1: Series](#)

[5.2: Taylor Series](#)

[5.3: Laurent Series](#)

[5.4: Classification of Singularities](#)

[5.5: The Mandelbrot Set](#)

[5.6: The Julia Set](#)

This page titled [5: Chapter 5](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

5.1: Series

Series

Convergence of Sequences

An infinite sequence $\{z_1, z_2, z_3 \dots\}$ of complex numbers has a limit z if, for each positive number ϵ , there exists a positive integer n_0 such that

$$|z_n - z| < \epsilon \quad \text{whenever} \quad n > n_0. \quad (5.1.1)$$

Geometrically, this means that for sufficiently large values of n , the points z_n lie in any given ϵ neighborhood of z (Figure 1). Since we can choose ϵ as small as we please, it follows that the points z_n become arbitrarily close to z as their subscripts increase. Note that the value of n_0 that is needed will, in general, depend on the value of ϵ .

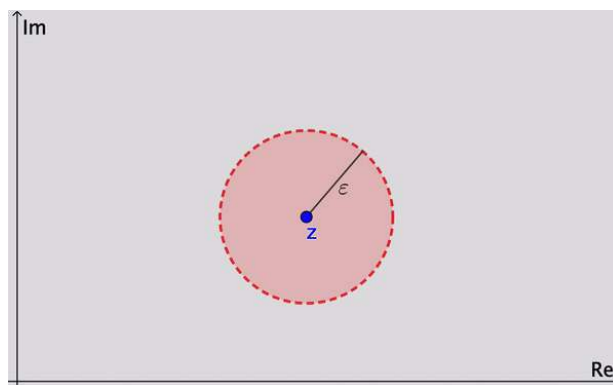


Figure 1: Geometric interpretation.

The sequence $\{z_n\}_{n=1}^{\infty}$ can have at most one limit. That is, a limit z is unique if it exists. When that limit exists, the sequence is said to *converge* to z ; and we write

$$\lim_{n \rightarrow \infty} z_n = z$$

If the sequence has no limit, it *diverges*.

Theorem 5.1.1

Suppose that $z_n = x_n + iy_n$ ($n = 1, 2, 3, \dots$) and $z = x + iy$. Then

$$\lim_{n \rightarrow \infty} z_n = z \quad (5.1.2)$$

if and only if

$$\lim_{n \rightarrow \infty} x_n = x \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = y. \quad (5.1.3)$$

Proof

To prove this theorem, we first assume that conditions (3) hold. That is, there exist, for each $\varepsilon > 0$, positive integers n_1 and n_2 such that

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_1$$

and

$$|y_n - y| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_2.$$

Hence if n_0 is the larger of the two integers n_1 and n_2 ,

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_0.$$

Since

$$|(x_n + iy_n) - (x + iy)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y|,$$

then

$$|z_n - z| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_0.$$

Therefore condition (2) holds.

Conversely, if we start with condition (2), we know that for each positive number ε , there exists a positive integer n_0 such that

$$|(x_n + iy_n) - (x + iy)| < \varepsilon \quad \text{whenever} \quad n > n_0.$$

However

$$|x_n - x| \leq |(x_n - x) + (y_n - y)| = |(x_n + iy_n) - (x + iy)|$$

and

$$|y_n - y| \leq |(x_n - x) + (y_n - y)| = |(x_n + iy_n) - (x + iy)|.$$

Consequently

$$|x_n - x| < \varepsilon \quad \text{and} \quad |y_n - y| < \varepsilon \quad \text{whenever} \quad n > n_0.$$

Therefore, conditions (3) are satisfied. ■

Convergence of Series

An infinite series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \dots \quad (5.1.4)$$

of complex numbers converges to the sum S if the sequence

$$\sum_{n=1}^N z_n = z_1 + z_2 + z_3 + \dots + z_N \quad (N = 1, 2, 3, \dots) \quad (5.1.5)$$

of partial sums converges to S ; we then write

$$\sum_{n=1}^{\infty} z_n = S.$$

Note that since a sequence can have at most one limit, a series can have at most one sum. When a series does not converge, we say that it diverges.

Theorem 5.1.2

Suppose that $z_n = x_n + iy_n$ ($n = 1, 2, 3, \dots$) and $S = X + iY$. Then

$$\sum_{n=1}^{\infty} z_n = S \quad (5.1.6)$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y. \quad (5.1.7)$$

Proof

To prove this theorem, we first write the partial sums (5) as

$$S_N = X_N + iY_N, \quad (5.1.8)$$

where

$$X_N = \sum_{n=1}^N x_n \quad \text{and} \quad Y_N = \sum_{n=1}^N y_n.$$

Now statement (6) is true if and only if

$$\lim_{N \rightarrow \infty} S_N = S; \quad (5.1.9)$$

and, in view of relation (8) and Theorem 1 on sequences, limit (9) holds if and only if

$$\lim_{N \rightarrow \infty} X_N = X \quad \text{and} \quad \lim_{N \rightarrow \infty} Y_N = Y. \quad (5.1.10)$$

Limits (10) therefore imply statement (6), and conversely. Since $X_N = X$ and $Y_N = Y$ are partial sums of the series (7), the theorem is proved. ■

This theorem can be useful in showing that a number of familiar properties of series in calculus carry over to series whose terms are complex numbers.

Property 1: If a series of complex numbers converges, the n -th term converges to zero as n tends to infinity.

It follows from **Property 1** that the terms of convergent series are bounded. That is, when series (4) converges, there exists a positive constant M such that

$$|z_n| \leq M \quad \text{for each positive integer } n.$$

Another important property of series of complex numbers that follows from a corresponding property in calculus is the following.

Property 2: The absolute convergence of a series of complex numbers implies the convergence of that series.

Recall that series (4) is said to be *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} \quad (z_n = x_n + iy_n)$$

of real numbers $\sqrt{x_n^2 + y_n^2}$ converges.

To establish the fact that the sum of a series is a given number S , it is often convenient to define the *remainder* ρ_N after N terms, using the partial sums:

$$\rho_N = S - S_N$$

Thus $S = S_N + \rho_N$. Now, since $|S_N - S| = |\rho_N - 0|$, then a series converges to a number S if and only if the sequence of remainders tends to zero.

Example 5.1.1

With the aid of remainders, it is easy to verify that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{whenever } |z| < 1$$

We need only recall the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

to write the partial sums

$$S_N(z) = \sum_{n=0}^N z^n = 1 + z + z^2 + \cdots + z^{N-1} \quad (z \neq 1) \quad (5.1.11)$$

as

$$S_N(z) = \frac{1 - z^{N+1}}{1 - z}.$$

If

$$S(z) = \frac{1}{1-z}$$

then

$$\rho_N(z) = S(z) - S_N(z) = \frac{z^{N+1}}{1-z} \quad (z \neq 1).$$

Thus

$$|\rho_N| = \frac{|z|^{N+1}}{|1-z|} \rightarrow 0 \quad \text{only when } |z| < 1.$$

In this case, it is clear that the remainders ρ_N tend to zero when $|z| < 1$ but not when $|z| \geq 1$

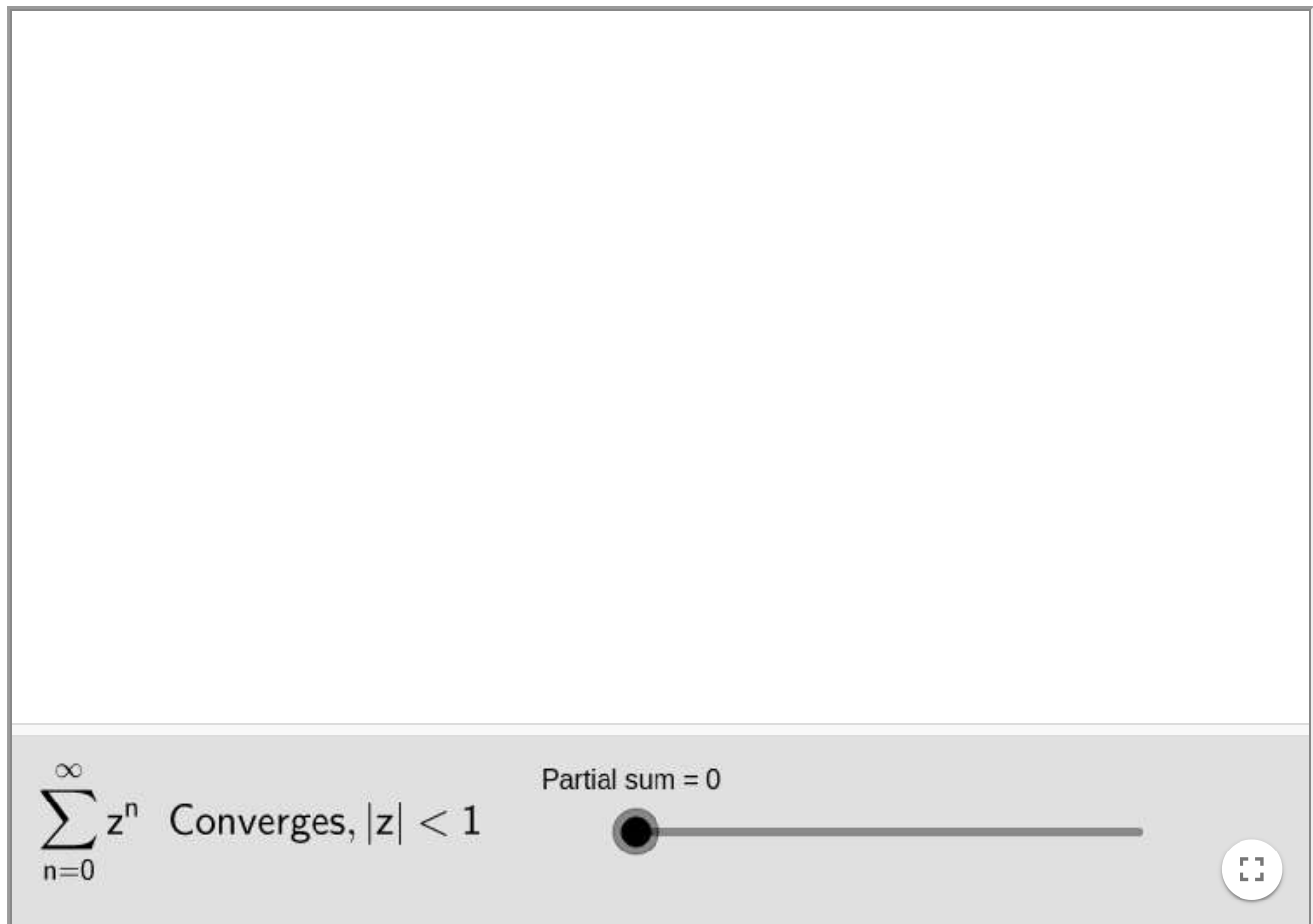
Geometric series exploration

The series introduced in the previous example

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{whenever } |z| < 1$$

is known as the *geometric series*.

Use the following applet to explore this series. Drag the point z around. Observe what happens when it is inside, outside or on the border of the unit circle. Drag the slider to show the partial sum.



Code

Enter the following script in [GeoGebra](#) to explore it yourself and make your own version. The symbol # indicates comments.

```
#Complex number

Z = 0.72 + i * 0.61

#Circle of radius 1

c = Circle((0,0), 1)

#Number of terms of the partial series

n = Slider(0, 250, 1, 1, 150, false, true, false, false)

SetValue(n, 250)

#Define the sequence  $z^n$ 

S = Join({0 + i * 0, 1 + i * 0}, Sequence(Z^j, j, 1, n))

#Define partial sum

SP = Sequence(Sum(S, j), j, 1, n + 2)

#Finally join the points of the partial sum

L = Sequence(Segment(Element(SP, j), Element(SP, j + 1)), j, 1, n + 1)
```

This page titled [5.1: Series](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

5.2: Taylor Series

For Real Functions

Let $a \in \mathbb{R}$ and $f(x)$ be an infinitely differentiable function on an interval I containing a . Then the one-dimensional Taylor series of f around a is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots$$

which can be written in the most compact form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

Recall that, in real analysis, Taylor's theorem gives an approximation of a k -times differentiable function around a given point by a k -th order Taylor polynomial.

