MATH 150: SURVEY OF MATH



Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia
Rio Hondo College



MA 150 Survey of Mathematics

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (https://LibreTexts.org) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide and accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by NICE CXone Expert and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptions contact info@LibreTexts.org or visit our main website at https://LibreTexts.org.

This text was compiled on 08/14/2025

TABLE OF CONTENTS

Authors

Licensing

1: Foundations

- 1.1: Introduction
- 1.2: Problem Solving and Estimating
  - 1.2.1: Exercises
- 1.3: Real Numbers
  - 1.3.1: Integers
    - 1.3.1.1: Exercises
  - 1.3.2: Rational and Irrational Numbers
    - 1.3.2.1: Exercises
  - 1.3.3: Associative, Commutative, and Distributive Properties
    - 1.3.3.1: Exercises
  - 1.3.4: Order of Operations
    - 1.3.4.1: Exercises
- 1.4: Exponents
  - 1.4.1: Rules of Exponents
    - 1.4.1.1: 1.4.1 and 1.4.2 Exercises
  - 1.4.2: The Power Rules for Exponents
    - 1.4.2.1: Exercises
  - 1.4.3: Scientific Notation
    - 1.4.3.1: Exercises
- 1.5: Percents
  - 1.5.1: Exercises
  - 1.5.2: Proportions and Rates
  - 1.5.3: Exercises

2: Finances

- 2.1: Introduction
- 2.2: Taxes
  - 2.2.1: Income Taxation
  - 2.2.2: Exercises
- 2.3: Simple Interest
  - 2.3.1: Exercises
- 2.4: Compound Interest
  - 2.4.1: Exercises
- 2.5: Annuities
  - 2.5.1: Exercises
- 2.6: Payout Annuities
  - 2.6.1: Exercises

- 2.7: Installment Loans
  - 2.7.1: Exercises
- 2.8: Credit Cards
  - 2.8.1: Exercises
- 2.9: Which equation to use?
  - 2.9.1: Exercises

3: Logic

- 3.1: Basics of Sets
  - 3.1.1: Union, Intersection, and Complement
    - 3.1.1.1: Exercises
  - 3.1.2: Venn Diagrams
    - 3.1.2.1: Exercises
- 3.2: Logic
  - 3.2.1: Boolean Logic
    - 3.2.1.1: Exercises
  - 3.2.2: Conditional Statements
    - 3.2.2.1: Exercises
  - 3.2.3: Quantified Statements
    - 3.2.3.1: Exercises
  - 3.2.4: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)
    - 3.2.4.1: Exercises
  - 3.2.5: Truth Tables- Conditional, Biconditional
    - 3.2.5.1: Exercises
- 3.3: Arguments
  - 3.3.1: Forms of Valid Arguments
    - 3.3.1.1: Exercises
  - 3.3.2: Evaluating Deductive Arguments with Euler Diagrams
    - 3.3.2.1: Exercises
- 3.4: Logical Fallacies in Common Language
  - 3.4.1: Exercises

4: Probability

- 4.1: Introduction to Probability
  - 4.1.1: Exercises
- 4.2: Basic Concepts
  - 4.2.1: Exercises
- 4.3: Working with Events
  - 4.3.1: Exercises
- 4.4: Two-Way Tables
  - 4.4.1: Exercises
- 4.5: Counting
  - 4.5.1: Exercises
- 4.6: Expected Value

- 4.6.1: Exercises
- 4.7: Additional Exercises

5: Statistics

- 5.1: What is Statistics?
  - 5.1.1: Exercises
- 5.2: Populations and Samples
  - 5.2.1: Exercises
- 5.3: Categorizing data
  - 5.3.1: Exercises
- 5.4: Sampling methods
  - 5.4.1: Exercises
- 5.5: Bias
  - 5.5.1: Exercises
- 5.6: Experiments
  - 5.6.1: Exercises
- 5.7: Presenting Data Graphically
  - 5.7.1: Exercises
- 5.8: Reading Graphs in Media
  - 5.8.1: Exercises
- 5.9: Correlation and Causation, Scatter Plots
  - 5.9.1: Exercises
- 5.10: Measures of Central Tendency
  - 5.10.1: Exercises

6: Measurement and Geometry

- 6.1: Measurement
  - 6.1.1: Measurement and the United States System
    - 6.1.1.1: Exercises
  - 6.1.2: The Metric System of Measurement
    - 6.1.2.1: Exercises
  - 6.1.3: Conversion Between the Metric and US Customary Systems of Measurement
    - 6.1.3.1: Exercises
  - 6.1.4: Temperature and Time
    - 6.1.4.1: Exercises
  - 6.1.5: Simplification of Denominate Numbers
    - 6.1.5.1: Exercises
- 6.2: Geometry
  - 6.2.1: Perimeter and Area of Geometric Figures
    - 6.2.1.1: Exercises
  - 6.2.2: Circles
    - 6.2.2.1: Exercises
  - 6.2.3: Solid Geometric Figures and Objects
    - 6.2.3.1: Exercises

- 6.3: Fractals
  - 6.3.1: Fractal Dimension
  - 6.3.2: Exercises

Index

Glossary

Detailed Licensing

Authors

We adapted the material in this open educational resource because we wanted a low-cost option for our students that supports the content of Rio Hondo College's Survey of Mathematics course.

Nancy Garcia graduated from California State University, Fullerton where she earned a bachelor's and master's degree in mathematics. She began teaching in 2015. She is not only passionate about math but also loves dogs and café hopping in search of the best cup of coffee

After completing a BA at Chapman College and an MS at California Polytechnic State University, San Luis Obispo, both in mathematics, **Leah Griffith** has spent more than 40 years teaching mathematics at the college level. She was a member of the faculty group that created Rio Hondo's Survey of Mathematics course. When not doing and learning math with her students Leah enjoys making quilts.

Veronica Holbrook began her STEM career as an engineer for FedEx after graduating with her BS degree in Electronic Engineering from ITT Technical Institute. She transitioned into education more than 20 years ago, first as a technology teacher in K-12, then a technology trainer for K-12 teachers while developing curriculum for their students. She then attended California State University, Fullerton to pursue a BA and MA in Mathematics. This allowed her to begin her third career as a mathematics teacher at the college level. She recently completed her EdD in Community College Leadership at California State University, Fullerton. For fun, Veronica enjoys East Coast swing dancing and participating at a local community garden.

Johnny Johnson has over 20 years of experience teaching mathematics in both high school and community college. He earned a bachelors and master's degree in mathematics from California State University, Long Beach. He completed his EdD in Adult Education from Capella University. When he's not spending quality time with his wife and children, Johnny indulges in his favorite pastimes: binge-watching his preferred TV shows and anime, playing games, and exploring the intriguing realms of AI, NFT's, and cryptocurrency.

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Foundations

1.1: Introduction
1.2: Problem Solving and Estimating
1.2.1: Exercises
1.3: Real Numbers
1.3.1: Integers
1.3.1.1: Exercises
1.3.2: Rational and Irrational Numbers
1.3.2.1: Exercises
1.3.3: Associative, Commutative, and Distributive Properties
1.3.3.1: Exercises
1.3.4: Order of Operations
1.3.4.1: Exercises
1.4: Exponents
1.4.1: Rules of Exponents
1.4.1.1: 1.4.1 and 1.4.2 Exercises
1.4.2: The Power Rules for Exponents
1.4.2.1: Exercises
1.4.3: Scientific Notation
1.4.3.1: Exercises
1.5: Percents
1.5.1: Exercises
1.5.2: Proportions and Rates
1.5.3: Exercises

This page titled 1: Foundations is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.1: Introduction

In previous math courses, you've no doubt run into the infamous "word problems." Unfortunately, these problems rarely resemble the type of problems we actually encounter in everyday life. In math books, you usually are told exactly which formula or procedure to use, and are given exactly the information you need to answer the question. In real life, problem solving requires identifying an appropriate formula or procedure, and determining what information you will need (and won't need) to answer the question.

In this chapter, we will review several basic but powerful algebraic ideas: percents, rates, and proportions. We will then focus on the problem solving process, and explore how to use these ideas to solve problems where we don't have perfect information.

This page titled 1.1: Introduction is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **1.1: Introduction** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

1.2: Problem Solving and Estimating

1.2 Learning Objectives

- Apply the problems solving process to real life problems
- Use dimensional analysis to solve problems
- Use estimates to solve real life problems
- Round decimals

In many problems, it is tempting to take the given information, plug it into whatever formulas you have handy, and hope that the result is what you were supposed to find. Chances are, this approach has served you well in other math classes.

This approach does not work well with real life problems. Instead, problem solving is best approached by first starting at the end: identifying exactly what you are looking for. From there, you then work backwards, asking "what information and procedures will I need to find this?" Very few interesting questions can be answered in one mathematical step; often times you will need to chain together a solution pathway, a series of steps that will allow you to answer the question.

Problem Solving Process

1. Identify the question you're trying to answer.
2. Work backwards, identifying the information you will need and the relationships you will use to answer that question.
3. Continue working backwards, creating a solution pathway.
4. If you are missing necessary information, look it up or estimate it. If you have unnecessary information, ignore it.
5. Solve the problem, following your solution pathway.
6. Check your answer. Does the answer make sense?

In most problems we work, we will be estimating a solution, because we will not have perfect information. We will begin with a few examples where we will be able to estimate the solution using basic knowledge from our lives.

Definition: Estimation

Estimation is the process of arriving at an approximate answer to a question. Usually by rounding answers to a place value that is easier to express in the situation.

Depending on the situation, you may be asked to round your answers to a specific place value. The tables below list more common values, but there are many more.

Place values of a whole number:

Place Value														
Trillions			Billions			Millions			Thousands			Ones		
Hundred Trillions	Ten Trillions	Trillions	Hundred Billions	Ten Billions	Billions	Hundred Millions	Ten Millions	Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Ones

Place values of a decimal number:

Decimal Place Value								
Tens	Ones	•	Tenths	Hundredths	Thousandths	Ten Thousandths	Hundred Thousandths	Millionths

Name and Write Decimals

Notice that “ten thousand” is a number larger than one, but “one ten-thousand**th**” is a number smaller than one. The “th” at the end of the name tells you that the number is smaller than one.

When we name a whole number, the name corresponds to the place value based on the powers of ten. We read 10,000 as “ten thousand” and 10,000,000 as “ten million.” Likewise, the names of the **decimal** places correspond to their fraction values.

We summarize the steps needed to name a **decimal** below.

1. Name the number to the left of the decimal point.
2. Write “and” for the decimal point.
3. Name the “number” part to the right of the decimal point as if it were a whole number.
4. Name the decimal place of the last digit.

Example 1

Name the decimal 4.3.

Solution

		4.3
Step 1. Name the number to the left of the decimal point.	4 is to the left of the decimal point.	four _____
Step 2. Write ‘and’ for the decimal point.		four and _____
Step 3. Name the ‘number’ part to the right of the decimal point as if it were a whole number.	3 is to the right of the decimal point.	four and three _____
Step 4. Name the decimal place.		four and three tenths

Try It Now 1

Name the decimal 6.7.

Answer

six and seven tenths

Try It Now 2

Name the decimal: -15.571.

Answer

	-15.571
Name the number to the left of the decimal point.	negative fifteen _____
Write “and” for the decimal point.	negative fifteen and _____
Name the number to the right of the decimal point.	negative fifteen and five hundred seventy-one _____

The 1 is in the thousandths place.	negative fifteen and five hundred seventy-one thousandths

When we write or read a check we interpret both the numerals and the name of the number. Let's see how to write the decimal from the name.

We summarize the steps to writing a **decimal**.

1. Look for the word "and"—it locates the decimal point.
   - Place a decimal point under the word "and." Translate the words before "and" into the whole number and place it to the left of the decimal point.
   - If there is no "and," write a "0" with a decimal point to its right.
2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
3. Translate the words after "and" into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
4. Fill in zeros for place holders as needed.

Example 2

Write "fourteen and twenty-four thousandths" as a decimal.

Solution

Step 1. Look for the word 'and'; it locates the decimal point. Place a decimal point under the word 'and'. Translate the words before 'and' into the whole number and place to the left of the decimal point.		fourteen and twenty-four thousandths fourteen and twenty-four thousandths ____ . _____ 14 . _____
Step 2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.	The last word is 'thousandths'.	14 . ____ ____ ____ tenths hundredths thousandths
Step 3. Translate the words after 'and' into the number to the right of the decimal point. Write the number in the spaces – putting the final digit in the last place.		14 . ____ 2 4
Step 4. Fill in zeros for empty place holders as needed.	Zeros are needed in the tenths place.	14 . 0 2 4 Fourteen and twenty-four thousandths is written 14.024.

Try It Now 3

Write as a decimal: thirteen and sixty-eight thousandths.

Answer

13.068

Round Decimals

Rounding decimals is very much like rounding whole numbers. We will round decimals with a method based on the one we used to round whole numbers.

We summarize the steps for rounding a **decimal** here.

1. Locate the given place value and mark it with an arrow.
2. Underline the digit to the right of the place value.
3. Is this digit greater than or equal to 5?
   - Yes—add 1 to the digit in the given place value.
   - No—do not change the digit in the given place value.
4. Rewrite the number, deleting all digits to the right of the rounding digit.

Example 3

Round 18.379 to the nearest hundredth.

Solution

Step 1. Locate the given place value and mark it with an arrow.		hundredths place ↓ 18.379
Step 2. Underline the digit to the right of the given place value.		hundredths place ↓ 18.379
Step 3. Is this digit greater than or equal to 5? **Yes:** Add 1 to the digit in the given place value. **No:** Do not change the digit in the given place value.	Because 9 is greater than or equal to 5, add 1 to the 7.	18.37 9 add 1 delete
Step 4. Rewrite the number, removing all digits to the right of the rounding digit.		18.38 18.38 is 18.379 rounded to the nearest hundredth.

Try It Now 4

Round 6.582 to the nearest

1. hundredth
2. tenth
3. whole number.

Answer

1. 6.58
2. 6.6
3. 7

We can use these estimation techniques to answer questions that involve some problem solving. Bringing the two concepts from above together, we apply them to several examples below.

Example 4

How many times does your heart beat in a year?

Solution

This question is asking for the rate of heart beats per year. Since a year is a long time to measure heart beats for, if we knew the rate of heart beats per minute, we could scale that quantity up to a year. So the information we need to answer this question is heart beats per minute. This is something you can easily measure by counting your pulse while watching a clock for a minute.

Suppose you count 80 beats in a minute. To convert this beats per year:

$$\frac{80 \text{ beats}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{365 \text{ days}}{1 \text{ year}} = 42,048,000 \text{ beats per year}$$

Example 5

How thick is a single sheet of paper? How much does it weigh?

Solution

While you might have a sheet of paper handy, trying to measure it would be tricky. Instead we might imagine a stack of paper, and then scale the thickness and weight to a single sheet. If you've ever bought paper for a printer or copier, you probably bought a ream, which contains 500 sheets. We could estimate that a ream of paper is about 2 inches thick and weighs about 5 pounds. Scaling these down,

$$\frac{2 \text{ inches}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.004 \text{ inches per sheet}$$

$$\frac{5 \text{ pounds}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.01 \text{ pounds per sheet, or 0.16 ounces per sheet.}$$

Example 6

A recipe for zucchini muffins states that it yields 12 muffins, with 250 calories per muffin. You instead decide to make mini-muffins, and the recipe yields 20 muffins. If you eat 4, how many calories will you consume?

Solution

There are several possible solution pathways to answer this question. We will explore one.

To answer the question of how many calories 4 mini-muffins will contain, we would want to know the number of calories in each mini-muffin. To find the calories in each mini-muffin, we could first find the total calories for the entire recipe, then divide it by the number of mini-muffins produced. To find the total calories for the recipe, we could multiply the calories per standard muffin by the number per muffin. Notice that this produces a multi-step solution pathway. It is often easier to solve a problem in small steps, rather than trying to find a way to jump directly from the given information to the solution.

We can now execute our plan:

$$12 \text{ muffins} \cdot \frac{250 \text{ calories}}{\text{muffin}} = 3000 \text{ calories for the whole recipe}$$

$$\frac{3000 \text{ calories}}{20 \text{ mini-muffins}} \text{ gives 150 calories per mini-muffin}$$

$$4 \text{ mini-muffins} \cdot \frac{150 \text{ calories}}{\text{mini-muffin}} \text{ totals 600 calories consumed.}$$

Example 7

You need to replace the boards on your deck. About how much will the materials cost?

Solution

There are two approaches we could take to this problem: 1) estimate the number of boards we will need and find the cost per board, or 2) estimate the area of the deck and find the approximate cost per square foot for deck boards. We will take the latter approach.

For this solution pathway, we will be able to answer the question if we know the cost per square foot for decking boards and the square footage of the deck. To find the cost per square foot for decking boards, we could compute the area of a single board, and divide it into the cost for that board. We can compute the square footage of the deck using geometric formulas. So first we need information: the dimensions of the deck, and the cost and dimensions of a single deck board.

Suppose that measuring the deck, it is rectangular, measuring 16 ft by 24 ft, for a total area of 384ft^2.

From a visit to the local home store, you find that an 8 foot by 4 inch cedar deck board costs about $7.50. The area of this board, doing the necessary conversion from inches to feet, is:

$$8 \text{ feet} \cdot 4 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = 2.667 \text{ft}^2 .$$

The cost per square foot is then

$$\frac{\$7.50}{2.667 \text{ft}^2} = \$2.8125 \text{ per ft}^2.$$

This will allow us to estimate the material cost for the whole 384ft^2 deck

$$384 \text{ ft}^2 \cdot \frac{\$2.8125}{\text{ft}^2} = \$1080 \text{ total cost.}$$

Of course, this cost estimate assumes that there is no waste, which is rarely the case. It is common to add at least 10% to the cost estimate to account for waste.

Example 8

Is it worth buying a Hyundai Sonata hybrid instead the regular Hyundai Sonata?

Solution

To make this decision, we must first decide what our basis for comparison will be. For the purposes of this example, we'll focus on fuel and purchase costs, but environmental impacts and maintenance costs are other factors a buyer might consider.

It might be interesting to compare the cost of gas to run both cars for a year. To determine this, we will need to know the miles per gallon both cars get, as well as the number of miles we expect to drive in a year. From that information, we can find the number of gallons required from a year. Using the price of gas per gallon, we can find the running cost.

From Hyundai's website, the 2013 Sonata will get 24 miles per gallon (mpg) in the city, and 35 mpg on the highway. The hybrid will get 35 mpg in the city, and 40 mpg on the highway.

An average driver drives about 12,000 miles a year. Suppose that you expect to drive about 75% of that in the city, so 9,000 city miles a year, and 3,000 highway miles a year.

We can then find the number of gallons each car would require for the year.

Sonata:

$$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{24 \text{ city miles}} + 3000 \text{ hightway miles} \cdot \frac{1 \text{ gallon}}{35 \text{ highway miles}} = 460.7 \text{ gallons}$$

Hybrid:

$$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{35 \text{ city miles}} + 3000 \text{ highway miles} \cdot \frac{1 \text{ gallon}}{40 \text{ highway miles}} = 332.1 \text{ gallons}$$

If gas in your area averages about $3.50 per gallon, we can use that to find the running cost:

Sonata: $460.7 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1612.45$

Hybrid: $332.1 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1162.35$

The hybrid will save $450.10 a year. The gas costs for the hybrid are about $\frac{\$450.10}{\$1612.45} = 0.279 = 27.9\%$ lower than the costs for the standard Sonata.

While both the absolute and relative comparisons are useful here, they still make it hard to answer the original question, since "is it worth it" implies there is some tradeoff for the gas savings. Indeed, the hybrid Sonata costs about $25,850, compared to the base model for the regular Sonata, at $20,895.

To better answer the "is it worth it" question, we might explore how long it will take the gas savings to make up for the additional initial cost. The hybrid costs $4965 more. With gas savings of $451.10 a year, it will take about 11 years for the gas savings to make up for the higher initial costs.

We can conclude that if you expect to own the car 11 years, the hybrid is indeed worth it. If you plan to own the car for less than 11 years, it may still be worth it, since the resale value of the hybrid may be higher, or for other non-monetary reasons. This is a case where math can help guide your decision, but it can't make it for you.

Try it Now 5

If traveling from Seattle, WA to Spokane WA for a three-day conference, does it make more sense to drive or fly?

Answer

There is not enough information provided to answer the question, so we will have to make some assumptions, and look up some values.

Assumptions:

a) We own a car. Suppose it gets 24 miles to the gallon. We will only consider gas cost.

b) We will not need to rent a car in Spokane, but will need to get a taxi from the airport to the conference hotel downtown and back.

c) We can get someone to drop us off at the airport, so we don't need to consider airport parking.

d) We will not consider whether we will lose money by having to take time off work to drive.

Values looked up (your values may be different)

a) Flight cost: $\$184$

b) Taxi cost: $\$25$ each way (estimate, according to hotel website)

c) Driving distance: 280 miles each way

d) Gas cost: $\$3.79$ a gallon

Cost for flying: $\$184$ flight cost $+\$50$ in taxi fares $=\$234$.

Cost for driving: 560 miles round trip will require 23.3 gallons of gas, costing $\$88.31$

Based on these assumptions, driving is cheaper. However, our assumption that we only include gas cost may not be a good one. Tax law allows you deduct $\$0.55$ (in 2012) for each mile driven, a value that accounts for gas as well as a portion of the car cost, insurance, maintenance, etc. Based on this number, the cost of driving would be $\$319$

This page titled 1.2: Problem Solving and Estimating is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **1.5: Problem Solving and Estimating** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.
- **1.8: Decimals** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

1.2.1: Exercises

1. Round 24,984.2649 up to the nearest hundredth.
2. Round 24,984.2649 down to the nearest hundredth.
3. Round 24,984.2649 to the nearest hundredth.
4. Round 1268.7269
   a. To the nearest hundredth
   b. To the nearest hundred
   c. To the nearest tenth
5. Assume that you drive 12,000 miles each year and you estimate the average fuel cost to be $4.28 per gallon. Your hybrid car will travel 430 miles on an 8-gallon tank of gas. Rounded to the nearest $100, estimate the annual fuel cost to drive this car.
6. If the pumps about 2.5 fluid ounces of blood every beat, how many gallons of blood does it pump in a day?
7. A car is driving at 100 kilometers per hour. How far does it travel in 2 seconds?
8. A crepe recipe calls for 2 eggs, 1 cup of flour, and 1 cup of milk. How much flour would you need if you use 5 eggs?
9. Four 3-megawatt wind turbines can supply enough electricity to power 3000 homes. How many turbines would be required to power 55,000 homes?
10. A highway had a landslide, where 3,000 cubic yards of material fell on the road, requiring 200 dump truck loads to clear. On another highway, a slide left 40,000 cubic yards on the road. How many dump truck loads would be needed to clear this slide?
11. A recipe for sabayon calls for 2 egg yolks, 3 tablespoons of sugar, and $\frac{1}{4}$ cup of white wine. After cracking the eggs, you start measuring the sugar, but accidentally put in 4 tablespoons of sugar. How can you compensate?
12. During the landing of the Mars Science Laboratory *Curiosity*, it was reported that the signal from the rover would take 14 minutes to reach earth. Radio signals travel at the speed of light, about 186,000 miles per second. How far was Mars from Earth when *Curiosity* landed?
13. It is estimated that a driver takes, on average, 1.5 seconds from seeing an obstacle to reacting by applying the brake or swerving. How far will a car traveling at 60 miles per hour travel (in feet) before the driver reacts to an obstacle?
14. The flash of lightning travels at the speed of light, which is about 186,000 miles per second. The sound of lightning (thunder) travels at the speed of sound, which is about 750 miles per hour.
    a. If you see a flash of lightning, then hear the thunder 4 seconds later, how far away is the lightning?
    b. Now let's generalize that result. Suppose it takes *n* seconds to hear the thunder after a flash of lightning. How far away is the lightning, in terms of n?
15. Sound travels about 750 miles per hour. If you stand in a parking lot near a building and sound a horn, you will hear an echo.
    a. Suppose it takes about $\frac{1}{2}$ a second to hear the echo. How far away is the building?
    b. Now let's generalize that result. Suppose it takes n seconds to hear the echo. How far away is the building, in terms of n?
16. Soda is often sold in 20 ounce bottles. The nutrition label for one of these bottles is shown below. A packet of sugar (the kind they have at restaurants for your coffee or tea) typically contain 4 grams of sugar in the U.S. Drinking a 20 oz soda is equivalent to eating how many packets of sugar?

Nutrition Facts

Serving Size: 8 fl oz (240 mL)
Servings Per Container: about 2.5

Amount Per Serving	
Calories 110	
	% Daily Value*
Total Fat 0g	0%
Sodium 70mg	3%
Total Carbohydrate 31g	10%
Sugars 30g	
Protein 0g	

17. The grocery store has bulk pecans on sale, which is great since you're planning on making 10 pecan pies for a wedding. Your recipe calls for 1¾ cups pecans per pie. However, in the bulk section there's only a scale available, not a measuring cup. You run over to the baking aisle and find a bag of pecans, and look at the nutrition label to gather some info. How many pounds of pecans should you buy?

Nutrition Facts

Serving Size: 1 cup, halves (99 g)
Servings per Container: about 2

Amount Per Serving	
Calories 684	Calories from Fat 596
	% Daily Value*
Total Fat 71g	110%
Saturated Fat 6g	31%
Trans Fat	
Cholesterol 0mg	0%

18. In the next 4 questions, estimate the values by making reasonable approximations for unknown values, or by doing some research to find reasonable values.
    a. Estimate how many gallons of water you drink in a year.
    b. Estimate how many times you blink in a day.
    c. How much does the water in a 6-person hot tub weigh?
    d. How many gallons of paint would be needed to paint a two-story house 40 ft long and 30 ft wide?

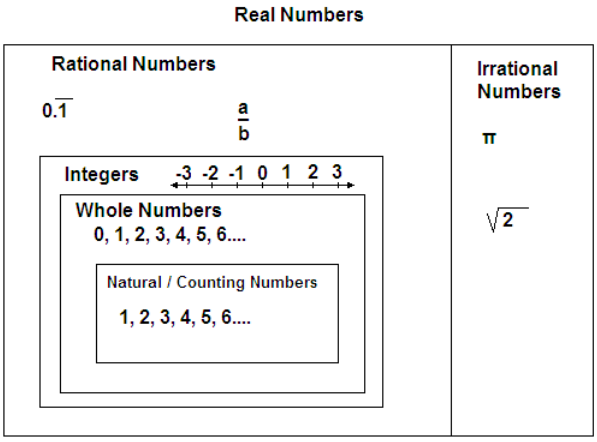
This page titled 1.2.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **5.4: Exercises** is licensed CC BY-NC-SA 4.0.
- **1.6: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

1.3: Real Numbers

This next section discusses the real number system. We will learn how to work with numbers in each of the subsets of the real numbers.

The diagram below shows the names and some representations of the types of numbers in each subset.



"File:The Real Number System.png" by Anzarta is licensed under CC BY-SA 3.0

This page titled 1.3: Real Numbers is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.3.1: Integers

1.3.1 Learning Objectives

- Recognize a real number
- Identify particular subsets of the real numbers
- Add, subtract, multiply, and divide signed numbers

Real Numbers

Real Number Line

The study of mathematics requires the use of several collections of numbers. The **real number line** allows us to visually display (graph) the numbers in which we are interested.

A line is composed of infinitely many points. To each point we can associate a unique number, and with each number, we can associate a particular point.

Definition: Real Number

A **real number** is any number that is the coordinate of a point on the real number line.

Definition: Coordinate

The *number* associated with a point on the number line is called the **coordinate** of the point.

Definition: Graph

The *point* on a number line that is associated with a particular number is called the **graph** of that number.

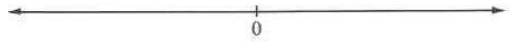
Constructing a Real Number Line

We construct a real number line as follows:

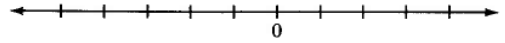
Draw a horizontal line.

Origin

Choose any point on the line and label it 0. This point is called the **origin**.

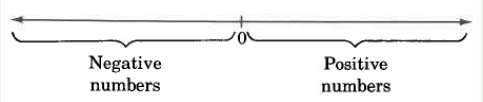


Choose a convenient length. Starting at 0, mark this length off repeatedly in both directions. When drawing the number line by hand, be careful to have the lengths look like they are the same.



Definition: Positive Number, Negative Numbers

Real numbers whose graphs are to the right of 0 are called *positive real numbers*, or more simply, **positive numbers**. Real numbers whose graphs appear to the left of 0 are called *negative real numbers*, or more simply, **negative numbers**.



The number 0 is neither positive nor negative.

THE NOTATION OF SIGNED NUMBERS

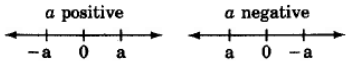
+ and − Notation
A number is denoted as **positive** if it is directly preceded by a plus sign or no sign at all.
A number is denoted as **negative** if it is directly preceded by a minus sign.

Opposites

Opposites
On the number line, each real number, other than zero, has an image on the opposite side of 0. For this reason, we say that each real number has an opposite. **Opposites** are the same distance from zero but have opposite signs.

The opposite of a real number is denoted by placing a negative sign directly in front of the number. Thus, if a is any real number, then $-a$ is its opposite.

If a is any real number, $-a$ is opposite a on the number line.



Reading Signed Numbers

The plus and minus signs now have *more than one* meaning*:*

The plus sign can denote the operation of addition *or* a positive number.

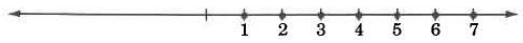
The minus sign can denote the operation of subtraction, a negative number, *or* the opposite of a number.

To avoid any confusion between "sign" and "operation," it is preferable to read the sign of a number as "positive," "negative," or "opposite of." When "+" is used as an operation sign, it is read as "plus." When "-" is used as an operation sign, it is read as "minus."

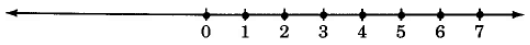
Subsets of Real Numbers

The set of real numbers has many subsets. Some of the subsets that are of interest in this course are listed below along with their notations and graphs.

Natural Numbers, Counting Numbers
The **natural** or **counting numbers** (N): 1, 2, 3, 4, . . . Read “and so on.”

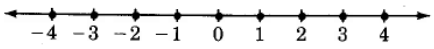


Whole Numbers
The **whole numbers** (W): 0, 1, 2, 3, 4, . . .



Notice that every natural number is a whole number.

Integers
The **integers** (Z): . . . -3, -2, -1, 0, 1, 2, 3, . . .



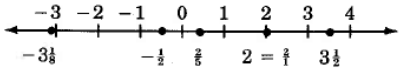
Notice that every whole number is an integer.

Rational Numbers (Fractions)
The **rational numbers** (Q): Rational numbers are sometimes called **fractions**. They are numbers that can be written as the *quotient* of two integers. They have decimal representations that either terminate or do not terminate but contain a repeating block

of digits. Some examples are

$$\underbrace{\frac{-3}{4} = -0.75}_{textTerminating} \quad \underbrace{8\frac{11}{27} = 8.407407407\ldots}_{\text{Nonterminating, but repeating.}}$$

Some rational numbers are graphed below.



Notice that *every integer is a rational number*.

There are still a great many points on the number line that have not yet been assigned a type of number. We will not examine all of these other types of numbers in this text. They are called irrational numbers. An example of these numbers is the number π, whose decimal representation does not terminate nor contain a repeating block of digits. An approximation for π is 3.14.

Absolute Value

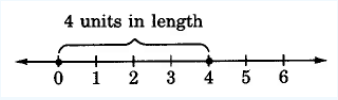
Absolute Value-Geometric Approach
Geometric definition of absolute value:
The **absolute value** of a number a, denoted $|a|$, is the distance from a to 0 on the number line.

Absolute value answers the question of "how far," and not "which way." The phrase "how far" implies "length" and *length is always a nonnegative quantity*. Thus, the absolute value of a number is a nonnegative number.

Example 1

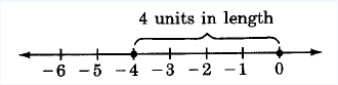
Determine each value.

$|4| = 4$



Example 2

$|-4| = 4$



Example 3

$|0| = 0$

Example 4

$-|5| = -5$. The quantity on the left side of the equal sign is read as "the opposite of the absolute value of 5." The absolute value of 5 is 5. Hence, the opposite of the absolute value of 5 is -5.

Example 5

$-|-3| = -3$. The quantity on the left side of the equal sign is read as "the opposite of the absolute value of -3." The absolute value of -3 is 3. Hence, the opposite of the absolute value of -3 is $-(3) = -3$.

Try It Now 1

By reasoning geometrically, determine the absolute value.

$|7|$

Answer

7

Addition of Numbers with Like Signs

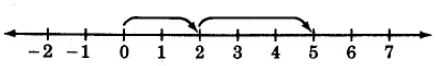
The addition of the *two positive numbers* 2 and 3 is performed on the number line as follows.

Begin at 0, the origin.

Since 2 is positive, move 2 units to the right.

Since 3 is positive, move 3 more units to the right.

We are now located at 5.



Thus, $2+3=5$.

Summarizing, we have

$(2 \text{ positive units})+(3 \text{ positive units})=(5 \text{ positive units})$

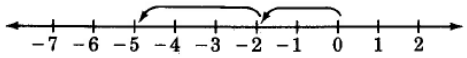
The addition of the *two negative numbers* -2 and -3 is performed on the number line as follows.

Begin at 0, the origin.

Since -2 is negative, move 2 units to the left.

Since -3 is negative, move 3 more units to the left.

We are now located at -5.



Thus, $(-2)+(-3)=-5$

Summarizing, we have

$(2 \text{ negative units})+(3 \text{ negative units})=(5 \text{ negative units})$

Observing these two examples, we can suggest these relationships:

$(\text{postitive number})+(\text{positive number})=(\text{positive number})$

$(\text{negative number})+(\text{negative number})=(\text{negative number})$

Adding Numbers with the Same Sign

Addition of numbers with like sign:

To add two real numbers that have the *same* sign, add the absolute values of the numbers and associate with the sum the common sign.

Example 6

Find the sums.

3 + 7

Solution

$|3| = 3$
$|7| = 7$ } Add these absolute values.

3 + 7 = 10

The common sign is “+.”

Thus, $3 + 7 = +10$, or $3 + 7 = 10$.

Example 7

Find the sums.

(-4) + (-9)

Solution

$|-4| = 4$
$|-9| = 9$ } Add these absolute values.

4 + 9 = 13

The common sign is “-.”

Thus, $(-4) + (-9) = -13$.

Addition With Zero

Addition with Zero
Notice that

$(0) + (\text{a positive number}) = (\text{that same positive number})$.
$(0) + (\text{a negative number}) = (\text{that same negative number})$.

Definition: The Additive Identity Is Zero

Since adding zero to a real number leaves that number unchanged, zero is called the **additive identity**.

Addition of Numbers with Unlike Signs

The addition $2 + (-6)$,

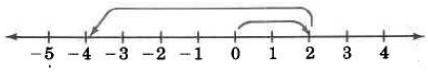
two numbers with unlike signs, can also be illustrated using the number line.

Begin at 0, the origin.

Since 2 is positive, move 2 units to the right.

Since -6 is negative, move, from 2, 6 units to the left.

We are now located at -4.



We can suggest a rule for adding two numbers that have *unlike signs* by noting that if the signs are disregarded, 4 can be obtained by subtracting 2 from 6. But 2 and 6 are precisely the absolute values of 2 and -6. Also, notice that the sign of the number with the larger absolute value is negative and that the sign of the resulting sum is negative.

Adding Numbers with Unlike Signs
Addition of numbers with unlike signs: To add two real numbers that have *unlike signs*, subtract the smaller absolute value from the larger absolute value and associate with this difference the sign of the number with the larger absolute value.

Example 8

Find the following sums.

7 + (-2)

Solution

$$\underbrace{|7| = 7}_{\text{Larger absolute value. Sign is positive.}} \qquad \underbrace{|-2| = 2}_{\text{Larger absolute value. Sign is positive.}}$$

Subtract absolute values: 7 - 2 = 5.

Attach the proper sign: "+."

Thus, $7 + (-2) = +5$ or $7 + (-2) = 5$.

Example 9

3 + (-11)

Solution

$$\underbrace{|3| = 3}_{\text{Smaller absolute value}} \qquad \underbrace{|-11| = 11}_{\text{Larger absolute value. Sign is negative.}}$$

Subtract absolute values: 11 - 3 = 8.

Attach the proper sign: "-."

Thus, $3 + (-11) = -8$.

Example 10

The morning temperature on a winter's day in Lake Tahoe was -12 degrees. The afternoon temperature was 25 degrees warmer. What was the afternoon temperature?

Solution

We need to find $-12 + 25$.

$$\underbrace{|-12| = 12}_{\text{Smaller absolute value}} \qquad \underbrace{|25| = 25}_{\text{Larger absolute value. Sign is positive.}}$$

Subtract absolute values: 25 - 12 = 16.

Attach the proper sign: "+."

Thus, $-12 + 25 = 13$.

Calculators

Calculators having the $\boxed{+/-}$ key can be used for finding sums of signed numbers.

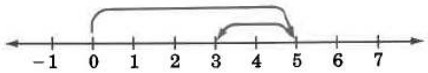
Example 11

Use a calculator to find the sum of -147 and 84.

		Display Reads	
Type	147	147	
Press	$+/-$	-147	This key changes the sign of the number, it is different than -.
Press	$+$	-147	
Type	84	84	
Press	$=$	-63	

Definition of Subtraction

We know from experience with arithmetic that the subtraction 5 - 2 produces 3, that is 5 - 2 = 3. We can suggest a rule for subtracting signed numbers by illustrating this process on the number line.



Begin at 0, the origin.

Since 5 is positive, move 5 units to the right.

Then, move *2 units to the left* to get to 6. (This reminds us of addition with a negative number.)

From this illustration we can see that 5 - 2 is the same as 5 + (-2). This leads us directly to the definition of subtraction.

Definition of Subtraction
If a and b are real numbers, $a-b$ is the same as $a+(-b)$, where $-b$ is the opposite of b.

The Process of Subtraction

From this definition, we suggest the following rule for subtracting signed numbers.

Subtraction of Signed Numbers
To perform the subtraction $a-b$, add the opposite of b to a, that is, change the sign of b and add.

Example 12

Perform the indicated subtractions.

1. 5 - 3 = 5 + (-3) = 2
2. 4 - 9 = 4 + (-9) = -5
3. -4 - 6 = -4 + (-6) = -10
4. -3 - (-12) = -3 + 12 = 9

Example 13

The high temperature today in Lake Tahoe was 26°F. The low temperature tonight is expected to be -7°F. How many degrees is the temperature expected to drop?

Solution

We need to find the difference between 26 and -7.

26 - (-7) = 26 + 7 = 33

Thus, the expected temperature drop is 33°F.

Example 14

$$\begin{aligned} -6-(-5)-10 &= -6+5+(-10) \\ &= (-6+5)+(-10) \\ &= -1+(-10) \\ &= -11 \end{aligned}$$

Try It Now 2

Perform the indicated subtractions.

9 − 6

Answer

3

Calculators

Calculators can be used for subtraction of signed numbers. The most efficient calculators are those with a [+/−] key.

Example 15

Use a calculator to find the difference.

-156 - (-211)

Solution

		Display Reads
Type	156	156
Press	+/−	−156
Type	−	−156
Press	211	211
Type	+/−	−211
Press	=	−55

Multiplication of Signed Numbers

Let us consider first, the product of two positive numbers. Multiply: $3 \cdot 5$.

$3 \cdot 5$ means $5+5+5=15$

This suggests that

$(\text{positive number}) \cdot (\text{positive number}) = (\text{positive number})$

More briefly,

(+) (+) = (+)

Now consider the product of a positive number and a negative number. Multiply: (3)(-5)

(3)(-5) means (-5) + (-5) + (-5) = -15

This suggests that

$(\text{positive number}) \cdot (\text{negative number}) = (\text{negative number})$

More briefly,

(+) (-) = (-)

By the commutative property of multiplication, we get

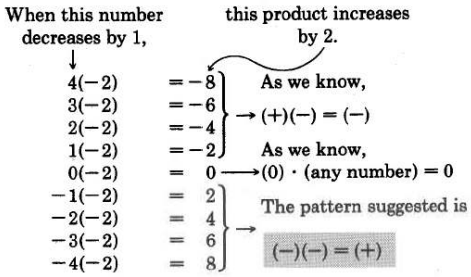
$$(\text{negative number}) \cdot (\text{positive number}) = (\text{negative number})$$

More briefly,

(-) (+) = (-)

The sign of the product of two negative numbers can be suggested after observing the following illustration.

Multiply -2 by, respectively, 4, 3, 2, 1, 0, -1, -2, -3, -4.



We have the following rules for multiplying signed numbers.

Rules for Multiplying Signed Numbers

Multiplying signed numbers:

1. To multiply two real numbers that have the *same sign*, multiply their absolute values. The product is positive.
   (+) (+) = (+)
   (-) (-) = (+)
2. To multiply two real numbers that have *opposite signs*, multiply their absolute values. The product is negative.
   (+) (-) = (-)
   (-) (+) = (-)

Example 16

Find the following products.

$8 \cdot 6$

Solution

$$\left.\begin{aligned} |8| &= 8 \\ |6| &= 6 \end{aligned}\right\} \text{Multiply these absolute values.}$$

$8 \cdot 6 = 48$

Since the numbers have the same sign, the product is positive.

Thus, $8 \cdot 6 = +48$, or $8 \cdot 6 = 48$.

Example 17

(-8)(-6)

Solution

$$\left.\begin{aligned} |-8| &= 8 \\ |-6| &= 6 \end{aligned}\right\} \text{Multiply these absolute values.}$$

$8 \cdot 6 = 48$

Since the numbers have the same sign, the product is positive.

Thus, $(-8)(-6) = +48$, or $(-8)(-6) = 48$.

Example 18

(-4)(7)

Solution

$$\left.\begin{aligned} |-4| &= 4 \\ |7| &= 7 \end{aligned}\right\}$$ Multiply these absolute values.

$4 \cdot 7 = 28$

Since the numbers have opposite signs, the product is negative.

Thus, (-4)(7) = -28.

Example 19

6(-3)

Solution

$$\left.\begin{aligned} |6| &= 6 \\ |-3| &= 3 \end{aligned}\right\}$$ Multiply these absolute values.

$6 \cdot 3 = 18$

Since the numbers have opposite signs, the product is negative.

Thus, 6(-3) = -18.

Division of Signed Numbers

To determine the signs in a division problem, recall that

$\frac{12}{3} = 4$ since $12 = 3 \cdot 4$

This suggests that

$\frac{(+)}{(+)} = (+)$

$\frac{(+)}{(+)} = (+)$ since (+) = (+) (+)

What is $\frac{12}{-3}$?

12 = (-3)(-4) suggests that $\frac{12}{-3} = -4$. That is,

$\frac{(+)}{(-)} = (-)$

(+) = (-) (-) suggests that $\frac{(+)}{(-)} = (-)$

What is $\frac{-12}{3}$?

-12 = (3)(-4) suggests that $\frac{-12}{3} = -4$. That is,

$\frac{(-)}{(+)} = (-)$

(-) = (+) (-) suggests that $\frac{(-)}{(+)} = (-)$

What is $\frac{-12}{-3}$?

-12 = (-3)(4) suggests that $\frac{-12}{-3} = 4$. That is,

$$\frac{(-)}{(-)} = (+)$$

(-) = (-)(+) suggests that $\frac{(-)}{(-)} = (+)$

We have the following rules for dividing signed numbers.

Rules for Dividing Signed Numbers
Dividing signed numbers:

1. To divide two real numbers that have the *same sign*, divide their absolute values. The quotient is positive.
   $\frac{(+)}{(+)} = (+) \frac{(-)}{(-)} = (+)$
2. To divide two real numbers that have *opposite signs*, divide their absolute values. The quotient is negative.
   $\frac{(-)}{(+)} = (-) \frac{(+)}{(-)} = (-)$

Example 20

Find the following quotients.

$\frac{-10}{2}$

Solution

$$\left.\begin{array}{rcl} |-10| & = & 10 \\ |2| & = & 2 \end{array}\right\} \text{Divide these absolute values.}$$

$\frac{10}{2} = 5$

Since the numbers have opposite signs, the quotient is negative.

Thus $\frac{-10}{2} = -5$.

Example 21

$\frac{-35}{-7}$

Solution

$$\left.\begin{array}{rcl} |-35| & = & 35 \\ |-7| & = & 7 \end{array}\right\} \text{Divide these absolute values.}$$

$\frac{35}{7} = 5$

Since the numbers have the same signs, the quotient is positive.

Thus $\frac{-35}{-7} = 5$.

Example 22

$\frac{18}{-9}$

Solution

$$\left.\begin{aligned} |18| &= 18 \\ |-9| &= 9 \end{aligned}\right\}$$ Divide these absolute values.

$\frac{18}{9} = 2$

Since the numbers have opposite signs, the quotient is negative.

Thus $\frac{18}{-9} = -2$.

Calculators

Calculators with the [+/−] key can be used for multiplying and dividing signed numbers.

Example 23

Use a calculator to find each quotient or product.

$(-186) \cdot (-43)$

Solution

Since this product involves a $(\text{negative}) \cdot (\text{negative})$, we know the result should be a positive number. We'll illustrate this on the calculator.

		Display Reads
Type	186	186
Press	$+/-$	-186
Press	$\times$	-186
Type	43	43
Press	$+/-$	-43
Press	$=$	7998

Thus, $(-186) \cdot (-43) = 7,998$

Example 24

$\frac{158.64}{-54.3}$. Round to one decimal place.

Solution

Since this product involves a $(\text{negative}) \cdot (\text{negative})$, we know the result should be a positive number. We'll illustrate this on the calculator.

		Display Reads
Type	158.64	158.64
Press	$\div$	158.64
Type	54.3	54.3
Press	$+/-$	-54.3
Press	$=$	-2.921546961

Rounding to one decimal place we get -2.9.

This page titled 1.3.1: Integers is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.3.1.1: Exercises

Section 1.3.1.1 Exercises

1. Find a whole number that is not a counting number.
2. Find an integer that is not a whole number.
3. Find a whole number that is not an integer.
4. Find an integer that is not negative.
5. Find a rational number that is not an integer.
6. Find a whole number that is not rational.
7. Identify which sets of numbers include the following numbers: counting numbers, whole numbers, integers, or none of the above
   a. $\frac{5}{8}$
   b. 0
   c. $\sqrt{2}$
   d. -4
   e. $\sqrt{-5}$
8. *True or False*: I can buy π pencils.
9. Between what two integers is each number below?
   a. $\sqrt{20}$
   b. -10
   c. $\sqrt{-15}$
10. Approximate the value of $\sqrt{5}$ by rounding to the nearest hundredth.
11. Approximate the value of $-\sqrt{12}$ by rounding to one decimal place.
12. Determine which of these statements are true, if any.
    a. $-3 > 0$
    b. $-(-3) > 0$
    c. $|-3| > 0$
    d. $-|3| > 0$
    e. $-|-3| > 0$
    f. $|-3| > -|-3|$
13. Evaluate
    a. -2 + -5
    b. -7 + 4
    c. 5 + -11
    d. 5 – 14
    e. -2 – 8
    f. 8 – (-8)
    g. -7 – (-2)
    h. (-44)(-2)
    i. 6 (-3)
    j. (-6.239)(-100)
    k. (-4)(6.25)
    l. $\frac{-3}{8} - \frac{5}{9}$
    m. -32 ÷ 8
    n. 12 ÷ (-4)
    o. $\frac{1}{3} \div -3$
    p. $-3 \div \frac{1}{3}$
    q. 0.3(-0.4)

r. 3.2(-5)

s. -2.4 ÷ (-6)

t. -4 $\frac{2}{3} \div \frac{-7}{12}$

14. Without actually doing the calculations, determine the sign of the product.

a. (-5)(-3)(6)(-7)

b. $(-7)^4$

c. (6)(-4)(7)(-5)(3)

d. (-5)(-6)(0)(-3)(4)(-7)(-2)

This page titled 1.3.1.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.3.2: Rational and Irrational Numbers

1.3.2 Learning Objectives

- Order real numbers using inequality symbols
- Add, subtract, multiply and divide fractions
- Name and write decimals
- Add, subtract, multiply, and divide decimals
- Add, subtract, multiply, and divide certain irrational numbers

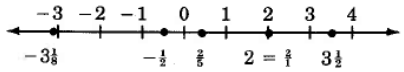
Rational Numbers

Rational numbers are a subset of the real numbers.

Rational numbers are sometimes called **fractions**. They are numbers that can be written as the *quotient* of two integers. They have decimal representations that either terminate or do not terminate but contain a repeating block of digits. Some examples are

$\frac{3}{4} = 0.75$ or $8\frac{11}{7} = 8.407407407\ldots$

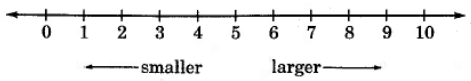
Some rational numbers are graphed below.



Notice that *every integer is a rational number*.

Ordering Real Numbers

Our number system is called an **ordered number system** because the numbers in the system can be placed in order from smaller to larger. This is easily seen on the number line.



On the number line, a number that appears to the right of another number is larger than that other number. For example, 5 is greater than 2 because 5 is located to the right of 2 on the number line. We may also say that 2 is less than 5.

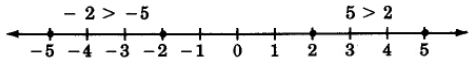
To make the inequality phrases "greater than" and "less than" more brief, mathematicians represent them with the symbols > and <, respectively.

Symbols for Greater Than > and Less Than <
> represents the phrase "greater than."
< represents the phrase "less than."

5 > 2 represents "5 is greater than 2."

2 < 5 represents "2 is less than 5."

Ordering Real Numbers
A real number b is said to be *greater* than a real number a, denoted $b > a$, if b is to the right of a on the number line. Thus, as we would expect, $5 > 2$ since 5 is to the right of 2 on the number line. Also, $-2 > -5$ since -2 is to the right of -5 on the number line.



If we let a and b represent two numbers, then a and b are related in exactly one of three ways: Either

Equality Symbol
$a = b$ a and b are equal (8 = 8)

Inequality Symbols

$a > b$	a is greater than b	$(8 > 5)$
$a < b$	a is less than b	$(5 < 8)$
	Some variations of these symbols are	
$a \neq b$	a is not equal to b	$(8 \neq 5)$
$a \geq b$	a is greater than or equal to b	$(a \geq 8)$
$a \leq b$	a is less than or equal to b	$(a \leq 8)$

Comparing Fractions

Recall that the fraction $\frac{4}{5}$ indicates that we have 4 of 5 parts of some whole quantity, and the fraction $\frac{3}{5}$ indicates that we have 3 of 5 parts. Since 4 of 5 parts is more than 3 of 5 parts, $\frac{4}{5}$ is greater than $\frac{3}{5}$; that is,

$$\frac{4}{5} > \frac{3}{5}$$

We have just observed that when two fractions have the same denominator, we can determine which is larger by comparing the numerators.

Comparing Fractions

If two fractions have the same denominators, the fraction with the larger numerator is the larger fraction.

Thus, to compare the sizes of two or more fractions, we need only convert each of them to equivalent fractions that have a common denominator. We then compare the numerators. It is convenient if the common denominator is the LCD. The fraction with the larger numerator is the larger fraction.

Equivalent Fractions

Fundamental Principle of Rational Numbers (Fractions)

If a fraction is multiplied by the same value to the numerator and the denominator, the fraction that results is called and equivalent fraction as long as the mulitplied number is not zero.

Example 1

Produce an equivalent fraction for $\frac{2}{3}$.

Solution

Choose any value except zero. If we choose 3, then we multiply both the numerator and denominator by that number.

$\frac{2 \times 3}{3 \times 3} = \frac{6}{9}$.

Therefore $\frac{6}{9}$ is an equivalent fraction to $\frac{2}{3}$.

Least Common Multiple (LCM) and Least Common Denominator (LCD)

The least common multiple (LCM) of a collection of numbers is the smallest value that all the numbers in a collection can produce with multiplication. If these numbers are used as denominators of fractions, we call the least common multiple, the least common denominator (LCD).

Example 2

Compare $\frac{8}{9}$ and $\frac{14}{15}$. Find the LCD. Convert each fraction to an equivalent fraction with the LCD as the denominator.

Solution

$$\left\{ \begin{array}{rcl} 9 & = & 3^2 \\ 15 & = & 3 \cdot 5 \end{array} \right\} \text{ The LCD } = 3^2 \cdot 5 = 9 \cdot 5 = 45$$

$\frac{8}{9} = \frac{8 \cdot 5}{45} = \frac{40}{45}$

$\frac{14}{15} = \frac{14 \cdot 3}{45} = \frac{42}{45}$

Since $40 < 42$.

$\frac{40}{45} < \frac{42}{45}$

Thus $\frac{8}{9} < \frac{14}{15}$.

Example 3

Write $\frac{5}{6}$, $\frac{7}{10}$, and $\frac{13}{15}$ in order from smallest to largest.

Find the LCD of the three fractions. Convert each fraction to an equivalent fraction with the LCD as the denominator.

Solution

$\left\{ \begin{array}{rcl} 6 & = & 2 \cdot 3 \\ 10 & = & 2 \cdot 5 \\ 15 & = & 3 \cdot 5 \end{array} \right\}$ The LCD $= 2 \cdot 3 \cdot 5 = 30$

$\frac{5}{6} = \frac{5 \cdot 5}{30} = \frac{25}{30}$

$\frac{7}{10} = \frac{7 \cdot 3}{30} = \frac{21}{30}$

$\frac{13}{15} = \frac{13 \cdot 2}{30} = \frac{26}{30}$

Since $21 < 25 < 26$.

$\frac{21}{30} < \frac{25}{30} < \frac{26}{30}$

$\frac{7}{10} < \frac{5}{6} < \frac{13}{15}$

Writing these numbers in order from smallest to largest, we get $\frac{7}{10}$, $\frac{5}{6}$, $\frac{13}{15}$.

Example 4

Compare $8\frac{6}{7}$ and $6\frac{3}{4}$.

Solution

To compare mixed numbers that have different whole number parts, we need only compare whole number parts. Since $6 < 8$,

$6\frac{3}{4} < 8\frac{6}{7}$

Example 5

Compare $4\frac{5}{8}$ and $4\frac{7}{12}$

Solution

To compare mixed numbers that have the same whole number parts, we need only compare fractional parts.

$\left\{ \begin{array}{rcl} 8 & = & 2 \cdot 3 \\ 12 & = & 2^2 \cdot 3 \end{array} \right\}$ The LCD $= 2^3 \cdot 3 = 8 \cdot 3 = 24$

$\frac{5}{8} = \frac{5 \cdot 3}{24} = \frac{15}{24}$

$\frac{7}{12} = \frac{7 \cdot 2}{24} = \frac{14}{24}$

Since $14 < 15$.

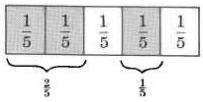
$\frac{14}{24} < \frac{15}{24}$

$\frac{7}{12} < \frac{5}{8}$

Hence, $4\frac{7}{12} < 4\frac{5}{8}$

Addition and Subtraction of Fractions With Like Denominators

Let's examine the following diagram.



2 one-fifths and 1 one fifth is shaded.

It is shown in the shaded regions of the diagram that

(2 one-fifths) + (1 one-fifth) = (3 one-fifths)

That is,

$\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$

From this observation, we can suggest the following rule.

Method of Adding Fractions Having Like Denominators

To add two or more fractions that have the same denominators, add the numerators and place the resulting sum over the common denominator. Simplify, if necessary.

Example 6

Find the following sums.

$\frac{3}{7} + \frac{2}{7}$. The denominators are the same. Add the numerators and place that sum over 7.

Solution

$\frac{3}{7} + \frac{2}{7} = \frac{3+2}{7} = \frac{5}{7}$

Example 7

$\frac{1}{8} + \frac{3}{8}$. The denominators are the same. Add the numerators and place the sum over 8. Simplify.

Solution

$\frac{1}{8} + \frac{3}{8} = \frac{1+3}{8} = \frac{4}{8} = \frac{1}{2}$

Example 8

To see what happens if we *mistakenly add the denominators* as well as the numerators, let's add

$$\frac{1}{2}+\frac{1}{2}$$

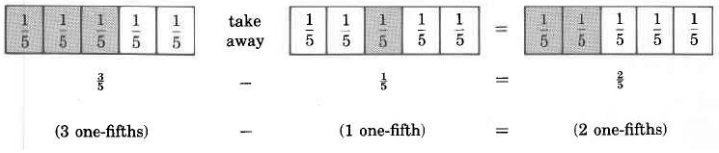
Adding the numerators and *mistakenly* adding the denominators produces

$$\frac{1}{2}+\frac{1}{2}=\frac{1+1}{2+2}=\frac{2}{4}=\frac{1}{2}$$

This means that two $\frac{1}{2}$'s is the same as one $\frac{1}{2}$. Preposterous! **We do not add denominators**.

Subtraction of Fractions With Like Denominators

We can picture the concept of subtraction of fractions in much the same way we pictured addition.



From this observation, we can suggest the following rule for subtracting fractions having like denominators:

Subtraction of Fractions with Like Denominators
To subtract two fractions that have like denominators, subtract the numerators and place the resulting difference over the common denominator. Simplify, if possible.

Example 9

Find the following differences.

$\frac{3}{5}-\frac{1}{5}$. The denominators are the same. Subtract the numerators. Place the difference over 5.

Solution

$$\frac{3}{5}-\frac{1}{5}=\frac{3-1}{5}=\frac{2}{5}$$

Example 10

$\frac{8}{6}-\frac{2}{6}$. The denominators are the same. Subtract the numerators. Place the difference over 6.

Solution

$$\frac{8}{6}-\frac{2}{6}=\frac{8-2}{6}=\frac{6}{6}=1$$

Example 11

To see what happens if we *mistakenly* subtract the denominators, let's consider

$$\frac{7}{15}-\frac{4}{15}=\frac{7-4}{15-15}=\frac{3}{0}$$

We get division by zero, which is undefined. **We do not subtract denominators.**

A Basic Rule

There is a basic rule that must be followed when adding or subtracting fractions.

A Basic Rule
Fractions can only be added or subtracted conveniently if they have like denominators.

To see why this rule makes sense, let's consider the problem of adding a quarter and a dime.

$1 \text{ quarter} + 1 \text{ dime} = 35 \text{ cents}$

Now,

$$\left\{ \begin{array}{l} 1 \text{ quarter } = \dfrac{25}{100} \\ 1 \text{ dime } = \dfrac{10}{100} \end{array} \right\} \text{ same denominations}$$

$35,\ ¢ = \dfrac{35}{100}$

$$\frac{25}{100} + \frac{10}{100} = \frac{25+10}{100} = \frac{35}{100}$$

In order to combine a quarter and a dime to produce 35¢, we convert them to quantities of the same denomination.

Same denomination $\rightarrow$ same denominator

Addition and Subtraction of Fractions with Unlike Denominators

Method of Adding or Subtracting Fractions with Unlike Denominators

To add or subtract fractions having unlike denominators, convert each fraction to an equivalent fraction having as a denominator the least common denominator (LCD) of the original denominators.

Example 12

Find the following sums and differences.

$\frac{1}{6} + \frac{3}{4}$. The denominators are not the same. Find the LCD of 6 and 4.

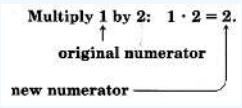
$$\left\{ \begin{array}{l} 6 = 2 \cdot 3 \\ 4 = 2^2 \end{array} \right\} \text{ The LCD} = 2^2 \cdot 3 = 4 \cdot 3 = 12$$

Write each of the original fractions as a new, equivalent fraction having the common denominator 12.

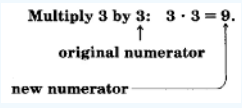
$$\frac{1}{6} + \frac{3}{4} = \frac{}{12} + \frac{}{12}$$

To find a new numerator, we divide the original denominator into the LCD. Since the original denominator is being multiplied by this quotient, we must multiply the original numerator by this quotient.

$12 \div 6 = 2$



$12 \div 4 = 3$



$$\begin{aligned} \frac{1}{6} + \frac{3}{4} &= \frac{1 \cdot 2}{12} + \frac{3 \cdot 3}{12} \\ &= \frac{2}{12} + \frac{9}{12} && \text{Now the denominators are the same.} \\ &= \frac{2+9}{12} && \text{Add the numerators and place the sum over the common denominator.} \\ &= \frac{11}{12} \end{aligned}$$

Example 13

$\frac{1}{2}+\frac{2}{3}$. The denominators are not the same. Find the LCD of 2 and 3.

$\text{LCD}=2\cdot 3=6$

Write each of the original fractions as a new, equivalent fraction having the common denominator 6.

$\frac{1}{2}+\frac{2}{3}=\frac{}{6}+\frac{}{6}$

To find a new numerator, we divide the original denominator into the LCD. Since the original denominator is being multiplied by this quotient, we must multiply the original numerator by this quotient.

$6\div 2=3$ Multiply the numerator 1 by 3.
$6\div 2=3$ Multiply the numerator 2 by 2.

$$\begin{aligned}\frac{1}{2}+\frac{2}{3} &= \frac{1\cdot 3}{6}+\frac{2\cdot 3}{6}\\ &= \frac{3}{6}+\frac{4}{6}\\ &= \frac{3+4}{6}\\ &= \frac{7}{6}\text{ or }1\frac{1}{6}\end{aligned}$$

Example 14

$\frac{5}{9}-\frac{5}{12}$. The denominators are not the same. Find the LCD of 9 and 12.

$$\left.\begin{cases}9=3\cdot 3=3^2\\ 12=2\cdot 6=2\cdot 2\cdot 3=2^2\cdot 3\end{cases}\right\}\text{LCD}=2^2\cdot 3^2=4\cdot 9=36$$

$\frac{5}{9}-\frac{5}{12}=\frac{}{36}-\frac{}{36}$

$36\div 9=4$ Multiply the numerator 5 by 4.
$36\div 12=3$ Multiply the numerator 5 by 3.

$$\begin{aligned}\frac{5}{9}-\frac{5}{12} &= \frac{5\cdot 4}{36}-\frac{5\cdot 3}{36}\\ &= \frac{20}{36}-\frac{15}{36}\\ &= \frac{20-15}{6}\\ &= \frac{5}{36}\end{aligned}$$

Multiplication of Fractions

When multiplying numbers in fraction form, the numerator and denominator are calculated separately. The numbers in the numerator are multiplied across and the numbers of the denominator are multiplied across. Again, the final answer should be reduced to the simplest form.

Example 15

Multiply the following rational numbers in fraction form.

1. $\frac{2}{9}\times\frac{5}{9}$
2. $\frac{7}{12}\times\frac{15}{16}$

Solution

1. $\frac{2}{9} \times \frac{5}{9}$

$\frac{2}{9} \times \frac{5}{9} = \frac{2 \times 5}{9 \times 9} = \frac{10}{81}$

$\frac{10}{81}$ cannot be simplified, so this is the final answer.

2. $\frac{7}{12} \times \frac{15}{16}$

In multiplication, simplification can also be done before multiplying. Notice in this example, 12 is in the denominator of the first fraction and 15 is in the numerator of the second fraction. Both of these numbers are divisible by 3. Since one is in the numerator and the other is in the denominator, this is the same as dividing by 3 in both places in the final step of the process above. Reduce those numbers then multiply.

$\frac{7}{12} \times \frac{15}{16} = \frac{7}{12 \div 3} \times \frac{15 \div 3}{16} = \frac{7}{4} \times \frac{5}{16} = \frac{7 \times 5}{4 \times 16} = \frac{35}{64}$

$\frac{35}{64}$ cannot be simplified, so this is the final answer.

Division of Fractions

Fractions divided by fractions are typically called complex fractions. The first step to division will be to convert the operation to multiplication. In order to convert the operation to multiplication, the reciprocal of the fraction in the denominator must be used. The numerator fraction will remain the same, switch the operation to multiplication, and then write the reciprocal of the denominator. Once the operation is converted use the above steps for multiplication to determine the final answer. Remember that whole numbers can be written as fractions and be careful not to simplify anything until after the operation is converted.

Example 16

Divide the following rational numbers.

1. $\frac{\frac{5}{14}}{\frac{7}{6}}$
2. $\frac{\frac{2}{9}}{\frac{5}{9}}$
3. $\frac{\frac{2}{9}}{\frac{11}{}}$

Solution

1. $\frac{\frac{5}{14}}{\frac{7}{6}}$

Keep, Switch, Flip first.

$\frac{\frac{5}{14}}{\frac{7}{6}} = \frac{5}{14} \times \frac{6}{7}$

Notice the numerator and denominator have numbers that can be simplified. Both the 14 and 6 are divisible by 2.

$\frac{5}{14} \times \frac{6}{7} = \frac{5}{14 \div 2} \times \frac{6 \div 2}{7} = \frac{5}{7} \times \frac{3}{7} = \frac{5 \times 3}{7 \times 7} = \frac{15}{49}$

$\frac{15}{49}$ cannot be simplified, so this is the final answer.

2. $\frac{\frac{2}{9}}{\frac{5}{9}}$

Keep, Switch, Flip first.

$\frac{\frac{2}{9}}{\frac{5}{9}} = \frac{2}{9} \times \frac{9}{5}$

Notice the numerator and denominator have a common factor of nine nines that can be reduced.

$\frac{2}{9} \times \frac{9}{5} = \frac{2}{9 \div 9} \times \frac{9 \div 9}{5} = \frac{2}{1} \times \frac{1}{5} = \frac{2 \times 1}{1 \times 5} = \frac{2}{5}$

$\frac{2}{5}$ cannot be simplified, so this is the final answer.

3. $\frac{\frac{2}{9}}{11}$

Keep, Switch, Flip first. Since 11 can be written as $\frac{11}{1}$, the reciprocal would be $\frac{1}{11}$.

$\frac{\frac{2}{9}}{11} = \frac{2}{9} \times \frac{1}{11} = \frac{2 \times 1}{9 \times 11} = \frac{2}{99}$

$\frac{2}{99}$ cannot be simplified, so this is the final answer.

Try It Now 2

Multiply or Divide the following.

1. $\frac{5}{14} \times \frac{7}{16}$
2. $\frac{\frac{2}{5}}{\frac{3}{4}}$

Answer

1. $\frac{5}{14} \times \frac{7}{16}$

   $\frac{5}{14} \times \frac{7}{16} = \frac{5 \times 7}{14 \times 16} = \frac{35}{224}$

   $\frac{35}{224}$ can be simplified since 35 and 224 are both divisible by 7.

   $\frac{35}{224} = \frac{35 \div 7}{224 \div 7} = \frac{5}{32}$

   $\frac{5}{32}$ cannot be simplified, so this is the final answer.

2. $\dfrac{\frac{2}{5}}{\frac{3}{4}}$

Remember keep the numerator, switch the operation, flip the denominator before doing anything else. "Keep, Switch, Flip."

$$\frac{\frac{2}{5}}{\frac{3}{4}} = \frac{2}{5} \times \frac{4}{3} = \frac{2 \times 4}{5 \times 3} = \frac{8}{15}$$

$\frac{8}{15}$ cannot be simplified, so this is the final answer.

Improper Fractions and Mixed Numbers

At times, a rational number is written with a numerator that is larger than the denominator. This is called an **improper fraction**. A fraction represents a part of a whole. When there is an improper fraction it represents a combination of whole numbers with extra pieces called mixed numbers.

An improper fraction can be converted to a mixed number to more clearly display the whole part with the fraction part.

Example 17

Convert the improper fractions to mixed numbers.

1. $\frac{3}{2}$
2. $\frac{7}{3}$
3. $\frac{26}{7}$

Solution

1. $\frac{3}{2}$ can be rewritten as $\frac{2}{2}+\frac{1}{2}=1+\frac{1}{2}=1\frac{1}{2}$
2. $\frac{7}{3}$ can also be viewed as division. $7 \div 3$. Using division we see that 3 goes into 7 two times with a remainder of 1. So the mixed number would be $2\frac{1}{3}$
3. $\frac{26}{7}$. Viewing it as several fractions or as division, we find that $\frac{26}{7} = 3\frac{5}{7}$

A mixed number can also be converted to an improper fraction to make mathematical calculations easier than working with mixed numbers. The whole number part of the mixed number is changed to the same denominator as the fraction part, and the two parts are added.

Example 18

Convert to improper fractions.

1. $4\ \frac{2}{5}$
2. $12\ \frac{3}{8}$

Solution

1. $4\ \frac{2}{5} = 4 + \frac{2}{5} = 4 \cdot \frac{5}{5} + \frac{2}{5} = \frac{20}{5} + \frac{2}{5} = \frac{22}{5}$
2. $12\ \frac{3}{8} = 12 + \frac{3}{8} = 12 \cdot \frac{8}{8} + \frac{3}{8} = \frac{96}{8} + \frac{3}{8} = \frac{99}{8}$

A shortcut method is to keep the denominator the same and to find the new numerator. To find the new numerator, multiply the whole number to the denominator and then add the original numerator.

1. $4\,\frac{2}{5}=\frac{4\cdot 5+2}{2}=\frac{22}{5}$
2. $12\,\frac{3}{8}=\frac{12\cdot 8+3}{8}=\frac{99}{8}$

Fractions in Decimal Form

Fractions can also be written as decimals that are terminating or have a repeating pattern.

The place value system extends to the right of the ones place, so 0.2 has 2 in the tenths place and is equal to the fraction $\frac{2}{10}$. 0.07 has 7 in the hundredths place, so it is the same as $\frac{7}{100}$. A decimal point is used to identify which place is the ones place.

Reading decimal numbers

Decimal numbers like 0.27 mean $\frac{2}{10}+\frac{7}{100}$. Using equivalent fractions, $\frac{2}{10}=\frac{20}{100}$, so $0.27=\frac{20}{100}+\frac{7}{100}$ or $\frac{27}{100}$. To read 0.27, we look at the digits to the right of the decimal point and read them as if they were a whole number: twenty seven. Then we identify the place value of the right-hand digit: hundredths. 0.27 is read "twenty-seven hundredths."

Mixed numbers are also written in decimal form such as 16.352. Just as with mixed fractions, we say the whole number part, followed by "and," followed by the fraction part. We read 16.352 as "sixteen AND three hundred fifty-two thousandths."

