MATH 142: PRECALCULUS II



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Precalculus II - Math 142

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Licensing

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CHAPTER OVERVIEW

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1.0: The Power to Learn

Questions to consider

- What actually happens to me when I learn something?
- Am I aware of different types of learning?
- Do I approach studying or practicing differently depending on the desired outcome?

Welcome to one of the most empowering chapters in this book! While each chapter focuses on showing you clear paths to success as a student, this one deals specifically with what is at the core of being a student: the act of learning.

We humans have been obsessed with how we learn and understand things since ancient times. Because of this, some of our earliest recorded philosophies have tried to explain how we take in information about the world around us, how we acquire new knowledge, and even how we can be certain what we learn is correct. This obsession has produced a large number of theories, ideas, and research into how we learn. There is a great deal of information out there on the subject—some of it is very good, and some of it, while well intentioned, has been a bit misguided.

Because of this obsession with learning, over the centuries, people have continually come up with new ideas about how we acquire knowledge. The result has been that commonly held “facts” about education have been known to change frequently. Often, what was once thought to be the newest, greatest discovery about learning was debunked later on. One well-known example of this is that of corporal punishment. For most of the time formal education has existed in our society, educators truly believed that beating students when they made a mistake actually helped them learn faster. Thankfully, *birching* (striking someone with a rod made from a birch tree) has fallen out of favor in education circles, and our institutions of learning have adopted different approaches. In this chapter, not only will you learn about current learning theories that are backed by neuroscience (something we did not have back in the days of birching), but you will also learn other learning theories that did not turn out to be as effective or as thoroughly researched as once thought. That does not mean those ideas about learning are useless. Instead, in these cases you find ways to separate the valuable parts from the myths to make good learning choices.

> ***"Research has shown that one of the most influential aids in learning is an understanding about learning itself."***

What Is the Nature of Learning?

To begin with, it is important to recognize that learning is work. Sometimes it is easy and sometimes it is difficult, but there is always work involved. For many years people made the error of assuming that learning was a passive activity that involved little more than just absorbing information. Learning was thought to be a lot like copying and pasting words in a document; the student’s mind was blank and ready for an instructor to teach them facts that they could quickly take in. As it turns out, learning is much more than that. In fact, at its most rudimentary level, it is an actual process that physically changes our brains. Even something as simple as learning the meaning of a new word requires the physical alteration of neurons and the creation of new paths to receptors. These new electrochemical pathways are formed and strengthened as we utilize, practice, or remember what we have learned. If the new skill or knowledge is used in conjunction with other things we have already learned, completely different sections of the brain, our nerves, or our muscles may be tied in as a part of the process. A good example of this would be studying a painting or drawing that depicts a scene from a story or play you are already familiar with. Adding additional connections, memories, and mental associations to things you already know something about expands your knowledge and understanding in a way that cannot be reversed. In essence, it can be said that every time we learn something new we are no longer the same.

In addition to the physical transformation that takes place during learning, there are also a number of other factors that can influence how easy or how difficult learning something can be. While most people would assume that the ease or difficulty would really depend on what is being learned, there are actually several other factors that play a greater role.

In fact, research has shown that one of the most influential factors in learning is a clear understanding about learning itself. This is not to say that you need to become neuroscientists in order to do well in school, but instead, knowing a thing or two about learning and how we learn in general can have strong, positive results for your own learning. This is called *metacognition* (i.e., thinking about thinking).

Some of the benefits to how we learn can be broken down into different areas such as

- attitude and motivation toward learning,
- types of learning,
- methods of learning, and
- your own preferences for learning.

In this chapter you will explore these different areas to better understand how they may influence your own learning, as well as how to make conscious decisions about your own learning process to maximize positive outcomes.

All Learning Is Not the Same

The first, fundamental point to understand about learning is that there are several types of learning. Different kinds of knowledge are learned in different ways. Each of these different types of learning can require different processes that may take place in completely different parts of our brain.

For example, simple memorization is a form of learning that does not always require deeper understanding. Children often learn this way when they memorize poems or verses they recite. An interesting example of this can be found in the music industry, where there have been several hit songs sung in English by vocalists who do not speak English. In these cases, the singers did not truly understand what they were singing, but instead they were taught to memorize the sounds of the words in the proper order.



Figure 1.0.2 Learning has many levels and forms. For example, collaborative learning and showing your work require different skills and produce different results than reading or notetaking on your own. (Credit: StartUpStockPhotos / Pexels)

Memorizing sounds is a very different type of learning than, say, acquiring a deep understanding of Einstein's general theory of relativity.

Notice in the comparative examples of music and physics that the different levels of learning are being defined by what they allow you to know or do. When classifying learning in this way, people usually agree on six different levels of learning. In this next section we will take a detailed look at each of these.

In the table below, the cells in the left column each contain one of the main levels of learning, categorized by what the learning allows you to do. To the right of each category are the "skill acquired" and a set of real-world examples of what those skills might be as applied to a specific topic. This set of categories is called Bloom's Taxonomy, and it is often used as a guide for educators when they are determining what students should learn within a course.

Table 2.1

Category of Learning	Skill acquired	Example 1: Musical ability	Example 2: Historical information on Charles the Bald

Category of Learning	Skill acquired	Example 1: Musical ability	Example 2: Historical information on Charles the Bald
Create	Produce new or original work	Compose a piece of music	Write a paper on Charles that draws a new conclusion about his reign
Evaluate	Justify or support an idea or decision	Make critical decisions about the notes that make up a melody—what works, what doesn't, and why	Make arguments that support the idea that Charles was a good ruler
Analyze	Draw connections	Play the specific notes that are found in the key of A	Compare and contrast the historical differences between the reign of Charles and his grandfather, Charlemagne
Apply	Use information in new ways	Use knowledge to play several notes that sound good together	Use the information to write a historical account on the reign of Charles
Understand/ Comprehend	Explain ideas or concepts	Understand the relationship between the musical notes and how to play each on a musical instrument	Explain the historical events that enabled Charles to become Emperor
Remember	Recall facts and basic concepts	Memorize notes on a musical scale	Recall that Charles the Bald was Holy Roman Emperor from 875–877 CE

A review of the above table shows that actions in the left column (or what you will be able to do with the new knowledge) has a direct influence over what needs to be learned and can even dictate the type of learning approach that is best. For example, *remembering* requires a type of learning that allows the person basic memorization. In the case of Charles the Bald and his reign, it is simply a matter of committing the dates to memory. When it comes to *understanding and comprehension*, being able to explain how Charles came to power requires not only the ability to recall several events, but also for the learner to be able to understand the cause and effect of those events and how they worked together to make Charles emperor. Another example would be the ability to *analyze*. In this particular instance the information learned would not only be about Charles, but also about other rulers, such as Charlemagne. The information would have to be of such a depth that the learner could compare the events and facts about each ruler.

When you engage in any learning activity, take the time to understand what you will do with the knowledge once you have attained it. This can help a great deal when it comes to making decisions on how to go about it. Using flashcards to help memorize angles does not really help you solve problems using geometry formulas. Instead, practicing problem-solving with the actual formulas is a much better approach. The key is to make certain the learning activity fits your needs.

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1.1: It's All in the Mindset

Questions to consider

- What is a growth mindset, and how does it affect my learning?
- What are performance goals versus learning goals?

In the previous sections of this chapter you have focused on a number of concepts and models about learning. One of the things they all have in common is that they utilize different approaches to education by presenting new ways to think about learning. In each of these, the common element has been a better understanding of yourself as a learner and how to apply what you know about yourself to your own learning experience. If you were to distill all that you have learned in this chapter so far down to a single factor, it would be about using your mindset to your best advantage. In this next section, you will examine how all of this works in a broader sense by learning about the significance of certain mindsets and how they can hinder or promote your own learning efforts.



Figure 1.1.7: Many fields of study and work create intersections of growth and fixed mindset. People may feel great ability to grow and learn in some areas, like art and communication, but feel more limited in others, such as planning and financials. Recognizing these intersections will help you approach new topics and tasks. (Credit: mentatdgt / Pexels)

Performance vs. Learning Goals

As you have discovered in this chapter, much of our ability to learn is governed by our motivations and goals. What has not yet been covered in detail has been how sometimes hidden goals or mindsets can impact the learning process. In truth, we all have goals that we might not be fully aware of, or if we are aware of them, we might not understand how they help or restrict our ability to learn. An illustration of this can be seen in a comparison of a student that has *performance*-based goals with a student that has *learning*-based goals.

If you are a student with strict performance goals, your primary psychological concern might be to appear intelligent to others. At first, this might not seem to be a bad thing for college, but it can truly limit your ability to move forward in your own learning. Instead, you would tend to play it safe without even realizing it. For example, a student who is strictly performance-goal-oriented will often only says things in a classroom discussion when they think it will make them look knowledgeable to the instructor or their classmates. For example, a performance-oriented student might ask a question that she knows is beyond the topic being covered (e.g., asking about the economics of Japanese whaling while discussing the book *Moby Dick* in an American literature course). Rarely will they ask a question in class because they actually do not understand a concept. Instead they will ask questions that make them look intelligent to others or in an effort to “stump the teacher.” When they do finally ask an honest question, it may be because they are more afraid that their lack of understanding will result in a poor performance on an exam rather than simply wanting to learn.

If you are a student who is driven by learning goals, your interactions in classroom discussions are usually quite different. You see the opportunity to share ideas and ask questions as a way to gain knowledge quickly. In a classroom discussion you can ask for

clarification immediately if you don't quite understand what is being discussed. If you are a person guided by learning goals, you are less worried about what others think since you are there to learn and you see that as the most important goal.

Another example where the difference between the two mindsets is clear can be found in assignments and other coursework. If you are a student who is more concerned about performance, you may avoid work that is challenging. You will take the "easy A" route by relying on what you already know. You will not step out of your comfort zone because your psychological goals are based on approval of your performance instead of being motivated by learning.

This is very different from a student with a learning-based psychology. If you are a student who is motivated by learning goals, you may actively seek challenging assignments, and you will put a great deal of effort into using the assignment to expand on what you already know. While getting a good grade is important to you, what is even more important is the learning itself.

If you find that you sometimes lean toward performance-based goals, do not feel discouraged. Many of the best students tend to initially focus on performance until they begin to see the ways it can restrict their learning. The key to switching to learning-based goals is often simply a matter of first recognizing the difference and seeing how making a change can positively impact your own learning.

What follows in this section is a more in-depth look at the difference between performance- and learning-based goals. This is followed by an exercise that will give you the opportunity to identify, analyze, and determine a positive course of action in a situation where you believe you could improve in this area.

WHAT STUDENTS SAY

1. In the past, did you feel like you had control over your own learning?
    a. No. Someone has always dictated how and what I learned.
    b. Yes. I always look for ways to take control of what and how I learned.
    c. I am uncertain. I never thought about it before.
2. Have you ever heard of learning styles or do you know your own learning style?
    a. No. I have never heard of learning styles.
    b. Yes. I have heard of learning styles and know my own.
    c. Yes. I have heard of learning styles, but I don't think they're accurate or relate to me.
3. Which factors other than intelligence do you think have the greatest influence on learning?
    a. Motivation
    b. Perseverance
    c. Understanding how I learn
    d. Good teachers and support

You can also take the anonymous What Students Say surveys to add your voice to this textbook. Your responses will be included in updates.

Students offered their views on these questions, and the results are displayed in the graphs below.

In the past, did you feel like you had control over your own learning?

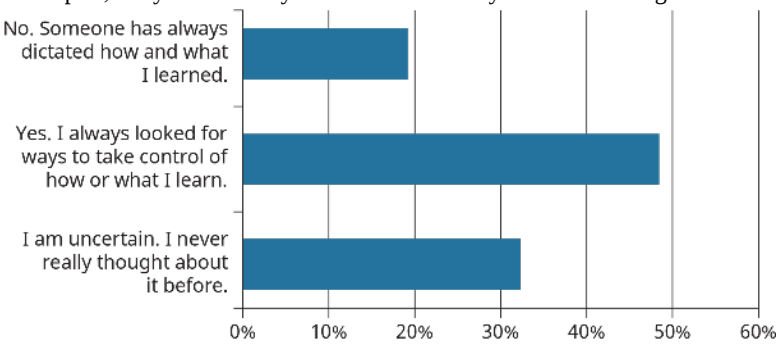


Figure 1.1.8

Have you ever heard of learning styles or do you know your own learning style?

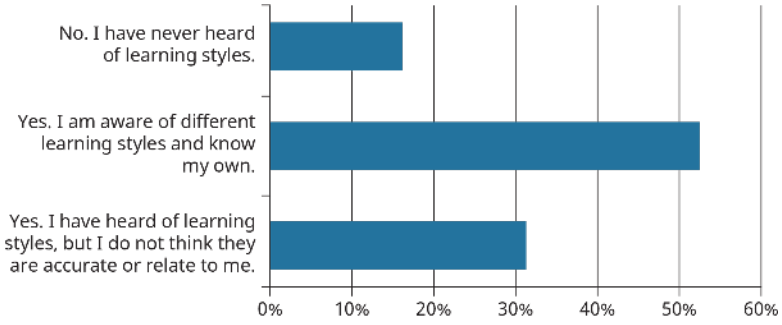


Figure 1.1.9

Which factors other than intelligence do you think have the greatest influence on learning?

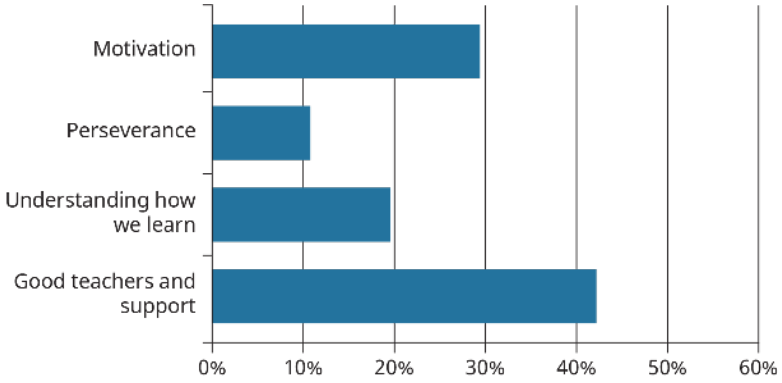


Figure 1.1.10

Fixed vs. Growth Mindset

The research-based model of these two mindsets and their influence on learning was presented in 1988 by Carol Dweck.[7] In Dr. Dweck's work, she determined that a student's perception about their own learning accompanied by a broader goal of learning had a significant influence on their ability to overcome challenges and grow in knowledge and ability. This has become known as the Fixed vs. Growth Mindset model. In this model, the *performance*-goal-oriented student is represented by the *fixed* mindset, while the *learning*-goal-oriented student is represented by the *growth* mindset.

In the following graphic, based on Dr. Dweck's research, you can see how many of the components associated with learning are impacted by these two mindsets.

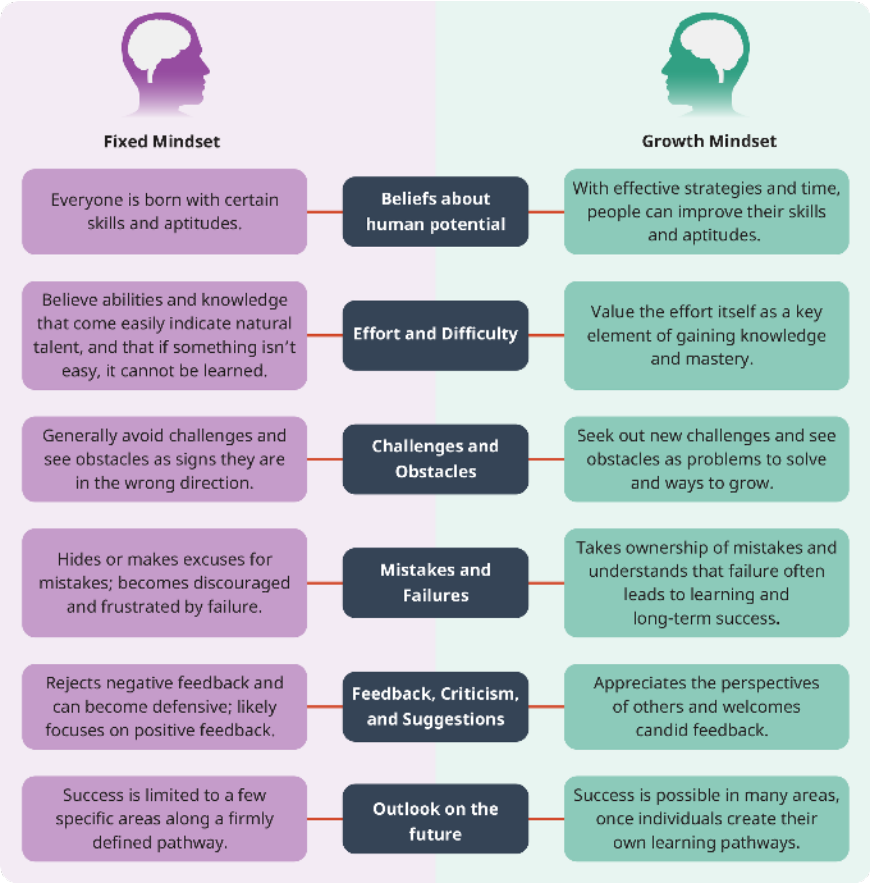


Figure 1.1.11: The differences between fixed and growth mindset are clear when aligned to key elements of learning and personality. (Credit: Based on work by Dr. Carol Dweck)

The Growth Mindset and Lessons About Failing

Something you may have noticed is that a growth mindset would tend to give a learner grit and persistence. If you had learning as your major goal, you would normally keep trying to attain that goal even if it took you multiple attempts. Not only that, but if you learned a little bit more with each try you would see each attempt as a success, even if you had not achieved complete mastery of whatever it was you were working to learn.

With that in mind, it should come as no surprise that Dr. Dweck found that those people who believed their abilities could change through learning (growth vs. a fixed mindset) readily accepted learning challenges and persisted despite early failures.

Improving Your Ability to Learn

As strange as it may seem, research into fixed vs. growth mindsets has shown that if you believe you can learn something new, you greatly improve your ability to learn. At first, this may seem like the sort of feel-good advice we often encounter in social media posts or quotes that are intended to inspire or motivate us (e.g., *believe in yourself!*), but in looking at the differences outlined between a fixed and a growth mindset, you can see how each part of the growth mindset path would increase your probability of success when it came to learning.

ACTIVITY

Very few people have a strict fixed or growth mindset all of the time. Often we tend to lean one way or another in certain situations. For example, a person trying to improve their ability in a sport they enjoy may exhibit all of the growth mindset traits and characteristics, but they find themselves blocked in a fixed mindset when they try to learn something in another area like computer programming or arithmetic.

In this exercise, do a little self-analysis and think of some areas where you may find yourself hindered by a fixed mindset. Using the outline presented below, in the far right column, write down how you can change your own behavior for each of the parts of the learning process. What will you do to move from a fixed to a growth mindset? For example, say you were trying to

learn to play a musical instrument. In the *Challenges* row, you might pursue a growth path by trying to play increasingly more difficult songs rather than sticking to the easy ones you have already mastered. In the *Criticism* row, you might take someone's comment about a weakness in timing as a motivation for you to practice with a metronome. For *Success of others* you could take inspiration from a famous musician that is considered a master and study their techniques.

Whatever it is that you decide you want to use for your analysis, apply each of the Growth characteristics to determine a course of action to improve.

Table 2.4

Parts of the learning process	Growth characteristic	What will you do to adopt a growth mindset?
Challenges	Embraces challenges	
Obstacles	Persists despite setbacks	
Effort	Sees effort as a path to success	
Criticism	Learns from criticism	
Success of Others	Finds learning and inspiration in the success of others	

Footnotes

- Dweck, C.S. & Leggett, E.L. (1988). A Social-Cognitive Approach to Motivation and Personality

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1.2: Failure Is Not an Option

Failure Is Not an Option

Nathan Wallace

In the movie Apollo 13, Ed Harris portrays NASA flight director Gene Kranz as he successfully guides the crew of a damaged spacecraft to safety. In a famous scene during which Kranz and his staff are attempting to overcome some extremely daunting challenges, Harris shouts, “Failure is not an option!” This singular statement perfectly articulated the determination of Kranz to bring the Apollo astronauts back to Earth.

This “failure is not an option” credo was perfect for the life and death situation that NASA was facing. Failure meant that the astronauts on Apollo 13 would never come home, and that outcome was unacceptable. Attending college, on the other hand, shouldn’t be a life or death experience, though it sometimes might feel like one. Failure, though never the intended outcome, can and sometimes does happen. Sometimes failure manifests itself in election results for a student government post, in a test score, or even in a final grade.

Throughout my life I have had many failures. In high school I drove my parents and teachers crazy because of my lack of academic achievement. I even managed to get an F- in Spanish on my report card. When I told my mom that it was a typo she responded, “So you didn’t get an F?” “No,” I said, “I definitely earned the F, but there’s no such thing as an F-.” To this day I’m not so sure that my reply was accurate. I might have earned that minus after all.

My failures in high school led to only one acceptance from of all the colleges I applied to attend. Furthermore, I was not accepted to the school’s main campus, but to their branch campus. During my first semester there my effort wasn’t much better than in high school, but since my parents were now paying for my education I did enough work to avoid academic probation. It wasn’t until my second semester that I found my niche as a Religious Studies major and started getting good grades, moved to the main campus, and eventually graduated with honors.

Since graduating from college, my career path has taken me into higher education as a Student Affairs administrator. This career has exposed me to many great theories regarding student success, and many of them gave me insight into my own college experience. But it was Stanford psychologist Carol Dweck who appeared to be thinking of me when she wrote the following about fixed mindsets in the introduction to her book titled *Mindset: The New Psychology of Success*:

Believing that your qualities are carved in stone—the fixed mindset—creates an urgency to prove yourself over and over. If you have only a certain amount of intelligence, a certain personality, and a certain moral character—well, then you’d better prove that you have a healthy dose of them. It simply wouldn’t do to look or feel deficient in these most basic characteristics. (Dweck, 2006)

This statement was a revelation to me. I finally understood my problem throughout high school and even in college. I earned good grades because I liked Religious Studies but never really challenged myself inside or outside of the classroom. My problem was that I had a fixed mindset about academic success. I believed that a person is either smart or they’re not, and nothing could be done to significantly change that. I also believed that I was one of the fortunate ones to be “gifted” with an abundance of intelligence.

One might think that having confidence in your intelligence is a whole lot better than thinking that you’re stupid, but the result was the same. My fixed mindset was holding me back because it led to a paralyzing fear of failure. Since as far back as I could remember, my family, friends, and teachers were always telling me how smart I was, and I believed them. But that belief was a double-edged sword. High school and college offered many occasions when self-confidence in my inherent intelligence could be threatened. If I fail on this test or in this course it means that I’m not the smart person I thought I was. If I fail, my family and friends will find out that they were wrong about me.

However, there was a way to avoid all of the risks of academic rigor. I could just not try. If I don’t try I’ll get bad marks on my report card, but those won’t be true indicators of my intelligence. By not putting forth any effort, my intelligence would never be disproven. I would always be able to say to myself and others that, “I could do the work and be a straight A student, but I’m just not interested.” Looking back on this time in my life, it is clear to me that this wasn’t a conscious decision to save face. It was fear, not logic, which was guiding my behavior.

After reading *Mindset* I have made a conscious effort to identify and thwart any remaining fixed mindset thoughts that I continue to hold. Dweck’s book acts as a manual for rooting out fixed mindset thoughts, because she explains that the idea of fixed mindsets is only half of her mindset theory. There is another kind of mindset, and she calls it growth mindset. Dweck writes that, “This growth

mindset is based on the belief that your basic qualities are things you can cultivate through your efforts" (Dweck, 2006). Dweck goes on to explain that we can choose to have a growth mindset about any type of ability, whether it's math, art, athletics, or any other skill that one wishes to cultivate.

I put this theory to the test not long after reading the book. A few years ago I attended a meeting only to find out that it wasn't any ordinary meeting. During this meeting we would be brainstorming solutions to a specific problem. This was going to be a true brainstorming session, led by a facilitator trained in the science of soliciting uninhibited ideas from an audience. As soon as I heard the word *brainstorming* I froze. I have always hated brainstorming. I'm the type of person that likes to think things through two or three times before expressing an opinion. My fear of failing at this task in front of my coworkers paralyzed my mind. I couldn't think.

That's when it hit me. This was fixed mindset thinking. My belief in my brainstorming inadequacies was preventing me from even trying. So I flipped this thinking on its head and decided the best way to improve my brainstorming abilities was to clear my mind and start firing out ideas. I gave it a shot, and though the ideas didn't come out at the prolific rate of some of my colleagues, I had never before had such a positive outcome and experience while brainstorming. Through this experience I found that I really could choose to have a growth mindset, and that this choice produces a greater chance of success. With a greater chance of success comes a smaller chance of failure.

Nevertheless, when it comes to academic success and success in all phases of life, failure is always an option. Though it can be painful, failure can lead to great learning and progress when a specific failure is analyzed through the lens of a growth mindset. By focusing more on effort than on outcomes anyone can learn and grow, regardless of their skill level. Therefore, to make the most of their time in college, students must seek out challenges that will stretch their abilities. These challenges can take many forms and they can occur in a variety of settings, both inside and outside of the classroom. When seeking out challenges there is always the possibility of agonizing defeat, but out of that defeat can be the seeds of great success in the future.

Reference

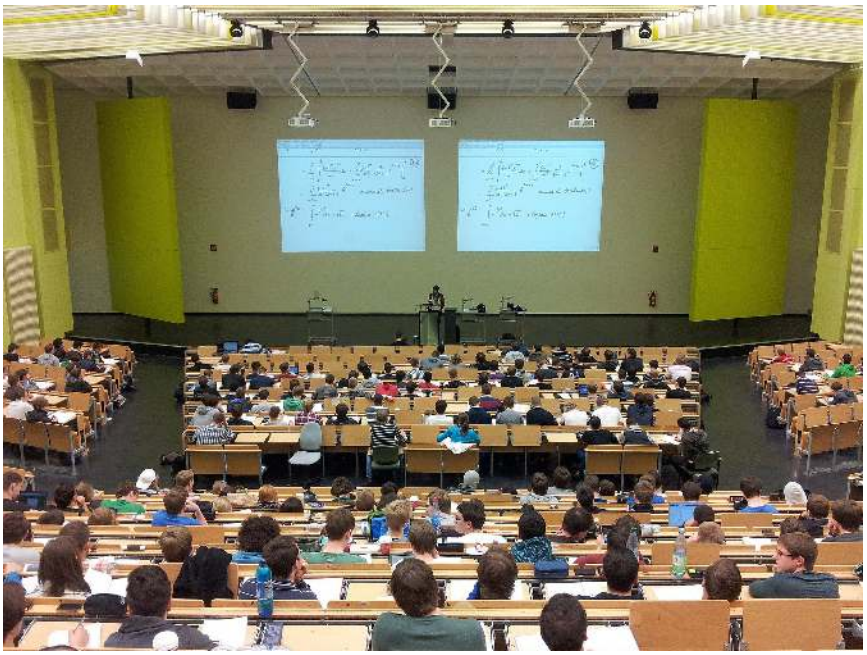
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1.3: Active Learning from Lectures

The learning of information and skills presented in class lectures or discussions should be viewed as a process: preparation to take in the new information, the act of taking in the new information, and then reviewing the information so that it is later accessible (recalled from memory) to use for a project, paper, or test.



The activities suggested for “Before Class” in Table 5-2 help students develop a mental framework into which this new information can be “filed” or organized which aids information recall (Bower, Clark, Lesgold, & Winzenz, 1969). It also makes note taking easier because you are familiar with the material and feel less panicked about getting everything noted in full detail.

Attending class and taking the notes is important primarily because the professor may be presenting information that is not presented in the text. It should also be seen as a structured opportunity to engage with the material that will further aid recall when needed. Oftentimes, students can also get clues as to what a professor perceives as important material from attending class that will aid in test preparation.

Table 5-2: Effective Techniques for Learning from Class

Before Class	During Class	After Class
Complete assigned readings to prepare for class. Take notes or mark your text for easy reference.	Take notes. Active engagement with the material (like writing down what you are hearing) helps improve memory and concentration and material to later review. Have a positive mindset about taking notes you are writing down answers to questions on the test.”	Review your notes and handouts within 24 hours of taking them. Fill in details that you remember but did not have time to write down. Write questions for the notes that you can use for self-testing later. Write a summary of the day’s notes.
Prepare questions that you may have about the reading.	Put a date on your note paper. It helps to also include the text chapter or general topic that might be related to these notes as well.	Compare your notes with a reliable classmate who may help give you ideas on what you missed that might be important or how to better organize the information. Discuss the notes to add another-layer of processing.

Before Class	During Class	After Class
Review notes from the previous class period to get a sense of where you may have left off.	Use active listening techniques. Refer to the informational box called “Ten Tips for Active Listening” for more information.	Follow up on questions that you have written down with your professor, the teaching assistant, or another reliable classmate.
Be sure you have your materials ready for class: text, three-ring notebook with paper, writing utensils, highlighters, charged laptop, etc.	Have your textbook accessible in case your faculty member refers to specific information. Write down that reference in your notes as it is probably important information.	Integrate your class notes with your text notes or markings. If you highlighted in your book, add some of that information to your notes for the “whole” picture.
Get a good night’s sleep. It is easier to concentrate on the lecture when you are alert.	Write down the main points a speaker is making along with any supple-mental details like examples or experiments. Do not expect to be able to write down every word the speaker says.	Develop tools that swill serve to help you review the material now, as well as later. Create visual diagrams* of concepts, develop ram cards* of concepts, reorganize your notes, etc. *Refer to informational boxes on “Visualize Your Information” and “Cram Cards for Long-Term Review.”
Make it a habit to attend every class. Put your classes on your schedule and treat them as a limited opportunity to engage with your professor and this material.	Use abbreviations for commonly used words to increase your speed in note taking. Create some abbreviations of your own.	Self-test your comprehension of the information on an ongoing basis. Do not just “look over” or read what you have written. Use the questions you have created. Cover up the “answer” and see if you can explain what is in the notes. Use the cram cards or visual diagrams to test yourself.
If you cannot attend class due to illness or an-other extraordinary reason, contact a classmate to arrange for notes. It is always a good idea to send an e-mail or call your professor as well.	Develop a system of organization for your notes that works for you. Common methods include the Cornell, Outlining, Mapping, Text-Class, and Sentence Methods. Refer to informational box called “Note Taking Methods: What Is Right for You?”	
	Leave space between main points so that you can add in new information that you may have missed or from the text.	
	If the professor writes it down or puts it on a PowerPoint slide, it is probably important, so write it down.	
	If the professor repeats it, it is important, so write it down.	
	Listen for cues like “very important,” “in summary,” or sequence words like “first,” “second,” etc. This information provides cues for what is important and possible note organization.	
	Ask questions you may have. If you are not comfortable talking in class, write them down and talk with your professor later.	

Adapted from Dembo Sell (2008) and Downing (2008).

Ten Tips for Active Listening

Effective note taking in class requires the use of active listening techniques for optimal results.

1. Accept responsibility as a listener.
2. Adopt a positive attitude toward listening. Listening is a choice.
3. Sit somewhere in the classroom that will minimize distractions and allow you to focus on the professor.
4. Maintain eye contact with your professor.
5. Focus on the content being presented rather than the delivery.
6. Ask questions in class.
7. Ask mental questions and search for the answer in the lecture or discussion.
8. Avoid emotional involvement that may impair your ability to concentrate or filter important information.
9. Clue in on non-verbal communication, such as exaggerated movements, excitement, etc., as it may serve as a cue to important information.
10. Monitor your concentration throughout the class period and continually refocus.

Adapted from Kline (2002) and Treuer (2006).



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1.4: Active Learning from Reading

As with learning from lectures and discussions, the learning of information and skills presented in readings, texts, and written text from a Web site should be viewed as a process: preparation to take in the new information, the act of taking in the new information, and then reviewing the information so that it is later accessible (recalled from memory) to use for a project, paper, or test.



Students who simply start and finish a reading with no further actions taken are really wasting their time. Reading a text needs to be approached properly in order to ensure comprehension and retention of the information. Planning for an active reading session by engaging in the "before reading" activities facilitates an active mindset to the reading session and discourages "auto-pilot reading" that results in scanning of the words but no real careful thought about what is being said.

Preparing for class by reading assigned texts related to the lecture material is extremely important to optimize the lecture experience. If you have a professor who tends to lecture at a fast pace, you have the foundational information and only need to focus on writing down unfamiliar information. If your professor expects class participation in discussions or utilizes active classroom techniques, such as case studies, small group activities and discussions, simulations, or demonstrations, you will understand what is happening and be able to contribute to the learning experience. Marking or highlighting your text for main ideas while reading not only keeps your mind focused on the reading but also serves to prepare you for easier exam review.

Table 5-3. Effective Techniques for Reading

Before Reading	During Reading	After Reading
Preview the reading headings to get a "big picture" of the outline of the reading. Look at the pictures and figures. Check out the bold and italicized words.	As you are reading, seek out the answers to the questions that the reading or you generated rather than passively scanning the words.	Take reading notes from the highlighted material from the text. Utilize visual organizers and summaries to capture information. Revisit the informational box on "Note-Taking Methods: What Is Right for You?" and visit "Visualize Your information and "Cram Cards for Long-term Review."
Preview any questions that the chapter may offer so that you can actively seek answers to those questions.	Monitor your concentration and comprehension. After each paragraph, ask yourself, "What was the main point of that paragraph?" After each section, summarize what you have read.	Answer the questions that a the reading may have provided or that you developed. Reread sections in which you remember little information.
Turn text headings into questions so that you can actively seek answers to those questions. Jot those questions down in the margin of the text or in your notes if you plan to take reading notes.	Mark your text. Highlight the main idea of a paragraph or write down the main idea of a paragraph in your notes. If you cannot mark your text, take reading notes on the main ideas. Refer to the informational box called "In Search of Main Ideas" for more information	Integrate your reading notes and lecture/discussion notes into one location for easier review for an exam.
Plan for a high energy time of the day to read. Consider breaking up a long reading assignment into multiple, shorter reading sessions.	Look up the definitions to words that you do not understand to help build your vocabulary and facilitate understanding of the topic.	Discuss readings with classmates on a regular basis. Take turns explaining to each other sections of the reading. Ask and answer reading questions. Jot down questions that are unclear to the study group to ask the professor in class or during office hours.
Be sure you have your materials ready for reading: text, notebook paper, writing utensils, high-lighters, charged laptop, etc.		

Adapted from Dembo and Seli (2008) and Downing (2008).

In Search of Main Ideas

Some students do not actively take notes or mark their readings and texts because they determine it is a lot of extra work to write down so much information. However, the problem is that students are usually writing down too much information and not really cluing in to the main ideas. Here are some tips for identifying the main ideas when reading:

1. The Table of Contents and chapter outlines provide a broad view of the main points that will be covered in a reading. Flesh out the outlines that are already provided for you.
2. Oftentimes in a textbook, the main idea is the first or last sentence of a paragraph. If it is not the first or the last sentence, then look back at the entire paragraph to see what the overall issue seems to be. Look for the overall pat- terns of your textbooks.
3. Titles, headings, and subheadings announce the major subject. Make these headings into questions, and the answers to the questions will likely be the main ideas.
4. Bold and italic words point to a main idea or key concept that you need to understand.
5. Repetition of key words or phrases throughout the text point to a main idea.
6. Chapter questions at the end of the chapter are about the main ideas of the text. Answer those questions and you will identify your main ideas.
7. Summaries presented at the end of the chapter also tend to restate the main ideas briefly. Flesh these ideas out with some supporting ideas, and you would have a good view of the entire chapter.
8. Stop and look at the visuals—pictures, diagrams, tables, etc. Oftentimes, the message depicted in the graph or picture is typically a main idea.
9. Detailed statistics, several examples in a row, and other details often signal that a main idea is being clarified, proven, supported, etc. Track back or ahead to find the main idea they are trying to illustrate.
10. Text that includes bullet points, numbering, or sequences is often a main idea.
11. Look for organizational patterns in the reading that might highlight the main ideas. For instance, are two issues being compared or contrasted? What was the effect of a certain event? Are problems and various solutions being presented? Is there a timeline of events that is important?
12. Be intentional about searching for the main ideas. Ask yourself at the end of each section or paragraph, "What is the point?" or "What is it that the author wants me to know?"
13. If you are reading a narrative, ask yourself questions like, "Who are the main characters?" "Why is this character important to this story?" "Why did the author chose to tell this part of the story?"

Finally, taking the time to think critically about the reading you have done further solidifies that information in your memory and can serve to prepare you for exam review. Summarizing material using visual organizers (refer to informational box "Visualize Your Information" and Figure 5-1) moves your thinking from low levels of thinking to high levels of thinking on Bloom's Taxonomy. Integrating text and lecture notes and discussing readings with your classmates and professor promotes memory storage of that information and thinking at higher levels as well.

Visualize Your Information

Summarizing main ideas of the readings or lectures can be difficult without the right tools. Visual organizers can help students condense important information into visual diagrams that can be easily reviewed for exams or help organize a writing assignment. Muskingum College's Center for Advancement and Learning (n.d.) hosts a useful Web site that summarizes and provides examples of various visual organizers (a.k.a. Information Organization Strategies) at

muskingum.edu/~cal/database/g...anization.html (Web site wants username and password now.)

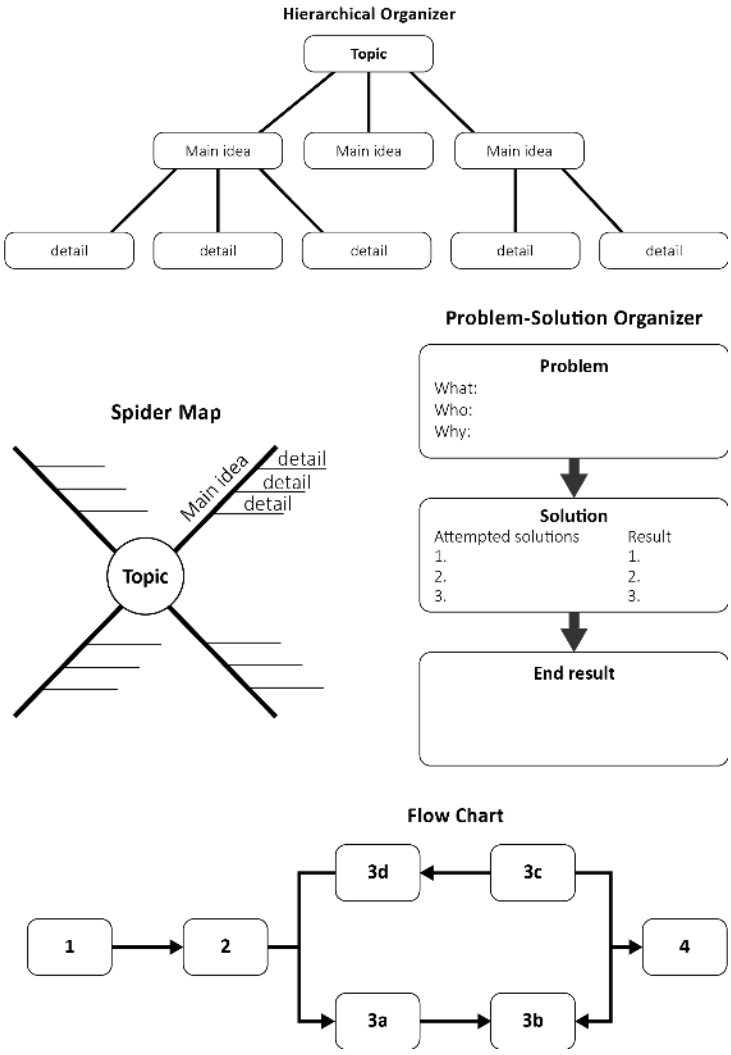
Consider how you might be able to apply several pages of text to a simple flowchart, hierarchical organizer, comparison-contrast organizer, spider map, etc. (See Figure 5-1.)

An additional benefit of using visual organizers is that you are thinking about the material at a deeper level as you identify the proper visual tool and apply the information. While at first it may take more time to determine the proper tool, it will get easier with practice and become second nature.

Activity 5-2

Utilizing what you have learned from the informational boxes "In Search of Main Ideas" and "Visualize Your Information," select a text chapter or a week of class notes and create some visual organizers that summarize the main ideas of that material. Ask a classmate to share in the same exercise and compare your work in a critical manner. What have you learned from this exercise and each other?

Figure 5-1. Examples of visual organizers



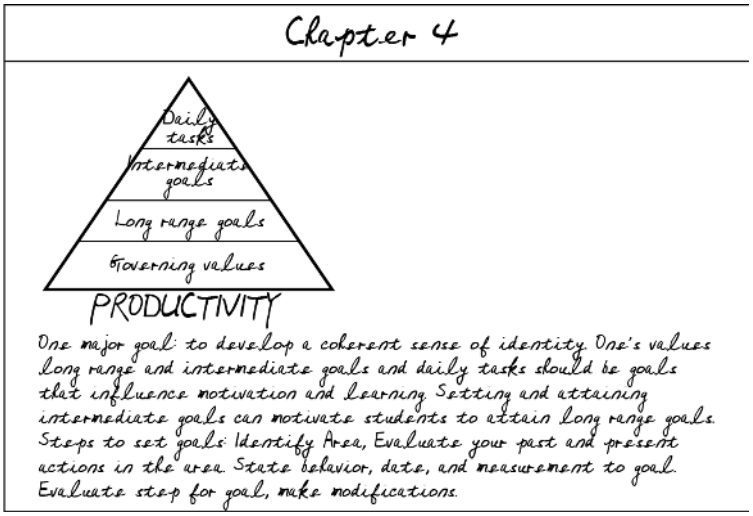
Source: Muskingum College's Center for Advancement and Learning (n.d)

Cram Cards for Long-term Review

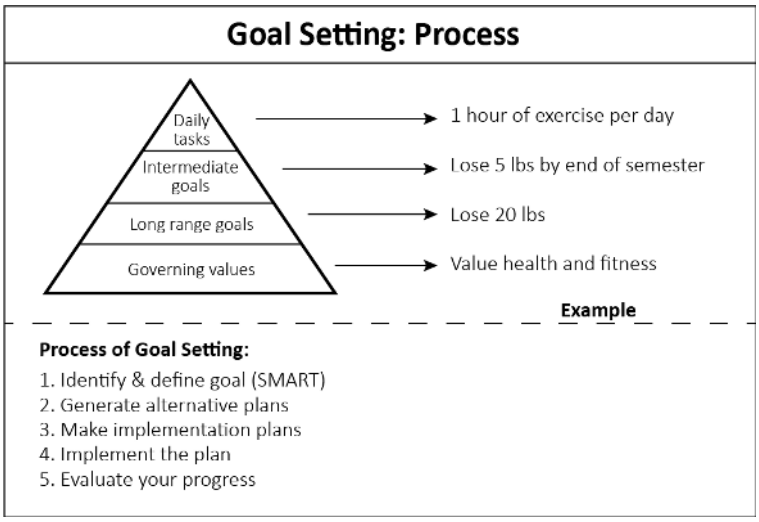
The use of "cram" in cram cards is deceiving. Cram cards are a study tool that promote the active learning of reading or lecture notes and allows for the easy and portable review of material over a longer period of time, therefore promoting later recall. It is a technique that will take some practice to hone your skills in selecting main ideas, but students who use cram cards indicate it is well worth their time.

Tips for Writing Cram Cards:

- Use 4 x 6 index cards (3 x 5 size tends to be too small)—some enthusiasts will even use the colored index cards for different sources of information (i.e., notes versus text reading) or topics (e.g., blue for English and green for Biology).
- Make one cram card per major concept or main idea.
- Do not simply take notes on the card; capture a summary of the main ideas maintaining the proper relationships of the concepts.
- Write the information in your own words.
- Include book examples or provide real-life examples for difficult concepts.
- Include key words.
- Be sure to write legibly.



Example of a Bad Cram Card



Example of a Good Cram Card

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1.5: Taking Notes

Questions to consider

- How can you prepare to take notes to maximize the effectiveness of the experience?
- What are some specific strategies you can employ for better notetaking?
- Why is annotating your notes after the notetaking session a critical step to follow?

Beyond providing a record of the information you are reading or hearing, notes help you organize the ideas and help you make meaning out of something about which you may not be familiar, so notetaking and reading are two compatible skill sets. Taking notes also helps you stay focused on the question at hand. Nanami often takes notes during presentations or class lectures so she can follow the speaker's main points and condense the material into a more readily usable format. Strong notes build on your prior knowledge of a subject, help you discuss trends or patterns present in the information, and direct you toward areas needing further research or reading.



Figure 1.5.9 Strong notes build on your prior knowledge of a subject, help you discuss trends or patterns present in the information, and direct you toward areas needing further research or reading.

It is not a good habit to transcribe every single word a speaker utters—even if you have an amazing ability to do that. Most of us don't have that court-reporter-esque skill level anyway, and if we try, we would end up missing valuable information. Learn to listen for main ideas and distinguish between these main ideas and details that typically support the ideas. Include examples that explain the main ideas, but do so using understandable abbreviations.

Think of all notes as potential study guides. In fact, if you only take notes without actively working on them after the initial notetaking session, the likelihood of the notes helping you is slim. Research on this topic concludes that without active engagement after taking notes, most students forget 60–75 percent of material over which they took the notes—within two days! That sort of defeats the purpose, don't you think? This information about memory loss was first brought to light by 19th-century German psychologist Hermann Ebbinghaus. Fortunately, you do have the power to thwart what is sometimes called the Ebbinghaus Forgetting Curve by reinforcing what you learned through review at intervals shortly after you take in the material and frequently thereafter.

If you are a musician, you'll understand this phenomenon well. When you first attempt a difficult piece of music, you may not remember the chords and notes well at all, but after frequent practice and review, you generate a certain muscle memory and cognitive recall that allows you to play the music more easily.

Notetaking may not be the most glamorous aspect of your higher-education journey, but it is a study practice you will carry throughout college and into your professional life. Setting yourself up for successful notetaking is almost as important as the actual taking of notes, and what you do after your notetaking session is equally significant. Well-written notes help you organize your thoughts, enhance your memory, and participate in class discussion, and they prepare you to respond successfully on exams. With all that riding on your notes, it would behoove you to learn how to take notes properly and continue to improve your notetaking skills.

Analysis Question

Do you currently have a preferred way to take notes? When did you start using it? Has it been effective? What other strategy might work for you?

Preparing to Take Notes

Preparing to take notes means more than just getting out your laptop or making sure you bring pen and paper to class. You'll do a much better job with your notes if you understand why we take notes, have a strong grasp on your preferred notetaking system, determine your specific priorities depending on your situation, and engage in some version of efficient shorthand.

Like handwriting and fingerprints, we all have unique and fiercely independent notetaking habits. These understandably and reasonably vary from one situation to the next, but you can only improve your skills by learning more about ways to take effective notes and trying different methods to find a good fit.

The very best notes are the ones you take in an organized manner that encourages frequent review and use as you progress through a topic or course of study. For this reason, you need to develop a way to organize all your notes for each class so they remain together and organized. As old-fashioned as it sounds, a clunky three-ring binder is an excellent organizational container for class notes. You can easily add to previous notes, insert handouts you may receive in class, and maintain a running collection of materials for each separate course. If the idea of carrying around a heavy binder has you rolling your eyes, then transfer that same structure into your computer files. If you don't organize your many documents into some semblance of order on your computer, you will waste significant time searching for improperly named or saved files.

You may be interested in relatively new research on what is the more effective notetaking strategy: handwriting versus typing directly into a computer. While individuals have strong personal opinions on this subject, most researchers agree that the format of student notes is less important than what students do with the notes they take afterwards. Both handwriting notes and using a computer for notetaking have pros and cons.

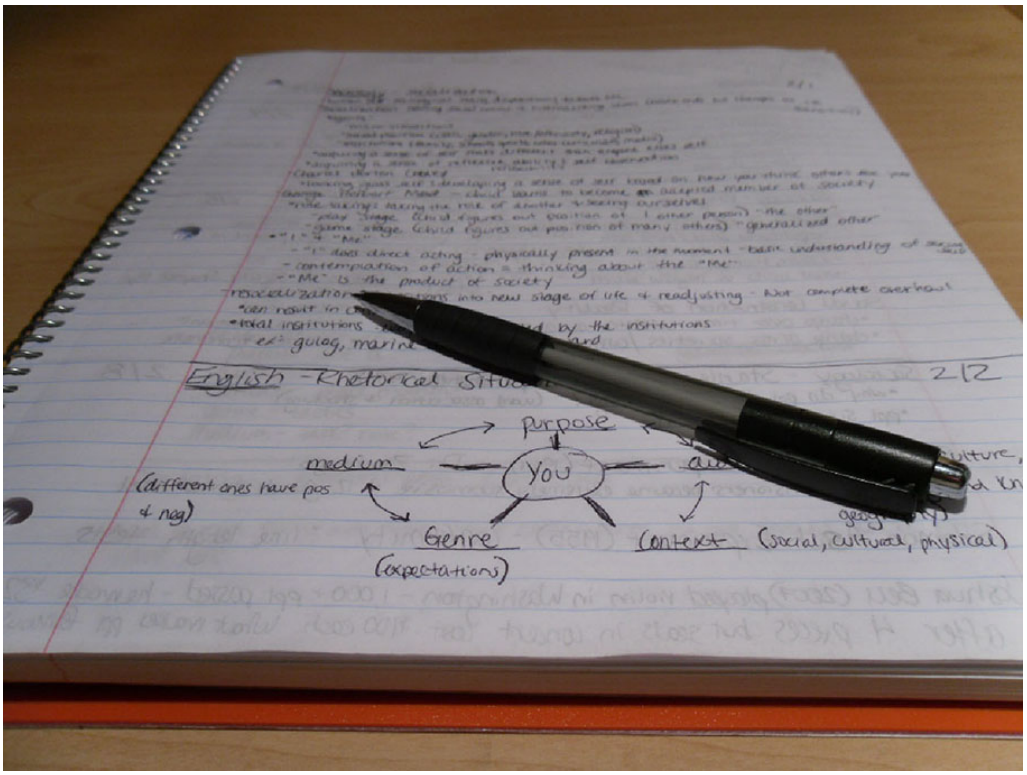


Figure 1.5.10 The best notes are the ones you take in an organized manner. Frequent review and further annotation are important to build a deep and useful understanding of the material. (Credit: English106 / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

Managing Notetaking Systems (Computer, Paper/Pen, Note Cards, Textbook)

Whichever of the many notetaking systems you choose (and new ones seem to come out almost daily), the very best one is the one that you will use consistently. The skill and art of notetaking is not automatic for anyone; it takes a great deal of practice, patience, and continuous attention to detail. Add to that the fact that you may need to master multiple notetaking techniques for different classes, and you have some work to do. Unless you are specifically directed by your instructor, you are free to combine the best parts of different systems if you are most comfortable with that hybrid system.

Just to keep yourself organized, all your notes should start off with an identifier, including at the very least the date, the course name, the topic of the lecture/presentation, and any other information you think will help you when you return to use the notes for further study, test preparation, or assignment completion. Additional, optional information may be the number of notetaking sessions about this topic or reminders to cross-reference class handouts, textbook pages, or other course materials. It's also always a good idea to leave some blank space in your notes so you can insert additions and questions you may have as you review the material later.

Notetaking Strategies

You may have a standard way you take all your notes for all your classes. When you were in high school, this one-size-fits-all approach may have worked. Now that you're in college, reading and studying more advanced topics, your general method may still work some of the time, but you should have some different strategies in place if you find that your method isn't working as well with college content. You probably will need to adopt different notetaking strategies for different subjects. The strategies in this section represent various ways to take notes in such a way that you are able to study after the initial notetaking session.

Cornell Method

One of the most recognizable notetaking systems is called the *Cornell Method*, a relatively simple way to take effective notes devised by Cornell University education professor Dr. Walter Pauk in the 1940s. In this system, you take a standard piece of note paper and divide it into three sections by drawing a horizontal line across your paper about one to two inches from the bottom of the page (the summary area) and then drawing a vertical line to separate the rest of the page above this bottom area, making the left side about two inches (the recall column) and leaving the biggest area to the right of your vertical line (the notes column). You may want to make one page and then copy as many pages as you think you'll need for any particular class, but one advantage of this

system is that you can generate the sections quickly. Because you have divided up your page, you may end up using more paper than you would if you were writing on the entire page, but the point is not to keep your notes to as few pages as possible. The Cornell Method provides you with a well-organized set of notes that will help you study and review your notes as you move through the course. If you are taking notes on your computer, you can still use the Cornell Method in Word or Excel on your own or by using a template someone else created.

Topic/Objective:	Name:
	Class/Period:
	Date:
Essential Question:	

Questions:	Notes:

Summary:

Figure 1.5.11: The Cornell Method provides a straightforward, organized, and flexible approach

Now that you have the notetaking format generated, the beauty of the Cornell Method is its organized simplicity. Just write on one side of the page (the right-hand notes column)—this will help later when you are reviewing and revising your notes. During your notetaking session, use the notes column to record information over the main points and concepts of the lecture; try to put the ideas into your own words, which will help you not transcribe the speaker's words verbatim. Skip lines between each idea in this column. Practice the shortcut abbreviations covered in the next section and avoid writing in complete sentences. Don't make your notes too cryptic, but you can use bullet points or phrases equally well to convey meaning—we do it all the time in conversation. If you know you will need to expand the notes you are taking in class but don't have time, you can put reminders directly in the notes by adding and underlining the word *expand* by the ideas you need to develop more fully.

As soon as possible after your notetaking session, preferably within eight hours but no more than twenty-four hours, read over your notes column and fill in any details you missed in class, including the places where you indicated you wanted to expand your notes. Then in the recall column, write any key ideas from the corresponding notes column—you can't stuff this smaller recall column as if you're explaining or defining key ideas. Just add the one- or two-word main ideas; these words in the recall column serve as cues to help you remember the detailed information you recorded in the notes column.

Once you are satisfied with your notes and recall columns, summarize this page of notes in two or three sentences using the summary area at the bottom of the sheet. This is an excellent time to get with another classmate or a group of students who all heard the same lecture to make sure you all understood the key points. Now, before you move onto something else, cover the large notes column, and quiz yourself over the key ideas you recorded in the recall column. Repeat this step often as you go along, not just immediately before an exam, and you will help your memory make the connections between your notes, your textbook reading, your in-class work, and assignments that you need to succeed on any quizzes and exams.

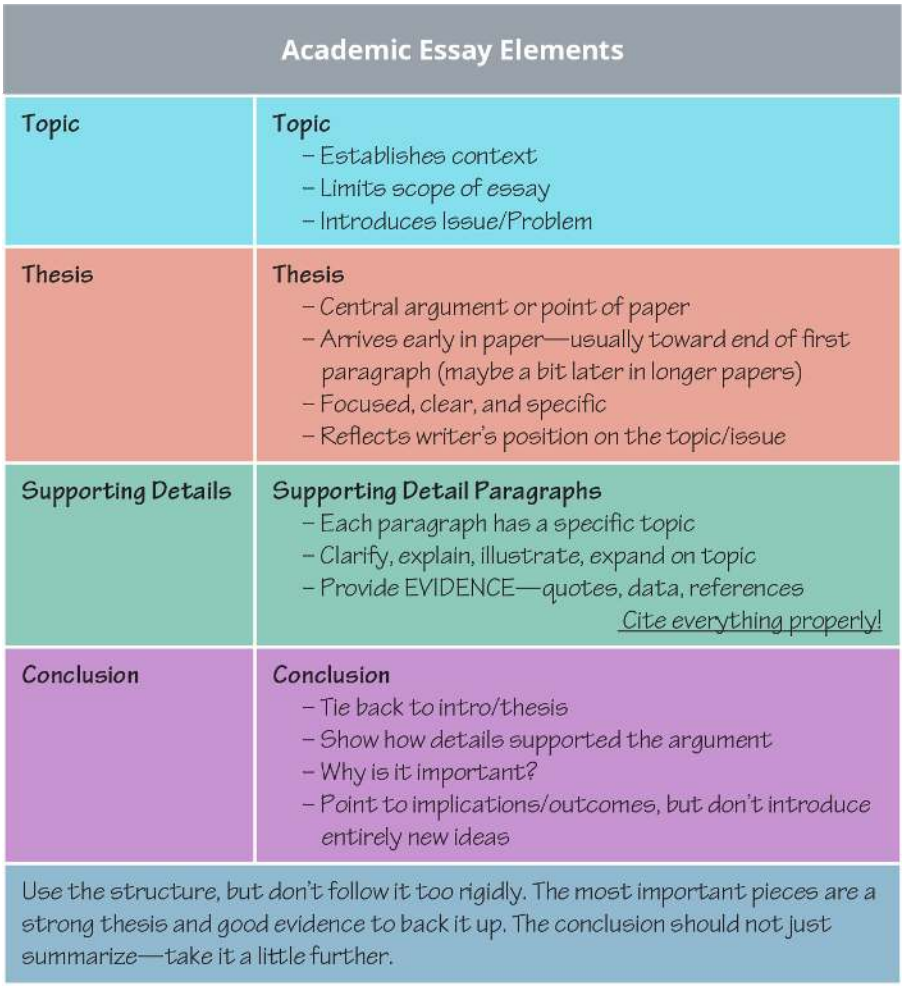


Figure 1.5.12 This sample set of notes in the Cornell Method is designed to make sense of a large amount of information. The process of organizing the notes can help you retain the information more effectively than less consistent methods.

The main advantage of the Cornell Method is that you are setting yourself up to have organized, workable notes. The neat format helps you move into study-mode without needing to re-copy less organized notes or making sense of a large mass of information you aren't sure how to process because you can't remember key ideas or what you meant. If you write notes in your classes without any sort of system and later come across something like "Napoleon—short" in the middle of a glob of notes, what can you do at this point? Is that important? Did it connect with something relevant from the lecture? How would you possibly know? You are your best advocate for setting yourself up for success in college.

Outlining

Other note organizing systems may help you in different disciplines. You can take notes in a formal outline if you prefer, using Roman numerals for each new topic, moving down a line to capital letters indented a few spaces to the right for concepts related to the previous topic, then adding details to support the concepts indented a few more spaces over and denoted by an Arabic numeral. You can continue to add to a formal outline by following these rules.

You don't absolutely have to use the formal numerals and letter, but you have to then be careful to indent so you can tell when you move from a higher level topic to the related concepts and then to the supporting information. The main benefit of an outline is how

organized it is. You have to be on your toes when you are taking notes in class to ensure you keep up the organizational format of the outline, which can be tricky if the lecture or presentation is moving quickly or covering many diverse topics.

The following formal outline example shows the basic pattern:

I. Dogs *(main topic–usually general)*
   A. German Shepherd *(concept related to main topic)*
      1. Protection *(supporting info about the concept)*
      2. Assertive
      3. Loyal
   B. Weimaraner *(concept related to main topic)*
      1. Family-friendly *(supporting info about the concept)*
      2. Active
      3. Healthy
II. Siamese

You would just continue on with this sort of numbering and indenting format to show the connections between main ideas, concepts, and supporting details. Whatever details you do not capture in your notetaking session, you can add after the lecture as you review your outline.

Chart or table

Similar to creating an outline, you can develop a chart to compare and contrast main ideas in a notetaking session. Divide your paper into four or five columns with headings that include either the main topics covered in the lecture or categories such as How?, What?, When used?, Advantages/Pros, Disadvantages/Cons, or other divisions of the information. You write your notes into the appropriate columns as that information comes to light in the presentation.

Example of a Chart to Organize Ideas and Categories

	Structure	Types	Functions in Body	Additional Notes
Carbohydrates				
Lipids				
Proteins				
Nucleic Acid				

This format helps you pull out the salient ideas and establishes an organized set of notes to study later. (If you haven't noticed that this *reviewing later* idea is a constant across all notetaking systems, you should…take note of that.) Notes by themselves that you never reference again are little more than scribblings. That would be a bit like compiling an extensive grocery list so you stay on budget when you shop, work all week on it, and then just throw it away before you get to the store. You may be able to recall a few items, but likely won't be as efficient as you could be if you had the notes to reference. Just as you cannot read all the many books, articles, and documents you need to peruse for your college classes, you cannot remember the most important ideas of all the notes you will take as part of your courses, so you must review.

Concept Mapping and Visual Notetaking

One final notetaking method that appeals to learners who prefer a visual representation of notes is called *mapping* or sometimes *mind mapping* or *concept mapping*, although each of these names can have slightly different uses. Variations of this method abound, so you may want to look for more versions online, but the basic principles are that you are making connections between main ideas through a graphic depiction; some can get rather elaborate with colors and shapes, but a simple version may be more useful at least to begin. Main ideas can be circled or placed in a box with supporting concepts radiating off these ideas shown with a connecting line and possibly details of the support further radiating off the concepts. You can present your main ideas vertically or horizontally, but turning your paper long-ways, or in landscape mode, may prove helpful as you add more main ideas.

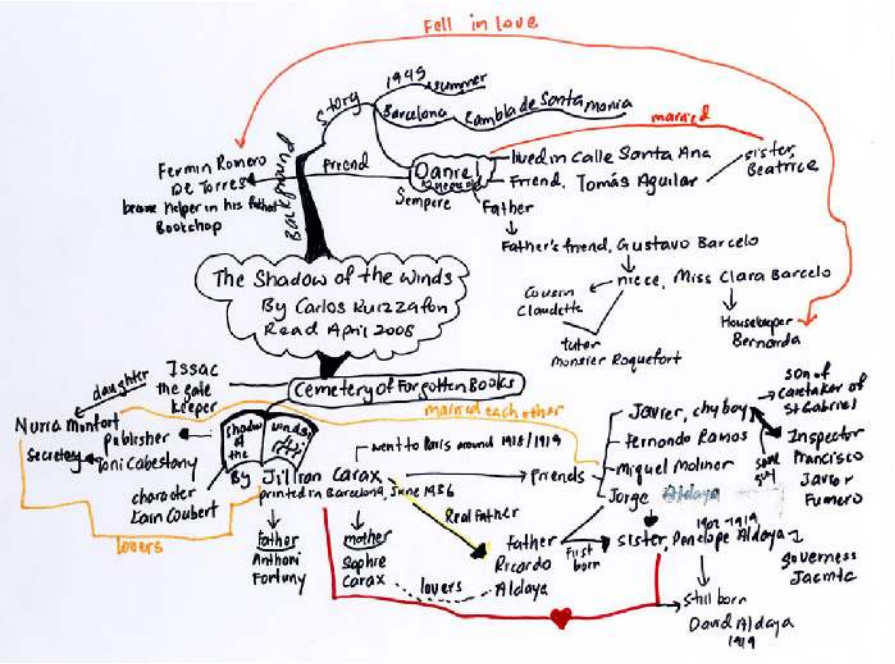


Figure 1.5.13 Concept mapping, sometimes referred to as mind mapping, can be an effective and very personalized approach to capturing information. (Credit: ArtistIvanChew / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

You may be interested in trying visual notetaking or adding pictures to your notes for clarity. Sometimes when you can't come up with the exact wording to explain something or you're trying to add information for complex ideas in your notes, sketching a rough image of the idea can help you remember. According to educator Sherrill Knezel in an article entitled "The Power of Visual Notetaking," this strategy is effective because "When students use images and text in notetaking, it gives them two different ways to pull up the information, doubling their chances of recall." Don't shy away from this creative approach to notetaking just because you believe you aren't an artist; the images don't need to be perfect. You may want to watch Rachel Smith's TEDx Talk called "Drawing in Class" to learn more about visual notetaking.

You can play with different types of notetaking suggestions and find the method(s) you like best, but once you find what works for you, stick with it. You will become more efficient with the method the more you use it, and your notetaking, review, and test prep will become, if not easier, certainly more organized, which can delete decrease your anxiety.

Practicing Decipherable Shorthand

Most college students don't take a class in shorthand, once the domain of secretaries and executive assistants, but maybe they should. That almost-lost art in the age of computers could come in very handy during intense notetaking sessions. Elaborate shorthand systems do exist, but you would be better served in your college notetaking adventures to hone a more familiar, personalized form of shorthand to help you write more in a shorter amount of time. Seemingly insignificant shortcuts can add up to ease the stress notetaking can induce—especially if you ever encounter an "I'm not going to repeat this" kind of presenter! Become familiar with these useful abbreviations:

Shortcut symbol	Meaning
w/, w/o, w/in	with, without, within
&	and
#	number
b/c	because
X, √	incorrect, correct
Diff	different, difference
etc.	and so on

ASAP	as soon as possible
US, UK	United States, United Kingdom
info	information
Measurements: ft, in, k, m	foot, inch, thousand, million
¶	paragraph or new paragraph
Math symbols: =, +, >, <, ÷	equal, plus, greater, less, divided by
WWI, WWII	World Wars I and II
impt	important
?, !, **	denote something is very significant; don't over use

Do you have any other shortcuts or symbols that you use in your notes? Ask your parents if they remember any that you may be able to learn.

Annotating Notes After Initial Notetaking Session

Annotating notes after the initial notetaking session may be one of the most valuable study skills you can master. Whether you are highlighting, underlining, or adding additional notes, you are reinforcing the material in your mind and memory.

Admit it—who can resist highlighting markers? Gone are the days when yellow was the star of the show, and you had to be very careful not to press too firmly for fear of obliterating the words you were attempting to emphasize. Students now have a veritable rainbow of highlighting options and can color-code notes and text passages to their hearts' content. Technological advances may be important, but highlighter color choice is monumental! Maybe.

The only reason to highlight anything is to draw attention to it, so you can easily pick out that ever-so-important information later for further study or reflection. One problem many students have is not knowing when to stop. If what you need to recall from the passage is a particularly apt and succinct definition of the term important to your discipline, highlighting the entire paragraph is less effective than highlighting just the actual term. And if you don't rein in this tendency to color long passages (possibly in multiple colors) you can end up with a whole page of highlighted text. Ironically, that is no different from a page that is not highlighted at all, so you have wasted your time. Your mantra for highlighting text should be *less is more*. Always read your text selection first before you start highlighting anything. You need to know what the overall message is before you start placing emphasis in the text with highlighting.

Another way to annotate notes after initial notetaking is underlying significant words or passages. Albeit not quite as much fun as its colorful cousin highlighting, underlining provides precision to your emphasis.

Some people think of annotations as only using a colored highlighter to mark certain words or phrases for emphasis. Actually, annotations can refer to anything you do with a text to enhance it for your particular use (either a printed text, handwritten notes, or other sort of document you are using to learn concepts). The annotations may include highlighting passages or vocabulary, defining those unfamiliar terms once you look them up, writing questions in the margin of a book, underlining or circling key terms, or otherwise marking a text for future reference. You can also annotate some electronic texts.

Realistically, you may end up doing all of these types of annotations at different times. We know that repetition in studying and reviewing is critical to learning, so you may come back to the same passage and annotate it separately. These various markings can be invaluable to you as a study guide and as a way to see the evolution of your learning about a topic. If you regularly begin a reading session writing down any questions you may have about the topic of that chapter or section and also write out answers to those questions at the end of the reading selection, you will have a good start to what that chapter covered when you eventually need to study for an exam. At that point, you likely will not have time to reread the entire selection especially if it is a long reading selection, but with strong annotations in conjunction with your class notes, you won't need to do that. With experience in reading discipline-specific texts and writing essays or taking exams in that field, you will know better what sort of questions to ask in your annotations

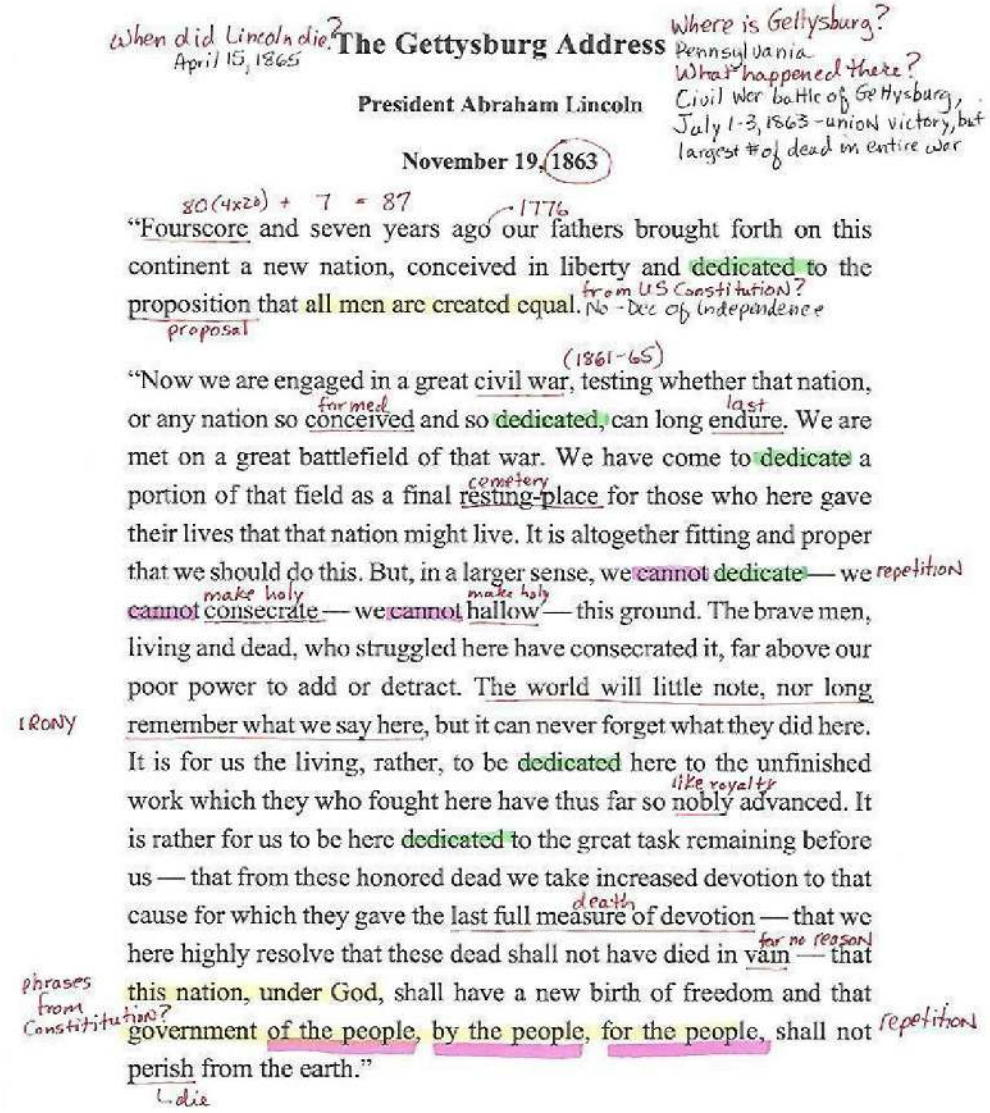


Figure 1.5.14 Annotations may include highlighting important topics, defining unfamiliar terms, writing questions in, underlining or circling key terms, or otherwise marking a text for future reference. Whichever approach you choose, try not to overdo it; neat, organized, and efficient notes are more effective than crowded or overdone notes.

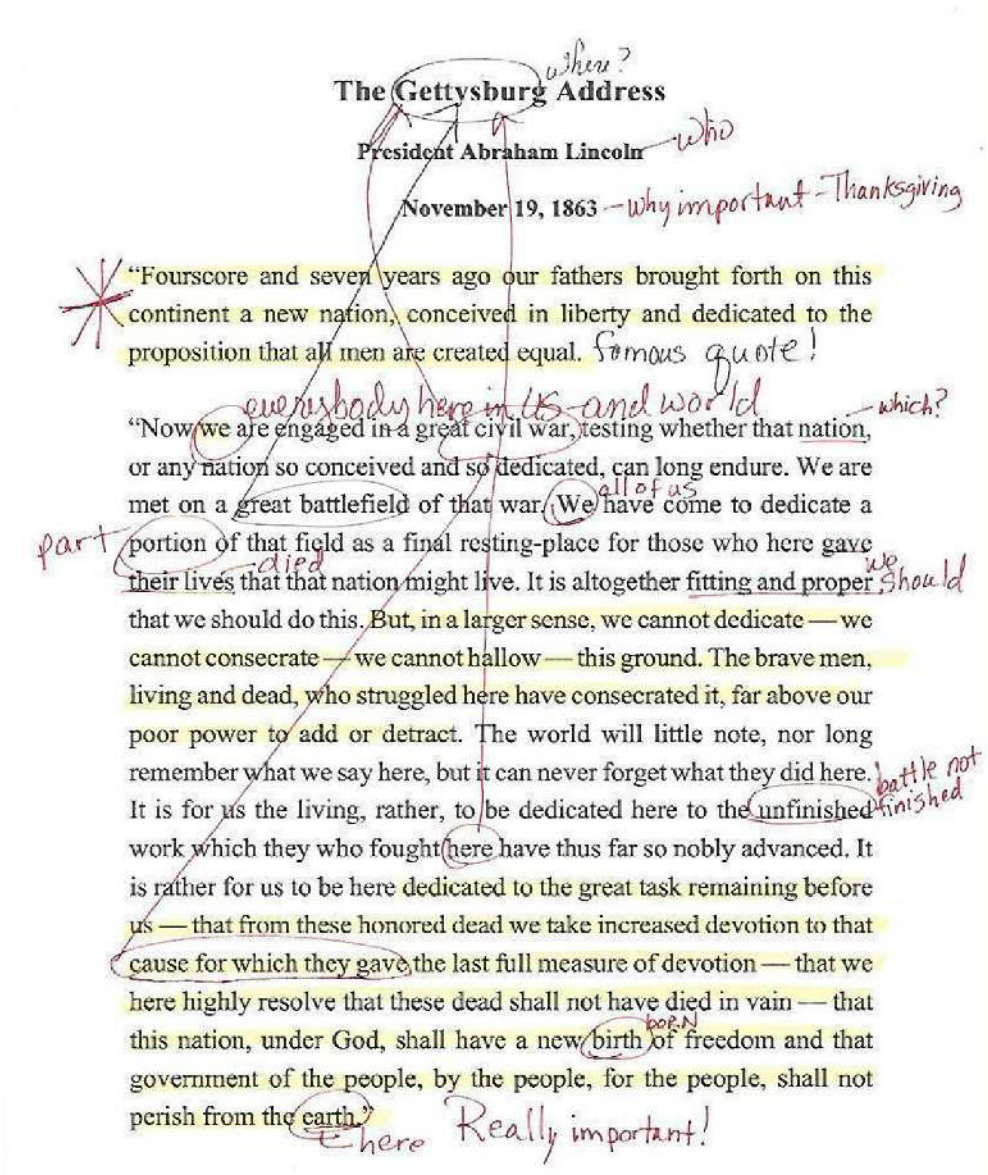


Figure 1.5.15 While these notes may be meaningful to the person who took them, they are neither organized nor consistent. For example, note that some of the more commonly used terms, like "we" and "unfinished," are defined, but less common ones -- "consecrate" and "hallow" -- are not.

What you have to keep in the front of your mind while you are annotating, especially if you are going to conduct multiple annotation sessions, is to not overdo whatever method you use. Be judicious about what you annotate and how you do it on the page, which means you must be neat about it. Otherwise, you end up with a mess of either color or symbols combined with some cryptic notes that probably took you quite a long time to create, but won't be worth as much to you as a study aid as they could be. This is simply a waste of time and effort.

You cannot eat up every smidgen of white space on the page writing out questions or summaries and still have a way to read the original text. If you are lucky enough to have a blank page next to the beginning of the chapter or section you are annotating, use this, but keep in mind that when you start writing notes, you aren't exactly sure how much space you'll need. Use a decipherable shorthand and write only what you need to convey the meaning in very small print. If you are annotating your own notes, you can make a habit of using only one side of the paper in class, so that if you need to add more notes later, you could use the other side. You can also add a blank page to your notes before beginning the next class date in your notebook so you'll end up with extra paper for annotations when you study.

Professional resources may come with annotations that can be helpful to you as you work through the various documentation requirements you'll encounter in college as well. Purdue University's Online Writing Lab (OWL) provides an annotated sample for

how to format a college paper according to guidelines in the Modern Language Association (MLA) manual that you can see, along with other annotations.

Adding Needed Additional Explanations to Notes

Marlon was totally organized and ready to take notes in a designated course notebook at the beginning of every philosophy class session. He always dated his page and indicated what the topic of discussion was. He had various colored highlighters ready to denote the different note purposes he had defined: vocabulary in pink, confusing concepts in green, and note sections that would need additional explanations later in yellow. He also used his own shorthand and an impressive array of symbols to indicate questions (red question mark), highly probable test material (he used a tiny bomb exploding here), additional reading suggestions, and specific topics he would ask his instructor before the next class. Doing everything so precisely, Marlon's methods seemed like a perfect example of how to take notes for success. Inevitably though, by the end of the hour-and-a-half class session, Marlon was frantically switching between writing tools, near to tears, and scouring his notes as waves of yellow teased him with uncertainty. What went wrong?

As with many of us who try diligently to do everything we know how to do for success or what we think we know because we read books and articles on success in between our course work, Marlon is suffering from trying to do too much simultaneously. It's an honest mistake we can make when we are trying to save a little time or think we can multitask and kill two birds with one stone.

Unfortunately, this particular error in judgement can add to your stress level exponentially if you don't step back and see it for what it is. Marlon attempted to take notes in class as well as annotate his notes to get them ready for his test preparation. It was too much to do at one time, but even if he could have done all those things during class, he's missing one critical point about notetaking.

As much as we may want to hurry and get it over with, notetaking in class is just the beginning. Your instructor likely gave you a pre-class assignment to read or complete before coming to that session. The intention of that preparatory lesson is for you to come in with some level of familiarity for the topic under consideration and questions of your own. Once you're in class, you may also need to participate in a group discussion, work with your classmates, or perform some other sort of lesson-directed activity that would necessarily take you away from taking notes. Does that mean you should ignore taking notes for that day? Most likely not. You may just need to indicate in your notes that you worked on a project or whatever other in-class event you experienced that date.

Very rarely in a college classroom will you engage in an activity that is not directly related to what you are studying in that course. Even if you enjoyed every minute of the class session and it was an unusual format for that course, you still need to take some notes. Maybe your first note could be to ask yourself why you think the instructor used that unique teaching strategy for the class that day. Was it effective? Was it worth using the whole class time? How will that experience enhance what you are learning in that course?

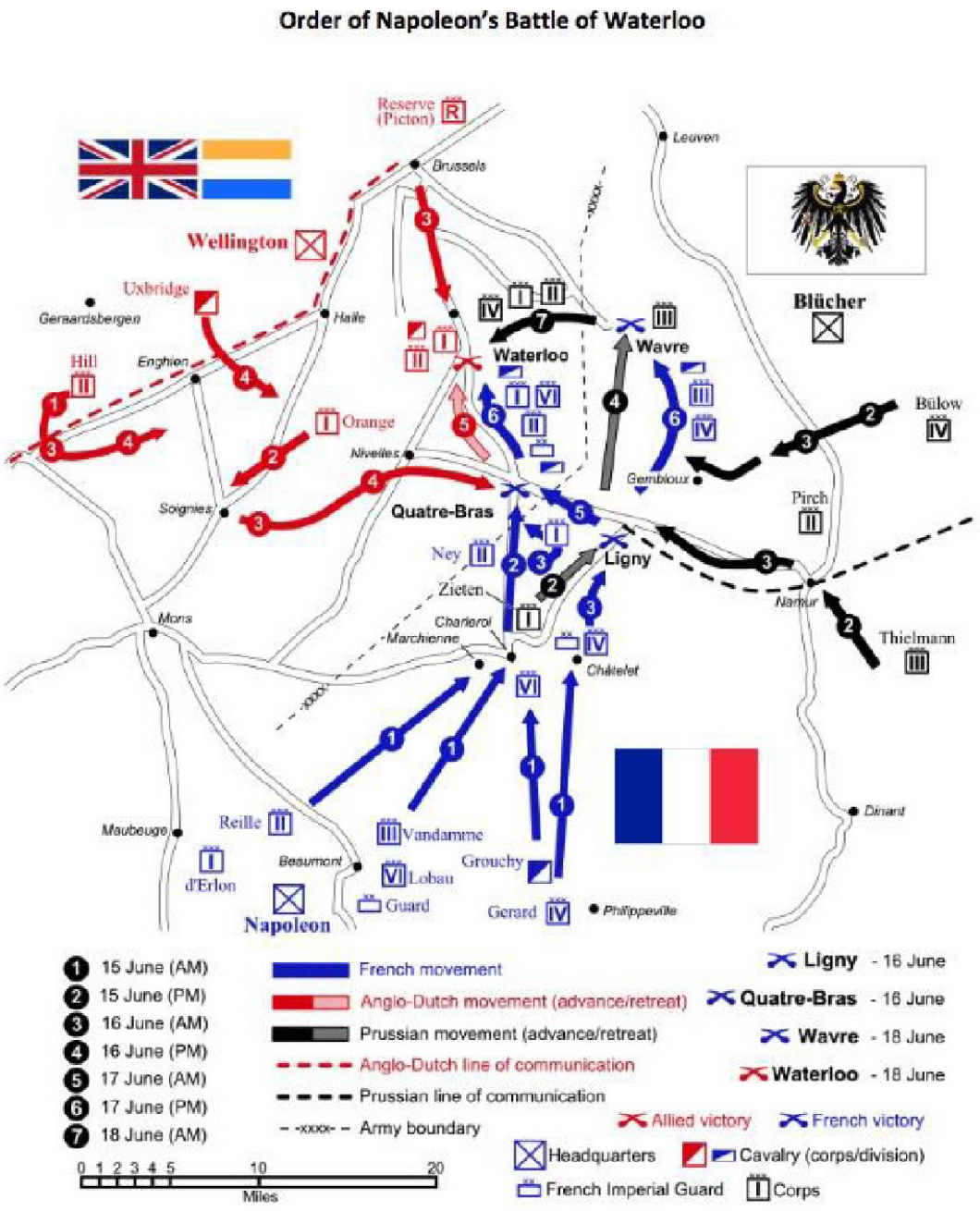
If you use an ereader or ebooks to read texts for class or read articles from the Internet on your laptop or tablet, you can still take effective notes. Depending on the features of your device, you have many choices. Almost all electronic reading platforms allow readers to highlight and underline text. Some devices allow you to add a written text in addition to marking a word or passage that you can collect at the end of your notetaking session. Look into the specific tools for your device and learn how to use the features that allow you to take notes electronically. You can also find apps on devices to help with taking notes, some of which you may automatically have installed when you buy the product. Microsoft's OneNote, Google Keep, and the Notes feature on phones are relatively easy to use, and you may already have free access to those.

Taking Notes on Non-Text Items (i.e., Tables, Maps, Figures, etc.)

You may also encounter situations as you study and read textbooks, primary sources, and other resources for your classes that are not actually texts. You can still take notes on maps, charts, graphs, images, and tables, and your approach to these non-text features is similar to when you prepare to take notes over a passage of text. For example, if you are looking at the following map, you may immediately come up with several questions. Or it may initially appear overwhelming. Start by asking yourself these questions:

What is the main point of this map?

- Who is the intended audience?
- Where is it?
- What time period does it depict?
- What does the map's legend (the explanation of symbols) include?
- What other information do I need to make sense of this map?



Source: Wikipedia Creative Commons, https://en.wikipedia.org/wiki/Order_of_battle_of_the_Waterloo_campaign

Figure 1.5.16 Graphics, charts, graphs, and other visual items are also important to annotate. Not only do they often convey important information, but they may appear on exams or in other situations where you'll need to use or demonstrate knowledge. Credit: "Lpankonin" / Wikipedia Commons / Attribution 3.0 Generic (CC BY 3.0)

You may want to make an extra copy of a graphic or table before you add annotations if you are dealing with a lot of information. Making sense of all the elements will take time, and you don't want to add to the confusion.

Returning to Your Notes

Later, as soon as possible after the class, you can go back to your notes and add in missing parts. Just as you may generate questions as you're reading new material, you may leave a class session or lecture or activities with many questions. Write those

down in a place where they won't get lost in all your other notes.

The exact timing of when you get back to the notes you take in class or while you are reading an assignment will vary depending on how many other classes you have or what other obligations you have in your daily schedule. A good starting place that is also easy to remember is to make every effort to review your notes within 24 hours of first taking them. Longer than that and you are likely to have forgotten some key features you need to include; must less time than that, and you may not think you need to review the information you so recently wrote down, and you may postpone the task too long.

Use your phone or computer to set reminders for all your note review sessions so that it becomes a habit and you keep on top of the schedule.

Your personal notes play a significant role in your test preparation. They should enhance how you understand the lessons, textbooks, lab sessions, and assignments. All the time and effort you put into first taking the notes and then annotating and organizing the notes will be for naught if you do not formulate an effective and efficient way to use them before sectional exams or comprehensive tests.

The whole cycle of reading, notetaking in class, reviewing and enhancing your notes, and preparing for exams is part of a continuum you ideally will carry into your professional life. Don't try to take short cuts; recognize each step in the cycle as a building block. Learning doesn't end, which shouldn't fill you with dread; it should help you recognize that all this work you're doing in the classroom and during your own study and review sessions is ongoing and cumulative. Practicing effective strategies now will help you be a stronger professional.

Activity

What resources can you find about reading and notetaking that will actually help you with these crucial skills? How do you go about deciding what resources are valuable for improving your reading and notetaking skills?

The selection and relative value of study guides and books about notetaking vary dramatically. Ask your instructors for recommendations and see what the library has available on this topic. The following list is not comprehensive, but will give you a starting point for books and articles on notetaking in college.

- *College Rules!: How to Study, Survive, and Succeed in College*, by Sherri Nist-Olejnik and Jodi Patrick Holschuh. More than just notetaking, this book covers many aspects of transitioning into the rigors of college life and studying.
- *Effective Notetaking*, by Fiona McPherson. This small volume has suggestions for using your limited time wisely before, during, and after notetaking sessions.
- *How to Study in College*, by Walter Pauk. This is the book that introduced Pauk's notetaking suggestions we now call the Cornell Method. It is a bit dated (from the 1940s), but still contains some valuable information.
- *Learn to Listen, Listen to Learn 2: Academic Listening and Note-taking*, by Roni S. Lebauer. The main point of this book is to help students get the most from college lectures by watching for clues to lecture organization and adapting this information into strong notes.
- *Study Skills: Do I Really Need this Stuff?*, by Steve Piscitelli. Written in a consistently down-to-earth manner, this book will help you with the foundations of strong study skills, including time management, effective notetaking, and seeing the big picture.
- "What Reading Does for the Mind," by Anne Cunningham and Keith Stanovich, 1998, https://www.aft.org/sites/default/files/periodicals/cunningham.pdf
- Adler, Mortimer J. and Charles Van Doren. *How to Read a Book: The Classic Guide to Intelligent Reading. NY: Simon & Schuster, 1940.*
- Berns, Gregory S., Kristina Blaine, Michael J. Prietula, and Brandon E. Pye. *Brain Connectivity.* Dec 2013.ahead of print http://doi.org/10.1089/brain.2013.0166

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1.6: How to Manage Time

Questions to consider:

- How can I use time-on-task estimates to improve time management?
- What behaviors can help or hinder when it comes to managing time?

In this next section you will learn about managing time and prioritizing tasks. This is not only a valuable skill for pursuing an education, but it can become an ability that follows you through the rest of your life, especially if your career takes you into a leadership role.



Figure 1.6.7 An online calendar is a very useful tool for keeping track of classes, meetings, and other events. Most learning management systems contain these features, or you can use a calendar application.

ANALYSIS QUESTION

Read each statement in the brief self-evaluation tool below, and check the answer that best applies to you. There are no right or wrong answers.

Table 3.3

	Always	Usually	Sometimes	Rarely	Never
I like to be given strict deadlines for each task. It helps me stay organized and on track.					
I would rather be 15 minutes early than 1 minute late.					
I like to improvise instead of planning everything out ahead of time.					

	Always	Usually	Sometimes	Rarely	Never
I prefer to be able to manage when and how I do each task.					
I have a difficult time estimating how long a task will take.					
I have more motivation when there is an upcoming deadline. It helps me focus.					
I have difficulty keeping priorities in the most beneficial order.					

This exercise is intended to help you recognize some things about your own time management style. The important part is for you to identify any areas where you might be able to improve and to find solutions for them. This chapter will provide some solutions, but there are many others that can be found by researching time management strategies.

After you have decided your best response to each statement, think about what they may mean in regard to potential strengths and/or challenges for you when it comes to time management in college. If you are a person that likes strict deadlines, what would you do if you took a course that only had one large paper due at the end? Would you set yourself a series of mini deadlines that made you more comfortable and that kept things moving along for you? Or, if you have difficulty prioritizing tasks, would it help you to make a list of the tasks to do and order them, so you know which ones must be finished first?

How to Manage Time

The simplest way to manage your time is to accurately plan for how much time it will take to do each task, and then set aside that amount of time. How you divide the time is up to you. If it is going to take you five hours to study for a final exam, you can plan to spread it over five days, with an hour each night, or you can plan on two hours one night and three hours the next. What you would not want to do is plan on studying only a few hours the night before the exam and find that you fell very short on the time you estimated you would need. If that were to happen, you would have run out of time before finishing, with no way to go back and change your decision. In this kind of situation, you might even be tempted to “pull an all-nighter,” which is a phrase that has been used among college students for decades. In essence it means going without sleep for the entire night and using that time to finish an assignment. While this method of trying to make up for poor planning is common enough to have a name, rarely does it produce the best work.

Table 3.4 Sample Time Estimate Table

Activity	Estimated Time	Actual Time
Practice Quiz	5 minutes	15 minutes
Lab Conclusions	20 minutes	35 minutes
Food shopping	45 minutes	30 minutes
Drive to work	20 minutes	20 minutes
Physical Therapy	1 hour	50 minutes

Of all the parts of time management, accurately predicting how long a task will take is usually the most difficult—and the most elusive. Part of the problem comes from the fact that most of us are not very accurate timekeepers, especially when we are busy applying ourselves to a task. The other issue that makes it so difficult to accurately estimate time on task is that our estimations must also account for things like interruptions or unforeseen problems that cause delays.?

When it comes to academic activities, many tasks can be dependent upon the completion of other things first, or the time a task takes can vary from one instance to another, both of which add to the complexity and difficulty of estimating how much time and effort are required.

For example, if an instructor assigned three chapters of reading, you would not really have any idea how long each chapter might take to read until you looked at them. The first chapter might be 30 pages long while the second is 45. The third chapter could be only 20 pages but made up mostly of charts and graphs for you to compare. By page count, it might seem that the third chapter would take the least amount of time, but actually studying charts and graphs to gather information can take longer than regular reading.?

To make matters even more difficult, when it comes to estimating time on task for something as common as reading, not all reading takes the same amount of time. Fiction, for example, is usually a faster read than a technical manual. But something like the novel *Finnegan's Wake* by James Joyce is considered so difficult that most readers never finish it.

ACTIVITY

To better understand how much time different kinds of material can take to read, try this experiment. You will use two examples of famous texts that are very close to being the same number of words: *The Gettysburg Address* and the opening paragraphs from *A Christmas Carol*. Before you begin, estimate how long it will take you to read each, and predict which you think will take longer. When you do the reading, use a stopwatch function on a device such as a phone or some other timer to see how long it actually takes.

Make certain that you are reading for understanding, not just skimming over words. If you must reread a section to better comprehend what is being said, that is appropriate. The goal here is to compare reading of different texts, not to see how fast you can sight-read the words on a page.

After you have finished *The Gettysburg Address*, read and time *A Christmas Carol* and compare both of your times.

The Gettysburg Address
Abraham Lincoln
Gettysburg, Pennsylvania November 19, 1863
Word count: 278

Four score and seven years ago our fathers brought forth on this continent, a new nation, conceived in Liberty, and dedicated to the proposition that all men are created equal.

Now we are engaged in a great civil war, testing whether that nation, or any nation so conceived and so dedicated, can long endure. We are met on a great battle-field of that war. We have come to dedicate a portion of that field, as a final resting place for those who here gave their lives that that nation might live. It is altogether fitting and proper that we should do this.

But, in a larger sense, we can not dedicate -- we can not consecrate -- we can not hallow -- this ground. The brave men, living and dead, who struggled here, have consecrated it, far above our poor power to add or detract. The world will little note, nor long remember what we say here, but it can never forget what they did here. It is for us the living, rather, to be dedicated here to the unfinished work which they who fought here have thus far so nobly advanced. It is rather for us to be here dedicated to the great task remaining before us -- that from these honored dead we take increased devotion to that cause for which they gave the last full measure of devotion -- that we here highly resolve that these dead shall not have died in vain -- that this nation, under God, shall have a new birth of freedom -- and that government of the people, by the people, for the people, shall not perish from the earth.

A Christmas Carol
Charles Dickens
Chapman & Hall, 1843
Word count: 260

Marley was dead: to begin with. There is no doubt whatever about that. The register of his burial was signed by the clergyman, the clerk, the undertaker, and the chief mourner. Scrooge signed it: and Scrooge's name was good upon 'Change, for anything he chose to put his hand to. Old Marley was as dead as a door-nail.

Mind! I don't mean to say that I know, of my own knowledge, what there is particularly dead about a door-nail. I might have been inclined, myself, to regard a coffin-nail as the deadest piece of ironmongery in the trade. But the wisdom of our ancestors

> is in the simile; and my unhallowed hands shall not disturb it, or the Country's done for. You will therefore permit me to repeat, emphatically, that Marley was as dead as a door-nail.
>
> Scrooge knew he was dead? Of course he did. How could it be otherwise? Scrooge and he were partners for I don't know how many years. Scrooge was his sole executor, his sole administrator, his sole assign, his sole residuary legatee, his sole friend, and sole mourner. And even Scrooge was not so dreadfully cut up by the sad event, but that he was an excellent man of business on the very day of the funeral, and solemnised it with an undoubted bargain.
>
> The mention of Marley's funeral brings me back to the point I started from. There is no doubt that Marley was dead. This must be distinctly understood, or nothing wonderful can come of the story I am going to relate.
>
> In comparing the two, was one or the other easier to understand or faster to read? Was it the piece you predicted you would read faster?
>
> It is important to note that in this case both readings were only three paragraphs long. While there may have only been half a minute or so between the reading of each, that amount of time would multiply greatly over an entire chapter.

Knowing Yourself

While you can find all sorts of estimates online as to how long a certain task may take, it is important to know these are only averages. People read at different speeds, people write at different speeds, and those numbers even change for each individual depending on the environment.

If you are trying to read in surroundings that have distractions (e.g., conversations, phone calls, etc.), reading 10 pages can take you a lot longer than if you are reading in a quiet area. By the same token, you may be reading in a quiet environment (e.g., in bed after everyone in the house has gone to sleep), but if you are tired, your attention and retention may not be what it would be if you were refreshed.

In essence, the only way you are going to be able to manage your time accurately is to know yourself and to know how long it takes you to do each task. But where to begin?

Below, you will find a table of common college academic activities. This list has been compiled from a large number of different sources, including colleges, publishers, and professional educators, to help students estimate their own time on tasks. The purpose of this table is to both give you a place to begin in your estimates and to illustrate how different factors can impact the actual time spent.

You will notice that beside each task there is a column for the *unit*, followed by the average *time on task*, and a column for notes. The *unit* is whatever is being measured (e.g., pages read, pages written, etc.), and the *time on task* is an average time it takes students to do these tasks. It is important to pay attention to the notes column, because there you will find factors that influence the time on task. These factors can dramatically change the amount of time the activity takes.

Table 3.5 Time on task for common college activities.

Time on Task			
Activity	**Unit**	**Time on task**	**Notes**
General academic reading (textbook, professional journals)	1 page	5–7 minutes	Be aware that your personal reading speed may differ and may change over time.
Technical reading (math, charts and data)	1 page	10–15 minutes	Be aware that your personal reading speed may differ and may change over time.
Simple Quiz or homework question: short answer—oriented toward recall or identification type answers	Per question	1–2 minutes	Complexity of question will greatly influence the time required.
Complex Quiz or homework question: short answer—oriented toward application, evaluation, or synthesis of knowledge	Per question	2–3 minutes	Complexity of question will greatly influence the time required.

Time on Task			
Math problem sets, complex	Per question	15 minutes	For example, algebra, complex equations, financial calculations
Writing: short, no research	Per page	60 minutes	Short essays, single-topic writing assignments, summaries, freewriting assignments, journaling—includes drafting, writing, proofing, and finalizing
Writing: research paper	Per page	105 minutes	Includes research time, drafting, editing, proofing, and finalizing (built into per-page calculation)
Study for quiz	Per chapter	60 minutes	45–90 minutes per chapter, depending upon complexity of material
Study for exam	Per exam	90 minutes	1–2 hours, depending upon complexity of material

Again, these are averages, and it does not mean anything if your times are a little slower or a little faster. There is no "right amount of time," only the time that it takes you to do something so you can accurately plan and manage your time.

There is also another element to look for in the table. These are differentiations in the similar activities that will also affect the time you spend. A good example of this can be found in the first four rows. Each of these activities involves reading, but you can see that depending on the material being read and its complexity, the time spent can vary greatly. Not only do these differences in time account for the different types of materials you might read (as you found in the comparative reading exercise earlier in this chapter), but also they also take into consideration the time needed to think about what you are reading to truly understand and comprehend what it is saying.

GET CONNECTED

Which apps help you best prepare for success when managing your time?

Do you have trouble keeping track of multiple tasks over the course of a term?

Trello lets you organize all your obligations in helpful boards. You can share them with others (project collaborators), set alerts as reminders, and mark tasks off as you complete them.

Do you use a particular app to help you manage your time?

Sticky note apps are available for PC, Mac, and mobile devices. They let you post quick reminders, reorganize them as needed, and view them separately or as a full to-do list.

What do you wish you could improve about your time management skills?

Toggl helps you keep track of how and where you are spending your time so you can budget better and make time management changes that free you up for the really important stuff.

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1.7: Enhanced Strategies for Time and Task Management

Questions to consider:

- What strategy helps me prioritize my top tasks?
- How do I make the best use of my time when prioritizing?
- How do I make sure I tackle unpleasant tasks instead of putting them off?
- What's the best way to plan for long-term tasks?
- How do I find time in a busy schedule?

Over the years, people have developed a number of different strategies to manage time and tasks. Some of the strategies have proven to be effective and helpful, while others have been deemed not as useful.

The good news is that the approaches that do not work very well or do not really help in managing time do not get passed along very often. But others, those which people find of value, do. What follows here are three unique strategies that have become staples of time management. While not everyone will find that all three work for them in every situation, enough people have found them beneficial to pass them along with high recommendations.

Daily Top Three

The idea behind the *daily top three* approach is that you determine which three things are the most important to finish that day, and these become the tasks that you complete. It is a very simple technique that is effective because each day you are finishing tasks and removing them from your list. Even if you took one day off a week and completed no tasks on that particular day, a *daily top three* strategy would have you finishing 18 tasks in the course of a single week. That is a good amount of things crossed off your list.

analysis question

Analysis: Think about what would be your top three tasks for today? What would you have on the list tomorrow?

Pomodoro Technique



Figure 1.7.14 The Pomodoro Technique is named after a type of kitchen timer, but you can use any clock or countdown timer. (Marco Verch /Flickr / Attribution 2.0 Generic (CC BY 2.0))

The Pomodoro Technique was developed by Francesco Cirillo. The basic concept is to use a timer to set work intervals that are followed by a short break. The intervals are usually about 25 minutes long and are called *pomodoros*, which comes from the Italian word for tomato because Cirillo used a tomato-shaped kitchen timer to keep track of the intervals.

In the original technique there are six steps:

1. Decide on the task to be done.
2. Set the timer to the desired interval.
3. Work on the task.
4. When the timer goes off, put a check mark on a piece of paper.
5. If you have fewer than four check marks, take a short break (3–5 minutes), then go to Step 1 or 2 (whichever is appropriate).
6. After four pomodoros, take a longer break (15–30 minutes), reset your check mark count to zero, and then go to Step 1 or 2.

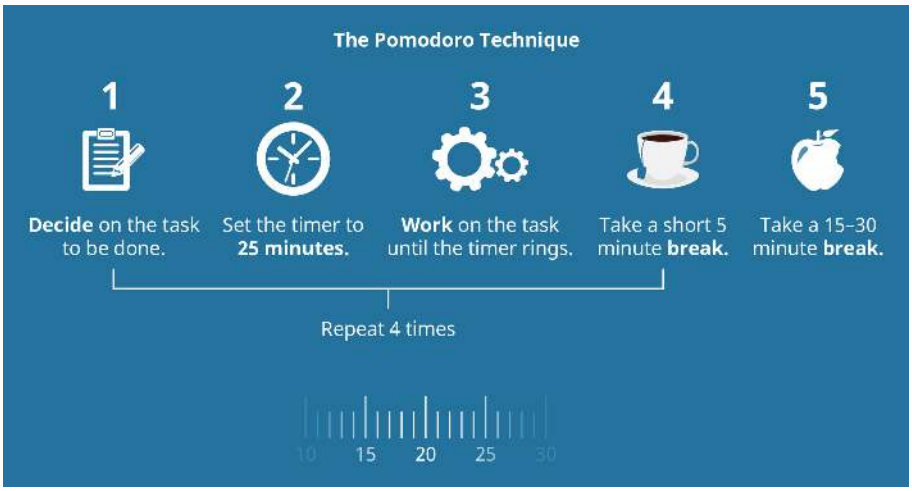


Figure 1.7.15 The Pomodoro Technique contains five defined steps.

There are several reasons this technique is deemed effective for many people. One is the benefit that is derived from quick cycles of work and short breaks. This helps reduce mental fatigue and the lack of productivity caused by it. Another is that it tends to encourage practitioners to break tasks down to things that can be completed in about 25 minutes, which is something that is usually manageable from the perspective of time available. It is much easier to squeeze in three 25-minute sessions of work time during the day than it is to set aside a 75- minute block of time.

Eat the Frog

Of our three quick strategies, *eat the frog* probably has the strangest name and may not sound the most inviting. The name comes from a famous quote, attributed to Mark Twain: “Eat a live frog first thing in the morning and nothing worse will happen to you the rest of the day.” *Eat the Frog* is also the title of a best-selling book by Brian Tracy that deals with time management and avoiding procrastination.

How this applies to time and task management is based on the concept that if a person takes care of the biggest or most unpleasant task first, everything else will be easier after that.

Although stated in a humorous way, there is a good deal of truth in this. First, we greatly underestimate how much worry can impact our performance. If you are continually distracted by anxiety over a task you are dreading, it can affect the task you are working on at the time. Second, not only will you have a sense of accomplishment and relief when the task you are concerned with is finished and out of the way, but other tasks will seem lighter and not as difficult.

Application: Try Three Time Management Strategies

Over the next two weeks, try each of these three methods to see which ones might work for you. Is there one you favor over the others? Might each of these three approaches serve you better in different situations or with different tasks? Do you have a creative alternative or possibly a way to use some combination of these techniques?

In addition to these three strategies, you could also develop whole new approaches from suggestions found earlier in this chapter. For example, you could apply some of the strategies for avoiding procrastination or for setting appropriate priorities and see how they work in combination with these techniques or on their own.

The key is to find which system works best for you.

Breaking Down the Steps and Spreading Them over Shorter Work Periods

Above, you read about several different tried-and-tested strategies for effective time management—approaches that have become staples in the professional world. In this section you will read about two more creative techniques that combine elements from these other methods to handle tasks when time is scarce and long periods of time are a luxury you just do not have.

The concept behind this strategy is to break tasks into smaller, more manageable units that do not require as much time to complete. As an illustration of how this might work, imagine that you are assigned a two-page paper that is to include references. You estimate that to complete the paper—start to finish—would take you between four and a half and five hours. You look at your

calendar over the next week and see that there simply are no open five-hour blocks (unless you decided to only get three hours of sleep one night). Rightly so, you decide that going without sleep is not a good option. While looking at your calendar, you do see that you can squeeze in an hour or so every night. Instead of trying to write the entire paper in one sitting, you break it up into much smaller components as shown in the table below:

Table 3.8: Breaking Down Projects into Manageable-Sized Tasks

Day/Time	Task	Time
Monday, 6:00 p.m.	Write outline; look for references.	60 minutes
Tuesday, 6:00 p.m.	Research references to support outline; look for good quotes.	60 minutes
Wednesday, 7:00 p.m.	Write paper introduction and first page draft.	60 minutes
Thursday, 6:00 p.m.	Write second page and closing draft.	60 minutes
Friday, 5:00 p.m.	Rewrite and polish final draft.	60 minutes
Saturday, 10:00 a.m.	*Only if needed—finish or polish final draft.*	60 minutes?

While this is a simple example, you can see how it would redistribute tasks to fit your available time in a way that would make completing the paper possible. In fact, if your time constraints were even more rigid, it would be possible to break these divided tasks down even further. You could use a variation of the Pomodoro Technique and write for three 20-minute segments each day at different times. The key is to look for ways to break down the entire task into smaller steps and spread them out to fit your schedule.

Table 3.9

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
8:00–10:00		Work		Work			
10:00–12:00	Algebra	Work	Algebra	Work	Algebra	10 a.m.–11 a.m. *Only if needed*	Work
12:00–2:00	Lunch/study	1 p.m. English Comp	Lunch/study	1 p.m. English Comp	Lunch/study	Family picnic	Work
2:00–4:00	History	English Comp	History	English Comp	History	Family picnic	
4:00–6:00	Study for Algebra quiz.	Grocery	Study for History exam.	Study for History exam.	5 p.m.–6 p.m. Rewrite and polish final draft.	Family picnic	Laundry
6:00–7:00	Write outline; look for references.	Research references to support outline; look for good quotes.	Research presentation project.	Write second page and closing draft	Create presentation.	Meet with Darcy.	Prepare school stuff for next week.
7:00–8:00	Free time	Free time	Write paper introduction and first page draft.	Research presentation project.	Create presentation.		Free time

STUDENT PROFILE

"Time management is probably one of the hardest things I had to pick up when I got to college. For starters, I didn't have anyone to come wake me up if I forgot to set an alarm or to tell me to get out of bed so that I wouldn't be late. I had to start placing my phone far away from my bed; so that way, I would have to get out of bed in order to turn the alarm off.

Accomplishing work on time can also be difficult. It's tough to find the fine balance between when you have to stay in and work on assignments and when is acceptable to go out and do leisure activities.

"I learned the 8-8-8 rule. Every day you spend eight hours working on school work or going to class, eight hours of free time to do what you want, and then eight hours to sleep at night so that you will get enough rest. Sleep is crucial for time management. I learned very quickly that you cannot focus or be productive if you are struggling to keep your head from falling over because you are so tired. Basically, I've learned that if you want to be successful in college, then you have to be on top of your game when it comes to time. It's something thing you cannot make up once it's gone."

—Preston Allen, University of Central Arkansas

Analyzing Your Schedule and Creating Time to Work

Of all the strategies covered in this chapter, this one may require the most discipline, but it can also be the most beneficial in time management. The fact is most of us waste time throughout the day. Some of it is due to a lack of awareness, but it can also be caused by the constraints of our current schedules. An example of this is when we have 15 to 20 minutes before we must leave to go somewhere. We don't do anything with that time because we are focused on leaving or where we are going, and we might not be organized enough to accomplish something in that short of a time period. In fact, a good deal of our 24- hour days are spent a few minutes at a time waiting for the next thing scheduled to occur. These small units of time add up to a fair amount each day.

The intent of this strategy is to recapture those lost moments and use them to your advantage. This may take careful observation and consideration on your part, but the results of using this as a method of time management are more than worth it.

The first step is to look for those periods of time that are wasted or that can be repurposed. In order to identify them, you will need to pay attention to what you do throughout the day and how much time you spend doing it. The example of waiting for the next thing in your schedule has already been given, but there are many others. How much time do you spend in activities after you have really finished doing them but are still lingering because you have not begun to do something else (e.g., letting the next episode play while binge-watching, reading social media posts or waiting for someone to reply, surfing the Internet, etc.)? You might be surprised to learn how much time you use up each day by just adding a few unproductive minutes here and there.

If you set a limit on how much time you spend on each activity, you might find that you can recapture time to do other things. An example of this would be limiting yourself to reading news for 30 minutes. Instead of reading the main things that interest you and then spending an additional amount of time just looking at things that you are only casually interested in because that is what you are doing at the moment, you could stop after a certain allotted period and use the extra time you have gained on something else.

After you identify periods of lost time, the next step will be to envision how you might restructure your activities to bring those extra minutes together into useful blocks of time. Using the following scenario as an illustration, we will see how this could be accomplished.



Figure 1.7.16 Sarah has to balance a lot of obligations.

On Tuesday nights, Sarah has a routine: After work, she does her shopping for the week (2 hours driving and shopping) and then prepares and eats dinner (1 hour). After dinner, she spends time on homework (1 hour) and catching up with friends, reading the news, and other Internet activities (1 hour), and then she watches television or reads before going to bed (1 hour). While it may seem that there is very little room for improvement in her schedule without cutting out something she enjoys, limiting the amount of time she spends on each activity and rethinking how she goes about each task can make a significant difference.

In this story, Sarah's Tuesday-night routine includes coming home from work, taking stock of which items in her home she might need to purchase, and then driving to the store. While at the store, she spends time picking out and selecting groceries as she plans for meals she will eat during the rest of the week. Then, after making her purchases, she drives home. Instead, if she took the time to make a list and plan for what she needed at the store before she arrived, she would not spend as much time looking for inspiration in each aisle. Also, if she had a prepared list, not only could she quickly pick up each item, but she could stop at the store on the way home from work, thus cutting out the extra travel time. If purchasing what she needed took 30 minutes less because she was more organized and she cut out an additional 20 minutes of travel time by saving the extra trip to the store from her house, she could recapture a significant amount of her Tuesday evening. If she then limited the time she spent catching up with friends and such to 30 minutes or maybe did some of that while she prepared dinner, she would find that she had added almost an extra hour and a half to the time available to her on that evening, without cutting out anything she needed to do or enjoys. If she decided to spend her time on study or homework, this would more than double the time she previously had available in her schedule for homework.

Analysis Question: Reflection

Analysis: Identify areas in the way you spend your day where you may be able to recapture and repurpose time. Are there things you can move around to gain more time? Are there ways you can combine tasks or reduce travel time?

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1.8: Goal Setting and Motivation

Questions to consider:

- How do I set motivational goals?
- What are SMART goals?
- What's the importance of an action plan?
- How do I keep to my plan?

Motivation often means the difference between success and failure. That applies to school, to specific tasks, and to life in general. One of the most effective ways to keep motivated is to set goals.

Goals can be big or small. A goal can range from *I am going to write one extra page tonight*, to *I am going to work to get an A in this course*, all the way to *I am going to graduate in the top of my class so I can start my career with a really good position*. The great thing about goals is that they can include and influence a number of other things that all work toward a much bigger picture. For example, if your goal is to get an A in a certain course, all the reading, studying, and every assignment you do for that course contributes to the larger goal. You have motivation to do each of those things and to do them well.

Setting goals is something that is frequently talked about, but it is often treated as something abstract. Like time management, goal setting is best done with careful thought and planning. This next section will explain how you can apply tested techniques to goal setting and what the benefits of each can be.

Set Goals That Motivate You

The first thing to know about goal setting is that a goal is a specific end result you desire. If the goal is not something you are really interested in, there is little motivational drive to achieve it. Think back to when you were much younger and some well-meaning adult set a goal for you—something that didn't really appeal to you at all. How motivated were you to achieve the goal? More than likely, if you were successful at all in meeting the goal, it was because you were motivated by earning the approval of someone or receiving a possible reward, or you were concerned with avoiding something adverse that might happen if you did not do what you were told. From an honest perspective in that situation, your real goal was based on something else, not the meeting of the goal set for you. To get the most from the goals you set, make sure they are things that you are interested in achieving.

That is not to say you shouldn't set goals that are supported by other motivations (e.g., If I finish studying by Friday, I can go out on Saturday), but the idea is to be intellectually honest with your goals.

Set SMART Goals

Goals should also be SMART. In this case, the word *smart* is not only a clever description of the type of goal, but it is also an acronym that stands for Specific, Measurable, Attainable, Relevant, and Time-bound. The reason these are all desirable traits for your goals is because they not only help you plan how to meet the goal, but they can also contribute to your decision-making processes during the planning stage.

What does it mean to create SMART goals?

- Specific—For a goal to be specific, it must be defined enough to actually determine the goal. A goal of *get a good job when I graduate* is too general. It doesn't define what a good job is. In fact, it doesn't even necessarily include a job in your chosen profession. A more specific goal would be something like *be hired as a nurse in a place of employment where it is enjoyable to work and that has room for promotion.*
- Measurable—The concept of *measurable* is one that is often overlooked when setting goals. What this means is that the goal should have clearly defined outcomes that are detailed enough to measure and can be used for planning of how you will achieve the goal. For example, setting a goal of *doing well in school* is a bit undefined, but making a goal of *graduating with a GPA above 3.0* is measurable and something you can work with. If your goal is measurable, you can know ahead of time how many points you will have to earn on a specific assignment to stay in that range or how many points you will need to make up in the next assignment if you do not do as well as you planned.
- Attainable—*Attainable* or *achievable* goals means they are reasonable and within your ability to accomplish. While a goal of *make an extra one million dollars by the end of the week* is something that would be nice to achieve, the odds that you could make that happen in a single week are not very realistic.

- Relevant—For goal setting, *relevant* means it applies to the situation. In relation to college, a goal of *getting a horse to ride* is not very relevant, but *getting dependable transportation* is something that would contribute to your success in school.
- Time-bound—Time-bound means you set a specific time frame to achieve the goal. *I will get my paper written by Wednesday* is time-bound. You know when you have to meet the goal. *I will get my paper written sometime soon* does not help you plan how and when you will accomplish the goal.

In the following table you can see some examples of goals that do and do not follow the SMART system. As you read each one, think about what elements make them SMART or how you might change those that are not.

Table 3.6

Goal	Is it SMART?	
I am going to be rich someday.	No	There is nothing really specific, measurable, or time-bound in this goal.
I will graduate with my degree, on time.	Yes	The statement calls out specific, measureable, and time-bound details. The other attributes of attainable and relevant are implied.
I am going to save enough money to buy a newer car by June.	Yes	All SMART attributes are covered in this goal.
I would like to do well in all my courses next semester.	No	While this is clearly time-bound and meets most of the SMART goal attributes, it is not specific or measurable without defining what “do well” means.
I am going to start being a nicer person.	No	While most of the SMART attributes are implied, there is nothing really measurable in this goal.
I will earn at least a 3.0 GPA in all my courses next semester.	Yes	All of the SMART attributes are present in this goal.
I am going to start being more organized.	No	While most of the SMART attributes are implied, there is nothing really measurable in this goal.

APPLICATION

Try writing two SMART goals—something with a one-week time frame and something that you will accomplish over the next year. Make certain that you include all the appropriate elements—Specific, Measurable, Attainable, Relevant, and Time-bound.

Make an Action Plan

Like anything else, making a step-by-step action plan of how you will attain your goals is the best way to make certain you achieve them. It doesn’t matter if it is a smaller goal with immediate results (e.g., finish all your homework due by Friday) or something bigger that takes several years to accomplish (graduate with my degree in the proper amount of time).

The planning techniques you use for time management and achieving goals can be similar. In fact, accurate goal setting is very much a part of time management if you treat the completion of each task as a goal.

What follows is an example of a simple action plan that lists the steps for writing a short paper. You can use something like this or modify it in a way that would better suit your own preferences.

Table 3.7

Action Plan		
Task	**Objective**	**When**

Action Plan		
Choose topic.	Select something interesting.	Needs to be done by Monday!
Write outline, look for references.	Create structure of paper and outline each part.	Monday, 6:00 p.m.
Research references to support outline, look for good quotes.	Strengthen paper and resources.	Tuesday, 6:00 p.m.
Write paper introduction and first page draft.	Get main ideas and thesis statement down.	Wednesday, 7:00 p.m.
Write second page and closing draft.	Finish main content and tie it all together.	Thursday, 6:00 p.m.
Rewrite and polish final draft.	Clean up for grammar, writing style, and effective communication.	Friday, 5:00 p.m.

Another useful approach to goal setting is to create SMART goals and then write them down. For most people there is a higher level of commitment when we write something down. If you have your goals written out, you can refer to each component of the SMART acronym and make certain you are on track to achieve it.

Stick with It!

As with anything else, the key to reaching goals is to keep at it, keep yourself motivated, and overcome any obstacles along the way. In the following graphic you will find seven methods that highly successful people use to accomplish this.



Figure 1.8.13 These seven ways to stay motivated are good suggestions from highly successful people. What other strategies would you suggest?

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1.9: Test Taking and Preparation

Students need to develop a mindset that every time they attend class or complete a reading assignment they are preparing for the next exam, because oftentimes so much of their grade rides on those exams. In college, it is not uncommon for your entire course grade to be based on two or three exams with no options for a retake or extra credit. Instead of an exam being over one or two chapters, exams are more likely to be over 5+ chapters of material, and in the case of a comprehensive exam, an entire semester's worth of material. Intentionally working through the process of preparing for, taking, and learning from the testing process will facilitate successful outcomes.



Table 5-4. Effective Technique for Test Preparation and Taking

Before the Test	During the Test	After the Test
Go to all of your classes and take lecture notes and read all of your assigned reading using techniques described in this chapter.	Mark your exam with the necessary identification. Read all instructions on the exam. Scan the exam for question types, point values, etc.	Conduct an error analysis for items missed on the test. Look up the correct answers. Determine the nature of the errors you made.
Develop a study plan and schedule the time to review for an exam over several days. Refer to "Create a Five-Day Study Plan for Exams."	Develop a plan for how you will use your time to complete all questions on the exam.	Talk with your professor about items you missed on the exam. Discuss adjustments in your test preparation methods if necessary.

Schedule study sessions with classmates in advance of the night before the exam.	Read each item carefully and fully before marking an answer. You might find clues to the answer and will be less likely to make a preventable mistake.	Analyze the test questions to see if patterns arise, such as,"Were most of the questions from the lecture, the book, or both?" "What types of questions were asked?" "What levels of thinking were being tested based on Bloom's Taxonomy?" etc.
Be sure you have materials you need for the exam ready at least the night before the exam, such as multiple writing utensils, scantrons, blue books, notebook paper, etc.	Skip difficult questions and come back to them if you are pressed for time. Caution: if using a scantron to record answers be very careful that you do not incorrectly record your answers on the wrong line and get off track.	Review the methods that you used to study for the exam and determine what adjustments you need to make in preparing for the next exam.
Get a good night's rest the night before the exam.	If time is available at the end of the exam, review questions you marked as "difficult" in order to check for errors.	Determine your overall grade in the course based on the new information that this grade has provided you. Talk with your professor and academic advisor if you have concerns about your overall grade.
Eat a good breakfast prior to the exam. Avoid caffeine prior to the exam.	If you have an essay question to write, take the time to briefly outline an answer to ensure that you are answering all aspects of the question and that you provide a well-organized answer for your faculty member to read.	Archive your exam for future reference.
To ensure that you are on time to your exam, set two alarm clocks or arrange to have a friend call you to make sure you are awake.	Try not to leave any answers blank. You might be awarded partial credit or even make a correct guess.	
	Take a moment to take some deep cleansing breaths (or any other quick relaxation techniques) if you feel stressed by a question or the exam as a whole.	

Preparation for an exam should begin the first day of class. Repeated exposure to material over a longer period of time (Baddeley, 1982; Bahrick & Hall, 1991) optimizes the recall of information when it is needed. Every assigned reading, every opportunity to take notes and engage in class activities, and a routine of reviewing notes should be considered activities toward exam preparation. The volume of information that students are responsible for on each exam is much larger than experienced in high school. Careful review of the material prior to each exam is required for optimal performance, and student reports tell us that the habit of "cramming" the night before the exam is not going to lead to positive, long-term results. Research of the brain supports this typical finding. Physical and neurochemical changes in the synapses of the brain occur when information is learned, a process called long-term potentiation (Kandel & Schwartz, 1982). Without proper time and repeated stimulation of the neural circuits, long-term learning of this information will not occur efficiently and may not be accessible when needed for a test. Planning for a review period over five days prior to the exam and healthy eating and sleep habits cannot be overemphasized.

Activity 5-4

Identify when your next substantial exam will be given in a class. Create a five-day study plan using the formula described in "Create a Five-Day Study Plan for Exams" and the "Five-Day Study Plan Worksheet" at the end of this chapter. Once you have created this plan, incorporate the schedule into your time management calendar for implementation.

Oftentimes, students jump right in to taking an exam feeling stressed and rushed with a need to regurgitate information immediately. However, approaching a test in a careful and methodical way can help ensure the best results. You can best utilize your exam time by taking the time to survey the exam directions and point values and planning how to approach sections of the

test. Strategy is particularly important for tests with mixed types of questions (i.e., multiple choice and essay) or tests with multiple essay questions. For ex- ample, if you were to spend too much time on the multiple choice questions and not allocate enough time to complete an essay question, you may have answered the majority of the total number of questions but still do poorly on the exam if the essay question was worth a larger portion of the exam's total points. Plan your time carefully and manage your stress.

Create a Five-Day Study Plan for Exams

Making a plan that will allow for the careful review of all of the assigned and presented material leads to less stress the day of an exam and the best conditions for recall and performance on an exam. Here is a simple method for creating a study plan:

1. Break the material for the exam into four manageable "chunks." If material can be logically divided by chapters, use that method. If not, make up your own chunks based on the content of the material. Note: It is not a good idea to break up information by the method of delivery (i.e., one chunk equals notes, one chunk equals readings, etc.). You want to include all information on specific topics together in one chunk (e.g., information about the topic of osmosis from the lecture notes and the textbook in a chunk).
2. Plan to spend about two hours studying on each of the five days.
3. You should work with the material on the exam in two ways, by preparing that material for review and by reviewing that material. Preparation of material incorporates study techniques discussed earlier in this chapter such as combining and clarifying lecture and textbook notes, creating visual organizers for topics, developing cram cards, solving practice problems, creating self-testing questions using multiple levels of Bloom's Taxonomy, etc. Once you get the hang of preparing material for review, you could do some of that along the way (for instance creating cram cards for readings and notes) to reduce the amount of time needed for preparation at this time, leaving more time for review.

Example of the Schedule for a Five-Day Study Plan

Day One
Prepare 1st Chunk – 2 hours

Day Two
Prepare 2nd Chunk – 2 hours
Review 1st Chunk – 30 minutes

Day Three
Prepare 3rd Chunk – 1.5 hours
Review 2nd Chunk – 30 minutes
Review 1st Chunk – 15 minutes

Day Four
Prepare 4th Chunk – 1 hour
Review 3rd Chunk – 30 minutes
Review 2nd Chunk – 15 minutes
Review 1st Chunk – 10 minutes

Day Five
Review 4th Chunk – 1 hour
Review 3rd Chunk – 30 minutes
Review 2nd Chunk – 30 minutes
Review 1st Chunk – 30 minutes

Taking some time to plan out what material will go into each chunk will ensure that you do not miss anything. Use the "Five-Day Study Plan Worksheet" at the end of this chapter as a guide for your planning.

This method typically works very well for courses that offer two or three tests throughout the semester; however, a similar method could be modified by the number of days or the amount of time spent each day if a course offers more exams with less material on each exam.

Tips for Better Test Taking

The types of test items on a test are as different as the professors who created the tests. "Objective" test questions refer to items in which students have to recognize the correct answer from a list of provided options (e.g., multiple choice, true-false, matching). In

most cases there is one best answer, though it is important to note that a professor could indicate in the directions that you can select more than one answer. “Subjective” test items are the fill-in-the-blank, short answer, or essay questions in which students must recall and produce the answer.

Multiple Choice Questions

- Read the question and all of the answer options first.
- Mark out any answers that you know are not correct.
- Once you have selected an answer, do not change it unless you misread the question and know the new answer is correct. Usually, your first answer is the correct one if you are making an educated guess.
- Many times, the correct answer has more information in it.
- If there is no penalty for guessing, do not leave any items blank—make an educated guess.
- If there is an “All of the Above” option and there are two options that are correct, select “All of the Above.”
- In a question with “All of the Above” and “None of the Above,” if you are certain that one of the answers is true, do not select “None of the Above” and, likewise, if one of the answers is false, then do not select “All of the Above.”

True-False Questions

- Read the statement carefully, but do not read too much into the statement. Base your answer on the information provided.
- With statements that have multiple facts, if any one of the facts is false the entire statement is false.
- Qualifying words like “always,” “never,” and “every” indicate that this would have to be true all of the time. If it is not true all of the time, then you should answer false.
- Qualifying words like “usually,” “sometimes,” and “generally” indicate that it could be true or false depending on the situation. Oftentimes the answer is true.

Essay Questions

- Most Important Rule: Make sure you understand what the question is asking you to do. An essay question is not an opportunity to simply regurgitate everything you know about a particular topic. Revisit Question Cues listed in the “Table 5-1: “Bloom’s Taxonomy: Cognitive Domain” and be sure you know what you are supposed to accomplish when you see those cues. Underline key words and let those words help guide you. If you are not sure, ask your professor for clarification on what he or she wants you to accomplish in your answer.
- Take a minute or so to plan out your answer and jot down a brief outline of key points to guide your writing before you start. A well-organized answer tends to score more points when graded.
- Formula for Essay Questions:
  - Introductory statements that define terms and describe what you will accomplish in your essay. This section should be brief.
  - State your first main idea, and then give supporting facts, examples, statistics, or details. Follow with your next main idea and continue the pattern until complete. The bulk of your time should be spent developing this section of the essay as it would likely include the details your professor wants to see in grading. Typically, each main idea would have its own paragraph. Use transitional works like “first, second, third…,” “next,” “also,” “however,” etc., to aid your reader.
  - End with a summary or final conclusion. This section is also likely to be brief.
- Budget time to proofread and revise if needed.
- Write legibly. Neater papers tend to receive higher marks.

Adapted from Dembo and Seli (2008) and TestTakingTips.com (2003–2009).

Once the exam has been completed, the learning experience is not over. Look over the test after you get it back from your professor and continue your review of the material that may be important for a comprehensive final or information that serves as a foundation for material to be learned in other courses. Even if only a scantron answer sheet is returned to you, professors are willing to give students access to review the test results during their office hours. Comprehensive post-test review not only includes reviewing your correct and incorrect answers to the question but also looking at where questions came from (i.e., lecture notes, readings, or both). Also, consider the study strategies you used and how they worked for you in order to help you prepare for future exams. Overall, it is important to emphasize the benefit of talking with your professor after each exam to gain insights into the information presented in the class as well as your learning and test strategies.

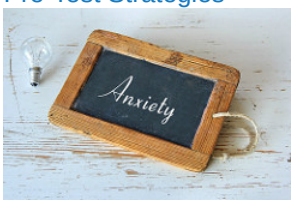
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1.10: Pre- Mid- and Post-Test-Taking Strategies

Phyllis Nissila

Pre-Test Strategies



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Q: When should you start preparing for the first test? Circle…

1. The night before.
2. The week prior.
3. The first day of classes.

If you answered "3. The first day of classes," you are correct. If you circled all three, you are also correct. Preparing to pass tests is something that begins when learning begins and continues all the way through to the final exam.

Many students, however, don't start thinking about test taking, whether weekly exams, mid-terms, or finals, until the day before when they engage in an all-nighter, or cramming. From the previous unit on memory, you might recall that the brain can only process an average of 5-7 new pieces of information at a time. Additionally, unless memory devices are used to aid memory and to cement information into long term memory (or at least until the test is over tomorrow!) chances are slim that students who cram will effectively learn and remember the information.

Additionally, a lot of students are unaware of the many strategies available to help with the test-taking experience before, during, and after. For starters, take a look at what has helped you so far.

EXERCISE 30-1

Pre-Test Taking Strategies

PART A:

Put a check mark next to the pre-test strategies you already employ.

____ Organize your notebook and other class materials the first week of classes.

____ Maintain your organized materials throughout the term.

____ Take notes on key points from lectures and other materials.

____ Make sure you understand the information as you go along.

____ Access your instructor's help and the help of a study group, as needed.

____ Organize a study group, if desired.

____ Create study tools such as flashcards, graphic organizers, etc. as study aids.

____ Complete all homework assignments on time.

____ Review likely test items several times beforehand.

____ Ask your instructor what items are likely to be covered on the test.

____ Ask your instructor if she or he can provide a study guide or practice test.

____ Ask your instructor if he/she gives partial credit for test items such as essays.

____ Maintain an active learner attitude.

____ Schedule extra study time in the days just prior to the test.

____ Gather all notes, handouts, and other materials needed before studying.

____ Review all notes, handouts, and other materials.

____ Organize your study area for maximum concentration and efficiency.

____ Create and use mnemonic devices to aid memory.

____ Put key terms, formulas, etc., on a single study sheet that can be quickly reviewed.

____ Schedule study times short enough (1-2 hours) so you do not get burned out.

____ Get plenty of sleep the night before.

____ Set a back-up alarm in case the first alarm doesn't sound or you sleep through it.

____ Have a good breakfast with complex carbs and protein to see you through.

____ Show up 5-10 minutes early to get completely settled before the test begins.

____ Use the restroom beforehand to minimize distractions.

PART B

By reviewing the pre-test strategies, above, you have likely discovered new ideas to add to what you already use. Make a list of them.

Mid-Test Strategies

Here is a list of the most common–and useful–strategies to survive this ubiquitous college experience.

- Scan the test, first, to get the big picture of how many test items there are, what types there are (multiple choice, matching, essay, etc.), and the point values of each item or group of items.
- Determine which way you want to approach the test: Some students start with the easy questions first, that is, the ones they immediately know the answers to, saving the difficult ones for later, knowing they can spend the remaining time on them. Some students begin with the biggest-point items first, to make sure they get the most points.
- Determine a schedule that takes into consideration how long you have to test, and the types of questions on the test. Essay questions, for example, will require more time than multiple choice, or matching questions.
- Keep your eye on the clock.
- If you can mark on the test, put a check mark next to items you are not sure of just yet. It is easy to go back and find them to answer later on. You might just find help in other test questions covering similar information.
- Sit where you are most comfortable. That said, sitting near the front has a couple of advantages: You may be less distracted by other students. If a classmate comes up with a question for the instructor and there is an important clarification given, you will be better able to hear it and apply it, if needed.
- Wear ear plugs, if noise distracts you.
- You do NOT have to start with #1! It you are unsure of it, mark it to come back to later on.
- Bring water…this helps calm the nerves, for one, and water is also needed for optimum brain function.
- If permitted, get up and stretch (or stretch in your chair) from time to time to relieve tension and assist the blood to the brain!
- Remember to employ strategies to reduce test-taking anxiety (covered in the next lesson)
- If despite all of your best efforts to prepare for a test you just cannot remember the answer to a given item for multiple choice, matching, and/or true/false questions, employ one or more of the following educated guessing (also known as "educated selection") techniques. By using these techniques, you have a *better* chance of selecting the correct answer. It is usually best to avoid selecting an extreme or all-inclusive answer (also known as 100% modifiers) such as "always," and "never". Choose, instead, words such as "usually," "sometimes," etc. (also known as in-between modifiers). If the answers are numbers, choose one of the middle numbers. If you have options such as "all of the above," or "both A and B," make sure each item is true before selecting those options. Choose the longest, or most inclusive, answer. Make sure to match the grammar of question and answer. For example, if the question indicates a plural answer, look for the plural answer. Regarding matching tests: count both sides to be matched. If there are more questions than answers, ask if you can use an answer more than once. Pay close attention to items that ask you to choose the "best" answer. This means one answer is better or more inclusive than a similar answer. Read all of the response options.