For example, the best linear approximation for $f(x)$ is

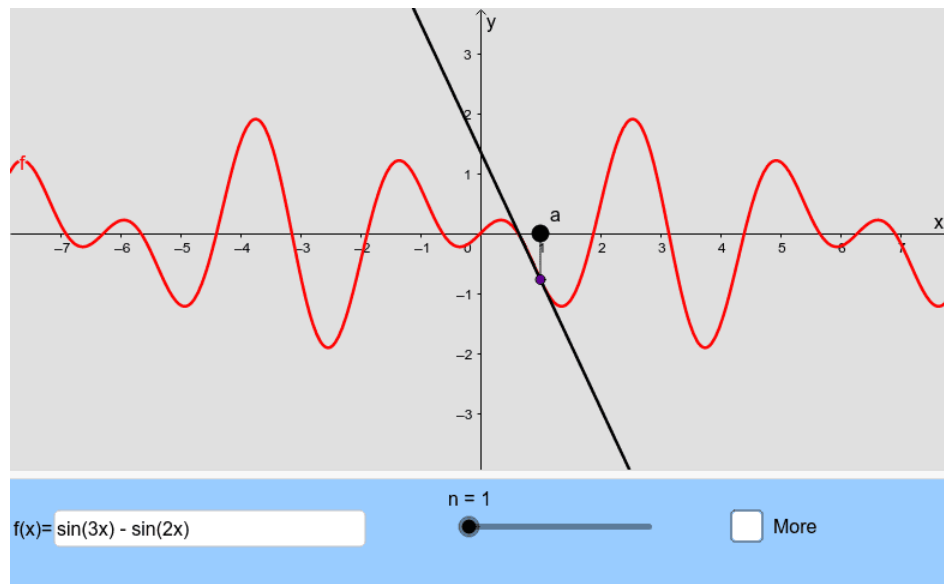
$$f(x) \approx f(a) + f'(a)(x-a).$$

This linear approximation fits $f(x)$ with a line through $x = a$ that matches the slope of f at a .

For a better approximation we can add other terms in the expansion. For instance, the best quadratic approximation is

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

The following applet shows the partial sums of the Taylor series for a given function. Drag the slider to show more terms of the series. Drag the point a or change the function.



INTERACTIVE GRAPH

For Complex Functions

Suppose that a function f is analytic throughout a disk $|z - z_0| < R$, centred at z_0 and with radius R_0 . Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R, \quad (5.2.1)$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \dots \quad (5.2.2)$$

That is, series (1) converges to $f(z)$ when z lies in the stated open disk.

Every complex power series (1) has a radius of convergence. Analogous to the concept of an interval of convergence for real power series, a complex power series (1) has a circle of convergence, which is the circle centered at z_0 of largest radius $R > 0$ for which (1) converges at every point within the circle $|z - z_0| = R$. A power series converges absolutely at all points z within its circle of convergence, that is, for all z satisfying $|z - z_0| < R$, and diverges at all points z exterior to the circle, that is, for all z satisfying $|z - z_0| > R$. The radius of convergence can be:

1. $R = 0$ (in which case (1) converges only at its center $z = z_0$),
2. R a finite positive number (in which case (1) converges at all interior points of the circle $|z - z_0| < R$, or
3. $R = \infty$ (in which case (1) converges for all z).

The radius of convergence can be calculated using the ratio test of convergence. For example, if:

1. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$, the radius of convergence is $R = \frac{1}{L}$;
2. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$, the radius of convergence is $R = \infty$;
3. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, the radius of convergence is $R = 0$.

Dynamic Exploration

Use the following applet to explore Taylor series representations and its radius of convergence which depends on the value of z_0 .

On the left side of the applet below, a phase portrait of a complex function is displayed. On the right side, you can see the approximation of the function through its Taylor polynomials at the blue base point z_0 . The complex function, the base point z_0 , the order of the polynomial (vertical slider) and the zoom (horizontal slider) can be modified.

INTERACTIVE GRAPH

Maclaurin series

A Taylor series with centre $z_0 = 0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

is referred to as *Maclaurin series*.

Some important Maclaurin series are:

$$\begin{aligned}\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n, \quad |z| < 1; \\ e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z| < \infty; \\ \sin z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{n!}, \quad |z| < \infty; \\ \cos z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{n!}, \quad |z| < \infty; \\ \sinh z &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{n!}, \quad |z| < \infty; \\ \cosh z &= \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}, \quad |z| < \infty;\end{aligned}$$

Exercise 5.2.1

Exercise: Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9}$$

and calculate the radius of convergence.

Note

The applet was originally written by [Aaron Montag](#) using [CindyJS](#). The source can be found at [GitHub](#).

This page titled [5.2: Taylor Series](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

5.3: Laurent Series

Recall that a function f of the complex variable z is *analytic* at a point z_0 if it has a derivative at each point in some neighbourhood of z_0 . An *entire* function is a function that is analytic at each point in the entire finite plane. If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighbourhood of z_0 , then z_0 is called a *singular point*, or *singularity*, of f .

Suppose that $f(z)$, or any single valued branch of $f(z)$, if $f(z)$ is multivalued, is analytic in the region $0 < |z - z_0| < R$ and not at the point z_0 . Then the point z_0 is called an *isolated singular point* of $f(z)$.

Now, recall also that any function which is analytic throughout a disk $|z - z_0| < R_0$ must have a Taylor series about z_0 . If the function fails to be analytic at a point z_0 , it is often possible to find a series representation for $f(z)$ involving both positive and negative powers of $z - z_0$. Formally speaking we have the following result:

Theorem 5.3.1

Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centred at z_0 , and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad (5.3.1)$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^{n+1}}, \quad n = 0, 1, 2, \dots \quad (5.3.2)$$

and

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^{-n+1}}, \quad n = 1, 2, \dots \quad (5.3.3)$$

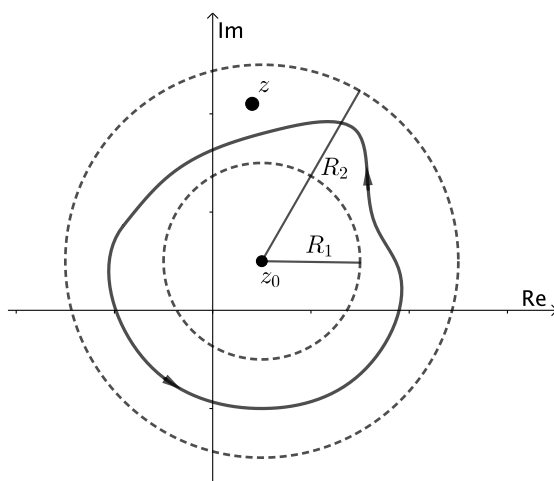


Figure 1: $f(z)$ is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$.

In practice, the above integral formulae (2) and (3) may not offer the most practical method for computing the coefficients a_n and b_n for a given function $f(z)$; instead, one often pieces together the Laurent series by combining known Taylor expansions. Because the Laurent expansion of a function is unique whenever it exists, any expression of this form that actually equals the given function $f(z)$ in some annulus must actually be the Laurent expansion of $f(z)$.

For example, consider the function

$$f(z) = \frac{1}{z(1+z^2)}$$

which has isolated singularities at $z = 0$ and $z = \pm i$. In this case, there is a Laurent series representation for the domain $0 < |z| < 1$ and also one for the domain $1 < |z| < \infty$, which is exterior to the circle $|z| = 1$. To find each of these Laurent series, we recall

the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

For the domain $0 < |z| < 1$, we have

$$\begin{aligned} f(z) &= \frac{1}{z} \frac{1}{1+z^2} = \frac{1}{z} \sum_{n=0}^{\infty} (-z^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n z^{2n-1} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{2n-1} \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1} + \frac{1}{z}. \end{aligned}$$

On the other hand, when $1 < |z| < \infty$,

$$\begin{aligned} f(z) &= \frac{1}{z^3} \frac{1}{1+\frac{1}{z^2}} = \frac{1}{z^3} \sum_{n=0}^{\infty} \left(-\frac{1}{z^2}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+3}} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{2n+1}} \end{aligned}$$

In this last part we use the fact that $(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1}$.

Exercise 5.3.1

Expand $f(z) = e^{3/z}$ in a Laurent series valid for $0 < |z| < \infty$.

This page titled [5.3: Laurent Series](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

5.4: Classification of Singularities

The portion

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \dots \quad (5.4.1)$$

of the **Laurent series**, involving negative powers of $z - z_0$, is called the *principal part* of $z - z_0$, at z_0 . The coefficient b_1 in equation (1), turns out to play a very special role in complex analysis. It is given a special name: the *residue* of the function $f(z)$. In this section we will focus on the principal part to identify the isolated singular point z_0 as one of three special types.

Poles

If the principal part of f at z_0 contains at least one nonzero term but the number of such terms is only finite, then there exists a integer $m \geq 1$ such that

$$b_m \neq 0 \quad \text{and} \quad b_k = 0 \quad \text{for} \quad k > m.$$

In this case, the isolated singular point z_0 is called a *pole of order m* . A pole of order $m = 1$ is usually referred to as a *simple pole*.

Examples

Consider the functions

$$f(z) = \frac{e^z - 1}{z^2}, \quad g(z) = \frac{\cos z}{z^2} \quad \text{and} \quad h(z) = \frac{\sinh z}{z^4},$$

with an isolated singularity at $z_0 = 0$. Figures 1, 2 and 3 show the **enhanced phase portraits** of these functions defined in the square $|\operatorname{Re} z| < 3$ and $|\operatorname{Im} z| < 3$.

 Pole order 1

Figure 1: $f(z) = \frac{e^z - 1}{z^2}$

 Pole order 2

Figure 2: $g(z) = \frac{\cos z}{z^2}$

 Pole order 3

Figure 3: $h(z) = \frac{\sinh z}{z^4}$

Now from the **enhanced phase portraits** we can observe that $z_0 = 0$ is in fact a pole which order can also be easily seen, it is just the number of isochromatic rays of one (arbitrarily chosen) color which meet at that point. Thus we can claim that f , g and h have poles of order 1, 2 and 3; respectively. To confirm this let's calculate the Laurent series representation centred at 0. First observe that

$$\begin{aligned} f(z) &= \frac{1}{z^2} \left[\left(1 + z + \frac{z^2}{2!} + \dots \right) - 1 \right] \\ &= \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots, \quad (0 < |z| < \infty). \end{aligned}$$

Thus we can see that f has a simple pole. On the other hand

$$\begin{aligned} g(z) &= \frac{1}{z^2} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \right) \\ &= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \dots, \quad (0 < |z| < \infty) \end{aligned}$$

then g has a pole of order 2. Finally, h has a pole of order 3 since

$$\begin{aligned} h(z) &= \frac{1}{z^4} \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \right) \\ &= \frac{1}{z^3} + \frac{1}{3!} \cdot \frac{1}{z} + \frac{z}{5!} + \frac{z^3}{7!} + \dots, \quad (0 < |z| < \infty). \end{aligned}$$

Removable singularity

When every b_n is zero, so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (0 < |z - z_0| < R_2). \quad (5.4.2)$$

In this case, z_0 is known as a *removable singular point*. Note that the residue at a removable singular point is always zero. If we define, or possibly redefine, f at z_0 so that $f(z_0) = a_0$, expansion (2) becomes valid throughout the entire disk $|z - z_0| < R_2$. Since a power series always represents an analytic function interior to its circle of convergence, it follows that f is analytic at z_0 when it is assigned the value a_0 there. The singularity z_0 is, therefore, removed.