Example 19

Write these decimal numbers in words

1. 487.59
2. 83.9403

Solution

1. four hundred eighty-seven and fifty-nine hundredths
2. eighty-three and nine thousand four hundred three ten-thousandths

Example 20

Write these decimal numbers using digits

1. twenty and forty-one hundredths
2. five and eighty-nine ten-thousandths
3. three hundred and thirty-six thousandths

Solution

1. 20.41
2. 5.0089
3. 300.036

Add and Subtract Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding **place values**. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

Steps for adding or subtracting decimals.

1. Write the numbers so the decimal points line up vertically.
2. Use zeros as place holders, as needed.
3. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.

Example 21

Add: $23.5 + 41.38$.

Solution

Write the numbers so that the decimal points line up vertically.

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline \end{array}$$

Put 0 as a placeholder after the 5 in 23.5. Remember, $\frac{5}{10} = \frac{50}{100}$, so $0.5 = 0.50$

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline \end{array}$$

Add the numbers as if they were whole numbers . Then place the decimal point in the sum.

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline 64.88 \end{array}$$

Try It Now 3

Add: $4.8 + 11.69$.

Answer

16.49

Multiply and Divide Decimals

Multiplying decimals is very much like multiplying whole numbers—we just have to determine where to place the decimal point. The procedure for multiplying decimals will make sense if we first convert them to fractions and then multiply.

So let's see what we would get as the **product** of decimals by converting them to fractions first. We will do two examples side-by-side. Look for a pattern!

Table 1.3.2.1

$(0.3)(0.7)$	$(0.2)(0.46)$
1 place 1 place	1 place 2 places

Convert to fractions.	$\frac{3}{10} \cdot \frac{7}{10}$	$\frac{2}{10} \cdot \frac{46}{100}$
Multiply.	$\frac{21}{100}$	$\frac{92}{1000}$
Convert to decimals.	$\underbrace{0.21}_{\text{2 places}}$	$\underbrace{0.092}_{\text{3 places}}$

Notice, in the first example, we multiplied two numbers that each had one digit after the decimal point and the product had two decimal places. In the second example, we multiplied a number with one decimal place by a number with two decimal places and the product had three decimal places.

We multiply the numbers just as we do whole numbers, temporarily ignoring the decimal point. We then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product.

The rules for multiplying positive and negative numbers apply to decimals, too, of course!

When *multiplying* two numbers,

- if their signs are the *same* the product is *positive*.
- if their signs are *different* the product is *negative*.

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. Finally, we write the product with the appropriate sign.

Steps for multiplying decimals.

1. Determine the sign of the product.
2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
4. Write the product with the appropriate sign.

Example 22

Multiply: $(-3.9)(4.075)$.

Solution

	$(-3.9)(4.075)$
The signs are different. The product will be negative.	
Write in vertical format, lining up the numbers on the right.	$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline \end{array}$
Multiply.	$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ 12225 \\ \hline 158925 \end{array}$
Add the number of decimal places in the factors $(1+3)$. (–3.9) (4.075) 1 place 3 places Place the decimal point 4 places from the right.	$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ 12225 \\ \hline 15.8925 \end{array}$ 4 places
The signs are different, so the product is negative.	$(-3.9)(4.075) = -15.8925$

Try It Now 4

Multiply: $-4.5(6.107)$

Answer

-27.4815

Just as with multiplication, **division** of decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed.

To divide decimals, determine what power of 10 to multiply the denominator by to make it a whole number. Then multiply the numerator by that same power of 10. Because of the equivalent fractions property, we haven't changed the value of the fraction! The effect is to move the decimal points in the numerator and denominator the same number of places to the right. For example:

$$\frac{0.8}{0.4}$$
$$\frac{0.8(10)}{0.4(10)}$$
$$\frac{8}{4}$$

We use the rules for dividing positive and negative numbers with decimals, too. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:

$$\underset{\text{dividend}}{a} \div \underset{\text{divisor}}{b} = \underset{\text{quotient}}{c} \qquad \begin{array}{r} \overset{\text{quotient}}{c} \\ \underset{\text{divisor}}{b}\,\overline{)\,a} \\ \text{dividend} \end{array}$$

We'll write the steps to take when dividing decimals, for easy reference.

Steps for dividing decimals.

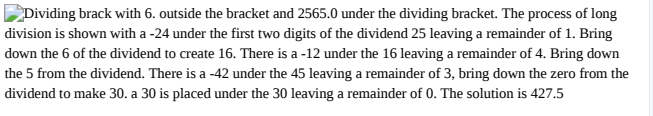
1. Determine the sign of the quotient.
2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.
3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
4. Write the quotient with the appropriate sign.

Example 23

Divide: $-25.65 \div (-0.06)$.

Solution

Remember, you can "move" the decimals in the divisor and dividend because of the Equivalent Fractions Property.

	$-25.65 \div (-0.06)$
The signs are the same.	The quotient is positive.
Make the divisor a whole number by "moving" the decimal point all the way to the right.	
"Move" the decimal point in the dividend the same number of places.	$0.06\overline{)25.65}$
Divide. Place the decimal point in the quotient above the decimal point in the dividend.	
Write the quotient with the appropriate sign.	$-25.65 \div (-0.06) = 427.5$

Try It Now 5

Divide: $-23.492 \div (-0.04)$.

Answer

687.3

Irrational Numbers

Irrational numbers are also a subset of the real numbers.

Irrational numbers are numbers with decimal representations that do not terminate or contain a repeating block of digits. Some examples are:

$\pi, e, \sqrt{2}$.

If you put these in your calculator, it will display the first several digits of the decimal approximation until it runs out of space. The decimal approximation of these numbers continues beyond what your calculator can display.

Operations with Irrational Numbers

Most often when working with irrational numbers, an exact number is more desirable than a decimal approximation. An exact number will contain the irrational number in its nondecimal form such as π or $\sqrt{2}$. It is important to pay attention to the coefficient of the irrational number to determine how to calculate the final answer.

Multiplication and Division

When an irrational number is the same type, for example both are square roots of a number, they will remain the same type while the multiplication or division is performed within the type.

Example 24

Multiply or divide the following irrational numbers. Write the answer in exact form.

1. $\pi \times \pi$
2. $\sqrt{5} \times \sqrt{5}$
3. $\sqrt{2} \times \sqrt{3}$
4. $\sqrt{2} \times \sqrt{8}$
5. $7\sqrt{2} \times \pi$
6. $\sqrt{2} \div \sqrt{8}$
7. $15\pi \div 3\pi$
8. $\sqrt{6} \div 2\pi$

Solution

1. $\pi \times \pi$

$\pi \times \pi = \pi^2$

2. $\sqrt{5} \times \sqrt{5}$

Since these are both square roots, do the multiplication or division within the square root. first Then simplify, if possible.

$\sqrt{5} \times \sqrt{5} = \sqrt{5^2} = 5$

3. $\sqrt{2} \times \sqrt{3}$

$\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6}$

4. $\sqrt{2} \times \sqrt{8}$

$\sqrt{2} \times \sqrt{8} = \sqrt{2 \times 8} = \sqrt{16} = 4$

5. $7\sqrt{2} \times \pi$

Since these are different types of irrational numbers they cannot be combined in exact form.

$7\sqrt{2} \times \pi = 7\sqrt{2}\pi$

6. $\sqrt{2} \div \sqrt{8}$

$$\sqrt{2} \div \sqrt{8} = \sqrt{2 \div 8} = \sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$$

7. $15\pi \div 3\pi$

Since both the numerator and denominator are divisible by π, the π will divide out.

$15\pi \div 3\pi = 15 \div 3 = 5$

8. $\sqrt{6} \div 2\pi$

Since these are different types of irrational numbers they cannot be combined in exact form. It can be rewritten as a fraction.

$$\frac{\sqrt{6}}{2\pi}$$

Addition and Subtraction

When adding and subtracting the irrational number is treated like a variable. Only "like terms" can be added or subtracted, meaning the terms must already have or be able to be converted to the same irrational number. Like term coefficients are what are added or subtracted, while the irrational number remains a part of the final answer.

Example 25

Add the following irrational numbers in exact form, if possible.

1. $2\sqrt{3}+5\sqrt{3}$
2. $7\pi+2\sqrt{5}$
3. $9\sqrt{2}-4\sqrt{2}$
4. $\sqrt{5}+7\sqrt{20}$

Solution

1. $2\sqrt{3}+5\sqrt{3}$

Since the irrational number $\sqrt{3}$ is in both terms, these can be added.

$2\sqrt{3}+5\sqrt{3}=(2+5)\sqrt{3}=7\sqrt{3}$

2. $7\pi+2\sqrt{5}$

These do not have the same irrational number, so they cannot be added in exact form.

3. $9\sqrt{2}-4\sqrt{2}$

Since the irrational number $\sqrt{2}$ is in both terms, these can be subtracted.

$9\sqrt{2}-4\sqrt{2}=(9-4)\sqrt{2}=5\sqrt{2}$

4. $\sqrt{5}+7\sqrt{20}$

At first glance, these look like they cannot be added, but 20 is divisible by 5 leaving a perfect square factor of 4. Also, when there is no visible coefficient, assume that the coefficient is 1. Rewrite the sum and then try again.

$\sqrt{5}+7\sqrt{20}=1\sqrt{5}+7\sqrt{4\times5}=1\sqrt{5}+7\sqrt{4}\sqrt{5}=1\sqrt{5}+(7\times2)\sqrt{5}=1\sqrt{5}+14\sqrt{5}=(1+14)\sqrt{5}=15\sqrt{5}$

This page titled 1.3.2: Rational and Irrational Numbers is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **1.8: Decimals** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

1.3.2.1: Exercises

Section 1.3.2.1 Exercises

1. Which is larger: $\frac{2}{3}$ or $\frac{5}{8}$? Which one would be on the right if they were graphed on a number line?
2. Write the fractions in order from smallest to largest.
3. $\frac{3}{4}, \frac{5}{8}, \frac{2}{3}, \frac{7}{6}, \frac{2}{5}$
4. Simplify each fraction by putting it in lowest terms.
   a. \(\dfrac{15}{35}
   b. \(\dfrac{56}{21}
   c. \(\dfrac{9}{20}
5. You survey students in your class and find that 8 students are left-handed and 28 are not. Write the ratio of left-handed students to all students as a fraction in simplest form.
6. Evaluate:
   a. $\frac{4}{9}+\frac{1}{3}$
   b. $\frac{3}{10}+\frac{5}{12}+\frac{1}{6}$
   c. $\frac{7}{12}-\frac{9}{16}$
   d. $8-\frac{5}{7}$
   e. $\frac{7}{8}-\frac{3}{4}+\frac{5}{4}-\frac{1}{2}$
   f. Evaluate:
   g. $\left(\frac{3}{6}\right)\cdot\left(4\frac{5}{8}\right)\cdot\left(\frac{2}{15}\right)$
   h. $\left(\frac{5}{6}\right)^2$
   i. $6\cdot 4\frac{2}{3}$
   j. $\left(12\cdot\frac{3}{4}\right)+\left(15\cdot\frac{1}{5}\right)$
   k. $\left(\frac{1}{2}\right)^2\cdot\left(\frac{5}{3}\right)^2$
   l. $\frac{1}{3}\div\frac{1}{2}$
   m. $\frac{13}{28}\div\frac{39}{14}$
   n. $\frac{4}{3}\div 6$
   o. $4+\frac{3}{8}\div\frac{3}{16}$
7. Rewrite as rational numbers in fraction form.
   a. $2\frac{3}{8}$
   b. $1\frac{1}{5}$
   c. $-4\frac{2}{7}$
   d. $5\frac{47}{100}$
8. Write each rational number as a mixed number.
   a. $\frac{5}{3}$

b. $-\frac{8}{5}$
c. $\frac{43}{10}$
d. $\frac{16}{3}$
e. $\frac{38}{6}$

9. *True or False.*
   a. $\sqrt{9}+\sqrt{16}=\sqrt{9+16}$
   b. $\sqrt{9\cdot 16}=\sqrt{9}\sqrt{6}$
   c. $\sqrt{\left(\frac{9}{16}\right)}=\frac{\sqrt{9}}{\sqrt{16}}$
10. Simplify each expression. Leave irrational numbers in exact form; do not approximate them.
    a. $6\sqrt{3}+4\sqrt{3}$
    b. $2(6\sqrt{3}+4\sqrt{3})$
    c. $\frac{6\sqrt{3}+4\sqrt{3}}{2\sqrt{3}}$
    d. $2\sqrt{3}\cdot\sqrt{3}$
    e. $2\sqrt{3}(6\sqrt{3}+4\sqrt{3})$
    f. $4\sqrt{3}^2$
    g. $(4\sqrt{3})^2$
    h. $5\pi+4\pi$
    i. $\frac{12\pi}{4\pi}$
    j. $2(5\pi+4\pi)$
    k. $2\cdot 5\pi^2$
    l. $2\cdot(5\pi)^2$
    m. $(2\cdot 5\pi)^2$
    n. $2\pi\cdot 5^2$
11. Compare the two numbers in each pair using the correct inequality symbol.
    a. -6 ____ -2
    b. -5 ____ -9
    c. -7 ____ 2
    d. -3/5 ____ -4/7
    e. -2.56 ____ - 3.4
    f. Evaluate these expressions
    g. -5/7 + 2/7
    h. 4.79 – 8.39
    i. -64 – 25.5
    j. 23 1/3 - 52 2/3
    k. 6/5 ÷ 3/8
    l. 4 2/5 - 2 7/8
    m. 3 1/2 • 4 1/3
12. How many quarter-pound burgers can you make from 7 ½ pounds of hamburger meat?
13. In a recent election on campus, 760 students voted. The winning candidate received 4/7 of the votes cast. How many students voted for this candidate?
14. Evaluate 48 – 2.8*x* for *x* = 10
15. Evaluate 1/2 h(b + B) for h = 3/2 , b = 4 ½ , B = 6
16. Write these rational numbers as decimal numbers.
    a. $\frac{2}{5}$

b. $\frac{3}{5}$
c. $\frac{7}{20}$
d. $\frac{7}{8}$
e. $\frac{17}{100}$
f. $\frac{5}{10}$
g. $\frac{13}{400}$
h. $\frac{1}{3}$
i. $\frac{5}{6}$
j. $\frac{4}{7}$
k. $\frac{8}{5}$
l. $\frac{5}{4}$
m. 5
n. $\frac{8}{3}$

17. Rewrite these decimal numbers as rational numbers in lowest terms.
    a. 0.6
    b. 0.48
    c. 0.028
    d. 0.89
    e. $0.\bar{6}$
    f. 2.5
    g. 12.25

This page titled 1.3.2.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.3.3: Associative, Commutative, and Distributive Properties

1.3.3 Learning Objectives

- Use the commutative and associative properties of addition and multiplication
- Use the identity and inverse properties of addition and multiplication
- Use the properties of zero
- Simplify expressions using the distributive property

Use the Commutative and Associative Properties

Think about adding two numbers, say 5 and 3. The order we add them doesn't affect the result, does it?

$$\begin{array}{cc} 5+3 & 3+5 \\ 8 & 8 \\ 5+3= & 3+5 \end{array}$$

The results are the same.

As we can see, the order in which we add does not matter!

What about multiplying 5 and 3?

$$\begin{array}{cc} 5\cdot 3 & 3\cdot 5 \\ 15 & 15 \\ 5\cdot 3= & 3\cdot 5 \end{array}$$

Again, the results are the same!

The order in which we multiply does not matter!

These examples illustrate the **commutative property**. When adding or multiplying, changing the *order* gives the same result.

Definition: Commutative Property

of Addition	If a, b are real numbers, then	$a+b=b+a$
of Multiplication	If a, b are real numbers, then	$a\cdot b=b\cdot a$

When adding or multiplying, changing the *order* gives the same result.

The commutative property has to do with order. If you change the order of the numbers when adding or multiplying, the result is the same.

What about subtraction? Does order matter when we subtract numbers? Does $7-3$ give the same result as $3-7$?

$$\begin{array}{cc} 7-3 & 3-7 \\ 4 & -4 \\ & 4\neq -4 \\ 7-3 & \neq 3-7 \end{array}$$

The results are not the same.

Since changing the order of the subtraction did not give the same result, we know that *subtraction is not commutative*.

Let's see what happens when we divide two numbers. Is **division** commutative?

$$\begin{array}{cc} 12\div 4 & 4\div 12 \\ \frac{12}{4} & \frac{4}{12} \\ 3 & \frac{1}{3} \\ & 3\neq\frac{1}{3} \\ 12\div 4 & \neq 4\div 12 \end{array}$$

The results are not the same.

Since changing the order of the division did not give the same result, *division is not commutative*. The commutative properties only apply to addition and multiplication!

- Addition and multiplication *are* commutative.
- Subtraction and Division *are not* commutative.

If you were asked to simplify this expression, how would you do it and what would your answer be?

$7+8+2$

Some people would think $7+8$ is 15 and then $15+2$ is 17. Others might start with $8+2$ makes 10 and then $7+10$ makes 17.

Either way gives the same result. Remember, we use parentheses as grouping symbols to indicate which operation should be done first.

Add $7+8$.	$(7+8)+2$
Add.	$15+2$
Add.	17
	$7+(8+2)$
Add $8+2$.	$7+10$
Add.	77

$(7+8)+2=7+(8+2)$

When adding three numbers, changing the grouping of the numbers gives the same result.

This is true for multiplication, too.

	$(5\cdot\frac{1}{3})\cdot 3$
Multiply. $5\cdot\frac{1}{3}$	$\frac{5}{3}\cdot 3$
Multiply.	5
	$5\cdot(\frac{1}{3}\cdot 3)$
Multiply. $\frac{1}{3}\cdot 3$	$5\cdot 1$
Multiply.	5

$(5\cdot\frac{1}{3})\cdot 3=5\cdot(\frac{1}{3}\cdot 3)$

When multiplying three numbers, changing the grouping of the numbers gives the same result.

You probably know this, but the terminology may be new to you. These examples illustrate the **associative property**.

Definition: Associative property

of Addition	If a, b, c are real numbers, then $(a+b)+c=a+(b+c)$
of Multiplication	If a, b, c are real numbers, then $(a\cdot b)\cdot c=a\cdot(b\cdot c)$

When adding or multiplying, changing the *grouping* gives the same result.

Let's think again about multiplying $5\cdot\frac{1}{3}\cdot 3$. We got the same result both ways, but which way was easier? Multiplying $\frac{1}{3}$ and 3 first, as shown above on the right side, eliminates the fraction in the first step. Using the associative property can make the math easier!

The **associative property** has to do with grouping. If we change how the numbers are grouped, the result will be the same. Notice it is the same three numbers in the same order—the only difference is the grouping.

We saw that subtraction and division were not commutative. They are not associative either.

When simplifying an expression, it is always a good idea to plan what the steps will be. In order to combine like terms in the next example, we will use the commutative property of addition to write the like terms together.

Example 1

Simplify: $18p+6q+15p+5q$.

Answer

	$18p+6q+15p+5q$
Use the commutative property of addition to re-order so that like terms are together.	$18p+15p+6q+5q$
Add like terms.	$33p+11q$

Try it Now 1

Simplify: $23r+14s+9r+15s$.

Answer

$32r+29s$

Try it Now 2

Simplify: $37m+21n+4m-15n$.

Answer

$41m+6n$

When we have to simplify algebraic **expression**s, we can often make the work easier by applying the commutative or associative property first, instead of automatically following the order of operations. When adding or subtracting fractions, combine those with a common denominator first.

Example 2

Simplify: $(\frac{5}{13}+\frac{3}{4})+\frac{1}{4}$

Answer

	$(\frac{5}{13}+\frac{3}{4})+\frac{1}{4}$
Notice that the last 2 terms have a common denominator, so change the grouping.	$\frac{5}{13}+(\frac{3}{4}+\frac{1}{4})$
Add in parentheses first.	$\frac{5}{13}+(\frac{4}{4})$
Simplify the fraction.	$\frac{5}{13}+1$
Add.	$1\frac{5}{13}$
Convert to an improper fraction.	$\frac{18}{13}$

Try It Now 3

Simplify: $(\frac{7}{15}+\frac{5}{8})+\frac{3}{8}$

Answer

$1\frac{7}{15}$

Try it Now 4

Simplify: $(\frac{2}{9}+\frac{7}{12})+\frac{5}{12}$

Answer

$1\frac{2}{9}$

Example 3

Use the associative property to simplify $6(3x)$.

Answer

Use the associative property of multiplication, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$, to change the grouping.

	$6(3x)$	
Change the grouping.	$(6\cdot 3)x$	(1.3.3.1)
Multiply in the parentheses.	18	

Notice that we can multiply $6\cdot 3$ but we could not multiply \(3x\) without having a value for \(x\).

Try it Now 5

Use the associative property to simplify $8(4x)$.

Answer

$32x$

Try it Now 6

Use the associative property to simplify $-9(7y)$.

Answer

$-63y$

Use the Identity and Inverse Properties of Addition and Multiplication

What happens when we add 0 to any number? Adding 0 doesn't change the value. For this reason, we call 0 the **additive identity**.

For example,

$13+0$	$-14+0$	$0+(-8)$
13	-14	-8

These examples illustrate the **Identity Property of Addition** that states that for any real number a, $a+0=a$ and $0+a=a$.

What happens when we multiply any number by one? Multiplying by 1 doesn't change the value. So we call 1 the **multiplicative identity**.

For example,

$43\cdot 1$	$-27\cdot 1$	$1\cdot\frac{3}{5}$
43	-27	$\frac{3}{5}$

These examples illustrate the **Identity Property of Multiplication** that states that for any real number a, $a\cdot 1=a$ and $1\cdot a=a$.

We summarize the Identity Properties below.

Definition: Identity property

of addition For any real number a: $a+0=a$ $0+a=a$
0 is the **additive identity**
of multiplication For any real number a: $a\cdot 1=a$ $1\cdot a=a$
1 is the **multiplicative identity**


In the top line of this figure, we have the question "What number added to 5 gives the additive identity, 0?" On the following line, we have 5 plus a blank space equals 0. Then it is stated that "We know 5 plus negative 5 equals 0." On the following line, we have the question "What number added to negative 6 gives the additive identity, 0?" On the following line, we have negative 6 plus a blank space equals 0. Then it is stated that "We know negative 6 plus 6 equals 0."

Figure 1.3.3.1

Notice that in each case, the missing number was the opposite of the number!

We call $-a$. The **additive inverse** of a. *The opposite of a number is its additive inverse.* A number and its opposite add to zero, which is the additive identity. This leads to the **Inverse Property of Addition** that states for any real number $a, a+(-a)=0$. Remember, a number and its opposite add to zero.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity, 1? In other words, $\frac{2}{3}$ times what results in 1?


We have the statement that 2/3 times a blank space equals 1. Then it is stated that "We know 2/3 times 3/2 equals 1."

Figure 1.3.3.2

What number multiplied by 2 gives the multiplicative identity, 1? In other words 2 times what results in 1?


We have the statement that 2 times a blank space equals 1. Then it is stated that "We know 2 times 1/2 equals 1."

Figure 1.3.3.3

Notice that in each case, the missing number was the **reciprocal** of the number!

We call $\frac{1}{a}$ the **multiplicative inverse** of a. *The reciprocal of aa number is its multiplicative inverse.* A number and its reciprocal multiply to one, which is the multiplicative identity. This leads to the **Inverse Property of Multiplication** that states that for any real number $a, a\neq 0, a\cdot\frac{1}{a}=1$.

We'll formally state the inverse properties here:

Definition: Inverse property

of addition	For any real number a, $-a$. is the **additive inverse** of a A number and its opposite add to zero.	$a+(-a)=0$
of multiplication	For any real number a, $a \neq 0$ $\frac{1}{a}$. is the **multiplicative inverse** of a A number and its reciprocal multiply to zero.	$a \cdot \frac{1}{a}=1$

Example 4

Find the additive inverse of

1. $\frac{5}{8}$
2. 0.6
3. -8
4. $-\frac{4}{3}$

Answer

To find the additive inverse, we find the opposite.

1. The additive inverse of $\frac{5}{8}$ is the opposite of $\frac{5}{8}$. The additive inverse of $\frac{5}{8}$ is $-\frac{5}{8}$
2. The additive inverse of 0.6 is the opposite of 0.6. The additive inverse of 0.6 is -0.6.
3. The additive inverse of -8 is the opposite of -8. We write the opposite of -8 as $-(-8)$, and then simplify it to 8. Therefore, the additive inverse of -8 is 8.
4. The additive inverse of $-\frac{4}{3}$ is the opposite of $-\frac{4}{3}$. We write this as $-(-\frac{4}{3})$, and then simplify to $\frac{4}{3}$. Thus, the additive inverse of $-\frac{4}{3}$ is $\frac{4}{3}$.

Try it Now 7

Find the additive inverse of

1. $\frac{7}{9}$
2. 1.2
3. -14
4. $-\frac{9}{4}$

Answer

1. $-\frac{7}{9}$
2. -1.2
3. 14
4. $\frac{9}{4}$

Try it Now 8

Find the additive inverse of

1. $\frac{7}{13}$
2. 8.4
3. -46
4. $-\frac{5}{2}$

Answer

1. $-\frac{7}{13}$
2. -8.4
3. 46
4. $\frac{5}{2}$

Example 5

Find the multiplicative inverse of

1. 9
2. $-\frac{1}{9}$
3. 0.9

Answer

To find the multiplicative inverse, we find the reciprocal.

1. The multiplicative inverse of 9 is the reciprocal of 9, which is $\frac{1}{9}$. Therefore, the multiplicative inverse of 9 is $\frac{1}{9}$.
2. The multiplicative inverse of $-\frac{1}{9}$ is the reciprocal of $-\frac{1}{9}$, which is -9. Thus, the multiplicative inverse of $-\frac{1}{9}$ is -9.
3. To find the multiplicative inverse of 0.9, we first convert 0.9 to a fraction, $\frac{9}{10}$. Then we find the reciprocal of the fraction. The reciprocal of $\frac{9}{10}$ is $\frac{10}{9}$. So the multiplicative inverse of 0.9 is $\frac{10}{9}$.

Try it Now 9

Find the multiplicative inverse of

1. 4
2. $-\frac{1}{7}$
3. 0.3

Answer

1. $\frac{1}{4}$
2. -7
3. $\frac{10}{3}$

Try it Now 10

Find the multiplicative inverse of

1. 18
2. $-\frac{4}{5}$
3. 0.6

Answer

1. $\frac{1}{18}$
2. $-\frac{5}{4}$
3. $\frac{5}{3}$

Definition: Multiplication by zero

For any real number a.

$a \cdot 0 = 0 \quad 0 \cdot a = 0$

The product of any real number and 0 is 0.

What about **division** involving zero? What is $0 \div 3$? Think about a real example: If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get? There are no cookies to share, so each person gets 0 cookies. So,

$0 \div 3 = 0$

We can check division with the related multiplication fact.

$12 \div 6 = 2$ because $2 \cdot 6 = 12$

So we know $0 \div 3 = 0$ because $0 \cdot 3 = 0$.

Definition: Division of Zero

For any real number a, except 0, $\frac{0}{a} = 0$ and $0 \div a = 0$.

Zero divided by any real number except zero is zero.

Now think about dividing *by* zero. What is the result of dividing 4 by 0? Think about the related multiplication fact: $4 \div 0 = ?$ means $? \cdot 0 = 4$. Is there a number that multiplied by 0 gives 4? Since any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4.

We conclude that there is no answer to $4 \div 0$ and so we say that division by 0 is undefined.

Definition: Division by Zero

For any real number a, except 0, $\frac{a}{0}$ and $a \div 0$ are undefined.

Division by zero is undefined.

We summarize the properties of zero below.

Definition: Properties of Zero

Multiplication by Zero: For any real number a,

$$a \cdot 0 = 0 \quad 0 \cdot a = 0 \quad \text{The product of any number and 0 is 0} \tag{1.3.3.2}$$

Division of Zero, Division by Zero: For any real number a, $a \neq 0$

$\frac{0}{a} = 0$	Zero divided by any real number, except itself is zero.
$\frac{a}{0}$ is undefined	Division by zero is undefined.

Example 6

Simplify:

1. $-8 \cdot 0$
2. $\frac{0}{-2}$
3. $\frac{-32}{0}$

Answer

1. The product of any real number and 0 is 0 $\quad -8 \cdot 0$ $\quad 0$

2. Zero divided by any real number, except itself, is 0 $\frac{0}{-2}$ 0
3. Division by 0 is undefined. $\frac{-32}{0}$ undefined

Try it Now 11

Simplify:

1. $-14 \cdot 0$
2. $\frac{0}{-6}$
3. $\frac{-2}{0}$

Answer

1. 0
2. 0
3. undefined

Try it Now 12

Simplify:

1. $0(-17)$
2. $\frac{0}{-10}$
3. $\frac{-5}{0}$

Answer

1. 0
2. 0
3. undefined

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

Example 7

Simplify:

1. $\frac{0}{n+5}$, where $n \neq -5$
2. $\frac{10-3p}{0}$ where $10-3p \neq 0$

Answer

1. Zero divided by any real number except itself is 0. $\frac{0}{n+5}$ 0
2. Division by 0 is undefined $\frac{10-3p}{0}$ undefined

Example 8

Simplify: $-84n+(-73n)+84n$.

Answer

	$-84n+(-73n)+84n$
Notice that the first and third terms are opposites; use the commutative property of addition to re-order the terms.	$-84n+84n+(-73n)$
Add left to right.	$0+(-73)$
Add.	$-73n$

Try it Now 13

Simplify: $-27a+(-48a)+27a$.

Answer

$-48a$

Try it Now 14

Simplify: $39x+(-92x)+(-39x)$.

Answer

$-92x$

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1.

Example 9

Simplify: $\frac{7}{15}\cdot\frac{8}{23}\cdot\frac{15}{7}$

Answer

	$\frac{7}{15}\cdot\frac{8}{23}\cdot\frac{15}{7}$
Notice that the first and third terms are reciprocals, so use the commutative property of multiplication to re-order the factors.	$\frac{7}{15}\cdot\frac{15}{7}\cdot\frac{8}{23}$
Multiply left to right.	$1\cdot\frac{8}{23}$
Multiply.	$\frac{8}{23}$

Try it Now 15

Simplify: $\frac{9}{16}\cdot\frac{5}{49}\cdot\frac{16}{9}$

Answer

$\frac{5}{49}$

Try it Now 16

Simplify: $\frac{6}{17} \cdot \frac{11}{25} \cdot \frac{17}{6}$

Answer

$\frac{11}{25}$

Try it Now 17

Simplify:

1. $\frac{0}{m+7}$, where $m \neq -7$
2. $\frac{18-6c}{0}$, where $18-6c \neq 0$

Answer

1. 0
2. undefined

Try it Now 18

Simplify:

1. $\frac{0}{d-4}$, where $d \neq 4$
2. $\frac{15-4q}{0}$, where $15-4q \neq 0$

Answer

1. 0
2. undefined

Example 10

Simplify: $\frac{3}{4} \cdot \frac{4}{3}(6x+12)$

Answer

	$\frac{3}{4} \cdot \frac{4}{3}(6x+12)$
There is nothing to do in the parentheses, so multiply the two fractions first—notice, they are reciprocals.	$1(6x+12)$
Simplify by recognizing the multiplicative identity.	$6x+12$

Try it Now 19

Simplify: $\frac{2}{5} \cdot \frac{5}{2}(20y+50)$

Answer

$20y+50$

Try it Now 20

Simplify: $\frac{3}{8} \cdot \frac{8}{3}(12z+16)$

Answer

$12z+16$

Simplify Expressions Using the Distributive Property

Suppose that three friends are going to the movies. They each need $9.25—that's 9 dollars and 1 quarter—to pay for their tickets. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times $9 so $27, and 3 times 1 quarter, so 75 cents. In total, they need $27.75. If you think about doing the math in this way, you are using the **distributive property**.

Definition: Distributive property

If a, b, c are real numbers, then $a(b+c) = ab+ac$

Also, $(b+c)a = ba+ca$

$a(b-c) = ab-ac$

$(b-c)a = ba-ca$

Back to our friends at the movies, we could find the total amount of money they need like this:

$$\begin{aligned} &3(9.25)\\ &3(9+0.25)\\ &3(9)+3(0.25)\\ &27+0.75\\ \\ &27.75 \end{aligned}$$

In algebra, we use the **distributive property** to remove parentheses as we simplify expressions.

For example, if we are asked to simplify the expression $3(x+4)$, the order of operations says to work in the parentheses first. But we cannot add x and 4, since they are not like terms. So we use the distributive property, as shown in Exercise 1.3.3.31.

Example 11

Simplify: $3(x+4)$.

Answer

	$3(x+4)$
Distribute.	$3 \cdot x+3 \cdot 4$
Multiply.	$3x+12$

Try it Now 21

Simplify: $4(x+2)$.

Answer

$4x+8$

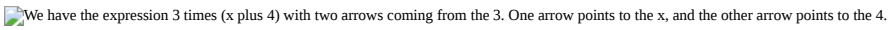
Try it Now 22

Simplify: $6(x+7)$.

Answer

$6x+42$

Some students find it helpful to draw in arrows to remind them how to use the distributive property. Then the first step in Exercise $1.3.3.31$would look like this:



Example 12

Simplify: $8(\frac{3}{8}x+\frac{1}{4})$.

Answer

Distribute.	
Multiply.	

Try it Now 23

Simplify: $6(\frac{5}{6}y+\frac{1}{2})$.

Answer

$5y+3$

Using the distributive property will be very useful when we solve money applications in later chapters.

Example 13

Simplify: $100(0.3+0.25q)$.

Answer

Distribute.	
Multiply.	

Try it Now 24

Simplify: $100(0.7+0.15p)$.

Answer

$70+15p$

When we distribute a negative number, we need to be extra careful to get the signs correct!

Example 14

Simplify: $-2(4y+1)$.

Answer

	
Distribute.	
Multiply.	

Try it Now 25

Simplify: $-3(6m+5)$.

Answer

$-18m-15)$

Exercise 1.3.3.46will show how to use the distributive property to find the opposite of an expression.

Example 15

Simplify: $-(y+5)$.

Answer

	$-(y+5)$	
Multiplying by -1 results in the opposite.	$-1(y+5)$	
Distribute.	$-1\cdot y+(-1)\cdot 5$	(1.3.3.3)
Simplify.	$-y+(-5)$	
	$-y-5$	

Try it Now 26

Simplify: $-(z-11)$.

Answer

$-z+11$

There will be times when we'll need to use the **distributive property** as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiplication using the distributive property, which removes the parentheses. The next two examples will illustrate this.

Example 16

Simplify: $8-2(x+3)$.

Be sure to follow the order of operations. Multiplication comes before subtraction, so we will distribute the 2 first and then subtract.

Answer

	$8-2(x+3)$	
Distribute.	$8-2\cdot x-2\cdot 3$	(1.3.3.4)
Multiply.	$8-2x-6$	
Combine like terms.	$-2x+2$	

Try it Now 27

Simplify: $9-3(x+2)$.

Answer

$3-3x$

All the properties of real numbers we have used in this chapter are summarized in below.

Commutative Property of Addition	If a, b are real numbers, then $a+b=b+a$
Commutative Property of Multiplication	If a, b are real numbers, then $a\cdot b=b\cdot a$
Addition	If a

b
a
r
e
r
e
a
l
n
u
m
b
e
r
s
,
t
h
e
n

o
f
m
u
l
t
i
p
l
i
c
a
t
i
o
n

I
f
a
,
b
a
r
e
r
e
a
l
n
u

m b e r s , t h e n	
A s s o c i a t i v e P r o p e r t y	
(a)	$(a+b)+c=a+(b+c)$
(b)	$(a \cdot b) \cdot c=a \cdot(b \cdot c)$
o f a d d i t i o n	
I f a , b , c a r e r	

e
a
l
n
u
m
b
e
r
s
,
t
h
e
n

o
f
m
u
l
t
i
p
l
i
c
a
t
i
o
n

I
f
a
,
b
,
c
a
r
e
r
e
a
l
n
u
m
b
e

r
s
,
t
h
e
n

D
i
s
t
r
i
b
u
t
i
v
e
P
r
o
p
e
r
t
y

https://math.libretexts.org/@go/page/90372

If a, b, c are real numbers, then $a(b+c) = ab+ac$	**Identity Property**
$a+0=a$ $0+a=a$	**additive identity**
$a \cdot 1 = a$ $1 \cdot a = a$	**multiplicative identity**

n

For any real number a:

0 is the **additive identity of mult**

i p l i c a t i o n

F o r a n y r e a l n u m b e r a :

1 i s t h e m u l t i p l i c a t i v e

identity	
Inverse Property	
of addition	$a+(-a)=0$
of multiplication	$a \cdot \frac{1}{a}=1$
Definition	For any real number a

,

$-a$ is the **additive inverse** of a

of multiplication

For a

n
y
r
e
a
l
n
u
m
b
e
r

$a, a \neq 0$

$\frac{1}{a}$
i
s
t
h
e
m
u
l
t
i
p
l
i
c
a
t
i
v
e
i
n
v
e
r
s
e
o
f
a

https://math.libretexts.org/@go/page/90372

Properties of Zero

$a \cdot 0 = 0$

$0 \cdot a = 0$

$\frac{0}{a} = 0$

$\frac{a}{0}$ is undefined

y r e a l n u m b e r a ,

F o r a n y r e a l n u m b e r

$a, a \neq 0$

For any real number $a, a \neq 0$

This page titled 1.3.3: Associative, Commutative, and Distributive Properties is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.3.3.1: Exercises

Section 1.3.3.1 Exercises

1. Identify the property of real numbers illustrated by each equation.
   a. $(2+7)+5=2+(7+5)$
   b. $\sqrt{2}+(\sqrt{7}+\sqrt{5})=\sqrt{2}+(\sqrt{5}+\sqrt{7})$
   c. $2(7+\sqrt{5})=14+2\sqrt{5}$
   d. $2\cdot\sqrt{7}=\sqrt{7}\cdot 2$
   e. $2(7+\sqrt{5})=(7+\sqrt{5})2$
   f. $2+0=2$
   g. $\sqrt{5}\cdot 1=\sqrt{5}$
   h. $17+(-17)=0$
2. Evaluate $8\left(\frac{3}{4}-\frac{1}{2}\right)$ in two ways using the distributive property. Which way do you think is easier?
3. A deposit of P dollars, invested at a simple interest rate, r, (in decimal form) will grow to amount A after t years.
   $A=P(1+rt)$
4. Rewrite the right side of this formula so that it has no parentheses.
5. Simplify: $12(\frac{1}{3}n+\frac{3}{4})$.
6. Simplify: $100(0.3+0.25q)$.
7. Simplify: $100(0.7+0.15p)$.
8. Simplify: $100(0.04+0.35d)$.
9. Simplify: $-2(4y+1)$.
10. Simplify: $-3(6m+5)$.
11. Simplify: $-6(8n+11)$.
12. Simplify: $-11(4-3a)$.
13. Simplify: $-5(2-3a)$.
14. Simplify: $-7(8-15y)$.
15. Simplify: $-(y+5)$.
16. Simplify: $-(z-11)$.
17. Simplify: $-(x-4)$.
18. Simplify: $8-2(x+3)$.
19. Simplify: $9-3(x+2)$.
20. Simplify: $7x-5(x+4)$.
21. Simplify: $4(x-8)-(x+3)$.
22. Simplify: $6(x-9)-(x+12)$.
23. Simplify: $8(x-1)-(x+5)$.

This page titled 1.3.3.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.3.4: Order of Operations

1.3.4 Learning Objectives

- Use the order of operations to evaluate a numerical expression
- Use a calculator to evaluate a numerical expression

Grouping Symbols

Grouping symbols are used to indicate that a particular collection of numbers and meaningful operations is to be grouped together and considered as one number. The grouping symbols commonly used in mathematics are the following:

( ), [ ], { }, ‾, $\sqrt{\ }$

Parentheses: ( )

Brackets: [ ]

Braces: { }

Fraction Bar: ‾

Radical: $\sqrt{\ }$

In a computation in which more than one operation is involved, grouping symbols indicate which operation to perform first. If possible, we perform operations *inside* grouping symbols first.

Example 1

If possible, determine the value of each of the following.

$9+(3\cdot 8)$

Solution

Since 3 and 8 are within parentheses, they are to be combined first.

$$\begin{aligned} 9+(3\cdot 8) &= 9+24 \\ &= 33 \end{aligned}$$

Thus,

$$9+(3\cdot 8)=33$$

Example 2

$(10\div 0)\cdot 6$

Solution

Since $10\div 0$ is undefined, this operation is meaningless, and we attach no value to it. We write, "undefined."

Multiple Grouping Symbols

When a set of grouping symbols occurs *inside* another set of grouping symbols, we perform the operations within the innermost set first.

Example 3

Determine the value of each of the following.

$2+(8\cdot 3)-(5+6)$

Solution

Combine 8 and 3 first, then combine 5 and 6.

$2+24-11$ Now combine left to right.
$26-11$
15

Example 4

$10+[30-(2\cdot 9)]$

Solution

Combine 2 and 9 since they occur in the innermost set of parentheses.

$10+[30-18]$ Now combine 30 and 18.
$10+12$
22

The Order of Operations

Sometimes there are no grouping symbols indicating which operations to perform first. For example, suppose we wish to find the value of $3+5\cdot 2$. We could do *either* of two things:

Add 3 and 5, then multiply this sum by 2.

$$\begin{aligned} 3+5\cdot 2 &= 8\cdot 2 \\ &= 16 \end{aligned}$$

Multiply 5 and 2, then add 3 to this product.

$$\begin{aligned} 3+5\cdot 2 &= 3+10 \\ &= 13 \end{aligned}$$

We now have two different values for one expression. To determine the correct value, we must use the *accepted order of operations*.

Order of Operations

1. Perform all operations *inside* grouping symbols, beginning with the innermost set, in the order 2, 3, 4 described below,
2. Perform all exponential and root operations.
3. Perform all multiplications and divisions, moving left to right.
4. Perform all additions and subtractions, moving left to right.

Example 5

Determine the value of each of the following.

$21+3\cdot 12$ Multiply first.
$21+36$ Add.
57

Example 6

$(15-8)+5\cdot(6+4).$ Simplify inside parentheses first.
$7+5\cdot 10$ Multiply.
$7+50$ Add.
57

Example 7

$63-(4+6\cdot 3)+76-4$	Simplify first within the parenthesis by multiplying, then adding.
$63-(4+18)+76-4$	
$63-22+76-4$	Now perform the additions and subtractions, moving left to right.
$41+76-4$	Add 41 and 76: $41+76=117$.
$117-4$	Subtract 4 from 117: 117 - 4 = 113.
113	

Example 8

$7\cdot 6-4^2+1^5$	Evaluate the exponential forms, moving left to right.
$7\cdot 6-16-1$	Multiply 7 and 6: 7 \cdot 6 = 42
$42-16+1$	Subtract 16 from 42: 42 - 16 = 26
$26+1$	Add 26 and 1: $26+1=27$
27	

Example 9

$6\cdot(3^2+2^2)+4^2$	Evaluate the exponential forms in the parentheses: $3^2=9$ and $2^2=4$
$6\cdot(9+4)+4^2$	Add the 9 and 4 in the parentheses: $9+4=13$
$6\cdot(13)+4^2$	Evaluate the exponential form: $4^2=16$
$6\cdot(13)+16$	Multiply 6 and 13: $6\cdot 13=78$
$78+16$	Add 78 and 16: $78+16=94$
94	

Example 10

$$\frac{6^2+2^2}{4^2+6\cdot 2^2}+\frac{1^2+8^2}{10^2-19\cdot 5}$$

Recall that the bar is a grouping symbol.

The fraction · is equivalent to $(6^2+2^2)\div(4^2+6\cdot 2^2)$

$$\frac{36+4}{16+6\cdot 4}+\frac{1+64}{100-19\cdot 5}$$

$$\frac{36+4}{16+24}+\frac{1+64}{100-95}$$

$$\frac{40}{40}+\frac{65}{5}$$

$$1+13$$

$$14$$

This page titled 1.3.4: Order of Operations is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.3.4.1: Exercises

Section 1.3.4.1 Exercises

1. Evaluate each expression using the Order of Operations.
   a. $9 – 2 \cdot 3 + 5$
   b. $52 \div 4 + 2$
   c. $52 + 4 \div 2$
   d. $6(3+5)^2$
   e. $5+2(1+4)^3$
   f. $14-3(8-5)$
   g. $6[3(4)-2(5)]$
   h. $2 \cdot 2^4 + 25 \div 5 - 3^2$
   i. $2+5[8-2(4-1)]$
   j. $\dfrac{5+5(3)}{16-12}$
   k. $\dfrac{52}{3+10}$
   l. \(\dfrac{9 + 11}{8 - 4}
   m. $5 + 12 \div 4 \; cdot \; 2 – 8$
   n. $24 – 3 \cdot 2^4$
   o. $10 – (20 – 4 \cdot 2) \div 3$
   p. $3 \cdot 2^4$ - 5^2
   q. $\dfrac{72}{2^3-2^2}$
   r. $32 \div (6 – 2) \cdot 5$
2. Do any of the expressions have the same value?
   a. $20(100 – 2)$
   b. $20 \cdot 100 – 2$
   c. $20 \cdot 98$
   d. $20(2 – 100)$
3. Evaluate the formula for the given values.
   a. $d = rt$ for $r = 15$ and $t = 4.5$
   b. $F = mv^{\frac{2}{r}}$ for $m = 250$, $v = 30$, and $r = 120$
   c. $A = (t_1 + t_2 + t_3)/3$ for $t_1 = 72$, $t_2 = 85$, and $t_3 = 81$
4. Approximate the value of the Golden Mean, $\dfrac{1+\sqrt{5}}{2}$, rounded to the nearest hundredth.
5. Approximate $\dfrac{1}{\sqrt{60}}$. Round to three decimal places, if needed.
6. Approximate $\sqrt{\dfrac{20736}{18}}$. Round to three decimal places, if needed.

This page titled 1.3.4.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.4: Exponents

1.4.1 Learning Objectives

- Read, interpret, and write exponential notation

Exponential Notation

Definition: Exponential Notation

Just as multiplication is a description of repeated addition, **exponential notation** is a description of repeated multiplication.

Suppose we have the repeated multiplication $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$.

The factor 8 is repeated 5 times.

Definition: Exponent

Exponential notation uses a *superscript* for the number of times the factor is repeated. The superscript is placed on the repeated factor, 8^5, in this case. The superscript is called an **exponent**. An exponent records the number of identical factors that are repeated in a multiplication.

If x is any real number and n is a natural number, then
$x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ factors of } x}$
An exponent records the number of identical factors in a multiplication.

Example 1

Write the following multiplications using exponents.

1. $3 \cdot 3$
2. $62 \cdot 62 \cdot 62 \cdot 62 \cdot 62 \cdot 62 \cdot 62 \cdot 62 \cdot 62$

Solution

1. Since the factor 3 appears 2 times, we record this as 3^2.
2. Since the factor 62 appears 9 times, we record this as 62^9.

Example 3

Expand (write without exponents) each number.

1. 12^4
2. 706^3
3. 15^1

Solution

1. The exponent 4 is recording 4 factors of 12 in a multiplication. Thus, $12^4 = 12 \cdot 12 \cdot 12 \cdot 12$.
2. The exponent 3 is recording 3 factors of 706 in a multiplication. Thus, $706^3 = 706 \cdot 706 \cdot 706$.
3. The exponent 1 is recording 1 factor of 15 in a multiplication. Thus, $15^1 = 15$.

Try It Now 1

Write the following using exponents.

$37 \cdot 37$

Answer

37^2

Reading Exponential Notation

Base Exponent Power

In x^n,

x is the **base**
n is the **exponent**
The number represented by x^n is called a **power**

The term x^n is read as "x to the nth."

In a number such as 8^5.

Base
8 is called the **base**.

Exponent, Power
5 is called the **exponent**, or **power**. 8^5 is read as "eight to the fifth power," or more simply as "eight to the fifth," or "the fifth power of eight."

Squared
When a number is raised to the second power, it is said to be **squared**. The number 5^2 can be read as

5 to the second power, or
5 to the second, or
5 squared.

Cubed
When a number is raised to the third power, it is said to be **cubed**. The number 5^3 can be read as

5 to the third power, or
5 to the third, or
5 cubed.

When a number is raised to the power of 4 or higher, we simply say that that number is raised to that particular power. The number 5^8 can be read as

5 to the eighth power, or just
5 to the eighth.

This page titled 1.4: Exponents is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **2.6: Rules of Exponents** by Denny Burzynski & Wade Ellis, Jr. is licensed CC BY 4.0.

1.4.1: Rules of Exponents

1.4.2 Learning Objectives

- Interpret exponential expression with negative exponents
- Multiply exponential expressions with the same base using the Product Rule for Exponents
- Divide exponential expressions with the same base using the Quotient Rule for Exponents
- Use zero as an exponent

Negative Exponents

The first rule we wist to develop is the rule for interpreting negative exponents. A negative exponent implies that the exponential quantity is in the wrong position. For example:

$$b^{-1} = \frac{1}{b^1} = \frac{1}{b}$$

$$\frac{1}{b^{-1}} = b^1 = b$$

Notice that the value of the base does not change. The exponent changes to a positive value when it is moved into the correct position, being in the numerator or the denominator.

Example 1

Rewrite these exponential quantities with positive exponents.

1. 2^{-1}
2. 5^{-2}
3. $\frac{1}{7^{-1}}$
4. $\frac{1}{3^{-4}}$

Solution

1. $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
2. $5^{-2} = \frac{1}{5^2}$
3. $\frac{1}{7^{-1}} = 7^1 = 7$
4. $\frac{1}{3^{-4}} = 3^4$

The Product Rule for Exponents

The second rule we wish to develop is the rule for multiplying two exponential quantities having the **same base** and natural number exponents. The following examples suggest this rule:

$$x^2 \cdot x^4 = \underbrace{xx}_{2} \cdot \underbrace{xxxx}_{4} = \underbrace{xxxxxx}_{6} = x^6$$

$$2 + 4 = 6$$

$$a \cdot a^2 = \underbrace{a}_{1} \cdot \underbrace{aa}_{2} = \underbrace{aaa}_{3} = a^3$$

$$1 + 2 = 3$$

Product Rule for Exponents

If x is a real number and n and m are natural numbers,

$x^n x^m = x^{n+m}$

To multiply two exponential quantities having the same base, add the exponents. Keep in mind that the exponential quantities being multiplied **must** have the **same base** for this rule to apply.

Example 2

Find the following products. All exponents are natural numbers.

1. $2^3 \cdot 2^5 = 2^{3+5} = 2^8$
2. $a^6 \cdot a^1 4 = a^{6+14} = a^2 0$
3. $y^5 \cdot y = y^5 \cdot y^1 = y^{5+1} = y^6$

Example 3

$x^3 y^4 \neq (xy)^{3+4}$

Since the bases are not the same, the product rule does not apply.

The Quotient Rule for Exponents

The third rule we wish to develop is the rule for dividing two exponential quantities having the same base and natural number exponents.
The following examples suggest a rule for dividing two exponential quantities having the same base and natural number exponents.

$\frac{x^5}{x^2} = \frac{xxxxx}{xx} = \frac{\cancel{(xx)}xxx}{\cancel{(xx)}} = xxx = x^3$. Notice that $5 - 2 = 3$

$\frac{a^8}{a^3} = \frac{aaaaaaaa}{aaa} = \frac{\cancel{(aaa)}aaaaa}{\cancel{(aaa)}} = aaaaa = a^5$. Notice that $8 - 3 = 5$

Quotient Rule for Exponents

If x is a real number and n and m are natural numbers,

$\frac{x^n}{x^m} = x^{n-m}, x \neq 0.$

Example 3

Find the following quotients. All exponents are natural numbers.

1. $\frac{3^5}{3^2} = 3^{5-2} = 3^3$
2. $\frac{27a^3b^6c^2}{3a^2bc} = 9a^{3-2}b^{6-1}c^{2-1} = 9ab^5c$

When we make the subtraction, $n - m$, in the division, $\frac{x^n}{x^m}$, there are three possibilities for the values of the exponents:

The exponent of the numerator is greater than the exponent of the denominator, that is, $n > m$. Thus, the exponent, $n - m$, is a natural number.

The exponents are the same, that is, $n = m$. Thus, the exponent, $n - m$, is zero, a whole number.

The exponent of the denominator is greater than the exponent of the numerator, that is, $n < m$. Thus, the exponent, $n - m$, is a negative integer.

Zero as an Exponent

In the examples above, the exponents of the numerators were greater than the exponents of the denominators. Let's study the case when the exponents are the same.

When the exponents are the same, say n, the subtraction $n - n$ produces 0.

Thus, by the second rule of exponents, $\frac{x^n}{x^n} = x^{n-n} = x^0$

But what real number, if any, does x^0 represent? Let's think for a moment about our experience with division in arithmetic. We know that any nonzero number divided by itself is one.

$\frac{8}{8} = 1, \frac{43}{43} = 1, \frac{258}{258} = 1$

Since the letter x represents some nonzero real number divided by itself. Then $\frac{x^n}{x^n} = 1$.

But we have also established that if $x \neq 0$, $\frac{x^n}{x^n} = x^0$. We now have that $\frac{x^n}{x^n} = x^0$ and $\frac{x^n}{x^n} = 1$. This implies that $x^0 = 1, x \neq 0$.

Exponents can now be natural numbers and zero. We have enlarged our collection of numbers that can be used as exponents from the collection of natural numbers to the collection of whole numbers.

Zero as an Exponent

If $x \neq 0, x^0 = 1$

Any number, other than 0, raised to the power of 0, is 1. 0^0 has no meaning (it does not represent a number).

Example 4

Find each value. Assume the base is not zero.

1. $6^0 = 1$
2. $246^0 = 1$
3. $4y^0 = 4 \cdot 1 = 4$
4. $\frac{y^6}{y^6} = y^0 = 1$

This page titled 1.4.1: Rules of Exponents is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **2.6: Rules of Exponents** by Denny Burzynski & Wade Ellis, Jr. is licensed CC BY 4.0.

1.4.1.1: 1.4.1 and 1.4.2 Exercises

Section 1.4.1 and 1.4.2 Exercises

1. Rewrite each expression using exponents.
   a. $3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 5$
   b. $(2.45)(2.45)(2.45)(2.45)(2.45)$
2. Rewrite these expressions with positive exponents only. Then evaluate the expression.
   a. 5^{-2}
   b. $\dfrac{1}{5^{-2}}$
   c. $\dfrac{3 \cdot 5}{5^{-2}}$
3. Evaluate:
   a. $47^0 + 5$
   b. $(5.2 + 5.97)^0$
   c. $-3(5^1 - 4 \cdot 2^0)$
4. Write these numbers in order by size: $-5^2, 5^{-2}, (5)^2, -5^{-2}$
5. Evaluate:
   a. $3^6 \cdot 3^{-3}$
   b. $\dfrac{4^{-2}}{4^{-3}}$
6. Evaluate. Write the results as a fraction and as a decimal.
   a. 10^{-5}
   b. 5^{-2}
   c. $(-2)^{-3}$
   d. 6×10^{-2}
   e. $\dfrac{16}{[(4 \times 10^{-2}) - (2 \times 10^{-2})]}$
7. Are any of these the same: 4^{-2} , $(-4)^{-2}$, -4^2 , $(-4)^2$, $(1/4)^{-2}$, $(-1/4)^2$
8. Evaluate for $x = 2, y = 3$, and $z = 5$.
   a. $(4x^2y^3) \cdot (3x^{-3}y^{-2})$
   b. $\dfrac{24x^{-2}y^3}{6x^3y^{-1}}$
   c. $\left(\dfrac{2x^{-2}y^{-3}}{z^2}\right)$
9. Complete the following:
   a. Show why $(2^3)^2 = 2^{3 \cdot 2}$.
   b. Show why $(2 \cdot 3)^2 = 2^2 \cdot 3^2$.
   c. Show why $\left(\dfrac{2}{3}\right)^2 = \dfrac{2^2}{3^2}$

This page titled 1.4.1.1: 1.4.1 and 1.4.2 Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.4.2: The Power Rules for Exponents

1.4.3 Learning Objectives

- Use the Power Rule for Powers to simplify expressions involving exponents
- Use the Power Rule for Products to simplify expressions involving exponents
- Use the Power Rule for Quotients to simplify expressions involving exponents

The Power Rule for Powers

The following examples suggest a rule for raising a power to a power:

$(a^2)^3 = a^2 \cdot a^2 \cdot a^2$
Using the product rule we get:
$(a^2)^3 = a^{2+2+2}$
$(a^2)^3 = a^{3 \cdot 2}$
$(a^2)^3 = a^6$

$(x^9)^4 = x^9 \cdot x^9 \cdot x^9 \cdot x^9$
$(x^9)^4 = x^{9+9+9+9}$
$(x^9)^4 = x^{4 \cdot 9}$
$(x^9)^4 = x^{36}$

Power Rule for Powers

If x is a real number and n and m are natural numbers,
$(x^n)^m = x^{n \cdot m}$

To raise a power to a power, multiply the exponents.

Example 1

Simplify each expression using the power rule for powers. All exponents are natural numbers.

1. $(7^3)^4 = 7^{3 \cdot 4} = 7^{12}$
2. $(y^5)^3 = y^{5 \cdot 3} = y^{15}$
3. $(d^{20})^6 = d^{20 \cdot 6} = d^{120}$

Try It Now 1

Simplify the expression using the power rule for powers.

$(8^5)^4$

Answer

8^{20}

Try It Now 2

Simplify the expression using the power rule for powers.

$(y^7)^7$

Answer

y^{49}

The Power Rule for Products

The following examples suggest a rule for raising a product to a power:

$(ab)^3 = ab \cdot ab \cdot ab$ Use the commutative property of multiplication.
$= aaabbb$
$= a^3b^3$

$(xy)^5 = xy \cdot xy \cdot xy \cdot xy \cdot xy$
$= xxxxx \cdot \text{ yyyyy}$
$= x^5y^5$

$(4xyz)^2 = 4xyz \cdot 4xyz$
$= 4 \cdot 4 \cdot xx \cdot yy \cdot zz$
$= 16x^2y^2z^2$

Power Rule for Products

If x and y are real numbers are n is a natural number,
$(xy)^n = x^n y^n$

To raise a product to a power, apply the exponent rule to each and every factor

Example 2

Make use of either the power rule for products or power rule for powers to simplify each expression.

1. $(2b)^7 = 2^7b^7$
2. $(axy)^4 = a^4x^4y^4$
3. $(3ab)^2 = 3^2a^2b^2 = 9a^2b^2$

   Don't forget to apply the exponent to the 3!

Example 3

Make use of both the power rule for products and power rule for powers to simplify each expression.

1. $(ab^3)^2 = a^2(b^3)^2 = a^2b^6$

   We used two rules here. First, the power rule for products. Second, the power rule for powers.

2. $(7a^4b^2c^8)^2 = 7^2(a^4)^2(b^2)^2(c^8)^2 = 49a^8b^4c^{16}$
3. $[2(x+1)^4]^6 = 2^6(x+1)^{24} = 64(x+1)^{24}$

Example 4

If $6a^3c^7 \neq 0$, then $(6a^3c^7)^0 = 1$. Recall that $x^0 = 1$ for $x \neq 0$

The Power Rule for Quotients

The following example suggests a rule for raising a quotient to a power.

$$\left(\frac{a}{b}\right)^3 = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \frac{a \cdot a \cdot a}{b \cdot b \cdot b} = \frac{a^3}{b^3}$$

Power Rule for Quotients

If x and y are real numbers and n is a natural number,

$$(\frac{x}{y})^n = \frac{x^n}{y^n}, y \neq 0$$

To raise a quotient to a power, distribute the exponent to both the numerator and denominator.

Example 5

Make use of the power rule for quotients, the power rule for products, the power rule for powers, or a combination of these rules to simplify each expression. All exponents are natural numbers.

1. $(\frac{x}{y})^6 = \frac{x^6}{y^6}$
2. $(\frac{2x}{b})^4 = \frac{(2x)^4}{b^4} = \frac{2^4x^4}{b^4} = \frac{16x^4}{b^4}$
3. $(\frac{a^3}{b^5})^7 = \frac{(a^3)^7}{(b^5)^7} = \frac{a^21}{b^35}$

This page titled 1.4.2: The Power Rules for Exponents is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **2.7: The Power Rules for Exponents** by Denny Burzynski & Wade Ellis, Jr. is licensed CC BY 4.0.

1.4.2.1: Exercises

Section 1.4.3.1 Exercises

1. True or False. If false, correct the statement.
   a. $(3ab)^3 = 9a^3b^3$
   b. $\dfrac{49x^2y^4}{36z^6} = \left(\dfrac{7xy^2}{6z^3}\right)^2$
   c. $\left(\dfrac{xy}{z}\right)^{-5} = \dfrac{z^5}{xy^5}$
2. Simplify the product.
   a. $(9ab^2)^4$
   b. $(10x)^2(5x)^3$
   c. $(-2p^3qr^2)^5$
3. Simplify the quotient. Write in simplest terms.
   a. $\left(\dfrac{5a^2b}{6ab^2}\right)^3$
   b. $\left(\dfrac{8p^3q^4}{-2pq}\right)^2$
   c. $\left(\dfrac{2xy}{4x^2}\right)^{-4}$
4. Simplify the expression. Write in simplest terms.
   a. $(7ab)^2(5bc)^3(2ac)^4$
   b. $\left(\dfrac{3pq^2}{4r^3}\right)^3\left(\dfrac{2p^3r}{qt^2}\right)^5$
   c. $\left(\dfrac{(2xy)(3x^2)^4}{4x^2y^{-3}}\right)^4$

This page titled 1.4.2.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.4.3: Scientific Notation

1.4.4 Learning Objectives

- Change numbers in standard form to scientific form
- Change numbers in scientific form to standard form
- Perform computations on numbers in scientific form

Standard Form to Scientific Form

Very large numbers such as $43,000,000,000,000,000,000,000$ (the number of different possible configurations of Rubik's cube) and very small numbers such as $0.000000000000000000000034$ (the mass of the amino acid tryptophan) are extremely inconvenient to write and read. Such numbers can be expressed more conveniently by writing them as part of a power of 10.

To see how this is done, let us start with a somewhat smaller number such as 2480. Notice that

$$\begin{aligned} \underbrace{2480}_{\text{Standard form}} &= 248.0 \times 10^1 \\ &= 24.80 \times 10^2 \\ &= \underbrace{2.480 \times 10^3}_{\text{Scientific form}} \end{aligned}$$

Scientific Form

The last form is called the **scientific form** of the number. There is **one** nonzero digit to the left of the decimal point and the absolute value of the exponent on 10 records the number of places the original decimal point was moved to the **left**.

$$\begin{aligned} 0.00059 &= \frac{0.0059}{10} = \frac{0.0059}{10^1} = 0.0059 \times 10^{-1} \\ &= \frac{0.059}{100} = \frac{0.059}{10^2} = 0.059 \times 10^{-2} \\ &= \frac{0.59}{1000} = \frac{0.59}{10^3} = 0.59 \times 10^{-3} \\ &= \frac{5.9}{10,000} = \frac{5.9}{10^4} = 5.9 \times 10^{-4} \end{aligned}$$

There is **one** nonzero digit to the left of the decimal point and the absolute value of the exponent of 10 records the number of places the original decimal point was moved to the **right**.

Scientific Notation

Numbers written in scientific form are also said to be written using scientific notation. In **scientific notation**, a number is written as the product of a number between and including 1 and 10 (1 is included,10 is not) and some power of 10.

Writing a Number in Scientific Notation

To write a number in scientific notation:

1. Move the decimal point so that there is one nonzero digit to its left.
2. Multiply the result by a power of 10 using an exponent whose absolute value is the number of places the decimal point was moved. Make the exponent positive if the decimal point was moved to the left and negative if the decimal point was moved to the right.

Example 1

Write 981 in scientific notation.

The number 981 is actually $981.$, and it is followed by a decimal point. In integers, the decimal point at the end is usually omitted.

$981 = 981. = 9.81 \times 10^2$

The decimal point is now two places to the left of its original position, and the power of 10 is 2.

Example 2

Write 54.066 in scientific notation.

$54.066 = 5.4066 \times 10^1 = 5.4066 \times 10$

The decimal point is one place to the left of its original position, and the power of 10 is 1

Example 3

Write 0.000000000004632 in scientific notation.

$0.000000000004632 = 4.632 \times 10^{-12}$

The decimal point is twelve places to the right of its original position, and the power of 10 is -12.

Example 4

Write 0.027 in scientific notation.

$0.027 = 2.7 \times 10^{-2}$

The decimal point is two places to the right of its original position, and the power of 10 is -2

Scientific Form to Standard Form

A number written in scientific notation can be converted to standard form by reversing the above process.

Converting from Scientific Notation

To convert a number written in scientific notation to a number in standard form, move the decimal point the number of places prescribed by the exponent on the 10.

Positive Exponent/Negative Exponent

Move the decimal point to the right when you have a positive exponent, and move the decimal point to the left when you have a negative exponent.

Example 5

Write 4.673×10^4 in standard form.

The exponent of 10 is 4 so we must move the decimal point to the right 4 places (including 0's, if necessary).

$4.6730 \times 10^4 = 46730$

Example 6

Write 2.9×10^7 in standard form.

The exponent of 10 is 7 so we must move the decimal point to the right 7 places (including 0's, if necessary).

$2.9 \times 10^7 = 29000000$

Example 7

Write 1×10^{27} in standard form.

The exponent of 10 is 27 so we must move the decimal point to the right 27 places (including 0's, if necessary).

$1 \times 10^{27} = 1,000,000,000,000,000,000,000,000,000$

Example 8

Write 4.21×10^{-5} in standard form.

The exponent of 10 is -5 so we must move the decimal point to the left 5 places (including 0's, if necessary).

$4.21 \times 10^{-5} = 0.0000421$

Example 9

Write 1.006×10^{-18} in standard form.

The exponent of 10 is -18 so we must move the decimal point to the left 18 places (including 0's, if necessary).

$1.006 \times 10^{-18} = 0.000000000000000001006$

Working with Numbers in Scientific Notation

Multiplying and Dividing Numbers Using Scientific Notation

There are many occasions (particularly in the sciences) when it is necessary to find the product or quotient of two numbers written in scientific notation. This is accomplished by using two of the basic rules of algebra.

For Multiplication:

Suppose we wish to find $(a \times 10^n)(b \times 10^m)$. Since the only operation is multiplication, we can use the commutative property of multiplication to rearrange the numbers.

$(a \times 10^n)(b \times 10^m) = (a \times b)(10^n \times 10^m)$

Then, by the rules of exponents, $10^n \times 10^m = 10^{n+m}$. Thus,

$(a \times 10^n)(b \times 10^m) = (a \times b) \times 10^{n+m}$

The product of $(a \times b)$ may not be between 1 and 10, so $(a \times b) \times 10^{n+m}$ may not be in scientific form. The decimal point in $(a \times b)$ may have to be moved. There is an example of this in the exercises below.

Example 10

$$\begin{aligned} \left(2 \times 10^3\right)\left(4 \times 10^8\right) &= (2 \times 4)\left(10^3 \times 10^8\right) \\ &= 8 \times 10^{3+8} \\ &= 8 \times 10^{11} \end{aligned}$$

Example 11

$$\begin{aligned} \left(5 \times 10^{17}\right)\left(8.1 \times 10^{-22}\right) &= (5 \times 8.1)\left(10^{17} \times 10^{-22}\right) \\ &= 40.5 \times 10^{17-22} \\ &= 40.5 \times 10^{-5} \end{aligned}$$

We need to move the decimal point one place to the left to put this number in scientific notation.
Thus, we must also change the exponent of 10.

40.5×10^{-5}
$4.05 \times 10^1 \times 10^{-5}$
$4.05 \times \left(10^1 \times 10^{-5}\right)$
$4.05 \times \left(10^{1+(-5)}\right)$
$4.05 \times \left(10^{1-5}\right)$
4.05×10^{-4}

Thus,
$\left(5 \times 10^{17}\right)\left(8.1 \times 10^{-22}\right) = 4.05 \times 10^{-4}$

For Division:

Suppose we wish to find $\frac{a \times 10^n}{b \times 10^m}$. Similar to multiplication we will separate the exponential terms from the constant terms using the multiplication property of fractions.

$$\frac{a \times 10^n}{b \times 10^m} = \frac{a}{b} \times \frac{10^n}{10^m}$$

Then, by the rules of exponents, $10^n \div 10^m = 10^{n-m}$. Thus,

$$\frac{a \times 10^n}{b \times 10^m} = (a \div b) \times 10^{n-m}$$

Be attentive to any negative exponents when applying the subtraction rule of exponents. Remember to include the negative sign when performing the operation to get the proper exponent. Review the examples below for different scenarios.

The quotient of $(a \div b)$ may not be between 1 and 10, so $(a \div b) \times 10^{n-m}$ may not be in scientific form. The decimal point in $(a \div b)$ may have to be moved. There is an example of this in the exercises below.

Example 12

$$\begin{aligned} \frac{9 \times 10^7}{5 \times 10^4} &= \\ &= \frac{9}{5} \times \frac{10^7}{10^4} \\ &= \frac{9}{5} \times 10^{7-4} \\ &= 1.8 \times 10^3 \end{aligned}$$

Example 13

$$\begin{aligned} \frac{8.7 \times 10^{-4}}{3.2 \times 10^3} &= \\ &= \frac{8.7}{3.2} \times \frac{10^{-4}}{10^3} \\ &= \frac{8.7}{3.2} \times 10^{-4-3} \\ &= 2.72 \times 10^{-7} \end{aligned}$$

Example 13

$$\begin{aligned} \frac{-5.3 \times 10^{-7}}{2 \times 10^{-3}} &= \\ &= \frac{-5.3}{2} \times \frac{10^{-7}}{10^{-3}} \\ &= \frac{-5.3}{2} \times 10^{-7-(-3)} \\ &= \frac{-5.3}{2} \times 10^{-7+3} \\ &= -2.65 \times 10^{-4} \end{aligned}$$

Example 14

$$\begin{aligned}\frac{1.2\times10^{7}}{4.4\times10^{9}} &= \\ &= \frac{1.2}{4.4}\times\frac{10^{7}}{10^{9}} \\ &= \frac{1.2}{4.4}\times10^{7-9} \\ &= 0.272\times10^{-2}\end{aligned}$$

We need to move the decimal point one place to the right to put this number in scientific notation.
Thus, we must also change the exponent of 10.

$$0.272\times10^{-2}$$
$$2.72\times10^{-1}\times10^{-2}$$
$$2.72\times\left(10^{-1}\times10^{-2}\right)$$
$$2.72\times\left(10^{-1+(-2)}\right)$$
$$2.72\times\left(10^{-1-2}\right)$$
$$2.72\times10^{-3}$$

Therefore,

$$\frac{1.2\times10^{7}}{4.4\times10^{9}} = 2.72\times10^{-3}$$

This page titled 1.4.3: Scientific Notation is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **3.8: Scientific Notation** by Denny Burzynski & Wade Ellis, Jr. is licensed CC BY 4.0.