Post-Test Strategies

In addition to sighing that big sigh of relief, here are a few suggestions to help with future tests.

- If you don't understand why you did not get an item right, ask the instructor. This is especially useful for quizzes that contain information that may be incorporated into more inclusive exams such as mid-terms and finals.
- Analyze your results to help you in the future. For example, see if most of your incorrect answers were small things such as failing to include the last step in a math item, or neglecting to double-check for simple errors in a short-answer or essay item. See where in the test you made the most errors: beginning, middle, or end. Also analyze which type of questions, true/false, multiple choice, essay, etc. And which topics were missed. This will help you pay closer attention to those sections in the future.

EXERCISE 30-2

Write a letter of advice to Chen incorporating 10 test-taking tips and strategies you think will help him.

Chen believes he is good at organization, and he usually is–for about the first two weeks of classes. He then becomes overwhelmed with all of the handouts and materials and tends to start slipping in the organization department. When it comes to tests, he worries that his notes might not cover all of the right topics and that he will not be able to remember all of the key terms and points–especially for his math class. During tests, he sometimes gets stuck on an item and tends to spend too much time there. He also sometimes changes answers but finds out later that his original selection was correct. Chen is also easily distracted by other students and noises which makes it hard for him to concentrate sometimes, and, unfortunately, he does admit to occasionally "cramming" the night before.

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Adaptions: Changed formatting, removed one exercise.

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1.11: Test-Taking Strategy Specifics

Dave Dillon

"By failing to prepare, you are preparing to fail."

– Benjamin Franklin

Discipline, Preparation and Execution

From what I have seen in my own educational experiences, along with coaching and teaching at the college and university level for over 20 years, test taking (with few exceptions) comes down to discipline, preparation and execution. Students wanting to be successful have to have the self-discipline to schedule time to study well in advance of the exam. They have to actually do the work: the preparation needed in order to have the best opportunity for success on the exam. Then they must execute: they have to be able to apply their preparation accordingly and perform well on the exam.

Preparation for an exam is not glamorous. It's easy to find other things to do that are more interesting and fun. Students need to keep themselves motivated with their "eyes on the prize." Think of it like this: if the most important event of your life was coming up and you wanted to perform to the best of your ability in that event, you would likely spend some time preparing for it, rehearsing for it, practicing it, etc. A student may argue that an exam they will be taking would not be the most important event of their life, but think back to the chapter on Passion. If you're already spending the time, effort, energy and money to attend college, why not do it to the best of your ability?

It would be beneficial to spread this preparation and practice out over time and prepare periodically rather than to wait until the last minute and binge study or cram. Your preparation would not be the same and this may affect your test score. Binge studying and cramming also are not healthy. Staying up late puts stress on our brain and body, and not getting adequate sleep places our bodies at risk for getting sick.

"One of the most important keys to success is having the discipline to do what you know you should do, even when you don't feel like doing it."

– Unknown

"The will to succeed is important, but what's more important is the will to prepare."

– Bobby Knight

Everyone wants to be successful. When the exam is passed out, everyone wants to perform well. But what often separates successful students and less successful students is the preparation time put in.

Studying the right thing is a process and a skill. As you gain more experience, you will learn how to become better at knowing what to study. It can be very frustrating to spend a lot of time preparing and studying and then finding out that what you studied was not on the exam. You will see a lot of variance with exams due to different instructors, classes and types of tests. The better you become at predicting what will be on the exam and study accordingly, the better you will perform on your exams. Try placing yourself in your instructor's shoes and design questions you think your instructor would ask. It's often an eye-opening experience for students and a great study strategy.

Preparation for Exam Strategies

Find out as much about the exam in advance as you can.

Some professors will tell you how many questions there will be, what format the exam will be in, how much time you will have, etc., and others will not. I encourage students to ask questions about the exam if there is not information given. I also encourage students to ask those questions before class, after class, in professors' office hours or via e-mail rather than during class.

Know the test

If you know how many questions, what the format is, and/or how much time you will have, you can start to mentally prepare for the exam much more so than if you are coming in with no information. There are two more important aspects that you may or may not know: a) what will be covered or asked on the exam; b) how the exam will be scored. Obviously, the more you know about what will be covered, the easier it is for you to be able to prepare for the exam. Most exam scoring is standardized, but not always.

Look for opportunities where some areas of the exam are worth more points than others. For example: An exam consists of 21 questions, with 10 being True/False, 10 being multiple choice, and one essay question. The T/F questions are worth 1 point each (10 points), the multiple-choice questions are worth 2 points each (20 points), and the essay question is worth 30 points. We know that the essay question is the most valuable (it is worth half of the value of the exam). And we should allocate our time for it accordingly. I would advise starting with the essay question. Do a quick analysis of time to be able to spend your time on the exam wisely. You want to spend some time with the exam question since it is so valuable, without sacrificing adequate time to ensure the T/F and multiple-choice questions are answered.

Often, the order of the exam in this scenario will be: T/F first, multiple choice second and essay third. Most students will go in the chronological order of the exam, but a savvy student would start with the essay. If an exam were to last for 30 minutes with this format of questions, I might recommend a student spent 15 minutes on the essay question, ten minutes on the multiple choice, three minutes on the T/F and two minutes reviewing their answer.

Also, look for situations where exams penalize students for incorrectly answering a question. This does not occur very often, but is the case with some exams. With the SAT for example, students are awarded one point for a correct answer and ¼ of a point subtracted for an incorrect answer. Points are not awarded nor subtracted for leaving a question blank. Thus, the strategy for a multiple-choice question is: if you can narrow down the potentially correct answer to two rather than four or five, it is statistically advantageous to answer the question and guess between the two answers; however, if a student had no idea if any of the answers were correct or incorrect, it would be best to leave the answer blank. Remember, this is rare, but it is important to understand the strategy when students take these exams.

In conclusion, the more information you have about the exam, the better you can prepare for content, allocation of time spent on aspects of the exam, and the more confident you will be in knowing how and when to attempt to answer questions.

Take care of your body

Before the exam, it is important to prepare your brain and body for optimal performance for your exam. Do not cram the night before. Get a good night's sleep. Make sure you eat (nutritiously) before the exam. I recommend exercising the day before and if possible a few hours before the exam.

Strategies for Specific Exam Formats

True or False Questions

Look for qualifiers. A qualifier is a word that is absolute. Examples are: all, never, no, always, none, every, only, entirely. They are often seen in false statements. This is because it is more difficult to create a true statement using a qualifier like never, no, always, etc. For example, "All cats chase mice." Cats may be known for chasing mice, but not all of them do so. The answer here is false and the qualifier "all" gave us a tip. Qualifiers such as: sometimes, many, some, most, often, and usually are commonly found in true statements. For example: "Most cats chase mice." This is true and the qualifier "most" gave us a tip.

Make sure to read the entire statement. All parts of a sentence must be true if the whole statement is to be true. If one part of it is false, the whole sentence is false. Long sentences are often false for this reason.

Students should guess on True or False questions they do not know the answer to unless there is a penalty for an incorrect answer.

Multiple Choice Questions

Think of multiple choice questions as four (or five) true or false statements in one. One of the statements is true (the correct answer) and the others will be false. Apply the same strategy toward qualifiers. If you see an absolute qualifier in one of the answer choices, it is probably false and not the correct answer. Try to identify the true statement. If you can do this, you have the answer as there is only one. If you cannot do this at first, try eliminating answers you know to be false.

If there is no penalty for incorrect answers, my suggestion is to guess if you are not certain of the answer. If there is a penalty for incorrect answers, common logic is to guess if you can eliminate two of the answers as incorrect (pending what the penalty is). If there's a penalty and you cannot narrow down the answers, it's best to leave it blank. You may wish to ask your instructor for clarification.

Answers that are strange and unrelated to the question are usually false. If two answers have a word that looks or sounds similar, one of those is usually correct. For example: abductor/ adductor. If you see these as two of the four or five choices, one of them is usually correct. Also look for answers that are grammatically incorrect. These are usually incorrect answers. If you have to completely guess, choose B or C. It is statistically proven to be correct more than 25 percent of the time. If there are four answers

for each question, and an exam had standardized the answers, each answer on the exam A, B, C and D would be equal. But most instructors do not standardize their answers, and more correct answers are found in the middle (B and C then the extremes A and D or E). “People writing isolated four-choice questions hide the correct answer in the two middle positions about 70% of the time.”[1] This is 20 percent more correct answers found in B or C than a standardized exam with equal correct answers for each letter.

Matching Questions

Although less common than the other types of exams, you will likely see some matching exams during your time in college. First, read the instructions and take a look at both lists to determine what the items are and their relationship. It is especially important to determine if both lists have the same number of items and if all items are to be used, and used only once.

Matching exams become much more difficult if one list has more items than the other or if items either might not be used or could be used more than once. If your exam instructions do not discern this, you may wish to ask your instructor for further clarification. I advise students to take a look at the whole list before selecting an answer because a more correct answer may be found further into the list. Mark items when you are sure you have a match (pending the number of items in the list this may eliminate answers for the future). Guessing (if needed) should take place once you have selected answers you are certain about.

Short-Answer Questions

Read all of the instructions first. Budget your time and then read all of the questions. Answer the ones you know best or feel the most confident with. Then go back to the other ones. If you do not know the answer and there is no penalty for incorrect answers, guess. Use common sense. Sometimes instructors will award partial credit for a logical answer that is related even if it is not the correct answer.

Essay Questions

Keep in mind that knowing the format of the exam can help you determine how to study. If I know that I am taking a True-False exam, I know that I will need to discern whether a statement is True or False. I will need to know subject content for the course. But if I am studying for short answer and especially for essay questions, I must know a lot more. For essay questions, I must have much greater content knowledge and be able to make a coherent argument that answers the question using information from textbooks, lectures or other course materials. I have to place a lot more time and thought into studying for an essay exam than for True-False or Multiple-Choice exams.

Read the essay question(s) and the instructions first. Plan your time wisely and organize your answer before you start to write. Address the answer to the question in your first or second sentence. It may help to restate the original question. Write clearly and legibly. Instructors have difficulty grading essays that they cannot read. Save some time for review when you have finished writing to check spelling, grammar and coherent thought in your answer. Make sure you have addressed all parts of the essay question.

During the Exam:

Always read the directions first. Read them thoroughly.

Preview the exam to help you allocate proper time for each area.

Skip questions if you do not know the answer but make a mark somewhere to ensure you are able to go back to those questions (you may need to reallocate your prepared time for this depending on how many there are).

Allocate some time to review your answers before submitting your exam or the exam time expiring.

One of the biggest mistakes that students make after they take an exam in a course is that they do not use the exam for the future. The exam contains a lot of information that can be helpful in studying for future exams. Students that perform well on an exam often put it away thinking they do not need it anymore. Students who do poorly on an exam often put it away, not wanting to think about it any further.

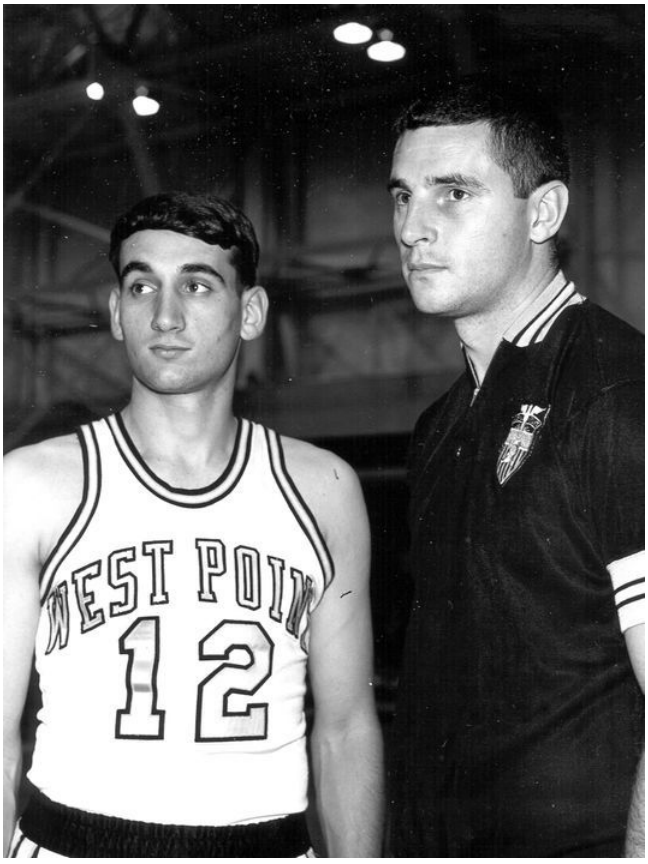
In both cases, students are missing out on the value of reviewing their exams. It is wise to review exams for three reasons: 1) students should review the answers that were correct because they may see those questions on future exams and it is important to reinforce learning; 2) students should review the answers that were incorrect in order to learn what the correct answer was and why. These questions also may appear on a future exam. In addition, occasionally an answer is marked incorrect, when it should have been marked correct. The student would never know this if they didn’t review their exam; and 3) there is value in reviewing the exam to try to predict what questions or what format will be used by a professor for a future exam in the same course.

Author's Story

What does it mean to be a "poor test taker?" Think about that. Does it mean that the student has put effort into studying but has difficulty under pressure? Does it mean that a student studies the wrong material? Is the student prepared but does not execute well? Could the student have a learning disability? Are they missing key strategies for taking tests? It could mean any of those things. And while I believe that it may be true that a student may be a "poor test taker," it does not by any means mean it is permanent. Students willing to work hard and learn can improve their test taking skills and raise their confidence.

In high school chemistry, I earned a D grade. In microeconomics at UC Santa Cruz, I struggled and barely passed. Both of these classes had grading systems that were heavily weighted by exams. I was an above average student in my other classes in both high school and college, and so I was told (and believed for a long time) that I was not a "good test taker." I no longer believe this. I will not make excuses. The responsibility for the grade I earned rests solely with me. I mention this because many students come to my office and complain that their professor grades in a way that is unfair to them because so much emphasis is on exams and that they understand the material but are "not good test takers." It may be the case that my grade or another students' grade would have been better had the course grade been determined with less weight levied to exams. That to me is irrelevant. It does not absolve me from being responsible for knowing what the grading metrics were at the beginning of the course and choosing to continue in the course after learning this knowledge. In the end, I realized I was not a poor test taker and that I needed to spend more time studying and preparing to perform well on exams. I changed my attitude from being afraid of exams to one of believing I was going to do well regardless of the class or the format of the exam. Once I had confidence that I had the necessary tools and was willing to work hard, it changed my entire perspective and experience with exams.

Note: if you think you may have a learning disability and would like to get assessed, contact your college to see what steps to follow to begin the assessment process.



Mike Krzyzewski, point guard at Army with Bob Knight, coach at Army.

Thinking about excuses reminds me of a story about Mike Krzyzewski, the successful Duke University men's basketball coach. Before Krzyzewski coached, he was a student at the United States Military Academy at West Point. One day in his first year, Krzyzweski and his roommate were walking and his roommate stepped in a puddle. Mud splashed onto Krzyzewski's uniform. Immediately, an upperclassman screamed at Krzyzewski about not knowing the rules of wearing a clean uniform. Plebes (first-year

students at military academies) are allowed three answers when asked a question: “Yes sir!” “No sir!” And “No excuse sir!” Krzyzewski repeated himself, “No excuse sir!” Despite wanting to explain what had happened and that it was not his fault, he realized that there was no excuse. It was his responsibility to keep a clean uniform.

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1. Yigal Attali and Maya Bar-Hillel, “Guess Where: The Position of Correct Answers in Multiple-Choice Test Items as a Psychometric Variable,” *Journal of Educational Measurement* 40 no. 2 (2003):109-128. ↵

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CHAPTER OVERVIEW

2: Introduction to Trigonometry

The trigonometric functions are functions of an angle. and relate the angles of a triangle to the lengths of its sides. They are important in the study of triangles and modeling periodic phenomena, among many other applications.

2.0: Prelude to Trigonometric Functions
2.1: Angles - Radians and Degrees
2.2: Unit Circle - Sine and Cosine Functions
2.3: Right Triangle Trigonometry
2.4: The Other Trigonometric Functions
2.5: Arcs, Angles, and Trig Function Values
2.E: Trigonometric Functions (Exercises)
2.R: Trigonometric Functions (Review)

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2.0: Prelude to Trigonometric Functions

Life is dense with phenomena that repeat in regular intervals. Each day, for example, the tides rise and fall in response to the gravitational pull of the moon. Similarly, the progression from day to night occurs as a result of Earth's rotation, and the pattern of the seasons repeats in response to Earth's revolution around the sun. Outside of nature, many stocks that mirror a company's profits are influenced by changes in the economic business cycle.



The tide rises and falls at regular, predictable intervals. (credit: Andrea Schaffer, Flickr)

In mathematics, a function that repeats its values in regular intervals is known as a periodic function. The graphs of such functions show a general shape reflective of a pattern that keeps repeating. This means the graph of the function has the same output at exactly the same place in every cycle. And this translates to all the cycles of the function having exactly the same length. So, if we know all the details of one full cycle of a true periodic function, then we know the state of the function's outputs at all times, future and past. In this chapter, we will investigate various examples of periodic functions.

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2.1: Angles - Radians and Degrees

Learning Objectives

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a *ray*. A ray consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its *endpoint* and any other point on it. The ray in Figure 2.1.1 can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

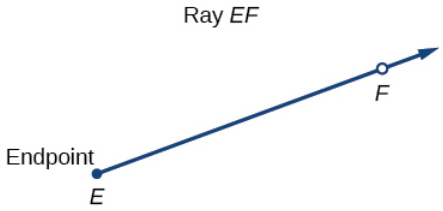


Figure 2.1.1

An *angle* is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in Figure 2.1.2 is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

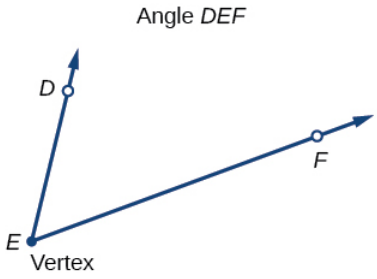


Figure 2.1.2

Greek letters are often used as variables for the measure of an angle. Table 2.1.1 is a list of Greek letters commonly used to represent angles, and a sample angle is shown in Figure 2.1.3.

Table 2.1.1

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

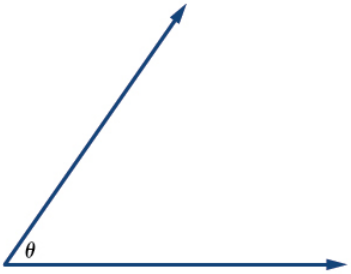


***Figure* 2.1.3**: *Angle theta, shown as* $\angle\theta$

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 2.1.4.

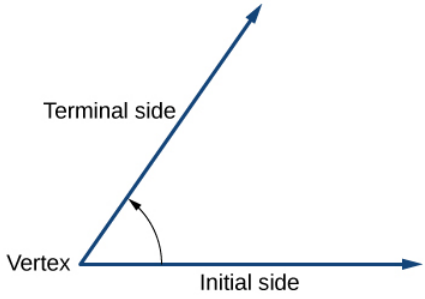


Figure 2.1.4

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol °. For example, 90 degrees = 90°.

To formalize our work, we will begin by drawing angles on an *x*-*y* coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See Figure 2.1.5.

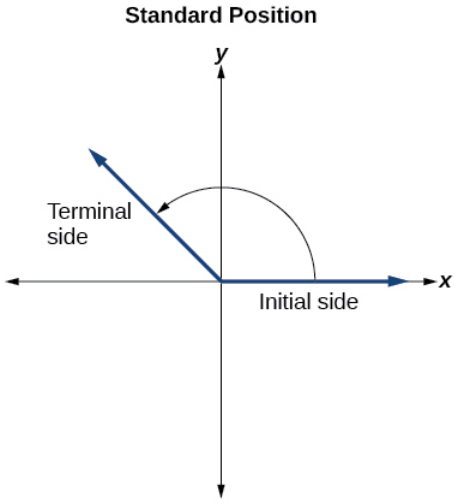


Figure 2.1.5

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90^\circ}{360^\circ} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive *x*-axis. To draw a 360° angle, we calculate that $\frac{360^\circ}{360^\circ} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive *x*-axis. In this case, the initial side and the terminal side overlap. See Figure 2.1.6.

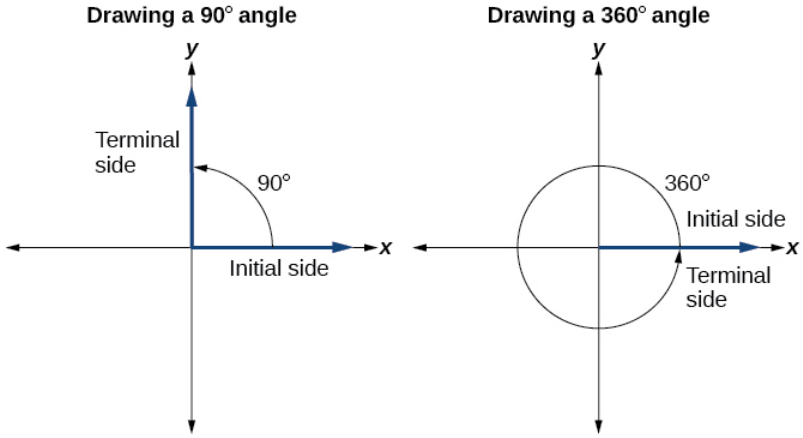


Figure 2.1.6

Since we define an angle in **standard position** by its initial side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See Figure 2.1.7.

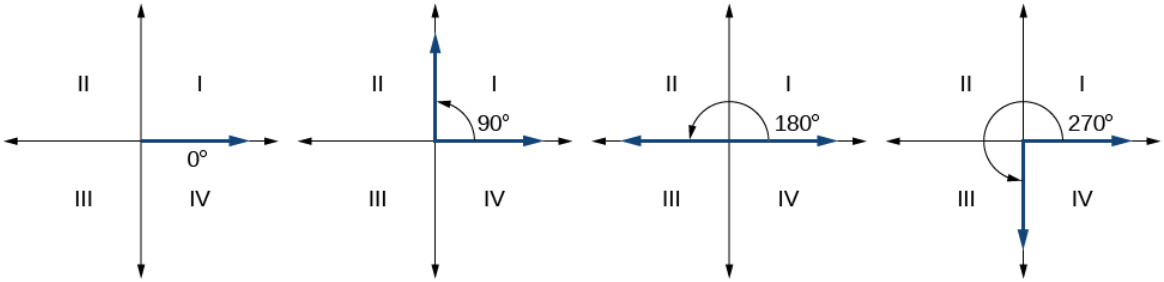


Figure 2.1.7: Quadrantal angles are angles in standard position whose terminal side lies along an axis. Examples are shown.

QUADRANTAL ANGLES

Quadrantal angles are angles in standard position whose terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

Given an angle measure in degrees, draw the angle in standard position

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 2.1.1: Drawing an Angle in Standard Position Measured in Degrees

1. Sketch an angle of 30° in standard position.
2. Sketch an angle of −135° in standard position.

Solution

1. Divide the angle measure by 360°.

$$\frac{30^\circ}{360^\circ} = \frac{1}{12} \tag{2.1.1}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}(\frac{1}{4}) \tag{2.1.2}$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in Figure 2.1.8.

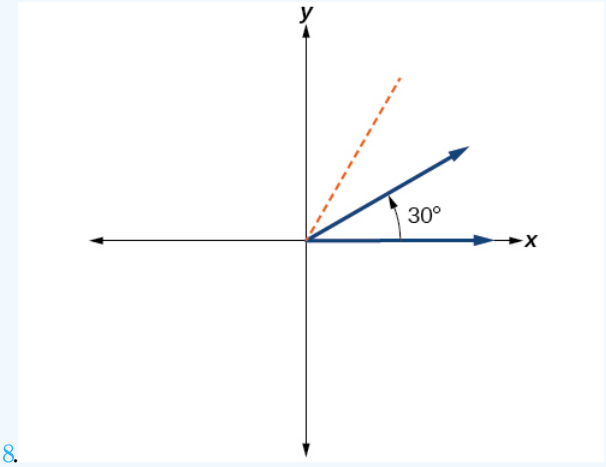


***Figure* 2.1.8**

2. Divide the angle measure by 360°.

$$\frac{-135^\circ}{360^\circ} = -\frac{3}{8} \tag{2.1.3}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}(\frac{1}{4}) \tag{2.1.4}$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in Figure 2.1.9.

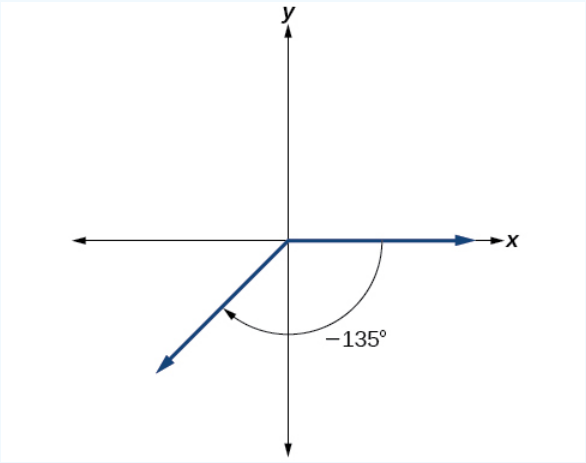


***Figure* 2.1.9**

2.1.1:

Show an angle of 240° on a circle in standard position.

Solution

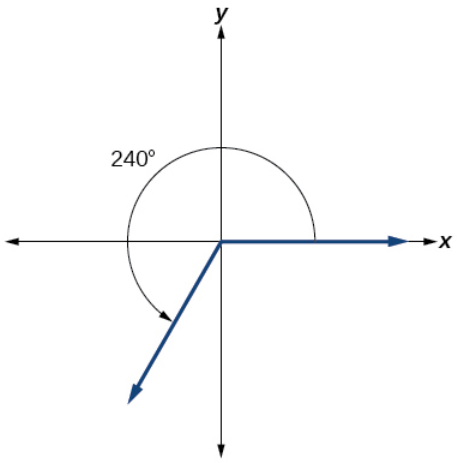


Figure 2.1.10

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the **circumference** of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in Figure 2.1.11.

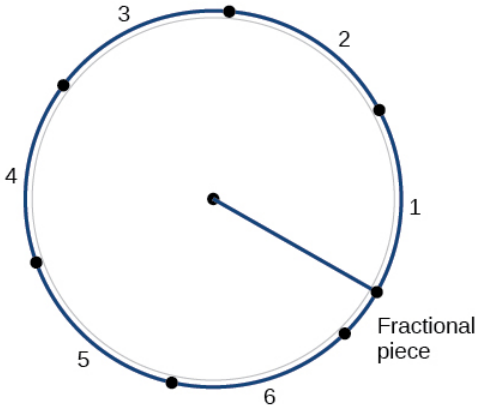


Figure 2.1.11

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians } = 360^\circ \tag{2.1.5}$$

$$\pi \text{ radians } = \frac{360^\circ}{2} = 180^\circ \tag{2.1.6}$$

$$1 \text{ radian } = \frac{180^\circ}{\pi} \approx 57.3^\circ \tag{2.1.7}$$

See Figure 2.1.12. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

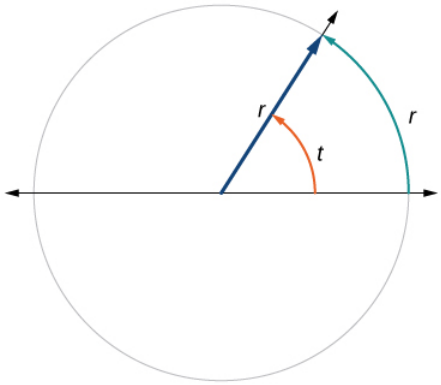


Figure 2.1.12: The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the **radian measure**, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s s to the radius r. See Figure 2.1.13.

$$s = r\theta \tag{2.1.8}$$

$$\theta = \frac{s}{r} \tag{2.1.9}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

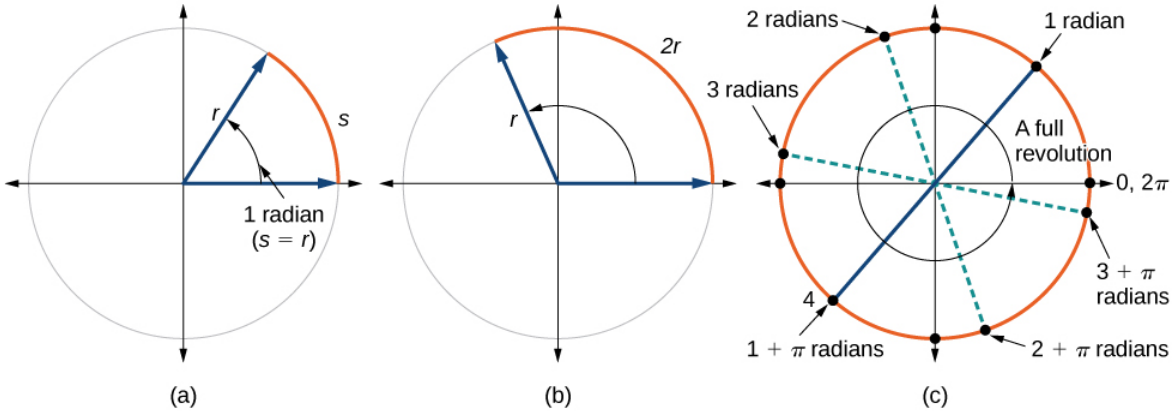


Figure 2.1.13: (a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$,where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in Figure 2.1.14.

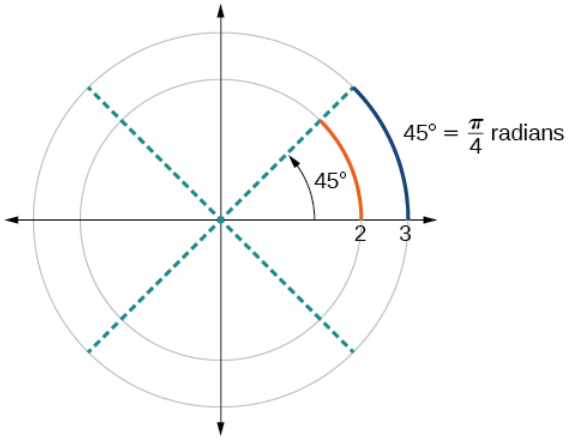


Figure 2.1.14: A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi \tag{2.1.10}$$

$$\text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi \tag{2.1.11}$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

RADIANS

One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π radians equals 360°, 1 radian equals $\frac{360^\circ}{2\pi} \approx 57.3^\circ$.

Using Radians

Because **radian** measure is the ratio of two lengths, it is a unitless measure. For example, in Figure 2.1.14, suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the **unit circle** (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{aligned} 1\text{rotation} &= 360^\circ &&= 2\pi \quad \text{radians} \\ \frac{1}{2}\text{rotation} &= 180^\circ &&= \pi \quad \text{radians} \\ \frac{1}{4}\text{rotation} &= 90^\circ &&= \frac{\pi}{2} \quad \text{radians} \end{aligned} \tag{2.1.12}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in Figure 2.1.15. Memorizing these angles will be very useful as we study the properties associated with angles.

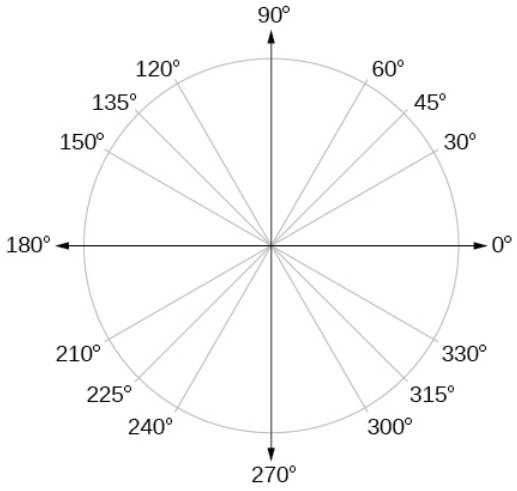


Figure 2.1.15: Commonly encountered angles measured in degrees

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in Figure 2.1.15, which are shown in Figure 2.1.16. Be sure you can verify each of these measures.

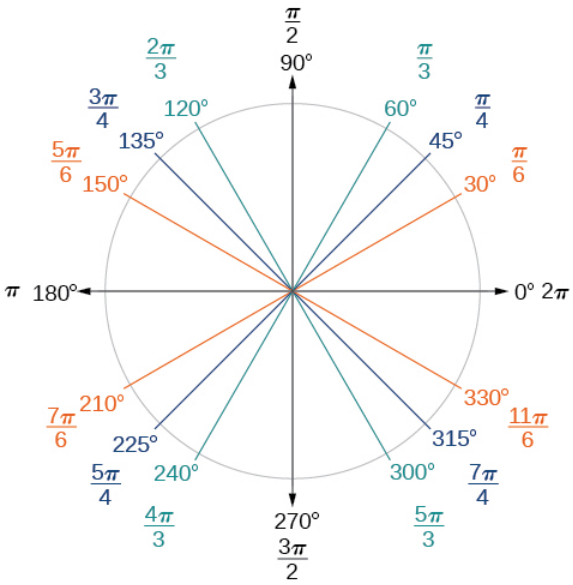


Figure 2.1.16: Commonly encountered angles measured in radians

Example 2.1.2: Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Solution

For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$1 \text{ rotation} = 2\pi r \tag{2.1.13}$$

So,

$$s = \frac{1}{3}(2\pi r) \tag{2.1.14}$$

$$= \frac{2\pi r}{3} \tag{2.1.15}$$

The radian measure would be the arc length divided by the radius.

$$\text{radian measure} = \frac{\frac{2\pi r}{3}}{r} \tag{2.1.16}$$

$$= \frac{2\pi r}{3r} \tag{2.1.17}$$

$$= \frac{2\pi}{3} \tag{2.1.18}$$

2.1.2

Find the radian measure of three-fourths of a full rotation.

Solution

$$\frac{3\pi}{2} \tag{2.1.19}$$

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{2.1.20}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{Radians}{\pi} \tag{2.1.21}$$

CONVERTING BETWEEN RADIANS AND DEGREES

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{2.1.22}$$

Example 2.1.3: Converting Radians to Degrees

Convert each radian measure to degrees.

1. $\frac{\pi}{6}$
2. 3

Solution

Because we are given radians and we want degrees, we should set up a proportion and solve it.

1. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{2.1.23}$$

$$\frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi} \tag{2.1.24}$$

$$\theta = \frac{180}{6} \tag{2.1.25}$$

$$\theta = 30^\circ \tag{2.1.26}$$

2. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{2.1.27}$$

$$\frac{\theta}{180} = \frac{3}{\pi} \tag{2.1.28}$$

$$\theta = \frac{3(180)}{\pi} \tag{2.1.29}$$

$$\theta \approx 172^\circ \tag{2.1.30}$$

2.1.3

Convert $-\frac{3\pi}{4}$ radians to degrees.

Solution

$-135°$

Example 2.1.4: Converting Degrees to Radians

Convert 15 degrees to radians.

Solution

In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{2.1.31}$$

$$\frac{15}{180} = \frac{\theta^R}{\pi} \tag{2.1.32}$$

$$\frac{15\pi}{180} = \theta^R \tag{2.1.33}$$

$$\frac{\pi}{12} = \theta^R \tag{2.1.34}$$

Analysis

Another way to think about this problem is by remembering that $30^\circ = \frac{\pi}{6}$. Because $15^\circ = \frac{1}{2}(30^\circ)$, ,we can find that $\frac{1}{2}(\frac{\pi}{6})$ is $\frac{\pi}{12}$.

2.1.4

Convert 126° to radians.

Solution

$\frac{7\pi}{10}$

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at Figure 2.1.17. The angle of 140° is a **positive angle**, measured counterclockwise. The angle of –220° is a **negative angle**, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are **coterminal** angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

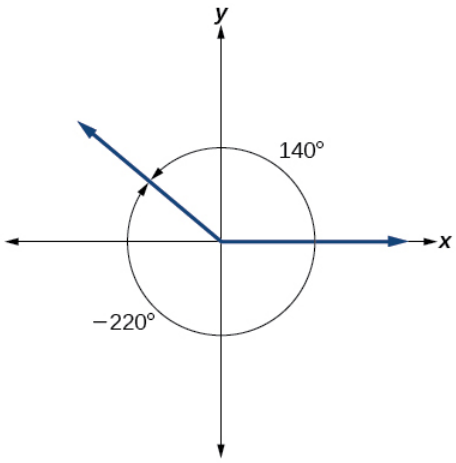


Figure 2.1.17: An angle of 140° and an angle of –220° are coterminal angles.

Any angle has infinitely many **coterminal angles** because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is −260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See Figure 2.1.18 for examples of reference angles for angles in different quadrants.

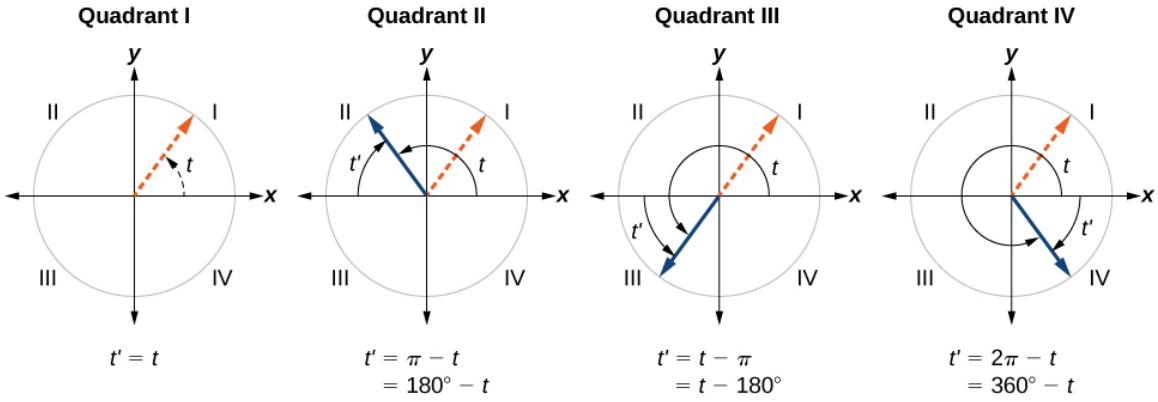


Figure 2.1.18

COTERMINAL AND REFERENCE ANGLES

- *Coterminal angles* are two angles in standard position that have the same terminal side.
- An angle's *reference angle* is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

Given an angle greater than 360°, find a coterminal angle between 0° and 360°

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 2.1.5: Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0^\circ \le \theta < 360^\circ$.

Solution

An angle with measure 800° is coterminal with an angle with measure 800 − 360 = 440°, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: 440 − 360 = 80°.

The angle $\theta = 80^\circ$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in Figure 2.1.19.

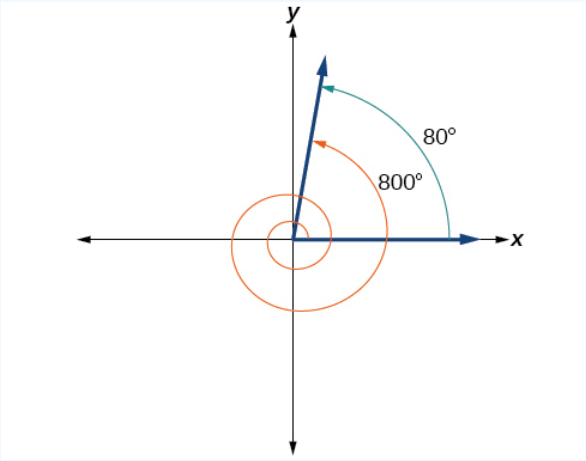


Figure 2.1.19

2.1.5

Find an angle α that is coterminal with an angle measuring 870°, where $0^\circ \le \alpha < 360^\circ$.

Solution

$\alpha = 150^\circ$

Given an angle with measure less than 0°, find a coterminal angle having a measure between 0° and 360°.

1. Add 360° to the given angle.
2. If the result is still less than 0°, add 360° again until the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 2.1.6: Finding an Angle Coterminal with an Angle Measuring Less Than 0°

Show the angle with measure −45° on a circle and find a positive coterminal angle α such that 0° ≤ α < 360°.

Solution

Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$$-45^\circ + 360^\circ = 315^\circ \tag{2.1.35}$$

We can then show the angle on a circle, as in Figure 2.1.20.

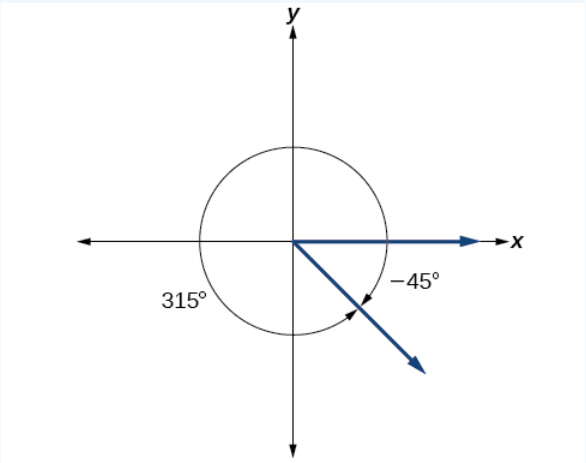


Figure 2.1.20

2.1.6

Find an angle β that is coterminal with an angle measuring −300° such that $0^\circ \le \beta < 360^\circ$.

Solution

$\beta = 60^\circ$

Finding Coterminal Angles Measured in Radians

We can find **coterminal** angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example 2.1.7: Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \le \beta < 2\pi$.

Solution

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} \tag{2.1.36}$$

$$= \frac{11\pi}{4} \tag{2.1.37}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} \tag{2.1.38}$$

$$= \frac{3\pi}{4} \tag{2.1.39}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in Figure 2.1.21.

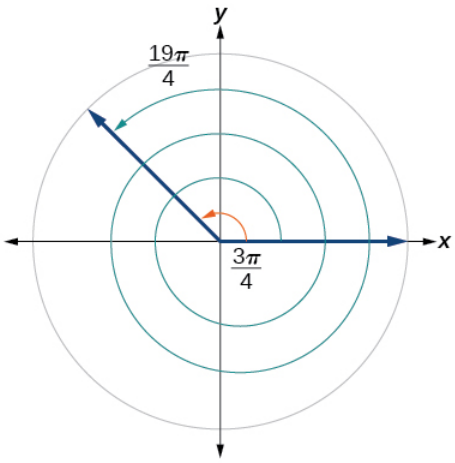


Figure 2.1.21

2.1.7

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Solution

$\frac{7\pi}{6}$

Determining the Length of an Arc

Recall that the **radian measure** θ of an angle was defined as the ratio of the **arc length** s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

ARC LENGTH ON A CIRCLE

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians, shown in Figure 2.1.22, is

$$s = r\theta \tag{2.1.40}$$

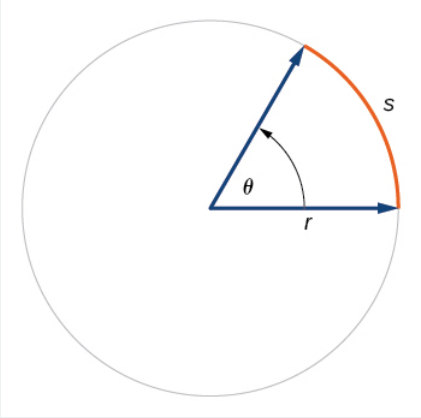


Figure 2.1.22

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ : $s = r\theta$.

Example 2.1.8: Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Solution

1. Let's begin by finding the circumference of Mercury's orbit.

$$C = 2\pi r \tag{2.1.41}$$
$$= 2\pi(36 \text{ million miles}) \tag{2.1.42}$$
$$\approx 226 \text{ million miles} \tag{2.1.43}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles} \tag{2.1.44}$$

2. Now, we convert to radians:

$$\text{radian} = \frac{\text{arc length}}{\text{radius}} \tag{2.1.45}$$
$$= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} \tag{2.1.46}$$
$$= 0.0717 \tag{2.1.47}$$

2.1.8

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Solution

$$\frac{215\pi}{18} = 37.525 \text{ units} \tag{2.1.48}$$

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a **sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\text{Area of sector} = (\frac{\theta}{2\pi})\pi r^2 \tag{2.1.49}$$
$$= \frac{\theta \pi r^2}{2\pi} \tag{2.1.50}$$
$$= \frac{1}{2}\theta r^2 \tag{2.1.51}$$

AREA OF A SECTOR

The area of a **sector of a circle** with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2 \tag{2.1.52}$$

See Figure 2.1.23.

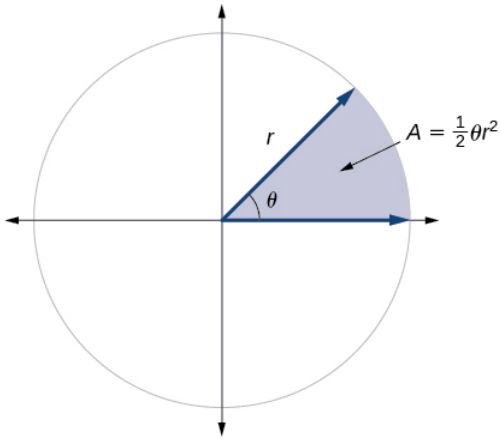


Figure 2.1.23: The area of the sector equals half the square of the radius times the central angle measured in radians.

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r : $A = \frac{1}{2}\theta r^2$.

Example 2.1.9: Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in Figure 2.1.24. What is the area of the sector of grass the sprinkler waters?

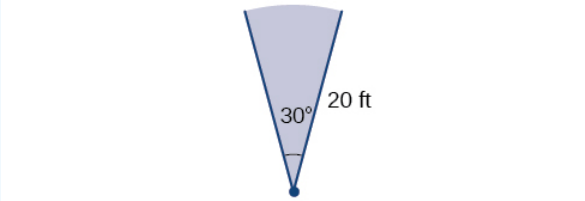


Figure 2.1.24: The sprinkler sprays 20 ft within an arc of 30°.

Solution

First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$30 \text{ degrees} = 30 \cdot \frac{\pi}{180} \tag{2.1.53}$$

$$= \frac{\pi}{6} \text{ radians} \tag{2.1.54}$$

The area of the sector is then

$$\text{Area} = \frac{1}{2}(\frac{\pi}{6})(20)^2 \tag{2.1.55}$$

$$\approx 104.72 \tag{2.1.56}$$

So the area is about 104.72 ft^2.

2.1.9

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Solution

1.88

Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its **displacement**) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$$v = \frac{s}{t} \tag{2.1.57}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}} = 90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t} \tag{2.1.58}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$s = r\theta \tag{2.1.59}$$
$$= r\omega t \tag{2.1.60}$$

Substituting this into the linear speed equation gives:

$$v = \frac{s}{t} \qquad = \frac{r\omega t}{t} = r\omega \tag{2.1.61}$$

ANGULAR AND LINEAR SPEED

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t} \tag{2.1.62}$$

The **linear speed.** v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v = \frac{s}{t} \tag{2.1.63}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\omega \tag{2.1.64}$$

This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

Given the amount of angle rotation and the time elapsed, calculate the angular speed

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Example 2.1.10: Finding Angular Speed

A water wheel, shown in Figure 2.1.25, completes 1 rotation every 5 seconds. Find the angular speed in radians per second.

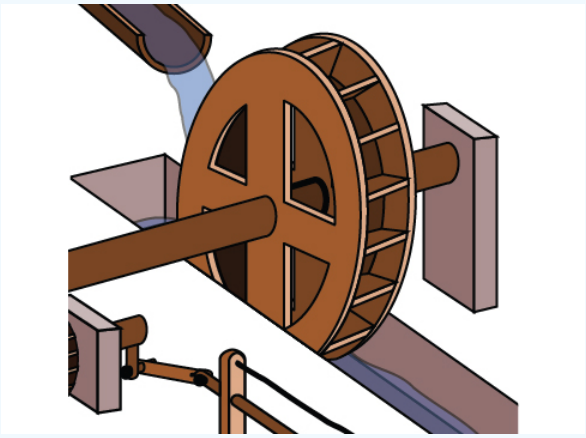


Figure 2.1.25

Solution

The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega = \frac{2\pi}{5} \approx 1.257$ radians per second.

2.1.10

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

Solution

$-\frac{3\pi}{2}$ rad/s

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \frac{\theta}{t}$.
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v = r\omega$.

Example 2.1.11: Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution

Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units to make this conversion:

$$180 \frac{\text{rotations}}{\text{minute}} \cdot \frac{2\pi \text{ radians}}{\text{rotation}} = 360\pi \frac{\text{radians}}{\text{minute}} \tag{2.1.65}$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

$$v = (14 \text{ inches})(360\pi \frac{\text{radians}}{\text{minute}}) \tag{2.1.66}$$

$$= 5040\pi \frac{\text{inches}}{\text{minute}} \tag{2.1.67}$$

Remember that radians are a unitless measure, so it is not necessary to include them.unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$5040\pi \frac{\cancel{\text{inches}}}{\cancel{\text{minute}}} \cdot \frac{1 \ \cancel{\text{feet}}}{12 \ \cancel{\text{inches}}} \cdot \frac{1 \ \cancel{\text{mile}}}{5280 \ \cancel{\text{feet}}} \cdot \frac{60 \ \cancel{\text{minutes}}}{1 \text{ hour}} \approx 14.99 \text{ miles per hour (mph)} \tag{2.1.68}$$

Try It 2.1.11

A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

Solution

1655 kilometers per hour

Media

Access these online resources for additional instruction and practice with angles, arc length, and areas of sectors.

- Angles in Standard Position
- Angle of Rotation
- Coterminal Angles
- Determining Coterminal Angles
- Positive and Negative Coterminal Angles
- Radian Measure
- Coterminal Angles in Radians
- Arc Length and Area of a Sector

Key Equations

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See **Example**.
- In addition to degrees, the measure of an angle can be described in radians. See **Example**.
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$. See Example and Example.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π. See Example and Example.

- Coterminal angles can be found using radians just as they are for degrees. See **Example**.
- The length of a circular arc is a fraction of the circumference of the entire circle. See **Example**.
- The area of sector is a fraction of the area of the entire circle. See **Example**.
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See **Example**.
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See **Example**.

Glossary

angle
the union of two rays having a common endpoint

angular speed
the angle through which a rotating object travels in a unit of time

arc length
the length of the curve formed by an arc

area of a sector
area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$ multiplied by the area of the entire circle

coterminal angles
description of positive and negative angles in standard position sharing the same terminal side

degree
a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

initial side
the side of an angle from which rotation begins

linear speed
the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle
the amount of rotation from the initial side to the terminal side

negative angle
description of an angle measured clockwise from the positive x-axis

positive angle
description of an angle measured counterclockwise from the positive x-axis

quadrantal angle
an angle whose terminal side lies on an axis

radian measure
the ratio of the arc length formed by an angle divided by the radius of the circle

radian
the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray
one point on a line and all points extending in one direction from that point; one side of an angle

reference angle

the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

standard position

the position of an angle having the vertex at the origin and the initial side along the positive *x*-axis

terminal side

the side of an angle at which rotation ends

vertex

the common endpoint of two rays that form an angle

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2.2: Unit Circle - Sine and Cosine Functions

Learning Objectives

- Find function values for the sine and cosine of $30°$ or $(\frac{\pi}{6})$,$45°$ or $(\frac{\pi}{4})$,and $60°$ or $(\frac{\pi}{3})$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.



Figure 2.2.1: The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in Figure 2.2.2. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a **unit circle**, $s = t$.

Recall that the *x*- and *y*-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

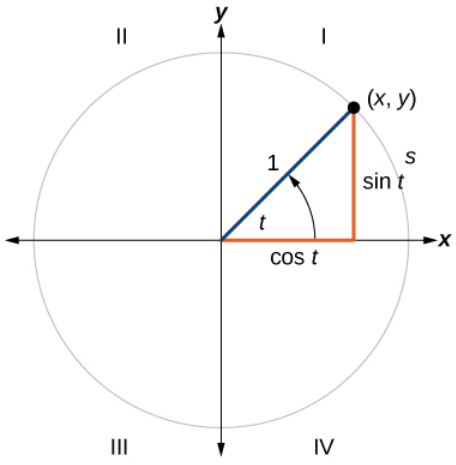


Figure 2.2.2: Unit circle where the central angle is t radians

UNIT CIRCLE

A unit circle has a center at $(0, 0)$ and radius 1. The length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the **arc length** and **angle**. The **sine function** relates a real number t to the y-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t. In Figure 2.2.3, the sine is equal to y. Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y-coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x-value of the endpoint on the unit circle of an arc of length t. In Figure 2.2.1, the cosine is equal to x.

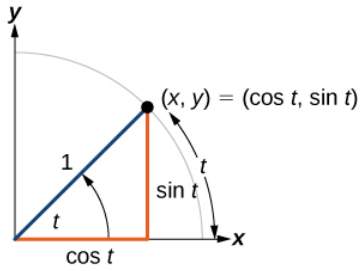


Figure 2.2.3

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

SINE AND COSINE FUNCTIONS

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t, then

$$\cos t = x \tag{2.2.1}$$
$$\sin t = y \tag{2.2.2}$$

How To: Given a point $P(x, y)$ on the unit circle corresponding to an angle of t, find the sine and cosine

1. The sine of t is equal to the y-coordinate of point P: $\sin t = y$.
2. The cosine of t is equal to the x-coordinate of point P: $\cos t = x$.

Example 2.2.1: Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in Figure 2.2.4. Find $\cos(t)$ and $\sin(t)$.

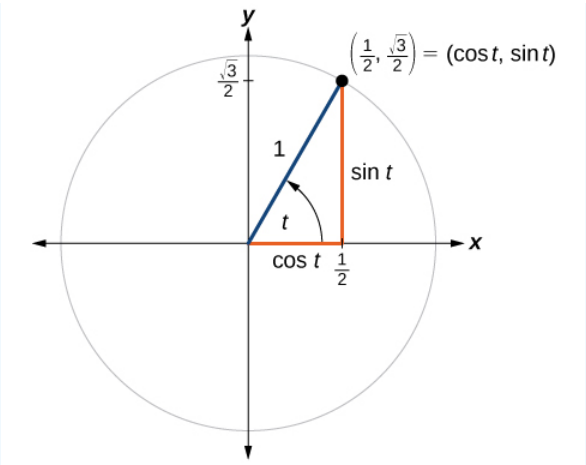


Figure 2.2.4

Solution

We know that $\cos\, t$ is the x-coordinate of the corresponding point on the unit circle and $\sin\, t$ is the y-coordinate of the corresponding point on the unit circle. So:

$$x = \cos\, t = \frac{1}{2}$$
$$y = \sin\, t = \frac{\sqrt{3}}{2}$$

Try It 2.2.1

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure 2.2.5. Find $\cos\, t$ and $\sin\, t$.

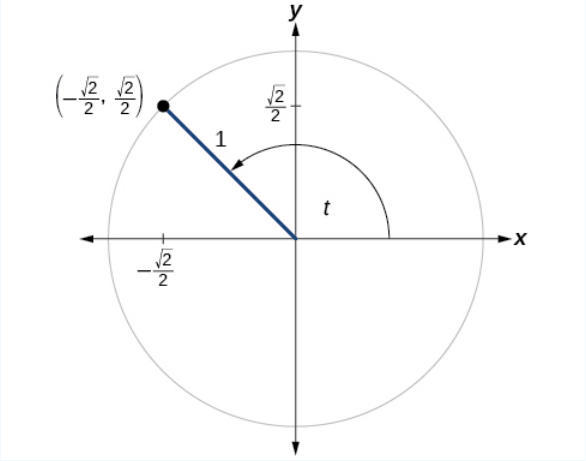


Figure 2.2.5

Answer

$$\cos(t) = -\frac{\sqrt{2}}{2}, \sin(t) = \frac{\sqrt{2}}{2}$$

Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the x- or y-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example 2.2.2: Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

Solution

Moving 90° counterclockwise around the unit circle from the positive x-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in Figure 2.2.6.

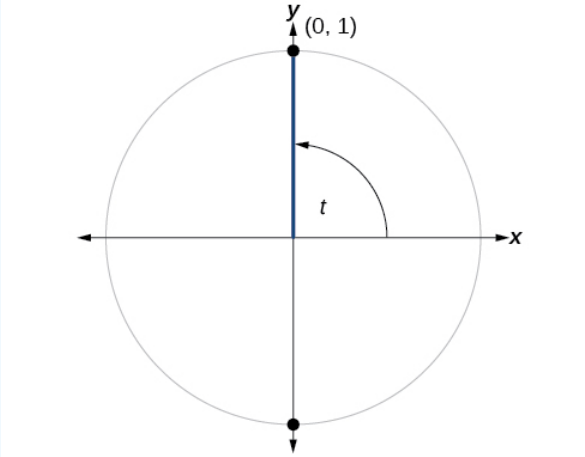


Figure 2.2.6

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90^\circ) = 0$$
$$y = \sin t = \sin(90^\circ) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

Try It 2.2.2

Find cosine and sine of the angle π.

Answer

$\cos(\pi) = -1,\ \sin(\pi) = 0$

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See Figure 2.2.7.

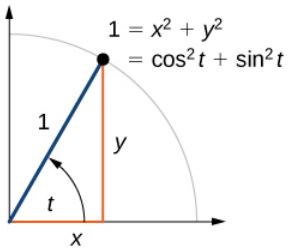


Figure 2.2.7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1 \tag{2.2.3}$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Example 2.2.3: Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution

If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See Figure 2.2.8.

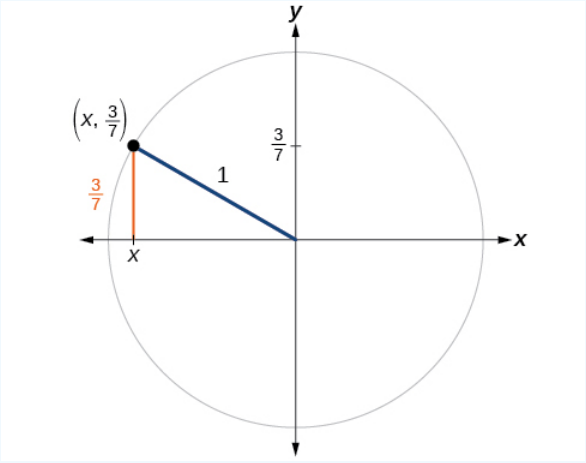


Figure 2.2.8

Substituting the known value for sine into the Pythagorean Identity,

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \cos^2(t) + \frac{9}{49} &= 1 \\ \cos^2(t) &= \frac{40}{49} \\ \cos(t) &= \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7} \end{aligned}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative. So

$$\cos(t) = -\frac{2\sqrt{10}}{7}$$

Try It 2.2.3

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Answer

$$\sin(t) = -\frac{7}{25}$$

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the **Pythagorean Identity** and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in Figure 2.2.9. A $45^\circ - 45^\circ - 90^\circ$ triangle is an isosceles triangle, so the x- and y-coordinates of the corresponding point on the circle are the same. Because the x- and y-values are the same, the sine and cosine values will also be equal.

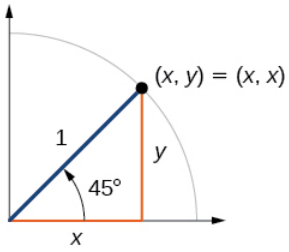


Figure 2.2.9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first **quadrantal angle**. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y (with $y = x$), and a radius = 1. See Figure 2.2.10.

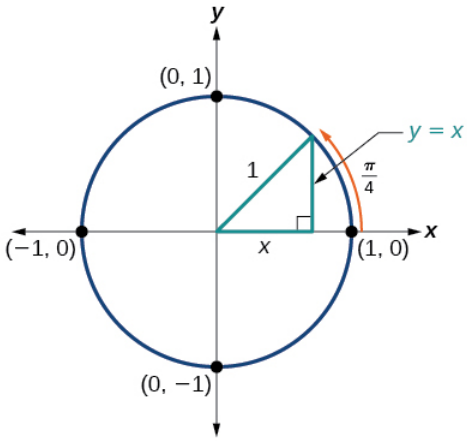


Figure 2.2.10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for x, we get

$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$$
$$\cos t = \frac{1}{\sqrt{2}},\ \sin t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\cos t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\sin t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30°, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of 30°, and another at an angle of -30°, as shown in Figure 2.2.11. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60°, as shown in Figure 2.2.12

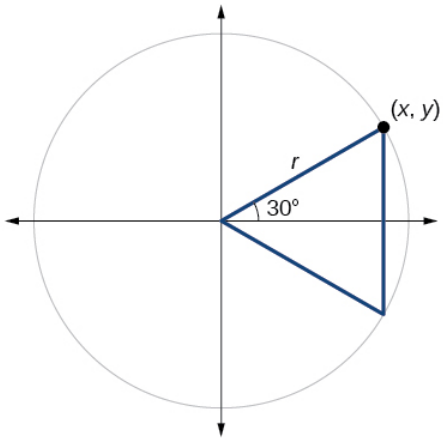


Figure 2.2.11

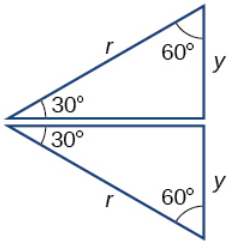


Figure 2.2.12

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

And since $r = 1$ in our **unit circle**,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1)\\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\begin{aligned}\cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right) &= 1\\ \cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 &= 1\\ \cos^2\left(\frac{\pi}{6}\right) &= \frac{3}{4} && \text{Use the square root property.}\\ \cos\left(\frac{\pi}{6}\right) &= \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} && \text{Since } y \text{ is positive, choose the positive root.}\end{aligned}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30° are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in Figure 2.2.13. Angle A has measure 60°. 60°. At point B, we draw an angle ABC with measure of 60°. We know the angles in a triangle sum to 180°, so the measure of angle C is also 60°. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

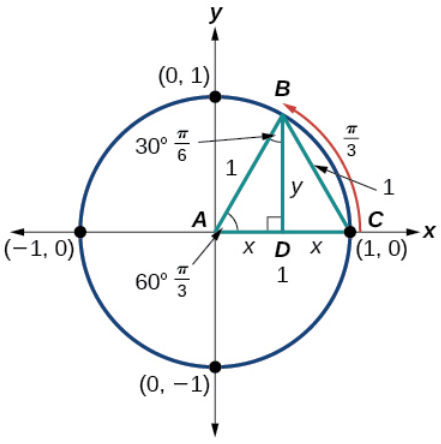


Figure 2.2.13

The measure of angle ABD is 30°. So, if double, angle ABC is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x-coordinate of point B, which is at the intersection of the 60°

angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2+y^2=1$$

Substituting $x=\frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2+y^2=1$$

Solving for y, we get

$$\begin{aligned}\frac{1}{4}+y^2&=1\\ y^2&=1-\frac{1}{4}\\ y^2&=\frac{3}{4}\\ y&=\pm\frac{\sqrt{3}}{2}\end{aligned}$$

Since $t=\frac{\pi}{3}$ has the terminal side in quadrant I where the y-coordinate is positive, we choose $y=\frac{\sqrt{3}}{2}$, the positive value.

At $t=\frac{\pi}{3}$ (60°), the (x,y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$$

$$x=\frac{1}{2},\ y=\frac{\sqrt{3}}{2}$$

$$\cos t=\frac{1}{2},\ \sin t=\frac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. Table 2.2.1 summarizes these values.

Table 2.2.1

Angle	0	$\frac{\pi}{6}$, or 30	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Figure 2.2.14 shows the common angles in the first quadrant of the unit circle.

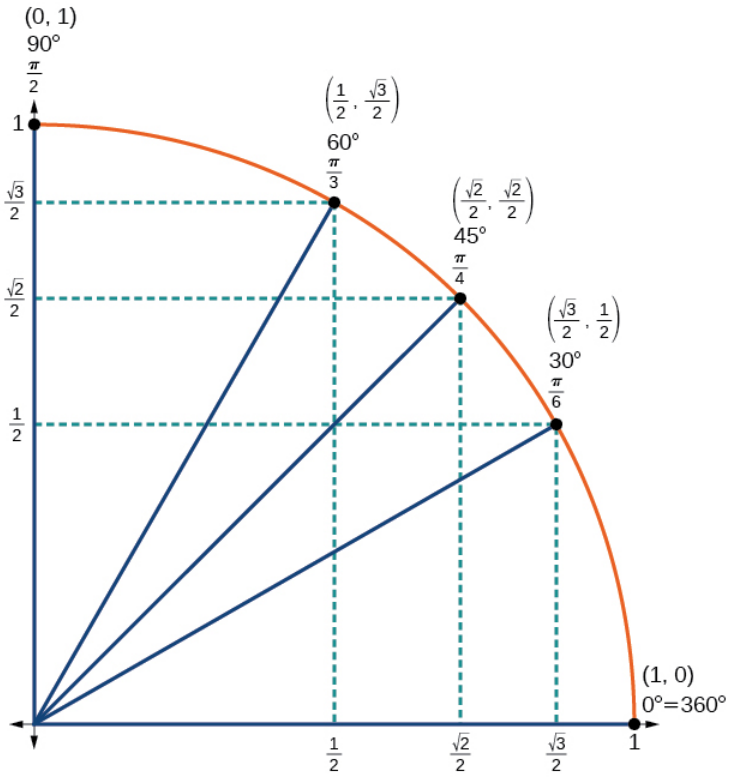


Figure 2.2.14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the **special angles**, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To: Given an angle in radians, use a graphing calculator to find the cosine

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Example 2.2.4: Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\dfrac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution

Enter the following keystrokes:

$$\text{COS}(5 \times \pi \div 3) \text{ ENTER}$$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of 20°, for example, by including the conversion factor to radians as part of the input:

$$\text{SIN}(20 \times \pi \div 180) \text{ ENTER}$$

Try It 2.2.4

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Answer

approximately 0.866025403

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the **unit circle**, as shown in Figure 2.2.15. The bounds of the x-coordinate are $[-1, 1]$. The bounds of the y-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

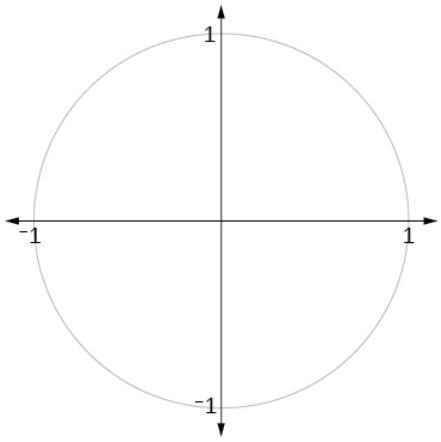


Figure 2.2.15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y-coordinate on the unit circle, the other angle with the same sine will share the same y-value, but have the opposite x-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x-value but will have the opposite y-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in Figure 2.2.16, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\sin(t) = \sin(\alpha) \quad \text{and} \quad \cos(t) = -\cos(\alpha)$$
$$\sin(t) = -\sin(\beta) \quad \text{and} \quad \cos(t) = \cos(\beta)$$

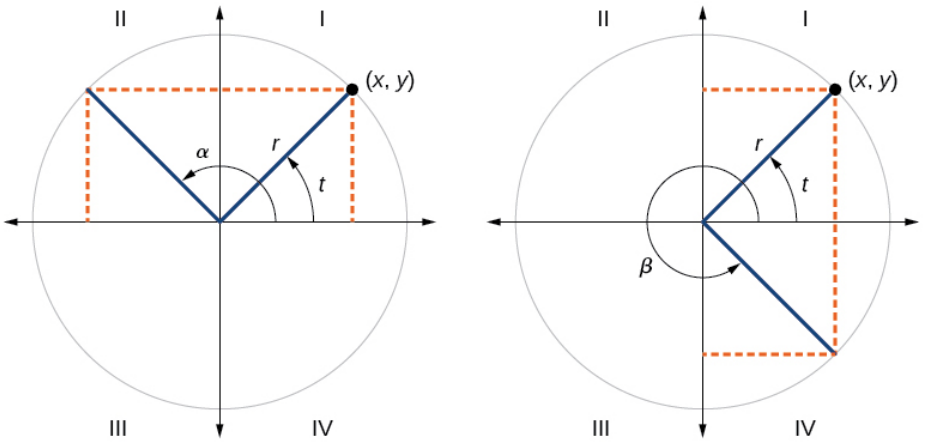


Figure 2.2.16

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90°, or 0 and $\frac{\pi}{2}$ radians. As we can see from Figure 2.2.17, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

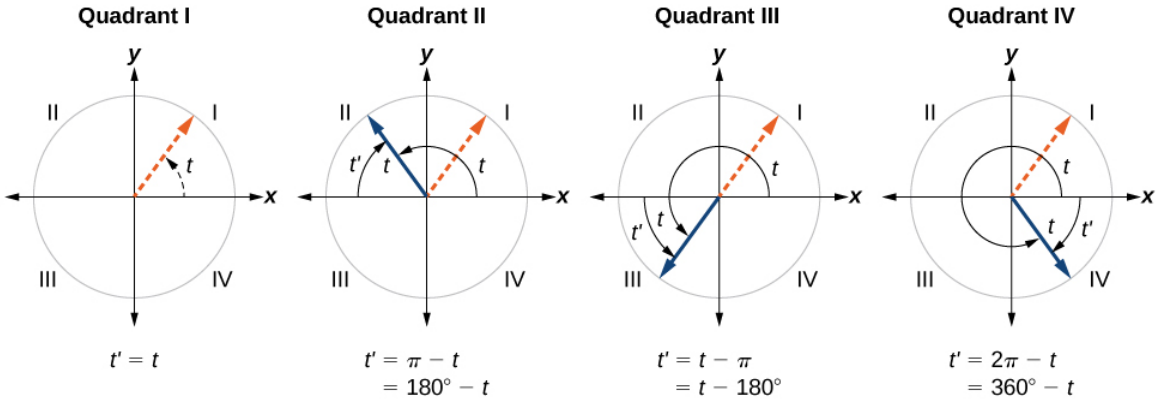


Figure 2.2.17

How To: Given an angle between 0 and 2π, find its reference angle

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180^\circ - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360^\circ - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example 2.2.5: Finding a Reference Angle

Find the reference angle of 225° as shown in Figure 2.2.18

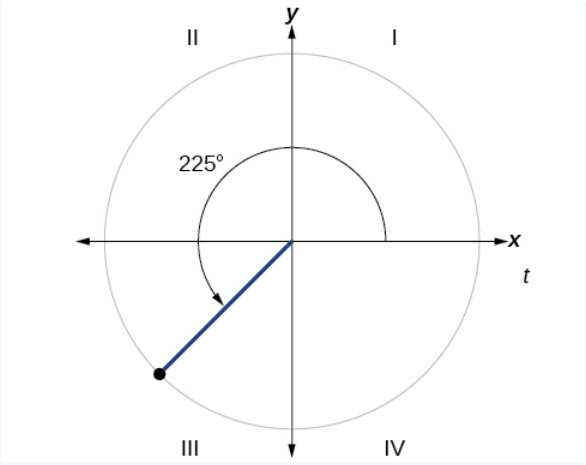


Figure 2.2.18

Solution

Because 225° is in the third quadrant, the reference angle is

$$|(180^\circ - 225^\circ)| = |-45^\circ| = 45^\circ$$

Try It 2.2.5

Find the reference angle of $\frac{5\pi}{3}$.

Answer

(\dfrac{π}{3}\)

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a **negative angle**. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the **reference angle** of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

USING REFERENCE ANGLES TO FIND COSINE AND SINE

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To: Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example 2.2.6: Using Reference Angles to Find Sine and Cosine

1. Using a reference angle, find the exact value of $\cos(150^\circ)$ and $\sin(150^\circ)$.
2. Using the reference angle, find $\cos\ \frac{5\pi}{4}$ and $\sin\ \frac{5\pi}{4}$.

Solution

This tells us that 150° has the same sine and cosine values as 30°, except for the sign. We know that

Since 150° is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. The y-coordinate is positive, so the sine value is positive.

$\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\left|\frac{5\pi}{4}-\pi\right|=\frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos\ \frac{5\pi}{4}=-\frac{\sqrt{2}}{2}\ \text{and}\ \sin\ \frac{5\pi}{4}=-\frac{\sqrt{2}}{2}$$

Try It 2.2.6

1. Use the reference angle of 315° to find $\cos(315^\circ)$ and $\sin(315^\circ)$.
2. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$

Answer

1. $\cos(315^\circ)=\frac{\sqrt{2}}{2}, \sin(315^\circ)=\frac{-\sqrt{2}}{2}$
2. $\cos\left(-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}, \sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in Figure 2.2.19. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

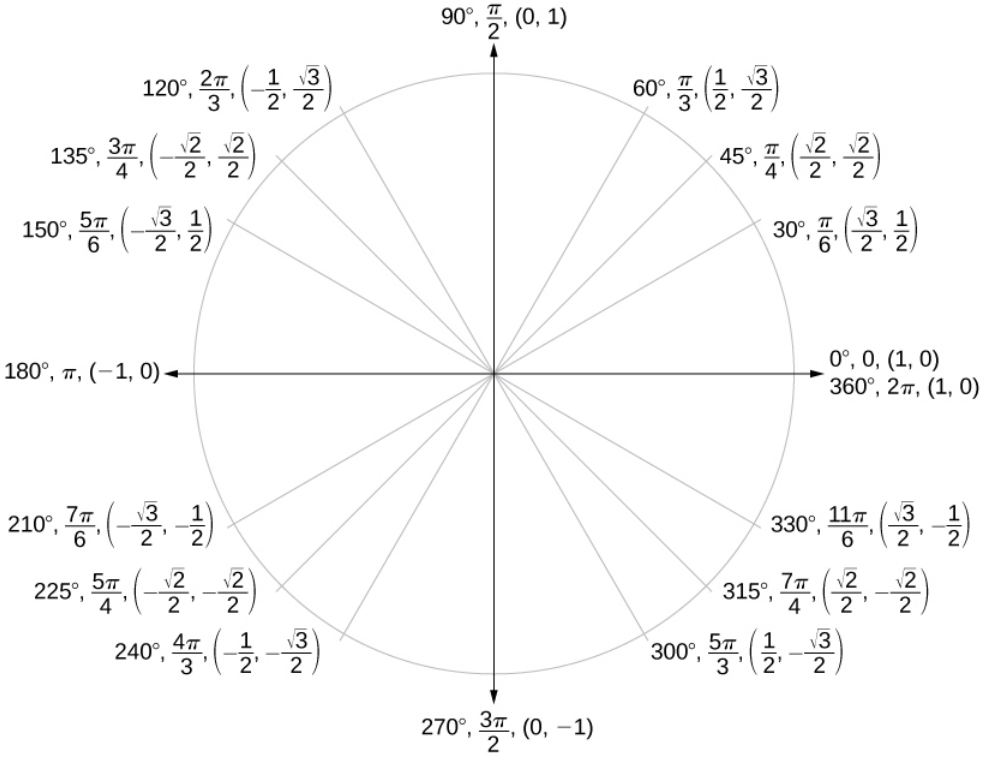


Figure 2.2.19: Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the **identities**

$$x = \cos t$$
$$y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the **reference angle**, and give them the signs corresponding to the y- and x-values of the quadrant.

How To: Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example 2.2.7: Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{3}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$
$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\,\theta$ and $y = \sin\,\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It 2.2.7

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Answer

$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

media

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle
- Sine and Cosine from the Unit Circle
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four
- Trigonometric Functions Using Reference Angles

Key Equations

Cosine	$\cos\,t = x$
Sine	$\sin\,t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint. See Example.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See Example.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See Example.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See Example.
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the *x*- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See Example.

- Reference angles can be used to find the sine and cosine of the original angle. See Example.
- Reference angles can also be used to find the coordinates of a point on a circle. See Example.

Glossary

cosine function

the x-value of the point on a unit circle corresponding to a given angle

Pythagorean Identity

a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function

the y-value of the point on a unit circle corresponding to a given angle

unit circle

a circle with a center at $(0, 0)$ and radius 1.

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2.3: Right Triangle Trigonometry

Learning Objectives

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°(\frac{\pi}{6})$,$45°(\frac{\pi}{4})$,and $60°(\frac{\pi}{3})$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We have previously defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos t = x$$
$$\sin t = y$$

In this section, we will see another way to define trigonometric functions using properties of *right triangles*.

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the *trigonometric functions*. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure $2.3.1$ shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the *x*-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

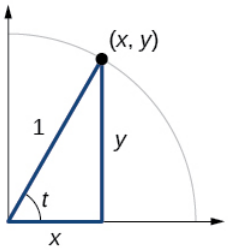


Figure 2.3.1

We know

$$\cos t = \frac{x}{1} = x \tag{2.3.1}$$

Likewise, we know

$$\sin t = \frac{y}{1} = y \tag{2.3.2}$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y,we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure $2.3.2$.

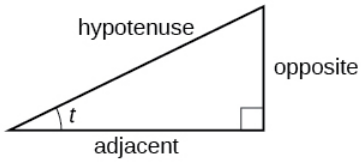


Figure 2.3.2: The sides of a right triangle in relation to angle t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{2.3.3}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{2.3.4}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \tag{2.3.5}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example 2.3.1: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure 2.3.3, find the value of $\cos\alpha$.

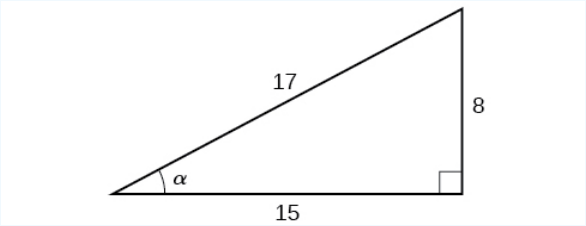


Figure 2.3.3

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so via Equation 2.3.4:

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{15}{17}$$

Exercise 2.3.1

Given the triangle shown in Figure 2.3.4, find the value of $\sin t$.

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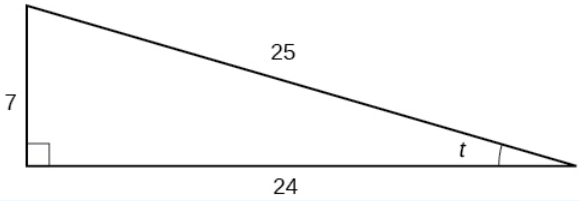


Figure 2.3.4

Answer

$\frac{7}{25}$

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure 2.3.5. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

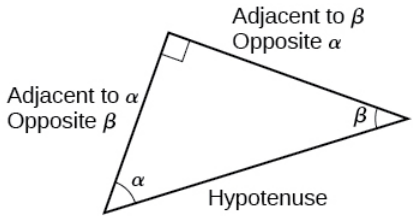


Figure 2.3.5: The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

how to: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 2.3.2: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure 2.3.6, evaluate $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

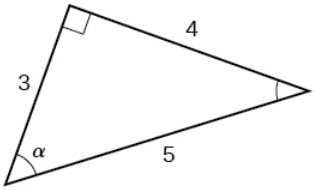


Figure 2.3.6

Solution

$$\sin\alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos\alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan\alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$
$$\sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$
$$\csc\alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$
$$\cot\alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Exercise 2.3.2

Using the triangle shown in Figure 2.3.7, evaluate $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

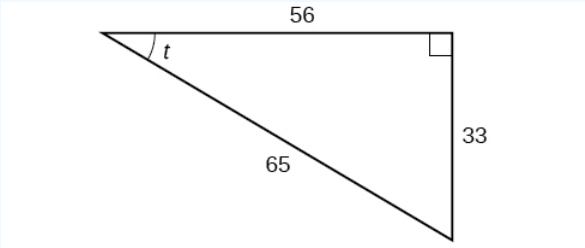


Figure 2.3.7

Answer

$$\sin t = \frac{33}{65}, \cos t = \frac{56}{65}, \tan t = \frac{33}{56},$$

$$\sec t = \frac{65}{56}, \csc t = \frac{65}{33}, \cot t = \frac{56}{33}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the **special angles** on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of $30^\circ, 60^\circ,$ and 45°, however, remember that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a $30^\circ, 60^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45^\circ, 45^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure 2.3.8.

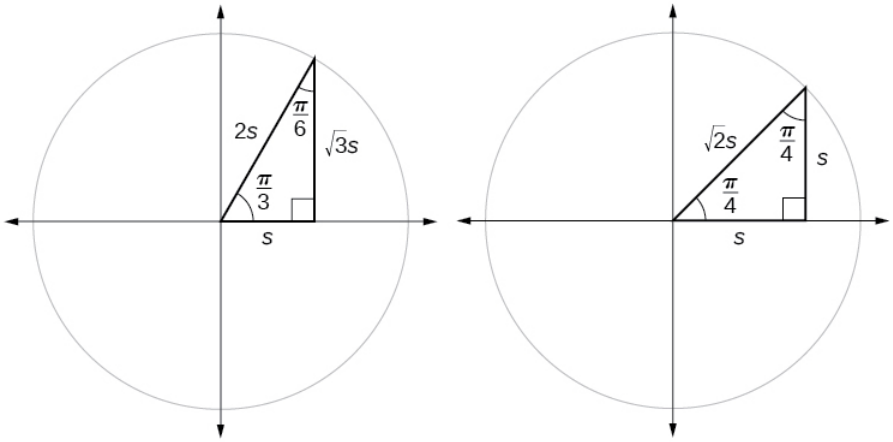


Figure 2.3.8: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 2.3.8 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 2.3.3: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin(\frac{\pi}{3}) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\cos(\frac{\pi}{3}) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$
$$\tan(\frac{\pi}{3}) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$
$$\sec(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot(\frac{\pi}{3}) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 2.3.3

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$ using side lengths.

Answer

$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{\pi}{4}) = 1,$
$\sec(\frac{\pi}{4}) = \sqrt{2}, \csc(\frac{\pi}{4}) = \sqrt{2}, \cot(\frac{\pi}{4}) = 1$

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the

sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure 2.3.9).

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

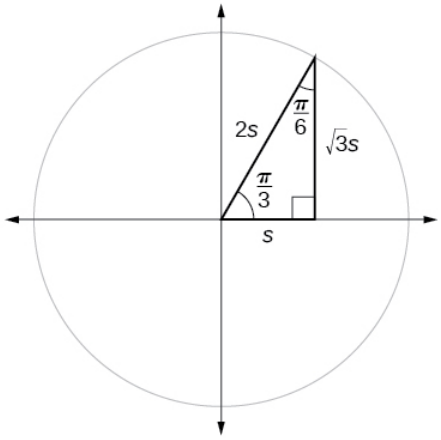
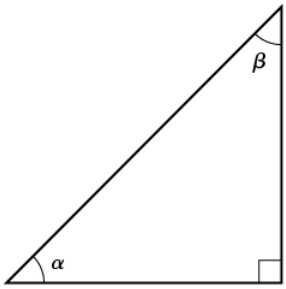


Figure 2.3.9: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure 2.3.9, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 2.3.10.



$\sin\alpha = \cos\beta$
$\sin\beta = \cos\alpha$

Figure 2.3.10: Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

COFUNCTION IDENTITIES

The **cofunction identities** in radians are listed in Table 2.3.1.

Table 2.3.1

$\cos t = \sin(\frac{\pi}{2} - t)$	$\sin t = \cos(\frac{\pi}{2} - t)$
$\tan t = \cot(\frac{\pi}{2} - t)$	$\cot t = \tan(\frac{\pi}{2} - t)$

$\sec t = \csc(\frac{\pi}{2} - t)$	$\csc t = \sec(\frac{\pi}{2} - t)$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 2.3.4: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $(\cos\frac{\pi}{2} - t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos(\frac{\pi}{2} - t).$$

So

$$\cos(\frac{\pi}{2} - t) = \frac{5}{12}.$$

Exercise 2.3.4

If $\csc(\frac{\pi}{6}) = 2$, find $\sec(\frac{\pi}{3})$.

Solution

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

how to: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 2.3.5: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure 2.3.11.

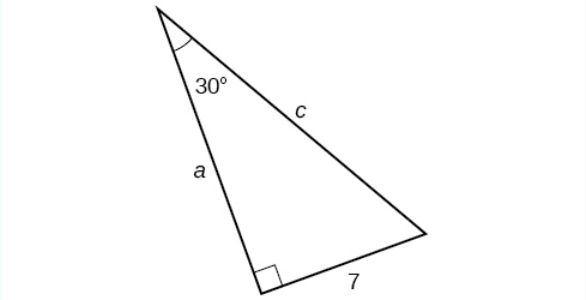


Figure 2.3.11

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30^\circ) = \frac{7}{a}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30^\circ)} \tag{2.3.6}$$

$$= 12.1 \tag{2.3.7}$$

We can use the sine to find the hypotenuse.

$$\sin(30^\circ) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30^\circ)} = 14$$

Exercise 2.3.5:

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Answer

$\text{adjacent} = 10; \text{opposite} = 10\sqrt{3}$; missing angle is $\frac{\pi}{6}$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure 2.3.12

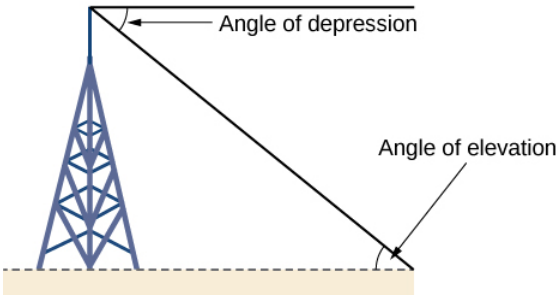


Figure 2.3.12

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.

4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 2.3.6: Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° 57° between a line of sight to the top of the tree and the ground, as shown in Figure 2.3.13. Find the height of the tree.

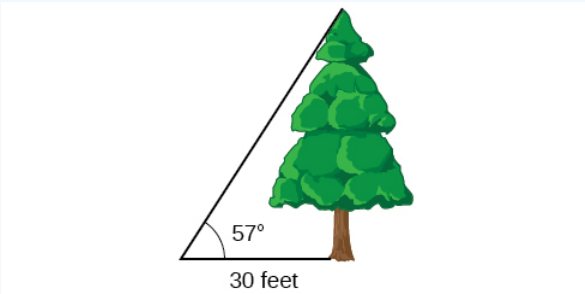


Figure 2.3.13

Solution

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57^\circ) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30\tan(57^\circ) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.} \end{aligned} \tag{2.3.8}$$

The tree is approximately 46 feet tall.

Exercise 2.3.6:

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Answer

About 52 ft

media:

Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator
- Finding Trig Functions Using a Right Triangle
- Relate Trig Functions to Sides of a Right Triangle
- Determine Six Trig Functions from a Triangle
- Determine Length of Right Triangle Side

Visit this website for additional practice questions from Learningpod.

Key Equations

Cofunction Identities

$$\cos t = \sin\left(\frac{\pi}{2} - t\right)$$
$$\sin t = \cos\left(\frac{\pi}{2} - t\right)$$
$$\tan t = \cot\left(\frac{\pi}{2} - t\right)$$
$$\cot t = \tan\left(\frac{\pi}{2} - t\right)$$
$$\sec t = \csc\left(\frac{\pi}{2} - t\right)$$
$$\csc t = \sec\left(\frac{\pi}{2} - t\right)$$

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See Example.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See Example.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See Example.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See Example.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See Example.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See Example.

Glossary

adjacent side
in a right triangle, the side between a given angle and the right angle

angle of depression
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side
in a right triangle, the side most distant from a given angle

hypotenuse
the side of a right triangle opposite the right angle

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2.4: The Other Trigonometric Functions

Learning Objectives

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t,as shown in Figure 2.4.1. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

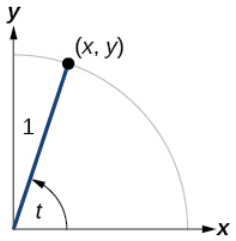


Figure 2.4.1

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle. In Figure 2.4.1, the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the *y*-value is equal to the sine of t, and the *x*-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}, \cos t \neq 0$. The tangent function is abbreviated as tan. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In Figure 2.4.1, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as sec.
- The **cotangent** function is the reciprocal of the tangent function. In Figure 2.4.1, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as cot.
- The **cosecant** function is the reciprocal of the sine function. In Figure 2.4.1, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as csc.

TANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0 \tag{2.4.1}$$

$$\sec t = \frac{1}{x}, x \neq 0 \tag{2.4.2}$$

$$\csc t = \frac{1}{y}, y \neq 0 \tag{2.4.3}$$

$$\cot t = \frac{x}{y}, y \neq 0 \tag{2.4.4}$$

Example 2.4.1: Finding Trigonometric Functions from a Point on the Unit Circle

The point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle, as shown in Figure 2.4.2. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

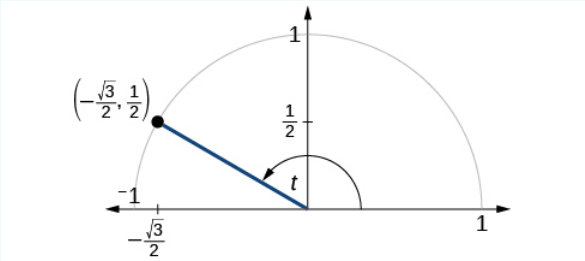


Figure 2.4.2

Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$

$$\cos t = x = -\frac{\sqrt{3}}{2}$$

$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

Exercise 2.4.1:

The point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ is on the unit circle, as shown in Figure 2.4.3. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

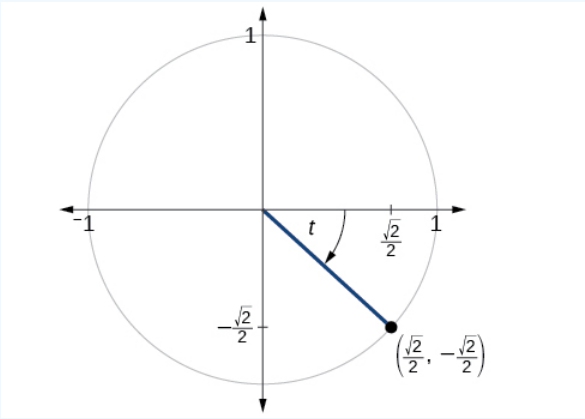


Figure 2.4.3

Solution

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = -1, \sec t = \sqrt{2}, \csc t = -\sqrt{2}, \cot t = -1$

Example 2.4.2: Finding the Trigonometric Functions of an Angle

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{6}$.

Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\begin{aligned}
\tan\frac{\pi}{6} &= \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} \\
&= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\
\sec\frac{\pi}{6} &= \frac{1}{\cos\frac{\pi}{6}} \\
&= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\
\csc\frac{\pi}{6} &= \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2 \\
\cot\frac{\pi}{6} &= \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} \\
&= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}
\end{aligned}$$

Exercise 2.4.2:

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{3}$.

Solution

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \tag{2.4.5}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \tag{2.4.6}$$

$$\tan\frac{\pi}{3} = \sqrt{3} \tag{2.4.7}$$

$$\sec\frac{\pi}{3} = 2 \tag{2.4.8}$$

$$\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3} \tag{2.4.9}$$

$$\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3} \tag{2.4.10}$$

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x x equal to the cosine and y y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in Table 2.4.1.

Table 2.4.1

Angle	**0**	$\frac{\pi}{6}$, or **30**$^\circ$	$\frac{\pi}{4}$, or **45**$^\circ$	$\frac{\pi}{3}$, or **60**$^\circ$	$\frac{\pi}{2}$, or **90**$^\circ$
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the **reference angle** formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by *x*- and *y*-values in the original quadrant. Figure 2.4.4 shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.

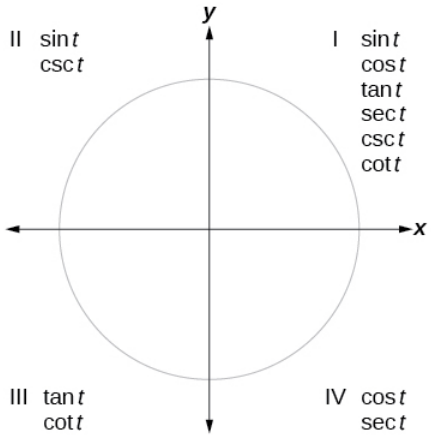


Figure 2.4.4

HOWTO: Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example 2.4.3: Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution

The angle between this angle's terminal side and the *x*-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos(-\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}, \sin(-\frac{5\pi}{6}) = -\frac{1}{2}, \tan(-\frac{5\pi}{6}) = \frac{\sqrt{3}}{3} \quad (2.4.11)$$

$$\sec(-\frac{5\pi}{6}) = -\frac{2\sqrt{3}}{3}, \csc(-\frac{5\pi}{6}) = -2, \cot(-\frac{5\pi}{6}) = \sqrt{3} \quad (2.4.12)$$

Exercise 2.4.3

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Solution

$\sin(-\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{-7\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{-7\pi}{4}) = 1,$

$\sec(\frac{-7\pi}{4}) = \sqrt{2}, \csc(\frac{-7\pi}{4}) = \sqrt{2}, \cot(\frac{-7\pi}{4}) = 1$

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f(x) = x^2$, shown in Figure 2.4.5. The graph of the function is symmetrical about the *y*-axis. All along the curve, any two points with opposite *x*-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2, (-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an **even function**, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

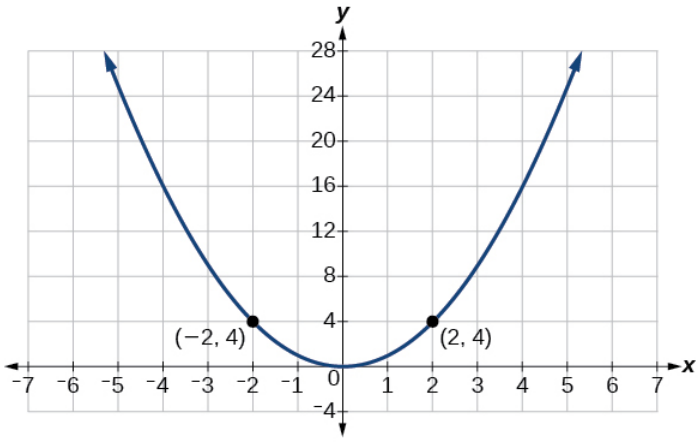


Figure 2.4.5: The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in Figure 2.4.6. The graph is not symmetrical about the *y*-axis. All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f(x) = x^3$ is an **odd function**, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

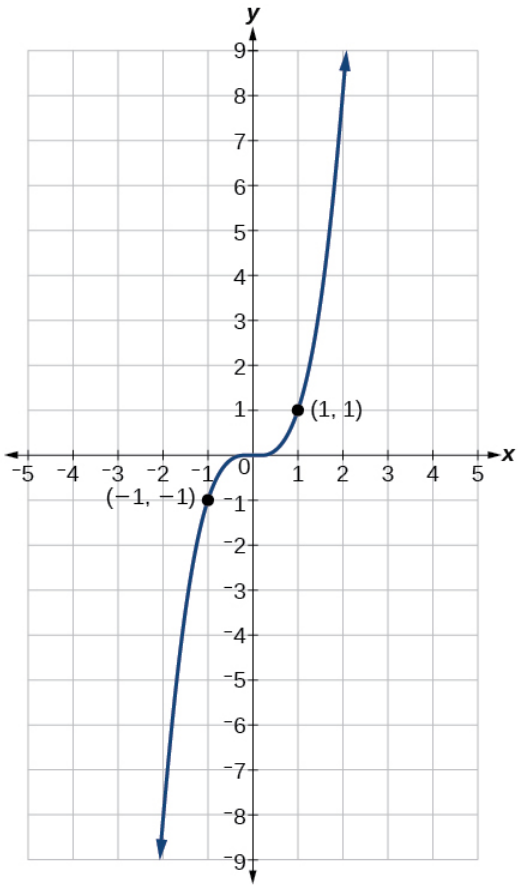


Figure 2.4.6: The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a **unit circle** with a positive and a negative angle, as in Figure 2.4.7. The sine of the positive angle is y. The sine of the negative angle is $-y$. The **sine function**, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in Table 2.4.2.

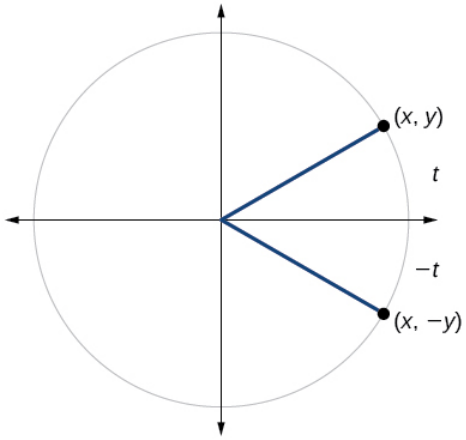


Figure 2.4.7

Table 2.4.2

$$\sin t = y \quad (2.4.13)$$
$$\sin(-t) = -y \quad (2.4.14)$$
$$\sin t \neq sin(-t) \quad (2.4.15)$$

$$\cos t = x \quad (2.4.16)$$
$$\cos(-t) = x \quad (2.4.17)$$
$$\cos t = \cos(-t) \quad (2.4.18)$$

$$\tan(t) = \frac{y}{x} \quad (2.4.19)$$
$$\tan(-t) = -\frac{y}{x} \quad (2.4.20)$$
$$\tan t \neq \tan(-t) \quad (2.4.21)$$

$$\sec t = \frac{1}{x} \quad (2.4.22)$$
$$\sec(-t) = \frac{1}{x} \quad (2.4.23)$$
$$\sec t = \sec(-t) \quad (2.4.24)$$
$$\csc t = \frac{1}{y} \quad (2.4.25)$$
$$\csc(-t) = \frac{1}{-y} \quad (2.4.26)$$
$$\csc t \neq \csc(-t) \quad (2.4.27)$$
$$\cot t = \frac{x}{y} \quad (2.4.28)$$
$$\cot(-t) = \frac{x}{-y} \quad (2.4.29)$$
$$\cot t \neq \cot(-t) \quad (2.4.30)$$

EVEN AND ODD TRIGONOMETRIC FUNCTIONS

- An **even function** is one in which $f(-x) = f(x)$.
- An **odd function** is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\cos(-t) = \cos t \quad (2.4.31)$$
$$\sec(-t) = \sec t \quad (2.4.32)$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t \quad (2.4.33)$$
$$\tan(-t) = -\tan t \quad (2.4.34)$$
$$\csc(-t) = -\csc t \quad (2.4.35)$$
$$\cot(-t) = -\cot t \quad (2.4.36)$$

Example 2.4.4: Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Exercise 2.4.4:

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Solution

$-\sqrt{3}$

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

FUNDAMENTAL IDENTITIES

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \quad (2.4.37)$$
$$\sec t = \frac{1}{\cos t} \quad (2.4.38)$$
$$\csc t = \frac{1}{\sin t} \quad (2.4.39)$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \tag{2.4.40}$$

Example 2.4.5: Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45^\circ) = \frac{\sqrt{2}}{2}, \cos(45^\circ) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45^\circ)$.

b. Given $\sin(\frac{5\pi}{6}) = \frac{1}{2}, \cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$, evaluate $\sec(\frac{5\pi}{6})$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.
$$\begin{aligned} \tan(45^\circ) &= \frac{\sin(45^\circ)}{\cos(45^\circ)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1 \end{aligned}$$

b.
$$\begin{aligned} \sec(\frac{5\pi}{6}) &= \frac{1}{\cos(\frac{5\pi}{6})} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \\ &= \frac{-2\sqrt{3}}{1} \\ &= \frac{-2}{\sqrt{3}} \\ &= -\frac{2\sqrt{3}}{3} \end{aligned}$$

Exercise 2.4.5

Evaluate $\csc(\frac{7\pi}{6})$.

Solution

-2

Example 2.4.6: Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution

We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{aligned} \frac{\sec t}{\tan t} &= \frac{1/\cos t}{\sin t/\cos t} && \text{To divide the functions, we multiply by the reciprocal.} \\ &= \frac{1}{\cos t}\frac{\cos t}{\sin t} && \text{Divide out the cosines.} \\ &= \frac{1}{\sin t} && \text{Simplify and use the identity.} \\ &= \csc t \end{aligned} \tag{2.4.41}$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$,we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Exercise 2.4.6

Simplify $(\tan t)(\cos t)$.

Solution

$\sin t$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the **Pythagorean Identity**, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \tag{2.4.42}$$

$$1 + \tan^2 t = \sec^2 t \tag{2.4.43}$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \tag{2.4.44}$$

$$\cot^2 t + 1 = \csc^2 t \tag{2.4.45}$$

ALTERNATE FORMS OF THE PYTHAGOREAN IDENTITY

$$1 + \tan^2 t = \sec^2 t \tag{2.4.46}$$

$$\cot^2 t + 1 = \csc^2 t \tag{2.4.47}$$

Example 2.4.7: Using Identities to Relate Trigonometric Functions

If cos(t)=1213 cos(t)=1213 and t t is in quadrant IV, as shown in Figure 2.4.8, find the values of the other five trigonometric functions.

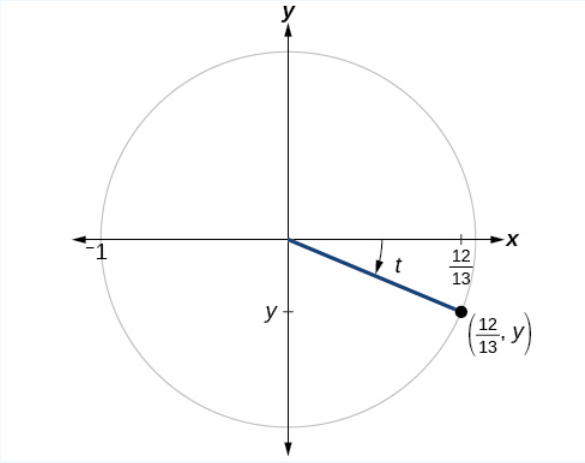


Figure 2.4.8

Solution

We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$(\frac{12}{13})^2+\sin^2 t=1 \tag{2.4.48}$$

$$\sin^2 t=1-(\frac{12}{13})^2 \tag{2.4.49}$$

$$\sin^2 t=1-\frac{144}{169} \tag{2.4.50}$$

$$\sin^2 t=\frac{25}{169} \tag{2.4.51}$$

$$\sin t=\pm\sqrt{\frac{25}{169}} \tag{2.4.52}$$

$$\sin t=\pm\frac{\sqrt{25}}{\sqrt{169}} \tag{2.4.53}$$

$$\sin t=\pm\frac{5}{13} \tag{2.4.54}$$

The sign of the sine depends on the *y*-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the *y*-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t=\frac{\sin t}{\cos t}=\frac{-\frac{5}{13}}{\frac{12}{13}}=-\frac{5}{12} \tag{2.4.55}$$

$$\sec t=\frac{1}{\cos t}=\frac{1}{\frac{12}{13}}=\frac{13}{12} \tag{2.4.56}$$

$$\csc t=\frac{1}{\sin t}=\frac{1}{-\frac{5}{13}}=-\frac{13}{5} \tag{2.4.57}$$

$$\cot t=\frac{1}{\tan t}=\frac{1}{-\frac{5}{12}}=-\frac{12}{5} \tag{2.4.58}$$

Exercise 2.4.7:

If $\sec(t)=-\frac{17}{8}$ and $0<t<\pi,$ find the values of the other five functions.

Solution

$\cos t=-\frac{8}{17}, \sin t=\frac{15}{17}, \tan t=-\frac{15}{8}$

$\csc t=\frac{17}{15}, \cot t=-\frac{8}{15}$

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a **periodic function**. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π,will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4)=f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

PERIOD OF A FUNCTION

The **period** P of a repeating function f f is the number representing the interval such that $f(x+P)=f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

Example 2.4.8: Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on Figure 2.4.9.

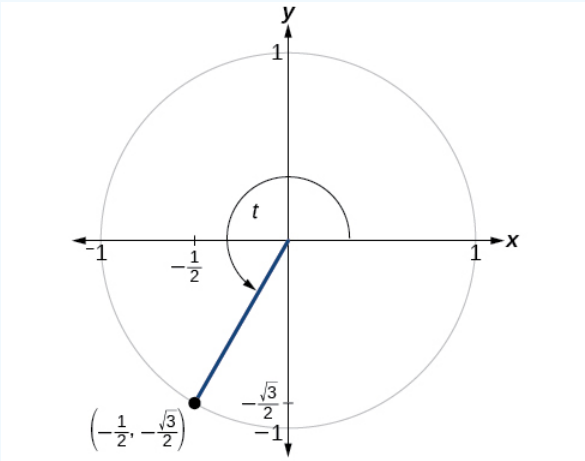


Figure 2.4.9

Solution

$$\sin t = y = -\frac{\sqrt{3}}{2}$$

$$\cos t = x = -\frac{1}{2}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 2.4.8

Find the values of the six trigonometric functions of angle t based on Figure 2.4.10.

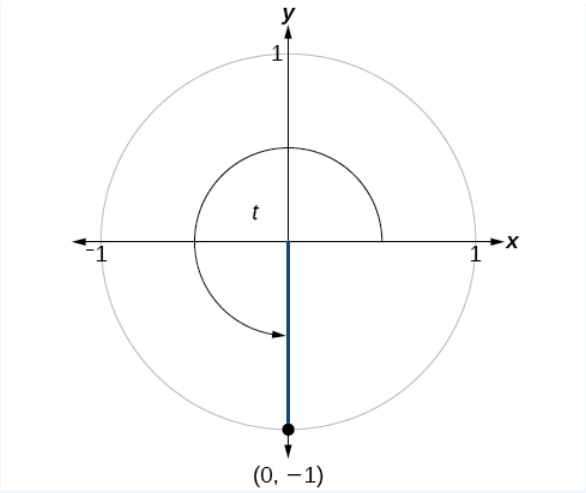


Figure 2.4.10

Solution

$$\sin t = -1, \cos t = 0, \tan t = \text{Undefined} \tag{2.4.59}$$

$$\tag{2.4.60}$$

$$sect = \text{Undefined}, \csc t = -1, \cot t = 0 \tag{2.4.61}$$

Example 2.4.9: Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$,find $\sec(t), \csc(t), \tan(t), \cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2 \tag{2.4.62}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} - \frac{2\sqrt{3}}{3} \tag{2.4.63}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \tag{2.4.64}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} \tag{2.4.65}$$

Exercise 2.4.9:

If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t), \csc(t), \tan(t)$, and $\cot(t)$.

Solution

$\sec t = \sqrt{2}, \csc t = \sqrt{2}, \tan t = 1, \cot t = 1$

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of 30°, we could press

$$(\text{for a scientific calculator}): \frac{1}{30 \times \frac{\pi}{180}} \text{COS} \tag{2.4.66}$$

or

$$(\text{for a graphing calculator}): \frac{1}{\cos(\frac{30\pi}{180})} \tag{2.4.67}$$

how to: Given an angle measure in radians, use a scientific calculator to find the cosecant

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

how to: Given an angle measure in radians, use a graphing utility/calculator to find the cosecant

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Press the SIN key.
4. Enter the value of the angle inside parentheses.
5. Press the ENTER key.

Example 2.4.10: Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1/(5 \times \pi/7)\text{SIN} = \tag{2.4.68}$$

$$\csc(\frac{5\pi}{7}) \approx 1.279 \tag{2.4.69}$$

Exercise 2.4.10:

Evaluate the cotangent of $-\frac{\pi}{8}$.

≈ -2.414

media

Access these online resources for additional instruction and practice with other trigonometric functions.

- Determining Trig Function Values
- More Examples of Determining Trig Functions
- Pythagorean Identities
- Trig Functions on a Calculator

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$

Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See Example.
- Trigonometric functions can also be found from an angle. See Example.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See Example.
- A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See Example.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See Example and Example.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See Example.
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f f is the smallest interval such that $f(x + P) = f(x)$ for any value of x.
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See Example.

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Glossary

cosecant

the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cotangent

the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

identities

statements that are true for all values of the input on which they are defined

period

the smallest interval P of a repeating function f such that $f(x + P) = f(x)$

secant

the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

tangent

the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

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2.5: Arcs, Angles, and Trig Function Values

Learning Objectives

After studying this section, we should understand the concepts motivated by the questions below and be able to write precise, coherent answers to these questions:

- How do we measure angles using degrees?
- What do we mean by the radian measure of an angle? How is the radian measure of an angle related to the length of an arc on the unit circle?
- Why is radian measure important?
- How do we convert from radians to degrees and from degrees to radians?
- How do we determine the values of both x and y when parameters on the domain are restricted?
- What trigonometric functions are positive in the first, second, third and fourth quadrants?
- How do we use one trigonometric function to represent or describe another?

The ancient civilization known as Babylonia was a cultural region based in southern Mesopotamia, which is present-day Iraq. Babylonia emerged as an independent state around 1894 BCE. The Babylonians developed a system of mathematics that was based on a sexigesimal (base 60) number system. This was the origin of the modern day usage of 60 minutes in an hour, 60 seconds in a minute, and 360 degrees in a circle.

Many historians now believe that for the ancient Babylonians, the year consisted of 360 days, which is not a bad approximation given the crudeness of the ancient astronomical tools. As a consequence, they divided the circle into 360 equal length arcs, which gave them a unit angle that was 1/360 of a circle or what we now know as a degree. Even though there are 365.24 days in a year, the Babylonian unit angle is still used as the basis for measuring angles in a circle. Figure 2.5.1 shows a circle divided up into 6 angles of 60 degrees each, which is also something that fit nicely with the Babylonian base-60 number system.

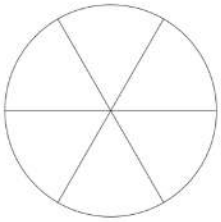


Figure 2.5.1: A circle with six 60-degree angles.

We often denote a line that is drawn through 2 points A and B by $\overleftrightarrow{AB}$. The portion of the line $\overleftrightarrow{AB}$ that starts at the point A and continues indefinitely in the direction point of point B is called **ray AB** and is denoted by $\overrightarrow{AB}$. The point A is the **initial point** of ray $\overrightarrow{AB}$. An **angle** is formed by rotating a ray about its endpoint. The ray in its initial position is called the **initial side** of the angle, and the position of the ray after it has been rotated is called the **terminal side** of the ray. The endpoint of the ray is called the **vertex of the angle.**

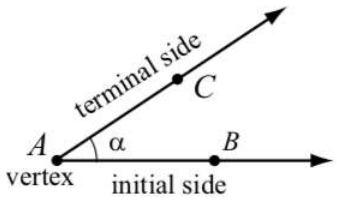


Figure 2.5.2: An angle including some notation.

Figure 2.5.2 shows the ray $\overrightarrow{AB}$ rotated about the point A to form an angle. The terminal side of the angle is the ray $\overrightarrow{AC}$. We often refer to this as angle BAC, which is abbreviated as $\angle BAC$. We can also refer to this angle as angle CAB or $\angle CAB$. If we want to

use a single letter for this angle, we often use a Greek letter such as α(alpha). We then just say the angleα. Other Greek letters that are often used are β(beta), γ(gamma), θ(theta), ϕ(phi), and ρ(rho).

Arcs and Angles

To define the trigonometric functions in terms of angles, we will make a simple connection between angles and arcs by using the standard position of an angle. When the vertex of an angle is at the origin in the xy-plane and the initial side lies along the positive x-axis, we see that the angle is in **standard position**. The terminal side of the angle is then in one of the four quadrants or lies along one of the axes. When the terminal side is in one of the four quadrants, the terminal side determines the so-called quadrant designation of the angle. See Figure 2.5.3.

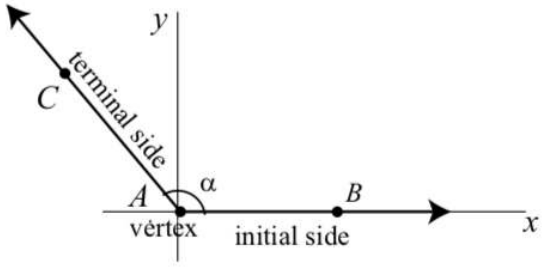


Figure 2.5.3: Standard position of an angle in the second quadrant.

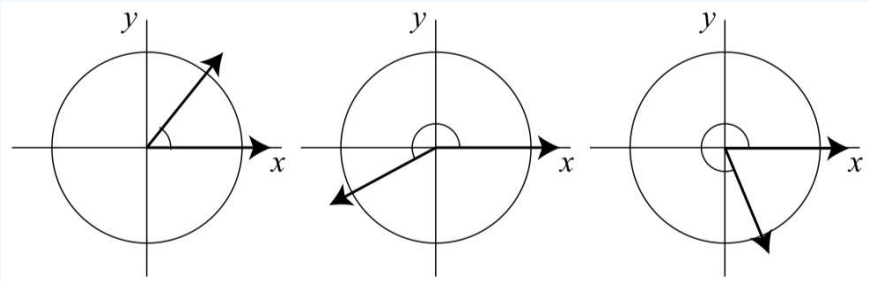
Exercise 2.5.1

Draw an angle in standard position in

a. the first quadrant;
b. the third quadrant; and
c. the fourth quadrant.

Answer

These graphs show positive angles in standard position. The one on the left has its terminal point in the first quadrant, the one in the middle has its terminal point in the third quadrant, and the one on the right has its terminal point in the fourth quadrant.



If an angle is in standard position, then the point where the terminal side of the angle intersects the unit circle marks the terminal point of an arc as shown in Figure 1.11. Similarly, the terminal point of an arc on the unit circle determines a ray through the origin and that point, which in turn defines an angle in standard position. In this case we say that the angle is *subtended* by the arc. So there is a natural correspondence between arcs on the unit circle and angles in standard position. Because of this correspondence, we can also define the trigonometric terminal side functions in terms of angles as well as arcs. Before we do this, however, we need to discuss two different ways to measure angles.

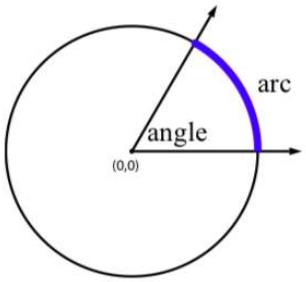


Figure 2.5.4: An arc and its corresponding angle.

Degrees Versus Radians - A Review (Because we ALL need one!)

There are two ways we will measure angles – in degrees and radians. When we measure the length of an arc, the measurement has a dimension (the length, be it inches, centimeters, or something else). As mentioned in the introduction, the Babylonians divided the circle into 360 regions. So one complete wrap around a circle is 360 degrees, denoted 360°. The unit measure of 1° is an angle that is 1/360 of the central angle of a circle. Figure 2.5.1 shows 6 angles of 60° each. The degree $^\circ$ is a dimension, just like a length. So to compare an angle measured in degrees to an arc measured with some kind of length, we need to connect the dimensions. We can do that with the radian measure of an angle.

Radians will be useful in that a radian is a dimensionless measurement. We want to connect angle measurements to arc measurements, and to do so we will directly define an angle of 1 radian to be an angle subtended by an arc of length 1 (the length of the radius) on the unit circle as shown in Figure 2.5.5.

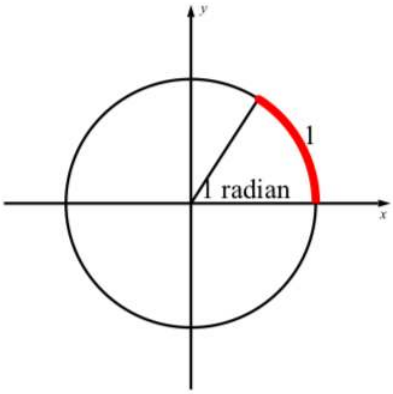


Figure 2.5.5: One radian.

Definition: Radian

An angle of **one radian** is the angle in standard position on the unit circle that is subtended by an arc of length 1 (in the positive direction).

This directly connects angles measured in radians to arcs in that we associate a real number with both the arc and the angle. So an angle of 2 radians cuts off an arc of length 2 on the unit circle, an angle of 3 radians cuts of an arc of length 3 on the unit circle, and so on. Figure 1.13 shows the terminal sides of angles with measures of 0 radians, 1 radian, 2 radians, 3 radians, 4 radians, 5 radians, and 6 radians. Notice that $2\pi \approx 6.2832$ and so $6 < 2\pi$ as shown in Figure 2.5.6.

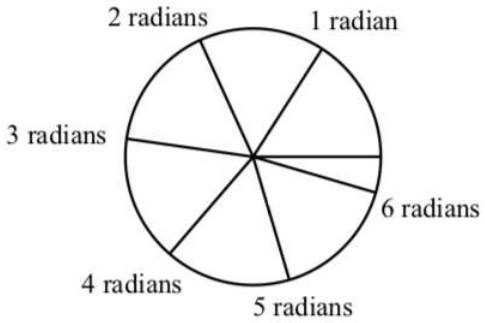


Figure 2.5.6: Angles with Radian Measure 1, 2, 3, 4, 5, and 6

We can also have angles whose radian measure is negative just like we have arcs with a negative length. The idea is simply to measure in the negative (clockwise) direction around the unit circle. So an angle whose measure is -1 radian is the angle in standard position on the unit circle that is subtended by an arc of length 1 in the negative (clockwise) direction.

So in general, an angle (in standard position) of t radians will correspond to an arc of length t on the unit circle. This allows us to discuss the sine and cosine of an angle measured in radians. That is, when we think of $\sin(t)$ and $\cos(t)$, we can consider t to be:

- a real number;
- the length of an arc with initial point $(1, 0)$ on the unit circle;
- the radian measure of an angle in standard position.

When we draw a picture of an angle in standard position, we often draw a small arc near the vertex from the initial side to the terminal side as shown in Figure 2.5.7, which shows an angle whose measure is $\frac{3}{4}\pi$ radians.

Example 2.5.1

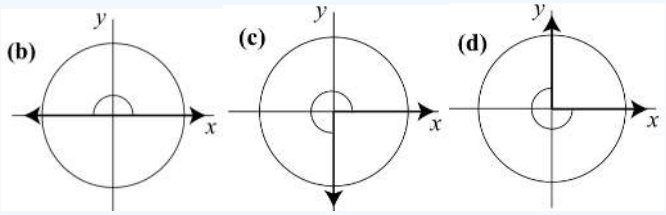
1. Draw a terminal ray in the quadrant defined by the given θ restrictions; measuring from standard position with a radian measure:

Let $\sin(\theta$

a. $\frac{\pi}{2}$
b. π radians.
c. $\frac{3\pi}{2}$ radians.
d. $\frac{-3\pi}{2}$ radians.

2. What is the degree measure of each of the angles in part (1)?

Answer



a. 90° (2.5.1)

b. 180° (2.5.2)

c. 270° (2.5.3)

d. -270° (2.5.4)

Conversion Between Radians and Degrees

Radian measure is the preferred measure of angles in mathematics for many reasons, the main one being that a radian has no dimensions. However, to effectively use radians, we will want to be able to convert angle measurements between radians and degrees.

Recall that one wrap of the unit circle corresponds to an arc of length 2π, and an arc of length 2π on the unit circle corresponds to an angle of 2π radians. An angle of 360° is also an angle that wraps once around the unit circle, so an angle of 360° is equivalent to an angle of 2π radians, or

- each degree is $\dfrac{\pi}{180}$ radians,
- each radian is $\dfrac{180}{\pi}$ degrees.

Notice that 1 radian is then $\dfrac{180}{\pi} \approx 57.3^\circ$, so a radian is quite large compared to a degree. These relationships allow us to quickly convert between degrees and radians.

Example 2.5.1

- If an angle has a degree measure of 35 degrees, then its radian measure can be calculated as follows:

$$35\ degrees \times \frac{\pi\ radians}{180\ degrees} = \frac{35\pi}{180}\ radians \tag{2.5.5}$$

Rewriting this fraction, we see that an angle with a measure of 35 degrees has a radian measure of $\dfrac{7\pi}{36}$ radians.

- If an angle has a radian measure of $\dfrac{3\pi}{10}$ radians, then its degree measure can be calculated as follows:

$$\frac{3\pi}{10}\ radians \times \frac{180\ degrees}{\pi\ radians} = \frac{540}{10}\ degrees \tag{2.5.6}$$

So an angle with a radian measure of $\dfrac{3\pi}{10}$ has an angle measure of 54°.

IMPORTANT NOTE

Since a degree is a dimension, we MUST include the degree mark $^\circ$ whenever we write the degree measure of an angle. A radian has no dimension so there is no dimension mark to go along with it. Consequently, if we write 2 for the measure of an angle we understand that the angle is measured in radians. If we really mean an angle of 2 degrees, then we *must* write 2°.

Exercise 2.5.3

Complete the following table to convert from degrees to radians and vice versa.

Answer

Angle in Radians	Angle in degrees
0	0°
$\dfrac{\pi}{6}$	30°
$\dfrac{\pi}{4}$	45°
$\dfrac{\pi}{3}$	60°

Angle in Radians	Angle in degrees
$\frac{\pi}{2}$	90°
$\frac{7\pi}{6}$	210°
$\frac{5\pi}{4}$	225°
$\frac{4\pi}{3}$	240°
$\frac{3\pi}{2}$	270°
$\frac{5\pi}{3}$	300°
$\frac{2\pi}{3}$	120°
$\frac{4\pi}{3}$	135°
$\frac{5\pi}{6}$	150°
π	180°
$\frac{7\pi}{4}$	315°
$\frac{11\pi}{6}$	330°
2π	360°

Calculators and the Trigonometric Functions

We have now seen that when we think of $\sin(t)$ or $\cos(t)$, we can think of (t) as a real number, the length of an arc, or the radian measure of an angle. We will see how to determine the exact values of the cosine and sine functions for a few special arcs (or angles). For example, we will see that $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$. However, the definition of cosine and sine as coordinates of points on the unit circle makes it difficult to find exact values for these functions *except at very special arcs*. While exact values are always best, technology plays an important role in allowing us to approximate the values of the circular (or trigonometric)functions. Most hand-held calculators, calculators in phone or tablet apps, and online calculators have a cosine key and a sine key that you can use to approximate values of these functions, but we must keep in mind that the calculator only provides an *approximation* of the value, not the exact value (except for a small collection of arcs). In addition, most calculators will approximate the sine and cosine of angles. Remember to read homework questions carefully - if it says anything about an exact number, it probably does NOT want a calculator approximation.

Angle in radians	Angle in degrees
0	0°
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	
$\frac{\pi}{2}$	90°

	120°
$\frac{3\pi}{4}$	135°
	150°
	180°
$\frac{7\pi}{6}$	
$\frac{5\pi}{4}$	
$\frac{4\pi}{3}$	
$\frac{3\pi}{2}$	270°
	300°
	315°
	330°
2π	360°

Table 1.1: Conversions between radians and degrees.

To do this, the calculator has two modes for angles: Radian and Degree. Because of the correspondence between real numbers, length of arcs, and radian measures of angles, for now, we will always put our calculators in radian mode. In fact, we have seen that an angle measured in radians subtends an arc of that radian measure along the unit circle. **So the cosine or sine of an angle measure in radians is the same thing as the cosine or sine of a real number when that real number is interpreted as the length of an arc along the unit circle.** (When we study the trigonometry of triangles, we will use the degree mode. In general, if a questions or application is given in degrees the solution is usually also given in degrees. Similarly, when working in radians the solution would also be given in radians, unless you have specifically been asked to convert from one to the other form.

Exercise 2.5.4

We can use the Geogebra Applet called *Terminal Points of Arcs on the Unit Circle* at http://gvsu.edu/s/JY to approximate the values of the cosine and sine functions at certain values. For example, we could find the following:

- $\cos(1) \approx 0.5403$, $\sin(1) \approx 0.8415$.
- $\cos(2) \approx -0.4161$, $\sin(2) \approx 0.9093$.
- $\cos(-4) \approx -0.6536$, $\sin(-4) \approx 0.7568$.
- $\cos(-15) \approx -0.7597$, $\sin(-15) \approx -0.6503$.

Use a calculator to determine these values of the cosine and sine functions and compare the values to the ones above. Are they the same? How are they different?

Answer

Using a calculator, we obtain the following results correct to ten decimal places:

$\cos(1) \approx 0.5403023059, \sin(1) \approx 0.8414709848 \cos(1) \approx 0:4161468365, \sin(1) \approx 0.9092974268$

The difference between these values and those obtained previously, is that these values are correct to 10 decimal places (and the others are correct to 4 decimal places). If we round off each of the values above to 4 decimal places, we get the same results.

- $\cos(1) \approx 0:6536436209, \sin(1) \approx 0.7568024953$
- $\cos(1) \approx 0:7596879129, \sin(1) \approx 0.6502878402$

Exercise 2.5.5

When working with sine and cosine we are reminded that these two functions can be used to generate the remaining 4 functions - tangent, secant, cosecant and cotangent.

Each of these trigonometric functions are positive in different quadrants of the unit circle. Now that we can calculate the values of sine and cosine at ANY terminal point along a unit circle, we now need to be able to do the same for the remaining 4 functions.

- In quadrant I **A**LL trigonometric functions are positive.
- In quadrant II the **S**INE function is positive.
- In quadrant III the **T**ANGENT function is positive.
- In quadrant IV the **C**OSINE function is positive.

Using the first letter from each of these quadrants and putting it into an easy-to-remember phrase can help you remember where functions are positive.

Here are some common phrases: "**A**ll **S**tudents **T**ake **C**alculus" "**A** **S**mart **T**rig **C**lass" or maybe you can think up a better one!

a. In what quadrant does the terminal ray reside if $sin(t) < 0$ and $cos(t) < 0$?
b. In what quadrant does the terminal ray reside if $tan(t) < 0$ and $cos(t) > 0$?
c. In what quadrant does the terminal ray reside if $sec(t) < 0$ and $tan(t) > 0$?
d. In what quadrant does the terminal ray reside if $csc(t) > 0$ and $cot(t) < 0$?

Answer

a. Sine is negative in quadrant III and IV, while cosine is negative in quadrant II and III. Therefore the overlap must be in **quadrant II** where both functions are negative.
b. Recall that tangent has an identity where $tan(t) = \dfrac{sin(t)}{cos(t)}$ is negative in quadrant II and IV, while cosine is positive in quadrant I and IV. Therefore the overlap must be in **quadrant IV** where these parameters are both met.
c. Recall that secant has an identity where $sec(t) = \dfrac{1}{cos(t)}$ this implies that the same quadrants will be positive for secant as for cosine. Secant is negative in quadrants II and III, while tangent is positive in quadrant I and III. Therefore the overlap must be in **quadrant III** where these parameters are both met.
d. Recall that cosecant has an identity where $csc(t) = \dfrac{1}{sin(t)}$ this implies that the same quadrants will be positive for cosecant as for sine. Recall that cotangent is the reciprocal of tangent and has an identity where $cot(t) = \dfrac{cos(t)}{sin(t)}$ is negative in quadrant II and IV. Cosecant is positive in quadrants I and II, while cotangent is negative in quadrant II and IV. Therefore, the overlap must be in **quadrant II** where these parameters are both met.

Exercise 2.5.6

Sometimes it will be important to write a given trigonometric function in terms of another.

The Pythagorean Theorem is $sin^2(\theta)+cos^2(\theta)=1$.

Dividing this initial identity by $sin^2(\theta)$ will generate another form for Pythagorean Theorem.

Dividing this initial identity by $cos^2(\theta)$ will generate a third form.

Use these three different forms to find the functions listed in terms of another function. Use the quadrant to determine if the function is positive or negative.

Remember from the previous exercise that the following are true:

- In quadrant I **A**LL trigonometric functions are positive.
- In quadrant II the **S**INE function is positive.
- In quadrant III the **T**ANGENT function is positive.
- In quadrant IV the **C**OSINE function is positive.

Use the Pythagorean Theorem and the phrase you have memorized to find the following trigonometric functions:

a. Write the $tan\theta$ in terms of $cos\theta$ for θ in Quadrant II.
b. Write the $sin\theta$ in terms of $cos\theta$ for θ in Quadrant III.
c. Write the $tan\theta$ in terms of $sec\theta$ for θ in Quadrant I.
d. Write the $sec\theta$ in terms of $cos\theta$ for θ in Quadrant IV.

Answer

a. We need to find a way to write $tan\theta$ equal to something on the right with only cosine functions included. If we divide $sin^2\theta$+$cos^2\theta$=1 by $cos^2\theta$ we will have a form for the Pythagorean Theorem in terms of tangent and cosine.

$$\frac{sin^2\theta}{cos^2\theta}+\frac{cos^2\theta}{cos^2\theta}=\frac{1}{cos^2\theta} \tag{2.5.7}$$

Now, solving for tangent we should have:

$$tan^2\theta=\frac{1}{cos^2\theta}-\frac{cos^2\theta}{cos^2\theta} \tag{2.5.8}$$

Simplify and take the square root of both sides. Remember, tangent is negative in Quadrant II, therefore our answer should be:

$$tan\theta=-\frac{\sqrt{1-cos^2\theta}}{cos\theta} \tag{2.5.9}$$

b. $$sin\theta=-\sqrt{1-cos^2\theta} \tag{2.5.10}$$

c. $$tan\theta=\sqrt{sec^2\theta-1} \tag{2.5.11}$$

d. $$tan\theta=-\frac{1}{\sqrt{1-cos^2\theta}} \tag{2.5.12}$$

Exercise 2.5.7

Throughout trigonometry - even in Calculus, Linear Algebra and some Differential Equations classes - we will be asked to find the terminal point given limited information and sometimes restrictions on your domain or quadrant. Use the Pythagorean Theorem to find the missing sides and recognize when the function will be positive or negative based on the quadrant. It is usually helpful to first restrict the quadrant to the one in which the values are true, then calculate the exact value for the remaining five functions based on the given information.

a. Given $sin\theta=-\frac{4}{5}$ find the remaining five values for the remaining five functions if θ is in Quadrant IV.
b. Given $cos\theta=-\frac{5}{6}$ find the remaining five values for the remaining five functions if $cot\theta>0$.
c. Given $csc\theta=3$ find the remaining five values for the remaining five functions in Quadrant I.
d. Given $cos\theta=\frac{7}{13}$ find the remaining five values for the remaining five functions if $sin\theta<0$.
e. Given $sec\theta=-\frac{1}{2}$ find the remaining five values for the remaining five functions if $csc\theta<0$.
f. Given $cos\theta=-\frac{2}{11}$ find the remaining five values for the remaining five functions if$tan\theta<0$.