Examples

Consider the functions

$$f(z) = \frac{1 - \cos z}{z^2}, \quad g(z) = \frac{\sin z}{z} \quad \text{and} \quad h(z) = \frac{z}{e^z - 1}.$$

Figure shows the enhanced phase portraits of these functions defined in the square $|\operatorname{Re} z| < 8$ and $|\operatorname{Im} z| < 8$.

 Removable singularity

Figure 4: $f(z) = \frac{1 - \cos z}{z^2}$

 Removable singularity

Figure 5: $g(z) = \frac{\sin z}{z}$

 Removable singularity

Figure 6: $h(z) = \frac{z}{e^z - 1}$

We notice that f has a singularity at $z_0 = 0$ but in this case the plot does not show isochromatic lines meeting at that point. This indicates that the singularity might be removable.

We can confirm this claim easily from the Laurent series representation:

$$\begin{aligned} f(z) &= \frac{1}{z^2} \left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \right) \right] \\ &= \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots, \quad (0 < |z| < \infty). \end{aligned}$$

In this case, when the value $f(0) = 1/2$ is assigned, f becomes entire. Furthermore, we can intuitively observe that since $z = 0$ is a removable singular point of f , then f must be analytic and bounded in some deleted neighbourhood $0 < |z| < \epsilon$.

Exercise 5.4.1

Find the Laurent series expansion for g and h to confirm that they have removable singularities at $z_0 = 0$.

Essential singularity

If an infinite number of the coefficients b_n in the principal part (1) are nonzero, then $z_0 = 0$ is said to be an *essential singular point* of f .

Examples

The function

$$f(z) = \exp\left(\frac{1}{z}\right)$$

has an essential singularity at $z_0 = 0$ since

$$\begin{aligned} f(z) &= 1 + \frac{1}{1!} \cdot \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \dots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n}, \quad (0 < |z| < \infty). \end{aligned}$$

Figure 7 shows the enhanced portrait of f in the square $|\operatorname{Re} z| < 0.5$ and $|\operatorname{Im} z| < 0.5$. The first thing we notice is that the behaviour of f near the essential singular point is quite irregular. Observe how the isochromatic lines, near $z_0 = 0$, form infinite self-contained

figure-eight shapes.

Essential singularity

Figure 7: defined on .

In fact, a neighbourhood of $z_0 = 0$ intersects infinitely many isochromatic lines of the phase portrait of one and the same colour [Wegert, 2012, p. 181]. This fact can be appreciated intuitively by plotting the simple phase portrait of $\exp(1/z)$ on a smaller region, as shown in Figure 8.

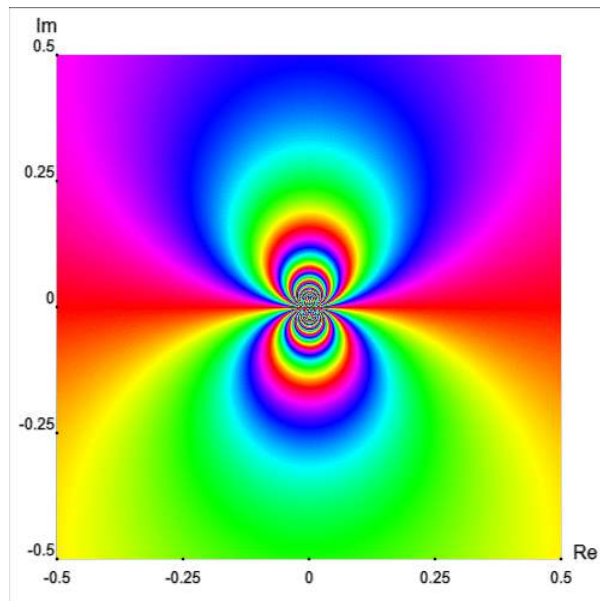


Figure 8: Simple phase portrait: A closer view to the essential singularity.

Another example with an essential singularity at the origin is the function

$$g(z) = (z - 1) \cos\left(\frac{1}{z}\right)$$

Figure 9 shows the enhanced phase portrait of g in the square $|\operatorname{Re} z| < 0.3$ and $|\operatorname{Im} z| < 0.3$.

Essential singularity

Figure 9: $(z - 1) \cos(1/z)$

Exercise 5.4.2

Find the Laurent series expansion for $(z - 1) \cos(1/z)$ to confirm that it has an essential singularity at $z_0 = 0$.

Final remark

Phase portraits are quite useful to understand the behaviour of functions near isolated singularities. Figures 7 and 9 indicate a rather wild behavior of these functions in a neighbourhood of essential singularities, in comparison with poles and removable singular points. In addition, they can be used to explore and comprehend, from a geometric point of view, more abstract mathematical results such as the [Great Picard Theorem](#), which tells us that any analytic function with an essential singularity at z_0 takes on all possible complex values (with at most a single exception) infinitely often in any neighbourhood of z_0 .

Essential singularity

Figure 10: $\sum_{k=-\infty}^{\infty} q^{k^2} z^k$

This page titled [5.4: Classification of Singularities](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

5.5: The Mandelbrot Set

Essentially, the Mandelbrot set is generated by iterating a simple function on the points of the complex plane. The points that produce a cycle (the same value over and over again) fall in the set, whereas the points that diverge (give ever-growing values) lie outside it. When plotted on a computer screen in many colors (different colors for different rates of divergence), the points outside the set can produce pictures of great beauty. The boundary of the Mandelbrot set is a fractal curve of infinite complexity, any portion of which can be blown up to reveal ever more outstanding detail, including miniature replicas of the whole set itself.

The Mandelbrot set is certainly the most popular fractal, and perhaps the most popular object of contemporary mathematics of all. Since [Benoît B. Mandelbrot](#) (1924-2010) discovered it in 1979-1980, while he was investigating the mapping $z \rightarrow z^2 + c$, it has been duplicated by tens of thousands of people around the world (including myself).

Constructing the Mandelbrot Set

Here is how the Mandelbrot set is constructed. Take a starting point z_0 in the complex plane. Then we use the quadratic recurrence equation

$$z_{n+1} = z_n^2 + z_0$$

to obtain a sequence of complex numbers z_n with $n = 0, 1, 2, \dots$. The points z_n are said to form the *orbit* of z_0 , and the Mandelbrot set, denoted by M , is defined as follows:

If the orbit z_n fails to go to infinity, we say that z_0 is contained within the set M . If the orbit z_n does go to infinity, we say that the point z_0 is outside M .

Take, for example, $z_0 = 1$. Then we have

$$\begin{aligned} z_0 &= 1 \\ z_1 &= 1^2 + 1 = 2 \\ z_2 &= 2^2 + 1 = 5 \\ z_3 &= 5^2 + 1 = 26 \\ z_4 &= 26^2 + 1 = 677 \\ &\vdots \end{aligned}$$

As you can see, z_n just keeps getting bigger and bigger. Thus $z_0 = 1$ is *not* in the Mandelbrot set. But if we choose different values for z_0 this won't always be the case. Consider now the value $z_0 = i$. In this case, we obtain:

$$\begin{aligned} z_0 &= i \\ z_1 &= i^2 + i = -1 + i \\ z_2 &= (-1 + i)^2 + i = -2i + i = -i \\ z_3 &= (-i)^2 + i = -1 + i \\ z_4 &= (-1 + i)^2 + i = -i \\ &\vdots \end{aligned}$$

It is clear that in this case further iterations will just repeat the values $-1 + i$ and $-i$. All of these complex numbers lie within distance 3 of the origin. So they stay in a bounded subset of the plane; they do not run out to infinity. So the number $z_0 = i$ is *in* the Mandelbrot set.

It is great fun to calculate elements of the Mandelbrot set and to plot them. The resulting set is endlessly complicated. And for this purpose we can use the power of the computer. In the applet below a point z_0 is defined on the complex plane. Since the computer can not handle infinity, it will be enough to calculate 500 iterations and use the number 10^8 (instead of infinity) to generate the Mandelbrot set:

If the orbit z_n is outside a disk of radius 10^8 , then z_0 is *not* in the Mandelbrot Set and its color will be *WHITE*. If the orbit z_n is inside that disk, then z_0 is *in* the Mandelbrot Set and its color will be *BLACK*.

Now explore the iteration orbits in the applet. Observe its behaviour while dragging the point. Activate the *Trace* box to sketch the Mandelbrot set or drag the slider.



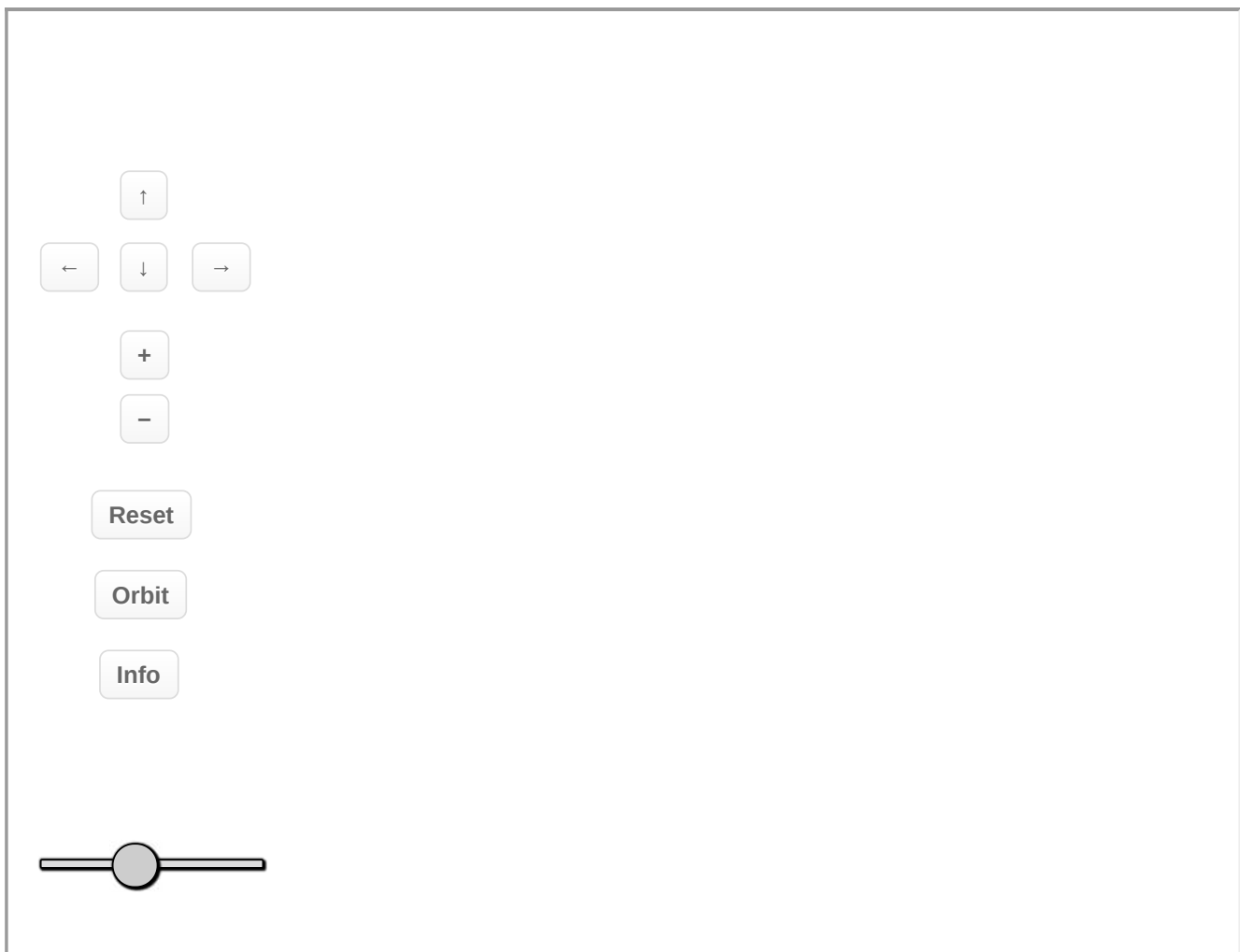
Colorful Mandelbrot Set

In the previous applet the Mandelbrot set is sketched using only one single point. However, it is possible to plot it considering a particular region of pixels on the screen. The simplest algorithm for generating a representation of the Mandelbrot set is known as the [escape time algorithm](#). A repeating calculation is performed for each x, y point in the plot area and based on the behavior of that calculation, a color is chosen for that pixel.