1.4.3.1: Exercises

Section 1.4.4.1 Exercises

1. Change from standard notation to scientific notation.
    1. 492,000
    2. 23,487,000,000,000
    3. 0.000098
    4. 0.00000000000938
    5. Change from scientific notation to standard notation.
        a. 6.32×10^{5}
        b. 8.3×10^{13}
        c. 5.26×10^{-3}
        d. 8.265×10^{-8}
    6. Multiply or divide the numbers in scientific form. Leave the answer in scientific form.
        a. $(3.7 \times 10^{8})(1.5 \times 10^{3})$
        b. $(4.2 \times 10^{4})(6.8 \times 10^{-7})$
        c. $(3.7 \times 10^{8}) \div (1.5 \times 10^{3})$
        d. $(4.2 \times 10^{4}) \div (6.8 \times 10^{-7})$
        e. $\dfrac{(6 \times 10^{7})(4.8 \times 10^{-6})}{2.4 \times 10^{5}}$
    7. If the U.S. annual production of sugar is 9 million tons, estimate the number of grains of sugar produced in a year in the U.S. Use scientific notation. (Note: There are 2,000 lbs per ton; assume there are 2,260,000 grains in a pound of sugar.)
    8. The national debt in 2020 was approximately $26.7 trillion. Write this number in scientific notation. Suppose, also, that in 2020 there were about 331,500,000 people in the U.S. If the debt is divided equally among these people, how much is each person's share?
    9. For the next few problems, you will need the following information.
        One astronomical unit (AU) is the average distance of the earth from the sun, or approximately 93,000,000 miles.
        Jupiter's average distance from the sun is 5.2 AU.

        The time it takes an object moving at a constant speed is given by the formula $t = \dfrac{d}{r}$, where d represents distance traveled, r represents speed of the object, and t represents time the object is in motion.
        a. Write the distance of the earth from the sun in scientific form.
        b. What is Jupiter's average distance from the sun in miles? Do the computation in both standard form and scientific form. Which is easier?
        c. How long does it take light from the sun to reach earth? How many minutes is this?
        d. How far will light travel in one year?

This page titled 1.4.3.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.5: Percents

1.5 Learning Objectives

- Find what percent one number is of another
- Change fractions to percents
- Rewrite decimals as percents
- Find a percent of a number
- Solve applications involving percents

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties[1]. Who is correct? How can we make sense of these numbers?

Percent literally means “per 100,” or “parts per hundred.” When we write 40%, this is equivalent to the fraction $\frac{40}{100}$ or the decimal 0.40. Notice that 80 out of 200 and 10 out of 25 are also 40%, since $\frac{80}{200} = \frac{10}{25} = \frac{40}{100}$.

Example 1

243 people out of 400 state that they like dogs. What percent is this?

Solution

$\frac{243}{400} = 0.6075 = \frac{60.75}{100}$. This is 60.75%.

Notice that the percent can be found from the equivalent decimal by moving the decimal point two places to the right.

Example 2

Write each as a percent: a) $\frac{1}{4}$ b) 0.02 c) 2.35

Solution

a) $\frac{1}{4} = 0.25 = 25\%$ b) $0.02 = 2\%$ c) $2.35 = 235\%$

Percents

If we have a *part* that is some *percent* of a *whole*, then

$\text{percent} = \frac{\text{part}}{\text{whole}}$, or equivalently, $\text{part} = \text{percent} \cdot \text{whole}$

To do the calculations, we write the percent as a decimal.

Notice that the equation translated to English would be "part is percent **of** the whole." So direct multiplication is used.

In some cases the question will ask for the percent **off** the whole, which is different.

If we have a *part* that is some *percent* off a *whole*, then

$\text{part} = (100\% - \text{percent}) \cdot \text{whole}$

Since the statement is the percent **off**, to find the correct value for the part, the percent must be subtracted from 100% first.

Example 3

The sales tax in a town is 9.4%. How much tax will you pay on a $140 purchase?

Solution

Here, $140 is the whole, and we want to find 9.4% *of* $140. We start by writing the percent as a decimal by moving the decimal point two places to the left (which is equivalent to dividing by 100). We can then compute:

$\text{tax} = 0.094(140) = \$13.16$ in tax.

Example 4

In the news, you hear "tuition is expected to increase by 7% next year." If tuition this year was $1200 per quarter, what will it be next year?

Solution

The tuition next year will be the current tuition plus an additional 7%, so it will be 107% of this year's tuition:

$\$1200(1.07) = \$1284$

Alternatively, we could have first calculated 7% of $1200: $\$1200(0.07) = \$84$

Notice this is *not* the expected tuition for next year (we could only wish). Instead, this is the expected *increase*, so to calculate the expected tuition, we'll need to add this change to the previous year's tuition:

$\$1200 + \$84 = \$1284$

Try it Now 1

A TV originally priced at $799 is on sale for 30% off. There is then a 9.2% sales tax. Find the price after including the discount and sales tax.

Answer

The sale price is $\$799(1 - 0.30) = \$799(0.70) = \$559.30$ After tax, the price is $\$559.30(1.092) = \$610.76$

Example 5

The value of a car dropped from $7400 to $6800 over the last year. What percent decrease is this?

Solution

To compute the percent change, we first need to find the dollar value change: $\$6800 - \$7400 = -\$600$ Often we will take the absolute value of this amount, which is called the **absolute change**: $|-600| = 600$.

Since we are computing the decrease relative to the starting value, we compute this percent out of $7400:

$\frac{600}{7400} = 0.081 = 8.1\%$ decrease. This is called a **relative change**.

Absolute and Relative Change

Given two quantities,

Absolute change = $|\text{ending quantity} - \text{starting quantity}|$

Relative change: $\dfrac{\text{absolute change}}{\text{starting quantity}}$

Absolute change has the same units as the original quantity.

Relative change gives a percent change.

The starting quantity is called the **base** of the percent change.

The base of a percent is very important. For example, while Nixon was president, it was argued that marijuana was a “gateway” drug, claiming that 80% of marijuana smokers went on to use harder drugs like cocaine. The problem is, this isn’t true. The true claim is that 80% of harder drug users first smoked marijuana. The difference is one of base: 80% of marijuana smokers using hard drugs, vs. 80% of hard drug users having smoked marijuana. These numbers are not equivalent. As it turns out, only one in 2,400 marijuana users actually go on to use harder drugs[2].

Example 6

There are about 75 QFC supermarkets in the U.S. Albertsons has about 215 stores. Compare the size of the two companies.

Solution

When we make comparisons, we must ask first whether an absolute or relative comparison. The absolute difference is $215-75=140$. From this, we could say “Albertsons has 140 more stores than QFC.” However, if you wrote this in an article or paper, that number does not mean much. The relative difference may be more meaningful. There are two different relative changes we could calculate, depending on which store we use as the base:

Using QFC as the base, $\frac{140}{75}=1.867$.

This tells us Albertsons is 186.7% larger than QFC.

Using Albertsons as the base, $\frac{140}{215}=0.651$.

This tells us QFC is 65.1% smaller than Albertsons.

Notice both of these are showing percent *differences*.

We could also calculate the size of Albertsons relative to QFC: 215/75, which means Albertsons is 2.867 or 286.7% times the size of QFC. Likewise, we could calculate the size of QFC relative to Albertsons: 75/215, which tells us that QFC is 34.9% of the size of Albertsons.

Example 7

Suppose a stock drops in value by 60% one week, then increases in value the next week by 75%. Is the value higher or lower than where it started?

Solution

To answer this question, suppose the value started at \$100. After one week, the value dropped by 60%:

$\$100-\$100(0.60)=\$100-\$60=\$40$.

In the next week, notice that base of the percent has changed to the new value, \$40. Computing the 75% increase:

$\$40+\$40(0.75)=\$40+\$30=\$70$.

In the end, the stock is still \$30 lower, or $\frac{\$30}{\$100}=30\%$ lower, valued than it started.

Try it Now 2

The U.S. federal debt at the end of 2001 was \$5.77 trillion, and grew to \$6.20 trillion by the end of 2002. At the end of 2005 it was \$7.91 trillion, and grew to \$8.45 trillion by the end of 2006[3]. Calculate the absolute and relative increase for 2001-2002 and 2005-2006. Which year saw a larger increase in federal debt?

Answer

2001-2002: Absolute change: \$0.43 trillion. Relative change: 7.45%

2005-2006: Absolute change: \$0.54 trillion. Relative change: 6.83%

2005-2006 saw a larger absolute increase, but a smaller relative increase.

Example 8

A Seattle Times article on high school graduation rates reported “The number of schools graduating 60 percent or fewer students in four years – sometimes referred to as “dropout factories” – decreased by 17 during that time period. The number of kids attending schools with such low graduation rates was cut in half.”

a) Is the “decrease by 17” number a useful comparison?

b) Considering the last sentence, can we conclude that the number of “dropout factories” was originally 34?

Solution

a) This number is hard to evaluate, since we have no basis for judging whether this is a large or small change relative to the number of such schools. If the number of “dropout factories” dropped from 20 to 3, that would be a very significant change, but if the number dropped from 217 to 200, that would be less of an improvement.

b) The last sentence provides relative change which helps put the first sentence in perspective. We can estimate that the number of “dropout factories” was probably previously around 34. However, it’s possible that students simply moved schools rather than the school improving, so that estimate might not be fully accurate.

Example 9

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties. Who is correct?

Solution

Without more information, it is hard for us to judge who is correct, but we can easily conclude that these two percents are talking about different things, so one does not necessarily contradict the other. Edward’s claim was a percent with coalition forces as the base of the percent, while Cheney’s claim was a percent with both coalition and Iraqi security forces as the base of the percent. It turns out both statistics are in fact fairly accurate.

Try it Now 3

In the 2012 presidential elections, one candidate argued that “the president’s plan will cut $716 billion from Medicare, leading to fewer services for seniors,” while the other candidate rebuts that “our plan does not cut current spending and actually expands benefits for seniors, while implementing cost saving measures.” Are these claims in conflict, in agreement, or not comparable because they’re talking about different things?

Answer

Without more information, it is hard to judge these arguments. This is compounded by the complexity of Medicare. As it turns out, the $716 billion is not a cut in current spending, but a cut in future increases in spending, largely reducing future growth in health care payments. In this case, at least the numerical claims in both statements could be considered at least partially true. Here is one source of more information if you’re interested: http://factcheck.org/2012/08/a-campaign-full-of-mediscare/

We’ll wrap up our review of percents with a few cautions.

First, when talking about a change in quantities that are already measured in percents, we have to be careful in how we describe the change.

Example 10

A politician’s support increases from 40% of voters to 50% of voters. Describe the change.

Solution

We could describe this using an absolute change: $|50\% - 40\%| = 10\%$. Notice that since the original quantities were percents, this change also has the units of percent. In this case, it is best to describe this as an increase of 10 **percentage points**.

In contrast, we could compute the percent change: $\frac{10\%}{40\%} = 0.25 = 25\%$ increase. This is the relative change, and we'd say the politician's support has increased by 25%.

Second, pay attention to shifting base values.

Example 11

An employer asks his employee to take a temporary 10% pay cut for 6 months because the company has hit some hard times. At the end of the 6 months the employer will give the employee a 10% raise. Will the employee make more, less, or the same amount as before the pay cut? Is that fair?

Solution

Suppose the employee makes $4500 per month. The employer is proposing to take 10% off the current salary.

$\text{part} = \$4500\,(1 - 0.10) = \$4500\,(0.90) = \$4050$

After 6 months pass the employer promises to give a 10% raise on the salary. The employee will retain 100% of what they have plus 10%.

$\text{part} = \$4050\,(1 + 0.10) = \$4050\,(1.10) = \$4455$

Since the value that the percent is based on has changed the employee would not be restored to their original salary in this scenario.

Last, beware of averaging percents.

Example 12

A basketball player scores on 40% of 2-point field goal attempts, and on 30% of 3-point of field goal attempts. Find the player's overall field goal percentage.

Solution

It is very tempting to average these values, and claim the overall average is 35%, but this is likely not correct, since most players make many more 2-point attempts than 3-point attempts. We don't actually have enough information to answer the question. Suppose the player attempted 200 2-point field goals and 100 3-point field goals. Then they made 200(0.40) = 80 2-point shots and $100(0.30) = 30$ 3-point shots. Overall, they made 110 shots out of 300, for a $\frac{110}{300} = 0.367 = 36.7\%$ overall field goal percentage.

---

[1] www.factcheck.org/cheney_edwards_mangle_facts.html

[2] http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics

[3] www.whitehouse.gov/sites/defa...s/hist07z1.xls

This page titled 1.5: Percents is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **1.2: Percents** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

1.5.1: Exercises

Section 1.5.1

1. Write each percent as a fraction.
   a. 87%
   b. 40%
   c. 110%
2. Write as a percent.
   a. 5858
   b. 4545
3. Write each percent as a decimal.
   a. 27%
   b. 8.328%
   c. 125%
   d. 0.5%
4. Change each decimal to a percent.
   a. 0.34
   b. 2.5
   c. 0.0045
5. 50 is 40% of what number?
6. A shirt that usually sells for $25 is on sale for $21. What is the percent decrease in the price of the shirt?
7. A car salesman has a commission rate of 12% on the cars sold. If the commission on one sale was $1836, what was the purchase price of the car?
8. You purchase an item originally priced at $240 that is discounted 30%. What is the sale price of the item?
9. You pay sales tax of 9.5% on your purchase. What is the total cost for your purchase with the discount and the sales tax?
10. The sales clerk accidentally added the sales tax based on the original price of your purchase before giving you a 30% discount. Would this error cause the item to cost you more or less or the same as what you figured in part b)?
11. A store advertises a 40% off clearance sale. When you arrived at the store, you received a coupon for an additional 10% off. The coupon is applied after the 40% discount.
    a. What is the total discount if the item is normally priced at $30?
    b. What is the total discount percent?
    c. Why do you think the store uses two discount percents instead of one? Explain.
12. Consider the following.
    a. A parenting blog states that 12-year-old girls typically weigh 200% of what 6-year-old girls weigh? Does that mean a girl's weight doubles between ages 6 and 12?
    b. Could it be true that an 8-year-old girl weighs 30% more than a 6-year- old girl?
13. Out of 230 racers who started the marathon, 212 completed the race, 14 gave up, and 4 were disqualified. What percentage did not complete the marathon?
14. Patrick left an $8 tip on a $50 restaurant bill. What percent tip is that?
15. Ireland has a 23% VAT (value-added tax, similar to a sales tax). How much will the VAT be on a purchase of a €250 item?
16. Employees in 2012 paid 4.2% of their gross wages towards social security (FICA tax), while employers paid another 6.2%.
    a. How much will someone earning $45,000 a year pay towards social security out of their gross wages?
    b. What is the total amount contributed towards the worker's social security?
17. A project on Kickstarter.com was aiming to raise $15,000 for a precision coffee press. They ended up with 714 supporters, raising 557% of their goal. How much did they raise?
18. Another project on Kickstarter for an iPad stylus raised 1,253% of their goal, raising a total of $313,490 from 7,511 supporters. What was their original goal?
19. The population of a town increased from 3,250 in 2008 to 4,300 in 2010. Find the absolute and relative (percent) increase.

20. The number of CDs sold in 2010 was 114 million, down from 147 million the previous year[1]. Find the absolute and relative (percent) decrease.
21. A company wants to decrease their energy use by at least 15%. If their next bill is $1,700 a month, were they successful? Why or why not?
22. A store is hoping an advertising campaign will increase their number of customers by at least 30%. They currently have about 80 customers a day.
    a. How many customers will they have if their campaign is successful?
    b. If they increase to 120 customers a day, were they successful? Why or why not?
23. An article reports “attendance dropped 6% this year, to 300.” What was the attendance before the drop?
24. An article reports “sales have grown by 30% this year, to $200 million.” What were sales before the growth?
25. The Walden University had 47,456 students in 2010, while Kaplan University had 77,966 students. Complete the following statements:
    a. Kaplan’s enrollment was ___% larger than Walden’s.
    b. Walden’s enrollment was ___% smaller than Kaplan’s.
    c. Walden’s enrollment was ___% of Kaplan’s.
26. In the 2012 Olympics, Usain Bolt ran the 100m dash in 9.63 seconds. Jim Hines won the 1968 Olympic gold with a time of 9.95 seconds.
    a. Bolt’s time was ___% faster than Hines’.
    b. Hines’ time was ___% slower than Bolt’s.
    c. Hines’ time was ___% of Bolt’s.
27. A store has clearance items that have been marked down by 60%. They are having a sale, advertising an additional 30% off clearance items. What percent of the original price do you end up paying?
28. Which is better: having a stock that goes up 30% on Monday than drops 30% on Tuesday, or a stock that drops 30% on Monday and goes up 30% on Tuesday? In each case, what is the net percent gain or loss?
29. Are these two claims equivalent, in conflict, or not comparable because they’re talking about different things?
    a. “16.3% of Americans are without health insurance”[2]
    b. “only 55.9% of adults receive employer provided health insurance”[3]
30. Are these two claims equivalent, in conflict, or not comparable because they’re talking about different things?
    a. “We mark up the wholesale price by 33% to come up with the retail price”
    b. “The store has a 25% profit margin”
31. Are these two claims equivalent, in conflict, or not comparable because they’re talking about different things?
    a. “Every year since 1950, the number of American children gunned down has doubled.”
    b. “The number of child gunshot deaths has doubled from 1950 to 1994.”
32. Are these two claims equivalent, in conflict, or not comparable because they’re talking about different things?[4]
    a. “75 percent of the federal health care law’s taxes would be paid by those earning less than $120,000 a year”
    b. “75 percent of those who would pay the penalty [health care law’s taxes] for not having insurance in 2016 would earn under $120,000”
33. Are these two claims equivalent, in conflict, or not comparable because they’re talking about different things?
    a. “The school levy is only a 0.1% increase of the property tax rate.”
    b. “This new levy is a 12% tax hike, raising our total rate to $9.33 per $1000 of value.”
34. A high school currently has a 30% dropout rate. They’ve been tasked to decrease that rate by 20%. Find the equivalent percentage point drop.
35. A politician’s support grew from 42% by 3 percentage points to 45%. What percent (relative) change is this?
36. Marcy has a 70% average in her class going into the final exam. She says "I need to get a 100% on this final so I can raise my score to 85%." Is she correct?
37. In a survey of households in the United States 9.4% of the households had 2 dogs. If there are about 117 million households in the U.S., about how many households have 2 dogs?
38. A typical 180-lb man is 111 lbs of water and 27 pounds of fat. What percent of the weight is water? What percent of the weight is fat?

39. Between the last two censuses, a town's population grew from 3200 to 3650. What is the percent increase in the population?
40. In 1901 life expectancy at birth was 49 years. At the end of the century life expectancy at birth was 77 years. What is the increase in life expectancy during the 20th century? Complete this sentence: Life expectancy in 2000 was _____ percent of life expectancy in 1901.
41. If you bought a stock for $250 and later sold it for $1250, by what percent did your stock increase in value? What percent of the purchase price was the sales price?
42. At work, you get a 2% cost-of-living adjustment (COLA). *True of False*: Over a 5-year period, your pay will increase by 10%. If this is false, by what percent will your pay increase over the 5-year period?
43. All merchandise at a store is on sale at 30% off. You have a coupon that gives you 10% off the purchase price. What will you pay for a jacket originally priced at $80? What percent off of the original price is your final price?
44. Your first three exam scores in a class are 74%, 89%, and 83%. What percent do you need to receive on the last exam to have an 85% overall in the class?
45. What is the markup percent for an item that has a wholesale price of $1.13 and a retail price of $1.69?
46. If you buy a refrigerator priced at $1325 and the sales tax rate is 6.5%, how much sales tax would you pay on the purchase? What would the total price of the refrigerator be?
47. Find the sales tax rate in a region where you paid $62.65 tax on a big screen TV priced at $895.
48. Why is this conclusion incorrect? "In a survey of a class, 70% want pizza for their class party and 20% want cake. Therefore, 90% want pizza or cake for their party."
49. Why is this reasoning incorrect? "You received a score of 80% on five tests in a class and 90% on one test. Therefore, your average test score is 85% for the six tests."
50. Suppose you have one quart of water/juice mix that is 50% juice, and you add 2 quarts of juice. What percent juice is the final mix?
51. During the 2004-2005 NBA, Kobe Bryant attempted 387 3-point shots and made 131 of them. Express his performance as a fraction, a decimal and a percent.
52. The following table describes the students enrolled at a small two-year college.

class	full-time	part-time
freshman	262	116
sophomore	214	239

    a. What percent of the freshmen are full-time students?
    b. What percent of the students are enrolled part-time?
    c. What percent of the part-time students are sophomores?

This page titled 1.5.1: Exercises is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

1.5.2: Proportions and Rates

1.5.1 Learning Objectives

- Compare 2 quantities as a rate or ratio
- Solve a proportion
- Solve applications of proportions
- Solve applications using dimensional analysis

If you wanted to power the city of Seattle using wind power, how many windmills would you need to install? Questions like these can be answered using rates and proportions.

Rates

A **ratio** compares two quantities with the same units by division.

A **rate** is the ratio (fraction) of two quantities with different units.

A **unit rate** is a rate with a numerator or a denominator of one.

Example 1

Your car can travel 300 miles on a tank of 15 gallons. Express this as a rate and as a unit rate.

Solution

Expressed as a rate, $\frac{300 \text{ miles}}{15 \text{ gallons}}$. We can divide to find a unit rate: $\frac{20 \text{ miles}}{1 \text{ gallon}}$, which we could also write as $20\frac{\text{miles}}{\text{gallon}}$, or just 20 miles per gallon.

Proportion Equation

A **proportion** equation is an equation showing the equivalence of two rates or ratios.

Example 2

Solve the proportion $\frac{5}{3} = \frac{x}{6}$ for the unknown value x.

Solution

This proportion is asking us to find a fraction with denominator 6 that is equivalent to the fraction $\frac{5}{3}$. We can solve this by multiplying both sides of the equation by 6, giving $x = \frac{5}{3} \cdot 6 = 10$.

You could also note that the denominator of the right-hand ratio is twice as large as the denominator of the left-hand ratio, so the numerator on the right must also be twice as large as the numerator on the left.

Example 3

A map scale indicates that $\frac{1}{2}$ inch on the map corresponds with 3 real miles. How many miles apart are two cities that are $2\frac{1}{4}$ inches apart on the map?

Solution

We can set up a proportion by setting equal two $\frac{\text{map inches}}{\text{real miles}}$ rates, and introducing a variable, x, to represent the unknown quantity – the mile distance between the cities.

$\frac{\frac{1}{2}\text{ map inch}}{3\text{ miles}} = \frac{2\frac{1}{4}\text{ map inches}}{x\text{ miles}}$	Multiply both sides by x and rewriting the mixed number
$\frac{\frac{1}{2}}{3} \cdot x = \frac{9}{4}$	Multiply both sides by 3
$\frac{1}{2}x = \frac{27}{4}$	Multiply both sides by 2 (or divide by $\frac{1}{2}$)
$x = \frac{27}{2} = 13\frac{1}{2}$ miles	

Many proportion problems can also be solved using **dimensional analysis**, the process of multiplying a quantity by rates to change the units.

Example 4

Your car can travel 300 miles on a tank of 15 gallons. How far can it travel on 40 gallons?

Solution

We could certainly answer this question using a proportion: $\frac{300\text{ miles}}{15\text{ gallons}} = \frac{x\text{ miles}}{40\text{ gallons}}$.

However, we earlier found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit "cancels" and we're left with a number of miles:

$$40\text{gallons} \cdot \frac{20\text{ miles}}{\text{gallon}} = \frac{40\ \cancel{\text{gallons}}}{1} \cdot \frac{20\text{ miles}}{\cancel{\text{gallon}}} = 800\text{ miles}$$

Notice if instead we were asked "how many gallons are needed to drive 50 miles?" we could answer this question by inverting the 20 mile per gallon rate so that the *miles* unit cancels and we're left with gallons:

$$50\text{miles} \cdot \frac{1\text{ gallon}}{20\text{ miles}} = \frac{50\ \cancel{\text{miles}}}{1} \cdot \frac{1\text{ gallon}}{20\ \cancel{\text{miles}}} = \frac{50\text{ gallons}}{20} = 2.5\text{ gallons}$$

Dimensional analysis can also be used to do unit conversions. Here are some unit conversions for reference.

Unit Conversions

Length

1 foot (ft) = 12 inches (in)	1 yard (yd) = 3 feet (ft)
1 mile = 5,280 feet	
1000 millimeters mm = 1 meter (m)	100 centimeters (cm) = 1 meter
1000 meters (m) = 1 kilometer (km)	2.54 centimeters (cm) = 1 inch

Weight and Mass

1 pound (lb) = 16 ounces (oz)	1 ton = 2000 pounds
1000 milligrams (mg) = 1 gram (g)	1000 grams = 1 kilogram (kg)
1 kilogram = 2.2 pounds (on earth)	

Capacity

1 cup = 8 fluid ounces (fl oz)*	1 pint = 2 cups
1 quart = 2 pints = 4 cups	1 gallon = 4 quarts = 16 cups
1000 milliliters (ml) = 1 liter (L)	

*Fluid ounces are a capacity measurement for liquids. 1 fluid ounce ≈ 1 ounce (weight) for water only.

Example 5

A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

Solution

To answer this question, we need to convert 20 seconds into hours. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don't, we will need to do additional unit conversions. We will need to know that 5280 ft = 1 mile. We might start by converting the 20 seconds into hours:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{180} \text{ hour}$$ Now we can multiply by the 15 miles/hr

$$\frac{1}{180} \text{ hour} \cdot \frac{15 \text{ miles}}{1 \text{ hour}} = \frac{1}{12} \text{ mile}$$ Now we can convert to feet

$$\frac{1}{12} \text{ mile} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

We could have also done this entire calculation in one long set of products:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{15 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

Try it Now 1

A 1000 foot spool of bare 12-gauge copper wire weighs 19.8 pounds. How much will 18 inches of the wire weigh, in ounces?

Answer

$$18 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{19.8 \text{ pounds}}{1000 \text{ feet}} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} \approx 0.475 \text{ ounces}$$

Notice that with the miles per gallon example, if we double the miles driven, we double the gas used. Likewise, with the map distance example, if the map distance doubles, the real-life distance doubles. This is a key feature of proportional relationships, and one we must confirm before assuming two things are related proportionally.

Example 6

Suppose you're tiling the floor of a 10 ft by 10 ft room, and find that 100 tiles will be needed. How many tiles will be needed to tile the floor of a 20 ft by 20 ft room?

Solution

In this case, while the width the room has doubled, the area has quadrupled. Since the number of tiles needed corresponds with the area of the floor, not the width, 400 tiles will be needed. We could find this using a proportion based on the areas of the rooms:

$$\frac{100 \text{ tiles}}{100\text{ft}^2} = \frac{n \text{ tiles}}{400\text{ft}^2}$$

Other quantities just don't scale proportionally at all.

Example 7

Suppose a small company spends $1000 on an advertising campaign, and gains 100 new customers from it. How many new customers should they expect if they spend $10,000?

Solution

While it is tempting to say that they will gain 1000 new customers, it is likely that additional advertising will be less effective than the initial advertising. For example, if the company is a hot tub store, there are likely only a fixed number of people interested in buying a hot tub, so there might not even be 1000 people in the town who would be potential customers.

Sometimes when working with rates, proportions, and percents, the process can be made more challenging by the magnitude of the numbers involved. Sometimes, large numbers are just difficult to comprehend.

Example 8

Compare the 2010 U.S. military budget of $683.7 billion to other quantities.

Here we have a very large number, about $683,700,000,000 written out. Of course, imagining a billion dollars is very difficult, so it can help to compare it to other quantities.

If that amount of money was used to pay the salaries of the 1.4 million Walmart employees in the U.S., each would earn over $488,000.

There are about 300 million people in the U.S. The military budget is about $2,200 per person.

If you were to put $683.7 billion in $100 bills, and count out 1 per second, it would take 216 years to finish counting it.

Example 9

Compare the electricity consumption per capita in China to the rate in Japan.

To address this question, we will first need data. From the CIA[1] website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank[2], we can find the population of China is 1,344,130,000, or 1.344 billion, and the population of Japan is 127,817,277, or 127.8 million.

Solution

Computing the consumption per capita for each country:

China: $\frac{4,693,000,000,000\text{KWH}}{1,344,130,000 \text{ people}} \approx 3491.5$ KWH per person

Japan: $\frac{859,700,000,000\text{KWH}}{127,817,277 \text{ people}} \approx 6726$ KWH per person

While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

[1] www.cia.gov/library/publicat.../2042rank.html

[2] http://data.worldbank.org/indicator/SP.POP.TOT

This page titled 1.5.2: Proportions and Rates is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **1.3: Proportions and Rates** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

1.5.3: Exercises

Section 1.5.2 Exercises

1. Solve:
   a. $\frac{2}{5} = \frac{6}{x}$
   b. $\frac{2x}{9} = \frac{3}{10}$
   c. $\frac{x}{12} = \frac{2}{3}$
   d. $\frac{4}{9x} = \frac{10}{13}$
   e. $\frac{n}{5} = \frac{16}{20}$.
2. Find a unit rate: You bought 10 pounds of potatoes for $4.
3. Find a unit rate: Joel ran 1500 meters in 4 minutes, 45 seconds.
4. Which is the more economical purchase: 32 oz of detergent for $6.29 or 48 oz of the same detergent for $8.29?
5. Solve the proportion.
   a. $\frac{3}{25} = \frac{x}{3000}$
   b. $\frac{5}{8} = \frac{200}{x}$
6. A student scored 32 out of 40 on a practice exam and 12 out of 15 on the actual exam. Did the student do better on the practice exam or on the actual exam?
7. A soft drink is sold in a six-pack of plastic bottles holding 101.4 fl oz for $2.69. The same soft drink can be purchased in a 12-pack of cans containing 144 fl oz for $3.79. Which is the better buy?

This page titled 1.5.3: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

CHAPTER OVERVIEW

2: Finances

2.1: Introduction
2.2: Taxes
2.2.1: Income Taxation
2.2.2: Exercises
2.3: Simple Interest
2.3.1: Exercises
2.4: Compound Interest
2.4.1: Exercises
2.5: Annuities
2.5.1: Exercises
2.6: Payout Annuities
2.6.1: Exercises
2.7: Installment Loans
2.7.1: Exercises
2.8: Credit Cards
2.8.1: Exercises
2.9: Which equation to use?
2.9.1: Exercises

This page titled 2: Finances is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

2.1: Introduction

We have to work with money every day. While balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more mathematics.

This page titled 2.1: Introduction is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.1: Introduction** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.2: Taxes

2.2 Learning Objectives

- Distinguish among flat tax, progressive tax and regressive tax
- Determine a flat tax amount

Governments collect taxes to pay for the services they provide. In the United States, federal income taxes help fund the military, the environmental protection agency, and thousands of other programs. Property taxes help fund schools. Gasoline taxes help pay for road improvements. While very few people enjoy paying taxes, they are necessary to pay for the services we all depend upon.

Taxes can be computed in a variety of ways, but are typically computed as a percentage of a sale, of one's income, or of one's assets.

Example 1

The sales tax rate in a city is 9.3%. How much sales tax will you pay on a $140 purchase?

Solution

The sales tax will be 9.3% of $140. To compute this, we multiply $140 by the percent written as a decimal: $\$140(0.093) = \$13.02$

When taxes are not given as a fixed percentage rate, sometimes it is necessary to calculate the **effective rate.**

Effective rate

The effective tax rate is the equivalent percent rate of the tax paid out of the dollar amount the tax is based on.

Example 2

Joan paid $3,200 in property taxes on her house valued at $215,000 last year. What is the effective tax rate?

Solution

We can compute the equivalent percentage: $\frac{3200}{215000} = 0.01488$, or about 1.49% effective rate.

Taxes are often referred to as progressive, regressive, or flat.

Tax categories

A **flat tax**, or proportional tax, charges a constant percentage rate.

A **progressive tax** increases the percent rate as the base amount increases.

A **regressive tax** decreases the percent rate as the base amount increases.

Example 3

The United States federal income tax on earned wages is an example of a progressive tax. People with a higher wage income pay a higher percent tax on their income.

Solution

For a single person in 2011, adjusted gross income (income after deductions) under $8,500 was taxed at 10%. Income over $8,500 but under $34,500 was taxed at 15%.

A person earning $10,000 would pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

$8500(0.10) = 850$ 10% of 8500
$1500(0.15) = 225$ 15% of the remaining $\$1500$ of income
$\text{Total tax: } = \$1075$

The effective tax rate paid is $\frac{1075}{10000} = 10.75\%$

A person earning $30,000 would also pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

$8500(0.10) = 850$ 10% of 8500
$21500(0.15) = 3225$ 15% of the remaining $\$21500$ of income
$\text{Total tax: } = \$4075$

The effective tax rate paid is $\frac{4075}{30000} = 13.58\%$.

Notice that the effective rate has increased with income, showing this is a progressive tax.

Example 4

A gasoline tax is a flat tax when considered in terms of consumption, a tax of, say, $0.30 per gallon is proportional to the amount of gasoline purchased. Someone buying 10 gallons of gas at $4 a gallon would pay $3 in tax, which is $\frac{\$3}{\$40} = 7.5\%$. Someone buying 30 gallons of gas at $4 a gallon would pay $9 in tax, which is $\frac{\$9}{\$120} = 7.5\%$, the same effective rate.

Solution

However, in terms of income, a gasoline tax is often considered a regressive tax. It is likely that someone earning $30,000 a year and someone earning $60,000 a year will drive about the same amount. If both pay $60 in gasoline taxes over a year, the person earning $30,000 has paid 0.2% of their income, while the person earning $60,000 has paid 0.1% of their income in gas taxes.

Try it Now 1

A sales tax is a fixed percentage tax on a person's purchases. Is this a flat, progressive, or regressive tax?

Answer

While sales tax is a flat percentage rate, it is often considered a regressive tax for the same reasons as the gasoline tax.

This page titled 2.2: Taxes is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **1.7: Extension - Taxes** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.2.1: Income Taxation

2.2.1 Learning Objectives

- Compare flat tax, modified flat tax, and progressive tax

Many people have proposed various revisions to the income tax collection in the United States. Some, for example, have claimed that a flat tax would be fairer. Others call for revisions to how different types of income are taxed, since currently investment income is taxed at a different rate than wage income.

The following two projects will allow you to explore some of these ideas and draw your own conclusions.

Project 1: Flat tax, Modified Flat Tax, and Progressive Tax.

Imagine the country is made up of 100 households. The federal government needs to collect $800,000 in income taxes to be able to function. The population consists of 6 groups:

Group A: 20 households that earn $12,000 each

Group B: 20 households that earn $29,000 each

Group C: 20 households that earn $50,000 each

Group D: 20 households that earn $79,000 each

Group E: 15 households that earn $129,000 each

Group F: 5 households that earn $295,000 each

This scenario is roughly proportional to the actual United States population and tax needs. We are going to determine new income tax rates.

The first proposal we'll consider is a flat tax – one where every income group is taxed at the same percentage tax rate.

1) Determine the total income for the population (all 100 people together)

2) Determine what flat tax rate would be necessary to collect enough money.

The second proposal we'll consider is a modified flat-tax plan, where everyone only pays taxes on any income over $20,000. So, everyone in group A will pay no taxes. Everyone in group B will pay taxes only on $9,000.

3) Determine the total *taxable* income for the whole population

4) Determine what flat tax rate would be necessary to collect enough money in this modified system

5) Complete this table for both the plans

		Flat Tax Plan		Modified Flat Tax Plan	
Group	**Income per household**	**Income tax per household**	**Income after taxes**	**Income tax per household**	**Income after taxes**
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				

The third proposal we'll consider is a progressive tax, where lower income groups are taxed at a lower percent rate, and higher income groups are taxed at a higher percent rate. For simplicity, we're going to assume that a household is taxed at the same rate on

all their income.

6) Set progressive tax rates for each income group to bring in enough money. There is no one right answer here – just make sure you bring in enough money!

Group	Income per household	Tax rate (%)	Income tax per household	Total tax collected for all households	Income after taxes per household
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				
				This better total to $800,000	

7) Discretionary income is the income people have left over after paying for necessities like rent, food, transportation, etc. The cost of basic expenses does increase with income, since housing and car costs are higher, however usually not proportionally. For each income group, estimate their essential expenses, and calculate their discretionary income. Then compute the effective tax rate for each plan relative to discretionary income rather than income.

Group	Income per household	Discretionary Income (estimated)	Effective rate, flat	Effective rate, modified	Effective rate, progressive
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				

8) Which plan seems the most fair to you? Which plan seems the least fair to you? Why?

Project 2: Calculating Taxes.

Visit www.irs.gov, and download the most recent version of forms 1040, and schedules A, B, C, and D.

Scenario 1: Calculate the taxes for someone who earned $60,000 in standard wage income (W-2 income), has no dependents, and takes the standard deduction.

Scenario 2: Calculate the taxes for someone who earned $20,000 in standard wage income, $40,000 in qualified dividends, has no dependents, and takes the standard deduction. (Qualified dividends are earnings on certain investments such as stocks.)

Scenario 3: Calculate the taxes for someone who earned $60,000 in small business income, has no dependents, and takes the standard deduction.

Based on these three scenarios, what are your impressions of how the income tax system treats these different forms of income (wage, dividends, and business income)?

Scenario 4: To get a more realistic sense for calculating taxes, you'll need to consider itemized deductions. Calculate the income taxes for someone with the income and expenses listed below.

Married with 2 children, filing jointly

Wage income: $50,000 combined

Paid sales tax in Washington State

Property taxes paid: $3200

Home mortgage interest paid: $4800

Charitable gifts: $1200

This page titled 2.2.1: Income Taxation is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **1.8: Income Taxation** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.2.2: Exercises

Section 2.2.2 Exercises

1. Laars earns an annual salary of $60,000. Determine his gross earnings per pay period under each of the following payment frequencies:
   a. Monthly
   b. Semi-monthly
   c. Biweekly
   d. Weekly
2. A city has a sales tax of 10.25%. What would be the sales tax on a $2575.57 purchase? Round to the nearest cent.
3. The tax on gas in California went up to $0.511 cents per gallon in July 2021. If an average gallon of gas costs $4.31 per gallon, what is the effective tax rate of the gas tax? Round the percent to two decimal places.
4. Samantha and Anahi carpool to work to split the cost of gas. They each pay $120 per month to cover the cost of the commute. If Samantha makes $3200 a month and Anahi makes $4100, what is the effective tax rate on their income? Would this be considered a progressive, regressive, or flat tax rate?
5. Alex bought a new gaming system with several games that retailed at $435 before tax. They noticed the tax charged was $42.41. What was the sales tax rate of that city?
6. Use the 2021 Federal Income Tax Brackets at the end of this page for the following questions.
   a. Determine the federal income tax on $23,325 for a single individual.
   b. Determine the federal income tax on $152,904 for married individuals filing jointly.
   c. Determine the federal income tax on $102,350 for a single individual.
   d. Adrian made $40,525 in 2021 and is filing as single. Melany made $40,526 in 2021 and is also filing as single. Melany thinks that since both she and Adrian make approximately the same amount of money, they should be in the same tax bracket. Using the tax bracket chart, determine if Melany's claim is true. Find the difference in the amount of taxes they paid.
   e. Brenda made $41,685 in 2021 and is filing single. Luz made $51,892 filing as Head of Household. Brenda claims to Luz that since she is making less money, that she will pay less in taxes. Using the tax chart below, determine if Brenda's claim is true. Find the difference in the amount of taxes they paid and state who paid more. What is each of their effective tax rates?
7. A lawmaker proposed that everyone will pay the same amount of taxes. Fill in the following table by writing out each rate. Determine if this is an example of a progressive, regressive, or flat tax rate.

Rate	For Single Individuals	Amount of Tax Paid
	$6,500	$2,500
	$9,951	$2,500
	$40,526	$2,500
	$86,376	$2,500
	$164,926	$2,500
	$209,426	$2,500
	$523,601	$2,500

2021 Federal Income Tax Brackets and Rates for Single Filers, Married Couples Filing Jointly, and Heads of Households

Rate	For Single Individuals	For Married Individuals Filing Joint Returns	For Heads of Households
10%	of any amount up to $9,950	of any amount up to $19,900	of any amount up to $14,200
12%	of the portion above $9,950 up to $40,525 plus $995	of the portion above $19,900 up to $81,050 plus $1990	of the portion above $14,200 up to $54,200 plus $1420

Rate	For Single Individuals	For Married Individuals Filing Joint Returns	For Heads of Households
22%	of the portion above $40,525 up to $86,375 plus $4664	of the portion above $81,050 up to $172,750 plus $9328	of the portion above $54,200 up to $86,350 plus $6220
24%	of the portion above $86,375 up to $164,925 plus $14751	of the portion above $172,750 to $329,850 plus $29502	of the portion above $86,350 up to $164,900 plus $13296
32%	of the portion above $164,925 up to $209,425 plus $33603	of the portion above $329,850 up to $418,850 plus $ 67206	of the portion above $164,900 up to $209,400 plus $32148
35%	of the portion above $209,425 up to $523,600 plus $47843	of the portion above $418,850 up to $628,300 plus $95686	of the portion above $209,400 up to $523,600 plus $46388
37%	of the portion above $523,600 or more plus $157804	of the portion above $628,300 or more plus $168994	of the portion above $523,600 or more plus 156358

This page titled 2.2.2: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.10: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.
- **4.E: Human Resources and Economic Applications (Exercises)** by Jean-Paul Olivier is licensed CC BY-NC-SA 4.0. Original source: https://open.bccampus.ca/browse-our-collection/find-open-textbooks/?uuid=16301119-8ec4-4241-b0f7-cc87ffc942d6.

2.3: Simple Interest

2.3 Learning Objectives

- Find the simple interest for a loan

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed $100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $\$100(0.05) = \$5$ The total amount you would repay would be $105, the original principal plus the interest.

Simple One-time Interest

$$I = P_0 r \tag{2.3.1}$$

$$A = P_0 + I = P_0 + P_0 r = P_0(1 + r) \tag{2.3.2}$$

where

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate (in decimal form. Example: $5\% = 0.05$)

Example 1

A friend asks to borrow $300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

Solution

$P_0 = \$300$	the principal
$r = 0.03$	3% rate
$I = \$300(0.03) = \$9.$	You will earn $9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a $1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Solution

Each year, you would earn 5% interest: $\$1000(0.05) = \$50$in interest. So over the course of five years, you would earn a total of $250 in interest. When the bond matures, you would receive back the $1,000 you originally paid, leaving you with a total of $1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time

$I = P_0 rt$

$A = P_0 + I = P_0 + P_0 rt = P_0(1 + rt)$

where

- I is the interest

- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate in decimal form
- t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

Definition: Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.
A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3 Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Solution

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$P_0 = \$1000$	the principal
$r = 0.02$	2% rate
$t = 8$	4 years $= 8$ half-years
$I = \$1000(0.02)(8) = \$160.$	You will earn $\$160$ interest total over the four years.

Try it Now 1

A loan company charges $30 interest for a one month loan of $500. Find the annual interest rate they are charging.

Answer

$I = \$30$ of interest

$P_0 = \$500$ principal

$r =$ unknown (annual)

$t = 1$ month (1/12 of a year)

This page titled 2.3: Simple Interest is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.2: Simple Interest** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.3.1: Exercises

Section 2.3.1 Exercises

1. You borrow $350 from your aunt and promise to pay her back in one month with 3% interest. How much extra will she get when you pay her back?
2. You were charged $487.50 in interest at the end of one year for money you borrowed from a bank that charged 6.5% interest. How much did you borrow?
3. You had to pay $618.12 in interest on a loan for $10,750 over the first year. What was the interest rate used? Round the percent to two decimal places.
4. A friend lends you $200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?
5. Your brother deposits $1200 in a savings account that pays 4% simple interest. How much will he have in the account in 4 years if he doesn't touch it.
6. Suppose you obtain a $3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?
7. Suppose you take a cash advance of $750 on your credit card. The service charge is $5.50. You pay off the cash advance after 45 days. Your credit card statement shows the interest on the cash advance was $29.59. For the interest alone, what is the APR?
8. If you buy a 3-year CD that pays 2.4% simple interest for $75,000, what will it be worth at the end of its term?
9. If you invest $1500 at 4.25% simple interest and the investment is later worth $1553.13, what was the term?
10. A T-bill is a type of bond that is sold at a discount over the face value. For example, suppose you buy a 13-week T-bill with a face value of $10,000 for $9,800. This means that in 13 weeks, the government will give you the face value, earning you $200. What annual interest rate have you earned?
11. Suppose you are looking to buy a $5000 face value 26-week T-bill. If you want to earn at least 1% annual interest, what is the most you should pay for the T-bill?

This page titled 2.3.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.10: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.4: Compound Interest

2.4 Learning Objectives

- Use the compound interest formula to compute beginning or ending balance in an account

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit $1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3\%}{12} = 0.25\%$ per month.

In the first month,

$P_0 = \$1000$

$r = 0.0025 (0.25\%)$

$I = \$1000(0.0025) = \$2.50$

$A = \$1000 + \$2.50 = \$1002.50$

In the first month, we will earn $2.50 in interest, raising our account balance to $1002.50.

In the second month,

$P_0 = \$1002.50$

$I = \$1002.50(0.0025) = \$2.51$ (rounded)

$A = \$1000 + \$2.50 = \$1002.50$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original $1000 we deposited, but we also earned interest on the $2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:

Month	Starting balance	Interest earned	Ending Balance
1	1000.00	2.50	1002.50
2	1002.50	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.10
7	1015.10	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73
10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85
12	1027.85	2.57	1030.42

To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

$P_0 = \$1000$

$P_m = (1 + 0.0025) P_{m-1}$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$P_0 = \$1000$

$P_1 = 1.0025P_0 = 1.0025(1000)$

$P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = 1.0025^2(1000)$

$P_3 = 1.0025P_2 = 1.0025\left(1.0025^2(1000)\right) = 1.0025^3(1000)$

$P_4 = 1.0025P_3 = 1.0025\left(1.0025^3(1000)\right) = 1.0025^4(1000)$

Observing a pattern, we could conclude

$P_m = (1.0025)^m(\$1000)$

Notice that the \$1000 in the equation was P_0, the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

$$A = P_0\left(1 + \frac{r}{n}\right)^m$$

In this formula:

m is the number of compounding periods (months in our example)

r is the annual interest rate

n is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If t is the number of years, then $m = tn$. Making this change gives us the standard formula for compound interest.

Compound Interest

$$A = P_0\left(1 + \frac{r}{n}\right)^{tn}$$

A is the balance in the account after N years.

P_0 is the starting balance of the account (also called initial deposit, or principal)

r is the annual interest rate in decimal form

n is the number of compounding periods in one year.

If the compounding is done annually (once a year), $n = 1$.

If the compounding is done quarterly, $n = 4$.

If the compounding is done monthly, $n = 12$.

If the compounding is done daily, $n = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

Example 1

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

Solution

In this example,

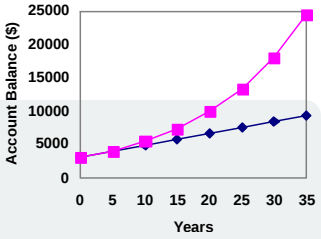
$P_0 = \$3000$	the initial deposit
$r = 0.06$	6% annual rate
$n = 12$	12 months in 1 year
$t = 20$	since we're looking for how much we'll have after 20 years

So $A = 3000\left(1 + \frac{0.06}{12}\right)^{20\times12} = \$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest ($15 per month)	6% compounded monthly $= 0.5\%$ each month.
5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13394.91
30	$8400	$18067.73
35	$9300	$24370.65

As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.



Evaluating exponents on the calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$. But when we need to calculate something like 1.005^{240}, it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

$[\wedge]$, $[y^x]$, or $[x^y]$

To evaluate 1.005^{240} we'd type 1.005 $[\wedge]$ 240, or 1.005 $[y^x]$ 240. Try it out - you should get something around 3.3102044758.

Example 2

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Solution

We're looking for P_0.

$r = 0.04$	4%
$n = 4$	4 quarters in 1 year
$t = 18$	Since we know the balance in 18 years
$A = \$40,000$	The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for P_0.

$$40000 = P_0\left(1 + \frac{0.04}{4}\right)^{18\times4}$$

$$40000 = P_0(2.0471)$$

$$P_0 = \frac{40000}{2.0471} = \$19539.84$$

So you would need to deposit $19,539.84 now to have $40,000 in 18 years.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better.

Example 3

To see why not over-rounding is so important, suppose you were investing $1000 at 5% interest compounded monthly for 30 years.

Solution

$P_0 = \$1000$	the initial deposit
$r = 0.05$	5%
$n = 12$	12 months in 1 year
$t = 30$	since we're looking for the amount after 30 years

If we first compute $\frac{r}{n}$, we find $\frac{0.05}{12} = 0.00416666666667$

Here is the effect of rounding this to different values:

r/n rounded to:	Gives P_{30} to be:	Error
0.004	$4208.59	$259.15
0.0042	$4521.45	$53.71
0.00417	$4473.09	$5.35
0.004167	$4468.28	$0.54
0.0041667	$4467.80	$0.06
no rounding	$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough - $5 off of $4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$$A = 1000\left(1 + \frac{0.05}{12}\right)^{12\times 30}$$

We can quickly calculate $12 \times 30 = 360$, giving $A = 1000\left(1 + \frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.

Type this	**Calculator shows**
0.05[÷]12[=]	0.00416666666667
[+]11[=]	1.00416666666667
[y^x]360[=]	4.46774431400613
[×]1000[=]	4467.74431400613

Using your calculator continued

The previous steps were assuming you have a “one operation at a time” calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 $[\times]$ (1 $[+]$ 0.05 $[\div]$ 12) $[y^x]$ 360 $[=]$.

This page titled 2.4: Compound Interest is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.3: Compound Interest** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.4.1: Exercises

Section 2.4.1 Exercises

1. You deposit $300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?
2. How much will $1000 deposited in an account earning 7% interest compounded annually be worth in 20 years?
3. You deposit $2000 in an account earning 3% interest compounded monthly.
   a. How much will you have in the account in 20 years?
   b. How much interest will you earn?
4. You deposit $10,000 in an account earning 4% interest compounded monthly.
   a. How much will you have in the account in 25 years?
   b. How much interest will you earn?
5. How much would you need to deposit in an account now in order to have $6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.
6. How much would you need to deposit in an account now in order to have $20,000 in the account in 4 years? Assume the account earns 5% interest compounded weekly.

This page titled 2.4.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.10: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.5: Annuities

2.5 Learning Objectives

- Use the annuity formula to determine the balance in an account
- Use the annuity formula to determine what regular deposits are needed to achieve a goal
- Determine when to use the compound interest formula and when to use the annuity formula

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m = \left(1 + \frac{r}{n}\right) P_{m-1}$$

For a savings annuity, we simply need to add a deposit, d, to the account with each compounding period:

$$P_m = \left(1 + \frac{r}{n}\right) P_{m-1} + d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit \$100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$r = 0.06$ (6%)

$n = 12$ (12 compounds/deposits per year)

$d = \$100$ (our deposit per month)

Writing out the recursive equation gives

$$P_m = \left(1 + \frac{0.06}{12}\right) P_{m-1} + 100 = (1.005)P_{m-1} + 100$$

Assuming we start with an empty account, we can begin using this relationship:

$P_0 = 0$

$P_1 = (1.005)P_0 + 100 = 100$

$P_2 = (1.005)P_1 + 100 = (1.005)(100) + 100 = 100(1.005) + 100$

$P_3 = (1.005)P_2 + 100 = (1.005)(100(1.005) + 100) + 100 = 100(1.005)^2 + 100(1.005) + 100$

Continuing this pattern, after m deposits, we'd have saved:

$$P_m = 100(1.005)^{m-1} + 100(1.005)^{m-2} + \cdots + 100(1.005) + 100$$

In other words, after m months, the first deposit will have earned compound interest for $m-1$ months. The second deposit will have earned interest for $m-2$ months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005P_m = 1.005\left(100(1.005)^{m-1} + 100(1.005)^{m-2} + \cdots + 100(1.005) + 100\right)$$

Distributing on the right side of the equation gives

$$1.005P_m = 100(1.005)^{m} + 100(1.005)^{m-1} + \cdots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \cdots + \quad 100(1.005)$$
$$P_m = \quad 100(1.005)^{m-1} + \cdots + \quad 100(1.005) + 100$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005P_m - P_m = 100(1.005)^m - 100$$

Solving for P_m

$$0.005P_m = 100\left((1.005)^m - 1\right)$$

$$P_m = \frac{100\left((1.005)^m - 1\right)}{0.005}$$

Replacing m months with $12t$, where t is measured in years, gives

$$P_t = \frac{100\left((1.005)^{12t} - 1\right)}{0.005}$$

Recall 0.005 was $\frac{r}{k}$ and 100 was the deposit d. 12 was k, the number of deposit each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula

$$A = \frac{d\left[\left(1 + \frac{r}{n}\right)^{tn} - 1\right]}{\left(\frac{r}{n}\right)}$$

A is the balance in the account after t years.

d is the regular deposit (the amount you deposit each year, each month, etc.)

r is the annual interest rate in decimal form.

n is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, $n = 12$.

If you make your deposits every year, use yearly compounding, $n = 1$.

If you make your deposits every quarter, use quarterly compounding, $n = 4$.

Etc.

When do you use this

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Example 1

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

Solution

In this example,

$d = \$100$	the monthly deposit
$r = 0.06$	6% annual rate
$n = 12$	since we're doing monthly deposits, we'll compound monthly
$t = 20$	we want the amount after 20 years

Putting this into the equation:

$$A = \frac{100\left[\left(1+\frac{0.06}{12}\right)^{20(12)}-1\right]}{\left(\frac{0.06}{12}\right)}$$

$$A = \frac{100\left((1.005)^{240}-1\right)}{(0.005)}$$

$$A = \frac{100(3.310-1)}{(0.005)}$$

$$A = \frac{100(2.310)}{(0.005)} = \$46200$$

The account will grow to $46,200 after 20 years.

Notice that you deposited into the account a total of $24,000 ($100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $\$46,200 - \$24,000 = \$22,200$

Example 2

You want to have $200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Solution

In this example,

We're looking for d.

$r = 0.08$	8% annual rate
$n = 12$	since we're doing monthly deposits, we'll compound monthly
$t = 30$	30 years
$A = \$200,000$	The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for d.

$$200,000 = \frac{d\left[\left(1+\frac{0.08}{12}\right)^{30(12)}-1\right]}{\left(\frac{0.08}{12}\right)}$$

$$200,000 = \frac{d\left((1.00667)^{360}-1\right)}{(0.00667)}$$

$$200,000 = d(1491.57)$$

$$d = \frac{200,000}{1491.57} = \$134.09$$

So you would need to deposit $\$134.09$ each month to have $\$200,000$ in 30 years if your account earns 8% interest

Try it Now 1

A more conservative investment account pays 3% interest. If you deposit $5 a day into this account, how much will you have after 10 years? How much is from interest?

Answer

$d = \$5$ the daily deposit
$r = 0.03$ 3% annual rate
$n = 365$ since we're doing daily deposits, we'll compound daily
$t = 10$ we want the amount after 10 years

$$A = \frac{5\left[\left(1+\frac{0.03}{365}\right)^{365\times 10}-1\right]}{\frac{0.03}{365}} = \$21,282.07$$

We would have deposited a total of $\$5 \cdot 365 \cdot 10 = \$18,250$ so $3,032.07 is from interest

---

This page titled 2.5: Annuities is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.4: Annuities** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.5.1: Exercises

Section 2.5.1 Exercises

1. You wish to have $3000 in 2 years to buy a fancy new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?
2. You deposit $200 each month into an account earning 3% interest compounded monthly.
   a. How much will you have in the account in 30 years?
   b. How much total money will you put into the account?
   c. How much total interest will you earn?
3. You deposit $1000 each year into an account earning 8% compounded annually.
   a. How much will you have in the account in 10 years?
   b. How much total money will you put into the account?
   c. How much total interest will you earn?
4. Raymond has determined he needs to have $800,000 for retirement in 30 years. His account earns 6% interest.
   a. How much would he need to deposit in the account each month?
   b. How much total money will he put into the account?
   c. How much total interest will he earn?

This page titled 2.5.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.10: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.6: Payout Annuities

2.6 Learning Objectives

- Use the payout annuity formula to determine the regular withdrawal amount
- Determine the amount needed in an account to allow for a given payout amount
- Determine when to use the annuity formula and when to use the payout formula

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved $500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$P_0 = \frac{d\left[1-\left(1+\frac{r}{n}\right)^{-tn}\right]}{\left(\frac{r}{n}\right)}$$

P_0 is the balance in the account at the beginning (starting amount, or principal).

d is the regular withdrawal (the amount you take out each year, each month, etc.)

r is the annual interest rate (in decimal form. Example: $5\% = 0.05$)

n is the number of compounding periods in one year.

t is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Example 1

After retiring, you want to be able to take $1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Solution

In this example,

$d = \$1000$ the monthly withdrawal
$r = 0.06$ 6% annual rate
$n = 12$ since we're doing monthly withdrawals, we'll compound monthly
$t = 20$ since we're taking withdrawals for 20 years

We're looking for P_0; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$P_0 = \frac{1000\left[1-\left(1+\frac{0.06}{12}\right)^{-20(12)}\right]}{\left(\frac{0.06}{12}\right)}$$

$$P_0 = \frac{1000 \times \left(1-(1.005)^{-240}\right)}{(0.005)}$$

$$P_0 = \frac{1000 \times (1-0.302)}{(0.005)} = \$139,600$$

You will need to have $139,600 in your account when you retire.

Notice that you withdrew a total of $240,000 ($1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is $\$240,000 - \$139,600 = \$100,400$ in interest.

Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this [(-)], some with [+/-] . The button is often near the = key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005^{-240} you'd type something like: $1.005[\wedge][(-)]240$

If your calculator only shows one value at a time, then usually you hit the (-) key after a number to negate it, so you'd hit: $1.005[y^x]240[(-)] =$

Give it a try - you should get $1.005^{-240} = 0.302096$

Example 2

You know you will have $500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

Solution

In this example,

We're looking for d.

$r = 0.08$ 8% annual rate
$n = 12$ since we're doing monthly withdrawals
$t = 30$ since were taking withdrawals for 30 years
$P_0 = \$500,000$ we are beginning with $\$500,000$

In this case, we're going to have to set up the equation, and solve for d.

$$500,000 = \frac{d\left[1-\left(1+\frac{0.08}{12}\right)^{-30(12)}\right]}{\left(\frac{0.08}{12}\right)}$$

$$500,000 = \frac{d\left(1-(1.00667)^{-360}\right)}{(0.00667)}$$

$$500,000 = d(136.232)$$

$$d = \frac{500,000}{136.232} = \$3670.21$$

You would be able to withdraw $3,670.21 each month for 30 years.

Try it Now 1

A donor gives $100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Answer

$d =$ unknown	
$r = 0.04$	4% annual rate
$n = 1$	since we're doing annual scholarships
$t = 20$	since were taking withdrawals for 20 years
$P_0 = \$100,000$	we are starting with $\$100,000$

$$100,000 = \frac{d\left[1-\left(1+\frac{0.04}{1}\right)^{-20\times 1}\right]}{\frac{0.04}{1}}$$

Solving for d gives $7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, $\$100,000(0.04) = \$4,000$ a year would have been available.

This page titled 2.6: Payout Annuities is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.5: Payout Annuities** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.6.1: Exercises

Section 2.6.1 Exercises

1. You want to be able to withdraw $30,000 each year for 25 years. Your account earns 8% interest compounded annually.
   a. How much do you need in your account at the beginning?
   b. How much total money will you pull out of the account?
   c. How much of that money is interest?
2. How much money will I need to have at retirement so I can withdraw $60,000 a year for 20 years from an account earning 8% compounded annually?
   a. How much do you need in your account at the beginning
   b. How much total money will you pull out of the account?
   c. How much of that money is interest?
3. You have $500,000 saved for retirement. Your account earns 6% interest compounded monthly. How much will you be able to pull out each month, if you want to be able to take withdrawals for 20 years?
4. Loren already knows that he will have $500,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 10% interest compounded monthly, how much could the annuity provide each month?

This page titled 2.6.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.10: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.7: Installment Loans

2.7 Learning Objectives

- Determine the payment on an installment loan
- Determine amount that can be borrowed on an installment loan given the amount of the loan payment

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had $10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests $10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula

$$P_0 = \frac{d\left[1-\left(1+\frac{r}{n}\right)^{-tn}\right]}{\left(\frac{r}{n}\right)}$$

P_0 is the balance in the account at the beginning (the principal, or amount of the loan).

d is your loan payment (your monthly payment, annual payment, etc)

r is the annual interest rate in decimal form.

n is the number of compounding periods in one year.

t is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Loans: Many payments

Example 1

You can afford $200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with $200 per month?

Solution

In this example,

$d = \$200$ the monthly loan payment
$r = 0.03$ 3% annual rate
$n = 12$ since we're doing monthly payments, we'll compound monthly
$t = 5$ since we're making monthly payments for 5 years

We're looking for P_0, the starting amount of the loan.

$$P_0 = \frac{200\left[1-\left(1+\frac{0.03}{12}\right)^{-5(2)}\right]}{\left(\frac{0.03}{12}\right)}$$

$$P_0 = \frac{200\left(1-(1.0025)^{-60}\right)}{(0.0025)}$$

$$P_0 = \frac{200(1-0.861)}{(0.0025)} = \$11,120$$

You can afford a $\$11,120$ loan.

You will pay a total of $12,000 ($200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying $\$12,000 - \$11,120 = \$880$ interest total.

Example 2

You want to take out a $140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Solution

In this example,

We're looking for d.

$r = 0.06$ 6% annual rate
$n = 12$ since we're doing monthly payments, we'll compound monthly
$t = 30$ since we're making monthly payments for 30 years
$P_0 = \$140,000$ the starting loan amount

In this case, we're going to have to set up the equation, and solve for d.

$$140,000 = \frac{d\left[1-\left(1+\frac{0.06}{12}\right)^{-30(12)}\right]}{\left(\frac{0.06}{12}\right)}$$

$$140,000 = \frac{d\left(1-(1.005)^{-360}\right)}{(0.005)}$$

$$140,000 = d(166.792)$$

$$d = \frac{140,000}{166.792} = \$839.37$$

You will make payments of $839.37 per month for 30 years.

You're paying a total of $\$302,173.20$ to the loan company: $\$839.37$ per month for 360 months. You are paying a total of $\$302,173.20 - \$140,000 = \$162,173.20$ in interest over the life of the loan.

Try it Now 1

Janine bought $3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Answer

$d =$ unknown

$r = 0.16$	16% annual rate
$n = 12$	since we're doing monthly payments, we'll compound monthly
$t = 2$	2 year to repay
$P_0 = 3,000$	the starting loan amount $3,000 loan

$$3,000 = \frac{d\left[1 - \left(1 + \frac{0.16}{12}\right)^{-2\times 12}\right]}{\frac{0.16}{12}}$$

Solving for d gives $146.89 as monthly payments.

In total, she will pay $3,525.36 to the store, meaning she will pay $525.36 in interest over the two years.

This page titled 2.7: Installment Loans is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.6: Loans** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.7.1: Exercises

Section 2.7.1 Exercises

1. You can afford a $700 per month mortgage payment. You've found a 30 year loan at 5% interest.
   a. How big of a loan can you afford?
   b. What would be the total payments made on the loan?
   c. How much of that money is interest?
2. Marie can afford a $250 per month car payment. She's found a 5 year loan at 7% interest.
   a. How expensive of a car can she afford?
   b. What would be the total payments made on the loan?
   c. How much of that money is interest?
3. You want to buy a $25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?
4. You decide finance a $12,000 car at 3% compounded monthly for 4 years. What will your monthly payments be? How much interest will you pay over the life of the loan?
5. You want to buy a $200,000 home. You plan to pay 10% as a down payment, and take out a 30 year loan for the rest.
   a. How much is the loan amount going to be?
   b. What will your monthly payments be if the interest rate is 5%?
   c. What will your monthly payments be if the interest rate is 6%?
6. Lynn bought a $300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.
   a. Find her monthly payments.
   b. How much interest will she pay over the life of the loan?
7. Emile bought a car for $24,000 three years ago. The loan had a 5 year term at 3% interest rate, making monthly payments. How much does he still owe on the car?
8. A friend bought a house 15 years ago, taking out a $120,000 mortgage at 6% for 30 years, making monthly payments. How much does she still owe on the mortgage?

This page titled 2.7.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.10: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.8: Credit Cards

2.8 Learning Objectives

- Determine the interest on a credit card loan using the Average Daily Balance method
- Understand revolving credit
- Analyze common myths about credit scores
- Define and calculate average daily balance

Understanding Credit Cards

Revolving credit is an installment loan that remains open to debt increases and credits. These loans require a regular payment schedule but do not have a fixed amount that is being paid off. The balance of the loan changes as the credit is used for products and services as decided by the account holder.

Credit cards are used by a financial institution to give users access to a loan based on revolving credit. When a credit card is used, the account holder is borrowing money from the institution to make a purchase. The maximum a user can borrow is set by the financial institution based on the credit worthiness of the account holder. Each billing period (typically 25-30 days) the account accrues interest for any remaining balance based on the **average daily balance**. The interest is compounded each billing period, which is why it can be easy to get into debt when using credit cards.

Debit cards are used like credit cards, but they are directly connected to your account. Since they do not involve borrowing money from a financial institution, they do not affect your credit score in the same way a credit card does.

Credit reports contain information about a person's current and past debt. A current account will show the amount available to be borrowed if it is revolving credit or the amount still owed if it is an installment loan. It will also show if payments are consistently paid on time or if there are late or missed payments. Closed accounts, bankruptcy, collections, and delinquencies will show on the credit report for 7 to 10 years depending on the situation. Other financial obligations that can affect a credit score are cell phone contracts, rental contracts, and home utilities. The three credit bureaus that track credit are TransUnion, Equifax, and Experian. You are entitled to a free annual report from each of these companies.

A **credit score** helps financial institutions decide how reliable an applicant is when dealing with financial obligations. The credit score is a single number that shows a person's credit worthiness to lenders. The two most commonly used credit scores are from FICO® and VantageScore.

Credit Card Myths

Myth 1: Carrying a credit card balance can increase your credit score.

> FALSE! Having a high balance on your cards can actually decrease your credit score since financial institutions see that you have a lot of debt. It is best to pay off the balance of the card every billing period if possible. If not, then keeping your total balance for all credit cards below 15% of the maximum you can borrow is the second-best idea.

Myth 2: Your income is a factor in your credit score.

> FALSE! No institution knows your income except you, your employer, and the IRS. The only way an institution knows your income is if you disclose it to them.

Myth 3: Credit repair companies can fix your credit.

> FALSE! Credit repair companies can only remove information from a credit report that is incorrect, which any user can do for free. Credit repair companies often will create a new loan to consolidate your debt into one payment. This does not necessarily improve the credit score.

Myth 4: Closing old credit cards can hurt your credit score.

> FALSE/TRUE This one is a bit of both.
>
> FALSE: If you have old credit cards that have high maximums that you do not use or excessive number of credit cards, it can be beneficial to close those since financial institutions look at how much debt you COULD get into if you maxed out

all your cards.

TRUE: If you have old cards that are not being used, they will show on the credit report as an account that is in good standing which can help your credit score. Some financial institutions will close accounts that have not been used for an extended period of time.

Myth 5: Checking your credit score can hurt your credit score.

FALSE! When you check your personal credit score it is considered a soft inquiry. These inquiries do not affect the credit score. Everyone can request a free credit report from the three main credit reporting agencies once a year.

Applying for a loan or a new service that requires a credit check is considered a hard inquiry. These can negatively affect your score if there are several hard inquiries within a short period of time for different types of loans. For example: if you are applying to purchase a car at several financial institutions to find the best rate that may not affect your score negatively. If you apply for several different credit cards in a short period of time that will negatively affect your score.

Determining Interest on a Revolving Loan

Since the balance of a revolving loan changes often, the typical way of determining interest cannot be used. A financial institution cannot charge interest for the entire billing period on money that was not borrowed at the beginning of that period. For this reason, financial institutions use the average balance over the length of the billing period.

$$\text{average daily balance} = \frac{\text{sum of unpaid balances for each day in the billing period}}{\text{number of days in the billing period}} \tag{2.8.1}$$

Example 1

Victor got a new credit card. He made the following charges in the first billing cycle. Determine the average daily balance for his first billing period.

March 1	The first billing cycle begins.
March 3	He bought a jacket for $55.
March 8	He went to the movies and spent $42.
March 15	He put gas in his car and spent $67.
March 21	He treated his family to dinner and spent $73.
March 25	The billing cycle ended.

Solution

To determine the average daily balance we first need to find the sum of unpaid balances for each day in the billing period. The calendar below lists the balances for each day of the billing cycle.

March 1 - $ 0.00	March 2 - $ 0.00	March 3 - $ 55.00	March 4 - $ 55.00	March 5 - $ 55.00
March 6 - $ 55.00	March 7 - $ 55.00	March 8 - $ 97.00	March 9 - $ 97.00	March 10 - $ 97.00
March 11 - $ 97.00	March 12 - $ 97.00	March 13 - $ 97.00	March 14 - $ 97.00	March 15 - $ 164.00
March 16 - $ 164.00	March 17 - $ 164.00	March 18 - $ 164.00	March 19 - $ 164.00	March 20 - $ 164.00
March 21 - $ 237.00	March 22 - $ 237.00	March 23 - $ 237.00	March 24 - $ 237.00	March 25 - $ 237.00

Begin on March 3rd since that is the first date with a balance to find the sum. The sum will be the numerator of the average daily balance fraction.

$$55+55+55+55+55+97+97+97+97+97+97+97+164+164+164+164+164+164+237+237+237+237+237=3123$$

The denominator will be the number of days in the billing cycle which, in this case, is 25 days.

$$\text{average daily balance} = \frac{3123}{25} = 125.92$$

Interest will be calculated on the amount $125.92 since that is the average daily balance for this billing cycle.