Answer

a. Rotating through to a terminal ray in Quadrant IV, we now can draw a triangle using the terminal ray as the hypotenuse of the triangle. Use a reference angle off the x axis to orient the opposite and adjacent sides. The missing side is the x-direction length. Using the Pythagorean Theorem the sides of the triangle should be: Opposite is -4, Adjacent is 3 and Hypotenuse is 5. Now, using the identities,

$$sin\theta = \frac{opposite}{hypotenuse} = \frac{-4}{5} \tag{2.5.13}$$

$$cos\theta = \frac{adjacent}{hypotenuse} = \frac{3}{5} \tag{2.5.14}$$

and

$$tan\theta = \frac{opposite}{adjacent} = \frac{sin\theta}{cos\theta} = \frac{-4}{3} \tag{2.5.15}$$

For the remaining three trig functions we can use what we found previously, just as reciprocals:

$$csc\theta = \frac{hypotenuse}{opposite} = \frac{5}{-4} \tag{2.5.16}$$

$$sec\theta = \frac{hypotenuse}{adjacent} = \frac{5}{3} \tag{2.5.17}$$

$$cot\theta = \frac{adjacent}{opposite} = \frac{cos\theta}{sin\theta} = \frac{3}{-4} \tag{2.5.18}$$

NOTE:From here on the answers will be simplified such that we bring the negative sign in front of the fraction and it is understood that it was associated with the negative side of the triangle. It is more formal to write it in front of the fraction instead of with the numerator or denominator.

b. Cotangent is only positive in Quadrants I and III. While Cosine is only negative in Quadrants II and III. Therefore, we know our values for all six trig functions must exist in Quadrant III!

$$sin\theta = \frac{opposite}{hypotenuse} = -\frac{\sqrt{11}}{6} \tag{2.5.19}$$

$$cos\theta = -\frac{5}{6} \tag{2.5.20}$$

$$tan\theta = \frac{opposite}{adjacent} = \frac{sin\theta}{cos\theta} = \frac{\sqrt{11}}{5} \tag{2.5.21}$$

For the remaining three trig functions we can use what we found previously, just as reciprocals:

$$csc\theta = \frac{hypotenuse}{opposite} = -\frac{6\sqrt{11}}{11} \tag{2.5.22}$$

$$sec\theta = \frac{hypotenuse}{adjacent} = -\frac{6}{5} \tag{2.5.23}$$

$$cot\theta = \frac{adjacent}{opposite} = \frac{cos\theta}{sin\theta} = \frac{5\sqrt{11}}{11} \tag{2.5.24}$$

c.

$$sin\theta = \frac{1}{3} \tag{2.5.25}$$

$$cos\theta = \frac{2\sqrt{2}}{3} \tag{2.5.26}$$

$$tan\theta = \frac{\sqrt{2}}{4} \tag{2.5.27}$$

$$csc\theta = 3 \tag{2.5.28}$$

$$sec\theta = \frac{3\sqrt{2}}{4} \tag{2.5.29}$$

$$cot\theta = 2\sqrt{2} \tag{2.5.30}$$

d.

$$sin\theta = -\frac{2\sqrt{30}}{13} \tag{2.5.31}$$

$$cos\theta = \frac{7}{13} \tag{2.5.32}$$

$$tan\theta = 2\frac{2\sqrt{30}}{7} \tag{2.5.33}$$

$$csc\theta = -\frac{13\sqrt{30}}{60} \tag{2.5.34}$$

$$sec\theta = \frac{13}{7} \tag{2.5.35}$$

$$cot\theta = -\frac{7\sqrt{30}}{60} \tag{2.5.36}$$

e. Algebraically, we can calculate the value of cosine because $sec\theta = -\frac{1}{2}$. Therefore, cosine is -2. If we try to calculate the values for the remaining 5 functions we notice that the hypotenuse is less than the adjacent side. This should be a warning sign that the values do not exist. Geometrically/Graphically, we could plot or graph the function $f(x) = sec\theta$ simultaneously with $f(x) = -\frac{1}{2}$ to see that there is never a point at which the two functions intersect. This should lead us to believe that there is NO SOLUTION, or a solution DOES NOT EXIST (sometimes annotated as DNE).

f.

$$sin\theta = \frac{3\sqrt{13}}{11} \tag{2.5.37}$$

$$cos\theta = -\frac{2}{1} \tag{2.5.38}$$

$$tan\theta = -\frac{3\sqrt{13}}{2} \tag{2.5.39}$$

$$csc\theta = \frac{11\sqrt{13}}{39} \tag{2.5.40}$$

$$sec\theta = -\frac{11}{2} \tag{2.5.41}$$

$$cot\theta = -\frac{2\sqrt{13}}{39} \tag{2.5.42}$$

Section Summary

In this section, we studied the following important concepts and ideas:

- An **angle** is formed by rotating a ray about its endpoint. The ray in its initial position is called the **initial side** of the angle, and the position of the ray after it has been rotated is called the **terminal side** of the ray. The endpoint of the ray is called the **vertex of the angle**.
- When the vertex of an angle is at the origin in the xy-plane and the initial side lies along the positive x-axis, we see that the angle is in **standard position**.
- There are two ways to measure angles. For degree measure, one complete wrap around a circle is 360 degrees, denoted 360 $^\circ$. The unit measure of 1° is an angle that is 1=360 of the central angle of a circle. An angle of **one radian** is the angle in standard position on the unit circle that is subtended by an arc of length 1 (in the positive direction).
- We convert the measure of an angle from degrees to radians by using the fact that each degree is radians. We convert the measure of an angle from 180 radians to degrees by using the fact that each radian is 180 degrees.
- We looked at the strategies to use when finding the range values for all 6 trigonometric functions.
- We imposed restriction on the domain and quadrant, then found the missing trig function value. This will come in handy in the following sections!

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2.E: Trigonometric Functions (Exercises)

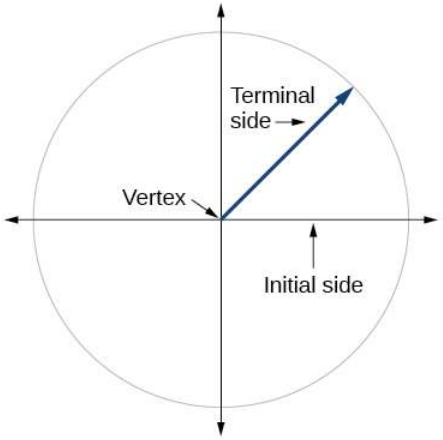
5.1: Angles

In this section, we will examine properties of angles.

Verbal

1) Draw an angle in standard position. Label the vertex, initial side, and terminal side.

Answer



2) Explain why there are an infinite number of angles that are coterminal to a certain angle.

3) State what a positive or negative angle signifies, and explain how to draw each.

Answer

Whether the angle is positive or negative determines the direction. A positive angle is drawn in the counterclockwise direction, and a negative angle is drawn in the clockwise direction.

4) How does radian measure of an angle compare to the degree measure? Include an explanation of 1 radian in your paragraph.

5) Explain the differences between linear speed and angular speed when describing motion along a circular path.

Answer

Linear speed is a measurement found by calculating distance of an arc compared to time. Angular speed is a measurement found by calculating the angle of an arc compared to time.

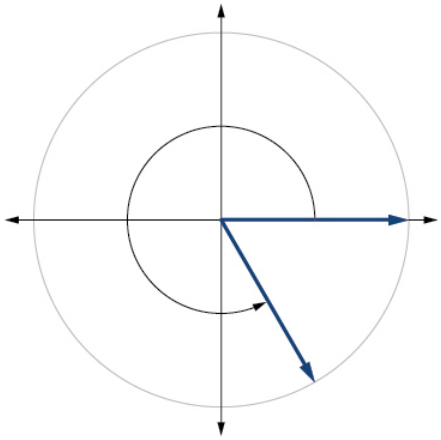
Graphical

For the exercises 6-21, draw an angle in standard position with the given measure.

6) 30°

7) 300°

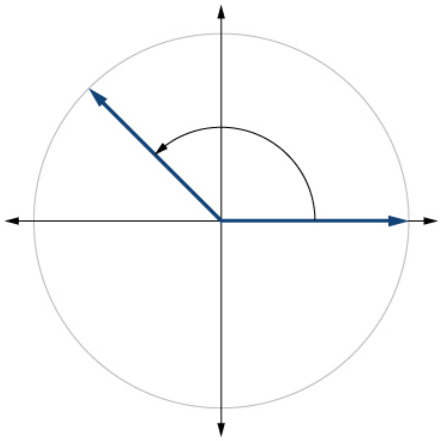
Answer



8) -80°

9) 135°

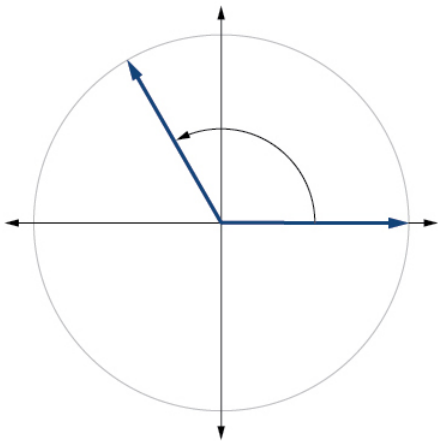
Answer



10) -150°

11) $\frac{2\pi}{3}$

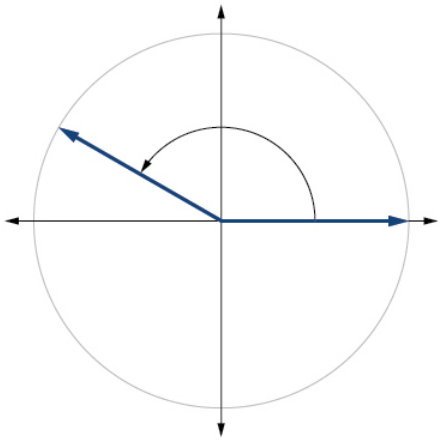
Answer



12) $\frac{7\pi}{4}$

13) $\frac{5\pi}{6}$

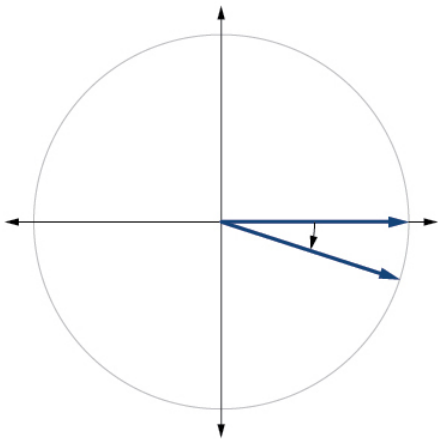
Answer



14) $\frac{\pi}{2}$

15) $-\frac{\pi}{10}$

Answer

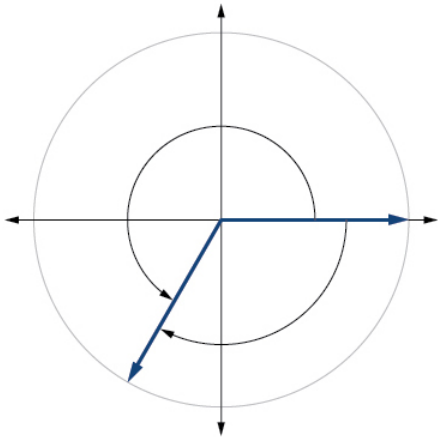


16) 415°

17) -120°

Answer

240°

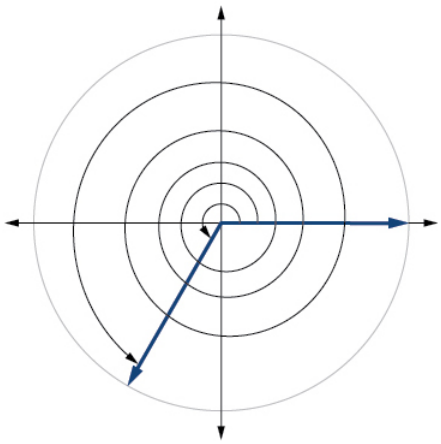


18) -315°

19) $\frac{22\pi}{3}$

Answer

$\frac{4\pi}{3}$

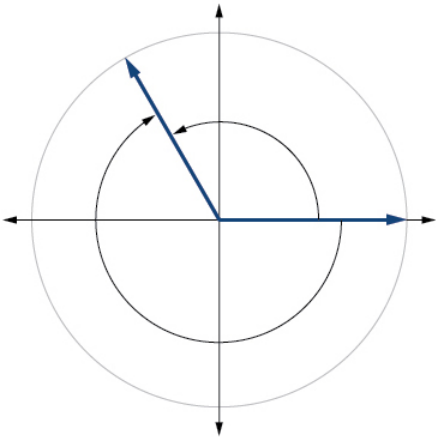


20) $-\frac{\pi}{6}$

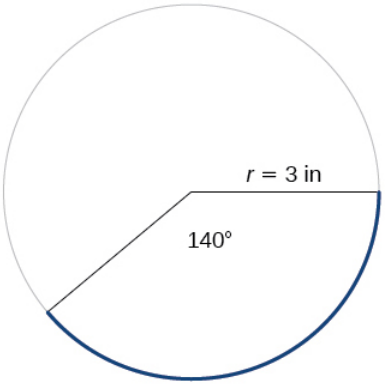
21) $-\frac{4\pi}{3}$

Answer

$\frac{2\pi}{3}$



For the exercises 22-23, refer to Figure below. Round to two decimal places.



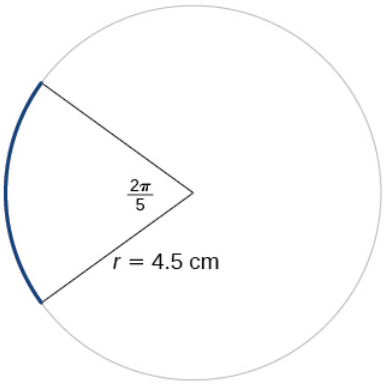
22) Find the arc length.

23) Find the area of the sector.

Answer

$\frac{27\pi}{2} \approx 11.00 \text{ in}^2$

For the exercises 24-25, refer to Figure below. Round to two decimal places.



24) Find the arc length.

25) Find the area of the sector.

Answer

$\frac{81\pi}{20} \approx 12.72 \text{ cm}^2$

Algebraic

For the exercises 26-32, convert angles in radians to degrees.

26) $\frac{3\pi}{4}$ radians

27) $\frac{\pi}{9}$ radians

Answer

20°

28) $-\frac{5\pi}{4}$ radians

29) $\frac{\pi}{3}$ radians

Answer

60°

30) $-\frac{7\pi}{3}$ radians

31) $-\frac{5\pi}{12}$ radians

Answer

-75°

32) $\frac{11\pi}{6}$ radians

For the exercises 33-39, convert angles in degrees to radians.

33) 90°

Answer

$\frac{\pi}{2}$ radians

34) 100°

35) -540°

Answer

-3π radians

36) -120°

37) 180°

Answer

π radians

38) -315°

39) 150°

Answer

$\frac{5\pi}{6}$ radians

For the exercises 40-45, use to given information to find the length of a circular arc. Round to two decimal places.

40) Find the length of the arc of a circle of radius 12 inches subtended by a central angle of $\frac{\pi}{4}$ radians.

41) Find the length of the arc of a circle of radius 5.02 miles subtended by the central angle of $\frac{\pi}{3}$.

Answer

$\frac{5.02\pi}{3} \approx 5.26$ miles

42) Find the length of the arc of a circle of diameter 14 meters subtended by the central angle of $\frac{5\pi}{6}$.

43) Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of 50°.

Answer

$\frac{25\pi}{9} \approx 8.73$ centimeters

44) Find the length of the arc of a circle of radius 5 inches subtended by the central angle of 220^{circ}.

45) Find the length of the arc of a circle of diameter 12 meters subtended by the central angle is 63^{circ}.

Answer

$\frac{21\pi}{10} \approx 6.60$ meters

For the exercises 46-49, use the given information to find the area of the sector. Round to four decimal places.

46) A sector of a circle has a central angle of 45° and a radius 6 cm.

47) A sector of a circle has a central angle of 30° and a radius of 20 cm.

Answer

$104.7198\ cm^2$

48) A sector of a circle with diameter 10 feet and an angle of $\frac{\pi}{2}$ radians.

49) A sector of a circle with radius of 0.7 inches and an angle of π radians.

Answer

$0.7697\ in^2$

For the exercises 50-53, find the angle between 0° and 360° that is coterminal to the given angle.

50) -40°

51) -110°

Answer

250°

52) 700°

53) 1400°

Answer

320°

For the exercises 54-57, find the angle between 0 and 2π in radians that is coterminal to the given angle.

54) $-\frac{\pi}{9}$

55) $\frac{10\pi}{3}$

Answer

$\frac{4\pi}{3}$

56) $\frac{13\pi}{6}$

57) $\frac{44\pi}{9}$

Answer

$\frac{8\pi}{9}$

Real-World Applications

58) A truck with 32-inch diameter wheels is traveling at 60 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

59) A bicycle with 24-inch diameter wheels is traveling at 15 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

Answer

$1320\text{rad}\ 210.085\text{RPM}$

60) A wheel of radius 8 inches is rotating $15^\circ/s$. What is the linear speed v, the angular speed in RPM, and the angular speed in rad/s?

61) A wheel of radius 14 inches is rotating 0.5rad/s. What is the linear speed v, the angular speed in RPM, and the angular speed in deg/s?

Answer

7 in./s, 4.77RPM, 28.65deg/s

62) A CD has diameter of 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed.

63) When being burned in a writable CD-R drive, the angular speed of a CD is often much faster than when playing audio, but the angular speed still varies to keep the linear speed constant where the disc is being written. When writing along the outer edge of the disc, the angular speed of one drive is about 4800 RPM (revolutions per minute). Find the linear speed if the CD has diameter of 120 millimeters.

Answer

$1,809,557.37\ \text{mm/min} = 30.16\ \text{m/s}$

64) A person is standing on the equator of Earth (radius 3960 miles). What are his linear and angular speeds?

65) Find the distance along an arc on the surface of Earth that subtends a central angle of 5 minutes $(1 \text{ minute} = \frac{1}{60} \text{ degree})$. The radius of Earth is 3960 miles.

Answer

5.76 miles

66) Find the distance along an arc on the surface of Earth that subtends a central angle of 7 minutes $(1 \text{ minute} = \frac{1}{60} \text{ degree})$. The radius of Earth is 3960 miles.

67) Consider a clock with an hour hand and minute hand. What is the measure of the angle the minute hand traces in 20 minutes?

Answer

120°

Extensions

68) Two cities have the same longitude. The latitude of city A is 9.00 degrees north and the latitude of city B is 30.00 degree north. Assume the radius of the earth is 3960 miles. Find the distance between the two cities.

69) A city is located at 40 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

Answer

794 miles per hour

70) A city is located at 75 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

71) Find the linear speed of the moon if the average distance between the earth and moon is 239,000 miles, assuming the orbit of the moon is circular and requires about 28 days. Express answer in miles per hour.

Answer

2,234 miles per hour

72) A bicycle has wheels 28 inches in diameter. A tachometer determines that the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

73) A car travels 3 miles. Its tires make 2640 revolutions. What is the radius of a tire in inches?

Answer

11.5 inches

74) A wheel on a tractor has a 24-inch diameter. How many revolutions does the wheel make if the tractor travels 4 miles?

5.2: Unit Circle - Sine and Cosine Functions

Verbal

1) Describe the unit circle.

Answer

The unit circle is a circle of radius 1 centered at the origin.

2) What do the x- and y-coordinates of the points on the unit circle represent?

3) Discuss the difference between a coterminal angle and a reference angle.

Answer

Coterminal angles are angles that share the same terminal side. A reference angle is the size of the smallest acute angle, t, formed by the terminal side of the angle t and the horizontal axis.

4) Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.

5) Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

Answer

The sine values are equal.

Algebraic

For the exercises 6-9, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

6) $\sin(t) < 0$ and $\cos(t) < 0$

7) $\sin(t) > 0$ and $\cos(t) > 0$

Answer

I

8) $\sin(t) > 0$ and $\cos(t) < 0$

9) $\sin(t) < 0$ and $\cos(t) > 0$

Answer

IV

For the exercises 10-22, find the exact value of each trigonometric function.

10) $\sin\frac{\pi}{2}$

11) $\sin\frac{\pi}{3}$

Answer

$\frac{\sqrt{3}}{2}$

12) $\cos\frac{\pi}{2}$

13) $\cos\frac{\pi}{3}$

Answer

$\frac{1}{2}$

14) $\sin\frac{\pi}{4}$

15) $\cos\frac{\pi}{4}$

Answer

$\frac{\sqrt{2}}{2}$

16) $\sin\frac{\pi}{6}$

17) $\sin\pi$

Answer

0

18) $\sin \frac{3\pi}{2}$

19) $\cos \pi$

Answer

-1

20) $\cos 0$

21) $cos \frac{\pi}{6}$

Answer

$\frac{\sqrt{3}}{2}$

22) $\sin 0$

Numeric

For the exercises 23-33, state the reference angle for the given angle.

23) 240°

Answer

60°

24) -170°

25) 100°

Answer

80°

26) -315°

27) 135°

Answer

45°

28) $\frac{5\pi}{4}$

29) $\frac{2\pi}{3}$

Answer

$\frac{\pi}{3}$

30) $\frac{5\pi}{6}$

31) $-\frac{11\pi}{3}$

Answer

$\frac{\pi}{3}$

32) $\frac{-7\pi}{4}$

33) $\frac{-\pi}{8}$

Answer

$\frac{\pi}{8}$

For the exercises 34-49, find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle. If the angle is not one of the angles on the unit circle, use a calculator and round to three decimal places.

34) 225°

35) 300°

Answer

60°, Quadrant IV, $\sin(300^\circ) = -\frac{\sqrt{3}}{2}, \cos(300^\circ) = \frac{1}{2}$

36) 320°

37) 135°

Answer

45°, Quadrant II, $\sin(135^\circ) = \frac{\sqrt{2}}{2}, \cos(135^\circ) = -\frac{\sqrt{2}}{2}$

38) 210°

39) 120°

Answer

60°, Quadrant II, $\sin(120^\circ) = \frac{\sqrt{3}}{2}, \cos(120^\circ) = -\frac{1}{2}$

40) 250°

41) 150°

Answer

30°, Quadrant II, $\sin(150^\circ) = \frac{1}{2}, \cos(150^\circ) = -\frac{\sqrt{3}}{2}$

42) $\frac{5\pi}{4}$

43) $\frac{7\pi}{6}$

Answer

$\frac{\pi}{6}$, Quadrant III, $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}, \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

44) $\frac{5\pi}{3}$

45) $\frac{3\pi}{4}$

Answer

$\frac{\pi}{4}$, Quadrant II, $\sin\left(\frac{3\pi}{4}\right)=\frac{\sqrt{2}}{2}$, $\cos\left(\frac{4\pi}{3}\right)=-\frac{\sqrt{2}}{2}$

46) $\frac{4\pi}{3}$

47) $\frac{2\pi}{3}$

Answer

$\frac{\pi}{3}$, Quadrant II, $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$, $\cos\left(\frac{2\pi}{3}\right)=-\frac{1}{2}$

48) $\frac{5\pi}{6}$

49) $\frac{7\pi}{4}$

Answer

$\frac{\pi}{4}$, Quadrant IV, $\sin\left(\frac{7\pi}{4}\right)=-\frac{\sqrt{2}}{2}$, $\cos\left(\frac{7\pi}{4}\right)=\frac{\sqrt{2}}{2}$

For the exercises 50-59, find the requested value.

50) If $\cos(t)=\frac{1}{7}$ and t is in the 4th quadrant, find $\sin(t)$.

51) If $\cos(t)=\frac{2}{9}$ and t is in the 1st quadrant, find $\sin(t)$.

Answer

$\frac{\sqrt{77}}{9}$

52) If $\sin(t)=\frac{3}{8}$ and t is in the 2nd quadrant, find $\cos(t)$.

53) If $\sin(t)=-\frac{1}{4}$ and t is in the 3rd quadrant, find $\cos(t)$.

Answer

$-\frac{\sqrt{15}}{4}$

54) Find the coordinates of the point on a circle with radius 15 corresponding to an angle of 220°.

55) Find the coordinates of the point on a circle with radius 20 corresponding to an angle of 120°.

Answer

$(-10, 10\sqrt{3})$

56) Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$.

57) Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9}$.

Answer

$(-2.778, 15.757)$

58) State the domain of the sine and cosine functions.

59) State the range of the sine and cosine functions.

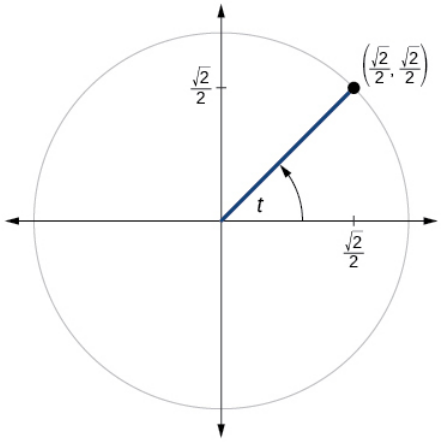
Answer

$[-1, 1]$

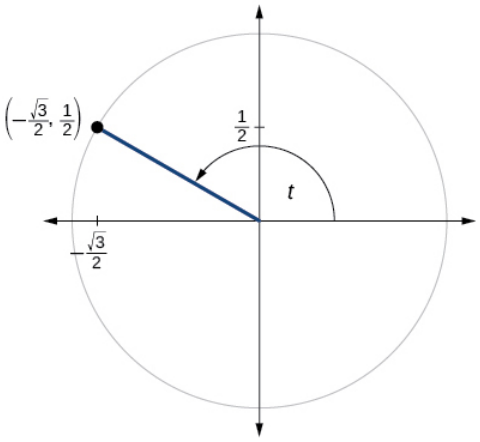
Graphical

For the exercises 60-79, use the given point on the unit circle to find the value of the sine and cosine of t.

60)



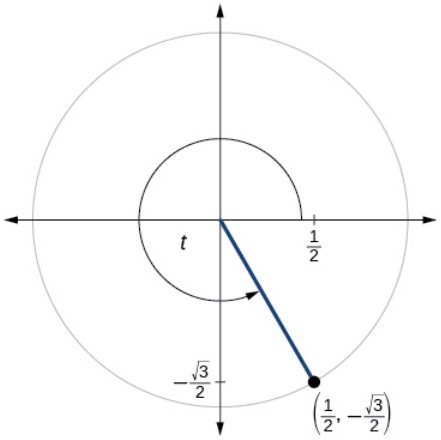
61)



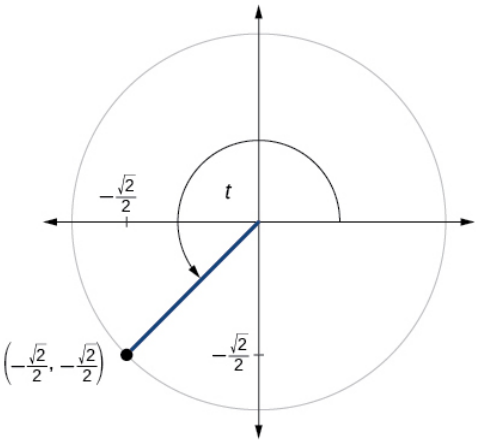
Answer

$\sin t = \frac{1}{2}, \cos t = -\frac{\sqrt{3}}{2}$

62)



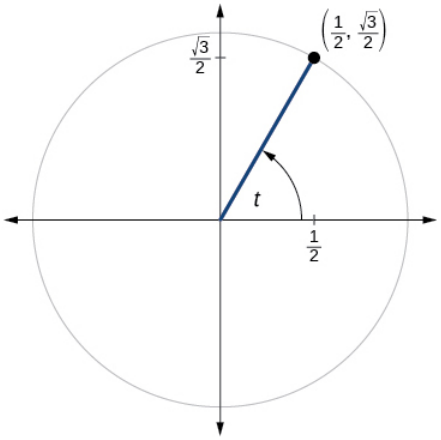
63)



Answer

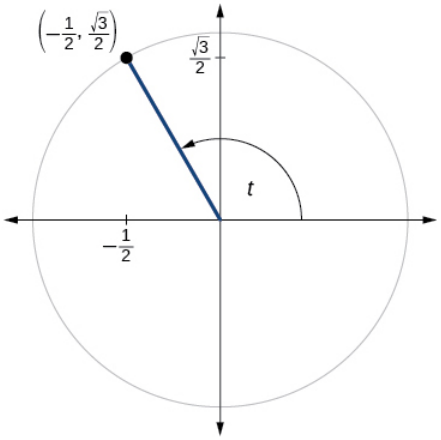
$$\sin t = -\frac{\sqrt{2}}{2}, \cos t = -\frac{\sqrt{2}}{2}$$

64)



65)

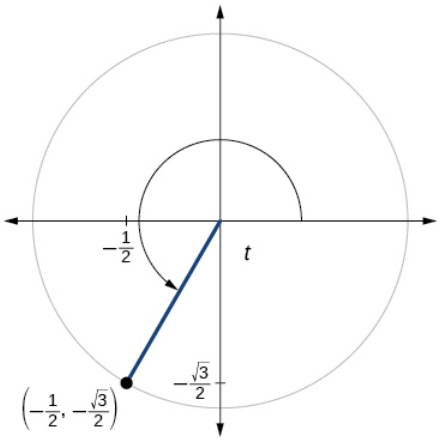
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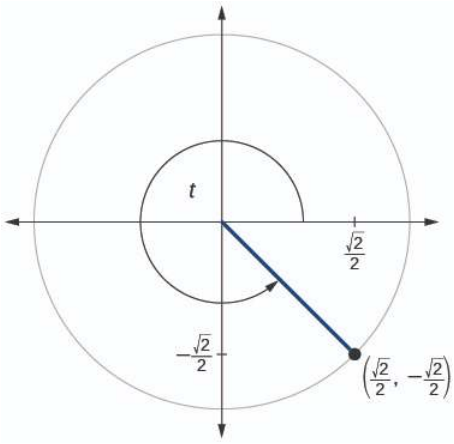
Answer

$$\sin t = \frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}$$

66)



67)

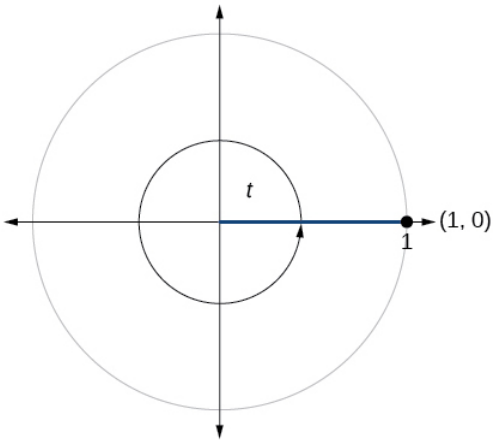


Answer

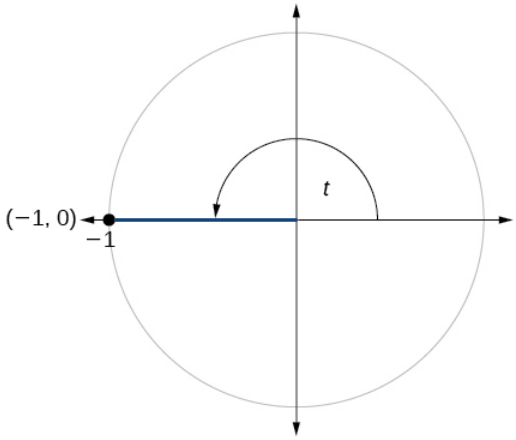
$$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}$$

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68)



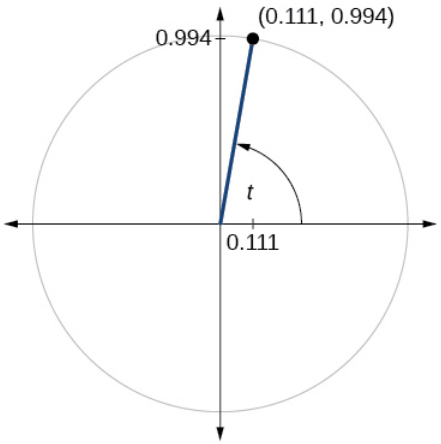
69)



Answer

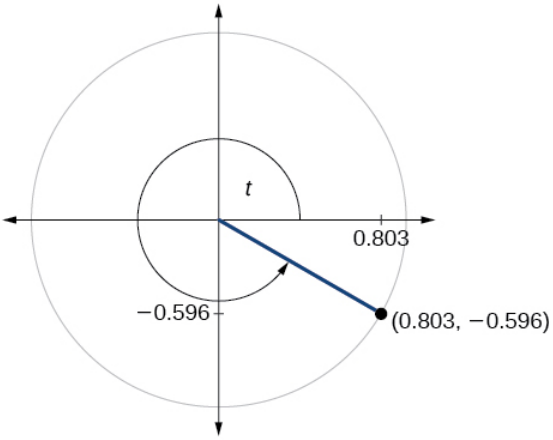
$\sin t = 0, \cos t = -1$

70)



71)

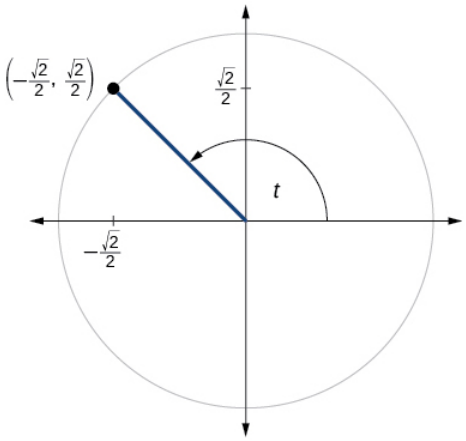
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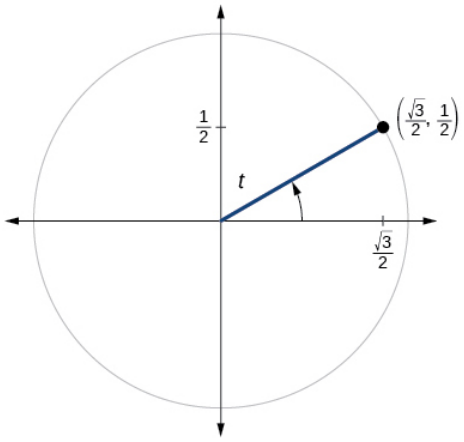
Answer

$\sin t = -0.596, \cos t = 0.803$

72)



73)

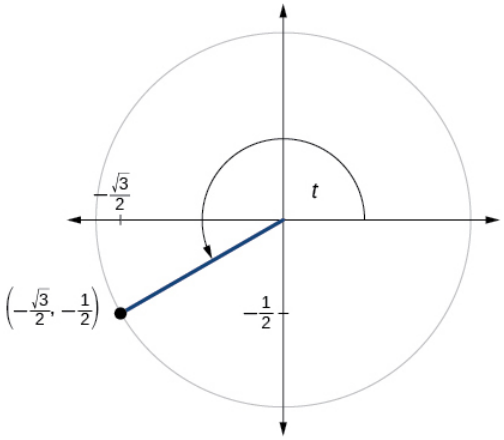


Answer

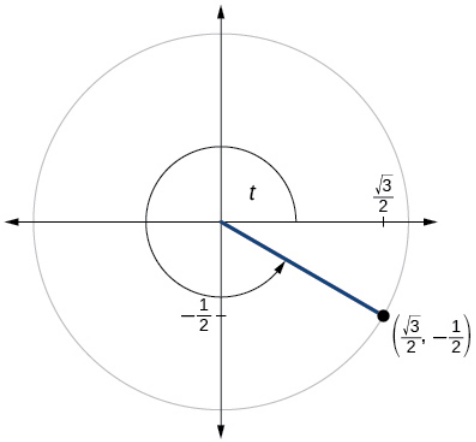
$\sin t = \frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$

74)

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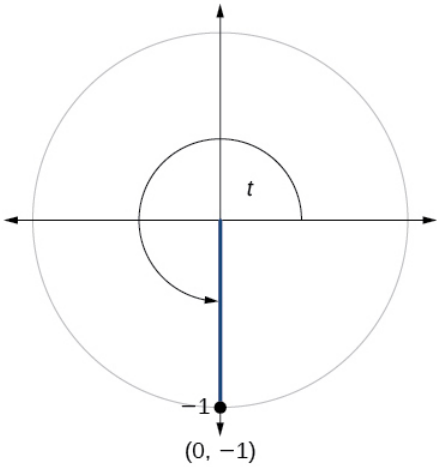
75)



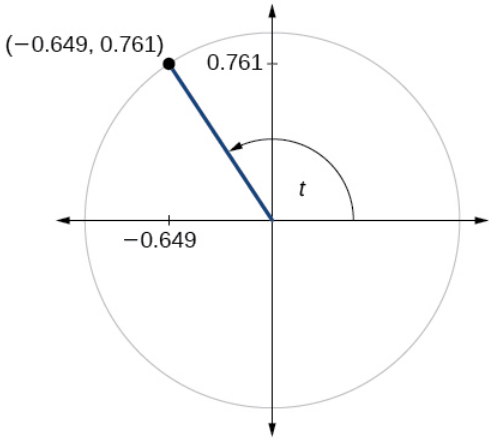
Answer

$$\sin t = -\frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$$

76)



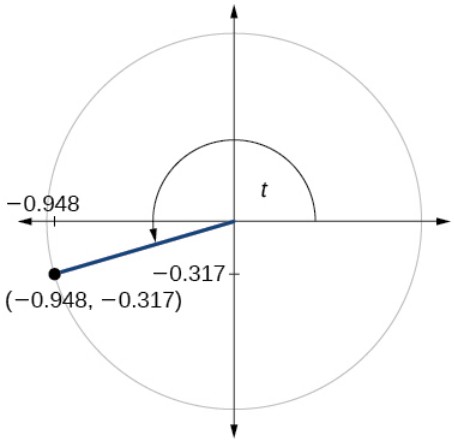
77)



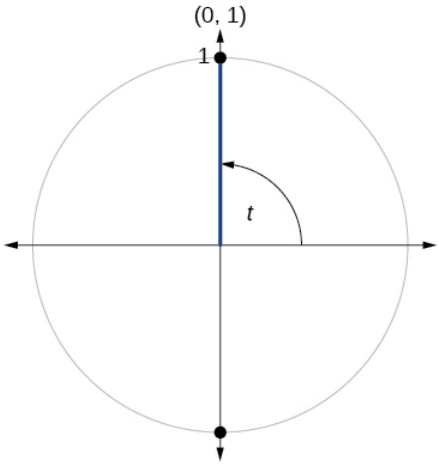
Answer

$\sin t = 0.761, \cos t = -0.649$

78)



79)



Answer

$\sin t = 1, \cos t = 0$

Technology

For the exercises 80-89, use a graphing calculator to evaluate.

80) $\sin\frac{5\pi}{9}$

81) $cos\frac{5\pi}{9}$

Answer

-0.1736

82) $\sin\frac{\pi}{10}$

83) $\cos\frac{\pi}{10}$

Answer

0.9511

84) $\sin\frac{3\pi}{4}$

85) $\cos\frac{3\pi}{4}$

Answer

-0.7071

86) $\sin 98^\circ$

87) $\cos 98^\circ$

Answer

-0.1392

88) $\cos 310^\circ$

89) $\sin 310^\circ$

Answer

-0.7660

Extensions

For the exercises 90-99, evaluate.

90) $\sin\left(\frac{11\pi}{3}\right)\cos\left(\frac{-5\pi}{6}\right)$

91) $\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right)$

Answer

$\frac{\sqrt{2}}{4}$

92) $\sin\left(-\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{2}\right)$

93) $\sin\left(\frac{-9\pi}{4}\right)\cos\left(\frac{-\pi}{6}\right)$

Answer

$-\frac{\sqrt{6}}{4}$

94) $\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{-\pi}{3}\right)$

95) $\sin\left(\frac{7\pi}{4}\right)\cos\left(\frac{-2\pi}{3}\right)$

Answer

$\frac{\sqrt{2}}{4}$

96) $\cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{2\pi}{3}\right)$

97) $\cos\left(\frac{-\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)$

Answer

$\frac{\sqrt{2}}{4}$

98) $\sin\left(\frac{-5\pi}{4}\right)\sin\left(\frac{11\pi}{6}\right)$

99) $\sin(\pi)\sin\left(\frac{\pi}{6}\right)$

Answer

0

Real-World Applications

For the exercises 100-104, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point $(0,1)$, that is, on the due north position. Assume the carousel revolves counter clockwise.

100) What are the coordinates of the child after 45 seconds?

101) What are the coordinates of the child after 90 seconds?

Answer

$(0,-1)$

102) What is the coordinates of the child after 125 seconds?

103) When will the child have coordinates $(0.707,-0.707)$if the ride lasts 6 minutes? (There are multiple answers.)

Answer

37.5seconds, 97.5seconds, 157.5seconds, 217.5seconds, 277.5seconds, 337.5seconds

104) When will the child have coordinates $(-0.866,-0.5)$if the ride last 6 minutes?

5.3: The Other Trigonometric Functions

Verbal

1) On an interval of $[0, 2\pi)$, can the sine and cosine values of a radian measure ever be equal? If so, where?

Answer

Yes, when the reference angle is $\frac{\pi}{4}$ and the terminal side of the angle is in quadrants I and III. Thus, at $x = \frac{\pi}{4}, \frac{5\pi}{4}$, the sine and cosine values are equal.

2) What would you estimate the cosine of π degrees to be? Explain your reasoning.

3) For any angle in quadrant II, if you knew the sine of the angle, how could you determine the cosine of the angle?

Answer

Substitute the sine of the angle in for y in the Pythagorean Theorem $x^2 + y^2 = 1$. Solve for x and take the negative solution.

4) Describe the secant function.

5) Tangent and cotangent have a period of π. What does this tell us about the output of these functions?

Answer

The outputs of tangent and cotangent will repeat every π units.

Algebraic

For the exercises 6-17, find the exact value of each expression.

6) $\tan\frac{\pi}{6}$

7) $\sec\frac{\pi}{6}$

Answer

$\frac{2\sqrt{3}}{3}$

8) $\csc\frac{\pi}{6}$

9) $\cot\frac{\pi}{6}$

Answer

$\sqrt{3}$

10) $\tan\frac{\pi}{4}$

11) $\sec\frac{\pi}{4}$

Answer

$\sqrt{2}$

12) $\csc\frac{\pi}{4}$

13) $\cot\frac{\pi}{4}$

Answer

1

14) $\tan\frac{\pi}{3}$

15) $\sec\frac{\pi}{3}$

Answer

2

16) $\csc\frac{\pi}{3}$

17) $\cot\frac{\pi}{3}$

Answer

$\frac{\sqrt{3}}{3}$

For the exercises 18-48, use reference angles to evaluate the expression.

18) $\tan\frac{5\pi}{6}$

19) $\sec\frac{7\pi}{6}$

Answer

$-\frac{2\sqrt{3}}{3}$

20) $\csc\frac{11\pi}{6}$

21) $\cot\frac{13\pi}{6}$

Answer

$\sqrt{3}$

22) $\tan\frac{7\pi}{4}$

23) $\sec\frac{3\pi}{4}$

Answer

$-\sqrt{2}$

24) $\csc\frac{5\pi}{4}$

25) $\cot\frac{11\pi}{4}$

Answer

-1

26) $\tan\frac{8\pi}{3}$

27) $\sec \frac{4\pi}{3}$

Answer

-2

28) $\csc \frac{2\pi}{3}$

29) $\cot \frac{5\pi}{3}$

Answer

$-\frac{\sqrt{3}}{3}$

30) $\tan 225^\circ$

31) $\sec 300^\circ$

Answer

2

32) $\csc 150^\circ$

33) $\cot 240^\circ$

Answer

$\frac{\sqrt{3}}{3}$

34) $\tan 330^\circ$

35) $\sec 120^\circ$

Answer

-2

36) $\csc 210^\circ$

37) $\cot 315^\circ$

Answer

-1

38) If $\sin t = \frac{3}{4}$, and t is in quadrant II, find $\cos t, \sec t, \csc t, \tan t, \cot t$.

39) If $\cos t = -\frac{1}{3}$, and t is in quadrant III, find $\sin t, \sec t, \csc t, \tan t, \cot t$.

Answer

If $\sin t = -\frac{2\sqrt{2}}{3}, \sec t = -3, \csc t = -\csc t = -\frac{3\sqrt{2}}{4}, \tan t = 2\sqrt{2}, \cot t = \frac{\sqrt{2}}{4}$

40) If $\tan t = \frac{12}{5}$, and $0 \le t < \frac{\pi}{2}$, find $\sin t, \cos t, \sec t, \csc t$, and $\cot t$.

41) If $\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$, find $\sec t, \csc t, \tan t$, and $\cot t$.

Answer

$\sec t = 2, \csc t = \csc t = \dfrac{2\sqrt{3}}{3}, \tan t = \sqrt{3}, \cot t = \dfrac{\sqrt{3}}{3}$

42) If $\sin 40^\circ \approx 0.643 \ \cos 40^\circ \approx 0.766 \ \sec 40^\circ, \csc 40^\circ, \tan 40^\circ,$ and $\cot 40.^\circ$

43) If $\sin t = \dfrac{\sqrt{2}}{2},$ what is the $\sin(-t)$?

Answer

$-\dfrac{\sqrt{2}}{2}$

44) If $\cos t = \dfrac{1}{2},$ what is the $\cos(-t)$?

45) If $\sec t = 3.1,$ what is the $\sec(-t)$?

Answer

3.1

46) If $\csc t = 0.34,$ what is the $\csc(-t)$?

47) If $\tan t = -1.4,$ what is the $\tan(-t)$?

Answer

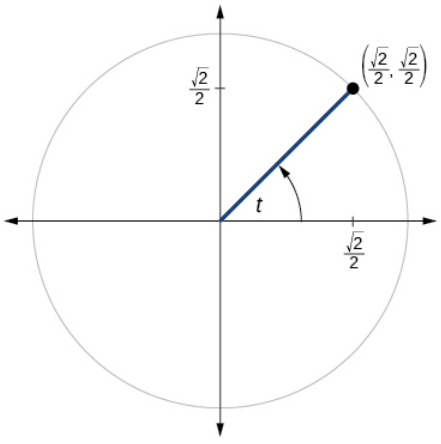
1.4

48) If $\cot t = 9.23,$ what is the $\cot(-t)$?

Graphical

For the exercises 49-51, use the angle in the unit circle to find the value of the each of the six trigonometric functions.

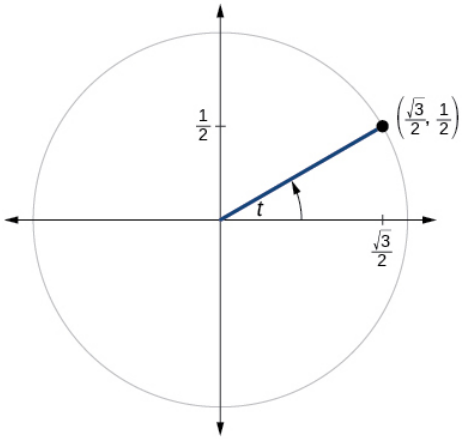
49)



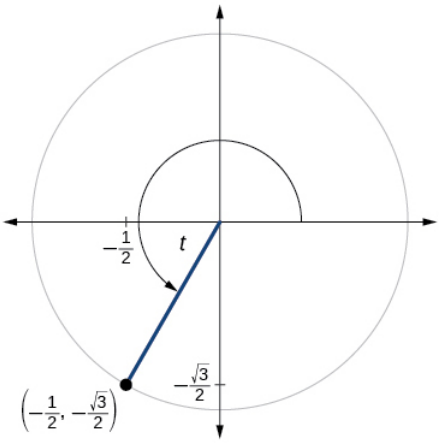
Answer

$\sin t = \dfrac{\sqrt{2}}{2}, \cos t = \dfrac{\sqrt{2}}{2}, \tan t = 1, \cot t = 1, \sec t = \sqrt{2}, \csc t = \csc t = \sqrt{2}$

50)



51)



Answer

$$\sin t = -\frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}, \tan t = \sqrt{3}, \cot t = \frac{\sqrt{3}}{3}, \sec t = -2, \csc t = -\csc t = -\frac{2\sqrt{3}}{3}$$

Technology

For the exercises 52-61, use a graphing calculator to evaluate.

52) $\csc \frac{5\pi}{9}$

53) $\cot \frac{4\pi}{7}$

Answer

-0.228

54) $\sec \frac{\pi}{10}$

55) $\tan \frac{5\pi}{8}$

Answer

-2.414

56) $\sec \frac{3\pi}{4}$

57) $\csc\frac{\pi}{4}$

Answer

1.414

58) $\tan 98^\circ$

59) $\cot 33^\circ$

Answer

1.540

60) $\cot 140^\circ$

61) $\sec 310^\circ$

Answer

1.556

Extensions

For the exercises 62-69, use identities to evaluate the expression.

62) If $\tan(t) \approx 2.7$, and $\sin(t) \approx 0.94$, find $\cos(t)$.

63) If $\tan(t) \approx 1.3$, and $\cos(t) \approx 0.61$, find $\sin(t)$.

Answer

$\sin(t) \approx 0.79$

64) If $\csc(t) \approx 3.2$, and $\csc(t) \approx 3.2$, and $\cos(t) \approx 0.95$, find $\tan(t)$.

65) If $\cot(t) \approx 0.58$, and $\cos(t) \approx 0.5$, find $\csc(t)$.

Answer

$\csc(t) \approx 1.16$

66) Determine whether the function $f(x) = 2\sin x \cos x$ is even, odd, or neither.

67) Determine whether the function $f(x) = 3\sin^2 x \cos x + \sec x$ is even, odd, or neither.

Answer

even

68) Determine whether the function $f(x) = \sin x - 2\cos^2 x$ is even, odd, or neither.

69) Determine whether the function $f(x) = \csc^2 x + \sec x$ is even, odd, or neither.

Answer

even

For the exercises 70-71, use identities to simplify the expression.

70) $\csc t \tan t$

71) $\frac{\sec t}{\csc t}$

Answer

$$\frac{\sin t}{\cos t} = \tan t$$

Real-World Applications

72) The amount of sunlight in a certain city can be modeled by the function $h = 15\cos\left(\frac{1}{600}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on February 10, the 42nd day of the year. State the period of the function.

73) The amount of sunlight in a certain city can be modeled by the function $h = 16\cos\left(\frac{1}{500}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on September 24, the 267th day of the year. State the period of the function.

Answer

13.77hours, period: 1000π

74) The equation $P = 20\sin(2\pi t) + 100$ models the blood pressure, P, where t represents time in seconds.

a. Find the blood pressure after 15 seconds.
b. What are the maximum and minimum blood pressures?

75) The height of a piston, h, in inches, can be modeled by the equation $y = 2\cos x + 6$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

Answer

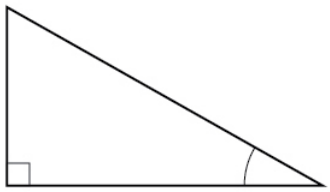
7.73inches

76) The height of a piston, h,in inches, can be modeled by the equation $y = 2\cos x + 5$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

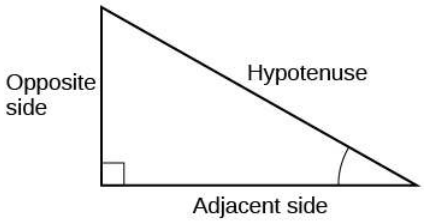
5.4: Right Triangle Trigonometry

Verbal

1) For the given right triangle, label the adjacent side, opposite side, and hypotenuse for the indicated angle.



Answer



2) When a right triangle with a hypotenuse of 1 is placed in the unit circle, which sides of the triangle correspond to the x- and y-coordinates?

3) The tangent of an angle compares which sides of the right triangle?

Answer

The tangent of an angle is the ratio of the opposite side to the adjacent side.

4) What is the relationship between the two acute angles in a right triangle?

5) Explain the cofunction identity.

Answer

For example, the sine of an angle is equal to the cosine of its complement; the cosine of an angle is equal to the sine of its complement.

Algebraic

For the exercises 6-9, use cofunctions of complementary angles.

6) $\cos(34^\circ) = \sin(\_\_^\circ)$

7) $\cos(\frac{\pi}{3}) = \sin(\_\_\_)$

Answer

$\frac{\pi}{6}$

8) $\csc(21^\circ) = \sec(\_\_\_^\circ)$

9) $\tan(\frac{\pi}{4}) = \cot(\_\_)$

Answer

$\frac{\pi}{4}$

For the exercises 10-16, find the lengths of the missing sides if side a is opposite angle A, side b is opposite angle B, and side c is the hypotenuse.

10) $\cos B = \frac{4}{5}, a = 10$

11) $\sin B = \frac{1}{2}, a = 20$

Answer

$b = \frac{20\sqrt{3}}{3}, c = \frac{40\sqrt{3}}{3}$

12) $\tan A = \frac{5}{12}, b = 6$

13) $\tan A = 100, b = 100$

Answer

$a = 10,000, c = 10,000.5$

14) $\sin B = \frac{1}{\sqrt{3}}, a = 2$

15) $a = 5, \measuredangle A = 60^\circ$

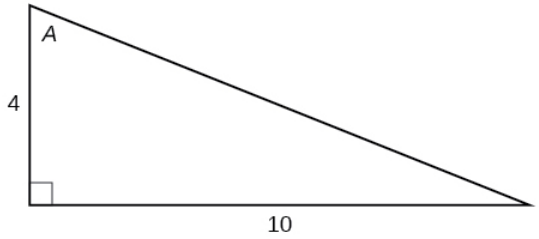
Answer

$b = \frac{5\sqrt{3}}{3}, c = \frac{10\sqrt{3}}{3}$

16) $c = 12, \measuredangle A = 45^\circ$

Graphical

For the exercises 17-22, use Figure below to evaluate each trigonometric function of angle A.



17) $\sin A$

Answer

$\frac{5\sqrt{29}}{29}$

18) $\cos A$

19) $\tan A$

Answer

$\frac{5}{2}$

20) $\csc A$

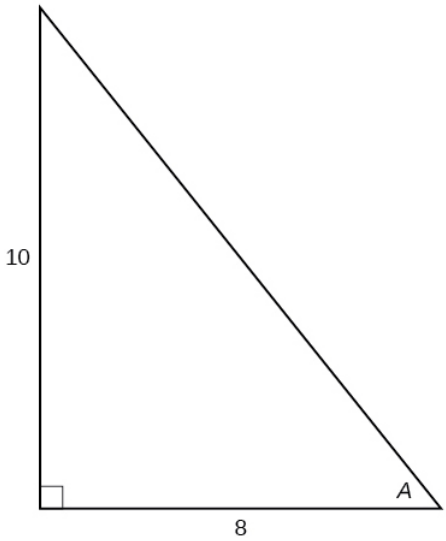
21) $\sec A$

Answer

$\frac{\sqrt{29}}{2}$

22) $\cot A$

For the exercises 23-,28 use Figure below to evaluate each trigonometric function of angle A.



23) $\sin A$

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Answer

$\frac{5\sqrt{41}}{41}$

24) $\cos A$

25) $\tan A$

Answer

$\frac{5}{4}$

26) $\csc A$

27) $\sec A$

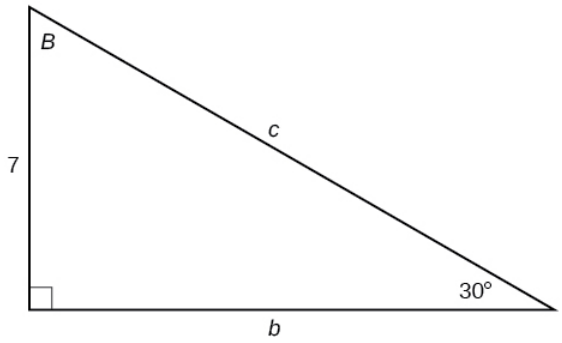
Answer

$\frac{\sqrt{41}}{4}$

28) $\cot A$

For the exercises 29-31, solve for the unknown sides of the given triangle.

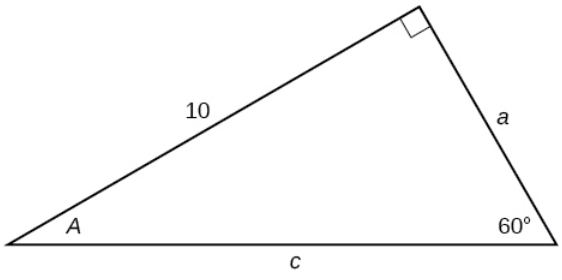
29)



Answer

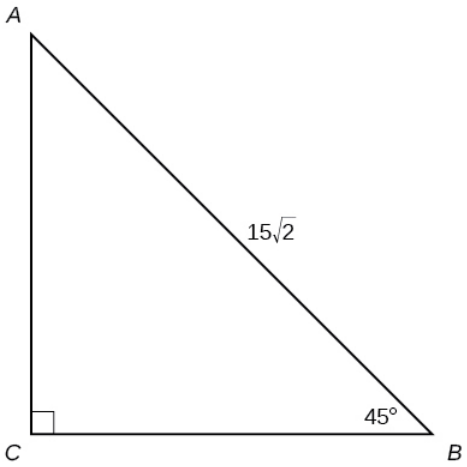
$c = 14, b = 7\sqrt{3}$

30)



31)

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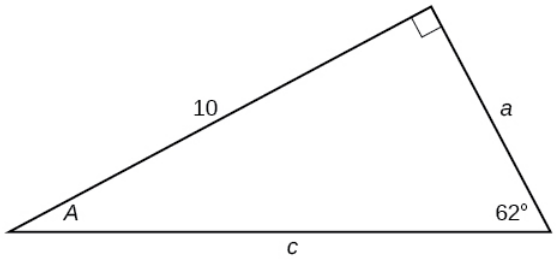
Answer

$a = 15, b = 15$

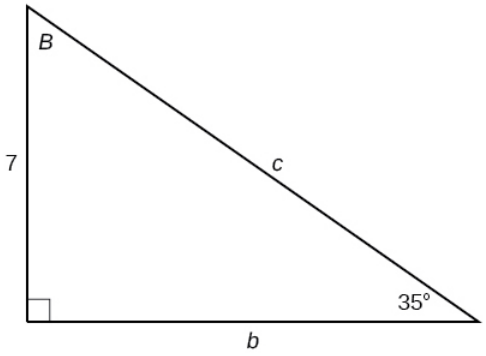
Technology

For the exercises 32-41, use a calculator to find the length of each side to four decimal places.

32)



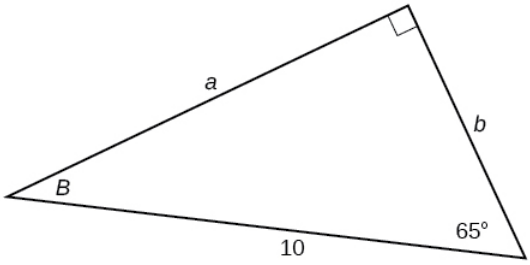
33)



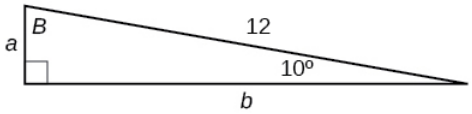
Answer

$b = 9.9970, c = 12.2041$

34)



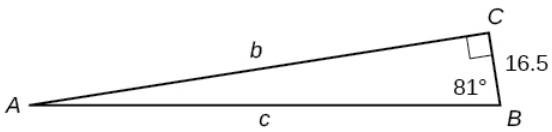
35)



Answer

$a = 2.0838, b = 11.8177$

36)



37) $b = 15, \angle B = 15^\circ$

Answer

$a = 55.9808, c = 57.9555$

38) $c = 200, \angle B = 5^\circ$

39) $c = 50, \angle B = 21^\circ$

Answer

$a = 46.6790, b = 17.9184$

40) $a = 30, \angle A = 27^\circ$

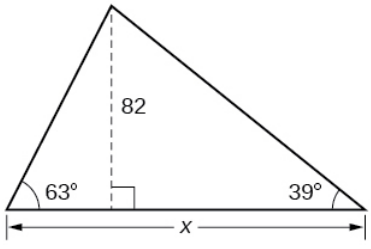
41) $b = 3.5, \angle A = 78^\circ$

Answer

$a = 16.4662, c = 16.8341$

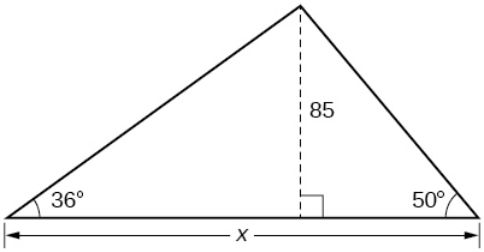
Extensions

42) Find x.



43) Find x.

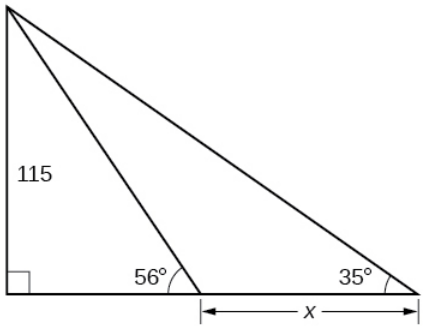
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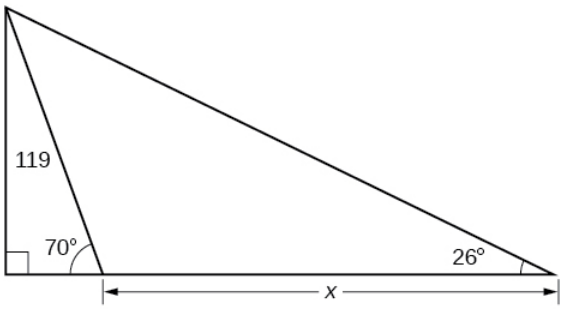
Answer

188.3159

44) Find x.



45) Find x.



Answer

200.6737

46) A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 36°, and that the angle of depression to the bottom of the tower is 23°. How tall is the tower?

47) A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 43°, and that the angle of depression to the bottom of the tower is 31°. How tall is the tower?

Answer

498.3471ft

48) A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 15°, and that the angle of depression to the bottom of the tower is 2°. How far is the person from the monument?

49) A 400-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 18°, and that the angle of depression to the bottom of the monument is 3°. How far is the person from the monument?

Answer

1060.09ft

50) There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40°. From the same location, the angle of elevation to the top of the antenna is measured to be 43°. Find the height of the antenna.

51) There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be 36°. From the same location, the angle of elevation to the top of the lightning rod is measured to be 38°. Find the height of the lightning rod.

Answer

27.372ft

Real-World Applications

52) A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

53) A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

Answer

22.6506ft

54) The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

55) The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

Answer

368.7633ft

56) Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be 60°, how far from the base of the tree am I?

5.5 Arcs, Angles, and Trig Function Values

Exercise $2.E.1$

In each of the following, the coordinates of a point P on the terminal side of an angle θ are given. For each of the following:

Plot the point P in a coordinate system and draw the terminal side of the angle.

Determine the radius r of the circle centered at the origin that passes through the point P.

Determine the values of the six trigonometric functions of the angle θ.

1. $P(3,3)$
2. $P(5,8)$
3. $P(-2,-2)$
4. $P(5,-2)$
5. $P(-1,-4)$
6. $P(2\sqrt{3},2)$
7. $P(-3,4)$
8. $P(3,-3\sqrt{3})$
9. $P(2,-1)$

Answer

(a) We see that $r = \sqrt{3^2 + 3^2} = \sqrt{18}$. So

$$\cos(\theta) = \frac{3}{\sqrt{18}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \tag{2.E.1}$$

$$\tan(\theta) = \frac{3}{3} = 1 \tag{2.E.2}$$

$$\sec(\theta) = \frac{\sqrt{18}}{3} = \sqrt{2} \tag{2.E.3}$$

$$\sin(\theta) = \frac{3}{\sqrt{18}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \tag{2.E.4}$$

$$\cot(\theta) = \frac{3}{3} = 1 \tag{2.E.5}$$

$$\csc(\theta) = \frac{\sqrt{18}}{3} = \sqrt{2} \tag{2.E.6}$$

(b)

We see that $r = \sqrt{5^2 + 8^2} = \sqrt{89}$. So

$$\cos(\theta) = \frac{5}{\sqrt{89}} \tag{2.E.7}$$

$$\tan(\theta) = \frac{8}{5} \tag{2.E.8}$$

$$\sec(\theta) = \frac{\sqrt{89}}{5} \tag{2.E.9}$$

$$\sin(\theta) = \frac{8}{\sqrt{89}} \tag{2.E.10}$$

$$\cot(\theta) = \frac{5}{8} = 1 \tag{2.E.11}$$

$$\csc(\theta) = \frac{\sqrt{89}}{8} \tag{2.E.12}$$

(e)

We see that $r = \sqrt{(-1)^2 + (-4)^2} = \sqrt{26}$. So

$$\cos(\theta) = \frac{-3}{\sqrt{26}} \tag{2.E.13}$$

$$\tan(\theta) = 4 \tag{2.E.14}$$

$$\sec(\theta) = -\frac{\sqrt{26}}{4} \tag{2.E.15}$$

$$\sin(\theta) = \frac{-3}{\sqrt{26}} \tag{2.E.16}$$

$$\cot(\theta) = \frac{1}{4} = 1 \tag{2.E.17}$$

$$\csc(\theta) = -\sqrt{26} \tag{2.E.18}$$

Exercise 2.E.2

For each of the following, draw the terminal side of the indicated angle on a coordinate system and determine the values of the six trigonometric functions of that angle

1. The terminal side of the angle α is in the first quadrant and $\sin(\alpha) = \dfrac{1}{\sqrt{3}}$
2. The terminal side of the angle β is in the second quadrant and $\cos(\beta) = -\dfrac{2}{3}$
3. The terminal side of the angle γ is in the second quadrant and $\tan(\gamma) = -\dfrac{1}{2}$
4. The terminal side of the angle θ is in the second quadrant and $\sin(\theta) = \dfrac{1}{3}$

Answer

2. We first use the Pythagorean Identity and obtain $\sin^2(\beta) = \dfrac{5}{9}$. Since the terminal side of β is in the second quadrant, $\sin(\beta) = \dfrac{\sqrt{5}}{3}$. In addition,

$$\tan(\beta) = -\frac{\sqrt{5}}{2} \tag{2.E.19}$$

$$\cot(\beta) = -\frac{2}{\sqrt{5}} \tag{2.E.20}$$

$$\sec(\beta) = -\frac{3}{2} \tag{2.E.21}$$

$$\csc(\beta) = \frac{3}{\sqrt{5}} \tag{2.E.22}$$

Exercise 2.E.3

For each of the following, determine an approximation for the angle θ in degrees (to three decimal places) when $0^\circ \leq \theta \leq 360^\circ$.

1. The point $(3, 5)$ is on the terminal side of θ.
2. The point $(2, -4)$ is on the terminal side of θ.
3. $\sin(\theta) = \dfrac{2}{3}$ and the terminal side of θ is in the second quadrant.
4. $\sin(\theta) = -\dfrac{2}{3}$ and the terminal side of θ is in the fourth quadrant.
5. $\cos(\theta) = -\dfrac{1}{4}$ and the terminal side of θ is in the second quadrant.
6. $\cos(\theta) = -\dfrac{3}{4}$ and the terminal side of θ is in the third quadrant.

Answer

3. Since the terminal side of θ is in the second quadrant, θ is not the inverse sine of $\dfrac{2}{3}$. So we let $\alpha = \arcsin(\dfrac{2}{3})$. Using α as the reference angle, we then see that

$$\theta = 180^\circ - \arcsin(\frac{2}{3}) \approx 138.190^\circ \tag{2.E.23}$$

5. $\theta = \arccos(-\frac{1}{4}) \approx 104.478^\circ$, or use \(\alpha\ = \arccos(\dfrac{1}{4})) for the reference angle.

$$\theta = 180^\circ - \arccos(\frac{1}{4}) \approx 104.478^\circ. \tag{2.E.24}$$

Exercise $2.E.4$

For each of the angles in Exercise (3), determine the radian measure of θ if $0 \leq \theta \leq 2\pi$.

Answer

3. $\theta = \pi - \arcsin(\frac{2}{3}) \approx 2.142.$

5. $\theta = \pi - \arccos(\frac{1}{4}) \approx 1.823.$

Exercise $2.E.5$

One angle in a right triangle is 55° and the side opposite that angle is 10 feet long. Determine the length of the other side, the length of the hypotenuse, and the measure of the other acute angle.

Exercise $2.E.6$

One angle in a right triangle is 37.8° and the length of the hypotenuse is 25 inches. Determine the length of the other two sides of the right triangle.

Exercise $2.E.7$

One angle in a right triangle is $27^\circ 12'$ and the length of the side adjacent to this angle is 4 feet. Determine the other acute angle in the triangle, the length of the side opposite this angle, and the length of the hypotenuse.

Note: The notation means that the angle is 27 degrees, 12 seconds. Remember that 1 second is $\frac{1}{60}$ of a degree.

Answer

The other acute angle is $64^\circ 48'$.

The side opposite the $27^\circ 12'$ angle is $4\tan(27^\circ 12') \approx 2.056$ feet.

The hypotenuse is $\frac{4}{\cos(27^\circ 12')} \approx 4.497$ feet.

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Note that the Pythagorean Theorem can be used to check the results by showing that $4^2 + 2.056^2 \approx 4.497^2$. The check will not be exact because the $2.056$and $4.497$are approximations of the exact values.

Exercise 2.E. 8

If we only know the measures of the three angles of a right triangle, explain why it is not possible to determine the lengths of the sides of this right triangle.

Exercise 2.E. 9

Suppose that we know the measure θ of one of the acute angles in a right triangle and we know the length x of the side opposite the angle θ. Explain how to determine the length of the side adjacent to the angle θ and the length of the hypotenuse.

Exercise 2.E. 10

In the diagram to the right, determine the values of a, b, and h to the nearest thousandth.
The given values are:

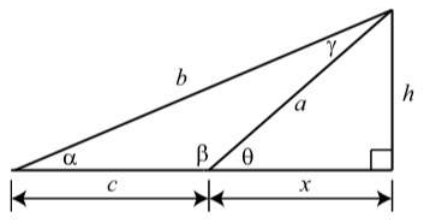


Figure 2.E. 1

$$\alpha = 23^\circ \tag{2.E.25}$$

$$\beta = 140^\circ \tag{2.E.26}$$

$$c = 8 \tag{2.E.27}$$

Answer

We first note that $\theta = 40^\circ$. We use the following two equations to determine x.

$$\tan(\alpha) = \frac{h}{c+x} \tag{2.E.28}$$

$$\tan(x) = \frac{h}{x} \tag{2.E.29}$$

Substituting $h = x\tan(\theta)$ into the first equation and solving for x gives

$$x = \frac{c\tan(\alpha)}{\tan(\theta) - \tan(\alpha)} \approx 8.190. \tag{2.E.30}$$

We can then use right triangles to obtain $h \approx 6.872\ ft, a \approx 10.691\ ft, and b \approx 17.588\ ft$

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Exercise 2.E.11

A tall evergreen tree has been damaged in a strong wind. The top of the tree is cracked and bent over, touching the ground as if the trunk were hinged. The tip of the tree touches the ground 20 feet 6 inches from the base of the tree (where the tree and the ground meet). The tip of the tree forms an angle of 17 degrees where it touches the ground. Determine the original height of the tree (before it broke) to the nearest tenth of a foot. Assume the base of the tree is perpendicular to the ground.

Exercise 2.E.12

Suppose a person is standing on the top of a building and that she has an instrument that allows her to measure angles of depression. There are two points that are 100 feet apart and lie on a straight line that is perpendicular to the base of the building.

Now suppose that she measures the angle of depression to the closest point to be 35.5° and that she measures the angle of depression to the other point to be 29.8°. Determine the height of the building.page206image32203024

Exercise 2.E.13

A company has a 35 foot ladder that it uses for cleaning the windows in their building. For safety reasons, the ladder must never make an angle of more than 50° with the ground.

1. What is the greatest height that the ladder can reach on the building if the angle it makes with the ground is no more than 50°.
2. Suppose the building is 40 feet high. Again, following the safety guidelines, what length of ladder is needed in order to have the ladder reach the top of the building?

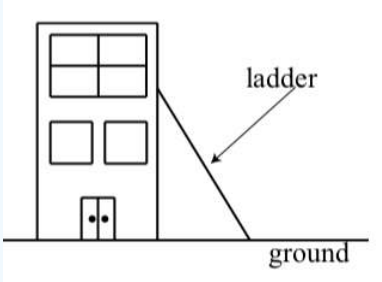


Figure 2.E.2

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2.R: Trigonometric Functions (Review)

5.1: Review Exercises

For the exercises 1-2, convert the angle measures to degrees.

1) $\frac{\pi}{4}$

Answer

45°

2) $-\frac{5\pi}{3}$

For the exercises 3-6, convert the angle measures to radians.

3) -210°

Answer

$-\frac{7\pi}{6}$

4) 180°

5) Find the length of an arc in a circle of radius 7 meters subtended by the central angle of 85°.

Answer

10.385 meters

6) Find the area of the sector of a circle with diameter 32 feet and an angle of $\frac{3\pi}{5}$ radians.

For the exercises 7-8, find the angle between 0° and 360° that is coterminal with the given angle.

7) 420°

Answer

60°

8) -80°

For the exercises 9-10, find the angle between 0 and 2π in radians that is coterminal with the given angle.

9) $-\frac{20\pi}{11}$

Answer

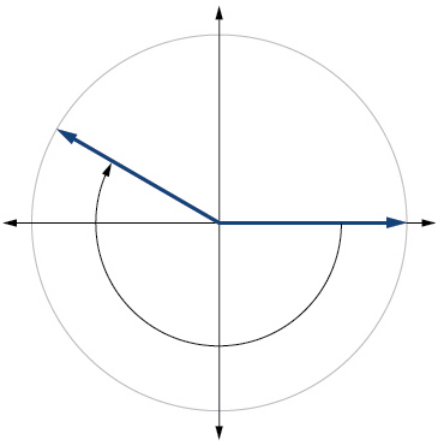
$\frac{2\pi}{11}$

10) $\frac{14\pi}{5}$

For the exercises 11-, draw the angle provided in standard position on the Cartesian plane.

11) -210°

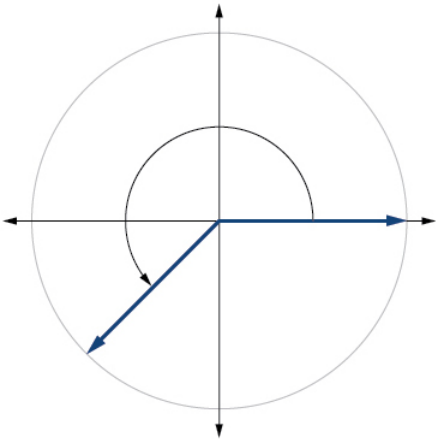
Answer



12) 75°

13) $\frac{5\pi}{4}$

Answer



14) $-\frac{\pi}{3}$

15) Find the linear speed of a point on the equator of the earth if the earth has a radius of $3,960$ miles and the earth rotates on its axis every 24 hours. Express answer in miles per hour.

Answer

1036.73 miles per hour

16) A car wheel with a diameter of 18 inches spins at the rate of 10 revolutions per second. What is the car's speed in miles per hour?

5.2: Review Exercises

1) Find the exact value of $\sin\frac{\pi}{3}$.

Answer

$\frac{\sqrt{3}}{2}$

2) Find the exact value of $\cos\frac{\pi}{4}$.

3) Find the exact value of $\cos\pi$.

Answer

-1

4) State the reference angle for 300°.

5) State the reference angle for $\frac{3\pi}{4}$.

Answer

$\frac{\pi}{4}$

6) Compute cosine of 330°.

7) Compute sine of $\frac{5\pi}{4}$.

Answer

$-\frac{\sqrt{2}}{2}$

8) State the domain of the sine and cosine functions.

9) State the range of the sine and cosine functions.

Answer

$[-1, 1]$

5.3: Review Exercises

For the exercises 1-4, find the exact value of the given expression.

1) $\cos\frac{\pi}{6}$

2) $\tan\frac{\pi}{4}$

Answer

1

3) $\csc\frac{\pi}{3}$

4) $\sec\frac{\pi}{4}$

Answer

$\sqrt{2}$

For the exercises 4-12, use reference angles to evaluate the given expression.

5) $\sec\frac{11\pi}{3}$

6) $\sec 315^\circ$

Answer

$\sqrt{2}$

7) If $\sec(t) = -2.5$, what is the $\sec(-t)$?

8) If $\tan(t) = -0.6$, what is the $\tan(-t)$?

Answer

0.6

9) If $\tan(t) = \frac{1}{3}$, find $\tan(t - \pi)$.

10) If $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sin(t + 2\pi)$.

Answer

$\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$

11) Which trigonometric functions are even?

12) Which trigonometric functions are odd?

Answer

sine, cosecant, tangent, cotangent

5.4: Review Exercises

For the exercises 1-5, use side lengths to evaluate.

1) $\cos\frac{\pi}{4}$

2) $\cot\frac{\pi}{3}$

Answer

$\frac{\sqrt{3}}{3}$

3) $\tan\frac{\pi}{6}$

4) $\cos(\frac{\pi}{2}) = \sin(\_\_^\circ)$

Answer

0

5) $\csc(18^\circ) = \sec(\_\_^\circ)$

For the exercises 6-7, use the given information to find the lengths of the other two sides of the right triangle.

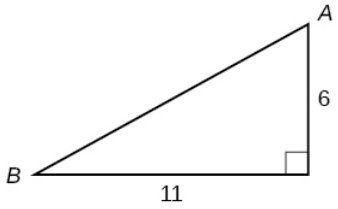
6) $\cos B = \frac{3}{5}, a = 6$

Answer

$b = 8, c = 10$

7) $\tan A = \frac{5}{9}, b = 6$

For the exercises 8-9, use Figure below to evaluate each trigonometric function.



8) $\sin A$

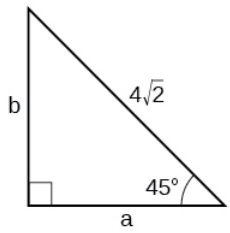
Answer

$$\frac{11\sqrt{157}}{157}$$

9) $\tan B$

For the exercises 10-11, solve for the unknown sides of the given triangle.

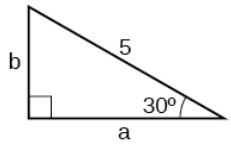
10)



Answer

$a = 4, b = 4$

11)



12) A 15-ft ladder leans against a building so that the angle between the ground and the ladder is 70°. How high does the ladder reach up the side of the building?

Answer

$14.0954\,\text{ft}$

13) The angle of elevation to the top of a building in Baltimore is found to be 4 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

Practice Test

1) Convert $\frac{5\pi}{6}$ radians to degrees.

Answer

150°

2) Convert -620° to radians.

3) Find the length of a circular arc with a radius 12 centimeters subtended by the central angle of 30°.

Answer

6.283 centimeters

4) Find the area of the sector with radius of 8 feet and an angle of $\frac{5\pi}{4}$ radians.

5) Find the angle between 0° and 360° that is coterminal with 375°.

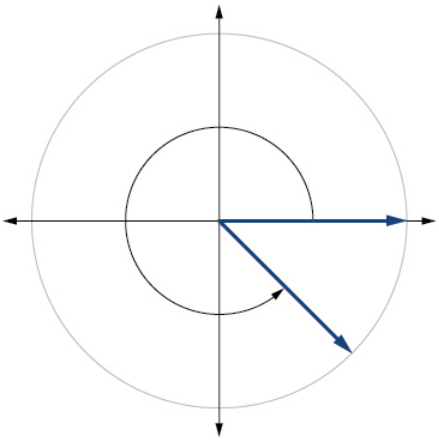
Answer

15°

6) Find the angle between 0 and 2π in radians that is coterminal with $-\frac{4\pi}{7}$.

7) Draw the angle 315° in standard position on the Cartesian plane.

Answer



8) Draw the angle $-\frac{\pi}{6}$ in standard position on the Cartesian plane.

9) A carnival has a Ferris wheel with a diameter of 80 feet. The time for the Ferris wheel to make one revolution is 75 seconds. What is the linear speed in feet per second of a point on the Ferris wheel? What is the angular speed in radians per second?

Answer

3.351 feet per second, $\frac{2\pi}{75}$ radians per second

10) Find the exact value of $\sin\frac{\pi}{6}$.

11) Compute sine of 240°.

Answer

$$-\frac{\sqrt{3}}{2}$$

12) State the domain of the sine and cosine functions.

13) State the range of the sine and cosine functions.

Answer

$$[-1, 1]$$

14) Find the exact value of $\cot\frac{\pi}{4}$.

15) Find the exact value of $\tan\frac{\pi}{3}$.

Answer

$$\sqrt{3}$$

16) Use reference angles to evaluate $\csc\frac{7\pi}{4}$.

17) Use reference angles to evaluate $\tan 210^\circ$.

Answer

$$\frac{\sqrt{3}}{3}$$

18) If $\csc t = 0.68$, what is the $\csc(-t)$?

19) If $\cos t = \frac{\sqrt{3}}{2}$, find $\cos(t - 2\pi)$.

Answer

$$\frac{\sqrt{3}}{2}$$

20) Which trigonometric functions are even?

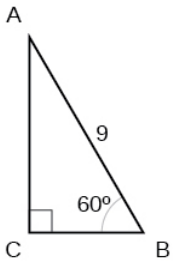
21) Find the missing angle: $\cos\left(\frac{\pi}{6}\right) = \sin(\ )$

Answer

$$\frac{\pi}{3}$$

22) Find the missing sides of the triangle ABC : $\sin B = \frac{3}{4}, c = 12$

23) Find the missing sides of the triangle.



Answer

$$a = \frac{9}{2}, b = \frac{9\sqrt{3}}{2}$$

24) The angle of elevation to the top of a building in Chicago is found to be 9 degrees from the ground at a distance of 2000 feet from the base of the building. Using this information, find the height of the building.

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CHAPTER OVERVIEW

3: Periodic Functions

In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

3.0: Prelude to Periodic Functions
3.1: Graphs of the Sine and Cosine Functions
3.2: Graphs of the Other Trigonometric Functions
3.3: Inverse Trigonometric Functions
3.E: Periodic Functions (Exercises)
3.R: Periodic Functions (Review)

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3.0: Prelude to Periodic Functions

Each day, the sun rises in an easterly direction, approaches some maximum height relative to the celestial equator, and sets in a westerly direction. The celestial equator is an imaginary line that divides the visible universe into two halves in much the same way Earth's equator is an imaginary line that divides the planet into two halves. The exact path the sun appears to follow depends on the exact location on Earth, but each location observes a predictable pattern over time.



Figure 3.0.1: (credit: "Maxxer_", Flickr)

The pattern of the sun's motion throughout the course of a year is a periodic function. Creating a visual representation of a periodic function in the form of a graph can help us analyze the properties of the function. In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

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3.1: Graphs of the Sine and Cosine Functions

Learning Objectives

- Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- Use phase shifts of sine and cosine curves.

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.



Figure 3.1.1: Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

Light waves can be represented graphically by the sine function. In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. Table 3.1.1 lists some of the values for the sine function on a unit circle.

Table 3.1.1

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the *x*-axis gives the shape of the sine function. See Figure 3.1.2.

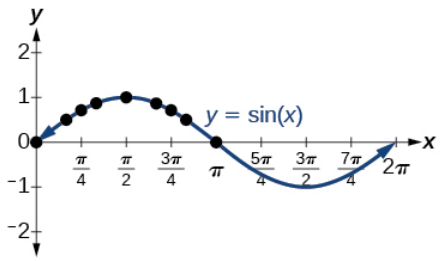


Figure 3.1.2: The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See Figure 3.1.3.

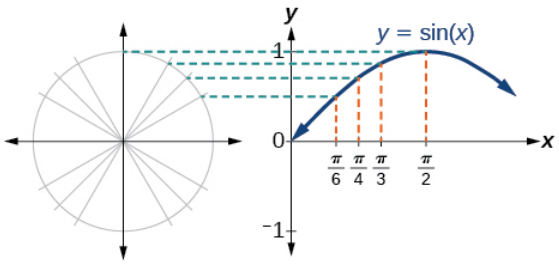


Figure 3.1.3: Plotting values of the sine function

Now let's take a similar look at the **cosine function**. Again, we can create a table of values and use them to sketch a graph. Table 3.1.2 lists some of the values for the cosine function on a unit circle.

Table 3.1.2

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

As with the sine function, we can plots points to create a graph of the cosine function as in Figure 3.1.4.

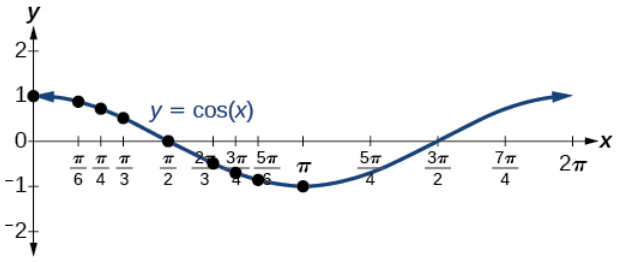


Figure 3.1.4: The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific **horizontal shift**, P, results in a function equal to the original function: $f(x+P) = f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P > 0$ the **period** of the function. Figure 3.1.5 shows several periods of the sine and cosine functions.

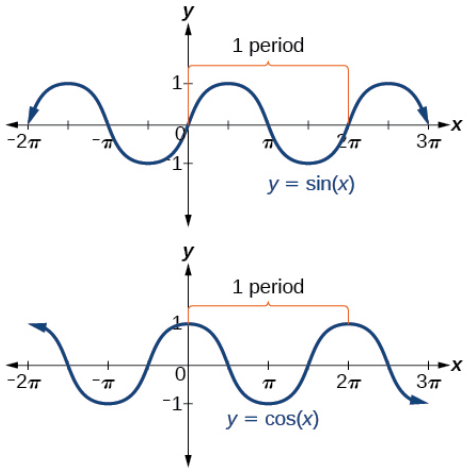


Figure 3.1.5

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in Figure 3.1.6, the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin\ x$. Now we can clearly see this property from the graph.

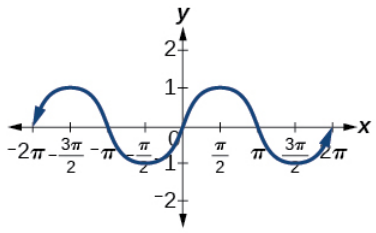


Figure 3.1.6: Odd symmetry of the sine function

Figure 3.1.7 shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos\ x$.

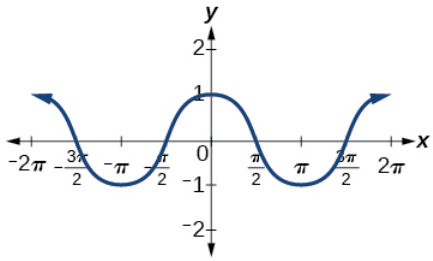


Figure 3.1.7: Even symmetry of the cosine function

CHARACTERISTICS OF SINE AND COSINE FUNCTIONS

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos\ x$ is symmetric about they- y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or **cosine function** is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D \tag{3.1.1}$$

and

$$y = A\cos(Bx - C) + D \tag{3.1.2}$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \dfrac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π,which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π

and the graph is compressed. If $f(x)=\sin\left(\frac{x}{2}\right)$, then $B=\frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in Figure 3.1.8 how the period is indirectly related to $|B|$.

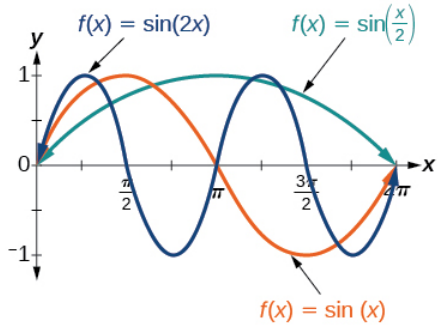


Figure 3.1.8

PERIOD OF SINUSOIDAL FUNCTIONS

If we let $C=0$ and $D=0$ in the general form equations of the sine and cosine functions, we obtain the forms

- $y=A\sin(Bx)$
- $y=A\cos(Bx)$

The period is $\frac{2\pi}{|B|}$.

Example 3.1.1: Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x)=\sin\left(\frac{\pi}{6}x\right)$.

Solution

Let's begin by comparing the equation to the general form $y=A\sin(Bx)$.

In the given equation, $B=\frac{\pi}{6}$, so the period will be

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{6}} \\ &= 2\pi\cdot\frac{6}{\pi} \\ &= 12 \end{aligned}$$

Exercise 3.1.1

Determine the period of the function $g(x)=\cos(\frac{x}{3})$.

Answer

6π

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the

graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the x-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4sinx$ is twice the amplitude of

$$f(x) = 2 \sin x$$

If $|A| < 1$, the function is compressed. Figure 3.1.9 compares several sine functions with different amplitudes.

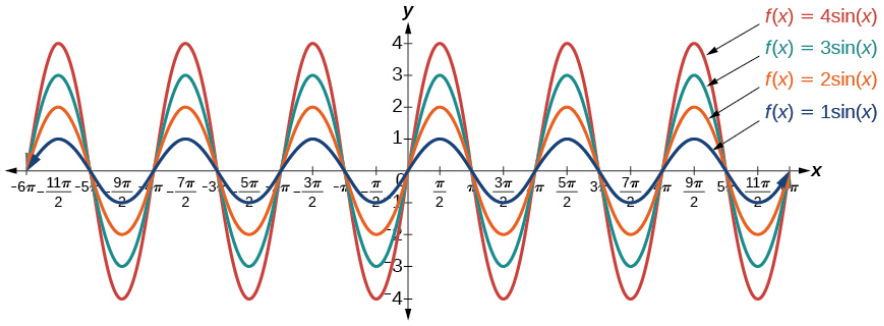


Figure 3.1.9

AMPLITUDE OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A \sin(Bx) \text{ and } y = A \cos(Bx) \tag{3.1.3}$$

The amplitude is A, and the vertical height from the midline is $|A|$. In addition, notice in the example that

$$|A| = amplitude = \frac{1}{2} \mid maximum - minimum| \tag{3.1.4}$$

Example 3.1.2: Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4 \sin(x)$? Is the function stretched or compressed vertically?

Solution

Let's begin by comparing the function to the simplified form $y = A \sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis

The negative value of A results in a reflection across the x-axis of the sine function, as shown in Figure 3.1.10.

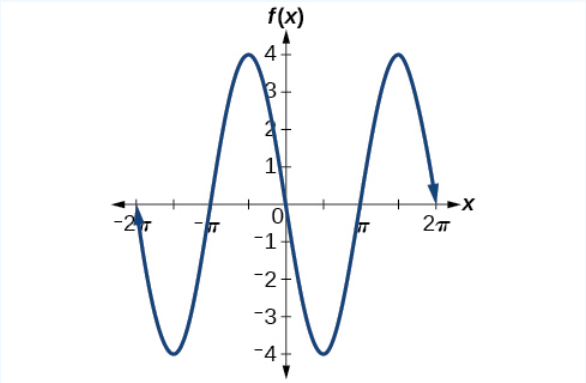


Figure 3.1.10

Exercise 3.1.2

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2} \sin(x)$? Is the function stretched or compressed vertically?

Answer

$\frac{1}{2}$ compressed

Analyzing Graphs of Variations of $y = \sin\ x$ and $y = \cos\ x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \qquad \text{and} \qquad y = A\cos(Bx - C) + D \tag{3.1.5}$$

or

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \qquad \text{and} \qquad y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D \tag{3.1.6}$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. Figure 3.1.11 shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

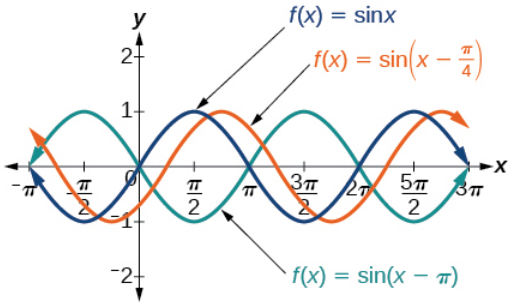


Figure 3.1.11

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See Figure 3.1.12 The function $y = \cos(x) + D$ has its midline at $y = D$.

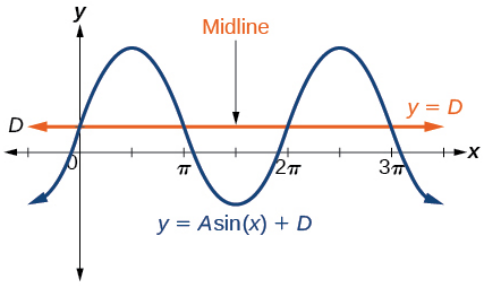


Figure 3.1.12

Any value of D other than zero shifts the graph up or down. Figure 3.1.13compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

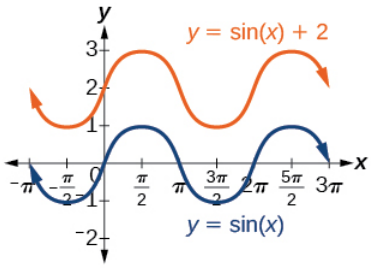


Figure 3.1.13

VARIATIONS OF SINE AND COSINE FUNCTIONS

Given an equation in the form $f(x)=A\sin(Bx-C)+D$ or $f(x)=A\cos(Bx-C)+D$, $\frac{C}{D}$ is the **phase shift** and D is the **vertical shift.**

Example 3.1.3: Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x)=\sin\left(x+\frac{\pi}{6}\right)-2$.

Solution

Let's begin by comparing the equation to the general form $y=A\sin(Bx-C)+D$.

In the given equation, notice that $B=1$ and $C=-\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned}\frac{C}{B}&=-\frac{\frac{\pi}{6}}{1}\\&=-\frac{\pi}{6}\end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x)=\sin(x+\frac{\pi}{6})-2$ can be rewritten as $f(x)=\sin\left(x-\left(-\frac{\pi}{6}\right)\right)-2$. If the value of C is negative, the shift is to the left.

Exercise 3.1.3

Determine the direction and magnitude of the phase shift for $f(x)=3\cos\left(x-\frac{\pi}{2}\right)$.

Answer

$\frac{\pi}{2}$; right

Example 3.1.4: Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x)=\cos(x)-3$.

Solution

Let's begin by comparing the equation to the general form $y=A\cos(Bx-C)+D$.

In the given equation, $D=-3$ so the shift is 3 units downward.

Exercise 3.1.4

Determine the direction and magnitude of the vertical shift for $f(x)=3\sin(x)+2$.

Answer

2 units up

How to: Given a sinusoidal function in the form $f(x)=A\sin(Bx-C)+D$,identify the midline, amplitude, period, and phase shift

1. Determine the amplitude as $|A|$.
2. Determine the period as $P=\frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y=D$.

Example 3.1.5: Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\dfrac{C}{B} = \dfrac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis

Inspecting the graph, we can determine that the period is π, the midline is $y = 1$, and the amplitude is 3. See Figure $3.1.14$

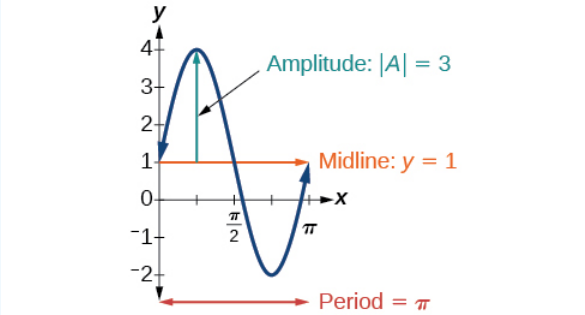


Figure 3.1.14

Exercise 3.1.5

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Answer

midline: $y = 0$; amplitude: $|A| = \frac{1}{2}$; period: $P = \frac{2\pi}{|B|} = 6\pi$; phase shift: $\frac{C}{B} = \pi$

Example 3.1.6: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in Figure $3.1.15$

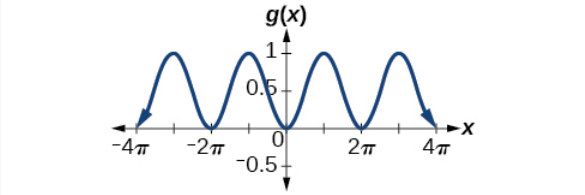


Figure 3.1.15

Solution

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin(Bx - C) + D$$

$$y = A\cos(Bx - C) + D$$

The graph could represent either a sine or a **cosine function** that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the x-axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

Exercise 3.1.6

Determine the formula for the sine function in Figure 3.1.16.

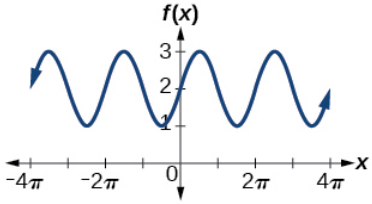


Figure 3.1.16

Answer

$f(x) = \sin(x) + 2$

Example 3.1.7: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in Figure 3.1.17.

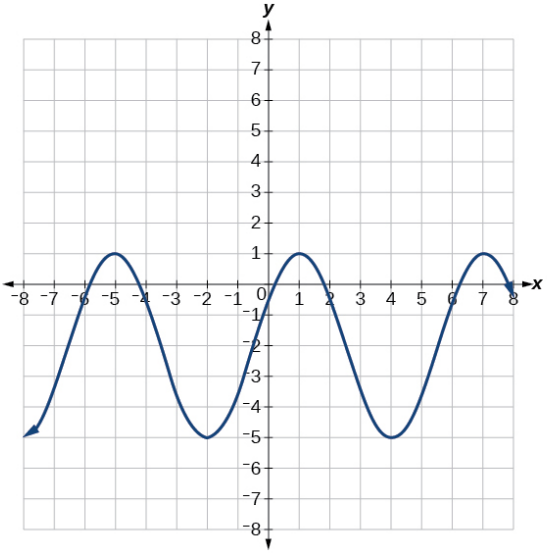


Figure 3.1.17

Solution

With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D = -2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A| = 3$.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$,or from the distance between the lowest points. Therefore, $P = \dfrac{2\pi}{|B|} = 6$. Using the positive value for B,we find that

$$\begin{aligned} B &= \frac{2\pi}{P} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

So far, our equation is either $y = 3\sin\left(\frac{\pi}{3}x - C\right) - 2$ or $y = 3\cos\left(\frac{\pi}{3}x - C\right) - 2$ .For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

$$\begin{aligned} y &= 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2 \qquad \text{or} \\ y &= -3\cos\left(\frac{\pi}{3}x + \frac{2\pi}{3}\right) - 2 \end{aligned}$$

Again, these functions are equivalent, so both yield the same graph.

Exercise 3.1.7

Write a formula for the function graphed in Figure 3.1.18.

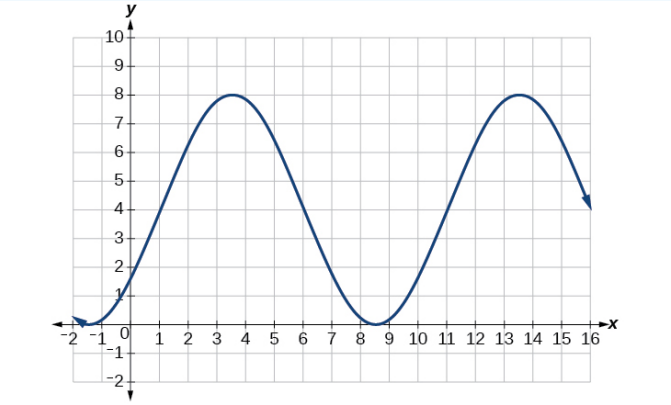


Figure 3.1.18

Answer

two possibilities: $y = 4\sin\left(\frac{\pi}{5}x - \frac{\pi}{5}\right) + 4$ or $y = -4\sin\left(\frac{\pi}{5}x + \frac{4\pi}{5}\right) + 4$

Graphing Variations of $y = \sin\ x$ and $y = \cos\ x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \dfrac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \dfrac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the *x*-axis at $x = \dfrac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \dfrac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the *x*-axis at $x = \dfrac{\pi}{2|B|}$.

Example 3.1.8: Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\dfrac{\pi x}{2}\right)$.

Solution

Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

- *Step 1.* We can see from the equation that $A = -2$, so the amplitude is 2.

  $|A| = 2$

- *Step 2.* The equation shows that $B = \dfrac{\pi}{2}$, so the period is

$$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$

- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4.* The x-intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. Figure 3.1.19 shows the graph of the function.

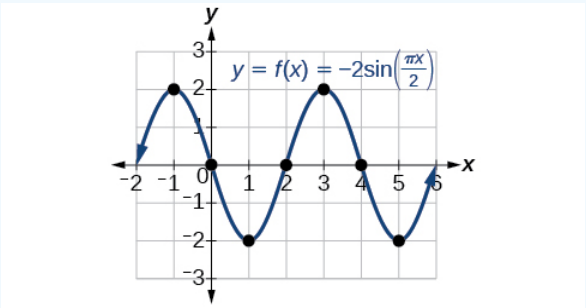


Figure 3.1.19

Exercise 3.1.8

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

Answer

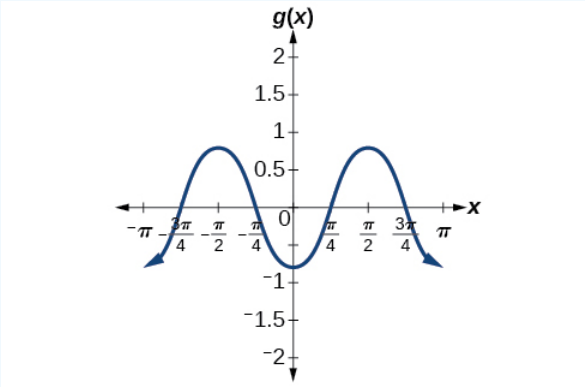


Figure 3.1.20

midline: $y = 0$; amplitude: $|A| = 0.8$; period: $P = \dfrac{2\pi}{|B|} = \pi$; phase shift: $\dfrac{C}{B} = 0$ or none

How to: Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \dfrac{2\pi}{|B|}$.
4. Identify the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\dfrac{C}{B}$ and up or down by D.

Example 3.1.9: Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\dfrac{\pi}{4x} - \dfrac{\pi}{4}\right)$.

Solution

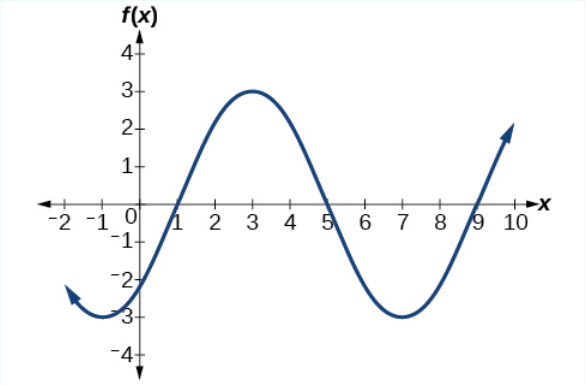


Figure 3.1.21: A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

- *Step 1.* The function is already written in general form: $f(x) = 3\sin\left(\dfrac{\pi}{4x} - \dfrac{\pi}{4}\right)$ .This graph will have the shape of a sine function, starting at the midline and increasing to the right.
- *Step 2.* $|A| = |3| = 3$. The amplitude is 3.

- *Step 3.* Since $|B| = |\frac{\pi}{4}| = \frac{\pi}{4}$, we determine the period as follows.

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{4}} \\ &= 2\pi \cdot \frac{4}{\pi} \\ &= 8 \end{aligned}$$

The period is 8.

- *Step 4.* Since $C = \frac{\pi}{4}$, the phase shift is

$$\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1 \tag{3.1.7}$$

.

The phase shift is 1 unit.

- *Step 5.* Figure 3.1.21 shows the graph of the function.

Exercise 3.1.9

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Answer

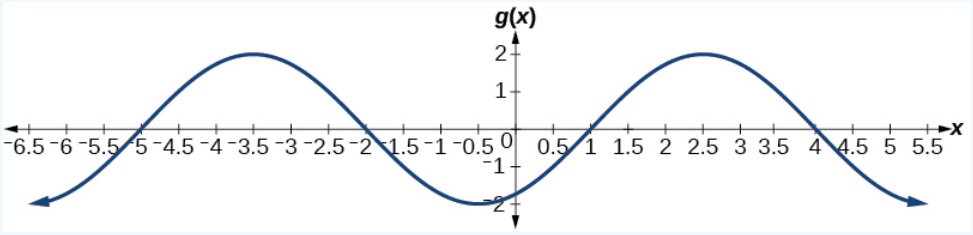


Figure 3.1.22

midline: $y = 0$; amplitude: $|A| = 2$; period: $P = \frac{2\pi}{|B|} = 6$; phase shift: $\frac{C}{B} = -\frac{1}{2}$

Example 3.1.10: Identifying the Properties of a Sinusoidal Function

Given $y = -2cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution

Begin by comparing the equation to the general form.

$$y = A\cos(Bx - C) + D$$

- *Step 1.* The function is already written in general form.
- *Step 2.* Since $A = -2$, the amplitude is $|A| = 2$.
- *Step 3.* $|B| = \frac{\pi}{2}$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$. The period is 4.

- *Step 4.* $C = -\pi$,so we calculate the phase shift as $\frac{C}{B} = -\frac{\pi}{\frac{\pi}{2}} = -\pi \cdot \frac{2}{\pi} = -2$. The phase shift is -2.
- *Step 5.* $D = 3$,so the midline is $y = 3$, and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the x-axis.

Figure $3.1.23$shows one cycle of the graph of the function.

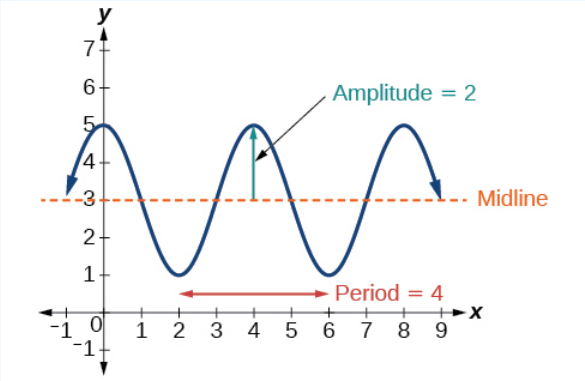


Figure 3.1.23

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter,**circular motion** can be modeled using either the sine or **cosine function**.

Example 3.1.11: Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in Figure $3.1.24$

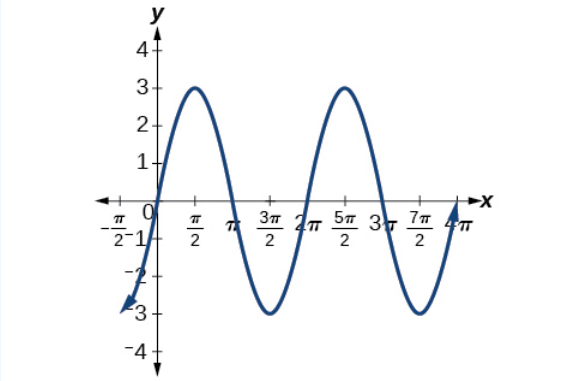


Figure 3.1.24

Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi$,...Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

Exercise 3.1.10

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Answer

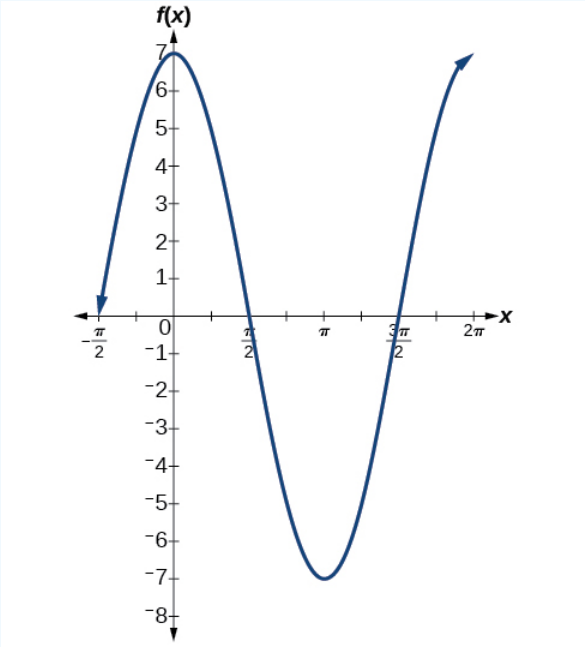


Figure 3.1.25

Example 3.1.12: Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in Figure 3.1.26. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

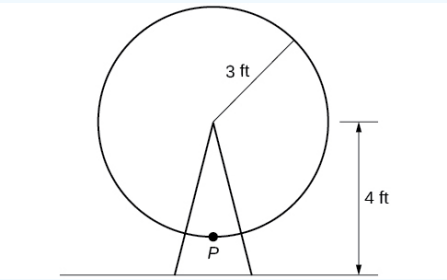


Figure 3.1.26

Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure 3.1.27.

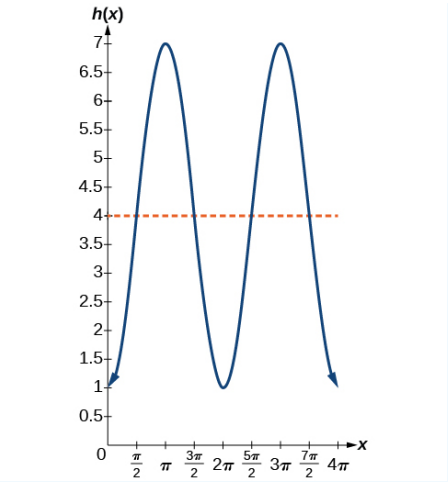


Figure 3.1.27

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a **sine function** starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

Exercise 3.1.11

A weight is attached to a spring that is then hung from a board, as shown in Figure 3.1.28. As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

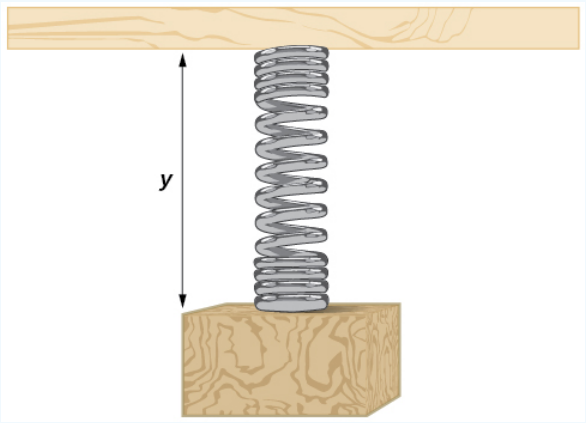


Figure 3.1.28

Answer

$y = 3\cos(x) - 4$

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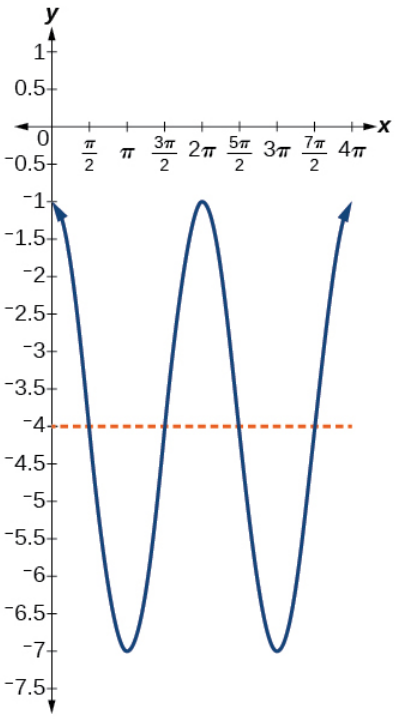


Figure 3.1.29

Example 3.1.13: Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5+2=69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A=67.5$
- Midline: 69.5, so $D=69.5$
- Period:30, so $B=\dfrac{2\pi}{30}=\dfrac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y=-67.5\cos\left(\frac{\pi}{15}t\right)+69.5$$

where t is in minutes and y is measured in meters.

Media

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine
- Translations of Sine and Cosine
- Graphing Sine and Cosine Transformations
- Graphing the Sine Function

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$ $f(x) = A\cos(Bx - C) + D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the *y*-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \dfrac{2\pi}{|B|}$. See Example 3.1.1.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See Example 3.1.2.
- The value $\dfrac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See Example 3.1.3.
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See Example 3.1.4.
- Combinations of variations of sinusoidal functions can be detected from an equation. See Example 3.1.5.
- The equation for a sinusoidal function can be determined from a graph. See Example 3.1.6 and Example 3.1.7.
- A function can be graphed by identifying its amplitude and period. See Example 3.1.8 and Example 3.1.9.
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See Example 3.1.10.
- Sinusoidal functions can be used to solve real-world problems. See Example 3.1.11, Example 3.1.12, and Example 3.1.13.

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3.2: Graphs of the Other Trigonometric Functions

Learning Objectives

- Analyze the graph of $y = \tan x$.
- Graph variations of $y = \tan x$.
- Analyze the graphs of $y = \sec x$ and $y = \csc x$.
- Graph variations of $y = \sec x$ and $y = \csc x$.
- Analyze the graph of $y = \cot x$.
- Graph variations of $y = \cot x$.

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan x = \frac{\sin x}{\cos x} \tag{3.2.1}$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned} \tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent} \\ &= \frac{-\sin x}{\cos x} && \text{Sine is an odd function, cosine is even} \\ &= -\frac{\sin x}{\cos x} && \text{The quotient of an odd and an even function is odd} \\ &= -\tan x && \text{Definition of tangent} \end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table 3.2.1.

Table 3.2.1

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in Table 3.2.2.

Table 3.2.2

x	1.3	1.5	1.55	1.56
$\tan x$	3.6	14.1	48.1	92.6

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in Table 3.2.3.

Table 3.2.3

x	−1.3	−1.5	−1.55	−1.56
$\tan x$	−3.6	−14.1	−48.1	−92.6

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. Figure 3.2.1 represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

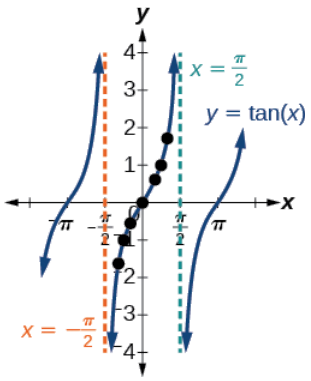


Figure 3.2.1: Graph of the tangent function

Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the *tangent function* can be described by a general equation.

$$y = A\tan(Bx)$$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

FEATURES OF THE GRAPH OF $Y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is all real numbers x,where $x \neq \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ where k is an integer.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the **tangent function** to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single **period** of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is

then the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\frac{P}{2}$ where $P = \frac{\pi}{B}$. On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x = -\frac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x = \frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right) = A\tan\left(B\frac{P}{4}\right) = A\tan\left(B\frac{\pi}{4B}\right) = A$$

because $\tan\left(\frac{\pi}{4}\right) = 1$.

Howto: Given the function $f(x) = A\tan(Bx)$, graph one period.

1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x = -\frac{P}{2}$ and $x = \frac{P}{2}$.
4. For $A > 0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A < 0$).
5. Plot reference points at $\left(\frac{P}{4}, A\right)$, $(0, 0)$, and $\left(-\frac{P}{4}, -A\right)$, and draw the graph through these points.

Example 3.2.1: Sketching a Compressed Tangent

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

Solution

First, we identify A and B.

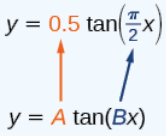


Figure 3.2.2

Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$, so the asymptotes are at $x = \pm 1$. At a quarter period from the origin, we have

$$\begin{aligned} f(0.5) &= 0.5\tan\left(\frac{0.5\pi}{2}\right) \\ &= 0.5\tan\left(\frac{\pi}{4}\right) \\ &= 0.5 \end{aligned}$$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$,and $(-0.5, -0.5)$. The only inflection point is at the origin. Figure 3.2.3 shows the graph of one period of the function.

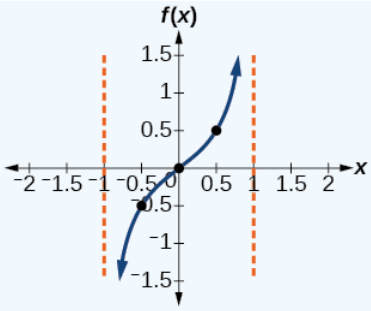


Figure 3.2.3

? Exercise 3.2.1

Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$.

Answer

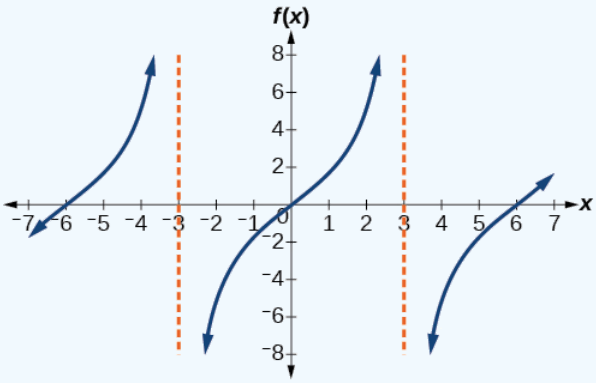


Figure 3.2.4

Graphing One Period of a Shifted Tangent Function

Now that we can graph a **tangent function** that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$$f(x) = A\tan(Bx - C) + D$$

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

FEATURES OF THE GRAPH OF $Y = A\tan(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\tan(Bx)$ is and odd function because it is the qoutient of odd and even functions(sin and cosine perspectively).

Howto: Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the **stretching/compressing** factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Example 3.2.2: Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

Solution

- *Step 1.* The function is already written in the form $y = A\tan(Bx - C) + D$.
- *Step 2.*\(A=−2\), so the stretching factor is $|A| = 2$.
- *Step 3.* \(B=\pi\), so the period is $P = \dfrac{\pi}{|B|} = \dfrac{\pi}{pi} = 1$.
- *Step 4.* \(C=−\pi\), so the phase shift is $CB = \dfrac{-\pi}{\pi} = -1$.
- *Step 5-7.* The asymptotes are at $x = -\dfrac{3}{2}$ and $x = -\dfrac{1}{2}$ and the three recommended reference points are $(-1.25, 1)$, $(-1, -1)$, and $(-0.75, -3)$. The graph is shown in Figure 3.2.5.

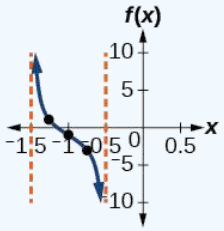


Figure 3.2.5

Analysis

Note that this is a decreasing function because $A < 0$.

Exercise 3.2.2

How would the graph in Example 3.2.2 look different if we made $A = 2$ instead of -2?

Answer

It would be reflected across the line $y = -1$, becoming an increasing function.

Howto: Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or *x*-intercepts.
2. Write $f(x) = A\tan\left(\dfrac{\pi}{P}x\right)$.
3. Determine a convenient point $(x, f(x))$ on the given graph and use it to determine A.

Example 3.2.3: Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in Figure 3.2.6.

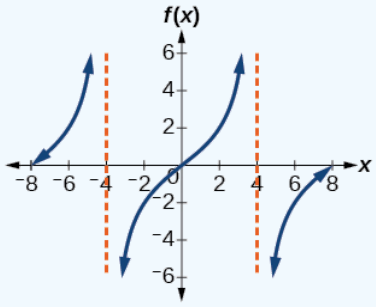


Figure 3.2.6: A stretched tangent function

Solution

The graph has the shape of a tangent function.

- *Step 1.* One cycle extends from -4 to 4, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{\pi}{P} = \frac{\pi}{8}$.
- *Step 2.* The equation must have the form $f(x) = A\tan\left(\frac{\pi}{8}x\right)$.
- *Step 3.* To find the vertical stretch A,we can use the point $(2, 2)$.

$$\begin{aligned} 2 &= A\tan\left(\frac{\pi}{8}\cdot 2\right) \\ &= A\tan\left(\frac{\pi}{4}\right) \end{aligned}$$

Because $\tan\left(\frac{\pi}{4}\right) = 1$, $A = 2$.

This function would have a formula $f(x) = 2\tan\left(\frac{\pi}{8}x\right)$.

Exercise 3.2.3

Find a formula for the function in Figure 3.2.7.

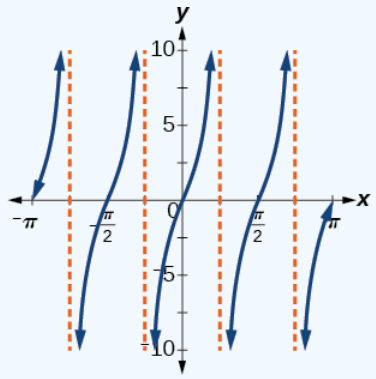


Figure 3.2.7

Answer

$g(x) = 4\tan(2x)$

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

The secant was defined by the reciprocal identity $sec\, x = \dfrac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\dfrac{\pi}{2}$, $\dfrac{3\pi}{2}$ etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure 3.2.8. The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

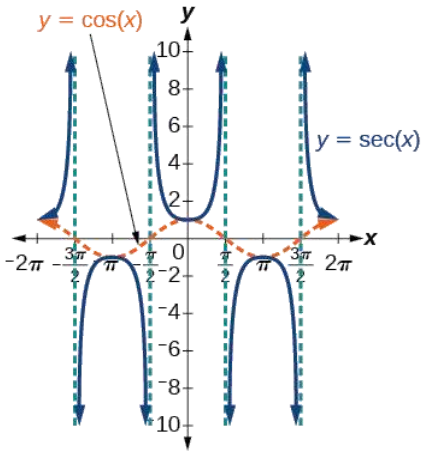


Figure 3.2.8: Graph of the secant function, $f(x) = \sec x = \dfrac{1}{\cos x}$

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

FEATURES OF THE GRAPH OF $Y = A\sec(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the **cosecant** is defined by the reciprocal identity $\csc x = \dfrac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure 3.2.7. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the **cosecant function** decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in Figure 3.2.9, is similar to the graph of secant.

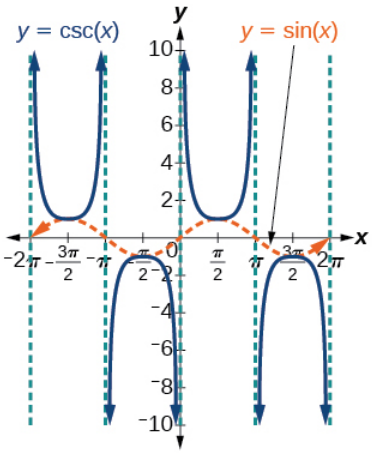


Figure 3.2.9: The graph of the cosecant function, $f(x) = \csc x = \frac{1}{\sin x}$

FEATURES OF THE GRAPH OF $Y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equations become the following.

$$y = A\sec(Bx - C) + D \tag{3.2.2}$$

$$y = A\csc(Bx - C) + D \tag{3.2.3}$$

FEATURES OF THE GRAPH OF $Y = A\sec(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- There is no amplitude.

- $y = A\sec(Bx)$ is an even function because cosine is an even function.

FEATURES OF THE GRAPH OF $Y = A\csc(Bx - C) + D$

1. The stretching factor is $|A|$.
2. The period is $\dfrac{2\pi}{|B|}$.
3. The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an integer.
4. The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
5. The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
6. There is no amplitude.
7. $y = A\csc(Bx)$ is an odd function because sine is an odd function.

HOWTO: Given a function of the form $y = A\sec(Bx)$, graph one period

1. Express the function given in the form $y = A\sec(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos(Bx)$.
5. Use the reciprocal relationship between $y = \cos\ x$ and $y = \sec\ x$ to draw the graph of $y = A\sec(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 3.2.4: Graphing a Variation of the Secant Function

Graph one period of $f(x) = 2.5\sec(0.4x)$

Solution

- *Step 1.* The given function is already written in the general form, $y = A\sec(Bx)$.
- *Step 2.* $A = 2.5$ so the stretching factor is 2.5.
- *Step 3.* $B = 0.4$ so $P = \dfrac{2\pi}{0.4} = 5\pi$. The period is 5π units.
- *Step 4.* Sketch the graph of the function $g(x) = 2.5\cos(0.4x)$.
- *Step 5.* Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.
- *Steps 6–7.* Sketch two asymptotes at $x = 1.25\pi$ and $x = 3.75\pi$. We can use two reference points, the local minimum at $(0, 2.5)$ and the local maximum at $(2.5\pi, -2.5)$ Figure $3.2.10$ shows the graph.

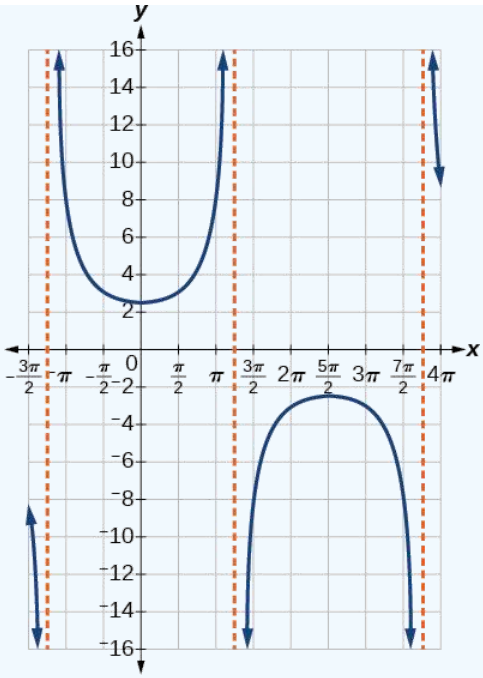


Figure 3.2.10

Exercise 3.2.4

Graph one period of $f(x) = -2.5\sec(0.4x)$.

Answer

This is a vertical reflection of the preceding graph because A is negative.

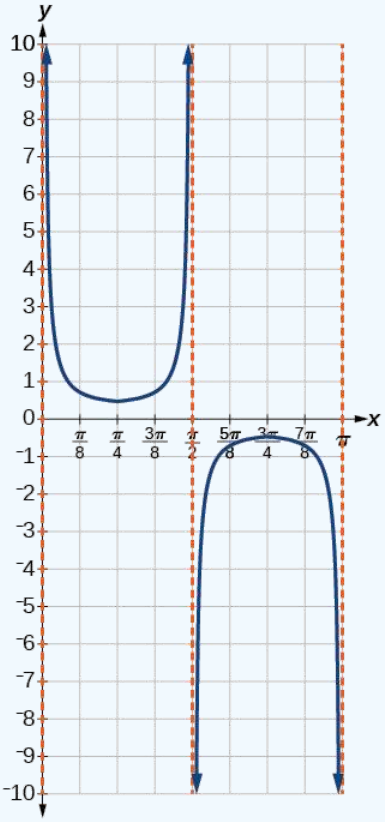


Figure 3.2.11

Q&A: Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

Howto: Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\sec(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\dfrac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\sec(Bx)$. but shift it to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.

Example 3.2.5: Graphing a Variation of the Secant Function

Graph one period of $y = 4\sec\left(\dfrac{\pi}{3x} - \dfrac{\pi}{2}\right) + 1$.

Solution

- *Step 1.* Express the function given in the form $y = 4\sec\left(\dfrac{\pi}{3x} - \dfrac{\pi}{2}\right) + 1$.
- *Step 2.* The stretching/compressing factor is $|A| = 4$.
- *Step 3.* The period is

$$\begin{aligned}\frac{2\pi}{|B|} &= \frac{2\pi}{\frac{\pi}{3}} \\ &= 2\pi \cdot \frac{3}{\pi} \\ &= 6\end{aligned}$$

- *Step 4.* The phase shift is

$$\begin{aligned}\frac{C}{B} &= \frac{\frac{\pi}{2}}{\frac{\pi}{3}} \\ &= \frac{\pi}{2} \cdot \frac{3}{\pi} \\ &= 1.5\end{aligned}$$

- *Step 5.* Draw the graph of $y = A\sec(Bx)$, but shift it to the right by $\frac{C}{B} = 1.5$ and up by $D = 6$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 3$, and $x = 6$. There is a local minimum at $(1.5, 5)$ and a local maximum at $(4.5, -3)$. Figure $3.2.12$ shows the graph.

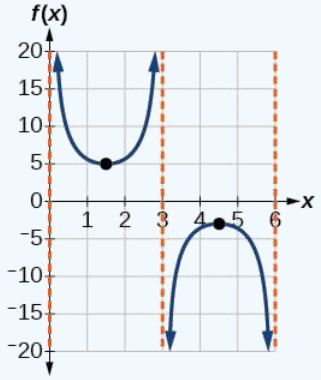


Figure 3.2.12

? Exercise 3.2.5

Graph one period of $f(x) = -6\sec(4x + 2) - 8$.

Answer

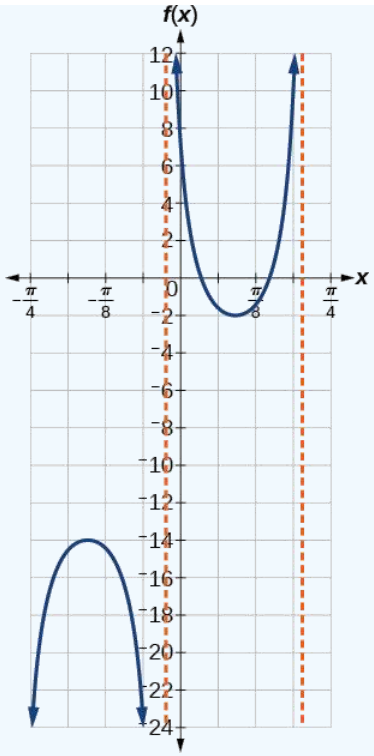


Figure 3.2.13

Q&A: The domain of $\csc\ x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \dfrac{C + k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

Howto: Given a function of the form $y = A\csc(Bx)$, graph one period.

1. Express the function given in the form $y = A\csc(Bx)$.
2. $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Draw the graph of $y = A\sin(Bx)$.
5. Use the reciprocal relationship between $y = sin\ x$ and $y = \csc\ x$ to draw the graph of $y = A\csc(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 3.2.6: Graphing a Variation of the Cosecant Function

Graph one period of $f(x) = -3\csc(4x)$.

Solution

- *Step 1.* The given function is already written in the general form, $y = A\csc(Bx)$.
- *Step 2.* $|A| = |-3| = 3$,so the stretching factor is 3.
- *Step 3.* $B = 4$,so $P = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$. The period is $\dfrac{\pi}{2}$ units.
- *Step 4.* Sketch the graph of the function $g(x) = -3\sin(4x)$.
- *Step 5.* Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.