In the following applet, the [HSV](#) color scheme is used and depends on the distance from point z_0 (in exterior or interior) to nearest point on the boundary of the Mandelbrot set. In other words, provided that the maximal number of iterations is sufficiently high, we can obtain a picture of the Mandelbrot set with the following properties:

1. Every pixel that contains a point of the Mandelbrot set is colored black.
2. Every pixel that does not contain a point of the Mandelbrot set is colored using [hue](#) values depending on how close that point is to the Mandelbrot set.

Now explore the Mandelbrot set. Zoom in or out in different regions. Change the number of iterations and observe what happens to the plot. You can also plot the orbit.



+: Zoom In -: Zoom Out **R**=Reset view **O**=Orbit **I**=Information & Frame

Reset

[Open in a separate tab](#)

Further reading

Although the Mandelbrot set is defined by a very simple rule, it possesses interesting and complex properties that can be seen graphically if we pay close attention to the computer-generated pictures. For example, a geometric question we can ask: Is it connected? That is, is it just of one piece? This turns out to be true, and was proved by [Adrien Douady and John H. Hubbard](#) in the 80's.

The Mandelbrot set has been widely studied and I do not intend to cover all its fascinating properties here. However, if you want to learn more details I recommend you to consult B. B. Mandelbrot's works:

- **The Fractal Geometry of Nature.** New York: W. H. Freeman, 1983.
- [Fractals and Chaos: The Mandelbrot Set and Beyond.](#) New York: Springer-Verlag, 2004.

I also recommend you these *Numberphile* videos:

- [The Mandelbrot Set](#) by **Holly Krieger**.
- [What's so special about the Mandelbrot Set?](#) by **Ben Sparks**.

The applets were made with [GeoGebra](#) and [p5.js](#). The source code is available in the following links:

- [Mandelbrot iteration orbits](#)
- [Mandelbrot set](#)

If you want to learn how to program it yourself, I recommend you this tutorial

- **Coding Challenge: Mandelbrot set**

Finally, if you are adept at programming, then you can easily translate the pseudocode below into C++, Python, JavaScript, or any other language. For each pixel on the screen perform this operation:

```
{
  x0 = x co-ordinate of pixel
  y0 = y co-ordinate of pixel

  x = 0
  y = 0

  iteration = 0
  max_iteration = 1000

  while ( x*x + y*y <= (2*2) AND iteration < max_iteration )
  {
    xtemp = x*x - y*y + x0
    y = 2*x*y + y0

    x = xtemp

    iteration = iteration + 1
  }

  if ( iteration == max_iteration )
  then
    color = black
  plot(x0,y0,color)
  else
    color = iteration

  plot(x0,y0,color)
}
```

This page titled [5.5: The Mandelbrot Set](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

5.6: The Julia Set

In the previous section we showed how the Mandelbrot set can be generated using the expression

$$z_{n+1} = z_n^2 + z_0.$$

This is a particular case of the quadratic recurrence equation

$$z_{n+1} = z_n^2 + c \quad (5.6.1)$$

with c a fixed complex number. The set we obtain with this equation is known as the Julia set. In fact, there is a different Julia set for almost every c .

Similarly as we did for the Mandelbrot set, we obtain a sequence of complex numbers z_n with $n = 0, 1, 2, \dots$. Again, the points z_n are said to form the orbit of z_0 , and the Julia set is defined as follows:

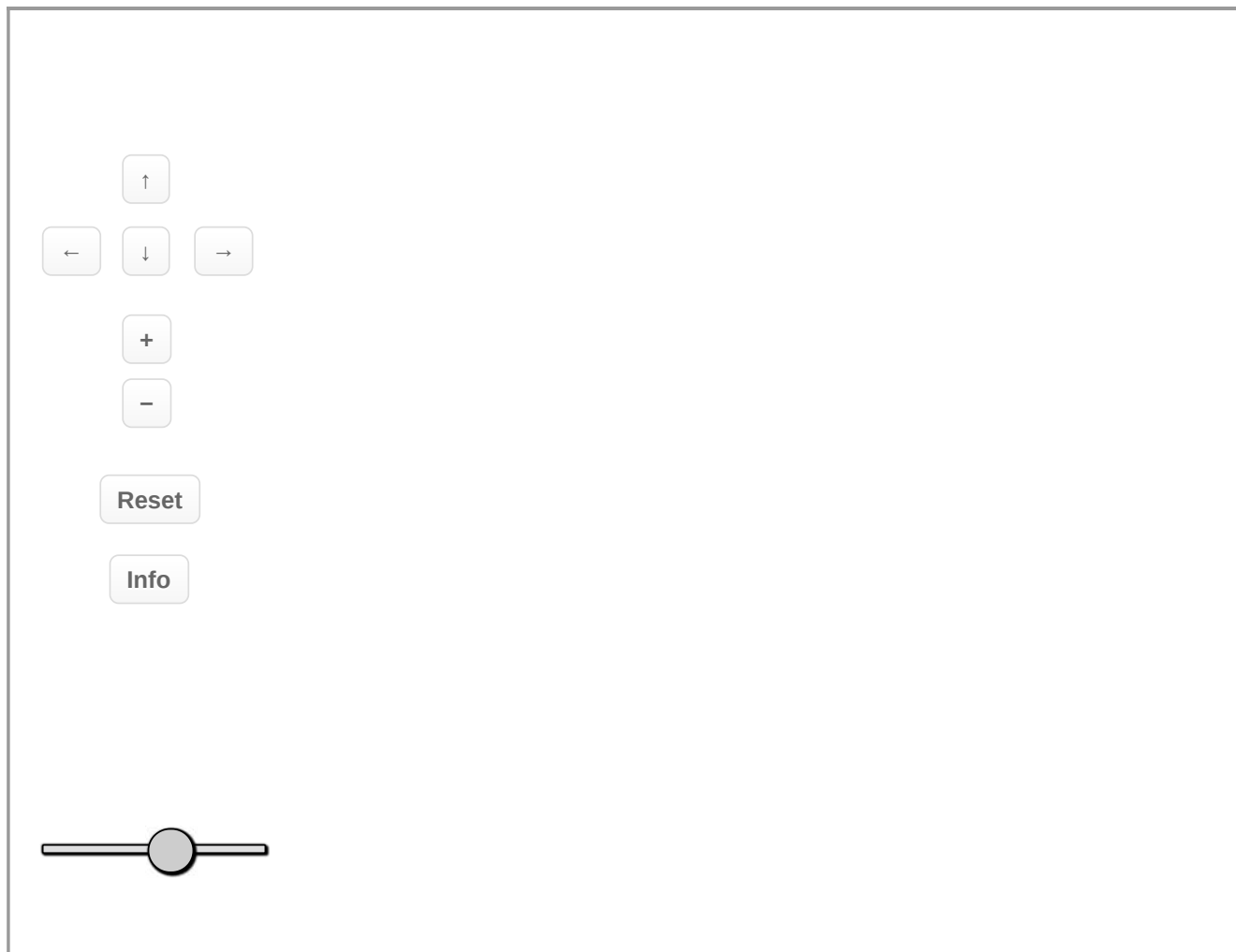
If the orbit z_n fails to escape to infinity, the initial z_0 is said to belong to the *filled-in Julia Set*.

The Julia set is named after the French mathematician [Gaston Julia](#) who investigated their properties in 1915 and culminated in his famous paper in 1918: *Mémoire sur l'itération des fonctions rationnelles*. While the Julia set is now associated with the quadratic polynomial in (1), Julia was interested in the iterative properties of a more general expression, namely

$$z^4 + \frac{z^3}{z-1} + \frac{z^2}{z^3+4z^2+5} + c.$$

The Julia sets, defined by the equation (1), can take all kinds of shapes, and a small change in c can change the Julia set very greatly. In 1979, with the help of computer, B. B. Mandelbrot studied the Julia sets and tried to classify all the possible shapes and came up with a new shape: the Mandelbrot Set.

Explore the Julia sets in the applet below. Zoom in or out in different regions. Change the number of iterations and observe what happens to the plot. Move the mouse around and observe the different Julia sets depending of the value of c .



+: Zoom In -: Zoom Out R=Reset view I=Information & Frame

Reset

The Mandelbrot and Julia Sets Connection

Due to the definition of the Mandelbrot set, there is a close correspondence between the geometry of the Mandelbrot set at a given point and the structure of the corresponding Julia set. In other words, the Mandelbrot set forms a kind of index into the Julia set. A Julia set is either connected or disconnected, values of c chosen from within the Mandelbrot set are connected while those from the outside of the Mandelbrot set are disconnected. The disconnected sets are often called *dust*, they consist of individual points no matter what resolution they are viewed at.

Explore the relationship between the Mandelbrot and Julia sets in the following applet. Move the mouse over to the Mandelbrot set to observe different Julia sets. Zoom in or out in different regions. Open the Controls menu to change the number of iterations or choose an specific value of c .



+: Zoom In -: Zoom Out R=Reset view I=Information & Frame

Reset

[Open in a separate tab](#)

Further reading

The applets were made with [p5.js](#) and the source code can be found:

- [Julia set](#)
- [Julia-Mandelbrot relationship](#)

If you want to learn how to program it yourself, I recommend you this tutorial:

- **Coding Challenge: Julia set**

Finally, I recommend you two of the most widely read basic introductions to the Mandelbrot and Julia sets:

- [The Beauty of Fractals](#) by Heinz-Otto Peitgen & Peter H. Richter, Munich: Springer-Verlang, 1986.
- [The Colours of Infinity](#), by Nigel Lesmoir-Gordon (ed), London: Springer-Verlang, 2010.

They both cover other fascinating fractals and contain many spectacular pictures.

This page titled [5.6: The Julia Set](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

CHAPTER OVERVIEW

6: Chapter 6

[6.1: Applications of Conformal Mappings](#)

[6.2: Complex Potential- Basic examples](#)

[6.3: Uniform Flow Around a Circle](#)

[6.4: Joukowski Airfoil](#)

This page titled [6: Chapter 6](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

6.1: Applications of Conformal Mappings

Hydrodynamics

If we have a (steady-state) incompressible, nonviscous fluid, we are interested in finding its velocity field

$$\mathbf{V}(x, y) = (u(x, y), v(x, y)).$$

From vector analysis we know that ‘incompressible’ means that the divergence $\text{div } \mathbf{V} = 0$. (We say $\mathbf{V}$ is *divergence free*.) We assume that $\mathbf{V}$ is also a *potential flow* and hence is circulation free; that is $\mathbf{V} = \text{grad } \phi$ for some ϕ called the *velocity potential*. Thus ϕ is harmonic because

$$\nabla^2 \phi = \text{div grad } \phi = \text{div } \mathbf{V} = 0.$$

Thus when we solve for ϕ we can obtain $\mathbf{V}$ by taking $\mathbf{V} = \text{grad } \phi$. That is

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}.$$

The conjugate ψ of the harmonic function ϕ (which will exist on any simple connected region) is called the *stream function*, and the analytic function

$$F = \phi + i\psi$$

is called the *complex potential*.

The stream function must satisfy

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

Finally, lines of constant ψ have $\mathbf{V}$ as their tangents, so lines of constant ψ may be interpreted as the *lines along which particles of fluid move*; hence the name stream function.