Example 2

Jasmine has had her credit card for several months. She only pays the minimum payment, which is 2% of the balance or $10, whichever is more. Review the transactions of her latest billing cycle to determine the average daily balance. Then find the interest that will be charged to her account with a 7.99% APR and the minimum payment required.

June 1	The billing cycle begins with a balance of $2750
June 5	She makes a payment of $55.
June 6	She got a coffee and snacks $27.
June 12	She put gas in her car and spent $33.
June 17	She bought tickets to a concert $119.
June 25	The billing cycle ended.

Solution

In this example, there is an existing balance and a payment is made. The payment will decrease the balance while purchases will increase the balance. Instead of using a calendar to determine the sum, we will use multiplication.

First determine when the balance changed and what the change was. Then determine how many days the account was at the balance.

June 1	$2750.00
June 5	$2750 − $55 = $2695
June 6	$2695 + $27 = $2722
June 12	$2722 + $33 = $2755
June 17	$2755 + $119 = $2874
June 30	The billing cycle ended.

4 days	$2750
1 day	$2695
6 days	$2722
5 days	$2755
14 days	$2874

This billing cycle has 30 days. The total number of days should add to 30. Now we can use multiplication to determine the sum of the daily balances.

$$\text{average daily balance} = \frac{4\times 2750+1\times 2695+6\times 2722+5\times 2755+14\times 2874}{30} = \frac{84038}{30} = 2801.27$$

To find the interest, use the $I = Prt$ formula, where P is the average daily balance, r is the APR, and t is the time in years. Since this is a monthly account $t = \frac{1}{12}$.

$$I = 2801.27 \times 0.0799 \times \frac{1}{12} = 18.65$$

The interest is added to the balance before the minimum payment is determined. Recall that the minimum payment is either 2% or $10, whichever is greater. The balance is the last value before the billing cycle ended. Do not use the average daily balance to calculate the minimum payment. Note: If the balance is paid in full by the end of the cycle, no interest is charged.

$$2874 + 18.65 = 2892.65 \times 0.02 = 57.85$$

The minimum payment due for this billing period is $57.85.

Try it Now 1

Refer to the following credit card transactions. Determine the average daily balance and the interest charged for this billing cycle for an account with a 24.99% APR. Find the final balance. Assume the account holder wanted to pay the credit card off in one year. Use the loan payment formula from the previous sections to determine how much should be paid each month if no additional charges are made to meet that goal.

May 1	The billing cycle begins with a balance of $4212
May 4	A payment of $25 is made.
May 12	A purchase of $148 is made.
May 16	A purchase of $16 is made.
May 17	A purchase of $96 is made.
May 30	The billing cycle ended.

Answer

Find the sum of the daily balances and divide by the number of days in the billing cycle.

$$\text{average daily balance} = \frac{3 \times 4212 + 8 \times 4187 + 4 \times 4335 + 1 \times 4351 + 13 \times 4447}{30} = \frac{125634}{30} = 4187.80$$

Determine the interest for the current billing cycle.

$$I = 4187.80 \times 0.2499 \times \frac{1}{12} = 87.21$$

Use the loan payment formula to determine how much should be paid each month to pay off the credit card in one year.

The final balance would be $4447 + 87.21 = 4534.21$ which will be used as P, $r = 24.99$, $t = 1$, and $n = 12$.

$$d = \frac{4534.24\left(\frac{0.2499}{12}\right)}{\left[1 - \left(1 + \frac{0.2499}{12}\right)^{(-12 \times 1)}\right]} = 430.93$$

It would require a monthly payment of $430.93 to pay off the balance of the given credit card in one year if no additional charges are made.

This page titled 2.8: Credit Cards is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

2.8.1: Exercises

Section 2.8.1 Exercises

1. During the previous 25 day billing cycle your credit card balance was at $15 for 5 days, $72 for 4 days, $95 for 7 days, $134 for 3 days, and $197 for 6 days. What was the average daily balance for that billing cycle?

2. Refer to the following credit card transactions. Determine the average daily balance for this billing cycle.

Sept. 1	The billing cycle begins with a balance of $975
Sept. 3	A payment of $100 is made.
Sept. 6	A purchase of $27 is made.
Sept. 17	A purchase of $133 is made.
Sept. 22	A purchase of $325 is made.
Sept. 30	The billing cycle ended.

3. Refer to the following credit card transactions. Determine the average daily balance and the interest charged for this billing cycle for an account with a 12.99% APR.

Feb. 1	The billing cycle begins with a balance of $1750.00
Feb. 2	A purchase of $842 is made.
Feb. 4	A payment of $100 is made.
Feb. 9	A purchase of $61 is made.
Feb. 12	A purchase of $15 is made.
Feb. 19	A purchase of $23 is made.
Feb. 25	The billing cycle ended.

4. Refer to the following credit card transactions. Determine the average daily balance and the interest charged for this billing cycle for an account with a 9.99% APR. Then find the minimum payment when it is either 3% of the balance or $15, whichever is greater.

July 1	The billing cycle begins with a balance of $314
July 2	A payment of $314 is made.
July 6	A purchase of $5 is made.
July 15	A purchase of $12 is made.
July 23	A purchase of $103 is made.
July 25	The billing cycle ended.

This page titled 2.8.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.10: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.9: Which equation to use?

2.10 Learning Objectives

- Determine which equation to use to solve financial problems

Determining Which Equation to Use

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan", "amortize" (the fancy word for loans), "finance (a car)", or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a basic Annuity problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a Payout Annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

Try it Now 1

For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loans problem. Then solve each problem.

a. Marcy received an inheritance of $20,000, and invested it at 6% interest. She is going to use it for college, withdrawing money for tuition and expenses each quarter. How much can she take out each quarter if she has 3 years of school left?
b. Paul wants to buy a new car. Rather than take out a loan, he decides to save $200 a month in an account earning 3% interest compounded monthly. How much will he have saved up after 3 years?
c. Keisha is managing investments for a non-profit company. They want to invest some money in an account earning 5% interest compounded annually with the goal to have $30,000 in the account in 6 years. How much should Keisha deposit into the account?
d. Miao is going to finance new office equipment at a 2% rate over a 4 year term. If she can afford monthly payments of $100, how much new equipment can she buy?
e. How much would you need to save every month in an account earning 4% interest to have $5,000 saved up in two years?

Answer

a. This is a payout annuity problem. She can pull out $1833.60 a quarter.
b. This is a savings annuity problem. He will have saved up $7,524.11.
c. This is compound interest problem. She would need to deposit $22,386.46.
d. This is a loans problem. She can buy $4,609.33 of new equipment.
e. This is a savings annuity problem. You would need to save $200.46 each month

This page titled 2.9: Which equation to use? is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.8: Which equation to use?** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

2.9.1: Exercises

Section 2.9.1 Exercises

1. Suppose you invest $50 a month for 5 years into an account earning 8% compounded monthly. After 5 years, you leave the money in the account for another 25 years without making additional deposits. How much will you have in the end?
2. Kendra wants to be able to make withdrawals of $60,000 a year for 30 years after retiring in 35 years. How much will she have to save each year up until retirement if her account earns 7% interest?
3. You have $2,000 to invest, and want it to grow to $3,000 in two years. What interest rate would you need to find to make this possible if the money is compounded annually?

Exploration

1. Pay day loans are short term loans that you take out against future paychecks: The company advances you money against a future paycheck. Either visit a pay day loan company, or look one up online. Be forewarned that many companies do not make their fees obvious, so you might need to do some digging or look at several companies.
   a. Explain the general method by which the loan works.
   b. We will assume that we need to borrow $500 and that we will pay back the loan in 14 days. Determine the total amount that you would need to pay back and the effective loan rate. The effective loan rate is the percentage of the original loan amount that you pay back. It is not the same as the APR (annual rate) that is probably published.
   c. If you cannot pay back the loan after 14 days, you will need to get an extension for another 14 days. Determine the fees for an extension, determine the total amount you will be paying for the now 28 day loan, and compute the effective loan rate.
2. Suppose that 10 years ago you bought a home for $110,000, paying 10% as a down payment, and financing the rest at 9% interest for 30 years.
   a. How much money did you pay as your down payment?
   b. How much money was your mortgage (loan) for?
   c. What is your current monthly payment?
   d. How much total interest will you pay over the life of the loan?
3. Referring to the same scenario from the problem above, suppose you check your loan balance on your mortgage. Only part of your payments have been going to pay down the loan; the rest has been going towards interest. You see that you still have $88,536 left to pay on your loan. Your house that you purchased for $110,000 is now valued at $150,000.
   a. How much of the loan have you paid off? (i.e., How much have you reduced the loan balance by? Keep in mind that interest is charged each month - it's not part of the loan balance.)
   b. How much money have you paid to the loan company so far?
   c. How much interest have you paid so far?
   d. How much equity do you have in your home? (Equity is value minus remaining debt.)
4. Since interest rates have dropped, you consider refinancing the above mortgage at a lower rate of 6%.
   a. If you took out a new 30 year mortgage at 6% for your remaining loan balance, what would your new monthly payments be?
   b. How much interest will you pay over the life of the new loan?
5. Notice that if you refinance the loan above, you would be making payments on your home for another 30 years. In addition to the 10 years you've already been paying, that's 40 years total.
   a. How much will you save each month because of the lower monthly payment?
   b. How much total interest will you be paying? (You need to consider the amount from 3c and 4b.)
   c. Does it make sense to refinance? (There isn't a correct answer to this question. Just give your opinion and your reasons.)
   d. Would your answer change if you could refinance with a new loan term of 20 years, thus keeping the original 30-year time line?

This page titled 2.9.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **9.10: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

CHAPTER OVERVIEW

3: Logic

3.1: Basics of Sets

3.1.1: Union, Intersection, and Complement

3.1.1.1: Exercises

3.1.2: Venn Diagrams

3.1.2.1: Exercises

3.2: Logic

3.2.1: Boolean Logic

3.2.1.1: Exercises

3.2.2: Conditional Statements

3.2.2.1: Exercises

3.2.3: Quantified Statements

3.2.3.1: Exercises

3.2.4: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

3.2.4.1: Exercises

3.2.5: Truth Tables- Conditional, Biconditional

3.2.5.1: Exercises

3.3: Arguments

3.3.1: Forms of Valid Arguments

3.3.1.1: Exercises

3.3.2: Evaluating Deductive Arguments with Euler Diagrams

3.3.2.1: Exercises

3.4: Logical Fallacies in Common Language

3.4.1: Exercises

This page titled 3: Logic is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

3.1: Basics of Sets

3.1 Learning Objectives

- Use the correct notation for elements of sets and subsets
- Find the cardinality of a set, including the empty set

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Definition: Set

A **set** is a collection of distinct objects, called **elements** of the set

A set is said to be **well-defined** if the contents of the set can be easily determined by the description.

Definition: Roster Method and Set-Builder

A set can be defined using word descriptions, roster method, and set-builder notation.

A word description could be: The days of the week.

The **roster method** lists the elements of the set enclosed in curly brackets.

$\{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday\}$

Set-builder notation describes what a general element should be.

$\{x|x = \text{a day of the week}\}$. In English this reads as: "The set of all *x* such that *x* is a day of the week."

Example 1

Some examples of sets defined by describing the contents:

a. The set of all even numbers
b. The set of all books written about travel to Chile

Some examples of sets defined by roster method:

a. {1, 3, 9, 12}
b. {red, orange, yellow, green, blue, indigo, purple}

Some examples of sets defined by set-builder notation:

a. $\{x|0 \leq x \leq 3 \text{ and } x \in \text{integers}\}$
b. $\{m|m = \text{ a car model}\}$

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

Notation

Commonly, we will use a capital letter to represent a set to make it easier to refer to that set later.

The symbol $\in$ means “is an element of”.

The symbol $\notin$ means “is not an element of”.

A set that contains no elements, {}, is called the **empty set** and is notated Ø or {}.

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is an element of the set, we would write $2 \in A$

To notate that 5 is *not* an element of the set, we would write $5 \notin A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Definition: Subset

A **subset** of a set A is another set that contains only elements from the set A but may not contain all the elements of A.

If B is a subset of A, we write $B \subseteq A$.

A **proper subset** is a subset that is not identical to the original set. There is at least one element in the original set that is not in the subset.

If B is a proper subset of A, we write $B \subset A$.

To indicate a set B is *not* a subset of A, we write $B \not\subseteq A$

The empty set is a subset of *every* set.

Example 3

Consider these three sets

$A =$ the set of all even numbers $\quad B = \{2, 4, 6\} \quad C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so it is an element of A.

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$

It is also true that $B \subset C$.

C is not a subset of A, since C contains an element, 3 , that is not contained in A

Example 4

Suppose a set contains the plays "Much Ado About Nothing", "MacBeth", and "A Midsummer's Night Dream". What is a larger set this might be a subset of?

Solution

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try it Now 1

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Answer

There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

Example 5

If A = {1, 2, 3} , B = {1,3} , and C = {1,2,3}, determine if:

a. $B \subset A$

b. $C \subset A$

Solution

a. Since B is not an identical set to A, we say that this is a proper subset of A.

b. Since C is an identical set to A, we say that C is not a proper subset of A, and should be written as $C \subseteq A$.

In some situations, we are interested in how many elements a set has. This number is called the **cardinality** of the set.

Definition: Cardinality

The **cardinality** of a set is the number of elements in the set.

n(A) is the notation used for the cardinality of the set A.

Example 6

Given A = { 1, 3, 5 }, find the cardinality for the set A.

Solution

Since there are 3 elements in the set A, then we say n(A) = 3.

Example 7

Given the set is empty, find the cardinality for the empty set.

Solution

Since there are no elements in the empty set { }, then we say n(Ø) = 0.

We say that two sets are equal only when they contain exactly the same elements. If they have the same cardinality but not the exact same elements, the sets are called equivalent.

Example 8

Determine if the sets are equal, equivalent, or neither.

a. {1, 3, 9, 12} and {12, 3, 1, 9}
b. {red, orange, yellow, green, blue, indigo, purple} and {blue, indigo, purple}
c. {a, e, i, o, u} and {p, q, r, s, t}

Solution

a. These sets are equal since it has the same elements even though they are in a different order.
b. This set is neither equal or equivalent.
c. This set is equivalent since it contains the same number of elements but they are different.

This page titled 3.1: Basics of Sets is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **13.1: Basics of Sets** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.1.1: Union, Intersection, and Complement

3.1.1 Learning Objectives

- Find the union and intersection of two sets
- Find the complement of a set relative to a universal set
- Use and interpret set notation for unions, intersections and complements of sets

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Definition: Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets). The word that you will often see that indicates a union is "or".

The union is notated $A \cup B$

More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets. The word that you will often see that indicates an intersection is "and".

The intersection is notated $A \cap B$

More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$

The **complement** of a set *A* contains everything that is *not* in the set *A*.

The complement is notated A, or A^c, or sometimes $\sim A$.Example 3.1.1.1

Example 1

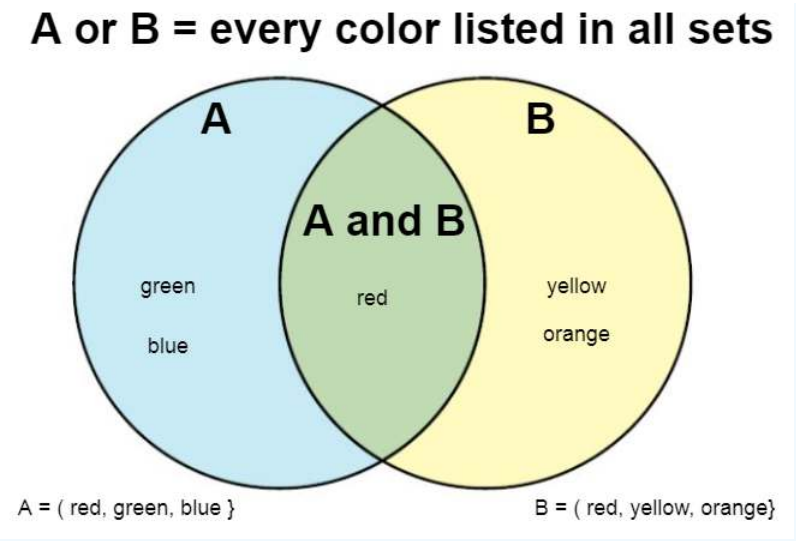
Consider the sets:

$$A = \{ \text{red, green, blue} \} \quad B = \{ \text{red, yellow, orange} \} \quad C = \{ \text{red, orange, yellow, green, blue, purple} \}$$

a. Find $A \cup B$ (Hint: This means the union of two sets, so it means elements could belong to set A, set B, or both sets, commonly referred as A or B)
b. Find $A \cap B$ (Hint: This means the intersection of two sets, so it means elements must be in common in both sets, commonly referred as A and B)
c. Find $A^c \cap C$

Solution

A **Venn diagram** can help organize each of these problems.

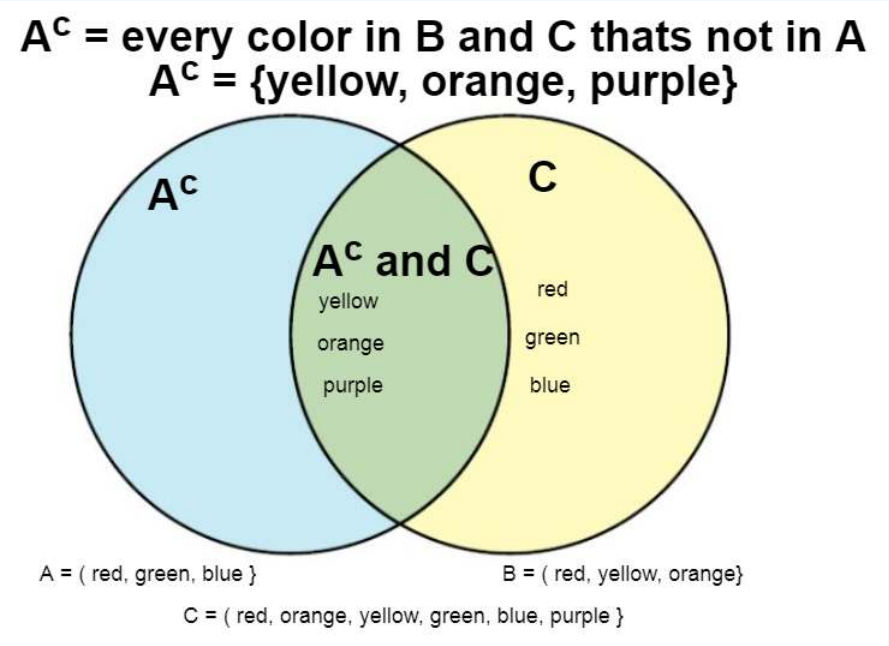


a) The union contains all the elements in either set: $A \cup B = \{ \text{red, green, blue, yellow, orange} \}$

Notice we only list red once.

b) The intersection contains all the elements in both sets: $A \cap B = \{ \text{red} \}$

c) Here we're looking for all the elements that are not in set A and are also in C.



$A^c \cap C = \{ \text{orange, yellow, purple} \}$

Try it Now 1

Using the sets from the previous example, find $A \cup C$ and $B^c \cap A$

Answer

$A \cup C = \{ \text{red, orange, yellow, green, blue purple} \}$

$B^c \cap A = \{ \text{green, blue} \}$

Notice that in the example above, it would be hard to just ask for A^c, since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Definition: Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^C contains all the elements in the universal set that are not in A.

Example 2

a. If we were discussing searching for books, the universal set might be all the books in the library.
b. If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
c. If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 3

Suppose the universal set is $U =$ all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then

$A^c = \{3, 5, 6, 7, 8, 9\}$

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic - to force an order of operations.

Example 4

Suppose

$H = \{ \text{cat, dog, rabbit, mouse} \}, F = \{ \text{dog, cow, duck, pig, rabbit} \} \quad W = \{ \text{duck, rabbit, deer, frog, mouse} \}$

a. Find $(H \cap P) \cup W$
b. Find $H \cap (F \cup W)$
c. Find $(H \cap P) \cap W$

Solution

a) We start with the intersection: $H \cap F = \{ \text{dog, rabbit} \}$

Now we union that result with W : $(H \cap F) \cup W = \{\text{dog, duck, rabbit, deer, frog, mouse} \}$

b) We start with the union: $F \cup W = \{\text{dog, cow, rabbit, duck, pig, deer, frog, mouse} \}$

Now we intersect that result with H : $H \cap (F \cup W) = \{ \text{dog, rabbit, mouse} \}$

c) We start with the intersection: $H \cap F = \{\text{dog, rabbit} \}$

Now we want to find the elements of W that are not in $\text{H} \cap F$

$(H \cap P)^c \cap W = \{ \text{duck, deer, frog, mouse} \}$

This page titled 3.1.1: Union, Intersection, and Complement is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **13.2: Union, Intersection, and Complement** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.1.1.1: Exercises

1. List the elements of the set “The letters of the word Mississipi”
2. List the elements of the set “Months of the year”
3. Use a word description, and set-builder notation for the set $\{3, 6, 9\}$
4. Use a word description, and set-builder notation for the set $\{a, i, e, o, u\}$
5. Is $\{1, 3, 5, 6\}$a subset of the set of odd integers?
6. Is $\{A, B, C\}$a subset of the set of letters of the alphabet?
7. Consider the sets below, and indicate if each statement is true or false.

   $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 3, 5\}$ $C = \{4, 6\}$ $U = \{\text{whole numbers from 0 to 10}\}$

   a. $3 \in B$
   b. $5 \in C$
   c. $B \subset A$
   d. $C \subset A$
   e. $C \subset B$
   f. $C \subset U$
8. Use the sets from above and treat U as the Universal set. Find each of the following:

   a. $A \cup B$
   b. $A \cup C$
   c. $A \cap C$
   d. $B \cap C$
   e. A^c
   f. B^c
9. Let $D = \{b, a, c, k\}$, $E = \{t, a, s, k\}$, $F = \{b, a, t, h\}$. Using these sets, find the following:

   a. $D^c \cap E$
   b. $F^c \cap D$
   c. $(D \cap E) \cup F$
   d. $D \cap (E \cup P)$
   e. $(F \cap E)^c \cap D$
   f. $(D \cup E)^c \cap F$

3.1.1.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

- **13.5: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.1.2: Venn Diagrams

3.1.2 Learning Objectives

- Use Venn diagrams to depict unions, intersections, and complements of sets
- Create an expression to describe a section of a Venn diagram

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations are now called **Venn Diagrams**.

Definition: Venn Diagram

A **Venn diagram** represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

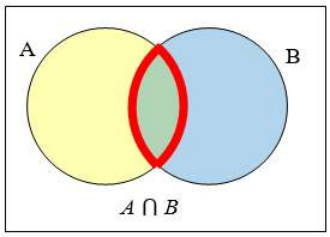
Example 1

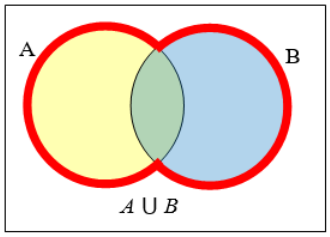
Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$

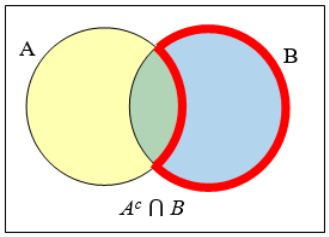
$A \cap B$ contains all elements in both sets - elements in A and B.

$A \cup B$ contains all elements in either set - elements in A or B (or both).

$A^c \cap B$ contains elements outside of A and also in B.



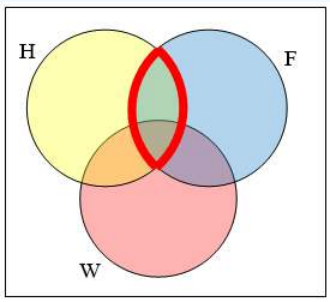




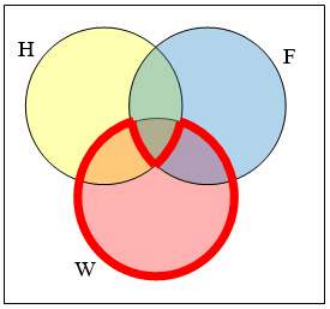
Example 2

Use a Venn diagram to illustrate $(H \cap F)^c \cap W$

We'll start by identifying everything in the set $H \cap F$

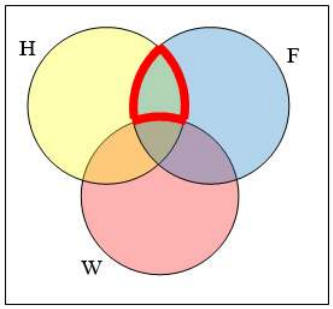


Now, $(H \cap F)^c \cap W$ will contain everything not in the set identified above that is also in set W



Example 3

Create an expression to represent the outlined part of the Venn diagram shown.

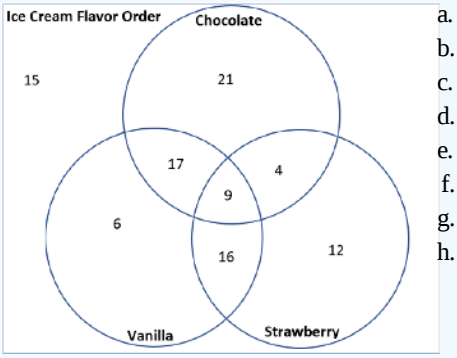


The elements in the outlined set are in sets H and F, but are not in set W. So we could represent this set as $H \cap F \cap W^c$

Sometimes Venn diagrams include numbers to show the cardinality of each region in the diagram.

Example 4

Use the Venn Diagram representing the number of people who ordered the given ice cream flavors to answer the following questions.



a. How many people ordered chocolate and strawberry?
b. How many people ordered chocolate or strawberry?
c. How many people ordered chocolate and strawberry but not vanilla?
d. How many people ordered only vanilla?
e. How many people ordered vanilla?
f. How many people did not order chocolate, vanilla, or strawberry?
g. How many people ordered all three flavors?
h. How many total orders were there?

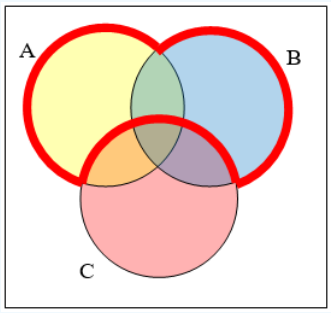
Solution

a. Look at the intersection of chocolate and strawberry to see that 13 people ordered that combination.
b. Look at the union of the chocolate circle with the strawberry circle to see that 79 people ordered chocolate or strawberry.
c. Look at the intersection of chocolate and strawberry and exclude those who ordered vanilla.
d. Look at the section of the vanilla circle that does not intersect with any other flavor to see that 6 people ordered only vanilla.
e. Look at all of the regions inside the vanilla circle to see that 48 people ordered vanilla.
f. Look outside the Venn diagram to see that 15 orders did not contain any of those flavors.
g. Look in the center where all three circles overlap to see that 9 people ordered all three flavors.

h. Add the numbers in each section of the Venn and the number outside the Venn to find that there were a total of 100 orders.

Try it Now 1

Create an expression to represent the outlined portion of the Venn diagram shown



Answer

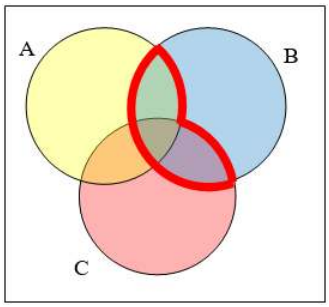
$A \cup B \cap C^c$

This page titled 3.1.2: Venn Diagrams is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

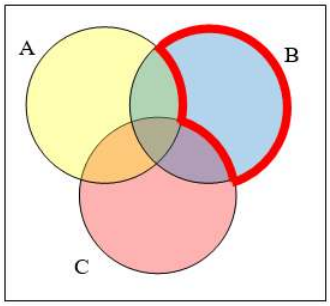
- **13.3: Venn Diagrams** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.
- **13.1: Basics of Sets** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.1.2.1: Exercises

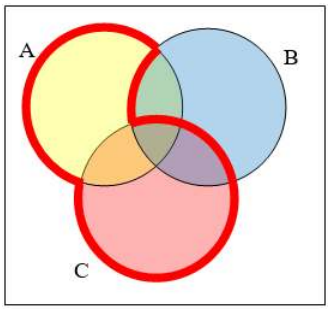
1. Create a Venn diagram to illustrate each of the following:
   a. $(F \cap E) \cup D$
   b. $(D \cup E)^c \cap F$
   c. $(F^c \cap E^c) \cap D$
   d. $(D \cup E) \cup F$
2. Write an expression for the region outlined in red for the following Venn diagrams.
   a.



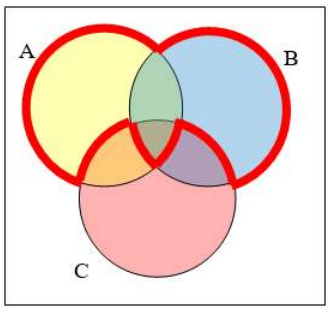
   b.



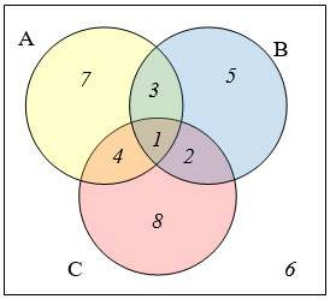
   c.



   d.



3. Let $A=\{1,2,3,4,5\}$ $B=\{1,3,5\}$ $C=\{4,6\}$ Find the cardinality of the given set.
   a. $\mathrm{n}(A)$
   b. $\mathrm{n}(B)$
   c. $\mathrm{n}(A\cup C)$
   d. $\mathrm{n}(A\cap C)$
   e. $\mathrm{n}(B\cap C)$
4. The Venn diagram here shows the cardinality of each region in the diagram. Use this to find the cardinality of each given set.



   a. $\mathrm{n}(A\cap C)$
   b. $\mathrm{n}(B\cup C)$
   c. $\mathrm{n}\left(A\cap B\cap C^{c}\right)$
   d. $\mathrm{n}\left(A\cap B^{c}\cap C\right)$
5. If $n(G)=20, n(H)=30, n(G\cap H)=5$, find $n(G\cup H)$
6. If $n(G)=5, n(H)=8, n(G\cap H)=4$, find $n(G\cup H)$
7. A survey was given asking people whether they watch movies at home from Netflix, Hulu, or Amazon. Use the results to determine how many people use Redbox.

52 only use Netflix	62 only use Hulu
24 only use Amazon	16 use only Amazon and Hulu
48 use only Netflix and Hulu	30 use only Amazon and Netflix
10 use all three	25 use none of these

8. A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?

5 said only color	8 said only size
16 said only brand	20 said only color and size
42 said only color and brand	53 said only size and brand
102 said all three	20 said none of these

9. Use the given information to complete a Venn diagram, then determine: a) how many students have seen exactly one of these movies, and b) how many had seen only *Star Wars*.

18 had seen The Matrix (M)	24 had seen Star Wars (SW)
20 had seen Lord of the Rings (LotR)	10 had seen M and SW
14 had seen LotR and SW	12 had seen M and LotR
6 had seen all three	

10. A survey asked people what alternative transportation modes they use. Using the data to complete a Venn diagram, then determine: a) what percent of people only ride the bus, and b) how many people don't use any alternate transportation.

30% use the bus	20% ride a bicycle
25% walk	5% use the bus and ride a bicycle
10% ride a bicycle and walk	12% use the bus and walk
2% use all three	

This page titled 3.1.2.1: Exercises is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **13.5: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2: Logic

Logic is, basically, the study of valid reasoning. When searching the internet, we use Boolean logic – terms like “and,” “or,” and "not" – to help us find specific web pages that fit the sets we are interested in. After exploring this form of logic, we will look at logical arguments and how we can determine the validity of a claim.

This page titled 3.2: Logic is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.1: Logic** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2.1: Boolean Logic

3.2.1 Learning Objectives

- Use “and, “or,” and “not” to describe sets meeting given requirements

We can often classify items as belonging to sets. If you went the library to search for a book and they asked you to express your search using unions, intersections, and complements of sets, that would feel a little strange. Instead, we typically using words like “and”, “or”, and “not” to connect our keywords together to form a search. These words, which form the basis of **Boolean logic**, are directly related to our set operations. (Boolean logic was developed by the 19th-century English mathematician George Boole.)

Definition: Boolean Logic

Boolean logic combines multiple statements that are either true or false into an expression that is either true or false.

In connection to sets, a search is true if the element is part of the set.

Suppose M is the set of all mystery books, and C is the set of all comedy books. If we search for “mystery”, we are looking for all the books that are an element of the set M; the search is true for books that are in the set.

When we search for “mystery *and* comedy”, we are looking for any book that is an element of both sets, in the intersection. If we were to search for “mystery *or* comedy”, we are looking for a book that is a mystery, a comedy, or both, which is the union of the sets. If we searched for “*not* comedy”, we are looking for any book in the library that is not a comedy, the complement of the set C.

Connection to Set Operations

A and B - elements in the intersection $A \cap B$

A or B - elements in the union $A \cup B$

not A - elements in the complement A^c

Notice here that *or* is not exclusive. This is a difference between the Boolean logic use of the word and common everyday use. When your significant other asks “do you want to go to the park or the movies?” they usually are proposing an exclusive choice – one option or the other, but not both. In Boolean logic, the *or* is not exclusive – more like being asked at a restaurant “would you like fries or a drink with that?” Answering “both, please” is an acceptable answer.

Example 1

Suppose we are searching a library database for Mexican universities. Express a reasonable search using Boolean logic.

Solution

We could start with the search “Mexico *and* university”, but would be likely to find results for the U.S. state New Mexico. To account for this, we could revise our search to read:

Mexico *and* university *not* “New Mexico”

In most internet search engines, it is not necessary to include the word *and*; the search engine assumes that if you provide two keywords you are looking for both. In Google’s search, the keyword *or* has be capitalized as OR, and a negative sign in front of a word is used to indicate *not*. Quotes around a phrase indicate that the entire phrase should be looked for. The search from the previous example on Google could be written:

Mexico university -“New Mexico”

Example 2

Describe the numbers that meet the condition:

even and less than 10 and greater than 0

Solution

The numbers that satisfy all three requirements are $\{2, 4, 6, 8\}$

Sometimes statements made in English can be ambiguous. For this reason, Boolean logic uses parentheses to show precedence, just like in algebraic order of operations.

Example 3

Describe the numbers that meet the condition:

odd number and less than 20 and greater than 0 and (multiple of 3 or multiple of 5)

Solution

The first three conditions limit us to the set $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

The last grouped conditions tell us to find elements of this set that are also either a multiple of 3 or a multiple of 5. This leaves us with the set $\{3, 5, 9, 15\}$

Notice that we would have gotten a very different result if we had written

(odd number and less than 20 and greater than 0 and multiple of 3) or multiple of 5

The first grouped set of conditions would give {3, 9, 15}. When combined with the last condition, though, this set expands without limits:

$\{3, 5, 9, 15, 20, 25, 30, 35, 40, 45, \ldots\}$

Example 4

The English phrase “Go to the store and buy me eggs and bagels or cereal” is ambiguous; it is not clear whether the requestors is asking for eggs always along with either bagels or cereal, or whether they’re asking for either the combination of eggs and bagels, or just cereal.

For this reason, using parentheses clarifies the intent:

Eggs and (bagels or cereal) means Option 1: Eggs and bagels, Option 2: Eggs and cereal

(Eggs and bagels) or cereal means Option 1: Eggs and bagels, Option 2: Cereal

Be aware that when a string of conditions is written without grouping symbols, it is often interpreted from the left to right, resulting in the latter interpretation.

This page titled 3.2.1: Boolean Logic is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.2: Boolean Logic** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2.1.1: Exercises

Boolean Logic

1. List the set of integers that satisfy the given conditions.
   a. A positive multiple of 5 and not a multiple of 2
   b. Greater than 12 and less than or equal to 18

3.2.1.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

- **17.13: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2.2: Conditional Statements

3.2.2 Learning Objectives

- Determine whether a conditional statement is true or false

In logic a **statement** is something that is either true or false. A statement like 3 < 5 is true; a statement like “a rat is a fish” is false. A statement like “$x < 5$” is true for some values of x and false for others. When an action is taken or not depending on the value of a statement, it forms a **conditional**.

Definition: Statement and Conditional

A **statement** is either true or false.

A **conditional** is a compound statement of the form

"if p then q" or "if p then q, else s"

where p and q are both statements.

Example 1

In common language, an example of a conditional statement would be “If it is raining, then we’ll go to the mall. Otherwise we’ll go for a hike.”

The statement “If it is raining” is the condition – this may be true or false for any given day. If the condition is true, then we will follow the first course of action, and go to the mall. If the condition is false, then we will use the alternative, and go for a hike.

"If p, then q" can be stated different ways and still mean the same thing. The following statements all could be rewritten using the if, then format. Here p represents "I receive my check tomorrow," and q represents "I pay off my debt."

If p, then q	If I receive my check tomorrow, then I will pay off my debt.
p is sufficient for q	Receiving my check tomorrow is sufficient for paying off my debt.
p will lead to q	Receiving my check tomorrow will lead to the paying off my debt.
p implies q	Receiving my check tomorrow implies paying off my debt.
p is necessary for q	Receiving my check tomorrow is necessary for paying off my debt .
q if p	I will pay off my debt if I receive my check tomorrow.
q whenever p	I will pay off my debt whenever I receive my check. .

As we did earlier, we can create more complex conditions by using the operators *and*, *or*, and *not* to join simpler conditions together.

Example 2

A parent might say to their child “if you clean your room and take out the garbage, then you can have ice cream.”

Here, there are two simpler conditions:

1) The child cleaning her room

2) The child taking out the garbage

Since these conditions were joined with *and*, the combined conditional will be true only if both simpler conditions are true; if either chore is not completed, then the parent’s condition is not met.

Notice that if the parent had said “if you clean your room *or* take out the garbage, then you can have ice cream”, then the child would need to complete only one chore to meet the condition.

This page titled 3.2.2: Conditional Statements is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.3: Conditional Statements** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2.2.1: Exercises

Conditional Statements

1. Consider the statement “If you are under age 17, then you cannot attend this movie.” Identify the two simpler statements used in this conditional statement.
   a. Write the converse.
   b. Write the inverse.
   c. Write the contrapositive.
2. Assume that the statement “If you swear, then you will get your mouth washed out with soap” is true. Which of the following statements must also be true?
   a. If you don’t swear, then you won’t get your mouth washed out with soap.
   b. If you don’t get your mouth washed out with soap, then you didn’t swear.
   c. If you get your mouth washed out with soap, then you swore.
3. Write the negation of each conditional statement.
   a. If you don’t look both ways before crossing the street, then you will get hit by a car.
   b. If Luke faces Vader, then Obi-Wan cannot interfere.
   c. If you weren’t talking, then you wouldn’t have missed the instructions.

3.2.2.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

- **17.13: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2.3: Quantified Statements

3.2.3 Learning Objectives

- Interpret the use of universal quantifiers and existential quantifiers correctly
- Negate a quantified statement

Words that describe an entire set, such as “all”, “every”, or “none”, are called **universal quantifiers** because the description applies to all elements in the set. In contrast, words or phrases such as “some”, “one”, or “at least one” are called **existential quantifiers** because they describe the existence of at least one element in a set.

Definition: Quantifiers

A **universal quantifier** states that an entire set of things share a characteristic.

An **existential quantifier** states that a set contains at least one element with that characteristic.

Something interesting happens when we **negate** – or state the opposite of – a quantified statement.

Example 1

Suppose your friend says “Everybody cheats on their taxes.” This statement uses a universal quantifier. What is the minimum amount of evidence you would need to prove your friend wrong?

To show that it is not true that everybody cheats on their taxes, all you need is one person who does not cheat on their taxes. It would be perfectly fine to produce more people who do not cheat, but one counterexample is all you need.

It is important to note that you do not need to show that absolutely nobody cheats on their taxes.

Example 2

Suppose your friend says “One of these six cartons of milk is leaking.” This statement uses an existential quantifier. What is the minimum amount of evidence you would need to prove your friend wrong?

Solution

In this case, you would need to check all six cartons and show that none of them is leaking. You cannot disprove your friend’s statement by checking only one of the cartons.

When we negate a statement with a universal quantifier, we get a statement with an existential quantifier, and vice-versa.

Negating a quantified statement

The negation of “all A are B” is “at least one A is not B”.

The negation of “no A are B” is “at least one A is B”.

The negation of “at least one A is B” is “no A are B”.

The negation of “at least one A is not B” is “all A are B”.

Example 3

“Somebody brought a flashlight.” Write the negation of this statement.

The negation is “Nobody brought a flashlight.”

Example 4

“There are no prime numbers that are even.” Write the negation of this statement.

The negation is “At least one prime number is even.”

Try it Now 1

Write the negation of “All Icelandic children learn English in school.”

Answer

At least one Icelandic child did not learn English in school.

This page titled 3.2.3: Quantified Statements is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.4: Quantified Statements** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.
- **13.2: Union, Intersection, and Complement** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2.3.1: Exercises

Quantified Statements

1. Write the negation of each quantified statement.
   a. Everyone failed the quiz today.
   b. Someone in the car needs to use the restroom.
   c. No one likes getting a shot.
   d. There is a person who does not like broccoli.

3.2.3.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

- **17.13: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2.4: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

3.2.4 Learning Objectives

- Translate statements involving “and, “or,” and “not” into symbolic notation and vice versa
- Use basic truth tables for conjunction, disjunction, and negation
- Build truth tables for more complex statements involving conjunction, disjunction, and negation

Before we focus on truth tables, we’re going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

Symbols

The symbol $\wedge$ is used for and: A and B is notated $A \wedge B$

The symbol $\vee$ is used for or: A or B is notated $A \vee B$

The symbol $\sim$ is used for not: not A is notated $\sim A$

You can remember the first two symbols by relating them to the shapes for the union and intersection. $A \wedge B$ would be the elements that exist in both sets, in $A \cap B$. Likewise, $A \vee B$ would be the elements that exist in either set, in $A \cup B$. When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

Example 1

Translate each statement into symbolic notation. Let P represent "I like Pepsi" and let C represent "I like Coke."

a. I like Pepsi or I like Coke.
b. I like Pepsi and I like Coke.
c. I do not like Pepsi.
d. It is not the case that I like Pepsi or Coke.
e. I like Pepsi and I do not like Coke.

Solution

a. $P \vee C$
b. $P \wedge C$
c. $\sim P$
d. $\sim (P \vee C)$
e. $P \wedge \sim C$

As you can see, we can use parentheses to organize more complicated statements.

Try it Now 1

Translate “We have carrots or we will not make soup” into symbols. Let C represent “we have carrots” and let S represent “we will make soup”.

Answer

$C \vee \sim S$

Because complex statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true or false.

Definition: Truth Table

A **truth table** is a table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 2

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise".

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use *S* to designate "is a sectional", and *C* to designate "has a chaise".

A truth table for this situation would look like this:

S	C	S or C
T	T	T
T	F	T
F	T	T
F	F	F

In the table, T is used for true, and F for false. In the first row, if *S* is true and *C* is also true, then the complex statement "*S* or *C*" is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that *or* in logic is not exclusive; if the couch has both features, it meets the condition.)

In the previous example about the couch, the truth table was really just summarizing what we already know about how the *or* statement works. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

Basic truth tables

Conjunction

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

A	$\sim A$
T	F
F	T

Truth tables really become useful when we analyze more complex Boolean statements.

Example 3

Create a truth table for the statement $A \vee \sim B$

Solution

When we create the truth table, we need to list all the possible truth value combinations for A and B. Notice how the first column contains 2 Ts followed by 2 Fs, and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.

A	B
T	T
T	F
F	T
F	F

After creating columns with those initial values, we create a third column for the expression $\sim B$. Now we will temporarily ignore the column for A and write the truth values for $\sim B$

A	B	$\sim B$
T	T	F
T	F	T
F	T	F
F	F	T

Next we can find the truth values of $A \vee \sim B$, using the first and third columns.

A	B	$\sim B$	$A \vee \sim B$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

The truth table shows that $A \vee \sim B$ is true in three cases and false in one case. If you're wondering what the point of this is, suppose it is the last day of the baseball season and two teams, who are not playing each other, are competing for the final playoff spot. Anaheim will make the playoffs if it wins its game or if Boston does not win its game. (Anaheim owns the tie-breaker; if both teams win, or if both teams lose, then Anaheim gets the playoff spot.) If $A =$ Anaheim wins its game and $B =$ Boston wins its game, then $A \vee \sim B$ represents the situation "Anaheim wins its game or Boston does not win its game". The truth table shows us the different scenarios related to Anaheim making the playoffs. In the first row, Anaheim wins its game and Boston wins its game, so it is true that Anaheim makes the playoffs. In the second row, Anaheim wins and Boston does not win, so it is true that Anaheim makes the playoffs. In the third row, Anaheim does not win its game and Boston wins its game, so it is false that Anaheim makes the playoffs. In the fourth row, Anaheim does not win and Boston does not win, so it is true that Anaheim makes the playoffs.

Try it Now 2

Create a truth table for this statement: $\sim A \wedge B$

Answer

A	B	$\sim A$	$\sim A \wedge B$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Example 4

Create a truth table for the statement $A \wedge \sim (B \vee C)$

Solution

It helps to work from the inside out when creating a truth table, and to create columns in the table for intermediate operations. We start by listing all the possible truth value combinations for A, B, and C. Notice how the first column contains 4 Ts followed by 4Fs, the second column contains 2Ts, 2Fs, then repeats, and the last column alternates $\mathrm{T}, \mathrm{F}, \mathrm{T}, \mathrm{F} \ldots$ This pattern ensures that all 8 combinations are considered. After creating columns with those initial values, we create a fourth column for

the innermost expression, $B \vee C$. Now we will temporarily ignore the column for A and focus on B and C, writing the truth values for $B \vee C$

A	B	C
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

A	B	C	$B \vee C$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Next we can find the negation of $B \vee C$, working off the $B \vee$ C column we just created. (Ignore the first three columns and simply negate the values in the $B \vee C$ column.)

A	B	C	$B \vee C$	$\sim (B \vee C)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	F
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

Finally, we find the values of A and $\sim (B \vee C)$. (Ignore the second, third, and fourth columns.)

A	B	C	$B \vee C$	$\sim B \vee C)$	$A \wedge \sim B \vee C$)
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

It turns out that this complex expression is true in only one case: when A is true, B is false, and C is false. To illustrate this situation, suppose that Anaheim will make the playoffs if: (1) Anaheim wins, and (2) neither Boston nor Cleveland wins. TFF is the only scenario in which Anaheim will make the playoffs.

Try it Now 3

Create a truth table for this statement: $(\sim A \wedge B) \vee \sim B$

Answer

A	B	$\sim A$	$\sim A \wedge B$	$\sim B$	$(\sim A \wedge B) \vee \sim B$
T	T	F	F	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	F	T	T

This page titled 3.2.4: Truth Tables- Conjunction (and), Disjunction (or), Negation (not) is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.5: Truth Tables: Conjunction (and), Disjunction (or), Negation (not)** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2.4.1: Exercises

Truth Tables

1. Translate each statement from symbolic notation into English sentences. Let *A* represent “Elvis is alive” and let *G* represent “Elvis gained weight”.
    1. $A \vee G$
    2. $\sim (A \wedge G)$
2. For questions 2 - 3, use the statements *A* and *G* from the previous problem and let statement *P* represent "Elvis is in Vegas". Translate each statement from symbolic notation into English sentences then create a truth table for each statement.
    1. $A \wedge \sim G$
    2. $\sim (\sim A \vee G)$
3. Complete the truth table for the exclusive or.

A	B	$A \vee B$
T	T	
T	F	
F	T	
F	F	

4. Complete the truth table for $(A \vee B) \wedge \sim (A \wedge B)$.

A	B	$A \vee B$	$A \wedge B$	$\sim (A \wedge B)$	$(A \vee B) \wedge \sim (A \wedge B)$
T	T				
T	F				
F	T				
F	F				

5. Compare your answers for questions 3 and 4. Can you explain the similarities?

3.2.4.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

- **17.13: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2.5: Truth Tables- Conditional, Biconditional

3.2.5 Learning Objectives

- Translate conditional and biconditional statements into symbolic notation and vice versa
- Use basic truth tables for conditional and biconditional statements
- Build truth tables for more complex statements involving conditional and biconditional statements
- Determine the truth value of the converse, inverse and contrapositive of a conditional statement
- Build truth tables for more complex statements involving conjunction, disjunction, negation, conditionals, and biconditionals

We discussed conditional statements earlier, in which we take an action based on the value of the condition. We are now going to look at another version of a conditional, sometimes called an implication, which states that the second part must logically follow from the first.

Definition: Conditional

A **conditional** is a logical compound statement in which a statement p, called the antecedent, implies a statement q, called the consequent.

A conditional is written as $p \rightarrow q$ and is translated as "if p, then q".

Example 1

The English statement “If it is raining, then there are clouds is the sky” is a conditional statement. It makes sense because if the antecedent “it is raining” is true, then the consequent “there are clouds in the sky” must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consequent becomes irrelevant.

Example 2

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday’s game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

There are four possible outcomes:

1) You pay for expedited shipping and receive the jersey by Friday

2) You pay for expedited shipping and don’t receive the jersey by Friday

3) You don’t pay for expedited shipping and receive the jersey by Friday

4) You don’t pay for expedited shipping and don’t receive the jersey by Friday

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don’t receive the jersey by Friday. The first outcome is exactly what was promised, so there’s no problem with that. The third outcome is not a lie because the website never said what would happen if you didn’t pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn’t make any promises about when the jersey would arrive if you didn’t pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the antecedent is false, we cannot make any judgment about the consequent. The website never said that paying for expedited shipping was the *only* way to receive the jersey by Friday.

Example 3

A friend tells you "If you upload that picture to Facebook, you'll lose your job." Under what conditions can you say that your friend was wrong?

There are four possible outcomes:

1) You upload the picture and lose your job

2) You upload the picture and don't lose your job

3) You don't upload the picture and lose your job

4) You don't upload the picture and don't lose your job

There is only one possible case in which you can say your friend was wrong: the second outcome in which you upload the picture but still keep your job. In the last two cases, your friend didn't say anything about what would happen if you didn't upload the picture, so you can't say that their statement was wrong. Even if you didn't upload the picture and lost your job anyway, your friend never said that you were guaranteed to keep your job if you didn't upload the picture; you might lose your job for missing a shift or punching your boss instead.

In traditional logic, a conditional is considered true as long as there are no cases in which the antecedent is true and the consequent is false.

Truth table for the conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Again, if the antecedent p is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true.

Example 4

Construct a truth table for the statement $(m \wedge \sim p) \rightarrow r$

Solution

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the antecedent, $m \wedge \sim p$.

m	p	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

m	p	r	$\sim p$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

m	p	r	$\sim p$	$m \wedge \sim p$
T	T	T	F	F
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	F	F
F	T	F	F	F
F	F	T	T	F
F	F	F	T	F

Now we can create a column for the conditional. Because it can be confusing to keep track of all the Ts and Fs, why don't we copy the column for r to the right of the column for $m \wedge \sim p$? This makes it a lot easier to read the conditional from left to right.

m	p	r	$\sim p$	$m \wedge \sim p$	r	$(m \wedge \sim p) \rightarrow r$
T	T	T	F	F	T	T
T	T	F	F	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	T	F	F	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	F	T

When m is true, p is false, and r is false- -the fourth row of the table-then the antecedent $m \wedge \sim p$ will be true but the consequent false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by $(m \wedge \sim p) \rightarrow r$, consider this: let $m =$ we order meatballs, $p =$ we order pasta, and $r =$ Rob is happy. The statement $(m \wedge \sim p) \rightarrow r$ is "if we order meatballs and don't order pasta, then Rob is happy". If m is true (we order meatballs), p is false (we don't order pasta), and r is false (Rob is not happy), then the statement is false, because we satisfied the antecedent but Rob did not satisfy the consequent.

For any conditional, there are three related statements, the converse, the inverse, and the contrapositive.

Related Statements

The original conditional is "if p, then q" $p \rightarrow q$

The converse is "if q, then p" $q \rightarrow p$

The inverse is "if not p, then not q" $\sim p \rightarrow \sim q$

The contrapositive is "if not q, then not p" $\sim q \rightarrow \sim p$

Example 5

Consider again the conditional "If it is raining, then there are clouds in the sky." It seems reasonable to assume that this is true.

The converse would be "If there are clouds in the sky, then it is raining." This is not always true.

The inverse would be "If it is not raining, then there are not clouds in the sky." Likewise, this is not always true.

The contrapositive would be "If there are not clouds in the sky, then it is not raining." This statement is true, and is equivalent to the original conditional.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Conditional	Converse	Inverse	Contrapositive
p	q	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalent

Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a conditional statement are logically equivalent.

In other words, the original statement and the contrapositive must agree with each other; they must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other; they must both be true, or they must both be false.

Be aware that symbolic logic cannot represent the English language perfectly. For example, we may need to change the verb tense to show that one thing occurred before another.

Example 6

Suppose this statement is true: "If I eat this giant cookie, then I will feel sick." Which of the following statements must also be true?

a. If I feel sick, then I ate that giant cookie.
b. If I don't eat this giant cookie, then I won't feel sick.
c. If I don't feel sick, then I didn't eat that giant cookie.

Solution

a. This is the converse, which is not necessarily true. I could feel sick for some other reason, such as drinking sour milk.
b. This is the inverse, which is not necessarily true. Again, I could feel sick for some other reason; avoiding the cookie doesn't guarantee that I won't feel sick.
c. This is the contrapositive, which is true, but we have to think somewhat backwards to explain it. If I ate the cookie, I would feel sick, but since I don't feel sick, I must not have eaten the cookie.

Notice again that the original statement and the contrapositive have the same truth value (both are true), and the converse and the inverse have the same truth value (both are false).

Try it Now 1

"If you microwave salmon in the staff kitchen, then I will be mad at you." If this statement is true, which of the following statements must also be true?

a. If you don't microwave salmon in the staff kitchen, then I won't be mad at you.
b. If I am not mad at you, then you didn't microwave salmon in the staff kitchen.

c. If I am mad at you, then you microwaved salmon in the staff kitchen.

Answer

Choice b is correct because it is the contrapositive of the original statement.

Consider the statement "If you park here, then you will get a ticket." What set of conditions would prove this statement false?

a. You don't park here and you get a ticket.
b. You don't park here and you don't get a ticket.
c. You park here and you don't get a ticket.

The first two statements are irrelevant because we don't know what will happen if you park somewhere else. The third statement, however contradicts the conditional statement "If you park here, then you will get a ticket" because you parked here but didn't get a ticket. This example demonstrates a general rule; the negation of a conditional can be written as a conjunction: "It is not the case that if you park here, then you will get a ticket" is equivalent to "You park here *and* you do not get a ticket."

The Negation of a Conditional

The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

$$\sim (p \rightarrow q) \text{ is equivalent to } p \wedge \sim q$$

Example 7

Which of the following statements is equivalent to the negation of "If you don't grease the pan, then the food will stick to it" ?

a. I didn't grease the pan and the food didn't stick to it.
b. I didn't grease the pan and the food stuck to it.
c. I greased the pan and the food didn't stick to it.

Solution

a. This is correct; it is the conjunction of the antecedent and the negation of the consequent. To disprove that not greasing the pan will cause the food to stick, I have to not grease the pan and have the food not stick.
b. This is essentially the original statement with no negation; the "if…then" has been replaced by "and".
c. This essentially agrees with the original statement and cannot disprove it.

Try it Now 2

"If you go swimming less than an hour after eating lunch, then you will get cramps." Which of the following statements is equivalent to the negation of this statement?

a. I went swimming more than an hour after eating lunch and I got cramps.
b. I went swimming less than an hour after eating lunch and I didn't get cramps.
c. I went swimming more than an hour after eating lunch and I didn't get cramps.

Answer

Choice b is equivalent to the negation; it keeps the first part the same and negates the second part.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider "If you submit your hours today, then you will be paid next Friday." What the payroll rep really means is "If you submit your hours today, then you will be paid next Friday, and if you don't submit your hours today, then you won't be paid next Friday." The conditional statement if *t*, then *p* also includes the inverse of the statement: if not *t*, then not *p*. A more compact way to express this statement is "You will be paid next Friday *if and only if* you submit your timesheet today." A statement of this form is called a **biconditional**.

Biconditional

A biconditional is a logical conditional statement in which the antecedent and consequent are interchangeable.

A biconditional is written as $p \leftrightarrow q$ and is translated as " p if and only if q".

Because a biconditional statement $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$, we may think of it as a conditional statement combined with its converse: if p, then q and if q, then p. The double-headed arrow shows that the conditional statement goes from left to right and from right to left. A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

Truth table for the biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Notice that the fourth row, where both components are false, is true; if you don't submit your timesheet and you don't get paid, the person from payroll told you the truth.

Example 8

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?

a. It is noon on Thursday and the garbage truck did not come down my street this morning.
b. It is Monday and the garbage truck is coming down my street.
c. It is Wednesday at 11:59PM and the garbage truck did not come down my street today.

Solution

a. This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.
b. This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.
c. This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

Try it Now 3

Suppose this statement is true: "I wear my running shoes if and only if I am exercising." Determine whether each of the following statements must be true or false.

a. I am exercising and I am not wearing my running shoes.
b. I am wearing my running shoes and I am not exercising.
c. I am not exercising and I am not wearing my running shoes.

Answer

Choices a & b are false; c is true.

Example 9

Create a truth table for the statement $(A \vee B) \leftrightarrow \sim C$

Solution

Whenever we have three component statements, we start by listing all the possible truth value combinations for A, B, and C. After creating those three columns, we can create a fourth column for the antecedent, $A \vee B$. Now we will temporarily ignore the column for C and focus on A and B, writing the truth values for $A \vee B$.

A	B	C
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

A	B	C	$A \vee B$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

Next we can create a column for the negation of C. (Ignore the $A \vee B$ column and simply negate the values in the C column.)

A	B	C	$A \vee B$	$\sim C$
T	T	T	T	F
T	T	F	T	T
T	F	T	T	F
T	F	F	T	T
F	T	T	T	F
F	T	F	T	T
F	F	T	F	F
F	F	F	F	T

Finally, we find the truth values of $(A \vee B) \leftrightarrow \sim C$. Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.

A	B	C	$A \vee B$	$\sim C$	$(A \vee B) \leftrightarrow \sim C$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	F

To illustrate this situation, suppose your boss needs you to do either project A or project B (or both, if you have the time). If you do one of the projects, you will not get a crummy review (C is for crummy). So $(A \vee B) \leftrightarrow \sim C$ means "You will not get a crummy review if and only if you do project A or project B." Looking at a few of the rows of the truth table, we can see how this works out. In the first row, A, B, and C are all true: you did both projects and got a crummy review, which is not what

your boss told you would happen! That is why the final result of the first row is false. In the fourth row, A is true, B is false, and C is false: you did project A and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row, A, B, and C are all false: you didn't do either project and did not get a crummy review. This is not what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)

This page titled 3.2.5: Truth Tables- Conditional, Biconditional is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.6: Truth Tables: Conditional, Biconditional** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.
- **13.2: Union, Intersection, and Complement** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.2.5.1: Exercises

Truth Tables

1. Translate each statement from symbolic notation into English sentences. Let *A* represent “Elvis is alive” and let *G* represent “Elvis gained weight”.
   a. $G \rightarrow \sim A$
   b. $A \leftrightarrow \sim G$
2. Use the statements *A* and *G* from the previous problem and let statement *P* represent "Elvis is in Vegas". Translate each statement from symbolic notation into English sentences then create a truth table for each statement.
   a. $(A \wedge G) \rightarrow P$
   b. $(A \vee G) \rightarrow \sim P$
3. Determine the truth value of the statement, “If you are under age 17, then you cannot attend this movie.” for a person under age 17. Show your answer using the appropriate truth table.
4. Determine the truth value of the statement, “If you are under age 17, then you cannot attend this movie.” for a person over age 17. Show your answer using the appropriate truth table.
5. Assume that the statement “If you swear, then you will get your mouth washed out with soap” is true. Which of the following statements must also have the same truth value? Show your answer using the appropriate truth table.
   a. If you don’t swear, then you won’t get your mouth washed out with soap.
   b. If you don’t get your mouth washed out with soap, then you didn’t swear.
   c. If you get your mouth washed out with soap, then you swore.
6. Assume that the biconditional statement “You will play in the game if and only if you attend all practices this week” is true. Which of the following statements must also have the same truth value? Show your answer using the appropriate truth table.
   a. You attended all practices this week and didn’t play in the game.
   b. You didn’t attend all practices this week and played in the game.
   c. You didn’t attend all practices this week and didn’t play in the game.

This page titled 3.2.5.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.13: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.3: Arguments

3.3 Learning Objectives

- Identify inductive and deductive argument
- Evaluate the truth of inductive and deductive arguments

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

Argument types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

Example 1

The argument "when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store" is an inductive argument.

The premises are:

I forgot my purse last week

I forgot my purse today

The conclusion is:

I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

Example 2

The argument "every day for the past year, a plane flies over my house at 2:00 P.M. A plane will fly over my house every day at 2:00 P.M." is a stronger inductive argument, since it is based on a larger set of evidence. While it is not necessarily true—the airline may have cancelled its afternoon flight—it is probably a safe bet.

Example 3

The argument "Edgar Allen Poe must be a writer since all poets are writers and he wrote poems."

The premises are:

All poets are writers.

Edgar Allen Poe wrote poems.

The conclusion is:

Edgar Allen Poe is a writer.

The argument uses the general premise that all poets are writers to conclude specifically that Poe is a writer since he wrote poems.

Evaluating inductive arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest that it may be true.

Since the argument cannot be proven to be true the conclusions are called **conjectures** or **hypotheses**.

If there is one example that contradicts the conclusion, it proves the argument is false. This example is called a **counterexample**.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. Common scientific theories, like Newton's theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence, such as when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes it easier to evaluate.

Evaluating deductive arguments

An argument is **valid** when the conclusion follows from the premises whether or not they are true.

A deductive argument is considered **sound** if, assuming that all the premises are true, the conclusion follows logically from those premises. In other words, when the premises are all true, the conclusion *must* be true.

An argument can be valid but not sound.

The argument becomes sound when it is valid and the premises are all true.

Example 4

Evaluate the deductive argument from above to determine if the argument is sound.

The premises are:

All poets are writers.

Edgar Allen Poe wrote poems.

The conclusion is:

Edgar Allen Poe is a writer.

First, determine if the argument is valid. Refer to the premises. Do those premises lead to the conclusion that is stated?

Yes they do. If all writers are poets and Edgar Allen Poe was a poet then the conclusion that he is a writer makes sense.

Second, determine if the argument is sound. Since the argument is valid, we just need to verify that the premises are true. The premises are true and the argument is valid, thus, this argument is sound.

This page titled 3.3: Arguments is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.8: Arguments** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.3.1: Forms of Valid Arguments

3.3.1 Learning Objectives

- Recognize and use several common forms of argument and fallacies to determine whether or not a conclusion is valid

Rather than making a truth table for every argument, we may be able to recognize certain common forms of arguments that are valid (or invalid). If we can determine that an argument fits one of the common forms, we can immediately state whether it is valid or invalid.

The Law of Detachment (*Modus Ponens*)

The law of detachment applies when a conditional and its antecedent are given as premises, and the consequent is the conclusion. The general form is:

Premise:	$p \to q$
Premise:	p
Conclusion:	q

The Latin name, *modus ponens*, translates to "mode that affirms".

Example 1

Consider this argument:

Premise:	If you bought bread, then you went to the store.
Premise:	You bought bread.
Conclusion:	You went to the store.

In symbolic form:

Premise:	$b \to s$
Premise:	b
Conclusion:	s

This argument has the structure described by the law of detachment. (The second premise and the conclusion are simply the two parts of the first premise detached from each other.) Instead of making a truth table, we can say that this argument is valid by stating that it satisfies the law of detachment.

The Law of Contraposition (*Modus Tollens*)

The law of contraposition applies when a conditional and the negation of its consequent are given as premises, and the negation of its antecedent is the conclusion. The general form is:

Premise:	$p \to q$
Premise:	$\sim q$
Conclusion:	$\sim p$

The Latin name, *modus tollens*, translates to "mode that denies".

Notice that the second premise and the conclusion look like the contrapositive of the first premise, $\sim q \to \sim p$, but they have been detached. You can think of the law of contraposition as a combination of the law of detachment and the fact that the contrapositive is logically equivalent to the original statement.

Example 2

Premise:	If I drop my phone into the swimming pool, my phone will be ruined.
Premise:	My phone isn't ruined.
Conclusion:	I didn't drop my phone into the swimming pool.

If we let $d =$ I drop the phone in the pool and $r =$ the phone is ruined, then we can represent the argument this way:

Premise:	$d \to r$
Premise:	$\sim r$
Conclusion:	$\sim d$

The form of this argument matches what we need to invoke the law of contraposition, so it is a valid argument.

Try it Now 1

Is this argument valid?

Premise:	If you brushed your teeth before bed, then your toothbrush will be wet.
Premise:	Your toothbrush is dry.
Conclusion:	You didn't brush your teeth before bed.

Answer

Let $b =$ brushed teeth and $w =$ toothbrush is wet.

Premise:	$b \to w$
Premise:	$\sim w$
Conclusion:	$\sim b$

This argument is valid by the Law of Contraposition.

The Transitive Property (Hypothetical Syllogism)

The transitive property has as its premises a series of conditionals, where the consequent of one is the antecedent of the next. The conclusion is a conditional with the same antecedent as the first premise and the same consequent as the final premise. The general form is:

Premise:	$p \to q$
Premise:	$q \to r$
Conclusion:	$p \to r$

The transitive property describes a chain reaction: if the first thing happens, then the second thing happens, and if the second thing happens, then the third thing happens. Therefore, if we want to ignore the second thing, we can say that if the first thing happens, then we know the third thing will happen. We don't have to mention the part in the middle; we can simply say that the first event leads to the final event. We could even have more than two premises; as long as they form a chain reaction, the transitive property will give us a valid argument.

Example 3

Premise:	If a soccer player commits a reckless foul, she will receive a yellow card.
Premise:	If Hayley receives a yellow card, she will be suspended for the next match.
Conclusion:	If Hayley commits a reckless foul, she will be suspended for the next match.

If we let $r =$ committing a reckless foul, $y =$ receiving a yellow card, and $s =$ being suspended, then our argument looks like this:

Premise:	$r \to y$
Premise:	$y \to s$
Conclusion:	$r \to s$

This argument has the exact structure required to use the transitive property, so it is a valid argument.

Try it Now 2

Is this argument valid?

Premise:	If the old lady swallows a fly, she will swallow a spider.
Premise:	If the old lady swallows a spider, she will swallow a bird.
Premise:	If the old lady swallows a bird, she will swallow a cat.
Premise:	If the old lady swallows a cat, she will swallow a dog.
Premise:	If the old lady swallows a dog, she will swallow a goat.
Premise:	If the old lady swallows a goat, she will swallow a cow.
Premise:	If the old lady swallows a cow, she will swallow a horse.
Premise:	If the old lady swallows a horse, she will die, of course.
Conclusion:	If the old lady swallows a fly, she will die, of course.

Answer

This argument is valid by the Transitive Property, which can involve more than two premises, as long as they continue the chain reaction. The premises $f \to s, s \to b, b \to c, c \to d \quad d \to g, g \to w, w \to h, h \to x$ can be reduced to $f \to x$. (Because we had already used c and d we decided to use w for cow and x for death. If the old lady swallows the fly, she will eventually eat a horse and die.

Disjunctive Syllogism

In a disjunctive syllogism, the premises consist of an *or* statement and the negation of one of the options. The conclusion is the other option. The general form is:

Premise:	$p \vee q$
Premise:	$\sim p$
Conclusion:	q

The order of the two parts of the disjunction isn't important. In other words, we could have the premises $p \vee q$ and $\sim q$, and the conclusion p

Example 4

Premise:	I can either drive or take the train.
Premise:	I refuse to drive.
Conclusion:	I will take the train.

If we let $d = I$ drive and $t = I$ take the train, then the symbolic representation of the argument is:

Premise:	$d \vee t$
Premise:	$\sim d$
Conclusion:	t

This argument is valid because it has the form of a disjunctive syllogism. I have two choices, and one of them is not going to happen, so the other one must happen.

Try it Now 3

Is this argument valid?

Premise:	Alison was required to write a 10-page paper or give a 5-minute speech.
Premise:	Alison did not give a 5-minute speech.
Conclusion:	Alison wrote a 10-page paper.

Answer

Let $p =$ wrote a paper and $s =$ gave a speech.

Premise: $p \vee s$
Premise: $-s$
Conclusion: p

This argument is valid by Disjunctive Syllogism. Alison had to do one or the other; she didn't choose the speech, so she must have chosen the paper.

Keep in mind that, when you are determining the validity of an argument, you must assume that the premises are true. If you don't agree with one of the premises, you need to keep your personal opinion out of it. Your job is to pretend that the premises are true and then determine whether they force you to accept the conclusion. You may attack the premises in a court of law or a political discussion, of course, but here we are focusing on the structure of the arguments, not the truth of what they actually say.

We have just looked at four forms of valid arguments; there are two common forms that represent *invalid* arguments, which are also called *fallacies*.

The Fallacy of the Converse

The fallacy of the converse arises when a conditional and its consequent are given as premises, and the antecedent is the conclusion. The general form is:

Premise: $p \to q$
Premise: q
Conclusion: p

Notice that the second premise and the conclusion look like the converse of the first premise, $q \to p$, but they have been detached. The fallacy of the converse incorrectly tries to assert that the converse of a statement is equivalent to that statement.

Example 5

Premise: If I drink coffee after noon, then I have a hard time falling asleep that night.
Premise: I had a hard time falling asleep last night.
Conclusion: I drank coffee after noon yesterday.