- *Steps 6–7.* Sketch three asymptotes at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We can use two reference points, the local maximum at $\left(\frac{\pi}{8}, -3\right)$ and the local minimum at $\left(\frac{3\pi}{8}, 3\right)$. Figure 3.2.14 shows the graph.

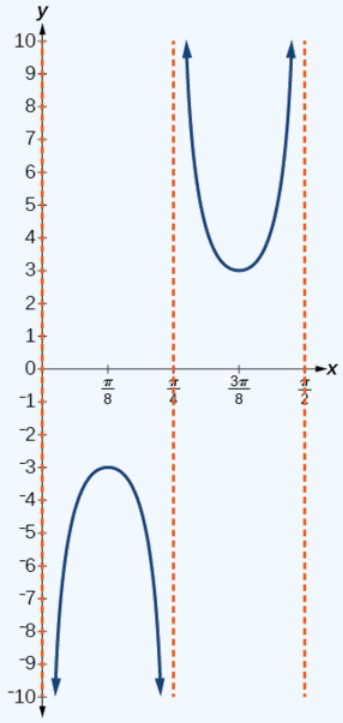


Figure 3.2.14

Exercise 3.2.6

Graph one period of $f(x) = 0.5 \csc(2x)$.

Answer

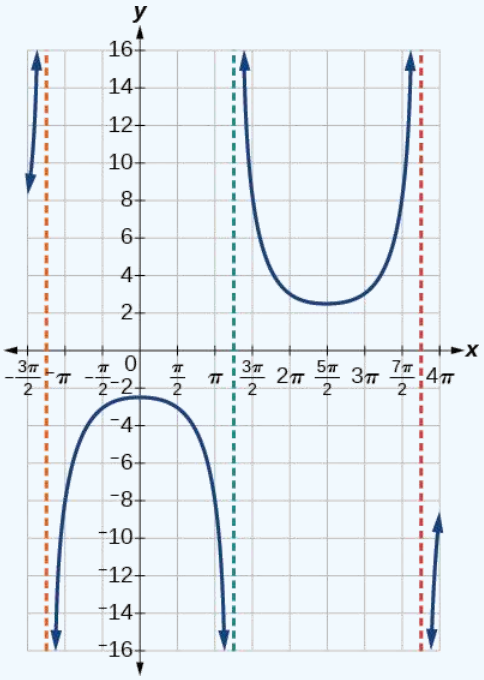


Figure 3.2.15

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Howto: Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period

1. Express the function given in the form $y = A\csc(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\csc(Bx)$ but shift it to the right by and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.

Example 3.2.7: Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$. What are the domain and range of this function?

Solution

- *Step 1.* Express the function given in the form $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$.
- *Step 2.* Identify the stretching/compressing factor, $|A| = 2$.
- *Step 3.* The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$.
- *Step 4.* The phase shift is $\frac{0}{\frac{\pi}{2}} = 0$.
- *Step 5.* Draw the graph of $y = A\csc(Bx)$ but shift it up $D = 1$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 2$, $x = 4$.

The graph for this function is shown in Figure 3.2.16.

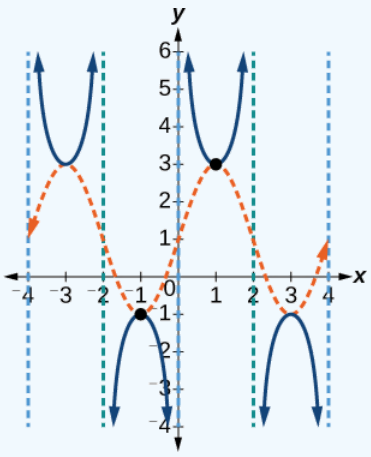


Figure 3.2.16: A transformed cosecant function

Analysis

The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$,shown as the orange dashed wave.

Exercise 3.2.7

Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in Figure 3.2.17, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

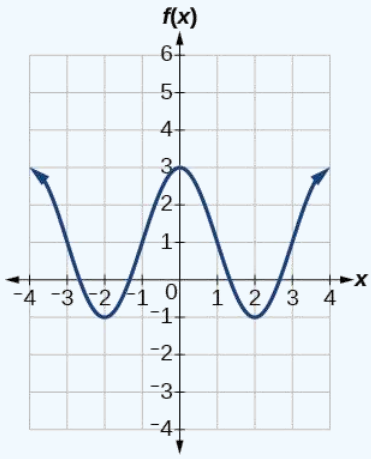


Figure 3.2.17

Answer

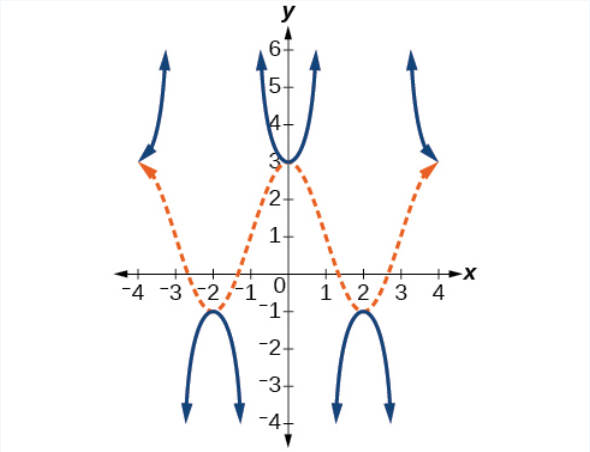


Figure 3.2.18

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $cot\, x = \dfrac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See Figure 3.2.19. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

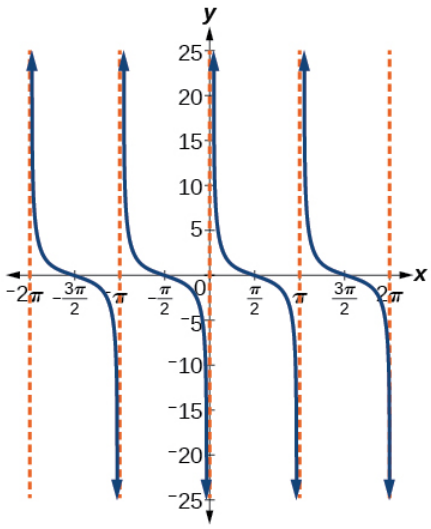


Figure 3.2.19: The cotangent function

FEATURES OF THE GRAPH OF $Y = A\cot(BX)$

- The stretching factor is $|A|$.
- The period is $P = \dfrac{\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D \tag{3.2.4}$$

PROPERTIES OF THE GRAPH OF $Y = A\cot(Bx - c) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{\pi}{|B|}$
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- There is no amplitude.
- $y = A\cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx)$,graph one period.

1. Express the function in the form $f(x) = A\cot(Bx)$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \dfrac{\pi}{|B|}$.
4. Draw the graph of $y = A\tan(Bx)$.
5. Plot any two reference points.

6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y = Acot(Bx)$.
7. Sketch the asymptotes.

Example 3.2.8: Graphing Variations of the Cotangent Function

Determine the stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph.

Solution

- *Step 1.* Expressing the function in the form $f(x) = A\cot(Bx)$ gives $f(x) = 3\cot(4x)$.
- *Step 2.* The stretching factor is $|A| = 3$.
- *Step 3.* The period is $P = \frac{\pi}{4}$.
- *Step 4.* Sketch the graph of $y = 3\tan(4x)$.
- *Step 5.* Plot two reference points. Two such points are $\left(\frac{\pi}{16}, 3\right)$ and $\left(\frac{3\pi}{16}, -3\right)$.
- *Step 6.* Use the reciprocal relationship to draw $y = 3\cot(4x)$.
- *Step 7.* Sketch the asymptotes, $x = 0$, $x = \frac{\pi}{4}$.

The orange graph in Figure 3.2.20 shows $y = 3\tan(4x)$ and the blue graph shows $y = 3\cot(4x)$.

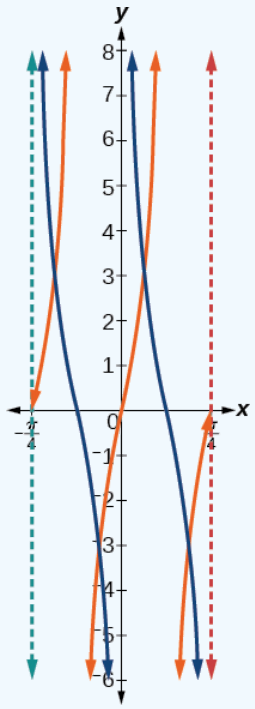


Figure 3.2.20

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
7. Plot any three reference points and draw the graph through these points.

Example 3.2.9: Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Solution

- *Step 1.* The function is already written in the general form $f(x) = A\cot(Bx - C) + D$.
- *Step 2.* $A = 4$,so the stretching factor is 4.
- *Step 3.* $B = \frac{\pi}{8}$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.
- *Step 4.* $C = \frac{\pi}{2}$,so the phase shift is $CB = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.
- *Step 5.* We draw $f(x) = 4\tan\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 6-7.* Three points we can use to guide the graph are $(6, 2)$, $(8, -2)$, and $(10, -6)$. We use the reciprocal relationship of tangent and cotangent to draw $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 8.* The vertical asymptotes are $x = 4$ and $x = 12$.

The graph is shown in Figure 3.2.21.

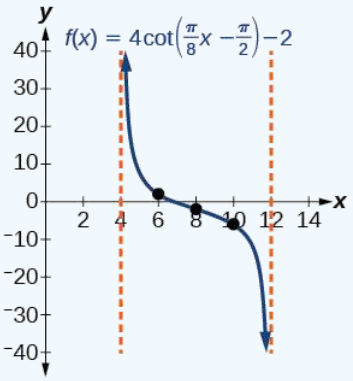


Figure 3.2.21: One period of a modified cotangent function

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the **tangent function**.

Example 3.2.10: Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan(\frac{\pi}{4}t)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.
b. Graph on the interval $[0, 5]$.
c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.

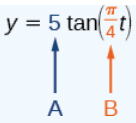


Figure 3.2.22

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See Figure 3.2.23

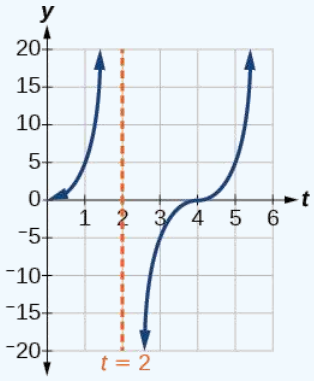


Figure 3.2.23

c. period: $f(1) = 5\tan(\frac{\pi}{4}(1)) = 5(1) = 5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Media

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent
- Graphing Cosecant and Secant
- Graphing the Cotangent

Key Equations

Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See Example 3.2.1, Example 3.2.2, and Example 3.2.3.
- The secant and cosecant are both periodic functions with a period of 2π. $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See Example 3.2.4 and Example 3.2.5.
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See Example 3.2.6 and Example 3.2.7.
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi, ...$.
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.

- The cotangent is zero at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$,....
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See Example $3.2.8$ and Example $3.2.9$.
- Real-world scenarios can be solved using graphs of trigonometric functions. See Example $3.2.10$.

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3.3: Inverse Trigonometric Functions

Learning Objectives

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in Figure 3.3.1.

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

Figure 3.3.1

For example, if $f(x) = \sin\ x$, then we would write $f^{-1}(x) = \sin^{-1} x$. Be aware that $\sin^{-1} x$ does not mean $\dfrac{1}{\sin\ x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$, then $\dfrac{\pi}{6} = \sin^{-1}\left(\dfrac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\dfrac{\pi}{4}\right) = 1$, then $\dfrac{\pi}{4} = \tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the **domain** of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. Figure 3.3.2 shows the graph of the sine function limited to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.

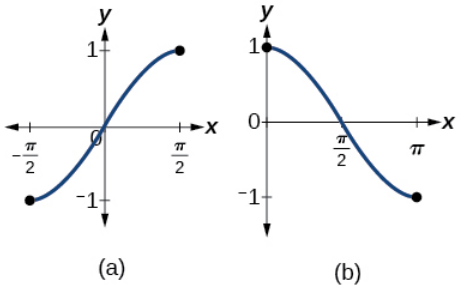


Figure 3.3.2: (a) Sine function on a restricted domain of $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$; (b) Cosine function on a restricted domain of $[0, \pi]$

Figure 3.3.3 shows the graph of the tangent function limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

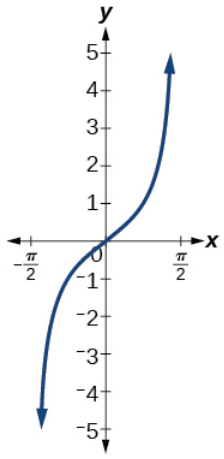


Figure 3.3.3: Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one **vertical asymptote** to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin\ y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin\ x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos\ y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos\ x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan\ y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan\ x$.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

The graphs of the inverse functions are shown in Figures 3.3.4 - 3.3.6. Notice that the output of each of these inverse functions is a *number*, an angle in radian measure. We see that $\sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

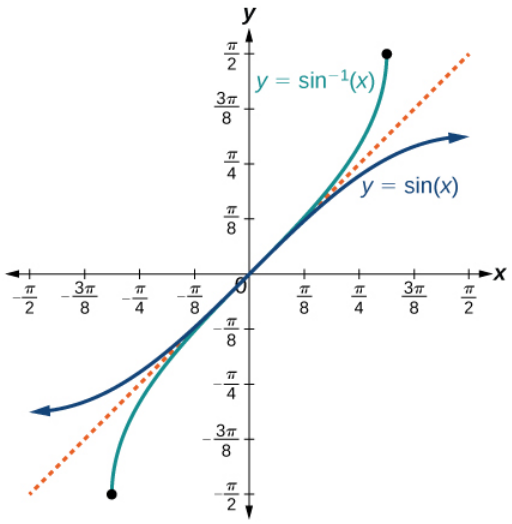


Figure 3.3.4: The sine function and inverse sine (or arcsine) function

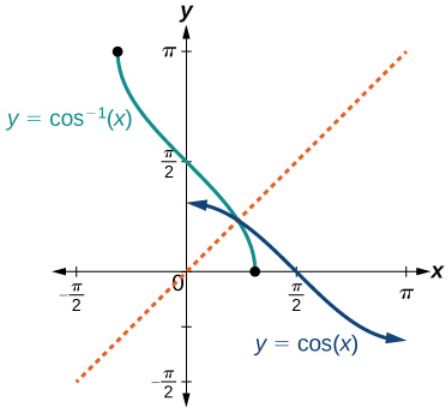


Figure 3.3.5: The cosine function and inverse cosine (or arccosine) function

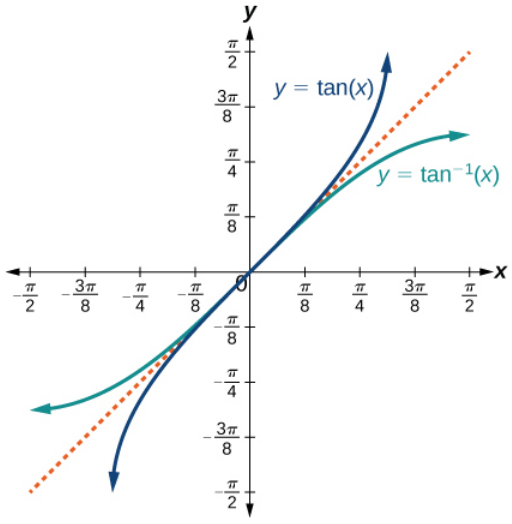


Figure 3.3.6: The tangent function and inverse tangent (or arctangent) function

RELATIONS FOR INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$,then $\tan^{-1} x = y$.

Example 3.3.1: Writing a Relation for an Inverse Function

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Solution

Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$

Exercise 3.3.1

Given $\cos(0.5) \approx 0.8776$,write a relation involving the inverse cosine.

Answer

$\arccos(0.8776) \approx 0.5$

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}(30°)$, $\frac{\pi}{4}(45°)$, and $\frac{\pi}{3}(60°)$, and their reflections into other quadrants.

Given a “special” input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x,depending on which corresponds to the given inverse function.

Example 3.3.2: Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$

b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

c. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

d. $\tan^{-1}(1)$

Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x) = \frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a

function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0, \pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1) = \frac{\pi}{4}$.

? Exercise 3.3.2

Evaluate each of the following.

a. $\sin^{-1}(-1)$
b. $\tan^{-1}(-1)$
c. $\cos^{-1}(-1)$
d. $\cos^{-1}\left(\frac{1}{2}\right)$

Answer a

$-\frac{\pi}{2}$

Answer b

$-\frac{\pi}{4}$

Answer c

π

Answer d

$\frac{\pi}{3}$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate **inverse trigonometric functions** that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN-1, ARCSIN, or ASIN.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

✓ Example 3.3.3: Evaluating the Inverse Sine on a Calculator

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Solution

Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93^\circ$. Note that in calculus and beyond we will use radians in almost all cases.

? Exercise 3.3.3

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer

$1.9823 \text{or } 113.578^\circ$

Given two sides of a right triangle like the one shown in Figure 8.4.7, find an angle.

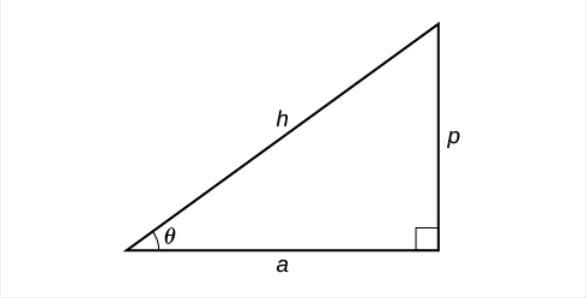


Figure 3.3.7

1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

✓ Example 3.3.4: Applying the Inverse Cosine to a Right Triangle

Solve the triangle in Figure 3.3.8 for the angle θ.

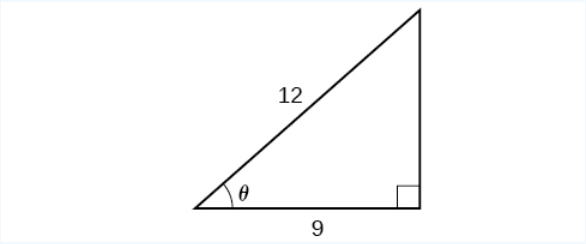


Figure 3.3.8

Solution

Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos\theta = \frac{9}{12}$$
$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \quad \text{Apply definition of the inverse}$$
$$\theta \approx 0.7227 \quad \text{or about } 41.4096^\circ \text{ Evaluate}$$

Exercise 3.3.4

Solve the triangle in Figure 3.3.9 for the angle θ.

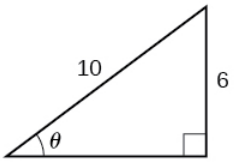


Figure 3.3.9

Answer

$\sin^{-1}(0.6) = 36.87^\circ = 0.6435$radians

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set{ $\sin(x)$,$\cos(x)$,$\tan(x)$ } and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function,$f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

COMPOSITIONS OF A TRIGONOMETRIC FUNCTION AND ITS INVERSE

$$\begin{aligned}
\sin(\sin^{-1}x) &= x \quad && \text{for } -1 \le x \le 1 \\
\cos(\cos^{-1}x) &= x \quad && \text{for } -1 \le x \le 1 \\
\tan(\tan^{-1}x) &= x \quad && \text{for } -\infty < x < \infty \\
\sin^{-1}(\sin x) &= x \quad && \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos^{-1}(\cos x) &= x \quad && \text{only for } 0 \le x \le \pi \\
\tan^{-1}(\tan x) &= x \quad && \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Q&A

Is it correct that $\sin^{-1}(\sin x) = x$?

No. This equation is correct ifx x belongs to the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but sine is defined for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain off f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Example 3.3.5: Using Inverse Trigonometric Functions

Evaluate the following:

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

c. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

d. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution

a. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

b. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $sin\left(\frac{2\pi}{3}\right) = sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$.

c. $\frac{2\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$.

d. $-\frac{\pi}{3}$ is not in $[0, \pi]$, but $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ because cosine is an even function. $\frac{\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

Exercise 3.3.5

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Answer

$\frac{\pi}{8}; \frac{2\pi}{9}$

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}(g(x))$. For special values of x,we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2} - \theta$.Consider the sine and cosine of each angle of the right triangle in Figure 3.3.10

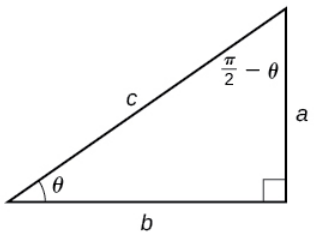


Figure 3.3.10: Right triangle illustrating the cofunction relationships

Because $\cos\theta = \frac{b}{c} = sin\left(\frac{\pi}{2} - \theta\right)$, we have $\sin^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$ if $0 \le \theta \le \pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$.Similarly, $\sin\theta = \frac{a}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$, so $\cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$ if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle y in $[0, \pi]$ such that $\cos y = \cos x$.

$$\sin^{-1}(\cos x) = \frac{\pi}{2} - y \tag{3.3.1}$$

3. If x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then find another angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \sin x$.

$$\cos^{-1}(\sin x) = \frac{\pi}{2} - y \tag{3.3.2}$$

Example 3.3.6: Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

a. by direct evaluation.
b. by the method described previously.

Solution

a. Here, we can directly evaluate the inside of the composition.

$$\begin{aligned}\cos\left(\frac{13\pi}{6}\right) &= \cos\left(\frac{\pi}{6} + 2\pi\right) \\ &= \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Now, we can evaluate the inverse function as we did earlier.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \tag{3.3.3}$$

b. We have $x = \frac{13\pi}{6}$, $y = \frac{\pi}{6}$, and

$$\begin{aligned}\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) &= \frac{\pi}{2} - \frac{\pi}{6} \\ &= \frac{\pi}{3}\end{aligned}$$

Exercise 3.3.6

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Answer

$\frac{3\pi}{4}$

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of

Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Example 3.3.7: Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution

Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 && \text{Use our known value for cosine} \\ \sin^2\theta + \left(\frac{4}{5}\right)^2 &= 1 && \text{Solve for sine} \\ \sin^2\theta &= 1 - \frac{16}{25} \\ \sin\theta &= \pm\frac{9}{25} \\ &= \pm\frac{3}{5} \end{aligned}$$

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in quadrant I, $\sin\theta$ must be positive, so the solution is 35. See Figure 3.3.11.

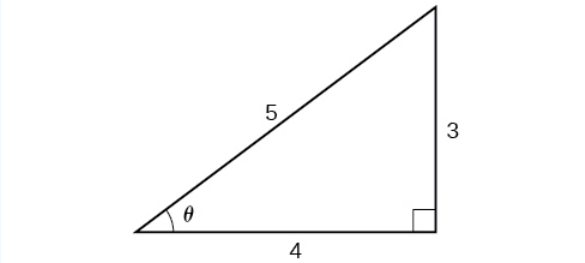


Figure 3.3.11: Right triangle illustrating that if $\cos\theta = \frac{4}{5}$, then $\sin\theta = \frac{3}{5}$

We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive; therefore $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \frac{3}{5}$.

Exercise 3.3.7

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Answer

$\frac{12}{13}$

Example 3.3.8: Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Solution

While we could use a similar technique as in Example 3.3.6, we will demonstrate a different technique here. From the inside, we know there is an angle such that $\tan\theta = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in Figure 3.3.12

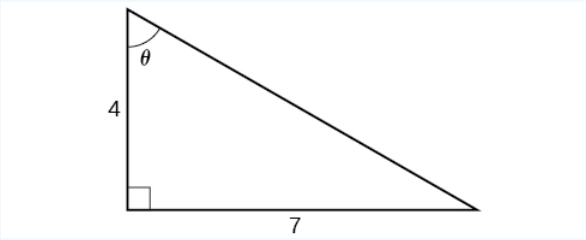


Figure 3.3.12: A right triangle with two sides known

Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

$$\begin{aligned} 4^2 + 7^2 &= hypotenuse^2 \\ hypotenuse &= \sqrt{65} \end{aligned}$$

Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{7}{\sqrt{65}}$$

This gives us our desired composition.

$$\begin{aligned} \sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) &= \sin\theta \\ &= \frac{7}{\sqrt{65}} \\ &= \frac{7\sqrt{65}}{65} \end{aligned}$$

? Exercise 3.3.8

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Answer

$\frac{4\sqrt{2}}{9}$

✓ Example 3.3.9: Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

Solution

We know there is an angle θ such that $\sin\theta = \frac{x}{3}$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \qquad \text{Use the Pythagorean Theorem} \\ \left(\frac{x}{3}\right)^2 + \cos^2\theta &= 1 \qquad \text{Solve for cosine} \\ \cos^2\theta &= 1 - \frac{x^2}{9} \\ \cos\theta &= \pm\sqrt{\frac{9-x^2}{9}} \\ &= \pm\sqrt{\frac{9-x^2}{3}} \end{aligned}$$

Because we know that the inverse sine must give an angle on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we can deduce that the cosine of that angle must be positive.

$cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \sqrt{\frac{9-x^2}{3}}$

Exercise 3.3.9

Find a simplified expression for $\sin(\tan^{-1}(4x))$ for $-\frac{1}{4} \leq x \leq \frac{1}{4}$.

Answer

$$\frac{4x}{\sqrt{16x^2+1}}$$

Media

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- Evaluate Expressions Involving Inverse Trigonometric Functions

Visit this website for additional practice questions from Learningpod.

Key Concepts

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f(x)$, if $x = f^{-1}(y)$, then $f(x) = y$. However, $f(x) = y$ only implies $x = f^{-1}(y)$ if x is in the restricted domain of f. See Example 3.3.1.
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4} = \tan^{-1}(1)$ and $\frac{\pi}{6} = \sin^{-1}(\frac{1}{2})$ .See Example 3.3.2.
- A calculator will return an angle within the restricted domain of the original trigonometric function. See Example 3.3.3.
- Inverse functions allow us to find an angle when given two sides of a right triangle. See Example 3.3.4.
- In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example,$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$. See Example 3.3.5.
- If the inside function is a trigonometric function, then the only possible combinations are $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ if $0 \leq x \leq \pi$ and $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$ if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. See Example 3.3.6 and Example 3.3.7.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See Example 3.3.8.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See Example 3.3.9.

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3.E: Periodic Functions (Exercises)

6.1: Graphs of the Sine and Cosine Functions

In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions

Verbal

1) Why are the sine and cosine functions called periodic functions?

Answer

The sine and cosine functions have the property that $f(x+P)=f(x)$ for a certain P. This means that the function values repeat for every P units on the x-axis.

2) How does the graph of $y=\sin x$ compare with the graph of $y=\cos x$? Explain how you could horizontally translate the graph of $y=\sin x$ to obtain $y=\cos x$.

3) For the equation $A\cos(Bx+C)+D$, what constants affect the range of the function and how do they affect the range?

Answer

The absolute value of the constant A (amplitude) increases the total range and the constant D (vertical shift) shifts the graph vertically.

4) How does the range of a translated sine function relate to the equation $y=A\sin(Bx+C)+D$?

5) How can the unit circle be used to construct the graph of $f(t)=\sin t$?

Answer

At the point where the terminal side of t intersects the unit circle, you can determine that the $\sin t$ equals the y-coordinate of the point.

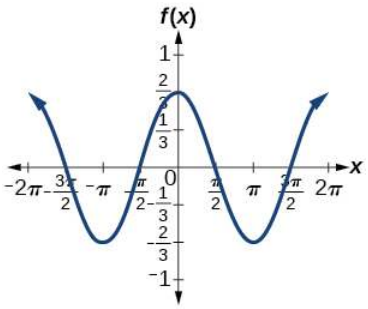
Graphical

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x>0$. Round answers to two decimal places if necessary.

6) $f(x)=2\sin x$

7) $f(x)=\frac{2}{3}\cos x$

Answer

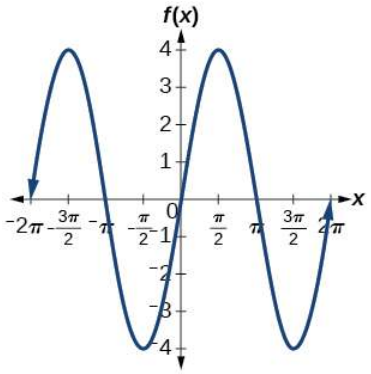


amplitude: $\frac{2}{3}$; period: 2π; midline: $y=0$; maximum: $y=\frac{2}{3}$ occurs at $x=0$; minimum: $y=-\frac{2}{3}$ occurs at $x=\pi$; for one period, the graph starts at 0 and ends at 2π.

8) $f(x) = -3\sin x$

9) $f(x) = 4\sin x$

Answer

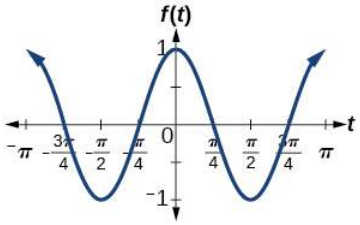


amplitude: 4; period: 2π; midline: $y = 0$; maximum $y = 4$ occurs at $x = \frac{\pi}{2}$; minimum: $y = -4$ occurs at $x = \frac{3\pi}{2}$; one full period occurs from $x = 0$ to $x = 2\pi$

10) $f(x) = 2\cos x$

11) $f(x) = \cos(2x)$

Answer

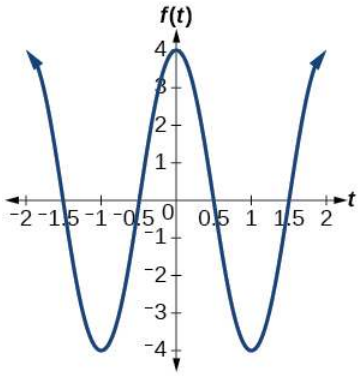


amplitude: 1; period: π; midline: $y = 0$; maximum: $y = 1$ occurs at $x = \pi$; minimum: $y = -1$ occurs at $x = \frac{\pi}{2}$; one full period is graphed from $x = 0$ to $x = \pi$

12) $f(x) = 2\sin\left(\frac{1}{2}x\right)$

13) $f(x) = 4\cos(\pi x)$

Answer

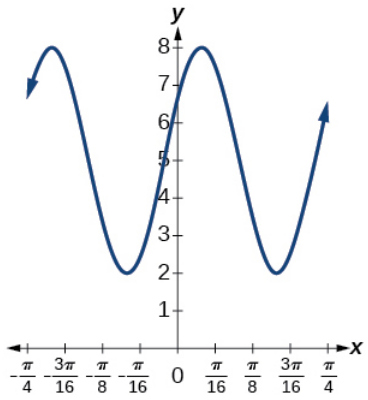


amplitude: 4; period: 2; midline: $y = 0$; maximum: $y = 4$ occurs at $x = 0$; minimum: $y = -4$ occurs at $x = 1$

14) $f(x) = 3\cos\left(\frac{6}{5}x\right)$

15) $y = 3\sin(8(x+4)) + 5$

Answer

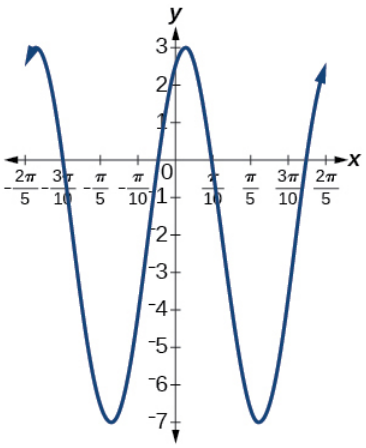


amplitude: 3; period: $\frac{\pi}{4}$; midline: $y = 5$;
maximum: $y = 8$ occurs at $x = -4 + \frac{21\pi}{16} \approx 0.123$;
minimum: $y = 2$ occurs at $x = -4 + \frac{23\pi}{16} \approx 0.516$;
horizontal shift: -4; vertical translation 5;
one period occurs from $x = -4 + \frac{22\pi}{16} \approx 0.320$ to $x = -4 + \frac{26\pi}{16} \approx 1.105$

16) $y = 2\sin(3x - 21) + 4$

17) $y = 5\sin(5x + 20) - 2$

Answer



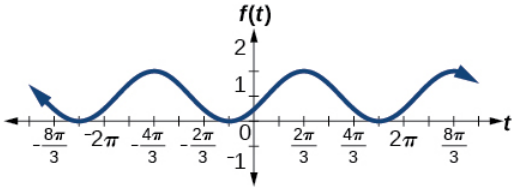
amplitude: 5; period: $\frac{2\pi}{5}$; midline: $y = -2$;
maximum: $y = 3$ occurs at $x = -4 + \frac{13\pi}{10} \approx 0.084$;
minimum: $y = -7$ occurs at $x = -4 + \frac{15\pi}{10} \approx 0.712$;
phase shift: -4; vertical translation: -2;
one full period can be graphed on $x = -4 + \frac{7\pi}{5} \approx 0.398$ to$x = -4 + \frac{9\pi}{5} \approx 1.655$

For the following exercises, graph one full period of each function, starting at $x = 0$.
For each function, state the amplitude, period, and midline.
State the maximum and minimum y-values and their corresponding x-values on one period for $x > 0$.
State the phase shift and vertical translation, if applicable.
Round answers to two decimal places if necessary.

18) $f(t) = 2\sin\left(t - \frac{5\pi}{6}\right)$

19) $f(t) = -\cos\left(t + \frac{\pi}{3}\right) + 1$

Answer

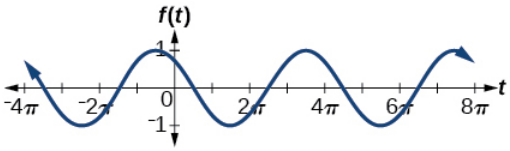


amplitude: 1; period: 2π; midline: $y = 1$;
maximum: $y = 2$ occurs at $t = \frac{2\pi}{3} \approx 2.094$;
minimum: $y = 0$ occurs at $t = \frac{2\pi}{3} \approx 5.24$;
phase shift: $-\frac{\pi}{3}$; vertical translation: 1;
one full period is from $t = \frac{2\pi}{3} \approx 2.094$ to $t = \frac{8\pi}{3} \approx 8.378$

20) $f(t) = 4\cos\left(2\left(t + \frac{\pi}{4}\right)\right) - 3$

21) $f(t) = -\sin\left(\frac{1}{2}t + \frac{5\pi}{3}\right)$

Answer



amplitude: 1; period: 4π; midline: $y = 0$;
maximum: $y = 1$ occurs at $t = \frac{11\pi}{3} \approx 11.52$;
minimum: $y = -1$ occurs at $t = \frac{5\pi}{3} \approx 5.24$;
phase shift: $-\dfrac{10\pi}{3}$; vertical shift: 0;
one full period is from $t = \frac{2\pi}{3} \approx 2.094$ to $t = \frac{14\pi}{3} \approx 14.661$

22) $f(x) = 4\sin\left(\dfrac{\pi}{2}(x-3)\right) + 7$

23) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

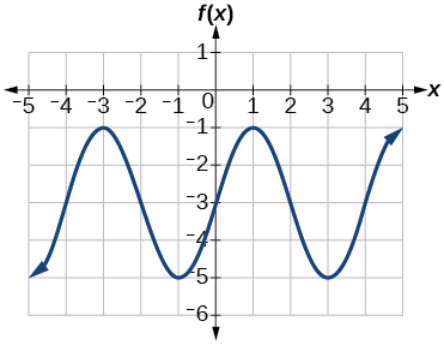


Figure $3.E.23$

Answer

23. amplitude: 2; midline: $y = -3$ period: 4; equation: $f(x) = 2\sin\left(\dfrac{\pi}{2}x\right) - 3$

24) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.

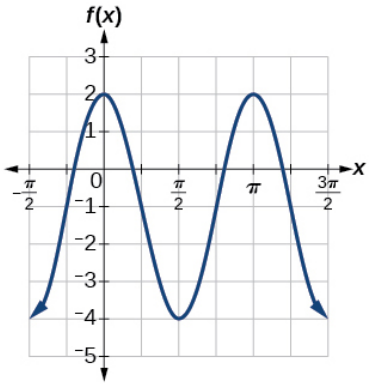


Figure $3.E.24$

25) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.

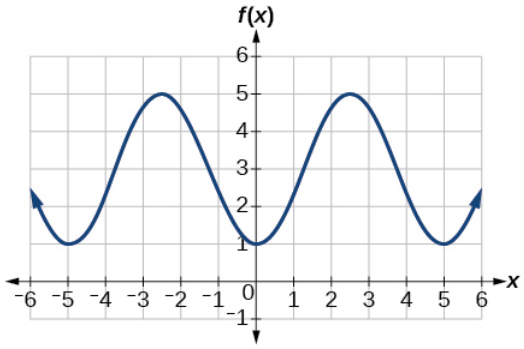


Figure $3.E.25$

Answer

25. amplitude: 2; period: 5; midline: $y = 3$ equation: $f(x) = -2\cos\left(\frac{2\pi}{5}x\right) + 3$

26) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

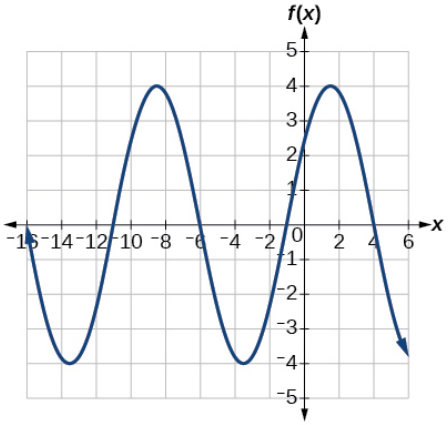


Figure $3.E.26$

27) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.



Figure $3.E.27$

Answer

27. amplitude: 4; period: 2; midline: $y = 0$; equation: $f(x) = -4\cos\left(\pi\left(x - \frac{\pi}{2}\right)\right)$

28) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

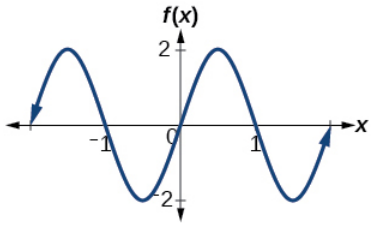


Figure 3.E.28

29) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.

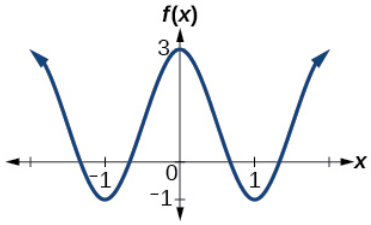


Figure 3.E.29

Answer

29. amplitude: 2; period: 2; midline $y = 1$ equation: $f(x) = 2\cos(\pi x) + 1$

30) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.



Figure 3.E.30

Algebraic

For the following exercises, let $f(x) = \sin x$.

31) On $[0, 2\pi)$, solve $f(x) = 0$.

32) On $[0, 2\pi)$, solve $f(x) = \dfrac{1}{2}$.

Answer

$\dfrac{\pi}{6}, \dfrac{5\pi}{6}$

33) Evaluate $f\left(\dfrac{\pi}{2}\right)$.

34) On $[0, 2\pi)$, $f(x) = \frac{\sqrt{2}}{2}$. Find all values of x.

Answer

$\frac{\pi}{4}, \frac{3\pi}{4}$

35) On $[0, 2\pi)$, the maximum value(s) of the function occur(s) at what x-value(s)?

36) On $[0, 2\pi)$, the minimum value(s) of the function occur(s) at what x-value(s)?

Answer

$\frac{3\pi}{2}$

37) Show that $f(-x) = -f(x)$. This means that $f(x) = \sin x$ is an odd function and possesses symmetry with respect to _____.

For the following exercises, let $f(x) = \cos x$

38) On $[0, 2\pi)$, solve the equation $f(x) = \cos x = 0$

Answer

$\frac{\pi}{2}, \frac{3\pi}{2}$

39) On $[0, 2\pi)$, solve $f(x) = \dfrac{1}{2}$.

40) On $[0, 2\pi)$, find the x-intercepts of $f(x) = \cos x$.

Answer

$\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

41) On $[0, 2\pi)$, find the x-values at which the function has a maximum or minimum value.

42) On $[0, 2\pi)$, solve the equation $f(x) = \dfrac{\sqrt{3}}{2}$.

Answer

$\dfrac{\pi}{6}, \dfrac{11\pi}{6}$

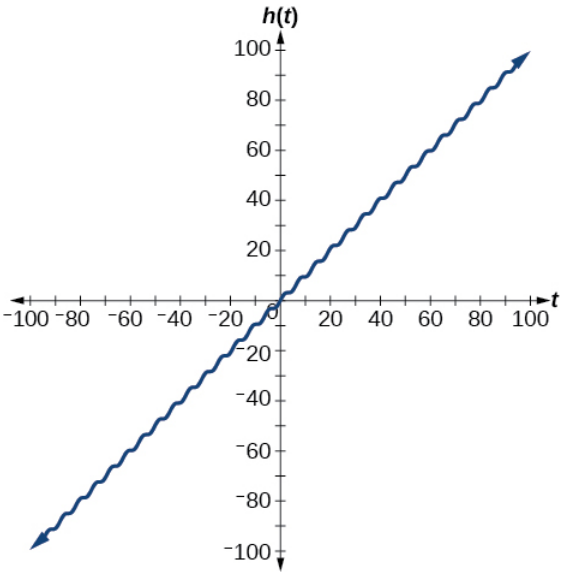
Technology

43) Graph $h(x) = x + \sin x$ on $[0, 2\pi]$. Explain why the graph appears as it does.

44) Graph $h(x) = x + \sin x$ on $[-100, 100]$. Did the graph appear as predicted in the previous exercise?

Answer

The graph appears linear. The linear functions dominate the shape of the graph for large values of x.

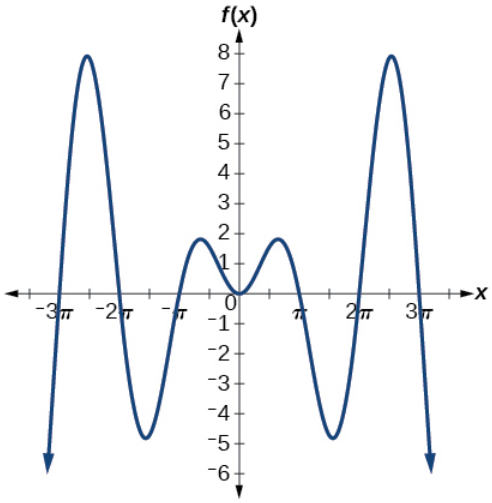


45) Graph $f(x) = x \sin x$ on $[0, 2\pi]$ and verbalize how the graph varies from the graph of $f(x) = x \sin x$.

46) Graph $f(x) = x \sin x$ on the window $[-10, 10]$ and explain what the graph shows.

Answer

The graph is symmetric with respect to the y-axis and there is no amplitude because the function is not periodic.



47) Graph $f(x) = \dfrac{\sin x}{x}$ on the window $[-5\pi, 5\pi]$and explain what the graph shows.

Real-World Applications

48) A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives a person's height in meters above the ground t minutes after the wheel begins to turn

a. Find the amplitude, midline, and period of $h(t)$.
b. Find a formula for the height function $h(t)$.
c. How high off the ground is a person after 5 minutes?

Answer

a. Amplitude: 12.5; period: 10; midline: $y = 13.5$
b. $h(t) = 12.5 \sin\left(\dfrac{\pi}{5}(t - 2.5)\right) + 13.5$
c. 26 ft

6.2: Graphs of the Other Trigonometric Functions

This section addresses the graphing of the Tangent, Cosecant, Secant, and Cotangent curves.

Verbal

1) Explain how the graph of the sine function can be used to graph $y = \csc x$.

Answer

Since $y = \csc x$ is the reciprocal function of $y = \sin x$, you can plot the reciprocal of the coordinates on the graph of $y = \sin x$ obtain the y-coordinates of $y = \csc x$. The x-intercepts of the graph $y = \sin x$ are the vertical asymptotes for the graph of $y = \csc x$.

2) How can the graph of $y = \cos x$ be used to construct the graph of $y = \sec x$?

3) Explain why the period of $y = \tan x$ is equal to π.

Answer

Answers will vary. Using the unit circle, one can show that $y = \tan(x + \pi) = \tan x$.

4) Why are there no intercepts on the graph of $y = \csc x$?

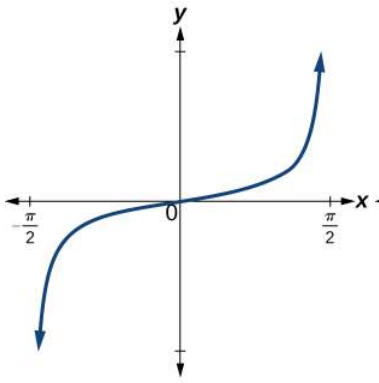
5) How does the period of $y = \csc x$ compare with the period of $y = \sin x$?

Answer

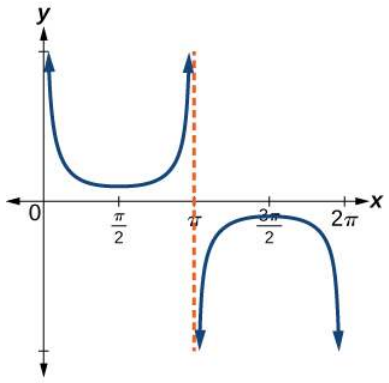
The period is the same: 2π

Algebraic

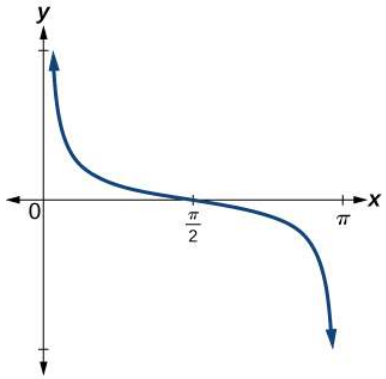
For the exercises 6-9, match each trigonometric function with one of the following graphs.


y
x
0
$\frac{\pi}{2}$
$\frac{\pi}{2}$

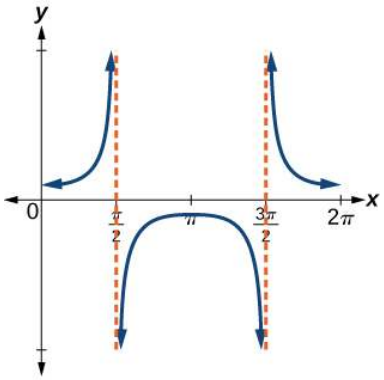
I


y
x
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

II


y
x
0
$\frac{\pi}{2}$
π

III


y
x
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

IV

6) $f(x) = \tan x$

7) $f(x) = \sec x$

Answer

IV

8) $f(x) = \csc x$

9) $f(x) = \cot x$

Answer

III

For the exercises 10-16, find the period and horizontal shift of each of the functions.

10) $f(x) = 2\tan(4x - 32)$

11) $h(x) = 2\sec\left(\frac{\pi}{4}(x + 1)\right)$

Answer

period: 8; horizontal shift: 1 unit to left

12) $m(x) = 6 \csc\left(\frac{\pi}{3}x + \pi\right)$

13) If $\tan x = -1.5$, find $\tan(-x)$.

Answer

1.5

14) If $\sec x = 2$, find $\sec(-x)$.

15) If $\csc x = -5$, find $\csc(-x)$.

Answer

5

16) If $x \sin x = 2$, find $(-x)\sin(-x)$.

For the exercises 17-18, rewrite each expression such that the argument x is positive.

17) $\cot(-x)\cos(-x)+\sin(-x)$

Answer

$-\cot x \cos x - \sin x$

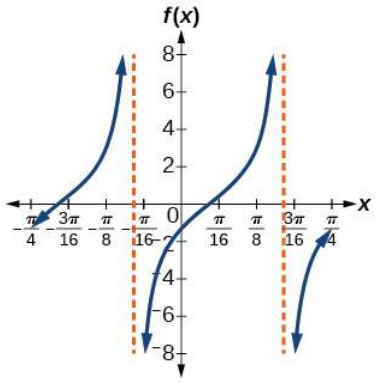
18) $\cos(-x) + \tan(-x)\sin(-x)$

Graphical

For the exercises 19-36, sketch two periods of the graph for each of the following functions. Identify the stretching factor, period, and asymptotes.

19) $f(x) = 2\tan(4x - 32)$

Answer

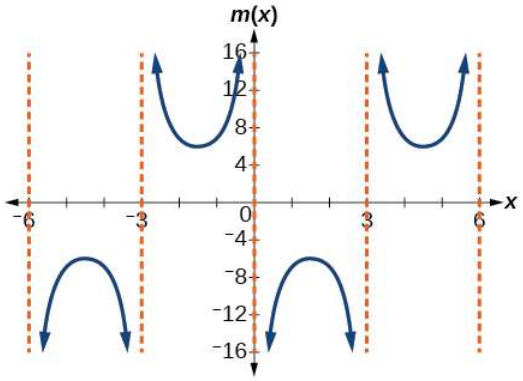


stretching factor: 2; period: $\frac{\pi}{3}$; asymptotes: $x = \frac{1}{4}\left(\frac{\pi}{2} + \pi k\right) + 8$, where k is an integer

20) $h(x) = 2\sec\left(\frac{\pi}{4}(x+1)\right)$

21) $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

Answer

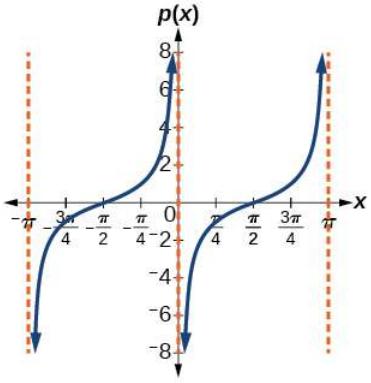


stretching factor: 6; period: 6; asymptotes: $x = k$, where k is an integer

22) $j(x) = \tan\left(\frac{\pi}{2}x\right)$

23) $p(x) = \tan\left(x - \frac{\pi}{2}\right)$

Answer

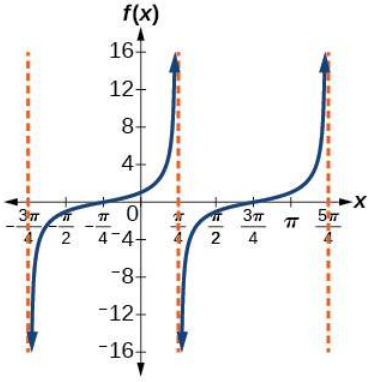


stretching factor: 1; period: π; asymptotes: $x = \pi k$, where k is an integer

24) $f(x) = 4\tan(x)$

25) $f(x) = \tan\left(x + \frac{\pi}{4}\right)$

Answer

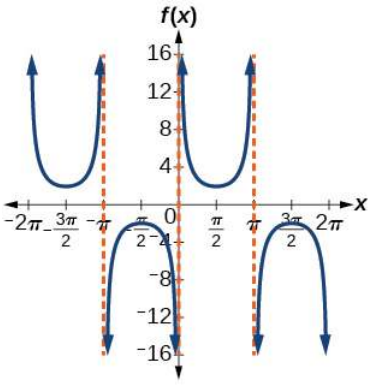


Stretching factor: 1; period: π; asymptotes: $x = \frac{\pi}{4} + \pi k$, where k is an integer

26) $f(x) = \pi \tan(\pi x - \pi) - \pi$

27) $f(x) = 2\csc(x)$

Answer

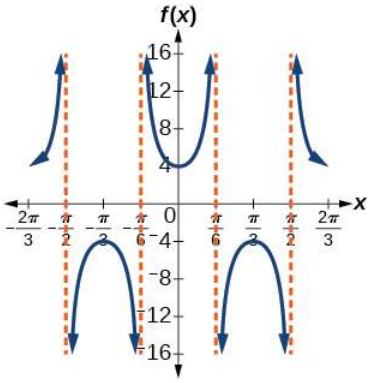


stretching factor: 2; period: 2π; asymptotes: $x = \pi k$, where k is an integer

28) $f(x) = -\frac{1}{4}\csc(x)$

29) $f(x) = 4\sec(3x)$

Answer

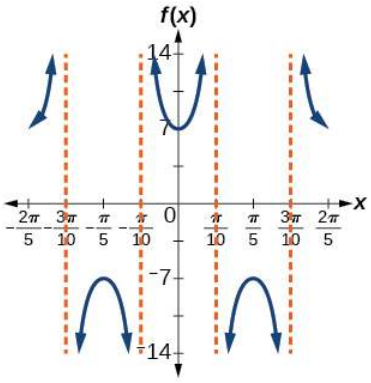


stretching factor: 4; period: $\frac{2\pi}{3}$; asymptotes: $x = \frac{\pi}{6}k$, where k is an odd integer

30) $f(x) = -3\cot(2x)$

31) $f(x) = 7\sec(5x)$

Answer

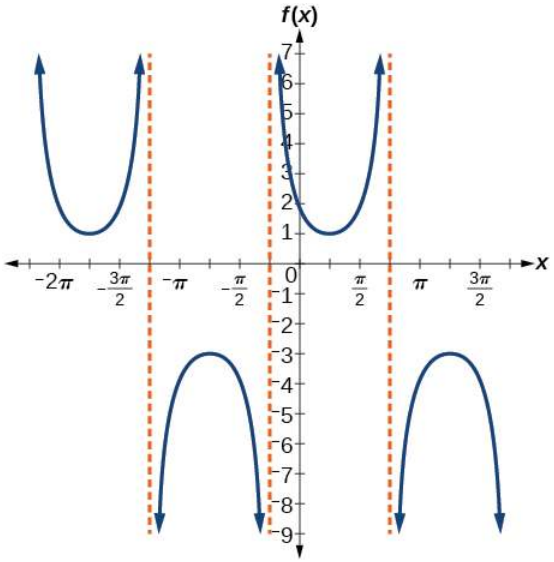


stretching factor: 7; period: $\frac{2\pi}{5}$; asymptotes: $x = \frac{\pi}{10}k$, where k is an odd integer

32) $f(x) = \frac{9}{10}\csc(\pi x)$

33) $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 1$

Answer

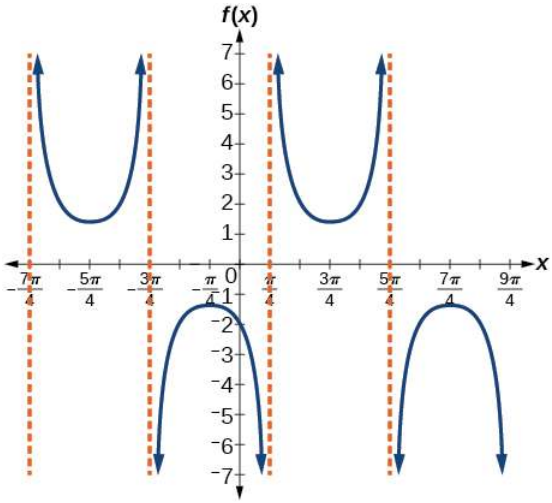


Stretching factor: 2; period: 2π ; asymptotes: $x = -\frac{\pi}{4} + \pi k$, where k is an integer

34) $f(x) = -\sec\left(x - \frac{\pi}{3}\right) - 2$

35) $f(x) = \frac{7}{5}\csc\left(x - \frac{\pi}{4}\right)$

Answer



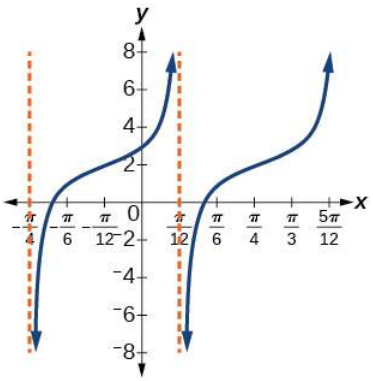
Stretching factor: $\frac{7}{5}$; period: 2π ; asymptotes: $x = \frac{\pi}{4} + \pi k$, where k is an integer

36) $f(x) = 5\left(\cot\left(x + \frac{\pi}{2}\right) - 3\right)$

37) A tangent curve, $A = 1$, period of $\frac{\pi}{3}$; and phase shift $(h, k) = \left(\frac{\pi}{4}, 2\right)$

Answer

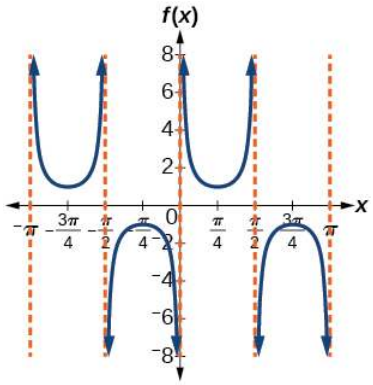
$$y = \tan\left(3\left(x - \frac{\pi}{4}\right)\right) + 2$$



38) A tangent curve, $A = -2$, period of $\frac{\pi}{4}$; and phase shift $(h, k) = \left(-\frac{\pi}{4}, -2\right)$

For the exercises 39-45, find an equation for the graph of each function.

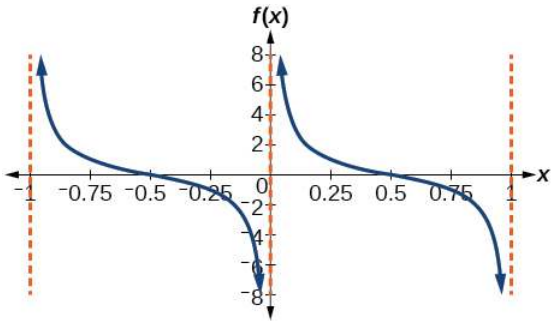
39)



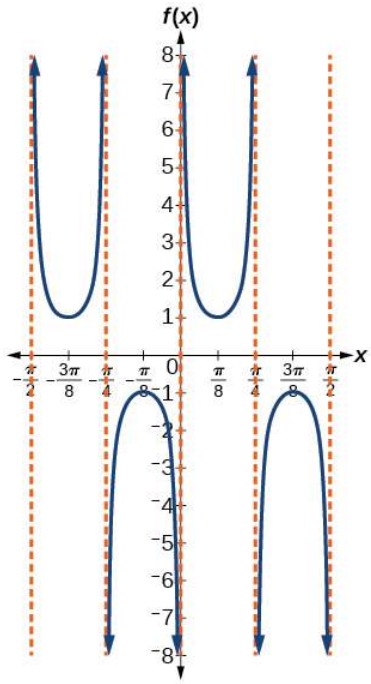
Answer

$f(x) = \csc(2x)$

40)



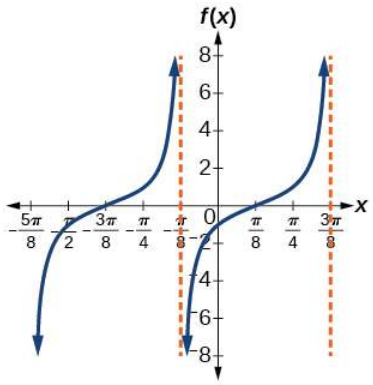
41)



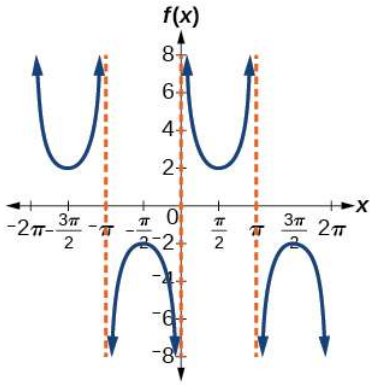
Answer

$f(x) = \csc(4x)$

42)


f(x)
8
6
4
2
0
-2
-4
-6
-8
x
$-\frac{5\pi}{8}$
$-\frac{\pi}{2}$
$-\frac{3\pi}{8}$
$-\frac{\pi}{4}$
$-\frac{\pi}{8}$
$\frac{\pi}{8}$
$\frac{\pi}{4}$
$\frac{3\pi}{8}$

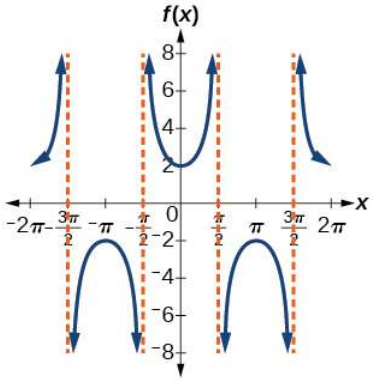
43)


f(x)
8
6
4
2
x
-2π
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
-2
-4
-6
-8

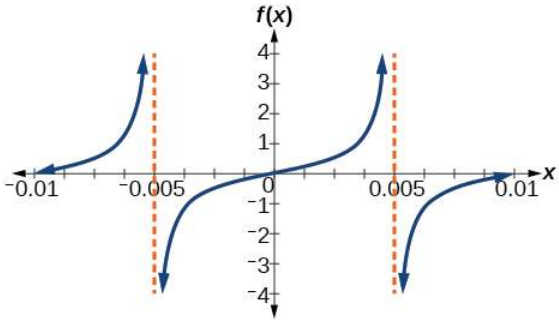
Answer

$f(x) = 2 \csc x$

44)


f(x)
8
6
4
2
x
-2π
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
-2
-4
-6
-8

45)



Answer

$f(x) = \frac{1}{2}\tan(100\pi x)$

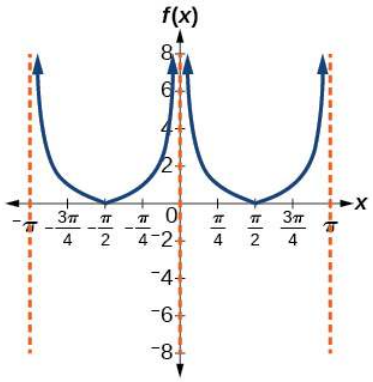
Technology

For the exercises 46-53, use a graphing calculator to graph two periods of the given function. Note: most graphing calculators do not have a cosecant button; therefore, you will need to input $\csc x$ as $\frac{1}{\sin x}$

46) $f(x) = |\csc(x)|$

47) $f(x) = |\cot(x)|$

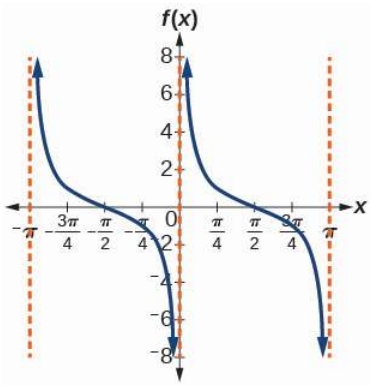
Answer



48) $f(x) = 2^{\csc(x)}$

49) $f(x) = \frac{\csc(x)}{\sec(x)}$

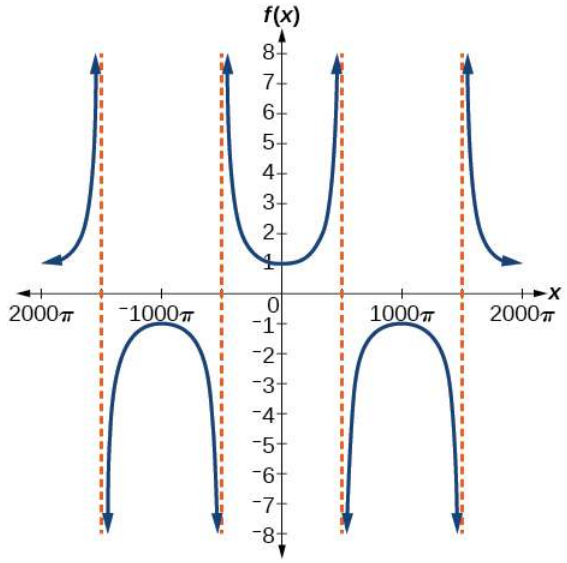
Answer



50) Graph $f(x) = 1 + \sec^2(x) - \tan^2(x)$. What is the function shown in the graph?

51) $f(x) = \sec(0.001x)$

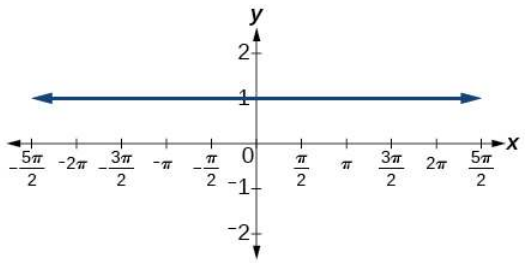
Answer



52) $f(x) = \cot(100\pi x)$

53) $f(x) = \sin^2 x + \cos^2 x$

Answer



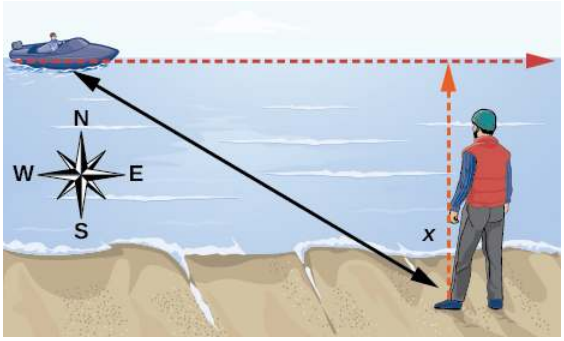
Real-World Applications

54) The function $f(x) = 20\tan\left(\frac{\pi}{10}x\right)$ marks the distance in the movement of a light beam from a police car across a wall for time x, in seconds, and distance $f(x)$, in feet.

a. Graph on the interval $[0, 5]$
b. Find and interpret the stretching factor, period, and asymptote.
c. Evaluate $f(10)$ and $f(2.5)$ and discuss the function's values at those inputs.

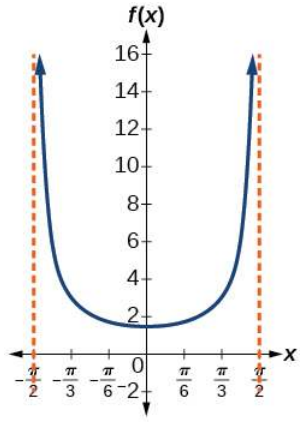
55) Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let x, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. (See Figure below.) The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$, in kilometers, from the fisherman to the boat is given by the function $d(x) = 1.5 \sec(x)$.

a. What is a reasonable domain for $d(x)$?
b. Graph $d(x)$ on this domain.
c. Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.
d. Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the second decimal place.
e. Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the second decimal place.
f. What is the minimum distance between the fisherman and the boat? When does this occur?



Answer

a. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b. 

c. $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$; the distance grows without bound as $|x|$ approaches $\frac{\pi}{2}$—i.e., at right angles to the line representing due north, the boat would be so far away, the fisherman could not see it;

d. 3; when $x = -\frac{\pi}{3}$, the boat is 3 km away;

e. 1.73; when $x = \frac{\pi}{6}$, the boat is about 1.73km away;

f. 1.5 km; when $x = 0$

56) A laser rangefinder is locked on a comet approaching Earth. The distance $g(x)$, in kilometers, of the comet after x days, for x in the interval 0 to 30 days, is given by $g(x) = 250,000 \csc\left(\frac{\pi}{30}x\right)$.

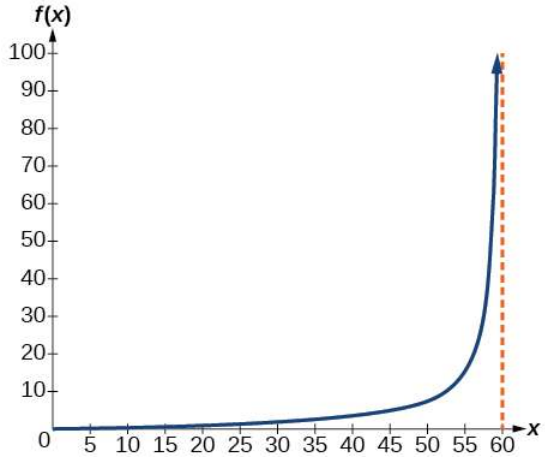
a. Graph $g(x)$ on the interval $[0, 35]$.
b. Evaluate $g(5)$ and interpret the information.
c. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?
d. Find and discuss the meaning of any vertical asymptotes.

57) A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after x seconds is $\frac{\pi}{120}x$.

a. Write a function expressing the altitude $h(x)$, in miles, of the rocket above the ground after x seconds. Ignore the curvature of the Earth.
b. Graph $h(x)$ on the interval $(0, 60)$.
c. Evaluate and interpret the values $h(0)$ and $h(30)$.
d. What happens to the values of $h(x)$ as x approaches 60 seconds? Interpret the meaning of this in terms of the problem.

Answer

a. $h(x) = 2\tan\left(\frac{\pi}{120}x\right)$

b. 

c. $h(0) = 0$: after 0 seconds, the rocket is 0 mi above the ground; $h(30) = 2$: after 30 seconds, the rockets is 2 mi high;

d. As x approaches 60 seconds, the values of $h(x)$ grow increasingly large. The distance to the rocket is growing so large that the camera can no longer track it.

6.3: Inverse Trigonometric Functions

In this section, we will explore the inverse trigonometric functions. Inverse trigonometric functions "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa.

Verbal

1) Why do the functions $f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x$ have different ranges?

Answer

The function $y = \sin x$ is one-to-one on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; thus, this interval is the range of the inverse function of $y = \sin x$, $f(x) = \sin^{-1} x$ The function $y = \cos x$ is one-to-one on $[0, \pi]$; thus, this interval is the range of the inverse function of $y = \cos x$, $f(x) = \cos^{-1} x$

2) Since the functions $y = \cos x$ and $y = \cos^{-1} x$ are inverse functions, why is $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ not equal to $-\frac{\pi}{6}$?

3) Explain the meaning of $\frac{\pi}{6} = \arcsin(0.5)$.

Answer

$\frac{\pi}{6}$ is the radian measure of an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is 0.5.

4) Most calculators do not have a key to evaluate $\sec^{-1}(2)$. Explain how this can be done using the cosine function or the inverse cosine function.

5) Why must the domain of the sine function, $\sin x$, be restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the inverse sine function to exist?

Answer

In order for any function to have an inverse, the function must be one-to-one and must pass the horizontal line test. The regular sine function is not one-to-one unless its domain is restricted in some way. Mathematicians have agreed to restrict the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that it is one-to-one and possesses an inverse.

6) Discuss why this statement is incorrect: $\arccos(\cos x) = x$ for all x.

7) Determine whether the following statement is true or false and explain your answer: $\arccos(-x) = \pi - \arccos x$

Answer

True . The angle, θ_1 that equals $\arccos(-x)$, $x > 0$, will be a second quadrant angle with reference angle, θ_2, where θ_2 equals $\arccos x$, $x > 0$. Since θ_2 is the reference angle for θ_1, $\theta_2 = \pi - \theta_1$ and $\arccos(-x) = \pi - \arccos x -$

Algebraic

For the exercises 8-16, evaluate the expressions.

8) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

9) $\sin^{-1}\left(-\frac{1}{2}\right)$

Answer

$-\frac{\pi}{6}$

10) $\cos^{-1}\left(-\frac{1}{2}\right)$

11) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Answer

$\frac{3\pi}{4}$

12) $\tan^{-1}(1)$

13) $\tan^{-1}(-\sqrt{3})$

Answer

$-\frac{\pi}{3}$

14) $\tan^{-1}(-1)$

15) $\tan^{-1}(\sqrt{3})$

Answer

$\frac{\pi}{3}$

16) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

For the exercises 17-21, use a calculator to evaluate each expression. Express answers to the nearest hundredth.

17) $\cos^{-1}(-0.4)$

Answer

1.98

18) $\arcsin(0.23)$

19) $\arccos\left(\frac{3}{5}\right)$

Answer

0.93

20) $\cos^{-1}(-0.8)$

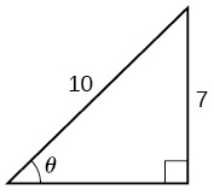
21) $\tan^{-1}(6)$

Answer

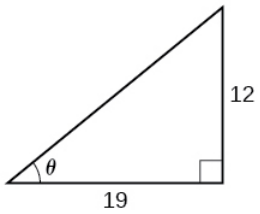
1.41

For the exercises 22-23, find the angle θ in the given right triangle. Round answers to the nearest hundredth.

22)



23)



Answer

0.56radians

For the exercises 24-36, find the exact value, if possible, without a calculator. If it is not possible, explain why.

24) $\sin^{-1}(\cos(\pi))$

25) $\tan^{-1}(\sin(\pi))$

Answer

0

26) $\cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

27) $\tan^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

Answer

0.71

28) $\sin^{-1}\left(\cos\left(\frac{-\pi}{2}\right)\right)$

29) $\tan^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

Answer

-0.71

30) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

31) $\tan^{-1}\left(\sin\left(\frac{-5\pi}{2}\right)\right)$

Answer

$-\frac{\pi}{4}$

32) $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

33) $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

Answer

0.8

34) $\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

35) $\cos\left(\tan^{-1}\left(\frac{12}{5}\right)\right)$

Answer

$\frac{5}{13}$

36) $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

For the exercises 37-41, find the exact value of the expression in terms of x with the help of a reference triangle.

37) $\tan\left(\sin^{-1}(x-1)\right)$

Answer

$\frac{x-1}{\sqrt{-x^2+2x}}$

38) $\sin\left(\sin^{-1}(1-x)\right)$

39) $\cos\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$

Answer

$\frac{\sqrt{x^2-1}}{x}$

40) $\cos\left(\tan^{-1}(3x-1)\right)$

41) $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$

Answer

$\frac{x+0.5}{\sqrt{-x^2-x+\frac{3}{4}}}$

Extensions

For the exercise 42, evaluate the expression without using a calculator. Give the exact value.

2) $\dfrac{\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\cos^{-1}(1)}{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(0)}$

For the exercises 43-47, find the function if $\sin t = \dfrac{x}{x+1}$

43) $\cos t$

Answer

$\dfrac{\sqrt{2x+1}}{x+1}$

44) $\sec t$

45) $\cot t$

Answer

$\dfrac{\sqrt{2x+1}}{x}$

46) $\cos\left(\sin^{-1}\left(\dfrac{x}{x+1}\right)\right)$

47) $\tan^{-1}\left(\dfrac{x}{\sqrt{2x+1}}\right)$

Answer

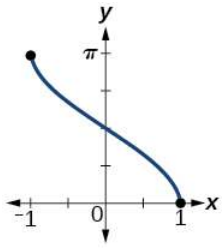
t

Graphical

48) Graph $y=\sin^{-1}x$ and state the domain and range of the function.

49) Graph $y=\arccos x$ and state the domain and range of the function.

Answer



domain $[-1, 1]$; range $[0, \pi]$

50) Graph one cycle of $y = \tan^{-1} x$ and state the domain and range of the function.

51) For what value of x does $\sin x = \sin^{-1} x$? Use a graphing calculator to approximate the answer.

Answer

approximately $x = 0.00$

52) For what value of x does $\cos x = \cos^{-1} x$? Use a graphing calculator to approximate the answer.

Real-World Applications

53) Suppose a 13-foot ladder is leaning against a building, reaching to the bottom of a second-floor window 12 feet above the ground. What angle, in radians, does the ladder make with the building?

Answer

0.395radians

54) Suppose you drive 0.6 miles on a road so that the vertical distance changes from 0 to 150 feet. What is the angle of elevation of the road?

55) An isosceles triangle has two congruent sides of length 9 inches. The remaining side has a length of 8 inches. Find the angle that a side of 9 inches makes with the 8-inch side.

Answer

1.11radians

56) Without using a calculator, approximate the value of $\arctan(10,000)$. Explain why your answer is reasonable.

57) A truss for the roof of a house is constructed from two identical right triangles. Each has a base of 12 feet and height of 4 feet. Find the measure of the acute angle adjacent to the 4-foot side.

Answer

1.25radians

58) The line $y = \frac{3}{5}x$ passes through the origin in the x, y-plane. What is the measure of the angle that the line makes with the positive x-axis?

59) The line $y = \frac{-3}{7}x$ passes through the origin in the x, y-plane. What is the measure of the angle that the line makes with the negative x-axis?

Answer

0.405radians

60) What percentage grade should a road have if the angle of elevation of the road is 4 degrees? (The percentage grade is defined as the change in the altitude of the road over a 100-foot horizontal distance. For example a 5% grade means that the road rises 5 feet for every 100 feet of horizontal distance.)

61) A 20-foot ladder leans up against the side of a building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?

Answer

No. The angle the ladder makes with the horizontal is 60 degrees.

62) Suppose a 15-foot ladder leans against the side of a house so that the angle of elevation of the ladder is 42 degrees. How far is the foot of the ladder from the side of the house?

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3.R: Periodic Functions (Review)

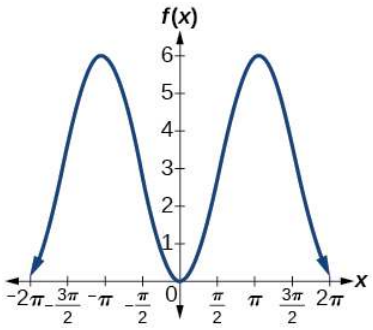
6.1: Graphs of the Sine and Cosine Functions

For the exercises 1-8, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

1) $f(x) = -3\cos x + 3$

Answer

amplitude: 3; period: 2π; midline: $y = 3$; no asymptotes

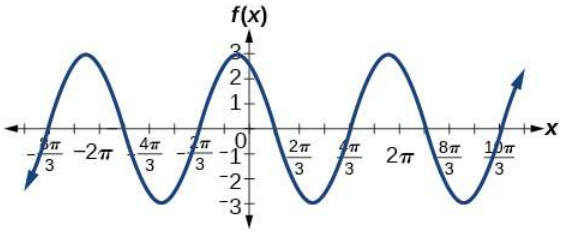


2) $f(x) = \frac{1}{4}\sin x$

3) $f(x) = 3\cos\left(x + \frac{\pi}{6}\right)$

Answer

amplitude: 3; period: 2π; midline: $y = 0$; no asymptotes

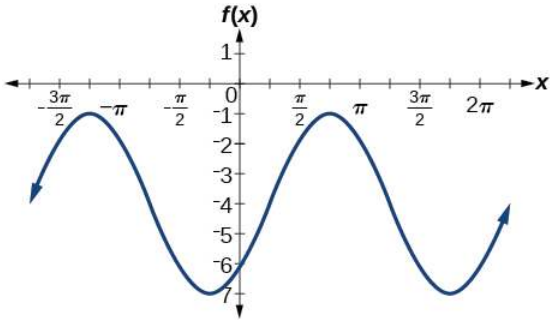


4) $f(x) = -2\sin\left(x - \frac{2\pi}{3}\right)$

5) $f(x) = 3\sin\left(x - \frac{\pi}{4}\right) - 4$

Answer

amplitude: 3; period: 2π; midline: $y = -4$; no asymptotes

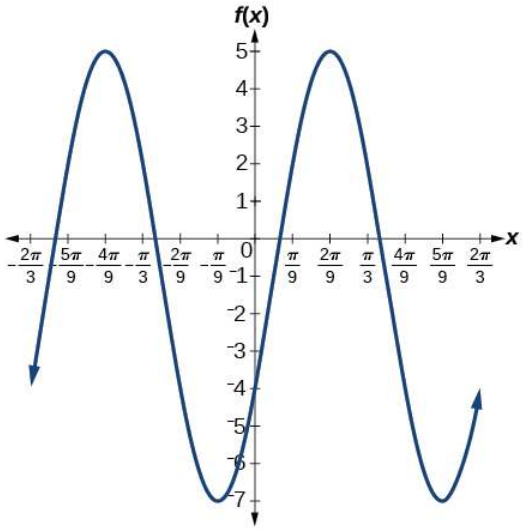


6) $f(x) = 2\left(\cos\left(x - \frac{4\pi}{3}\right) + 1\right)$

7) $f(x) = 6\sin\left(3x - \frac{\pi}{6}\right) - 1$

Answer

amplitude: 6; period: $dfrac2\pi3$; midline: $y = -1$; no asymptotes



8) $f(x) = -100\sin(50x - 20)$

6.2: Graphs of the Other Trigonometric Functions

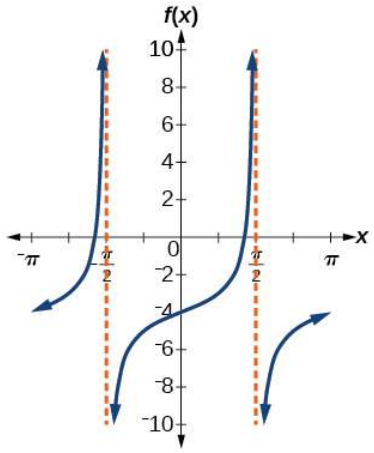
For the exercises 1-4, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

1) $f(x) = \tan x - 4$

Answer

stretching factor: none; period: π; midline: $y = -4$; asymptotes: $x = \frac{\pi}{2} + \pi k$, where k is an integer

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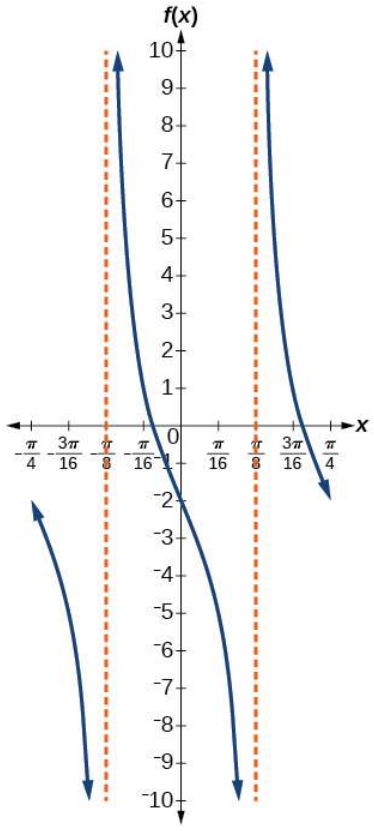


2) $f(x) = 2\tan\left(x - \frac{\pi}{6}\right)$

3) $f(x) = -3\tan(4x) - 2$

Answer

stretching factor: 3; period: $\frac{\pi}{4}$; midline: $y = -2$; asymptotes: $x = \frac{\pi}{8} + \frac{\pi}{4}k$, where k is an integer



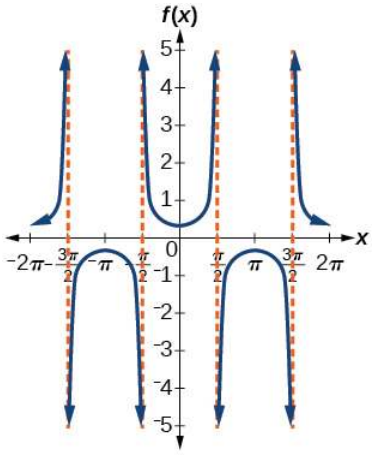
4) $f(x) = 0.2\cos(0.1x) + 0.3$

For the exercises 5-10, graph two full periods. Identify the period, the phase shift, the amplitude, and asymptotes.

5) $f(x) = \frac{1}{3}\sec x$

Answer

amplitude: none; period: 2π; no phase shift; asymptotes: $x = \frac{\pi}{2}k$, where k is an integer

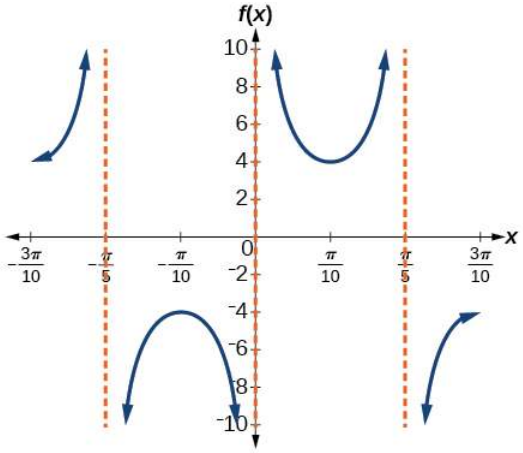


6) $f(x) = 3\cot x$

7) $f(x) = 4\csc(5x)$

Answer

amplitude: none; period: $\frac{2\pi}{5}$; no phase shift; asymptotes: $x = \frac{\pi}{5}k$, where k is an integer

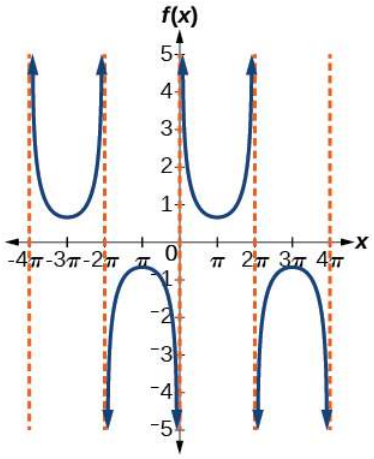


8) $f(x) = 8\sec\left(\frac{1}{4}x\right)$

9) $f(x) = \frac{2}{3}\csc\left(\frac{1}{2}x\right)$

Answer

amplitude: none; period: 4π; no phase shift; asymptotes: $x = 2\pi k$, where k is an integer



10) $f(x) = -\csc(2x + \pi)$

For the exercises 11-15, use this scenario: The population of a city has risen and fallen over a 20-year interval. Its population may be modeled by the following function: $y = 12{,}000 + 8{,}000\sin(0.628x)$, where the domain is the years since 1980 and the range is the population of the city.

11) What is the largest and smallest population the city may have?

Answer

largest: $20,000$; smallest: $4,000$

12) Graph the function on the domain of $[0, 40]$.

13) What are the amplitude, period, and phase shift for the function?

Answer

amplitude: $8,000$; period: 10; phase shift: 0

14) Over this domain, when does the population reach $18,000$? $13,000$?

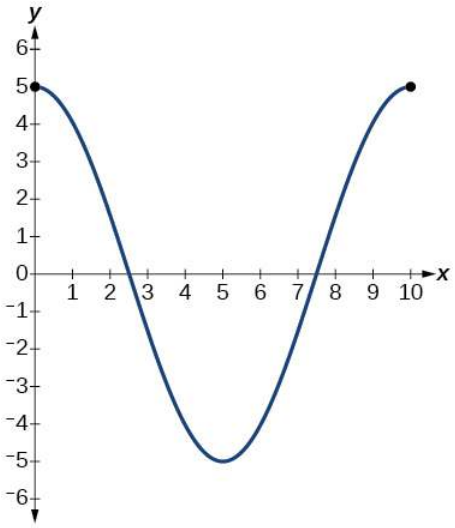
15) What is the predicted population in 2007? 2010?

Answer

In 2007, the predicted population is $4,413$. In 2010, the population will be $11,924$.

For the exercises 16a-16d, suppose a weight is attached to a spring and bobs up and down, exhibiting symmetry.

16) Suppose the graph of the displacement function is shown in the Figure below, where the values on the x-axis represent the time in seconds and the y-axis represents the displacement in inches.



a. Give the equation that models the vertical displacement of the weight on the spring.
b. At $\text{time} = 0$, what is the displacement of the weight?

Answer

5 in.

c. At what time does the displacement from the equilibrium point equal zero?
d. What is the time required for the weight to return to its initial height of 5 inches? In other words, what is the period for the displacement function?

Answer

10 seconds

6.3: Inverse Trigonometric Functions

For the exercises 1-11, find the exact value without the aid of a calculator.

1) $\sin^{-1}(1)$

2) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Answer

$\frac{\pi}{6}$

3) $\tan^{-1}(-1)$

4) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Answer

$\frac{\pi}{4}$

5) $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

6) $\sin^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)$

Answer

$\frac{\pi}{3}$

7) $\cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

8) $\sin\left(\sec^{-1}\left(\frac{3}{5}\right)\right)$

Answer

No solution

9) $\cot\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

10) $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$

Answer

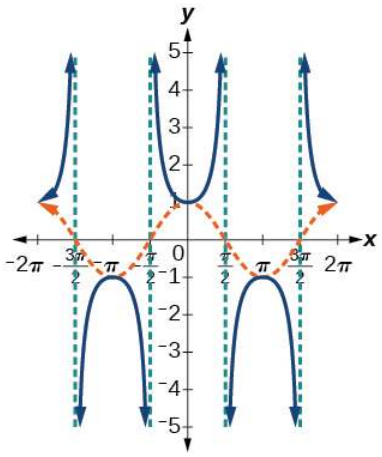
$\frac{12}{5}$

11) $\sin\left(\cos^{-1}\left(\frac{x}{x+1}\right)\right)$

12) Graph $f(x)=\cos x$ and $f(x)=\sec x$ on the interval $[0,2\pi)$ and explain any observations.

Answer

The graphs are not symmetrical with respect to the line $y=x$. They are symmetrical with respect to the y-axis.



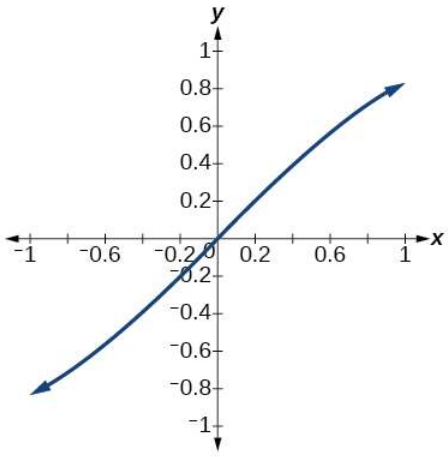
13) Graph $f(x) = \sin x$ and $f(x) = \csc x$ and explain any observations.

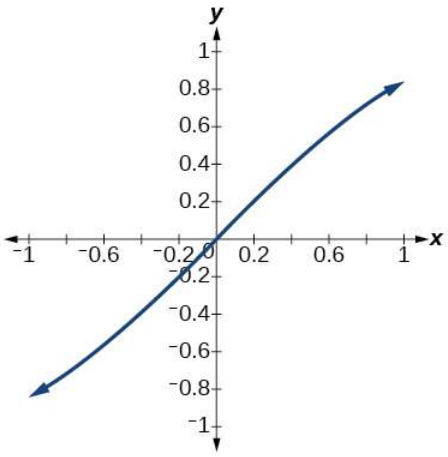
14) Graph the function $f(x) = \dfrac{x}{1} - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}$ on the interval $[-1, 1]$ and compare the graph to the graph of $f(x) = \sin x$ on the same interval. Describe any observations.

Answer

The graphs appear to be identical.

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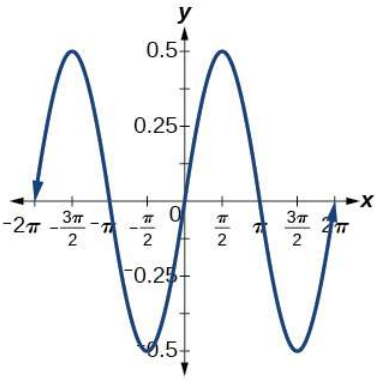
Practice Test

For the exercises 1-13, sketch the graph of each function for two full periods. Determine the amplitude, the period, and the equation for the midline.

1) $f(x) = 0.5 \sin x$

Answer

amplitude: 0.5; period: 2π; midline $y = 0$



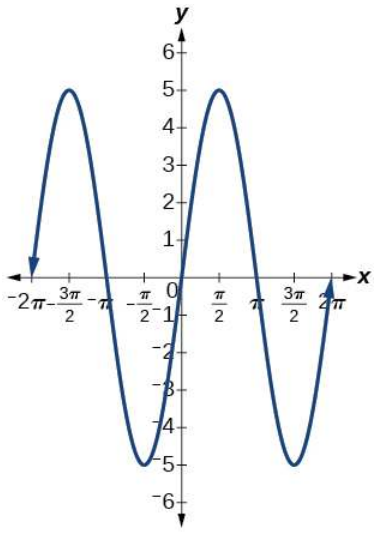
y=0 y=0

2) $f(x) = 5\cos x$

3) $f(x) = 5\sin x$

Answer

amplitude: 0.5; period: 2π; midline $y = 0$

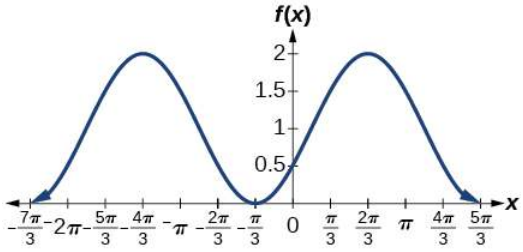


4) $f(x) = \sin(3x)$

5) $f(x) = -\cos\left(x + \dfrac{\pi}{3}\right) + 1$

Answer

amplitude: 1; period: 2π; midline $y = 1$

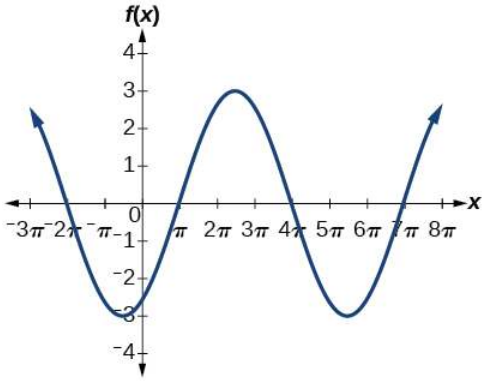


6) $f(x) = 5\sin\left(3\left(x - \dfrac{\pi}{6}\right)\right) + 4$

7) $f(x) = 3\cos\left(\dfrac{1}{3}x - \dfrac{5\pi}{6}\right)$

Answer

amplitude: 3; period: 6π; midline $y = 0$

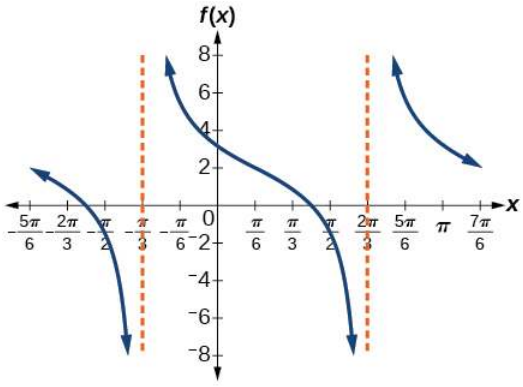


8) $f(x) = \tan(4x)$

9) $f(x) = -2\tan\left(x - \frac{7\pi}{6}\right) + 2$

Answer

amplitude: none; period: π; midline $y = 0$, asymptotes: $x = \frac{2\pi}{3} + \pi k$, where k is an integer



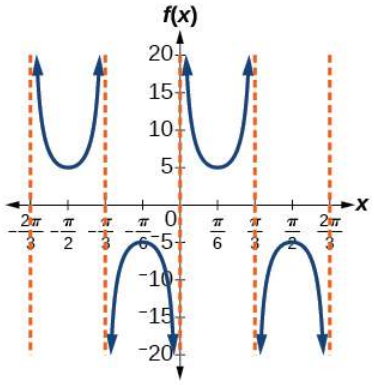
10) $f(x) = \pi\cos(3x + \pi)$

11) $f(x) = 5\csc(3x)$

Answer

amplitude: none; period: $\frac{2\pi}{3}$; midline $y = 0$, asymptotes: $x = \frac{\pi}{3}k$, where k is an integer

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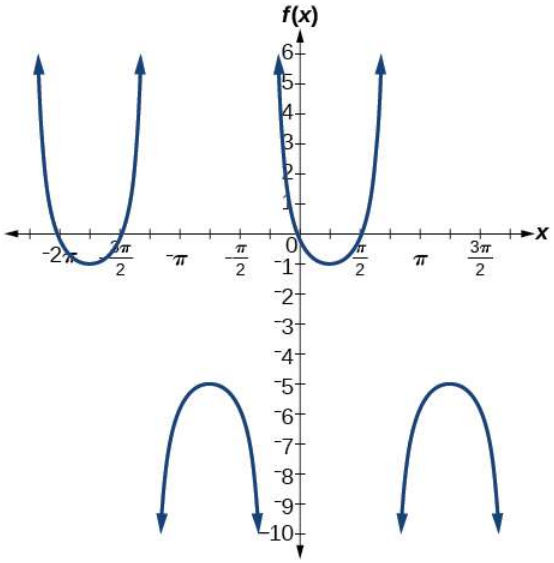


12) $f(x) = \pi \sec\left(\frac{\pi}{2}x\right)$

13) $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 3$

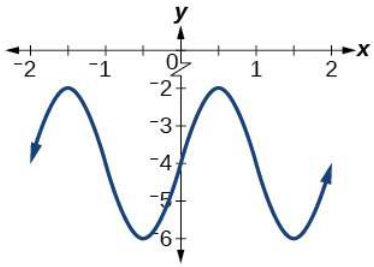
Answer

amplitude: none; period: 2π; midline $y = -3$

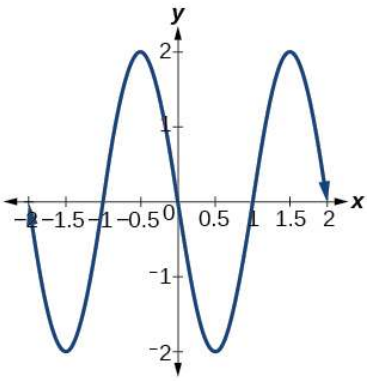


For the exercises 14-16, determine the amplitude, period, and midline of the graph, and then find a formula for the function.

14) Give in terms of a sine function.



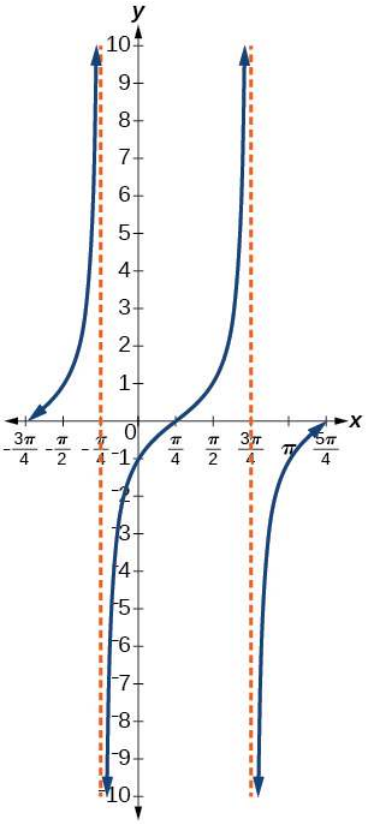
15) Give in terms of a sine function.



Answer

amplitude: 2; period: 2; midline: $y = 0$; $f(x) = 2\sin(\pi(x-1))$

16) Give in terms of a tangent function.



For the exercises 17-20, find the amplitude, period, phase shift, and midline.

17) $y = \sin\left(\frac{\pi}{6}x + \pi\right) - 3$

Answer

amplitude: 1; period: 12; phase shift: -6; midline: $y = -3$

18) $y = 8\sin\left(\frac{7\pi}{6}x + \frac{7\pi}{2}\right) + 6$

19) The outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68°F at midnight and the high and low temperatures during the day are 80°F and 56°F, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

Answer

$D(t) = 68 - 12\sin\left(\frac{\pi}{12}x\right)$

20) Water is pumped into a storage bin and empties according to a periodic rate. The depth of the water is 3 feet at its lowest at 2:00 a.m. and 71 feet at its highest, which occurs every 5 hours. Write a cosine function that models the depth of the water as a function of time, and then graph the function for one period.

For the exercises 21-25, find the period and horizontal shift of each function.

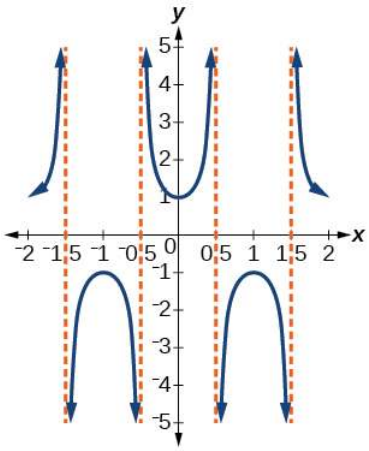
21) $g(x) = 3\tan(6x + 42)$

Answer

period: $\frac{\pi}{6}$; horizontal shift: -7

22) $n(x) = 4\csc\left(\frac{5\pi}{3}x - \frac{20\pi}{3}\right)$

23) Write the equation for the graph in the Figure below in terms of the secant function and give the period and phase shift.



Answer

$f(x) = \sec(\pi x)$; period: 2; phase shift: 0

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24) If $\tan x = 3$, find $\tan(-x)$.

25) If $\sec x = 4$, find $\sec(-x)$.

Answer

4

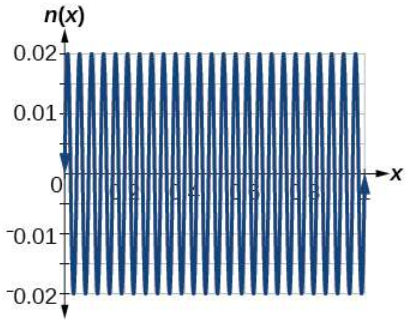
For the exercises 26-28, graph the functions on the specified window and answer the questions.

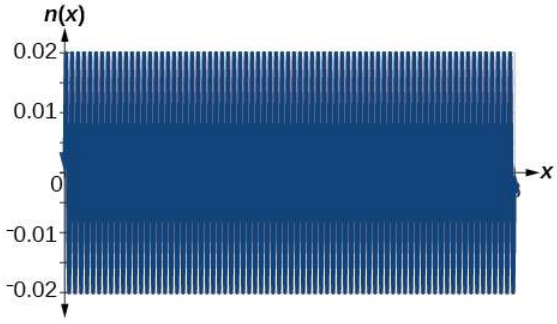
26) Graph $m(x) = \sin(2x) + \cos(3x)$ on the viewing window $[-10, 10]$ by $[-3, 3]$. Approximate the graph's period.

27) Graph $n(x) = 0.02\sin(50\pi x)$ on the following domains in x : $[0, 1]$ and$[0, 3]$. Suppose this function models sound waves. Why would these views look so different?

Answer

The views are different because the period of the wave is 125. Over a bigger domain, there will be more cycles of the graph.





28) Graph $f(x) = \dfrac{\sin x}{x}$ on $[-0.5, 0.5]$ and explain any observations.

For the exercises 29-31, let $f(x) = \dfrac{3}{5}\cos(6x)$.

29) What is the largest possible value for $f(x)$?

Answer

$\frac{3}{5}$

30) What is the smallest possible value for $f(x)$?

31) Where is the function increasing on the interval $[0, 2\pi]$?

Answer

On the approximate intervals $(0.5, 1), (1.6, 2.1), (2.6, 3.1), (3.7, 4.2), (4.7, 5.2), (5.6, 6.28)$

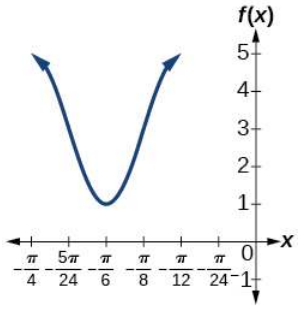
For the exercises 32-33, find and graph one period of the periodic function with the given amplitude, period, and phase shift.

32) Sine curve with amplitude 3, period $\frac{\pi}{3}$, and phase shift $(h, k) = \left(\frac{\pi}{4}, 2\right)$

33) Cosine curve with amplitude 2, period \(\dfrac{\pi }{6}\), and phase shift $(h, k) = \left(-\frac{\pi}{4}, 3\right)$

Answer

$f(x) = 2\cos\left(12\left(x + \frac{\pi}{4}\right)\right) + 3$



For the exercises 34-35, graph the function. Describe the graph and, wherever applicable, any periodic behavior, amplitude, asymptotes, or undefined points.

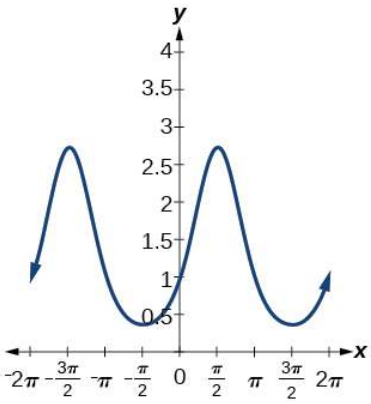
34) $f(x) = 5\cos(3x) + 4\sin(2x)$

35) $f(x) = e^{(sint)}$

Answer

This graph is periodic with a period of 2π

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For the exercises 36-43, find the exact value.

36) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

37) $\tan^{-1}\left(\sqrt{3}\right)$

Answer

$\frac{\pi}{3}$

38) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

39) $\cos^{-1}(\sin(\pi))$

Answer

$\frac{\pi}{2}$

40) $\cos^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right)$

41) $\cos(\sin^{-1}(1-2x))$

Answer

$\sqrt{1-(1-2x)^2}$

42) $\cos^{-1}(-0.4)$

43) $\cos\left(\tan^{-1}\left(x^2\right)\right)$

Answer

$\frac{1}{\sqrt{1+x^4}}$

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For the exercises 44-46, suppose $\sin t = \dfrac{x}{x+1}$. Evaluate the following expressions.

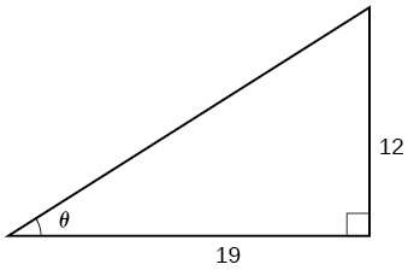
44) $\tan t$

45) $csct$

Answer

$\dfrac{x+1}{x}$

46) Given Figure, find the measure of angle θ to three decimal places. Answer in radians.



For the exercises 47-49, determine whether the equation is true or false.

47) $\arcsin\left(\sin\left(\dfrac{5\pi}{6}\right)\right) = \dfrac{5\pi}{6}$

Answer

False

48) $\arccos\left(\cos\left(\dfrac{5\pi}{6}\right)\right) = \dfrac{5\pi}{6}$

49) The grade of a road is 7%. This means that for every horizontal distance of 100 feet on the road, the vertical rise is 7 feet. Find the angle the road makes with the horizontal in radians.

Answer

approximately 0.07 radians

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CHAPTER OVERVIEW

4: Trigonometric Identities and Equations

In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

4.0: Prelude to Trigonometric Identities and Equations
4.1: Basic Trigonometric Identities
4.2: Sum and Difference Identities
4.3: Double-Angle, Half-Angle, and Reduction Formulas
4.4: Sum-to-Product and Product-to-Sum Formulas
4.5: Solving Trigonometric Equations
4.6: Modeling with Trigonometric Equations
4.E: Trigonometric Identities and Equations (Exercises)
4.R: Trigonometric Identities and Equations (Review)

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4.0: Prelude to Trigonometric Identities and Equations

Math is everywhere, even in places we might not immediately recognize. For example, mathematical relationships describe the transmission of images, light, and sound. The sinusoidal graph in Figure 4.0.1 models music playing on a phone, radio, or computer. Such graphs are described using trigonometric equations and functions. In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

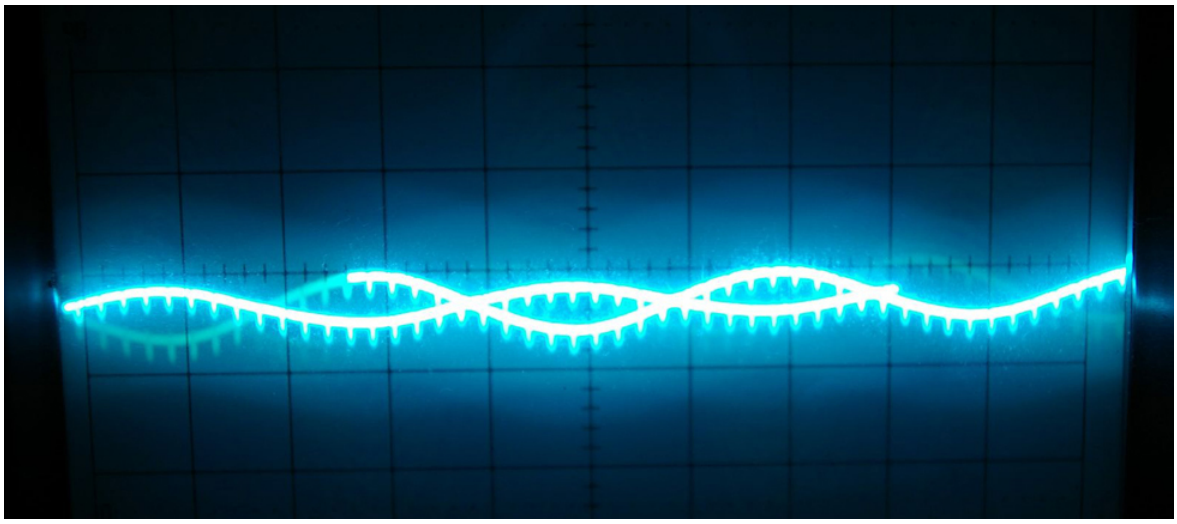


Figure 4.0.1: A sine wave models disturbance. (credit: modification of work by Mikael Altemark, Flickr).

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4.1: Basic Trigonometric Identities

Learning Objectives

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports—there are many ways to represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.



Figure 4.1.1: International passports and travel documents

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the **Pythagorean identities**, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean identities** (Table 4.1.1), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identifies, but now we will also use additional identities.

Table 4.1.1: Pythagorean Identities

$\sin^2\theta+\cos^2\theta=1$	$1+\cot^2\theta=\csc^2\theta$	$1+\tan^2\theta=\sec^2\theta$

The second and third identities can be obtained by manipulating the first. The identity $1+\cot^2\theta=\csc^2\theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1+\cot^2\theta=\csc^2\theta$

$$\begin{aligned} 1+\cot^2\theta &= (1+\frac{\cos^2}{\sin^2}) && \text{Rewrite the left side} \\ &= \left(\frac{\sin^2}{\sin^2}\right)+\left(\frac{\cos^2}{\sin^2}\right) && \text{Write both terms with the common denominator} \\ &= \frac{\sin^2+\cos^2}{\sin^2} \\ &= \frac{1}{\sin^2} \\ &= \csc^2 \end{aligned}$$

Similarly,$1+\tan^2\theta=\sec^2\theta$ can be obtained by rewriting the left side of this identity in terms of sine and cosine. This gives

$$\begin{aligned} 1+\tan^2\theta &= 1+\left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Rewrite left side} \\ &= \left(\frac{\cos\theta}{\cos\theta}\right)^2+\left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Write both terms with the common denominator} \\ &= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} \\ &= \sec^2\theta \end{aligned}$$

Recall that we determined which trigonometric functions are odd and which are even. The next set of fundamental identities is the set of **even-odd identities.** The **even-odd identities** relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle (Table 4.1.2).

Table 4.1.2: Even-Odd Identities

$\tan(-\theta)=-\tan\theta$	$\sin(-\theta)=-\sin\theta$	$\cos(-\theta)=\cos\theta$
$\cot(-\theta)=-\cot\theta$	$\csc(-\theta)=-\csc\theta$	$\sec(-\theta)=\sec\theta$

Recall that an **odd function** is one in which $f(-x)=-f(x)$ for all x in the domain off. f. The sine function is an odd function because $\sin(-\theta)=-\sin\theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is opposite the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$\begin{aligned} \sin\left(\frac{\pi}{2}\right) &= 1 \\ \sin\left(-\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

This is shown in Figure 4.1.2.

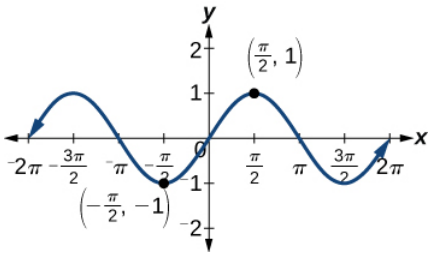


Figure 4.1.2: Graph of $y=\sin\theta$

Recall that an **even function** is one in which

$f(-x)=f(x)$ for all x in the domain of f

The graph of an even function is symmetric about the *y*-axis. The cosine function is an even function because $\cos(-\theta) = \cos\theta$. For example, consider corresponding inputs $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. The output of $\cos\left(\frac{\pi}{4}\right)$ is the same as the output of $\cos\left(-\frac{\pi}{4}\right)$. Thus,

$$\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$
$$\approx 0.707$$

See Figure 4.1.3.

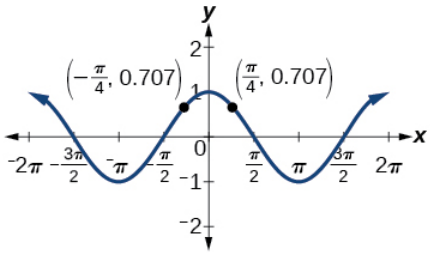


Figure 4.1.3: Graph of $y = \cos\theta$

For all θ in the domain of the sine and cosine functions, respectively, we can state the following:

- Since $\sin(-\theta) = -\sin\theta$,sine is an odd function.
- Since $\cos(-\theta) = \cos\theta$,cosine is an even function.

The other even-odd identities follow from the even and odd nature of the sine and cosine functions. For example, consider the tangent identity,$\tan(-\theta) = -\tan\theta$. We can interpret the tangent of a negative angle as

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$$

Tangent is therefore an odd function, which means that $\tan(-\theta) = -\tan(\theta)$ for all θ in the domain of the **tangent function.**

The cotangent identity, $\cot(-\theta) = -\cot\theta$,also follows from the sine and cosine identities. We can interpret the cotangent of a negative angle as

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$$

Cotangent is therefore an odd function, which means that $\cot(-\theta) = -\cot(\theta)$ for all θ in the domain of the **cotangent function.**

The **cosecant function** is the reciprocal of the sine function, which means that the cosecant of a negative angle will be interpreted as

$$\csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta.$$

The cosecant function is therefore odd.

Finally, the secant function is the reciprocal of the cosine function, and the secant of a negative angle is interpreted as

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta.$$

The secant function is therefore even.

To sum up, only two of the trigonometric functions, cosine and secant, are even. The other four functions are odd, verifying the even-odd identities.

The next set of fundamental identities is the set of reciprocal identities, which, as their name implies, relate trigonometric functions that are reciprocals of each other. (Table 4.1.3). Recall that we first encountered these identities when defining trigonometric functions from right angles in Right Angle Trigonometry.

Table 4.1.3: Reciprocal Identities

$\sin\theta = \dfrac{1}{\csc\theta}$	$\csc\theta = \dfrac{1}{\sin\theta}$
$\cos\theta = \dfrac{1}{\sec\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$
$\tan\theta = \dfrac{1}{\cot\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$

The final set of identities is the set of **quotient identities,** which define relationships among certain trigonometric functions and can be very helpful in verifying other identities (Table 4.1.4).

Table 4.1.4: Quotient Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

The reciprocal and quotient identities are derived from the definitions of the basic trigonometric functions.

SUMMARIZING TRIGONOMETRIC IDENTITIES

The Pythagorean identities are based on the properties of a right triangle.

$$\cos^2\theta + \sin^2\theta = 1 \tag{4.1.1}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{4.1.2}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{4.1.3}$$

The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle.

$$\tan(-\theta) = -\tan\theta \tag{4.1.4}$$

$$\cot(-\theta) = -\cot\theta \tag{4.1.5}$$

$$\sin(-\theta) = -\sin\theta \tag{4.1.6}$$

$$\csc(-\theta) = -\csc\theta \tag{4.1.7}$$

$$\cos(-\theta) = \cos\theta \tag{4.1.8}$$

$$\sec(-\theta) = \sec\theta \tag{4.1.9}$$

The reciprocal identities define reciprocals of the trigonometric functions.

$$\sin\theta = \frac{1}{\csc\theta} \tag{4.1.10}$$

$$\cos\theta = \frac{1}{\sec\theta} \tag{4.1.11}$$

$$\tan\theta = \frac{1}{\cot\theta} \tag{4.1.12}$$

$$\csc\theta = \frac{1}{\sin\theta} \tag{4.1.13}$$

$$\sec\theta = \frac{1}{\cos\theta} \tag{4.1.14}$$

$$\cot\theta = \frac{1}{\tan\theta} \tag{4.1.15}$$

The **quotient identities** define the relationship among the trigonometric functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{4.1.16}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{4.1.17}$$

Example 4.1.1: Graphing the Equations of an Identity

Graph both sides of the identity $\cot\theta = \dfrac{1}{\tan\theta}$. In other words, on the graphing calculator, graph $y = \cot\theta$ and $y = \dfrac{1}{\tan\theta}$.

Solution

See Figure 4.1.4.

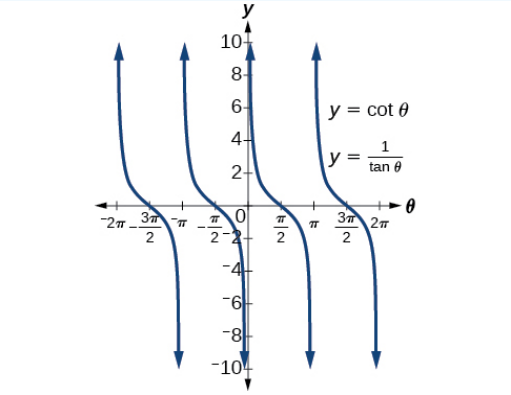


Figure 4.1.4

Analysis

We see only one graph because both expressions generate the same image. One is on top of the other. This is a good way to prove any identity. If both expressions give the same graph, then they must be identities.

How to: Given a trigonometric identity, verify that it is true.

1. Work on one side of the equation. It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. Noting which functions are in the final expression, look for opportunities to use the identities and make the proper substitutions.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

Example 4.1.2: Verifying a Trigonometric Identity

Verify $\tan\theta\cos\theta = \sin\theta$.

Solution

We will start on the left side, as it is the more complicated side:

$$\begin{aligned}\tan\theta\cos\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta \\ &= \sin\theta\end{aligned}$$

Analysis

This identity was fairly simple to verify, as it only required writing $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

? Exercise 4.1.1

Verify the identity $\csc\theta\cos\theta\tan\theta = 1$.

Answer

$$\begin{aligned}\csc\theta\cos\theta\tan\theta &= \left(\frac{1}{\sin\theta}\right)\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\cos\theta}{\sin\theta}\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\\ &= 1\end{aligned}$$

✓ Example 4.1.3A: Verifying the Equivalency Using the Even-Odd Identities

Verify the following equivalency using the even-odd identities:

$$(1+\sin x)[1+\sin(-x)] = \cos^2 x$$

Solution

Working on the left side of the equation, we have

$$(1+\sin x)[1+\sin(-x)] = (1+\sin x)(1-\sin x)$$

Since

$$\begin{aligned}\sin(-x) &= -\sin x\\ [5pt] &= 1-\sin^2 x \qquad \text{Difference of squares}\\ [5pt] &= \cos^2 x\\ \cos^2 x &= 1-\sin^2 x\end{aligned}$$

✓ Example 4.1.3B: Verifying a Trigonometric Identity Involving $\sec^2\theta$

Verify the identity $\dfrac{\sec^2\theta-1}{\sec^2\theta} = \sin^2\theta$

Solution

As the left side is more complicated, let's begin there.

$$\begin{aligned}\frac{\sec^2\theta-1}{\sec^2\theta} &= \frac{(\tan^2\theta+1)-1}{\sec^2\theta}\\ \sec^2\theta &= \tan^2\theta+1\\ &= \frac{\tan^2\theta}{\sec^2\theta}\\ &= \tan^2\theta\left(\frac{1}{\sec^2\theta}\right)\\ &= \tan^2\theta\left(\cos^2\theta\right)\\ \cos^2\theta &= \frac{1}{\sec^2\theta}\\ &= \left(\frac{\sin^2\theta}{\cos^2\theta}\right)\\ \tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta}\\ &= \sin^2\theta\end{aligned}$$

There is more than one way to verify an identity. Here is another possibility. Again, we can start with the left side.

$$\begin{aligned}\frac{\sec^2\theta-1}{\sec^2\theta} &= \frac{\sec^2\theta}{\sec^2\theta}-\frac{1}{\sec^2\theta}\\ &= 1-\cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

Analysis

In the first method, we used the identity $\sec^2\theta=\tan^2\theta+1$ and continued to simplify. In the second method, we split the fraction, putting both terms in the numerator over the common denominator. This problem illustrates that there are multiple ways we can verify an identity. Employing some creativity can sometimes simplify a procedure. As long as the substitutions are correct, the answer will be the same.

Exercise 4.1.2

Show that $\dfrac{\cot\theta}{\csc\theta}=\cos\theta$.

Answer

$$\begin{aligned}\frac{\cot\theta}{\csc\theta} &= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}}\\ &= \frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1}\\ &= \cos\theta\end{aligned}$$

Example 4.1.4: Creating and Verifying an Identity

Create an identity for the expression $2\tan\theta\sec\theta$ by rewriting strictly in terms of sine.

Solution

There are a number of ways to begin, but here we will use the quotient and reciprocal identities to rewrite the expression:

$$\begin{aligned}2\tan\theta\sec\theta &= 2\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right)\\ &= \frac{2\sin\theta}{\cos^2\theta}\\ &= \frac{2\sin\theta}{1-\sin^2\theta} \qquad \text{Substitute } 1-\sin^2\theta \text{ for } \cos^2\theta\end{aligned}$$

Thus,

$$2\tan\theta\sec\theta=\frac{2\sin\theta}{1-\sin^2\theta}$$

Example 4.1.5: Verifying an Identity Using Algebra and Even/Odd Identities

Verify the identity:

$$\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}=\cos\theta-\sin\theta$$

Solution

Let's start with the left side and simplify:

$$\begin{aligned}\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} &= \frac{[\sin(-\theta)]^2-[\cos(-\theta)]^2}{\sin(-\theta)-\cos(-\theta)} \\ &= \frac{(-\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} \quad ,\sin(-x)=-\sin\ x \text{ and } \cos(-x)=\cos\ x \\ &= \frac{(\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} \quad \text{Difference of squares} \\ &= \frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{-(\sin\theta+\cos\theta)} \\ &= \cos\theta-\sin\theta\end{aligned}$$

? Exercise 4.1.3

Verify the identity $\dfrac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta}=\dfrac{\sin\theta+1}{\tan\theta}$.

Answer

$$\begin{aligned}\frac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta} &= \frac{(\sin\theta+1)(\sin\theta-1)}{\tan\theta(\sin\theta-1)} \\ &= \frac{\sin\theta+1}{\tan\theta}\end{aligned}$$

✓ Example 4.1.6: Verifying an Identity Involving Cosines and Cotangents

Verify the identity: $(1-\cos^2x)(1+\cot^2x)=1$.

Solution

$$\begin{aligned}(1-\cos^2x)(1+\cot^2x) &= (1-\cos^2x)\left(1+\frac{\cos^2x}{\sin^2x}\right) \\ &= (1-\cos^2x)\left(\frac{\sin^2x}{\sin^2x}+\frac{\cos^2x}{\sin^2x}\right) \quad \text{Find the common denominator} \\ &= (1-\cos^2x)\left(\frac{\sin^2x+\cos^2x}{\sin^2x}\right) \\ &= (\sin^2x)\left(\frac{1}{\sin^2x}\right) \\ &= 1\end{aligned}$$

Using Algebra to Simplify Trigonometric Expressions

We have seen that algebra is very important in verifying trigonometric identities, but it is just as critical in simplifying trigonometric expressions before solving. Being familiar with the basic properties and formulas of algebra, such as the difference of squares formula, the perfect square formula, or substitution, will simplify the work involved with trigonometric expressions and equations.

For example, the equation $(\sin x+1)(\sin x-1)=0$ resembles the equation $(x+1)(x-1)=0$, which uses the factored form of the difference of squares. Using algebra makes finding a solution straightforward and familiar. We can set each factor equal to zero and solve. This is one example of recognizing algebraic patterns in trigonometric expressions or equations.

Another example is the difference of squares formula, $a^2-b^2=(a-b)(a+b)$, which is widely used in many areas other than mathematics, such as engineering, architecture, and physics. We can also create our own identities by continually expanding an expression and making the appropriate substitutions. Using algebraic properties and formulas makes many trigonometric equations easier to understand and solve.

Example 4.1.7A: Writing the Trigonometric Expression as an Algebraic Expression

Write the following trigonometric expression as an algebraic expression: $2\cos^2\theta+\cos\theta-1$.

Solution

Notice that the pattern displayed has the same form as a standard quadratic expression,ax^2+bx+c . Letting $\cos\theta=x$,we can rewrite the expression as follows:

$$2x^2+x-1$$

This expression can be factored as $(2x+1)(x-1)$. If it were set equal to zero and we wanted to solve the equation, we would use the zero factor property and solve each factor for x. At this point, we would replace x with $\cos\theta$ and solve for θ.

Example 4.1.7B: Rewriting a Trigonometric Expression Using the Difference of Squares

Rewrite the trigonometric expression using the difference of squares: $4cos^2\theta-1$.

Solution

Notice that both the coefficient and the trigonometric expression in the first term are squared, and the square of the number 1 is 1. This is the difference of squares.

$$\begin{aligned}4\cos^2\theta-1 &= (2\cos\theta)^2-1\\ &= (2\cos\theta-1)(2\cos\theta+1)\end{aligned}$$

Analysis

If this expression were written in the form of an equation set equal to zero, we could solve each factor using the zero factor property. We could also use substitution like we did in the previous problem and let $\cos\theta=x$, rewrite the expression as $4x^2-1$, and factor $(2x-1)(2x+1)$. Then replace x with $\cos\theta$ and solve for the angle.

Exercise 4.1.4

Rewrite the trigonometric expression using the difference of squares: $25-9\sin^2\theta$.

Answer

This is a difference of squares formula: $25-9\sin^2\theta=(5-3\sin\theta)(5+3\sin\theta)$.

Example 4.1.8: Simplify by Rewriting and Using Substitution

Simplify the expression by rewriting and using identities:

$$\csc^2\theta-\cot^2\theta$$

Solution

We can start with the Pythagorean identity.

$$1+\cot^2\theta=\csc^2\theta$$

Now we can simplify by substituting $1+\cot^2\theta$ for $\csc^2\theta$

$$\begin{aligned}\csc^2\theta-\cot^2\theta &= 1+\cot^2\theta-\cot^2\theta\\ &= 1\end{aligned}$$

Exercise 4.1.5

Use algebraic techniques to verify the identity: $\dfrac{\cos\theta}{1+\sin\theta}=\dfrac{1-\sin\theta}{\cos\theta}$.

(Hint: Multiply the numerator and denominator on the left side by $1-\sin\theta$.)

Answer

$$\begin{aligned}\frac{\cos\theta}{1+\sin\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right) &= \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta} \\ &= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta} \\ &= \frac{1-\sin\theta}{\cos\theta}\end{aligned}$$

Media

Access these online resources for additional instruction and practice with the fundamental trigonometric identities.

- Fundamental Trigonometric Identities
- Verifying Trigonometric Identities

Key Equations

Pythagorean identities	$\cos^2\theta+\sin^2\theta=1$ $1+\cot^2\theta=\csc^2\theta$ $1+\tan^2\theta=\sec^2\theta$
Even-odd identities	$\tan(-\theta)=-\tan\theta$ $\cot(-\theta)=-\cot\theta$ $\sin(-\theta)=-\sin\theta$ $\csc(-\theta)=-\csc\theta$ $\cos(-\theta)=\cos\theta$ $\sec(-\theta)=\sec\theta$
Reciprocal identities	$\sin\theta=\frac{1}{\csc\theta}$ $\cos\theta=\frac{1}{\sec\theta}$ $\tan\theta=\frac{1}{\cot\theta}$ $\csc\theta=\frac{1}{\sin\theta}$ $\sec\theta=\frac{1}{\cos\theta}$ $\cot\theta=\frac{1}{\tan\theta}$
Quotient identities	$\tan\theta=\frac{\sin\theta}{\cos\theta}$ $\cot\theta=\frac{\cos\theta}{\sin\theta}$

Key Concepts

- There are multiple ways to represent a trigonometric expression. Verifying the identities illustrates how expressions can be rewritten to simplify a problem.
- Graphing both sides of an identity will verify it. See Example 4.1.1.
- Simplifying one side of the equation to equal the other side is another method for verifying an identity. See Example 4.1.2 and Example 4.1.3.
- The approach to verifying an identity depends on the nature of the identity. It is often useful to begin on the more complex side of the equation. See Example 4.1.4.
- We can create an identity and then verify it. See Example 4.1.5.
- Verifying an identity may involve algebra with the fundamental identities. See Example 4.1.6 and Example 4.1.7.

- Algebraic techniques can be used to simplify trigonometric expressions. We use algebraic techniques throughout this text, as they consist of the fundamental rules of mathematics. See Example 4.1.8, Example 4.1.9, and Example 4.1.10.

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4.2: Sum and Difference Identities

Learning Objectives

- Use sum and difference formulas for cosine.
- Use sum and difference formulas for sine.
- Use sum and difference formulas for tangent.
- Use sum and difference formulas for cofunctions.
- Use sum and difference formulas to verify identities.

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.



Figure 4.2.1: Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown in Figure 4.2.2.

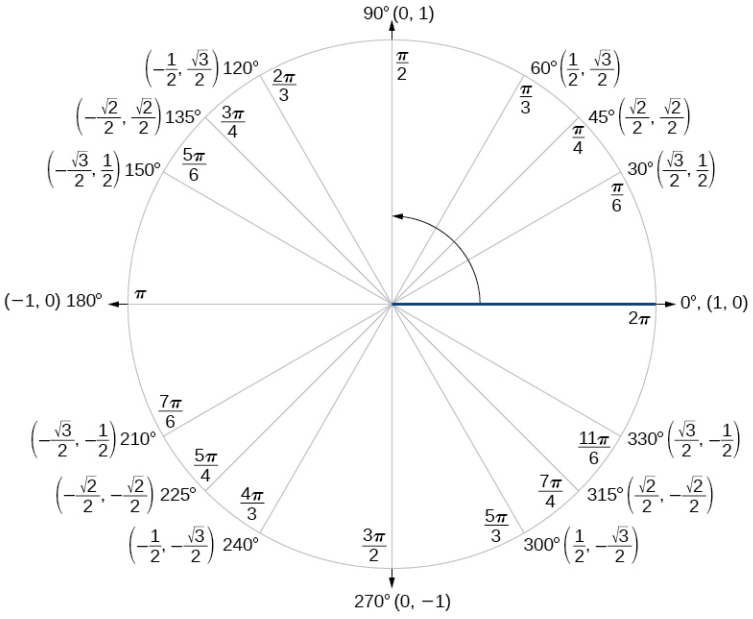


Figure 4.2.2: The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles (Table 4.2.1).

Table 4.2.1

Sum formula for cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle (Figure 4.2.3). Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha,\sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta,\sin\beta)$. Note the measure of angle POQ is $\alpha-\beta$.

Label two more points: A at an angle of $(\alpha-\beta)$ from the positive x-axis with coordinates $(\cos(\alpha-\beta),\sin(\alpha-\beta))$; and point B with coordinates $(1,0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.