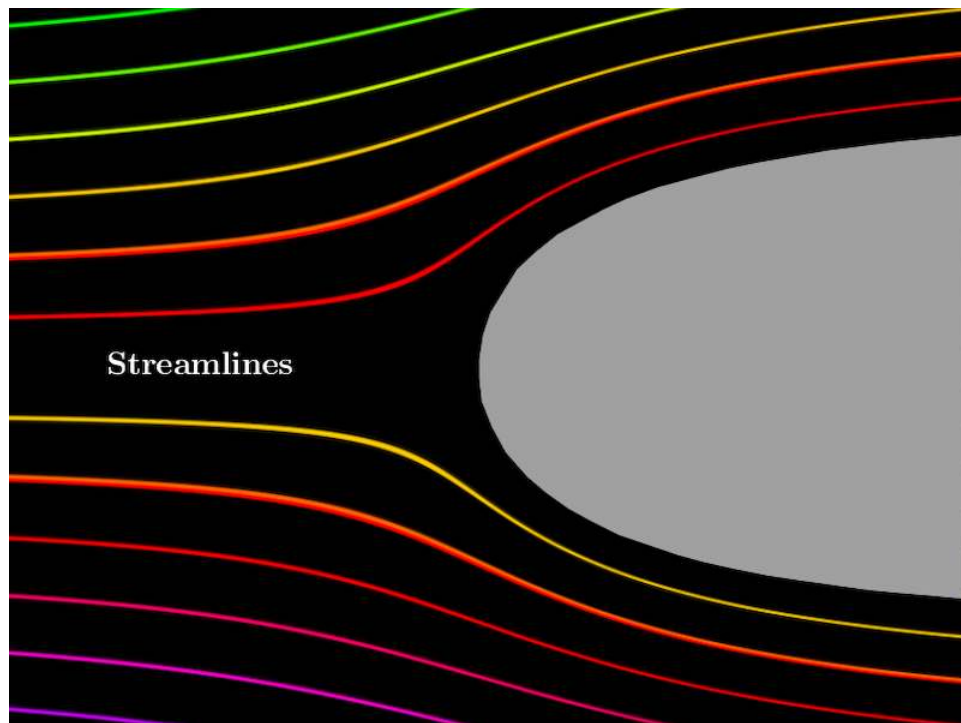


Figure 1: Streamlines.

This page titled [6.1: Applications of Conformal Mappings](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

6.2: Complex Potential- Basic examples

Basic examples

Uniform flow

The complex potential

$$F(z) = U e^{-i\alpha} z \quad (6.2.1)$$

corresponds to uniform flow at speed U in a direction making an angle with the x -axis.

Here we are interested in finding the velocity field

$$\mathbf{V} = (u(x, y), v(x, y)).$$

But first we need to obtain the stream function ψ , which is the imaginary component of (1).

Rewriting (1) we obtain

$$\begin{aligned} F(z) &= U e^{-i\alpha} z \\ &= U (\cos \alpha - i \sin \alpha) (x + iy) \\ &= U (x \cos \alpha + y \sin \alpha) + iU (y \cos \alpha - x \sin \alpha). \end{aligned}$$

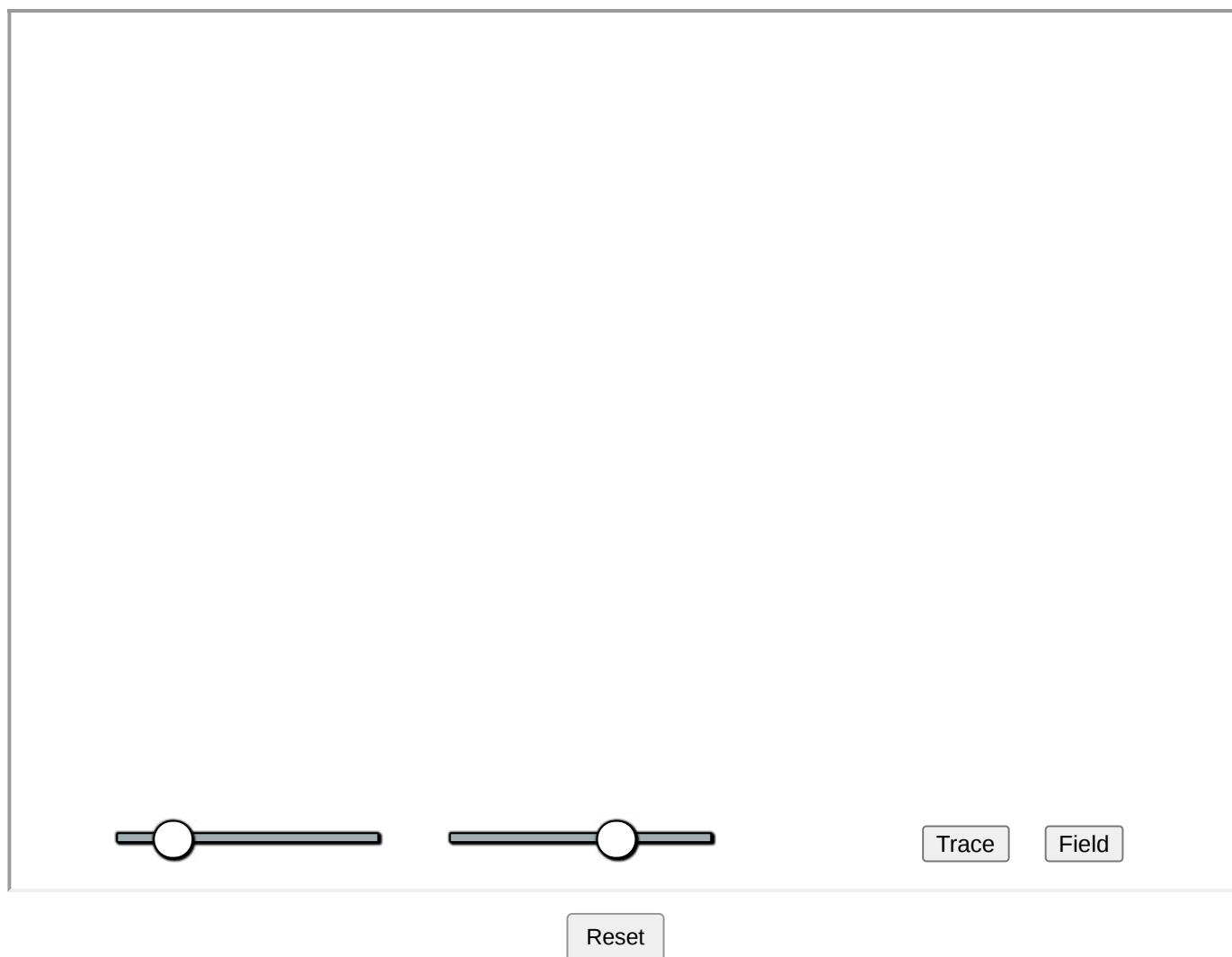
Thus

$$\psi = U (y \cos \alpha - x \sin \alpha).$$

Finally, since $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$, we have that

$$u = U \cos \alpha, \quad v = U \sin \alpha.$$

The applet below shows a simulation of the uniform flow. Drag the sliders to change parameters. Click on **Trace** button to show streamlines. Click on **Field** button to show vector field.

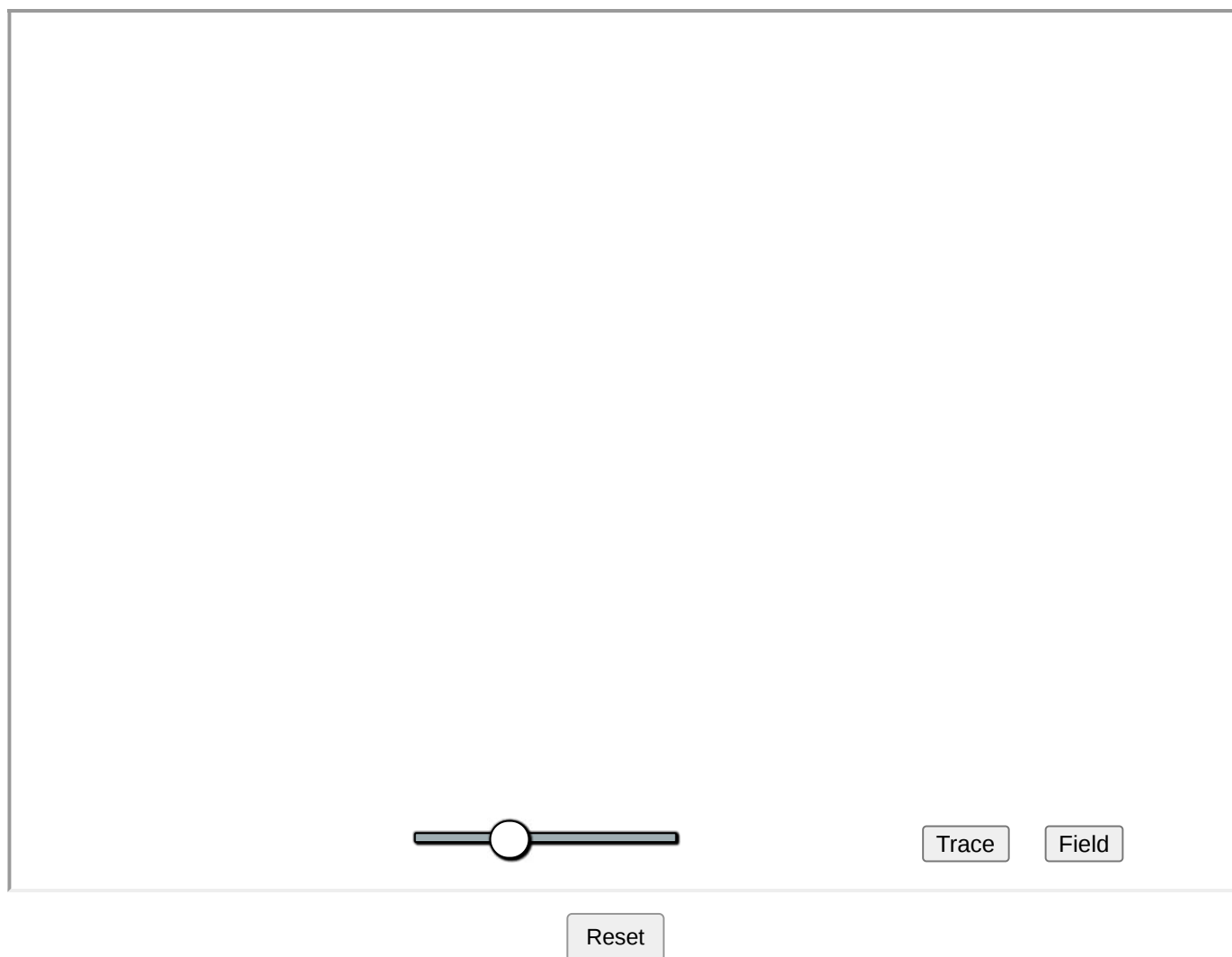


Stagnation point flow

The complex potential

$$F(z) = \frac{kz^2}{2}$$

corresponds to the stagnation point flow with strength $k \geq 0$.