If we let $c = \text{I}$ drink coffee after noon and $h = \text{I}$ have a hard time falling asleep, then our argument looks like this:

Premise: $c \to h$
Premise: h
Conclusion: c

This argument uses converse reasoning, so it is an invalid argument. There could be plenty of other reasons why I couldn't fall asleep: I could be worried about money, my neighbors might have been setting off fireworks, …

Try it Now 4

Is this argument valid?

Premise: If you pull that fire alarm, you will get in big trouble.
Premise: You got in big trouble.
Conclusion: You must have pulled the fire alarm.

Answer

Let $f =$ pulled fire alarm and $t =$ got in big trouble.

Premise: $f \to t$
Premise: t
Conclusion: f

The Fallacy of the Inverse

The fallacy of the inverse occurs when a conditional and the negation of its antecedent are given as premises, and the negation of the consequent is the conclusion. The general form is:

Premise:	$p \to q$
Premise:	$\sim p$
Conclusion:	$\sim q$

Again, notice that the second premise and the conclusion look like the inverse of the first premise, $\sim p \to \sim q$, but they have been detached. The fallacy of the inverse incorrectly tries to assert that the inverse of a statement is equivalent to that statement.

Example 6

Premise:	If you listen to the Grateful Dead, then you are a hippie.
Premise:	Sky doesn't listen to the Grateful Dead.
Conclusion:	Sky is not a hippie.

If we let $g =$ listen to the Grateful Dead and $h =$ is a hippie, then this is the argument:

Premise:	$g \to h$
Premise:	$\sim g$
Conclusion:	$\sim h$

This argument is invalid because it uses inverse reasoning. The first premise does not imply that all hippies listen to the Grateful Dead; there could be some hippies who listen to Phish instead.

Try it Now 5

Is this argument valid?

Premise:	If a hockey player trips an opponent, he will be assessed a 2-minute penalty.
Premise:	Alexei did not trip an opponent.
Conclusion:	Alexei will not be assessed a 2-minute penalty.

Answer

Let $t =$ tripped and $p =$ got a penalty.

Premise:	$t \to p$
Premise:	$\sim t$
Conclusion:	$\sim p$

This argument is invalid because it has the form of the Fallacy of the Inverse. Alexei may have gotten a penalty for an infraction other than tripping.

Of course, arguments are not limited to these six basic forms; some arguments have more premises, or premises that need to be rearranged before you can see what is really happening. There are plenty of other forms of arguments that are invalid. If an argument doesn't seem to fit the pattern of any of these common forms, though, you may want to use a Venn diagram or a truth table instead.

Lewis Carroll, author of *Alice's Adventures in Wonderland*, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected using syllogisms. The following example is one such puzzle.

Example 7

Solve the puzzle. In other words, find a logical conclusion from these premises.

All babies are illogical.

Nobody is despised who can manage a crocodile.

Illogical persons are despised.

Let $b =$ is a baby, $d =$ is despised, $i =$ is illogical, and $m =$ can manage a crocodile.

Then we can write the premises as:

$b \rightarrow i$

$m \rightarrow \sim d$

$i \rightarrow d$

Writing the second premise correctly can be a challenge; it can be rephrased as "If you can manage a crocodile, then you are not despised."

Using the transitive property with the first and third premises, we can conclude that $b \rightarrow d$, that all babies are despised. Using the contrapositive of the second premise, $d \rightarrow \sim m$, we can then use the transitive property with $b \rightarrow d$ to conclude that $b \rightarrow \sim m$, that babies cannot manage crocodiles. While it is silly, this is a logical conclusion from the given premises.

Example 8

Premise:	If I work hard, I'll get a raise.
Premise:	If I get a raise, I'll buy a boat.
Conclusion:	If I don't buy a boat, I must not have worked hard.

If we let $h =$ working hard, $r =$ getting a raise, and $b =$ buying a boat, then we can represent our argument symbolically:

Premise:	$h \rightarrow r$
Premise:	$r \rightarrow b$
Conclusion:	$\sim b \rightarrow \sim h$

Using the transitive property with the two premises, we can conclude that $h \rightarrow b$, if I work hard, then I will buy a boat. When we learned about the contrapositive, we saw that the conditional statement $h \rightarrow b$ is equivalent to $\sim b \rightarrow \sim h$. Therefore, the conclusion is indeed a logical syllogism derived from the premises.

Try it Now 6

Is this argument valid?

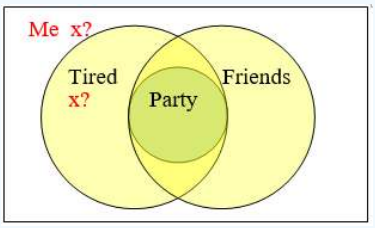
Premise:	If I go to the party, I'll be really tired tomorrow.
Premise:	If I go to the party, I'll get to see friends.
Conclusion:	If I don't see friends, I won't be tired tomorrow.

Answer

Let $p =$ go to party, $t =$ be tired, and $f =$ see friends.

Premise:	$p \rightarrow t$
Premise:	$p \rightarrow f$
Conclusion:	$-f \rightarrow \sim t$

We could try to rewrite the second premise using the contrapositive to state $\sim f \rightarrow \sim p$, but that does not allow us to form a syllogism. If I don't see friends, then I didn't go the party, but that is not sufficient to claim I won't be tired tomorrow. Maybe I stayed up all night watching movies.



A Venn diagram can help, if we set it up correctly. The "party" circle must be completely contained within the intersection of the other circles. We know that I am somewhere outside the "friends" circle, but we cannot determine whether I am in the "tired" circle. All we really know for sure is that I didn't go to the party.

This page titled 3.3.1: Forms of Valid Arguments is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.11: Forms of Valid Arguments** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.3.1.1: Exercises

For questions 1-9, determine if inductive or deductive reasoning was used to arrive at the conclusion of each scenario.

1. At Wonderbrain University, accountant majors must have at least a 3.0 GPA in their core courses to stay with the program. Cameron is an accountant major at Wonderbrain University and has a 3.2 GPA in all his core courses. Therefore, he is qualified to enroll for the next semester and continue his program.
2. Danny notices that his neighbor Amy trims her plants every Monday. Today is Monday. Danny concludes that Amy will trim her plants.
3. If a number is divisible by 6, then it is divisible by 2 and 3. The number 36 is divisible by 6. Therefore, 36 is divisible by 2 and 3.
4. All public beach resorts do not have entrance fees. DiveIn is a public beach resort. Therefore, DiveIn does not have an entrance fee.
5. Leslie was asked to make a conjecture about the sum of two perfect square numbers. She showed to class the following examples.

   $1 + 4 = 5$ $4 + 9 = 13$ $25 + 36 = 61$

   She then concludes that the sum of any two square numbers is always an odd number.
6. The left-handed people I know are geniuses. Therefore, all left-handed people are geniuses.
7. At Superstrict High School, if you are late by at least 15 minutes, you will no longer be allowed to attend your classes. On Monday, Sean overslept and was 20 minutes late. Therefore, he will not be allowed to attend his classes.
8. Medical practitioners say that if you are fully vaccinated against flu virus, then your chance of getting infected is lower than those who are not vaccinated. Shannon received her full dose of flu vaccine, one month ago. Therefore, she will have a lesser chance of getting infected with the virus.
9. All US passport holders are allowed to travel to more than 180 countries without a visa. Jeremy is a US passport holder. Therefore, he can be allowed entry to more than 180 countries without a visa.

3.3.1.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

3.3.2: Evaluating Deductive Arguments with Euler Diagrams

3.3.2 Learning Objectives

- Evaluate deductive arguments with Euler diagrams
- Use Euler diagrams to determine if an argument is valid

We can interpret a deductive argument visually with an Euler diagram, which is essentially the same thing as a Venn diagram. This can make it easier to determine whether the argument is valid or invalid.

Example 1

Consider the deductive argument "All cats are mammals and a tiger is a cat, so a tiger is a mammal." Is this argument valid?

Solution

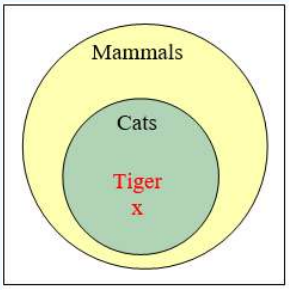
The premises are:

All cats are mammals.

A tiger is a cat.

The conclusion is:

A tiger is a mammal.



Both the premises are true. To see that the premises must logically lead to the conclusion, we can use an Euler diagram. From the first premise, we draw the set of cats as a subset of the set of mammals. From the second premise, we are told that a tiger is contained within the set of cats. From that, we can see in the Venn diagram that the tiger must also be inside the set of mammals, so the conclusion is valid.

Analyzing arguments with Euler diagrams

To analyze an argument with an Euler diagram:

1) Draw an Euler diagram based on the premises of the argument

2) The argument is valid if the diagram cannot be drawn to make the conclusion false

3) The argument is invalid if there is a way to draw the diagram that makes the conclusion false

4) The argument is invalid if the premises are insufficient to determine the location of an element or a set mentioned in the conclusion

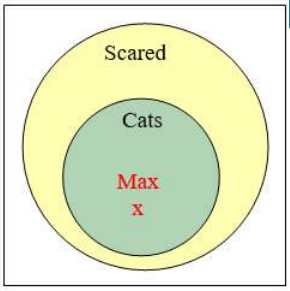
Try it Now 1

Determine the validity of this argument:

Premise:	All cats are scared of vacuum cleaners.
Premise:	Max is a cat.
Conclusion:	Max is scared of vacuum cleaners.

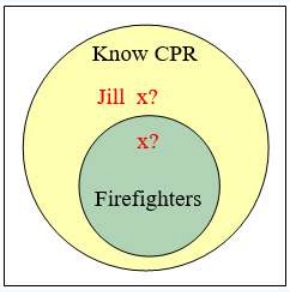
Answer

Valid. Cats are a subset of creatures that are scared by vacuum cleaners. Max is in the set of cats, so he must also be in the set of creatures that are scared by vacuum cleaners.



Example 2

Premise: All firefighters know CPR.
Premise: Jill knows CPR.
Conclusion: Jill is a firefighter.



Solution

From the first premise, we know that firefighters all lie inside the set of those who know CPR. (Firefighters are a subset of people who know CPR.) From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know whether she also is a member of the smaller subset that is firefighters.

Since the conclusion does not necessarily follow from the premises, this is an invalid argument. It's possible that Jill is a firefighter, but the structure of the argument doesn't allow us to conclude that she definitely is.

It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are concerned with whether the premises are enough to prove the conclusion.

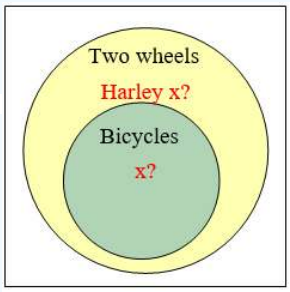
Try it Now 2

Determine the validity of this argument:

Premise: All bicycles have two wheels.
Premise: This Harley-Davidson has two wheels.
Conclusion: This Harley-Davidson is a bicycle.

Answer

Invalid. The set of bicycles is a subset of the set of vehicles with two wheels; the Harley-Davidson is in the set of two-wheeled vehicles but not necessarily in the smaller circle.



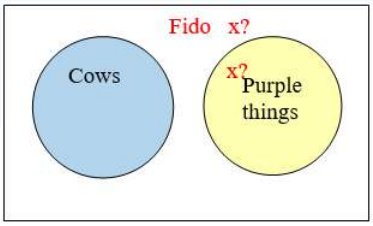
Try it Now 3

Determine the validity of this argument:

Premise: No cows are purple.
Premise: Fido is not a cow.
Conclusion: Fido is purple.

Answer

Invalid. Since no cows are purple, we know there is no overlap between the set of cows and the set of purple things. We know Fido is not in the cow set, but that is not enough to conclude that Fido is in the purple things set.



In addition to these categorical style premises of the form "all ___", "some ____", and "no ____", it is also common to see premises that are conditionals.

Example 3

Premise: If you live in Seattle, you live in Washington.
Premise: Marcus does not live in Seattle.
Conclusion: Marcus does not live in Washington.

Solution

From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether Marcus lives in Washington or not. This is an invalid argument.

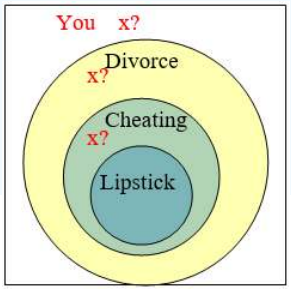
Try it Now 4

Determine the validity of this argument:

Premise:	If you have lipstick on your collar, then you are cheating on me.
Premise:	If you are cheating on me, then I will divorce you.
Premise:	You do not have lipstick on your collar.
Conclusion:	I will not divorce you.

Answer

Invalid. Lipstick on your collar is a subset of scenarios in which you are cheating, and cheating is a subset of the scenarios in which I will divorce you. Although it is wonderful that you don't have lipstick on your collar, you could still be cheating on me, and I will divorce you. In fact, even if you aren't cheating on me, I might divorce you for another reason. You'd better shape up.



This page titled 3.3.2: Evaluating Deductive Arguments with Euler Diagrams is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.9: Evaluating Deductive Arguments with Euler Diagrams** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.3.2.1: Exercises

For questions 1-5, use an Euler diagram to decide whether each argument is valid or invalid.

1. If a person is on this reality show, they must be self-absorbed. Laura is not self-absorbed. Therefore, Laura cannot be on this reality show.
2. If you are a triathlete, then you have outstanding endurance. LeBron James is not a triathlete. Therefore, LeBron does not have outstanding endurance.
3. Some of these kids are rude. Jimmy is one of these kids. Therefore, Jimmy is rude!
4. Every student brought something to write with. Marcie is a student. Therefore, Marcie brought something to write with.
5. If a creature is a chimpanzee, then it is a primate. If a creature is a primate, then it is a mammal. Bobo is a mammal. Therefore, Bobo is a chimpanzee.

This page titled 3.3.2.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.13: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.4: Logical Fallacies in Common Language

3.4 Learning Objectives

- Recognize some forms of logical fallacies in common language

In the previous discussion, we saw that logical arguments can be invalid when the premises are not true, when the premises are not sufficient to guarantee the conclusion, or when there are invalid chains in logic. There are a number of other ways in which arguments can be invalid, a sampling of which are given here.

Definition: Ad Hominem

An **ad hominem** argument attacks the person making the argument, ignoring the argument itself.

This can also be called Personal Attack.

Example 1

"Jane says that whales aren't fish, but she's only in the second grade, so she can't be right."

Here the argument is attacking Jane, not the validity of her claim, so this is an ad hominem argument.

Example 2

"Jane says that whales aren't fish, but everyone knows that they're really mammals. She's so stupid."

This certainly isn't very nice, but it is *not* ad hominem since a valid counterargument is made along with the personal insult.

Definition: Appeal to Ignorance

This type of argument assumes something is true because it hasn't been proven false.

Example 3

"Nobody has proven that photo isn't of Bigfoot, so it must be Bigfoot."

Definition: Appeal to Authority

These arguments attempt to use the authority of a person to prove a claim. While often authority can provide strength to an argument, problems can occur when the person's opinion is not shared by other experts, or when the authority is irrelevant to the claim.

Example 4

"A diet high in bacon can be healthy; Doctor Atkins said so."

Here, an appeal to the authority of a doctor is used for the argument. This generally would provide strength to the argument, except that the opinion that eating a diet high in saturated fat runs counter to general medical opinion. More supporting evidence would be needed to justify this claim.

Example 5

"Jennifer Hudson lost weight with Weight Watchers, so their program must work."

Here, there is an appeal to the authority of a celebrity. While her experience does provide evidence, it provides no more than any other person's experience would.

Appeal to consequence

An appeal to consequence concludes that a premise is true or false based on whether the consequences are desirable or not.

Example 6

“Humans will travel faster than light: faster-than-light travel would be beneficial for space travel.”

Definition: Appeal to Popularity

This type of argument assumes something it true because many people believe it is true.

Example 7

“This face wash is the best for acne. I've seen several TikTok videos that say so.”

Definition: False Dilemma

A **false dilemma** argument falsely frames an argument as an “either or” choice, without allowing for additional options.

This can also be called Limited Choice.

Example 8

“Either those lights in the sky were an airplane or aliens. There are no airplanes scheduled for tonight, so it must be aliens.”

This argument ignores the possibility that the lights could be something other than an airplane or aliens.

Definition: Circular Reasoning

Circular reasoning is an argument that relies on the conclusion being true for the premise to be true.

Example 9

“I shouldn’t have gotten a C in that class; I’m an A student!”

In this argument, the student is claiming that because they’re an A student, though shouldn’t have gotten a C. But because they got a C, they’re not an A student.

Definition: Post hoc (post hoc ergo propter hoc)

A **post hoc argument** claims that because two things happened sequentially, then the first must have caused the second.

This can also be called False Cause.

Example 10

“Today I wore a red shirt, and my football team won! I need to wear a red shirt every time they play to make sure they keep winning.”

Definition: Straw Man

A **straw man argument** involves misrepresenting the argument in a less favorable way to make it easier to attack.

Example 11

“Senator Jones has proposed reducing military funding by 10%. Apparently he wants to leave us defenseless against attacks by terrorists”

Here the arguer has represented a 10% funding cut as equivalent to leaving us defenseless, making it easier to attack Senator Jones’ position.

Correlation implies causation

Similar to post hoc, but without the requirement of sequence, this fallacy assumes that just because two things are related one must have caused the other. Often there is a third variable not considered.

This can also be called Hasty Generalization.

Example 12

"Months with high ice cream sales also have a high rate of deaths by drowning. Therefore, ice cream must be causing people to drown."

This argument is implying a causal relation, when really both are more likely dependent on the weather; that ice cream and drowning are both more likely during warm summer months.

Definition: Red Herring

This type of argument assumes something is true because it has an argument about another related topic.

Example 13

" We should not fund cloning research because there are ethical issues involved. We don't want to be a society without ethical foundations."

Try it Now 1

Identify the logical fallacy in each of the arguments

a. Only an untrustworthy person would run for office. The fact that politicians are untrustworthy is proof of this.
b. Since the 1950s, both the atmospheric carbon dioxide level and obesity levels have increased sharply. Hence, atmospheric carbon dioxide causes obesity.
c. The oven was working fine until you started using it, so you must have broken it.
d. You can't give me a D in the class because I can't afford to retake it.
e. The senator wants to increase support for food stamps. He wants to take the taxpayers' hard-earned money and give it away to lazy people. This isn't fair, so we shouldn't do it.

Answer

a. Circular
b. Correlation does not imply causation
c. Post hoc
d. Appeal to consequence
e. Straw man

It may be difficult to identify one particular fallacy for an argument. Consider this argument: "Emma Watson says she's a feminist, but she posed for these racy pictures. I'm a feminist, and no self-respecting feminist would do that." Could this be ad hominem, saying that Emma Watson has no self-respect? Could it be appealing to authority because the person making the argument claims to be a feminist? Could it be a false dilemma because the argument assumes that a woman is either a feminist or not, with no gray area in between?

This page titled 3.4: Logical Fallacies in Common Language is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.12: Logical Fallacies in Common Language** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

3.4.1: Exercises

For questions 1-18, name the type of logical fallacy being used.

1. Dr. Oz was a featured health expert on the “Oprah Winfrey Show” and says that a specific brand of weight loss product is effective at burning fats and calories. Therefore, it must be true.

2. I thought you cared about homeless people, but you did not attend the “concert for a cause” to support homeless people in Canada.

3. Every country is fighting the COVID-19 virus because it is a global pandemic.

4. It rained so hard when the power went off. The rain has caused the power to go off.

5. Family cohesion is correlated to volume of crimes. Therefore, family cohesion causes crime.

6. If you cannot justify that mermaids don’t exist, then they must exist.

7. Get enough sleep daily and be healthy or continue to go to bed late each night and be sick. These are the only two options.

8. You must follow what your parents told you because parents are always right.

9. Andy was accused of a crime. There was no solid evidence to prove that Andy committed the crime. Therefore, the accuser was wrong.

10. The pandemic will be over this year; a lot of people will lose their jobs and loved ones if it won’t.

11. You obey my house rules, or you cannot go out with your friends.

12. Mickey Simpson and Bugsy Humperdinck are running for President. Mickey argues that women won’t possibly vote for Bugsy as he has a religious conviction that tends him to become pro-life.

13. A commercial claim that a certain brand of toothpaste is the world’s leading brand as more than 80% of dentists recommended its use.

14. I do not accept the claim that human beings came from apes. Otherwise, my biological predecessors would be monkeys. Therefore, Charles Darwin’s theory of evolution must be wrong.

15. A pregnant woman tells the unmarried obstetrician-gynecologist that she doesn’t believe in her judgment because she has never been pregnant.

16. If you don’t want to drive from Boston to New York, then you will have to take the train.

17. New England Patriots quarterback Tom Brady likes his footballs slightly underinflated. The “Cheatriots” have a history of bending or breaking the rules, so Brady must have told the equipment manager to make sure that the footballs were underinflated.

18. Whenever our smoke detector beeps, my kids eat cereal for dinner. Therefore, the loud beeping sound must make them want to eat cereal.

This page titled 3.4.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **17.13: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

CHAPTER OVERVIEW

4: Probability

4.1: Introduction to Probability

4.1.1: Exercises

4.2: Basic Concepts

4.2.1: Exercises

4.3: Working with Events

4.3.1: Exercises

4.4: Two-Way Tables

4.4.1: Exercises

4.5: Counting

4.5.1: Exercises

4.6: Expected Value

4.6.1: Exercises

4.7: Additional Exercises

This page titled 4: Probability is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

4.1: Introduction to Probability

4.1 Learning Objectives

- Describe theoretical, empirical, and subjective probability
- Distinguish among the three uses of probability

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability.

One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says "there is a 10% chance of rain tomorrow," she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Example 1

Conduct an experiment to determine the probability of the spinner landing on 1?



Solution

Using the experimental method, suppose out of 10,000 spins, 1,711 of those landed on 1. This is normally done on a computer application that can randomly generate outcomes of each spin to avoid someone having to spin the spinner 10,000 times.

To calculate this probability, divide the number of times 1 has occurred which is 1,711 by the number of times the spinner was spun which was 10,000.

The result is 0.1711 or approximately 17% of the time. So 17 times out of 100 spins one would expect the spinner to land on the number 1.

Another view would be **subjective** in nature, in other words an educated guess or opinion. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

Example 2

Determine the probability that the Seattle Mariners would win their next baseball game.

Solution

It would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions.

Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance the Mariners will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games.

Definition: Probabilities

Empirical Probability uses the results of an experiment to predict the percent chance an event could occur.

Subjective Probability uses intuition or guesswork to predict the percent chance an event could occur.

Theoretical Probability uses the number of possible desired outcomes of an event compared to the number of all possible outcomes of an event to predict the percent chance an event could occur.

Theoretical Probability is defined mathematically as follows:

Suppose there is a situation with n equally likely possible outcomes and that m of those n outcomes correspond to a particular event; then the **probability** of that event is defined as $\frac{m}{n}$.

We will return to the empirical and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability.

This page titled 4.1: Introduction to Probability is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.1: Introduction to Probability** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

4.1.1: Exercises

1. In a sample of 50 people who were stating their favorite fruit, 21 liked apples the most, 18 people like peaches the most, 5 people liked bananas the most and 6 liked pineapple the most. The probability that all people like bananas the most is 10%. What type of probability is this?
2. A person wants to determine the probability of winning a raffle. One winner is selected out of 84 entrants. What type of probability is used to answer this question?
3. A mother thinks there is a 100% chance that her son will love the jelly donut because he loves jelly, and he loves donuts. What kind of probability was used here?

This page titled 4.1.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

4.2: Basic Concepts

4.2 Learning Objectives

- Find the probability of an event
- Find the probabilities certain and impossible events

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, you are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Definition: Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further

The **sample space** is the set of all possible simple events.

Example 1

If we roll a standard 6-sided die, describe the sample space and some simple events.

The sample space is the set of all possible simple events: $\{1, 2, 3, 4, 5, 6\}$

Some examples of simple events:

We roll a 1

We roll a 5

Some compound events:

We roll a number bigger than 4

We roll an even number



Two dice One die

Definition: Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event E using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event E}}{\text{Total number of equally likely outcomes}} \tag{4.2.1}$$

Example 2

If we roll a 6-sided die, calculate

a. P(rolling a 1)
b. P(rolling a number bigger than 4)

Solution

Recall that the sample space is $\{1, 2, 3, 4, 5, 6\}$

a. There is one outcome corresponding to "rolling a 1", so the probability is $\frac{1}{6}$
b. There are two outcomes bigger than a 4, so the probability is $\frac{2}{6} = \frac{1}{3}$

Probabilities are essentially fractions, and can be simplified to lower terms like fractions. Probabilities are also written as decimals or percents.

Example 3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

Solution

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20} = \frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Try it Now 1

At some random moment, you look at your clock and note the minutes showing.

a. What is probability the minutes reading is 15?
b. What is the probability the minutes reading is 15 or less?

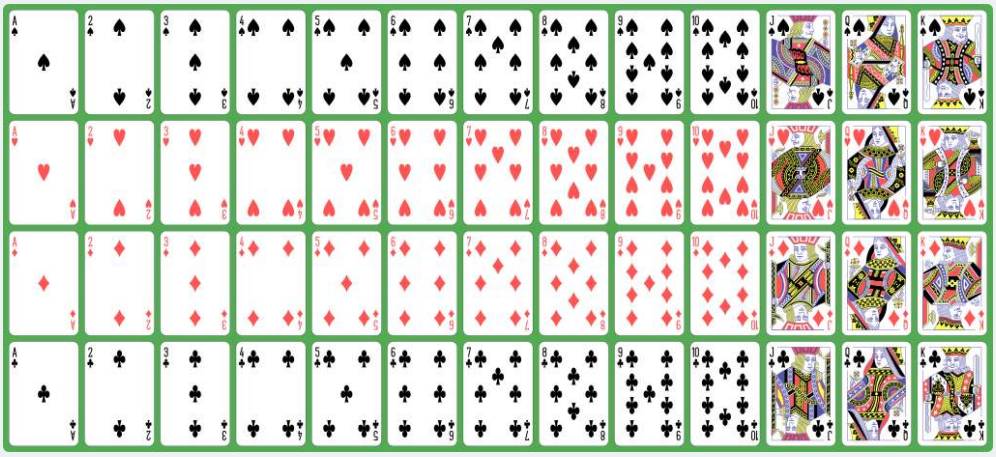
Answer

There are 60 possible readings, from 00 to 59.

a. $\frac{1}{60}$
b. $\frac{16}{60}$ (counting 00 through 15)

Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.



Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

Solution

There are 52 cards in the deck and 4 Aces so

$$P(Ace) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – if there are no outcomes that correspond with the event. The largest possible probability is 1 – if all possible outcomes correspond with the event.

Definition: Certain and Impossible Events

An **impossible event** has a probability of 0.

A **certain event** has a probability of 1.

The probability of any event must be $0 \leq P(E) \leq 1$.

In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

This page titled 4.2: Basic Concepts is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.2: Basic Concepts** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

4.2.1: Exercises

1. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of the given event.
   a. A red ball is drawn
   b. A white ball is drawn
2. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. If you draw one slip of paper from the hat at random, what is the probability of getting:
   a. A consonant?
   b. A vowel?
3. A group of people were asked if they had run a red light in the last year. 150 responded "yes", and 185 responded "no". Find the probability that if a person is chosen at random, they have run a red light in the last year.
4. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dogs, and 40 indicated they don't enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.
5. Compute the probability of tossing a six-sided die (with sides numbered 1 through 6) and getting a 5.
6. Compute the probability of tossing a six-sided die and getting a 7.
7. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female.

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

8. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person had no credit cards.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

9. Compute the probability of tossing a six-sided die and getting an even number.
10. Compute the probability of tossing a six-sided die and getting a number less than 3.
11. If you pick one card at random from a standard deck of cards, what is the probability it will be a King?
12. If you pick one card at random from a standard deck of cards, what is the probability it will be a Diamond

This page titled 4.2.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.7: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

4.3: Working with Events

4.3 Learning Objectives

- Find the probability that an event does not happen
- Determine whether or not two events are independent
- Find P(A and B) for independent events
- Determine whether or not two events are mutually exclusive
- Find P(A or B)
- Find P(A|B)
- Determine probabilities from two-ways tables
- Determine odds for and against an event

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six}) = \frac{1}{6}$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = \frac{5}{6}$. Notice that

$$P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1 \tag{4.3.1}$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event E that corresponds to m of these outcomes. Then the remaining $n - m$ outcomes correspond to E not happening, thus

$$P(\text{not } E) = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E) \tag{4.3.2}$$

Complement of an Event

The **complement** of an event is the event "E doesn't happen"

The notation $\bar{E}$ is used for the complement of event E.

We can compute the probability of the complement using $P(\bar{E}) = 1 - P(E)$

Notice also that $P(E) = 1 - P(\bar{E})$

Example 1

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

Solution

There are 13 hearts in the deck, so

$$P(\text{heart}) = \frac{13}{52} = \frac{1}{4}.$$

The probability of *not* drawing a heart is the complement:

$$P(\text{ not heart }) = 1 - P(\text{ heart }) = 1 - \frac{1}{4} = \frac{3}{4}$$

Probability of Two Independent Events

Example 2

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

Solution

We could list all possible outcomes: $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

Notice there are $2 \cdot 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example was looking at two independent events.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 3

Are these events independent?

a. A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.
b. The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).
c. You draw a red card from a deck, then draw a second red card without replacing the first.

Solution

a. The probability that a head comes up on the second toss is $\frac{1}{2}$ regardless of whether or not a head came up on the first toss, so these events are independent.
b. These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.
c. The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

$P(A \text{ and } B)$ for Independent Events

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B) \tag{4.3.3}$$

where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 4

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

Solution

The probability of choosing a white pair of socks is $\frac{6}{10}$.

The probability of choosing a white tee shirt is $\frac{3}{7}$.

The probability of both being white is $\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}$

Try it Now 1

A card is pulled a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

Answer

since the second draw is made after replacing the first card, these events are independent The probability of an ace on each draw is $\frac{4}{52}=\frac{1}{13}$, so the probability of an Ace on both draws is $\frac{1}{13}\cdot\frac{1}{13}=\frac{1}{169}$

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Example 5

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Solution

Here, there are still 12 possible outcomes:

$\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are $H1, H2, H3, H4, H5, H6, T6$), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2}+\frac{1}{6}=\frac{6}{12}+\frac{2}{12}=\frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12}-\frac{1}{12}=\frac{7}{12}$.

$P(A \text{ or } B)$

Inclusive OR: Either event could occur or both could occur at the same time.

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B)=P(A)+P(B)-P(A \text{ and } B) \tag{4.3.4}$$

Mutually Exclusive OR: Either event could occur but they could not occur at the same time.

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B)=P(A)+P(B) \tag{4.3.5}$$

Example 6

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

Solution

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes.

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen})=0$ so this is a mutually exclusive event. Using our probability rule, the probability of drawing a Queen or a King is:

$$P(\text{ King or Queen})=P(\text{ King })+P(\text{ Queen})-P(\text{ King and Queen})=\frac{4}{52}+\frac{4}{52}-0=\frac{8}{52}$$

Example 7

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King? Simplify your answers.

Solution

- Half the cards are red, so $P(\text{red}) = \frac{26}{52}$.
- There are four kings, so $P(\text{ King }) = \frac{4}{52}$.
- There are two red kings, so $P(\text{ Red and King}) = \frac{2}{52}$.

We can then calculate:

$$P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$$

Try it Now 2

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

Answer

$$P(\text{ white socks and white tee }) = \left(\frac{6}{10}\right)\left(\frac{3}{7}\right) = \frac{9}{35}$$

$$P(\text{ white socks or white tee }) = \frac{6}{10} + \frac{3}{7} - \frac{9}{35} = \frac{27}{35}$$

Example 8

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a. Has a red car *and* got a speeding ticket
b. Has a red car *or* got a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Solution

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $\frac{15}{665} \approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665} \approx 0.2932$.

We also could have found this probability by:

$$\text{P(had a red car)} + \text{P(got a speeding ticket)} - \text{P(had a red car and got a speeding ticket)} = \frac{150}{665} + \frac{60}{665} - \frac{15}{665}$$

$$= \frac{195}{665}$$

Conditional Probability

Often it is required to compute the probability of an event given that another event has occurred.

Example 9

What is the probability that two cards drawn at random from a deck of playing cards without replacement will both be aces?

Solution

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case the "condition" is that the first card is an ace. Symbolically, we write this as:

P(ace on second draw | an ace on the first draw).

The vertical bar "|" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw." What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $\frac{3}{51} = \frac{1}{17}$.

Thus, the probability of both cards being aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$.

Conditional Probability

The probability the event B occurs, given that event A has happened, is represented as

$$P(B|A) = \frac{\text{number of outcomes common to A and B}}{\text{number of outcomes in A}} \tag{4.3.6}$$

This is read as “the probability of B given A”

Example 10

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

Solution

These are two independent events, so the probability of the die rolling a 6 is $\frac{1}{6}$, regardless of the result of the coin flip.

Example 11

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a) Has a speeding ticket *given* they have a red car

b) Has a red car *given* they have a speeding ticket

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Solution

a) Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

P(ticket | red car) = $\frac{15}{150} = \frac{1}{10} = 0.1$

b) Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

P(red car | ticket) = $\frac{15}{60} = \frac{1}{4} = 0.25$.

Notice from the last example that $P(B|A)$ is **not** equal to $P(A|B)$.

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

Conditional Probability Formula

If events A and B are not independent, then

$$P(A \text{ and } B) = P(A)P(B|A) \tag{4.3.7}$$

Example 12

If you pull 2 cards out of a deck, what is the probability that both are spades?

Solution

The probability that the first card is a spade is $\frac{13}{52}$.

The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck, and one less total cards.

The probability that both cards are spades is $\left(\frac{13}{52}\right)\left(\frac{12}{51}\right) = \frac{156}{2652} \approx 0.0588$

Example 13

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

Solution

You can satisfy this condition by having Case A or Case B, as follows:

Case A) you can get the Ace of Diamonds first and then a black card or

Case B) you can get a black card first and then the Ace of Diamonds.

Let's calculate the probability of **Case A**. The probability that the first card is the Ace of Diamonds is $\frac{1}{52}$. The probability that the second card is black given that the first card is the Ace of Diamonds is $\frac{26}{51}$ because 26 of the remaining 51 cards are black. The probability is therefore $\left(\frac{1}{52}\right)\left(\frac{26}{51}\right) = \frac{1}{102}$.

Now for **Case B:** the probability that the first card is black is $\frac{26}{52} = \frac{1}{2}$. The probability that the second card is the Ace of Diamonds given that the first card is black is $\frac{1}{51}$. The probability of Case B is therefore $\left(\frac{1}{2}\right)\left(\frac{1}{51}\right) = \frac{1}{102}$, the same as the probability of Case 1.

Recall that the probability of A or B is $P(\text{A}) + P(\text{B}) - P(\text{A and B})$. In this problem, $P(\text{A and B}) = 0$ since the first card cannot be the Ace of Diamonds and be a black card. Therefore, the probability of Case A or Case B is

$\frac{1}{102}+\frac{1}{102}=\frac{2}{102}=\frac{1}{51}$. The probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is $\frac{1}{51}$.

Try it Now 3

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

Answer

$\left(\frac{6}{10}\right)\left(\frac{5}{9}\right)=\frac{30}{90}=\frac{1}{3}$

Example 14

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

a) P(not pregnant | positive test result)

b) P(positive test result | not pregnant)

	Positive test	Negative test	Total
Pregnant	70	4	74
Not Pregnant	5	14	19
Total	75	18	93

Solution

a) Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant.

P(not pregnant | positive test result) = $\frac{5}{75}\approx 0.067$.

b) Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test.

P(positive test result | not pregnant) = $\frac{5}{19}\approx 0.263$.

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

Odds

If we know the probability, we can determine the odds in favor of or against an event happening.

Probability to Odds

For any event, E

To convert from probability to odds *for* an event,

$$\text{Odds for E}=\frac{P(E)}{P(\bar{E})} \tag{4.3.8}$$

To convert from probability to odd *against* an event,

$$\text{Odds against E}=\frac{P(\bar{E})}{P(E)} \tag{4.3.9}$$

Odd can be written as

a to b, a:b, or $\frac{a}{b}$

Example 15

If you roll two die, what is the probability you will roll a sum of 10?

Solution

First determine the probability of the event happening. Since there are 36 possible rolls and only 3 ways to roll a sum of 10, the probability for the event happening is $\frac{3}{36} = \frac{1}{12}$

Next determine the probability of the event *not* happening. Since the probability of the event happening is $\frac{1}{12}$. The probability of an event *not* happening is $1 - \frac{1}{12} = \frac{12}{12} - \frac{1}{12} = \frac{11}{12}$

Now, use the given formula to determine the odds.

Since the probability is *for* rolling a sum a 10, we will use 4.3.8

$$\text{Odds for sum of } 10 = \frac{\frac{1}{12}}{\frac{11}{12}} = \frac{1}{11}$$

So the odds you will roll a sum of 10 are 1 to 11. This can be read as, the odds are that out of 12 rolls 1 will be a sum of 10 and the other 11 will not. We can also say 1 out of 12 rolls will be a sum a 10.

Odds to Probability

If the odds in favor of an event E are x to y, then the probability of the event occurring is given by

$$P(E) = \frac{x}{x+y} \tag{4.3.10}$$

Example 16

The horse named Bolt has 10:2 odds against winning at the race track. What is the probability Bolt will win the race?

Solution

Referring to the formula 4.3.10 determine what the value for x and y is. The odds 10:2 can be rewrittens as 10 to 2 against Bolt winning. This means the odds in favor of Bolt winning would be 2 to 10. Therefore, $x = 2$ and $y = 10$. The probability Bolt wins would be

$$P(\text{Bolt wins}) = \frac{2}{2+10} = \frac{2}{12} = \frac{1}{6} = 0.1667 \times 100 = 16.67$$

This page titled 4.3: Working with Events is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.3: Working with Events** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

4.3.1: Exercises

1. Compute the probability of rolling a 12-sided die and getting a number other than 8.
2. If you pick one card at random from a standard deck of cards, what is the probability it is not the Ace of Spades?
3. Referring to the grade table below, what is the probability that a student chosen at random did NOT earn a C?

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

4. Referring to the credit card table below, what is the probability that a person chosen at random has at least one credit card?

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

5. A six-sided die is rolled twice. What is the probability of showing a 6 on both rolls?
6. A fair coin is flipped twice. What is the probability of showing heads on both flips?
7. A six-sided die is rolled twice. What is the probability of showing a 5 on the first roll and an even number on the second roll?
8. Suppose that 21% of people own dogs. If you pick two people at random, what is the probability that they both own a dog?
9. Suppose a jar contains 17 red marbles and 32 blue marbles. If you reach in the jar and pull out 2 marbles at random without replacement, find the probability that both are red.
10. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. If you pull out two slips at random, find the probability that both are vowels.
11. Bert and Ernie each have a well-shuffled standard deck of 52 cards. They each draw one card from their own deck. Compute the probability that:
    a. Bert and Ernie both draw an Ace.
    b. Bert draws an Ace but Ernie does not.
    c. neither Bert nor Ernie draws an Ace.
    d. Bert and Ernie both draw a heart.
    e. Bert gets a card that is not a Jack and Ernie draws a card that is not a heart.
12. Bert has a well-shuffled standard deck of 52 cards, from which he draws one card; Ernie has a 12-sided die, which he rolls at the same time Bert draws a card. Compute the probability that:
    a. Bert gets a Jack and Ernie rolls a five.
    b. Bert gets a heart and Ernie rolls a number less than six.
    c. Bert gets a face card (Jack, Queen or King) and Ernie rolls an even number.
    d. Bert gets a red card and Ernie rolls a fifteen.
    e. Bert gets a card that is not a Jack and Ernie rolls a number that is not twelve.
13. Compute the probability of drawing a King from a deck of cards and then drawing a Queen without replacement.
14. Compute the probability of drawing two spades from a deck of cards without replacement.
15. A math class consists of 25 students, 14 freshmen and 11 sophomores. Two students are selected at random to participate in a probability experiment. Compute the probability that
    a. a sophomore is selected, then a freshman.
    b. a freshman is selected, then a sophomore.
    c. two sophomores are selected.
    d. two freshmen are selected.
    e. no sophomores are selected.
16. A math class consists of 25 students, 14 freshmen and 11 sophomores. Three students are selected at random to participate in a probability experiment. Compute the probability that
    a. a sophomore is selected, then two freshmen.

b. a freshmen is selected, then two sophomores.
c. two freshmen are selected, then one sophomore.
d. three sophomores are selected.
e. three freshmen are selected.

17. Giving a test to a group of students, the grades and class are summarized below. If one student was chosen at random, find the probability that the student was a freshman and earned an A.

	A	B	C	Total
Sophomore	8	18	13	39
Freshman	10	4	12	26
Total	18	22	25	65

18. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person was male and had two or more credit cards.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

19. A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd-numbered.
20. A jar contains 4 red marbles numbered 1 to 4 and 10 blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability the marble is blue or even-numbered.
21. Referring to the grade table below, find the probability that a student chosen at random is a freshman or earned a B.

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

22. Referring to the table below, find the probability that a person chosen at random is male or has no credit cards.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

23. Compute the probability of drawing the King of hearts or a Queen from a deck of cards.
24. Compute the probability of drawing a King or a heart from a deck of cards.
25. A jar contains 5 red marbles numbered 1 to 5 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. Even-numbered given that the marble is red.
    b. Red given that the marble is even-numbered.
26. A jar contains 4 red marbles numbered 1 to 4 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. Odd-numbered given that the marble is blue.
    b. Blue given that the marble is odd-numbered.
27. Compute the probability of flipping a coin and getting heads, given that the previous flip was tails.
28. Find the probability of rolling a “1” on a fair die, given that the last 3 rolls were all ones.
29. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student speaks French, given that the student is female.
30. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student is male, given that the student speaks French.

31. A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 90% of the time if the person has the virus and 10% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".
    a. Find the probability that a person has the virus given that they have tested positive, i.e. find $P(A|B)$.
    b. Find the probability that a person does not have the virus given that they test negative, i.e. find $P(\text{ not } A|\text{ not } B)$.
32. A certain virus infects one in every 2000 people. A test used to detect the virus in a person is positive 96% of the time if the person has the virus and 4% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".
    a. Find the probability that a person has the virus given that they have tested positive, i.e. find $P(A|B)$.
    b. Find the probability that a person does not have the virus given that they test negative, i.e. find $P(\text{ not } A|\text{ not } B)$.

This page titled 4.3.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

4.4: Two-Way Tables

4.4 Learning Objectives

- Use two-way tables to find conditional probabilities

In this section we concentrate on the more complex conditional probability problems we began to look at in the last section.

Example 1

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

Solution

Let's break down the information in the problem piece by piece, using a two-way table to record our results.

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). The percentage 0.1% can be converted to a decimal number by moving the decimal place two places to the left, to get 0.001. In turn, 0.001 can be rewritten as a fraction: 1/1000. This tells us that about 1 in every 1000 people has the disease. (If we wanted we could write *P* (disease)=0.001.)

A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it). This part is fairly straightforward: everyone who has the disease will test positive, or alternatively everyone who tests negative does not have the disease. (We could also say *P* (positive | disease)=1.)

The false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). This is even more straightforward. Another way of looking at it is that of every 100 people who are tested and do not have the disease, 5 will test positive even though they do not have the disease. (We could also say that P(positive | no disease)=0.05.)

Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease? Here we want to compute P(disease|positive). We already know that P(positive|disease)=1, but remember that conditional probabilities are not equal if the conditions are switched.

Rather than thinking in terms of all these probabilities we have developed, let's create a hypothetical situation and apply the facts as set out above. First, suppose we randomly select 1000 people and administer the test. How many do we expect to have the disease? Since about 1/1000 of all people are afflicted with the disease, $\frac{1}{1000}$ of 1000 people is 1. (Now you know why we chose 1000.) Only 1 of 1000 test subjects actually has the disease; the other 999 do not.

	Has Disease	Does Not Have Disease	Total
Tests Positive			
Tests Negative			
Total	1	999	1000

We also know that 5% of all people who do not have the disease will test positive. There are 999 disease-free people, so we would expect $(0.05)(999) = 49.95$(so, about 50) people to test positive who do not have the disease.

	Has Disease	Does Not Have Disease	Total
Tests Positive	1	50	
Tests Negative	0	949	
Total	1	999	1000

Now back to the original question, computing P (disease|positive). There are 51 people who test positive in our example (the one unfortunate person who actually has the disease, plus the 50 people who tested positive but don't). Only one of these people has the disease, so

$$\text{P(disease | positive)} \approx \frac{1}{51} \approx 0.0196$$

or less than 2%. This means that of all people who test positive, over 98% *do not have the disease*. Does this surprise you?

The answer we got was slightly approximate, since we rounded 49.95 to 50.

But back to the surprising result. *Of all people who test positive, over 98% do not have the disease*. If your guess for the probability a person who tests positive has the disease was wildly different from the right answer (2%), don't feel bad. The exact same problem was posed to doctors and medical students at the Harvard Medical School 25 years ago and the results revealed in a 1978 *New England Journal of Medicine* article. Only about 18% of the participants got the right answer. Most of the rest thought the answer was closer to 95% (perhaps they were misled by the false positive rate of 5%).

The significance of this finding and similar results from other studies in the intervening years lies in the possibly catastrophic consequences it might have for patient care. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. Or worse, as in the early days of the AIDS crisis when being HIV-positive was often equated with a death sentence, the patient might take a drastic action and commit suicide.

As we have seen in this hypothetical example, the most responsible course of action for treating a patient who tests positive would be to counsel the patient that they most likely do *not* have the disease and to order further, more reliable, tests to verify the diagnosis.

Example 2

A certain disease has an incidence rate of 2%. If the false negative rate is 10% and the false positive rate is 1%, compute the probability that a person who tests positive actually has the disease.

Solution

Imagine 10,000 people who are tested. Of these 10,000, 200 will have the disease since the disease as an incident rate of 2%.

	Has Disease	Does Not Have Disease	Total
Tests Positive			
Tests Negative			
Total	200	9800	10000

Of those who have the disease, 10% of them, or 20, will test negative and the remaining 180 will test positive. Of the 9800 who do not have the disease, 1% of them, or 98, will test positive. So of the 278 total people who test positive, 180 will have the disease.

	Has Disease	Does Not Have Disease	Total
Tests Positive	180	98	278
Tests Negative	20	9702	
Total	200	9800	10000

Thus, $P(\text{ disease} \mid \text{positive }) = \frac{180}{278} \approx 0.647$.

So, about 65% of the people who test positive will have the disease.

Try it Now 1

A certain disease has an incidence rate of 0.5%. If there are no false negatives and if the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

Answer

Out of 100,000 people, 500 would have the disease since the disease as an incidence rate of 0.5%.

	Has Disease	Does Not Have Disease	Total
Tests Positive			
Tests Negative			
Total	500	99500	100000

Of those, all 500 would test positive since there are no false negatives. Of the 99,500 without the disease 3%, or 2,985, would falsely test positive and the other 96,515 would test negative.

	Has Disease	Does Not Have Disease	Total
Tests Positive	500	2985	3485
Tests Negative	0	96515	96515
Total	500	99500	100000

$$\mathrm{P}(\text{ disease } \mid \text{ positive }) = \frac{500}{500+2985} = \frac{500}{3485} \approx 14.3\%$$

This page titled 4.4: Two-Way Tables is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.4: Bayes Theorem** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

4.4.1: Exercises

1. A certain disease has an incidence rate of 0.3%. If the false negative rate is 6% and the false positive rate is 4%. Set up a two way table with a sample size of 10,000 people.
   a. Compute the probability that a person who tests positive actually has the disease.
   b. Compute the probability that a person who tests negative actually has the disease.
   c. Compute the probability that a person who has the disease tests negative.
   d. Compute the probability that a person who does not have the disease tests negative.
2. A certain disease has an incidence rate of 0.1%. If the false negative rate is 8% and the false positive rate is 3%. Set up a two way table with a sample size of 10,000 people.
   a. Compute the probability that a person who tests positive does NOT have the disease.
   b. Compute the probability that a person who tests negative does NOT have the disease.
   c. Compute the probability that a person who has the disease tests positive.
   d. Compute the probability that a person who does not have the disease test positive.
3. A certain group of symptom-free women between the ages of 40 and 50 are randomly selected to participate in mammography screening. The incidence rate of breast cancer among such women is 0.8%. The false negative rate for the mammogram is 10%. The false positive rate is 7%. If a the mammogram results for a particular woman are positive (indicating that she has breast cancer), what is the probability that she actually has breast cancer?
4. About 0.01% of men with no known risk behavior are infected with HIV. The false negative rate for the standard HIV test 0.01% and the false positive rate is also 0.01%. If a randomly selected man with no known risk behavior tests positive for HIV, what is the probability that he is actually infected with HIV?

This page titled 4.4.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.7: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

4.5: Counting

4.5 Learning Objectives

- Use the Fundamental Counting Rule to determine the number of outcomes
- Use permutations to determine the number of outcomes
- Use combinations to determine the number of outcomes
- Determine whether a counting situation is permutations or combinations
- Use counting techniques to find probabilities

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting *efficiently*. When we get to the probability situations a bit later in this chapter we will need to count some *very* large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

Basic Counting

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example 1

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution

Solution 1: One way to solve this problem would be to systematically list each possible meal:

soup + hamburger	soup + sandwich	soup + quiche
soup + fajita	soup + pizza	salad + hamburger
salad + sandwich	salad + quiche	salad + fajita
salad + pizza	breadsticks + hamburger	breadsticks + sandwich
breadsticks + quiche	breadsticks + fajita	breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

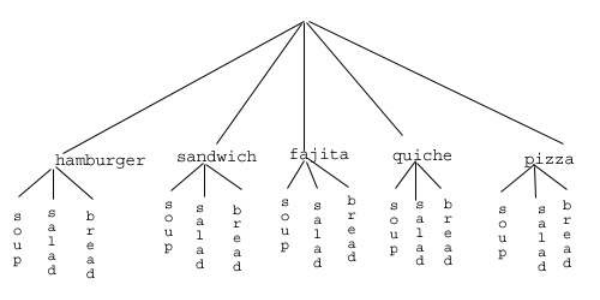
Solution 2: Another way to solve this problem would be to list all the possibilities in a table:

	hamburger	**sandwich**	**quiche**	**fajita**	**pizza**
soup	Soup + burger				
salad	Salad + burger				
bread	*etc.*				

In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care *what* the possible meals are, only *how many* possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

Solution 3: We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose form instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a **tree diagram**:



This is called a "tree" diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5) to get 15. Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the *fundamental counting rule*:

Fundamental Counting Rule

If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is

$$\text{number of choices} = m \cdot n \tag{4.5.1}$$

This is sometimes called the multiplication counting rule.

Example 2

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

Solution

There are 21 choices from the first category and 18 for the second, so there are $21 \cdot 18 = 378$ possibilities.

The Fundamental Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example 3

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution

There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are $3 \cdot 5 \cdot 2 = 30$ possibilities.

Example 4

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

Solution

There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in $2 \cdot 2 \cdot 2 = 8$ different ways. Recall that another way to write $2 \cdot 2 \cdot 2$ is 2^3 which is much more compact.

Try it Now 1

Suppose at a particular restaurant you have eight choices for an appetizer, eleven choices for a main course and five choices for dessert. If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Answer

$8 \cdot 11 \cdot 5 = 440$ menu combinations

Permutations

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

Example 5

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

Solution

This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is: the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to spell a code word with the letters of the word MATH.

In example 5, we needed to calculate $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. This calculation shows up often in mathematics, and is called the **factorial**, and is notated $n!$

Factorial

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 \tag{4.5.2}$$

0! = 1 by definition.

Example 6

How many ways can five different door prizes be distributed among five people?

Solution

There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

Now we will consider some slightly different examples.

Example 7

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of $100, the second is worth $25 and the third is worth $10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Solution

Using the Fundamental Counting Rule, there are 25 choices for the person who receives the $100 certificate, 24 remaining choices for the $25 certificate and 23 choices for the $10 certificate, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways in which the prizes can be awarded.

Example 8

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Solution

Using the Fundamental Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are $8 \cdot 7 \cdot 6 = 336$ ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the $100 prize, or a different goldmedalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a $10 gift certificate; in this case the order of selection is *not* important since each of the three people receive the same prize. Situations where the order is *not* important will be discussed in the next section.)

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order r items out of n possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

If you don't see why $(n-r+1)$ is the right number to use for the last factor, just think back to the first example in this section, where we calculated $25 \cdot 24 \cdot 23$ to get $13,800$. In this case $n = 25$ and $r = 3$, so $n-r+1 = 25-3+1 = 23$, which is exactly the right number for the final factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Fundamental Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often, we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* and where the *order of selection is important*. In this situation we write:

Permutations

$$ {}_nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

We say that there are ${}_nP_r$ **permutations** of size r that may be selected from among n choices *without replacement* when *order matters*.

It turns out that we can express this result more simply using factorials.

$$ {}_nP_r = \frac{n!}{(n-r)!} \tag{4.5.3}$$

In practice, we usually use technology rather than factorials or repeated multiplication to compute permutations.

Example 9

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Solution

Since we are choosing 4 paintings out of 9 *without replacement* where the *order of selection is important* there are ${}_9P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024$ permutations.

Example 10

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

Solution

We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is ${}_{16}P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43,680$.

Try it Now 2

How many 5 character passwords can be made using the letters A through Z

a. if repeats are allowed
b. if no repeats are allowed

Answer

There are 26 characters.

a. $26^5 = 11,881,376$
b. ${}_{26}\mathrm{P}_5 = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7,893,600$

In some cases, some of the choices available are identical. Suppose someone was given the chance to choose 5 gift cards out of 10 available gift cards and they notice that 3 of the gift cards is from their favorite coffee shop. They might choose those 3 first and then decide on the other two. The number of choices changes when there are duplicate items available.

Permutations with Duplicate Items

The number of permutations of n items when p are identical, q are identical, r are identical, and so on... is given by,

$$\text{choices} = \frac{n!}{p!q!r!\cdots} \tag{4.5.4}$$

Example 11

How many ways could you arrange the letters in the word MISSISSIPPI?

Solution

Consider first the the total number of items. There are 11 letters in the word MISSISSIPPI. This means $n = 11$

Next consider the items. There are four of the letter I, four of the letter S, and two of the letter P. If we arrange the four S's, it would not matter exactly which S was moved to another position.

$$\text{arrangements} = \frac{11!}{4!4!2!\cdots} = 34650$$

In comparison, if we were asked to arrange the letters in an 11 letter word with no duplicate letters the answer would be

$${}_{11}P_{11} = \frac{11!}{(11-11)!} = 11! = 39,916,800$$

Combinations

In the previous section we considered the situation where we chose r items out of n possibilities *without replacement* and where the *order of selection was important*. We now consider a similar situation in which the order of selection is *not* important.

Example 12

A charity benefit is attended by 25 people at which three $50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Solution

Using the Fundamental Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get $50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

ABC ACB BAC BCA CAB CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are $3 \cdot 2 \cdot 1 = 6$ ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be $\frac{13,800}{6} = 2300$.

We can generalize the situation in this example above to any problem of choosing a collection of items *without replacement* where the *order of selection is* ***not*** *important*. If we are choosing r items out of n possibilities (instead of 3 out of 25 as in the previous examples), the number of possible choices will be given by this formula ${}_nC_r = \frac{n!}{(n-r)!r!}$.

Combinations

We say that there are ${}_nC_r$ **combinations** of size r that may be selected from among n choices *without replacement where order doesn't matter*.

We can write the combinations formula in terms of factorials:

$$ {}_nC_r = \frac{n!}{(n-r)!r!} \tag{4.5.5}$$

Example 13

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Solution

Since we are choosing 4 people out of 35 *without replacement* where the *order of selection is* ***not*** *important* there are ${}_{35}C_4 = \frac{35 \cdot 34 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1} = 52,360$ combinations.

Try it Now 3

Suppose the United States Senate Appropriations Committee consists of 29 members, and the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

Answer

Order does not matter. ${}_{29}\mathrm{C}_{19} = 20,030,010$possible subcommittees

In the preceding Try it Now problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same of course would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form:

Example 14

Suppose the United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats, and the Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

Solution

In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are ${}_{15}C_{10} = 3003$ ways to choose the 10 Republicans and ${}_{14}C_{9} = 2002$ ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Fundamental Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be $3003 \cdot 2002 = 6,012,006$possible ways of selecting the members of the Defense Subcommittee.

Probability using Permutations and Combinations

We can use permutations and combinations to help us answer more complex probability questions.

Example 15

A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

Solution

There are 10 possible values for each digit of the PIN (namely: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$), so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \cdot 9 \cdot 8 \cdot 7$, or notice that this is the same as the permutation ${}_{10}P_4 = 5040$.

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is $\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504$

Example 16

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

Solution

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$$

Example 17

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

Solution

As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by ${}_6C_5 = 6$ and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1 = 42$. Thus the number of favorable outcomes is then given by the Fundamental Counting Rule: ${}_6C_5 \cdot {}_{42}C_1 = 6 \cdot 42 = 252$. So the probability of winning the second prize is.

$$\frac{({}_6C_5)\,({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$$

Try it Now 4

A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

Answer

There are $5^{10} = 9,765,625$ different ways the exam can be answered. There are 10 possible locations for the one missed question, and in each of those locations there are 4 wrong answers, so there are 40 ways the test could be answered with one wrong answer.

$\mathrm{P}(9 \text{ answers correct}) = \frac{40}{5^{10}} \approx 0.0000041$ chance

Example 18

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

Solution

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be ${}_{48}C_4$ ways to select the four non-Aces. Now we use the Fundamental Counting Rule to calculate that there will be ${}_4C_1 \cdot {}_{48}C_4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$$P(\text{one Ace}) = \frac{({}_4C_1)\,({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$$

Example 19

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

Solution

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{two Aces}) = \frac{({}_4C_2)({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now 5

Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Answer

$$P(\text{three Aces and two Kings}) = \frac{({}_4C_3)({}_4C_2)}{{}_{52}C_5} = \frac{24}{2598960} \approx 0.0000092$$

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer ($\frac{30}{365}$, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example 20

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

Solution

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves "What is the alternative to having at least one shared birthday?" In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to "at least one" is having **none**. In other words, since this is a complementary event,

$$P(\text{at least one}) = 1 - P(\text{none})$$

We will start, then, by computing the probability that there is no shared birthday. Let's imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let's ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is $\frac{364}{365}$. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is $\frac{363}{365}$.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{no shared birthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday}) = 1 - 0.9918 = 0.0082 \ .$$

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

Example 21

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Solution

Continuing the pattern of the previous example, the answer should be

$$P(\text{shared birthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

Example 21

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Solution

Here we can calculate

$$P(\text{ shared birthday }) = 1 - \frac{{}_{365}P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page: http://statweb.stanford.edu/~susan/surprise/Birthday.html, and once the applet has loaded, select 30 birthdays and then keep clicking Start and Reset. If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

Try it Now 6

Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

Answer

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_{10}}{365^{10}} \approx 0.117$$

This page titled 4.5: Counting is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.5: Counting** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

4.5.1: Exercises

1. A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?
2. At a restaurant you can choose from 3 appetizers, 8 entrees, and 2 desserts. How many different three-course meals can you have?
3. How many three-letter "words" can be made from 4 letters "FGHI" if
   a. repetition of letters is allowed
   b. repetition of letters is not allowed
4. How many four-letter "words" can be made from 6 letters "AEBWDP" if
   a. repetition of letters is allowed
   b. repetition of letters is not allowed
5. All of the license plates in a particular state feature three letters followed by three digits (e.g. ABC 123). How many different license plate numbers are available to the state's Department of Motor Vehicles? Letters and digits may be repeated.
6. A computer password must be eight characters long. How many passwords are possible if only the 26 letters of the alphabet are allowed? A letter may not be used more than once.
7. A pianist plans to play 4 pieces at a recital. In how many ways can she arrange these pieces in the program?
8. In how many ways can first, second, and third prizes be awarded in a contest with 210 contestants?
9. Seven Olympic sprinters are eligible to compete in the 4 x 100 m relay race for the USA Olympic team. How many four-person relay teams can be selected from among the seven athletes?
10. A computer user has downloaded 25 songs using an online file-sharing program and wants to create a CD-R with ten songs to use in his portable CD player. If the order that the songs are placed on the CD-R is important to him, how many different CD-Rs could he make from the 25 songs available to him?
11. In western music, an octave is divided into 12 pitches. For the film *Close Encounters of the Third Kind*, director Steven Spielberg asked composer John Williams to write a five-note theme, which aliens would use to communicate with people on Earth. Disregarding rhythm and octave changes, how many five-note themes are possible if no note is repeated?
12. In the early twentieth century, proponents of the Second Viennese School of musical composition (including Arnold Schönberg, Anton Webern and Alban Berg) devised the twelve-tone technique, which utilized a tone row consisting of all 12 pitches from the chromatic scale in any order, but with no pitches repeated in the row. Disregarding rhythm and octave changes, how many tone rows are possible?
13. In how many ways can 4 pizza toppings be chosen from 12 available toppings?
14. At a baby shower 17 guests are in attendance and 5 of them are randomly selected to receive a door prize. If all 5 prizes are identical, in how many ways can the prizes be awarded?
15. In the 6/50 lottery game, a player picks six numbers from 1 to 50. How many different choices does the player have if order doesn't matter?
16. In a lottery daily game, a player picks three numbers from 0 to 9. How many different choices does the player have if order doesn't matter?
17. A jury pool consists of 27 people. How many different ways can 11 people be chosen to serve on a jury and one additional person be chosen to serve as the jury foreman?
18. Suppose the United States Senate Committee on Commerce, Science, and Transportation consists of 23 members, 12 Republicans and 11 Democrats and that the Surface Transportation and Merchant Marine Subcommittee consists of 8 Republicans and 7 Democrats. How many ways can members of the Subcommittee be chosen from the Committee?
19. You own 16 vinyl albums. You want to randomly arrange 5 of them in a album rack. What is the probability that the rack ends up in alphabetical order?
20. A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.
21. In a lottery game, a player picks six numbers from 1 to 48. If 5 of the 6 numbers match those drawn, they player wins second prize. What is the probability of winning this prize?
22. In a lottery game, a player picks six numbers from 1 to 48. If 4 of the 6 numbers match those drawn, they player wins third prize. What is the probability of winning this prize?
23. Compute the probability that a 5-card poker hand is dealt to you that contains all hearts.
24. Compute the probability that a 5-card poker hand is dealt to you that contains four Aces.

This page titled 4.5.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.7: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

4.6: Expected Value

4.6 Learning Objectives

- Determine the expected value of an event

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 1

[1]In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?



Solution

Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won" $-\$2/\$38 \approx -\$0.053$ In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the **expected value** of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

Outcome	Probability of outcome
\$35	$\frac{1}{38}$
$-\$1$	$\frac{37}{38}$

Notice that if we multiply each outcome by its corresponding probability we get $\$35 \cdot \frac{1}{38} = 0.9211$ and $-\$1 \cdot \frac{37}{38} = -0.9737$, and if we add these numbers we get $0.9211 + (-0.9737) \approx -0.053$ which is the expected value we computed above.

Expected Value

Expected Value is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

Try it Now 1

You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

Answer

$$(\$3,995) \cdot \frac{1}{2000} + (-\$5) \cdot \frac{1999}{2000} \approx -\$3.00$$

Example 2

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. If they match 5 numbers, then win $1,000. It costs $1 to buy a ticket. Find the expected value.

Solution

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$ for all 6 numbers,

$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$ for 5 numbers.

Our probabilities and outcome values are:

Outcome	Probability of outcome
$\$999,999$	$\frac{1}{12271512}$
$\$999$	$\frac{252}{12271512}$
$-\$1$	$1-\frac{253}{12271512}=\frac{12271259}{12271512}$

The expected value, then is:

$(\$999,999)\cdot\frac{1}{12271512}+(\$999)\cdot\frac{252}{12271512}+(-\$1)\cdot\frac{12271259}{12271512}\approx -\$0.898$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose $1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

Try it Now 2

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him $1. If any two dice match values, you get $2. What is the expected value of this game? Would you play?

Answer

Suppose you roll the first die. The probability the second will be different is $\frac{5}{6}$. The probability that the third roll is different than the previous two is $\frac{4}{6}$, so the probability that the three dice are different is $\frac{5}{6}\cdot\frac{4}{6}=\frac{20}{36}$. The probability that two dice will match is the complement, $1-\frac{20}{36}=\frac{16}{36}$.

The expected value is: $(\$2)\cdot\frac{16}{36}+(-\$1)\cdot\frac{20}{36}=\frac{12}{36}\approx\$0.33$. Yes, it is in your advantage to play. On average, you'd win $0.33 per play.

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example 3

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year[2]. An insurance company charges $275 for a life-insurance policy that pays a $100,000 death benefit. What is the expected value for the person buying the insurance?

Solution

The probabilities and outcomes are

Outcome	Probability of outcome
$\$100,000 - \$275 = \$99,725$	0.00242
$-\$275$	$1 - 0.00242 = 0.99758$

The expected value is $(\$99,725)(0.00242) + (-\$275)(0.99758) = -\$33$

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

---

[1] Photo CC-BY-SA http://www.flickr.com/photos/stoneflower/

[2] According to the estimator at www.numericalexample.com/inde...=article&id=91

This page titled 4.6: Expected Value is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.6: Expected Value** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

4.6.1: Exercises

1. A bag contains 3 gold marbles, 6 silver marbles, and 28 black marbles. Someone offers to play this game: You randomly select on marble from the bag. If it is gold, you win $3. If it is silver, you win $2. If it is black, you lose $1. What is your expected value if you play this game?
2. A friend devises a game that is played by rolling a single six-sided die once. If you roll a 6, he pays you $3; if you roll a 5, he pays you nothing; if you roll a number less than 5, you pay him $1. Compute the expected value for this game. Should you play this game?
3. In a lottery game, a player picks six numbers from 1 to 23. If the player matches all six numbers, they win $30,000. Otherwise, they lose $1. Find the expected value of this game.
4. A game is played by picking two cards from a deck. If they are the same value, then you win $5, otherwise you lose $1. What is the expected value of this game?
5. A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of $350. If they offer a 2 year extended warranty for $48, what is the company's expected value of each warranty sold?
6. An insurance company estimates the probability of an earthquake in the next year to be 0.0013. The average damage done by an earthquake it estimates to be $60,000. If the company offers earthquake insurance for $100, what is their expected value of the policy?

This page titled 4.6.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.7: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

4.7: Additional Exercises

Exploration

Some of these questions were adapted from puzzles at mindyourdecisions.com.

1. A small college has been accused of gender bias in its admissions to graduate programs.
   a. Out of 500 men who applied, 255 were accepted. Out of 700 women who applied, 240 were accepted. Find the acceptance rate for each gender. Does this suggest bias?
   b. The college then looked at each of the two departments with graduate programs, and found the data below. Compute the acceptance rate within each department by gender. Does this suggest bias?
   c. Looking at our results from Parts *a* and *b*, what can you conclude? Is there gender bias in this college's admissions? If so, in which direction?
2. A bet on "black" in Roulette has a probability of 18/38 of winning. If you win, you double your money. You can bet anywhere from $1 to $100 on each spin.
   a. Suppose you have $10, and are going to play until you go broke or have $20. What is your best strategy for playing?
   b. Suppose you have $10, and are going to play until you go broke or have $30. What is your best strategy for playing?
3. Your friend proposes a game: You flip a coin. If it's heads, you win $1. If it's tails, you lose $1. However, you are worried the coin might not be fair coin. How could you change the game to make the game fair, without replacing the coin?
4. Fifty people are in a line. The first person in the line to have a birthday matching someone in front of them will win a prize. Of course, this means the first person in the line has no chance of winning. Which person has the highest likelihood of winning?
5. Three people put their names in a hat, then each draws a name, as part of a randomized gift exchange. What is the probability that no one draws their own name? What about with four people?
6. How many different "words" can be formed by using all the letters of each of the following words exactly once?
   a. "ALICE"
   b. "APPLE"
7. How many different "words" can be formed by using all the letters of each of the following words exactly once?
   a. "GRUMPS"
   b. "TEETER"
8. The *Monty Hall problem* is named for the host of the game show *Let's make a Deal*. In this game, there would be three doors, behind one of which there was a prize. The contestant was asked to choose one of the doors. Monty Hall would then open one of the other doors to show there was no prize there. The contestant was then asked if they wanted to stay with their original door, or switch to the other unopened door. Is it better to stay or switch, or does it matter?
9. Suppose you have two coins, where one is a fair coin, and the other coin comes up heads 70% of the time. What is the probability you have the fair coin given each of the following outcomes from a series of flips?
   a. 5 Heads and 0 Tails
   b. 8 Heads and 3 Tails
   c. 10 Heads and 10 Tails
   d. 3 Heads and 8 Tails
10. Suppose you have six coins, where five are fair coins, and one coin comes up heads 80% of the time. What is the probability you have a fair coin given each of the following outcomes from a series of flips?
    a. 5 Heads and 0 Tails
    b. 8 Heads and 3 Tails
    c. 10 Heads and 10 Tails
    d. 3 Heads and 8 Tails
11. We'll now consider a simplification of a case from the 1960s. In the area, about 26% of the jury eligible population was black. In the court case, there were 100 men on the juror panel, of which 8 were black. Does this provide evidence of racial bias in jury selection?
12. In this problem, we will explore probabilities from a series of events.
    a. If you flip 20 coins, how many would you *expect* to come up "heads", on average? Would you expect *every* flip of 20 coins to come up with exactly that many heads?

b. If you were to flip 20 coins, what would you consider a “usual” result? An “unusual” result?
c. Flip 20 coins (or one coin 20 times) and record how many come up “heads”. Repeat this experiment 9 more times. Collect the data from the entire class.
d. When flipping 20 coins, what is the theoretic probability of flipping 20 heads?
e. Based on the class’s experimental data, what appears to be the probability of flipping 10 heads out of 20 coins?
f. The formula ${}_nC_x p^x (1-p)^{n-x}$ will compute the probability of an event with probability p occurring x times out of n, such as flipping x heads out of n coins where the probability of heads is $p = \frac{1}{2}$. Use this to compute the theoretic probability of flipping 10 heads out of 20 coins.
g. If you were to flip 20 coins, based on the class’s experimental data, what range of values would you consider a “usual” result? What is the combined probability of these results? What would you consider an “unusual” result? What is the combined probability of these results?