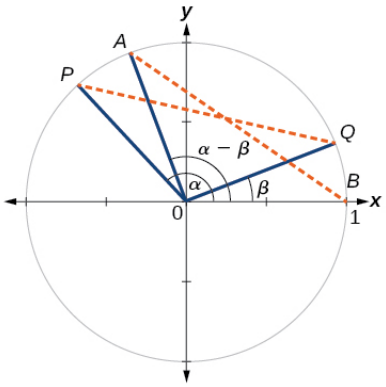


Figure 4.2.3

We can find the distance from P to Q using the distance formula.

$$\begin{aligned} d_{PQ} &= \sqrt{(\cos\alpha-\cos\beta)^2+(sin\alpha-sin\beta)^2} \\ &= \sqrt{\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta} && \text{Apply Pythagorean identity and simplify.} \\ &= \sqrt{(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \\ &= \sqrt{1+1-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \\ &= \sqrt{2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \end{aligned}$$

Similarly, using the distance formula we can find the distance from A to B.

$$\begin{aligned} d_{AB} &= \sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2} \\ &= \sqrt{\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)} && \text{Apply Pythagorean identity and simplify} \\ &= \sqrt{(\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta))-2\cos(\alpha-\beta)+1} \\ &= \sqrt{1-2\cos(\alpha-\beta)+1} \\ &= \sqrt{2-2\cos(\alpha-\beta)} && \text{Subtract 2 from both sides and divide both sides by } -2. \end{aligned}$$

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta = \cos(\alpha-\beta)$$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

SUM AND DIFFERENCE FORMULAS FOR COSINE

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \tag{4.2.1}$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \tag{4.2.2}$$

How to: Given two angles, find the cosine of the difference between the angles

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Example 4.2.1: Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right)$.

Solution

Begin by writing the formula for the cosine of the difference of two angles. Then substitute the given values.

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ \cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= -\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in radian mode.

Exercise 4.2.1:

Find the exact value of $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$.

Answer

$\frac{\sqrt{2}+\sqrt{6}}{4}$

Example 4.2.2: Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of $\cos(75^\circ)$.

Solution

As $75^\circ = 45^\circ + 30^\circ$,we can evaluate $\cos(75^\circ)$ as $\cos(45^\circ + 30^\circ)$.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\ \cos(45^\circ+30^\circ) &= \cos(45^\circ)\cos(30^\circ)-\sin(45^\circ)\sin(30^\circ)\\ &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in degree mode.

Analysis

Note that we could have also solved this problem using the fact that $75^\circ = 135^\circ - 60^\circ$.

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ \cos(135^\circ-60^\circ) &= \cos(135^\circ)\cos(60^\circ)+\sin(135^\circ)\sin(60^\circ)\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= -\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

? Exercise 4.2.2

Find the exact value of $\cos(105^\circ)$.

Answer

$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

SUM AND DIFFERENCE FORMULAS FOR SINE

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \tag{4.2.3}$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \tag{4.2.4}$$

How to: Given two angles, find the sine of the difference between the angles

1. Write the difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

Example 4.2.3: Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45^\circ-30^\circ)$
b. $\sin(135^\circ-120^\circ)$

Solution

a. Let's begin by writing the formula and substitute the given angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(45^\circ-30^\circ) &= \sin(45^\circ)\cos(30^\circ)-\cos(45^\circ)\sin(30^\circ)\end{aligned}$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45^\circ)=\tfrac{\sqrt{2}}{2},\qquad \cos(30^\circ)=\tfrac{\sqrt{3}}{2},\qquad \cos(45^\circ)=\tfrac{\sqrt{2}}{2},\qquad \sin(30^\circ)=\tfrac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(45^\circ-30^\circ) &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b. Again, we write the formula and substitute the given angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(135^\circ-120^\circ) &= \sin(135^\circ)\cos(120^\circ)-\cos(135^\circ)\sin(120^\circ)\end{aligned}$$

Next, we find the values of the trigonometric expressions.

$$\sin(135^\circ)=\tfrac{\sqrt{2}}{2},\qquad \cos(120^\circ)=-\tfrac{1}{2},\qquad \cos(135^\circ)=\tfrac{\sqrt{2}}{2},\qquad \sin(120^\circ)=\tfrac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(135^\circ-120^\circ) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example 4.2.4: Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right)$. Then check the answer with a graphing calculator.

Solution

The pattern displayed in this problem is $\sin(\alpha+\beta)$. Let $\alpha=\cos^{-1}\frac{1}{2}$ and $\beta=\sin^{-1}\frac{3}{5}$. Then we can write

$$\begin{aligned}\cos\alpha &= \frac{1}{2}, \quad 0\le\alpha\le\pi\\ \sin\beta &= \frac{3}{5}, \quad -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}\end{aligned}$$

We will use the Pythagorean identities to find $\sin\alpha$ and $\cos\beta$

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha}\\ &= \sqrt{1-\frac{1}{4}}\\ &= \sqrt{\frac{3}{4}}\\ &= \frac{\sqrt{3}}{2}\\ \cos\beta &= \sqrt{1-\sin^2\beta}\\ &= \sqrt{1-\frac{9}{25}}\\ &= \sqrt{\frac{16}{25}}\\ &= \frac{4}{5}\end{aligned}$$

Using the sum formula for sine,

$$\begin{aligned}\sin\left(\cos^{-1}\tfrac{1}{2}+\sin^{-1}\tfrac{3}{5}\right) &= \sin(\alpha+\beta)\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \frac{\sqrt{3}}{2}\cdot\frac{4}{5}+\frac{1}{2}\cdot\frac{3}{5}\\ &= \frac{4\sqrt{3}+3}{10}\end{aligned}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x=\dfrac{\sin x}{\cos x}$, when $\cos x\neq 0$.

Let's derive the sum formula for tangent.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\ &= \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\end{aligned}$$

We can derive the difference formula for tangent in a similar way.

SUM AND DIFFERENCE FORMULAS FOR TANGENT

The sum and difference formulas for tangent are:

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{4.2.5}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{4.2.6}$$

How to: Given two angles, find the tangent of the sum of the angles

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

Example 4.2.5: Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$.

Solution

Let's first write the sum formula for tangent and then substitute the given angles into the formula.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\tan\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{4}\right)}{1-\left(\tan\left(\frac{\pi}{6}\right)\right)\left(\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual function values within the formula:

$$\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}},\quad \text{and}\quad \tan\left(\frac{\pi}{4}\right)=1 \tag{4.2.7}$$

So we have,

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\frac{1}{\sqrt{3}}+1}{1-\left(\frac{1}{\sqrt{3}}\right)(1)}$$

$$=\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$$=\frac{1+\sqrt{3}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}-1}$$

$$=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

? Exercise 4.2.5:

Find the exact value of $\tan\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)$.

Answer

$\frac{1-\sqrt{3}}{1+\sqrt{3}}$

✓ Example 4.2.6: Finding Multiple Sums and Differences of Angles

Given $\sin\alpha=\frac{3}{5}, \quad 0<\alpha<\frac{\pi}{2}$, and $\cos\beta=-\frac{5}{13}, \quad \pi<\beta<\frac{3\pi}{2}$,

find

a. $\sin(\alpha+\beta)$
b. $\cos(\alpha+\beta)$
c. $\tan(\alpha+\beta)$
d. $\tan(\alpha-\beta)$

Solution

We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

a. To find $\sin(\alpha+\beta)$, we begin with $\sin\alpha=\frac{3}{5}$ and $0<\alpha<\frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See Figure 4.2.4. Using the Pythagorean Theorem, we can find the length of side a:

$$a^2+3^2=5^2$$
$$a^2=16$$
$$a=4$$

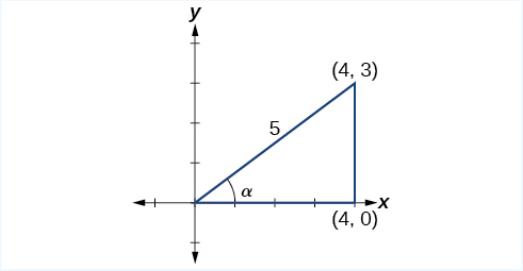


Figure 4.2.4

Since $\cos\beta=-\frac{5}{13}$ and $\pi<\beta<\frac{3\pi}{2}$,the side adjacent to β is -5,the hypotenuse is 13, and β is in the third quadrant. See Figure 4.2.5. Again, using the Pythagorean Theorem, we have

$$\begin{aligned}(-5)^2+a^2 &= 13^2\\ 25+a^2 &= 169\\ a^2 &= 144\\ a &= \pm 12\end{aligned}$$

Since β is in the third quadrant, $a = -12$.

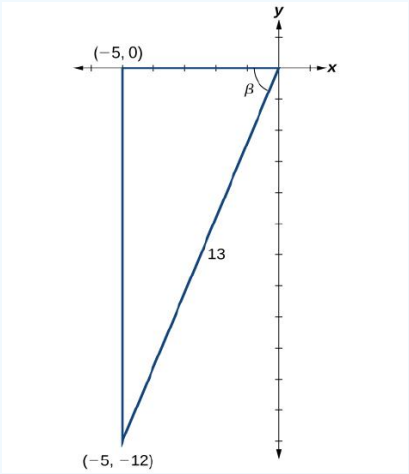


Figure 4.2.5

The next step is finding the cosine of α and the sine of β. The cosine ofα α is the adjacent side over the hypotenuse. We can find it from the triangle in Figure 4.2.5: $\cos\alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in Figure 4.2.5, as opposite side over the hypotenuse: $\sin\beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha+\beta)$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)+\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)\\ &= -\frac{15}{65}-\frac{48}{65}\\ &= -\frac{63}{65}\end{aligned}$$

b. We can find $\cos(\alpha+\beta)$ in a similar manner. We substitute the values according to the formula.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right)-\left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)\\ &= -\frac{20}{65}+\frac{36}{65}\\ &= \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha+\beta)$,if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, then

$$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

If $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$, then

$$\tan\beta = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$$

Then,

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{\frac{63}{20}}{-\frac{16}{20}}\\ &= -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha-\beta)$, we have the values we need. We can substitute them in and evaluate.

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4}-\frac{12}{5}}{1+\frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{-\frac{33}{20}}{\frac{56}{20}}\\ &= -\frac{33}{56}\end{aligned}$$

Analysis

A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that

$$\tan(\alpha+\beta) = \sin(\alpha+\beta)\cos(\alpha+\beta)$$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from Right Triangle Trigonometry that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In Figure 4.2.6, notice that if one of the acute angles is labeled as θ, then the other acute angle must be labeled $\frac{\pi}{2}-\theta$.

Notice also that $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$,which is opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

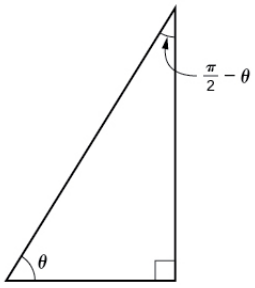


Figure 4.2.6

From these relationships, the cofunction identities are formed. Recall that you first encountered these identities in The Unit Circle: Sine and Cosine Functions.

COFUNCTION IDENTITIES

The cofunction identities are summarized in Table 4.2.2.

Table 4.2.2

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Notice that the formulas in the table may also justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

We can write

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta \\ &= \sin\theta\end{aligned}$$

Example 4.2.7: Finding a Cofunction with the Same Value as the Given Expression

Write $\tan\frac{\pi}{9}$ in terms of its cofunction.

Solution

The cofunction of $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$. Thus,

$$\begin{aligned}\tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{9}\right) \\ &= \cot\left(\frac{9\pi}{18} - \frac{2\pi}{18}\right) \\ &= \cot\left(\frac{7\pi}{18}\right)\end{aligned}$$

Exercise 4.2.7

Write $\sin\frac{\pi}{7}$ in terms of its cofunction.

Answer

$\cos\left(\frac{5\pi}{14}\right)$

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules presented earlier may help simplify the process of verifying an identity.

How to: Given an identity, verify using sum and difference formulas

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Example 4.2.8A: Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

Solution

We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ &= 2\sin\alpha\cos\beta\end{aligned}$$

We see that the identity is verified.

Example 4.2.8B: Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\tan\beta$$

Solution

We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned}\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} &= \frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\\ &= \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} && \text{Rewrite using a common denominator}\\ &= \frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta} && \text{Cancel}\\ &= \tan\alpha-\tan\beta && \text{Rewrite in terms of tangent}\end{aligned}$$

Exercise 4.2.8:

Verify the identity: $\tan(\pi-\theta)=-\tan\theta$.

Answer

$$\begin{aligned}\tan(\pi-\theta) &= \frac{\tan(\pi)-\tan\theta}{1+\tan(\pi)\tan\theta}\\ &= \frac{0-\tan\theta}{1+0\cdot\tan\theta}\\ &= -\tan\theta\end{aligned}$$

Example 4.2.9A: Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2. See Figure 4.2.7. Show that

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan\theta_1=m_1$ and $\tan\theta_2=m_2$.)

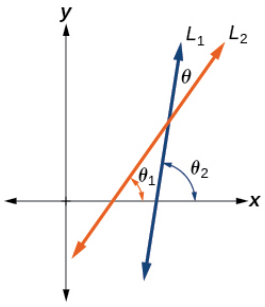


Figure 4.2.7

Solution

Using the difference formula for tangent, this problem does not seem as daunting as it might.

$$\begin{aligned}\tan\theta &= \tan(\theta_2 - \theta_1)\\ &= \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_1\tan\theta_2}\\ &= \frac{m_2 - m_1}{1+m_1 m_2}\end{aligned}$$

Example 4.2.9B: Investigating a Guy-wire Problem

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See Figure 4.2.8.

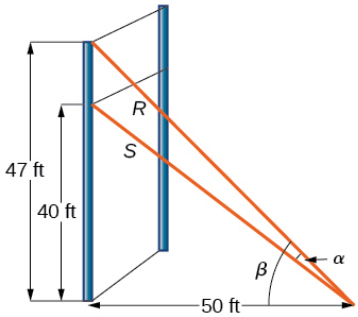


Figure 4.2.8

Solution

Let's first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan\beta = \frac{47}{50}$, and $\tan(\beta-\alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha} \tag{4.2.8}$$

Now, substituting the values we know into the formula, we have,

$$\frac{4}{5} = \frac{\frac{47}{50} - \tan\alpha}{1+\frac{47}{50}\tan\alpha}$$

$$4\left(1+\tfrac{47}{50}\tan\alpha\right) = 5\left(\tfrac{47}{50} - \tan\alpha\right)$$

Use the distributive property, and then simplify the functions.

$$\begin{aligned} 4(1)+4\left(\tfrac{47}{50}\right)\tan\alpha &= 5\left(\tfrac{47}{50}\right)-5\tan\alpha \\ 4+3.76\tan\alpha &= 4.7-5\tan\alpha \\ 5\tan\alpha+3.76\tan\alpha &= 0.7 \\ 8.76\tan\alpha &= 0.7 \\ \tan\alpha &\approx 0.07991 \\ \tan^{-1}(0.07991) &\approx .079741 \end{aligned}$$

Now we can calculate the angle in degrees.

$$\begin{aligned} \alpha &\approx 0.079741\left(\frac{180}{\pi}\right) \\ &\approx 4.57^\circ \end{aligned}$$

Analysis

Occasionally, when an application appears that includes a right triangle, we may think that solving is a matter of applying the Pythagorean Theorem. That may be partially true, but it depends on what the problem is asking and what information is given.

Media

Access these online resources for additional instruction and practice with sum and difference identities.

- Sum and Difference Identities for Cosine
- Sum and Difference Identities for Sine
- Sum and Difference Identities for Tangent

Key Equations

Sum Formula for Cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference Formula for Cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
Sum Formula for Sine	$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
Difference Formula for Sine	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
Sum Formula for Tangent	$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
Difference Formula for Tangent	$\tan(\alpha-\beta)=\dfrac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$
Cofunction identities	$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$ $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$ $\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right)$ $\cot\theta=\tan\left(\frac{\pi}{2}-\theta\right)$ $\sec\theta=\csc\left(\frac{\pi}{2}-\theta\right)$ $\csc\theta=\sec\left(\frac{\pi}{2}-\theta\right)$

Key Concepts

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle. See Example 4.2.1 and Example 4.2.2.
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle. See Example 4.2.3.
- The sum and difference formulas for sine and cosine can also be used for inverse trigonometric functions. See Example 4.2.4.

- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles. The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles. See Example 4.2.5.
- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles. See Example 4.2.6.
- The cofunction identities apply to complementary angles and pairs of reciprocal functions. See Example 4.2.7.
- Sum and difference formulas are useful in verifying identities. See Example 4.2.8 and Example 4.2.9.
- Application problems are often easier to solve by using sum and difference formulas. See Example 4.2.10 and Example 4.2.11.

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4.3: Double-Angle, Half-Angle, and Reduction Formulas

Learning Objectives

- Use double-angle formulas to find exact values
- Use double-angle formulas to verify identities
- Use reduction formulas to simplify an expression
- Use half-angle formulas to find exact values

Bicycle ramps made for competition (see Figure 4.3.1) must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be θ such that $\tan\theta = \frac{5}{3}$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

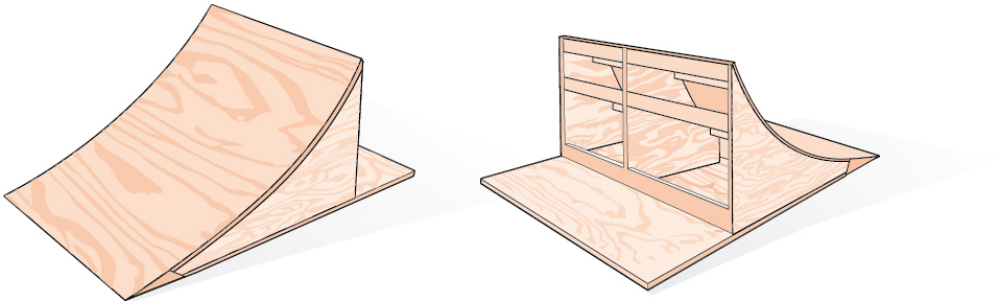


Figure 4.3.1: Bicycle ramps for advanced riders have a steeper incline than those designed for novices.

Using Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The **double-angle formulas** are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{4.3.1}$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned}\sin(\theta+\theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ \sin(2\theta) &= 2\sin\theta\cos\theta\end{aligned}$$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha+\beta) = \cos\,\alpha\,\cos\,\beta - \sin\,\alpha\,\sin\,\beta$,and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned}\cos(\theta+\theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more variations. The first variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= (1-\sin^2\theta) - \sin^2\theta\end{aligned}$$

The second variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1\end{aligned}$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

DOUBLE-ANGLE FORMULAS

The *double-angle formulas* are summarized as follows:

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{4.3.2}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{4.3.3}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} \tag{4.3.4}$$

How to: Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 4.3.1: Using a Double-Angle Formula to Find the Exact Value Involving Tangent

Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$
b. $\cos(2\theta)$
c. $\tan(2\theta)$

Solution

If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4}$,such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the *x*-axis and is negative. Use the **Pythagorean Theorem** to find the length of the hypotenuse:

$$\begin{aligned}(-4)^2 + (3)^2 &= c^2 \\ 16 + 9 &= c^2 \\ 25 &= c^2 \\ c &= 5\end{aligned}$$

Now we can draw a triangle similar to the one shown in Figure 4.3.2.

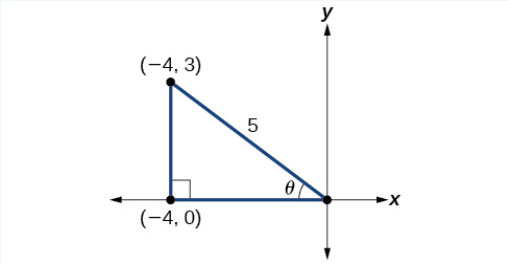


Figure 4.3.2

a. Let's begin by writing the double-angle formula for sine.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

We see that we to need to find $\sin\theta$ and $\cos\theta$. Based on Figure 4.3.2, we see that the hypotenuse equals 5, so $\sin\theta = 35$, $\sin\theta = 35$, and $\cos\theta = -45$. Substitute these values into the equation, and simplify.

Thus,

$$\begin{aligned}\sin(2\theta) &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

b. Write the double-angle formula for cosine.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Again, substitute the values of the sine and cosine into the equation, and simplify.

$$\begin{aligned}\cos(2\theta) &= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2\\ &= \frac{16}{25} - \frac{9}{25}\\ &= \frac{7}{25}\end{aligned}$$

c. Write the double-angle formula for tangent.

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

In this formula, we need the tangent, which we were given as $\tan\theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2}\\ &= \frac{-\frac{3}{2}}{1-\frac{9}{16}}\\ &= -\frac{3}{2}\left(\frac{16}{7}\right)\\ &= -\frac{24}{7}\end{aligned}$$

Exercise 4.3.1

Given $\sin\alpha = \frac{5}{8}$,with θ in quadrant I, find $\cos(2\alpha)$.

Answer

$$\cos(2\alpha) = \frac{7}{32}$$

Example 4.3.2: Using the Double-Angle Formula for Cosine without Exact Values

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $cos(3x)$.

Solution

$$\begin{aligned}\cos(6x) &= \cos(3x+3x)\\ &= \cos 3x\cos 3x - \sin 3x\sin 3x\\ &= \cos^2 3x - \sin^2 3x\end{aligned}$$

Analysis

This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.

Using Double-Angle Formulas to Verify Identities

Establishing identities using the double-angle formulas is performed using the same steps we used to derive the sum and difference formulas. Choose the more complicated side of the equation and rewrite it until it matches the other side.

Example 4.3.3: Using the Double-Angle Formulas to Verify an Identity

Verify the following identity using double-angle formulas:

$$1+\sin(2\theta) = (\sin\theta+\cos\theta)^2$$

Solution

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\\ &= (\sin^2\theta+\cos^2\theta) + 2\sin\theta\cos\theta\\ &= 1 + 2\sin\theta\cos\theta\\ &= 1+\sin(2\theta)\end{aligned}$$

Analysis

This process is not complicated, as long as we recall the perfect square formula from algebra:

$$(a\pm b)^2 = a^2 \pm 2ab + b^2$$

where $a = \sin\theta$ and $b = \cos\theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

Exercise 4.3.2

Verify the identity: $\cos^4\theta - \sin^4\theta = \cos(2\theta)$.

Answer

$$\cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = \cos(2\theta)$$

Example 4.3.4: Verifying a Double-Angle Identity for Tangent

Verify the identity: $\tan(2\theta) = 2\cot\theta - \tan\theta$

Solution

In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}
\tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} && \text{Double-angle formula} \\
&= \frac{2\tan\theta\left(\frac{1}{\tan\theta}\right)}{(1-\tan^2\theta)\left(\frac{1}{\tan\theta}\right)} && \text{Multiply by a term that results in desired numerator} \\
&= \frac{2}{\frac{1}{\tan\theta} - \frac{\tan^2\theta}{\tan\theta}} \\
&= \frac{2}{\cot\theta - \tan\theta} && \text{Use reciprocal identity for } \frac{1}{\tan\theta}
\end{aligned}$$

Analysis

Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{2}{\cot\theta - \tan\theta}$$

Lets work on the right side

$$\begin{aligned}
\frac{2}{\cot\theta - \tan\theta} &= \frac{2}{\frac{1}{\tan\theta} - \tan\theta}\left(\frac{\tan\theta}{\tan\theta}\right) \\
&= \frac{2\tan\theta}{\frac{1}{\tan\theta}(\tan\theta) - \tan\theta(\tan\theta)} \\
&= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned}$$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

Exercise 4.3.3

Verify the identity: $\cos(2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$.

Answer

$$\cos(2\theta)\cos\,\theta = (\cos^2\theta - \sin^2\theta)\cos\,\theta = \cos^3\theta - \cos\,\theta\sin^2\theta$$

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the **reduction formulas,** which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\,\sin^2\theta$. Solve for $\sin^2\theta$:

$$\begin{aligned} \cos(2\theta) &= 1 - 2\sin^2\theta \\ 2\sin^2\theta &= 1 - \cos(2\theta) \\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2} \end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2\,\cos^2\theta - 1$. Solve for $\cos^2\theta$:

$$\begin{aligned} \cos(2\theta) &= 2\cos^2\theta - 1 \\ 1 + \cos(2\theta) &= 2\cos^2\theta \\ \frac{1+\cos(2\theta)}{2} &= \cos^2\theta \end{aligned}$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned} \tan^2\,\theta &= \frac{\sin^2\,\theta}{\cos^2\,\theta} \\ &= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} && \text{(Substitute the reduction formulas)} \\ &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{2}{1+\cos(2\theta)}\right) \\ &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \end{aligned}$$

REDUCTION FORMULAS

The reduction formulas are summarized as follows:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \tag{4.3.5}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \tag{4.3.6}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \tag{4.3.7}$$

Example 4.3.5: Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution

We will apply the reduction formula for cosine twice.

$$\begin{aligned}\cos^4 x &= \left(\cos^2 x\right)^2 \\ &= \left(\frac{1+\cos(2x)}{2}\right)^2 \qquad \text{Substitute reduction formula} \\ &= \frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{8}+\frac{1}{8}\cos(4x) \\ &= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\end{aligned}$$

Analysis

The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.

Example 4.3.6: Using the Power-Reducing Formulas to Prove an Identity

Use the power-reducing formulas to prove $\sin^3(2x) = \left[\frac{1}{2}\sin(2x)\right][1-\cos(4x)]$

Solution

We will work on simplifying the left side of the equation:

$$\begin{aligned}\sin^3(2x) &= [\sin(2x)][\sin^2(2x)] \\ &= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right] \qquad \text{Substitute the power-reduction formula.} \\ &= \sin(2x)\left(\frac{1}{2}\right)[1-\cos(4x)] \\ &= \frac{1}{2}[\sin(2x)][1-\cos(4x)]\end{aligned}$$

Analysis

Note that in this example, we substituted $\frac{1-\cos(4x)}{2}$ for $\sin^2(2x)$. The formula states $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

We let $\theta = 2x$, so $2\theta = 4x$.

Exercise 4.3.4

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4}+5\cos(2x)+\frac{5}{4}\cos(4x)$.

Answer

$$
\begin{aligned}
10\cos^4 x &= 10(\cos^2 x)^2 \\
&= 10\left[\frac{1+\cos(2x)}{2}\right]^2 \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}[1+2\cos(2x)+\cos^2(2x)] \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{8}+\frac{10}{8}\cos(4x) \\
&= \frac{30}{8}+5\cos(2x)+\frac{10}{8}\cos(4x) \\
&= \frac{15}{4}+5\cos(2x)+\frac{5}{4}\cos(4x)
\end{aligned}
$$

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$,the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$
\begin{aligned}
\sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\
\sin^2\left(\frac{\alpha}{2}\right) &= \frac{1-\left(\cos 2\cdot\frac{\alpha}{2}\right)}{2} \\
&= \frac{1-\cos\alpha}{2} \\
\sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{2}}
\end{aligned}
$$

To derive the half-angle formula for cosine, we have

$$
\begin{aligned}
\cos^2\theta &= \frac{1+\cos(2\theta)}{2} \\
\cos^2\left(\frac{\alpha}{2}\right) &= \frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2} \\
&= \frac{1+\cos\alpha}{2} \\
\cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}}
\end{aligned}
$$

For the tangent identity, we have

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

HALF-ANGLE FORMULAS

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}} \tag{4.3.8}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \tag{4.3.9}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} \tag{4.3.10}$$

Example 4.3.7

Using a Half-Angle Formula to Find the Exact Value of a Sine Function. Find $\sin(15^\circ)$ using a half-angle formula.

Solution

Since $15^\circ = \frac{30^\circ}{2}$,we use the half-angle formula for sine (Equation 4.3.8):

$$\begin{aligned}\sin\frac{30^\circ}{2} &= \sqrt{\frac{1-\cos 30^\circ}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Remember that we can check the answer with a graphing calculator.

Analysis

Notice that we used only the positive root because $\sin(15^\circ)$ is positive.

Howto: Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 4.3.8: Finding Exact Values Using Half-Angle Identities

Given that $\tan\alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$
b. $\cos\left(\frac{\alpha}{2}\right)$
c. $\tan\left(\frac{\alpha}{2}\right)$

Solution

Using the given information, we can draw the triangle shown in Figure 4.3.3. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin\alpha = -\frac{8}{17}$ and $\cos\alpha = -\frac{15}{17}$.

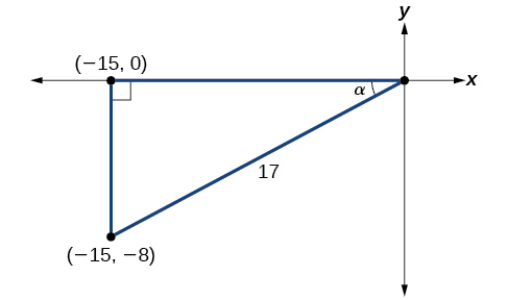


Figure 4.3.3

a. Before we start, we must remember that if α is in quadrant III, then $180^\circ < \alpha < 270^\circ$,so $\frac{180^\circ}{2} < \frac{\alpha}{2} < \frac{270^\circ}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90^\circ < \frac{\alpha}{2} < 135^\circ$. To find $\sin\frac{\alpha}{2}$,we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in Figure 4.3.3 and simplify.

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
&= \pm\sqrt{\frac{1-(-\frac{15}{17})}{2}} \\
&= \pm\sqrt{\frac{\frac{32}{17}}{2}} \\
&= \pm\sqrt{\frac{32}{17}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{16}{17}} \\
&= \pm\frac{4}{\sqrt{17}} \\
&= \frac{4\sqrt{17}}{17}
\end{aligned}$$

We choose the positive value of $\sin\frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos\frac{\alpha}{2}$,we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in Figure 4.3.3, and simplify.

$$\begin{aligned}\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}}\\ &= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}}\\ &= \pm\sqrt{\frac{\frac{2}{17}}{2}}\\ &= \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}}\\ &= \pm\sqrt{\frac{1}{17}}\\ &= -\frac{\sqrt{17}}{17}\end{aligned}$$

We choose the negative value of $\cos\frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan\frac{\alpha}{2}$,we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in Figure 4.3.3 and simplify.

$$\begin{aligned}\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\\ &= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}}\\ &= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}}\\ &= \pm\sqrt{\frac{32}{2}}\\ &= -\sqrt{16}\\ &= -4\end{aligned}$$

We choose the negative value of $\tan\frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

? Exercise 4.3.5

Given that $\sin\alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.

Answer

$$-\frac{2}{\sqrt{5}}$$

Example 4.3.9: Finding the Measurement of a Half Angle

Now, we will return to the problem posed at the beginning of the section. A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $tan\ \theta = 53$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution

Since the angle for novice competition measures half the steepness of the angle for the high level competition, and $\tan\theta = \frac{5}{3}$ for high competition, we can find $\cos\theta$ from the right triangle and the Pythagorean theorem so that we can use the half-angle identities. See Figure 4.3.4.

$$3^2 + 5^2 = 34$$
$$c = \sqrt{34}$$

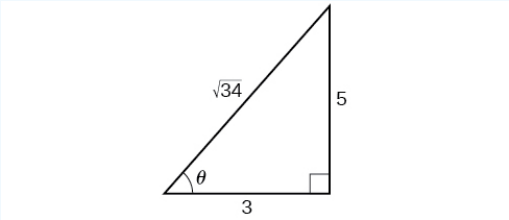


Figure 4.3.4

We see that $\cos\theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$. We can use the half-angle formula for tangent: $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$. Since $\tan\theta$ is in the first quadrant, so is $\tan\frac{\theta}{2}$.

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\frac{3\sqrt{34}}{34}}{1+\frac{3\sqrt{34}}{34}}}$$
$$= \sqrt{\frac{\frac{34-3\sqrt{34}}{34}}{\frac{34+3\sqrt{34}}{34}}}$$
$$= \sqrt{\frac{34-3\sqrt{34}}{34+3\sqrt{34}}}$$
$$\approx 0.57$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7^\circ$. So the angle of the ramp for novice competition is $\approx 29.7^\circ$.

Media

Access these online resources for additional instruction and practice with double-angle, half-angle, and reduction formulas.

- Double-Angle Identities
- Half-Angle Identities

Key Equations

Double-angle formulas	$\sin(2\theta) = 2\sin\,\theta\,\cos\,\theta$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $= 1 - 2\sin^2\theta$ $= 2\cos^2\theta - 1$ $\tan(2\theta) = \dfrac{2\tan\,\theta}{1-\tan^2\theta}$
Reduction formulas	$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$ $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$ $\tan^2\theta = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Half-angle formulas	$\sin\,\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\,\alpha}{2}}$ $\cos\,\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\,\alpha}{2}}$ $\tan\,\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\,\alpha}{1+\cos\,\alpha}}$ $= \dfrac{\sin\,\alpha}{1+\cos\,\alpha}$ $= \dfrac{1-\cos\,\alpha}{\sin\,\alpha}$

Key Concepts

- Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. See Example 4.3.1, Example 4.3.2, Example 4.3.3, and Example 4.3.4.
- Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. See Example 4.3.5 and Example 4.3.6.
- Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not. See Example 4.3.7, Example 4.3.8, and Example 4.3.9.

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4.4: Sum-to-Product and Product-to-Sum Formulas

Learning Objectives

- Express products as sums.
- Express sums as products.

A band marches down the field creating an amazing sound that bolsters the crowd. That sound travels as a wave that can be interpreted using trigonometric functions.



Figure* 4.4.1*: *The UCLA marching band (credit: Eric Chan, Flickr).*

For example, Figure 4.4.2 represents a sound wave for the musical note A. In this section, we will investigate trigonometric identities that are the foundation of everyday phenomena such as sound waves.

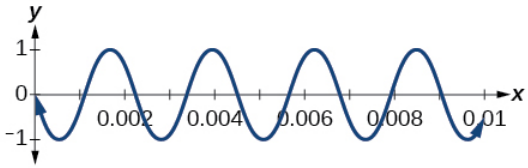


***Figure* 4.4.2**

Expressing Products as Sums

We have already learned a number of formulas useful for expanding or simplifying trigonometric expressions, but sometimes we may need to express the product of cosine and sine as a sum. We can use the **product-to-sum formulas,** which express products of trigonometric functions as sums. Let's investigate the cosine identity first and then the sine identity.

Expressing Products as Sums for Cosine

We can derive the product-to-sum formula from the sum and difference identities for **cosine**. If we add the two equations, we get:

$$\begin{aligned} \cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos(\alpha-\beta) \\ +\cos\alpha\cos\beta-\sin\alpha\sin\beta &= \cos(\alpha+\beta) \\ 2\cos\alpha\cos\beta &= \cos(\alpha-\beta)+\cos(\alpha+\beta) \end{aligned}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{4.4.1}$$

How to: Given a product of cosines, express as a sum

1. Write the formula for the product of cosines.
2. Substitute the given angles into the formula.
3. Simplify.

Example 4.4.1: Writing the Product as a Sum Using the Product-to-Sum Formula for Cosine

Write the following product of cosines as a sum: $2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)$.

Solution

We begin by writing the formula for the product of cosines (Equation 4.4.1):

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

We can then substitute the given angles into the formula and simplify.

$$\begin{aligned}2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right) &= 2\left(\frac{1}{2}\right)\left[\cos\left(\frac{7x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{7x}{2}+\frac{3x}{2}\right)\right]\\ &=\cos\left(\frac{4x}{2}\right)+\cos\left(\frac{10x}{2}\right)\\ &=\cos 2x+\cos 5x\end{aligned}$$

Exercise 4.4.1

Use the product-to-sum formula (Equation 4.4.1) to write the product as a sum or difference: $\cos(2\theta)\cos(4\theta)$.

Answer

$\frac{1}{2}(\cos 6\theta+\cos 2\theta)$

Expressing the Product of Sine and Cosine as a Sum

Next, we will derive the product-to-sum formula for sine and cosine from the sum and difference formulas for **sine**. If we add the sum and difference identities, we get:

$$\begin{aligned}\cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos(\alpha-\beta)\\ +\cos\alpha\cos\beta-\sin\alpha\sin\beta &= \cos(\alpha+\beta)\\ \hline 2\cos\alpha\cos\beta &= \cos(\alpha-\beta)+\cos(\alpha+\beta)\end{aligned}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

Example 4.4.2: Writing the Product as a Sum Containing only Sine or Cosine

Express the following product as a sum containing only sine or cosine and no products: $\sin(4\theta)\cos(2\theta)$.

Solution

Write the formula for the product of sine and cosine. Then substitute the given values into the formula and simplify.

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin(4\theta)\cos(2\theta) = \frac{1}{2}[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)]$$

$$= \frac{1}{2}[\sin(6\theta)+\sin(2\theta)]$$

Exercise 4.4.2

Use the product-to-sum formula to write the product as a sum: $\sin(x+y)\cos(x-y)$.

Answer

$\frac{1}{2}(\sin 2x+\sin 2y)$

Expressing Products of Sines in Terms of Cosine

Expressing the product of sines in terms of **cosine** is also derived from the sum and difference identities for cosine. In this case, we will first subtract the two cosine formulas:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\underline{-\cos(\alpha+\beta) = -(\cos\alpha\cos\beta-\sin\alpha\sin\beta)}$$

$$\cos(\alpha-\beta)-\cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

Then, we divide by 2 to isolate the product of sines:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

Similarly we could express the product of cosines in terms of sine or derive other product-to-sum formulas.

THE PRODUCT-TO-SUM FORMULAS

The **product-to-sum** formulas are as follows:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{4.4.2}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \tag{4.4.3}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \tag{4.4.4}$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)] \tag{4.4.5}$$

Example 4.4.3: Express the Product as a Sum or Difference

Write $\cos(3\theta)\cos(5\theta)$ as a sum or difference.

Solution

We have the product of cosines, so we begin by writing the related formula. Then we substitute the given angles and simplify.

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\cos(3\theta)\cos(5\theta) = \frac{1}{2}[\cos(3\theta-5\theta)+\cos(3\theta+5\theta)]$$

$$= \frac{1}{2}[\cos(2\theta)+\cos(8\theta)] \qquad \text{Use even-odd identity}$$

? Exercise 4.4.3

Use the product-to-sum formula to evaluate $\cos\frac{11\pi}{12}\cos\frac{\pi}{12}$.

Answer

$$\frac{-2-\sqrt{3}}{4}$$

Expressing Sums as Products

Some problems require the reverse of the process we just used. The **sum-to-product formulas** allow us to express sums of sine or cosine as products. These formulas can be derived from the product-to-sum identities. For example, with a few substitutions, we can derive the sum-to-product identity for **sine**. Let $\frac{u+v}{2}=\alpha$ and $\frac{u-v}{2}=\beta$.

Then,

$$\alpha+\beta = \frac{u+v}{2}+\frac{u-v}{2}$$
$$= \frac{2u}{2}$$
$$= u$$
$$\alpha-\beta = \frac{u+v}{2}-\frac{u-v}{2}$$
$$= \frac{2v}{2}$$
$$= v$$

Thus, replacing α and β in the product-to-sum formula with the substitute expressions, we have

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \frac{1}{2}[\sin u+\sin v] \qquad \text{Substitute for } (\alpha+\beta) \text{ and } (\alpha\beta)$$

$$2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \sin u+\sin v$$

The other sum-to-product identities are derived similarly.

SUM-TO-PRODUCT FORMULAS

The **sum-to-product formulas** are as follows:

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{4.4.6}$$

$$\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{4.4.7}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{4.4.8}$$

$$\cos\alpha + \cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{4.4.9}$$

Example 4.4.4: Writing the Difference of Sines as a Product

Write the following difference of sines expression as a product: $\sin(4\theta) - \sin(2\theta)$.

Solution

We begin by writing the formula for the difference of sines.

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

Substitute the values into the formula, and simplify.

$$\begin{aligned}\sin(4\theta) - \sin(2\theta) &= 2\sin\left(\frac{4\theta-2\theta}{2}\right)\cos\left(\frac{4\theta+2\theta}{2}\right)\\ &= 2\sin\left(\frac{2\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right)\\ &= 2\sin\theta\cos(3\theta)\end{aligned}$$

Exercise 4.4.4

Use the sum-to-product formula to write the sum as a product: $\sin(3\theta) + \sin(\theta)$.

Answer

$2\sin(2\theta)\cos(\theta)$

Example 4.4.5: Evaluating Using the Sum-to-Product Formula

Evaluate $\cos(15^\circ) - \cos(75^\circ)$. Check the answer with a graphing calculator.

Solution

We begin by writing the formula for the difference of cosines.

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Then we substitute the given angles and simplify.

$$\begin{aligned}\cos(15^\circ) - \cos(75^\circ) &= -2\sin\left(\frac{15^\circ+75^\circ}{2}\right)\sin\left(\frac{15^\circ-75^\circ}{2}\right)\\ &= -2\sin(45^\circ)\sin(-30^\circ)\\ &= -2\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Example 4.4.6: Proving an Identity

Prove the identity:

$$\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)}=-\tan t \tag{4.4.10}$$

Solution

We will start with the left side, the more complicated side of the equation, and rewrite the expression until it matches the right side.

$$\begin{aligned}\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} &= \frac{-2\sin\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)}{2\sin\left(\frac{4t+2t}{2}\right)\cos\left(\frac{4t-2t}{2}\right)}\\ &= \frac{-2\sin(3t)\sin t}{2\sin(3t)\cos t}\\ &= -\frac{\sin t}{\cos t}\\ &= -\tan t\end{aligned}$$

Analysis

Recall that verifying trigonometric identities has its own set of rules. The procedures for solving an equation are not the same as the procedures for verifying an identity. When we prove an identity, we pick one side to work on and make substitutions until that side is transformed into the other side.

Example 4.4.7: Verifying the Identity Using Double-Angle Formulas and Reciprocal Identities

Verify the identity $\csc^2\theta-2=\cos(2\theta)\sin 2\theta$.

Solution

For verifying this equation, we are bringing together several of the identities. We will use the double-angle formula and the reciprocal identities. We will work with the right side of the equation and rewrite it until it matches the left side.

$$\begin{aligned}\cos(2\theta)\sin 2\theta &= \frac{1-2\sin^2\theta}{\sin^2\theta}\\ &= \frac{1}{\sin^2\theta}-\frac{2\sin^2\theta}{\sin^2\theta}\\ &= \csc^2\theta-2\end{aligned}$$

Exercise 4.4.5

Verify the identity $\tan\theta\cot\theta-\cos^2\theta=\sin^2\theta$.

Answer

$$\begin{aligned}\tan\theta\cot\theta-\cos^2\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)-\cos^2\theta\\ &= 1-\cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

Media

Access these online resources for additional instruction and practice with the product-to-sum and sum-to-product identities.

- Sum to Product Identities
- Sum to Product and Product to Sum Identities

Key Equations

Product-to-sum Formulas

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

Sum-to-product Formulas

$$\sin\alpha+\sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$$

$$\sin\alpha-\sin\beta = 2\sin(\frac{\alpha-\beta}{2})\cos(\frac{\alpha+\beta}{2})$$

$$\cos\alpha-\cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$$

$$\cos\alpha+\cos\beta = 2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$$

Key Concepts

- From the sum and difference identities, we can derive the product-to-sum formulas and the sum-to-product formulas for sine and cosine.
- We can use the product-to-sum formulas to rewrite products of sines, products of cosines, and products of sine and cosine as sums or differences of sines and cosines. See Example 4.4.1, Example 4.4.2, and Example 4.4.3.
- We can also derive the sum-to-product identities from the product-to-sum identities using substitution.
- We can use the sum-to-product formulas to rewrite sum or difference of sines, cosines, or products sine and cosine as products of sines and cosines. See Example 4.4.4.
- Trigonometric expressions are often simpler to evaluate using the formulas. See Example 4.4.5.
- The identities can be verified using other formulas or by converting the expressions to sines and cosines. To verify an identity, we choose the more complicated side of the equals sign and rewrite it until it is transformed into the other side. See Example 4.4.6 and Example 4.4.7.

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4.5: Solving Trigonometric Equations

Learning Objectives

- Solve linear trigonometric equations in sine and cosine.
- Solve equations involving a single trigonometric function.
- Solve trigonometric equations using a calculator.
- Solve trigonometric equations that are quadratic in form.
- Solve trigonometric equations using fundamental identities.
- Solve trigonometric equations with multiple angles.
- Solve right triangle problems.

Thales of Miletus (circa 625–547 BC) is known as the founder of geometry. The legend is that he calculated the height of the Great Pyramid of Giza in Egypt using the theory of *similar triangles*, which he developed by measuring the shadow of his staff. Based on proportions, this theory has applications in a number of areas, including fractal geometry, engineering, and architecture. Often, the angle of elevation and the angle of depression are found using similar triangles.



Figure 4.5.1: Egyptian pyramids standing near a modern city. (credit: Oisin Mulvihill)

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Solving Linear Trigonometric Equations in Sine and Cosine

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The **period** of both the sine function and the cosine function is 2π. In other words, every 2π units, the y-values repeat. If we need to find all possible solutions, then we must add $2\pi k$,where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin\theta = \sin(\theta \pm 2k\pi) \tag{4.5.1}$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence. We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

Example 4.5.1A: Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

Solution

From the unit circle, we know that

$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3},\ \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Example 4.5.1B: Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

Solution

Solving for all possible values of t means that solutions include angles beyond the period of 2π. From the section on Sum and Difference Identities, we can see that the solutions are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. But the problem is asking for all possible values that solve the equation. Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where k is an integer.

Howto: Given a trigonometric equation, solve using algebra

1. Look for a pattern that suggests an algebraic property, such as the difference of squares or a factoring opportunity.
2. Substitute the trigonometric expression with a single variable, such as x or u.
3. Solve the equation the same way an algebraic equation would be solved.
4. Substitute the trigonometric expression back in for the variable in the resulting expressions.
5. Solve for the angle.

Example 4.5.2: Solve the Linear Trigonometric Equation

Solve the equation exactly: $2\cos\theta - 3 = -5$, $0 \le \theta < 2\pi$.

Solution

Use algebraic techniques to solve the equation.

$$\begin{aligned} 2\cos\theta - 3 &= -5 \\ 2\cos\theta &= -2 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Exercise 4.5.2

Solve exactly the following linear equation on the interval $[0, 2\pi)$: $2\sin x + 1 = 0$.

Answer

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Solving Equations Involving a Single Trigonometric Function

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and the unit circle (see [link]). We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems involving the reciprocals of the primary trigonometric functions need to be viewed from an algebraic perspective. In other words, we will write the reciprocal function, and solve for the angles using the function. Also, an equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π,not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$,unless, of course, a problem places its own restrictions on the domain.

Solving a Problem Involving a Single Trigonometric Function

Solve the problem exactly: $2\sin^2\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution

As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$\begin{aligned} 2\sin^2\theta - 1 &= 0 \\ 2\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{2} \\ \sqrt{\sin^2\theta} &= \pm\sqrt{\frac{1}{2}} \\ \sin\theta &= \pm\frac{1}{\sqrt{2}} \\ &= \pm\frac{\sqrt{2}}{2} \\ \theta &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

Example 4.5.3B: Solving a Trigonometric Equation Involving Cosecant

Solve the following equation exactly: $\csc\theta = -2$, $0 \le \theta < 4\pi$.

Solution

We want all values of θ for which $\csc\theta = -2$ over the interval $0 \le \theta < 4\pi$.

$$\begin{aligned} \csc\theta &= -2 \\ \frac{1}{\sin\theta} &= -2 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \end{aligned}$$

Analysis

As $\sin\theta = -\frac{1}{2}$, notice that all four solutions are in the third and fourth quadrants.

Example 4.5.3C: Solving an Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta - \frac{\pi}{2}\right) = 1$, $0 \le \theta < 2\pi$.

Solution

Recall that the tangent function has a period of π. On the interval $[0, \pi)$, and at the angle of $\frac{\pi}{4}$, the tangent has a value of 1. However, the angle we want is $\left(\theta - \frac{\pi}{2}\right)$. Thus, if $\tan\left(\frac{\pi}{4}\right) = 1$, then

$$\begin{aligned} \theta - \frac{\pi}{2} &= \frac{\pi}{4} \\ \theta &= \frac{3\pi}{4} \pm k\pi \end{aligned}$$

Over the interval $[0, 2\pi)$, we have two solutions:

$\theta = \frac{3\pi}{4}$ and $\theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$

Exercise 4.5.3

Find all solutions for $\tan x = \sqrt{3}$.

Answer

$\frac{\pi}{3} \pm \pi k$

Example 4.5.4: Identify all Solutions to the Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x + 3) = 5 + \tan x$, $0 \le x < 2\pi$.

Solution

We can solve this equation using only algebra. Isolate the expression $\tan x$ on the left side of the equals sign.

$$\begin{aligned} 2(\tan x) + 2(3) &= 5 + \tan x \\ 2\tan x + 6 &= 5 + \tan x \\ 2\tan x - \tan x &= 5 - 6 \\ \tan x &= -1 \end{aligned}$$

There are two angles on the unit circle that have a tangent value of -1: $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Solve Trigonometric Equations Using a Calculator

Not all functions can be solved exactly using only the unit circle. When we must solve an equation involving an angle other than one of the special angles, we will need to use a calculator. Make sure it is set to the proper mode, either degrees or radians, depending on the criteria of the given problem.

Example 4.5.5A: Using a Calculator to Solve a Trigonometric Equation Involving Sine

Use a calculator to solve the equation $\sin\theta = 0.8$, where θ is in radians.

Solution

Make sure mode is set to radians. To find θ, use the inverse sine function. On most calculators, you will need to push the 2ND button and then the SIN button to bring up the $\sin^{-1}$ function. What is shown on the screen is $\sin^{-1}$. The calculator is ready for

the input within the parentheses. For this problem, we enter $\sin^{-1}(0.8)$, and press ENTER. Thus, to four decimals places,

$\sin^{-1}(0.8) \approx 0.9273$

The solution is

$\theta \approx 0.9273 \pm 2\pi k$

The angle measurement in degrees is

$$\begin{aligned} \theta &\approx 53.1^\circ \\ \theta &\approx 180^\circ - 53.1^\circ \\ &\approx 126.9^\circ \end{aligned}$$

Analysis

Note that a calculator will only return an angle in quadrants I or IV for the sine function, since that is the range of the inverse sine. The other angle is obtained by using $\pi - \theta$.

Example $4.5.5B$: Using a Calculator to Solve a Trigonometric Equation Involving Secant

Use a calculator to solve the equation $\sec\theta = -4$, giving your answer in radians.

Solution

We can begin with some algebra.

$$\begin{aligned} \sec\theta &= -4 \\ \frac{1}{\cos\theta} &= -4 \\ \cos\theta &= -\frac{1}{4} \end{aligned}$$

Check that the MODE is in radians. Now use the inverse cosine function

$$\begin{aligned} \cos^{-1}\left(-\frac{1}{4}\right) &\approx 1.8235 \\ \theta &\approx 1.8235 + 2\pi k \end{aligned}$$

Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$,1.8235 is between these two numbers, thus $\theta \approx 1.8235$ is in quadrant II. Cosine is also negative in quadrant III. Note that a calculator will only return an angle in quadrants I or II for the cosine function, since that is the range of the inverse cosine. See Figure $4.5.2$.

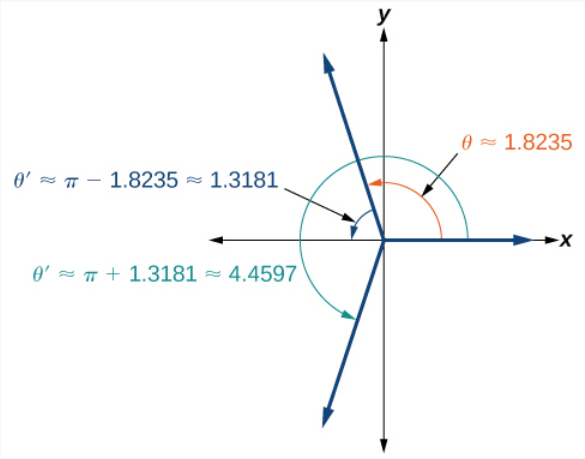


Figure 4.5.2

So, we also need to find the measure of the angle in quadrant III. In quadrant III, the reference angle is $\theta' \approx \pi - 1.8235 \approx 1.3181$. The other solution in quadrant III is $\theta' \approx \pi + 1.3181 \approx 4.4597$.

The solutions are $\theta \approx 1.8235 \pm 2\pi k$ and $\theta \approx 4.4597 \pm 2\pi k$.

? Exercise 4.5.5

Solve $\cos\theta = -0.2$.

Answer

$\theta \approx 1.7722 \pm 2\pi k$ and $\theta \approx 4.5110 \pm 2\pi k$

Solving Trigonometric Equations in Quadratic Form

Solving a *quadratic equation* may be more complicated, but once again, we can use algebra as we would for any quadratic equation. Look at the pattern of the equation. Is there more than one trigonometric function in the equation, or is there only one? Which trigonometric function is squared? If there is only one function represented and one of the terms is squared, think about the standard form of a quadratic. Replace the trigonometric function with a variable such as x or u. If substitution makes the equation look like a quadratic equation, then we can use the same methods for solving quadratics to solve the trigonometric equations.

✓ Example 4.5.6A: Solving a Trigonometric Equation in Quadratic Form

Solve the equation exactly: $\cos^2\theta + 3\cos\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution

We begin by using substitution and replacing $\cos\theta$ with x. It is not necessary to use substitution, but it may make the problem easier to solve visually. Let $\cos\theta = x$. We have

$$x^2 + 3x - 1 = 0$$

The equation cannot be factored, so we will use the **quadratic formula:** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2} \\ &= \frac{-3 \pm \sqrt{13}}{2} \end{aligned}$$

Replace x with $\cos\theta$ and solve.

$$\begin{aligned} \cos\theta &= \frac{-3 \pm \sqrt{13}}{2} \\ \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \end{aligned}$$

Note that only the + sign is used. This is because we get an error when we solve $\theta = \cos^{-1}\left(\dfrac{-3 - \sqrt{13}}{2}\right)$ on a calculator, since the domain of the inverse cosine function is $[-1, 1]$. However, there is a second solution:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \\ &\approx 1.26 \end{aligned}$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\begin{aligned} \theta &= 2\pi - \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \\ &\approx 5.02 \end{aligned}$$

Example 4.5.6B: Solving a Trigonometric Equation in Quadratic Form by Factoring

Solve the equation exactly: $2\sin^2\theta - 5\sin\theta + 3 = 0$, $0 \le \theta \le 2\pi$.

Solution

Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta = u$,or imagine it, as we factor:

$$\begin{aligned} 2\sin^2\theta - 5\sin\theta + 3 &= 0 \\ (2\sin\theta - 3)(\sin\theta - 1) &= 0 \qquad \text{Now set each factor equal to zero.} \\ 2\sin\theta - 3 &= 0 \\ 2\sin\theta &= 3 \\ \sin\theta &= \frac{3}{2} \\ \sin\theta - 1 &= 0 \\ \sin\theta &= 1 \end{aligned}$$

Next solve for θ: $\sin\theta \neq \frac{3}{2}$, as the range of the sine function is $[-1, 1]$. However, $\sin\theta = 1$, giving the solution $\theta = \frac{\pi}{2}$.

Analysis

Make sure to check all solutions on the given domain as some factors have no solution.

Exercise 4.5.6

Solve $\sin^2\theta = 2\cos\theta + 2$, $0 \le \theta \le 2\pi$. [Hint: Make a substitution to express the equation only in terms of cosine.]

Answer

$\cos\theta = -1$, $\theta = \pi$

Example 4.5.7A: Solving a Trigonometric Equation Using Algebra

Solve exactly: $2\sin^2\theta + \sin\theta = 0;\ 0 \le \theta < 2\pi$

Solution

This problem should appear familiar as it is similar to a quadratic. Let $\sin\theta = x$. The equation becomes $2x^2 + x = 0$. We begin by factoring:

$$\begin{aligned} 2x^2 + x &= 0 \\ x(2x + 1) &= 0 \qquad \text{Set each factor equal to zero.} \\ x &= 0 \\ 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

Then, substitute back into the equation the original expression $\sin\theta$ for x. Thus,

$$\begin{aligned} \sin\theta &= 0 \\ \theta &= 0, \pi \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

The solutions within the domain $0 \le \theta < 2\pi$ are $\theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

If we prefer not to substitute, we can solve the equation by following the same pattern of factoring and setting each factor equal to zero.

$$\begin{aligned} \sin^2\theta+\sin\theta &= 0 \\ \sin\theta(2\sin\theta+1) &= 0 \\ \sin\theta &= 0 \\ \theta &= 0,\pi \\ 2\sin\theta+1 &= 0 \\ 2\sin\theta &= -1 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6},\frac{11\pi}{6} \end{aligned}$$

Analysis

We can see the solutions on the graph in Figure 4.5.3. On the interval $0 \le \theta < 2\pi$,the graph crosses the x-axis four times, at the solutions noted. Notice that trigonometric equations that are in quadratic form can yield up to four solutions instead of the expected two that are found with quadratic equations. In this example, each solution (angle) corresponding to a positive sine value will yield two angles that would result in that value.

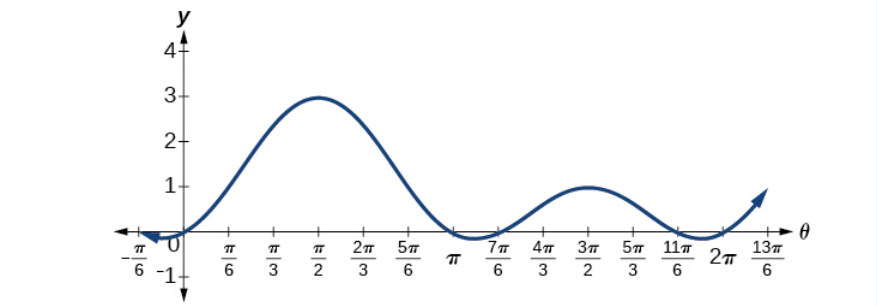


Figure 4.5.3

We can verify the solutions on the unit circle in via the result in the section on Sum and Difference Identities as well.

Example 4.5.7B: Solving a Trigonometric Equation Quadratic in Form

Solve the equation quadratic in form exactly: $2\sin^2\theta-3\sin\theta+1=0$, $0 \le \theta < 2\pi$.

Solution

We can factor using grouping. Solution values of θ can be found on the unit circle.

$$\begin{aligned} (2\sin\theta-1)(\sin\theta-1) &= 0 \\ 2\sin\theta-1 &= 0 \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6},\frac{5\pi}{6} \\ \sin\theta &= 1 \\ \theta &= \frac{\pi}{2} \end{aligned}$$

Exercise 4.5.7

Solve the quadratic equation $2\cos^2\theta+\cos\theta=0$.

Answer

$\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

Solving Trigonometric Equations Using Fundamental Identities

While algebra can be used to solve a number of trigonometric equations, we can also use the fundamental identities because they make solving equations simpler. Remember that the techniques we use for solving are not the same as those for verifying identities. The basic rules of algebra apply here, as opposed to rewriting one side of the identity to match the other side. In the next example, we use two identities to simplify the equation.

Example 4.5.8A: Use Identities to Solve an Equation

Use identities to solve exactly the trigonometric equation over the interval $0 \le x < 2\pi$.

$$\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$$

Solution

Notice that the left side of the equation is the difference formula for cosine.

$$\begin{aligned} \cos x \cos(2x) + \sin x \sin(2x) &= \frac{\sqrt{3}}{2} \\ \cos(x - 2x) &= \frac{\sqrt{3}}{2} && \text{Difference formula for cosine} \\ \cos(-x) &= \frac{\sqrt{3}}{2} && \text{Use the negative angle identity.} \\ \cos x &= \frac{\sqrt{3}}{2} \end{aligned}$$

From the unit circle in the section on Sum and Difference Identities, we see that $\cos x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}, \frac{11\pi}{6}$.

Example 4.5.8B: Solving the Equation Using a Double-Angle Formula

Solve the equation exactly using a double-angle formula: $\cos(2\theta) = \cos\theta$.

Solution

We have three choices of expressions to substitute for the double-angle of cosine. As it is simpler to solve for one trigonometric function at a time, we will choose the double-angle identity involving only cosine:

$$\begin{aligned} \cos(2\theta) &= \cos\theta \\ 2\cos^2\theta - 1 &= \cos\theta \\ 2\cos^2\theta - \cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta - 1) &= 0 \\ 2\cos\theta + 1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \cos\theta - 1 &= 0 \\ \cos\theta &= 1 \end{aligned}$$

So, if $\cos\theta = -\frac{1}{2}$,then $\theta = \frac{2\pi}{3} \pm 2\pi k$ and $\theta = \frac{4\pi}{3} \pm 2\pi k$; if $\cos\theta = 1$,then $\theta = 0 \pm 2\pi k$.

Example 4.5.8C: Solving an Equation Using an Identity

Solve the equation exactly using an identity: $3\cos\theta + 3 = 2\sin^2\theta$, $0 \le \theta < 2\pi$.

Solution

If we rewrite the right side, we can write the equation in terms of cosine:

$$\begin{aligned} 3\cos\theta+3 &= 2\sin^2\theta \\ 3\cos\theta+3 &= 2(1-\cos^2\theta) \\ 3\cos\theta+3 &= 2-2\cos^2\theta \\ 2\cos^2\theta+3\cos\theta+1 &= 0 \\ (2\cos\theta+1)(\cos\theta+1) &= 0 \\ 2\cos\theta+1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3},\ \frac{4\pi}{3} \\ \cos\theta+1 &= 0 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Our solutions are $\theta = \frac{2\pi}{3},\ \frac{4\pi}{3},\ \pi$.

Solving Trigonometric Equations with Multiple Angles

Sometimes it is not possible to solve a trigonometric equation with identities that have a multiple angle, such as $\sin(2x)$ or $\cos(3x)$. When confronted with these equations, recall that $y = \sin(2x)$ is a horizontal compression by a factor of 2 of the function $y = \sin x$. On an interval of 2π,we can graph two periods of $y = \sin(2x)$,as opposed to one cycle of $y = \sin x$. This compression of the graph leads us to believe there may be twice as many x-intercepts or solutions to $\sin(2x) = 0$ compared to $\sin x = 0$. This information will help us solve the equation.

Example 4.5.9: Solving a Multiple Angle Trigonometric Equation

Solve exactly: $\cos(2x) = \frac{1}{2}$ on $[0, 2\pi)$.

Solution

We can see that this equation is the standard equation with a multiple of an angle. If $\cos(\alpha) = \frac{1}{2}$,we know α is in quadrants I and IV. While $\theta = \cos^{-1}\frac{1}{2}$ will only yield solutions in quadrants I and II, we recognize that the solutions to the equation $\cos\theta = \frac{1}{2}$ will be in quadrants I and IV.

Therefore, the possible angles are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. So, $2x = \frac{\pi}{3}$ or $2x = \frac{5\pi}{3}$, which means that $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Does this make sense? Yes, because $\cos\left(2\left(\frac{\pi}{6}\right)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Are there any other possible answers? Let us return to our first step.

In quadrant I, $2x = \frac{\pi}{3}$, so $x = \frac{\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{\pi}{3} + 2\pi \\ &= \frac{\pi}{3} + \frac{6\pi}{3} \\ &= \frac{7\pi}{3} \\ x &= \frac{7\pi}{6} \end{aligned}$$

One more rotation yields

$$\begin{aligned} 2x &= \frac{\pi}{3} + 4\pi \\ &= \frac{\pi}{3} + \frac{12\pi}{3} \\ &= \frac{13\pi}{3} \end{aligned}$$

$x = \frac{13\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

In quadrant IV, $2x = \frac{5\pi}{3}$, so $x = \frac{5\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 2\pi \\ &= \frac{5\pi}{3} + \frac{6\pi}{3} \\ &= \frac{11\pi}{3} \end{aligned}$$

so $x = \frac{11\pi}{6}$.

One more rotation yields

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 4\pi \\ &= \frac{5\pi}{3} + \frac{12\pi}{3} \\ &= \frac{17\pi}{3} \end{aligned}$$

$x = \frac{17\pi}{6} > 2\pi$,so this value for x is larger than 2π,so it is not a solution on $[0, 2\pi)$.

Our solutions are $x = \frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Note that whenever we solve a problem in the form of $sin(nx) = c$, we must go around the unit circle n times.

Solving Right Triangle Problems

We can now use all of the methods we have learned to solve problems that involve applying the properties of right triangles and the Pythagorean Theorem. We begin with the familiar Pythagorean Theorem,

$$a^2 + b^2 = c^2 \tag{4.5.2}$$

and model an equation to fit a situation.

Example 4.5.10A: Using the Pythagorean Theorem to Model an Equation

One of the cables that anchors the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris wheel is 69.5 meters above the ground, and the second anchor on the ground is 23 meters from the base of the Ferris wheel. Approximately how long is the cable, and what is the angle of elevation (from ground up to the center of the Ferris wheel)? See Figure 4.5.4.

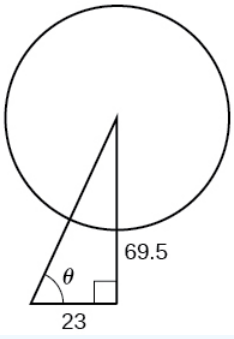


Figure 4.5.4

Solution

Use the Pythagorean Theorem (Equation 4.5.2) and the properties of right triangles to model an equation that fits the problem. Using the information given, we can draw a right triangle. We can find the length of the cable with the Pythagorean Theorem.

$$\begin{aligned} a^2+b^2 &= c^2 \\ (23)^2+(69.5)^2 &\approx 5359 \\ \sqrt{5359} &\approx 73.2\ m \end{aligned}$$

The angle of elevation is θ,formed by the second anchor on the ground and the cable reaching to the center of the wheel. We can use the tangent function to find its measure. Round to two decimal places.

$$\begin{aligned} \tan\theta &= 69.523 \\ \tan^{-1}(69.523) &\approx 1.2522 \\ &\approx 71.69^\circ \end{aligned}$$

The angle of elevation is approximately 71.7°, and the length of the cable is 73.2 meters.

Example $4.5.10B$: Using the Pythagorean Theorem to Model an Abstract Problem

OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length. Find the angle that a ladder of any length forms with the ground and the height at which the ladder touches the wall.

Solution

For any length of ladder, the base needs to be a distance from the wall equal to one fourth of the ladder's length. Equivalently, if the base of the ladder is "a" feet from the wall, the length of the ladder will be $4a$ feet. See Figure $4.5.5$.

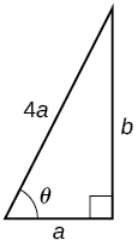


Figure 4.5.5

The side adjacent to θ is a and the hypotenuse is $4a$. Thus,

$$\begin{aligned} \cos\theta &= \frac{a}{4a} \\ &= \frac{1}{4} \\ \cos^{-1}\left(\frac{1}{4}\right) &\approx 75.5^\circ \end{aligned}$$

The elevation of the ladder forms an angle of 75.5° with the ground. The height at which the ladder touches the wall can be found using the Pythagorean Theorem:

$$\begin{aligned} a^2+b^2 &= (4a)^2 \\ b^2 &= (4a)^2-a^2 \\ b^2 &= 16a^2-a^2 \\ b^2 &= 15a^2 \\ b &= a\sqrt{15} \end{aligned}$$

Thus, the ladder touches the wall at $a\sqrt{15}$ feet from the ground.

Media

Access these online resources for additional instruction and practice with solving trigonometric equations.

- Solving Trigonometric Equations I
- Solving Trigonometric Equations II
- Solving Trigonometric Equations III
- Solving Trigonometric Equations IV
- Solving Trigonometric Equations V
- Solving Trigonometric Equations VI

Key Concepts

- When solving linear trigonometric equations, we can use algebraic techniques just as we do solving algebraic equations. Look for patterns, like the difference of squares, quadratic form, or an expression that lends itself well to substitution. See Example 4.5.1, Example 4.5.2, and Example 4.5.3.
- Equations involving a single trigonometric function can be solved or verified using the unit circle. See Example 4.5.4, Example 4.5.5, and Example 4.5.6, and Example 4.5.7.
- We can also solve trigonometric equations using a graphing calculator. See Example 4.5.8 and Example 4.5.9.
- Many equations appear quadratic in form. We can use substitution to make the equation appear simpler, and then use the same techniques we use solving an algebraic quadratic: factoring, the quadratic formula, etc. See Example 4.5.10, Example 4.5.11, Example 4.5.12, and Example 4.5.13.
- We can also use the identities to solve trigonometric equation. See Example 4.5.14, Example 4.5.15, and Example 4.5.16.
- We can use substitution to solve a multiple-angle trigonometric equation, which is a compression of a standard trigonometric function. We will need to take the compression into account and verify that we have found all solutions on the given interval. See Example 4.5.17.
- Real-world scenarios can be modeled and solved using the Pythagorean Theorem and trigonometric functions. See Example 4.5.18.

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4.6: Modeling with Trigonometric Equations

Suppose we charted the average daily temperatures in New York City over the course of one year. We would expect to find the lowest temperatures in January and February and highest in July and August. This familiar cycle repeats year after year, and if we were to extend the graph over multiple years, it would resemble a periodic function.



Figure 4.6.1: The hands on a clock are periodic: they repeat positions every twelve hours. (credit: “zoutedrop”/Flickr)

Many other natural phenomena are also periodic. For example, the phases of the moon have a period of approximately 28 days, and birds know to fly south at about the same time each year. So how can we model an equation to reflect periodic behavior? First, we must collect and record data. We then find a function that resembles an observed pattern. Finally, we make the necessary alterations to the function to get a model that is dependable. In this section, we will take a deeper look at specific types of periodic behavior and model equations to fit data.

Determining the Amplitude and Period of a Sinusoidal Function

Any motion that repeats itself in a fixed time period is considered **periodic motion** and can be modeled by a **sinusoidal function**. The amplitude of a sinusoidal function is the distance from the midline to the maximum value, or from the midline to the minimum value. The midline is the average value. Sinusoidal functions oscillate above and below the midline, are periodic, and repeat values in set cycles. Recall from Graphs of the Sine and Cosine Functions that the **period** of the sine function and the cosine function is 2π. In other words, for any value of x,

$$\sin(x \pm 2\pi k) = \sin x \text{ and } \cos(x \pm 2\pi k) = \cos x \tag{4.6.1}$$

where k is an integer.

STANDARD FORM OF SINUSOIDAL EQUATIONS

The general forms of a sinusoidal equation are given as

$$y = A\sin(Bt - C) + D \tag{4.6.2}$$

or

$$y = A\cos(Bt - C) + D \tag{4.6.3}$$

where $\text{amplitude} = |A|$, B is related to period such that the $\text{period} = \frac{2\pi}{B}$, C is the phase shift such that $\frac{C}{B}$ denotes the horizontal shift, and D represents the vertical shift from the graph’s parent graph.

Note that the models are sometimes written as

$$y = a\sin(\omega t \pm C) + D \tag{4.6.4}$$

or

$$y = a\cos(\omega t \pm C) + D, \tag{4.6.5}$$

with a period that is given as $\frac{2\pi}{\omega}$.

The difference between the sine and the cosine graphs is that the sine graph begins with the average value of the function and the cosine graph begins with the maximum or minimum value of the function.

Example 4.6.1: Showing How the Properties of a Trigonometric Function Can Transform a Graph

Show the transformation of the graph of $y = \sin x$ into the graph of $y = 2\sin(4x - \frac{\pi}{2}) + 2$.

Solution

Consider the series of graphs in Figure 4.6.1 and the way each change to the equation changes the image.

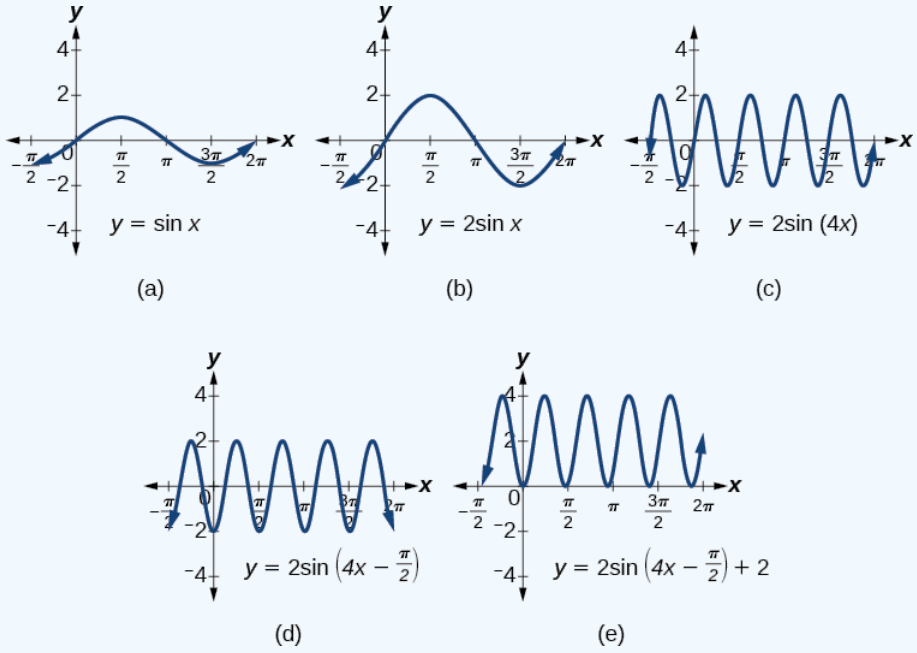


Figure 4.6.2:

a. The basic graph of $y = \sin x$
b. Changing the amplitude from 1 to 2 generates the graph of $y = 2\sin x$.
c. The period of the sine function changes with the value of B, such that period $= \frac{2\pi}{B}$. Here we have $B = 4$, which translates to a period of $\frac{\pi}{2}$. The graph completes one full cycle in $\frac{\pi}{2}$ units.
d. The graph displays a horizontal shift equal to $\frac{C}{B}$, or $\frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$.
e. Finally, the graph is shifted vertically by the value of D. In this case, the graph is shifted up by 2 units.

Example 4.6.2: Finding the Amplitude and Period of a Function

Find the amplitude and period of the following functions and graph one cycle.

a. $y = 2\sin(\frac{1}{4}x)$
b. $y = -3\sin(2x + \frac{\pi}{2})$
c. $y = \cos x + 3$

Solution

We will solve these problems according to the models.

a. $y = 2\sin(\frac{1}{4}x)$ involves sine, so we use the form

$$y = A\sin(Bt + C) + D$$

We know that $|A|$ is the amplitude, so the amplitude is 2. Period is $\frac{2\pi}{B}$ so the period is

$$\frac{2\pi}{B} = \frac{2\pi}{\frac{1}{4}}$$
$$= 8\pi$$

See the graph in Figure 4.6.3.

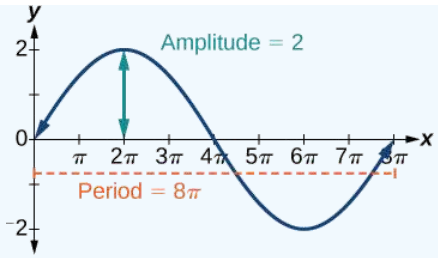


Figure 4.6.3:

b. $y = -3\sin(2x + \frac{\pi}{2})$ involves sine, so we use the form

$$y = A\sin(Bt - C) + D$$

Amplitude is $|A|$, so the amplitude is $|-3| = 3$. Since A is negative, the graph is reflected over the x-axis. Period is $\frac{2\pi}{B}$,so the period is

$$\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

The graph is shifted to the left by $\frac{C}{B} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$ units. See Figure 4.6.4.

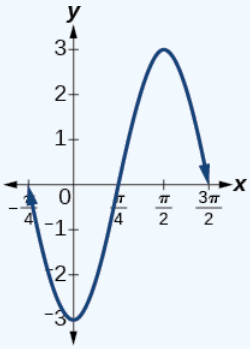


Figure 4.6.4:

c. $y = \cos x + 3$ involves cosine, so we use the form

d.
$$y = A\cos(Bt \pm C) + D$$

Amplitude is $|A|$, so the amplitude is 1 and the period is 2π (Figure 4.6.5. This is the standard cosine function shifted up three units.

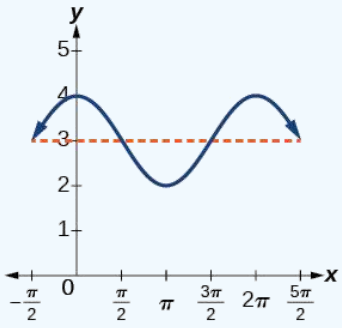


Figure 4.6.5

Exercise 4.6.1:

What are the amplitude and period of the function $y = 3\cos(3\pi x)$?

Answer

The amplitude is 3, and the period is $\frac{2}{3}$.

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Finding Equations and Graphing Sinusoidal Functions

One method of graphing sinusoidal functions is to find five key points. These points will correspond to intervals of equal length representing $\frac{1}{4}$ of the period. The key points will indicate the location of maximum and minimum values. If there is no vertical shift, they will also indicate x-intercepts. For example, suppose we want to graph the function $y = cos\theta$. We know that the period is 2π, so we find the interval between key points as follows.

$$\frac{2\pi}{4} = \frac{\pi}{2}$$

Starting with $\theta = 0,$ we calculate the first y-value, add the length of the interval $\frac{\pi}{2}$ to 0, and calculate the second y-value. We then add $\frac{\pi}{2}$ repeatedly until the five key points are determined. The last value should equal the first value, as the calculations cover one full period. Making a table similar to Table 4.6.1, we can see these key points clearly on the graph shown in Figure 4.6.6.

Table 4.6.1

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \cos\theta$	1	0	−1−1	0	1

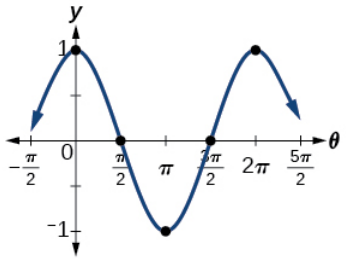


Figure 4.6.6

Example 4.6.3: Graphing Sinusoidal Functions Using Key Points

Graph the function $y = -4\cos(\pi x)$ using amplitude, period, and key points.

Solution

The amplitude is $|-4| = 4$. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$. (Recall that we sometimes refer to B as ω.) One cycle of the graph can be drawn over the interval $[0, 2]$. To find the key points, we divide the period by 4. Make a table similar to Figure 4.6.1:, starting with $x = 0$ and then adding $\frac{1}{2}$ successively to x and calculate y. See the graph in Figure 4.6.7.

Table 4.6.1

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$y = -4\cos(\pi x)$	−4	0	4	0	−4

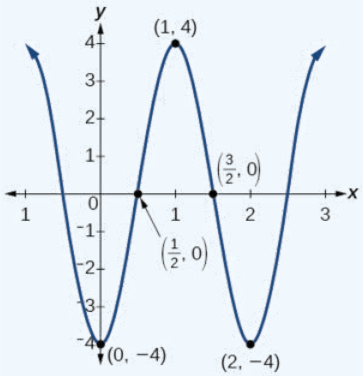


Figure 4.6.7

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Exercise 4.6.2:

Graph the function $y = 3\sin(3x)$ using the amplitude, period, and five key points.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$3\sin(3x)$	0	3	0	-3	0

Answer

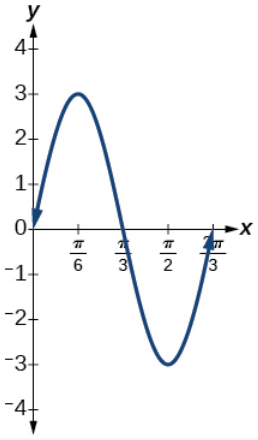


Figure 4.6.8

Modeling Periodic Behavior

We will now apply these ideas to problems involving periodic behavior.

Example 4.6.4: Modeling an Equation and Sketching a Sinusoidal Graph to Fit Criteria

The average monthly temperatures for a small town in Oregon are given in Table 4.6.1. Find a sinusoidal function of the form $y = A\sin(Bt - C) + D$ that fits the data (round to the nearest tenth) and sketch the graph.

Table 4.6.2

Month	Temperature, ^{o}F
January	42.5
February	44.5
March	48.5
April	52.5
May	58
June	63
July	68.5
August	69
September	64.5
October	55.5
November	46.5
December	43.5

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Solution

Recall that amplitude is found using the formula

$$A = \frac{\text{largest value} - \text{smallest value}}{2} \tag{4.6.6}$$

Thus, the amplitude is

$$\begin{aligned} |A| &= \frac{69 - 42.5}{2} \\ &= 13.25 \end{aligned}$$

The data covers a period of 12 months, so $\frac{2\pi}{B} = 12$ which gives $B = \frac{2\pi}{12} = \frac{\pi}{6}$.

The vertical shift is found using the following equation.

$$D = \frac{\text{highest value} + \text{lowest value}}{2} \tag{4.6.7}$$

Thus, the vertical shift is

$$D = \frac{69 + 42.5}{2} \qquad = 55.8$$

So far, we have the equation $y = 13.3\sin(\frac{\pi}{6}x - C) + 55.8$.

To find the horizontal shift, we input the x and y values for the first month and solve for C.

$$\begin{aligned} 42.5 &= 13.3\sin(\frac{\pi}{6}(1) - C) + 55.8 \\ -13.3 &= 13.3\sin(\frac{\pi}{6} - C) \\ -1 &= \sin(\frac{\pi}{6} - C) \qquad \sin\theta = -1 \to \theta = -\frac{\pi}{2} \\ \frac{\pi}{6} - C &= -\frac{\pi}{2} \\ \frac{\pi}{6} + \frac{\pi}{2} &= C \\ \frac{2\pi}{3} &= C \end{aligned}$$

We have the equation $y = 13.3\sin(\frac{\pi}{6}x - \frac{2\pi}{3}) + 55.8$. See the graph in Figure 4.6.9.

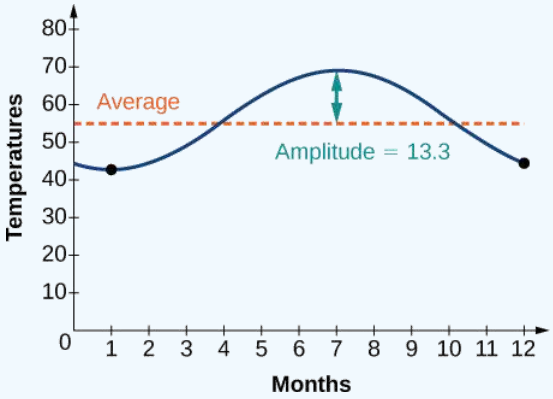


Figure 4.6.9

Example 4.6.5: Describing Periodic Motion

The hour hand of the large clock on the wall in Union Station measures 24 inches in length. At noon, the tip of the hour hand is 30 inches from the ceiling. At 3 PM, the tip is 54 inches from the ceiling, and at 6 PM, 78 inches. At 9 PM, it is again 54 inches from the ceiling, and at midnight, the tip of the hour hand returns to its original position 30 inches from the ceiling. Let

y equal the distance from the tip of the hour hand to the ceiling x hours after noon. Find the equation that models the motion of the clock and sketch the graph.

Solution

Begin by making a table of values as shown in Table 4.6.3.

Table 4.6.3

x	y	Points to plot
Noon	30 in	$(0, 30)$
3 PM	54 in	$(3, 54)$
6 PM	78 in	$(6, 78)$
9 PM	54 in	$(9, 54)$
Midnight	30 in	$(12, 30)$

To model an equation, we first need to find the amplitude.

$$\begin{aligned} |A| &= \left|\frac{78-30}{2}\right| \\ &= 24 \end{aligned}$$

The clock's cycle repeats every 12 hours. Thus,

$$\begin{aligned} B &= \frac{2\pi}{12} \\ &= \frac{\pi}{6} \end{aligned}$$

The vertical shift is

$$\begin{aligned} D &= \frac{78+30}{2} \\ &= 54 \end{aligned}$$

There is no horizontal shift, so $C = 0$. Since the function begins with the minimum value of y when $x = 0$ (as opposed to the maximum value), we will use the cosine function with the negative value for A. In the form $y = A\cos(Bx \pm C) + D$, the equation is

$$y = -24\cos\left(\frac{\pi}{6}x\right) + 54 \tag{4.6.8}$$

See Figure 4.6.10.

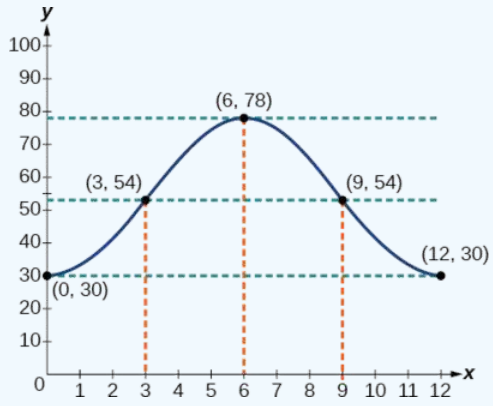


Figure 4.6.10

Example 4.6.6: Determining a Model for Tides

The height of the tide in a small beach town is measured along a seawall. Water levels oscillate between 7 feet at low tide and 15 feet at high tide. On a particular day, low tide occurred at 6 AM and high tide occurred at noon. Approximately every 12 hours, the cycle repeats. Find an equation to model the water levels.

Solution

As the water level varies from 7 ft to 15 ft, we can calculate the amplitude as

$$\begin{aligned} |A| &= |\frac{(15-7)}{2}| \\ &= 4 \end{aligned}$$

The cycle repeats every 12 hours; therefore, B is

$$\frac{2\pi}{12} = \frac{\pi}{6}$$

There is a vertical translation of $\frac{(15+8)}{2} = 11.5$. Since the value of the function is at a maximum at $t = 0$, we will use the cosine function, with the positive value for A.

$$y = 4\cos(\frac{\pi}{6})t + 11$$

See Figure 4.6.11.

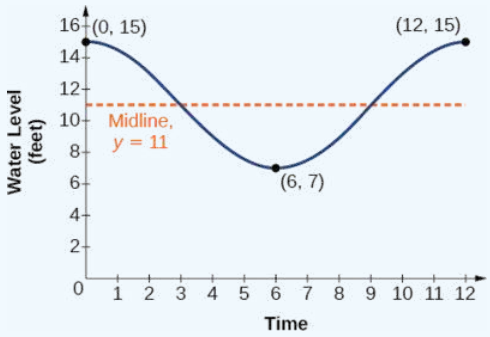


Figure 4.6.11

Exercise 4.6.3:

The daily temperature in the month of March in a certain city varies from a low of $24^\circ F$ to a high of $40^\circ F$. Find a sinusoidal function to model daily temperature and sketch the graph. Approximate the time when the temperature reaches the freezing point $32^\circ F$. Let $t = 0$ correspond to noon.

Answer

$$y = 8\sin(\frac{\pi}{12}t) + 32$$

The temperature reaches freezing at noon and at midnight.

Exercise 4.6.4: Interpreting the Periodic Behavior Equation

The average person's blood pressure is modeled by the function $f(t) = 20\sin(160\pi t) + 100$, where $f(t)$ represents the blood pressure at time t, measured in minutes. Interpret the function in terms of period and frequency. Sketch the graph and find the blood pressure reading.

Analysis

Blood pressure of $\frac{120}{80}$ is considered to be normal. The top number is the maximum or systolic reading, which measures the pressure in the arteries when the heart contracts. The bottom number is the minimum or diastolic reading, which measures the pressure in the arteries as the heart relaxes between beats, refilling with blood. Thus, normal blood pressure can be modeled by a periodic function with a maximum of 120 and a minimum of 80.

Example 4.6.7: Interpreting the Periodic Behavior Equation

The average person's blood pressure is modeled by the function $f(t) = 20\sin(160\pi t) + 100$, where $f(t)$ represents the blood pressure at time t, measured in minutes. Interpret the function in terms of period and frequency. Sketch the graph and find the blood pressure reading.

Solution

The period is given by

$$\begin{aligned} \frac{2\pi}{\omega} &= \frac{2\pi}{160\pi} \\ &= \frac{1}{80} \end{aligned}$$

In a blood pressure function, frequency represents the number of heart beats per minute. Frequency is the reciprocal of period and is given by

$$\begin{aligned} \frac{\omega}{2\pi} &= \frac{160\pi}{2\pi} \\ &= 80 \end{aligned}$$

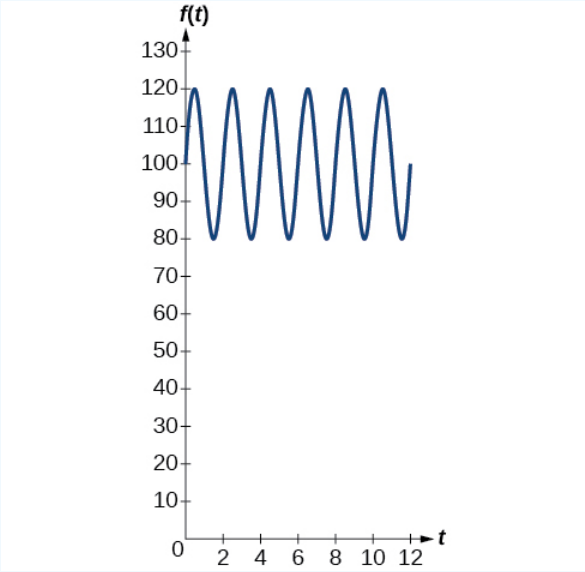


Figure 4.6.12: The blood pressure reading on the graph is $\frac{120}{80}\left(\frac{\text{maximum}}{\text{minimum}}\right)$.

Analysis

Blood pressure of $\frac{120}{80}$ is considered to be normal. The top number is the maximum or systolic reading, which measures the pressure in the arteries when the heart contracts. The bottom number is the minimum or diastolic reading, which measures the pressure in the arteries as the heart relaxes between beats, refilling with blood. Thus, normal blood pressure can be modeled by a periodic function with a maximum of 120 and a minimum of 80.

Modeling Harmonic Motion Functions

Harmonic motion is a form of periodic motion, but there are factors to consider that differentiate the two types. While general **periodic motion** applications cycle through their periods with no outside interference, **harmonic motion** requires a restoring force. Examples of harmonic motion include springs, gravitational force, and magnetic force.

Simple Harmonic Motion

A type of motion described as **simple harmonic motion** involves a restoring force but assumes that the motion will continue forever. Imagine a weighted object hanging on a spring, When that object is not disturbed, we say that the object is at rest, or in equilibrium. If the object is pulled down and then released, the force of the spring pulls the object back toward equilibrium and harmonic motion begins. The restoring force is directly proportional to the displacement of the object from its equilibrium point. When $t = 0, d = 0$.

SIMPLE HARMONIC MOTION

We see that **simple harmonic motion** equations are given in terms of displacement:

$$d = a\cos(\omega t) \text{ or } d = a\sin(\omega t) \tag{4.6.9}$$

where $|a|$ is the amplitude, $\frac{2\pi}{\omega}$ is the period, and $\frac{\omega}{2\pi}$ is the frequency, or the number of cycles per unit of time.

Exercise 4.6.5: Finding the Displacement, Period, and Frequency, and Graphing a Function

For each the given functions:

a. $y = 5\sin(3t)$
b. $y = 6\cos(\pi t)$
c. $y = 5\cos(\frac{\pi}{2})t$

address the following questions:

1. Find the maximum displacement of an object.
2. Find the period or the time required for one vibration.
3. Find the frequency.
4. Sketch the graph.

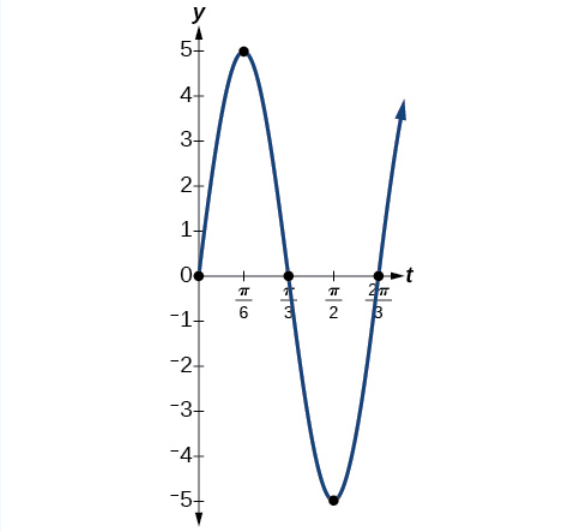


Figure 4.6.13

Answer a

1. The maximum displacement is equal to the amplitude,$|a|$, which is 5.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{3}$.
3. The frequency is given as $\frac{\omega}{2\pi} = \frac{3}{2\pi}$.
4. See Figure 4.6.13 The graph indicates the five key points.

Answer b

1. The maximum displacement is 6.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$.

3. The frequency is $\frac{\omega}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}$.
4. See Figure 4.6.14

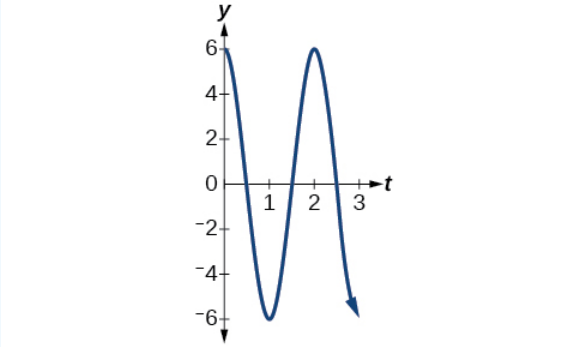


Figure 4.6.14

Answer c

1. The maximum displacement is 5.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$.
3. The frequency is $\frac{1}{4}$.
4. See Figure 4.6.15

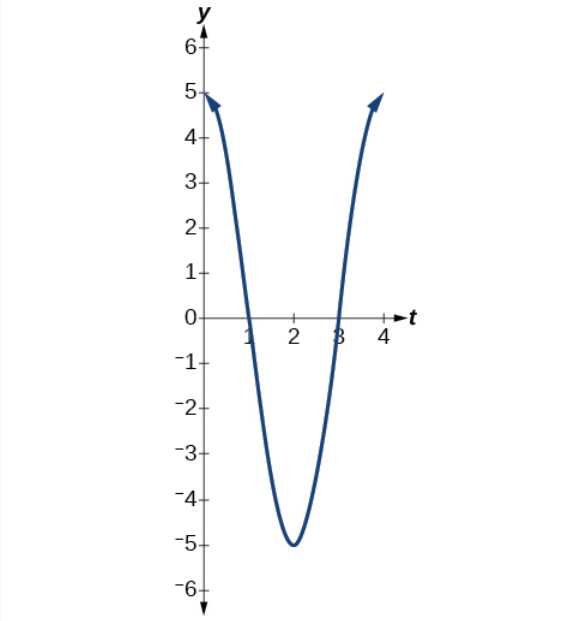


Figure 4.6.15

Damped Harmonic Motion

In reality, a pendulum does not swing back and forth forever, nor does an object on a spring bounce up and down forever. Eventually, the pendulum stops swinging and the object stops bouncing and both return to equilibrium. Periodic motion in which an energy-dissipating force, or damping factor, acts is known as **damped harmonic motion**. Friction is typically the damping factor.

In physics, various formulas are used to account for the damping factor on the moving object. Some of these are calculus-based formulas that involve derivatives. For our purposes, we will use formulas for basic damped harmonic motion models.

Definition: DAMPED HARMONIC MOTION

In *damped harmonic motion*, the displacement of an oscillating object from its rest position at time t is given as

$$f(t) = ae^{-ct}\sin(\omega t) \text{ or } f(t) = ae^{-ct}\cos(\omega t) \tag{4.6.10}$$

where c is a damping factor, $|a|$ is the initial displacement and $\frac{2\pi}{\omega}$ is the period.

Example 4.6.8: Modeling Damped Harmonic Motion

Model the equations that fit the two scenarios and use a graphing utility to graph the functions: Two mass-spring systems exhibit damped harmonic motion at a frequency of 0.5 cycles per second. Both have an initial displacement of 10 cm. The first has a damping factor of 0.5 and the second has a damping factor of 0.1.

Solution

At time $t = 0$, the displacement is the maximum of 10 cm, which calls for the cosine function. The cosine function will apply to both models.

We are given the frequency $f = \frac{\omega}{2\pi}$ of 0.5 cycles per second. Thus,

$$\begin{aligned} \frac{\omega}{2\pi} &= 0.5 \\ \omega &= (0.5)2\pi \\ &= \pi \end{aligned}$$

The first spring system has a damping factor of $c = 0.5$. Following the general model for damped harmonic motion, we have

$$f(t) = 10e^{-0.5t}\cos(\pi t)$$

Figure 4.6.16 models the motion of the first spring system.

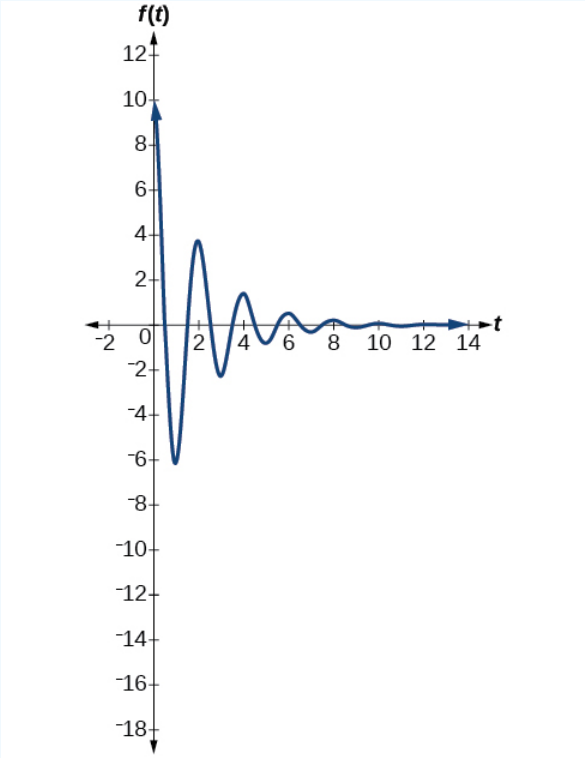


Figure 4.6.16

The second spring system has a damping factor of $c = 0.1$ and can be modeled as

$$f(t) = 10e^{-0.1t}\cos(\pi t) \tag{4.6.11}$$

Figure $4.6.17$ models the motion of the second spring system.

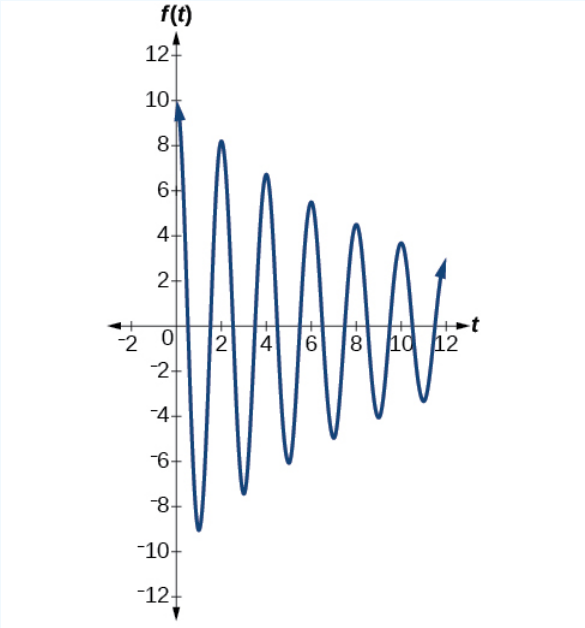


Figure 4.6.17

Analysis

Notice the differing effects of the damping constant. The local maximum and minimum values of the function with the damping factor $c = 0.5$ decreases much more rapidly than that of the function with $c = 0.1$.

Exercise $4.6.6$: Finding a Cosine Function that Models Damped Harmonic Motion

Find and graph a function of the form $y = ae^{-ct}\cos(\omega t)$ that models the information given.

a. $a = 20, c = 0.05, p = 4$
b. $a = 2, c = 1.5, f = 3$

Solution

Substitute the given values into the model. Recall that period is $\frac{2\pi}{\omega}$ and frequency is $\frac{\omega}{2\pi}$.

a. $y = 20e^{-0.05t}\cos(\frac{\pi}{2}t)$. See Figure $4.6.18$

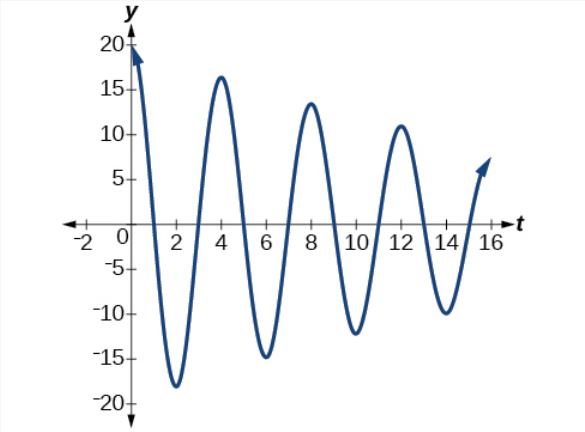


Figure 4.6.18

b. $y = 2e^{-1.5t}\cos(6\pi t)$. See Figure $4.6.19$

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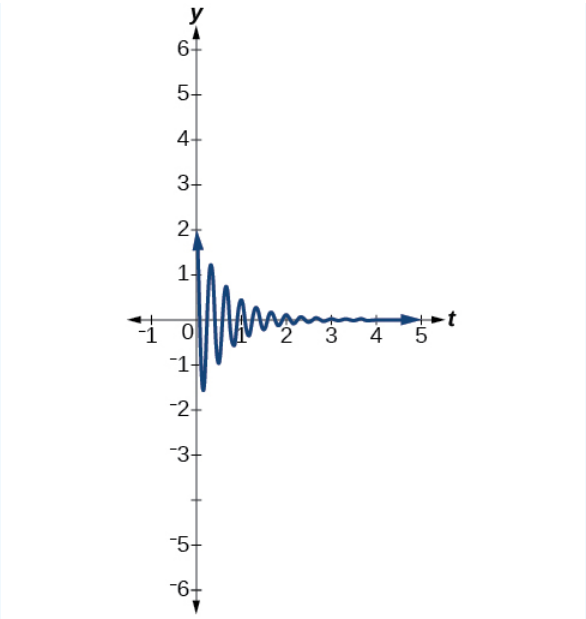


Figure 4.6.19

Exercise 4.6.7

The following equation represents a damped harmonic motion model: $f(t) = 5e^{-6t}\cos(4t)$ Find the initial displacement, the damping constant, and the frequency.

Answer

initial displacement =6, damping constant = -6, frequency = $\frac{2}{\pi}$

Example 4.6.9: Finding a Sine Function that Models Damped Harmonic Motion

Find and graph a function of the form $y = ae^{-ct}\sin(\omega t)$ that models the information given.

a. $a = 7, c = 10, p = \frac{\pi}{6}$
b. $a = 0.3, c = 0.2, f = 20$

Solution

Calculate the value of ω and substitute the known values into the model.

a. As period is $\frac{2\pi}{\omega}$, we have

$$\frac{\pi}{6} = \frac{2\pi}{\omega}$$
$$\omega\pi = 6(2\pi)$$
$$\omega = 12$$

The damping factor is given as 10 and the amplitude is 7. Thus, the model is $y = 7e^{-10t}\sin(12t)$. See Figure 4.6.20

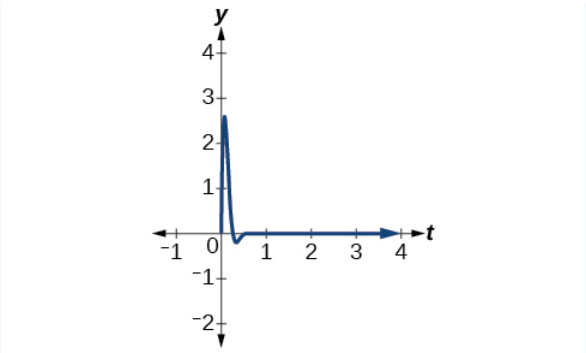


Figure 4.6.20

b. As frequency is $\frac{\omega}{2\pi}$, we have

$$\begin{aligned} 20 &= \frac{\omega}{2\pi} \\ 40\pi &= \omega \end{aligned}$$

The damping factor is given as 0.2 and the amplitude is 0.3. The model is $y = 0.3e^{-0.2t}\sin(40\pi t)$. See Figure 4.6.21.

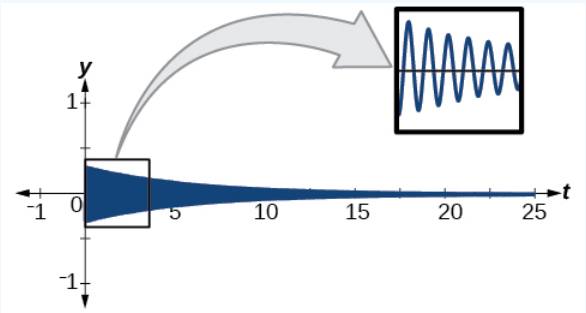


Figure 4.6.21

Analysis

A comparison of the last two examples illustrates how we choose between the sine or cosine functions to model sinusoidal criteria. We see that the cosine function is at the maximum displacement when $t = 0$, and the sine function is at the equilibrium point when $t = 0$. For example, consider the equation $y = 20e^{-0.05t}\cos(\frac{\pi}{2}t)$ from Example. We can see from the graph that when $t = 0, y = 20$, which is the initial amplitude. Check this by setting $t = 0$ in the cosine equation:

$$\begin{aligned} y &= 20e^{-0.05(0)}\cos(\frac{\pi}{2})(0) \\ &= 20(1)(1) \\ &= 20 \end{aligned}$$

Using the sine function yields

$$\begin{aligned} y &= 20e^{-0.05(0)}\sin(\frac{\pi}{2})(0) \\ &= 20(1)(0) \\ &= 0 \end{aligned}$$

Thus, cosine is the correct function.

Exercise 4.6.8:

Write the equation for damped harmonic motion given $a = 10, c = 0.5$, and $p = 2$.

Answer

$y = 10e^{-0.5t}\cos(\pi t)$

Example 4.6.10: Modeling the Oscillation of a Spring

A spring measuring 10 inches in natural length is compressed by 5 inches and released. It oscillates once every 3 seconds, and its amplitude decreases by 30% every second. Find an equation that models the position of the spring t seconds after being released.

Solution

The amplitude begins at 5 in. and deceases 30% each second. Because the spring is initially compressed, we will write A as a negative value. We can write the amplitude portion of the function as

$$A(t) = 5(1-0.30)^t$$

We put $(1-0.30)^t$ in the form e^{ct} as follows:

$$\begin{aligned} 0.7 &= e^c \\ c &= \ln .7 \\ c &= -0.357 \end{aligned}$$

Now let's address the period. The spring cycles through its positions every 3 seconds, this is the period, and we can use the formula to find omega.

$$\begin{aligned} 3 &= \frac{2\pi}{\omega} \\ \omega &= \frac{2\pi}{3} \end{aligned}$$

The natural length of 10 inches is the midline. We will use the cosine function, since the spring starts out at its maximum displacement. This portion of the equation is represented as

$$y = \cos(\frac{2\pi}{3}t) + 10$$

Finally, we put both functions together. Our the model for the position of the spring at t seconds is given as

$$y = -5e^{-0.357t}\cos(\frac{2\pi}{3}t) + 10$$

See the graph in Figure 4.6.22

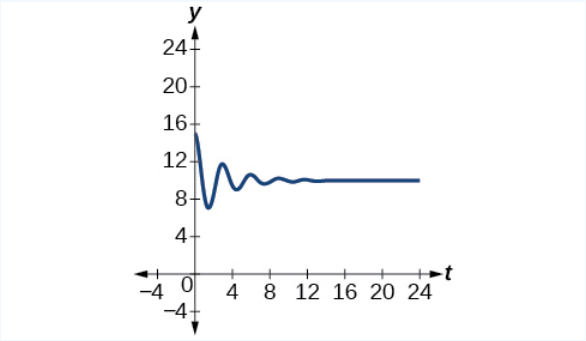


Figure 4.6.22

Exercise 4.6.9

A mass suspended from a spring is raised a distance of 5 cm above its resting position. The mass is released at time $t = 0$ and allowed to oscillate. After $\frac{1}{3}$ second, it is observed that the mass returns to its highest position. Find a function to model this motion relative to its initial resting position.

Answer

$y = 5\cos(6\pi t)$

According to the Given Criteria

A guitar string is plucked and vibrates in damped harmonic motion. The string is pulled and displaced 2 cm from its resting position. After 3 seconds, the displacement of the string measures 1 cm. Find the damping constant.

Solution

The displacement factor represents the amplitude and is determined by the coefficient ae^{-ct} in the model for damped harmonic motion. The damping constant is included in the term e^{-ct}. It is known that after 3 seconds, the local maximum measures one-half of its original value. Therefore, we have the equation

$$ae^{-c(t+3)} = \frac{1}{2}ae^{-ct} \tag{4.6.12}$$

Use algebra and the laws of exponents to solve for c.

$$\begin{aligned} ae^{-c(t+3)} &= \tfrac{1}{2}ae^{-ct} \\ e^{-ct}\cdot e^{-3c} &= \tfrac{1}{2}e^{-ct} && \text{Divide out } a. \\ e^{-3c} &= \tfrac{1}{2} && \text{Divide out } e^{-ct}. \\ e^{3c} &= 2 && \text{Take reciprocals.} \end{aligned} \tag{4.6.13}$$

Then use the laws of logarithms.

$$\begin{aligned} e^{3c} &= 2 \\ 3c &= \ln 2 \\ c &= \frac{\ln 2}{3} \end{aligned}$$

The damping constant is $\frac{\ln 2}{3}$.

Bounding Curves in Harmonic Motion

Harmonic motion graphs may be enclosed by bounding curves. When a function has a varying **amplitude**, such that the amplitude rises and falls multiple times within a period, we can determine the bounding curves from part of the function.

Example 4.6.12: Graphing an Oscillating Cosine Curve

Graph the function $f(x) = \cos(2\pi x)\cos(16\pi x)$.

Solution

The graph produced by this function will be shown in two parts. The first graph will be the exact function $f(x)$ (see Figure $4.6.23; top$), and the second graph is the exact function $f(x)$ plus a bounding function (see Figure $4.6.23; bottom$). The graphs look quite different.



Figure_07_07_016.jpg" />

***Figure* 4.6.23**

Analysis

The curves $y = \cos(2\pi x)$ and $y = -\cos(2\pi x)$ are bounding curves: they bound the function from above and below, tracing out the high and low points. The harmonic motion graph sits inside the bounding curves. This is an example of a function whose amplitude not only decreases with time, but actually increases and decreases multiple times within a period.

Key Equations

Standard form of sinusoidal equation	$y = A\sin(Bt - C) + D$ or $y = A\cos(Bt - C) + D$
Simple harmonic motion	$d = a\cos(\omega t)$ or $d = a\sin(\omega t)$
Damped harmonic motion	$f(t) = ae^{-ct}\sin(\omega t)$ or $f(t) = ae^{-ct}\cos(\omega t)$

Key Concepts

- Sinusoidal functions are represented by the sine and cosine graphs. In standard form, we can find the amplitude, period, and horizontal and vertical shifts. See Example and Example.
- Use key points to graph a sinusoidal function. The five key points include the minimum and maximum values and the midline values. See Example.
- Periodic functions can model events that reoccur in set cycles, like the phases of the moon, the hands on a clock, and the seasons in a year. See Example, Example, Example and Example.
- Harmonic motion functions are modeled from given data. Similar to periodic motion applications, harmonic motion requires a restoring force. Examples include gravitational force and spring motion activated by weight. See Example.
- Damped harmonic motion is a form of periodic behavior affected by a damping factor. Energy dissipating factors, like friction, cause the displacement of the object to shrink. See Example, Example, Example, Example, and Example.
- Bounding curves delineate the graph of harmonic motion with variable maximum and minimum values. See Example.

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4.E: Trigonometric Identities and Equations (Exercises)

7.1: Solving Trigonometric Equations with Identities

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verbal

1) We know $g(x) = \cos x$ is an even function, and $f(x) = \sin x$ and $h(x) = \tan x$are odd functions. What about $G(x) = \cos^2 x$, $F(x) = \sin^2 x$ and $H(x) = \tan^2 x$? Are they even, odd, or neither? Why?

Answer

All three functions, F, G, and H, are even.

This is because

$F(-x) = \sin(-x)\sin(-x) = (-\sin x)(-\sin x) = \sin^2 x = F(x), G(-x) = \cos(-x)\cos(-x) = \cos x \cos x = cos^2 x = H(-x) = \tan(-x)\tan(-x) = (-\tan x)(-\tan x) = \tan 2x = H(x)$

2) Examine the graph of $f(x) = \sec x$ on the interval $[-\pi, \pi]$. How can we tell whether the function is even or odd by only observing the graph of $f(x) = \sec x$?

3) After examining the reciprocal identity for $\sec t$, explain why the function is undefined at certain points.

Answer

When $\cos t = 0$, then $\sec t = 10$, which is undefined.

4) All of the Pythagorean identities are related. Describe how to manipulate the equations to get from $\sin^2 t + \cos^2 t = 1$ to the other forms.

Algebraic

For the exercises 5-15, use the fundamental identities to fully simplify the expression.

5) $\sin x \cos x \sec x$

Answer

$\sin x$

6) $\sin(-x)\cos(-x)\csc(-x)$

7) $\tan x \sin x + \sec x \cos^2 x$

Answer

$\sec x$

8) $\csc x + \cos x \cot(-x)$

9) $\dfrac{\cot t + \tan t}{\sec(-t)}$

Answer

$\csc x$

10) $3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t$

11) $-\tan(-x)\cot(-x)$

Answer

-1

12) $\dfrac{-\sin(-x)\cos x \sec x \csc x \tan x}{\cot x}$

13) $\dfrac{1+\tan^2\theta}{\csc^2\theta} + \sin^2\theta + \dfrac{1}{\sec^2\theta}$

Answer

$\sec^2 x$

14) $\left(\dfrac{\tan x}{\csc^2 x}+\dfrac{\tan x}{\sec^2 x}\right)\left(\dfrac{1+\tan x}{1+\cot x}\right)-\dfrac{1}{\cos^2 x}$

15) $\dfrac{1-\cos^2 x}{\tan^2 x}+2\sin^2 x$

Answer

$\sin^2 x+1$

For the exercises 16-28, simplify the first trigonometric expression by writing the simplified form in terms of the second expression.

16) $\dfrac{\tan x+\cot x}{\csc x};\cos x$

17) $\dfrac{\sec x+\csc x}{1+\tan x};\sin x$

Answer

$\dfrac{1}{\sin x}$

18) $\dfrac{\cos x}{1+\sin x}+\tan x;\cos x$

19) $\dfrac{1}{\sin x\cos x}-\cot x;\cot x$

Answer

$\dfrac{1}{\cot x}$

20) $\dfrac{1}{1-\cos x}-\dfrac{\cos x}{1+\cos x};\csc x$

21) $(\sec x+\csc x)(\sin x+\cos x)-2-\cot x;\tan x$

Answer

$\tan x$

22) $\dfrac{1}{\csc x-\sin x};\sec x \text{ and } \tan x$

23) $\dfrac{1-\sin x}{1+\sin x}-\dfrac{1+\sin x}{1-\sin x};\sec x \text{ and } \tan x$

Answer

$-4\sec x\tan x$

24) $\tan x;\sec x$

25) $\sec x;\cot x$

Answer

$\pm\sqrt{\dfrac{1}{\cot^2 x}+1}$

26) $\sec x;\sin x$

27) $\cot x;\sin x$

Answer

$\dfrac{\pm\sqrt{1-\sin^2 x}}{\sin x}$

28) $\cot x; \csc x$

For the exercises 29-33, verify the identity.

29) $\cos x - \cos^3 x = \cos x \sin^2 x$

Answer

Answers will vary. Sample proof:

$$\begin{aligned}\cos x - \cos^3 x &= \cos x(1 - \cos^2 x) \\ &= \cos x \sin^x\end{aligned}$$

30) $\cos x(\tan x - \sec(-x)) = \sin x - 1$

31) $\dfrac{1+\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x} = 1 + 2\tan^2 x$

Answer

Answers will vary. Sample proof:

$$\begin{aligned}\frac{1+\sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \\ &= \sec^2 x + \tan^2 x \\ &= \tan^2 x + 1 + \tan^2 x \\ &= 1 + 2\tan^2 x\end{aligned}$$

32) $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

33) $\cos^2 x - \tan^2 x = 2 - \sin^2 x - \sec^2 x$

Answer

Answers will vary. Sample proof:

$$\begin{aligned}\cos^2 x - \tan^2 x &= 1 - \sin^2 x - \left(\sec^2 x - 1\right) \\ &= 1 - \sin^2 x - \sec^2 x + 1 \\ &= 2 - \sin^2 x - \sec^2 x\end{aligned}$$

Extensions

For the exercises 34-39, prove or disprove the identity.

34) $\dfrac{1}{1+\cos x} - \dfrac{1}{1-\cos(-x)} = -2\cot x \csc x$

35) $\csc^2 x(1 + \sin^2 x) = \cot^2 x$

Answer

False

36) $\left(\dfrac{\sec^2(-x) - \tan^2 x}{\tan x}\right)\left(\dfrac{2+2\tan x}{2+2\cot x}\right) - 2\sin^2 x = \cos(2x)$

37) $\dfrac{\tan x}{\sec x}\sin(-x) = \cos^2 x$

Answer

False

38) $\dfrac{\sec(-x)}{\tan x + \cot x} = -\sin(-x)$

39) $\dfrac{1+\sin x}{\cos x} = \dfrac{\cos x}{1+\sin(-x)}$

Answer

Proved with negative and Pythagorean identities

For the exercises 40-, determine whether the identity is true or false. If false, find an appropriate equivalent expression.

40) $\dfrac{\cos^2\theta-\sin^2\theta}{1-\tan^2\theta}=\sin^2\theta$

41) $3\sin^2\theta+4\cos^2\theta=3+\cos^2\theta$

Answer

True

$$\begin{aligned}3\sin^2\theta+4\cos^2\theta &= 3\sin^2\theta+3\cos^2\theta+\cos^2\theta\\ &= 3\left(\sin^2\theta+\cos^2\theta\right)+\cos^2\theta\\ &= 3+\cos^2\theta\end{aligned}$$

42) $\dfrac{\sec\theta+\tan\theta}{\cot\theta+\cos^{\theta}}=\sec^2\theta$

7.2: Sum and Difference Identities

In this section, we will learn techniques that will enable us to solve useful problems. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word identity.

Verbal

1) Explain the basis for the cofunction identities and when they apply.

Answer

The cofunction identities apply to complementary angles. Viewing the two acute angles of a right triangle, if one of those angles measures x, the second angle measures $\dfrac{\pi}{2}-x$. Then $\sin x=\cos\left(\dfrac{\pi}{2}-x\right)$. The same holds for the other cofunction identities. The key is that the angles are complementary.

2) Is there only one way to evaluate $\cos\left(\dfrac{5\pi}{4}\right)$? Explain how to set up the solution in two different ways, and then compute to make sure they give the same answer.

3) Explain to someone who has forgotten the even-odd properties of sinusoidal functions how the addition and subtraction formulas can determine this characteristic for $f(x)=\sin(x)$ and $g(x)=\cos(x)$. (Hint: $0-x=-x$)

Answer

$\sin(-x)=-\sin x$, so $\sin x$ is odd. $\cos(-x)=\cos(0-x)=\cos x$, so $\cos x$ is even.

Algebraic

For the exercises 4-9, find the exact value.

4) $\cos\left(\dfrac{7\pi}{12}\right)$

5) $\cos\left(\dfrac{\pi}{12}\right)$

Answer

$\dfrac{\sqrt{2}+\sqrt{6}}{4}$

6) $\sin\left(\dfrac{5\pi}{12}\right)$

7) $\sin\left(\dfrac{11\pi}{12}\right)$

Answer

$\dfrac{\sqrt{6}-\sqrt{2}}{4}$

8) $\tan\left(-\dfrac{\pi}{12}\right)$

9) $\tan\left(\frac{19\pi}{12}\right)$

Answer

$-2-\sqrt{3}$

For the exercises 10-13, rewrite in terms of $\sin x$ and $\cos x$

10) $\sin\left(x+\frac{11\pi}{6}\right)$

11) $\sin\left(x-\frac{3\pi}{4}\right)$

Answer

$-\frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x$

12) $\cos\left(x-\frac{5\pi}{6}\right)$

13) $\cos\left(x+\frac{2\pi}{3}\right)$

Answer

$-\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$

For the exercises 14-19, simplify the given expression.

14) $\csc\left(\frac{\pi}{2}-t\right)$

15) $\sec\left(\frac{\pi}{2}-\theta\right)$

Answer

$\csc\theta$

16) $\cot\left(\frac{\pi}{2}-x\right)$

17) $\tan\left(\frac{\pi}{2}-x\right)$

Answer

$\cot x$

18) $\sin(2x)\cos(5x)-\sin(5x)\cos(2x)$

19) $\frac{\tan\left(\frac{3}{2}x\right)-\tan\left(\frac{7}{5}x\right)}{1+\tan\left(\frac{3}{2}x\right)\tan\left(\frac{7}{5}x\right)}$

Answer

$\tan\left(\frac{x}{10}\right)$

For the exercises 20-21, find the requested information.

20) Given that $\sin a=\frac{2}{3}$ and $\cos b=-\frac{1}{4}$, with a and b both in the interval $\left[\frac{\pi}{2},\pi\right)$, find $\sin(a+b)$ and $\cos(a-b)$.

21) Given that $\sin a=\frac{4}{5}$ and $\cos b=\frac{1}{3}$, with a and b both in the interval $\left[0,\frac{\pi}{2}\right)$, find $\sin(a-b)$ and $\cos(a+b)$.

Answer

$$\sin(a-b)=\left(\frac{4}{5}\right)\left(\frac{1}{3}\right)-\left(\frac{3}{5}\right)\left(\frac{2\sqrt{2}}{3}\right)=\frac{4-6\sqrt{2}}{15}$$

$$\cos(a+b)=\left(\frac{3}{5}\right)\left(\frac{1}{3}\right)-\left(\frac{4}{5}\right)\left(\frac{2\sqrt{2}}{3}\right)=\frac{3-8\sqrt{2}}{15}$$

For the exercises 22-24, find the exact value of each expression.

22) $\sin\left(\cos^{-1}(0)-\cos^{-1}\left(\frac{1}{2}\right)\right)$

23) $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

Answer

$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

24) $\tan\left(\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{1}{2}\right)\right)$

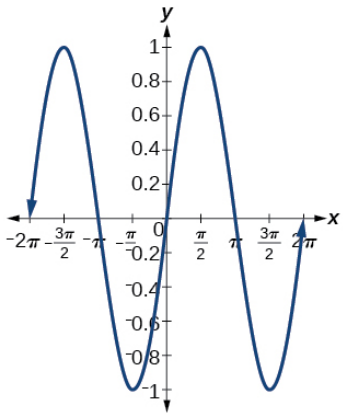
Graphical

For the exercises 25-32, simplify the expression, and then graph both expressions as functions to verify the graphs are identical.

25) $\cos\left(\frac{\pi}{2}-x\right)$

Answer

$\sin x$

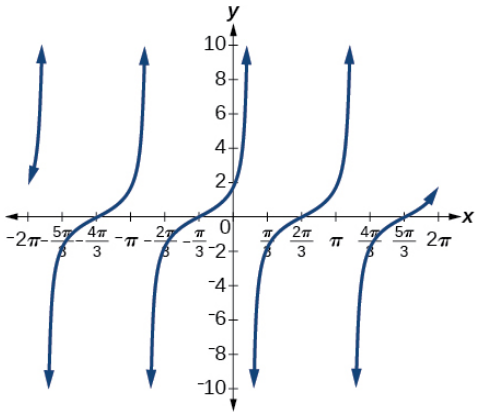


26) $\sin(\pi-x)$

27) $\tan\left(\frac{\pi}{3}+x\right)$

Answer

$\cot\left(\frac{\pi}{6}-x\right)$

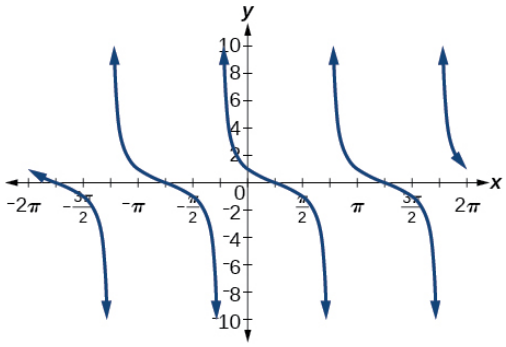


28) $\sin\left(\frac{\pi}{3}+x\right)$

29) $\tan\left(\frac{\pi}{4}-x\right)$

Answer

$\cot\left(\frac{\pi}{4}+x\right)$

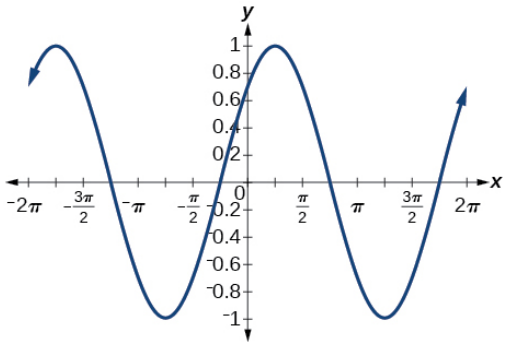


30) $\cos\left(\frac{7\pi}{6}+x\right)$

31) $\sin\left(\frac{\pi}{4}+x\right)$

Answer

$\frac{\sin x}{\sqrt{2}}+\frac{\cos x}{\sqrt{2}}$



32) $\cos\left(\frac{5\pi}{4}+x\right)$

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For the exercises 33-41, use a graph to determine whether the functions are the same or different. If they are the same, show why. If they are different, replace the second function with one that is identical to the first. (Hint: think $2x = x + x$)

33) $f(x) = \sin(4x) - \sin(3x)\cos x, g(x) = \sin x \cos(3x)$

Answer

They are the same.

34) $f(x) = \cos(4x) + \sin x \sin(3x), g(x) = -\cos x \cos(3x)$

35) $f(x) = \sin(3x)\cos(6x), g(x) = -\sin(3x)\cos(6x)$

Answer

They are different, try $g(x) = \sin(9x) - \cos(3x)\sin(6x)$

36) $f(x) = \sin(4x), g(x) = \sin(5x)\cos x - \cos(5x)\sin x$

37) $f(x) = \sin(2x), g(x) = 2\sin x \cos x$

Answer

They are the same.

38) $f(\theta) = \cos(2\theta), g(\theta) = \cos^2\theta - \sin^2\theta$

39) $f(\theta) = \tan(2\theta), g(\theta) = \dfrac{\tan\theta}{1 + \tan^2\theta}$

Answer

They are different, try $g(\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

40) $f(x) = \sin(3x)\sin x, g(x) = \sin^2(2x)\cos^2 x - \cos^2(2x)\sin 2x$

41) $f(x) = \tan(-x), g(x) = \dfrac{\tan x - \tan(2x)}{1 - \tan x \tan(2x)}$

Answer

They are different, try $g(x) = \dfrac{\tan x - \tan(2x)}{1 + \tan x \tan(2x)}$

Technology

For the exercises 42-46, find the exact value algebraically, and then confirm the answer with a calculator to the fourth decimal point.

42) $\sin(75^\circ)$

43) $\sin(195^\circ)$

Answer

$-\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, or -0.2588

44) $\cos(165^\circ)$

45) $\cos(345^\circ)$

Answer

$\dfrac{1+\sqrt{3}}{2\sqrt{2}}$, or -0.9659

46) $\tan(-15^\circ)$

Extensions

For the exercises 47-51, prove the identities provided.

47) $\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+1}{1-\tan x}$

Answer

$$\tan\left(x+\frac{\pi}{4}\right)=$$
$$\frac{\tan x+\tan\left(\frac{\pi}{4}\right)}{1-\tan x\tan\left(\frac{\pi}{4}\right)}=$$
$$\frac{\tan x+1}{1-\tan x(1)}=\frac{\tan x+1}{1-\tan x}$$

48) $\frac{\tan(a+b)}{\tan(a-b)}=\frac{\sin a\cos a+\sin b\cos b}{\sin a\cos a-\sin b\cos b}$

49) $\frac{\cos(a+b)}{\cos a\cos b}=1-\tan a\tan b$

Answer

$$\frac{\cos(a+b)}{\cos a\cos b}=$$
$$\frac{\cos a\cos b}{\cos a\cos b}-\frac{\sin a\sin b}{\cos a\cos b}=1-\tan a\tan b$$

50) $\cos(x+y)\cos(x-y)=\cos^2 x-\sin^2 y$

51) $\frac{\cos(x+h)-\cos(x)}{h}=\cos x\frac{\cos h-1}{h}-\sin x\frac{\sin h}{h}$

Answer

$$\frac{\cos(x+h)-\cos(x)}{h}=$$
$$\frac{\cos x\cosh-\sin x\sinh-\cos x}{h}=$$
$$\frac{\cos x(\cosh-1)-\sin x(\sinh-1)}{h}=\cos x\frac{\cos h-1}{h}-\sin x\frac{\sin h}{h}$$

For the exercises 52-, prove or disprove the statements.

52) $\tan(u+v)=\frac{\tan u+\tan v}{1-\tan u\tan v}$

53) $\tan(u-v)=\frac{\tan u-\tan v}{1+\tan u\tan v}$

Answer

True

54) $\frac{\tan(x+y)}{1+\tan x\tan x}=\frac{\tan x+\tan y}{1-\tan^2 x\tan^2 y}$

55) If α, β, and γ are angles in the same triangle, then prove or disprove $\sin(\alpha+\beta)=\sin\gamma$.

Answer

True. Note that $\sin(\alpha+\beta)=\sin(\pi-\gamma)$ and expand the right hand side.

56) If α, β, and γ are angles in the same triangle, then prove or disprove $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$.

7.3: Double-Angle, Half-Angle, and Reduction Formulas

In this section, we will investigate three additional categories of identities. Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not.

Verbal

1) Explain how to determine the reduction identities from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$

Answer

Use the Pythagorean identities and isolate the squared term.

2) Explain how to determine the double-angle formula for $\tan(2x)$ using the double-angle formulas for $\cos(2x)$ and $\sin(2x)$.

3) We can determine the half-angle formula for $\tan\left(\frac{x}{2}\right) = \frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}$ by dividing the formula for $\sin\left(\frac{x}{2}\right)$ by $\cos\left(\frac{x}{2}\right)$. Explain how to determine two formulas for $\tan\left(\frac{x}{2}\right)$ that do not involve any square roots.

Answer

$\frac{1-\cos x}{\sin x}$, $\frac{\sin x}{1+\cos x}$, multiplying the top and bottom by $\sqrt{1-\cos x}$ and $\sqrt{1+\cos x}$, respectively.