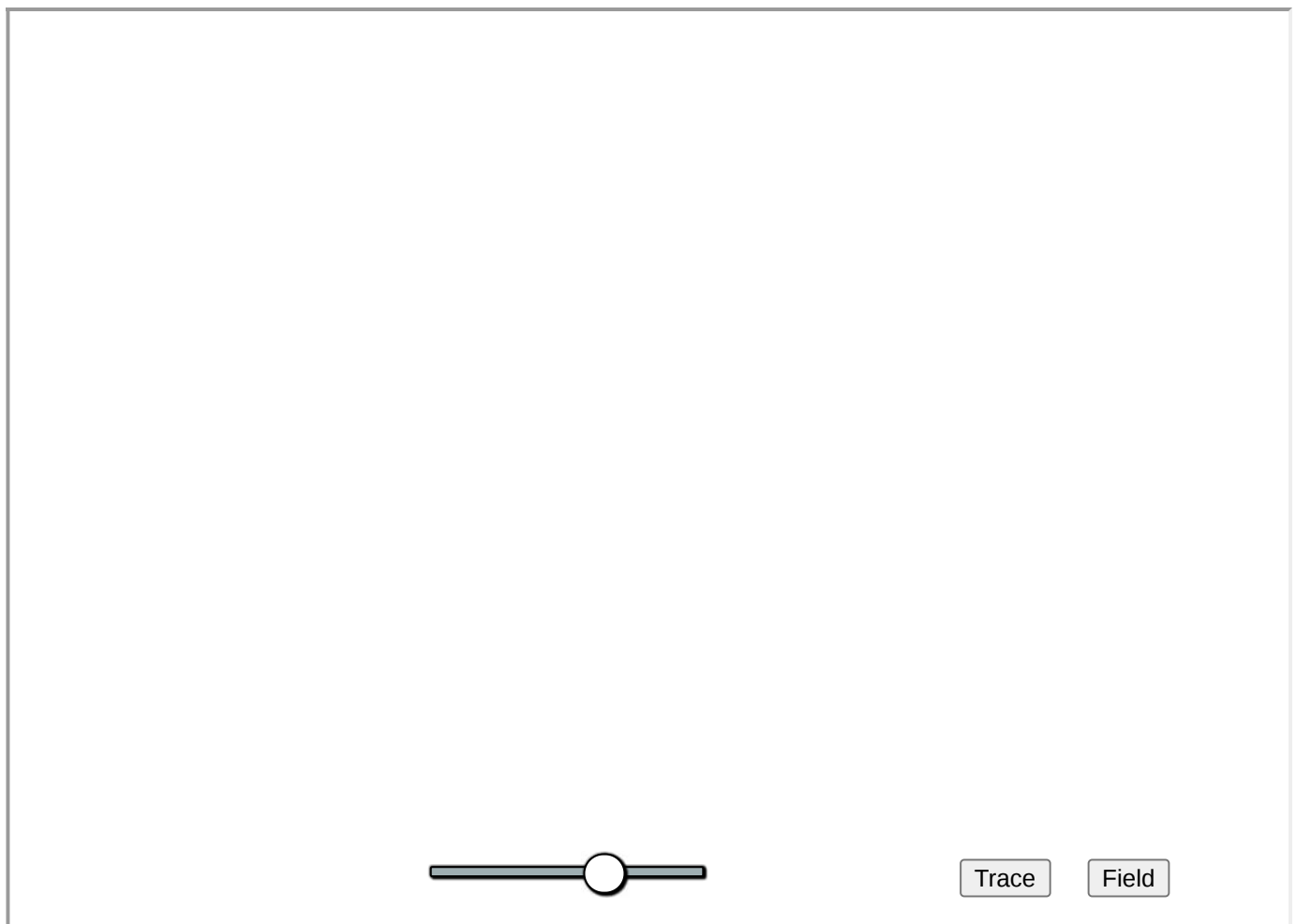


Source & Sink

A source of strength $Q > 0$ at the origin is represented by the complex potential

$$F(z) = \frac{Q}{2\pi} \log z. \quad (6.2.2)$$

Note that this is a multi-valued function, with a branch point at the origin. If $Q < 0$, then the complex potential corresponds to a sink.



Reset

It is easy to generalise (2) for an arbitrary point (a, b) in the complex plane. The required complex potential is

$$F(z) = \frac{Q}{2\pi} \log(z - c).$$

where $c = a + ib$.

Vortex

A vortex of strength C at the origin is represented by the complex potential

$$F(z) = \frac{-iC}{2\pi} \log z.$$

This is again a multi-valued function. For $C > 0$, rotation is anticlockwise, and for $C < 0$ rotation is clockwise.



A vortex at an arbitrary point $c \in \mathbb{C}$ is represented by the complex potential

$$F(z) = \frac{-iC}{2\pi} \log(z - c).$$

where $c = a + ib$.

Exercise 6.2.1

Find the velocity fields of the Stagnation point, Source & Sink and Vortex flows.

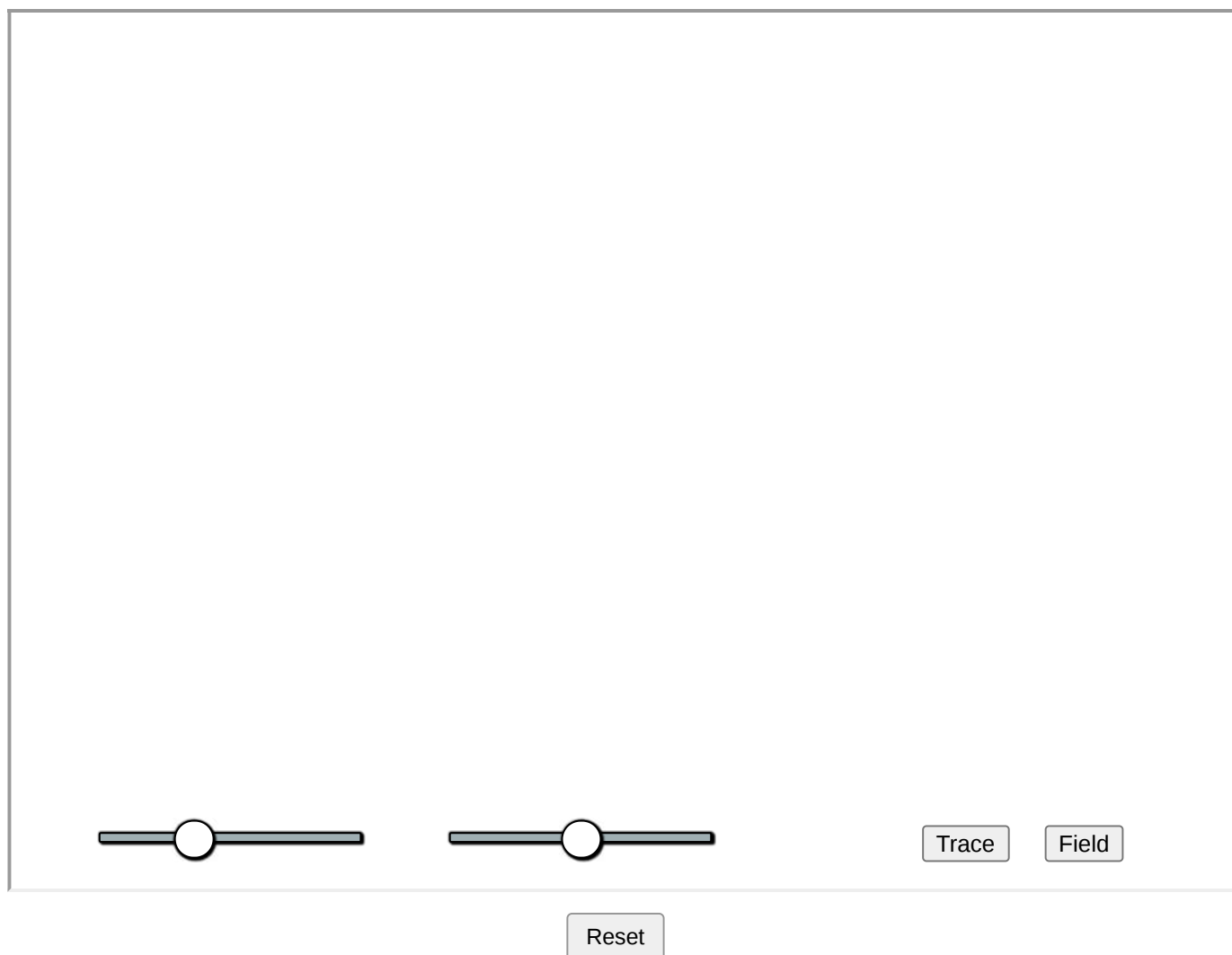
Combining complex potentials

The basic flows presented above can be combined by simply superimposing the corresponding complex potentials.

For example, consider a uniform flow Uz , with speed $U \geq 0$, and a source $\frac{Q}{2\pi} \log z$, with $Q \geq 0$. Thus we can produce the complex potential

$$F(z) = Uz + \frac{Q}{2\pi} \log z. \quad (6.2.3)$$

The following applet shows the flow produced by (3). Drag the sliders to change parameters.



Exercise 6.2.2

Find the velocity field of the flow produced by a source of strength Q in a uniform flow at speed U in the x -direction.

This page titled [6.2: Complex Potential- Basic examples](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

6.3: Uniform Flow Around a Circle

Consider a uniform flow at speed U in the x -direction, with complex potential $F(z) = Uz$. If we insert an impermeable circular obstacle, at $|z| = a$ say, then the flow will be disturbed as illustrated in the image below. The challenge here is to calculate the disturbed flow.



Here we can use a well known result in fluid dynamics established by the English mathematician [L. M. Milne-Thompson](#):

Circle Theorem

Suppose a complex potential $F(z)$ is given such that any singularities in $F(z)$ occur in $|z| > a$. Then the potential

$$w = F(z) + \overline{F\left(\frac{a^2}{\bar{z}}\right)} \quad (6.3.1)$$

(where the bar denotes complex conjugate) has the same singularities as $F(z)$ in $|z| > a$ and the circle $|z| = a$ is a streamline.

Proof: Let $F(z)$ be a complex potential such that any singularities occur only in the region $|z| > a$ (with $a > 0$) and define

$$w = F(z) + \overline{F\left(\frac{a^2}{\bar{z}}\right)}. \quad (6.3.2)$$

Notice that if $|z| > a$, then $|a^2/\bar{z}| < a$. Thus, since $F(z)$ has no singularities in $|z| \leq a$, it follows that the second term in (2) has no singularities in $|z| > a$. This means that F and w have the same singularities in $|z| > a$.

Now we are interested to know what happens on the circular boundary $|z| = a$. In this case we have that $\bar{z} \cdot z = a^2$. That is

$$z = \frac{a^2}{\bar{z}}.$$

Thus

$$w|_{|z|=a} = F(z) + \overline{F\left(\frac{a^2}{\bar{z}}\right)} = F(z) + \overline{F(z)} = 2\operatorname{Re}(f(z)),$$

which is entirely real. Therefore on the boundary $|z| = a$

$$\psi = \text{Im } w = 0.$$

This shows that the circle $|z| = a$ is a streamline. ■

Notice that the complex potential $F(z) = Uz$ satisfies the hypothesis of the Circle Theorem. We can therefore obtain the complex potential of the uniform flow around a circle by substituting $F(z) = Uz$ into equation (1):

$$w = Uz + \frac{\overline{Ua^2}}{\bar{z}} = Uz + \frac{Ua^2}{z}. \quad (6.3.3)$$

Consequently, the streamfunction is just the imaginary part of (3), namely

$$\psi = Uy \left(1 - \frac{a^2}{x^2 + y^2} \right).$$

and we can see that the circle $x^2 + y^2 = a^2$ is indeed a streamline, with $\psi = 0$. The resulting flow is shown in Figure 2 with $a = 1$.

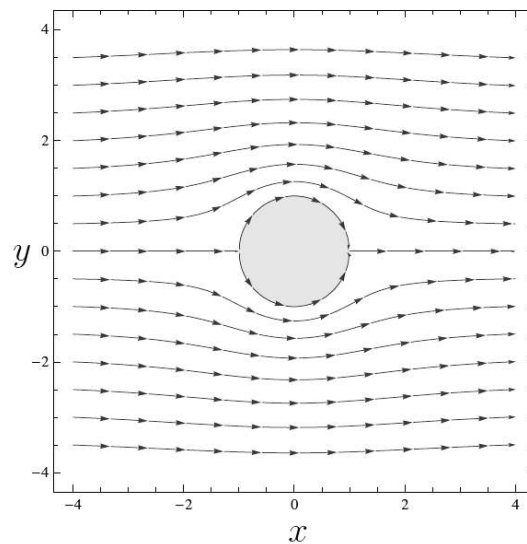


Figure 1: Uniform flow around a circle (right).