This page titled 4.7: Additional Exercises is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **12.7: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

CHAPTER OVERVIEW

5: Statistics

5.1: What is Statistics?
5.1.1: Exercises
5.2: Populations and Samples
5.2.1: Exercises
5.3: Categorizing data
5.3.1: Exercises
5.4: Sampling methods
5.4.1: Exercises
5.5: Bias
5.5.1: Exercises
5.6: Experiments
5.6.1: Exercises
5.7: Presenting Data Graphically
5.7.1: Exercises
5.8: Reading Graphs in Media
5.8.1: Exercises
5.9: Correlation and Causation, Scatter Plots
5.9.1: Exercises
5.10: Measures of Central Tendency
5.10.1: Exercises

This page titled 5: Statistics is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

5.1: What is Statistics?

5.1 Learning Objectives

- Describe what statistics is
- Identify and describe two branches of statistics

Like most people, you probably feel that it is important to "take control of your life." But what does this mean? Partly it means being able to properly evaluate the data and claims that bombard you every day. If you cannot distinguish good from faulty reasoning, then you are vulnerable to manipulation and to decisions that are not in your best interest. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, Statistics is one of the most important things that you can study.

To be more specific, here are some claims that we have heard on several occasions. (We are *not* saying that each one of these claims is true!)

- 4 out of 5 dentists recommend Dentyne.
- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- Condoms are effective 94% of the time.
- Native Americans are significantly more likely to be hit crossing the streets than are people of other ethnicities.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one's life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 80% chance that in a room full of 30 people that at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

All of these claims are statistical in character. We suspect that some of them sound familiar; if not, we bet that you have heard other claims like them. Notice how diverse the examples are; they come from psychology, health, law, sports, business, etc. Indeed, data and data-interpretation show up in discourse from virtually every facet of contemporary life.

Definition: Statistics

Statistics is the science of collecting, organizing, analyzing, and interpreting data in order to make a decision.

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading, and push you into decisions that you might find cause to regret. For these reasons, learning about statistics is a long step towards taking control of your life. (It is not, of course, the only step needed for this purpose.) This chapter will help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

You can take the first step right away. To be an intelligent consumer of statistics, your first reflex must be to question the statistics that you encounter. The British Prime Minister Benjamin Disraeli famously said, "There are three kinds of lies -- lies, damned lies, and statistics." This quote reminds us why it is so important to understand statistics. So let us invite you to reform your statistical habits from now on. No longer will you blindly accept numbers or findings. Instead, you will begin to think about the numbers, their sources, and most importantly, the procedures used to generate them.

We have put the emphasis on defending ourselves against fraudulent claims wrapped up as statistics. Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must also learn to recognize statistical evidence that supports a stated conclusion. When a research team is testing a new treatment for a disease, statistics allows them to conclude based on a relatively small trial that there is good evidence their drug is effective. Statistics allowed prosecutors in the 1950's and 60's to demonstrate racial bias existed in jury panels. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases.

There are two main branches of statistics; descriptive statistics and inferential statistics.

Definition: Branches of Statistics

Descriptive statistics involves the organization, the summary, and how the data is displayed.

Inferential statistics involves using a sample to draw conclusions about a population.

Example 1

Twenty of the 80 students enrolled in Math 150 were asked if they prefer to do an in-class examination or a take-home project for their second assessment. Fifteen, or 75% of the students polled, indicated their preference is to have a take-home project as a second assessment. What type of statistics was performed?

Solution

This was descriptive statistics since the data collected displays a summary of the results. It is not inferential statistics because we are not extending it to a much larger population.

Example 2

From data collected in the years past elections, it is predicted that 67% of registered voters will vote in the next presidential election. What type of statistics is this?

Solution

This was inferential statistics because data was collected from a sample of previous elections, and used it to draw conclusions for the entire population.

This page titled 5.1: What is Statistics? is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **10.1: Introduction** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.1.1: Exercises

Determine the branch of statistics that is being used.

1. A survey conducted to 54 various donut shops found that 38% of their customers like buttermilk glazed donut the most. Which branch of statistics is being used here?
2. A survey conducted to 54 various donut shops found that 38% of their customers like buttermilk glazed donut the most, and then the researcher concludes that 38% of all donut-purchasing customers will love the buttermilk glazed donut the most. Which branch of statistics is being used here?
3. A survey was given to movie goers at 100 movie theaters to estimate what percent of all movie goers arrive later than the published start time to avoid the previews. Which branch of statistics is being used here?
4. A survey was given to students at a local community college to determine how they liked their experience with online registration. They determined the 76% of the students were satisfied with their experience. Which branch of statistics is being used here?
5. A study measured the lifespan of 1000 golden retrievers and used that to predict the lifespan of all golden retrievers. What branch of statistics is being used here?

This page titled 5.1.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

5.2: Populations and Samples

5.2 Learning Objectives

- Identify the population of a study
- Identify the sample of a study
- Explain the difference between a parameter and a statistic

Before we begin gathering and analyzing data we need to characterize the **population** we are studying. If we want to study the amount of money spent on textbooks by a typical first-year college student, our population might be all first-year students at your college. Or it might be:

- All first-year community college students in the state of Washington.
- All first-year students at public colleges and universities in the state of Washington.
- All first-year students at all colleges and universities in the state of Washington.
- All first-year students at all colleges and universities in the entire United States.
- And so on.

Definition: Population

The **population** of a study is the group the collected data is intended to describe.

Sometimes the intended population is called the **target population** since if we design our study badly, the collected data might not actually be representative of the intended population.

Why is it important to specify the population? We might get different answers to our question as we vary the population we are studying. First-year students at the University of Washington might take slightly more diverse courses than those at your college, and some of these courses may require less popular textbooks that cost more; or, on the other hand, the University Bookstore might have a larger pool of used textbooks, reducing the cost of these books to the students. Whichever the case (and it is likely that some combination of these and other factors are in play), the data we gather from your college will probably not be the same as that from the University of Washington. Particularly when conveying our results to others, we want to be clear about the population we are describing with our data.

Example 1

A newspaper website contains a poll asking people their opinion on a recent news article. What is the population?

Solution

While the target (intended) population may have been all people, the real population of the survey is readers of the website.

If we were able to gather data on every member of our population, say the average (we will define "average" more carefully in a subsequent section) amount of money spent on textbooks by each first-year student at your college during the 2009-2010 academic year, the resulting number would be called a **parameter**.

Definition: Parameter

A **parameter** is a value (average, percentage, etc.) calculated using all the data from a population

We seldom see parameters, however, since surveying an entire population is usually very time-consuming and expensive, unless the population is very small or we already have the data collected.

Definition: Census

A survey of an entire population is called a **census**.

You are probably familiar with two common censuses: the official government Census that attempts to count the population of the U.S. every ten years, and voting, which asks the opinion of all eligible voters in a district. The first of these demonstrates one

additional problem with a census: the difficulty in finding and getting participation from everyone in a large population, which can bias, or skew, the results.

There are occasionally times when a census is appropriate, usually when the population is fairly small. For example, if the manager of Starbucks wanted to know the average number of hours her employees worked last week, she should be able to pull up payroll records or ask each employee directly.

Since surveying an entire population is often impractical, we usually select a **sample** to study;

Definition: Sample

A **sample** is a smaller subset of the entire population, ideally one that is fairly representative of the whole population.

We will discuss sampling methods in greater detail in a later section. For now, let us assume that samples are chosen in an appropriate manner. If we survey a sample, say 100 first-year students at your college, and find the average amount of money spent by these students on textbooks, the resulting number is called a **statistic**.

Definition: Statistic

A **statistic** is a value (average, percentage, etc.) calculated using the data from a sample.

Example 2

A researcher wanted to know how citizens of Tacoma felt about a voter initiative. To study this, she goes to the Tacoma Mall and randomly selects 500 shoppers and asks them their opinion. 60% indicate they are supportive of the initiative. What is the sample and population? Is the 60% value a parameter or a statistic?

Solution

The sample is the 500 shoppers questioned. The population is less clear. While the intended population of this survey was Tacoma citizens, the effective population was mall shoppers. There is no reason to assume that the 500 shoppers questioned would be representative of all Tacoma citizens.

The 60% value was based on the sample, so it is a statistic.

Try it Now 1

To determine the average length of trout in a lake, researchers catch 20 fish and measure them. What is the sample and population in this study?

Answer

The sample is the 20 fish caught. The population is all fish in the lake. The sample may be somewhat unrepresentative of the population since not all fish may be large enough to catch the bait.

Try it Now 2

A college reports that the average age of their students is 28 years old. Is this a statistic or a parameter?

Answer

This is a parameter, since the college would have access to data on all students (the population)

This page titled 5.2: Populations and Samples is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **10.2: Populations and Samples** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.2.1: Exercises

1. A political scientist surveys 28 of the current 106 representatives in a state's congress. Of them, 14 said they were supporting a new education bill, 12 said there were not supporting the bill, and 2 were undecided.
   a. What is the population of this survey?
   b. What is the size of the population?
   c. What is the sample of this survey?
   d. Find the sample statistic for the proportion of representatives surveyed who said they were supporting the education bill.
   e. Based on this sample, we might expect how many of the representatives to support the education bill?
2. The city of Raleigh has 9500 registered voters. There are two candidates for city council in an upcoming election: Brown and Feliz. The day before the election, a telephone poll of 350 randomly selected registered voters was conducted. 112 said they'd vote for Brown, 207 said they'd vote for Feliz, and 31 were undecided.
   a. What is the population of this survey?
   b. What is the size of the population?
   c. What is the sample of this survey?
   d. Find the sample statistic for the proportion of voters surveyed who said they'd vote for Brown.
   e. Based on this sample, we might expect how many of the 9500 voters to vote for Brown?
3. A community college wants to determine how old its 30,000 students are. Officials use student records to conclude that the mean age of students enrolled at the college is 28.2.
   a. What is the population of this survey?
   b. What is the size of the population?
   c. Is this a census or a sample?
   d. Is the mean of 28.2 a statistic or a parameter?
   e. Is the conclusion part of descriptive or inferential statistics?
4. Officials at the same community college also want to know how many of its students work at least 20 hours a week. They survey 120 students at random. 59% of those surveyed say they work at least 20 hours a week. They conclude that 59% of all the students enrolled at the college work at least 20 hours a week.
   a. What is the population of this survey?
   b. What is the sample of this survey?
   c. What is the size of the sample?
   d. Is 59% a statistic or a parameter?
   e. Is the conclusion part of descriptive or inferential statistics?

This page titled 5.2.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **10.7: Exercise** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.3: Categorizing data

5.3 Learning Objectives

- Classify data as categorical or numerical

Once we have gathered data, we might wish to classify it. Roughly speaking, data can be classified as **categorical data** or **quantitative data**.

Definition: Quantitative and Categorical Data

Categorical (qualitative) data are pieces of information that allow us to classify the objects under investigation into various categories.

Quantitative data are responses that are numerical in nature with which we can perform meaningful arithmetic calculations.

Example 1

We might conduct a survey to determine the name of the favorite movie that each person in a math class saw in a movie theater.

When we conduct such a survey, the responses would look like: *Finding Nemo*, *The Hulk*, or *Terminator 3: Rise of the Machines*. We might count the number of people who give each answer, but the answers themselves do not have any numerical values: we cannot perform computations with an answer like "*Finding Nemo*." This would be categorical data.

Example 2

A survey could ask the number of movies you have seen in a movie theater in the past 12 months (0, 1, 2, 3, 4, ...)

This would be quantitative data.

Other examples of quantitative data would be the running time of the movie you saw most recently (104 minutes, 137 minutes, 104 minutes, ...) or the amount of money you paid for a movie ticket the last time you went to a movie theater ($5.50, $7.75, $9, ...).

Sometimes, determining whether or not data is categorical or quantitative can be a bit trickier.

Example 3

Suppose we gather respondents' ZIP codes in a survey to track their geographical location.

ZIP codes are numbers, but we can't do any meaningful mathematical calculations with them (it doesn't make sense to say that 98036 is "twice" 49018 — that's like saying that Lynnwood, WA is "twice" Battle Creek, MI, which doesn't make sense at all), so ZIP codes are really categorical data.

Example 4

A survey about the movie you most recently attended includes the question "How would you rate the movie you just saw?" with these possible answers:

1 - it was awful
2 - it was just OK
3 - I liked it
4 - it was great
5 - best movie ever!

Again, there are numbers associated with the responses, but we can't really do any calculations with them: a movie that rates a 4 is not necessarily twice as good as a movie that rates a 2, whatever that means; if two people see the movie and one of them thinks it stinks and the other thinks it's the best ever it doesn't necessarily make sense to say that "on average they liked it."

As we study movie-going habits and preferences, we shouldn't forget to specify the population under consideration. If we survey 3-7 year-olds the runaway favorite might be *Finding Nemo*. 13-17 year-olds might prefer *Terminator 3*. And 33-37 year-olds might prefer...well, *Finding Nemo*.

Try it Now 1

Classify each measurement as categorical or quantitative

a. Eye color of a group of people
b. Daily high temperature of a city over several weeks
c. Annual income

Answer

a. Categorical
b. Quantitative
c. Quantitative

This page titled 5.3: Categorizing data is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **10.3: Categorizing data** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.3.1: Exercises

1. In a study, you ask the subjects their age in years. Is this data qualitative or quantitative?
2. In a study, you ask the subjects their gender. Is this data qualitative or quantitative?
3. A business collects the residence ZIP codes of its customers. Is this data qualitative or quantitative?
4. A business collects information about the amount of each customer's purchase. Is this data qualitative or quantitative?
5. A class studies bags of Skittles (R).
    a. They count the number of Skittles of each color found in a bag. Is this data qualitative or quantitative?
    b. They count the number of Skittles in each bag. Is this data qualitative or quantitative?
    c. They find the weight of each bag. Is this data qualitative or quantitative?
6. Researchers are studying trends in education. They collect information for 2018 on the number of college degrees of each type awarded in the U.S. Is this data qualitative or quantitative?

This page titled 5.3.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

5.4: Sampling methods

5.4 Learning Objectives

- Identify the sampling method used in a study

There are various types of sampling so it is important to understand the differences. We would not anticipate very accurate results if we drew all of our samples from among the customers at a Starbucks, nor would we expect that a sample drawn entirely from the membership list of the local Elks club would provide a useful picture of district-wide support for our candidate. Drawing samples in this way is called **convenience sampling**.

One way to ensure that the sample has a reasonable chance of mirroring the population we are interested in is to employ *randomness*. The most basic random method is simple random sampling.

A **random sample** is one in which each member of the population has an equal probability of being chosen. A **simple random sample** is one in which every member of the population and any group of members has an equal probability of being chosen.

Example 1

If we could somehow identify all likely voters in the state, put each of their names on a piece of paper, toss the slips into a (very large) hat and draw 1000 slips out of the hat, we would have a simple random sample.

In practice, computers are better suited for this sort of endeavor than millions of slips of paper and extremely large headgear.

It is always possible, however, that even a random sample might end up not being totally representative of the population. If we repeatedly take samples of 1000 people from among the population of likely voters in the state of Washington, some of these samples might tend to have a slightly higher percentage of Democrats (or Republicans) than does the general population; some samples might include more older people and some samples might include more younger people; etc. In most cases, this **sampling variability** is not significant.

Definition: Sampling variability

The natural variation of samples is called **sampling variability**.

This is unavoidable and expected in random sampling, and in most cases is not an issue.

To help account for variability, pollsters might instead use a **stratified sample**.

Definition: Stratified sampling

In **stratified sampling**, a population is divided into a number of subgroups (or strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

Example 2

Suppose in a particular state that previous data indicated that the electorate was comprised of 39% Democrats, 37% Republicans and 24% independents. In a sample of 1000 people, they would then expect to get about 390 Democrats, 370 Republicans and 240 independents. To accomplish this, they could randomly select 390 people from among those voters known to be Democrats, 370 from those known to be Republicans, and 240 from those with no party affiliation.

Stratified sampling can also be used to select a sample with people in desired age groups, a specified mix ratio of males and females, etc. A variation on this technique is called **quota sampling**.

Definition: Quota Sampling

Quota sampling is a variation on stratified sampling, wherein samples are collected in each subgroup until the desired quota is met.

Example 3

Suppose the pollsters call people at random, but once they have met their quota of 390 Democrats, they only gather people who do not identify themselves as a Democrat.

You may have had the experience of being called by a telephone pollster who started by asking you your age, income, etc. and then thanked you for your time and hung up before asking any "real" questions. Most likely, they already had contacted enough people in your demographic group and were looking for people who were older or younger, richer or poorer, etc. Quota sampling is usually a bit easier than stratified sampling, but also does not ensure the same level of randomness.

Another sampling method is **cluster sampling**, in which the population is divided into groups, and one or more groups are randomly selected to be in the sample.

Cluster sampling

In **cluster sampling**, the population is divided into subgroups (clusters), and a set of subgroups are selected to be in the sample

Example 4

If the college wanted to survey students, since students are already divided into classes, they could randomly select 10 classes and give the survey to all the students in those classes. This would be cluster sampling.

Other sampling methods include **systematic sampling**.

Systematic sampling

In **systematic sampling**, every n^{th} member of the population is selected to be in the sample.

Example 5

To select a sample using systematic sampling, a pollster calls every 100th name in the phone book.

Solution

Systematic sampling is not as random as a simple random sample (if your name is Albert Aardvark and your sister Alexis Aardvark is right after you in the phone book, there is no way you could both end up in the sample) but it can yield acceptable samples.

Perhaps the worst types of sampling methods are **convenience samples** and **voluntary response samples**.

Convenience sampling and voluntary response sampling

Convenience sampling is samples chosen by selecting whoever is convenient.

Voluntary response sampling is allowing the sample to volunteer.

Example 6

A pollster stands on a street corner and interviews the first 100 people who agree to speak to him. This is a convenience sample.

Example 7

A website has a survey asking readers to give their opinion on a tax proposal. This is a self-selected sample, or voluntary response sample, in which respondents volunteer to participate.

Usually voluntary response samples are skewed towards people who have a particularly strong opinion about the subject of the survey or who just have way too much time on their hands and enjoy taking surveys.

Try it Now 1

In each case, indicate what sampling method was used

a. Every 4th person in the class was selected
b. A sample was selected to contain 25 men and 35 women
c. Viewers of a new show are asked to vote on the show's website
d. A website randomly selects 50 of their customers to send a satisfaction survey to
e. To survey voters in a town, a polling company randomly selects 10 city blocks, and interviews everyone who lives on those blocks.

Answer

a. Systematic
b. Stratified or Quota
c. Voluntary response
d. Simple random
e. Cluster

This page titled 5.4: Sampling methods is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **10.4: Sampling methods** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.4.1: Exercises

1. In a study, the sample is chosen by separating all cars by size, and selecting 10 of each size grouping. What is the sampling method?
2. In a study, the sample is chosen by writing everyone’s name on a playing card, shuffling the deck, then choosing the top 20 cards. What is the sampling method?
3. A city government needs to know if its residents will support the construction of a new hospital. Which procedure would be the most appropriate for creating a sample of the city's population?
   a. Survey a random sample of doctors who live in the city.
   b. Survey a random sample of persons within each neighborhood of the city.
   c. Survey a random sample of people who live within a 5-mile radius of the proposed hospital site.
   d. Survey every 100th person who visits one of the other hospitals in the city.
4. The math department wants to know students' opinions of how much math homework they should be doing. Which procedures would be the most appropriate for getting an unbiased sample?
   a. Survey a selection of students enrolled at the college.
   b. Survey enrolled students chosen from the first 1000 students listed alphabetically.
   c. Survey a selection of students enrolled in a math course.
   d. Survey a selection of students walking through the Science Building.
5. A city government wants to conduct a survey among the city's homeless to discover their opinion about the recent ban on camping in certain public areas.
   a. They decide to do a random survey of people residing in homeless shelters. Does this seem like a good sample? Why or why not?
   b. They decide to survey all the homeless people contacted by the police. Does this seem like a good sample? Why or why not?
   c. They leave signs at several homeless encampments asking people to text their answers to some questions. Does this seem like a good sample? Why or why not?
   d. Design a better sampling process than any of those already given.
6. A local supermarket is considering the addition of a coffee bar to its premises. They want to find out how much people might patronize the coffee bar. Design a survey for the supermarket to use.

This page titled 5.4.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

5.5: Bias

5.5 Learning Objectives

- Identify possible sources of bias in a study

Definition: Bias in Statistics

Bias is the tendency to overestimate or underestimate a population parameter. Bias can skew the results of statistical research.

The first thing we should do before conducting a survey is to identify the population that we want to study. Suppose we are hired by a politician to determine the amount of support he has among the electorate should he decide to run for another term. What population should we study? Every person in the district? Not every person is eligible to vote, and regardless of how strongly someone likes or dislikes the candidate, they don't have much to do with him being re-elected if they are not able to vote.

What about eligible voters in the district? That might be better, but if someone is eligible to vote but does not register by the deadline, they won't have any say in the election either. What about registered voters? Many people are registered but choose not to vote. What about "likely voters?"

This is the criteria used in much political polling, but it is sometimes difficult to define a "likely voter." Is it someone who voted in the last election? In the last general election? In the last presidential election? Should we consider someone who just turned 18 a "likely voter?" They weren't eligible to vote in the past, so how do we judge the likelihood that they will vote in the next election?

In November 1998, former professional wrestler Jesse "The Body" Ventura was elected governor of Minnesota. Up until right before the election, most polls showed he had little chance of winning. There were several contributing factors to the polls not reflecting the actual intent of the electorate:

- Ventura was running on a third-party ticket and most polling methods are better suited to a two-candidate race.
- Many respondents to polls may have been embarrassed to tell pollsters that they were planning to vote for a professional wrestler.
- The mere fact that the polls showed Ventura had little chance of winning might have prompted some people to vote for him in protest to send a message to the major-party candidates.

But one of the major contributing factors was that Ventura recruited a substantial amount of support from young people, particularly college students, who had never voted before and who registered specifically to vote in the gubernatorial election. The polls did not deem these young people likely voters (since in most cases young people have a lower rate of voter registration and a turnout rate for elections) and so the polling samples were subject to **sampling bias**: they omitted a portion of the electorate that was weighted in favor of the winning candidate.

Definition: Sampling bias

A sampling method is biased if every member of the population doesn’t have equal likelihood of being in the sample.

So even identifying the population can be a difficult job, but once we have identified the population, how do we choose an appropriate sample? Remember, although we would prefer to survey all members of the population, this is usually impractical unless the population is very small, so we choose a sample. There are many ways to sample a population, but there is one goal we need to keep in mind: we would like the sample to be *representative of the population*.

There are number of ways that a study can be ruined before you even start collecting data. The first we have already explored – **sampling** or **selection bias**, which is when the sample is not representative of the population. One example of this is **voluntary response bias**, which is bias introduced by only collecting data from those who volunteer to participate. This is not the only potential source of bias.

Sources of bias

- **Sampling bias** – when the sample is not representative of the population
- **Voluntary response bias** – the sampling bias that often occurs when the sample is volunteers
- **Self-interest study** – bias that can occur when the researchers have an interest in the outcome
- **Response bias** – when the responder gives inaccurate responses for any reason

- **Perceived lack of anonymity** – when the responder fears giving an honest answer might negatively affect them
- **Loaded questions** – when the question wording influences the responses
- **Non-response bias** – when people refusing to participate in the study can influence the validity of the outcome

Example 1

Consider a recent study which found that chewing gum may raise math grades in teenagers[1]. This study was conducted by the Wrigley Science Institute, a branch of the Wrigley chewing gum company. This is an example of a **self-interest study**; one in which the researches have a vested interest in the outcome of the study. While this does not necessarily ensure that the study was biased, it certainly suggests that we should subject the study to extra scrutiny

Example 2

A survey asks people "when was the last time you visited your doctor?" This might suffer from **response bias**, since many people might not remember exactly when they last saw a doctor and give inaccurate responses.

Sources of response bias may be innocent, such as bad memory, or as intentional as pressuring by the pollster. Consider, for example, how many voting initiative petitions people sign without even reading them.

Example 3

A survey asks participants a question about their interactions with members of other races. Here, a **perceived lack of anonymity** could influence the outcome. The respondent might not want to be perceived as racist even if they are, and give an untruthful answer.

Example 4

An employer puts out a survey asking their employees if they have a drug abuse problem and need treatment help. Here, answering truthfully might have consequences; responses might not be accurate if the employees do not feel their responses are anonymous or fear retribution from their employer.

Example 5

A survey asks "do you support funding research of alternative energy sources to reduce our reliance on high-polluting fossil fuels?" This is an example of a **loaded** or **leading question** – questions whose wording leads the respondent towards an answer.

Loaded questions can occur intentionally by pollsters with an agenda, or accidentally through poor question wording. Also a concern is **question order**, where the order of questions changes the results. A psychology researcher provides an example[2]:

"My favorite finding is this: we did a study where we asked students, 'How satisfied are you with your life? How often do you have a date?' The two answers were not statistically related - you would conclude that there is no relationship between dating frequency and life satisfaction. But when we reversed the order and asked, 'How often do you have a date? How satisfied are you with your life?' the statistical relationship was a strong one. You would now conclude that there is nothing as important in a student's life as dating frequency."

Example 6

A telephone poll asks the question "Do you often have time to relax and read a book?", and 50% of the people called refused to answer the survey. It is unlikely that the results will be representative of the entire population. This is an example of **non-response bias**, introduced by people refusing to participate in a study or dropping out of an experiment. When people refuse to participate, we can no longer be so certain that our sample is representative of the population.

Try it Now 1

In each situation, identify a potential source of bias

a. A survey asks how many sexual partners a person has had in the last year
b. A radio station asks readers to phone in their choice in a daily poll.
c. A substitute teacher wants to know how students in the class did on their last test. The teacher asks the 10 students sitting in the front row to state their latest test score.
d. High school students are asked if they have consumed alcohol in the last two weeks.
e. The Beef Council releases a study stating that consuming red meat poses little cardiovascular risk.
f. A poll asks "Do you support a new transportation tax, or would you prefer to see our public transportation system fall apart?"

Answer

Add texts here. Do not delete this text first.

---

[1] Reuters. news.yahoo.com/s/nm/20090423/...k_gum_learning. Retrieved 4/27/09

[2] Swartz, Norbert. www.umich.edu/~newsinfo/MT/01...01/mt6f01.html. Retrieved 3/31/2009

This page titled 5.5: Bias is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **10.4: Sampling methods** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.
- **10.5: How to mess things up before you start** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.5.1: Exercises

1. Identify the most relevant source of bias in this situation: A survey asks the following: Should the mall prohibit loud and annoying rock music in clothing stores catering to teenagers?
2. Identify the most relevant source of bias in this situation: To determine opinions on voter support for a downtown renovation project, a surveyor randomly questions people working in downtown businesses.
3. Identify the most relevant source of bias in this situation: A survey asks people to report their actual income and the income they reported on their IRS tax form.
4. Identify the most relevant source of bias in this situation: A survey randomly calls people from the phone book and asks them to answer a long series of questions.
5. Identify the most relevant source of bias in this situation: A survey asks the following: Should the death penalty be permitted if innocent people might die?
6. Identify the most relevant source of bias in this situation: A study seeks to investigate whether a new pain medication is safe to market to the public. They test by randomly selecting 300 men from a set of volunteers.
7. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 40 AIDS patients from the start of treatment until their deaths.
   a. What is the population of this study?
   b. Would you expect the data to be identical? Why or why not?
   c. List two reasons why the data may differ.
   d. If the first researcher collected her data by randomly selecting 40 states, then selecting 1 person from each of those states. What sampling method is that?
   e. If the second researcher collected his data by choosing 40 patients he knew. What sampling method would that researcher have used? What concerns would you have about this data set, based upon the data collection method?

This page titled 5.5.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **10.7: Exercise** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.6: Experiments

5.6 Learning Objectives

- Describe the difference between an observational study and an experimental study
- Identify possible confounding variables in an experimental study
- Describe blind and double blind experimental studies

So far, we have primarily discussed **observational studies** – studies in which conclusions would be drawn from observations of a sample or the population. In some cases these observations might be unsolicited, such as studying the percentage of cars that turn right at a red light even when there is a "no turn on red" sign. In other cases the observations are solicited, like in a survey or a poll.

In contrast, it is common to use **experiments** when exploring how subjects react to an outside influence. In an experiment, some kind of **treatment** is applied to the subjects and the results are measured and recorded.

Definition: Observational Study and Experiment

An **observational study** is a study based on observations or measurements.

An **experiment** is a study in which the effects of a **treatment** are measured.

Here are some examples of experiments:

Example 1

a. A pharmaceutical company tests a new medicine for treating Alzheimer's disease by administering the drug to 50 elderly patients with recent diagnoses. The treatment here is the new drug.
b. A gym tests out a new weight loss program by enlisting 30 volunteers to try out the program. The treatment here is the new program.
c. You test a new kitchen cleaner by buying a bottle and cleaning your kitchen. The new cleaner is the treatment.
d. A psychology researcher explores the effect of music on temperament by measuring people's temperament while listening to different types of music. The music is the treatment.

Try it Now 1

Is each scenario describing an observational study or an experiment?

a. The weights of 30 randomly selected people are measured
b. Subjects are asked to do 20 jumping jacks, and then their heart rates are measured
c. Twenty coffee drinkers and twenty tea drinkers are given a concentration test

Answer

a. Observational study
b. Experiment; the treatment is the jumping jacks
c. Experiment; the treatments are coffee and tea

When conducting experiments, it is essential to isolate the treatment being tested.

Example 2

Suppose a middle school (junior high) finds that their students are not scoring well on the state's standardized math test. They decide to run an experiment to see if an alternate curriculum would improve scores. To run the test, they hire a math specialist to come in and teach a class using the new curriculum. To their delight, they see an improvement in test scores.

The difficulty with this scenario is that it is not clear whether the curriculum is responsible for the improvement, or whether the improvement is due to a math specialist teaching the class. This is called **confounding** – when it is not clear which factor or factors caused the observed effect. Confounding is the downfall of many experiments, though sometimes it is hidden.

Definition: Confounding

Confounding occurs when there are two or more potential variables that could have caused the outcome and it is not possible to determine which actually caused the result.

Example 3

A drug company study about a weight loss pill might report that people lost an average of 8 pounds while using their new drug. However, in the fine print you find a statement saying that participants were encouraged to also diet and exercise. It is not clear in this case whether the weight loss is due to the pill, to diet and exercise, or a combination of both. In this case confounding has occurred.

Example 4

Researchers conduct an experiment to determine whether students will perform better on an arithmetic test if they listen to music during the test. They first give the student a test without music, then give a similar test while the student listens to music. In this case, the student might perform better on the second test, regardless of the music, simply because it was the second test and they were warmed up.

There are a number of measures that can be introduced to help reduce the likelihood of confounding. The primary measure is to use a **control group**.

Definition: Control Group

When using a control group, the participants are divided into two or more groups, typically a **control group** and a treatment group. The treatment group receives the treatment being tested; the control group does not receive the treatment.

Ideally, the groups are otherwise as similar as possible, isolating the treatment as the only potential source of difference between the groups. For this reason, the method of dividing groups is important. Some researchers attempt to ensure that the groups have similar characteristics (same number of females, same number of people over 50, etc.), but it is nearly impossible to control for every characteristic. Because of this, random assignment is very commonly used.

Example 5

To determine if a two-day prep course would help high school students improve their scores on the SAT test, a group of students was randomly divided into two subgroups. The first group, the treatment group, was given a two day prep course. The second group, the control group, was not given the prep course. Afterward, both groups were given the SAT.

Example 6

A company testing a new plant food grows two crops of plants in adjacent fields, the treatment group receiving the new plant food and the control group not. The crop yield would then be compared. By growing them at the same time in adjacent fields, they are controlling for weather and other confounding factors.

Sometimes not giving the control group anything does not completely control for confounding variables. For example, suppose a medicine study is testing a new headache pill by giving the treatment group the pill and the control group nothing. If the treatment group showed improvement, we would not know whether it was due to the medicine in the pill, or a response to have taken any pill. This is called a **placebo effect**.

Definition: Placebo Effect

The **placebo effect** is when the effectiveness of a treatment is influenced by the patient's perception of how effective they think the treatment will be, so a result might be seen even if the treatment is ineffectual.

Example 7

A study found that when doing painful dental tooth extractions, patients told they were receiving a strong painkiller while actually receiving a saltwater injection found as much pain relief as patients receiving a dose of morphine.[1]

To control for the placebo effect, a **placebo**, or dummy treatment, is often given to the control group. This way, both groups are truly identical except for the specific treatment given.

Definition: Placebo and Placebo-controlled experiments

A **placebo** is a dummy treatment given to control for the placebo effect. An experiment that gives the control group a placebo is called a **placebo controlled experiment**.

Example 8

a. In a study for a new medicine that is dispensed in a pill form, a sugar pill could be used as a placebo.
b. In a study on the effect of alcohol on memory, a non-alcoholic beer might be given to the control group as a placebo.
c. In a study of a frozen meal diet plan, the treatment group would receive the diet food, and the control could be given standard frozen meals stripped of their original packaging.

In some cases, it is more appropriate to compare to a conventional treatment than a placebo. For example, in a cancer research study, it would not be ethical to deny any treatment to the control group or to give a placebo treatment. In this case, the currently acceptable medicine would be given to the control group, sometimes called a **comparison group** in this case. In our SAT test example, the non-treatment group would most likely be encouraged to study on their own, rather than be asked to not study at all, to provide a meaningful comparison.

When using a placebo, it would defeat the purpose if the participant knew they were receiving the placebo.

Definition: Blind studies

A **blind study** is one in which the participant does not know whether or not they are receiving the treatment or a placebo. A **double-blind study** is one in which those interacting with the participants don't know who is in the treatment group and who is in the control group.

Example 9

In a study about anti-depression medicine, you would not want the psychological evaluator to know whether the patient is in the treatment or control group either, as it might influence their evaluation, so the experiment should be conducted as a double-blind study.

It should be noted that not every experiment needs a control group.

Example 10

If a researcher is testing whether a new fabric can withstand fire, she simply needs to torch multiple samples of the fabric – there is no need for a control group.

Try it Now 2

To test a new lie detector, two groups of subjects are given the new test. One group is asked to answer all the questions truthfully, and the second group is asked to lie on one set of questions. The person administering the lie detector test does not know what group each subject is in.

Does this experiment have a control group? Is it blind, double-blind, or neither?

Answer

The truth-telling group could be considered the control group, but really both groups are treatment groups here, since it is important for the lie detector to be able to correctly identify lies, and also not identify truth telling as lying. This study is blind, since the person running the test does not know what group each subject is in.

---

[1] Levine JD, Gordon NC, Smith R, Fields HL. (1981) Analgesic responses to morphine and placebo in individuals with postoperative pain. Pain. 10:379-89.

This page titled 5.6: Experiments is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **10.6: Experiments** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.6.1: Exercises

1. Does this describe an observational study or an experiment: The temperature on randomly selected days throughout the year was measured.
2. Does this describe an observational study or an experiment? A group of students are told to listen to music while taking a test and their results are compared to a group not listening to music.
3. A teacher wishes to know whether the males in their class have more conservative attitudes than the females. A questionnaire is distributed assessing attitudes.
   a. Is this a sampling or a census?
   b. Is this an observational study or an experiment?
   c. Are there any possible sources of bias in this study?
4. A study is conducted to determine whether people learn better with spaced or massed practice. Subjects volunteer from an introductory psychology class. At the beginning of the semester 12 subjects volunteer and are assigned to the massed-practice group. At the end of the semester 12 subjects volunteer and are assigned to the spaced-practice condition.
   a. Is this a sampling or a census?
   b. Is this an observational study or an experiment?
   c. This study involves two kinds of non-random sampling: (1) Subjects are not randomly sampled from some specified population and (2) Subjects are not randomly assigned to groups. Which problem is more serious? What affect on the results does each have?
5. A team of researchers is testing the effectiveness of a new HPV vaccine. They randomly divide the subjects into two groups. Group 1 receives new HPV vaccine, and Group 2 receives the existing HPV vaccine. The patients in the study do not know which group they are in.
   a. Which is the treatment group?
   b. Which is the control group (if there is one)?
   c. Is this study blind, double-blind, or neither?
6. For the clinical trials of a weight loss drug containing *Garcinia cambogia t*he subjects were randomly divided into two groups. The first received an inert pill along with an exercise and diet plan, while the second received the test medicine along with the same exercise and diet plan. The patients do not know which group they are in, nor do the fitness and nutrition advisors.
   a. Which is the treatment group?
   b. Which is the control group (if there is one)?
   c. Is this study blind, double-blind, or neither?
7. A farmer believes that playing Barry Manilow songs to his peas will increase their yield. Describe a controlled experiment the farmer could use to test his theory.
8. A sports psychologist believes that people are more likely to be extroverted as adults if they played team sports as children. Describe two possible studies to test this theory. Design one as an observational study and the other as an experiment. Which is more practical?
9. Find a newspaper or magazine article, or the online equivalent, describing the results of a recent study (the results of a poll are not sufficient). Give a summary of the study's findings, then analyze whether the article provided enough information to determine the validity of the conclusions. If not, produce a list of things that are missing from the article that would help you determine the validity of the study. Look for the things discussed in the text: population, sample, randomness, blind, control, placebos, etc.

This page titled 5.6.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **10.7: Exercise** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.7: Presenting Data Graphically

5.7 Learning Objectives

- Create a frequency table from a data set
- Create bar graphs, pareto charts, pie charts, histograms and frequency polygons from data sets
- Describe the shape of a histogram

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table.**

Definition: Frequency Table

A **frequency table** is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

Example 1

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. The data is summarized in the frequency table below.

Color	Frequency
Blue	25
Green	52
Red	41
White	36
Black	39
Grey	23

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. In this section we will work with bar graphs that display categorical data; the next section will be devoted to bar graphs that display quantitative data.

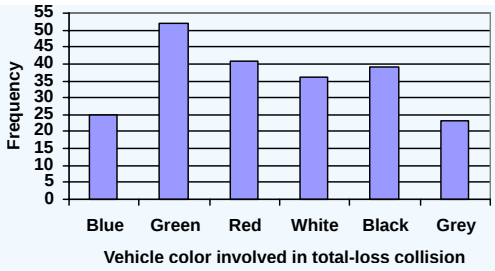
Definition: Bar Graph

A **bar graph** is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

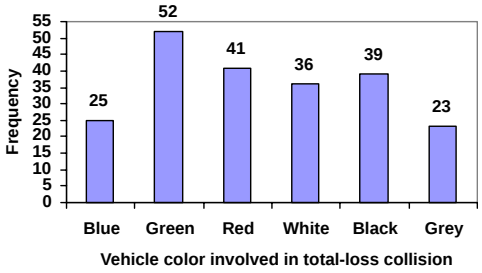
To construct a bar graph, we need to draw a vertical axis and a horizontal axis. The vertical direction will have a scale and measure the frequency of each category; the horizontal axis has no scale in this instance. The construction of a bar chart is most easily described by use of an example.

Example 2

Using our car data from above, note the highest frequency is 52, so our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units:



Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal gridlines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the gridlines will already be in place, or using technology. Instead of gridlines, we might also list the frequencies at the top of each bar, like this:



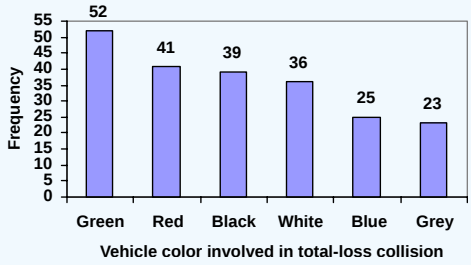
In this case, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart, even without gridlines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart**.

Definition: Pareto Chart

A **Pareto chart** is a bar graph ordered from highest to lowest frequency.

Example 3

Transforming our bar graph from earlier into a Pareto chart, we get:



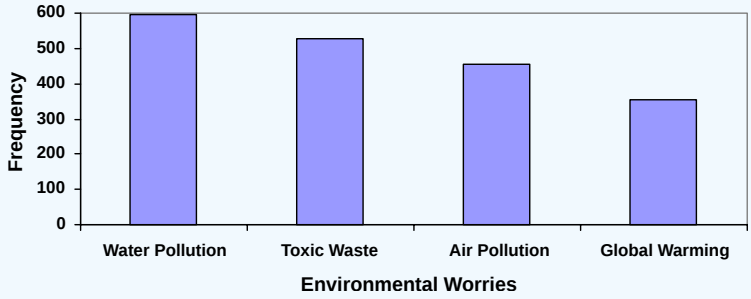
Example 4

In a survey[1], adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of 1012 surveyed) who indicated that they worried "a great deal" about some selected concerns are summarized below.

Environmental Issue	Frequency
Pollution of drinking water	597
Contamination of soil and water by toxic waste	526
Air pollution	455
Global warming	354

Solution

This data could be shown graphically in a bar graph:



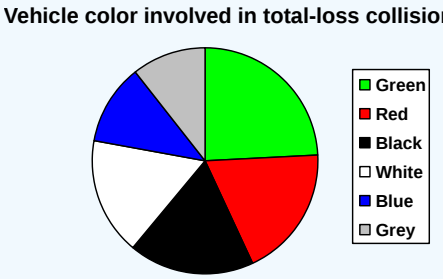
To show relative sizes, it is common to use a pie chart.

Definition: Pie Chart

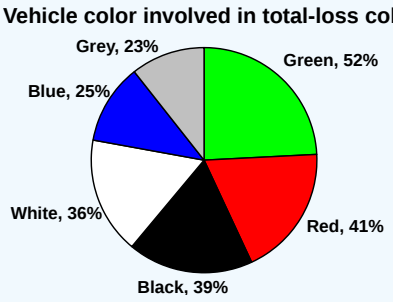
A **pie chart** is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies of the categories.

Example 5

For our vehicle color data, a pie chart might look like this:



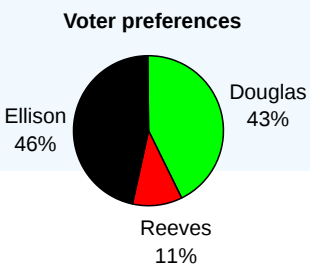
Pie charts can often benefit from including frequencies or relative frequencies (percents) in the chart next to the pie slices. Often having the category names next to the pie slices also makes the chart clearer.



Example 6

The pie chart to the right shows the percentage of voters supporting each candidate running for a local senate seat.

If there are 20,000 voters in the district, the pie chart shows that about 11% of those, about 2,200 voters, support Reeves.



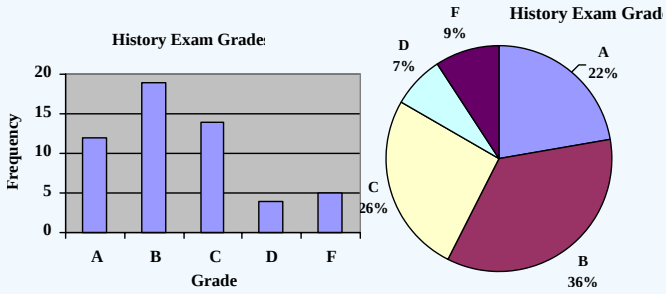
Pie charts look nice, but are harder to draw by hand than bar charts since to draw them accurately we would need to compute the angle each wedge cuts out of the circle, then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org Write or Calc, or Google Docs are able to create bar graphs, pie charts, and other graph types. There are also numerous online tools that can create graphs[2].

Try it Now 1

Create a bar graph and a pie chart to illustrate the grades on a history exam below.

A: 12 students, B: 19 students, C: 14 students, D: 4 students, F: 5 students

Answer



Quantitative, or numerical, data can also be summarized into frequency tables.

Example 7

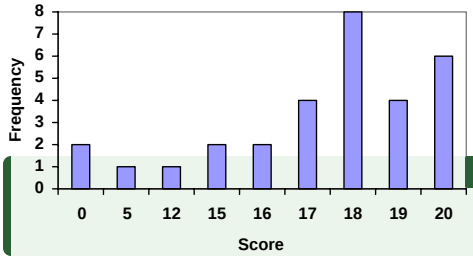
A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

These scores could be summarized into a frequency table by grouping like values:

Score	Frequency
0	2
5	1
12	1
15	2
16	2
17	4
18	8
19	4
20	6

Using this table, it would be possible to create a standard bar chart from this summary, like we did for categorical data:



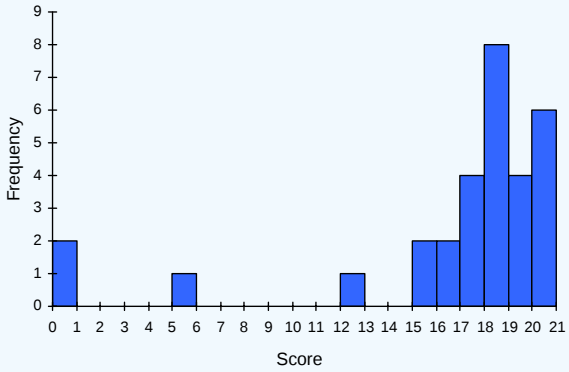
However, since the scores are numerical values, this chart doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be more correct to treat the horizontal axis as a number line. This type of graph is called a **histogram**.

Definition: Histogram

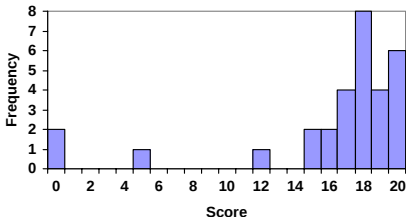
A histogram is like a bar graph, but where the horizontal axis is a number line.

Example 8

For the values above, a histogram would look like:



Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. Some people choose to have bars start at ½ values to avoid this ambiguity.



Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table, and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals** or **bins**.

Definition: Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that:

- Each interval is equal in size. For example, if the first class contains values from 120-129, the second class should include values from 130-139.
- We have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.
- The class interval width is the lowest value in one class minus the lowest value in the previous class.

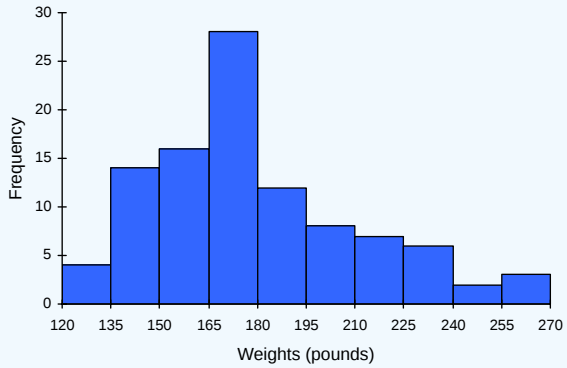
Example 9

Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a range of 263-121 = 142. We could create 7 intervals with a width of around 20, 14 intervals with a width of around 10, or somewhere in between. Often time we have to experiment with a few possibilities to find something that represents the data well.

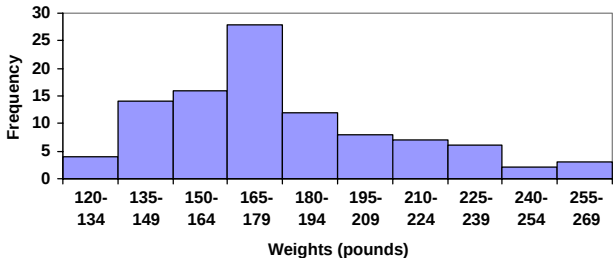
Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number. With a class width of 15, the next class begins at 120 + 15 or 135, the following one at 135 + 15 or 150, and so on until there are classes for all the data.

Interval	Frequency
$120-134$	4
$135-149$	14
$150-164$	16
$165-179$	28
$180-194$	12
$195-209$	8
$210-224$	7
$225-239$	6
$240-254$	2
$255-269$	3

A histogram of this data would look like:

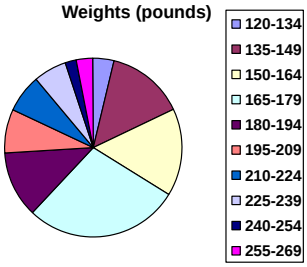


In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.



Histograms also display something very important called **shape**. In this particular example above, you can see a majority of the data is the the left, and it gets smaller and smaller the more you go to the right. We consider this the tail-end to the right and therefore call this **skewed to the right**. If a majority of the data was on the right and its tail-end was to the left, we would call this **skewed to the left**. If there was approximately equally distributed data to both the right and left, we would call this **symmetric**. If all rectangles in a histogram had the same height, we would call this **uniform**.

Other graph types such as pie charts are possible for quantitative data. The usefulness of different graph types will vary depending upon the number of intervals and the type of data being represented. For example, a pie chart of our weight data is difficult to read because of the quantity of intervals we used.



Try it Now 3

The total cost of textbooks for the term was collected from 36 students. Create a histogram for this data.

$140 $160 $160 $165 $180 $220 $235 $240 $250 $260 $280 $285

$285 $285 $290 $300 $300 $305 $310 $310 $315 $315 $320 $320

$330 $340 $345 $350 $355 $360 $360 $380 $395 $420 $460 $460

Answer

Using a class intervals of size 55, we can group our data into six intervals:

cost interval	Frequency
$140 – 194	5
$195 – 249	3
$250 – 304	9
$305 – 359	12
$360 – 414	4
$415 – 469	3

We can use the frequency distribution to generate the histogram.

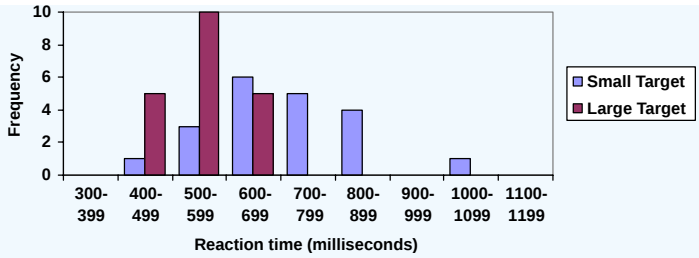
When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

Example 10

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

Interval (milliseconds)	Frequency small target	Frequency large target
300 – 399	0	0
400 – 499	1	5
500 – 599	3	10
600 – 699	6	5
700 – 799	5	0
800 – 899	4	0
900 – 999	0	0
1000 – 1099	1	0
1100 – 1199	0	0

One option to represent this data would be a comparative histogram or bar chart, in which bars for the small target group and large target group are placed next to each other.

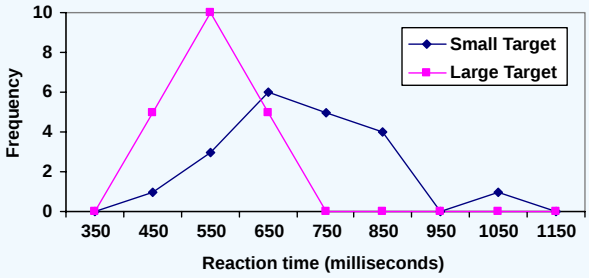


Definition: Fequency Polygon

An alternative representation is a **frequency polygon**. A frequency polygon starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval at height equal to the frequency. Typically the points are connected with straight lines to emphasize the distribution of the data.

Example 11

This graph makes it easier to see that reaction times were generally shorter for the larger target, and that the reaction times for the smaller target were more spread out.



[1] Gallup Poll. March 5-8, 2009. http://www.pollingreport.com/enviro.htm

[2] For example: http://nces.ed.gov/nceskids/createAgraph/ or http://docs.google.com

[3]CNN/Opinion Research Corporation Poll. Dec 19-21, 2008, from http://www.pollingreport.com/civil.htm

This page titled 5.7: Presenting Data Graphically is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **11.2: Presenting Categorical Data Graphically** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.
- **11.3: Presenting Quantitative Data Graphically** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.7.1: Exercises

1. The table below shows scores on a Math test.

80	50	50	90	70	70	100	60	70	80	70	50
90	100	80	70	30	80	80	70	100	60	60	50

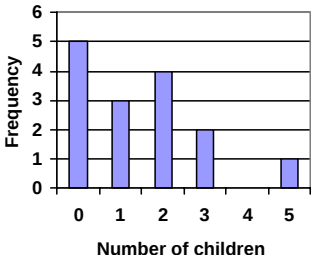
   a. Complete a frequency table for the Math test scores.
   b. Construct a bar graph of the data. Label both axes.
   c. Construct a pie chart of the data. Label the sectors.

2. A group of adults where asked how many cars they had in their household. Their responses are listed below.

1	4	2	2	1	2	3	3	1	4	2	2
1	2	1	3	2	2	1	2	1	1	1	2

   a. Complete a frequency table for the number of cars data.
   b. Construct a bar graph of the data. Label the axes.
   c. Construct a Pareto chart of the data. Label the axes.
   d. Construct a pie chart of the data. Label the sectors.

3. A group of adults were asked how many children they have in their families. The bar graph to the right shows the number of adults who indicated each number of children.



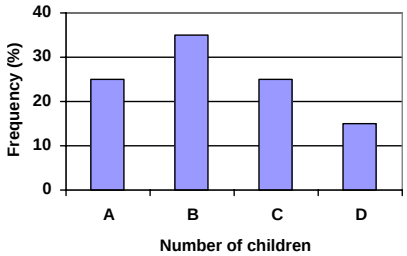
   a. How many adults where questioned?
   b. What percentage of the adults questioned had 0 children?
   c. Create a pie graph for the number of children in these families. Label the sectors.

4. Jasmine was interested in how many days it would take an order from Amazon to arrive at her door. The graph below shows the data she collected.



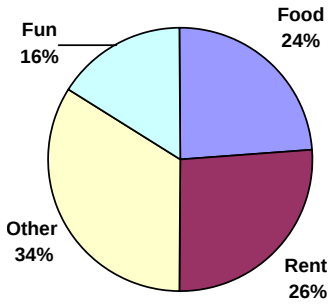
   a. How many items did she order?
   b. What percentage of the items arrived in one day?
   c. Create a pie graph for her data. Label the sectors.

5. The bar graph below shows the *percentage* of students who received each letter grade on their last English paper. The class contains 20 students. What number of students earned an A on their paper?

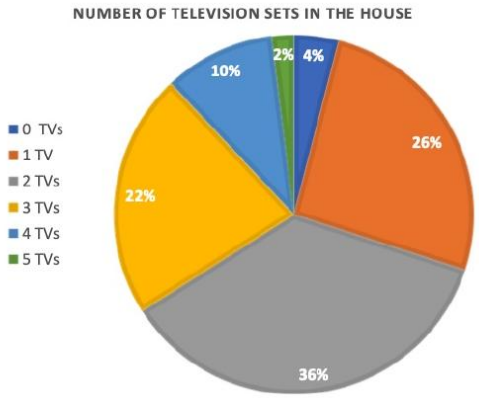


   a. How many students earned an A on their papers?
   b. Create a frequency bar graph for these data. How do the two bar graphs compare?

6. Kori categorized her spending for this month into four categories: Rent, Food, Fun, and Other. The percents she spent in each category are pictured here. If she spent a total of $2600 this month, how much did she spend on rent?



7. 50 people were asked about the number of television sets in their houses. The pie chart below shows their responses.



a. How many television sets did the largest number of people have?
b. How many television sets did the fewest number of people have?
c. How many people had no television sets?
d. How many people had at least 3 television sets?
e. What percent of the people surveyed had at most 2 television sets?
f. How many people had not more than 2 television sets?

8. In 2018, 487,890 people in California graduated from college with various degrees. The numbers are shown in the chart below.

Associate's Degree	170,890
Bachelor's Degree	216,810
Master's Degree	80,480
Doctorate or Professional Degree	19,710

a. Is the data categorical or numerical?
b. Which type(s) of graphs would be appropriate way(s) to display this information? Bar Graph, Pareto Chart, Pie Chart, Frequency Histogram, or Frequency Polygon

9. The ages of the U.S.Presidents who were inaugurated in the 20th century are given.

42; 51; 56; 55; 51; 54; 51; 60; 62; 43; 55; 56; 61; 52; 69; 64; 46

a. Create a grouped frequency table for these scores starting at 40 with a class width of 10.
b. Create a frequency histogram for these scores.
c. Describe the shape of the frequency histogram.

10. Results for 40 students on a recent math exam are given.

83; 76; 98; 75; 56; 47; 67; 92; 69; 74; 72; 84; 96; 66; 51; 64; 58; 80; 85; 78; 58; 88; 90; 76; 74; 82; 76; 66; 50; 84; 66; 79; 83; 43; 54; 93; 77; 67; 54; 76

a. Create a grouped frequency table for these scores starting at 40 with a class width of 10.
b. Create a frequency histogram for these scores.

c. Describe the shape of the frequency histogram.
d. Make a frequency polygon for this data.

11. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. A researcher follows a set of 40 AIDS patients from the start of treatment until their deaths. The following data (in months) are collected.

    3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34

    a. Create a frequency table starting at 0 with a class width of 6 months.
    b. What percent of the AIDS patients in the study survived at least 3 years after beginning treatment?
    c. What percent of the AIDS patients in the study survived less than 18 months?
    d. Create a frequency histogram using the above data. Label the axes.

12. As part of a study, the weights of 42 medium-sized dogs were measured in pounds. The results are shown below.

26.1	16.2	21.1	34.7	45.3	18.5	41.6
38.8	22.9	48.1	27.1	22.4	30.6	39.1
62.8	25.1	25.0	38.6	29.9	31.7	28.9
20.3	56.1	60.5	24.0	61.7	28.6	32.9
33.5	18.0	23.5	27.9	46.8	30.0	34.3
62.2	49.3	59.7	19.6	20.8	23.2	24.4

    a. Create a frequency table using classes. Start at 16 and use a class width of 8.
    b. What percent of the dogs weigh no more than 24 pounds?
    c. What percent of the dogs studied weigh at least 48 pounds?
    d. What percent of the dogs studied weigh at least 24 pounds, but do not exceed 40 pounds?

13. People were asked how long they waited in line to ride a popular attraction at a local theme park. The frequency table below shows the results.

x = Time (minutes)	Frequency
$0 \le x < 5$	32
$5 \le x < 10$	47
$10 \le x < 15$	36
$15 \le x < 20$	22
$20 \le x < 25$	13
$25 \le x < 30$	10

    a. Construct a histogram to represent the data. Label the axes of your histogram.
    b. Describe the shape of the histogram.

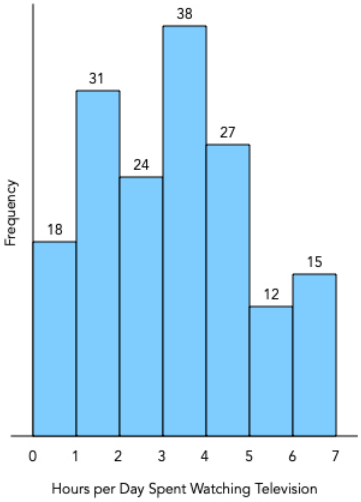
14. Hourly wages of workers at a local company are shown in the frequency table below.

x = Hourly Wage (dollars)	Frequency
$13.00 \le x < 14.50$	21
$14.50 \le x < 16.00$	35
$16.00 \le x < 17.50$	42
$17.50 \le x < 19.00$	27
$19.00 \le x < 20.50$	18

$20.50 \le x < 22.00$	9

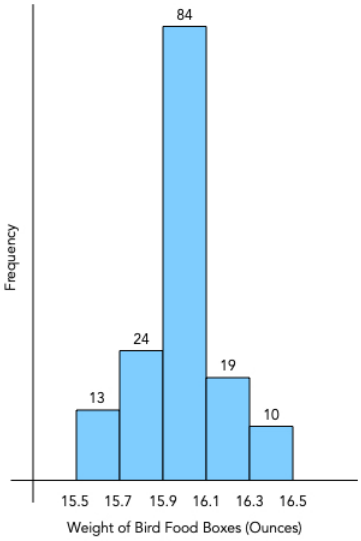
a. Construct a histogram to represent the data. Label the axes of your histogram.
b. Describe the shape of the histogram.

15. The frequency histogram below shows the number of hours per day a random sample of teenagers spent watching television.



a. How many teenagers were included in the sample?
b. What percent of the teenagers in the sample spent less than 4 hours watching television?
c. What percent of the teenagers in the sample spent at least 5 hours watching television?
d. What percent of the teenagers in the sample spent at least 1 hour watching television?
e. What percent of the teenagers in the sample spent less than 2 hours watching television?
f. What percent of the teenagers in the sample spent at least 2 hours but less than 4 hours watching television?
g. What percent of the teenagers in the sample spent more than 3.5 hours watching television?

16. A pet food company is testing the accuracy of the machine that fills its boxes with bird food. The weights of the sample boxes are shown in the frequency histogram.



a. How many boxes did the company test?
b. How many of the boxes tested weighed at least 16.1 ounces?
c. What percent of the boxes tested weighed less than 15.9 ounces?
d. What percent of the boxes tested weighed within 0.1 ounces of the 16 ounce weight stated on the box? That is,$15.9 \le w < 16.1$ where w is the weight of the box.

e. If boxes weighing within 0.1 ounces of the 16-ounce stated weight are acceptable, are there more underweight boxes or overweight boxes among the boxes that do not meet the weight standard?

This page titled 5.7.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **11.7: Exercises** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

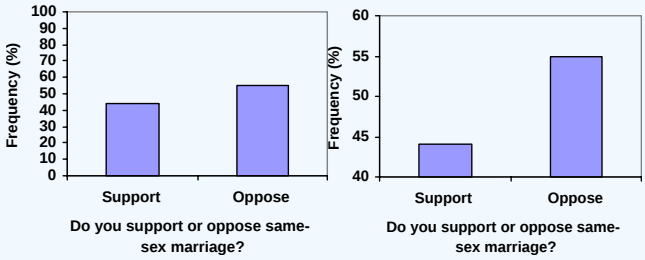
5.8: Reading Graphs in Media

Don't get fancy with graphs! People sometimes add features to graphs that don't help to convey their information. For example, 3-dimensional bar charts are usually not as effective as their two-dimensional counterparts.

Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

Example 1

Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December 2008[3]. The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages; the second graph makes it look like twice as many people oppose marriage rights as support it.



Here is another way that fanciness can lead to trouble. Instead of plain bars, it is tempting to substitute meaningful images. This type of graph is called a **pictogram**.

Pictogram

A **pictogram** is a statistical graphic in which the size of the picture is intended to represent the frequencies or size of the values being represented.

Example

A labor union might produce the graph to the right to show the difference between the average manager salary and the average worker salary.

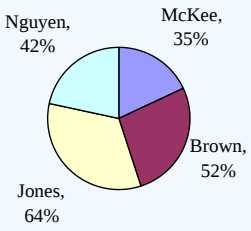
Looking at the picture, it would be reasonable to guess that the manager salaries is 4 times as large as the worker salaries – the area of the bag looks about 4 times as large. However, the manager salaries are in fact only twice as large as worker salaries, which were reflected in the picture by making the manager bag twice as tall.



Manager Salaries **Worker Salaries**

Try it Now 2

A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart present a good representation of this data? Explain.



Answer

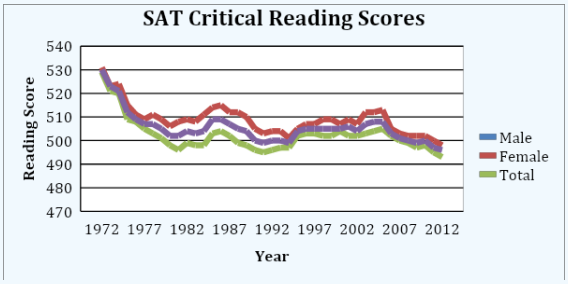
While the pie chart depicts the relative size of the people agreeing with each candidate as compared to the other candidates, the chart is confusing. Percents on a pie chart usually represent the percentage out of 100% that the slice represents.

There are many other types of graphs you will encounter in the media. The following example is a **multiple line graph**.

Example 3

The average score for SAT critical reading has been going down over the years. You can also see that the average score for male students is higher than the average score for female students for all of the years. Also, the average scores for male and female used to be closer to each other, then they separated fairly far, and look to be getting closer to each other again. What do you think could be a reason for this?

Multiple Line Graph for SAT Critical Reading Scores in AZ



(College Board: Arizona, 2012)

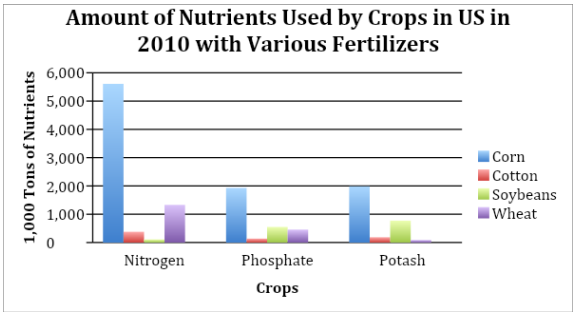
Solution

In this case the vertical axis did not start at zero, because if it did start at zero, the different lines would be very close together and difficult to see. When at all possible, the scale on the vertical axis should start at zero. Always carefully consider that if a vertical scale of a graph does not start at zero, the researchers could be attempting to exaggerate insignificant differences in the data.

Multiple Bar Graphs:

Sometimes you have information for multiple variables and instead of putting the information on different bar graphs, you can put them all on one so that you can compare the variables. The following is an example of where you might use this. The data is the amount of nutrients used by crops with various fertilizers. By creating a graph that has all of the fertilizers and all of the crops, you can see that corn with nitrogen uses the most nutrients, and soybeans with nitrogen uses the least amount of nutrients. You can also see that phosphate seems to use low amounts of nutrients for all of its crops. So, a multiple bar graph is useful to make all of these observations.

Multiple Bar Graph for the Amount of Nutrients Used by Crops: 2010

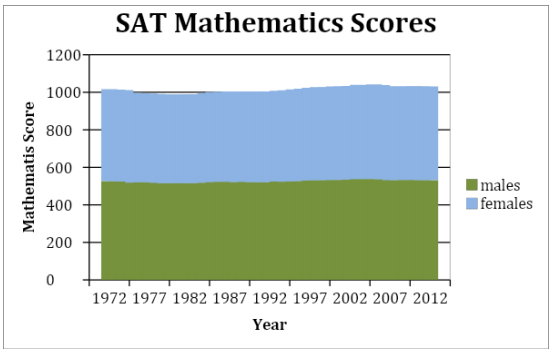


(United States Department of Agriculture [USDA], 2010)

Stack Plots:

A stack plot is basically a multiple bar graph, but with the bars separated (or stacked) on top of each other instead of next to each other. This can be useful when it is difficult to interpret a multiple bar graph since the bar heights are so close to one another. To read a given bar on a stack plot, you must subtract the height of the bar from the height of the bar below it. In the example of a stack plot below, you can see that the SAT Mathematics score for males in 1972 is about 530 while the SAT Mathematics score for females in 1972 is about 1010 – 530 = 480.

Stack Plot of SAT Mathematics Scores



(College Board: Arizona, 2012)

Geographical Graphs:

Weather maps, topographic maps, population distribution maps, gravity maps, and vegetation maps are examples of geographical graphs. They allow you to see a trend of information over a geographic area. The following is an example of a weather map showing temperatures. As you can see, the different colors represent certain temperature ranges. From this graph, you can see that on this date, the red in the south means that the temperature was in the 80s and 90s there, and the blue in the Rockies area means that the temperature was in the 40s there.

Geographical Graph

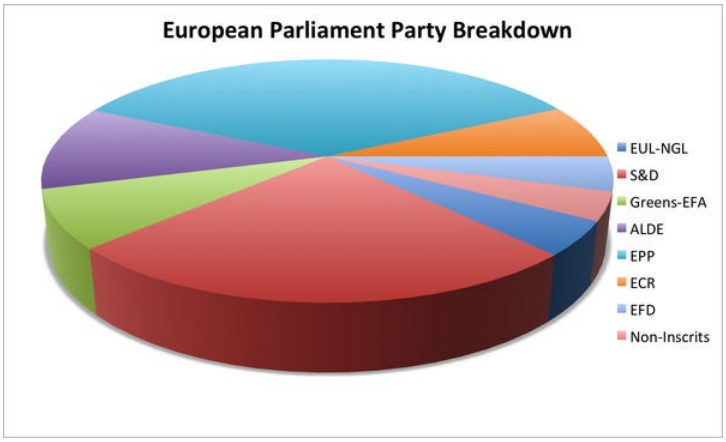


(Weather Channel, 2013)

Three-Dimensional Graphics

Some people like to show graphs in three-dimensions. This type of graph can be misleading when the media presents it this way. The following is misleading because the red party looks almost as equivalent to the teal party, which is not true.

Three-Dimensional Graph

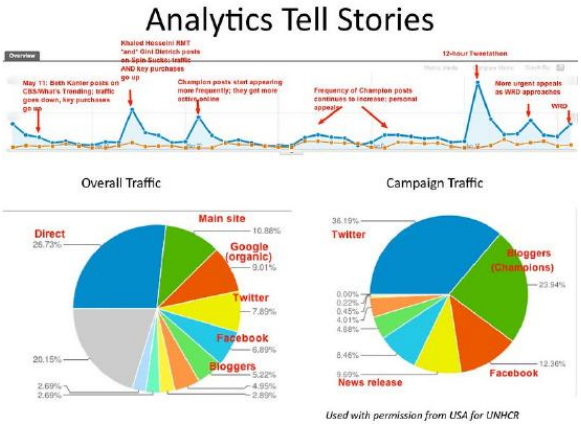


(Business Insider, 2013)

Combination Graphics

Some graphs are created so that they combine the data table and a graph or combine two types of graphs in one. The advantage is that you can see a graphical representation of the data, and still have the data to find exact values. The disadvantage is that they are busier, and usually people show graphical representation of data because people do not like looking at the data.

Combination Graph



Used with permission from USA for UNHCR

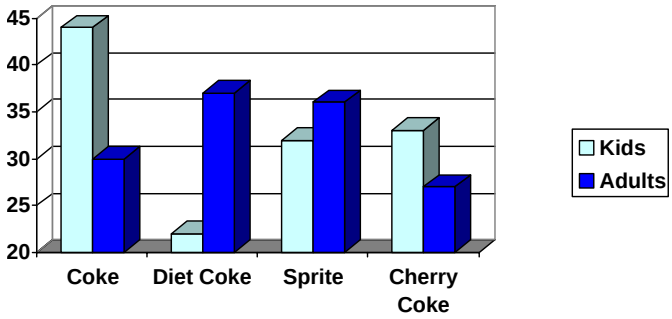
These are just a few of the different types of graphs that exist in the world. There are many other ones. A quick Google search on statistical graphs will show you many more. Just open up a newspaper, magazine, or website and you are likely to see others. The most important thing to remember is that you need to look at the graph objectively, and interpret for yourself what it says. Also, do not read any cause and effect into what you see which will be discussed in more depth in the next section.