4) For the half-angle formula given in the previous exercise for $\tan\left(\frac{x}{2}\right)$, explain why dividing by 0 is not a concern. (Hint: examine the values of $\cos x$ necessary for the denominator to be 0.)

Algebraic

For the exercises 5-8, find the exact values of a) $\sin(2x)$, b) $\cos(2x)$, and c) $\tan(2x)$ without solving for x.

5) If $\sin x = \frac{1}{8}$, and x is in quadrant I.

Answer

a. $\frac{3\sqrt{7}}{32}$
b. $\frac{31}{32}$
c. $\frac{3\sqrt{7}}{31}$

6) If $\cos x = \frac{2}{3}$, and x is in quadrant I.

7) If $\cos x = -\frac{1}{2}$, and x is in quadrant III.

Answer

a. $\frac{\sqrt{3}}{2}$
b. $-\frac{1}{2}$
c. $-\sqrt{3}$

8) If $\tan x = -8$, and x is in quadrant IV.

For the exercises 9-10, find the values of the six trigonometric functions if the conditions provided hold.

9) $\cos(2\theta) = \frac{3}{5}$ and $90^\circ \le \theta \le 180^\circ$

Answer

$\cos\theta = -\frac{2\sqrt{5}}{5}, \sin\theta = \frac{\sqrt{5}}{5}, \tan\theta = -\frac{1}{2}, \csc\theta = \sqrt{5}, \sec\theta = -\frac{\sqrt{5}}{2}, \cot\theta = -2$

10) $\cos(2\theta) = \frac{1}{\sqrt{2}}$ and $180^\circ \le \theta \le 270^\circ$

For the exercises 11-12, simplify to one trigonometric expression.

11) $2\sin\left(\frac{\pi}{4}\right)2\cos\left(\frac{\pi}{4}\right)$

Answer

$2\sin\left(\frac{\pi}{2}\right)$

12) $4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

For the exercises 13-19, find the exact value using half-angle formulas.

13) $\sin\left(\frac{\pi}{8}\right)$

Answer

$\frac{\sqrt{2-\sqrt{2}}}{2}$

14) $\cos\left(-\frac{11\pi}{12}\right)$

15) $\sin\left(\frac{11\pi}{12}\right)$

Answer

$\frac{\sqrt{2-\sqrt{3}}}{2}$

16) $\cos\left(\frac{7\pi}{8}\right)$

17) $\tan\left(\frac{5\pi}{12}\right)$

Answer

$2+\sqrt{3}$

18) $\tan\left(-\frac{3\pi}{12}\right)$

19) $\tan\left(-\frac{3\pi}{8}\right)$

Answer

$-1-\sqrt{2}$

For the exercises 20-23, find the exact values of a) $\sin\left(\frac{x}{2}\right)$ b) $\cos\left(\frac{x}{2}\right)$, and c) $\tan\left(\frac{x}{2}\right)$ without solving for x, when $0° \le \theta \le 360°$

20) If $\tan x = -\frac{4}{3}$, and x is in quadrant IV.

21) If $\sin x = -\frac{12}{13}$, and x is in quadrant III.

Answer

a. $\frac{3\sqrt{13}}{13}$

b. $-\frac{2\sqrt{13}}{13}$

c. $-\frac{3}{2}$

22) If $\csc x = 7$, and x is in quadrant II.

23) If $\sec x = -4$, and x is in quadrant II.

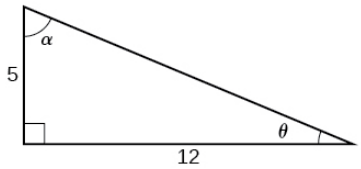
Answer

a. $\frac{\sqrt{10}}{4}$

b. $\frac{\sqrt{6}}{4}$

c. $\frac{\sqrt{15}}{3}$

For the exercises 24-27, use Figure below to find the requested half and double angles.



24) Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

25) Find $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$.

Answer

$\frac{120}{169}, -\frac{119}{169}, -\frac{120}{119}$

26) Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$.

27) Find $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$.

Answer

$\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}, \frac{2}{3}$

For the exercises 28-33, simplify each expression. Do not evaluate.

28) $\cos^2(28^\circ) - \sin^2(28^\circ)$

29) $2\cos^2(37^\circ) - 1$

Answer

$\cos(74^\circ)$

30) $1 - 2\sin^2(17^\circ)$

31) $\cos^2(9x) - \sin^2(9x)$

Answer

$\cos(18x)$

32) $4\sin(8x)\cos(8x)$

33) $6\sin(5x)\cos(5x)$

Answer

$3\sin(10x)$

For the exercises 34-37, prove the identity given.

34) $(\sin t - \cos t)^2 = 1 - \sin(2t)$

35) $\sin(2x) = -2\sin(-x)\cos(-x)$

Answer

$-2\sin(-x)\cos(-x) = -2(-\sin(x)\cos(x)) = \sin(2x)$

36) $\cot x - \tan x = 2\cot(2x)$

37) $\frac{1+\cos(2\theta)}{\sin(2\theta)}\tan^2\theta = \tan\theta$

Answer

$$\frac{\sin(2\theta)}{1+\cos(2\theta)}\tan^2\theta = \frac{2\sin(\theta)\cos(\theta)}{1+\cos^2\theta-\sin^2\theta}\tan^2\theta =$$

$$\frac{2\sin(\theta)\cos(\theta)}{2\cos^2\theta}\tan^2\theta = \frac{\sin(\theta)}{\cos(\theta)}\tan^2\theta =$$

$$\cot(\theta)\tan^2\theta = \tan\theta$$

For the exercises 38-44, rewrite the expression with an exponent no higher than 1.

38) $\cos^2(5x)$

39) $\cos^2(6x)$

Answer

$$\frac{1+\cos(12x)}{2}$$

40) $\sin^4(8x)$

41) $\sin^4(3x)$

Answer

$$\frac{3+\cos(12x)-4\cos(6x)}{8}$$

42) $\cos^2 x \sin^4 x$

43) $\cos^4 x \sin^2 x$

Answer

$$\frac{2+\cos(2x)-2\cos(4x)-\cos(6x)}{32}$$

44) $\tan^2 x \sin^2 x$

Technology

For the exercises 45-52, reduce the equations to powers of one, and then check the answer graphically.

45) $\tan^4 x$

Answer

$$\frac{3+\cos(4x)-4\cos(2x)}{3+\cos(4x)+4\cos(2x)}$$

46) $\sin^2(2x)$

47) $\sin^2 x \cos^2 x$

Answer

$$\frac{1-\cos(4x)}{8}$$

48) $\tan^2 x \sin x$

49) $\tan^4 x \cos^2 x$

Answer

$$\frac{3+\cos(4x)-4\cos(2x)}{4(\cos(2x)+1)}$$

50) $\cos^2 x \sin(2x)$

51) $\cos^2(2x)\sin x$

Answer

$$\frac{(1+\cos(4x))\sin x}{2}$$

52) $\tan^2\left(\frac{x}{2}\right)\sin x$

For the exercises 53-54, algebraically find an equivalent function, only in terms of $\sin x$ and/or $\cos x$, and then check the answer by graphing both equations.

53) $\sin(4x)$

Answer

$$4\sin x\cos x(\cos^2 x-\sin^2 x)$$

54) $\cos(4x)$

Extensions

For the exercises 55-63, prove the identities.

55) $\sin(2x)=\dfrac{2\tan x}{1+\tan^2 x}$

Answer

$$\frac{2\tan x}{1+\tan^2 x}=\frac{\frac{2\sin x}{\cos x}}{1+\frac{\sin^2 x}{\cos^2 x}}=\frac{\frac{2\sin x}{\cos x}}{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}=\frac{2\sin x}{\cos x}\cdot\frac{\cos^2 x}{1}=2\sin x\cos x=\sin(2x)$$

56) $\cos(2\alpha)=\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$

57) $\tan(2x)=\dfrac{2\sin x\cos x}{2\cos^2 x-1}$

Answer

$$\frac{2\sin x\cos x}{2\cos^2 x-1}=\frac{\sin(2x)}{\cos(2x)}=\tan(2x)$$

58) $(\sin^2 x-1)^2=\cos(2x)+\sin^4 x$

59) $\sin(3x)=3\sin x\cos^2 x-\sin^3 x$

Answer

$$\begin{aligned}\sin(x+2x)&=\sin x\cos(2x)+\sin(2x)\cos x\\&=\sin x(\cos^2 x-\sin^2 x)+2\sin x\cos x\cos x\\&=\sin x\cos^2 x-\sin^3 x+2\sin x\cos^2 x\\&=3\sin x\cos^2 x-\sin^3 x\end{aligned}$$

60) $\cos(3x)=\cos^3 x-3\sin^2 x\cos x$

61) $\dfrac{1+\cos(2t)}{\sin(2t)-\cos t}=\dfrac{2\cos t}{2\sin t-1}$

Answer

$$\begin{aligned}\frac{1+\cos(2t)}{\sin(2t)-\cos t}&=\frac{1+2\cos^2 t-1}{2\sin t\cos t-\cos t}\\&=\frac{2\cos^2 t}{\cos t(2\sin t-1)}\\&=\frac{2\cos t}{2\sin t-1}\end{aligned}$$

62) $\sin(16x)=16\sin x\cos x\cos(2x)\cos(4x)\cos(8x)$

63) $\cos(16x)=(\cos^2(4x)-\sin^2(4x)-\sin(8x))(\cos^2(4x)-\sin^2(4x)+\sin(8x))$

Answer

$$\begin{aligned}(\cos^2(4x)-\sin^2(4x)-\sin(8x))(\cos^2(4x)-\sin^2(4x)+\sin(8x)) &= (\cos(8x)-\sin(8x))(\cos(8x)+\sin(8x)) \\ &= \cos^2(8x)-\sin^2(8x) \\ &= \cos(16x)\end{aligned}$$

7.4: Sum-to-Product and Product-to-Sum Formulas

From the sum and difference identities, we can derive the product-to-sum formulas and the sum-to-product formulas for sine and cosine. The product-to-sum formulas can rewrite products of sines, products of cosines, and products of sine and cosine as sums or differences of sines and cosines. We can also derive the sum-to-product identities from the product-to-sum identities using substitution. The sum-to-product formulas are used to rewrite sum or difference as products of sines and cosines.

Verbal

1) Starting with the product to sum formula $\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$, explain how to determine the formula for $\cos\alpha\sin\beta$.

Answer

Substitute α into cosine and β into sine and evaluate.

2) Explain two different methods of calculating $\cos(195^\circ)\cos(105^\circ)$, one of which uses the product to sum. Which method is easier?

3) Explain a situation where we would convert an equation from a sum to a product and give an example.

Answer

Answers will vary. There are some equations that involve a sum of two trig expressions where when converted to a product are easier to solve. For example: $\frac{\sin(3x)+\sin x}{\cos x}=1$. When converting the numerator to a product the equation becomes: $\frac{2\sin(2x)\cos x}{\cos x}=1$

4) Explain a situation where we would convert an equation from a product to a sum, and give an example.

Algebraic

For the exercises 5-10, rewrite the product as a sum or difference.

5) $16\sin(16x)\sin(11x)$

Answer

$8(\cos(5x)-\cos(27x))$

6) $2\cos(36t)\cos(6t)$

7) $2\sin(5x)\cos(3x)$

Answer

$\sin(2x)+\sin(8x)$

8) $10\cos(5x)\sin(10x)$

9) $\sin(-x)\sin(5x)$

Answer

$\frac{1}{2}(\cos(6x)-\cos(4x))$

10) $\sin(3x)\cos(5x)$

For the exercises 11-16, rewrite the sum or difference as a product.

11) $\cos(6t)+\cos(4t)$

Answer

$2\cos(5t)\cos t$

12) $\sin(3x)+\sin(7x)$

13) $\cos(7x)+\cos(-7x)$

Answer

$2\cos(7x)$

14) $\sin(3x)-\sin(-3x)$

15) $\cos(3x)+\cos(9x)$

Answer

$2\cos(6x)\cos(3x)$

16) $\sin h-\sin(3h)$

For the exercises 17-21, evaluate the product for the following using a sum or difference of two functions.

17) $\cos(45^\circ)\cos(15^\circ)$

Answer

$\frac{1}{4}(1+\sqrt{3})$

18) $\cos(45^\circ)\sin(15^\circ)$

19) $\sin(-345^\circ)\sin(-15^\circ)$

Answer

$\frac{1}{4}(\sqrt{3}-2)$

20) $\sin(195^\circ)\cos(15^\circ)$

21) $\sin(-45^\circ)\sin(-15^\circ)$

Answer

$\frac{1}{4}(\sqrt{3}-1)$

For the exercises 22-26, evaluate the product using a sum or difference of two functions. Leave in terms of sine and cosine.

22) $\cos(23^\circ)\sin(17^\circ)$

23) $2\sin(100^\circ)\sin(20^\circ)$

Answer

$\cos(80^\circ)-\cos(120^\circ)$

24) $2\sin(-100^\circ)\sin(-20^\circ)$

25) $\sin(213^\circ)\cos(8^\circ)$

Answer

$\frac{1}{2}(\sin(221^\circ)+\sin(205^\circ))$

26) $2\cos(56^\circ)\cos(47^\circ)$

For the exercises 27-31, rewrite the sum as a product of two functions. Leave in terms of sine and cosine.

27) $\sin(76^\circ)+\sin(14^\circ)$

Answer

$\sqrt{3}\cos(31^\circ)$

28) $\cos(58^\circ)-\cos(12^\circ)$

29) $\sin(101^\circ)-\sin(32^\circ)$

Answer

$2\cos(66.5^\circ)\sin(34.5^\circ)$

30) $\cos(100^\circ)+\cos(200^\circ)$

31) $\sin(-1^\circ)+\sin(-2^\circ)$

Answer

$2\sin(-1.5^\circ)\cos(0.5^\circ)$

For the exercises 32-38, prove the identity.

32) $\dfrac{\cos(a-b)}{\cos(a+b)}=\dfrac{1-\tan a\tan b}{1+\tan a\tan b}$

33) $4\sin(3x)\cos(4x)=2\sin(7x)-2\sin x$

Answer

$$\begin{aligned}2\sin(7x)-2\sin x &= 2\sin(4x+3x)-2\sin(4x-3x)\\ &= 2(\sin(4x)\cos(3x)+\sin(3x)\cos(4x))-2(\sin(4x)\cos(3x)-\sin(3x)\cos(4x))\\ &= 2\sin(4x)\cos(3x)+2\sin(3x)\cos(4x))-2\sin(4x)\cos(3x)+2\sin(3x)\cos(4x))\\ &= 4\sin(3x)\cos(4x)\end{aligned}$$

34) $\dfrac{6\cos(8x)\sin(2x)}{\sin(-6x)}=-3\sin(10x)\csc(6x)+3$

35) $\sin x+\sin(3x)=4\sin x\cos^2 x$

Answer

$$\begin{aligned}\sin x+\sin(3x) &= 2\sin\left(\frac{4x}{2}\right)\cos\left(-\frac{2x}{2}\right)\\ &= 2\sin(2x)\cos x\\ &= 2(2\sin x\cos x)\cos x\\ &= 4\sin x\cos^2 x\end{aligned}$$

36) $2(\cos^3 x-\cos x\sin^2 x)=\cos(3x)+\cos x$

37) $2\tan x\cos(3x)=\sec x(\sin(4x)-\sin(2x))$

Answer

$$\begin{aligned}2\tan x\cos(3x) &= \frac{2\sin x\cos(3x)}{\cos x}\\ &= \frac{2(.5(\sin(4x)-\sin(2x)))}{\cos x}\\ &= \frac{1}{\cos x}(\sin(4x)-\sin(2x))\\ &= \sec x(\sin(4x)-\sin(2x))\end{aligned}$$

38) $\cos(a+b)+\cos(a-b)=2\cos a\cos b$

Numeric

For the exercises 39-43, rewrite the sum as a product of two functions or the product as a sum of two functions. Give your answer in terms of sines and cosines. Then evaluate the final answer numerically, rounded to four decimal places.

39) $\cos(58^\circ)+\cos(12^\circ)$

Answer

$2\cos(35^\circ)\cos(23^\circ), 1.5081$

40) $\sin(2^\circ)-\sin(3^\circ)$

41) $\cos(44^\circ)-\cos(22^\circ)$

Answer

$-2\sin(33^\circ)\sin(11^\circ), -0.2078$

42) $\cos(176^\circ)\sin(9^\circ)$

43) $\sin(-14^\circ)\sin(85^\circ)$

Answer

$\frac{1}{2}(\cos(99^\circ)-\cos(71^\circ)), -0.2410$

Technology

For the exercises 44-48, algebraically determine whether each of the given expressions is a true identity. If it is not an identity, replace the right-hand side with an expression equivalent to the left side. Verify the results by graphing both expressions on a calculator.

44) $2\sin(2x)\sin(3x)=\cos x-\cos(5x)$

45) $\frac{\cos(10\theta)+\cos(6\theta)}{\cos(6\theta)-\cos(10\theta)}=\cot(2\theta)\cot(\theta)$

Answer

It is an identity.

46) $\frac{\sin(3x)-\sin(5x)}{\cos(3x)+\cos(5x)}=\tan x$

47) $2\cos(2x)\cos x+\sin(2x)\sin x=2\sin x$

Answer

It is not an identity, but $2\cos^3 x$ is.

48) $\frac{\sin(2x)+\sin(4x)}{\sin(2x)-\sin(4x)}=-\tan(3x)\cot x$

For the exercises 49-53, simplify the expression to one term, then graph the original function and your simplified version to verify they are identical.

49) $\frac{\sin(9t)-\sin(3t)}{\cos(9t)+\cos(3t)}$

Answer

$\tan(3t)$

50) $2\sin(8x)\cos(6x)-\sin(2x)$

51) $\frac{\sin(3x)-\sin(x)}{\sin x}$

Answer

$2\cos(2x)$

52) $\frac{\cos(5x)+\cos(3x)}{\sin(5x)+\sin(3x)}$

53) $\sin x\cos(15x)-\cos x\sin(15x)$

Answer

$-\sin(14x)$

Extensions

For the exercises 54-55, prove the following sum-to-product formulas.

54) $\sin x-\sin y=2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

55) $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

Answer

Start with $\cos x + \cos y$. Make a substitution and let $x = \alpha + \beta$ and let $y = \alpha - \beta$, so $\cos x + \cos y$ becomes:

$\cos(\alpha+\beta) + \cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta = 2\cos\alpha\cos\beta$

Since $x = \alpha + \beta$ and $y = \alpha - \beta$, we can solve for α and β in terms of x and y and substitute in for $2\cos\alpha\cos\beta$ and get

$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

For the exercises 56-63, prove the identity.

56) $\frac{\sin(6x)+\sin(4x)}{\sin(6x)-\sin(4x)} = \tan(5x)\cot x$

57) $\frac{\cos(3x)+\cos x}{\cos(3x)-\cos(4x)} = -\cot(2x)\cot x$

Answer

$$\frac{\cos(3x)+\cos x}{\cos(3x)-\cos(4x)} = \frac{2\cos(2x)\cos x}{-2\sin(2x)\sin x} = -\cot(2x)\cot x$$

58) $\frac{\cos(6y)+\cos(8y)}{\sin(6y)-\sin(4y)} = \cot y\cos(7y)\sec(5y)$

59) $\frac{\cos(2y)-\cos(4y)}{\sin(2y)+\sin(4y)} = \tan y$

Answer

$$\begin{aligned}\frac{\cos(2y)-\cos(4y)}{\sin(2y)+\sin(4y)} &= \frac{-2\sin(3y)\sin(-y)}{2\sin(3y)\cos y}\\ &= \frac{2\sin(3y)\sin(y)}{2\sin(3y)\cos y}\\ &= \tan y\end{aligned}$$

60) $\frac{\sin(10x)-\sin(2x)}{\cos(10x)+\cos(2x)} = \tan(4x)$

61) $\cos x - \cos(3x) = 4\sin^2 x\cos x$

Answer

$$\begin{aligned}\cos x - \cos(3x) &= -2\sin(2x)\sin(-x)\\ &= 2(2\sin x\cos x)\sin x\\ &= 4\sin^2 x\cos x\end{aligned}$$

62) $(\cos(2x)-\cos(4x))^2 + (\sin(4x)+\sin(2x))^2 = 4\sin^2(3x)$

63) $\tan\left(\frac{\pi}{4}-t\right) = \frac{1-\tan t}{1+\tan t}$

Answer

$$\tan\left(\frac{\pi}{4}-t\right) = \frac{\tan\left(\frac{\pi}{4}\right)-\tan t}{1+\tan\left(\frac{\pi}{4}\right)\tan t} = \frac{1-\tan t}{1+\tan t}$$

7.5: Solving Trigonometric Equations

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Verbal

1) Will there always be solutions to trigonometric function equations? If not, describe an equation that would not have a solution. Explain why or why not.

Answer

There will not always be solutions to trigonometric function equations. For a basic example, $\cos(x) = -5$.

2) When solving a trigonometric equation involving more than one trig function, do we always want to try to rewrite the equation so it is expressed in terms of one trigonometric function? Why or why not?

3) When solving linear trig equations in terms of only sine or cosine, how do we know whether there will be solutions?

Answer

If the sine or cosine function has a coefficient of one, isolate the term on one side of the equals sign. If the number it is set equal to has an absolute value less than or equal to one, the equation has solutions, otherwise it does not. If the sine or cosine does not have a coefficient equal to one, still isolate the term but then divide both sides of the equation by the leading coefficient. Then, if the number it is set equal to has an absolute value greater than one, the equation has no solution.

Algebraic

For the exercises 4-12, find all solutions exactly on the interval $0 \le \theta < 2\pi$

4) $2\sin\theta = -\sqrt{2}$

5) $2\sin\theta = \sqrt{3}$

Answer

$\frac{\pi}{3}, \frac{2\pi}{3}$

6) $2\cos\theta = 1$

7) $2\cos\theta = -\sqrt{2}$

Answer

$\frac{3\pi}{4}, \frac{5\pi}{4}$

8) $\tan\theta = -1$

9) $\tan x = 1$

Answer

$\frac{\pi}{4}, \frac{5\pi}{4}$

10) $\cot x + 1 = 0$

11) $4\sin^2 x - 2 = 0$

Answer

$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

12) $\csc^2 x - 4 = 0$

For the exercises 13-22, solve exactly on $[0, 2\pi)$

13) $2\cos\theta = \sqrt{2}$

Answer

$\frac{\pi}{4}, \frac{7\pi}{4}$

14) $2\cos\theta = -1$

15) $2\sin\theta = -1$

Answer

$\frac{7\pi}{6}, \frac{11\pi}{6}$

16) $2\sin\theta = -\sqrt{3}$

17) $2\sin(3\theta) = 1$

Answer

$\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

18) $2\sin(2\theta) = \sqrt{3}$

19) $2\cos(3\theta) = -\sqrt{2}$

Answer

$\frac{3\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \frac{21\pi}{12}$

20) $\cos(2\theta) = -\frac{\sqrt{3}}{2}$

21) $2\sin(\pi\theta) = 1$

Answer

$\frac{1}{6}, \frac{5}{6}, \frac{13}{6}, \frac{17}{6}, \frac{25}{6}, \frac{29}{6}, \frac{37}{6}$

22) $2\cos\left(\frac{\pi}{5}\theta\right) = \sqrt{3}$

For the exercises 23-32, find all exact solutions on $[0, 2\pi)$

23) $\sec(x)\sin(x) - 2\sin(x) = 0$

Answer

$0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

24) $\tan(x) - 2\sin(x)\tan(x) = 0$

25) $2\cos^2 t + \cos(t) = 1$

Answer

$\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

26) $2\tan^2(t) = 3\sec(t)$

27) $2\sin(x)\cos(x) - \sin(x) + 2\cos(x) - 1 = 0$

Answer

$\frac{\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

28) $\cos^2\theta = \frac{1}{2}$

29) $\sec^2 x = 1$

Answer

$0, \pi$

30) $\tan^2(x) = -1 + 2\tan(-x)$

31) $8\sin^2(x) + 6\sin(x) + 1 = 0$

Answer

$$\pi - \sin^{-1}\left(-\frac{1}{4}\right), \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi + \sin^{-1}\left(-\frac{1}{4}\right)$$

32) $\tan^5(x) = \tan(x)$

For the exercises 33-40, solve with the methods shown in this section exactly on the interval $[0, 2\pi)$

33) $\sin(3x)\cos(6x) - \cos(3x)\sin(6x) = -0.9$

Answer

$$\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{\pi}{3} - \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{2\pi}{3} + \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \pi - \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{4\pi}{3} + \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{5\pi}{3} -$$

34) $\sin(6x)\cos(11x) - \cos(6x)\sin(11x) = -0.1$

35) $\cos(2x)\cos x + \sin(2x)\sin x = 1$

Answer

0

36) $6\sin(2t) + 9\sin t = 0$

37) $9\cos(2\theta) = 9\cos^2\theta - 4$

Answer

$$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

38) $\sin(2t) = \cos t$

39) $\cos(2t) = \sin t$

Answer

$$\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

40) $\cos(6x) - \cos(3x) = 0$

For the exercises 41-49, solve exactly on the interval $[0, 2\pi)$. Use the quadratic formula if the equations do not factor.

41) $\tan^2 x - \sqrt{3}\tan x = 0$

Answer

$$0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}$$

42) $\sin^2 x + \sin x - 2 = 0$

43) $\sin^2 x - 2\sin x - 4 = 0$

Answer

There are no solutions.

44) $5\cos^2 x + 3\cos x - 1 = 0$

45) $3\cos^2 x - 3\cos x - 2 = 0$

Answer

$$\cos^{-1}\left(\frac{1}{3}\left(1 - \sqrt{7}\right)\right), 2\pi - \cos^{-1}\left(\frac{1}{3}\left(1 - \sqrt{7}\right)\right)$$

46) $5\sin^2 x + 2\sin x - 1 = 0$

47) $\tan^2 x + 5\tan x - 1 = 0$

Answer

$$\tan^{-1}\left(\frac{1}{2}\left(\sqrt{29}-5\right)\right), \pi+\tan^{-1}\left(\frac{1}{2}\left(-\sqrt{29}-5\right)\right), \pi+\tan^{-1}\left(\frac{1}{2}\left(\sqrt{29}-5\right)\right), 2\pi+\tan^{-1}\left(\frac{1}{2}\left(-\sqrt{29}-5\right)\right)$$

48) $\cot^2 x = -\cot x$

49) $-\tan^2 x - \tan x - 2 = 0$

Answer

There are no solutions.

For the exercises 50-65, find exact solutions on the interval $[0, 2\pi)$. Look for opportunities to use trigonometric identities.

50) $\sin^2 x - \cos^2 x - \sin x = 0$

51) $\sin^2 x + \cos^2 x = 0$

Answer

There are no solutions.

52) $\sin(2x) - \sin x = 0$

53) $\cos(2x) - \cos x = 0$

Answer

$$0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

54) $\dfrac{2\tan x}{2-\sec^2 x} - \sin^2 x = \cos^2 x$

55) $1 - \cos(2x) = 1 + \cos(2x)$

Answer

$$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

56) $\sec^2 x = 7$

57) $10 \sin x \cos x = 6 \cos x$

Answer

$$\sin^{-1}\left(\frac{3}{5}\right), \frac{\pi}{2}, \pi - \sin^{-1}\left(\frac{3}{5}\right), \frac{3\pi}{2}$$

58) $-3 \sin t = 15 \cos t \sin t$

59) $4 \cos^2 x - 4 = 15 \cos x$

Answer

$$\cos^{-1}\left(-\frac{1}{4}\right), 2\pi - \cos^{-1}\left(-\frac{1}{4}\right)$$

60) $8 \sin^2 x + 6 \sin x + 1 = 0$

61) $8 \cos^2 \theta = 3 - 2 \cos\theta$

Answer

$$\frac{\pi}{3}, \cos^{-1}\left(-\frac{3}{4}\right), 2\pi - \cos^{-1}\left(-\frac{3}{4}\right), \frac{5\pi}{3}$$

62) $6 \cos^2 x + 7 \sin x - 8 = 0$

63) $12 \sin^2 t + \cos t - 6 = 0$

Answer

$$\cos^{-1}\left(\frac{3}{4}\right), \cos^{-1}\left(-\frac{2}{3}\right), 2\pi - \cos^{-1}\left(-\frac{2}{3}\right), 2\pi - \cos^{-1}\left(\frac{3}{4}\right)$$

64) $\tan x = 3\sin x$

65) $\cos^3 t = \cos t$

Answer

$$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

Graphical

For the exercises 66-72, algebraically determine all solutions of the trigonometric equation exactly, then verify the results by graphing the equation and finding the zeros.

66) $6\sin^2 x - 5\sin x + 1 = 0$

67) $8\cos^2 x - 2\cos x - 1 = 0$

Answer

$$\frac{\pi}{3}, \cos^{-1}\left(-\frac{1}{4}\right), 2\pi - \cos^{-1}\left(-\frac{1}{4}\right), \frac{5\pi}{3}$$

68) $100\tan^2 x + 20\tan x - 3 = 0$

69) $2\cos^2 x - \cos x + 15 = 0$

Answer

There are no solutions.

70) $20\sin^2 x - 27\sin x + 7 = 0$

71) $2\tan^2 x + 7\tan x + 6 = 0$

Answer

$$\pi + \tan^{-1}(-2), \pi + \tan^{-1}\left(-\frac{3}{2}\right), 2\pi + \tan^{-1}(-2), 2\pi + \tan^{-1}\left(-\frac{3}{2}\right)$$

72) $130\tan^2 x + 69\tan x - 130 = 0$

Technology

For the exercises 73-76, use a calculator to find all solutions to four decimal places.

73) $\sin x = 0.27$

Answer

$$2\pi k + 0.2734, 2\pi k + 2.8682$$

74) $\sin x = -0.55$

75) $\tan x = -0.34$

Answer

$$\pi k - 0.3277$$

76) $\cos x = 0.71$

77) $\tan^2 x + 3\tan x - 3 = 0$

Answer

$$0.6694, 1.8287, 3.8110, 4.9703$$

78) $6\tan^2 x + 13\tan x = -6$

79) $\tan^2 x - \sec x = 1$

Answer

$1.0472, 3.1416, 5.2360$

80) $\sin^2 x - 2\cos^2 x = 0$

81) $2\tan^2 x + 9\tan x - 6 = 0$

Answer

$0.5326, 1.7648, 3.6742, 4.9064$

82) $4\sin^2 x + \sin(2x)\sec x - 3 = 0$

Extensions

For the exercises 83-92, find all solutions exactly to the equations on the interval $[0, 2\pi)$.

83) $\csc^2 x - 3\csc x - 4 = 0$

Answer

$\sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\left(\frac{1}{4}\right), \frac{3\pi}{2}$

84) $\sin^2 x - \cos^2 x - 1 = 0$

85) $\sin^2 x(1 - \sin^2 x) + \cos^2 x(1 - \sin^2 x) = 0$

Answer

$\frac{\pi}{2}, \frac{3\pi}{2}$

86) $3\sec^2 x + 2 + \sin^2 x - \tan^2 x + \cos^2 x = 0$

87) $\sin^2 x - 1 + 2\cos(2x) - \cos^2 x = 1$

Answer

There are no solutions.

88) $\tan^2 x - 1 - \sec^3 x \cos x = 0$

89) $\frac{\sin(2x)}{\sec^2 x} = 0$

Answer

$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

90) $\frac{\sin(2x)}{2\csc^2 x} = 0$

91) $2\cos^2 x - \sin^2 x - \cos x - 5 = 0$

Answer

There are no solutions.

92) $\frac{1}{\sec^2 x} + 2 + \sin^2 x + 4\cos^2 x = 4$

Real-World Applications

93) An airplane has only enough gas to fly to a city 200 miles northeast of its current location. If the pilot knows that the city is 25 miles north, how many degrees north of east should the airplane fly?

Answer

7.2°

94) If a loading ramp is placed next to a truck, at a height of 4 feet, and the ramp is 15 feet long, what angle does the ramp make with the ground?

95) If a loading ramp is placed next to a truck, at a height of 2 feet, and the ramp is 20 feet long, what angle does the ramp make with the ground?

Answer

5.7°

96) A woman is watching a launched rocket currently 11 miles in altitude. If she is standing 4 miles from the launch pad, at what angle is she looking up from horizontal?

97) An astronaut is in a launched rocket currently 15 miles in altitude. If a man is standing 2 miles from the launch pad, at what angle is she looking down at him from horizontal? (Hint: this is called the angle of depression.)

Answer

82.4°

98) A woman is standing 8 meters away from a 10-meter tall building. At what angle is she looking to the top of the building?

99) A man is standing 10 meters away from a 6-meter tall building. Someone at the top of the building is looking down at him. At what angle is the person looking at him?

Answer

31.0°

100) A 20-foot tall building has a shadow that is 55 feet long. What is the angle of elevation of the sun?

101) A 90-foot tall building has a shadow that is 2 feet long. What is the angle of elevation of the sun?

Answer

88.7°

102) A spotlight on the ground 3 meters from a 2-meter tall man casts a 6 meter shadow on a wall 6 meters from the man. At what angle is the light?

103) A spotlight on the ground 3 feet from a 5-foot tall woman casts a 15-foot tall shadow on a wall 6 feet from the woman. At what angle is the light?

Answer

59.0°

For the exercises 104-106, find a solution to the word problem algebraically. Then use a calculator to verify the result. Round the answer to the nearest tenth of a degree.

104) A person does a handstand with his feet touching a wall and his hands 1.5 feet away from the wall. If the person is 6 feet tall, what angle do his feet make with the wall?

105) A person does a handstand with her feet touching a wall and her hands 3 feet away from the wall. If the person is 5 feet tall, what angle do her feet make with the wall?

Answer

36.9°

106) A 23-foot ladder is positioned next to a house. If the ladder slips at 7 feet from the house when there is not enough traction, what angle should the ladder make with the ground to avoid slipping?

7.6: Modeling with Trigonometric Equations

Many natural phenomena are also periodic. For example, the phases of the moon have a period of approximately 28 days, and birds know to fly south at about the same time each year. So how can we model an equation to reflect periodic behavior? First, we must collect and record data.

We then find a function that resembles an observed pattern and alter the function to get a dependable model. Here. we will take a deeper look at specific types of periodic behavior and model equations to fit data.

Verbal

1) Explain what types of physical phenomena are best modeled by sinusoidal functions. What are the characteristics necessary?

Answer

Physical behavior should be periodic, or cyclical.

2) What information is necessary to construct a trigonometric model of daily temperature? Give examples of two different sets of information that would enable modeling with an equation.

3) If we want to model cumulative rainfall over the course of a year, would a sinusoidal function be a good model? Why or why not?

Answer

Since cumulative rainfall is always increasing, a sinusoidal function would not be ideal here.

4) Explain the effect of a damping factor on the graphs of harmonic motion functions.

Algebraic

For the exercises 5-13, find a possible formula for the trigonometric function represented by the given table of values.

5)

x	y
0	-4
3	-1
6	2
9	-1
12	-4
15	-1
18	2

Answer

$y = -3\cos\left(\frac{\pi}{6}x\right) - 1$

6)

x	y
0	5
2	1
4	-3
6	1
8	5
10	1
12	-3

7)

x	y
0	2
$\frac{\pi}{4}$	7
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	-3
π	2
$\frac{5\pi}{4}$	7
$\frac{3\pi}{2}$	2

Answer

$5\sin(2x)+2$

8)

x	y
0	2
$\frac{\pi}{4}$	7
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	-3
π	2
$\frac{5\pi}{4}$	7
$\frac{3\pi}{2}$	2

9)

x	y
0	1
1	-3
2	-7
3	-3
4	1
5	-3
6	-7

Answer

$4\cos\left(\frac{x\pi}{2}\right)-3$

10)

x	y
0	-2
1	4
2	10
3	4

x	y
4	-2
5	4
6	10

11)

x	y
0	5
1	-3
2	5
3	13
4	5
5	-3
6	5

Answer

$5-8\sin\left(\frac{x\pi}{2}\right)$

12)

x	y
-3	$-1-\sqrt{2}$
-2	-1
-1	$1-\sqrt{2}$
0	0
1	$\sqrt{2}-1$
2	1
3	$\sqrt{2}+1$

13)

x	y
-1	$\sqrt{3}-2$
0	0
1	$2-\sqrt{3}$
2	$\frac{\sqrt{3}}{3}$
3	1
4	$\sqrt{3}$
5	$2+\sqrt{3}$

Answer

$\tan\left(\frac{x\pi}{12}\right)$

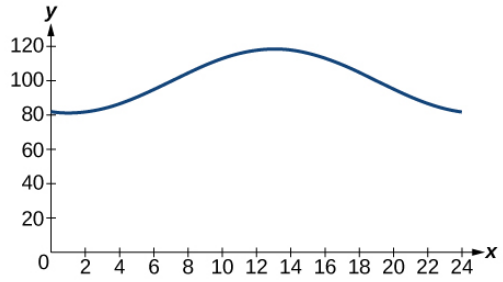
Graphical

For the exercises 14-16, graph the given function, and then find a possible physical process that the equation could model.

14) $f(x) = -30\cos\left(\frac{x\pi}{6}\right) - 20\cos^2\left(\frac{x\pi}{6}\right) + 80$ $[0, 12]$

15) $f(x) = -18\cos\left(\frac{x\pi}{12}\right) - 5\sin\left(\frac{x\pi}{12}\right) + 100$ on the interval $[0, 24]$

Answer



Answers will vary. Sample answer: This function could model temperature changes over the course of one very hot day in Phoenix, Arizona.

16) $f(x) = 10 - \sin\left(\frac{x\pi}{6}\right) + 24\tan\left(\frac{x\pi}{240}\right)$ on the interval $[0, 80]$

Technology

For the exercise 17, construct a function modeling behavior and use a calculator to find desired results.

17) A city's average yearly rainfall is currently 20 inches and varies seasonally by 5 inches. Due to unforeseen circumstances, rainfall appears to be decreasing by 15% each year. How many years from now would we expect rainfall to initially reach 0 inches? Note, the model is invalid once it predicts negative rainfall, so choose the first point at which it goes below 0.

Answer

9 years from now

Real-World Applications

For the exercises 18-29, construct a sinusoidal function with the provided information, and then solve the equation for the requested values.

18) Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the high temperature of $105^\circ F$ occurs at 5PM and the average temperature for the day is $85^\circ F$. Find the temperature, to the nearest degree, at 9AM.

19) Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the high temperature of $84^\circ F$ occurs at 6PM and the average temperature for the day is $70^\circ F$. Find the temperature, to the nearest degree, at 7AM.

Answer

$56^\circ F$

20) Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the temperature varies between $47^\circ F$ and $63^\circ F$ during the day and the average daily temperature first occurs at 10 AM. How many hours after midnight does the temperature first reach $51^\circ F$?

21) Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the temperature varies between $64^\circ F$ and $86^\circ F$ during the day and the average daily temperature first occurs at 12 AM. How many hours after midnight does the temperature first reach $70^\circ F$?

Answer

1.8024hours

22) A Ferris wheel is 20 meters in diameter and boarded from a platform that is 2 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 6 minutes. How much of the ride, in minutes and seconds, is spent higher than 13 meters above the ground?

23)A Ferris wheel is 45 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. How many minutes of the ride are spent higher than 27 meters above the ground? Round to the nearest second.

Answer

4 : 30

24) The sea ice area around the North Pole fluctuates between about 6 million square kilometers on September 1 to 14 million square kilometers on March 1. Assuming a sinusoidal fluctuation, when are there less than 9 million square kilometers of sea ice? Give your answer as a range of dates, to the nearest day.

25) The sea ice area around the South Pole fluctuates between about 18 million square kilometers in September to 3 million square kilometers in March. Assuming a sinusoidal fluctuation, when are there more than 15 million square kilometers of sea ice? Give your answer as a range of dates, to the nearest day.

Answer

From July 8 to October 23

26) During a 90-day monsoon season, daily rainfall can be modeled by sinusoidal functions. If the rainfall fluctuates between a low of 2 inches on day 10 and 12 inches on day 55, during what period is daily rainfall more than 10 inches?

27) During a 90-day monsoon season, daily rainfall can be modeled by sinusoidal functions. A low of 4 inches of rainfall was recorded on day 30, and overall the average daily rainfall was 8 inches. During what period was daily rainfall less than 5 inches?

Answer

From day 19 through day 40

28) In a certain region, monthly precipitation peaks at 8 inches on June 1 and falls to a low of 1 inch on December 1. Identify the periods when the region is under flood conditions (greater than 7 inches) and drought conditions (less than 2 inches). Give your answer in terms of the nearest day.

29) In a certain region, monthly precipitation peaks at 24 inches in September and falls to a low of 4 inches in March. Identify the periods when the region is under flood conditions (greater than 22 inches) and drought conditions (less than 5 inches). Give your answer in terms of the nearest day.

Answer

Floods: July 24 through October 7. Droughts: February 4 through March 27

For the exercises 30-32, find the amplitude, period, and frequency of the given function.

30) The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = 8\sin(6\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

31) The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = 11\sin(12\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

Answer

Amplitude: 11, period: $\frac{1}{6}$, frequency: 6 Hz

32) The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = 4\cos\left(\frac{\pi}{2}t\right)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

For the exercise 33, construct an equation that models the described behavior.

33) The displacement $h(t)$, in centimeters, of a mass suspended by a spring is modeled by the function $h(t) = -5\cos(60\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

Answer

Amplitude: 5, period: $\frac{1}{30}$, frequency: 30 Hz

For the exercises 34-41, construct an equation that models the described behavior.

34) A deer population oscillates 19 above and below average during the year, reaching the lowest value in January. The average population starts at 800 deer and increases by 160 each year. Find a function that models the population, P, in terms of months since January, t.

35) A rabbit population oscillates 15 above and below average during the year, reaching the lowest value in January. The average population starts at 650 rabbits and increases by 110 each year. Find a function that models the population, P, in terms of months since January, t.

Answer

$$P(t) = -15\cos\left(\frac{\pi}{6}t\right) + 650 + \frac{55}{6}t$$

36) A muskrat population oscillates 33 above and below average during the year, reaching the lowest value in January. The average population starts at 900 muskrats and increases by 7% each month. Find a function that models the population, P, in terms of months since January, t.

37) A fish population oscillates 40 above and below average during the year, reaching the lowest value in January. The average population starts at 800 fish and increases by 4% each month. Find a function that models the population, P, in terms of months since January, t.

Answer

$$P(t) = -40\cos\left(\frac{\pi}{6}t\right) + 800(1.04)^t$$

38) A spring attached to the ceiling is pulled 10 cm down from equilibrium and released. The amplitude decreases by 15% each second. The spring oscillates 18 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

39) A spring attached to the ceiling is pulled 7 cm down from equilibrium and released. The amplitude decreases by 11% each second. The spring oscillates 20 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

Answer

$$D(t) = 7(0.89)^t\cos(40\pi t)$$

40) A spring attached to the ceiling is pulled 17 cm down from equilibrium and released. After 3 seconds, the amplitude has decreased to 13 cm. The spring oscillates 14 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

41) A spring attached to the ceiling is pulled 19 cm down from equilibrium and released. After 4 seconds, the amplitude has decreased to 14 cm. The spring oscillates 13 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

Answer

$$D(t) = 19(0.9265)^t\cos(26\pi t)$$

For the exercises 42-47, create a function modeling the described behavior. Then, calculate the desired result using a calculator.

42) A certain lake currently has an average trout population of 20,000. The population naturally oscillates above and below average by 2,000 every year. This year, the lake was opened to fishermen. If fishermen catch 3,000 fish every year, how long will it take for the lake to have no more trout?

43) Whitefish populations are currently at 500 in a lake. The population naturally oscillates above and below by 25 each year. If humans overfish, taking 4% of the population every year, in how many years will the lake first have fewer than 200 whitefish?

Answer

20.1 years

44) A spring attached to a ceiling is pulled down 11 cm from equilibrium and released. After 2 seconds, the amplitude has decreased to 6 cm. The spring oscillates 8 times each second. Find when the spring first comes between -0.1 and 0.1 cm, effectively at rest.

45) A spring attached to a ceiling is pulled down 21 cm from equilibrium and released. After 6 seconds, the amplitude has decreased to 4 cm. The spring oscillates 20 times each second. Find when the spring first comes between -0.1 and 0.1 cm, effectively at rest.

Answer

17.8 seconds

46) Two springs are pulled down from the ceiling and released at the same time. The first spring, which oscillates 8 times per second, was initially pulled down 32 cm from equilibrium, and the amplitude decreases by 50% each second. The second spring, oscillating 18 times per second, was initially pulled down 15 cm from equilibrium and after 4 seconds has an amplitude of 2 cm. Which spring comes to rest first, and at what time? Consider "rest" as an amplitude less than 0.1 cm.

47) Two springs are pulled down from the ceiling and released at the same time. The first spring, which oscillates 14 times per second, was initially pulled down 2 cm from equilibrium, and the amplitude decreases by 8% each second. The second spring, oscillating 22 times per second, was initially pulled down 10 cm from equilibrium and after 3 seconds has an amplitude of 2 cm. Which spring comes to rest first, and at what time? Consider "rest" as an amplitude less than 0.1 cm.

Answer

Spring 2 comes to rest first after 8.0 seconds.

Extensions

48) A plane flies 1 hour at 150 mph at 22° east of north, then continues to fly for 1.5 hours at 120 mph, this time at a bearing of 112° east of north. Find the total distance from the starting point and the direct angle flown north of east.

49) A plane flies 2 hours at 200 mph at a bearing of 60°,then continues to fly for 1.5 hours at the same speed, this time at a bearing of 150°. Find the distance from the starting point and the bearing from the starting point. (**Hint:** bearing is measured counterclockwise from north.)

Answer

500 miles, at 90°

For the exercises 50-52, find a function of the form $y = ab^x \sin\left(\frac{\pi}{2}x\right) + c$ that fits the given data.

50)

x	0	1	2	3
y	6	29	96	379

51)

x	0	1	2	3
y	6	34	150	746

Answer

$y = 6(5)^x + 4\sin\left(\frac{\pi}{2}x\right)$

52)

x	0	1	2	3
y	4	0	16	-40

For the exercises 53-54, find a function of the form $y = ab^x \cos\left(\frac{\pi}{2}x\right) + c$ that fits the given data.

53)

x	0	1	2	3
y	11	3	1	3

Answer

$y = 8\left(\frac{1}{2}\right)^x \cos\left(\frac{\pi}{2}x\right) + 3$

54)

x	0	1	2	3
y	4	1	−11	1

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4.R: Trigonometric Identities and Equations (Review)

7.1: Solving Trigonometric Equations with Identities

For the exercises 1-6, find all solutions exactly that exist on the interval $[0, 2\pi)$.

1) $\csc^2 t = 3$

Answer

$\sin^{-1}\left(\frac{\sqrt{3}}{3}\right), \pi - \sin^{-1}\left(\frac{\sqrt{3}}{3}\right), \pi + \sin^{-1}\left(\frac{\sqrt{3}}{3}\right), 2\pi - \sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

2) $\cos^2 x = \frac{1}{4}$

3) $2\sin\theta = -1$

Answer

$\frac{7\pi}{6}, \frac{11\pi}{6}$

4) $\tan x \sin x + \sin(-x) = 0$

5) $9\sin\omega - 2 = 4\sin^2\omega$

Answer

$\sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\left(\frac{1}{4}\right)$

6) $1 - 2\tan(\omega) = \tan^2(\omega)$

For the exercises 7-8, use basic identities to simplify the expression.

7) $\sec x \cos x + \cos x - \frac{1}{\sec x}$

Answer

1

8) $\sin^3 x + \cos^2 x \sin x$

For the exercises 9-10, determine if the given identities are equivalent.

9) $\sin^2 x + \sec^2 x - 1 = \frac{(1 - \cos^2 x)(1 + \cos^2 x)}{\cos^2 x}$

Answer

Yes

10) $\tan^3 x \csc^2 x \cot^2 x \cos x \sin x = 1$

7.2: Sum and Difference Identities

For the exercises 1-4, find the exact value.

1) $\tan\left(\frac{7\pi}{12}\right)$

Answer

$-2 - \sqrt{3}$

2) $\cos\left(\frac{25\pi}{12}\right)$

3) $\sin(70^\circ)\cos(25^\circ)-\cos(70^\circ)\sin(25^\circ)$

Answer

$\frac{\sqrt{2}}{2}$

4) $\cos(83^\circ)\cos(23^\circ)+\sin(83^\circ)\sin(23^\circ)$

For the exercises 5-6, prove the identity.

5) $\cos(4x)-\cos(3x)\cos x=\sin^2 x-4\cos^2 x\sin^2 x$

Answer

$$\begin{aligned}\cos(4x)-\cos(3x)\cos x &= \cos(2x+2x)-\cos(x+2x)\cos x\\ &= \cos(2x)\cos(2x)-\sin(2x)\sin(2x)-\cos x\cos(2x)\cos x+\sin x\sin(2x)\cos x\\ &= (\cos^2 x-\sin^2 x)^2-4\cos^2 x\sin^2 x-\cos^2 x(\cos^2 x-\sin^2 x)+\sin x(2)\sin x\cos x\cos x\\ &= (\cos^2 x-\sin^2 x)^2-4\cos^2 x\sin^2 x-\cos^2 x(\cos^2 x-\sin^2 x)+2\sin^2 x\cos^2 x\\ &= \cos^4 x-2\cos^2 x\sin^2 x+\sin^4-\cos^2 x\sin^2 x-\cos^4 x+\cos^2 x\sin^2 x+2\sin^2 x\cos^2 x\\ &= \sin^4 x-4\cos^2 x\sin^2 x+\cos^2 x\sin^2 x\\ &= \sin^2 x(\sin^2 x+\cos^2 x)-4\cos^2 x\sin^2 x\\ &= \sin^2 x-4\cos^2 x\sin^2 x\end{aligned}$$

6) $\cos(3x)-\cos^3 x=-\cos x\sin^2 x-\sin x\sin(2x)$

For exercise 7, simplify the expression.

7) $\frac{\tan\left(\frac{1}{2}x\right)+\tan\left(\frac{1}{8}x\right)}{1-\tan\left(\frac{1}{8}x\right)\tan\left(\frac{1}{2}x\right)}$

Answer

$\tan\left(\frac{5}{8}x\right)$

For the exercises 8-9, find the exact value.

8) $\cos\left(\sin^{-1}(0)-\cos^{-1}\left(\frac{1}{2}\right)\right)$

9) $\tan\left(\sin^{-1}(0)-\sin^{-1}\left(\frac{1}{2}\right)\right)$

Answer

$\frac{\sqrt{3}}{3}$

7.3: Double-Angle, Half-Angle, and Reduction Formulas

For the exercises 1-4, find the exact value.

1) Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cos\theta=-\frac{1}{3}$ and θ is in the interval $\left[\frac{\pi}{2},\pi\right]$.

2) Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\sec\theta=-\frac{5}{3}$ and θ is in the interval $\left[\frac{\pi}{2},\pi\right]$.

Answer

$$-\frac{24}{25}, -\frac{7}{25}, \frac{24}{7}$$

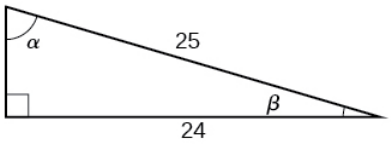
3) $\sin\left(\frac{7\pi}{8}\right)$

4) $\sec\left(\frac{3\pi}{8}\right)$

Answer

$$\sqrt{2(2+\sqrt{2})}$$

For the exercises 5-6, use Figure below to find the desired quantities.



5) $\sin(2\beta), \cos(2\beta), \tan(2\beta), \sin(2\alpha), \cos(2\alpha), \tan(2\alpha)$

6) $\sin\left(\frac{\beta}{2}\right), \cos\left(\frac{\beta}{2}\right), \tan\left(\frac{\beta}{2}\right), \sin\left(\frac{\alpha}{2}\right), \cos\left(\frac{\alpha}{2}\right), \tan\left(\frac{\alpha}{2}\right)$

Answer

$$\frac{\sqrt{2}}{10}, \frac{7\sqrt{2}}{10}, \frac{1}{7}, \frac{3}{5}, \frac{4}{5}, \frac{3}{4}$$

For the exercises 7-8, prove the identity.

7) $\frac{2\cos(2x)}{\sin(2x)} = \cot x - \tan x$

8) $\cot x \cos(2x) = -\sin(2x) + \cot x$

Answer

$$\begin{aligned}\cot x \cos(2x) &= \cot x(1 - 2\sin^2 x) \\ &= \cot x - \frac{\cos x}{\sin x}(2)\sin^2 x \\ &= -2\sin x \cos \\ &= -\sin(2x) + \cot x\end{aligned}$$

For the exercises 9-10, rewrite the expression with no powers.

9) $\cos^2 x \sin^4(2x)$

10) $\tan^2 x \sin^3 x$

Answer

$$\frac{10\sin x - 5\sin(3x) + \sin(5x)}{8(\cos(2x)+1)}$$

7.4: Sum-to-Product and Product-to-Sum Formulas

For the exercises 1-3, evaluate the product for the given expression using a sum or difference of two functions. Write the exact answer.

1) $\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

2) $2\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{5\pi}{6}\right)$

Answer

$\frac{\sqrt{3}}{2}$

3) $2\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{3}\right)$

For the exercises 4-5, evaluate the sum by using a product formula. Write the exact answer.

4) $\sin\left(\frac{\pi}{12}\right)-\sin\left(\frac{7\pi}{12}\right)$

Answer

$-\frac{\sqrt{2}}{2}$

5) $\cos\left(\frac{5\pi}{12}\right)+\cos\left(\frac{7\pi}{12}\right)$

For the exercises 6-9, change the functions from a product to a sum or a sum to a product.

6) $\sin(9x)\cos(3x)$

Answer

$\frac{1}{2}(\sin(6x)+\sin(12x))$

7) $\cos(7x)\cos(12x)$

8) $\sin(11x)+\sin(2x)$

Answer

$2\sin\left(\frac{13}{2}x\right)\cos\left(\frac{9}{2}x\right)$

9) $\cos(6x)+\cos(5x)$

7.5: Solving Trigonometric Equations

For the exercises 1-2, find all exact solutions on the interval $[0,2\pi)$.

1) $\tan x+1=0$

Answer

$\frac{3\pi}{4},\frac{7\pi}{4}$

2) $2\sin(2x)+\sqrt{2}=0$

For the exercises 3-7, find all exact solutions on the interval $[0,2\pi)$.

3) $2\sin^2 x-\sin x=0$

Answer

$0,\frac{\pi}{6},\frac{5\pi}{6},\pi$

4) $\cos^2 x-\cos x-1=0$

5) $2\sin^2 x + 5\sin x + 3 = 0$

Answer

$$\frac{3\pi}{2}$$

6) $\cos x - 5\sin(2x) = 0$

7) $\dfrac{1}{\sec^2 x} + 2 + \sin^2 x + 4\cos^2 x = 0$

Answer

No solution.

For the exercises 8-9, simplify the equation algebraically as much as possible. Then use a calculator to find the solutions on the interval $[0, 2\pi)$. Round to four decimal places.

8) $\sqrt{3}\cot^2 x + \cot x = 1$

9) $\csc^2 x - 3\csc x - 4 = 0$

Answer

$0.2527, 2.8889, 4.7124$

For the exercises 10-11, graph each side of the equation to find the zeroes on the interval $[0, 2\pi)$.

10) $20\cos^2 x + 21\cos x + 1 = 0$

11) $\sec^2 x - 2\sec x = 15$

Answer

$1.3694, 1.9106, 4.3726, 4.9137$

7.6: Modeling with Trigonometric Equations

For the exercises 1-3, graph the points and find a possible formula for the trigonometric values in the given table.

1)

x	0	1	2	3	4	5
y	1	6	11	6	1	6

2)

x	y
0	-2
1	1
2	-2
3	-5
4	-2
5	1

Answer

$$3\sin\left(\frac{x\pi}{2}\right) - 2$$

3)

x	y
-3	$3+2\sqrt{2}$
-2	3
-1	$2\sqrt{2}-1$
0	1
1	$3-2\sqrt{2}$
2	-1
3	$-1-2\sqrt{2}$

4) A man with his eye level 6 feet above the ground is standing 3 feet away from the base of a 15-foot vertical ladder. If he looks to the top of the ladder, at what angle above horizontal is he looking?

Answer

71.6°

5) Using the ladder from the previous exercise, if a 6-foot-tall construction worker standing at the top of the ladder looks down at the feet of the man standing at the bottom, what angle from the horizontal is he looking?

For the exercises 6-7, construct functions that model the described behavior.

6) A population of lemmings varies with a yearly low of 500 in March. If the average yearly population of lemmings is 950, write a function that models the population with respect to t, the month.

Answer

$P(t)=950-450\sin\left(\frac{\pi}{6}t\right)$

7) Daily temperatures in the desert can be very extreme. If the temperature varies from 90°F to 30°F and the average daily temperature first occurs at 10 AM, write a function modeling this behavior.

For the exercises 8-9, find the amplitude, frequency, and period of the given equations.

8) $y=3\cos(x\pi)$

Answer

Amplitude: 3, period: 2, frequency: $\frac{1}{2}$Hz

9) $y=-2\sin(16x\pi)$

For the exercises 10-11, model the described behavior and find requested values.

10) An invasive species of carp is introduced to Lake Freshwater. Initially there are 100 carp in the lake and the population varies by 20 fish seasonally. If by year 5, there are 625 carp, find a function modeling the population of carp with respect to t, the number of years from now.

Answer

$C(t)=20\sin(2\pi t)+100(1.4427)^t$

11) The native fish population of Lake Freshwater averages 2500 fish, varying by 100 fish seasonally. Due to competition for resources from the invasive carp, the native fish population is expected to decrease by 5% each year. Find a function modeling the population of native fish with respect to t, the number of years from now. Also determine how many years it will take for the carp to overtake the native fish population.

Practice Test

For the exercises 1-2, simplify the given expression.

1) $\cos(-x)\sin x\cot x+\sin^2 x$

Answer

1

2) $\sin(-x)\cos(-2x)-\sin(-x)\cos(-2x)$

For the exercises 3-6, find the exact value.

3) $\cos\left(\frac{7\pi}{12}\right)$

Answer

$\frac{\sqrt{2}-\sqrt{6}}{4}$

4) $\tan\left(\frac{3\pi}{8}\right)$

5) $\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\tan^{-1}\sqrt{3}\right)$

Answer

$-\sqrt{2}-\sqrt{3}$

6) $2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$

For the exercises 7-16, find all exact solutions to the equation on $[0,2\pi)$.

7) $\cos^2 x-\sin^2 x-1=0$

Answer

$0,\pi$

8) $\cos^2 x=\cos x$

Answer

$\sin^{-1}\left(\frac{1}{4}(\sqrt{13}-1)\right),\pi-\sin^{-1}\left(\frac{1}{4}(\sqrt{13}-1)\right)$

9) $\cos(2x)+\sin^2 x=0$

10) $2\sin^2 x-\sin x=0$

Answer

$0,\frac{\pi}{6},\frac{5\pi}{6},\pi$

11) Rewrite the expression as a product instead of a sum: $\cos(2x)+\cos(-8x)$

12) Find all solutions of $\tan(x)-\sqrt{3}=0$.

Answer

$\frac{\pi}{3}+k\pi$

13) Find the solutions of $\sec^2 x - 2\sec x = 15$ on the interval $[0, 2\pi)$ algebraically; then graph both sides of the equation to determine the answer.

14) Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cot\theta = -\frac{3}{4}$ and θ is on the interval $\left[\frac{\pi}{2}, \pi\right]$.

Answer

$$-\frac{24}{25}, -\frac{7}{25}, \frac{24}{7}$$

15) Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$ given $\cos\theta = -\frac{7}{25}$ and θ is in quadrant IV.

16) Rewrite the expression $\sin^4 x$ with no powers greater than 1.

Answer

$$\frac{1}{8}(3 + \cos(4x) - 4\cos(2x))$$

For the exercises 17-19, prove the identity.

17) $\tan^3 x - \tan x \sec^2 x = \tan(-x)$

18) $\sin(3x) - \cos x \sin(2x) = \cos^2 x \sin x - \sin^3 x$

Answer

$$\begin{aligned} \sin(3x) - \cos x \sin(2x) &= \\ \sin(x + 2x) - \cos x(2\sin x \cos x) &= \\ \sin x \cos(2x) + \sin(2x)\cos x - 2\sin x \cos^2 x &= \\ \sin x(\cos^2 x - \sin^2 x) + 2\sin x \cos x \cos x - 2\sin x \cos^2 x &= \\ \sin x \cos^2 x - \sin^3 x + 0 &= \\ \cos^2 x \sin x - \sin^3 x &= \cos^2 x \sin x - \sin^3 x \end{aligned}$$

19) $\frac{\sin(2x)}{\sin x} - \frac{\cos(2x)}{\cos x} = \sec x$

20) Plot the points and find a function of the form $y = A\cos(Bx + C) + D$ that fits the given data.

x	0	1	2	3	4	5
y	-2	2	-2	2	-2	2

Answer

$$y = 2\cos(\pi x + \pi)$$

21) The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = \frac{1}{4}\sin(120\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

22) A woman is standing 300 feet away from a 2000-foot building. If she looks to the top of the building, at what angle above horizontal is she looking? A bored worker looks down at her from the 15th floor (1500 feet above her). At what angle is he looking down at her? Round to the nearest tenth of a degree.

Answer

$$81.5^\circ, 78.7^\circ$$

23) Two frequencies of sound are played on an instrument governed by the equation $n(t) = 8\cos(20\pi t)\cos(1000\pi t)$. What are the period and frequency of the "fast" and "slow" oscillations? What is the amplitude?

24) The average monthly snowfall in a small village in the Himalayas is 6 inches, with the low of 1 inch occurring in July. Construct a function that models this behavior. During what period is there more than 10 inches of snowfall?

Answer

$6+5\cos\left(\frac{\pi}{6}(1-x)\right)$. From November 23 to February 6.

25) A spring attached to a ceiling is pulled down 20 cm. After 3 seconds, wherein it completes 6 full periods, the amplitude is only 15 cm. Find the function modeling the position of the spring t seconds after being released. At what time will the spring come to rest? In this case, use 1 cm amplitude as rest.

26) Water levels near a glacier currently average 9 feet, varying seasonally by 2 inches above and below the average and reaching their highest point in January. Due to global warming, the glacier has begun melting faster than normal. Every year, the water levels rise by a steady 3 inches. Find a function modeling the depth of the water t months from now. If the docks are 2 feet above current water levels, at what point will the water first rise above the docks?

Answer

$D(t)=2\cos\left(\frac{\pi}{6}t\right)+108+\frac{1}{4}t$, 93.5855 months (or 7.8 years) from now

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CHAPTER OVERVIEW

5: Applications of Trigonometry - Oblique Triangles and Polar Coordinates

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in Trigonometric Functions and investigate applications more deeply and meaningfully.

5.0: Prelude to Applications of Trigonometry
5.1: Non-right Triangles - Law of Sines
5.2: Non-right Triangles - Law of Cosines
5.3: Applications of Trigonometry - Area
5.4: Polar Coordinates
5.5: Polar Coordinates - Graphs
5.6: Polar Form of Complex Numbers
5.7: Parametric Equations
5.8: Parametric Equations - Graphs
5.E: Further Applications of Trigonometry (Exercises)
5.R: Further Applications of Trigonometry (Review)

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5.0: Prelude to Applications of Trigonometry

The world's largest tree by volume, named General Sherman, stands 274.9 feet tall and resides in Northern California. Just how do scientists know its true height? A common way to measure the height involves determining the angle of elevation, which is formed by the tree and the ground at a point some distance away from the base of the tree. This method is much more practical than climbing the tree and dropping a very long tape measure.



Figure 5.0.1: General Sherman, the world's largest living tree. (credit: Mike Baird, Flickr)

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in Trigonometric Functions and investigate applications more deeply and meaningfully.

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5.1: Non-right Triangles - Law of Sines

Learning Objectives

- Use the Law of Sines to solve oblique triangles.
- Find the area of an oblique triangle using the sine function.
- Solve applied problems using the Law of Sines.

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the **angle of elevation** measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in Figure 5.1.1 that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving **non-right triangles**.

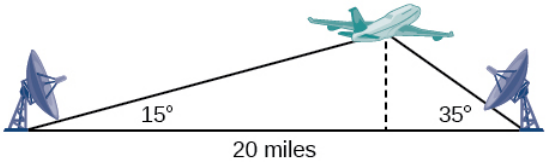


***Figure* 5.1.1**

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

ASA (angle-side-angle) We know the measurements of two angles and the included side. See Figure 5.1.2.

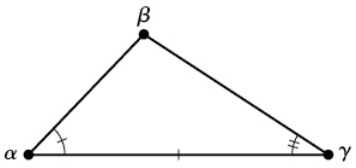


***Figure* 5.1.2**

AAS (angle-angle-side) We know the measurements of two angles and a side that is not between the known angles. See Figure 5.1.3.

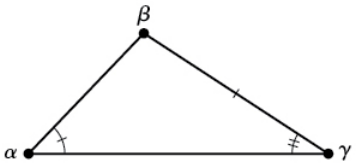


***Figure* 5.1.3**

SSA (side-side-angle) We know the measurements of two sides and an angle that is not between the known sides. See Figure 5.1.4.

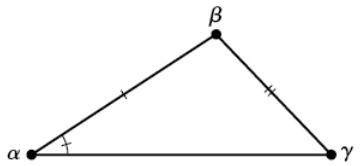


***Figure* 5.1.4**

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in Figure 5.1.5.

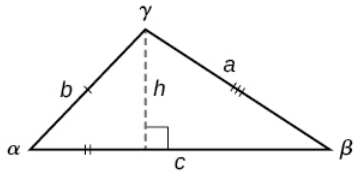


***Figure* 5.1.5**

Using the right triangle relationships, we know that $\sin\alpha = \dfrac{h}{b}$ and $\sin\beta = \dfrac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin\alpha \text{ and } h = a\sin\beta$$

We then set the expressions equal to each other.

$$\begin{aligned} b\sin\alpha &= a\sin\beta \\ \left(\frac{1}{ab}\right)(b\sin\alpha) &= (a\sin\beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab} \\ \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \end{aligned}$$

Similarly, we can compare the other ratios.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \text{ and } \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Collectively, these relationships are called the **Law of Sines**.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See Figure 5.1.6.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

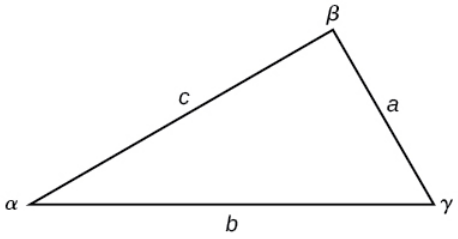


***Figure* 5.1.6**

Note: LAW OF SINES

Given a triangle with angles and opposite sides labeled as in Figure 5.1.6, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c} \tag{5.1.1}$$

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma} \tag{5.1.2}$$

To solve an oblique triangle, use any pair of applicable ratios.

Example 5.1.1: Solving for Two Unknown Sides and Angle of an AAS Triangle

Solve the triangle shown in Figure 5.1.7 to the nearest tenth.

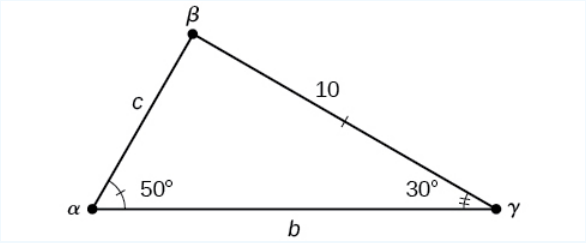


***Figure* 5.1.7**

Solution

The three angles must add up to 180 degrees. From this, we can determine that

$$\begin{aligned}\beta &= 180^\circ-50^\circ-30^\circ\\ &= 100^\circ\end{aligned}$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha=50^\circ$ and its corresponding side $a=10$. We can use the following proportion from the Law of Sines to find the length of c.

$$\begin{aligned}\frac{\sin(50^\circ)}{10}&=\frac{\sin(30^\circ)}{c}\\ c\frac{\sin(50^\circ)}{10}&=\sin(30^\circ) && \text{Multiply both sides by } c\\ c&=\sin(30^\circ)\frac{10}{\sin(50^\circ)} && \text{Multiply by the reciprocal to isolate } c\\ c&\approx 6.5\end{aligned}$$

Similarly, to solve for b, we set up another proportion.

$$\begin{aligned}\frac{\sin(50^\circ)}{10}&=\frac{\sin(100^\circ)}{b}\\ b\sin(50^\circ)&=10\sin(100^\circ) && \text{Multiply both sides by } b\\ b&=\frac{10\sin(100^\circ)}{\sin(50^\circ)} && \text{Multiply by the reciprocal to isolate } b\\ b&\approx 12.9\end{aligned}$$

Therefore, the complete set of angles and sides is

$$\begin{aligned}\alpha&=50^\circ & a&=10\\ \beta&=100^\circ & b&\approx 12.9\\ \gamma&=30^\circ & c&\approx 6.5\end{aligned}$$

Exercise 5.1.1

Solve the triangle shown in Figure 5.1.8 to the nearest tenth.

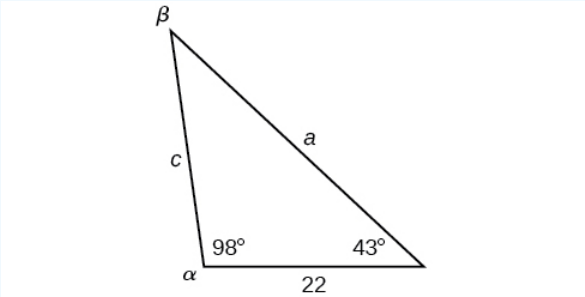


***Figure* 5.1.8**

Answer

$\alpha = 98^\circ \quad a = 34.6$
$\beta = 39^\circ \quad b = 22$
$\gamma = 43^\circ \quad c = 23.8$

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an ambiguous case. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

Note: POSSIBLE OUTCOMES FOR SSA TRIANGLES

Oblique triangles in the category SSA may have four different outcomes. Figure 5.1.9 illustrates the solutions with the known sides a and b and known angle α.

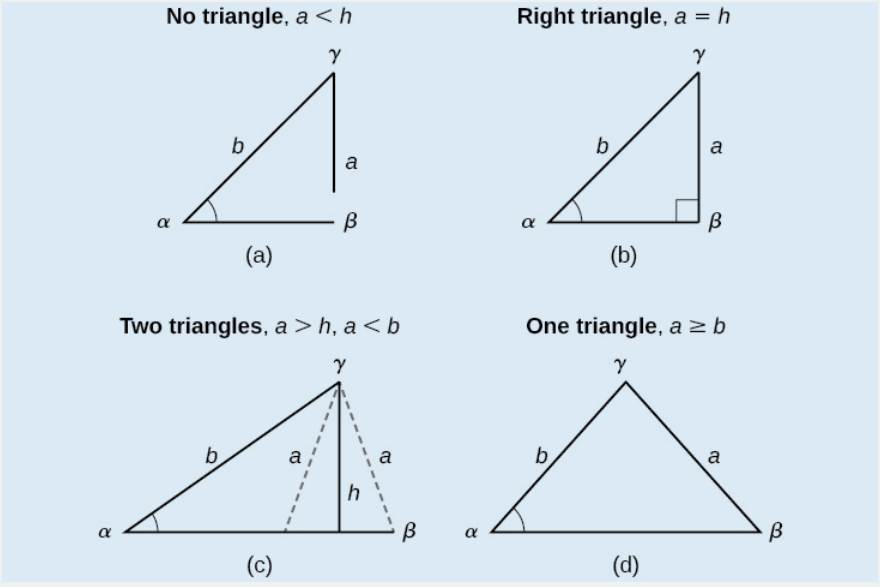


***Figure* 5.1.9**

https://math.libretexts.org/@go/page/69159

Example 5.1.2: Solving an Oblique SSA Triangle

Solve the triangle in Figure 5.1.10 for the missing side and find the missing angle measures to the nearest tenth.

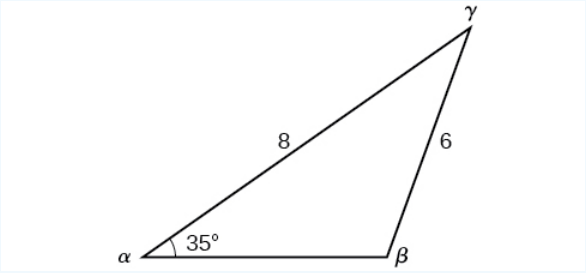


***Figure* 5.1.10**

Solution

Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\begin{aligned}\frac{\sin\alpha}{a} &= \frac{\sin\beta}{b}\\ \frac{\sin(35^\circ)}{6} &= \frac{\sin\beta}{8}\\ \frac{8\sin(35^\circ)}{6} &= \sin\beta\\ 0.7648 &\approx \sin\beta\\ \sin^{-1}(0.7648) &\approx 49.9^\circ\\ \beta &\approx 49.9^\circ\end{aligned}$$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than 90°. How did we get an acute angle, and how do we find the measurement of β? Let's investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have Figure 5.1.11. It appears that there may be a second triangle that will fit the given criteria.

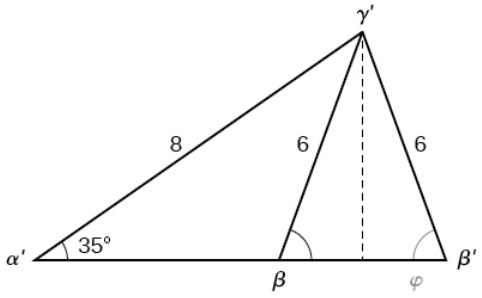


***Figure* 5.1.11**

The angle supplementary to β is approximately equal to 49.9°, which means that $\beta = 180^\circ - 49.9^\circ = 130.1^\circ$. (Remember that the sine function is positive in both the first and second quadrants.) Solving for γ, we have

$$\begin{aligned}\gamma &= 180^\circ - 35^\circ - 130.1^\circ\\ &\approx 14.9^\circ\end{aligned}$$

We can then use these measurements to solve the other triangle. Since β' is supplementary to β, we have

$$\begin{aligned}\gamma' &= 180^\circ - 35^\circ - 49.5^\circ\\ &\approx 95.1^\circ\end{aligned}$$

Now we need to find c and c'.

We have

$$\frac{c}{\sin(14.9^\circ)} = \frac{6}{\sin(35^\circ)}$$
$$c = \frac{6\sin(14.9^\circ)}{\sin(35^\circ)}$$
$$\approx 2.7$$

Finally,

$$\frac{c'}{\sin(95.1^\circ)} = \frac{6}{\sin(35^\circ)}$$
$$c' = \frac{6\sin(95.1^\circ)}{\sin(35^\circ)}$$
$$\approx 10.4$$

To summarize, there are two triangles with an angle of 35°, an adjacent side of 8, and an opposite side of 6, as shown in Figure 5.1.12

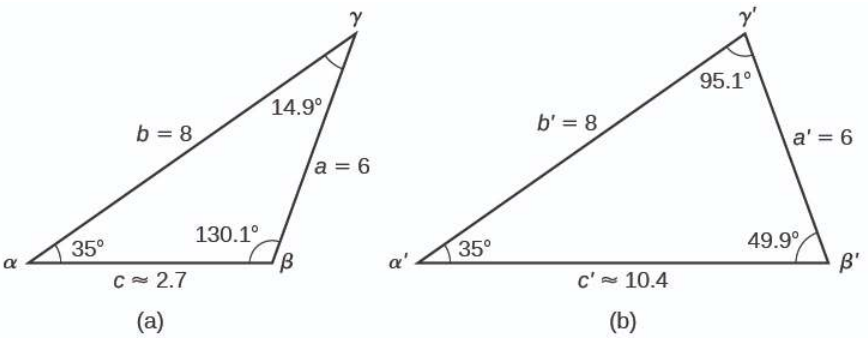


***Figure* 5.1.12**

However, we were looking for the values for the triangle with an obtuse angle β. We can see them in the first triangle (a) in Figure 5.1.12

? Exercise 5.1.2

Given $\alpha = 80^\circ$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.

Answer

Solution 1

$\alpha = 80^\circ$ $\quad a = 120$
$\beta \approx 83.2^\circ$ $\quad b = 121$
$\gamma \approx 16.8^\circ$ $\quad c \approx 35.2$

Solution 2

$\alpha' = 80^\circ$ $\quad a' = 120$
$\beta' \approx 96.8^\circ$ $\quad b' = 121$
$\gamma' \approx 3.2^\circ$ $\quad c' \approx 6.8$

✓ Example 5.1.3: Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in Figure 5.1.13, solve for the unknown side and angles. Round your answers to the nearest tenth.

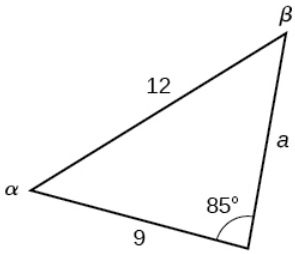


***Figure* 5.1.13**

Solution

In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle, $\gamma = 85^\circ$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\frac{\sin(85^\circ)}{12} = \frac{\sin\beta}{9} \qquad \text{Isolate the unknown.}$$
$$\frac{9\sin(85^\circ)}{12} = \sin\beta$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85^\circ)}{12}\right)$$
$$\beta \approx \sin^{-1}(0.7471)$$
$$\beta \approx 48.3^\circ$$

In this case, if we subtract β from 180°, we find that there may be a second possible solution. Thus, $\beta = 180^\circ - 48.3^\circ \approx 131.7^\circ$. To check the solution, subtract both angles, 131.7° and 85°, from 180°. This gives

$$\begin{aligned}\alpha &= 180^\circ - 85^\circ - 131.7^\circ \\ &\approx -36.7^\circ\end{aligned}$$

which is impossible, and so $\beta \approx 48.3^\circ$.

To find the remaining missing values, we calculate $\alpha = 180^\circ - 85^\circ - 48.3^\circ \approx 46.7^\circ$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85^\circ)}{12} = \frac{\sin(46.7^\circ)}{a}$$
$$a\frac{\sin(85^\circ)}{12} = \sin(46.7^\circ)$$
$$a = \frac{12\sin(46.7^\circ)}{\sin(85^\circ)}$$
$$\approx 8.8$$

The complete set of solutions for the given triangle is

$$\begin{aligned}\alpha &\approx 46.7^\circ & a &\approx 8.8 \\ \beta &\approx 48.3^\circ & b &= 9 \\ \gamma &= 85^\circ & c &= 12\end{aligned}$$

? Exercise 5.1.3

Given $\alpha = 80^\circ$, $a = 100$, $b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Answer

$\beta \approx 5.7^\circ, \gamma \approx 94.3^\circ, c \approx 101.3$

Example 5.1.4: Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Solution

Using the given information, we can solve for the angle opposite the side of length 10. See Figure 5.1.14

$$\frac{\sin\alpha}{10} = \frac{\sin(50^\circ)}{4}$$
$$\sin\alpha = \frac{10\sin(50^\circ)}{4}$$
$$\sin\alpha \approx 1.915$$

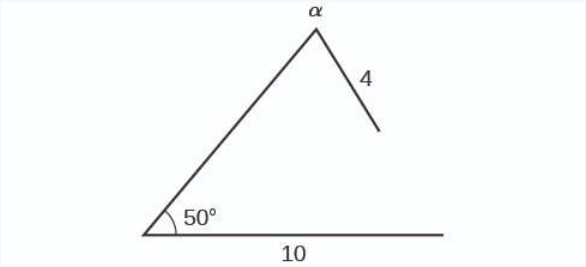


***Figure* 5.1.14**

We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$in a graphing calculator generates an ERROR DOMAIN. Therefore, no triangles can be drawn with the provided dimensions.

Exercise 5.1.4

Determine the number of triangles possible given $a = 31, b = 26, \beta = 48^\circ$.

Answer

two

Finding the Area of an Oblique Triangle Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as $Area = \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in Figure 5.1.15, one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin\alpha = \frac{opposite}{hypotenuse}$ to write an equation for area in oblique triangles. In the acute triangle, we have $\sin\alpha = \frac{h}{c}$ or $c\sin\alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is α', or $180 - \alpha$.

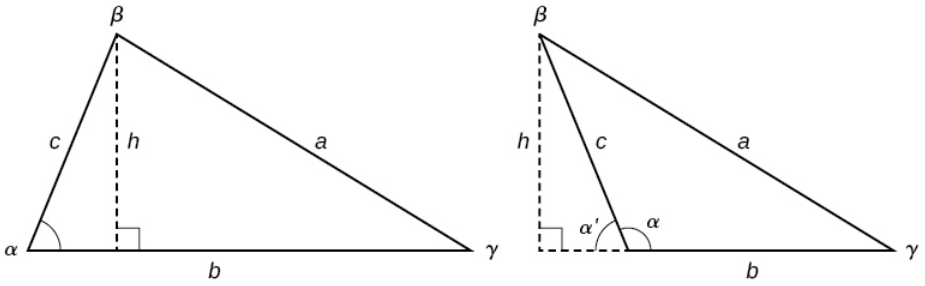


***Figure* 5.1.15**

Thus,

$$Area = \frac{1}{2}(base)(height) = \frac{1}{2}b(c\sin\alpha)$$

Similarly,

$$Area = \frac{1}{2}a(b\sin\gamma) = \frac{1}{2}a(c\sin\beta)$$

Note: AREA OF AN OBLIQUE TRIANGLE

The formula for the area of an oblique triangle is given by

$$Area = \frac{1}{2}bc\sin\alpha \tag{5.1.3}$$

$$Area = \frac{1}{2}ac\sin\beta \tag{5.1.4}$$

$$Area = \frac{1}{2}ab\sin\gamma \tag{5.1.5}$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Example 5.1.5: Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90$, $b = 52$, and angle $\gamma = 102^\circ$. Round the area to the nearest integer.

Solution

Using the formula, we have

$$Area = \frac{1}{2}ab\sin\gamma$$
$$Area = \frac{1}{2}(90)(52)\sin(102^\circ)$$
$$Area \approx 2289 \text{ square units}$$

Exercise 5.1.5

Find the area of the triangle given $\beta = 42^\circ, a = 7.2\ ft, c = 3.4\ ft$. Round the area to the nearest tenth.

Answer

about 8.2 square feet

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Example 5.1.6: Finding an Altitude

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in Figure 5.1.16. Round the altitude to the nearest tenth of a mile.

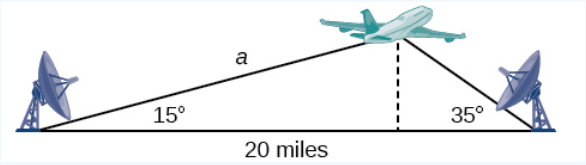


***Figure* 5.1.16**

Solution

To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side a, and then use right triangle relationships to find the height of the aircraft, h.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180^\circ - 15^\circ - 35^\circ = 130^\circ$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\begin{aligned} \frac{\sin(130^\circ)}{20} &= \frac{\sin(35^\circ)}{a} \\ a\sin(130^\circ) &= 20\sin(35^\circ) \\ a &= \frac{20\sin(35^\circ)}{\sin(130^\circ)} \\ a &\approx 14.98 \end{aligned}$$

The distance from one station to the aircraft is about 14.98 miles.

Now that we know a, we can use right triangle relationships to solve for h.

$$\begin{aligned} \sin(15^\circ) &= \frac{opposite}{hypotenuse} \\ \sin(15^\circ) &= \frac{h}{a} \\ \sin(15^\circ) &= \frac{h}{14.98} \\ h &= 14.98\sin(15^\circ) \\ h &\approx 3.88 \end{aligned}$$

The aircraft is at an altitude of approximately 3.9 miles.

Exercise 5.1.6

The diagram shown in Figure 5.1.17 represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point A, is 70°, the angle of elevation from the northern end zone, point B, is 62°, and the distance between the viewing points of the two end zones is 145 yards.

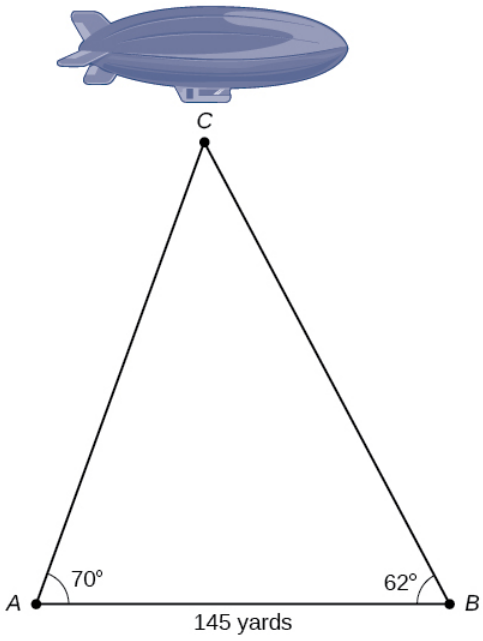


***Figure* 5.1.17**

Answer

161.9yd.

Media

Access these online resources for additional instruction and practice with trigonometric applications.

- Law of Sines: The Basics
- Law of Sines: The Ambiguous Case

Key Equations

Law of Sines	$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$ $\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$
Area for oblique triangles	$Area=\frac{1}{2}bc\sin\alpha$ $=\frac{1}{2}ac\sin\beta$ $=\frac{1}{2}ab\sin\gamma$

Key Concepts

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.
- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See Example 5.1.1.
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See Example 5.1.2 and Example 5.1.3.
- The Law of Sines can be used to solve triangles with given criteria. See Example 5.1.4.

- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value. See Example 5.1.5.
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See Example 5.1.6.

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5.2: Non-right Triangles - Law of Cosines

Learning Objectives

In this section, you will:

- Use the Law of Cosines to solve oblique triangles.
- Solve applied problems using the Law of Cosines.
- Use Heron's formula to find the area of a triangle.

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in Figure 5.2.1 How far from port is the boat?

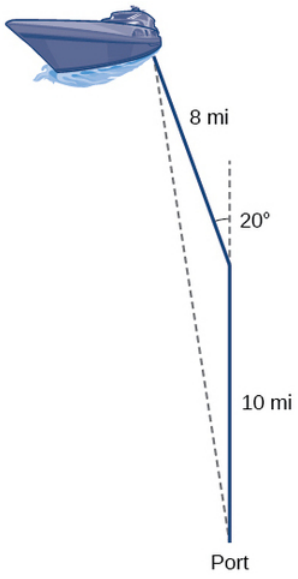


Figure 5.2.1

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the *Law of Cosines*, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the x-axis, and vertex C located at some point (x, y) in the plane, as illustrated in Figure 5.2.2. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

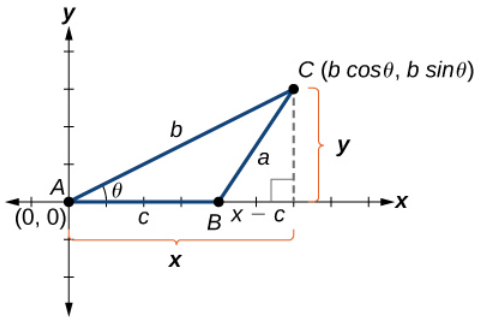


Figure 5.2.2

We can drop a perpendicular from C to the x-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x(adjacent)}{b(hypotenuse)} \text{ and } \sin\theta = \frac{y(opposite)}{b(hypotenuse)}$$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta, b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned} a^2 &= (x-c)^2 + y^2 \\ &= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y \\ &= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta && \text{Expand the perfect square.} \\ &= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1 \\ &= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{Factor out } b^2 \\ a^2 &= b^2 + c^2 - 2bc\cos\theta \end{aligned}$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

The LAW OF COSINES

The Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle.

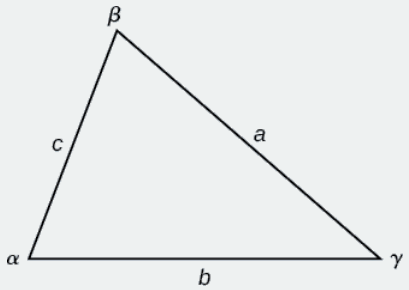


Figure 5.2.3

For triangles labeled as in Figure 5.2.3, with angles α, β and γ, and opposite corresponding sides a, b, and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \tag{5.2.1}$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \tag{5.2.2}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{5.2.3}$$

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \tag{5.2.4}$$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \tag{5.2.5}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \tag{5.2.6}$$

How to: Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Example 5.2.1: Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in Figure 5.2.4.

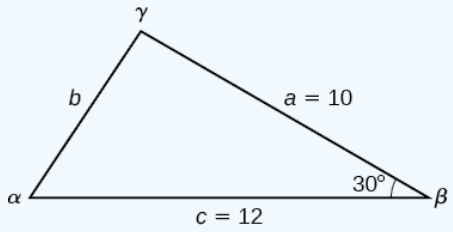


Figure 5.2.4

Solution

First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$b^2 = 10^2 + 12^2 - 2(10)(12)\cos(30^\circ) \quad \text{Substitute the measurements for the known quantities.}$$

$$b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right) \quad \text{Evaluate the cosine and begin to simplify.}$$

$$b^2 = 244 - 120\sqrt{3}$$

$$b = \sqrt{244 - 120\sqrt{3}} \quad \text{Use the square root property.}$$

$$b \approx 6.013$$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle α, we have

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{10}=\frac{\sin(30^\circ)}{6.013}$$

$$\sin\alpha=\frac{10\sin(30^\circ)}{6.013}\qquad\text{Multiply both sides of the equation by 10.}$$

$$\alpha=\sin^{-1}\left(\frac{10\sin(30^\circ)}{6.013}\right)\qquad\text{Find the inverse sine of }\frac{10\sin(30^\circ)}{6.013}.$$

$$\alpha\approx 56.3^\circ$$

The other possibility for α would be $\alpha=180^\circ-56.3^\circ\approx 123.7^\circ$. In the original diagram,α is adjacent to the longest side, so α is an acute angle and, therefore, 123.7° does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between 0° and 180°. Proceeding with $\alpha\approx 56.3^\circ$, we can then find the third angle of the triangle.

$$\begin{aligned}\gamma &=180^\circ-30^\circ-56.3^\circ\\ &\approx 93.7^\circ\end{aligned}$$

The complete set of angles and sides is

$$\alpha\approx 56.3^\circ\ a=10$$

$$\beta=30^\circ\ b\approx 6.013$$

$$\gamma\approx 93.7^\circ\ c=12$$

? Exercise 5.2.1

Find the missing side and angles of the given triangle: $\alpha=30^\circ$, $b=12$, $c=24$.

Answer

$a\approx 14.9$, $\beta\approx 23.8^\circ$, $\gamma\approx 126.2^\circ$.

✓ Example 5.2.2: Solving for an Angle of a SSS Triangle

Find the angle α for the given triangle if side $a=20$, side $b=25$, and side $c=18$.

Solution

For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$a^2=b^2+c^2-2bc\cos\alpha$$

$$20^2=25^2+18^2-2(25)(18)\cos\alpha\qquad\text{Substitute the appropriate measurements.}$$

$$400=625+324-900\cos\alpha\qquad\text{Simplify in each step.}$$

$$400=949-900\cos\alpha$$

$$-549=-900\cos\alpha\qquad\text{Isolate }\cos\alpha.$$

$$0.61\approx\cos\alpha\qquad\text{Find the inverse cosine.}$$

$$\alpha\approx 52.4^\circ$$

See Figure 5.2.5.

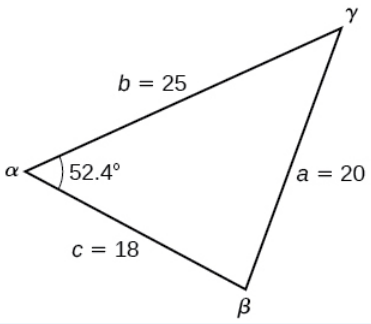


Figure 5.2.5

Analysis

Because the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases using this method.

Exercise 5.2.2

Given $a = 5$, $b = 7$, and $c = 10$, find the missing angles.

Answer

$\alpha \approx 27.7^\circ, \beta \approx 40.5^\circ, \gamma \approx 111.8^\circ$

Solving Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

Example 5.2.3A: Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Solution

For simplicity, we start by drawing a diagram similar to Figure $5.2.6$ and labeling our given information.

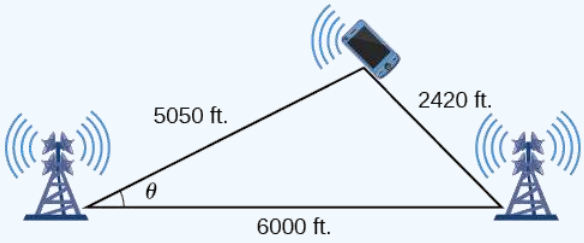


Figure 5.2.6

Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a = 2420$, $b = 5050$, and $c = 6000$. Thus, θ corresponds to the opposite side $a = 2420$.

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos\theta \\
(2420)^2 &= (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta \\
\cos\theta &\approx 0.9183 \\
\theta &\approx \cos^{-1}(0.9183) \\
\theta &\approx 23.3^\circ
\end{aligned}
$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in Figure 5.2.7. This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.

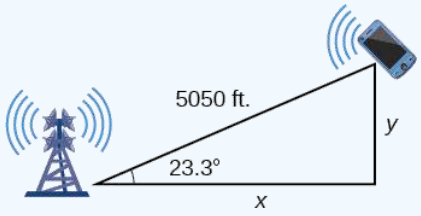


Figure 5.2.7

Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions. Thus

$$
\begin{aligned}
\cos(23.3^\circ) &= \frac{x}{5050} \\
x &= 5050\cos(23.3^\circ) \\
x &\approx 4638.15\ feet \\
\sin(23.3^\circ) &= \frac{y}{5050} \\
y &= 5050\sin(23.3^\circ) \\
y &\approx 1997.5\ feet
\end{aligned}
$$

The cell phone is approximately 4638 feet east and 1998 feet north of the first tower, and 1998 feet from the highway.

Example 5.2.3B: Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in Figure 5.2.8.

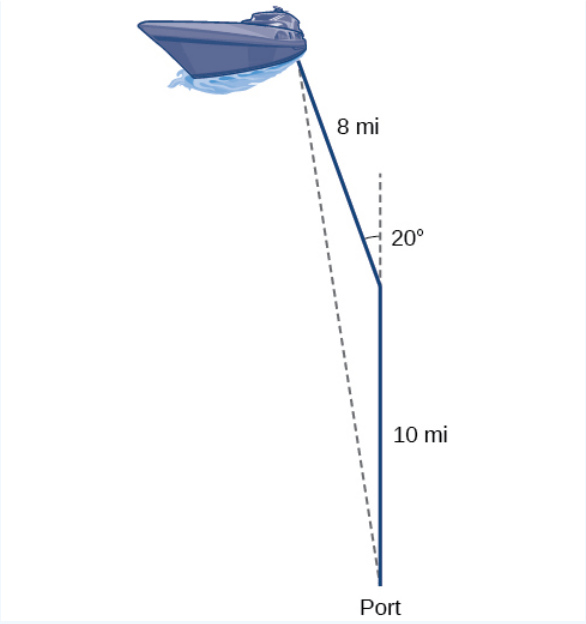


Figure 5.2.8

Solution

The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180^\circ - 20^\circ = 160^\circ$. With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle—the distance of the boat to the port.

$$\begin{aligned} x^2 &= 8^2 + 10^2 - 2(8)(10)\cos(160^\circ) \\ x^2 &= 314.35 \\ x &= \sqrt{314.35} \\ x &\approx 17.7\, miles \end{aligned}$$

The boat is about 17.7 miles from port.

Using Heron's Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use **Heron's formula** instead of finding the height. **Heron of Alexandria** was a geometer who lived during the first century A.D. He discovered a formula for finding the area of oblique triangles when three sides are known.

HERON'S FORMULA

Heron's formula finds the area of oblique triangles in which sides a, b,and c are known.

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \tag{5.2.7}$$

where $s = \dfrac{(a+b+c)}{2}$ is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

Example 5.2.4: Using Heron's Formula to Find the Area of a Given Triangle

Find the area of the triangle in Figure $5.2.9$ using Heron's formula.

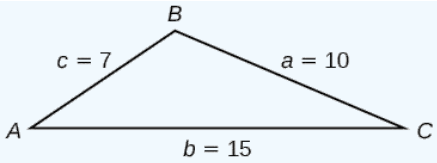


Figure 5.2.9

Solution

First, we calculate s.

$$s = \frac{(a+b+c)}{2}$$
$$s = \frac{(10+15+7)}{2}$$
$$= 16$$

Then we apply the formula.

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$
$$Area = \sqrt{16(16-10)(16-15)(16-7)}$$
$$Area \approx 29.4$$

The area is approximately 29.4 square units.

Exercise 5.2.3

Use Heron's formula to find the area of a triangle with sides of lengths $a = 29.7$ ft, $b = 42.3$ ft, and $c = 38.4$ ft.

Answer

Area = 552 square feet

Example 5.2.5: Applying Heron's Formula to a Real-World Problem

A Chicago city developer wants to construct a building consisting of artist's lofts on a triangular lot bordered by Rush Street, Wabash Avenue, and Pearson Street. The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters. How many square meters are available to the developer? See Figure $5.2.10$ for a view of the city property.



Figure 5.2.10

Solution

Find the measurement for s, which is one-half of the perimeter.

$$s = \frac{(62.4 + 43.5 + 34.1)}{2}$$
$$s = 70\ m$$

Apply Heron's formula.

$$Area = \sqrt{70(70 - 62.4)(70 - 43.5)(70 - 34.1)}$$
$$Area = \sqrt{506,118.2}$$
$$Area \approx 711.4$$

The developer has about 711.4 square meters.

Exercise 5.2.4

Find the area of a triangle given $a = 4.38$ ft , $b = 3.79$ ft, and $c = 5.22$ ft.

Answer

about 8.15 square feet

Media

Access these online resources for additional instruction and practice with the Law of Cosines.

- Law of Cosines
- Law of Cosines: Applications
- Law of Cosines: Applications 2

Key Equations

Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$ $b^2 = a^2 + c^2 - 2ac\cos\beta$ $c^2 = a^2 + b^2 - 2ab\cos\gamma$
Heron's formula	$Area = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{(a+b+c)}{2}$

Key Concepts

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See Example $5.2.1$ and Example $5.2.2$.
- The Law of Cosines is useful for many types of applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law of Cosines to find a solution. See Example $5.2.3$ and Example $5.2.4$.
- Heron's formula allows the calculation of area in oblique triangles. All three sides must be known to apply Heron's formula. See Example $5.2.5$ and See Example $5.2.6$.

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5.3: Applications of Trigonometry - Area

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we use the Law of Sines and the Law of Cosines to help solve applied problems that involve triangles?
- How do we determine the area of a triangle?
- What is Heron's Law for the area of a triangle?

In Section 3.2, we used right triangles to solve some applied problems. It should then be no surprise that we can use the Law of Sines and the Law of Cosines to solve applied problems involving triangles that are not right triangles.

In most problems, we will first get a rough diagram or picture showing the triangle or triangles involved in the problem. We then need to label the known quantities. Once that is done, we can see if there is enough information to use the Law of Sines or the Law of Cosines. Remember that each of these laws involves four quantities. If we know the value of three of those four quantities, we can use that law to determine the fourth quantity.

We begin with the example in Exercise 5.3.1. The solution of this problem involved some complicated work with right triangles and some algebra. We will now solve this problem using the results from Section 3.3.

Example 5.3.0: Height to the Top of a Flagpole

Suppose that the flagpole sits on top a hill and that we cannot directly measure the length of the shadow of the flagpole as shown in Figure 5.3.2.

Some quantities have been labeled in the diagram. Angles α and β are angles of elevation to the top of the flagpole from two different points on level ground. These points are d feet apart and directly in line with the flagpole. The problem is to determine h, the height from level ground to the top of the flagpole. The following measurements have been recorded.

$$\alpha = 43.2^\circ \tag{5.3.1}$$

$$\beta = 34.7^\circ \tag{5.3.2}$$

$$d = 22.75 feet \tag{5.3.3}$$

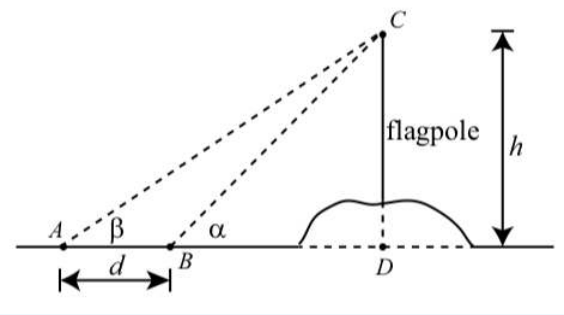


Figure 5.3.1: Flagpole on a hill

We notice that if we knew either length BC or BD in $\triangle BDC$, then we could use right triangle trigonometry to determine the length BC, which is equal to h. Now look at $\triangle ABC$. We are given one angle β. However, we also know the measure of angle α. Because they form a straight angle, we have

$$\angle ABC + \alpha = 180^\circ \tag{5.3.4}$$

Hence,

$$\angle ABC = 180^\circ - 43.2^\circ = 136.8^\circ \tag{5.3.5}$$

. We now know two angles in $\triangle ABC$ and hence, we can determine the third angle as follows:

$$\beta + \angle ABC + \angle ACB = 180^\circ \tag{5.3.6}$$

$$34.7^\circ + 136.8^\circ + \angle ACB = 180^\circ \tag{5.3.7}$$

$$\angle ACB = 8.5^\circ \tag{5.3.8}$$

We now know all angles in $\triangle ABC$ and the length of one side. We can use the Law of Sines. We have

$$\frac{AC}{\sin(34.7^\circ)} = \frac{22.75^\circ}{\sin(8.5^\circ)} \tag{5.3.9}$$

$$AC = \frac{22.75 \sin(34.7^\circ)}{\sin(8.5^\circ)} \approx 87.620 \tag{5.3.10}$$

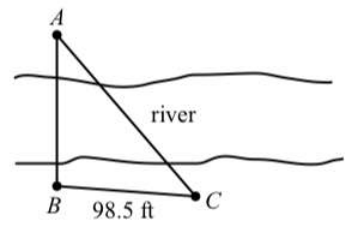
We can now use the right triangle $\triangle BDC$ to determine h as follows:

$$\frac{h}{AC} = \sin(43.2^\circ) \tag{5.3.11}$$

$$h = AC \cdot \sin(43.2^\circ) \approx 59.980 \tag{5.3.12}$$

So the top of the flagpole is 59.980 feet above the ground. This is the same answer we obtained in Exercise 5.3.1.

Exercise 5.3.1



A bridge is to be built across a river. The bridge will go from point A to point B in the diagram on the right. Using a transit (an instrument to mea- sure angles), a surveyor measures angle ABC to be 94.2° and measures angle BCA to be 48.5°. In addition, the distance from B to C is measured to be 98.5 feet. How long will the bridge from point B to point A be?

Answer

We first note that $\angle BAC = 180^\circ - 94.2^\circ - 48.5^\circ$ and so $\angle BAC = 37.3^\circ$. We can then use the Law of Sines to determine the length from A to B as follows:

$$\frac{AB}{\sin(48.5^\circ)} = \frac{98.5}{\sin(37.3^\circ)} \tag{5.3.13}$$

$$AB = \frac{98.5 \sin(48.5^\circ)}{\sin(37.3^\circ)} \tag{5.3.14}$$

$$AB \approx 121.7 \tag{5.3.15}$$

The bridge from point B to point A will be approximately 121.7feet long.

Area of a Triangle

We will now develop a few different ways to calculate the area of a triangle. Perhaps the most familiar formula for the area is the following:

The triangles in Figure $5.3.2$ illustrate the use of the variables in this formula.

The area A of a triangle is

$$A = \frac{1}{2}bh. \tag{5.3.16}$$

where b is the length of the base of a triangle and h is the length of the altitude that is perpendicular to that base.

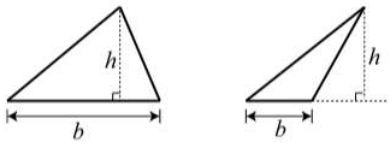
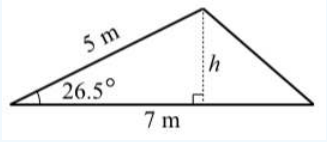


Figure 5.3.2: Diagrams for the Formula for the Area of a Triangle

A proof of this formula for the area of a triangle depends on the formula for the area of a parallelogram and is included in Appendix C.

Exercise 5.3.2



Suppose that the length of two sides of a triangle are 5 meters and 7 meters and that the angle formed by these two sides is $26.5°$. See the diagram on the right.

For this problem, we are using the side of length 7 meters as the base. The altitude of length h that is perpendicular to this side is shown.

1. Use right triangle trigonometry to determine the value of h.
2. Determine the area of this triangle.

Answer

Using the right triangle, we see that $\sin(26.5°) = \frac{h}{5}$. So $h = 5\sin(26.5°)$, and the area of the triangle is

$$A = \frac{1}{2}(7)[5\sin(26.5°)] = \frac{35}{2}\sin(26.5°) \approx 7.8085 \tag{5.3.17}$$

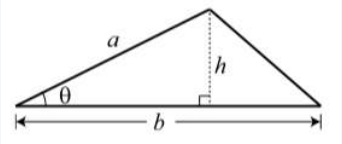
The area of the triangle is approximately 7.8085square meters.

The purpose of Progress Check 3.21 was to illustrate that if we know the length of two sides of a triangle and the angle formed by these two sides, then we can determine the area of that triangle.

The Area of a Triangle

The area of a triangle equals one-half the product of two of its sides times the sine of the angle formed by these two sides.

Exercise 5.3.3



In the diagram on the right, b is the length of the base of a triangle, a is the length of another side, and θ is the angle formed by these two sides. We let A be the area of the triangle.

Follow the procedure illustrated in Progress Check 3.21 to prove that

$$A = \frac{1}{2}ab\sin(\theta) \tag{5.3.18}$$

Explain why this proves the formula for the area of a triangle.

Answer

Using the right triangle, we see that $\sin(\theta) = \frac{h}{a}$. So $h = a\sin(\theta$, and the area of the triangle is

$$A = \frac{1}{2}b(a\sin(\theta)) = \frac{1}{2}ab\sin(\theta) \tag{5.3.19}$$

There is another common formula for the area of a triangle known as Heron's Formula named after Heron of Alexandria (circa 75 CE). This formula shows that the area of a triangle can be computed if the lengths of the three sides of the triangle are known.

Heron's Formula

The area A of a triangle with sides of length a, b, and c is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{5.3.20}$$

where $s = \frac{1}{2}(a+b+c)$.

For example, suppose that the lengths of the three sides of a triangle are $a = 3ft$, $b = 5ft$, and $c = 6ft$. Using Heron's Formula (Equation 5.3.20), we get

$$s = \frac{1}{2}(a+b+c) \tag{5.3.21}$$

$$s = 7 \tag{5.3.22}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{5.3.23}$$

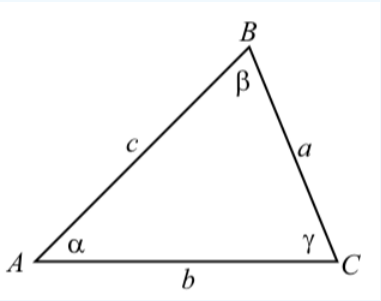
$$A = \sqrt{7(7-3)(7-5)(7-6)} \tag{5.3.24}$$

$$A = \sqrt{42} \tag{5.3.25}$$

This fairly complex formula is actually derived from the previous formula for the area of a triangle and the Law of Cosines. We begin our exploration of the proof of this formula in Progress Check

Exercise 5.3.4

Suppose we have a triangle as shown in the diagram below.



1. Use the Law of Cosines that involves the angle γ and solve this formula for $\cos(\gamma)$. This gives a formula for $\cos(\gamma)$ in terms of a, b, and c.
2. Use the Pythagorean Identity $\cos^2(\gamma) + \sin^2(\gamma) = 1$ 1 to write $\sin(\gamma)$ in terms of $\cos^2(\gamma)$. Substitute for $\cos^2(\gamma)$ using the formula in (1). This gives a formula for $\sin(\gamma)$ in terms of a, b, and c. (Do not do any algebraic simplification.)

3. We also know that a formula for the area of this triangle is $A = \frac{1}{2}ab\sin(\gamma)$

Substitute for $\sin(\gamma)$ using the formula in (2). (Do not do any algebraic simplification.) This gives a formula for the area A in terms of a, b, and c.

The formula obtained in Progress Check 3.23 was

$$A = \frac{1}{2}ab\sqrt{1-(\frac{a^2+b^2-c^2}{2ab})^2} \tag{5.3.26}$$

This is a formula for the area of a triangle in terms of the lengths of the three sides of the triangle. It does not look like Heron's Formula, but we can use some substantial algebra to rewrite this formula to obtain Heron's Formula. This algebraic work is completed in the appendix for this section.

Answer

1. Using the Law of Cosines, we see that

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma) \tag{5.3.27}$$

$$2ab\cos(\gamma) = a^2 + b^2 - c^2 \tag{5.3.28}$$

$$\cos(\gamma) = \frac{a^2+b^2-c^2}{2ab} \tag{5.3.29}$$

2. We see that

$$\sin^2(\gamma) = 1 - \cos^2(\gamma). \tag{5.3.30}$$

Since γ is between 0° and 180°, we know that $\sin(\gamma) > 0$ and so

\[\sin(\gamma) = \sqrt{\dfrac{a^{2} + b^{2} - c^{2}}{2ab}}\]

3. Substituting the equation in part (2) into the formula $A = \frac{1}{ab}\sin(\gamma)$, we obtain

\[A = \dfrac{1}{2}ab\sin(\gamma) = \dfrac{1}{2}ab\sqrt{1 - (\dfrac{a^{2} + b^{2} - c^{2}}{2ab})^{2}}\]

Appendix – Proof of Heron's Formula

The formula for the area of a triangle obtained in Progress Check 3.23 was

$$A = \frac{1}{2}ab\sqrt{1-(\frac{a^2+b^2-c^2}{2ab})^2} \tag{5.3.31}$$

We now complete the algebra to show that this is equivalent to Heron's formula. The first step is to rewrite the part under the square root sign as a single fraction.

$$A = \frac{1}{2}ab\sqrt{1-(\frac{a^2+b^2-c^2}{2ab})^2} \tag{5.3.32}$$

$$= \frac{1}{2}ab\sqrt{\frac{(2ab)^2-(a^2+b^2-c^2)^2}{(2ab)^2}} \tag{5.3.33}$$

$$= \frac{1}{2}ab\frac{\sqrt{(2ab)^2-(a^2+b^2-c^2)^2}}{2ab} \tag{5.3.34}$$

$$= \frac{\sqrt{(2ab)^2-(a^2+b^2-c^2)^2}}{4} \tag{5.3.35}$$

Squaring both sides of the last equation, we obtain

$$A^2 = \frac{(2ab)^2-(a^2+b^2-c^2)^2}{16} \tag{5.3.36}$$

The numerator on the right side of the last equation is a difference of squares. We will now use the difference of squares formula, $x^2-y^2=(x-y)(x+y)$ to factor the numerator.

$$A^2=\frac{(2ab)^2-(a^2+b^2-c^2)^2}{16} \tag{5.3.37}$$

$$=\frac{(2ab-(a^2+b^2-c^2))(2ab+(a^2+b^2-c^2))}{16} \tag{5.3.38}$$

$$=\frac{(-a^2+2ab-b^2+c^2)(a^2+2ab+b^2-c^2)}{16} \tag{5.3.39}$$

We now notice that $-a^2+2ab-b^2=-(a-b)^2$ and $a^2+2ab+b^2=(a+b)^2$. So using these in the last equation, we have

$$A^2=\frac{(-(a-b)^2+c^2)((a+b)^2-c^2)}{16} \tag{5.3.40}$$

$$=\frac{(-[(a-b)^2-c^2])((a+b)^2-c^2)}{16} \tag{5.3.41}$$

We can once again use the difference of squares formula as follows:

$$(a-b)^2-c^2=(a-b-c)(a-b+c) \tag{5.3.42}$$

$$(a+b)^2-c^2=(a+b-c)(a+b+c) \tag{5.3.43}$$

Substituting this information into the last equation for A^2, we obtain

$$A^2=\frac{-(a-b-c)(a-b+c)(a+b-c)(a+b+c)}{16} \tag{5.3.44}$$

Since $s=\frac{1}{2}(a+b+c)$, $2s=a+b+c$ Now notice that

$$-(a-b-c)=-a+b+c=a+b+c-2a=2s-2a \tag{5.3.45}$$

$$a-b+c=a+b+c-2b=2s-2b \tag{5.3.46}$$

$$a+b+c=a+b+c-2c=2s-2c \tag{5.3.47}$$

$$a+b+c=2s \tag{5.3.48}$$

so

$$A^2=\frac{-(a-b-c)(a-b+c)(a+b-c)(a+b+c)}{16} \tag{5.3.49}$$

$$=\frac{(2s-2a)(2s-2b)(2s-2c)(2s)}{16} \tag{5.3.50}$$

$$=\frac{16s(s-a)(s-b)(s-c)}{16} \tag{5.3.51}$$

$$=s(s-a)(s-b)(s-c) \tag{5.3.52}$$

$$A=\sqrt{s(s-a)(s-b)(s-c)} \tag{5.3.53}$$

This completes the proof of Heron's formula.

Area of a Sector & Area of a Segment

In geometry you learned that the area of a circle of radius r is πr^2. We will now learn how to find the area of a **sector** of a circle. A sector is the region bounded by a central angle and its intercepted arc.

Let θ be a central angle in a circle of radius r and let A be the area of its sector. Similar to arc length, the ratio of A to the area of the entire circle is the same as the ratio of θ to one revolution. In other words, again using radian measure,

$$\frac{\text{area of sector}}{\text{area of entire circle}} = \frac{\text{sector angle}}{\text{one revolution}} \quad \Rightarrow \quad \frac{A}{\pi\, r^2} = \frac{\theta}{2\pi}\,.$$

Solving for A in the above equation, we get a formula.

Definition: Area of a Sector

In a circle of radius r, the area A of the sector inside a central angle θ is

$$A = \tfrac{1}{2}\, r^2\, \theta\,, \tag{5.3.54}$$

where θ is measured in radians.

Applying the Equation

Using $\theta = \frac{\pi}{5}$ and $r = 4$ in Equation 5.3.54, the area A of the sector is

$$A = \tfrac{1}{2}\, r^2\, \theta = \tfrac{1}{2}\,(4)^2 \cdot \tfrac{\pi}{5} = \boxed{\tfrac{8\pi}{5}\ \text{cm}^2}\,.$$

Example 8.3.1

Find the area of a sector whose angle is 117° in a circle of radius $3.5\,\text{m}$.

Solution:

As with arc length, we have to make sure that the angle is measured in radians or else the answer will be way off. So converting $\theta = 117^\circ$ to radians and using $r = 3.5$ in Equation 5.3.54 for the area A of the sector, we get

$$\theta = 117^\circ = \frac{\pi}{180} \cdot 117 = 2.042\ \text{rad} \quad \Rightarrow \quad A = \tfrac{1}{2}\, r^2\, \theta = \tfrac{1}{2}\,(3.5)^2\,(2.042) = 12.51\ \text{m}^2\,.$$

For a sector whose angle is θ in a circle of radius r, the length of the arc cut off by that angle is $s = r\,\theta$. Thus, by Equation 5.3.54 the area A of the sector can be written as:

$$A = \tfrac{1}{2}\, rs \tag{5.3.55}$$

Note: The central angle θ that intercepts an arc is sometimes called the angle **subtended** by the arc.

Example 8.3.2

Using $s = 6$ and $r = 9$ in Equation 5.3.55 for the area A, we get

$$A = \tfrac{1}{2}\, rs = \ = \tfrac{1}{2}\,(9)\,(6) = \boxed{27\ \text{cm}^2}\,.$$

Note that the angle subtended by the arc is $\theta = \frac{s}{r} = \frac{2}{3}$ rad.

So far we have dealt with the area cut off by a central angle. How would you find the area of a region cut off by an inscribed angle, such as the shaded region in Figure 8.3.1? In this picture, the center of the circle is inside the inscribed angle, and the lengths a and

b of the two chords are given, as is the radius r of the circle. Drawing line segments from the center of the circle to the endpoints of the chords indicates how to solve this problem: add up the areas of the two triangles and the sector formed by the central angle. The areas and angles of the two triangles can be determined (since all three sides are known) using methods from pervious sections. Also, note that a central angle has twice the measure of any inscribed angle which intercepts the same arc. In the exercises you will be asked to solve problems like this (including the cases where the center of the circle is outside or on the inscribed angle).

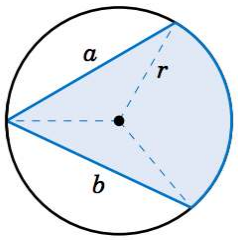


Figure 8.3.1

Another type of region we can consider is a **segment** of a circle, which is the region between a **chord** and the **arc** it cuts off. In Figure 8.3.2 the segment formed by the chord $\overline{AB}$ is the shaded region between the arc $\overset{\frown}{AB}$ and the triangle $\triangle OAB$. For the area of a triangle given two sides and their included angle, we know that

$$\text{area of } \triangle OAB = \tfrac{1}{2}(r)(r)\sin\theta = \tfrac{1}{2}r^2\sin\theta\,.$$

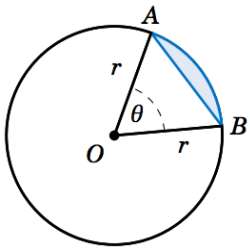


Figure 8.3.2

Thus, since the area K of the segment is the area of the sector AOB minus the area of the triangle $\triangle OAB$, we have

$$\text{area } K \text{ of segment } AB = \tfrac{1}{2}r^2\,\theta - \tfrac{1}{2}r^2\sin\theta = \tfrac{1}{2}r^2(\theta - \sin\theta)\,. \tag{5.3.56}$$

Note that as a consequence of Equation 5.3.56 we must have $\theta > \sin\theta$ for $0 < \theta \leq \pi$ (measured in radians), since the area of a segment is positive for those angles. See Figure 8.3.2.

Example 8.3.3

Given Figure 8.3.3, with radius 2 and chord 3, find the segment.

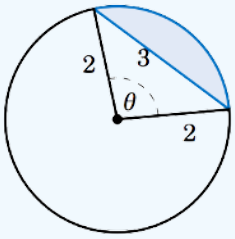


Figure 8.3.3

Solution

Figure 8.3.3 shows the segment formed by a chord of length 3 in a circle of radius $r = 2$. We can use the Law of Cosines to find the subtended central angle θ:

$$\cos\,\theta\,=\,\frac{2^2+2^2-3^2}{2\,(2)\,(2)}\,=\,-0.125\quad\Rightarrow\quad\theta\,=\,1.696\,\text{rad}$$

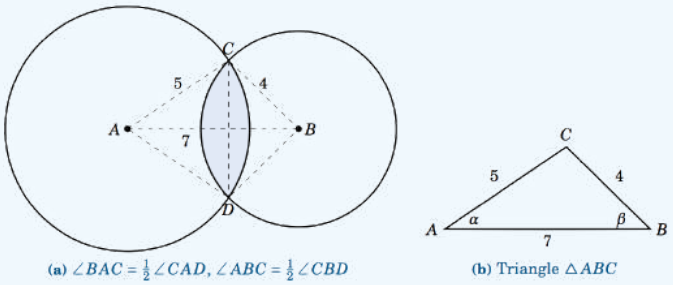
Thus, by Equation 5.3.56 the area K of the segment is:

$$K\,=\,\tfrac{1}{2}\,r^2\,(\theta-\sin\,\theta)\,=\,\tfrac{1}{2}\,(2)^2\,(1.696-\sin\,1.696)\,=\,\boxed{1.408}$$

Example 8.3.4

The centers of two circles are 7 cm apart, with one circle having a radius of 5 cm and the other a radius of 4 cm. Find the area K of their intersection.

Solution:



(a) $\angle BAC=\frac{1}{2}\angle CAD,\angle ABC=\frac{1}{2}\angle CBD$ (b) Triangle $\triangle ABC$

Figure 8.3.4

By symmetry, we see that $\angle\,BAC=\frac{1}{2}\angle\,CAD$ and $\angle\,ABC=\frac{1}{2}\angle\,CBD$. So let $\alpha=\angle\,BAC$ and $\beta=\angle\,ABC$, as in Figure 4.3.6(b). By the Law of Cosines, we have

$$\cos\,\alpha\,=\,\frac{7^2+5^2-4^2}{2\,(7)\,(5)}=0.8286\quad\Rightarrow\quad\alpha=0.594\,\text{rad}\quad\Rightarrow\quad\angle\,CAD=2\,(0.594)=1.188\,\text{rad}$$

$$\cos\,\beta\,=\,\frac{7^2+4^2-5^2}{2\,(7)\,(4)}=0.7143\quad\Rightarrow\quad\beta=0.775\,\text{rad}\quad\Rightarrow\quad\angle\,CBD=2\,(0.775)=1.550\,\text{rad}$$

Thus, the area K is

$$\begin{aligned}K\,&=\,(\text{Area of segment }CD\text{ in circle at }A)\,+\,(\text{Area of segment }CD\text{ in circle at }B)\\&=\,\tfrac{1}{2}\,(5)^2\,(1.188-\sin\,1.188)\,+\,\tfrac{1}{2}\,(4)^2\,(1.550-\sin\,1.550)\\&=\,\boxed{7.656\,\text{cm}^2}\,.\end{aligned}$$

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Summary

In this section, we studied the following important concepts and ideas:

- How to use right triangle trigonometry, the Law of Sines, and the Law of Cosines to solve applied problems involving triangles.
- Three ways to determine the area A of a triangle.
- How to find the area of a sector and some additional creative ways to calculate areas of overlap between two circles and **segments**.

$A=\dfrac{1}{2}bh$, where b is the length of the base and h is the length of the altitude.

$A = \frac{1}{2}ab$, where a and b are the lengths of two sides of the triangle and θ is the angle formed by the sides of length a and b.

Heron's Formula. If a, b, and c are the lengths of the sides of a triangle and $s = \frac{1}{2}(a+b+c)$, then

$$A = \sqrt{s(s-a)(s-b)(s-c)} \ . \tag{5.3.57}$$

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5.4: Polar Coordinates

Learning Objectives

- Plot points using polar coordinates.
- Convert from polar coordinates to rectangular coordinates.
- Convert from rectangular coordinates to polar coordinates.
- Transform equations between polar and rectangular forms.
- Identify and graph polar equations by converting to rectangular equations.

Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see Figure 5.4.1). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.

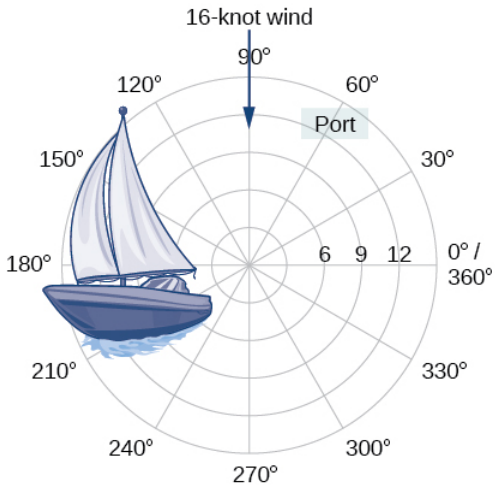
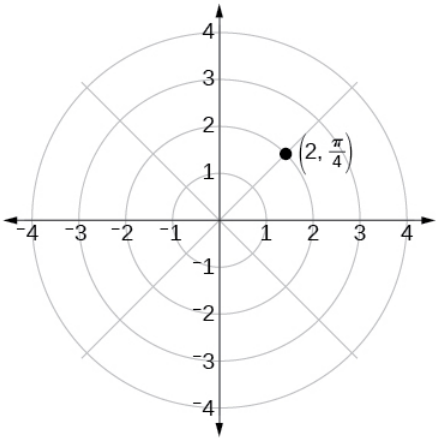


***Figure* 5.4.1**

Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of *rectangular coordinates* (x, y) in the *Cartesian coordinate plane*. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to polar coordinates, which are points labeled (r, θ) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the pole, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive x-axis now viewed as the polar axis and the origin as the pole. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ, measured in radians, indicates the direction of r. We move counterclockwise from the polar axis by an angle of θ,and measure a directed line segment the length of r in the direction of θ. Even though we measure θ first and then r, the polar point is written with the r-coordinate first. For example, to plot the point $\left(2, \frac{\pi}{4}\right)$,we would move $\frac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in Figure 5.4.2.



Polar Grid

***Figure* 5.4.2**

Example 5.4.1: Plotting a Point on the Polar Grid

Plot the point $\left(3, \frac{\pi}{2}\right)$ on the polar grid.

Solution

The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction 90° from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$ direction, as shown in Figure 5.4.3.

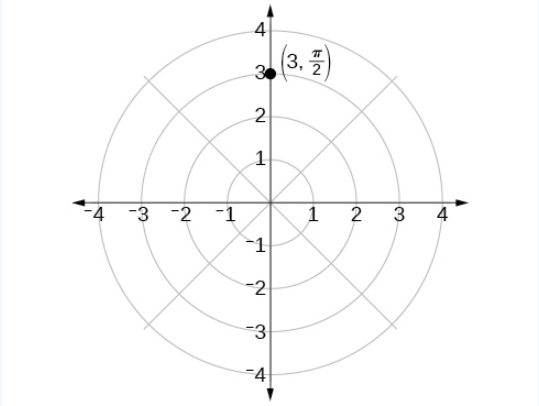


***Figure* 5.4.3**

Exercise 5.4.1

Plot the point $\left(2, \frac{\pi}{3}\right)$ in the polar grid.

Answer

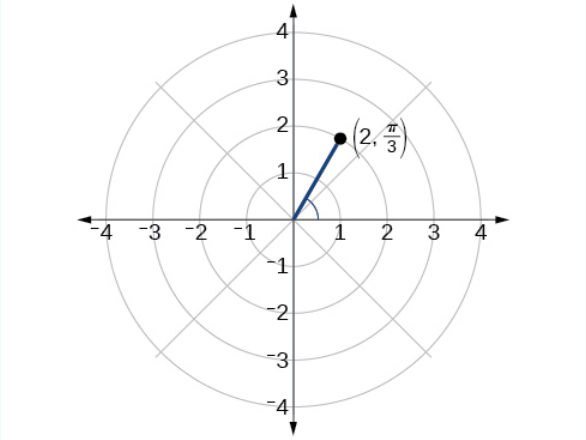


Figure 5.4.4

Example 5.4.2: Plotting a Point in the Polar Coordinate System with a Negative Component

Plot the point $\left(-2, \frac{\pi}{6}\right)$ on the polar grid.

Solution

We know that $\frac{\pi}{6}$ is located in the first quadrant. However, $r = -2$. We can approach plotting a point with a negative r in two ways:

1. Plot the point $\left(2, \frac{\pi}{6}\right)$ by moving $\frac{\pi}{6}$ in the counterclockwise direction and extending a directed line segment 2 units into the first quadrant. Then retrace the directed line segment back through the pole, and continue 2 units into the third quadrant;
2. Move $\frac{\pi}{6}$ in the counterclockwise direction, and draw the directed line segment from the pole 2 units in the negative direction, into the third quadrant.

See Figure 5.4.5*a*. Compare this to the graph of the polar coordinate $(2, \pi 6)$ shown in Figure 5.4.5*b*

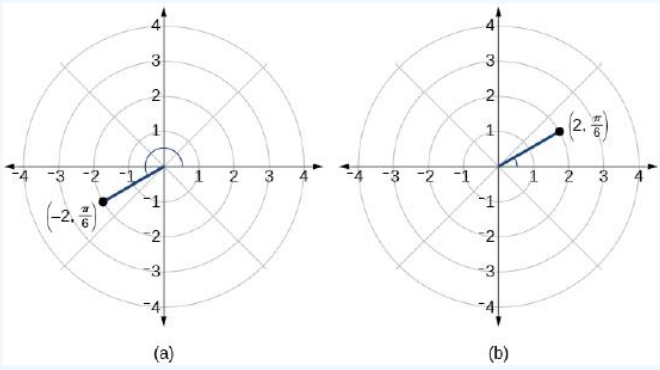


Figure **5.4.5**

Exercise 5.4.2

Plot the points $\left(3, -\frac{\pi}{6}\right)$ and $\left(2, \frac{9\pi}{4}\right)$ on the same polar grid.

Answer

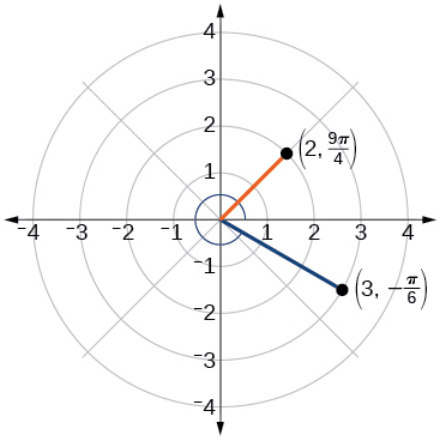


Figure 5.4.6

Converting from Polar Coordinates to Rectangular Coordinates

When given a set of **polar coordinates**, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables x, y, r, and θ.

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$$

Dropping a perpendicular from the point in the plane to the *x*-axis forms a right triangle, as illustrated in Figure 5.4.7. An easy way to remember the equations above is to think of $\cos\theta$ as the adjacent side over the hypotenuse and $\sin\theta$ as the opposite side over the hypotenuse.

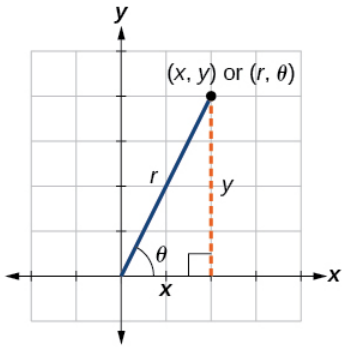


Figure **5.4.7**

CONVERTING FROM POLAR COORDINATES TO RECTANGULAR COORDINATES

To convert polar coordinates (r, θ) to rectangular coordinates (x, y), let

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta \tag{5.4.1}$$

$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta \tag{5.4.2}$$

How to: Given polar coordinates, convert to rectangular coordinates.

1. Given the polar coordinate (r, θ), write $x = r\cos\theta$ and $y = r\sin\theta$.
2. Evaluate $\cos\theta$ and $\sin\theta$.
3. Multiply $\cos\theta$ by r to find the x-coordinate of the rectangular form.

4. Multiply $\sin\theta$ by r to find the y-coordinate of the rectangular form.

Example $5.4.3A$: Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $\left(3, \frac{\pi}{2}\right)$ as rectangular coordinates.