You probably have noticed that (3) has a singularity at $z = 0$. This kind of singularity is known as a *doublet* and corresponds to the function Ua^2/z . The singularity at the origin is inside the obstacle and thus does not affect the external flow. The full streamline pattern, including the doublet inside the circle, is shown in Figure 3.

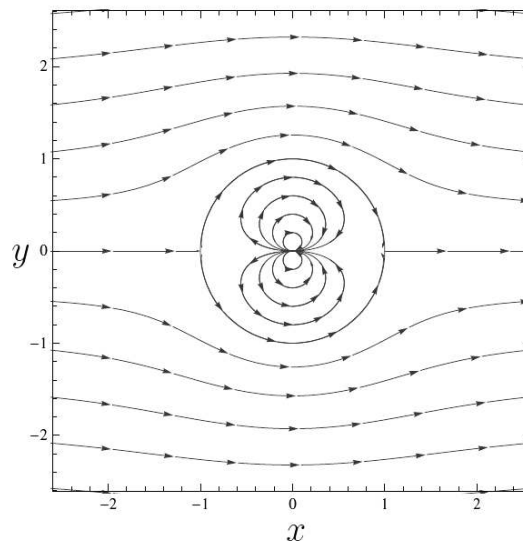


Figure 3: Streamlines caused by a doublet in a uniform flow.

Exercise 6.3.1

Show that the components of velocity field $\mathbf{V} = (u, v)$ for the uniform flow around a circle are given by

$$u = U \left(1 - a^2 \frac{x^2 - y^2}{(x^2 + y^2)^2} \right), \quad v = -2Ua^2 \frac{xy}{(x^2 + y^2)^2}$$

where U is the speed and a is the radius.

Uniform flow around circle with circulation

If we add a vortex to the complex potential defined in (3), we obtain a uniform flow around a circle with circulation:

$$w = Uz + \frac{Ua^2}{z} - \frac{iC}{2\pi} \log z, \quad (6.3.4)$$

where $C \in \mathbb{R}$ represents the *circulation* about the circle.

In this case, the streamfunction is

$$\psi = Uy \left(1 - \frac{a^2}{x^2 + y^2} \right) - \frac{C}{4\pi} \log(x^2 + y^2).$$

Notice that the circle $x^2 + y^2 = a^2$ is still a streamline, with $\psi = -(C/2\pi) \log a$. Any stagnation point in the flow satisfies the equation

$$0 = U - \frac{Ua^2}{z^2} - \frac{iC}{2\pi z},$$

which can be rearranged to the quadratic equation

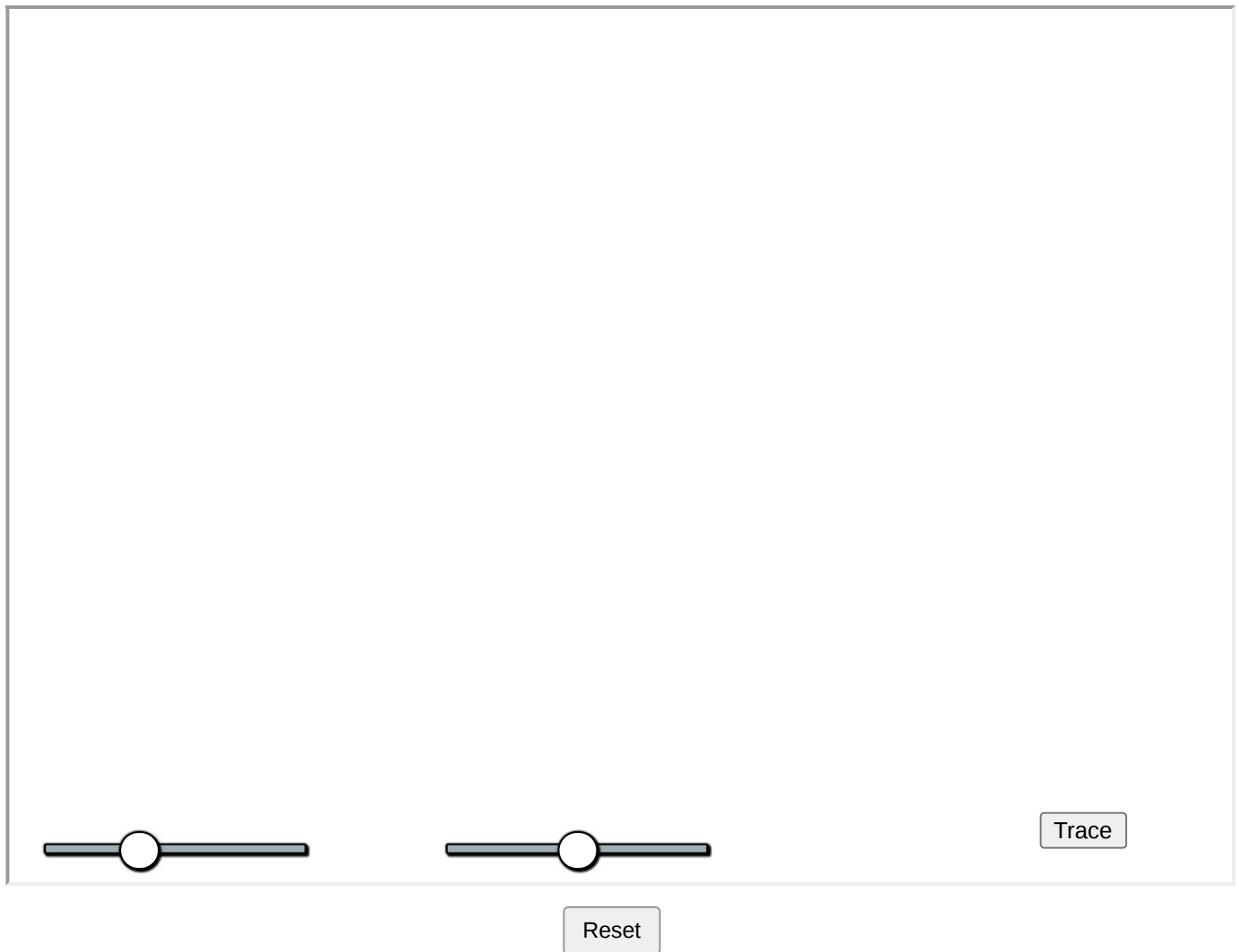
$$z^2 - 2i\gamma az - a^2 = 0, \quad \text{with } \gamma = \frac{C}{4\pi Ua}.$$

The roots of this equation are

$$\frac{z}{a} = i\gamma \pm \sqrt{1 - \gamma^2}.$$

Thus when $\gamma = 0$, there is no circulation with stagnation points at $z = \pm a$. As γ increases, the anticlockwise circulation causes the stagnation points to move upwards around the circle. When it reaches the value 1, the two stagnation points coalesce at the top of the cylinder $z = ia$. If $\gamma > 1$, then one stagnation point moves into the flow; the other one is inside the circle.

Explore all the cases in the applet below which shows the flow and a circle of radius 1. Drag the sliders U and C to change speed and circulation, respectively.



Exercise 6.3.2

Show that the components of velocity field $\mathbf{V} = (u, v)$ for the uniform flow around a circle with circulation are given by

$$u = U \left(1 - a^2 \frac{x^2 - y^2}{(x^2 + y^2)^2} \right) - \frac{C}{2\pi} \frac{y}{x^2 + y^2},$$

$$v = -2U^2 a^2 \frac{xy}{(x^2 + y^2)^2} + \frac{C}{2\pi} \frac{x}{x^2 + y^2}$$

where U is the speed, a is the radius and C is the circulation.

This page titled [6.3: Uniform Flow Around a Circle](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

6.4: Joukowski Airfoil

The Joukowski map

A well known example of a conformal function is the *Joukowski map*

$$w = z + 1/z. \quad (6.4.1)$$

It was first used in the study of flow around airplane wings by the pioneering Russian aero and hydrodynamics researcher **Nikolai Zhukovskii** (Joukowski).

Since

$$\frac{d}{dz}w = 1 - \frac{1}{z^2} = 0 \quad \text{if and only if} \quad z = \pm 1,$$

the function (1) is conformal except at the critical points $z = \pm 1$ as well as the singularity $z = 0$, where it is not defined.

If $z = e^{i\theta}$ lies on the unit circle, then

$$w = e^{i\theta} + e^{-i\theta} = 2 \cos \theta,$$

lies on the real axis, with $-2 \leq w \leq 2$. Thus, the Joukowski map squashes the unit circle down to the real line segment $[-2, 2]$. The images of points outside the unit circle fill the rest of the w plane, as do the images of the (nonzero) points inside the unit circle. Indeed, if we solve (1) for z , we have

$$z = \frac{1}{2} (w \pm \sqrt{w^2 - 4}).$$

We see that every w except ± 2 comes from two different points z ; for w not on the critical line segment $[-2, 2]$, one point (with the minus sign) lies inside and one (with the plus sign) lies outside the unit circle, whereas if $-2 < w < 2$, both points lie on the unit circle and a common vertical line.

Therefore, the Joukowski map defines a one-to-one conformal mapping from $|z| > 1$, the exterior of the unit circle, onto the exterior of the line segment $[-2, 2]$, i. e. $\mathbb{C} \setminus [-2, 2]$

In Figure 4 we can observe that the concentric circles $|z| = r > 1$ are mapped to ellipses with foci at ± 2 in the w -plane.

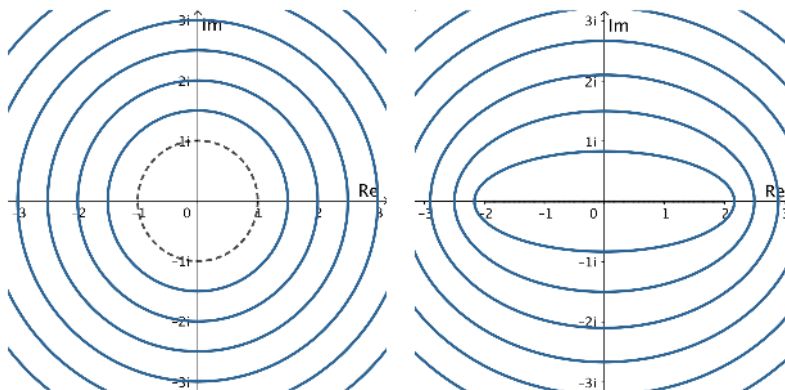


Figure 4: The Joukowski map applied to $|z| = r \geq 1$.

The effect on circles not centered at the origin is more interesting. The image curves take in a wide variety of shapes. When the circle passes through the singular point $z = 1$, then its image is no longer smooth, but has a cusp at $w = 2$ and when the circle passes through $z = -1$ the cusp is at $w = -2$. Some of the image curves assume the shape of the famous cross-section through an idealized airplane wing or *airfoil*, also known as the *Joukowski airfoil*.

You can explore the Joukowski map in the applet below. Drag around the center of the circle. Drag sliders to apply the mapping or change the radius. Click button to see predefined values.