This page titled 5.8: Reading Graphs in Media is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **11.2: Presenting Categorical Data Graphically** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.
- **1.6: Graphics in the Media** by Maxie Inigo, Jennifer Jameson, Kathryn Kozak, Maya Lanzetta, & Kim Sonier is licensed CC BY-SA 4.0. Original source: https://www.coconino.edu/open-source-textbooks#college-mathematics-for-everyday-life-by-inigo-jameson-kozak-lanzetta-and-sonier.

5.8.1: Exercises

1. A graph appears below showing the number of adults and children who prefer each type of soda. There were 130 adults and kids surveyed. Discuss some ways in which the graph below could be improved



This page titled 5.8.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **11.7: Exercises** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.9: Correlation and Causation, Scatter Plots

5.9 Learning Objectives

- Create a scatter plot from bi-variate data
- Describe the type of linear correlation in a scatter plot
- Explain why correlation is not the same as causation

The label on a can of Planters Cocktail Peanuts says, “Scientific evidence suggest but does not prove that eating 1.5 ounces per day of most nuts, such as peanuts, as part of a diet low in saturated fat and cholesterol & not resulting in increased caloric intake may reduce the risk of heart disease. See nutritional information for fat content (1.5 oz. is about 53 pieces).” Why is it written this way and what does this statement mean? There are many studies that exist that show that two variables are related to one another. The strength of a relationship between two variables is called **correlation**. Variables that are strongly related to each other have strong correlation. However, if two variables are correlated it does not mean that one variable caused the other variable to occur. The above example from the Planters Cocktail Peanuts label is an example of this. There is a strong correlation between eating a diet that is low in saturated fat and cholesterol and heart disease. But that correlation does not mean that eating a diet that is low in saturated fat and cholesterol will cause your risk of heart disease to go down. There could be many different variables that could cause both variables in question to go down or up. One example is that a person’s genetic makeup could make them not want to eat fatty food and also not develop heart disease. No matter how strong a correlation is between two variables, you can never know for sure if one variable causes the other variable to occur without conducting experimentation. The only way to find out if eating a diet low in saturated fat and cholesterol actually lowers the risk of heart disease is to do an experiment. This is where you tell one group of people that they have to eat a diet low in saturated fat and cholesterol and another group of people that they have to eat a diet high in saturated fat and cholesterol, and then observe what happens to both groups over the years. You cannot morally do this experiment, so there is no way to prove the statement. That is why the word “may” is in the statement. We see many correlations like this one. Always be sure not to make a correlation statement into a causation statement.

Example 5.9.1: Correlation vs Causation

For each of the following scenarios answer the question and give an example of another variable that could explain the correlation.

1. There is a negative correlation between number of children a woman has and her life expectancy. Does that mean that having children causes a woman to die earlier?

A correlation between two variables does not mean that one causes the other. A possible cause for both variables could be better health care. If there is better health care, then life expectancy goes up, and also with better health care birth control is more readily available.

2. There is a positive correlation between ice cream sales and the number of drownings at the beach. Does that mean that eating ice cream can cause a person to drown?

A correlation between two variables does not mean that one causes the other. The cause for both could be that the temperature is going up. The higher the temperature, the more likely someone will buy ice cream and the more people at the beach.

3. There is a correlation between waist measures and wrist measures. Does this mean that your waist measurement causes your wrist measurement to change?

A correlation between two variables does not mean that one causes the other. The cause of both could be a person’s genetics, eating habits, exercise habits, etc.

How do we tell if there is a correlation between two variables? The easiest way is to graph the two variables together as ordered pairs on a graph called a **scatter plot**. To create a scatter plot, consider that one variable is the independent variable and the other is the dependent variable. This means that the dependent variable depends on the independent variable. We usually set up these two variables as ordered pairs where the independent variable is first and the dependent variable is second. Thus, when graphed, the independent variable is graphed along the horizontal axis and the dependent variable is graphed along the vertical axis. You do not connect the dots after plotting these ordered pairs. Instead look to see if there is a pattern, such as a line, that fits the data well. Here are some examples of scatter plots and how strong the linear correlation is between the two variables.

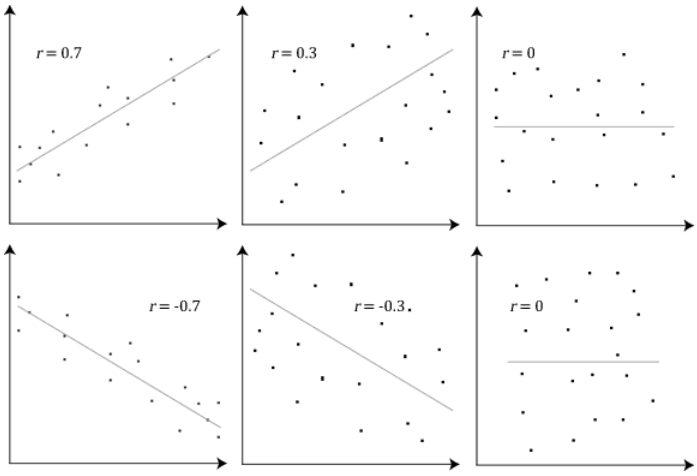


Figure 5.9.1: Scatter Plots Showing Types of Linear Correlation

Creating a scatter plot is not difficult. Just make sure that you set up your axes with scaling before you start to plot the ordered pairs.

Example 5.9.2: Creating a Scatter Plot

Data has been collected on the life expectancy and the fertility rate in different countries ("World health rankings," 2013). A random sample of 10 countries was taken, and the data is:

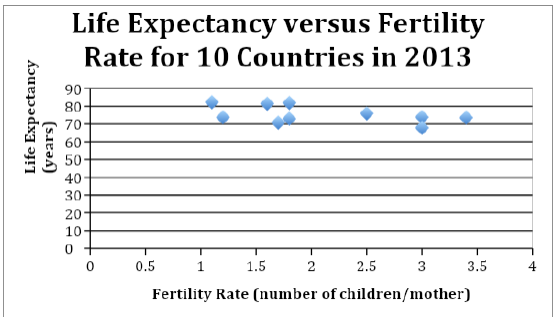
Table 5.9.2: Life Expectancy and Fertility Rate in 2013

Country	Life Expectancy (years)	Fertility Rate (number of children per mother)
SINGAPORE	82.3	1.1
MONACO	81.9	1.8
CANADA	81.5	1.6
ECUADOR	76	2.5
MALAYSIA	73.9	3
LITHUANIA	73.8	1.2
BELIZE	73.6	3.4
ALGERIA	73	1.8
TRINIDAD/TOB.	70.8	1.7
TAJIKISTAN	67.9	3

To make the scatter plot, you have to decide which variable is the independent variable and which one is the dependent variable. Sometimes it is obvious which variable is which, and in some case it does not seem to be obvious. In this case, it seems to make more sense to predict what the life expectancy is doing based on fertility rate, so choose life expectancy to be the dependent variable and fertility rate to be the independent variable. The horizontal axis needs to encompass 1.1 to 3.4, so have it range from zero to four, with tick marks every one unit. The vertical axis needs to encompass the numbers 70.8 to 81.9, so have it range from zero to 90, and have tick marks every 10 units.

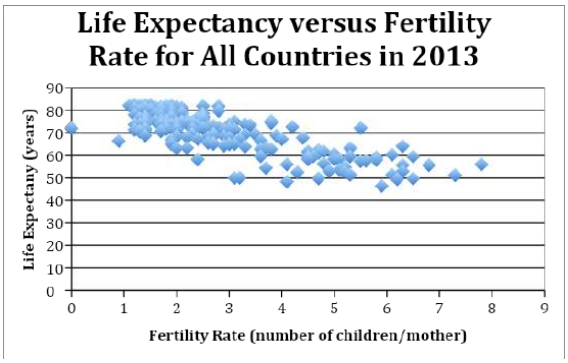
Note: Always start the vertical axis at zero to avoid exaggeration of the data.

Graph 2.5.3: Scatter Plot of Life Expectancy versus Fertility Rate



From the graph, you can see that there is somewhat of a downward trend, but it is not prominent. What this says is that as fertility rate increases, life expectancy decreases. The trend is not strong which could be due to not having enough data or this could represent the actual relationship between these two variables. Let's see what the scatter plot looks like with data from all countries in 2013 ("World health rankings," 2013).

Graph 2.5.4: Scatter Plot of Life Expectancy versus Fertility Rate for All Countries in 2013



Again, there is a downward trend. It looks a little stronger than the previous scatter plot and the trend looks more obvious. This correlation would probably be considered moderate negative correlation. It appears that there is a trend that the higher the fertility rate, the lower the life expectancy. Caution: just because there is a correlation between higher fertility rate and lower life expectancy, do not assume that having fewer children will mean that a person lives longer. The fertility rate does not necessarily cause the life expectancy to change. There are many other factors that could influence both, such as medical care and education. Remember a correlation does not imply causation.

This page titled 5.9: Correlation and Causation, Scatter Plots is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **2.5: Correlation and Causation, Scatter Plots** by Maxie Inigo, Jennifer Jameson, Kathryn Kozak, Maya Lanzetta, & Kim Sonier is licensed CC BY-SA 4.0. Original source: https://www.coconino.edu/open-source-textbooks#college-mathematics-for-everyday-life-by-inigo-jameson-kozak-lanzetta-and-sonier.

5.9.1: Exercises

The table shows the GPA of 15 randomly selected students and the number of hours spent on studying their lessons before an exam. Use this information for problems 1 - 4.

Student	Number of Hours	GPA
1	3	2.5
2	3	2.4
3	6	3.5
4	4	3
5	5	3.2
6	6	3.8
7	3	2.7
8	4	2.6
9	2	2.8
10	5	3.4
11	5	3
12	4	3.3
13	5	3
14	4	2.7
15	8	4

1. Draw a scatter plot of the data. Label the axes.
2. Based on the information provided, which best describes the relationship between students' GPA and study habits?
   a. Negative
   b. Positive
   c. No correlation
3. Based on the scatter plot, which statement accurately describes the information provided?
   a. The correlation is positive. As the number of hours spent on studying increases, the students' GPA tends to increase.
   b. The correlation is negative. As the number of hours spent on studying increases, the students' GPA tends to increase.
   c. The correlation is negative. As the number of hours spent on studying decreases, the students' GPA tends to increase.
   d. The data show no relationship between students' GPA and study habits.
4. Do the data imply that the length of time spent on studying before an exam causes the change in students' GPA? Why or why not?
   a. Yes. Since hours spent on studying and GPA are correlated, it means the time spent on studying causes the GPA to increase.
   b. No. The increase in GPA could be attributed to other factors such as gender, class size, socioeconomic status, faculty qualification, and many others.
   c. No. The two variables are not correlated. Hence, the increase in time spent on studying does not affect the GPA.
   d. Yes. There is sufficient evidence to support the effect of study habits on students' GPA.
5. If there is a strong correlation between marijuana use and use of other drugs, can we conclude that using marijuana leads to using other drugs?

6. Describe whether the scatter plot of the following variables will show a positive correlation, a negative correlation, or no correlation.
   a. Height and weight
   b. Height and grade in a course
   c. Grade in a course and shoe size
   d. Hours of television watched and grade in a course
   e. Height and shoe size
   f. Hours of television watched and weight
7. The table below shows the number of students in a class and the absences in a week.

Class Size	15	18	20	20	25	25	30	35	35	40
Number of Absences	5	2	10	4	8	12	2	5	7	10

   a. Draw a scatter plot and determine the relationship that best describes the data.
   b. Is the correlation between class size and the number of absences positive, negative, or zero? Fill in the blank: As the class size increases, the number of absences _____.
8. The table below shows the recorded maximum average monthly temperature readings of a certain state in the US and the shipment of residential storage water heaters.

Month	Avg. Monthly Temperature °C	Shipment of water heaters (in units)
Jan	-1 °C	$384,213$
Feb	0 °C	$745,570$
Mar	9 °C	$1,162,074$
Apr	15 °C	$1,473,757$
May	22 °C	$1,806,054$
June	27 °C	$2,217,780$
July	30 °C	$2,610,644$
Aug	26 °C	$3,011,506$
Sep	24 °C	$3,377,667$
Oct	16 °C	$3,803,921$
Nov	12 °C	$4,162,667$
Dec	5 °C	$4,584,367$

   a. Draw a scatterplot.
   b. Describe the correlation that best describes the data provided.
9. In order to determine whether or not there is a relationship between the temperature and the chirp rate of a certain variety of cricket, the following data were collected. Let x = ˚C and y = # chirps per second.

x	31.4	22.0	34.1	29.1	27.0	24.0	20.9	27.8	20.8	28.5	26.4	28.1	27.9	28.6	24.5
y	20.0	16.0	19.8	18.4	17.1	15.5	14.7	17.1	15.4	16.2	15.0	17.2	16.0	17.0	14.4

   a. Draw a scatterplot of the data given.
   b. Is the correlation between temperature and chirp rate positive, negative, or zero?
   c. If the correlation is strong, can we conclude that temperature affects the chirp rate?
10. During the spring and summer, an ice cream shop kept track of the average daily temperature (in ˚F) and the amount of ice cream sales (in dollars).

Temperature (˚F)	52	55	61	66	72	75	77	84	90	94	97
Ice Cream Sales ($)	3500	4200	5000	5300	6600	6800	7200	8000	8400	9100	9500

   a. Create a scatterplot with this data.
   b. Use your scatterplot to make a prediction for the ice cream sales when the temperature is 70˚F.

c. If ice cream sales were $7500, what do you think the average temperature was that day?

This page titled 5.9.1: Exercises is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

5.10: Measures of Central Tendency

5.10 Learning Objectives

- Find the mean, median, and mode of a data set.

It is often desirable to use a few numbers to summarize a distribution. One important aspect of a distribution is where its center is located. Measures of central tendency are discussed here.

Let's begin by trying to find the most "typical" value of a data set.

Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average".

Mean

The **mean** of a set of data is the sum of the data values divided by the number of values.

Example 1

Marci's exam scores for her last math class were: 79, 86, 82, 94. The mean of these values would be:

Solution

$$\frac{79+86+82+94}{4} = 85.25.$$

Typically we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3.

Example 2

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20

20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Solution

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get $\frac{634}{31} = 20.4516$. It would be appropriate to round this to 20.5.

It would be most correct for us to report that "The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes," but it is not uncommon to see the more casual word "average" used in place of "mean.

Try it Now 1

The price of a jar of peanut butter at 5 stores was: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the mean price.

Answer

Add texts here. Do not delete this text first.

Example 3

The one hundred families in a particular neighborhood are asked their annual household income, to the nearest $5 thousand dollars. The results are summarized in a frequency table below.

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\frac{15+\cdots+15+20+\cdots+20+25+\cdots+25+\cdots}{100}$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as $15 \cdot 6 = 90$. Using this simplification, we get

$$\frac{15 \cdot 6+20 \cdot 8+25 \cdot 11+30 \cdot 17+35 \cdot 19+40 \cdot 20+45 \cdot 12+50 \cdot 7}{100} = \frac{3390}{100} = 33.9$$

The mean household income of our sample is 33.9 thousand dollars ($33,900).

Example 4

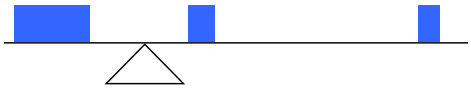
Extending the last example, suppose a new family moves into the neighborhood example that has a household income of $5 million ($5000 thousand). Adding this to our sample, our mean is now:

Solution

$$\frac{15 \cdot 6+20 \cdot 8+25 \cdot 11+30 \cdot 17+35 \cdot 19+40 \cdot 20+45 \cdot 12+50 \cdot 7+5000 \cdot 1}{101} = \frac{8390}{101} = 83.069$$

While 83.1 thousand dollars ($83,069) is the correct mean household income, it no longer represents a “typical” value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



If we graph our household data, the $5 million data value is so far out to the right that the mean has to adjust up to keep things in balance



For this reason, when working with data that have **outliers** – values far outside the primary grouping – it is common to use a different measure of center, the **median**.

Median

To find the **median**, begin by listing the data in order from smallest to largest, or largest to smallest.

If the number of data values, N, is odd, then the median is the middle data value. This value can be found by rounding $\frac{N}{2}$ up to the next whole number.

If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values $\frac{N}{2}$ and $\frac{N}{2}+1$)

Example 5

Returning to the football touchdown data, we would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20

20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Solution

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value (\frac{31}{2} = 15.5\), round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Example 6

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7

We start by listing the data in order: 2 4 5 5 6 7 7 8 8 10

Solution

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get $\frac{6+7}{2} = 6.5$.

The median quiz score was 6.5.

Try it Now 2

The price of a jar of peanut butter at 5 stores were: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the median price.

Answer

First we put the data in order: $3.29, $3.59, $3.75, $3.79, $3.99. Since there are an odd number of data, the median will be the middle value, $3.75.

Example 7

Let us return now to our original household income data

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values - the 50th and 51st data values. To find these, we start counting up from the bottom:

$$\begin{aligned}
&\text{There are 6 data values of \$15, so} && \text{Values 1 to 6 are \$15 thousand} \\
&\text{The next 8 data values are \$20, so} && \text{Values 7 to } (6+8)=14 \text{ are \$20 thousand} \\
&\text{The next 11 data values are \$25, so} && \text{Values 15 to } (14+11)=25 \text{ are \$25 thousand} \\
&\text{The next 17 data values are \$30, so} && \text{Values 26 to } (25+17)=42 \text{ are \$30 thousand} \\
&\text{The next 19 data values are \$35, so} && \text{Values 43 to } (42+19)=61 \text{ are \$35 thousand}
\end{aligned} \tag{5.10.1}$$

From this we can tell that values 50 and 51 will be \$35 thousand, and the mean of these two values is \$35 thousand. The median income in this neighborhood is \$35 thousand.

Example 8

If we add in the new neighbor with a \$5 million household income, then there will be 101 data values, and the 51st value will be the median. As we discovered in the last example, the 51st value is \$35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode**.

Mode

The **mode** is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example 9

In our vehicle color survey, we collected the data

Color	Frequency
Blue	3
Green	5
Red	4
White	3
Black	2
Grey	3

For this data, Green is the mode, since it is the data value that occurred the most frequently.

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

Try it Now 3

Reviewers were asked to rate a product on a scale of 1 to 5. Find

a. The mean rating
b. The median rating
c. The mode rating

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

Answer

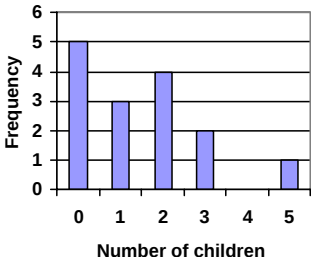
a. The mean is $\frac{1 \cdot 4 + 2 \cdot 8 + 3 \cdot 7 + 4 \cdot 3 + 5 \cdot 1}{23} \approx 2.5$
b. There are 23 data values, so the median will be the 12th data value. Ratings of 1 are the first 4 values, while a rating of 2 are the next 8 values, so the 12th value will be a rating of 2. The median is 2.
c. The mode is the most frequent rating. The mode rating is 2.

This page titled 5.10: Measures of Central Tendency is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **11.5: Measures of Central Tendency** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

5.10.1: Exercises

1. Is it possible to have a data set where the mean and median are not one of the data values?
2. Diners were asked how much they would pay for a meal. Their responses were: $7.50, $8.25, $9.00, $8.00, $7.25, $7.50, $8.00, $7.00.
   a. Find the mean of the prices diners would be willing to pay for a meal.
   b. Find the median of the prices.
   c. Write the mode price, if any.
3. You recorded the time in seconds it took for 8 participants to solve a puzzle. The times were: 15.2, 18.8, 19.3, 19.7, 20.2, 21.8, 22.1, 29.4.
   a. Find the mean of the times it took participants to solve a puzzle.
   b. Find the median of the times.
   c. Write the mode time, if any.
4. A group of adults were asked how many children they have in their families. The bar graph to the right shows the number of adults who indicated each number of children.



   a. How many adults were surveyed?
   b. Compute the mean number of children for each adult in the group surveyed.
   c. Compute the median number of children for the group surveyed.
   d. Write the mode number of children, if any.
5. Jasmine was interested in how many days it would take an order from Amazon to arrive at her door. The graph below shows the data she collected.



   a. How many Amazon orders did Jasmine make?
   b. Compute the mean number of shipping days for Jasmine's orders.
   c. Compute the median number of shipping days.
   d. Write the mode number of shipping days, if any.
6. The ages of the U.S. Presidents who were inaugurated in the 20th century are given.

   42; 51; 56; 55; 51; 54; 51; 60; 62; 43; 55; 56; 61; 52; 69; 64; 46

   a. Find the mean, median, and mode of these ages.
   b. What was a typical age for a U.S. President in the 20th century?
7. Results for 40 students on a recent English exam are given.

   83; 76; 98; 75; 56; 47; 67; 92; 69; 74; 72; 84; 96; 66; 51; 64; 58; 80; 85; 78; 58; 88; 90; 76; 74; 82; 76; 66; 50; 84; 66; 79; 83; 43; 54; 93; 77; 67; 54; 76

   a. Find the mean, median, and mode of these data values.
   b. If everyone whose score is at least average on this exam gets a C or higher, which average should be used?
   c. How many students had scores below average?
8. Find the mean, median and mode for the following data set.

   1.8; 4.0; 5.2; 2.9; 4.4; 4.4; 3.4; 4.9; 3.8; 2.7; 2.7

9. The table shows the salaries of employees of a small company.

Salary	Frequency
$\$40,000$	4
$\$43,000$	3
$\$45,000$	2
$\$75,000$	1

a. Find the mean, median, and mode of these salaries.
b. b. Which number best represents a typical salary for the employees at the company?

10. A class had the following scores on an exam.

Test Score	Frequency
90	2
80	4
70	9
60	5
50	3
40	1
30	2
0	4

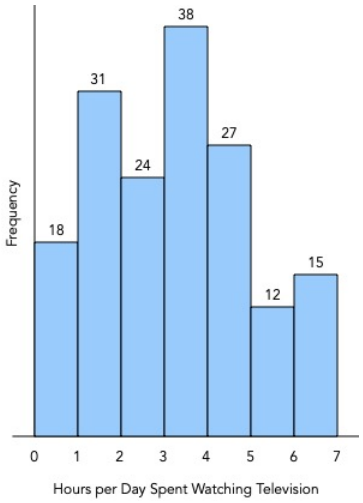
a. Find the mean, median, and mode score for the class.
b. Which value do you think best represents how the class did?

11. Comparable printers from three different companies were tested for printing speed in pages per minute (ppm). Each printer manufacturer claims that its printer has a faster average printing speed than its competitors. Results are shown in the table below.

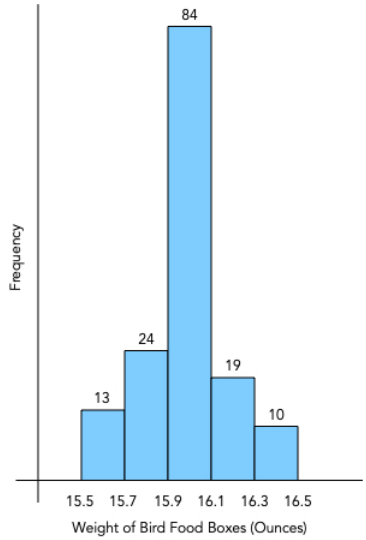
	Test 1	Test 2	Test 3	Test 4	Test 5
Printer X (ppm)	48	52	50	47	52
Printer Y (ppm)	50	48	54	48	52
Printer Z (ppm)	48	51	52	47	51

a. Which average would the manufacturer of Printer X use to support its claim?
b. Which average would the manufacturer of Printer Y use to support its claim?
c. Which average would the manufacturer of Printer Z use to support its claim?

12. The frequency histogram below shows the number of hours per day a random sample of teenagers spent watching television. Find the mean, median, and mode number of hours of television watched by these teenagers.



13. A pet food company is testing the accuracy of the machine that fills its boxes with bird food. The weights of the sample boxes are shown in the frequency histogram. Find the mean, median, and mode of the weights of the bird food boxes.



This page titled 5.10.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **11.7: Exercises** by David Lippman & Jeff Eldridge is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

CHAPTER OVERVIEW

6: Measurement and Geometry

6.1: Measurement

6.1.1: Measurement and the United States System

6.1.1.1: Exercises

6.1.2: The Metric System of Measurement

6.1.2.1: Exercises

6.1.3: Conversion Between the Metric and US Customary Systems of Measurement

6.1.3.1: Exercises

6.1.4: Temperature and Time

6.1.4.1: Exercises

6.1.5: Simplification of Denominate Numbers

6.1.5.1: Exercises

6.2: Geometry

6.2.1: Perimeter and Area of Geometric Figures

6.2.1.1: Exercises

6.2.2: Circles

6.2.2.1: Exercises

6.2.3: Solid Geometric Figures and Objects

6.2.3.1: Exercises

6.3: Fractals

6.3.1: Fractal Dimension

6.3.2: Exercises

Thumbnail: A two-dimensional perspective projection of a sphere (CC BY-3.0; Geek3 via Wikipedia).

This page titled 6: Measurement and Geometry is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1: Measurement

Measurement

There are two major systems of measurement in use today. They are the *United States system* and the *metric system*.

The **metric system of measurement** is the decimal system of measure developed in the 1790s in France. This system which is also referred to as SI is the official unit used by every country except the United States. These units use a standard set of prefixes such as milli, centi, kilo, deci

The **US Customary Units** is a system of measurement primarily used in the United States which was validated in 1832. This system of measurement came from the English units, the units of measure used in the British Empire before the formulation of the Imperial system on January 1, 1826, and before American Independence. The customary units are widely used in the areas of commerce and industry. Consumer products and industrial manufacturing materials are weighed or measured using the customary units as evident in their labels.

The common customary units of measure used are:

- inch, foot, yard, and mile for lengths
- acre for area
- cubic inch, cubic foot, and cubic yard for volume and capacity
- ounce, pint, quart, and gallon for fluid volume
- pint and quart for dry volume

This section begins by defining measurement and its common uses.

This page titled 6.1: Measurement is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1.1: Measurement and the United States System

6.1.1 Learning Objectives

- Define measurement.
- Perform arithmetic calculations on units of length, weight, and capacity.
- Convert from one unit of measure in the United States system to another unit of measure.
- Solve application problems using each form of measurement.

Measurement

Before we work the *United States measurement system* and the *metric system*, let's gain a clear understanding of the concept of measurement.

Definition: Measurement

Measurement is comparison to some standard.

Definition: Standard Unit of Measure

The concept of measurement is based on the idea of direct comparison. This means that measurement is the result of the comparison of two quantities. The quantity that is used for comparison is called the **standard unit of measure**.

Over the years, standards have changed. Quite some time in the past, the standard unit of measure was determined by a king. For example,

1 inch was the distance between the tip of the thumb and the knuckle of the king.
1 inch was also the length of 16 barley grains placed end to end.

Today, standard units of measure rarely change. Standard units of measure are the responsibility of the Bureau of Standards in Washington D.C.

Some desirable properties of a standard are the following:

1. *Accessibility*. We should have access to the standard so we can make comparisons.
2. *Invariance*. We should be confident that the standard is not subject to change.
3. *Reproducibility*. We should be able to reproduce the standard so that measurements are convenient and accessible to many people.

Definition: Length, Weight, Capacity

Length is a linear measurement to describe the distance between two objects.

Weight is used to describe how heavy or light an object is.

Capacity is used to describe the amount of liquid (or other pourable substance) that will fill an object.

The United States System of Measurement

Some of the common units (along with their abbreviations) for the United States system of measurement are listed in the following table.

Unit Conversion Table	
Length	1 foot (ft) = 12 inches (in) 1 yard (yd) = 3 feet (ft) 1 mile (mi) = 5,280 feet

Weight	1 pound (lb) =16 ounces (oz) 1 ton (T) = 2,000 pounds
Liquid Capacity	1 tablespoon (tbsp) = 3 teaspoons (tsp) 1 fluid ounce (fl oz) = 2 tablespoons 1 cup (c) = 8 fluid ounces 1 pint (pt) = 2 cups 1 quart (qt) = 2 pints 1 gallon (gal) = 4 quarts

Conversions in the United States System

It is often convenient or necessary to convert from one unit of measure to another. For example, it may be convenient to convert a measurement of length that is given in feet to one that is given in inches. Such conversions can be made using *unit fractions*.

Definition: Unit Fraction

A **unit fraction** is a fraction with a value of 1 in the numerator or the denominator.

Unit fractions are formed by using two equal measurements. One measurement is placed in the numerator of the fraction, and the other in the denominator. **Placement depends on the desired conversion**.

Placement of Units
Place the unit being converted *to* in the **numerator**.
Place the unit being converted *from* in the **denominator**.

For example,

Equal Measurements	Unit Fraction
1 ft = 12 in	$\frac{1\text{ ft}}{12\text{ in}}$ or $\frac{12\text{ in}}{1\text{ ft}}$
1 pt = 16 fl oz	$\frac{1\text{ pt}}{16\text{ fl oz}}$ or $\frac{16\text{ fl oz}}{1\text{ pt}}$
1 wk = 7 da	$\frac{7\text{ da}}{1\text{ wk}}$ or $\frac{1\text{ wk}}{7\text{ da}}$

Example 1

Make the following conversions. If a fraction occurs, convert it to a decimal rounded to two decimal places.

Convert 11 yards to feet.

Solution

Looking in the unit conversion table under *length*, we see that $1\text{ yd} = 3\text{ ft}$. There are two corresponding unit fractions, $\frac{1\text{ yd}}{3\text{ ft}}$ and $\frac{3\text{ ft}}{1\text{ yd}}$. Which one should we use? Look to see which unit we wish to convert to. Choose the unit fraction with this unit in the *numerator*. We will choose $\frac{3\text{ ft}}{1\text{ yd}}$ since this unit fraction has feet in the numerator. Now, multiply 11 yd by the unit fraction. Notice that since the unit fraction has the value of 1, multiplying by it does not change the value of 11 yd.

$$
\begin{aligned}
11\text{ yd} &= \frac{11\text{ yd}}{1}\cdot\frac{3\text{ ft}}{1\text{ yd}} && \text{Divide out common units.}\\
&= \frac{11\ \cancel{\text{yd}}}{1}\cdot\frac{3\text{ ft}}{1\ \cancel{\text{yd}}} && \text{(Units can be added, subtracted, multiplied, and divided, just as numbers can.)}\\
&= \frac{11\cdot 3\text{ ft}}{1}\\
&= 33\text{ ft}
\end{aligned}
$$

Thus, $11\text{ yd} = 33\text{ ft}$.

Example 2

Convert 36 fl oz to pints.

Solution

Looking in the unit conversion table under *liquid volume*, we see that $1\text{ pt} = 16\text{ fl oz}$. Since we are to convert to pints, we will construct a unit fraction with pints in the numerator.

$$
\begin{aligned}
36\text{ fl oz} &= \frac{36\text{ fl oz}}{1}\cdot\frac{1\text{ pt}}{16\text{ fl oz}} && \text{Divide out common units.}\\
&= \frac{36\ \cancel{\text{fl oz}}}{1}\cdot\frac{1\text{ pt}}{16\ \cancel{\text{fl oz}}}\\
&= \frac{36\cdot 1\text{ pt}}{16}\\
&= \frac{36\text{ pt}}{16} && \text{Reduce}\\
&= \frac{9}{4}\text{ pt} && \text{Convert to decimals: } \frac{9}{4} = 2.25
\end{aligned}
$$

Thus, $36\text{ fl oz} = 2.25\text{ pt}$.

Example 5

Your want to lay out a red carpet walkway for your grandma's 75th birthday. The distance from the dropoff area to the entrance of the party venue is 18 feet. The fabric you want is only sold in yards. How many yards of fabric do you need?

Solution

The unit fraction needed for converting from feet to yards is $\frac{1\text{ yd}}{3\text{ ft}}$.

$$
\begin{aligned}
18\text{ ft} &= \frac{18\text{ ft}}{1}\cdot\frac{1\text{ yd}}{3\text{ ft}} && \text{Divide out common units.}\\
&= \frac{18\ \cancel{\text{ft}}}{1}\cdot\frac{1\text{ yd}}{3\ \cancel{\text{ft}}}\\
&= \frac{18\text{ yd}}{1\cdot 3}\\
&= 6\text{ yd}
\end{aligned}
$$

Thus, $18\text{ ft} = 6\text{ yd}$. You would need to purchase 6 yards of fabric for the red carpet.

Try It Now 1

Make the following conversions. If a fraction occurs, convert it to a decimal rounded to two decimal places.

1. Convert 2 mi to feet.
2. Convert 26 ft to yards.
3. Convert 9 qt to pints.

Answer

1. 10,560 ft
2. 8.67 yd.
3. 18 pt

This page titled 6.1.1: Measurement and the United States System is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1.1.1: Exercises

1. In the U.S. system, abbreviations can be difficult to associate with the correct unit of measurement. What unit do these abbreviations belong to?
    a. lb
    b. oz
    c. ft
    d. mi
    e. yd
    f. pt
    g. qt
2. Convert the following:
    a. 52 inches to feet
    b. 257 miles to inches
    c. 11 feet to yards
    d. 47 ounces to pounds
    e. 2.3 tons to ounces
    f. 7 gallons to quarts
    g. 10 cups to quarts
    h. 3.6 quarts to cups
    i. 29 cups to gallons
    j. 3 pints to fluid ounces
    k. 417 ounces to ton
    l. $8\frac{1}{2}$ gallons to cups
    m. 2 mi to inches
    n. 3 cups to tablespoons
    o. 16 tsp to cups
3. A recipe calls for 3 teaspoons of soy sauce. You have a 5 pt container. How many teaspoons could you get from your container of soy sauce?
4. The coach of a local sports team brought 6 gallons of water that needs to be put into bottles that only hold one pint each. How many bottles will they be able to fill?
5. An 8 ft length of 4 inch wide crown molding costs $14. How much will it cost to buy 40 ft of crown molding?
6. A chain weighs 10 pounds per foot. How many ounces will 4 inches weigh?
7. Sugar contains 15 calories per teaspoon. How many calories are in 1 cup of sugar?
8. The store is selling lemons at 2 for $1. Each yields about 2 tablespoons of juice. How much will it cost to buy enough lemons to make a 9-inch lemon pie requiring $\frac{1}{2}$ cup of lemon juice?

This page titled 6.1.1.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1.2: The Metric System of Measurement

6.1.2 Learning Objectives

- Describe some of the advantages of the base ten number system
- Define prefixes of the metric measures
- Convert from one unit of measure in the metric system to another unit of measure

The Advantages of the Base Ten Number System

The metric system of measurement takes advantage of our base ten number system. The advantage of the metric system over the United States system is that in the metric system it is possible to convert from one unit of measure to another simply by multiplying or dividing the given number by a power of 10. This means we can make a conversion simply by moving the decimal point to the right or the left.

Prefixes

Common units of measure in the metric system are the meter (for length), the liter (for capacity), and the gram (for mass). The units are abbreviated m for meter, L for liter, and g for gram. To each of the units a prefix can be attached. The **metric prefixes** along with their meaning are listed below.

Metric Prefixes and Abbreviations

giga (g): billion
mega (m): million
kilo (k): thousand
hecto (h): hundred
deka (dk): ten
deci (d): tenth
centi (c): hundredth
milli (m): thousandth

For example, if length is being measured,

1 kilometer is equivalent to 1000 meters. 1 km = 1000 m
1 centimeter is equivalent to one hundredth of a meter. 1 cm = 0.01 m
1 millimeter is equivalent to one thousandth of a meter. 1 mm = 0.001 m

Conversion from One Unit to Another Unit

Let's note three characteristics of the metric system that occur in the metric table of measurements.

1. In each category, the prefixes are the same.
2. We can move from a *larger to a smaller* unit of measure by moving the decimal point to the *right*. (or multiplying by 10 for each step)
3. We can move from a *smaller to a larger unit* of measure by moving the decimal point to the *left*. (or dividing by 10 for each step)

The following table provides a summary of the relationship between the basic unit of measure (meter, gram, liter) and each prefix, and how many places the decimal point is moved and in what direction.

kilo hecto deka unit deci centi milli

Basic Unit to Prefix		
unit to deka	1 to 10	Divide by 10
unit to hecto	1 to 100	Divide by 10^2
unit to kilo	1 to 1,000	Divide by 10^3

unit to deci	1 to 0.1	Multiply by 10
unit to centi	1 to 0.01	Multiply by 10^2
unit to milli	1 to 0.001	Multiply by 10^3

Conversion Table

Listed below, in the unit conversion table, are some of the common metric units of measure.

Unit Conversion Table		
Length	$1\text{ kilometer (km)} = 1{,}000\text{ meters } (m)$	$1,000 \times 1\text{m}$
	$1\text{ hectometer (hm)} = 100\text{ meters}$	$100 \times 1\text{m}$
	$1\text{ dekameter (dam)} = 10\text{ meters}$	$10 \times 1\text{m}$
	1 meter (m)	$1 \times 1\text{m}$
	$1\text{ decimeter (dm)} = \frac{1}{10}\text{ meter}$	$.1 \times 1\text{m}$
	$1\text{ centimeter (cm)} = \frac{1}{100}\text{ meter}$	$.01 \times 1\text{m}$
	$1\text{ millimeter (mm)} = \frac{1}{1,000}\text{ meter}$	$.001 \times 1\text{m}$
Mass	$1\text{ kilogram (kg)} = 1{,}000\text{ grams } (g)$	$1,000 \times 1\text{g}$
	$1\text{ hectogram (hg)} = 100\text{ grams}$	$100 \times 1\text{g}$
	$1\text{ dekagram (dag)} = 10\text{ grams}$	$10 \times 1\text{g}$
	1 gram (g)	$1 \times 1\text{g}$
	$1\text{ decigram (dg)} = \frac{1}{10}\text{ gram}$	$.1 \times 1\text{g}$
	$1\text{ centigram (cg)} = \frac{1}{100}\text{ gram}$	$.01 \times 1\text{g}$
	$1\text{ milligram (mg)} = \frac{1}{1,000}\text{ gram}$	$.001 \times 1\text{g}$
Volume	$1\text{ kiloliter (kL)} = 1{,}000\text{ liters } (L)$	$1,000 \times 1\text{L}$
	$1\text{ hectoliter (hL)} = 100\text{ liters}$	$100 \times 1\text{L}$
	$1\text{ dekaliter (daL)} = 10\text{ liters}$	$10 \times 1\text{L}$
	1 liter (L)	$1 \times 1\text{L}$
	$1\text{ deciliter (dL)} = \frac{1}{10}\text{ liter}$	$.1 \times 1\text{L}$
	$1\text{ centiliter (cL)} = \frac{1}{100}\text{ liter}$	$.01 \times 1\text{L}$
	$1\text{ milliliter (mL)} = \frac{1}{1,000}\text{ liter}$	$.001 \times 1\text{L}$

Distinction Between Mass and Weight

There is a distinction between mass and weight. The **weight** of a body is related to gravity whereas the mass of a body is not. For example, your weight on the earth is different than it is on the moon, but your mass is the same in both places. **Mass** is a measure of a body's resistance to motion. The more massive a body, the more resistant it is to motion. Also, more massive bodies weigh more than less massive bodies.

Converting Metric Units

To convert from one metric unit to another metric unit:

1. Determine the location of the original number on the metric scale (pictured in each of the following examples).
2. Multiply or divide the original number by 10 raised to the same number of places as is necessary to move to the metric unit you wish to go to. (If changing from a larger to a smaller unit, multiply by 10. If changing from a smaller to a larger unit, divide by 10)

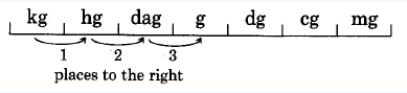
We can also convert from one metric unit to another using unit fractions. Both methods are shown in the following example.

Example 1

Convert 3 kilograms to grams.

Solution

a. 3 kg can be written as 3.0 kg. Then,



$3.0\ \text{kg} = 3\underset{1\,2\,3}{000}.\ \text{g}$

Since we are moving from a larger unit to a smaller unit. Multiply the original number by 10 three times.

$3 \times 10^3 = 3000$ grams

Thus, $3\ \text{kg} = 3{,}000\ \text{g}$.

b. We can also use unit fractions to make this conversion.

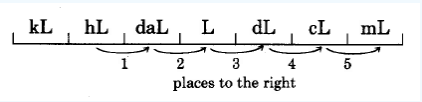
Since we are converting to grams, and $1{,}000\ \text{g} = 1\ \text{kg}$. we choose the unit fraction $\dfrac{1{,}000\ \text{g}}{1\ \text{kg}}$ since grams is in the numerator.

$$\begin{aligned} 3\ \text{kg} &= 3\ \text{kg} \cdot \frac{1{,}000\ \text{g}}{1\ \text{kg}} \\ &= 3\ \cancel{\text{kg}} \cdot \frac{1{,}000\ \text{g}}{1\ \cancel{\text{kg}}} \\ &= 3 \cdot 1{,}000\ \text{g} \\ &= 3{,}000\ \text{g} \end{aligned}$$

Example 2

Convert 67.2 hectoliters to milliliters.

Solution



$67.2\ \text{hL} = 67\underset{1\,2\,3\,4\,5}{20000}.\ \text{mL}$

Since we are moving from a larger unit to a smaller unit. Multiply the original number by 10 five times.

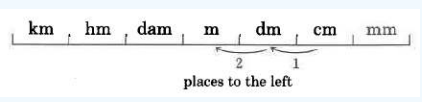
$67.2 \times 10^5 = 6{,}720{,}000$ milliliters

Thus, $67.2\ \text{hL} = 6{,}720{,}000\ \text{mL}$

Example 3

Convert 100.07 centimeters to meters.

Solution



$100.07\ \text{cm} = 1\underset{2\,1}{0007}\ \text{m}$

Since we are moving from a smaller unit to a larger unit. Divide the original number by 10 two times.

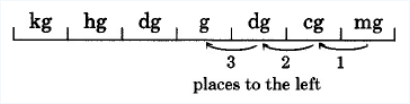
$100.07 \div 10^2 = 1.0007$ meters

Thus, 100.07 cm $= 1.0007$m

Example 4

Convert 0.16 milligrams to grams.

Solution



0.16 mg = 0.000 16 g
3 2 1

Since we are moving from a smaller unit to a larger unit. Divide the original number by 10 three times.

$0.16 \div 10^3 = 0.00016$ grams

Thus, 0.16 mg $= 0.00016$

Example 5

A medication is sold in 5 milligram size capsules. The pharmacist receives the medication that that is used to fill the capsules in 3.5 kilogram bags. How many capsules can be produced?

Solution

First we need to convert from kilograms to milligrams. Since we are moving from a larger to a smaller unit. Multiply the original number by 10 six times.

$3.5 \times 10^6 = 3,500,000$ milligrams

Each capuse holds 5 milligrams, so we divide the number we just found by 5 to determine the number of capsules that can be filled.

$3,500,000 \div 5 = 700,000$capsules

Thus, 3.5 kilograms of the medication produce 700,000 capsules.

Try It Now 1

Make the following conversions. If a fraction occurs, convert it to a decimal rounded to two decimal places.

1. 411 kilograms to grams
2. 5.626 liters to centiliters
3. 80 milliliters to kiloliters
4. 150 milligrams to centigrams
5. 2.5 centimeters to meters

Answer

1. 411,000 g
2. 562.6 cL
3. 0.00008 kL
4. 15 cg
5. 0.025 m

This page titled 6.1.2: The Metric System of Measurement is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1.2.1: Exercises

1. In the metric system, the prefix tells what power of ten the basic unit is being multiplied by. What power of ten does the prefix mean?
    a. milli
    b. kilo
    c. hecto
    d. centi
    e. giga
    f. mega
    g. deci
2. State the symbol or abbreviation for each prefix listed in number 1.
3. Convert the following:
    a. 9 meters to centimeters
    b. 520 milliliters to liters
    c. 48.66 liters to deciliters
    d. 2 millimeters to hectometers
    e. 211 centimeters to meters
    f. 6325 milliliters to liters
    g. 1278 grams to kilograms
    h. 3.1 kilograms to grams
    i. 67 millimeters to centimeters
    j. 1.24 kilograms to milligrams
    k. 6 kilograms to grams.
    l. 99.04 dekaliters to centiliters
4. A wire costs $2 per meter. How much will 3 kilometers of wire cost?

This page titled 6.1.2.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1.3: Conversion Between the Metric and US Customary Systems of Measurement

6.1.3 Learning Objectives

- To estimate equivalence between the Metric Units and US Customary Units
- To convert Metric Units to US Customary Units
- To convert US Customary Units to Metric Units

Estimate equivalence between the Metric Units and US Customary Units

Before converting metric units to US customary units and vice versa, you need to familiarize the approximate metric equivalents of common customary units. The table below summarizes the commonly used metric and US customary units and their approximate equivalence. The notations enclosed are the abbreviations of each unit.

Metric Units	US Customary Units
Lengths	
1 meter (m)	3.28 feet (ft. or ’)
1 kilometer (km)	0.62 mile (m)
1 centimeter (cm)	0.39 inch (in. or ”)
0.9144 meter (m)	1 yard (yd.)
1.852 kilometers (km)	1 nautical mile (nmi. or NM)
Weight	
1 kilogram (kg)	2.2 pounds (lb. or #)
28.3 grams (g)	1 ounce (oz.)
1 milligram (mg)	0.002 teaspoon (tsp.)
Fluid Capacity	
29.57 milliliters (mL)	1 fluid ounce (fl. Oz.)
1 liter (L)	4.23 cups (c)
3.79 liters (L)	1 gallon (gal.)
473.176 milliliters (mL)	1 pint (pt.)
0.95 liters (L)	1 quart (qt.)
158.987 liters (L)	1 barrel (bbl.)

Additional Common Conversions

Metric Units	US Customary Units
Area	
6.45 square centimeters (cm^2)	1 square inch (in^2)
1 square meter (m^2)	1.196 square yard (yd^2)
4046.873 square meters (m^2)	1 acre (ac.)

Volume and Capacity	
16.39 milliliter (mL)	1 cubic inch (in^3)
28.32 liters (L)	1 cubic foot (ft^3)
764.55 liters (L)	1 cubic yard (yd^3)
1233.482 cubic meters (m^3)	1 cubic foot (ft^3)

Convert Metric Units to US Customary Units

You have familiarized the commonly used equivalence between metric and US customary units. At this point, you will learn to convert metric units to US customary units.

To convert metric units to US customary units, use their estimate equivalence. Write the equivalence as a ratio, and multiply it to the given measurement (in metric) to convert. Be sure that the units of measure to convert cancels out. For instance, to convert liters to gallons, be sure to cancel out liters and retain gallons as the final unit.

Now, let us consider some examples.

Example 1

The official regulators of a shipment company classify a parcel weighing more than 51 pounds (lbs.) as a heavy shipment. Andrea packed 24 terracotta plant containers to be shipped to her customers. If each plant container weighs 1.3 kilograms (kg), will her package be classified as a heavy shipment?

Solution

To determine whether the parcel is a heavy shipment or not, you need to convert the total weight of the plant containers to pounds.

Compute for the total weight.

$$\begin{aligned}\text{Total Weight } &= 24 \times 1.3 \text{ kg} \\ &= 31.2 \text{ kg}\end{aligned}$$

Convert the total weight to pounds. Use the metric to US customary units' equivalence: 1 kg = 2.2 lbs.

$$31.2 \ \cancel{\text{kg}} \times \frac{2.2 \text{ lbs}}{1 \ \cancel{\text{kg}}} = 68.64 \text{ lbs}$$

Since the total weight is more than 51 lbs., the shipment is considered heavy.

Try it Now 1

Angelica who lives in Las Vegas, Nevada, visited her friend in Canada. She knows that Canada uses the metric system, so their speed limits are posted in kilometers per hour. On her way, she sees a road sign that says, "max 60 km/h". She knows the sign means "a maximum speed of 60 kilometers per hour" but she is unsure if this is over the speed limit in Las Vegas, Nevada. How can Angelica convert 60 km/h to miles per hour? Use the metric to US customary units' equivalence: 1 kilometer (km) = 0.62 mile.

Answer

In Las Vegas, the limit is 65 miles per hour (mph). Since 37.2 mph is below this limit, 60km/h is not over the speed limit.

Example 2

Half of a meter of ribbon is required for wrapping presents enclosed in a rectangular box. How many presents using the same box size can be wrapped with 10 yards of ribbon?

Solution

Convert half a meter (0.5 m) to yards. Use the metric to US customary units' equivalence: 1 yard = 0.9144 meter.

$$0.5 \ \cancel{\text{m}} \times \frac{1 \text{ yd}}{0.9144 \ \cancel{\text{m}}} = 0.5468 \text{ yd}$$

The result means that one present requires 0.5468 yards of ribbon. Calculate the number of presents by dividing 10 yards by 0.5468 yards.

$$10 \text{ yds} \div \frac{0.5468 \text{ yd}}{1 \text{ present}} = 10 \ \cancel{\text{yd}} \times \frac{1 \text{ present}}{0.5468 \ \cancel{\text{yd}}} = 18.29 \text{ presents}$$

Approximately, 18 presents can be wrapped with 10 yards of ribbons.

Try it Now 2

Crystal and her friends are planning for a road trip, which they have determined to be 2,000 kilometers (km) long. They want to rent a van that can travel 100 km using 9 liters of gas. Suppose the gas price is $2.74 a gallon (gal.), how many gallons of gas they will need? How much money will the group of friends need to save for gas? Use the metric to US customary units' equivalence: 4.405 liters (L)= 1 gallon (gal.).

Answer

The team needs 40.86 gallons of gas for the trip, which will cost them $111.96.

Try it Now 3

Christine plans to put up a short-course swimming pool at her house. A short-course swimming pool usually measures 22.86 meters in length, at least 18.29 meters in width, and 2.5 meters in depth. Convert the dimensions of the pool to inches. Use the metric to US customary units' equivalence: 1 centimeter (cm) = 0.39 inch (in.) and 1 centimeter (cm) = 0.01 meter (m).

Answer

Approximately, the pool measures 891.54 in by 713.31 in by 97.5 in.

Try It Now 4

Drug dosage for children is based on body weights, milligrams per pound (mg/lb). A 5-year-old child with Meningitis who weighs 20 kilograms was prescribed Ceftriaxone. The dosage requirement is 220 milligrams per pound per day (mg/lb/day) once daily. The drug is prediluted in 40 mg/mL concentration. Calculate the dose in teaspoons. Use the metric to US customary units' equivalence: 1 kilogram (kg) = 2.2 pounds (lb.) and 1 mg = 0.0002 tsp.

Answer

The dose is approximately 2 teaspoons.

Convert US Customary Units to Metric Units

You have learned how to convert metric units to US customary units. Now, you will learn to convert US customary units to metric units.

Example 3

Daniel drove 40 miles in two hours. What distance (in meters) did he cover in one minute? Use the US customary to metric units' equivalence: 0.62 mile (mi.) = 1 kilometer (km), 1 km is equal to 1,000 meters (m), and 1 hour = 60 minutes.

Solution

Convert 40 miles in two hours to kilometers per hour.

$$\frac{40 \text{ mi}}{2 \text{ hrs}} \times \frac{1 \text{ km}}{0.62 \text{ mi}} = 32.2581 \text{ km/hr}$$

Convert 32.2581 kilometers per hour to meters per hour.

$$\frac{32.2581 \text{ km}}{1 \text{ hr}} \times \frac{1000 \text{ m}}{1 \text{ km}} = 32,258.1 \text{ m/hr}$$

Convert 32,258.1 meters per hour to meters per minute.

$$\frac{32,258.1 \text{ m}}{1 \text{ hr}} \times \frac{1 \text{ hr}}{60 \text{ min}} = 537.635 \text{ m/min}$$

Daniel has covered 537.635 meters in one minute.

Try It Now 5

A supermarket has a car parking spot that is 9 feet wide and 18 feet long. Its ground-level parking has 50 parallel parking spots (side by side) of the same sizes. What is the area of the parking lot in square centimeters? Use the US customary units' equivalence: 1 foot (ft.) = 12 inches (in.) and 1 square inch (in^2) = 6.45 square centimeters (cm^2). (Hint: You will need to use the formula to find the area of a rectangle)

Answer

The area of the parking lot is 7,523,280 square centimeters.

Example 4

Allison wants to bake a chocolate cake. The recipe requires 1¾ cups of all-purpose flour. A supermarket sells all-purpose flour in grams instead of cups. All-purpose flour comes in the following sizes: 80 grams, 400 grams, and 800 grams. If Allison plans to buy just enough flour for 1 recipe, which size will she buy? Use the US customary units equivalence: 1 cup (c.) = 8 ounces (oz.) and 1 ounce (oz.) = 28.3 grams (g).

Solution

Converting 1¾ to a fraction or decimal will help when converting cups to ounces. In this example 1¾ = 1.75 will be used.

$$1.75 \text{ cups} \times \frac{8 \text{ oz}}{1 \text{ cup}} = 14 \text{ oz}$$

Convert ounces to grams.

$$14 \text{ oz} \times \frac{28.3 \text{ g}}{1 \text{ oz}} = 396.2 \text{ g}$$

Since 80 grams is less than the required recipe amount and 800 grams is much higher than 396.2 grams, Allison must buy the 400 gram package.

Try It Now 6

Melody's house garage is 45 feet long while David's garage is 10 yards and 10 feet long. Which house has a longer garage? How much is the difference (in meters)? Use the US customary to metric equivalence: 3.28 feet (ft.) = 1 meter (m) and 1 foot (ft.) = 0.3333 yard (yd.)

Answer

Melody's garage is longer by 1.52 meters.

Try it Now 7

How many liters of water are needed to fill an Olympic swimming pool with a total volume of 3,300 cubic yards? Assume that the water level is as deep as the pool. Use the US customary to metric units' equivalence: 1 cubic yard (yd^3) = 746.56 liters (L).

Answer

At most 2,463,648 liters of water are needed to fill the swimming pool.

Try It Now 8

Robert's new car requires at most 57 liters of fuel. If the current average price for regular gas is $2.864 per gallon, how much will it cost him to fill his car's tank? Use the US customary to metric units' equivalence: 1 gallon = 3.79 liters

Answer

It will cost Robert approximately $43.32 to fill his car's tank.

This page titled 6.1.3: Conversion Between the Metric and US Customary Systems of Measurement is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1.3.1: Exercises

Section 6.1.3 Exercises

1. Convert the following between U.S. and Metric measurement systems.
    a. 52 inches to centimeters
    b. 2 miles to meters
    c. 11 feet to decimeters
    d. 47 ounces to grams
    e. 4580 milliliters to cups
    f. 22 millimeters to inches
    g. 2.3 tons to kilograms
    h. 2,610 grams to pounds
    i. 7 gallons to liters
    j. 10 cups to quarts
    k. 3.6 quarts to cups
    l. 29 cups to gallons
    m. 3 pints to milliliters
    n. 8 meters to yards
2. If you drove 100 km/hour on the freeway, were you exceeding the speed limit?
3. Would you expect to pay more for 10 gallons (40 quarts) of gasoline or for 40 liters of gasoline?
4. Which is the better deal: apples priced at $1.69 per pound or the same apples priced $3.72 per kilogram?
5. If you ran 100 yards in the same time as I ran 100 meters, who ran faster?

This page titled 6.1.3.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1.4: Temperature and Time

6.1.4 Learning Objectives

- Perform arithmetic calculations on temperature and time
- Convert from one unit of measure to another unit of measure.
- Solve application problems using each form of measurement.

Definition: Time, Temperature

Temperature is used to describe the level of heat or cold of an object or environment.

Time is used to describe the amount of existence between events.

Some of the common units (along with their abbreviations) for temperature and time are listed in the following table.

Unit Conversion Table	
Temperature	Fahrenheit ($^\circ F$)$= \frac{9}{5}{}^\circ$C$+32$ Kelvin (K)$=^\circ$C$+273.15$ Celsius ($^\circ$C)$= \frac{5}{9}(^\circ$F$-32)$ Celsius ($^\circ$C)$=$K-273.15
Time	1 minute (min) = 60 seconds (sec) 1 hour (hr) = 60 minutes 1 day (da) = 24 hours 1 week (wk) = 7 days 1 year (yr)= 52 weeks

Example 1

Convert 73°F to Celsius.

Solution

To convert from Fahrenheit to Celsius, we need the equation ($^\circ$C)$= \frac{5}{9}$ $^\circ$F$-32)$. Substitute in the given value for Fahrenheit to find the answer. Round to the nearest degree.

($^\circ$C)$= \frac{5}{9}{}^\circ$F$-32($=

\dfrac{5}{9}$73^\circ F$- 32

=\dfrac{5}{9} \times 41=\dfrac{41 \times 5}{9}=23.7\approx 24^\circ F

Thus, 73°F$\approx 24^\circ$C\)

Example 2

Convert 3763 K to Fahrenheit.

Solution

First we need to covert to Celsius, then we can convert to Fahrenheit. Round to two decimal places.

($^\circ$C)$=$K$-273.15 = 3763 + 273.15 = 3489.85^\circ$C

($^\circ$F)$= \frac{9}{5}$ $^\circ$C$+32 = \frac{9}{5}3489.85 + 32 = \frac{9 \times 3489.85}{5} + 32 = 6281.73 + 32 = 6313.73^\circ$ F

Thus, 3763 K$= 6313.73^\circ$ F

Example 3

Convert 2,016 hr to weeks.

Solution

Looking in the unit conversion table under *time*, we see that $1\text{ wk} = 7\text{ da}$ and that $1\text{ da} = 24\text{ hr}$. To convert from hours to weeks, we must first convert from hours to days and then from days to weeks. We need two unit fractions.

The unit fraction needed for converting from hours to days is $\frac{1\text{ da}}{24\text{ hr}}$. The unit fraction needed for converting from days to weeks is $\frac{1\text{ wk}}{7\text{ da}}$.

$$\begin{aligned} 2{,}016\text{ hr} &= \frac{2{,}016\text{ hr}}{1} \cdot \frac{1\text{ da}}{24\text{ hr}} \cdot \frac{1\text{ wk}}{7\text{ da}} && \text{Divide out common units.} \\ &= \frac{2{,}016\ \cancel{\text{hr}}}{1} \cdot \frac{1\ \cancel{\text{da}}}{24\ \cancel{\text{hr}}} \cdot \frac{1\text{ wk}}{7\ \cancel{\text{da}}} \\ &= \frac{2{,}016 \cdot 1\text{ wk}}{24 \cdot 7} && \text{Simplify} \\ &= 12\text{ wk} \end{aligned}$$

Thus, $2{,}016\text{ hr} = 12\text{ wk}$.

Example 4

Your employer tracks your accumulated vacation time in hours. You want to take a two week vacation and have accumulated 412 hours of paid time off. Do you have enough hours for your vacation?

Solution

To solve this we need to covert 412 hours to weeks.

The unit fraction needed for converting from hours to days is $\frac{1\text{ da}}{24\text{ hr}}$. The unit fraction needed for converting from days to weeks is $\frac{1\text{ wk}}{7\text{ da}}$.

$$\begin{aligned} 412\text{ hr} &= \frac{412\text{ hr}}{1} \cdot \frac{1\text{ da}}{24\text{ hr}} \cdot \frac{1\text{ wk}}{7\text{ da}} && \text{Divide out common units.} \\ &= \frac{412\ \cancel{\text{hr}}}{1} \cdot \frac{1\ \cancel{\text{da}}}{24\ \cancel{\text{hr}}} \cdot \frac{1\text{ wk}}{7\ \cancel{\text{da}}} \\ &= \frac{412 \cdot 1\text{ wk}}{24 \cdot 7} && \text{Simplify} \\ &= 2.5\text{ wk} \end{aligned}$$

Thus, $412\text{ hr} = 2.5\text{ wk}$.

Yes, you have enough time for a 2.5 week vacation.

Try It Now 1

Make the following conversions. If a fraction occurs, convert it to a decimal rounded to two decimal places.

1. Convert 32°C to °F
2. Convert 1732°F to K
3. Convert 52 min to hours.

Answer

1. 89.6°F
2. 1217.59 K
3. 0.87 hr

This page titled 6.1.4: Temperature and Time is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1.4.1: Exercises

1. What unit of measurement do these abbreviations belong to?
    a. K
    b. da
    c. wk
    d. F
    e. C
    f. yr
    g. sec
    h. min
    i. hr
2. Convert the following:
    a. 21°C to °F
    b. 1736 K to °C
    c. 81°F to °C
    d. 789°C to K
    e. 235,000 sec to yr
    f. 67 hr to da
    g. 19 wk to hr
    h. 4 da to min
3. Australia uses Celsius to measure temperature. Your travel agent said it will be between 23° C and 27° C during your visit. What is the temperature range in Fahrenheit?
4. You want to binge watch a TV show that has 11 episodes that are 45 minutes each. How many hours do you need to set aside this weekend to watch the entire series?
5. A friend from Switzerland is coming to visit the U.S. in December. You tell them it will be about 35° C when they're here. You forgot to tell them that the measurement was in degrees Fahrenheit. They are used to Celsius and assume that is the temperature you told them. Are they planning for cold or hot weather? What Celsius temperature, should you have told them?
6. Your aunt always tells you her new child's age in weeks. The last time you saw them she said her child was 93 weeks old. How much is that in years?
7. To tell your colleague in Europe the temperature you are currently experiencing, you need to tell it to them in Celsius. If it is currently 64°F, what would you tell them?
8. The photosphere of the Sun is approximately 5778 K. How much is that in Celsius?

This page titled 6.1.4.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1.5: Simplification of Denominate Numbers

6.1.5 Learning Objectives

- Convert an unsimplified unit of measure to a simplified unit of measure
- Perform arithmetic calculations with denominate numbers

Converting to Multiple Units

Definition: Denominate Numbers

Numbers that have units of measure associated with them are called **denominate numbers**. It is often convenient, or even necessary, to simplify a denominate number.

Definition: Simplified Denominate Numbers

A denominate number is **simplified** when the number of standard units of measure associated with it does not exceed the next higher type of unit.

The denominate number 55 min is simplified since it is smaller than the next higher type of unit, 1 hr. The denominate number 65 min is *not* simplified since it is not smaller than the next higher type of unit, 1 hr. The denominate number 65 min can be simplified to 1 hr 5 min. The denominate number 1 hr 5 min is simplified since the next higher type of unit is day, and 1 hr does not exceed 1 day.

Example 1

Simplify 19 in.

Solution

Since $12 \text{ in} = 1 \text{ ft.}$, and $19 = 12 + 7$,

$$\begin{aligned} 19 \text{ in} &= 12 \text{ in} + 7 \text{ in} \\ &= 1 \text{ ft} + 7 \text{ in} \\ &= 1 \text{ ft } 7 \text{ in} \end{aligned}$$

Example 2

Simplify 4 gal 5 qt.

Solution

Since $4 \text{ qt} = 1 \text{ gal}$, and $5 = 4 + 1$,

$$\begin{aligned} 4 \text{ gal } 5 \text{ qt} &= 4 \text{ gal} + 4 \text{ qt} + 1 \text{ qt} \\ &= 4 \text{ gal} + 1 \text{ gal} + 1 \text{ qt} \\ &= 5 \text{ gal} + 1 \text{ qt} \\ &= 5 \text{ gal } 1 \text{ qt} \end{aligned}$$

Example 3

Simplify 2 hr 75 min.

Solution

Since $60 \text{ min} = 1 \text{ hr}$, and $75 = 60 + 15$,

$$\begin{aligned} 2 \text{ hr } 75 \text{ min} &= 2 \text{ hr} + 60 \text{ min} + 15 \text{ min} \\ &= 2 \text{ hr} + 1 \text{ hr} + 15 \text{ min} \\ &= 3 \text{ hr} + 15 \text{ min} \\ &= 3 \text{ hr } 15 \text{ min} \end{aligned}$$

Example 4

Simplify 43 fl oz.

Solution

Since $8 \text{ fl oz} = 1 \text{ c}$ (1 cup), and $43 \div 8 = 5\text{R}3$,

$$\begin{aligned} 43 \text{ fl oz} &= 40 \text{ fl oz} + 3 \text{ fl oz} \\ &= 5 \cdot 8 \text{ fl oz} + 3 \text{ fl oz} \\ &= 5 \cdot 1 \text{ c} + 3 \text{ fl oz} \\ &= 5 \text{ c} + 3 \text{ fl oz} \end{aligned}$$

But, $2\text{c} = 1 \text{ pt}$ and $5 \div 2 = 2\text{R}1$. So,

$$\begin{aligned} 5 \text{ c} + 3 \text{ fl oz} &= 2 \cdot 2 \text{ c} + 1 \text{ c} + 3 \text{ fl oz} \\ &= 2 \cdot 1 \text{ pt} + 1 \text{ c} + 3 \text{ fl oz} \\ &= 2 \text{ pt} + 1 \text{ c} + 3 \text{ fl oz} \end{aligned}$$

But, $2 \text{ pt} = 1 \text{ qt}$, so

$2 \text{ pt} + 1 \text{ c} + 3 \text{ fl oz} = 1 \text{ qt } 1 \text{ c } 3 \text{ fl oz}$

Try It Now 1

Simplify each denominate number. Refer to the conversion tables given in [link], if necessary.

1. 18 in
2. 8 gal 9 qt
3. 5 hr 80 min
4. 8 wk 11 days
5. 86 days

Answer

1. 1 ft 6 in
2. 10 gal 1 qt
3. 6 hr 20 min
4. 9 wk 4 days
5. 12 wk 2 days

Adding and Subtracting Denominate Numbers

Adding and Subtracting Denominate Numbers

Denominate numbers can be added or subtracted by:

1. writing the numbers vertically so that the like units appear in the same column.
2. adding or subtracting the number parts, carrying along the unit.
3. simplifying the sum or difference.

Example 5

Add 6 ft 8 in to 2 ft 9 in.

Solution

$$\begin{array}{r} 6 \text{ ft } 8 \text{ in} \\ +\,2 \text{ ft } 9 \text{ in} \\ \hline 8 \text{ ft } 17 \text{ in} \end{array}$$ Simplify this denominate number.

Since $12 \text{ in} = 1 \text{ ft}$,

$$\begin{aligned} 8\text{ ft} + 12\text{ in} + 5\text{ in} &= 8\text{ ft} + 1\text{ ft} + 5\text{ in} \\ &= 9\text{ ft} + 5\text{ in} \\ &= 9\text{ ft }5\text{ in} \end{aligned}$$

Example 6

Subtract 5 da 3 hr from 8 da 11 hr.

Solution

$$\begin{array}{r} 8\text{ da }11\text{ hr} \\ \underline{-\ 5\text{ da }3\text{ hr}} \\ 3\text{ da }8\text{ hr} \end{array}$$

Example 7

Subtract 3 lb 14 oz from 5 lb 3 oz.

Solution

$$\begin{array}{r} 5\text{ lb }3\text{ oz} \\ \underline{-\ 3\text{ lb }14\text{ oz}} \end{array}$$

We cannot directly subtract 14 oz from 3 oz, so we must borrow 16 oz from the pounds.

$$\begin{aligned} 5\text{ lb }3\text{ oz} &= 5\text{ lb} + 3\text{ oz} \\ &= 4\text{ lb} + 1\text{ lb} + 3\text{ oz} \\ &= 4\text{ lb} + 16\text{ oz} + 3\text{ oz (Since 1 lb} = 16\text{ oz)} \\ &= 4\text{ lb} + 19\text{ oz} \\ &= 4\text{ lb }19\text{ oz} \end{aligned}$$

$$\begin{array}{r} 4\text{ lb }19\text{ oz} \\ \underline{-\ 3\text{ lb }14\text{ oz}} \\ 1\text{ lb }5\text{ oz} \end{array}$$

Example 8

Subtract 4 da 9 hr 21 min from 7 da 10 min.

Solution

$$\begin{array}{r} 7\text{ da }0\text{ hr }10\text{ min} \\ \underline{-\ 4\text{ da }9\text{ hr }21\text{ min}} \end{array}$$ Borrow 1 da from the 7 da.

$$\begin{array}{r} 6\text{ da }24\text{ hr }10\text{ min} \\ \underline{-\ 4\text{ da }9\text{ hr }21\text{ min}} \end{array}$$ Borrow 1 hr from the 24 hr.

$$\begin{array}{r} 6\text{ da }23\text{ hr }70\text{ min} \\ \underline{-\ 4\text{ da }9\text{ hr }21\text{ min}} \\ 2\text{ da }14\text{ hr }49\text{ min} \end{array}$$

Try It Now 2

Perform each operation. Simplify when possible.

1. Add 4 gal 3 qt to 1 gal 2 qt.
2. Add 9 hr 48 min to 4 hr 26 min.
3. Subtract 2 ft 5 in from 8 ft 7 in.
4. Subtract 15 km 460 m from 27 km 800 m.
5. Subtract 8 min 35 sec from 12 min 10 sec.

Answer

1. 6 gal 1 qt
2. 14 hr 14 min
3. 6 ft 2in
4. 12 km 340 m
5. 3 min 35 sec

Multiplying a Denominate Number by a Whole Number

Let's examine the repeated sum

$$\underbrace{4 \text{ ft } 9 \text{ in} + 4 \text{ ft } 9 \text{ in} + 4 \text{ ft } 9 \text{ in}}_{3 \text{ times}} = 12 \text{ ft } 27 \text{ in}$$

Recalling that multiplication is a description of repeated addition, by the distributive property we have

$$\begin{aligned} 3(4 \text{ ft } 9 \text{ in}) &= 3\,(4\text{ft} + 9 \text{ in}) \\ &= 3 \cdot 4 \text{ ft } + 3 \cdot 9 \text{ in} \\ &= 12 \text{ ft} + 27 \text{ in. Now, } 27 \text{ in} = 2 \text{ ft } 3 \text{ in} \\ &= 12 \text{ ft} + 2 \text{ ft} + 3 \text{ in} \\ &= 14 \text{ ft} + 3 \text{ in} \\ &= 14 \text{ ft } 3 \text{ in} \end{aligned}$$

From these observations, we can suggest the following rule.

Multiplying a Denominate Number by a Whole Number
To multiply a denominate number by a whole number, multiply the number part of each unit by the whole number and affix the unit to this product.