Solution

Use the equivalent relationships.

$$\begin{aligned} x &= r\cos\theta \\ x &= 3\cos\frac{\pi}{2} \\ &= 0 \\ y &= r\sin\theta \\ y &= 3\sin\frac{\pi}{2} \\ &= 3 \end{aligned}$$

The rectangular coordinates are $(0, 3)$. See Figure $5.4.8$.

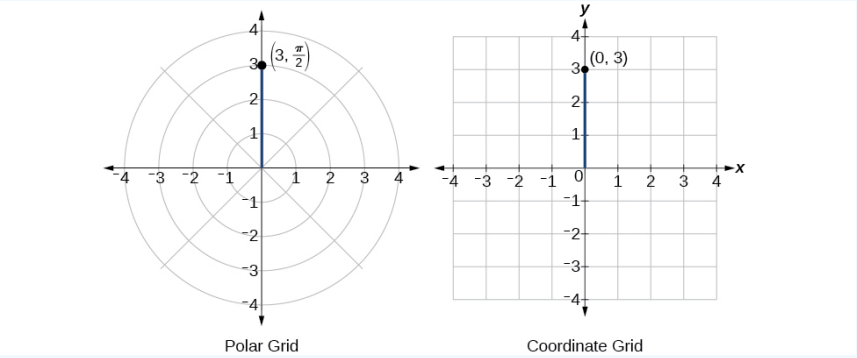


***Figure* 5.4.8**

Example $5.4.3B$: Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

Solution

See Figure $5.4.9$. Writing the polar coordinates as rectangular, we have

$$\begin{aligned} x &= r\cos\theta \\ x &= -2\cos(0) \\ &= -2 \\ y &= r\sin\theta \\ y &= -2\sin(0) \\ &= 0 \end{aligned}$$

The rectangular coordinates are also $(-2, 0)$.

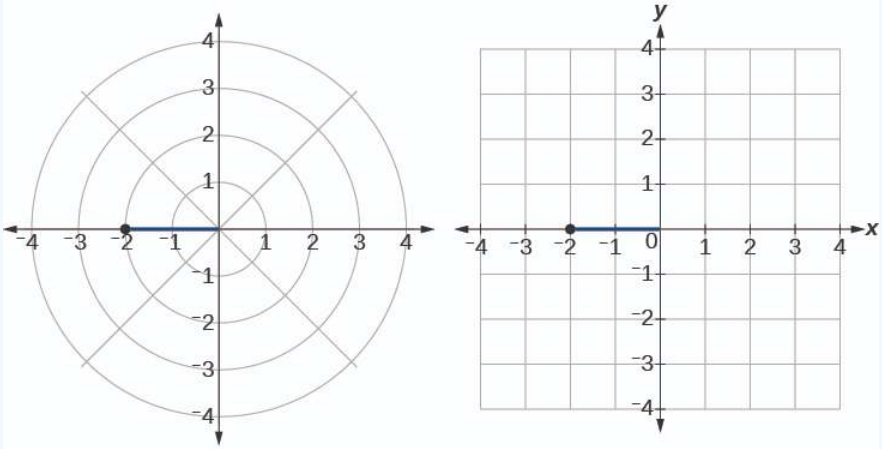


Figure **5.4.9**

? Exercise 5.4.3

Write the polar coordinates $\left(-1, \frac{2\pi}{3}\right)$ as rectangular coordinates.

Answer

$$(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

CONVERTING FROM RECTANGULAR COORDINATES TO POLAR COORDINATES

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in Figure 5.4.10

$$\cos\theta = \frac{x}{r} \text{ or } x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \text{ or } y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

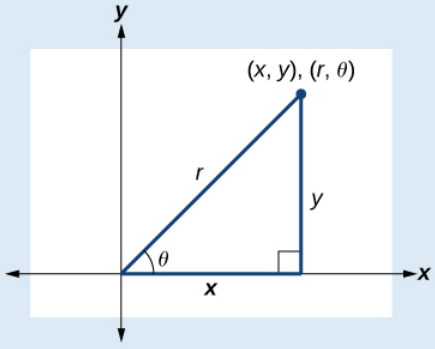


Figure **5.4.10**

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Example 5.4.4: Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates $(3, 3)$ to polar coordinates.

Solution

We see that the original point $(3, 3)$ is in the first quadrant. To find θ, use the formula $\tan\theta = \dfrac{y}{x}$. This gives

$$\begin{aligned} \tan\theta &= \frac{3}{3} \\ \tan\theta &= 1 \\ \tan^{-1}(1) &= \frac{\pi}{4} \end{aligned}$$

To find r, we substitute the values for x and y into the formula $r = \sqrt{x^2 + y^2}$. We know that r must be positive, as $\dfrac{\pi}{4}$ is in the first quadrant. Thus

$$\begin{aligned} r &= \sqrt{3^2 + 3^2} \\ r &= \sqrt{9 + 9} \\ r &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

So, $r = 3\sqrt{2}$ and $\theta = \dfrac{\pi}{4}$, giving us the polar point $(3\sqrt{2}, \dfrac{\pi}{4})$. See Figure 5.4.11.

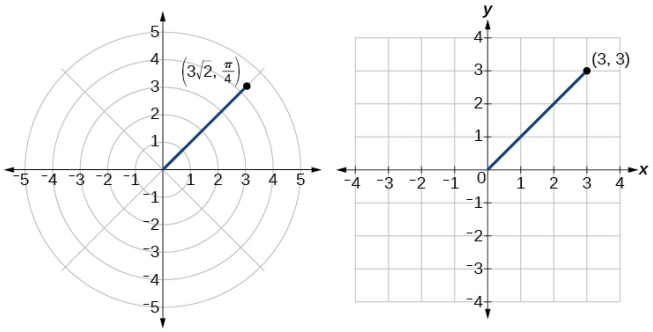


***Figure* 5.4.11**

Analysis

There are other sets of polar coordinates that will be the same as our first solution. For example, the points $\left(-3\sqrt{2}, \dfrac{5\pi}{4}\right)$ and $\left(3\sqrt{2}, -\dfrac{7\pi}{4}\right)$ will coincide with the original solution of $\left(3\sqrt{2}, \dfrac{\pi}{4}\right)$. The point $\left(-3\sqrt{2}, \dfrac{5\pi}{4}\right)$ indicates a move further counterclockwise by π, which is directly opposite $\dfrac{\pi}{4}$. The radius is expressed as $-3\sqrt{2}$. However, the angle $\dfrac{5\pi}{4}$ is located in the third quadrant and, as r is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as $\left(3\sqrt{2}, \dfrac{\pi}{4}\right)$. The point $\left(3\sqrt{2}, -\dfrac{7\pi}{4}\right)$ is a move further clockwise by $-\dfrac{7\pi}{4}$, from $\dfrac{\pi}{4}$. The radius, $3\sqrt{2}$, is the same.

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

How to: Given an equation in polar form, graph it using a graphing calculator

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: r_1, r_2,..., r_6.
3. Enter the polar equation, set equal to r.
4. Press **GRAPH**.

Example 5.4.5A: Writing a Cartesian Equation in Polar Form

Write the Cartesian equation $x^2+y^2=9$ in polar form.

Solution

The goal is to eliminate x and y from the equation and introduce r and θ. Ideally, we would write the equation r as a function of θ. To obtain the polar form, we will use the relationships between (x,y) and (r,θ). Since $x=r\cos\theta$ and $y=r\sin\theta$, we can substitute and solve for r.

$$\begin{aligned}(r\cos\theta)^2+(r\sin\theta)^2&=9\\ r^2\cos^2\theta+r^2\sin^2\theta&=9\\ r^2(\cos^2\theta+\sin^2\theta)&=9\\ r^2(1)&=9 \qquad \text{Substitute } \cos^2\theta+\sin^2\theta=1\\ r&=\pm 3 \qquad \text{Use the square root property.}\end{aligned}$$

Thus, $x^2+y^2=9$,$r=3$,and $r=-3$ should generate the same graph. See Figure 5.4.12

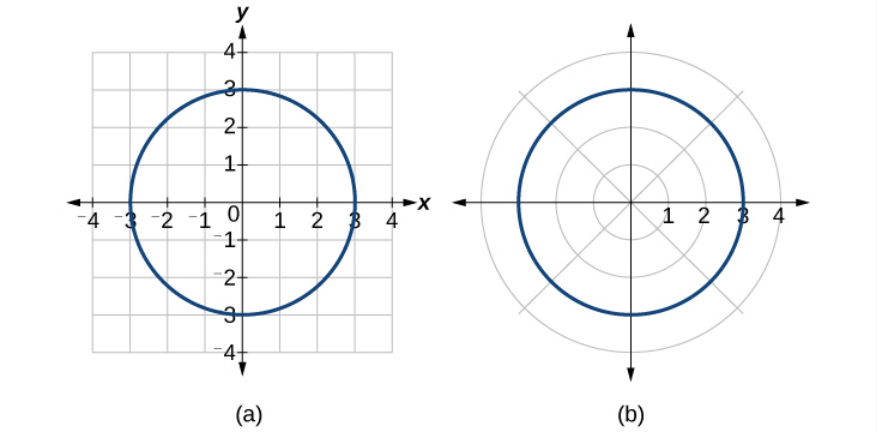


***Figure* 5.4.12:** *(a) Cartesian form $x^2+y^2=9$ (b) Polar form $r=3$*

To graph a circle in rectangular form, we must first solve for y.

$$\begin{aligned}x^2+y^2&=9\\ y^2&=9-x^2\\ y&=\pm\sqrt{9-x^2}\end{aligned}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1=\sqrt{9-x^2}$ and $Y_2=-\sqrt{9-x^2}$. Press **GRAPH.**

Example 5.4.5B: Rewriting a Cartesian Equation as a Polar Equation

Rewrite the **Cartesian equation** $x^2+y^2=6y$ as a polar equation.

Solution

This equation appears similar to the previous example, but it requires different steps to convert the equation.

We can still follow the same procedures we have already learned and make the following substitutions:

$r^2 = 6y$	Use $x^2 + y^2 = r^2$.
$r^2 = 6r\sin\theta$	Substitute $y = r\sin\theta$.
$r^2 - 6r\sin\theta = 0$	Set equal to 0.
$r(r - 6\sin\theta) = 0$	Factor and solve.
$r = 0$	We reject $r = 0$, as it only represents one point, $(0, 0)$.
or $r = 6\sin\theta$	

Therefore, the equations $x^2 + y^2 = 6y$ and $r = 6\sin\theta$ should give us the same graph. See Figure 5.4.13

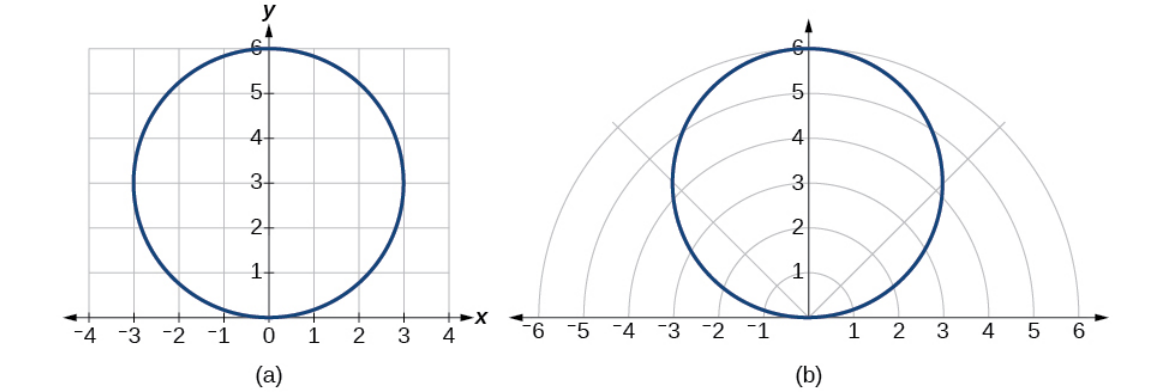


***Figure* 5.4.13:** *(a) Cartesian form $x^2 + y^2 = 6y$ (b) polar form $r = 6\sin\theta$*

The Cartesian or rectangular equation is plotted on the rectangular grid, and the **polar equation** is plotted on the polar grid. Clearly, the graphs are identical.

? Exercise 5.4.4*A*:

Rewriting a Cartesian Equation in Polar Form

Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

Answer

We will use the relationships $x = r\cos\theta$ and $y = r\sin\theta$.

$$
\begin{aligned}
y &= 3x + 2 \\
r\sin\theta &= 3r\cos\theta + 2 \\
r\sin\theta - 3r\cos\theta &= 2 \\
r(\sin\theta - 3\cos\theta) &= 2 \quad \text{Isolate } r. \\
r &= \frac{2}{\sin\theta - 3\cos\theta} \quad \text{Solve for } r.
\end{aligned}
$$

? Exercise 5.4.4*B*:

Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form.

Answer

$r = \sqrt{3}$

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example 5.4.6A: Graphing a Polar Equation by Converting to a Rectangular Equation

Covert the polar equation $r = 2\sec\theta$ to a rectangular equation, and draw its corresponding graph.

Solution

The conversion is

$$\begin{aligned} r &= 2\sec\theta \\ r &= \frac{2}{\cos\theta} \\ r\cos\theta &= 2 \\ x &= 2 \end{aligned}$$

Notice that the equation $r = 2\sec\theta$ drawn on the polar grid is clearly the same as the vertical line $x = 2$ drawn on the rectangular grid (see Figure 5.4.14). Just as $x = c$ is the standard form for a vertical line in rectangular form, $r = c\sec\theta$ is the standard form for a vertical line in polar form.

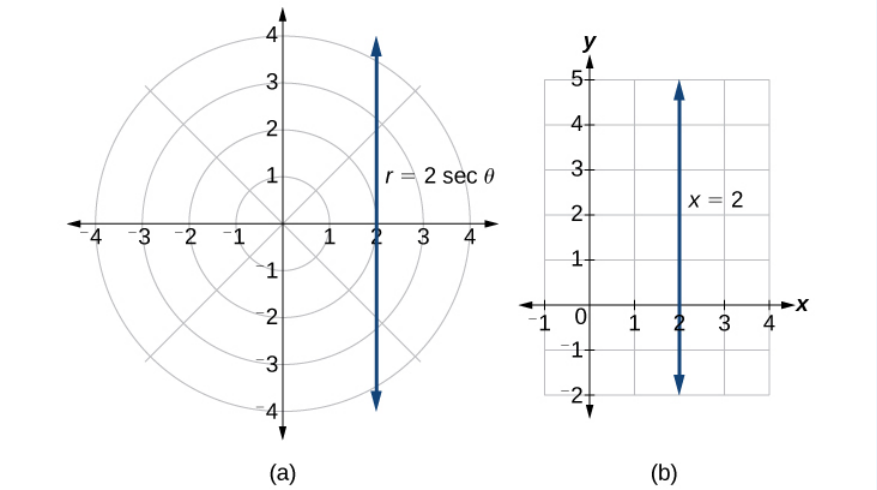


***Figure* 5.4.14:** *(a) Polar grid (b) Rectangular coordinate system*

A similar discussion would demonstrate that the graph of the function $r = 2\csc\theta$ will be the horizontal line $y = 2$. In fact, $r = c\csc\theta$ is the standard form for a horizontal line in polar form, corresponding to the rectangular form $y = c$.

Example 5.4.6B: Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \dfrac{3}{1-2\cos\theta}$ as a Cartesian equation.

Solution

The goal is to eliminate θ and r,and introduce x and y. We clear the fraction, and then use substitution. In order to replace r with x and y, we must use the expression $x^2 + y^2 = r^2$.

$$\begin{aligned} &= \frac{3}{1-2\cos\theta} \\ r(1-2\cos\theta) &= 3 \\ r\left(1-2\left(\frac{x}{r}\right)\right) &= 3 \quad \text{Use } \cos\theta = \frac{x}{r} \text{ to eliminate } \theta. \\ r - 2x &= 3 \\ r &= 3 + 2x \quad \text{Isolate } r. \\ r^2 &= (3+2x)^2 \quad \text{Square both sides.} \\ x^2 + y^2 &= (3+2x)^2 \quad \text{Use } x^2 + y^2 = r^2. \end{aligned}$$

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The Cartesian equation is $x^2+y^2=(3+2x)^2$. However, to graph it, especially using a graphing calculator or computer program, we want to isolate y.

$$\begin{aligned} x^2+y^2 &= (3+2x)^2 \\ y^2 &= (3+2x)^2-x^2 \\ y &= \pm(3+2x)^2-x^2 \end{aligned}$$

When our entire equation has been changed from r and θ to x and y, we can stop, unless asked to solve for y or simplify. See Figure 5.4.15

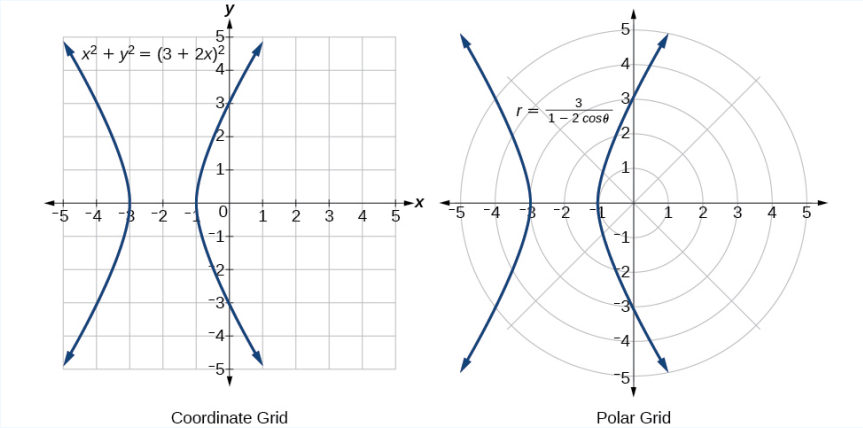


***Figure* 5.4.15**

The "hour-glass" shape of the graph is called a *hyperbola*. Hyperbolas have many interesting geometric features and applications, which we will investigate further in Analytic Geometry.

Analysis

In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However, the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola's standard form. To do this, we can start with the initial equation.

$$\begin{aligned} &x^2+y^2=(3+2x)^2 \\ &x^2+y^2-(3+2x)^2=0 \\ &x^2+y^2-(9+12x+4x^2)=0 \\ &x^2+y^2-9-12x-4x^2=0 \\ &-3x^2-12x+y^2=9 && \text{Multiply through by } -1. \\ &3x^2+12x-y^2=-9 \\ &3(x^2+4x)-y2=-9 && \text{Organize terms to complete the square for } x. \\ &3(x^2+4x+4)-y^2=-9+12 \\ &3(x+2)^2-y^2=3 \\ &(x+2)^2-\frac{y^2}{3}=1 \end{aligned}$$

Exercise 5.4.5

Rewrite the polar equation $r=2\sin\theta$ in Cartesian form.

Answer

$x^2+y^2=2y$ or, in the standard form for a circle, $x^2+(y-1)^2=1$

Example 5.4.7: Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

Solution

$$
\begin{aligned}
r &= \sin(2\theta) && \text{Use the double angle identity for sine.}\\
r &= 2\sin\theta\cos\theta && \text{Use } \cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}.\\
r &= 2\frac{x}{r})(\frac{y}{r}) && \text{Simplify.}\\
r &= \frac{2xy}{r^2} && \text{Multiply both sides by } r^2.\\
r^3 &= 2xy &&\\
(x^2+y^2)^{\,3} &= 2xy && \text{As } x^2+y^2 = r^2, r = \sqrt{x^2+y^2}.
\end{aligned}
$$

This equation can also be written as

$$(x^2+y^2)^{\frac{3}{2}} = 2xy \text{ or } x^2+y^2 = (2xy)^{\frac{2}{3}}$$

Media

Access these online resources for additional instruction and practice with polar coordinates.

- Introduction to Polar Coordinates
- Comparing Polar and Rectangular Coordinates

Key Equations

Conversion formulas	$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$ $\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$ $r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$

Key Concepts

- The polar grid is represented as a series of concentric circles radiating out from the pole, or origin.
- To plot a point in the form (r, θ), $\theta > 0$, move in a counterclockwise direction from the polar axis by an angle of θ, and then extend a directed line segment from the pole the length of r in the direction of θ. If θ is negative, move in a clockwise direction, and extend a directed line segment the length of r in the direction of θ. See Example 5.4.1.
- If r is negative, extend the directed line segment in the opposite direction of θ. See Example 5.4.2.
- To convert from polar coordinates to rectangular coordinates, use the formulas $x = r\cos\theta$ and $y = r\sin\theta$. See Example 5.4.3 and Example 5.4.4.
- To convert from rectangular coordinates to polar coordinates, use one or more of the formulas: $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, $\tan\theta = \frac{y}{x}$, and $r = \sqrt{x^2+y^2}$. See Example 5.4.5.
- Transforming equations between polar and rectangular forms means making the appropriate substitutions based on the available formulas, together with algebraic manipulations. See Example 5.4.6, Example 5.4.7, and Example 5.4.8.
- Using the appropriate substitutions makes it possible to rewrite a polar equation as a rectangular equation, and then graph it in the rectangular plane. See Example 5.4.9, Example 5.4.10, and Example 5.4.11.

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5.5: Polar Coordinates - Graphs

Learning Objectives

- Test polar equations for symmetry.
- Graph polar equations by plotting points.

Kepler's First Law of Planetary Motion argues that the planets move through space in elliptical, periodic orbits about the sun, as shown in Figure 5.5.1. They are in constant motion, so fixing an exact position of any planet is valid only for a moment. In other words, we can fix only a planet's instantaneous position. This is one application of polar coordinates, represented as (r, θ). We interpret r as the distance from the sun and θ as the planet's angular bearing, or its direction from a fixed point on the sun. In this section, we will focus on the polar system and the graphs that are generated directly from polar coordinates.

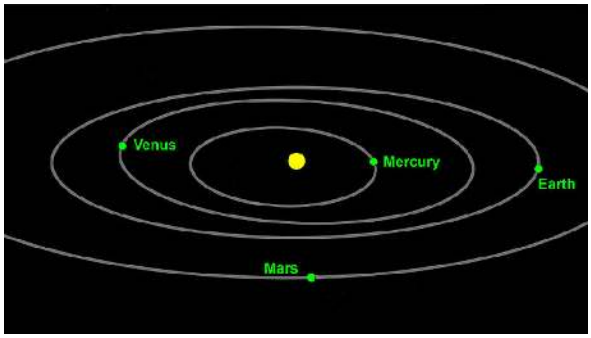


Figure 5.5.1: Planets follow elliptical paths as they orbit around the Sun. (credit: modification of work by NASA/JPL-Caltech)

Testing Polar Equations for Symmetry

Just as a rectangular equation such as $y = x^2$ describes the relationship between x and y on a Cartesian grid, a **polar equation** describes a relationship between r and θ on a polar grid. Recall that the coordinate pair (r, θ) indicates that we move counterclockwise from the polar axis (positive x-axis) by an angle of θ, and extend a ray from the pole (origin) r units in the direction of θ. All points that satisfy the polar equation are on the graph.

Symmetry is a property that helps us recognize and plot the graph of any equation. If an equation has a graph that is symmetric with respect to an axis, it means that if we folded the graph in half over that axis, the portion of the graph on one side would coincide with the portion on the other side. By performing three tests, we will see how to apply the properties of symmetry to polar equations. Further, we will use symmetry (in addition to plotting key points, zeros, and maximums of r) to determine the graph of a polar equation.

In the first test, we consider symmetry with respect to the line $\theta = \frac{\pi}{2}$ (y-axis). We replace (r, θ) with $(-r, -\theta)$ to determine if the new equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin\theta$;

$$\begin{aligned} r &= 2\sin\theta \\ -r &= 2\sin -\theta \quad \text{Replace } (r, \theta) \text{ with } (-r, -\theta). \\ -r &= -2\sin\theta \quad \text{Identity: } \sin(-\theta) = -\sin\theta. \\ r &= 2\sin\theta \quad \text{Multiply both sides by } -1 \end{aligned}$$

This equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$.

In the second test, we consider symmetry with respect to the polar axis (x-axis). We replace (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ to determine equivalency between the tested equation and the original. For example, suppose we are given the equation $r = 1 - 2\cos\theta$.

$$\begin{aligned} r &= 1 - 2\cos\theta \\ r &= 1 - 2\cos(-\theta) \quad \text{Replace } (r, \theta) \text{ with } (r, -\theta). \\ r &= 1 - 2\cos\theta \quad \text{Even/Odd identity} \end{aligned}$$

The graph of this equation exhibits symmetry with respect to the polar axis.

In the third test, we consider symmetry with respect to the pole (origin). We replace (r, θ) with $(-r, \theta)$ to determine if the tested equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin(3\theta)$.

$$\begin{aligned} r &= 2\sin(3\theta) \\ -r &= 2\sin(3\theta) \end{aligned}$$

The equation has failed the symmetry test, but that does not mean that it is not symmetric with respect to the pole. Passing one or more of the symmetry tests verifies that symmetry will be exhibited in a graph. However, failing the symmetry tests does not necessarily indicate that a graph will not be symmetric about the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole. In these instances, we can confirm that symmetry exists by plotting reflecting points across the apparent axis of symmetry or the pole. Testing for symmetry is a technique that simplifies the graphing of polar equations, but its application is not perfect.

Note: SYMMETRY TESTS

A polar equation describes a curve on the polar grid. The graph of a polar equation can be evaluated for three types of symmetry, as shown in Figure 5.5.2.

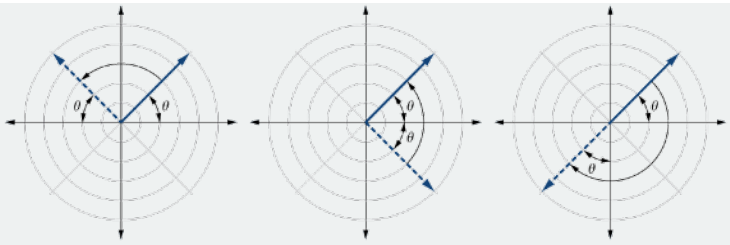


Figure 5.5.2: (a) A graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$ (y-axis) if replacing (r, θ) with $(-r, -\theta)$ yields an equivalent equation. (b) A graph is symmetric with respect to the polar axis (x-axis) if replacing (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ yields an equivalent equation. (c) A graph is symmetric with respect to the pole (origin) if replacing (r, θ) with $(-r, \theta)$ yields an equivalent equation.

How to: Given a polar equation, test for symmetry.

1. Substitute the appropriate combination of components for (r, θ): $(-r, -\theta)$ for $\theta = \frac{\pi}{2}$ symmetry; $(r, -\theta)$ for polar axis symmetry; and $(-r, \theta)$ for symmetry with respect to the pole.
2. If the resulting equations are equivalent in one or more of the tests, the graph produces the expected symmetry.

Example 5.5.1: Testing a Polar Equation for Symmetry

Test the equation $r = 2\sin\theta$ for symmetry.

Solution

Test for each of the three types of symmetry.

1) Replacing (r, θ) with $(-r, -\theta)$ yields the same result. Thus, the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.	$-r = 2\sin(-\theta)$ $-r = -2\sin\theta$ Even-odd identity $r = 2\sin\theta$ Multiply by -1 Passed
2) Replacing θ with $-\theta$ does not yield the same equation. Therefore, the graph fails the test and may or may not be symmetric with respect to the polar axis.	$r = 2\sin(-\theta)$ $r = -2\sin\theta$ Even-odd identity $r = -2\sin\theta \neq 2\sin\theta$ Failed
3) Replacing r with $-r$ changes the equation and fails the test. The graph may or may not be symmetric with respect to the pole.	$-r = 2\sin\theta$ $r = -2\sin\theta \neq 2\sin\theta$ Failed

Analysis

Using a graphing calculator, we can see that the equation $r = 2\sin\theta$ is a circle centered at $(0, 1)$ with radius $r = 1$ and is indeed symmetric to the line $\theta = \frac{\pi}{2}$. We can also see that the graph is not symmetric with the polar axis or the pole. See Figure 5.5.3.

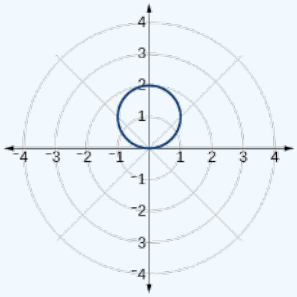


Figure 5.5.3

Exercise 5.5.1

Test the equation for symmetry: $r = -2\cos\theta$.

Answer

The equation fails the symmetry test with respect to the line $\theta = \frac{\pi}{2}$ and with respect to the pole. It passes the polar axis symmetry test.

Graphing Polar Equations by Plotting Points

To graph in the rectangular coordinate system we construct a table of x and y values. To graph in the polar coordinate system we construct a table of θ and r values. We enter values of θ into a polar equation and calculate r. However, using the properties of symmetry and finding key values of θ and r means fewer calculations will be needed.

Finding Zeros and Maxima

To find the zeros of a polar equation, we solve for the values of θ that result in $r = 0$. Recall that, to find the zeros of polynomial functions, we set the equation equal to zero and then solve for x. We use the same process for polar equations. Set $r = 0$, and solve for θ.

For many of the forms we will encounter, the maximum value of a polar equation is found by substituting those values of θ into the equation that result in the maximum value of the trigonometric functions. Consider $r = 5\cos\theta$; the maximum distance between the curve and the pole is 5 units. The maximum value of the cosine function is 1 when $\theta = 0$, so our polar equation is $5\cos\theta$, and the value $\theta = 0$ will yield the maximum $|r|$.

Similarly, the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, and if our polar equation is $r = 5\sin\theta$, the value $\theta = \frac{\pi}{2}$ will yield the maximum $|r|$. We may find additional information by calculating values of r when $\theta = 0$. These points would be polar axis intercepts, which may be helpful in drawing the graph and identifying the curve of a polar equation.

Example 5.5.2: Finding Zeros and Maximum Values for a Polar Equation

Using the equation in Example 5.5.1, find the zeros and maximum $|r|$ and, if necessary, the polar axis intercepts of $r = 2\sin\theta$.

Solution

To find the zeros, set r equal to zero and solve for θ.

$$\begin{aligned} 2\sin\theta &= 0 \\ \sin\theta &= 0 \\ \theta &= \sin^{-1} 0 \\ \theta &= n\pi \qquad \text{where n is an integer} \end{aligned}$$

Substitute any one of the θ values into the equation. We will use 0.

$$\begin{aligned} r &= 2\sin(0) \\ r &= 0 \end{aligned}$$

The points $(0, 0)$ and $(0, \pm n\pi)$ are the zeros of the equation. They all coincide, so only one point is visible on the graph. This point is also the only polar axis intercept.

To find the maximum value of the equation, look at the maximum value of the trigonometric function $\sin\theta$, which occurs when $\theta = \frac{\pi}{2} \pm 2k\pi$ resulting in $\sin\left(\frac{\pi}{2}\right) = 1$. Substitute $\frac{\pi}{2}$ for θ.

$$\begin{aligned} r &= 2\sin\left(\frac{\pi}{2}\right) \\ r &= 2(1) \\ r &= 2 \end{aligned}$$

Analysis

The point $\left(2, \frac{\pi}{2}\right)$ will be the maximum value on the graph. Let's plot a few more points to verify the graph of a circle. See Table 5.5.1 and Figure 5.5.4.

Table 5.5.1

θ	$r = 2\sin\theta$	r
0	$r = 2\sin(0) = 0$	0
$\frac{\pi}{6}$	$r = 2\sin\left(\frac{\pi}{6}\right) = 1$	1
$\frac{\pi}{3}$	$r = 2\sin\left(\frac{\pi}{3}\right) \approx 1.73$	1.73
$\frac{pi}{2}$	$r = 2\sin\left(\frac{\pi}{2}\right) = 2$	2
$\frac{2\pi}{3}$	$r = 2\sin\left(\frac{2\pi}{3}\right) \approx 1.73$	1.73
$\frac{5\pi}{6}$	$r = 2\sin\left(\frac{5\pi}{6}\right) = 1$	1
π	$r = 2\sin(\pi) = 0$	0

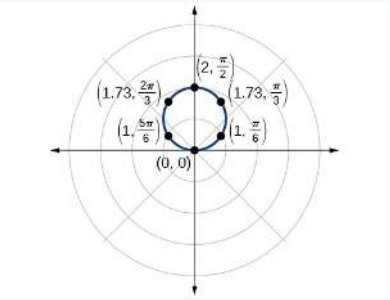


Figure 5.5.4

Exercise 5.5.2

Without converting to Cartesian coordinates, test the given equation for symmetry and find the zeros and maximum values of $|r|$: $r = 3\cos\theta$.

Answer

Tests will reveal symmetry about the polar axis. The zero is $\left(0, \frac{\pi}{2}\right)$, and the maximum value is $(3, 0)$.

Investigating Circles

Now we have seen the equation of a circle in the polar coordinate system. In the last two examples, the same equation was used to illustrate the properties of symmetry and demonstrate how to find the zeros, maximum values, and plotted points that produced the graphs. However, the circle is only one of many shapes in the set of polar curves.

There are five classic polar curves**: cardioids**, **limaçons, lemniscates, rose curves**, and **Archimedes' spirals**. We will briefly touch on the polar formulas for the circle before moving on to the classic curves and their variations.

FORMULAS FOR THE EQUATION OF A CIRCLE

Some of the formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$, whereа a is the diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is $\frac{|a|}{2}$, or one-half the diameter. For $r = a\cos\theta$, the center is $\left(\frac{a}{2}, 0\right)$. For $r = a\sin\theta$, the center is $\left(\frac{a}{2}, \pi\right)$. Figure 5.5.5 shows the graphs of

these four circles.

0. The center is at (a/2,0). Second is r=acos(theta), a<0. The center is at (a/2,0). Third is r=asin(theta), a>0. The center is at (a/2, pi). Fourth is r=asin(theta), a<0. The center is at (a/2, pi)." src="/api/deki/files/7429/CNX_Precalc_Figure_08_04_005new.jpg">

Figure 5.5.5

Example 5.5.3: Sketching the Graph of a Polar Equation for a Circle

Sketch the graph of $r = 4\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph is symmetric about the polar axis. Next, we find the zeros and maximum $|r|$ for $r = 4\cos\theta$. First, set $r = 0$, and solve for θ. Thus, a zero occurs at $\theta = \frac{\pi}{2} \pm k\pi$. A key point to plot is $\left(0, \frac{\pi}{2}\right)$.

To find the maximum value of r, note that the maximum value of the cosine function is 1 when $\theta = 0 \pm 2k\pi$. Substitute $\theta = 0$ into the equation:

$$\begin{aligned} r &= 4\cos\theta \\ r &= 4\cos(0) \\ r &= 4(1) \\ &= 4 \end{aligned}$$

The maximum value of the equation is 4. A key point to plot is $(4, 0)$.

As $r = 4\cos\theta$ is symmetric with respect to the polar axis, we only need to calculate r-values for θ over the interval $[0, \pi]$. Points in the upper quadrant can then be reflected to the lower quadrant. Make a table of values similar to Table 5.5.2. The graph is shown in Figure 5.5.6.

Table 5.5.2

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	3.46	2.83	2	0	-2	-2.83	-3.46	4

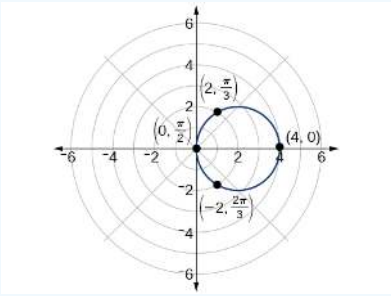


Figure 5.5.6

Investigating Cardioids

While translating from polar coordinates to Cartesian coordinates may seem simpler in some instances, graphing the classic curves is actually less complicated in the polar system. The next curve is called a cardioid, as it resembles a heart. This shape is often included with the family of curves called limaçons, but here we will discuss the cardioid on its own.

FORMULAS FOR A CARDIOID

The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. The cardioid graph passes through the pole, as we can see in Figure 5.5.7.

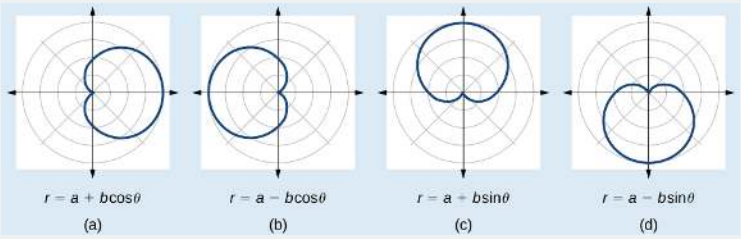


Figure 5.5.7

How to: Given the polar equation of a cardioid, sketch its graph

1. Check equation for the three types of symmetry.
2. Find the zeros. Set $r = 0$.
3. Find the maximum value of the equation according to the maximum value of the trigonometric expression.
4. Make a table of values for r and θ.
5. Plot the points and sketch the graph.

Example 5.5.4: Sketching the Graph of a Cardioid

Sketch the graph of $r = 2 + 2\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph of this equation will be symmetric about the polar axis. Next, we find the zeros and maximums. Setting $r = 0$, we have $\theta = \pi + 2k\pi$. The zero of the equation is located at $(0, \pi)$. The graph passes through this point.

The maximum value of $r = 2 + 2\cos\theta$ occurs when $\cos\theta$ is a maximum, which is when $\cos\theta = 1$ or when $\theta = 0$. Substitute $\theta = 0$ into the equation, and solve for r.

$$\begin{aligned} r &= 2 + 2\cos(0) \\ r &= 2 + 2(1) \\ &= 4 \end{aligned}$$

The point $(4, 0)$ is the maximum value on the graph.

We found that the polar equation is symmetric with respect to the polar axis, but as it extends to all four quadrants, we need to plot values over the interval $[0, \pi]$. The upper portion of the graph is then reflected over the polar axis. Next, we make a table of values, as in Table 5.5.3, and then we plot the points and draw the graph. See Figure 5.5.8.

Table 5.5.3

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	3.41	2	1	0

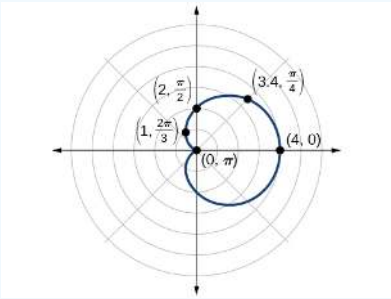


Figure 5.5.8

Investigating Limaçons

The word *limaçon* is Old French for "snail," a name that describes the shape of the graph. As mentioned earlier, the cardioid is a member of the limaçon family, and we can see the similarities in the graphs. The other images in this category include the one-loop limaçon and the two-loop (or inner-loop) limaçon. **One-loop limaçons** are sometimes referred to as dimpled limaçons when $1 < \frac{a}{b} < 2$ and convex limaçons when $\frac{a}{b} \geq 2$.

FORMULAS FOR ONE-LOOP LIMAÇONS

The formulas that produce the graph of a dimpled one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $1 < ab < 2$. All four graphs are shown in Figure 5.5.9.

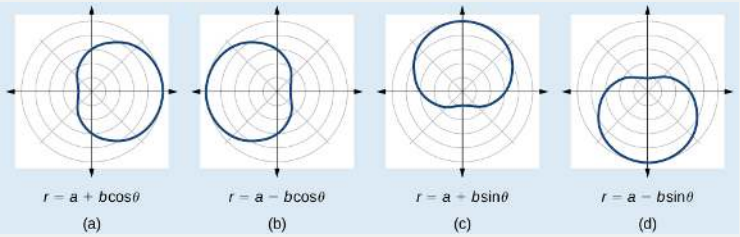


Figure 5.5.9

How to: Given a polar equation for a one-loop limaçon, sketch the graph

1. Test the equation for symmetry. Remember that failing a symmetry test does not mean that the shape will not exhibit symmetry. Often the symmetry may reveal itself when the points are plotted.
2. Find the zeros.
3. Find the maximum values according to the trigonometric expression.
4. Make a table.
5. Plot the points and sketch the graph.

Example 5.5.5: Sketching the Graph of a One-Loop Limaçon

Graph the equation $r = 4 - 3\sin\theta$.

Solution

First, testing the equation for symmetry, we find that it fails all three symmetry tests, meaning that the graph may or may not exhibit symmetry, so we cannot use the symmetry to help us graph it. However, this equation has a graph that clearly displays symmetry with respect to the line $\theta = \frac{\pi}{2}$, yet it fails all the three symmetry tests. A graphing calculator will immediately illustrate the graph's reflective quality.

Next, we find the zeros and maximum, and plot the reflecting points to verify any symmetry. Setting $r = 0$ results in θ being undefined. What does this mean? How could θ be undefined? The angle θ is undefined for any value of $\sin\theta > 1$. Therefore, θ is undefined because there is no value of θ for which $\sin\theta > 1$. Consequently, the graph does not pass through the pole. Perhaps the graph does cross the polar axis, but not at the pole. We can investigate other intercepts by calculating r when $\theta = 0$.

$$\begin{aligned} r(0) &= 4 - 3\sin(0) \\ r &= 4 - 3 \cdot 0 \\ &= 4 \end{aligned}$$

So, there is at least one polar axis intercept at $(4, 0)$.

Next, as the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, we will substitute $\theta = \frac{\pi}{2}$ into the equation and solve for r. Thus, $r = 1$.

Make a table of the coordinates similar to Table 5.5.4.

Table 5.5.4

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	4	2.5	1.4	1	1.4	2.5	4	5.5	6.6	7	6.6	5.5	4

The graph is shown in Figure 5.5.10.

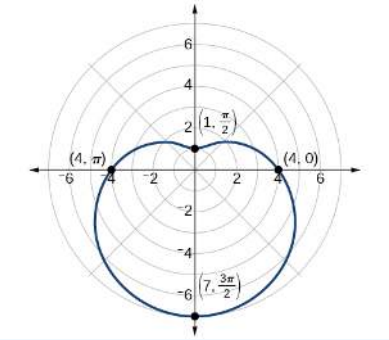


Figure 5.5.10: One-loop limaçon

Analysis

This is an example of a curve for which making a table of values is critical to producing an accurate graph. The symmetry tests fail; the zero is undefined. While it may be apparent that an equation involving $\sin\theta$ is likely symmetric with respect to the line $\theta = \frac{\pi}{2}$, evaluating more points helps to verify that the graph is correct.

Exercise 5.5.3

Sketch the graph of $r = 3 - 2\cos\theta$.

Answer

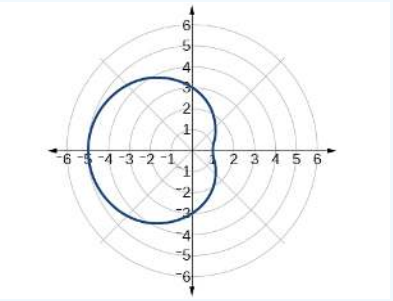


Figure 5.5.11

Another type of limaçon, the **inner-loop limaçon**, is named for the loop formed inside the general limaçon shape. It was discovered by the German artist Albrecht Dürer(1471-1528), who revealed a method for drawing the inner-loop limaçon in his 1525 book *Underweysung der Messing*. A century later, the father of mathematician Blaise Pascal, Étienne Pascal(1588-1651), rediscovered it.

FORMULAS FOR INNER-LOOP LIMAÇONS

The formulas that generate the inner-loop limaçons are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $a < b$. The graph of the inner-loop limaçon passes through the pole twice: once for the outer loop, and once for the inner loop. See Figure 10.5.12 for the graphs.

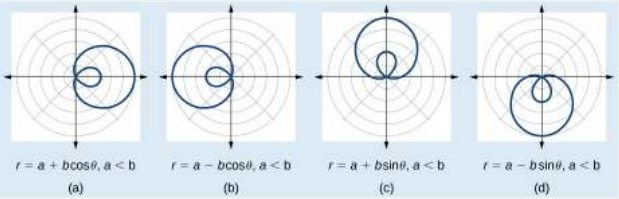


Figure 5.5.12

Example 5.5.6: Sketching the Graph of an Inner-Loop Limaçon

Sketch the graph of $r = 2 + 5\cos\theta$.

Solution

Testing for symmetry, we find that the graph of the equation is symmetric about the polar axis. Next, finding the zeros reveals that when $r = 0$, $\theta = 1.98$. The maximum $|r|$ is found when $\cos\theta = 1$ or when $\theta = 0$. Thus, the maximum is found at the point $(7, 0)$.

Even though we have found symmetry, the zero, and the maximum, plotting more points will help to define the shape, and then a pattern will emerge. See Table 5.5.5.

Table 5.5.5

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

r	7	6.3	4.5	2	-0.5	-2.3	-3	-2.3	-0.5	2	4.5	6.3	7

As expected, the values begin to repeat after $\theta = \pi$. The graph is shown in Figure 5.5.13.

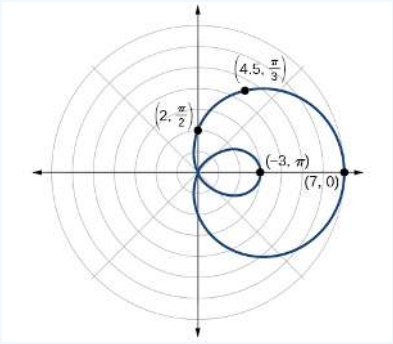


Figure 5.5.13: Inner-loop limaçon

Investigating Lemniscates

The lemniscate is a polar curve resembling the infinity symbol ∞ or a figure 8. Centered at the pole, a lemniscate is symmetrical by definition.

FORMULAS FOR LEMNISCATES

The formulas that generate the graph of a lemniscate are given by $r^2 = a^2 \cos 2\theta$ and $r^2 = a^2 \sin 2\theta$ where $a \neq 0$. The formula $r^2 = a^2 \sin 2\theta$ is symmetric with respect to the pole. The formula $r^2 = a^2 \cos 2\theta$ is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis. See Figure 5.5.14 for the graphs.

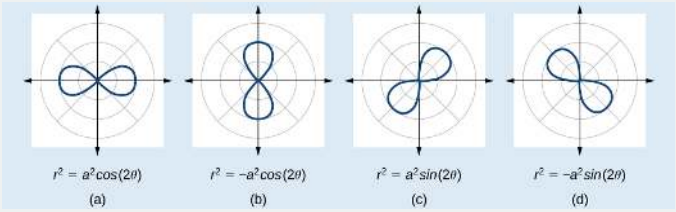


Figure 5.5.14

Example 5.5.7: Sketching the Graph of a Lemniscate

Sketch the graph of $r^2 = 4 \cos 2\theta$.

Solution

The equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, and the pole.

Let's find the zeros. It should be routine by now, but we will approach this equation a little differently by making the substitution $u = 2\theta$.

$$\begin{aligned} 0 &= 4\cos 2\theta \\ 0 &= 4\cos u \\ 0 &= \cos u \\ \cos^{-1} 0 &= \frac{\pi}{2} \\ u &= \frac{\pi}{2} \qquad \text{Substitute } 2\theta \text{ back in for } u. \\ 2\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{4} \end{aligned}$$

So, the point $\left(0, \frac{\pi}{4}\right)$ is a zero of the equation.

Now let's find the maximum value. Since the maximum of $\cos u = 1$ when $u = 0$, the maximum $\cos 2\theta = 1$ when $2\theta = 0$. Thus,

$$\begin{aligned} r^2 &= 4\cos(0) \\ r^2 &= 4(1) \\ r^2 &= 4 \\ r &= \pm 4 \\ &= 2 \end{aligned}$$

We have a maximum at $(2, 0)$. Since this graph is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis, we only need to plot points in the first quadrant.

Make a table similar to Table 5.5.6.

Table 5.5.6

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$\sqrt{2}$	0

Plot the points on the graph, such as the one shown in Figure 5.5.15.

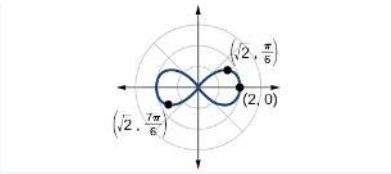


Figure 5.5.15: Lemniscate

Analysis

Making a substitution such as $u = 2\theta$ is a common practice in mathematics because it can make calculations simpler. However, we must not forget to replace the substitution term with the original term at the end, and then solve for the unknown.

Some of the points on this graph may not show up using the Trace function on the TI-84 graphing calculator, and the calculator table may show an error for these same points of r. This is because there are no real square roots for these values of θ. In other words, the corresponding r-values of $\sqrt{4\cos(2\theta)}$ are complex numbers because there is a negative number under the radical.

Investigating Rose Curves

The next type of polar equation produces a petal-like shape called a rose curve. Although the graphs look complex, a simple polar equation generates the pattern.

ROSE CURVES

The formulas that generate the graph of a **rose curve** are given by $r = a\cos n\theta$ and $r = a\sin n\theta$ where $a \neq 0$. If n is even, the curve has $2n$ petals. If n is odd, the curve has n petals. See Figure 5.5.16

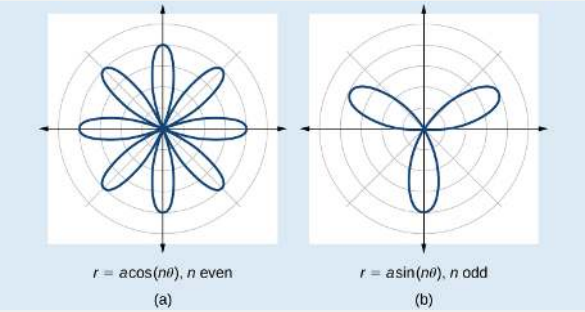


Figure 5.5.16

Example 5.5.8: Sketching the Graph of a Rose Curve (n Even)

Sketch the graph of $r = 2\cos 4\theta$.

Solution

Testing for symmetry, we find again that the symmetry tests do not tell the whole story. The graph is not only symmetric with respect to the polar axis, but also with respect to the line $\theta = \frac{\pi}{2}$ and the pole.

Now we will find the zeros. First make the substitution $u = 4\theta$.

$$\begin{aligned} 0 &= 2\cos 4\theta \\ 0 &= \cos 4\theta \\ 0 &= \cos u \\ \cos^{-1} 0 &= u \\ u &= \frac{\pi}{2} \\ 4\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{8} \end{aligned}$$

The zero is $\theta = \frac{\pi}{8}$. The point $\left(0, \frac{\pi}{8}\right)$ is on the curve.

Next, we find the maximum $|r|$. We know that the maximum value of $\cos u = 1$ when $\theta = 0$. Thus,

$$\begin{aligned} r &= 2\cos(4 \cdot 0) \\ r &= 2\cos(0) \\ r &= 2(1) \\ &= 2 \end{aligned}$$

The point $(2, 0)$ is on the curve.

The graph of the rose curve has unique properties, which are revealed in Table 5.5.7.

Table 5.5.7

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$5\pi 8$	$3\pi 4$
r	2	0	-2	0	2	0	-2

As $r = 0$ when $\theta = \frac{\pi}{8}$, it makes sense to divide values in the table by $\frac{\pi}{8}$ units. A definite pattern emerges. Look at the range of r-values: $2, 0, -2, 0$ and so on. This represents the development of the curve one petal at a time. Starting at $r = 0$, each petal extends out a distance of $r = 2$, and then turns back to zero $2n$ times for a total of eight petals. See the graph in Figure 5.5.17.

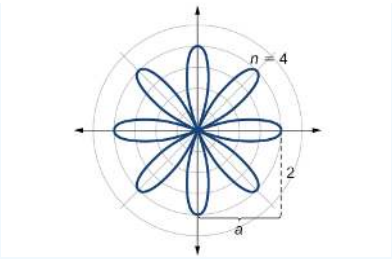


Figure 5.5.17: Rose curve, n even

Analysis

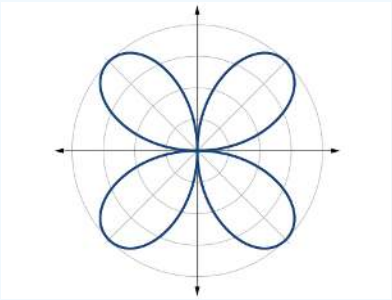
When these curves are drawn, it is best to plot the points in order, as in the Table 5.5.7. This allows us to see how the graph hits a maximum (the tip of a petal), loops back crossing the pole, hits the opposite maximum, and loops back to the pole. The action is continuous until all the petals are drawn.

? Exercise 5.5.4

Sketch the graph of $r = 4\sin(2\theta)$.

Answer

The graph is a rose curve, n even


Figure 5.5.18

✓ Example 5.5.9: Sketching the Graph of a Rose Curve (n Odd)

Sketch the graph of $r = 2\sin(5\theta)$.

Solution

The graph of the equation shows symmetry with respect to the line $\theta = \frac{\pi}{2}$. Next, find the zeros and maximum. We will want to make the substitution $u = 5\theta$.

$$\begin{aligned} 0 &= 2\sin(5\theta) \\ 0 &= \sin u \\ \sin^{-1} 0 &= 0 \\ u &= 0 \\ 5\theta &= 0 \\ \theta &= 0 \end{aligned}$$

The maximum value is calculated at the angle where $\sin\theta$ is a maximum. Therefore,

$$\begin{aligned} r &= 2\sin\left(5 \cdot \frac{\pi}{2}\right) \\ r &= 2(1) \\ &= 2 \end{aligned}$$

Thus, the maximum value of the polar equation is 2. This is the length of each petal. As the curve for n odd yields the same number of petals as n, there will be five petals on the graph. See Figure 5.5.19.

Create a table of values similar to Table 5.5.8.

Table 5.5.8

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	\(0\0	1	**−1.73**	2	**−1.73**	1	0

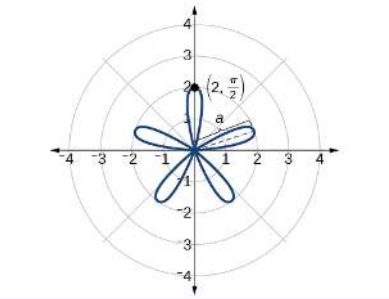


Figure 5.5.19: Rose curve, n odd

Exercise 5.5.5

Sketch the graph of $r = 3\cos(3\theta)$.

Answer

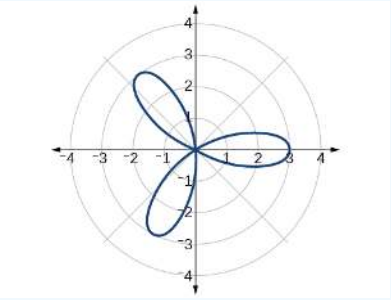


Figure 5.5.20

Rose curve, n odd

Investigating the Archimedes' Spiral

The final polar equation we will discuss is the Archimedes' spiral, named for its discoverer, the Greek mathematician Archimedes (c. 287 BCE - c. 212 BCE), who is credited with numerous discoveries in the fields of geometry and mechanics.

ARCHIMEDES' SPIRAL

The formula that generates the graph of the Archimedes' spiral is given by $r = \theta$ for $\theta \geq 0$. As θ increases, r increases at a constant rate in an ever-widening, never-ending, spiraling path. See Figure 5.5.21.

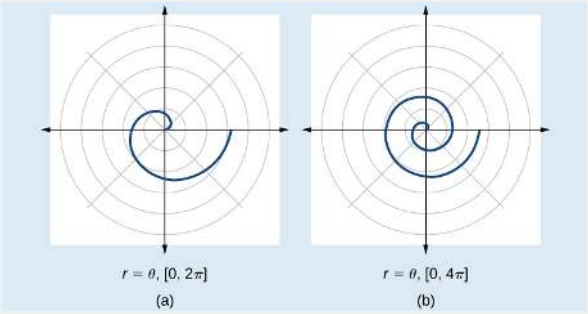


Figure 5.5.21

How to: Given an Archimedes' spiral over $[0, 2\pi]$, sketch the graph

1. Make a table of values for r and θ over the given domain.
2. Plot the points and sketch the graph.

Example 5.5.10: Sketching the Graph of an Archimedes' Spiral

Sketch the graph of $r = \theta$ over $[0, 2\pi]$.

Solution

As r is equal to θ, the plot of the Archimedes' spiral begins at the pole at the point $(0, 0)$. While the graph hints of symmetry, there is no formal symmetry with regard to passing the symmetry tests. Further, there is no maximum value, unless the domain is restricted.

Create a table such as Table 5.5.9.

Table 5.5.9

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0.785	1.57	3.14	4.71	5.50	6.28

Notice that the r-values are just the decimal form of the angle measured in radians. We can see them on a graph in Figure 5.5.22

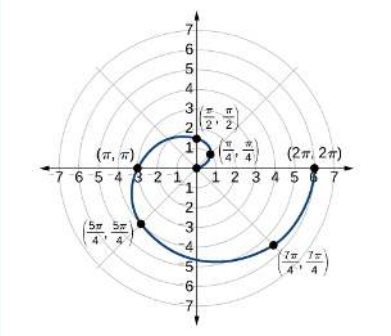


Figure 5.5.22: Archimedes' spiral

Analysis

The domain of this polar curve is $[0, 2\pi]$. In general, however, the domain of this function is $(-\infty, \infty)$. Graphing the equation of the Archimedes' spiral is rather simple, although the image makes it seem like it would be complex.

Exercise 5.5.6

Sketch the graph of $r = -\theta$ over the interval $[0, 4\pi]$.

Answer

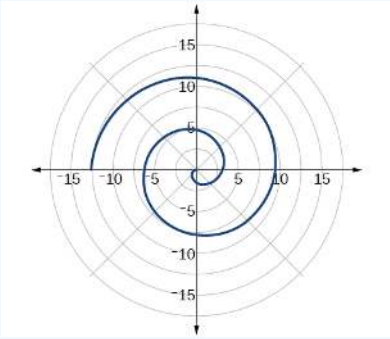


Figure 5.5.23

Summary of Curves

We have explored a number of seemingly complex polar curves in this section. Figure 5.5.24and Figure 5.5.25summarize the graphs and equations for each of these curves.

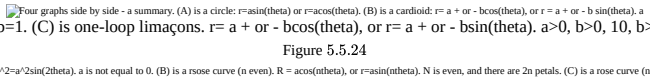
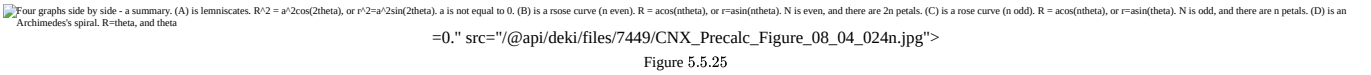

Four graphs side by side - a summary. (A) is a circle: r=asin(theta) or r=acos(theta). (B) is a cardioid: r= a + or - bcos(theta), or r = a + or - b sin(theta). a
0, b>0, a/b=1. (C) is one-loop limaçons. r= a + or - bcos(theta), or r= a + or - bsin(theta). a>0, b>0, 10, b>0, a<b.">
Figure 5.5.24


Figure 5.5.25

Media

Access these online resources for additional instruction and practice with graphs of polar coordinates.

- Graphing Polar Equations Part 1
- Graphing Polar Equations Part 2
- Animation: The Graphs of Polar Equations
- Graphing Polar Equations on the TI-84

Key Concepts

- It is easier to graph polar equations if we can test the equations for symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole.
- There are three symmetry tests that indicate whether the graph of a polar equation will exhibit symmetry. If an equation fails a symmetry test, the graph may or may not exhibit symmetry. See Example 5.5.1.
- Polar equations may be graphed by making a table of values for θ and r.
- The maximum value of a polar equation is found by substituting the value θ that leads to the maximum value of the trigonometric expression.
- The zeros of a polar equation are found by setting $r = 0$ and solving for θ. See Example 5.5.2.
- Some formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$. See Example 5.5.3.
- The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, for $a > 0$, $b > 0$, and $ab = 1$. See Example 5.5.4.
- The formulas that produce the graphs of a one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $1 < ab < 2$. See Example 5.5.5.
- The formulas that produce the graphs of an inner-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $a > 0$, $b > 0$, and $a < b$. See Example 5.5.6.
- The formulas that produce the graphs of a lemniscates are given by $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$, where $a \neq 0$.See Example 5.5.7.
- The formulas that produce the graphs of rose curves are given by $r = a\cos n\theta$ and $r = a\sin n\theta$, where $a \neq 0$; if n is even, there are $2n$ petals, and if n is odd, there aren n petals. See Example 5.5.8 and Example 5.5.9.
- The formula that produces the graph of an Archimedes' spiral is given by $r = \theta$, $\theta \geq 0$. See Example 5.5.10.

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5.6: Polar Form of Complex Numbers

Learning Objectives

- Plot complex numbers in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

"God made the integers; all else is the work of man." This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

We first encountered complex numbers in the section on Complex Numbers. In this section, we will focus on the mechanics of working with complex numbers: translation of complex numbers from polar form to rectangular form and vice versa, interpretation of complex numbers in the scheme of applications, and application of De Moivre's Theorem.

Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a+bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, a, and the vertical axis represents the imaginary part of the number, bi.

How to: Given a complex number $a+bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary axis.*
2. Plot the point in the complex plane by moving a units in the horizontal direction and b units in the vertical direction.

Example 5.6.1: Plotting a Complex Number in the Complex Plane

Plot the complex number $2-3i$ in the complex plane.

Solution

From the origin, move two units in the positive horizontal direction and three units in the negative vertical direction. See Figure 5.6.1.

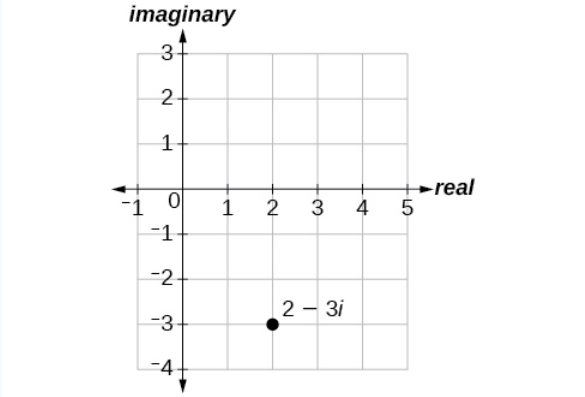


Figure 5.6.1

Exercise 5.6.1

Plot the point $1+5i$ in the complex plane.

Answer

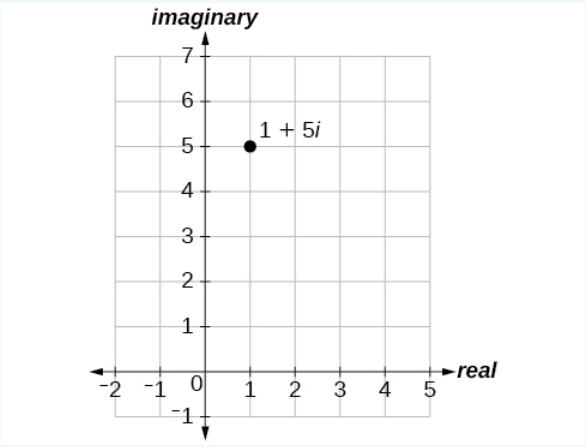


Figure 5.6.2

Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z=2+4i$, in Figure 5.6.3, shows $|z|$.

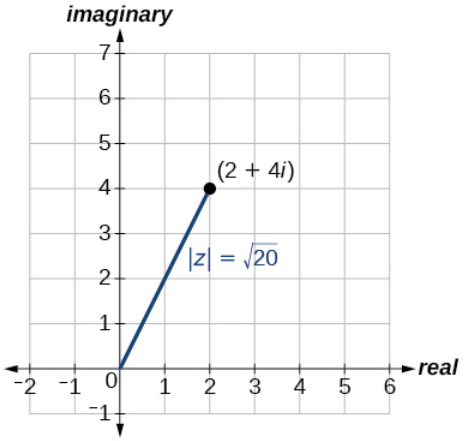


Figure 5.6.3

ABSOLUTE VALUE OF A COMPLEX NUMBER

Given $z=x+yi$, a complex number, the absolute value of z is defined as

$$|z|=\sqrt{x^2+y^2} \tag{5.6.1}$$

It is the distance from the origin to the point (x,y).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, $(0,0)$.

Example 5.6.2: Finding the Absolute Value of a Complex Number with a Radical

Find the absolute value of $z = \sqrt{5} - i$.

Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(\sqrt{5})^2 + (-1)^2}$$
$$|z| = \sqrt{5 + 1}$$
$$|z| = \sqrt{6}$$

See Figure 5.6.4.

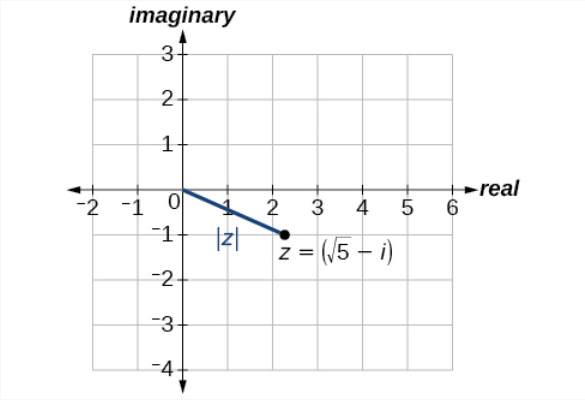


Figure 5.6.4

Exercise 5.6.2

Find the absolute value of the complex number $z = 12 - 5i$.

Answer

13

Example 5.6.3: Finding the Absolute Value of a Complex Number

Given $z = 3 - 4i$, find $|z|$.

Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(3)^2 + (-4)^2}$$
$$|z| = \sqrt{9 + 16}$$
$$|z| = \sqrt{25}$$
$$|z| = 5$$

The absolute value z is 5. See Figure 5.6.5.

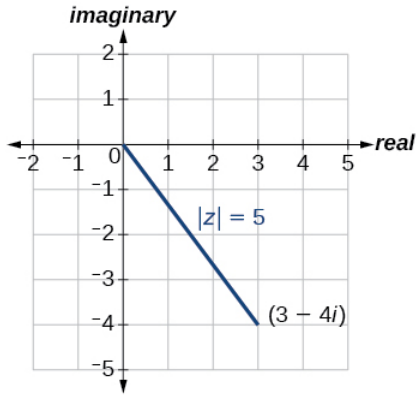


Figure 5.6.5

Exercise 5.6.3

Given $z = 1 - 7i$, find $|z|$.

Answer

$|z| = \sqrt{50} = 5\sqrt{2}$

Writing Complex Numbers in Polar Form

The polar form of a complex number expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

We review these relationships in Figure 5.6.6.

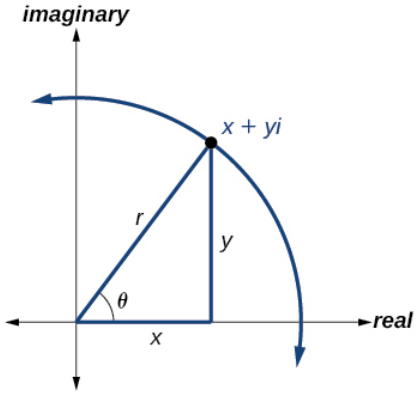


Figure 5.6.6

We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point (x, y). The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$\begin{aligned} z &= x + yi \\ z &= r\cos\theta + (r\sin\theta)i \\ z &= r(\cos\theta + i\sin\theta) \end{aligned}$$

POLAR FORM OF A COMPLEX NUMBER

Writing a complex number in polar form involves the following conversion formulas:

$$x = r\cos\theta \tag{5.6.2}$$
$$y = r\sin\theta \tag{5.6.3}$$
$$r = \sqrt{x^2+y^2} \tag{5.6.4}$$

Making a direct substitution, we have

$$z = x + yi \tag{5.6.5}$$
$$z = (r\cos\theta) + i(r\sin\theta) \tag{5.6.6}$$
$$z = r(\cos\theta + i\sin\theta) \tag{5.6.7}$$

where r is the modulus and θ is the argument. We often use the abbreviation $r\ cis\theta$ to represent $r(\cos\theta + i\sin\theta)$.

Example 5.6.4: Expressing a Complex Number Using Polar Coordinates

Express the complex number $4i$ using polar coordinates.

Solution

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{0^2+4^2}$$
$$r = \sqrt{16}$$
$$r = 4$$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \dfrac{\pi}{2}$. In polar coordinates, the complex number $z = 0 + 4i$ can be written as $z = 4\left(\cos\left(\dfrac{\pi}{2}\right) + i\sin\left(\dfrac{\pi}{2}\right)\right)$ or $4\ cis\left(\dfrac{\pi}{2}\right)$. See Figure 5.6.7.

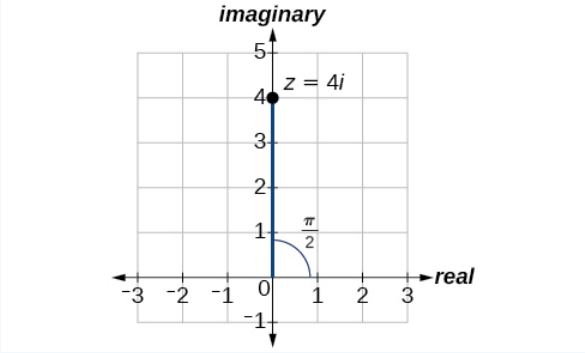


Figure 5.6.7

Exercise 5.6.4

Express $z = 3i$ as $r\ cis\theta$ in polar form.

Answer

$$z = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

Example 5.6.5: Finding the Polar Form of a Complex Number

Find the polar form of $-4+4i$.

Solution

First, find the value of r.

$$r=\sqrt{x^2+y^2}$$
$$r=\sqrt{(-4)^2+(4^2)}$$
$$r=\sqrt{32}$$
$$r=4\sqrt{2}$$

Find the angle θ using the formula:

$$\cos\theta=\frac{x}{r}$$
$$\cos\theta=\frac{-4}{4\sqrt{2}}$$
$$\cos\theta=-\frac{1}{\sqrt{2}}$$
$$\theta=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$
$$=\frac{3\pi}{4}$$

Thus, the solution is $4\sqrt{2}\,cis\left(\frac{3\pi}{4}\right)$.

Exercise 5.6.5

Write $z=\sqrt{3}+i$ in polar form.

Answer

$$z=2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given $z=r(\cos\theta+i\sin\theta)$, first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$. Then, multiply through by r.

Example 5.6.6A: Converting from Polar to Rectangular Form

Convert the polar form of the given complex number to rectangular form:

$$z=12\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

Solution

We begin by evaluating the trigonometric expressions.

$$\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\text{ and }\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

After substitution, the complex number is

$$z=12\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

We apply the distributive property:

$$\begin{aligned} z &= 12\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) \\ &= (12)\frac{\sqrt{3}}{2}+(12)\frac{1}{2}i \\ &= 6\sqrt{3}+6i \end{aligned}$$

The rectangular form of the given point in complex form is $6\sqrt{3}+6i$.

Example 5.6.6B: Finding the Rectangular Form of a Complex Number

Find the rectangular form of the complex number given $r=13$ and $\tan\theta=\frac{5}{12}$.

Solution

If $\tan\theta=\frac{5}{12}$, and $\tan\theta=\frac{y}{x}$, we first determine $r=\sqrt{x^2+y^2}=\sqrt{122+52}=13$. We then find $\cos\theta=\frac{x}{r}$ and $\sin\theta=\frac{y}{r}$.

$$\begin{aligned} z &= 13\left(\cos\theta+i\sin\theta\right) \\ &= 13\left(\frac{12}{13}+\frac{5}{13}i\right) \\ &= 12+5i \end{aligned}$$

The rectangular form of the given number in complex form is $12+5i$.

Exercise 5.6.6

Convert the complex number to rectangular form:

$$z=4\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$$

Answer

$z=2\sqrt{3}-2i$

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham **de Moivre** (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

PRODUCTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)] \tag{5.6.8}$$

$$z_1z_2 = r_1r_2\ cis(\theta_1+\theta_2) \tag{5.6.9}$$

Notice that the product calls for multiplying the moduli and adding the angles.

Example 5.6.7: Finding the Product of Two Complex Numbers in Polar Form

Find the product of $z_1 z_2$, given $z_1 = 4(\cos(80^\circ) + i\sin(80^\circ))$ and $z_2 = 2(\cos(145^\circ) + i\sin(145^\circ))$.

Solution

Follow the formula

$$\begin{aligned} z_1 z_2 &= 4 \cdot 2[\cos(80^\circ + 145^\circ) + i\sin(80^\circ + 145^\circ)] \\ z_1 z_2 &= 8[\cos(225^\circ) + i\sin(225^\circ)] \\ z_1 z_2 &= 8\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right] \\ z_1 z_2 &= 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ z_1 z_2 &= -4\sqrt{2} - 4i\sqrt{2} \end{aligned}$$

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

QUOTIENTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],\ z_2 \neq 0 \tag{5.6.10}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,cis(\theta_1 - \theta_2),\ z_2 \neq 0 \tag{5.6.11}$$

Notice that the moduli are divided, and the angles are subtracted.

How to: Given two complex numbers in polar form, find the quotient

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r(\cos\theta + i\sin\theta)$. Replace r with $\frac{r_1}{r_2}$, and replace θ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by r.

Example 5.6.8: Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2(\cos(213^\circ) + i\sin(213^\circ))$ and $z_2 = 4(\cos(33^\circ) + i\sin(33^\circ))$.

Solution

Using the formula, we have

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2}{4}[\cos(213^\circ - 33^\circ) + i\sin(213^\circ - 33^\circ)] \\ \frac{z_1}{z_2} &= \frac{1}{2}[\cos(180^\circ) + i\sin(180^\circ)] \\ \frac{z_1}{z_2} &= \frac{1}{2}[-1 + 0i] \\ \frac{z_1}{z_2} &= -\frac{1}{2} + 0i \\ \frac{z_1}{z_2} &= -\frac{1}{2} \end{aligned}$$

Exercise 5.6.8

Find the product and the quotient of $z_1 = 2\sqrt{3}(\cos(150^\circ) + i\sin(150^\circ))$ and $z_2 = 2(\cos(30^\circ) + i\sin(30^\circ))$.

Answer

$$z_1 z_2 = -4\sqrt{3}\,;\ \frac{z_1}{z_2} = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$$

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using **De Moivre's Theorem**. It states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the argument by n. It is the standard method used in modern mathematics.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] \tag{5.6.12}$$
$$z^n = r^n\, cis(n\theta) \tag{5.6.13}$$

where n is a positive integer.

Example 5.6.9: Evaluating an Expression Using De Moivre's Theorem

Evaluate the expression $(1+i)^5$ using De Moivre's Theorem.

Solution

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1+i)$ in polar form. Let us find r.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(1)^2 + (1)^2}$$
$$r = \sqrt{2}$$

Then we find θ. Using the formula $\tan\theta = \frac{y}{x}$ gives

$$\tan\theta = \frac{1}{1}$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

Use De Moivre's Theorem to evaluate the expression.

$$(a+bi)^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$
$$(1+i)^5 = (\sqrt{2})^5\left[\cos\left(5\cdot\frac{\pi}{4}\right) + i\sin\left(5\cdot\frac{\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$(1+i)^5 = -4 - 4i$$

Finding Roots of Complex Numbers in Polar Form

To find the nth root of a complex number in polar form, we use the nth Root Theorem or De Moivre's Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding n^{th} roots of complex numbers in polar form.

THE N^{TH} ROOT THEOREM

To find the nth root of a complex number in polar form, use the formula given as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right] \tag{5.6.14}$$

where $k = 0, 1, 2, 3, \ldots, n-1$. We add $\dfrac{2k\pi}{n}$ to $\dfrac{\theta}{n}$ in order to obtain the periodic roots.

th Root of a Complex Number

Evaluate the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)$.

Solution

We have

$$z^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)\right]$$
$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)\right]$$

There will be three roots: $k = 0, 1, 2$. When $k = 0$, we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)$$

When $k = 1$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)\right] \qquad \text{Add } \frac{2(1)\pi}{3} \text{ to each angle.}$$
$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)\right)$$

When $k = 2$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)\right] \qquad \text{Add } \frac{2(2)\pi}{3} \text{ to each angle.}$$
$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{14\pi}{9}\right)+i\sin\left(\frac{14\pi}{9}\right)\right)$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k = 1$, the angle simplification is

$$\frac{\frac{2\pi}{3}}{3}+\frac{2(1)\pi}{3} = \frac{2\pi}{3}\left(\frac{1}{3}\right)+\frac{2(1)\pi}{3}\left(\frac{3}{3}\right)$$
$$= \frac{2\pi}{9}+\frac{6\pi}{9}$$
$$= \frac{8\pi}{9}$$

Exercise 5.6.10

Find the four fourth roots of $16(\cos(120^\circ)+i\sin(120^\circ))$.

Answer

$z_0 = 2(\cos(30^\circ)+i\sin(30^\circ))$

$z_1 = 2(\cos(120^\circ)+i\sin(120^\circ))$

$z_2 = 2(\cos(210^\circ)+i\sin(210^\circ))$

$z_3 = 2(\cos(300^\circ)+i\sin(300^\circ))$

Media

Access these online resources for additional instruction and practice with polar forms of complex numbers.

- The Product and Quotient of Complex Numbers in Trigonometric Form
- De Moivre's Theorem

Key Concepts

- Complex numbers in the form $a+bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the x-axis as the real axis and the y-axis as the *imaginary* axis. See Example 5.6.1.
- The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2+b^2}$. See Example 5.6.2 and Example 5.6.3.
- To write complex numbers in polar form, we use the formulas $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2+y^2}$. Then, $z = r(\cos\theta+i\sin\theta)$. See Example 5.6.4 and Example 5.6.5.
- To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by r. See Example 5.6.6 and Example 5.6.7.
- To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See Example 5.6.8.
- To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See Example 5.6.9.
- To find the power of a complex number z^n, raise r to the power n, and multiply θ by n. See Example 5.6.10.
- Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See Example 5.6.11.

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5.7: Parametric Equations

Learning Objectives

- Parameterize a curve.
- Eliminate the parameter.
- Find a rectangular equation for a curve defined parametrically.
- Find parametric equations for curves defined by rectangular equations.

Consider the path a moon follows as it orbits a planet, which simultaneously rotates around the sun, as seen in Figure 5.7.1. At any moment, the moon is located at a particular spot relative to the planet. But how do we write and solve the equation for the position of the moon when the distance from the planet, the speed of the moon's orbit around the planet, and the speed of rotation around the sun are all unknowns? We can solve only for one variable at a time.

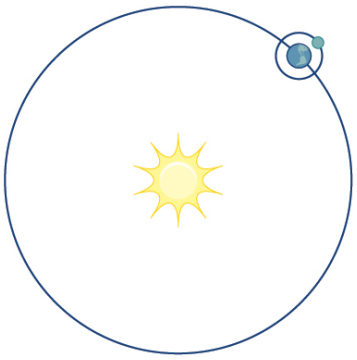


Figure 5.7.1

In this section, we will consider sets of equations given by $x(t)$ and $y(t)$ where t is the independent variable of time. We can use these parametric equations in a number of applications when we are looking for not only a particular position but also the direction of the movement. As we trace out successive values of t, the orientation of the curve becomes clear. This is one of the primary advantages of using parametric equations: we are able to trace the movement of an object along a path according to time. We begin this section with a look at the basic components of parametric equations and what it means to parameterize a curve. Then we will learn how to eliminate the parameter, translate the equations of a curve defined parametrically into rectangular equations, and find the parametric equations for curves defined by rectangular equations.

Parameterizing a Curve

When an object moves along a curve—or curvilinear path—in a given direction and in a given amount of time, the position of the object in the plane is given by the x-coordinate and the y-coordinate. However, both x and y vary over time and so are functions of time. For this reason, we add another variable, the parameter, upon which both x and y are dependent functions. In the example in the section opener, the parameter is time, t. The x position of the moon at time, t, is represented as the function $x(t)$, and the y position of the moon at time, t, is represented as the function $y(t)$. Together, $x(t)$ and $y(t)$ are called parametric equations, and generate an ordered pair $(x(t), y(t))$. Parametric equations primarily describe motion and direction.

When we parameterize a curve, we are translating a single equation in two variables, such as x and y,into an equivalent pair of equations in three variables, x, y, and t. One of the reasons we parameterize a curve is because the parametric equations yield more information: specifically, the direction of the object's motion over time.

When we graph parametric equations, we can observe the individual behaviors of x and of y. There are a number of shapes that cannot be represented in the form $y = f(x)$, meaning that they are not functions. For example, consider the graph of a circle, given as $r^2 = x^2 + y^2$. Solving for y gives $y = \pm\sqrt{r^2 - x^2}$, or two equations: $y_1 = \sqrt{r^2 - x^2}$ and $y_2 = -\sqrt{r^2 - x^2}$. If we graph y_1 and y_2 together, the graph will not pass the vertical line test, as shown in Figure 5.7.2. Thus, the equation for the graph of a circle is not a function.

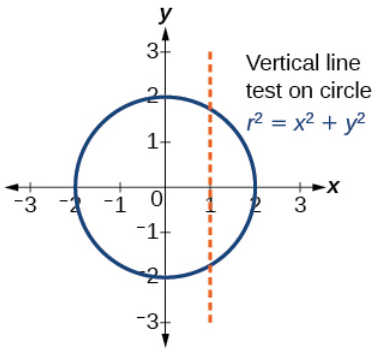


Figure 5.7.2

However, if we were to graph each equation on its own, each one would pass the vertical line test and therefore would represent a function. In some instances, the concept of breaking up the equation for a circle into two functions is similar to the concept of creating parametric equations, as we use two functions to produce a non-function. This will become clearer as we move forward.

PARAMETRIC EQUATIONS

Suppose t is a number on an interval, I. The set of ordered pairs, $(x(t), y(t))$, where $x = f(t)$ and $y = g(t)$,forms a plane curve based on the parameter t. The equations $x = f(t)$ and $y = g(t)$ are the parametric equations.

Example 5.7.1: Parameterizing a Curve

Parameterize the curve $y = x^2 - 1$ letting $x(t) = t$. Graph both equations.

Solution

If $x(t) = t$, then to find $y(t)$ we replace the variable x with the expression given in $x(t)$. In other words, $y(t) = t^2 - 1$ .Make a table of values similar to Table 5.7.1, and sketch the graph.

Table 5.7.1

t	$x(t)$	$y(t)$
-4	-4	$y(-4) = (-4)^2 - 1 = 15$
-3	-3	$y(-3) = (-3)^2 - 1 = 8$
-2	-2	$y(-2) = (-2)^2 - 1 = 3$
-1	-1	$y(-1) = (-1)^2 - 1 = 0$
0	0	$y(0) = (0)^2 - 1 = -1$
1	1	$y(1) = (1)^2 - 1 = 0$
2	2	$y(2) = (2)^2 - 1 = 3$
3	3	$y(3) = (3)^2 - 1 = 8$
4	4	$y(4) = (4)^2 - 1 = 15$

See the graphs in Figure $5.7.3$. It may be helpful to use the TRACE feature of a graphing calculator to see how the points are generated as t increases.

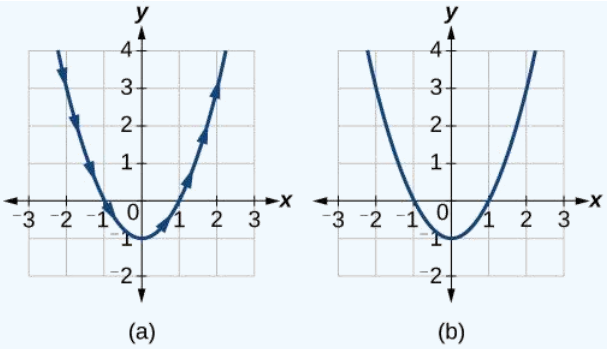


Figure 5.7.3: (a) Parametric $y(t) = t^2 - 1$ (b) Rectangular $y = x^2 - 1$

Analysis

The arrows indicate the direction in which the curve is generated. Notice the curve is identical to the curve of $y = x^2 - 1$.

Exercise 5.7.1

Construct a table of values and plot the parametric equations: $x(t) = t - 3$, $y(t) = 2t + 4$; $-1 \leq t \leq 2$.

Answer

t	$x(t)$	$y(t)$
-1	-4	2
0	-3	4
1	-2	6
2	-1	8

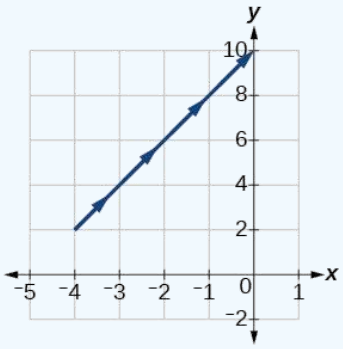


Figure 5.7.4

Example 5.7.2: Finding a Pair of Parametric Equations

Find a pair of parametric equations that models the graph of $y = 1 - x^2$, using the parameter $x(t) = t$. Plot some points and sketch the graph.

Solution

If $x(t) = t$ and we substitute t for x into the y equation, then $y(t) = 1 - t^2$. Our pair of parametric equations is

$$x(t) = t$$
$$y(t) = 1 - t^2$$

To graph the equations, first we construct a table of values like that in Table 5.7.2. We can choose values around $t = 0$, from $t = -3$ to $t = 3$. The values in the $x(t)$ column will be the same as those in the t column because $x(t) = t$. Calculate values

for the column $y(t)$.

Table 5.7.2

\(t)	$x(t)=t$	$y(t)=1-t^2$
-3	-3	$y(-3)=1-(-3)^2=-8$
-2	-2	$y(-2)=1-(-2)^2=-3$
-1	-1	$y(-1)=1-(-1)^2=0$
0	0	$y(0)=1-0=1$
1	1	$y(1)=1-(1)^2=0$
2	2	$y(2)=1-(2)^2=-3$
3	3	$y(3)=1-(3)^2=-8$

The graph of $y=1-t^2$ is a parabola facing downward, as shown in Figure 5.7.5. We have mapped the curve over the interval $[-3,3]$, shown as a solid line with arrows indicating the orientation of the curve according to t. Orientation refers to the path traced along the curve in terms of increasing values of t. As this parabola is symmetric with respect to the line $x=0$, the values of x are reflected across the *y*-axis.

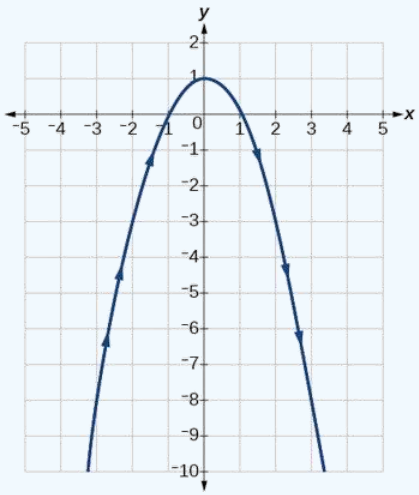


Figure 5.7.5

? Exercise 5.7.2

Parameterize the curve given by $x=y^3-2y$.

Answer

$x(t)=t^3-2t$

$y(t)=t$

✓ Example 5.7.3: Finding Parametric Equations That Model Given Criteria

An object travels at a steady rate along a straight path $(-5,3)$ to $(3,-1)$ in the same plane in four seconds. The coordinates are measured in meters. Find parametric equations for the position of the object.

Solution

The parametric equations are simple linear expressions, but we need to view this problem in a step-by-step fashion. The *x*-value of the object starts at -5 meters and goes to 3 meters. This means the distance x has changed by 8 meters in 4 seconds, which

is a rate of $\frac{8\ m}{4\ s}$, or $2\ m/s$. We can write the x-coordinate as a linear function with respect to time as $x(t) = 2t - 5$. In the linear function template $y = mx + b$, $2t = mx$ and $-5 = b$.

Similarly, the y-value of the object starts at 3 and goes to -1, which is a change in the distance y of -4 meters in 4 seconds, which is a rate of $\frac{-4\ m}{4\ s}$, or $-1\ m/s$. We can also write the y-coordinate as the linear function $y(t) = -t + 3$. Together, these are the parametric equations for the position of the object, where x and y are expressed in meters and t represents time:

$$\begin{aligned} x(t) &= 2t - 5 \\ y(t) &= -t + 3 \end{aligned}$$

Using these equations, we can build a table of values for t, x, and y (see Table 5.7.3). In this example, we limited values of t to non-negative numbers. In general, any value of t can be used.

Table 5.7.3

t	$x(t) = 2t - 5$	$y(t) = -t + 3$
0	$x = 2(0) - 5 = -5$	$y = -(0) + 3 = 3$
1	$x = 2(1) - 5 = -3$	$y = -(1) + 3 = 2$
2	$x = 2(2) - 5 = -1$	$y = -(2) + 3 = 1$
3	$x = 2(3) - 5 = 1$	$y = -(3) + 3 = 0$
4	$x = 2(4) - 5 = 3$	$y = -(4) + 3 = -1$

From this table, we can create three graphs, as shown in Figure 5.7.6.

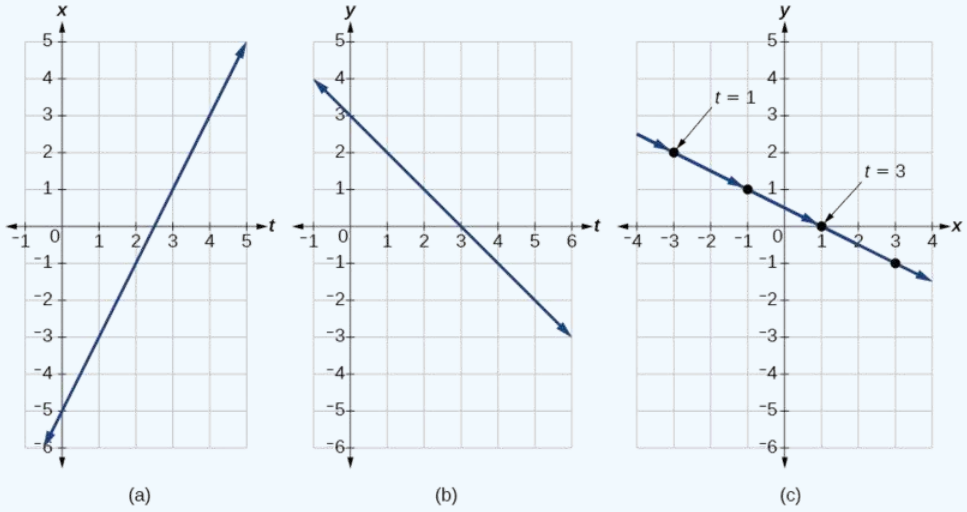


Figure 5.7.6 : (a) A graph of x vs. t, representing the horizontal position over time. (b) A graph of y vs. t, representing the vertical position over time. (c) A graph of y vs. x, representing the position of the object in the plane at time t.

Analysis

Again, we see that, in Figure 5.7.6 (c), when the parameter represents time, we can indicate the movement of the object along the path with arrows.

Eliminating the Parameter

In many cases, we may have a pair of parametric equations but find that it is simpler to draw a curve if the equation involves only two variables, such as x and y. Eliminating the parameter is a method that may make graphing some curves easier. However, if we are concerned with the mapping of the equation according to time, then it will be necessary to indicate the orientation of the curve as well. There are various methods for eliminating the parameter t from a set of parametric equations; not every method works for every type of equation. Here we will review the methods for the most common types of equations.

Eliminating the Parameter from Polynomial, Exponential, and Logarithmic Equations

For polynomial, exponential, or logarithmic equations expressed as two parametric equations, we choose the equation that is most easily manipulated and solve for t. We substitute the resulting expression for t into the second equation. This gives one equation in x and y.

Example 5.7.4: Eliminating the Parameter in Polynomials

Given $x(t)=t^2+1$ and $y(t)=2+t$, eliminate the parameter, and write the parametric equations as a Cartesian equation.

Solution

We will begin with the equation for y because the linear equation is easier to solve for t.

$$\begin{aligned} y &= 2+t \\ y-2 &= t \end{aligned}$$

Next, substitute $y-2$ for t in $x(t)$.

$$\begin{aligned} x &= t^2+1 \\ x &= (y-2)^2+1 \qquad \text{Substitute the expression for } t \text{ into } x. \\ x &= y^2-4y+4+1 \\ x &= y^2-4y+5 \\ x &= y^2-4y+5 \end{aligned}$$

The Cartesian form is $x=y^2-4y+5$.

Analysis

This is an equation for a parabola in which, in rectangular terms, x is dependent on y. From the curve's vertex at $(1,2)$, the graph sweeps out to the right. See Figure 5.7.7. In this section, we consider sets of equations given by the functions $x(t)$ and $y(t)$, where t is the independent variable of time. Notice, both x and y are functions of time; so in general y is not a function of x.

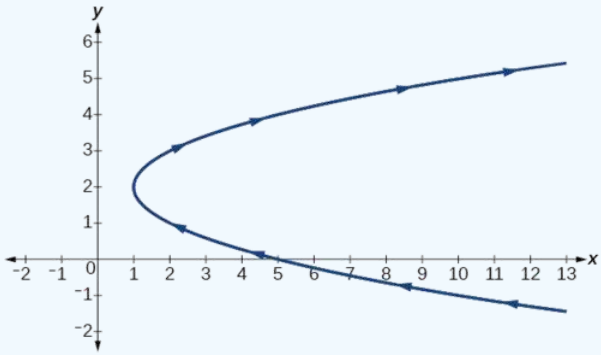


Figure 5.7.7

Exercise 5.7.3

Given the equations below, eliminate the parameter and write as a rectangular equation for y as a function of x.

$$\begin{aligned} x(t) &= 2t^2+6 \\ y(t) &= 5-t \end{aligned}$$

Answer

$$y=5-\sqrt{\tfrac{1}{2}x-3}$$

Example 5.7.5: Eliminating the Parameter in Exponential Equations

Eliminate the parameter and write as a Cartesian equation: $x(t) = e^{-t}$ and $y(t) = 3e^{t}, t > 0$.

Solution

Isolate e^{t}.

$$x = e^{-t}$$
$$e^{t} = \frac{1}{x}$$

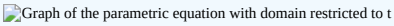
Substitute the expression into $y(t)$.

$$y = 3e^{t}$$
$$y = 3\left(\frac{1}{x}\right)$$
$$y = \frac{3}{x}$$

The Cartesian form is $y = \frac{3}{x}$.

Analysis

The graph of the parametric equation is shown in Figure 5.7.8*a*. The domain is restricted to $t > 0$. The Cartesian equation, $y = \frac{3}{x}$ is shown in Figure 5.7.8*b* and has only one restriction on the domain, $x \neq 0$.



0, and a graph of that parametric equation in polar coordinates with domain only restricted to x not equal to 0. The Cartesian coordinate version has an extra reflection of the function across the origin in Q 3 (original was just in Q 1). " src="/@api/deki/files/12489/imageedit_21_3487926125.png">

Figure 5.7.8

Example 5.7.6: Eliminating the Parameter in Logarithmic Equations

Eliminate the parameter and write as a Cartesian equation: $x(t) = \sqrt{t} + 2$ and $y(t) = \log(t)$.

Solution

Solve the first equation for t.

$$\begin{aligned} x &= \sqrt{t} + 2 \\ x - 2 &= \sqrt{t} \\ (x-2)^2 &= t \qquad \text{Square both sides.} \end{aligned}$$

Then, substitute the expression for t into the y equation.

$$y = \log(t)$$
$$y = \log\left(x-2\right)^2$$

The Cartesian form is $y = \log\left(x-2\right)^2$.

Analysis

To be sure that the parametric equations are equivalent to the Cartesian equation, check the domains. The parametric equations restrict the domain on $x = \sqrt{t} + 2$ to $t > 0$; we restrict the domain on x to $x > 2$. The domain for the parametric equation $y = \log(t)$ is restricted to $t > 0$; we limit the domain on $y = \log\left(x-2\right)^2$ to $x > 2$.

Exercise 5.7.4

Eliminate the parameter and write as a **rectangular equation.**

$$x(t) = t^2$$
$$y(t) = \ln t, t > 0$$

Answer

$y = \ln \sqrt{x}$

Eliminating the Parameter from Trigonometric Equations

Eliminating the parameter from trigonometric equations is a straightforward substitution. We can use a few of the familiar trigonometric identities and the Pythagorean Theorem.

First, we use the identities:

$$x(t) = a\cos t$$
$$y(t) = b\sin t$$

Solving for $\cos t$ and $\sin t$, we have

$$\frac{x}{a} = \cos t$$
$$\frac{y}{b} = \sin t$$

Then, use the Pythagorean Theorem:

$$\cos^2 t + \sin^2 t = 1$$

Substituting gives

$$\cos^2 t + \sin^2 t = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Example 5.7.7: Eliminating the Parameter from a Pair of Trigonometric Parametric Equations

Eliminate the parameter from the given pair of **trigonometric equations** where $0 \le t \le 2\pi$ and sketch the graph.

$$x(t) = 4\cos t$$
$$y(t) = 3\sin t$$

Solution

Solving for $\cos t$ and $\sin t$, we have

$$x = 4\cos t$$
$$\frac{x}{4} = \cos t$$
$$y = 3\sin t$$
$$\frac{y}{3} = \sin t$$

Next, use the Pythagorean identity and make the substitutions.

$$\cos^2 t + \sin^2 t = 1$$
$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The graph for the equation is shown in Figure 5.7.9.

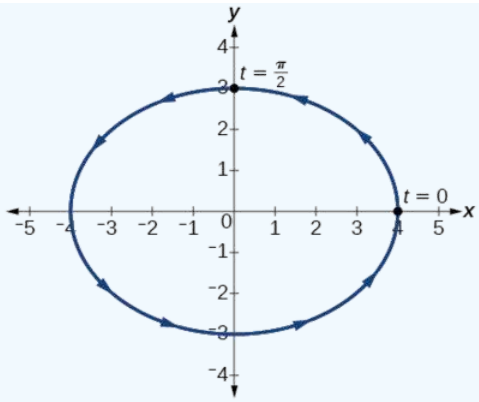


Figure 5.7.9

Analysis

Applying the general equations for conic sections (introduced in Analytic Geometry, we can identify $\frac{x^2}{16}+\frac{y^2}{9}=1$ as an ellipse centered at $(0,0)$. Notice that when $t=0$ the coordinates are $(4,0)$, and when $t=\frac{\pi}{2}$ the coordinates are $(0,3)$. This shows the orientation of the curve with increasing values of t.

? Exercise 5.7.5

Eliminate the parameter from the given pair of parametric equations and write as a Cartesian equation: $x(t)=2\cos t$ and $y(t)=3\sin t$.

Answer

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

Finding Cartesian Equations from Curves Defined Parametrically

When we are given a set of parametric equations and need to find an equivalent Cartesian equation, we are essentially "eliminating the parameter." However, there are various methods we can use to rewrite a set of parametric equations as a Cartesian equation. The simplest method is to set one equation equal to the parameter, such as $x(t)=t$. In this case, $y(t)$ can be any expression. For example, consider the following pair of equations.

$$\begin{aligned} x(t) &= t \\ y(t) &= t^2-3 \end{aligned}$$

Rewriting this set of parametric equations is a matter of substituting x for t. Thus, the Cartesian equation is $y=x^2-3$.

✓ Example 5.7.8: Finding a Cartesian Equation Using Alternate Methods

Use two different methods to find the Cartesian equation equivalent to the given set of parametric equations.

$$\begin{aligned} x(t) &= 3t-2 \\ y(t) &= t+1 \end{aligned}$$

Solution

Method 1. First, let's solve the x equation for t. Then we can substitute the result into the y equation.

$$\begin{aligned} x &= 3t-2 \\ x+2 &= 3t \\ \frac{x+2}{3} &= t \end{aligned}$$

Now substitute the expression for t into the y equation.

$$y = t+1$$
$$y = \left(\frac{x+2}{3}\right)+1$$
$$y = \frac{x}{3}+\frac{2}{3}+1$$
$$y = \frac{1}{3}x+\frac{5}{3}$$

Method 2. Solve the y equation for t and substitute this expression in the x equation.

$$\begin{aligned} y &= t+1 \\ y-1 &= t \end{aligned}$$

Make the substitution and then solve for y.

$$\begin{aligned} x &= 3(y-1)-2 \\ x &= 3y-3-2 \\ x &= 3y-5 \\ x+5 &= 3y \\ \frac{x+5}{3} &= y \\ y &= \frac{1}{3}x+\frac{5}{3} \end{aligned}$$

? Exercise 5.7.6

Write the given parametric equations as a Cartesian equation: $x(t) = t^3$ and $y(t) = t^6$.

Answer

$y = x^2$

Finding Parametric Equations for Curves Defined by Rectangular Equations

Although we have just shown that there is only one way to interpret a set of parametric equations as a rectangular equation, there are multiple ways to interpret a rectangular equation as a set of parametric equations. Any strategy we may use to find the parametric equations is valid if it produces equivalency. In other words, if we choose an expression to represent x, and then substitute it into the y equation, and it produces the same graph over the same domain as the rectangular equation, then the set of parametric equations is valid. If the domain becomes restricted in the set of parametric equations, and the function does not allow the same values for x as the domain of the rectangular equation, then the graphs will be different.

✓ Example 5.7.9: Finding a Set of Parametric Equations for Curves Defined by Rectangular Equations

Find a set of equivalent parametric equations for $y = (x+3)^2+1$.

Solution

An obvious choice would be to let $x(t) = t$. Then $y(t) = (t+3)^2+1$. But let's try something more interesting. What if we let $x = t+3$? Then we have

$$y = (x+3)^2+1$$
$$y = ((t+3)+3)^2+1$$
$$y = (t+6)^2+1$$

The set of parametric equations is

$$x(t) = t+3$$
$$y(t) = (t+6)^2+1$$

See Figure 5.7.10.

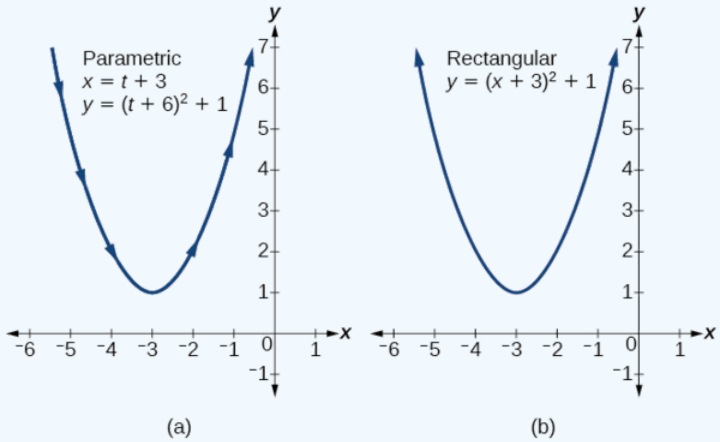


Figure 5.7.10

Media

Access these online resources for additional instruction and practice with parametric equations.

- Introduction to Parametric Equations
- Converting Parametric Equations to Rectangular Form

Key Concepts

- Parameterizing a curve involves translating a rectangular equation in two variables, x and y, into two equations in three variables, x, y, and t. Often, more information is obtained from a set of parametric equations. See Example 5.7.1, Example 5.7.2, and Example 5.7.3.
- Sometimes equations are simpler to graph when written in rectangular form. By eliminating t, an equation in x and y is the result.
- To eliminate t, solve one of the equations for t, and substitute the expression into the second equation. See Example 5.7.4, Example 5.7.5, Example 5.7.6, and Example 5.7.7.
- Finding the rectangular equation for a curve defined parametrically is basically the same as eliminating the parameter. Solve for t in one of the equations, and substitute the expression into the second equation. See Example 5.7.8.
- There are an infinite number of ways to choose a set of parametric equations for a curve defined as a rectangular equation.
- Find an expression for x such that the domain of the set of parametric equations remains the same as the original rectangular equation. See Example 5.7.9.

Contributors

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5.8: Parametric Equations - Graphs

Learning Objectives

- Graph plane curves described by parametric equations by plotting points.
- Graph parametric equations.

It is the bottom of the ninth inning, with two outs and two men on base. The home team is losing by two runs. The batter swings and hits the baseball at 140 feet per second and at an angle of approximately 45° to the horizontal. How far will the ball travel? Will it clear the fence for a game-winning home run? The outcome may depend partly on other factors (for example, the wind), but mathematicians can model the path of a projectile and predict approximately how far it will travel using parametric equations. In this section, we'll discuss parametric equations and some common applications, such as projectile motion problems.



Figure* 5.8.1*: *Parametric equations can model the path of a projectile. (credit: Paul Kreher, Flickr)*

Graphing Parametric Equations by Plotting Points

In lieu of a graphing calculator or a computer graphing program, plotting points to represent the graph of an equation is the standard method. As long as we are careful in calculating the values, point-plotting is highly dependable.

How to: Given a pair of parametric equations, sketch a graph by plotting points

1. Construct a table with three columns: t, $x(t)$,and $y(t)$.
2. Evaluate x and y for values of tt over the interval for which the functions are defined.
3. Plot the resulting pairs (x, y).

Example 5.8.1: Sketching the Graph of a Pair of Parametric Equations by Plotting Points

Sketch the graph of the parametric equations $x(t) = t^2 + 1$, $y(t) = 2 + t$.

Solution

Construct a table of values for t, $x(t)$, and $y(t)$, as in Table 5.8.1, and plot the points in a plane.

Table 5.8.1

t	$x(t) = t^2 + 1$	$y(t) = 2 + t$
-5	26	-3
-4	17	-2
-3	10	-1
-2	5	0
-1	2	1
0	1	2
1	2	3
2	5	4
3	10	5
4	17	6
5	26	7

The graph is a parabola with vertex at the point $(1, 2)$,opening to the right. See Figure 5.8.2.

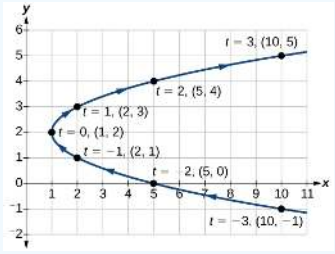


***Figure* 5.8.2**

Analysis

As values for t progress in a positive direction from 0 to 5, the plotted points trace out the top half of the parabola. As values oft t become negative, they trace out the lower half of the parabola. There are no restrictions on the domain. The arrows indicate direction according to increasing values of t. The graph does not represent a function, as it will fail the vertical line test. The graph is drawn in two parts: the positive values for t, and the negative values for t.

Exercise 5.8.1

Sketch the graph of the parametric equations $x = \sqrt{t}$, $y = 2t + 3$, $0 \le t \le 3$.

Answer

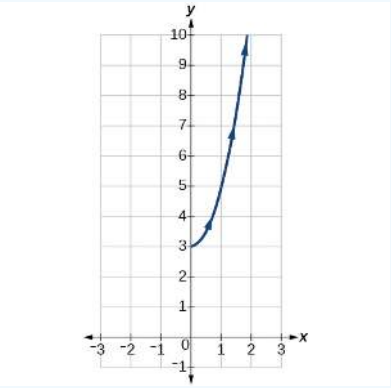


Figure 5.8.3

Example 5.8.2: Sketching the Graph of Trigonometric Parametric Equations

Construct a table of values for the given parametric equations and sketch the graph:

$$x = 2\cos t$$
$$y = 4\sin t$$

Solution

Construct a table like that in Table 5.8.2 using angle measure in radians as inputs for t, and evaluating x and y. Using angles with known sine and cosine values for t makes calculations easier.

Table 5.8.2

t	$x = 2\cos t$	$y = 4\sin t$
0	$x = 2\cos(0) = 2$	$y = 4\sin(0) = 0$
$\frac{\pi}{6}$	$x = 2\cos(\frac{\pi}{6}) = \sqrt{3}$	$y = 4\sin(\frac{\pi}{6}) =$
$\frac{\pi}{3}$	$x = 2\cos(\frac{\pi}{3}) = 1$	$y = 4\sin(\frac{\pi}{3}) = 2$
$\frac{\pi}{2}$	$x = 2\cos(\frac{\pi}{2}) = 0$	$y = 4\sin(\frac{\pi}{2}) =$
$\frac{2\pi}{3}$	$x = 2\cos(\frac{2\pi}{3}) = -1$	$y = 4\sin(\frac{2\pi}{3}) = 2$
$\frac{5\pi}{6}$	$x = 2\cos(\frac{5\pi}{6}) = -\sqrt{3}$	$y = 4\sin(\frac{5\pi}{6}) =$
π	$x = 2\cos(\pi) = -2$	$y = 4\sin(\pi) = 0$
$\frac{7\pi}{6}$	$x = 2\cos(\frac{7\pi}{6}) = -\sqrt{3}$	$y = 4\sin(\frac{7\pi}{6}) =$
$\frac{4\pi}{3}$	$x = 2\cos(\frac{4\pi}{3}) = -1$	$y = 4\sin(\frac{4\pi}{3}) =$
$\frac{3\pi}{2}$	$x = 2\cos(\frac{3\pi}{2}) = 0$	$y = 4\sin(\frac{3\pi}{2}) =$
$\frac{5\pi}{3}$	$x = 2\cos(\frac{5\pi}{3}) = 1$	$y = 4\sin(\frac{5\pi}{3}) =$
$\frac{11\pi}{6}$	$x = 2\cos(\frac{11\pi}{6}) = \sqrt{3}$	$y = 4\sin(\frac{11\pi}{6}) =$
2π	$x = 2\cos(2\pi) = 2$	$y = 4\sin(2\pi) =$

Figure 5.8.4 shows the graph.

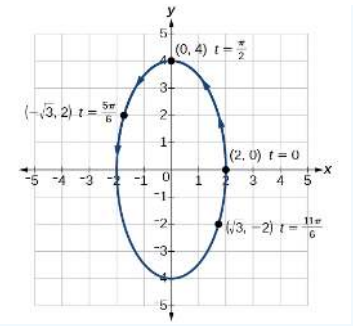


***Figure* 5.8.4**

By the symmetry shown in the values of x and y, we see that the parametric equations represent an ellipse. The ellipse is mapped in a counterclockwise direction as shown by the arrows indicating increasing t values.

Analysis

We have seen that parametric equations can be graphed by plotting points. However, a graphing calculator will save some time and reveal nuances in a graph that may be too tedious to discover using only hand calculations. Make sure to change the mode on the calculator to parametric (PAR). To confirm, the $Y =$ window should show

$$X_{1T} =$$
$$Y_{1T} =$$

instead of $Y_1 =$.

? Exercise 5.8.2

Graph the parametric equations: $x = 5\cos t$, $y = 3\sin t$.

Answer

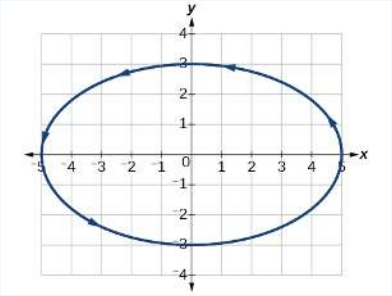


***Figure* 5.8.5**

✓ Example 5.8.3: Graphing Parametric Equations and Rectangular Form Together

Graph the parametric equations $x = 5\cos t$ and $y = 2\sin t$. First, construct the graph using data points generated from the parametric form. Then graph the rectangular form of the equation. Compare the two graphs.

Solution

Construct a table of values like that in Table 5.8.3.

Table 5.8.3

t	$x = 5\cos t$	$y = 2\sin t$
0	$x = 5\cos(0) = 5$	$y = 2\sin(0) = 0$
1	$x = 5\cos(1) \approx 2.7$	$y = 2\sin(1) \approx 1.7$
2	$x = 5\cos(2) \approx -2.1$	$y = 2\sin(2) \approx 1.8$
3	$x = 5\cos(3) \approx -4.95$	$y = 2\sin(3) \approx 0.28$
4	$x = 5\cos(4) \approx -3.3$	$y = 2\sin(4) \approx -1.5$
5	$x = 5\cos(5) \approx 1.4$	$y = 2\sin(5) \approx -1.9$
-1	$x = 5\cos(-1) \approx 2.7$	$y = 2\sin(-1) \approx -1.$
-2	$x = 5\cos(-2) \approx -2.1$	$y = 2\sin(-2) \approx -1.$
-3	$x = 5\cos(-3) \approx -4.95$	$y = 2\sin(-3) \approx -0.$
-4	$x = 5\cos(-4) \approx -3.3$	$y = 2\sin(-4) \approx 1.5$
-5	$x = 5\cos(-5) \approx 1.4$	$y = 2\sin(-5) \approx 1.9$

Plot the (x, y) values from the table (Figure 5.8.6).

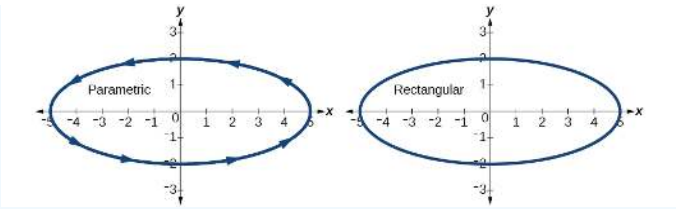


***Figure* 5.8.6**

Next, translate the parametric equations to rectangular form. To do this, we solve for t in either $x(t)$ or $y(t)$, and then substitute the expression for t in the other equation. The result will be a function $y(x)$ if solving for t as a function of x, or $x(y)$ if solving for t as a function of y.

$$x = 5\cos t$$
$$\frac{x}{5} = \cos t$$

Solve for $\cos t$.

$$y = 2\sin t$$

Solve for $\sin t$.

$$\frac{y}{2} = \sin t$$

Then, use the Pythagorean Theorem.

$$\cos^2 t + \sin^2 t = 1$$
$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$
$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

Analysis

In Figure 5.8.7, the data from the parametric equations and the rectangular equation are plotted together. The parametric equations are plotted in blue; the graph for the rectangular equation is drawn on top of the parametric in a dashed style colored red. Clearly, both forms produce the same graph.

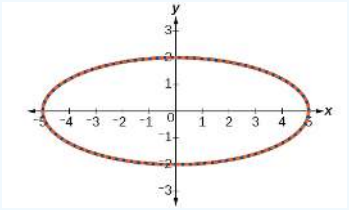


***Figure* 5.8.7**

Example 5.8.4: Graphing Parametric Equations and Rectangular Equations on the Coordinate System

Graph the parametric equations $x = t + 1$ and $y = \sqrt{t}, t \geq 0$, and the rectangular equivalent $y = \sqrt{x - 1}$ on the same coordinate system.

Solution

Construct a table of values for the parametric equations, as we did in the previous example, and graph $y = \sqrt{t}, t \geq 0$ on the same grid, as in Figure 5.8.8.

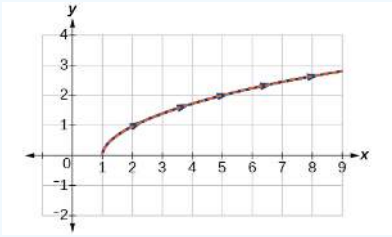


***Figure* 5.8.8**

Analysis

With the domain on t restricted, we only plot positive values of t. The parametric data is graphed in blue and the graph of the rectangular equation is dashed in red. Once again, we see that the two forms overlap.

Exercise 5.8.3

Sketch the graph of the parametric equations $x = 2\cos\theta$ and $y = 4\sin\theta$, along with the rectangular equation on the same grid.

Answer

The graph of the parametric equations is in red and the graph of the rectangular equation is drawn in blue dots on top of the parametric equations.

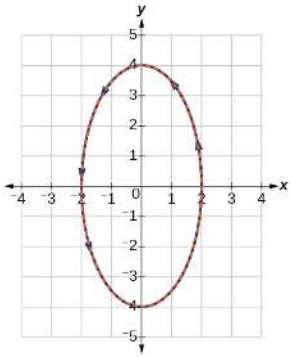


Figure 5.8.9

Applications of Parametric Equations

Many of the advantages of parametric equations become obvious when applied to solving real-world problems. Although rectangular equations in x and y give an overall picture of an object's path, they do not reveal the position of an object at a specific time. Parametric equations, however, illustrate how the values of x and y change depending on t, as the location of a moving object at a particular time.

A common application of parametric equations is solving problems involving projectile motion. In this type of motion, an object is propelled forward in an upward direction forming an angle of θ to the horizontal, with an initial speed of v_0, and at a height h above the horizontal.

The path of an object propelled at an inclination of θ to the horizontal, with initial speed v_0, and at a height h above the horizontal, is given by

$$\begin{aligned} x &= (v_0 \cos\theta)t \\ y &= -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + h \end{aligned}$$

where g accounts for the effects of **gravity** and h is the initial height of the object. Depending on the units involved in the problem, use $g = 32 ft/s^2$ or $g = 9.8 m/s^2$. The equation for x gives horizontal distance, and the equation for y gives the vertical distance.

How to: Given a projectile motion problem, use parametric equations to solve.

1. The horizontal distance is given by $x = (v_0 \cos\theta)t$. Substitute the initial speed of the object for v_0.
2. The expression $\cos\theta$ indicates the angle at which the object is propelled. Substitute that angle in degrees for $\cos\theta$.
3. The vertical distance is given by the formula $y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + h$. The term $-\frac{1}{2}gt^2$ represents the effect of gravity. Depending on units involved, use $g = 32 ft/s^2$ or $g = 9.8 m/s^2$. Again, substitute the initial speed for v_0, and the height at which the object was propelled for h.
4. Proceed by calculating each term to solve for t.

Example 5.8.5: Finding the Parametric Equations to Describe the Motion of a Baseball

Solve the problem presented at the beginning of this section. Does the batter hit the game-winning home run? Assume that the ball is hit with an initial velocity of 140 feet per second at an angle of 45° to the horizontal, making contact 3 feet above the ground.

1. Find the parametric equations to model the path of the baseball.
2. Where is the ball after 2 seconds?
3. How long is the ball in the air?
4. Is it a home run?

Solution

1. Use the formulas to set up the equations. The horizontal position is found using the parametric equation for x. Thus,

$$\begin{aligned} x &= (v_0 \cos\theta)t \\ x &= (140 \cos(45^\circ))t \end{aligned}$$

The vertical position is found using the parametric equation for y. Thus,

$$\begin{aligned} y &= -16t^2 + (v_0 \sin\theta)t + h \\ y &= -16t^2 + (140 \sin(45^\circ))t + 3 \end{aligned}$$

2. Substitute 2 into the equations to find the horizontal and vertical positions of the ball.

$$\begin{aligned} x &= (140 \cos(45^\circ))(2) \\ x &= 198 \ feet \\ y &= -16(2)^2 + (140 \sin(45^\circ))(2) + 3 \\ y &= 137 \ feet \end{aligned}$$

After 2 seconds, the ball is 198 feet away from the batter's box and 137 feet above the ground.

3. To calculate how long the ball is in the air, we have to find out when it will hit ground, or when $y = 0$. Thus,

$$\begin{aligned} y &= -16t^2 + (140 \sin(45\circ))t + 3 \\ y &= 0 \text{ Set } y(t) = 0 \text{ and solve the quadratic.} \\ t &= 6.2173 \end{aligned}$$

When $t = 6.2173$ seconds, the ball has hit the ground. (The quadratic equation can be solved in various ways, but this problem was solved using a computer math program.)

4. We cannot confirm that the hit was a home run without considering the size of the outfield, which varies from field to field. However, for simplicity's sake, let's assume that the outfield wall is 400 feet from home plate in the deepest part of the park. Let's also assume that the wall is 10 feet high. In order to determine whether the ball clears the wall, we need to calculate how high the ball is when \(x = 400\) feet. So we will set \(x = 400\), solve for t, and input tt into y.

$$
\begin{aligned}
x &= (140\cos(45^\circ))t \\
400 &= (140\cos(45^\circ))t \\
t &= 4.04 \\
y &= -16(4.04)^2 + (140\sin(45^\circ))(4.04) + 3 \\
y &= 141.8
\end{aligned}
$$

The ball is 141.8 feet in the air when it soars out of the ballpark. It was indeed a home run. See Figure $5.8.10$.

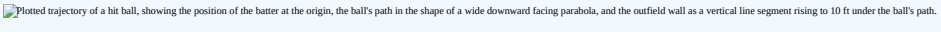


***Figure* 5.8.10**

Media

Access the following online resource for additional instruction and practice with graphs of parametric equations.

- Graphing Parametric Equations on the TI-84

Key Concepts

- When there is a third variable, a third parameter on which x and y depend, parametric equations can be used.
- To graph parametric equations by plotting points, make a table with three columns labeled t, $x(t)$, and $y(t)$. Choose values fort t in increasing order. Plot the last two columns for x and y. See Example $5.8.1$ and Example $5.8.2$.
- When graphing a parametric curve by plotting points, note the associated *t*-values and show arrows on the graph indicating the orientation of the curve. See Example $5.8.3$ and Example $5.8.4$.
- Parametric equations allow the direction or the orientation of the curve to be shown on the graph. Equations that are not functions can be graphed and used in many applications involving motion. See Example $5.8.5$.
- Projectile motion depends on two parametric equations: $x = (v_0\cos\theta)t$ and $y = -16t^2 + (v_0\sin\theta)t + h$. Initial velocity is symbolized as v_0. θ represents the initial angle of the object when thrown, and h represents the height at which the object is propelled.

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5.E: Further Applications of Trigonometry (Exercises)

8.1: Non-right Triangles: Law of Sines

Verbal

1) Describe the altitude of a triangle.

Answer

The altitude extends from any vertex to the opposite side or to the line containing the opposite side at a 90° angle.

2) Compare right triangles and oblique triangles.

3) When can you use the Law of Sines to find a missing angle?

Answer

When the known values are the side opposite the missing angle and another side and its opposite angle.

4) In the Law of Sines, what is the relationship between the angle in the numerator and the side in the denominator?

5) What type of triangle results in an ambiguous case?

Answer

A triangle with two given sides and a non-included angle.

Algebraic

For the exercises 6-10, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

6) $\alpha = 43^\circ, \gamma = 69^\circ, a = 20$

7) $\alpha = 35^\circ, \gamma = 73^\circ, c = 20$

Answer

$\beta = 72^\circ, a \approx 12.0, b \approx 19.9$

8) $\alpha = 60^\circ, \beta = 60^\circ, \gamma = 60^\circ$

9) $a = 4, \alpha = 60^\circ, \beta = 100^\circ$

Answer

$\gamma = 20^\circ, b \approx 4.5, c \approx 1.6$

10) $b = 10, \beta = 95^\circ, \gamma = 30^\circ$

11) Find side b when $A = 37^\circ, B = 49^\circ, c = 5$

Answer

$b \approx 3.78$

12) Find side a when $A = 132^\circ, C = 23^\circ, b = 10$

13) Find side c when $B = 37^\circ, C = 21^\circ, b = 23$

Answer

$c \approx 13.70$

For the exercises 14-23, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Determine whether there is no triangle, one triangle, or two triangles. Then solve each triangle, if possible. Round each answer to the nearest tenth.

14) $\alpha = 119^\circ, a = 14, b = 26$

15) $\gamma = 113^\circ, b = 10, c = 32$

Answer

one triangle, $\alpha \approx 50.3^\circ, \beta \approx 16.7^\circ, a \approx 26.7$

16) $b = 3.5, c = 5.3, \gamma = 80^\circ$

17) $a = 12, c = 17, \alpha = 35^\circ$

Answer

two triangles, $\gamma \approx 54.3^\circ, \beta \approx 90.7^\circ, b \approx 20.9$ or $\gamma' \approx 125.7^\circ, \beta' \approx 19.3^\circ, b' \approx 6.9$

18) $a = 20.5, b = 35.0, \beta = 25^\circ$

19) $a = 7, c = 9, \alpha = 43^\circ$

Answer

two triangles, $\beta \approx 75.7^\circ, \gamma \approx 61.3^\circ, b \approx 9.9$ or $\beta' \approx 18.3^\circ, \gamma' \approx 118.7^\circ, b' \approx 3.2$

20) $a = 7, b = 3, \beta = 24^\circ$

21) $b = 13, c = 5, \gamma = 10^\circ$

Answer

two triangles, $\alpha \approx 143.2^\circ, \beta \approx 26.8^\circ, a \approx 17.3$ or $\alpha' \approx 16.8^\circ, \beta' \approx 153.2^\circ, a' \approx 8.3$

22) $a = 2.3, c = 1.8, \gamma = 28^\circ$

23) $\beta = 119^\circ, b = 8.2, a = 11.3$

Answer

no triangle possible

For the exercises 24-26, use the Law of Sines to solve, if possible, the missing side or angle for each triangle or triangles in the ambiguous case. Round each answer to the nearest tenth.

24) Find angle A when $a = 24, b = 5, B = 22^\circ$

25) Find angle A when $a = 13, b = 6, B = 20^\circ$

Answer

$A \approx 47.8^\circ$ or $A' \approx 132.2^\circ$

26) Find angle B when $A = 12^\circ, a = 2, b = 9$

For the exercises 27-30, find the area of the triangle with the given measurements. Round each answer to the nearest tenth.

27) $a = 5, c = 6, \beta = 35^\circ$

Answer

8.6

28) $b = 11, c = 8, \alpha = 28^\circ$

29) $a = 32, b = 24, \gamma = 75^\circ$

Answer

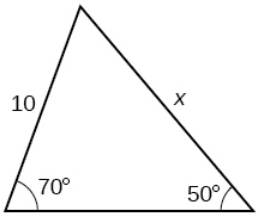
370.9

30) $a = 7.2, b = 4.5, \gamma = 43^\circ$

Graphical

For the exercises 31-36, find the length of side x. Round to the nearest tenth.

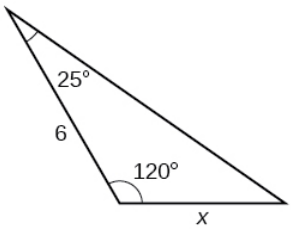
31)



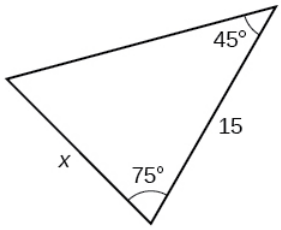
Answer

12.3

32)



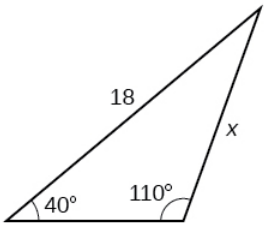
33)



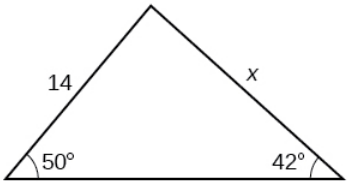
Answer

12.2

34)



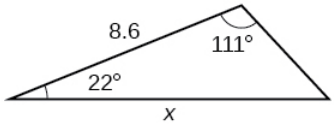
35)



Answer

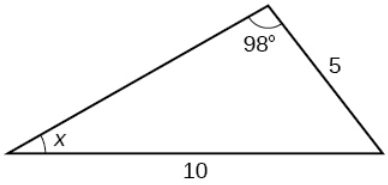
16.0

36)



For the exercises 37-,42 find the measure of angle x, if possible. Round to the nearest tenth.

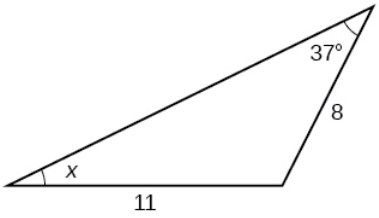
37)



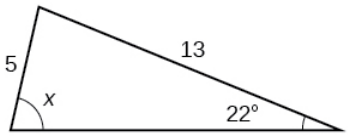
Answer

29.7°

38)



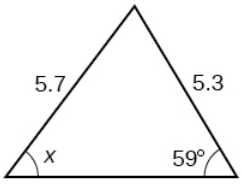
39)



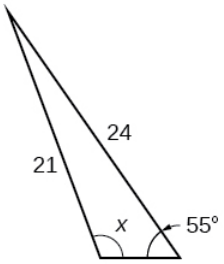
Answer

$x = 76.9^{\circ}$ or $x = 103.1^{\circ}$

40)



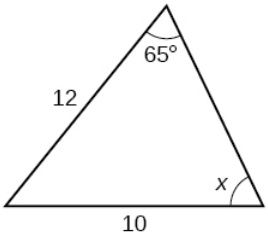
41) Notice that x is an obtuse angle.



Answer

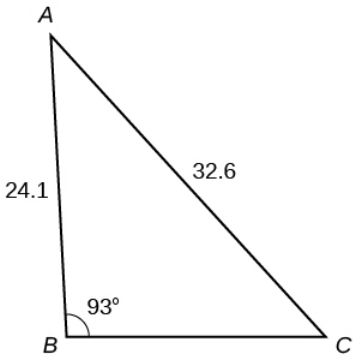
110.6°

42)



For the exercises 43-49, find the area of each triangle. Round each answer to the nearest tenth.

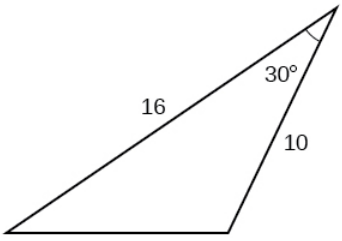
43)



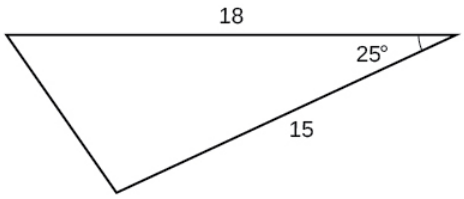
Answer

$A \approx 39.4, C \approx 47.6, BC \approx 20.7$

44)



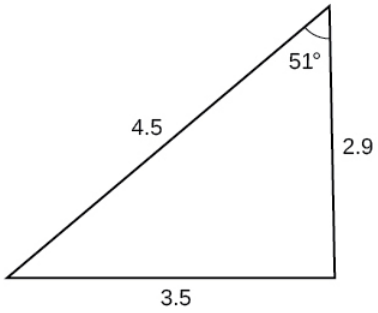
45)



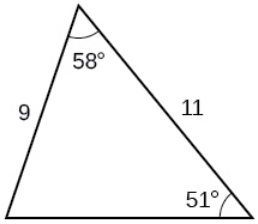
Answer

57.1

46)



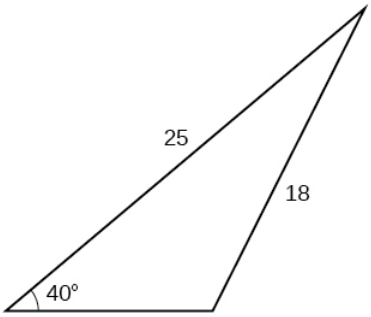
47)



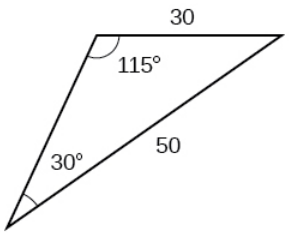
Answer

42.0

48)



49)



Answer

430.2

Extensions

50) Find the radius of the circle in the Figure below. Round to the nearest tenth.