Flow around the Joukowski airfoil

Consider now the uniform flow around the unit circle with circulation C and speed $U > 0$ given by the complex potential

$$F(z) = Uz + \frac{U}{z} - \frac{iC}{2\pi} \log z. \quad (6.4.2)$$

We can use the linear transformation

$$T(z) = -0.15 + 0.23i + 0.23\sqrt{13 \cdot 2}z$$

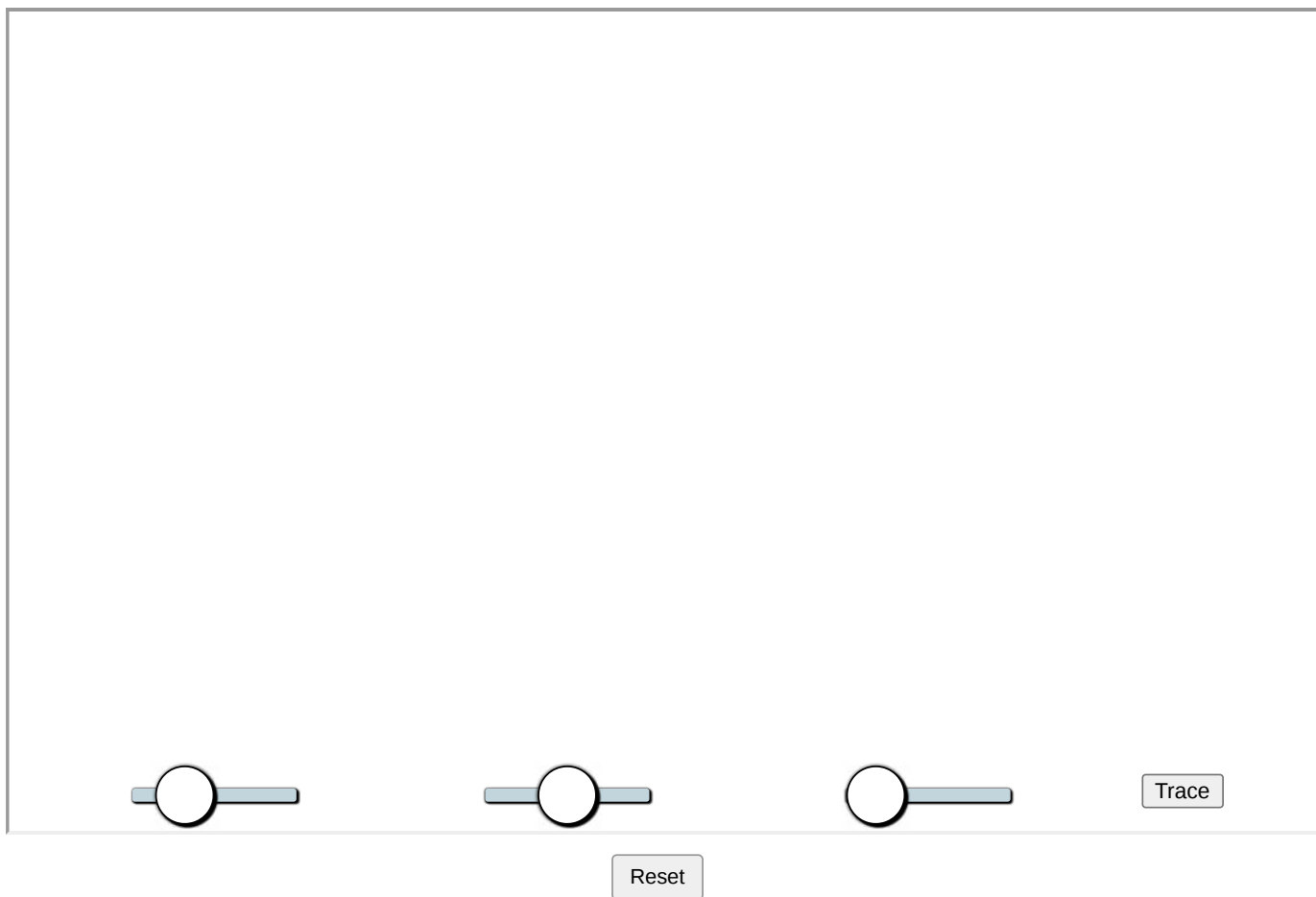
to map this flow around $|z| = 1$ onto the flow around the circle c_1 with center $z_1 = -0.15 + 0.23i$ and radius $r = 0.23\sqrt{13 \cdot 2}$

Finally, by applying the Joukowski map (1), we can obtain a uniform flow with circulation around the *Joukowski airfoil*.

The following simulation shows the uniform flow past the circular cylinder c_1 and its transformation to the *Joukowski airfoil*. Drag the sliders to explore:

- Slider U = speed.
- Slider C = circulation.
- Slider T = apply transformation.

Press the **Trace** button to show *streamlines*.



This page titled [6.4: Joukowski Airfoil](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).

Index

C

contour integrals

[4.2: Complex Integration](#)

P

principal argument

[1.4: The Principal Argument](#)

Glossary

Sample Word 1 | Sample Definition 1

Bibliography

1. Ablowitz, M. J., Fokas, A. S. (2003). *Complex variables: introduction and applications* (2nd ed). Cambridge University Press.
2. Brown, J. W., Churchill, R. V. (2009). *Complex Variables and Applications*. 8th Edition. New York: McGraw-Hill Higher Education.
3. Crone, L. (n.d.) *Color graphs of complex functions*.
4. Farris, F. (1997). *Visualizing complex-valued functions in the plane*.
5. Farris, F. (1998). Review of Visual Complex Analysis. By Tristan Needham. *The American Mathematical Monthly*, 105(6), 570-576.
6. Farris, F. A. (2017). Domain Coloring and the Argument Principle. *PRIMUS* 27:8-9, 827-844, DOI: **10.1080/10511970.2016.1234526**
7. Liste, R. L. (2014). El color dinámico de GeoGebra. *Gaceta De La Real Sociedad Matematica Española*, 17, 525–547, Madrid.
8. Lundmark, H. (2004). *Visualizing complex analytic functions using domain coloring*.
9. Needham, T. (1997). *Visual Complex Analysis*. Oxford University Press, Oxford.
10. Marsden, J. E. & Hoffman, M. J. (1987) *Basic Complex Analysis*. (2nd ed.) W. H. Freeman and Co, New York.
11. Poelke, K. & Polthier, K. (2012) Domain coloring of complex functions: An implementation-oriented introduction. *IEEE Computer Graphics Math & Applications* 32(5), pp. 90-97, DOI: **10.1109/MCG.2012.100**
12. Ponce Campuzano, J. C. (2019) The use of phase portraits to visualize and investigate isolated singular points of complex functions *Journal International Journal of Mathematical Education in Science and Technology*. 50(7), pp. 999-1010, DOI: **10.1080/0020739X.2019.1656829**
13. Thaller, B. (1998). Visualization of complex functions, *Mathematica J.*, Vol. 7, issue 2.
14. Farris, F. A. (1998). Review of Visual Complex Analysis. By Tristan Needham. *American Mathematical Monthly* 105 (1998), 570–576.
15. Wahlström, D. (2018). *The Great Picard Theorem*. Bachelor's Thesis. Sweeden: UMEÅ University.
16. Wegert, E. (2010). Phase Diagrams of Meromorphic Functions. *Comput. Methods Funct. Theory* 10, 639-661.
17. Wegert, E. (2012). *Visual Complex Functions: An introduction with phase portraits*. New York: Springer Basel.
18. Wegert, E. (2016). Visual exploration of complex functions. In T. Qian and L. R. Rodino (eds.), *Mathematical analysis, probability and applications – Plenary Lectures SPMS* (Vol. 177. pp. 253–279). Switzerland: Springer International Publishing.
19. Wegert, E. & Semmler, G. (2011). Phase plots of complex functions: a journey in illustration. *Notices Amer. Math. Soc.* 58, 768-780.
20. Wikipedia: *Domain Coloring*

That's it! If you found this book useful and help you in your journey to understand *Complex Analysis*, please **let me know!** You can also support this project with a direct donation. Your support will be appreciated!

 [Donate](#)

Detailed Licensing

Overview

Title: [Complex Analysis - A Visual and Interactive Introduction \(Ponce Campuzano\)](#)

Webpages: 49

Applicable Restrictions: Noncommercial

All licenses found:

- [CC BY-NC-SA 4.0](#): 81.6% (40 pages)
- [Undeclared](#): 18.4% (9 pages)

By Page

- [Complex Analysis - A Visual and Interactive Introduction \(Ponce Campuzano\)](#) - [CC BY-NC-SA 4.0](#)
 - [Front Matter](#) - [Undeclared](#)
 - [TitlePage](#) - [Undeclared](#)
 - [InfoPage](#) - [Undeclared](#)
 - [Table of Contents](#) - [Undeclared](#)
 - [Introduction](#) - [CC BY-NC-SA 4.0](#)
 - [Licensing](#) - [Undeclared](#)
 - [License](#) - [CC BY-NC-SA 4.0](#)
 - [Acknowledgements](#) - [CC BY-NC-SA 4.0](#)
 - [1: Chapter 1](#) - [CC BY-NC-SA 4.0](#)
 - [1.1: A Brief History](#) - [CC BY-NC-SA 4.0](#)
 - [1.2: Terminology and Notation](#) - [CC BY-NC-SA 4.0](#)
 - [1.3: Geometric Interpretation of the Arithmetic Operations](#) - [CC BY-NC-SA 4.0](#)
 - [1.4: The Principal Argument](#) - [CC BY-NC-SA 4.0](#)
 - [1.5: Roots of Complex Numbers](#) - [CC BY-NC-SA 4.0](#)
 - [1.6: Topology of the Complex Plane](#) - [CC BY-NC-SA 4.0](#)
 - [2: Chapter 2](#) - [CC BY-NC-SA 4.0](#)
 - [2.1: Complex functions](#) - [CC BY-NC-SA 4.0](#)
 - [2.2: Riemann Sphere](#) - [CC BY-NC-SA 4.0](#)
 - [2.3: Complex Differentiation](#) - [CC BY-NC-SA 4.0](#)
 - [2.4: The Logarithmic Function](#) - [CC BY-NC-SA 4.0](#)
 - [2.5: Riemann Surfaces](#) - [CC BY-NC-SA 4.0](#)
 - [3: Chapter 3](#) - [CC BY-NC-SA 4.0](#)
 - [3.1: Mappings](#) - [CC BY-NC-SA 4.0](#)
 - [3.2: The Transformation \$w=1/z\$](#) - [CC BY-NC-SA 4.0](#)
 - [3.3: Analytic Landscapes](#) - [CC BY-NC-SA 4.0](#)
 - [3.4: Domain coloring](#) - [CC BY-NC-SA 4.0](#)
 - [3.5: The Complex Power Function](#) - [CC BY-NC-SA 4.0](#)
 - [4: Chapter 4](#) - [CC BY-NC-SA 4.0](#)
 - [4.1: Curves in the Complex Plane](#) - [CC BY-NC-SA 4.0](#)
 - [4.2: Complex Integration](#) - [CC BY-NC-SA 4.0](#)
 - [4.3: Integrals of Functions with Branch Cuts](#) - [CC BY-NC-SA 4.0](#)
 - [5: Chapter 5](#) - [CC BY-NC-SA 4.0](#)
 - [5.1: Series](#) - [CC BY-NC-SA 4.0](#)
 - [5.2: Taylor Series](#) - [CC BY-NC-SA 4.0](#)
 - [5.3: Laurent Series](#) - [CC BY-NC-SA 4.0](#)
 - [5.4: Classification of Singularities](#) - [CC BY-NC-SA 4.0](#)
 - [5.5: The Mandelbrot Set](#) - [CC BY-NC-SA 4.0](#)
 - [5.6: The Julia Set](#) - [CC BY-NC-SA 4.0](#)
 - [6: Chapter 6](#) - [CC BY-NC-SA 4.0](#)
 - [6.1: Applications of Conformal Mappings](#) - [CC BY-NC-SA 4.0](#)
 - [6.2: Complex Potential- Basic examples](#) - [CC BY-NC-SA 4.0](#)
 - [6.3: Uniform Flow Around a Circle](#) - [CC BY-NC-SA 4.0](#)
 - [6.4: Joukowski Airfoil](#) - [CC BY-NC-SA 4.0](#)
 - [Back Matter](#) - [Undeclared](#)
 - [Index](#) - [Undeclared](#)
 - [Glossary](#) - [Undeclared](#)
 - [Bibliography](#) - [CC BY-NC-SA 4.0](#)
 - [Detailed Licensing](#) - [Undeclared](#)