Example 9

Perform the following multiplications. Simplify if necessary.

$$\begin{aligned} 6 \cdot (2 \text{ ft } 4 \text{ in}) &= 6 \cdot 2 \text{ ft} + 6 \cdot 4\text{in} \\ &= 12 \text{ ft} + 24 \text{ in} \end{aligned}$$

Since $3 \text{ ft} = 1 \text{ yd}$ and $12 \text{ in} = 1 \text{ ft}$,

$$\begin{aligned} 12 \text{ ft} + 24 \text{ in} &= 4 \text{ yd} + 2 \text{ ft} \\ &= 4 \text{ yd } 2 \text{ ft} \end{aligned}$$

Example 10

$$\begin{aligned} 8 \cdot (5 \text{ hr } 21 \text{ min } 55 \text{ sec}) &= 8 \cdot 5 \text{ hr} + 8 \cdot 21 \text{ min} + 8 \cdot 55 \text{ sec} \\ &= 40 \text{ hr} + 168 \text{ min} + 440 \text{ sec} \\ &= 40 \text{ hr} + 168 \text{ min} + 7 \text{ min} + 20 \text{ sec} \\ &= 40 \text{ hr} + 175 \text{ min} + 20 \text{ sec} \\ &= 40 \text{ hr} + 2 \text{ hr} + 55 \text{ min} + 20 \text{ sec} \\ &= 42 \text{ hr} + 55 \text{ min} + 20 \text{ sec} \\ &= 24 \text{ hr} + 18 \text{ hr} + 55 \text{ min} + 20 \text{ sec} \\ &= 1 \text{ da} + 18 \text{ hr} + 55 \text{ min} + 20 \text{ sec} \\ &= 1 \text{ da } 18 \text{ hr } 55 \text{ min } 20 \text{ sec} \end{aligned}$$

Try It Now 3

Perform the following multiplications. Simplify.

1. $2 \cdot (10 \text{ min})$
2. $5 \cdot (3 \text{ qt})$

3. $4 \cdot (5 \text{ ft } 8 \text{ in})$

Answer

1. 20 min
2. 15 qt = 3 gal 3 qt
3. 20 ft 32 in = 7 yd 1 ft 8 in

Dividing a Denominate Number by a Whole Number

Dividing a Denominate Number by a Whole Number

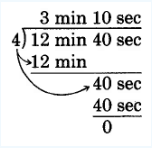
To divide a denominate number by a whole number, divide the number part of each unit by the whole number beginning with the largest unit. Affix the unit to this quotient. Carry any remainder to the next unit.

Example 11

Perform the following divisions. Simplify if necessary.

$(12 \text{ min } 40 \text{ sec}) \div 4$

Solution

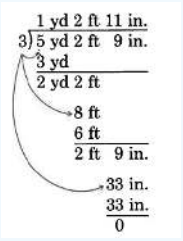


Thus $(12 \text{ min } 40 \text{ sec}) \div 4 = 3 \text{ min } 10 \text{ sec}$.

Example 12

$(5 \text{ yd } 2 \text{ ft } 9 \text{ in}) \div 3$

Solution



Convert to feet: $2 \text{ yd } 2 \text{ ft} = 8 \text{ ft}$

Convert to inches: $2 \text{ ft } 9 \text{ in.} = 33 \text{ in}$

Thus $(5 \text{ yd } 2 \text{ ft } 9 \text{ in}) \div 3 = 1 \text{ yd } 2 \text{ ft } 11 \text{ in}$.

Try It Now 4

Perform the following divisions. Simplify if necessary.

1. $(18 \text{ hr } 36 \text{ min}) \div 9$
2. $(36 \text{ hr } 8 \text{ min}) \div 8$
3. $(13 \text{ yd } 7 \text{ in}) \div 5$

Answer

1. 2 hr 4 min
2. 4 hr 18 min
3. 2 yd 1 ft 11 in

This page titled 6.1.5: Simplification of Denominate Numbers is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.1.5.1: Exercises

1. Simplify the following:
    a. 3,100 lb
    b. 1 ton 1100 lbs
    c. 135 min
    d. 2300 mm
    e. 14780 dL
    f. 1050 m
    g. 16 in
    h. 19 ft
    i. 90 min
    j. 17 da
    k. 240 oz
    l. 26 qt
    m. 300 sec
    n. 135 oz
    o. 14 tsp
    p. 18 pt
    q. 3,500 m
2. Perform the indicated operations. Simplify, if possible.
    a. Add 8 min 50 sec to 5 min 25 sec.
    b. Add 3 wk 3 da to 2 wk 5 da.
    c. Subtract 4 gal 3 qt from 5 gal 2 qt.
    d. Subtract 2 gal 3 qt 1pt from 8 gal 2 qt.
    e. Subtract 5 wk 4 da 21 hr from 12 wk 3 da 14 hr.
    f. Subtract 2 T 1,850 lb from 10 T 1,700 lb.
    g. Subtract the sum of 2 wk 3 da 15 hr and 5 wk 2 da 9 hr from 10 wk.
    h. Subtract the sum of 20 hr 15 min and 18 hr 18 min from the sum of 8 da 1 hr 16 min 5 sec.
    i. Multiply 5 times 5 ft 7 in.
    j. Multiply 2 qt 4 pt by 8.
    k. Divide 9 mi 6 ft by 6.
    l. Divide 22 qt 4 pt by 7.

This page titled 6.1.5.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.2: Geometry

Geometry

Geometry is used in our every day when we decorate, design, and arrange objects in a space.

The most commonly used aspects of geometry are:

- Perimeter and circumference, when dealing with the length of the edges of an object.
- Area, when dealing with the amount of surface an object covers.
- Volume, when dealing with how much an object can hold.

Each of these will be explored in the following sections.

This page titled 6.2: Geometry is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.2.1: Perimeter and Area of Geometric Figures

6.2.1 Learning Objectives

- Define polygon
- Find the perimeter of a polygon
- Find the areas of common polygons

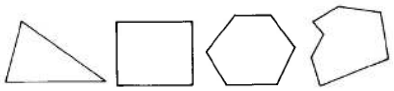
Polygons

We can make use of conversion skills with denominate numbers to make measurements of geometric figures such as rectangles, triangles, and circles. To make these measurements we need to be familiar with several definitions.

Definition: Polygon

A **polygon** is a closed plane (flat) figure whose sides are line segments (portions of straight lines).

Polygons



Not polygons



Perimeter

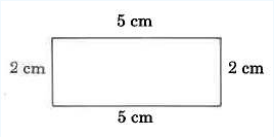
Definition: Perimeter

The **perimeter** of a polygon is the distance around the polygon.

To find the perimeter of a polygon, we simply add up the lengths of all the sides.

Example 1

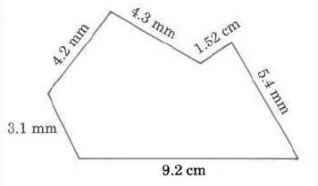
Find the perimeter of the rectangle.



Solution

$$\begin{aligned} \text{Perimeter} &= 2\text{ cm} + 5\text{ cm} + 2\text{ cm} + 5\text{ cm} \\ &= 14\text{ cm} \end{aligned}$$

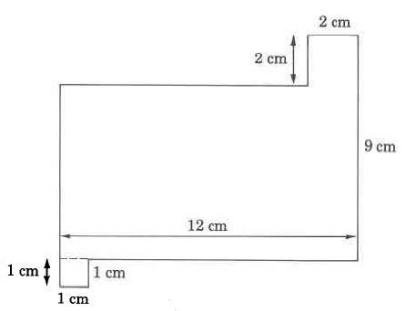
Example 2



Solution

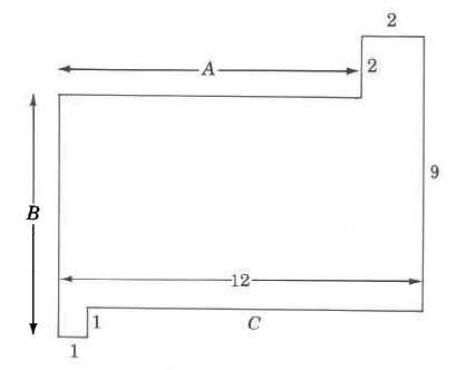
$$\begin{aligned}\text{Perimeter} &= 3.1\text{ mm}\\ &\quad 4.2\text{ mm}\\ &\quad 4.3\text{ mm}\\ &\quad 1.52\text{ mm}\\ &\quad 5.4\text{ mm}\\ &\quad +\,9.2\text{ mm}\\ &\quad 27.72\text{ mm}\end{aligned}$$

Example 3



Solution

Our first observation is that three of the dimensions are missing. However, we can determine the missing measurements using the following process. Let A, B, and C represent the missing measurements. Visualize

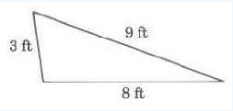


$A = 12\text{m} - 2\text{m} = 10\text{m}$
$B = 9\text{m} + 1\text{m} - 2\text{m} = 8\text{m}$
$C = 12\text{m} - 1\text{m} = 11\text{m}$

$$\begin{aligned}\text{Perimeter} &= 8\text{ m}\\ &\quad 10\text{ m}\\ &\quad 2\text{ m}\\ &\quad 2\text{ m}\\ &\quad 9\text{ m}\\ &\quad 11\text{ m}\\ &\quad 1\text{ m}\\ &\quad +\,1\text{ m}\\ &\quad 44\text{ m}\end{aligned}$$

Try It Now 1

Find the perimeter of this triangle.

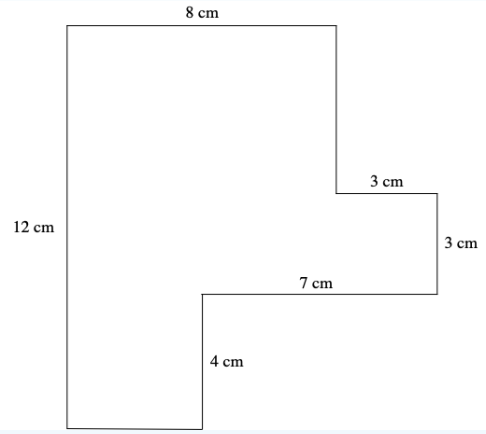


Answer

20 ft

Try It Now 2

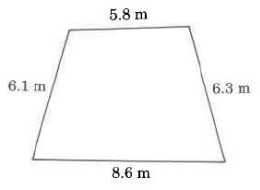
Find the perimeter of this polygon.



Answer

46 cm

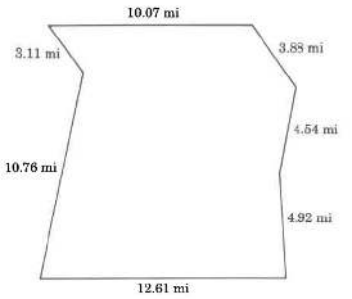
Try It Now 3



Answer

26.8 m

Try It Now 4



Answer

49.89 mi

Quite often it is necessary to multiply one denominate number by another. To do so, we multiply the number parts together and the unit parts together. For example,

$$\begin{aligned} 8\text{ in}\cdot 8\text{ in} &= 8\cdot 8\cdot \text{in}\cdot\text{in} \\ &= 64\text{ in}^2 \end{aligned}$$

$$\begin{aligned} 4\text{ mm}\cdot 4\text{ mm}\cdot 4\text{ mm} &= 4\cdot 4\cdot 4\cdot \text{mm}\cdot\text{mm}\cdot\text{mm} \\ &= 64\text{ mm}^3 \end{aligned}$$

Sometimes the product of units has a physical meaning. In this section, we will examine the meaning of the products $(\text{length unit})^2$ and $(\text{length unit})^3$

The Meaning and Notation for Area

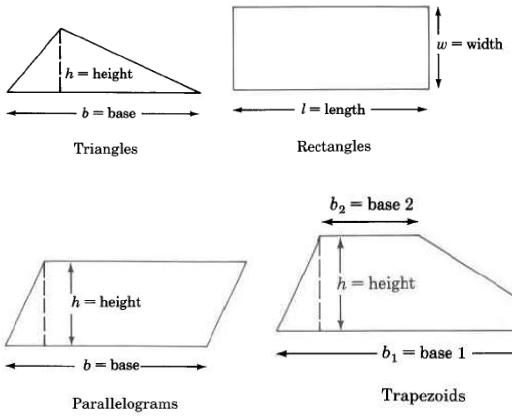
The product $(\text{length unit})\cdot(\text{length unit}) = (\text{length unit})^2$, or, square length unit (sq length unit), can be interpreted physically as the *area* of a surface.

Area

The **area** of a surface is the number of square length units contained in the surface.

For example, 3 sq in means that 3 squares, 1 inch on each side, can be placed precisely on some surface. (The squares may have to be cut and rearranged so they match the shape of the surface.)

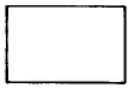
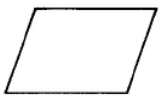
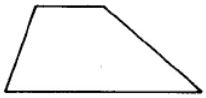
We will examine the area of the following geometric figures.



Area Formulas

We can determine the areas of these geometric figures using the following formulas.

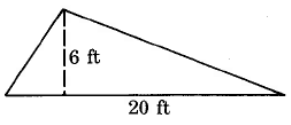
	Figure	Area Formula	Statement

	Triangle	$A_T = \frac{1}{2} \cdot b \cdot h$	Area of a triangle is one half the base times the height.
	Rectangle	$A_R = l \cdot w$	Area of a rectangle is the length times the width.
	Parallelogram	$A_P = b \cdot h$	Area of a parallelogram is base times the height.
	Trapezoid	$A_{Trap} = \frac{1}{2} \cdot (b_1 + b_2) \cdot h$	Area of a trapezoid is one half the sum of the two bases times the height.

Finding Areas of Some Common Geometric Figures

Example 4

Find the area of the triangle.



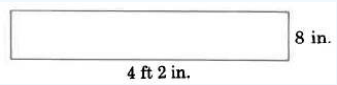
Solution

$$\begin{aligned} A_T &= \frac{1}{2} \cdot b \cdot h \\ &= \frac{1}{2} \cdot 20 \cdot 5 \text{ sq ft} \\ &= 10 \cdot 6 \text{ sq ft} \\ &= 60 \text{ sq ft} \\ &= 60 \text{ ft}^2 \end{aligned}$$

The area of this triangle is 60 sq ft, which is often written as 60 ft^2.

Example 5

Find the area of the rectangle.



Solution

Let's first convert 4 ft 2 in to inches. Since we wish to convert to inches, we'll use the unit fraction $\frac{12 \text{ in}}{1 \text{ ft}}$ since it has inches in the numerator. Then,

$$\begin{aligned} 4 \text{ ft} &= \frac{4 \text{ ft}}{1} \cdot \frac{12 \text{ in}}{1 \text{ ft}} \\ &= \frac{4 \cancel{\text{ ft}}}{1} \cdot \frac{12 \text{ in}}{1 \cancel{\text{ ft}}} \\ &= 48 \text{ in} \end{aligned}$$

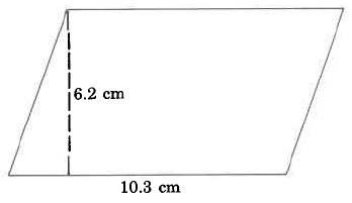
Thus, $4 \text{ ft } 2 \text{ in} = 48 \text{ in} + 2 \text{ in} = 50 \text{ in}$

$$\begin{aligned} A_R &= l \cdot w \\ &= 50 \text{ in} \cdot 8 \text{ in} \\ &= 400 \text{ sq in} \end{aligned}$$

The area of this rectangle is 400 sq in.

Example 6

Find the area of the parallelogram.



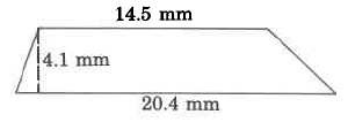
Solution

$$\begin{aligned} A_P &= b \cdot h \\ &= 10.3 \text{ cm} \cdot 6.2 \text{ cm} \\ &= 63.86 \text{ sq cm} \end{aligned}$$

The area of this parallelogram is 63.86 sq cm.

Example 7

Find the area of the trapezoid.



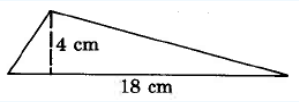
Solution

$$\begin{aligned} A_{Trap} &= \frac{1}{2} \cdot (b_1 + b_2) \cdot h \\ &= \frac{1}{2} \cdot (14.5 \text{ mm} + 20.4 \text{ mm}) \cdot (4.1 \text{ mm}) \\ &= \frac{1}{2} \cdot (34.9 \text{ mm}) \cdot (4.1 \text{ mm}) \\ &= \frac{1}{2} \cdot (143.09 \text{ sq mm}) \\ &= 71.545 \text{ sq mm} \end{aligned}$$

The area of this trapezoid is 71.545 sq mm.

Try It Now 5

Find the area of each of the following geometric figures.

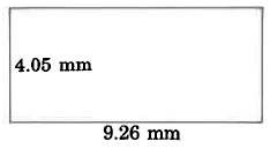


Answer

36 sq cm

Try It Now 6

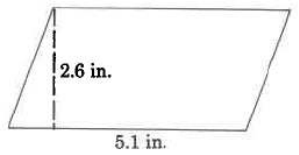
Find the area of the rectangle



Answer

37.503 sq mm

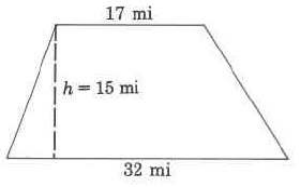
Try It Now 7



Answer

13.26 sq in.

Try It Now 8



Answer

367.5 sq mi

Composite Figures

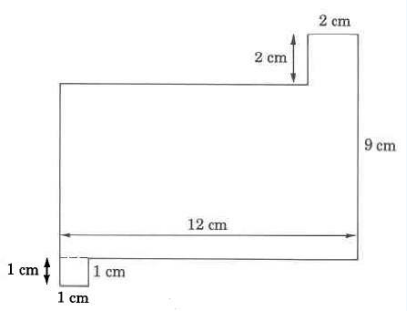
Definition: Composite Figures

A **composite figure** is a figure made up of two or more geometric figures.

When determining the area of a composite figure it is best to determine what shapes the composite figure is made of first. Once you have determined the shapes, find the area for those individual shapes and add them together.

Example 8

Find the area of the composite figure.



Solution

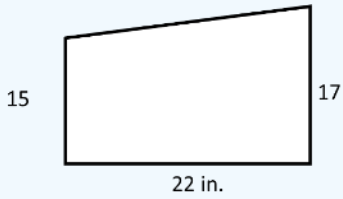
This figure is made up of two squares and one rectangle. There is a square with 1 cm sides and one with 2 cm sides. The rectangle has a 12 cm side, but to figure out the other side we need to subtract 9 cm$-$2 cm $=$7 cm.

$$\begin{aligned} A_{figure} &= l \cdot w + l \cdot w + l \cdot w \\ &= 1 \text{ cm} \cdot 1 \text{ cm} + 2 \text{ cm} \cdot 2 \text{ cm} + 12 \text{ cm} \cdot 7 \text{ cm} \\ &= 1 \text{ cm}^2 + 2 \text{ cm}^2 + 84 \text{ cm}^2 \\ &= 87 \text{sq cm} \end{aligned}$$

The area of the composite figure is 87 sq. cm.

Try It Now 9

Find the area of the composite figure.



Answer

This shape is composed of a rectangle and a triangle.

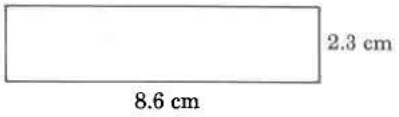
The area of the composite figure is 352 sq in.

This page titled 6.2.1: Perimeter and Area of Geometric Figures is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

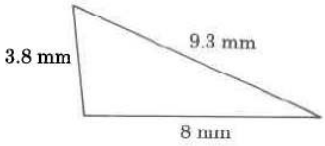
6.2.1.1: Exercises

1. Find each perimeter.

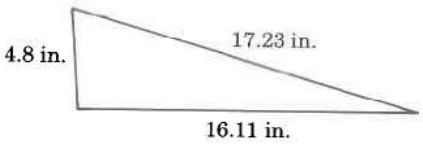
a.



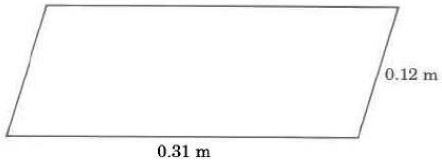
b.



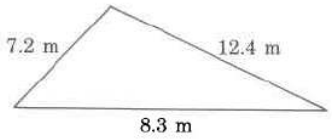
c.



d.

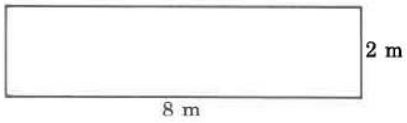


e.

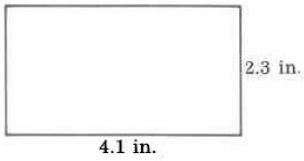


2. Find the area for the following:

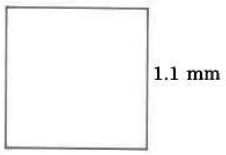
a.



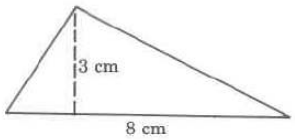
b.



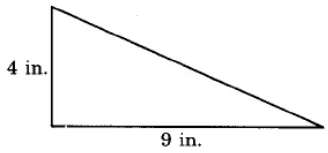
c.



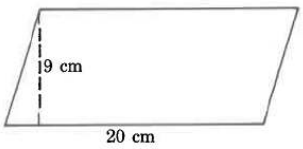
d.



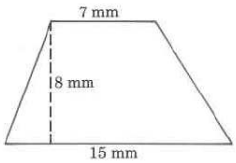
e.



f.

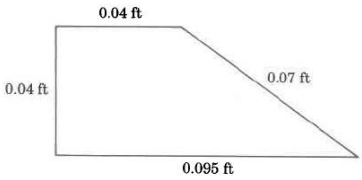


g.

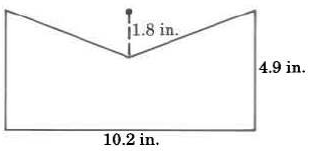


3. Find the area for the following figures below:

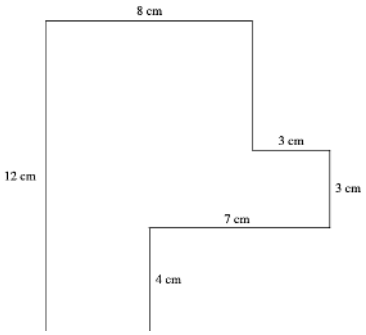
a.



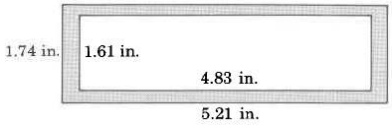
b.



c.



4. Find the area of the shaded region.



This page titled 6.2.1.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.2.2: Circles

6.2.2 Learning Objectives

- Identify the radius and diameter of a circle.
- Use π or its approximate value in formulas.
- Find the circumference of a circle.
- Find the area of a circle.
- Find the perimeter or area of composite figures involving circles/

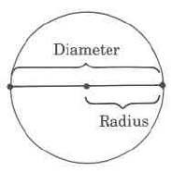
Circumference/Diameter/Radius

Definition: Part of a Circle

The **circumference** of a circle is the distance around the circle.

A **diameter** of a circle is any line segment that passes through the center of the circle and has its endpoints on the circle.

A **radius** of a circle is any line segment having as its endpoints the center of the circle and a point on the circle.
The radius is one half the diameter.



The Number π

The symbol π, read "pi," represents the nonterminating, nonrepeating decimal number 3.14159 … . This number has been computed to millions of decimal places without the appearance of a repeating block of digits.

For computational purposes, π is often approximated as 3.14. We will write $\pi \approx 3.14$ to denote that π is approximately equal to 3.14. The symbol "≈" means "approximately equal to." Use the π key on your calculator when evaluating for the best approximation.

Formulas

To find the circumference of a circle, we need only know its diameter or radius. We then use a formula for computing the circumference of the circle.

Definition: Formula

A **formula** is a rule or method for performing a task. In mathematics, a formula is a rule that directs us in computations.

Formulas are usually composed of letters that represent important, but possibly unknown, quantities.

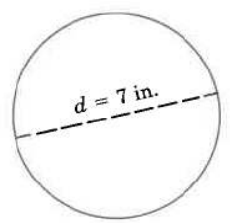
If C, d, and r represent, respectively, the circumference, diameter, and radius of a circle, then the following two formulas give us directions for computing the circumference of the circle.

Circumference Formulas

1. $C = \pi d$
2. $C = 2\pi r$

Example 1

Find the exact circumference of the circle.



Solution

Use the formula $C = \pi d$.

$C = \pi \cdot 7 \text{ in}$

By commutativity of multiplication,

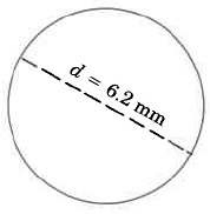
$C = 7 \text{ in} \cdot \pi$

$C = 7\pi \text{ in}$, exactly

This result is exact since π has not been approximated.

Example 2

Find the approximate circumference of the circle.



Solution

Use the formula $C = \pi d$.

$C \approx (3.14)(6.2)$

$C \approx 19.648 \text{ mm}$

This result is approximate since π has been approximated by 3.14.

Example 3

Find the approximate circumference of a circle with radius 18 inches.

Solution

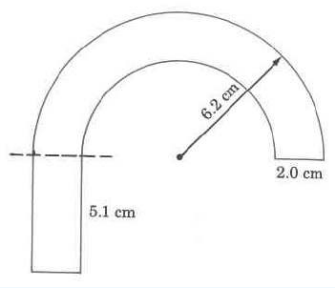
Since we're given that the radius, r, is 18 in, we'll use the formula $C = 2\pi r$.

$C \approx (2)(3.14)(18 \text{ in})$

$C \approx 113.04 \text{ in}$

Example 4

Find the approximate area of the figure.



Solution

We notice that we have two semicircles (half circles).

The larger radius is 6.2 cm.

The smaller radius is $6.2\text{ cm} - 2.0\text{ cm} = 4.2\text{ cm}$.

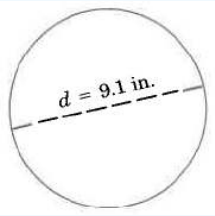
The width of the bottom part of the rectangle is 2.0 cm.

$$
\begin{array}{rrll}
\text{Perimeter} & = & 2.0\text{ cm} & \\
 & & 5.1\text{ cm} & \\
 & & 2.0\text{ cm} & \\
 & & 5.1\text{ cm} & \\
 & & (0.5)\cdot(2)\cdot(3.14)\cdot(6.2\text{ cm}) & \text{Circumference of outer semicircle.} \\
 & + & \underline{(0.5)\cdot(2)\cdot(3.14)\cdot(4.2\text{ cm})} & \text{Circumference of inner semicircle.} \\
 & & & 6.2\text{ cm} - 2.0\text{ cm} = 4.2\text{ cm} \\
 & & & \text{The } 0.5 \text{ appears because we want the} \\
 & & & \text{perimeter of only half a circle.}
\end{array}
$$

$$
\begin{array}{rrr}
\text{Perimeter} & \approx & 2.0\text{ cm} \\
 & & 5.1\text{ cm} \\
 & & 2.0\text{ cm} \\
 & & 5.1\text{ cm} \\
 & & 19.468\text{ cm} \\
 & & \underline{+13.188\text{ cm}} \\
 & & 48.856\text{ cm}
\end{array}
$$

Try It Now 1

Find the exact circumference of the circle.

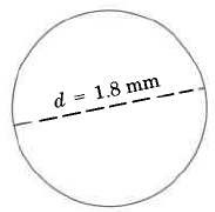


Answer

$9.1\pi\text{ in}$

Try It Now 2

Find the approximate circumference of the circle.



Answer

5.652 mm

Try It Now 3

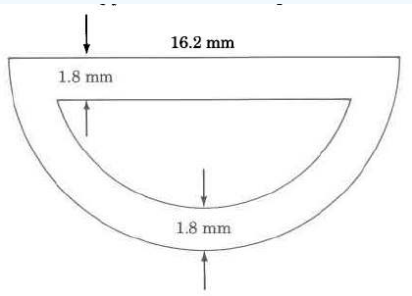
Find the approximate circumference of the circle with radius 20.1 m.

Answer

126.228 m

Try It Now 4

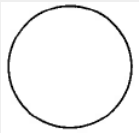
Find the approximate outside perimeter of



Answer

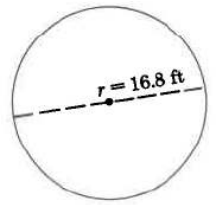
41.634 mm

Area of a Circle

	Figure	Area Formula	Statement
	Circle	$A_c = \pi r^2$	Area of a circle is π times the square of the radius.

Example 5

Find the approximate area of the circle.



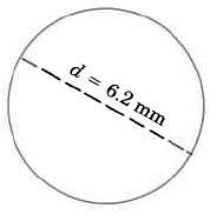
Solution

$$\begin{aligned} A_c &= \pi \cdot r^2 \\ &\approx (3.14) \cdot (16.8 \text{ ft})^2 \\ &\approx (3.14) \cdot (282.24 \text{ sq ft}) \\ &\approx 888.23 \text{ sq ft} \end{aligned}$$

The area of this circle is approximately 886.23 sq ft.

Example 6

Find the approximate area of the circle.



Solution

In this case we are given the diameter instead of the radius. We need to find the radius before we can calculate the area.

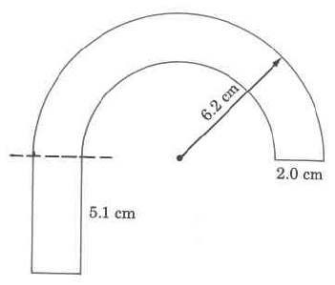
$$r = \frac{d}{2} = \frac{6.2}{2} = 3.1\text{mm}$$

$$\begin{aligned} A_c &= \pi \cdot r^2 \\ &\approx (3.14) \cdot (6.2 \text{ ft})^2 \\ &\approx (3.14) \cdot (38.44 \text{ sq mm}) \\ &\approx 120.7 \text{ sq mm} \end{aligned}$$

The area of this circle is approximately 120.7 sq mm.

Example 4

Find the approximate area of the figure.



Solution

We notice that we have two semicircles (half circles) and a small rectangle

The larger radius is 6.2 cm.

The smaller radius is 6.2 cm - 2.0 cm $= 4.2$ cm.

The width of the bottom part of the rectangle is 2.0 cm.

The area of this shape will be $A =$ area of the rectangle $+$ area of the outer semicircle $-$ area of the inner semicircle

Area of the rectangle: $A = 5.1 \cdot 2 = 10.2$ cm

Area of the outer semicircle $A = \dfrac{\pi(6.2)^2}{2} = 9.7$ cm

Area of the inner semicircle $A = \dfrac{\pi(4.2)^2}{2} = 6.6$ cm

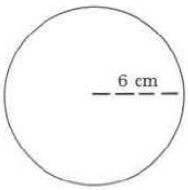
$A =$ area of the rectangle $+$ area of the outer semicircle $-$ area of the inner semicircle $= 10.2\text{cm} + 9.7\text{cm} - 6.6\text{cm} = 13.3\text{cm}$

This page titled 6.2.2: Circles is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

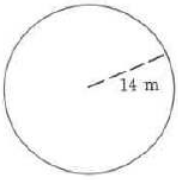
6.2.2.1: Exercises

1. Find the area and circumference of the following figures.

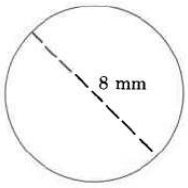
a.



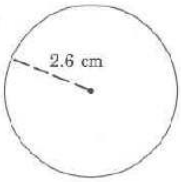
b.



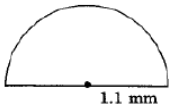
c.



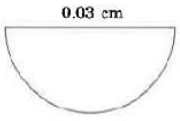
d.



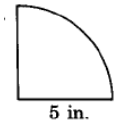
e.



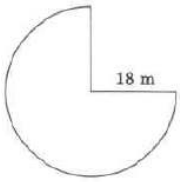
f.



g.

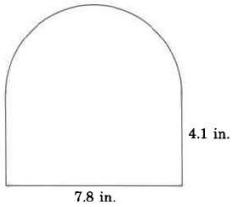


h.

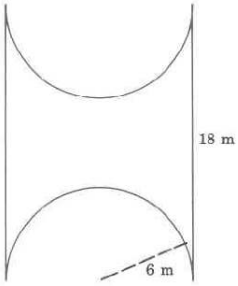


2. Find the area and perimeter of the following composite figures

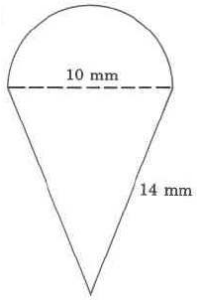
a.



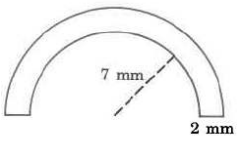
b.



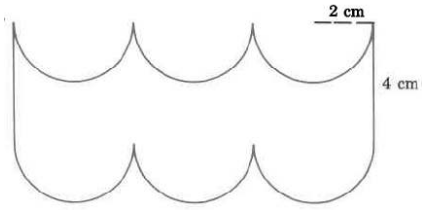
c.



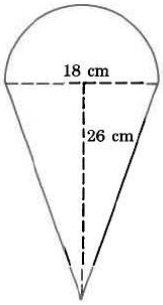
d.



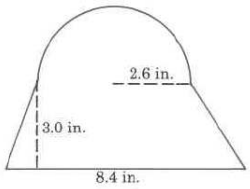
e.



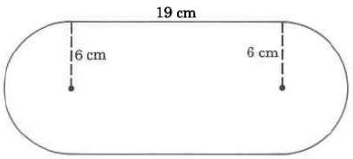
f.



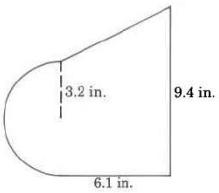
g.



h.



i.



This page titled 6.2.2.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

6.2.3: Solid Geometric Figures and Objects

6.2.3 Learning Objectives

- Find the volume of some common geometric objects

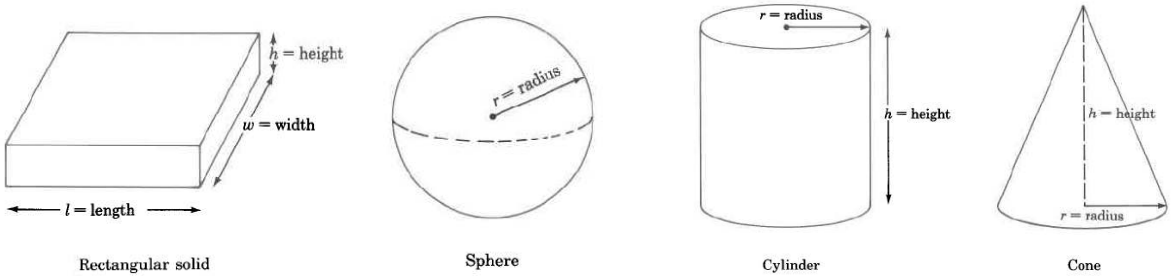
The Meaning and Notation for Volume

The product $(\text{length unit})(\text{length unit})(\text{length unit}) = (\text{length unit})^3$, or cubic length unit (cu length unit), can be interpreted physically as the *volume* of a three-dimensional object.

Definition: Volume

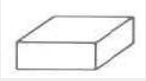
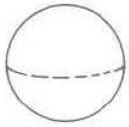
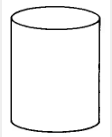
The **volume** of an object is the amount of cubic length units contained in the object.

For example, 4 cu mm means that 4 cubes, 1 mm on each side, would precisely fill some three-dimensional object. (The cubes may have to be cut and rearranged so they match the shape of the object.)



Rectangular solid | Sphere | Cylinder | Cone

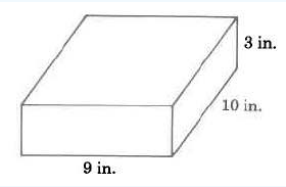
Volume Formulas

	Figure	Volume Formula	Statement
	Rectangular solid	$V_R = l \cdot w \cdot h$ $= (\text{area of base}) \cdot (\text{height})$	The volume of a rectangular solid is the length times the width times the height.
	Sphere	$V_s = \frac{4}{3} \cdot \pi \cdot r^3$	The volume of a sphere is $\frac{4}{3}$ times π times the cube of the radius.
	Cylinder	$V_{Cyl} = \pi \cdot r^2 \cdot h$ $= (\text{area of base}) \cdot (\text{height})$	The volume of a cylinder is π times the square of the radius times the height.
	Cone	$V_c = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$ $= (\text{area of base}) \cdot (\text{height})$	The volume of a cone is $\frac{1}{3}$ times π times the square of the radius times the height.

Finding Volumes of Some Common Geometric Objects

Example 1

Find the volume of the rectangular solid.



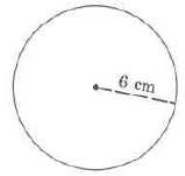
Solution

$$\begin{aligned} V_R &= l \cdot w \cdot h \\ &= 9 \text{ in} \cdot 10 \text{ in.} \cdot 3 \text{ in} \\ &= 270 \text{ cu in} \\ &= 270 \text{ in}^3 \end{aligned}$$

The volume of this rectangular solid is 270 cu in.

Example 2

Find the approximate volume of the sphere.



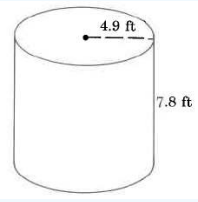
Solution

$$\begin{aligned} V_S &= \frac{4}{3} \cdot \pi \cdot r^3 \\ &\approx (\frac{4}{3}) \cdot (3.14) \cdot (6 \text{ cm})^3 \\ &\approx (\frac{4}{3}) \cdot (3.14) \cdot (216 \text{ cu cm}) \\ &\approx 904.32 \text{ cu cm} \end{aligned}$$

The approximate volume of this sphere is 904.32 cu cm, which is often written as 904.32 cm^3.

Example 3

Find the approximate volume of the cylinder.



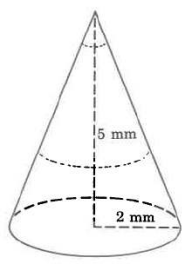
Solution

$$\begin{aligned} V_{Cyl} &= \pi \cdot r^2 \cdot h \\ &\approx (3.14) \cdot (4.9 \text{ ft})^2 \cdot (7.8 \text{ ft}) \\ &\approx (3.14) \cdot (24.01 \text{ sq ft}) \cdot (7.8 \text{ ft}) \\ &\approx (3.14) \cdot (187.278 \text{ cu ft}) \\ &\approx 588.05292 \text{ cu ft} \end{aligned}$$

The volume of this cylinder is approximately 588.05292 cu ft. The volume is approximate because we approximated π.

Example 4

Find the approximate volume of the cone. Round to two decimal places.



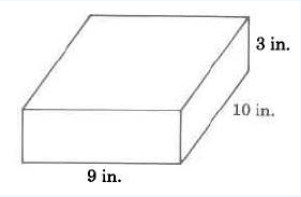
Solution

$$\begin{aligned} V_c &= \frac{1}{3} \cdot \pi \cdot r^2 \cdot h \\ &\approx (\frac{1}{3}) \cdot (3.14) \cdot (2 \text{ mm})^2 \cdot (5 \text{ mm}) \\ &\approx (\frac{1}{3}) \cdot (3.14) \cdot (4 \text{ sq mm}) \cdot (5 \text{ mm}) \\ &\approx (\frac{1}{3}) \cdot (3.14) \cdot (20 \text{ cu mm}) \\ &\approx 20.9\overline{3} \text{ cu mm} \\ &\approx 20.93 \text{ cu mm} \end{aligned}$$

The volume of this cone is approximately 20.93 cu mm. The volume is approximate because we approximated π.

Try It Now 1

Find the volume of the rectangular solid.

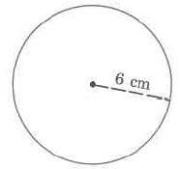


Answer

21 cu in

Try It Now 2

Find the volume of the sphere. Use the π key on your calculator to find the approximate volume.

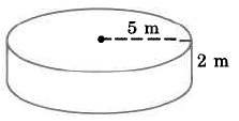


Answer

904.32 cu ft

Try It Now 3

Find the volume of the cylinder. Use the π key on your calculator to find the approximate volume.

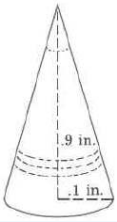


Answer

157 cu m

Try It Now 4

Find the volume of the cone. Use the π key on your calculator to find the approximate volume.



Answer

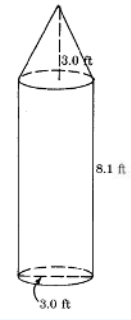
0.00942 cu in

Finding the Volume of Composite Figures

Similar to finding the area of composite figures, we will first determine the separate shapes that the figure is made of. Then use the formulas for the volumes of the individual shapes and add them together.

Example 5

Find the approximate volume of the composite figure. Use the π key on your calculator to find the approximate volume if necessary. Round to two decimal places.



Solution

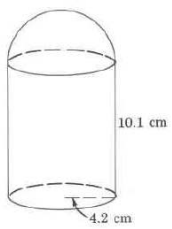
This shape is made up of a cone and a cylinder. We can tell from the base of the cylinder, that the diameter of the cone is 3.0 ft.

$$\begin{aligned} V &= V_{cone} + V_{cylinder} \\ &= \frac{1}{3} \cdot \pi \cdot r^2 \cdot h + \pi \cdot r^2 \cdot h \\ &\approx (\frac{1}{3}) \cdot \pi \cdot (1.5 \text{ ft})^2 \cdot (3.0 \text{ ft}) + \pi \cdot 1.5 \text{ ft}^2 \cdot 8.1 \text{ ft}. \\ &\approx (\frac{1}{3}) \cdot \pi \cdot (2.25 \text{ ft}^2) \cdot (3.0 \text{ ft}) + \pi \cdot 2.25 \text{ ft}^2 \cdot 8.1 \text{ ft}. \\ &\approx (\frac{1}{3}) \cdot \pi \cdot (6.75 \text{ cu ft}) + \pi \cdot 18.225 \text{ cu ft} \\ &\approx 7.06858 \text{ cu ft} + 57.25553 \text{ cu ft} \\ &\approx 64.32 \text{ cu ft} \end{aligned}$$

The volume of this figure is approximately 64.32 cu ft. The volume is approximate because we approximated π.

Try It Now 6

Find the volume of the composite figure.



Answer

This shape is composed of a hemisphere and a cylinder.

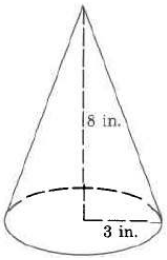
$V \approx 870$ cu cm

This page titled 6.2.3: Solid Geometric Figures and Objects is shared under a CC BY license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

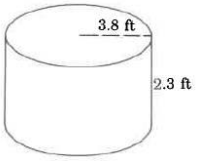
6.2.3.1: Exercises

1. Find the approximate volume

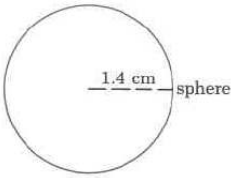
a.



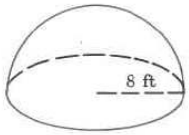
b.



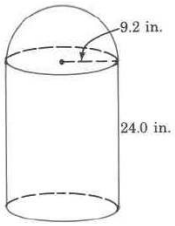
c.



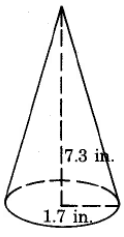
d.



e.

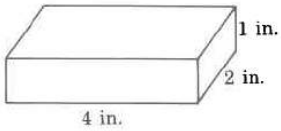


f.

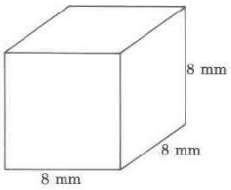


2. Find the exact volume.

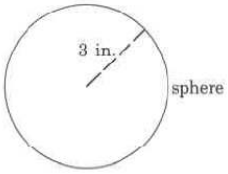
a.


1 in.
2 in.
4 in.

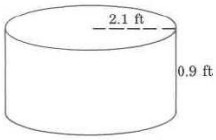
b.


8 mm
8 mm
8 mm

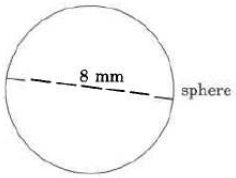
c.


3 in.
sphere

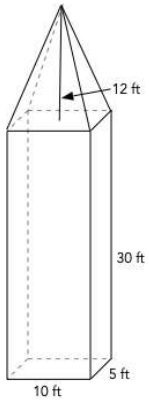
d.


2.1 ft
0.9 ft

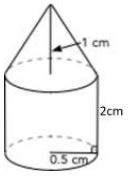
e.


8 mm
sphere

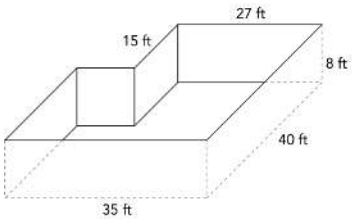
f.


12 ft
30 ft
5 ft
10 ft

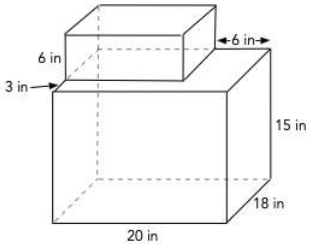
g.



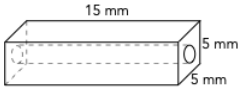
3. Sketch a rectangular prism.
4. How much soil must be removed from a lot to create a basement for a new house?



5. Two boxes in the shape of rectangular prisms are stacked one on top of the other. What is the combined volume of the boxes?



6. A glass bead used to make jewelry is a rectangular prism with square ends. The stringing hole is 1.5 mm in diameter. What is the volume of the glass bead?



This page titled 6.2.3.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

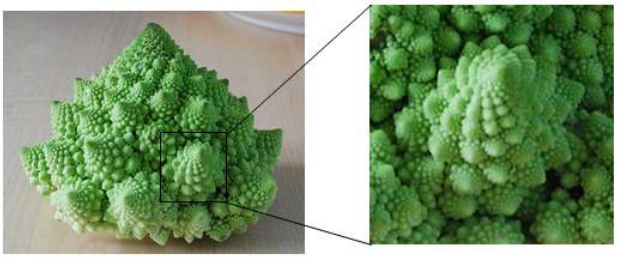
6.3: Fractals

Fractals are mathematical sets, usually obtained through recursion, that exhibit interesting dimensional properties. We'll explore what that sentence means through the rest of the chapter. For now, we can begin with the idea of self-similarity, a characteristic of most fractals.

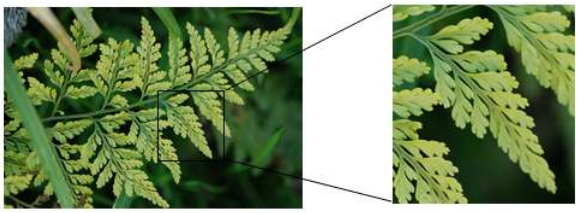
Definition: Self-Similarity

A shape is **self-similar** when it looks essentially the same from a distance as it does closer up.

Self-similarity can often be found in nature. In the Romanesco broccoli pictured below[1], if we zoom in on part of the image, the piece remaining looks similar to the whole.



Likewise, in the fern frond below[2], one piece of the frond looks similar to the whole.



Similarly, if we zoom in on the coastline of Portugal[3], each zoom reveals previously hidden detail, and the coastline, while not identical to the view from further way, does exhibit similar characteristics.



This self-similar behavior can be replicated through recursion: repeating a process over and over.

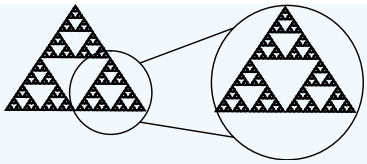
Example 1

Suppose that we start with a filled-in triangle. We connect the midpoints of each side and remove the middle triangle. We then repeat this process.



Initial Step 1 Step 2 Step 3

If we repeat this process, the shape that emerges is called the Sierpinski gasket. Notice that it exhibits self-similarity – any piece of the gasket will look identical to the whole. In fact, we can say that the Sierpinski gasket contains three copies of itself, each half as tall and wide as the original. Of course, each of those copies also contains three copies of itself.



We can construct other fractals using a similar approach. To formalize this a bit, we're going to introduce the idea of initiators and generators.

Definition: Initiators and Generators

- An **initiator** is a starting shape
- A **generator** is an arranged collection of scaled copies of the initiator

To generate fractals from initiators and generators, we follow a simple rule:

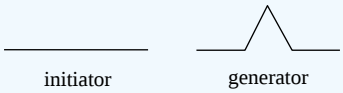
Fractal Generation Rule

At each step, replace every copy of the initiator with a scaled copy of the generator, rotating as necessary

This process is easiest to understand through example.

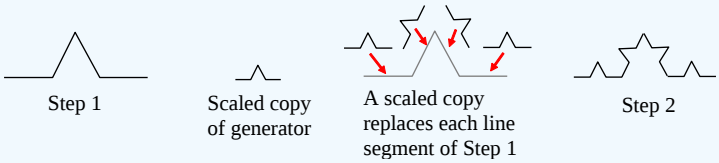
Example 2

Use the initiator and generator shown to create the iterated fractal.

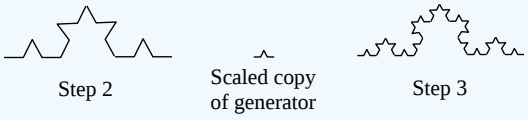


Solution

This tells us to, at each step, replace each line segment with the spiked shape shown in the generator. Notice that the generator itself is made up of 4 copies of the initiator. In step 1, the single line segment in the initiator is replaced with the generator. For step 2, each of the four line segments of step 1 is replaced with a scaled copy of the generator:



This process is repeated to form Step 3. Again, each line segment is replaced with a scaled copy of the generator.



Notice that since Step 0 only had 1 line segment, Step 1 only required one copy of Step 0.

Since Step 1 had 4 line segments, Step 2 required 4 copies of the generator.

Step 2 then had 16 line segments, so Step 3 required 16 copies of the generator.

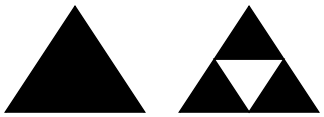
Step 4, then, would require $16 \times 4 = 64$ copies of the generator.

The shape resulting from iterating this process is called the **Koch curve**, named for Helge von Koch who first explored it in 1904.



Koch curve

Notice that the Sierpinski gasket can also be described using the initiator-generator approach



Initiator Generator

Example 3

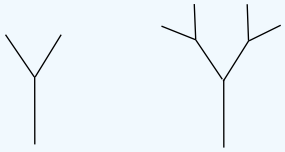
Use the initiator and generator below, however only iterate on the "branches." Sketch several steps of the iteration.



initiator generator

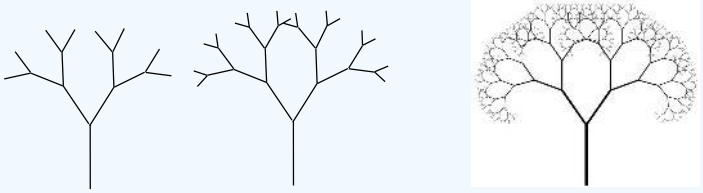
Solution

We begin by replacing the initiator with the generator. We then replace each "branch" of Step 1 with a scaled copy of the generator to create Step 2.



Step 1 Step 2

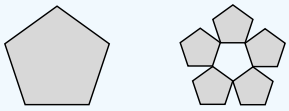
We can repeat this process to create later steps. Repeating this process can create intricate tree shapes[1].



Step 3 Step 4 Final shape

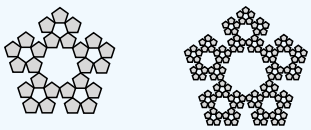
Try it Now 1

Use the initiator and generator shown to produce the next two stages



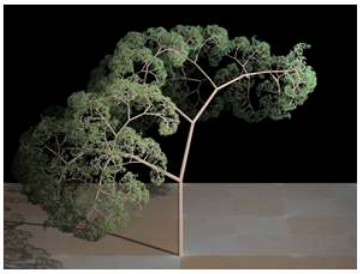
Initiator Generator

Answer



Step 2 Step 3

Using iteration processes like those above can create a variety of beautiful images evocative of nature[2][3].





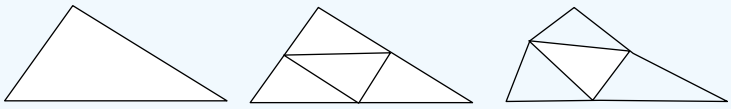
More natural shapes can be created by adding in randomness to the steps.

Example 4

Create a variation on the Sierpinski gasket by randomly skewing the corner points each time an iteration is made.

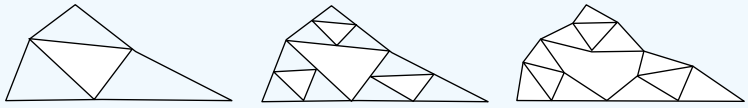
Solution

Suppose we start with the triangle below. We begin, as before, by removing the middle triangle. We then add in some randomness.



Step 0 | Step 1 | Step 1 with randomness

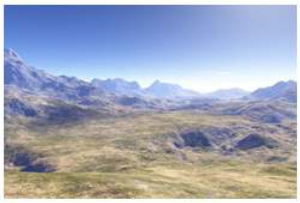
We then repeat this process.



Step 1 with randomness | Step 2 | Step 2 with randomness

Continuing this process can create mountain-like structures.

The landscape below was created using fractals, then colored and textured.



[1] http://www.flickr.com/photos/visualarts/5436068969/

[2] en.Wikipedia.org/wiki/File:Fr...e_b_-_2%29.jpg

[3] en.Wikipedia.org/wiki/File:Ba...-_4_states.PNG

[4] en.Wikipedia.org/wiki/File:Fr...lLandscape.jpg

[1] en.Wikipedia.org/wiki/File:Ca...ractal_AVM.JPG

[2] http://www.flickr.com/photos/cjewel/3261398909/

[3] Openstreetmap.org, CC-BY-SA

This page titled 6.3: Fractals is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **15.1: Fractals** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.
- **15.2: Iterated Fractals** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

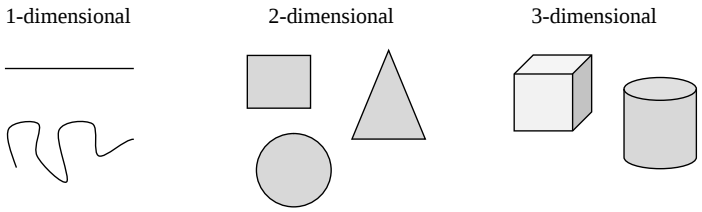
6.3.1: Fractal Dimension

6.3.2 Learning Objectives

- Determine the dimension of a fractal

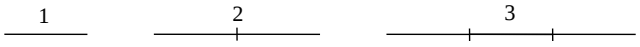
In addition to visual self-similarity, fractals exhibit other interesting properties. For example, notice that each step of the Sierpinski gasket iteration removes one quarter of the remaining area. If this process is continued indefinitely, we would end up essentially removing all the area, meaning we started with a 2-dimensional area, and somehow end up with something less than that, but seemingly more than just a 1-dimensional line.

To explore this idea, we need to discuss dimension. Something like a line is 1-dimensional; it only has length. Any curve is 1-dimensional. Things like squares and circles are 2-dimensional, since they have length and width, describing an area. Objects like boxes and cylinders have length, width, and depth, describing a volume, and are 3-dimensional.

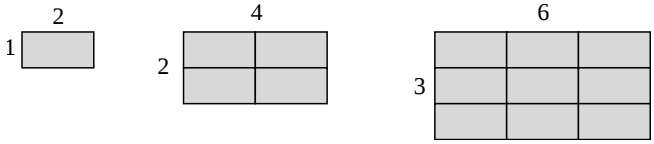


Certain rules apply for scaling objects, related to their dimension.

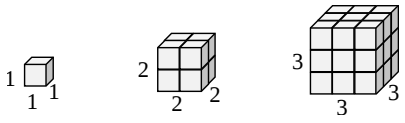
If I had a line with length 1, and wanted scale its length by 2, I would need two copies of the original line. If I had a line of length 1, and wanted to scale its length by 3, I would need three copies of the original.



If I had a rectangle with length 2 and height 1, and wanted to scale its length and width by 2, I would need four copies of the original rectangle. If I wanted to scale the length and width by 3, I would need nine copies of the original rectangle.



If I had a cubical box with sides of length 1, and wanted to scale its length and width by 2, I would need eight copies of the original cube. If I wanted to scale the length and width by 3, I would need 27 copies of the original cube.



Notice that in the 1-dimensional case, copies needed = scale.

- In the 2-dimensional case, copies needed $=$ scale^2.
- In the 3-dimensional case, copies needed $=$ scale^3.

From these examples, we might infer a pattern.

Scaling-Dimension Relation

To scale a D-dimensional shape by a scaling factor S, the number of copies C of the original shape needed will be given by:

Copies $=$ Scale^{Dimension}, or $C = S^D$

Example 5

Use the scaling-dimension relation to determine the dimension of the Sierpinski gasket.

Solution

Suppose we define the original gasket to have side length 1. The larger gasket shown is twice as wide and twice as tall, so has been scaled by a factor of 2.



1 2

Notice that to construct the larger gasket, 3 copies of the original gasket are needed.

Using the scaling-dimension relation $C = S^D$, we obtain the equation $3 = 2^D$

since $2^1 = 2$ and $2^2 = 4$, we can immediately see that D is somewhere between 1 and 2; the gasket is more than a 1 -dimensional shape, but we've taken away so much area its now less than 2-dimensional.

Solving the equation $3 = 2^D$ requires logarithms. If you studied logarithms earlier, you may recall how to solve this equation (if not, just skip to the box below and use that formula):

$$\begin{aligned} 3 &= 2^D && \text{Take the logarithm of both sides} \\ \log(3) &= \log\left(2^D\right) && \text{Use the exponent property of logs} \\ \log(3) &= D\log(2) && \text{Divide by } \log(2) \\ D &= \frac{\log(3)}{\log(2)} \approx 1.585 && \text{The dimension of the gasket is about } 1.585 \end{aligned} \tag{6.3.1.1}$$

Scaling-Dimension Relation, to find Dimension

To find the dimension D of a fractal, determine the scaling factor S and the number of copies C of the original shape needed, then use the formula

$$D = \frac{\log(C)}{\log(S)}$$

Try it Now 2

Determine the fractal dimension of the fractal produced using the initiator and generator



Initiator Generator

Answer



1 3

Scaling the fractal by a factor of 3 requires 5 copies of the original. $D = \frac{\log(5)}{\log(3)} \approx 1.465$

This page titled 6.3.1: Fractal Dimension is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **15.3: Fractal Dimension** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

6.3.2: Exercises

Section 6.2.3 Exercises

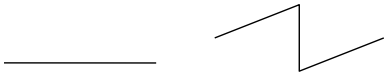
1. Using the initiator and generator shown, draw the next two stages of the iterated fractal.

a.



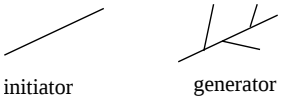
initiator generator

b.



initiator generator

c.



initiator generator

d.



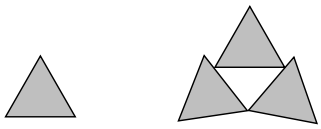
initiator generator

e.



initiator generator

f.



initiator generator

2. Create your own version of Sierpinski gasket with added randomness.
3. Create a version of the branching tree fractal from example #3 with added randomness.
4. Determine the fractal dimension of the Koch curve.
5. Determine the fractal dimension of the curve generated in exercise #1
6. Determine the fractal dimension of the Sierpinski carpet generated in exercise #5
7. Determine the fractal dimension of the Cantor set generated in exercise #4

This page titled 6.3.2: Exercises is shared under a CC BY-SA license and was authored, remixed, and/or curated by Leah Griffith, Veronica Holbrook, Johnny Johnson & Nancy Garcia.

- **15.6: Exercises** by David Lippman is licensed CC BY-SA 3.0. Original source: http://www.opentextbookstore.com/mathinsociety.

Index

A

arithmetic mean
5.10: Measures of Central Tendency

B

blind study
5.6: Experiments

C

categorical data
5.3: Categorizing data
census
5.2: Populations and Samples
complement
3.1.1: Union, Intersection, and Complement
Complementary events
4.3: Working with Events
confounding
5.6: Experiments
control group
5.6: Experiments
Correlation
5.9: Correlation and Causation, Scatter Plots
cube root
1.4: Exponents

D

denominate numbers
6.1.5: Simplification of Denominate Numbers
double blind study
5.6: Experiments

E

elements
3.1: Basics of Sets
experiment
5.6: Experiments
exponential notation
1.4: Exponents
exponents
1.4: Exponents

F

fractal dimension
6.3.1: Fractal Dimension
frequency table
5.7: Presenting Data Graphically

I

intersection
3.1.1: Union, Intersection, and Complement

M

mean
5.10: Measures of Central Tendency
mode
5.10: Measures of Central Tendency
mutually exclusive
4.3: Working with Events

O

observational study
5.6: Experiments
outliers
5.10: Measures of Central Tendency

P

Pareto chart
5.7: Presenting Data Graphically
percent
1.5: Percents
population
5.2: Populations and Samples
principal
2.3: Simple Interest
probability
4.1: Introduction to Probability
proper subset
3.1: Basics of Sets

Q

Quantitative Data
5.3: Categorizing data
Quota sampling
5.4: Sampling methods

R

radical sign
1.4: Exponents
random sample
5.4: Sampling methods
roots
1.4: Exponents

S

Sampling Bias
5.4: Sampling methods
set
3.1: Basics of Sets
simple interest
2.3: Simple Interest
square root
1.4: Exponents
stratified sampling
5.4: Sampling methods
subset
3.1: Basics of Sets

T

target population
5.2: Populations and Samples

U

union
3.1.1: Union, Intersection, and Complement
universal set
3.1.1: Union, Intersection, and Complement

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: Math 150: Survey of Mathematics

Webpages: 135

All licenses found:

- Undeclared: 53.3% (72 pages)
- CC BY-SA 4.0: 36.3% (49 pages)
- CC BY 4.0: 10.4% (14 pages)

By Page

- Math 150: Survey of Mathematics - *Undeclared*
  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Authors - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Foundations - *Undeclared*
    - 1.1: Introduction - *CC BY-SA 4.0*
    - 1.2: Problem Solving and Estimating - *CC BY-SA 4.0*
      - 1.2.1: Exercises - *Undeclared*
    - 1.3: Real Numbers - *Undeclared*
      - 1.3.1: Integers - *CC BY 4.0*
        - 1.3.1.1: Exercises - *Undeclared*
      - 1.3.2: Rational and Irrational Numbers - *CC BY 4.0*
        - 1.3.2.1: Exercises - *Undeclared*
      - 1.3.3: Associative, Commutative, and Distributive Properties - *CC BY 4.0*
        - 1.3.3.1: Exercises - *Undeclared*
      - 1.3.4: Order of Operations - *CC BY 4.0*
        - 1.3.4.1: Exercises - *Undeclared*
    - 1.4: Exponents - *CC BY 4.0*
      - 1.4.1: Rules of Exponents - *CC BY 4.0*
        - 1.4.1.1: 1.4.1 and 1.4.2 Exercises - *Undeclared*
      - 1.4.2: The Power Rules for Exponents - *CC BY 4.0*
        - 1.4.2.1: Exercises - *Undeclared*
      - 1.4.3: Scientific Notation - *CC BY 4.0*
        - 1.4.3.1: Exercises - *Undeclared*
    - 1.5: Percents - *CC BY-SA 4.0*
      - 1.5.1: Exercises - *CC BY-SA 4.0*
      - 1.5.2: Proportions and Rates - *CC BY-SA 4.0*
      - 1.5.3: Exercises - *Undeclared*
  - 2: Finances - *Undeclared*
    - 2.1: Introduction - *CC BY-SA 4.0*
    - 2.2: Taxes - *CC BY-SA 4.0*
      - 2.2.1: Income Taxation - *CC BY-SA 4.0*
      - 2.2.2: Exercises - *Undeclared*
    - 2.3: Simple Interest - *CC BY-SA 4.0*
      - 2.3.1: Exercises - *Undeclared*
    - 2.4: Compound Interest - *CC BY-SA 4.0*
      - 2.4.1: Exercises - *Undeclared*
    - 2.5: Annuities - *CC BY-SA 4.0*
      - 2.5.1: Exercises - *Undeclared*
    - 2.6: Payout Annuities - *CC BY-SA 4.0*
      - 2.6.1: Exercises - *Undeclared*
    - 2.7: Installment Loans - *CC BY-SA 4.0*
      - 2.7.1: Exercises - *Undeclared*
    - 2.8: Credit Cards - *Undeclared*
      - 2.8.1: Exercises - *Undeclared*
    - 2.9: Which equation to use? - *CC BY-SA 4.0*
      - 2.9.1: Exercises - *Undeclared*
  - 3: Logic - *Undeclared*
    - 3.1: Basics of Sets - *CC BY-SA 4.0*
      - 3.1.1: Union, Intersection, and Complement - *CC BY-SA 4.0*
        - 3.1.1.1: Exercises - *Undeclared*
      - 3.1.2: Venn Diagrams - *CC BY-SA 4.0*
        - 3.1.2.1: Exercises - *CC BY-SA 4.0*
    - 3.2: Logic - *CC BY-SA 4.0*
      - 3.2.1: Boolean Logic - *CC BY-SA 4.0*
        - 3.2.1.1: Exercises - *Undeclared*
      - 3.2.2: Conditional Statements - *CC BY-SA 4.0*
        - 3.2.2.1: Exercises - *Undeclared*
      - 3.2.3: Quantified Statements - *CC BY-SA 4.0*
        - 3.2.3.1: Exercises - *Undeclared*

- 3.2.4: Truth Tables- Conjunction (and), Disjunction (or), Negation (not) - *CC BY-SA 4.0*
  - 3.2.4.1: Exercises - *Undeclared*
- 3.2.5: Truth Tables- Conditional, Biconditional - *CC BY-SA 4.0*
  - 3.2.5.1: Exercises - *Undeclared*
- 3.3: Arguments - *CC BY-SA 4.0*
  - 3.3.1: Forms of Valid Arguments - *CC BY-SA 4.0*
    - 3.3.1.1: Exercises - *Undeclared*
  - 3.3.2: Evaluating Deductive Arguments with Euler Diagrams - *CC BY-SA 4.0*
    - 3.3.2.1: Exercises - *Undeclared*
- 3.4: Logical Fallacies in Common Language - *CC BY-SA 4.0*
  - 3.4.1: Exercises - *Undeclared*

4: Probability - *Undeclared*
- 4.1: Introduction to Probability - *CC BY-SA 4.0*
  - 4.1.1: Exercises - *Undeclared*
- 4.2: Basic Concepts - *CC BY-SA 4.0*
  - 4.2.1: Exercises - *Undeclared*
- 4.3: Working with Events - *CC BY-SA 4.0*
  - 4.3.1: Exercises - *Undeclared*
- 4.4: Two-Way Tables - *CC BY-SA 4.0*
  - 4.4.1: Exercises - *Undeclared*
- 4.5: Counting - *CC BY-SA 4.0*
  - 4.5.1: Exercises - *Undeclared*
- 4.6: Expected Value - *CC BY-SA 4.0*
  - 4.6.1: Exercises - *Undeclared*
- 4.7: Additional Exercises - *CC BY-SA 4.0*

5: Statistics - *Undeclared*
- 5.1: What is Statistics? - *CC BY-SA 4.0*
  - 5.1.1: Exercises - *Undeclared*
- 5.2: Populations and Samples - *CC BY-SA 4.0*
  - 5.2.1: Exercises - *Undeclared*
- 5.3: Categorizing data - *CC BY-SA 4.0*
  - 5.3.1: Exercises - *Undeclared*
- 5.4: Sampling methods - *CC BY-SA 4.0*
  - 5.4.1: Exercises - *Undeclared*
- 5.5: Bias - *CC BY-SA 4.0*
  - 5.5.1: Exercises - *Undeclared*
- 5.6: Experiments - *CC BY-SA 4.0*
  - 5.6.1: Exercises - *Undeclared*
- 5.7: Presenting Data Graphically - *CC BY-SA 4.0*
  - 5.7.1: Exercises - *Undeclared*
- 5.8: Reading Graphs in Media - *CC BY-SA 4.0*
  - 5.8.1: Exercises - *Undeclared*
- 5.9: Correlation and Causation, Scatter Plots - *CC BY-SA 4.0*
  - 5.9.1: Exercises - *CC BY-SA 4.0*
- 5.10: Measures of Central Tendency - *CC BY-SA 4.0*
  - 5.10.1: Exercises - *Undeclared*

6: Measurement and Geometry - *CC BY 4.0*
- 6.1: Measurement - *Undeclared*
  - 6.1.1: Measurement and the United States System - *CC BY 4.0*
    - 6.1.1.1: Exercises - *Undeclared*
  - 6.1.2: The Metric System of Measurement - *CC BY 4.0*
    - 6.1.2.1: Exercises - *Undeclared*
  - 6.1.3: Conversion Between the Metric and US Customary Systems of Measurement - *Undeclared*
    - 6.1.3.1: Exercises - *Undeclared*
  - 6.1.4: Temperature and Time - *Undeclared*
    - 6.1.4.1: Exercises - *Undeclared*
  - 6.1.5: Simplification of Denominate Numbers - *CC BY 4.0*
    - 6.1.5.1: Exercises - *Undeclared*
- 6.2: Geometry - *Undeclared*
  - 6.2.1: Perimeter and Area of Geometric Figures - *CC BY 4.0*
    - 6.2.1.1: Exercises - *Undeclared*
  - 6.2.2: Circles - *Undeclared*
    - 6.2.2.1: Exercises - *Undeclared*
  - 6.2.3: Solid Geometric Figures and Objects - *CC BY 4.0*
    - 6.2.3.1: Exercises - *Undeclared*
- 6.3: Fractals - *CC BY-SA 4.0*
  - 6.3.1: Fractal Dimension - *CC BY-SA 4.0*
  - 6.3.2: Exercises - *CC BY-SA 4.0*

Back Matter - *Undeclared*
- Index - *Undeclared*
- Glossary - *Undeclared*
- Detailed Licensing - *Undeclared*