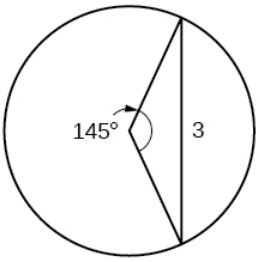
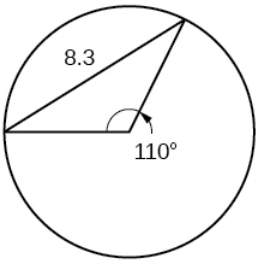


Figure 50.

51) Find the diameter of the circle in the Figure below. Round to the nearest tenth.



Answer

10.1

52) Find $m\angle ADC$ in the Figure below. Round to the nearest tenth.

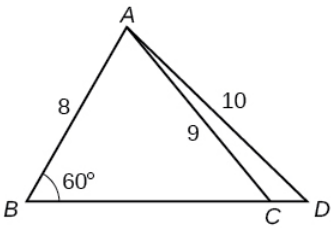
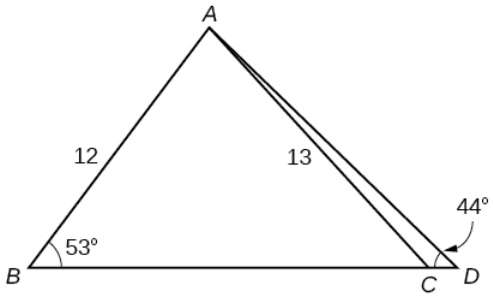


Figure 52.

53) Find side AD in the Figure below. Round to the nearest tenth.



Answer

$AD \approx 13.8$

54) Solve both triangles in the Figure below. Round each answer to the nearest tenth.

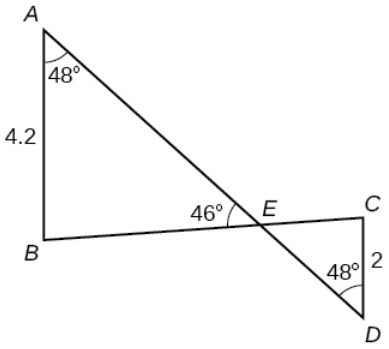
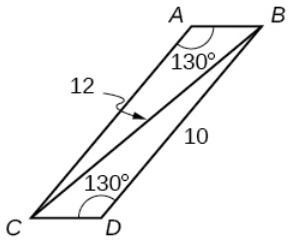


Figure 54.

55) Find side AB in the parallelogram shown below. Round to the nearest tenth.



Answer

$AB \approx 2.8$

56) Solve the triangle in the Figure below. (**Hint:** Draw a perpendicular from H to JK. Round each answer to the nearest tenth.

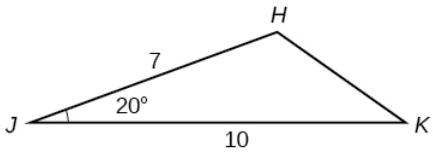
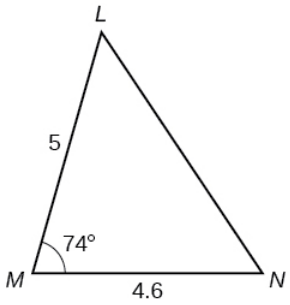


Figure 56.

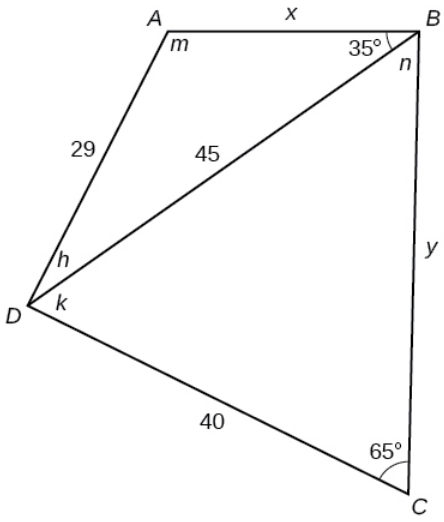
57) Solve the triangle in the Figure below. (Hint: Draw a perpendicular from N to LM. Round each answer to the nearest tenth.



Answer

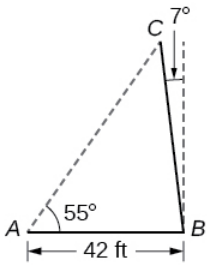
$L \approx 49.7^\circ, N \approx 56.3^\circ, LN \approx 5.8$

58) In the Figure below, $ABCD$ is not a parallelogram. $\angle m$ is obtuse. Solve both triangles. Round each answer to the nearest tenth.



Real-World Applications

59) A pole leans away from the sun at an angle of 7° to the vertical, as shown in the Figure below. When the elevation of the sun is 55°, the pole casts a shadow 42 feet long on the level ground. How long is the pole? Round the answer to the nearest tenth.



Answer

51.4ft

60) To determine how far a boat is from shore, two radar stations 500 feet apart find the angles out to the boat, as shown in the Figure below. Determine the distance of the boat from station A and the distance of the boat from shore. Round your answers to the nearest whole foot.

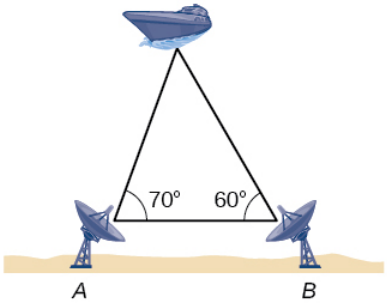
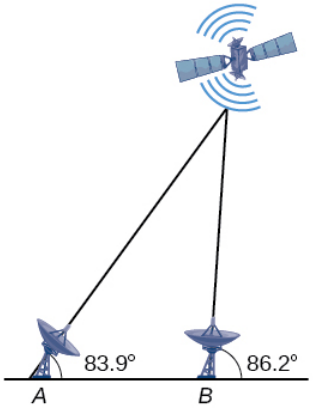


Figure 60.

61) The Figure below shows a satellite orbiting Earth. The satellite passes directly over two tracking stations A and B, which are 69 miles apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be $86.2°$ and $83.9°$ respectively. How far is the satellite from station A and how high is the satellite above the ground? Round answers to the nearest whole mile.



Answer

The distance from the satellite to station A is approximately 1716 miles. The satellite is approximately 1706 miles above the ground.

62) A communications tower is located at the top of a steep hill, as shown in the Figure below. The angle of inclination of the hill is $67°$. A guy wire is to be attached to the top of the tower and to the ground, 165 meters downhill from the base of the tower. The angle formed by the guy wire and the hill is $16°$. Find the length of the cable required for the guy wire to the nearest whole meter.

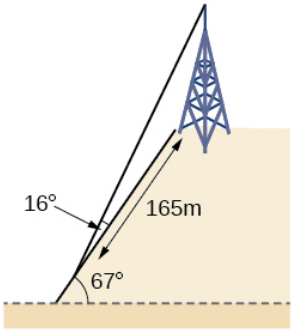
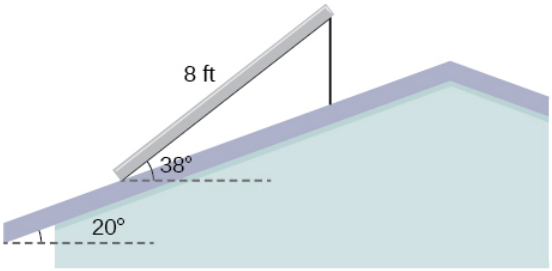


Figure 62.

63) The roof of a house is at a $20°$ angle. An 8-foot solar panel is to be mounted on the roof and should be angled $38°$ relative to the horizontal for optimal results. (See the Figure below). How long does the vertical support holding up the back of the panel need to be? Round to the nearest tenth.



Answer

2.6 ft

64) Similar to an angle of elevation, an *angle of depression* is the acute angle formed by a horizontal line and an observer's line of sight to an object below the horizontal. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 6.6 km apart, to be 37° and 44° as shown in the Figure below. Find the distance of the plane from point A to the nearest tenth of a kilometer.

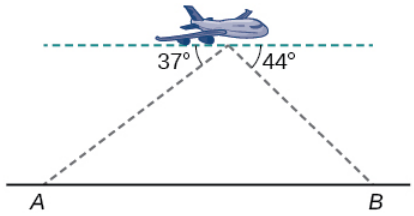
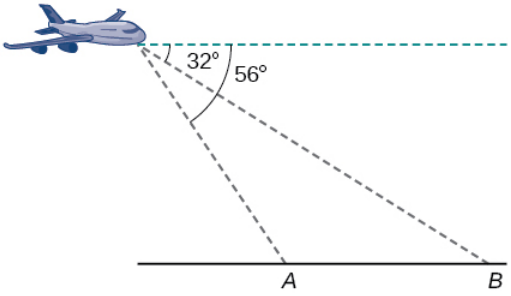


Figure 64.

65) A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 4.3 km apart, to be 32° and 56°, as shown in the Figure below. Find the distance of the plane from point A to the nearest tenth of a kilometer.



Answer

5.6 km

66) In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 39°. They then move 300 feet closer to the building and find the angle of elevation to be 50°. Assuming that the street is level, estimate the height of the building to the nearest foot.

67) In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 35°. They then move 250 feet closer to the building and find the angle of elevation to be 53°. Assuming that the street is level, estimate the height of the building to the nearest foot.

Answer

371 ft

68) Points A and B are on opposite sides of a lake. Point C is 97 meters from A. The measure of $\angle BAC$ is determined to be 101°, and the measure of $\angle ACB$ is determined to be 53°. What is the distance from A to B, rounded to the nearest whole meter?

69) A man and a woman standing $3\frac{1}{2}$ miles apart spot a hot air balloon at the same time. If the angle of elevation from the man to the balloon is 27°, and the angle of elevation from the woman to the balloon is 41°, find the altitude of the balloon to the nearest foot.

Answer

5936ft

70) Two search teams spot a stranded climber on a mountain. The first search team is 0.5 miles from the second search team, and both teams are at an altitude of 1 mile. The angle of elevation from the first search team to the stranded climber is 15°. The angle of elevation from the second search team to the climber is 22°. What is the altitude of the climber? Round to the nearest tenth of a mile.

71) A street light is mounted on a pole. A 6-foot-tall man is standing on the street a short distance from the pole, casting a shadow. The angle of elevation from the tip of the man's shadow to the top of his head of 28°. A 6-foot-tall woman is standing on the same street on the opposite side of the pole from the man. The angle of elevation from the tip of her shadow to the top of her head is 28°. If the man and woman are 20 feet apart, how far is the street light from the tip of the shadow of each person? Round the distance to the nearest tenth of a foot.

Answer

24.1ft

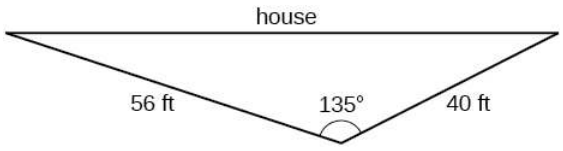
72) Three cities, A, B, and C, are located so that city A is due east of city B. If city C is located 35° west of north from city B and is 100 miles from city A and 70 miles from city B, how far is city A from city B? Round the distance to the nearest tenth of a mile.

73) Two streets meet at an 80° angle. At the corner, a park is being built in the shape of a triangle. Find the area of the park if, along one road, the park measures 180 feet, and along the other road, the park measures 215 feet.

Answer

$19,056$ft^2

74) Brian's house is on a corner lot. Find the area of the front yard if the edges measure 40 and 56 feet, as shown in the Figure below.



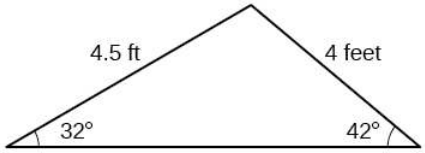
75) The Bermuda triangle is a region of the Atlantic Ocean that connects Bermuda, Florida, and Puerto Rico. Find the area of the Bermuda triangle if the distance from Florida to Bermuda is 1030 miles, the distance from Puerto Rico to Bermuda is 980 miles, and the angle created by the two distances is 62°.

Answer

$445,624$square miles

76) A yield sign measures 30 inches on all three sides. What is the area of the sign?

77) Naomi bought a modern dining table whose top is in the shape of a triangle. Find the area of the table top if two of the sides measure 4 feet and 4.5 feet, and the smaller angles measure 32° and 42°, as shown in the Figure below.



Answer

8.65ft^2

8.2: Non-right Triangles - Law of Cosines

Verbal

1) If you are looking for a missing side of a triangle, what do you need to know when using the Law of Cosines?

Answer

two sides and the angle opposite the missing side.

2) If you are looking for a missing angle of a triangle, what do you need to know when using the Law of Cosines?

3) Explain what s represents in Heron's formula.

Answer

s is the semi-perimeter, which is half the perimeter of the triangle.

4) Explain the relationship between the Pythagorean Theorem and the Law of Cosines.

5) When must you use the Law of Cosines instead of the Pythagorean Theorem?

Answer

The Law of Cosines must be used for any oblique (non-right) triangle.

Algebraic

For the exercises 6-15, assume α is opposite side a, β is opposite side b, and γ is opposite side c. If possible, solve each triangle for the unknown side. Round to the nearest tenth.

6) $\gamma = 41.2^\circ, a = 2.49, b = 3.13$

7) $\alpha = 120^\circ, b = 6, c = 7$

Answer

11.3

8) $\beta = 58.7^\circ, a = 10.6, c = 15.7$

9) $\alpha = 115^\circ, a = 18, b = 23$

Answer

34.7

10) $\alpha = 119^\circ, a = 26, b = 14$

11) $\gamma = 113^\circ, b = 10, c = 32$

Answer

26.7

12) $\beta = 67^\circ, a = 49, b = 38$

13) $\alpha = 43.1^\circ, a = 184.2, b = 242.8$

Answer

257.4

14) $\alpha = 36.6^\circ, a = 186.2, b = 242.2$

15) $\beta = 50^\circ, a = 105, b = 45$

Answer

not possible

For the exercises 16-20, use the Law of Cosines to solve for the missing angle of the oblique triangle. Round to the nearest tenth.

16) $a = 42, b = 19, c = 30$; find angle A.

17) $a = 14, b = 13, c = 20$; find angle C.

Answer

95.5°

18) $a = 16, b = 31, c = 20$; find angle B.

19) $a = 13, b = 22, c = 28$; find angle A.

Answer

26.9°

20) $a = 108, b = 132, c = 160$; find angle C.

For the exercises 21-26, solve the triangle. Round to the nearest tenth.

21) $A = 35^\circ, b = 8, c = 11$

Answer

$B \approx 45.9^\circ, C \approx 99.1^\circ, a \approx 6.4$

22) $B = 88^\circ, a = 4.4, c = 5.2$

23) $C = 121^\circ, a = 21, b = 37$

Answer

$A \approx 20.6^\circ, B \approx 38.4^\circ, c \approx 51.1$

24) $a = 13, b = 11, c = 15$

25) $a = 3.1, b = 3.5, c = 5$

Answer

$A \approx 37.8^\circ, B \approx 43.8^\circ, C \approx 98.4$

26) $a = 51, b = 25, c = 29$

For the exercises 27-,31 use Heron's formula to find the area of the triangle. Round to the nearest hundredth.

27) Find the area of a triangle with sides of length 18 in, 21 in, and 32 in. Round to the nearest tenth.

Answer

177.56in^2

28) Find the area of a triangle with sides of length 20 cm, 26 cm, and 37 cm. Round to the nearest tenth.

29) $a = \frac{1}{2}$ m, $b = \frac{1}{3}$ m, $c = \frac{1}{4}$ m

Answer

0.04m^2

30) $a = 12.4$ ft, $b = 13.7$ ft, $c = 20.2$ ft

31) $a = 1.6$ yd, $b = 2.6$ yd, $c = 4.1$ yd

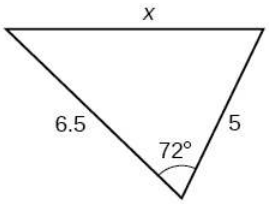
Answer

0.91yd^2

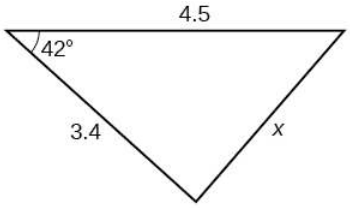
Graphical

For the exercises 32-37, find the length of side x. Round to the nearest tenth.

32)



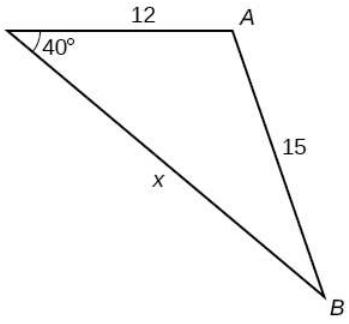
33)



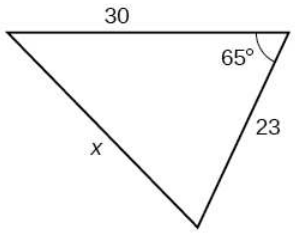
Answer

3.0

34)



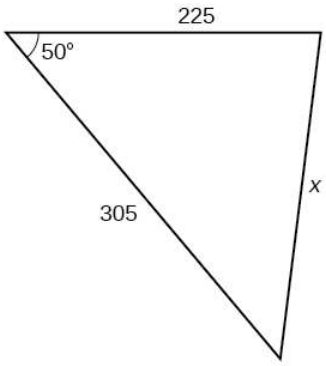
35)



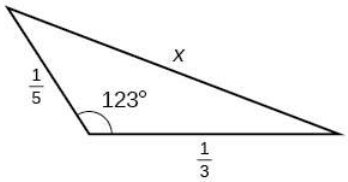
Answer

29.1

36)



37)

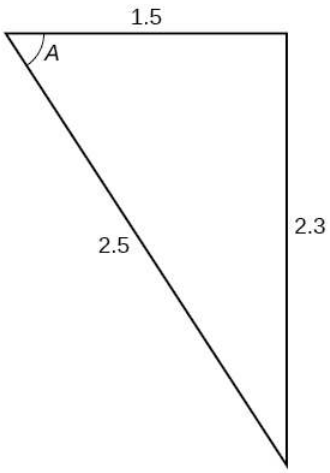


Answer

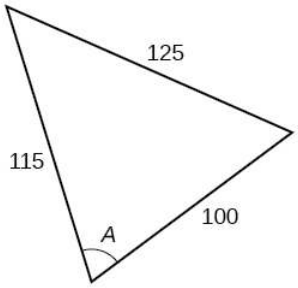
0.5

For the exercises 38-41, find the measurement of angle A

38)



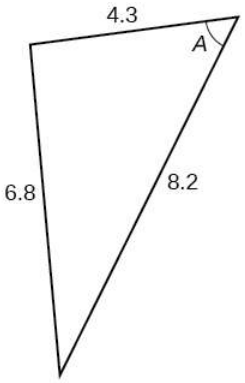
39)



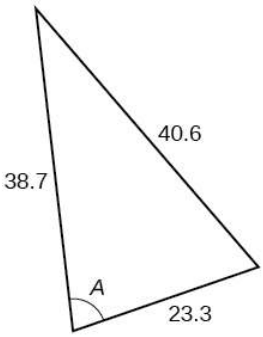
Answer

70.7°

40)



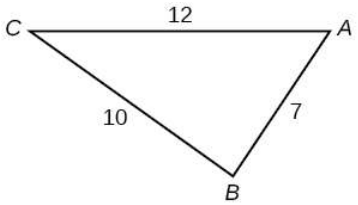
41)



Answer

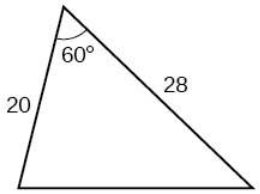
77.4°

42) Find the measure of each angle in the triangle shown in the Figure below. Round to the nearest tenth.



For the exercises 43-46, solve for the unknown side. Round to the nearest tenth.

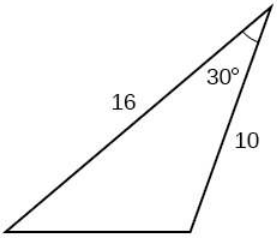
43)



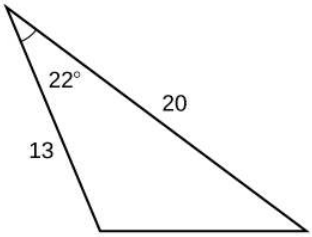
Answer

25.0

44)



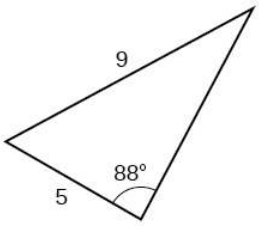
45)



Answer

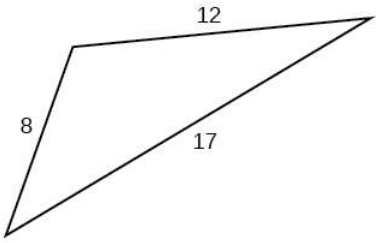
9.3

46)



For the exercises 47-51, find the area of the triangle. Round to the nearest hundredth.

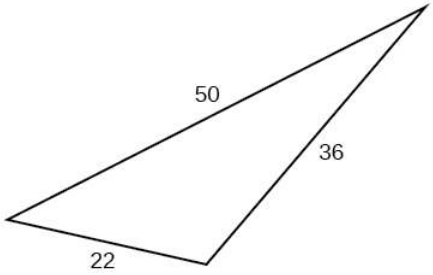
47)



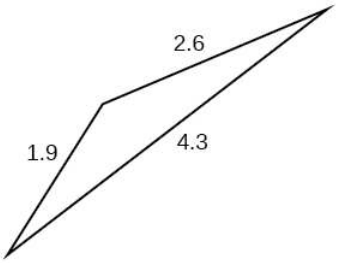
Answer

43.52

48)



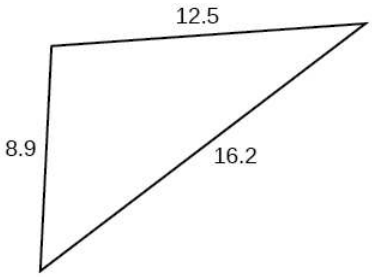
49)



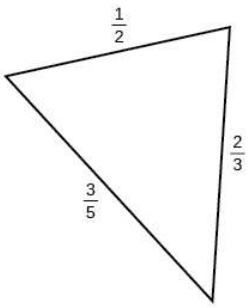
Answer

1.41

50)



51)



Answer

0.14

Extensions

52) A parallelogram has sides of length 16 units and 10 units. The shorter diagonal is 12 units. Find the measure of the longer diagonal.

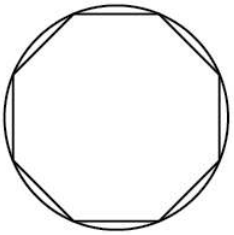
53) The sides of a parallelogram are 11 feet and 17 feet. The longer diagonal is 22 feet. Find the length of the shorter diagonal.

Answer

18.3

54) The sides of a parallelogram are 28 centimeters and 40 centimeters. The measure of the larger angle is100°. Find the length of the shorter diagonal.

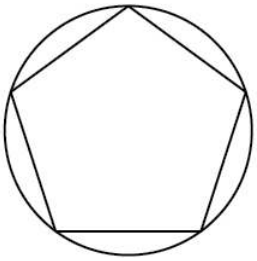
55) A regular octagon is inscribed in a circle with a radius of 8 inches. (See Figure below.) Find the perimeter of the octagon.



Answer

48.98

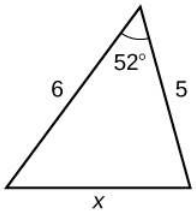
56) A regular pentagon is inscribed in a circle of radius 12 cm. (See Figure below.) Find the perimeter of the pentagon. Round to the nearest tenth of a centimeter.



For the exercises 57-58, suppose that $x^2 = 25 + 36 - 60\cos(52)$ represents the relationship of three sides of a triangle and the cosine of an angle.

57) Draw the triangle.

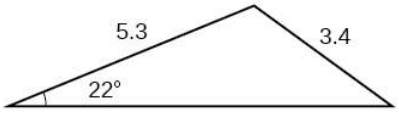
Answer



58) Find the length of the third side.

For the exercises 59-61, find the area of the triangle.

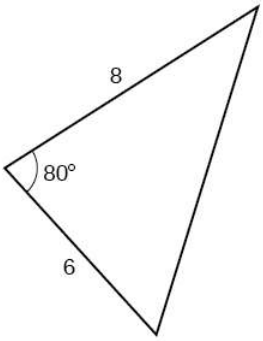
59)



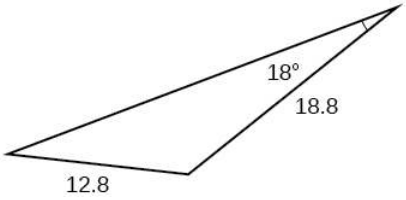
Answer

7.62

60)



61)



Answer

85.1

Real-World Applications

62) A surveyor has taken the measurements shown in the Figure below. Find the distance across the lake. Round answers to the nearest tenth.

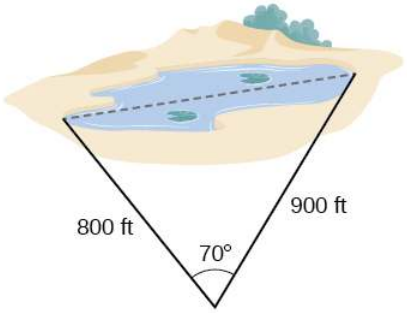
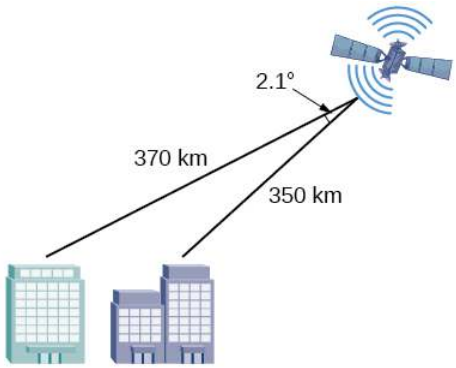


Figure below (not to scale). Find the distance between the two cities. Round answers to the nearest tenth.

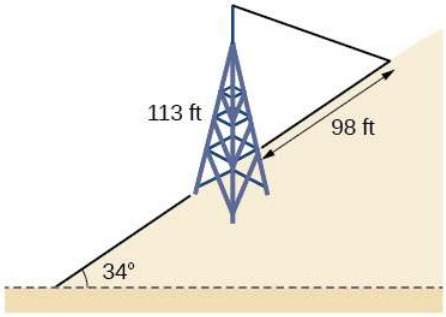


Answer

24.0km

64) An airplane flies 220 miles with a heading of 40°, and then flies 180 miles with a heading of 170°. How far is the plane from its starting point, and at what heading? Round answers to the nearest tenth.

65) A 113-foot tower is located on a hill that is inclined 34° to the horizontal, as shown in the Figure below. A guy-wire is to be attached to the top of the tower and anchored at a point 98 feet uphill from the base of the tower. Find the length of wire needed.

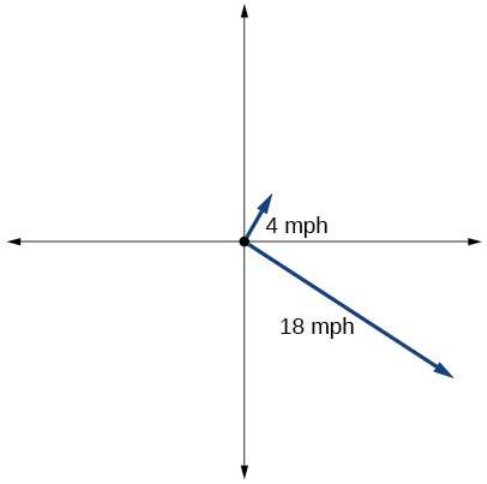


Answer

99.9ft

66) Two ships left a port at the same time. One ship traveled at a speed of 18 miles per hour at a heading of 320°. The other ship traveled at a speed of 22 miles per hour at a heading of 194°. Find the distance between the two ships after 10 hours of travel.

67) The graph in the Figure below represents two boats departing at the same time from the same dock. The first boat is traveling at 18 miles per hour at a heading of 327° and the second boat is traveling at 4 miles per hour at a heading of 60°. Find the distance between the two boats after 2 hours.



Answer

37.3miles

68) A triangular swimming pool measures 40 feet on one side and 65 feet on another side. These sides form an angle that measures 50°. How long is the third side (to the nearest tenth)?

69) A pilot flies in a straight path for 1 hour 30 min. She then makes a course correction, heading 10° to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?

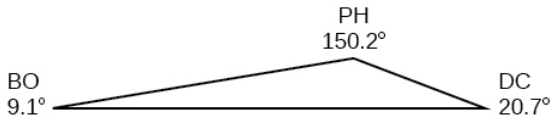
Answer

2371miles

70) Los Angeles is $1,744$ miles from Chicago, Chicago is 714 miles from New York, and New York is $2,451$ miles from Los Angeles. Draw a triangle connecting these three cities, and find the angles in the triangle.

71) Philadelphia is 140 miles from Washington, D.C., Washington, D.C. is 442 miles from Boston, and Boston is 315 miles from Philadelphia. Draw a triangle connecting these three cities and find the angles in the triangle.

Answer



72) Two planes leave the same airport at the same time. One flies at 20° east of north at 500 miles per hour. The second flies at 30° east of south at 600 miles per hour. How far apart are the planes after 2 hours?

73) Two airplanes take off in different directions. One travels 300 mph due west and the other travels 25° north of west at 420 mph. After 90 minutes, how far apart are they, assuming they are flying at the same altitude?

Answer

599.8miles

74) A parallelogram has sides of length 15.4 units and 9.8 units. Its area is 72.9 square units. Find the measure of the longer diagonal.

75) The four sequential sides of a quadrilateral have lengths 4.5 cm, 7.9 cm, 9.4 cm, and 12.9 cm. The angle between the two smallest sides is 117°. What is the area of this quadrilateral?

Answer

65.4cm^2

76) The four sequential sides of a quadrilateral have lengths 5.7 cm, 7.2 cm, 9.4 cm, and 12.8 cm. The angle between the two smallest sides is 106°. What is the area of this quadrilateral?

77) Find the area of a triangular piece of land that measures 30 feet on one side and 42 feet on another; the included angle measures 132°. Round to the nearest whole square foot.

Answer

$468\,\text{ft}^2$

78) Find the area of a triangular piece of land that measures 110 feet on one side and 250 feet on another; the included angle measures 85°. Round to the nearest whole square foot.

8.3: Polar Coordinates

Verbal

1) How are polar coordinates different from rectangular coordinates?

Answer

For polar coordinates, the point in the plane depends on the angle from the positive x-axis and distance from the origin, while in Cartesian coordinates, the point represents the horizontal and vertical distances from the origin. For each point in the coordinate plane, there is one representation, but for each point in the polar plane, there are infinite representations

2) How are the polar axes different from the x- and y-axes of the Cartesian plane?

3) Explain how polar coordinates are graphed.

Answer

Determine θ for the point, then move r units from the pole to plot the point. If r is negative, move r units from the pole in the opposite direction but along the same angle. The point is a distance of r away from the origin at an angle of θ from the polar axis.

4) How are the points $\left(3, \frac{\pi}{2}\right)$ and $\left(-3, \frac{\pi}{2}\right)$ related?

5) Explain why the points $\left(-3, \frac{\pi}{2}\right)$ and $\left(3, -\frac{\pi}{2}\right)$ are the same.

Answer

The point $\left(-3, \frac{\pi}{2}\right)$ has a positive angle but a negative radius and is plotted by moving to an angle of $\frac{\pi}{2}$ and then moving 3 units in the negative direction. This places the point 3 units down the negative y-axis. The point $\left(3, -\frac{\pi}{2}\right)$ has a negative angle and a positive radius and is plotted by first moving to an angle of $-\frac{\pi}{2}$ and then moving 3 units down, which is the positive direction for a negative angle. The point is also 3 units down the negative y-axis.

Algebraic

6) $\left(7, \frac{7\pi}{6}\right)$

7) $(5, \pi)$

Answer

$(-5, 0)$

8) $\left(6, -\frac{\pi}{4}\right)$

9) $\left(-3, \frac{\pi}{6}\right)$

Answer

$\left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

10) $\left(4, \frac{7\pi}{4}\right)$

For the exercises 11-15, convert the given Cartesian coordinates to polar coordinates with $r > 0$ and $0 \leq \theta \leq 2\pi$. Remember to consider the quadrant in which the given point is located.

11) $(4, 2)$

Answer

$(2\sqrt{5}, 0.464)$

12) $(-4, 6)$

13) $(3, -5)$

Answer

$(\sqrt{34}, 5.253)$

14) $(-10, -13)$

15) $(8, 8)$

Answer

$\left(8\sqrt{2}, \frac{\pi}{4}\right)$

For the exercises 16-27, convert the given Cartesian equation to a polar equation.

16) $x = 3$

17) $y = 4$

Answer

$r = 4 \csc \theta$

18) $y = 4x^2$

19) $y = 2x^4$

Answer

$r = \sqrt[3]{\frac{\sin \theta}{2 \cos^4 \theta}}$

20) $x^2 + y^2 = 4y$

21) $x^2 + y^2 = 3x$

Answer

$r = 3 \cos \theta$

22) $x^2 - y^2 = x$

23) $x^2 - y^2 = 3y$

Answer

$$r = \frac{3\sin\theta}{\cos(2\theta)}$$

24) $x^2 + y^2 = 9$

25) $x^2 = 9y$

Answer

$$r = \frac{9\sin\theta}{\cos^2\theta}$$

26) $y^2 = 9x$

27) $9xy = 1$

Answer

$$r = \sqrt{\frac{1}{9\cos\theta\sin\theta}}$$

For the exercises 28-39, convert the given polar equation to a Cartesian equation. Write in the standard form of a conic if possible, and identify the conic section represented.

28) $r = 3\sin\theta$

29) $r = 4\cos\theta$

Answer

$x^2 + y^2 = 4x$ or $\frac{(x-2)^2}{4} + \frac{y^2}{4} = 1$; circle

30) $r = \frac{4}{\sin\theta + 7\cos\theta}$

31) $r = \frac{6}{\cos\theta + 3\sin\theta}$

Answer

$3y + x = 6$; line

32) $r = 2\sec\theta$

33) $r = 3\csc\theta$

Answer

$y = 3$; line

34) $r = \sqrt{r\cos\theta + 2}$

35) $r^2 = 4\sec\theta\csc\theta$

Answer

$xy = 4$; hyperbola

36) $r = 4$

37) $r^2 = 4$

Answer

$x^2 + y^2 = 4$; circle

38) $r = \dfrac{1}{4\cos\theta - 3\sin\theta}$

39) $r = \dfrac{3}{\cos\theta - 5\sin\theta}$

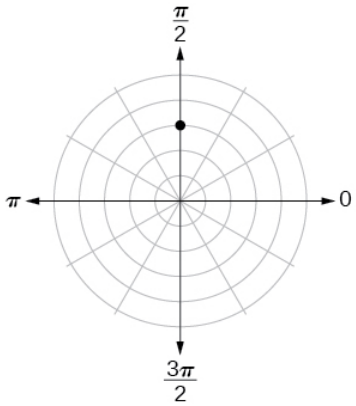
Answer

$x - 5y = 3$; line

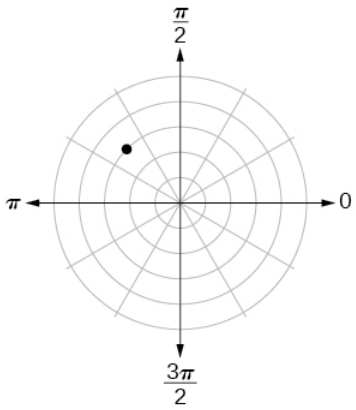
Graphical

For the exercises 40-44, find the polar coordinates of the point.

40)



41)

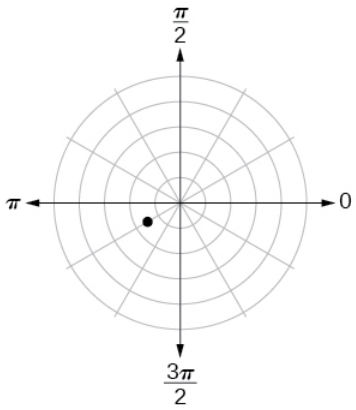


Answer

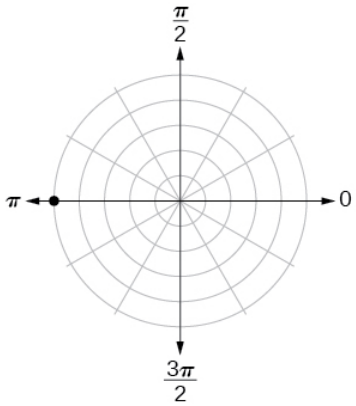
$\left(3, \dfrac{3\pi}{4}\right)$

42)

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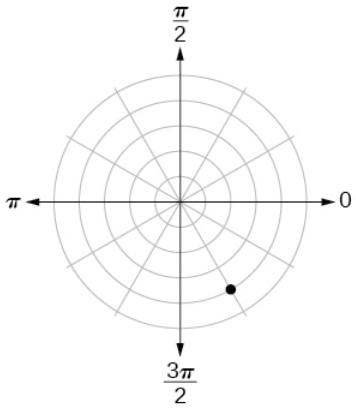
43)



Answer

$(5, \pi)$

44)

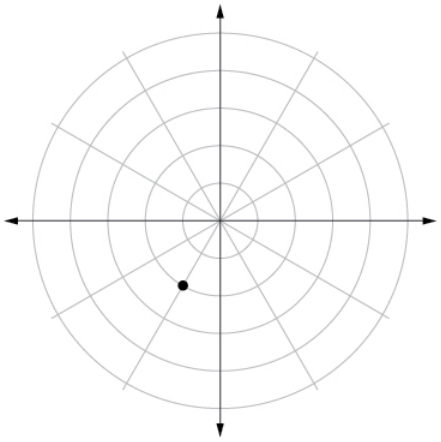


For the exercises 45-54, plot the points.

45) $\left(-2, \frac{\pi}{3}\right)$

Answer

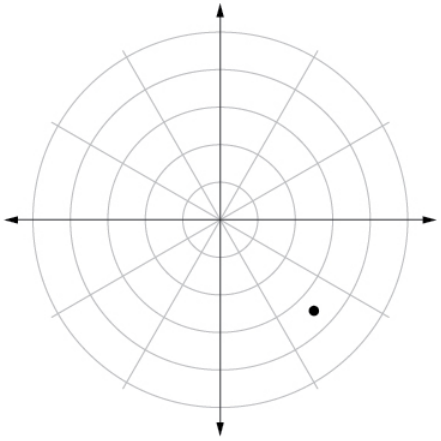
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46) $\left(-1, -\frac{\pi}{2}\right)$

47) $\left(3.5, \frac{7\pi}{4}\right)$

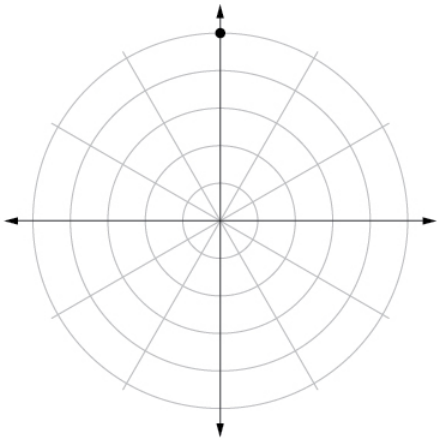
Answer



48) $\left(-4, \frac{\pi}{3}\right)$

49) $\left(5, \frac{\pi}{2}\right)$

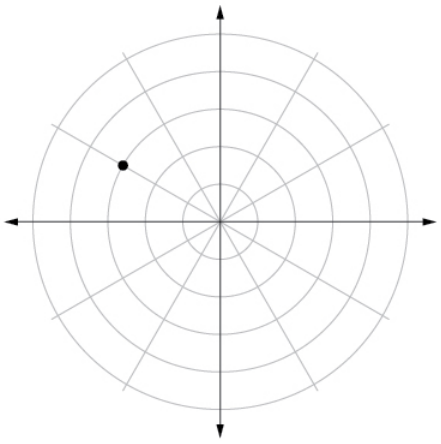
Answer



50) $\left(4, \frac{-5\pi}{4}\right)$

51) $\left(3, \frac{5\pi}{6}\right)$

Answer

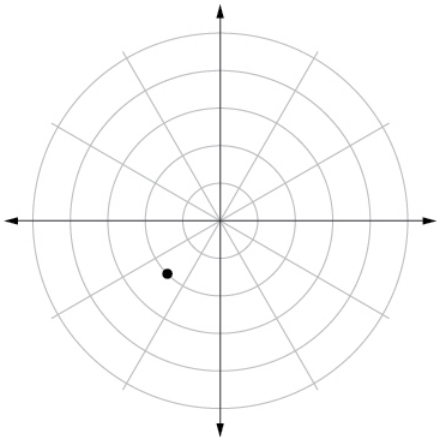


52) $\left(-1.5, \frac{7\pi}{6}\right)$

53) $\left(-2, \frac{\pi}{4}\right)$

Answer

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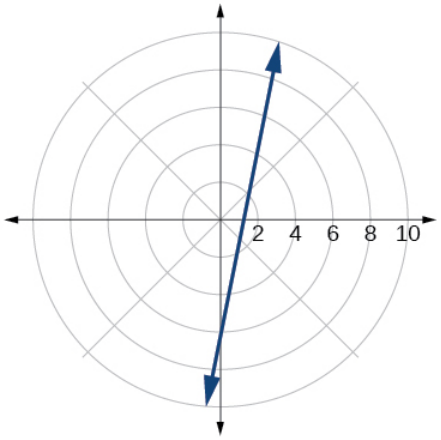
54) $\left(1, \frac{3\pi}{2}\right)$

For the exercises 55-61, convert the equation from rectangular to polar form and graph on the polar axis.

55) $5x - y = 6$

Answer

$$r = \frac{6}{5\cos\theta - \sin\theta}$$



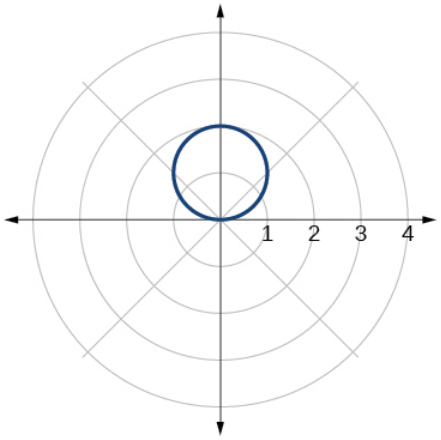
56) $2x + 7y = -3$

57) $x^2 + (y-1)^2 = 1$

Answer

$$r = 2\sin\theta$$

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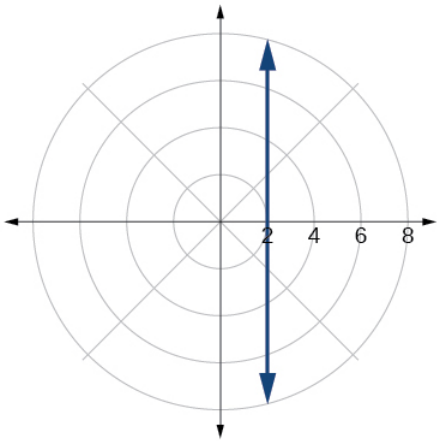


58) $(x+2)^2+(y+3)^2=13$

59) $x=2$

Answer

$r=\dfrac{2}{\cos\theta}$



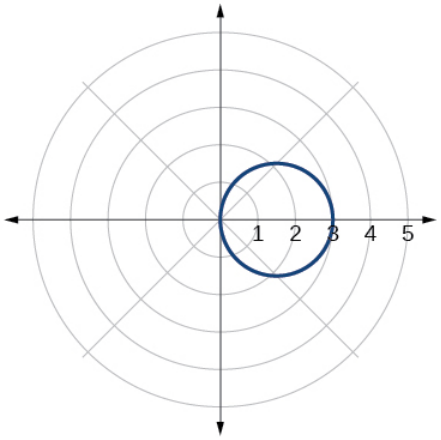
60) $x^2+y^2=5y$

61) $x^2+y^2=3x$

Answer

$r=3\cos\theta$

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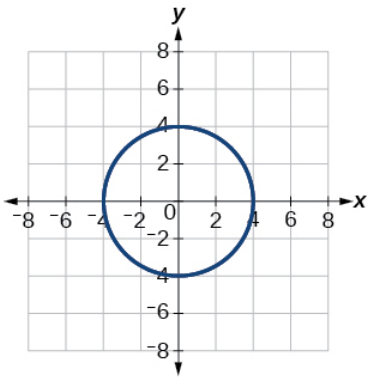
For the exercises 62-68, convert the equation from polar to rectangular form and graph on the rectangular plane.

62) $r = 6$

63) $r = -4$

Answer

$x^2 + y^2 = 16$

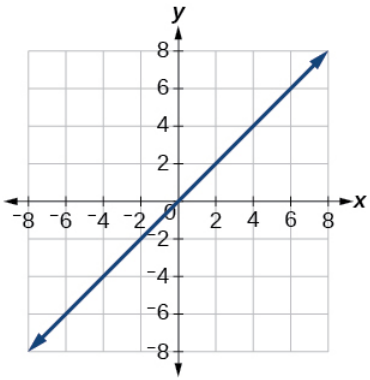


64) $\theta = -\dfrac{2\pi}{3}$

65) $\theta = \dfrac{\pi}{4}$

Answer

$y = x$

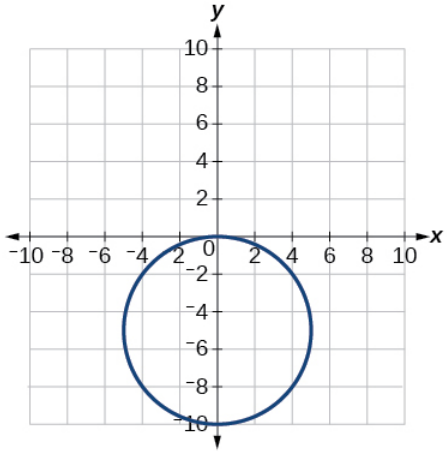


66) $r = \sec\theta$

67) $r = -10\sin\theta$

Answer

$x^2 + (y+5)^2 = 25$



68) $r = 3\cos\theta$

Technology

69) Use a graphing calculator to find the rectangular coordinates of $\left(2, -\frac{\pi}{5}\right)$. Round to the nearest thousandth.

Answer

$(1.618, -1.176)$

70) Use a graphing calculator to find the rectangular coordinates of $\left(-3, \frac{3\pi}{7}\right)$. Round to the nearest thousandth.

71) Use a graphing calculator to find the polar coordinates of $(-7, 8)$ in degrees. Round to the nearest thousandth.

Answer

$(10.630, 131.186^\circ)$

72) Use a graphing calculator to find the polar coordinates of $(3, -4)$ in degrees. Round to the nearest hundredth.

73) Use a graphing calculator to find the polar coordinates of $(-2, 0)$ in degrees. Round to the nearest hundredth.

Answer

$(2, 3.14)$or $(2, \pi)$

Extensions

74) Describe the graph of $r = a\sec\theta$; $a > 0$.

75) Describe the graph of $r = a\sec\theta$; $a < 0$.

Answer

A vertical line with a units left of the y-axis.

76) Describe the graph of $r = a\csc\theta$; $a > 0$.

77) Describe the graph of $r = a\csc\theta$; $a < 0$.

Answer

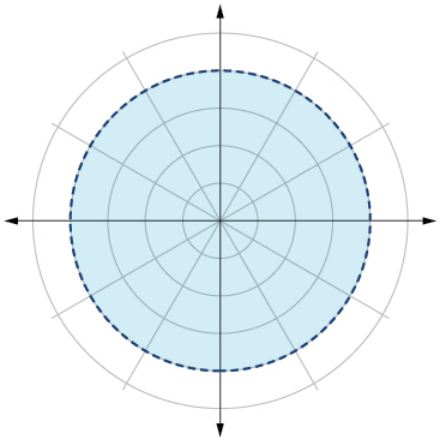
A horizontal line with a units below the x-axis.

78) What polar equations will give an oblique line?

For the exercises 79-84, graph the polar inequality.

79) $r < 4$

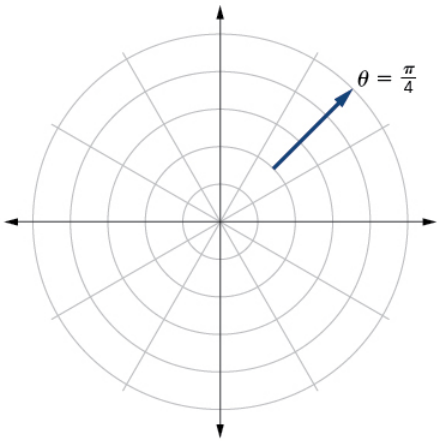
Answer



80) $0 \leq \theta \leq \frac{\pi}{4}$

81) $\theta = \frac{\pi}{4}, r \geq 2$

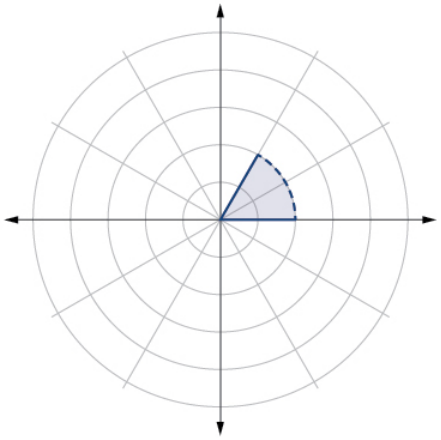
Answer



82) $\theta = \frac{\pi}{4}, r \geq -3$

83) $0 \leq \theta \leq \frac{\pi}{3}, r < 2$

Answer



84) $\frac{-\pi}{6} < \theta \leq \frac{\pi}{3}, -3 < r < 2$

8.4: Polar Coordinates - Graphs

Verbal

1) Describe the three types of symmetry in polar graphs, and compare them to the symmetry of the Cartesian plane.

Answer

Symmetry with respect to the polar axis is similar to symmetry about the x-axis, symmetry with respect to the pole is similar to symmetry about the origin, and symmetric with respect to the line $\theta = \frac{\pi}{2}$ is similar to symmetry about the y-axis.

2) Which of the three types of symmetries for polar graphs correspond to the symmetries with respect to the x-axis, y-axis, and origin?

3) What are the steps to follow when graphing polar equations?

Answer

Test for symmetry; find zeros, intercepts, and maxima; make a table of values. Decide the general type of graph, cardioid, limaçon, lemniscate, etc., then plot points at $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$ and sketch the graph.

4) Describe the shapes of the graphs of cardioids, limaçons, and lemniscates.

5) What part of the equation determines the shape of the graph of a polar equation?

Answer

The shape of the polar graph is determined by whether or not it includes a sine, a cosine, and constants in the equation.

Graphical

For the exercises 6-15, test the equation for symmetry.

6) $r = 5\cos 3\theta$

7) $r = 3 - 3\cos\theta$

Answer

symmetric with respect to the polar axis

8) $r = 3 + 2\sin\theta$

9) $r = 3\sin 2\theta$

Answer

symmetric with respect to the polar axis, symmetric with respect to the line $\theta = \frac{\pi}{2}$, symmetric with respect to the pole

10) $r = 4$

11) $r = 2\theta$

Answer

no symmetry

12) $r = 4\cos\frac{\theta}{2}$

13) $r = \frac{2}{\theta}$

Answer

no symmetry

14) $r = 3\sqrt{1-\cos^2\theta}$

15) $r = \sqrt{5\sin 2\theta}$

Answer

symmetric with respect to the pole

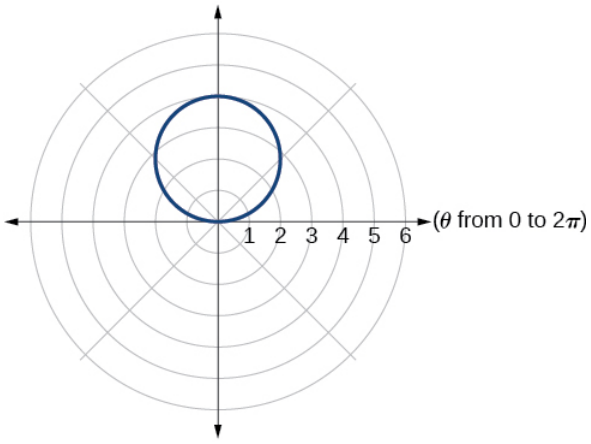
For the exercises 16-43, graph the polar equation. Identify the name of the shape.

16) $r = 3\cos\theta$

17) $r = 4\sin\theta$

Answer

circle

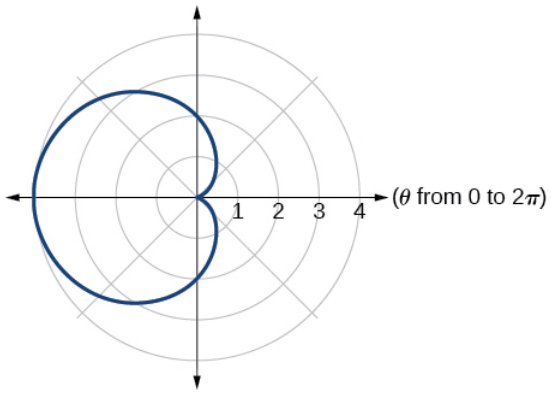


18) $r = 2 + 2\cos\theta$

19) $r = 2 - 2\cos\theta$

Answer

cardioid

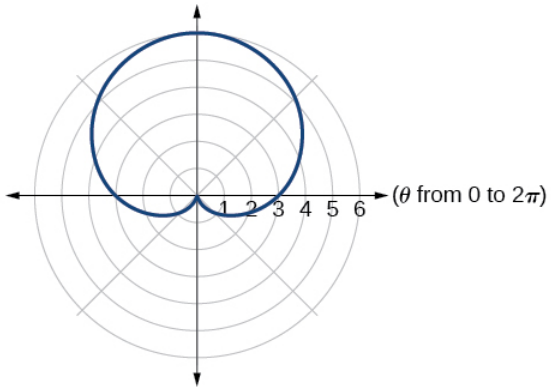


20) $r = 5 - 5\sin\theta$

21) $r = 3 + 3\sin\theta$

Answer

cardioid

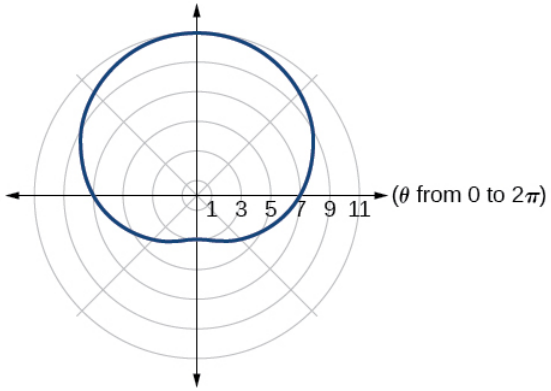


22) $r = 3 + 2\sin\theta$

23) $r = 7 + 4\sin\theta$

Answer

one-loop/dimpled limaçon

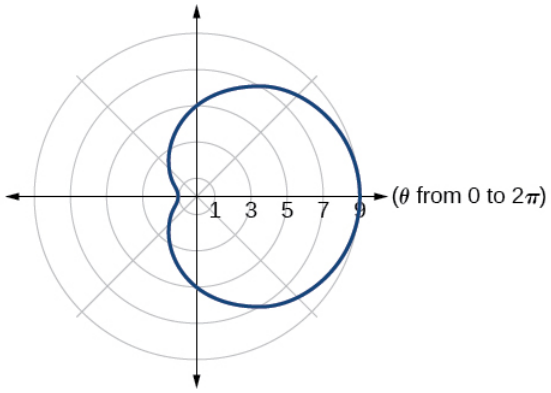


24) $r = 4 + 3\cos\theta$

25) $r = 5 + 4cos\theta$

Answer

one-loop/dimpled limaçon

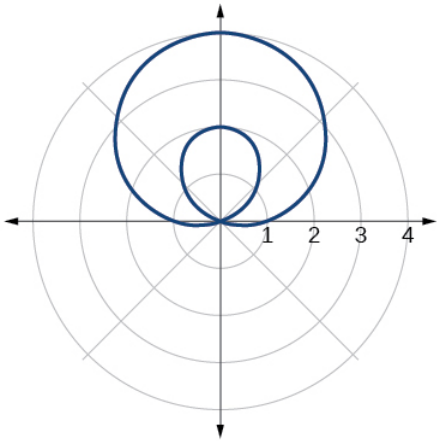


26) $r = 10 + 9\cos\theta$

27) $r = 1 + 3\sin\theta$

Answer

inner loop/ two-loop limaçon

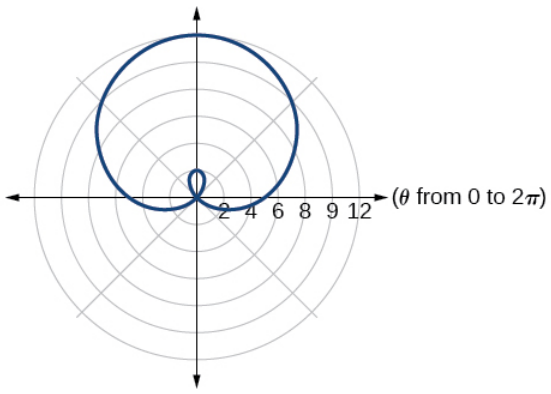


28) $r = 2 + 5\sin\theta$

29) $r = 5 + 7\sin\theta$

Answer

inner loop/ two-loop limaçon

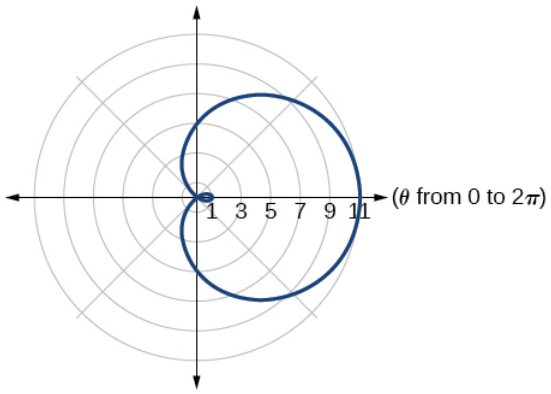


30) $r = 2 + 4\cos\theta$

31) $r = 5 + 6\cos\theta$

Answer

inner loop/ two-loop limaçon

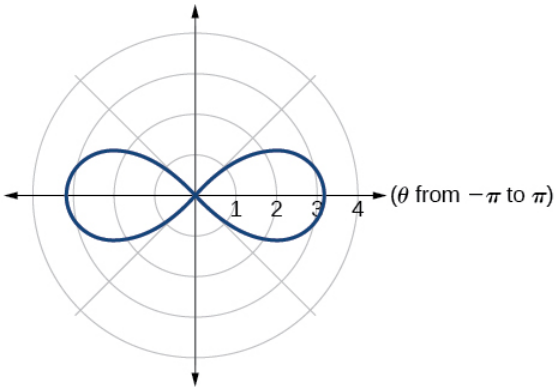


32) $r^2 = 36\cos(2\theta)$

33) $r^2 = 10\cos(2\theta)$

Answer

lemniscate



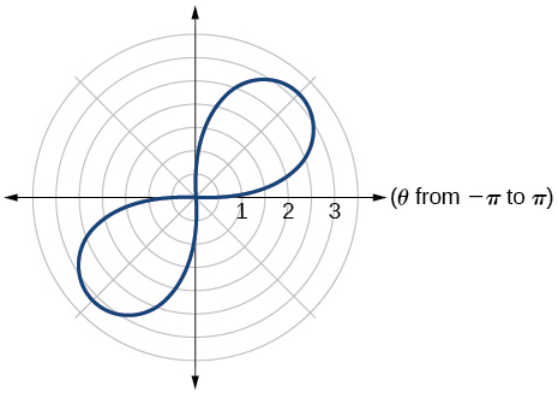
34) $r^2 = 4\sin(2\theta)$

35) $r^2 = 10\sin(2\theta)$

Answer

lemniscate

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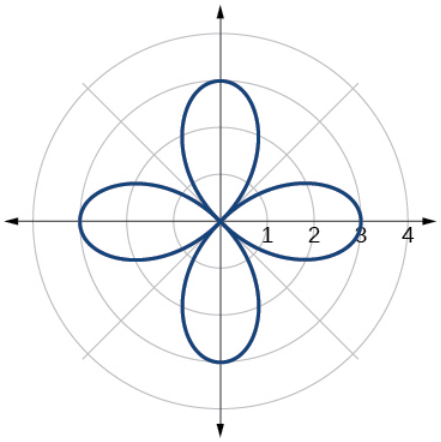


36) $r = 3\sin(2\theta)$

37) $r = 3\cos(2\theta)$

Answer

rose curve

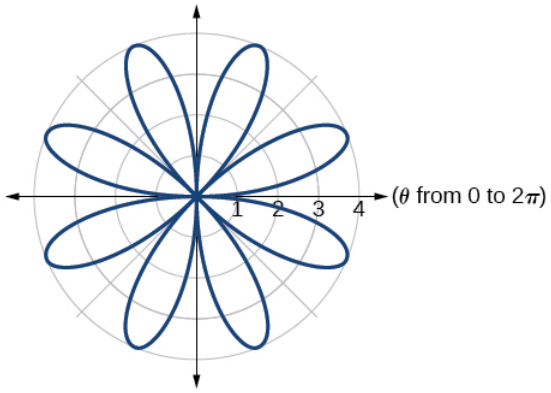


38) $r = 5\sin(3\theta)$

39) $r = 4\sin(4\theta)$

Answer

rose curve

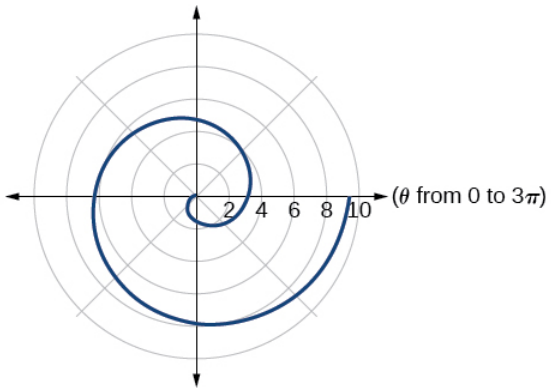


40) $r = 4\sin(5\theta)$

41) $r = -\theta$

Answer

Archimedes' spiral

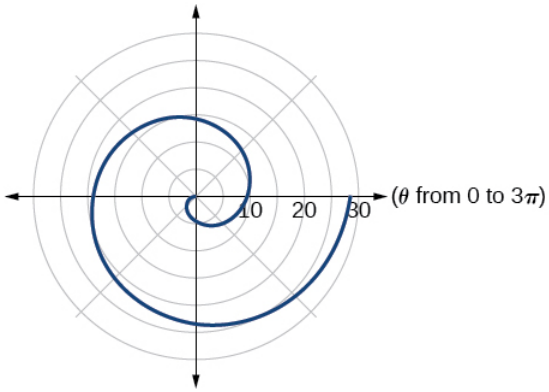


42) $r = 2\theta$

43) $r = -3\theta$

Answer

Archimedes' spiral



Technology

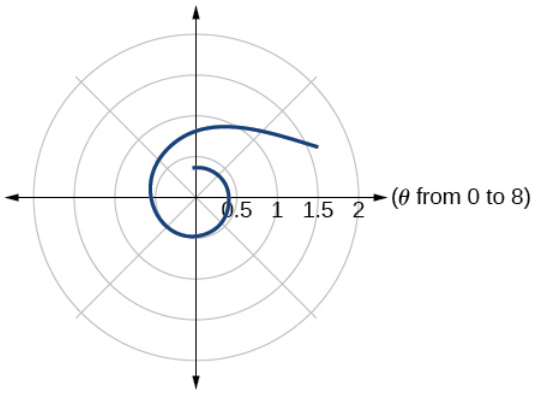
For the exercises 44-53, use a graphing calculator to sketch the graph of the polar equation.

44) $r = \dfrac{1}{\theta}$

45) $r = \dfrac{1}{\sqrt{\theta}}$

Answer

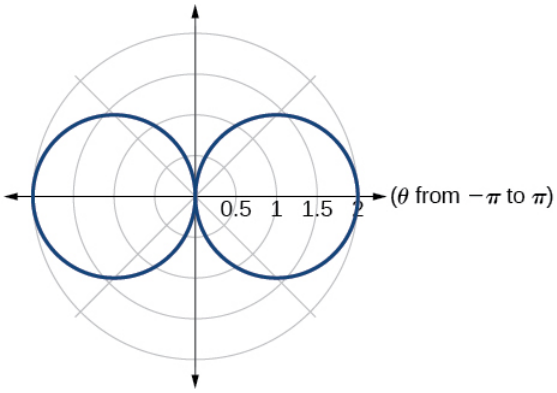
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46) $r = 2\sin\theta\tan\theta$, a cissoid

47) $r = 2\sqrt{1-\sin^2\theta}$, a hippopede

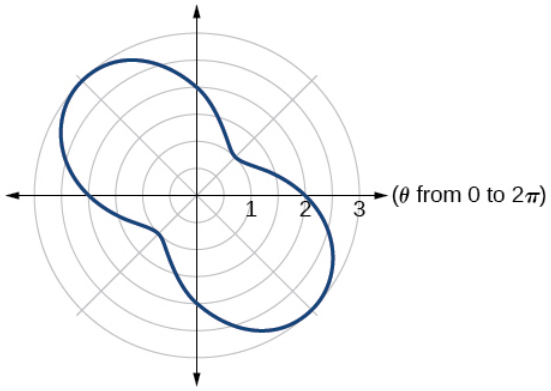
Answer



48) $r = 5 + \cos(4\theta)$

49) $r = 2 - \sin(2\theta)$

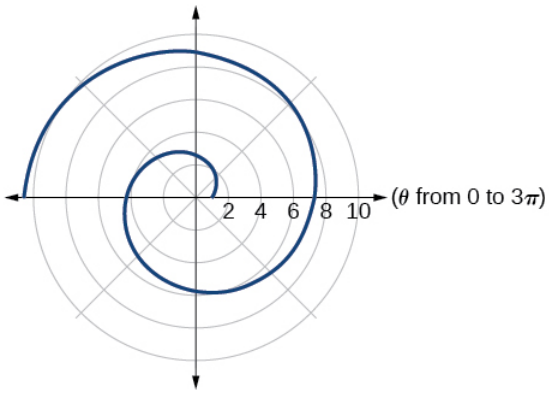
Answer



50) $r = \theta^2$

51) $r = \theta + 1$

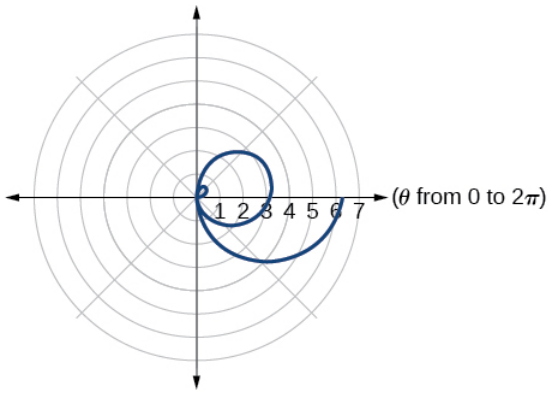
Answer



52) $r = \theta \sin \theta$

53) $r = \theta \cos \theta$

Answer



For the exercises 54-63, use a graphing utility to graph each pair of polar equations on a domain of $[0, 4\pi]$ and then explain the differences shown in the graphs.

54) $r = \theta, r = -\theta$

55) $r = \theta, r = \theta + \sin \theta$

Answer

They are both spirals, but not quite the same.

56) $r = \sin \theta + \theta, r = \sin \theta - \theta$

57) $r = 2 \sin\left(\frac{\theta}{2}\right), r = \theta \sin\left(\frac{\theta}{2}\right)$

Answer

Both graphs are curves with 2 loops. The equation with a coefficient of θ has two loops on the left, the equation with a coefficient of 2 has two loops side by side. Graph these from 0 to 4π to get a better picture.

58) $r = \sin(\cos(3\theta)), r = \sin(3\theta)$

59) On a graphing utility, graph $r = \sin\left(\frac{16}{5}\theta\right)$ on $[0, 4\pi]$, $[0, 8\pi]$, $[0, 12\pi]$ and $[0, 16\pi]$ Describe the effect of increasing the width of the domain.

Answer

When the width of the domain is increased, more petals of the flower are visible.

60) On a graphing utility, graph and sketch $r = \sin\theta + \left(\sin\left(\frac{5}{2}\theta\right)\right)^3$ on $[0, 4\pi]$.

61) On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3\sin(3\theta)$$
$$r_2 = 2\sin(3\theta)$$
$$r_3 = \sin(3\theta)$$

Answer

The graphs are three-petal, rose curves. The larger the coefficient, the greater the curve's distance from the pole.

62) On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3 + 3\cos(\theta)$$
$$r_2 = 2 + 2\cos(\theta)$$
$$r_3 = 1 + \cos(\theta)$$

63) On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3\theta$$
$$r_2 = 2\theta$$
$$r_3 = \theta$$

Answer

The graphs are spirals. The smaller the coefficient, the tighter the spiral.

Extensions

For the exercises 64-72, draw each polar equation on the same set of polar axes, and find the points of intersection.

64) $r_1 = 3 + 2\sin\theta, r_2 = 2$

65) $r_1 = 6 - 4\cos\theta, r_2 = 4$

Answer

$\left(4, \frac{\pi}{3}\right), \left(4, \frac{5\pi}{3}\right)$

66) $r_1 = 1 + \sin\theta, r_2 = 3\sin\theta$

67) $r_1 = 1 + \cos\theta, r_2 = 3\cos\theta$

Answer

$\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right)$

68) $r_1 = \cos(2\theta), r_2 = \sin(2\theta)$

69) $r_1 = \sin^2(2\theta), r_2 = 1 - \cos(4\theta)$

Answer

$\left(0, \frac{\pi}{2}\right), (0, \pi), \left(0, \frac{3\pi}{2}\right), (0, 2\pi)$

70) $r_1 = \sqrt{3}, r_2 = 2\sin(\theta)$

71) $r_1^2 = \sin\theta, r_2^2 = \cos\theta$

Answer

$\left(\frac{\sqrt[4]{8}}{2}, \frac{\pi}{4}\right), \left(\frac{\sqrt[4]{8}}{2}, \frac{5\pi}{4}\right)$, and at $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ since r is squared

72) $r_1 = 1 + \cos\theta, r_2 = 1 - \sin\theta$

8.5: Polar Form of Complex Numbers

Verbal

1) A complex number is $a + bi$. Explain each part.

Answer

a is the real part, b is the imaginary part, and $i = \sqrt{-1}$

2) What does the absolute value of a complex number represent?

3) How is a complex number converted to polar form?

Answer

Polar form converts the real and imaginary part of the complex number in polar form using $x = r\cos\theta$ and $y = r\sin\theta$.

4) How do we find the product of two complex numbers?

5) What is De Moivre's Theorem and what is it used for?

Answer

$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

It is used to simplify polar form when a number has been raised to a power.

Algebraic

For the exercises 6-11, find the absolute value of the given complex number.

6) $5 + 3i$

7) $-7 + i$

Answer

$5\sqrt{2}$

8) $-3 - 3i$

9) $\sqrt{2} - 6i$

Answer

$\sqrt{38}$

10) $2i$

11) $2.2 - 3.1i$

Answer

$\sqrt{14.45}$

For the exercises 12-16, write the complex number in polar form.

12) $2+2i$

13) $8-4i$

Answer

$4\sqrt{5}\textbf{cis}(333.4^\circ)$

14) $-\frac{1}{2}-\frac{1}{2}i$

15) $\sqrt{3}+i$

Answer

$2\textbf{cis}\left(\frac{\pi}{6}\right)$

16) $3i$

For the exercises 17-22, convert the complex number from polar to rectangular form.

17) $z=7\textbf{cis}\left(\frac{\pi}{6}\right)$

Answer

$\frac{7\sqrt{3}}{2}+i\frac{7}{2}$

18) $z=2\textbf{cis}\left(\frac{\pi}{3}\right)$

19) $z=4\textbf{cis}\left(\frac{7\pi}{6}\right)$

Answer

$-2\sqrt{3}-2i$

20) $z=7\textbf{cis}(25^\circ)$

21) $z=3\textbf{cis}(240^\circ)$

Answer

$-1.5-i\frac{3\sqrt{3}}{2}$

22) $z=\sqrt{2}\textbf{cis}(100^\circ)$

For the exercises 23-28, find $z_1 z_2$ in polar form.

23) $z_1=2\sqrt{3}\textbf{cis}(116^\circ); z_2=2\textbf{cis}(82^\circ)$

Answer

$4\sqrt{3}\textbf{cis}(198^\circ)$

24) $z_1=\sqrt{2}\textbf{cis}(205^\circ); z_2=2\sqrt{2}\textbf{cis}(118^\circ)$

25) $z_1=3\textbf{cis}(120^\circ); z_2=\frac{1}{4}\textbf{cis}(60^\circ)$

Answer

$\frac{3}{4}\textbf{cis}(180^\circ)$

26) $z_1 = 3\mathbf{cis}\left(\frac{\pi}{4}\right); z_2 = 5\mathbf{cis}\left(\frac{\pi}{6}\right)$

27) $z_1 = \sqrt{5}\mathbf{cis}\left(\frac{5\pi}{8}\right); z_2 = \sqrt{15}\mathbf{cis}\left(\frac{\pi}{12}\right)$

Answer

$5\sqrt{3}\mathbf{cis}\left(\frac{17\pi}{24}\right)$

28) $z_1 = 4\mathbf{cis}\left(\frac{\pi}{2}\right); z_2 = 2\mathbf{cis}\left(\frac{\pi}{4}\right)$

For the exercises 29-,34 find $\frac{z_1}{z_2}$ in polar form.

29) $z_1 = 21\mathbf{cis}(135^\circ); z_2 = 3\mathbf{cis}(65^\circ)$

Answer

$7\mathbf{cis}(70^\circ)$

30) $z_1 = \sqrt{2}\mathbf{cis}(90^\circ); z_2 = 2\mathbf{cis}(60^\circ)$

31) $z_1 = 15\mathbf{cis}(120^\circ); z_2 = 3\mathbf{cis}(40^\circ)$

Answer

$5\mathbf{cis}(80^\circ)$

32) $z_1 = 6\mathbf{cis}\left(\frac{\pi}{3}\right); z_2 = 2\mathbf{cis}\left(\frac{\pi}{4}\right)$

33) $z_1 = 5\sqrt{2}\mathbf{cis}(\pi); z_2 = \sqrt{2}\mathbf{cis}\left(\frac{2\pi}{3}\right)$

Answer

$5\mathbf{cis}\left(\frac{\pi}{3}\right)$

34) $z_1 = 2\mathbf{cis}\left(\frac{3\pi}{5}\right); z_2 = 3\mathbf{cis}\left(\frac{\pi}{4}\right)$

For the exercises 35-40, find the powers of each complex number in polar form.

35) Find z^3 when $z = 5\mathbf{cis}(45^\circ)$.

Answer

$125\mathbf{cis}(135^\circ)$

36) Find z^4 when $z = 2\mathbf{cis}(70^\circ)$.

37) Find z^2 when $z = 3\mathbf{cis}(120^\circ)$.

Answer

$9\mathbf{cis}(240^\circ)$

38) Find z^2 when $z = 4\mathbf{cis}\left(\frac{\pi}{4}\right)$.

39) Find z^4 when $z = \mathbf{cis}\left(\frac{3\pi}{16}\right)$.

Answer

$\mathbf{cis}\left(\frac{3\pi}{4}\right)$

40) Find z^3 when $z = 3\mathbf{cis}\left(\frac{5\pi}{3}\right)$.

For the exercises 41-45, evaluate each root.

41) Evaluate the cube root of z when $z = 27\mathbf{cis}(240^\circ)$.

Answer

$3\mathbf{cis}(80^\circ), 3\mathbf{cis}(200^\circ), 3\mathbf{cis}(320^\circ)$

42) Evaluate the square root of z when $z = 16\mathbf{cis}(100^\circ)$.

43) Evaluate the cube root of z when $z = 32\mathbf{cis}\left(\frac{2\pi}{3}\right)$.

Answer

$2\sqrt[3]{4}\mathbf{cis}\left(\frac{2\pi}{9}\right), 2\sqrt[3]{4}\mathbf{cis}\left(\frac{8\pi}{9}\right), 2\sqrt[3]{4}\mathbf{cis}\left(\frac{14\pi}{9}\right)$

44) Evaluate the square root of z when $z = 32\mathbf{cis}(\pi)$.

45) Evaluate the cube root of z when $z = 8\mathbf{cis}\left(\frac{7\pi}{4}\right)$.

Answer

$2\sqrt{2}\mathbf{cis}\left(\frac{7\pi}{8}\right), 2\sqrt{2}\mathbf{cis}\left(\frac{15\pi}{8}\right)$

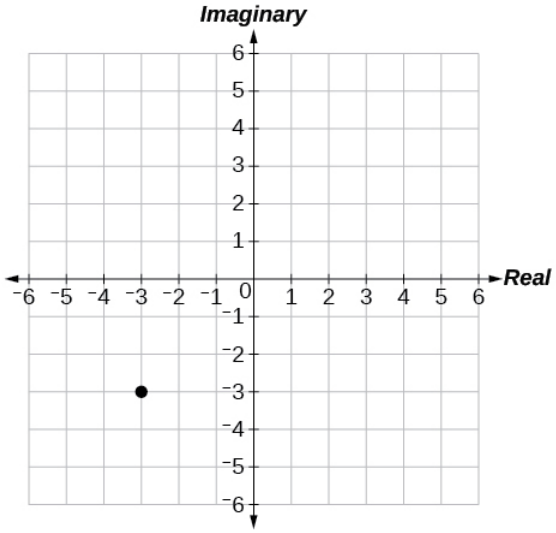
Graphical

For the exercises 46-55, plot the complex number in the complex plane.

46) $2 + 4i$

47) $-3 - 3i$

Answer

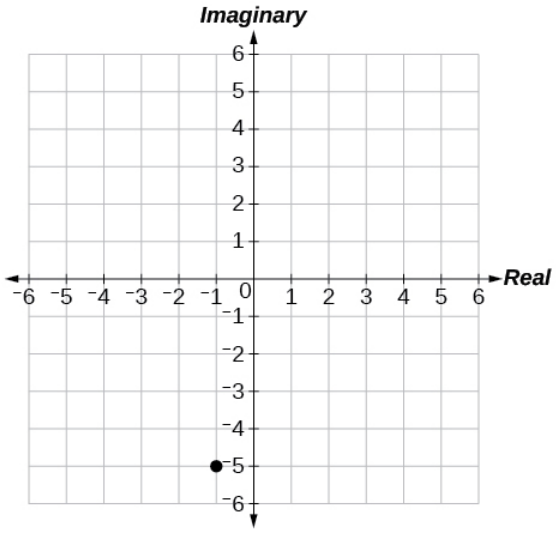


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48) $5-4i$

49) $-1-5i$

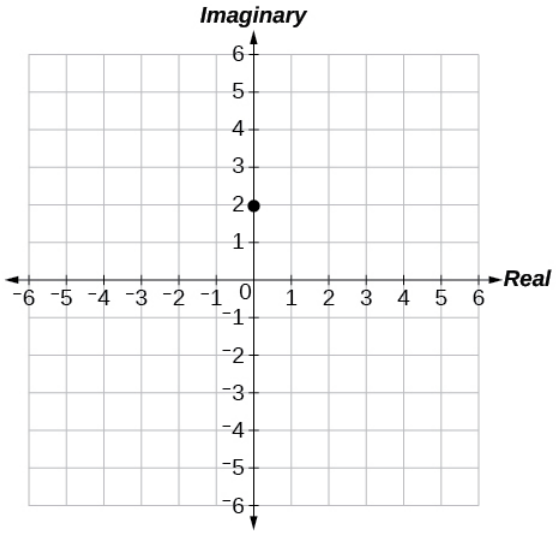
Answer



50) $3+2i$

51) $2i$

Answer

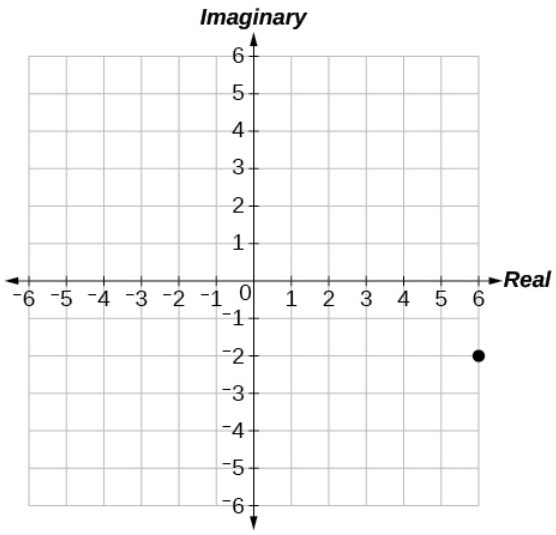


52) -4

53) $6-2i$

Answer

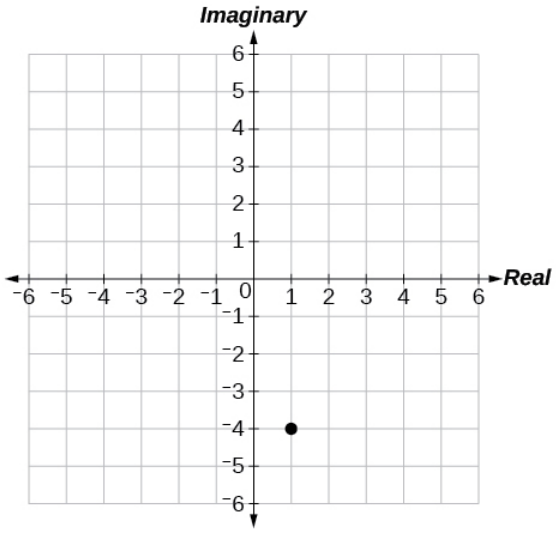
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54) $-2+i$

55) $1-4i$

Answer



Technology

For the exercises 56-, find all answers rounded to the nearest hundredth.

56) Use the rectangular to polar feature on the graphing calculator to change $5+5i$ to polar form.

57) Use the rectangular to polar feature on the graphing calculator to change $3-2i$ to polar form.

Answer

$3.61e^{-0.59i}$

58) Use the rectangular to polar feature on the graphing calculator to change $-3-8i$ to polar form.

59) Use the polar to rectangular feature on the graphing calculator to change $4\mathbf{cis}(120^\circ)$ to rectangular form.

Answer

$-2+3.46i$

60) Use the polar to rectangular feature on the graphing calculator to change $2\mathbf{cis}(45^\circ)$ to rectangular form.

61) Use the polar to rectangular feature on the graphing calculator to change $5\mathbf{cis}(210^\circ)$ to rectangular form.

Answer

$-4.33-2.50i$

8.6: Parametric Equations

Verbal

1) What is a system of parametric equations?

Answer

A pair of functions that is dependent on an external factor. The two functions are written in terms of the same parameter. For example, $x=f(t)$ and $y=f(t)$.

2) Some examples of a third parameter are time, length, speed, and scale. Explain when time is used as a parameter.

3) Explain how to eliminate a parameter given a set of parametric equations.

Answer

Choose one equation to solve for t, substitute into the other equation and simplify.

4) What is a benefit of writing a system of parametric equations as a Cartesian equation?

5) What is a benefit of using parametric equations?

Answer

Some equations cannot be written as functions, like a circle. However, when written as two parametric equations, separately the equations are functions.

6) Why are there many sets of parametric equations to represent on Cartesian function?

Algebraic

For the exercises 7-25, eliminate the parameter t to rewrite the parametric equation as a Cartesian equation.

7) $\begin{cases} x(t)=5-t \\ y(t)=8-2t \end{cases}$

Answer

$y=-2+2x$

8) $\begin{cases} x(t)=6-3t \\ y(t)=10-t \end{cases}$

9) $\begin{cases} x(t)=2t+1 \\ y(t)=3\sqrt{t} \end{cases}$

Answer

$y=3\sqrt{\dfrac{x-1}{2}}$

10) $\begin{cases} x(t)=3t-1 \\ y(t)=2t^2 \end{cases}$

11) $\begin{cases} x(t)=2e^t \\ y(t)=1-5t \end{cases}$

Answer

$$x = 2e^{\frac{1-y}{5}} \text{ or } y = 1 - 5\ln\left(\frac{x}{2}\right)$$

12) $\begin{cases} x(t) = e^{-2t} \\ y(t) = 2e^{-t} \end{cases}$

13) $\begin{cases} x(t) = 4\log(t) \\ y(t) = 3 + 2t \end{cases}$

Answer

$$x = 4\log\left(\frac{y-3}{2}\right)$$

14) $\begin{cases} x(t) = \log(2t) \\ y(t) = \sqrt{t-1} \end{cases}$

15) $\begin{cases} x(t) = t^3 - t \\ y(t) = 2t \end{cases}$

Answer

$$x = \left(\frac{y}{2}\right)^3 - \frac{y}{2}$$

16) $\begin{cases} x(t) = t - t^4 \\ y(t) = t + 2 \end{cases}$

17) $\begin{cases} x(t) = e^{2t} \\ y(t) = e^{6t} \end{cases}$

Answer

$$y = x^3$$

18) $\begin{cases} x(t) = t^5 \\ y(t) = t^{10} \end{cases}$

19) $\begin{cases} x(t) = 4\cos t \\ y(t) = 5\sin t \end{cases}$

Answer

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$

20) $\begin{cases} x(t) = 3\sin t \\ y(t) = 6\cos t \end{cases}$

21) $\begin{cases} x(t) = 2\cos^2 t \\ y(t) = -\sin t \end{cases}$

Answer

$$y^2 = 1 - \frac{1}{2}x$$

22) $\begin{cases} x(t) = \cos t + 4 \\ y(t) = 2\sin^2 t \end{cases}$

23) $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$

Answer

$y = x^2 + 2x + 1$

24) $\begin{cases} x(t) = -t \\ y(t) = t^3 + 1 \end{cases}$

25) $\begin{cases} x(t) = 2t - 1 \\ y(t) = t^3 - 2 \end{cases}$

Answer

$y = \left(\frac{x+1}{2}\right)^3 - 2$

For the exercises 26-29, rewrite the parametric equation as a Cartesian equation by building an $x - y$ table.

26) $\begin{cases} x(t) = 2t - 1 \\ y(t) = t + 4 \end{cases}$

27) $\begin{cases} x(t) = 4 - t \\ y(t) = 3t + 2 \end{cases}$

Answer

$y = -3x + 14$

28) $\begin{cases} x(t) = 2t - 1 \\ y(t) = 5t \end{cases}$

29) $\begin{cases} x(t) = 4t - 1 \\ y(t) = 4t + 2 \end{cases}$

Answer

$y = x + 3$

For the exercises 30-33, parameterize (write parametric equations for) each Cartesian equation by setting $x(t) = t$ or by setting $y(t) = t$.

30) $y(x) = 3x^2 + 3$

31) $y(x) = 2\sin x + 1$

Answer

$\begin{cases} x(t) = t \\ y(t) = 2\sin t + 1 \end{cases}$

32) $x(y) = 3\log(y) + y$

33) $x(y) = \sqrt{y} + 2y$

Answer

$\begin{cases} x(t) = \sqrt{t} + 2t \\ y(t) = t \end{cases}$

For the exercises 34-41, parameterize (write parametric equations for) each Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$. Identify the curve.

34) $\frac{x^2}{4} + \frac{x^2}{9} = 1$

35) $\frac{x^2}{16} + \frac{x^2}{36} = 1$

Answer

$\begin{cases} x(t) = 4\cos t \\ y(t) = 6\sin t \end{cases}$; Ellipse

36) $x^2 + y^2 = 16$

37) $x^2 + y^2 = 10$

Answer

$\begin{cases} x(t) = \sqrt{10}\cos t \\ y(t) = \sqrt{10}\sin t \end{cases}$; Circle

38) Parameterize the line from $(3, 0)$ to $(-2, -5)$ so that the line is at $(3, 0)$ at $t = 0$, and at $(-2, -5)$ at $t = 1$.

39) Parameterize the line from $(-1, 0)$ to $(3, -2)$ so that the line is at $(-1, 0)$ at $t = 0$, and at $(3, -2)$ at $t = 1$.

Answer

$\begin{cases} x(t) = -1 + 4t \\ y(t) = -2t \end{cases}$

40) Parameterize the line from $(-1, 5)$ to $(2, 3)$ so that the line is at $(-1, 5)$ at $t = 0$, and at $(2, 3)$ at $t = 1$.

41) Parameterize the line from $(4, 1)$ to $(6, -2)$ so that the line is at $(4, 1)$ at $t = 0$, and at $(6, -2)$ at $t = 1$.

Answer

$\begin{cases} x(t) = 4 + 2t \\ y(t) = 1 - 3t \end{cases}$

Technology

For the exercises 42-43, use the table feature in the graphing calculator to determine whether the graphs intersect.

42) $\begin{cases} x_1(t) = 3t \\ y_1(t) = 2t - 1 \end{cases}$ and $\begin{cases} x_2(t) = t + 3 \\ y_2(t) = 4t - 4 \end{cases}$

43) $\begin{cases} x_1(t) = t^2 \\ y_1(t) = 2t - 1 \end{cases}$ and $\begin{cases} x_2(t) = -t + 6 \\ y_2(t) = t + 1 \end{cases}$

Answer

yes, at $t = 2$

For the exercises 44-46, use a graphing calculator to complete the table of values for each set of parametric equations.

44) $\begin{cases} x_1(t) = 3t^2 - 3t + 7 \\ y_1(t) = 2t + 3 \end{cases}$

t	x	y
-1		
0		
1		

45) $\begin{cases} x_1(t) = t^2 - 4 \\ y_1(t) = 2t^2 - 1 \end{cases}$

t	x	y

t	x	y
1		
2		
3		

Answer

t	x	y
1	-3	1
2	0	7
3	5	17

46) $\begin{cases} x_1(t) = t^4 \\ y_1(t) = t^3 + 4 \end{cases}$

t	x	y
-1		
0		
1		
2		

Extensions

47) Find two different sets of parametric equations for $y = (x+1)^2$.

Answer

answers may vary: $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$ and $\begin{cases} x(t) = t + 1 \\ y(t) = (t+2)^2 \end{cases}$

48) Find two different sets of parametric equations for $y = 3x - 2$.

49) Find two different sets of parametric equations for $y = x^2 - 4x + 4$.

Answer

answers may vary: $\begin{cases} x(t) = t \\ y(t) = t^2 - 4t + 4 \end{cases}$ and $\begin{cases} x(t) = t + 2 \\ y(t) = t^2 \end{cases}$

8.7: Parametric Equations - Graphs

Verbal

1) What are two methods used to graph parametric equations?

Answer

plotting points with the orientation arrow and a graphing calculator

2) What is one difference in point-plotting parametric equations compared to Cartesian equations?

3) Why are some graphs drawn with arrows?

Answer

The arrows show the orientation, the direction of motion according to increasing values of t.

4) Name a few common types of graphs of parametric equations.

5) Why are parametric graphs important in understanding projectile motion?

Answer

The parametric equations show the different vertical and horizontal motions over time.

Graphical

For the exercises 6-11, graph each set of parametric equations by making a table of values. Include the orientation on the graph.

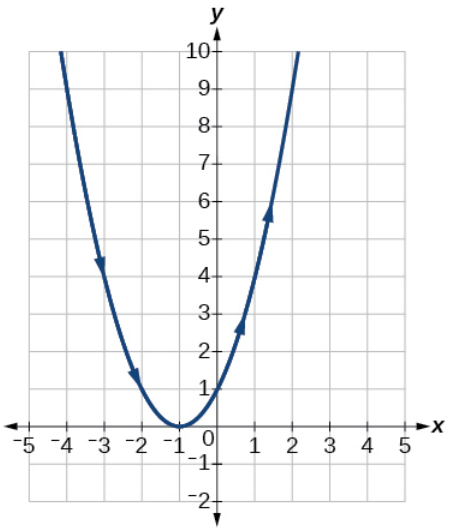
6) $\begin{cases} x(t) = t \\ y(t) = t^2 - 1 \end{cases}$

t	x	y
-3		
-2		
-1		
0		
1		
2		
3		

7) $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$

t	-3	-2	-1	0	1	2
x						
y						

Answer



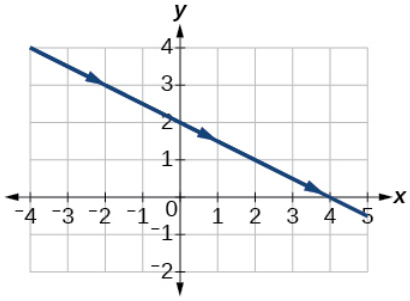
8) $\begin{cases} x(t) = 2 + t \\ y(t) = 3 - 2t \end{cases}$

t	-2	-1	0	1	2	3
x						
y						

9) $\begin{cases} x(t) = -2 - 2t \\ y(t) = 3 + t \end{cases}$

t	-3	-2	-1	0	1
x					
y					

Answer



10) $\begin{cases} x(t) = t^3 \\ y(t) = t + 2 \end{cases}$

t	-2	-1	0	1	2

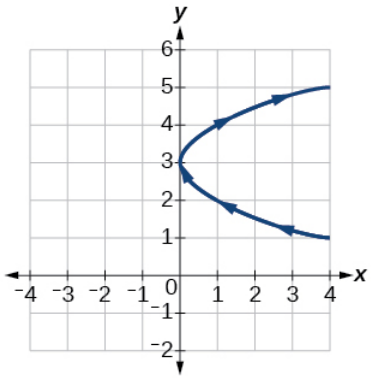
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x					
y					

11) $\begin{cases} x(t) = t^2 \\ y(t) = t + 3 \end{cases}$

t	-2	-1	0	1	2
x					
y					

Answer

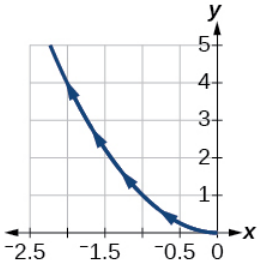


For the exercises 12-22, sketch the curve and include the orientation.

12) $\begin{cases} x(t) = t \\ y(t) = \sqrt{t} \end{cases}$

13) $\begin{cases} x(t) = -\sqrt{t} \\ y(t) = t \end{cases}$

Answer



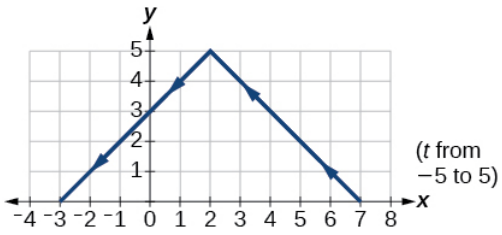
(t from -1 to 5)

14) $\begin{cases} x(t) = 5 - |t| \\ y(t) = t + 2 \end{cases}$

15) $\begin{cases} x(t) = -t + 2 \\ y(t) = 5 - |t| \end{cases}$

Answer

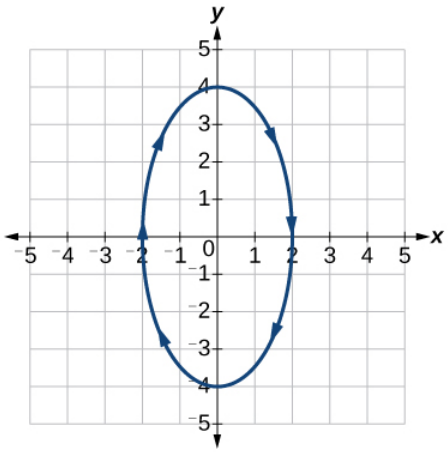
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16) $\begin{cases} x(t) = 4\sin t \\ y(t) = 2\cos t \end{cases}$

17) $\begin{cases} x(t) = 2\sin t \\ y(t) = 4\cos t \end{cases}$

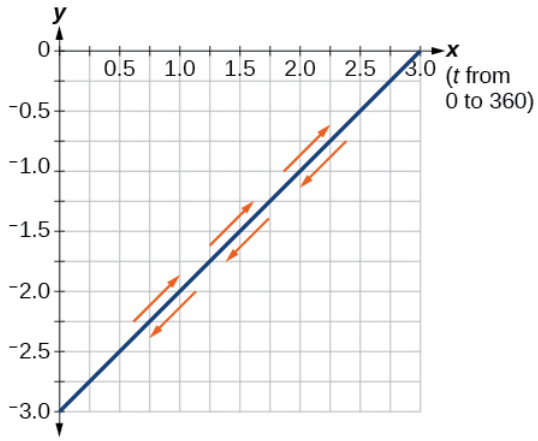
Answer



18) $\begin{cases} x(t) = 3\cos^2 t \\ y(t) = -3\sin t \end{cases}$

19) $\begin{cases} x(t) = 3\cos^2 t \\ y(t) = -3\sin^2 t \end{cases}$

Answer

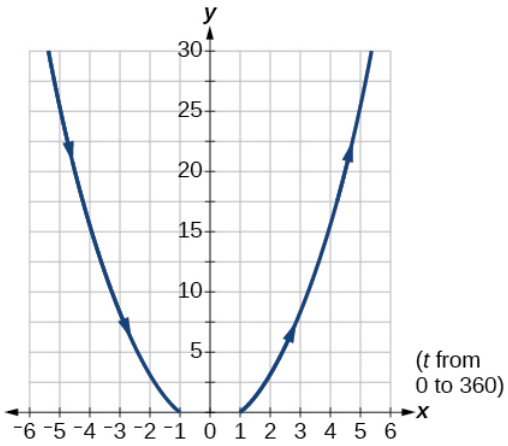


20) $\begin{cases} x(t) = \sec t \\ y(t) = \tan t \end{cases}$

21) $\begin{cases} x(t) = \sec t \\ y(t) = \tan^2 t \end{cases}$

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Answer

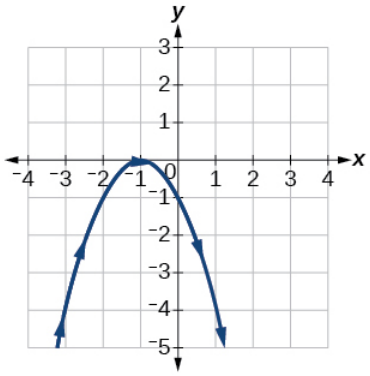


22) $\begin{cases} x(t) = \dfrac{1}{e^{2t}} \\ y(t) = e^{-t} \end{cases}$

For the exercises 23-27, graph the equation and include the orientation. Then, write the Cartesian equation.

23) $\begin{cases} x(t) = t - 1 \\ y(t) = -t^2 \end{cases}$

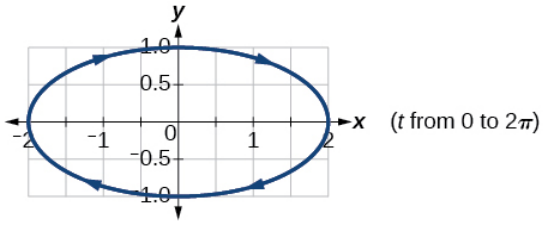
Answer



24) $\begin{cases} x(t) = t^3 \\ y(t) = t + 3 \end{cases}$

25) $\begin{cases} x(t) = 2\cos t \\ y(t) = -\sin t \end{cases}$

Answer

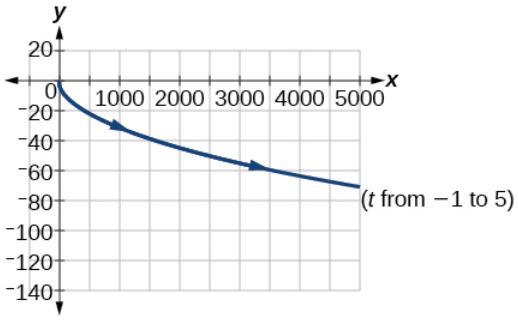


26) $\begin{cases} x(t) = 7\cos t \\ y(t) = 7\sin t \end{cases}$

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27) $\begin{cases} x(t) = e^{2t} \\ y(t) = -e^{t} \end{cases}$

Answer

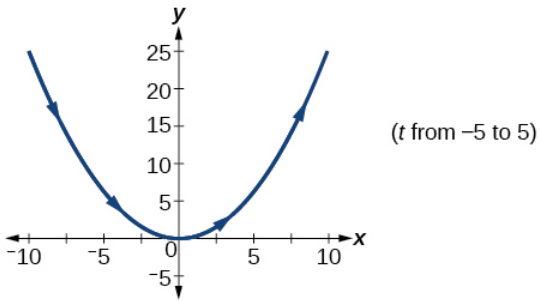


For the exercises 28-33, graph the equation and include the orientation.

28) $x = t^2, y = 3t, 0 \le t \le 5$

29) $x = 2t, y = t^2, -5 \le t \le 5$

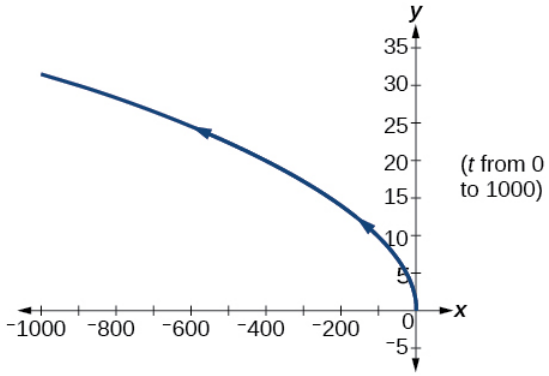
Answer



30) $x = t, y = \sqrt{25 - t^2}, 0 < t \le 5$

31) $x(t) = -t, y(t) = \sqrt{t}, t \ge 0$

Answer

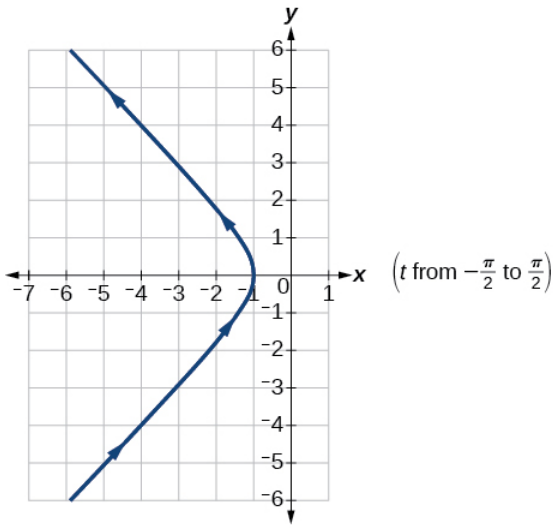


32) $x = -2\cos t, y = 6\sin t, 0 \le t \le \pi$

33) $x = -\sec t, y = \tan t, -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

Answer

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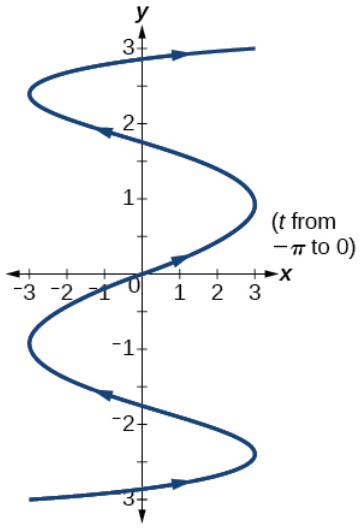
For the exercises 34-41, use the parametric equations for integers a and b:

$$x(t) = a\cos((a+b)t)$$
$$y(t) = a\cos((a-b)t)$$

34) Graph on the domain $[-\pi, 0]$, where $a = 2$ and $b = 1$, and include the orientation.

35) Graph on the domain $[-\pi, 0]$, where $a = 3$ and $b = 2$, and include the orientation.

Answer

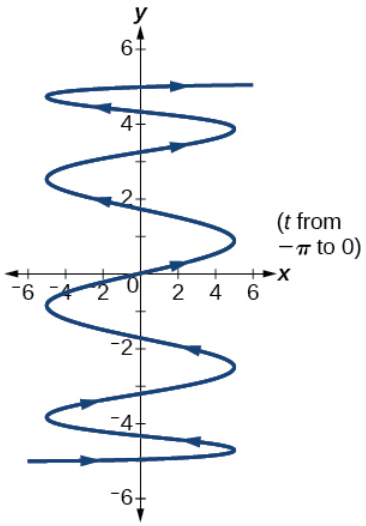


36) Graph on the domain $[-\pi, 0]$, where $a = 4$ and $b = 3$, and include the orientation.

37) Graph on the domain $[-\pi, 0]$, where $a = 5$ and $b = 4$, and include the orientation.

Answer

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38) If a is 1 more than b, describe the effect the values of a and b have on the graph of the parametric equations.

39) Describe the graph if $a = 100$ and $b = 99$.

Answer

There will be 100 back-and-forth motions.

40) What happens if b is 1 more than a? Describe the graph.

41) If the parametric equations $x(t) = t^2$ and $y(t) = 6 - 3t$ have the graph of a horizontal parabola opening to the right, what would change the direction of the curve?

Answer

Take the opposite of the $x(t)$ equation.

For the exercises 42-46, describe the graph of the set of parametric equations.

42) $x(t) = -t^2$ and $y(t)$ is linear

43) $y(t) = t^2$ and $x(t)$ is linear

Answer

The parabola opens up.

44) $y(t) = -t^2$ and $x(t)$ is linear

45) Write the parametric equations of a circle with center $(0, 0)$, radius 5, and a counterclockwise orientation.

Answer

$$\begin{cases} x(t) = 5\cos t \\ y(t) = 5\sin t \end{cases}$$

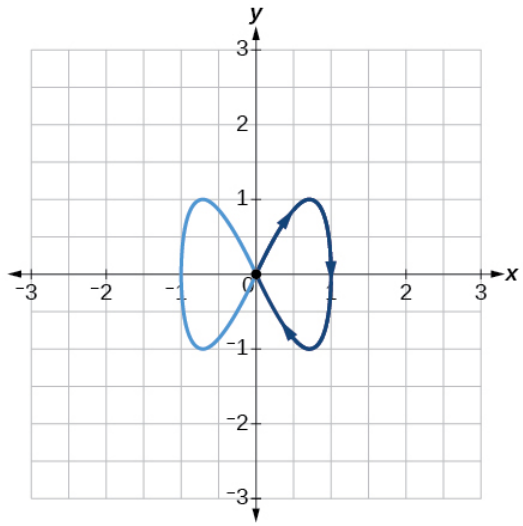
46) Write the parametric equations of an ellipse with center $(0, 0)$,major axis of length 10, minor axis of length 6, and a counterclockwise orientation.

For the exercises 47-52, use a graphing utility to graph on the window $[-3, 3]$ by $[-3, 3]$ on the domain $[0, 2\pi)$ for the following values of a and b, and include the orientation.

$$\begin{cases} x(t) = \sin(at) \\ y(t) = \sin(bt) \end{cases}$$

47) $a = 1, b = 2$

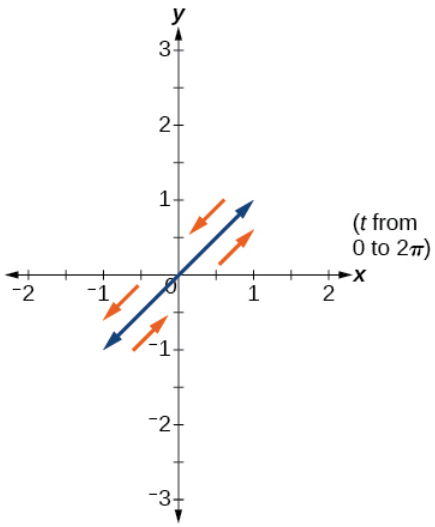
Answer



48) $a = 2, b = 1$

49) $a = 3, b = 3$

Answer

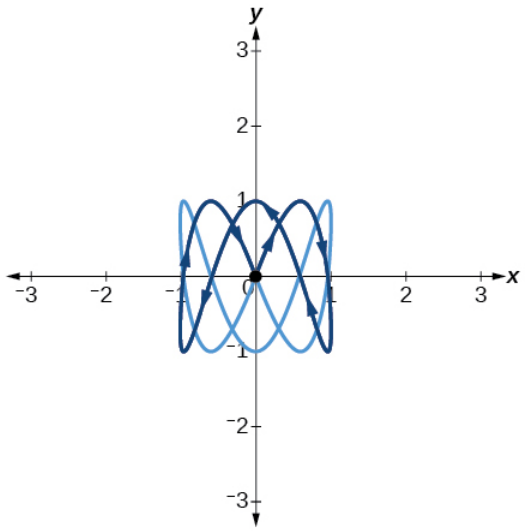


50) $a = 5, b = 5$

51) $a = 2, b = 5$

Answer

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52) $a = 5, b = 2$

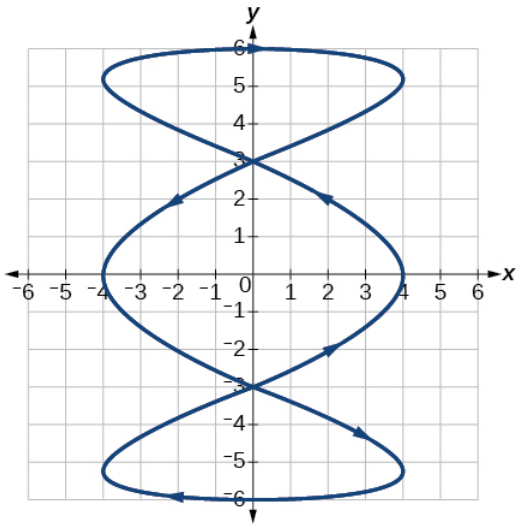
Technology

For the exercises 53-56, look at the graphs that were created by parametric equations of the form

$$\begin{cases} x(t) = a\cos(bt) \\ y(t) = c\sin(dt) \end{cases}$$

Use the parametric mode on the graphing calculator to find the values of a, b, c, and d to achieve each graph.

53)

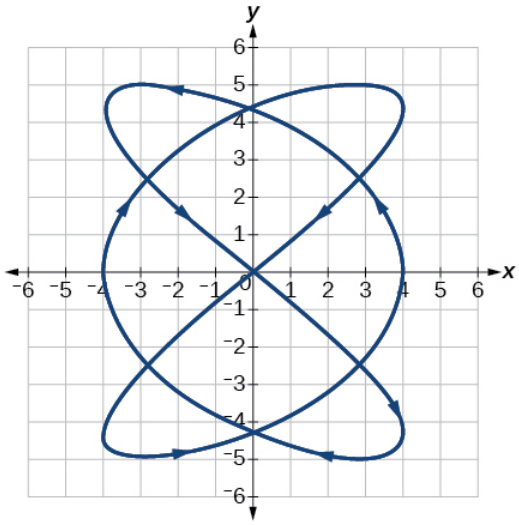


Answer

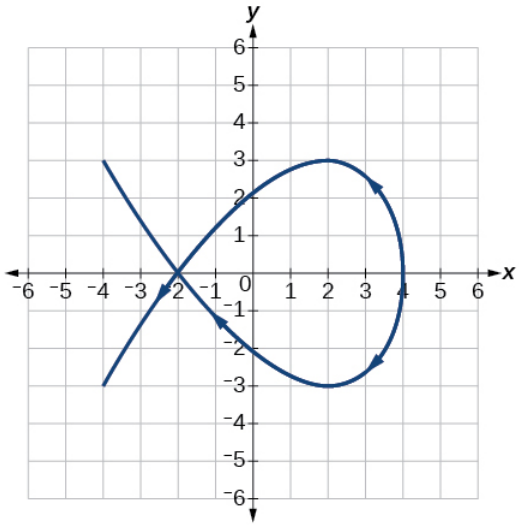
$a = 4, b = 3, c = 6, d = 1$

54)

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55)

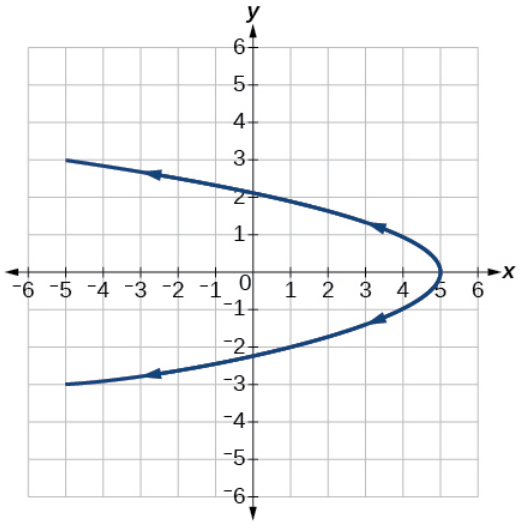


Answer

$a = 4, b = 2, c = 3, d = 3$

56)

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For the exercises 57-62, use a graphing utility to graph the given parametric equations.

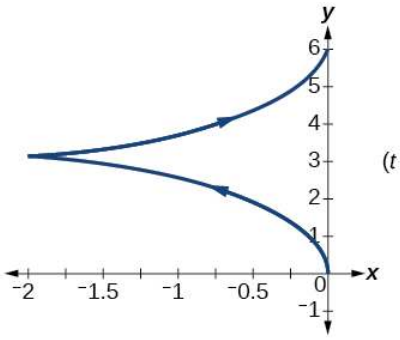
$$\begin{cases} x(t) = \cos t - 1 \\ y(t) = \sin t + t \end{cases}$$

$$\begin{cases} x(t) = \cos t + t \\ y(t) = \sin t - 1 \end{cases}$$

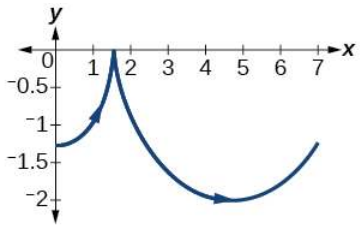
$$\begin{cases} x(t) = t - \sin t \\ y(t) = \cos t - 1 \end{cases}$$

57) Graph all three sets of parametric equations on the domain $[0, 2\pi]$.

Answer

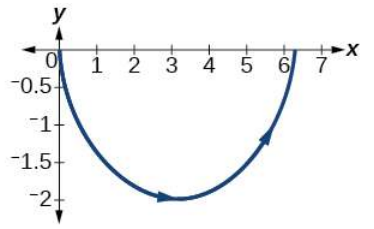


(t from 0 to 2π)



(t from 0 to 2π)

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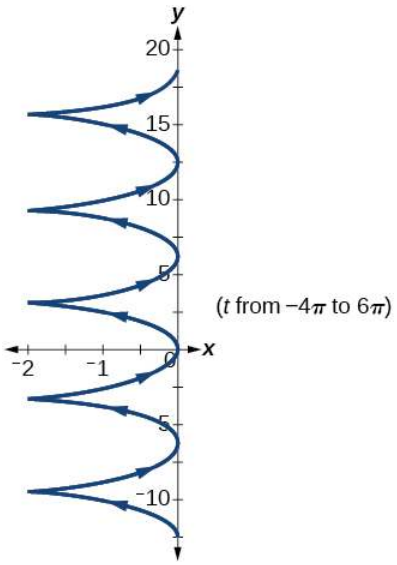


(t from 0 to 2π)

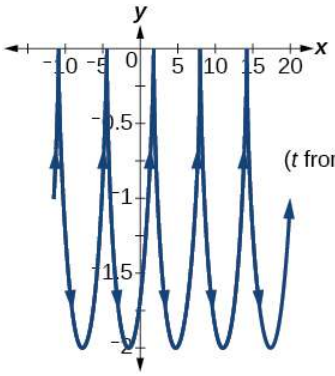
58) Graph all three sets of parametric equations on the domain $[0, 4\pi]$.

59) Graph all three sets of parametric equations on the domain $[-4\pi, 6\pi]$.

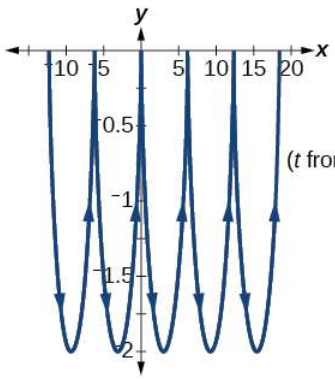
Answer



(t from -4π to 6π)



(t from -4π to 6π)



(t from -4π to 6π)

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60) The graph of each set of parametric equations appears to “creep” along one of the axes. What controls which axis the graph creeps along?

61) Explain the effect on the graph of the parametric equation when we switched $\sin t$ and $\cos t$.

Answer

The y-intercept changes.

62) Explain the effect on the graph of the parametric equation when we changed the domain.

Extensions

63) An object is thrown in the air with vertical velocity of 20 ft/s and horizontal velocity of 15 ft/s. The object’s height can be described by the equation $y(t) = -16t^2 + 20t$, while the object moves horizontally with constant velocity 15 ft/s. Write parametric equations for the object’s position, and then eliminate time to write height as a function of horizontal position.

Answer

$$y(x) = -16\left(\frac{x}{15}\right)^2 + 20\left(\frac{x}{15}\right)$$

64) A skateboarder riding on a level surface at a constant speed of 9 ft/s throws a ball in the air, the height of which can be described by the equation $y(t) = -16t^2 + 10t + 5$. Write parametric equations for the ball’s position, and then eliminate time to write height as a function of horizontal position.

For the exercises 65-69, use this scenario: A dart is thrown upward with an initial velocity of 65 ft/s at an angle of elevation of 52°. Consider the position of the dart at any time t. Neglect air resistance.

65) Find parametric equations that model the problem situation.

Answer

$$\begin{cases} x(t) = 64t\cos(52^\circ) \\ y(t) = -16t^2 + 64t\sin(52^\circ) \end{cases}$$

66) Find all possible values of x that represent the situation.

67) When will the dart hit the ground?

Answer

approximately 3.2 seconds

68) Find the maximum height of the dart.

69) At what time will the dart reach maximum height?

Answer

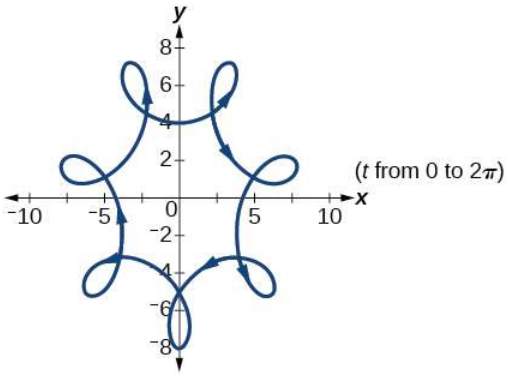
1.6 seconds

For the exercises 70-73, look at the graphs of each of the four parametric equations. Although they look unusual and beautiful, they are so common that they have names, as indicated in each exercise. Use a graphing utility to graph each on the indicated domain.

70) An epicycloid $\begin{cases} x(t) = 14\cos t - \cos(14t) \\ y(t) = 14\sin t + \sin(14t) \end{cases}$ on the domain $[0, 2\pi]$

71) An hypocycloid $\begin{cases} x(t) = 6\sin t + 2\sin(6t) \\ y(t) = 6\cos t - 2\cos(6t) \end{cases}$ on the domain $[0, 2\pi]$

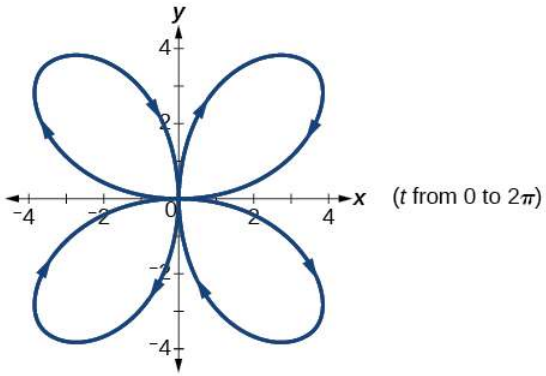
Answer



72) An hypotrochoid $\begin{cases} x(t) = 2\sin t + 5\cos(6t) \\ y(t) = 5\cos t - 2\sin(6t) \end{cases}$ on the domain $[0, 2\pi]$

73) A rose $\begin{cases} x(t) = 5\sin(2t)\sin t \\ y(t) = 5\sin(2t)\cos t \end{cases}$ on the domain $[0, 2\pi]$

Answer



8.8: Vectors

Verbal

1) What are the characteristics of the letters that are commonly used to represent vectors?

Answer

lowercase, bold letter, usually u, v, w

2) How is a vector more specific than a line segment?

3) What are i and j, and what do they represent?

Answer

They are unit vectors. They are used to represent the horizontal and vertical components of a vector. They each have a magnitude of 1.

4) What is component form?

5) When a unit vector is expressed as $\langle a, b \rangle$ which letter is the coefficient of the i and which the j?

Answer

The first number always represents the coefficient of the i and the second represents the j.

6) Given a vector with initial point $(5,2)$ and terminal point $(-1,-3)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b\rangle$.

7) Given a vector with initial point $(-4,2)$ and terminal point $(3,-3)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b\rangle$.

Answer

$\langle 7,-5\rangle$

8) Given a vector with initial point $(7,-1)$ and terminal point $(-1,-7)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b\rangle$.

For the exercises 9-15, determine whether the two vectors u and v are equal, where u has an initial point P_1 and a terminal point P_2 and v has an initial point P_3 and a terminal point P_4.

9) $P_1=(5,1), P_2=(3,-2), P_3=(-1,3), P4=(9,-4)$

Answer

not equal

10) $P_1=(2,-3), P_2=(5,1), P_3=(6,-1), P_4=(9,3)$

11) $P_1=(-1,-1), P_2=(-4,5), P_3=(-10,6), P_4=(-13,12)$

Answer

equal

12) $P_1=(3,7), P_2=(2,1), P_3=(1,2), P_4=(-1,-4)$

13) $P_1=(8,3), P_2=(6,5), P_3=(11,8), P4=(9,10)$

Answer

equal

14) Given initial point $P_1=(-3,1)$ and terminal point $P_2=(5,2)$, write the vector v in terms of i and j.

15) Given initial point $P_1=(6,0)$ and terminal point $P_2=(-1,-3)$, write the vector v in terms of i and j.

Answer

$7i-3j$

For the exercises 16-17, use the vectors $u=i+5j, v=-2i-3j, w=4i-j$

16) Find $u+(v-w)$

17) Find $4v+2u$

Answer

$-6i-2j$

For the exercises 18-21, use the given vectors to compute $u+v, u-v, 2u-3v$

18) $u=\langle 2,-3\rangle, v=\langle 1,5\rangle$

19) $u=\langle -3,4\rangle, v=\langle -2,1\rangle$

Answer

$u+v=\langle -5,5\rangle, u-v=\langle -1,3\rangle, 2u-3v=\langle 0,5\rangle$

20) Let $v=-4i+3j$. Find a vector that is half the length and points in the same direction as v.

21) Let $v=5i+2j$. Find a vector that is twice the length and points in the opposite direction as v.

Answer

$-10i-4j$

For the exercises 22-27, find a unit vector in the same direction as the given vector.

22) $a=3i+4j$

23) $b=-2i+5j$

Answer

$-\frac{2\sqrt{29}}{29}i+\frac{5\sqrt{29}}{29}j$

24) $c=10i-j$

25) $d=-\frac{1}{3}i+\frac{5}{2}j$

Answer

$-\frac{2\sqrt{229}}{229}i+\frac{15\sqrt{229}}{229}j$

26) $u=100i+200j$

27) $u=-14i+2j$

Answer

$-\frac{7\sqrt{2}}{10}i+\frac{\sqrt{2}}{10}j$

For the exercises 28-35, find the magnitude and direction of the vector, $0\le\theta<2\pi$.

28) $\langle 0,4\rangle$

29) $\langle 6,5\rangle$

Answer

$|v|=7.810, \theta=39.806^\circ$

30) $\langle 2,-5\rangle$

31) $\langle -4,-6\rangle$

Answer

$|v|=7.211, \theta=236.310^\circ$

32) Given $u=3i-4j$ and $v=-2i+3j$, calculate $u\cdot v$.

33) Given $u=-i-j$ and $v=i+5j$, calculate $u\cdot v$.

Answer

-6

34) Given $u = \langle -2, 4 \rangle$ and $v = \langle -3, 1 \rangle$, calculate $u \cdot v$.

35) Given $u = \langle -1, 6 \rangle$ and $v = \langle 6, -1 \rangle$,calculate $u \cdot v$.

Answer

-12

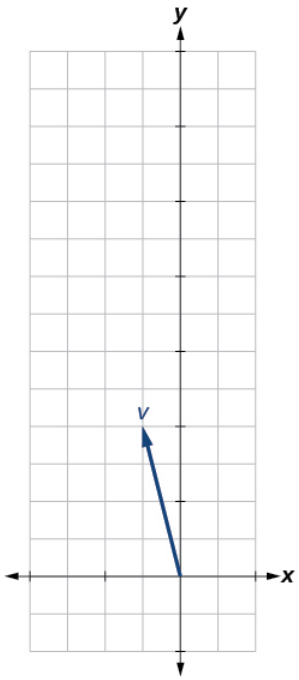
Graphical

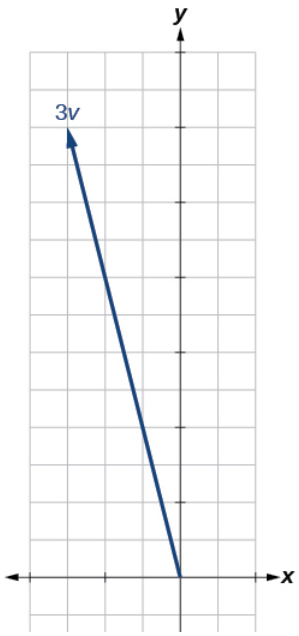
For the exercises 36-,38 given v, draw v, $3v$, and $\frac{1}{2}v$.

36) $\langle 2, -1 \rangle$

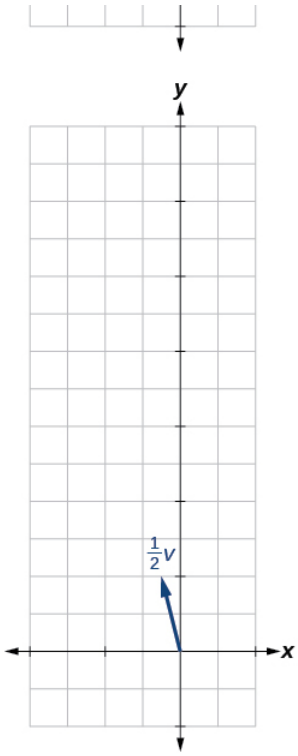
37) $\langle -1, 4 \rangle$

Answer





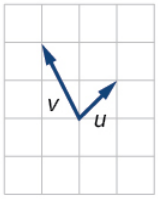
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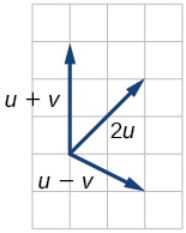
38) $\langle -3, -2 \rangle$

For the exercises 39-41, use the vectors shown to sketch $u + v$, $u - v$, and $2u$.

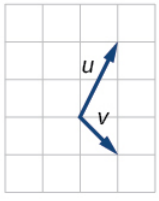
39)



Answer

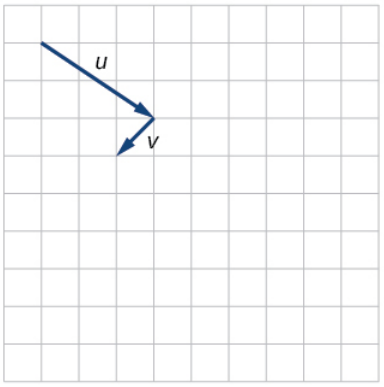


40)

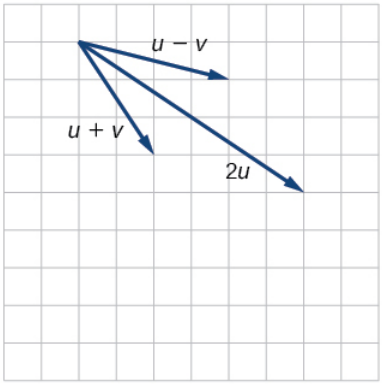


41)

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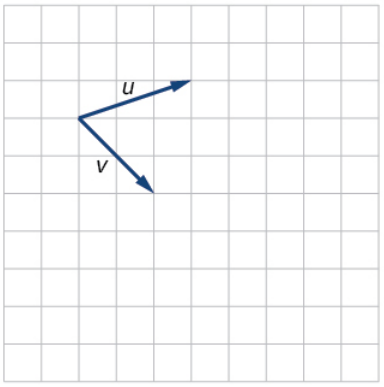


Answer



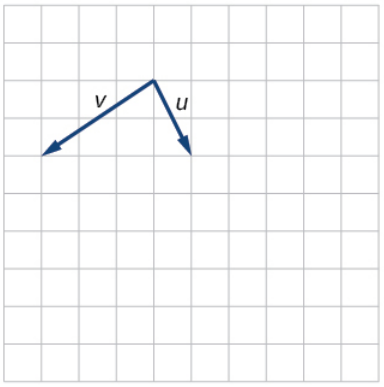
For the exercises 42-43, use the vectors shown to sketch $2u + v$.

42)



43)

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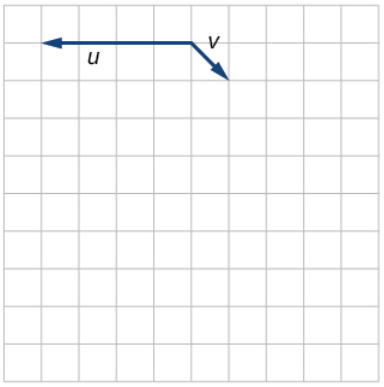


Answer



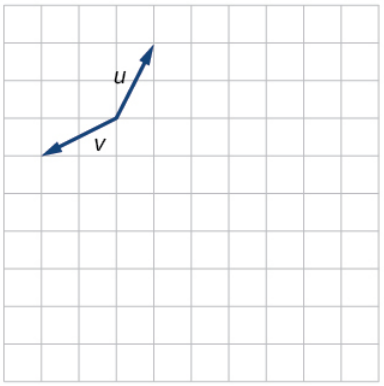
For the exercises 44-45, use the vectors shown to sketch $u - 3v$.

44)

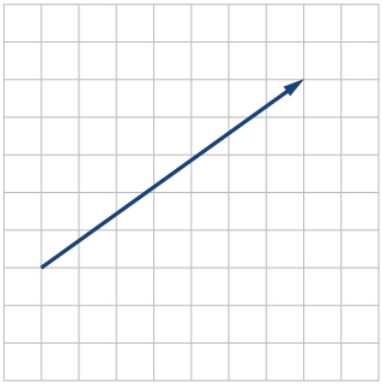


45)

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Answer

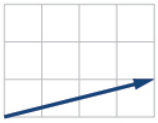


For the exercises 46-47, write the vector shown in component form.

46)



47)



Answer

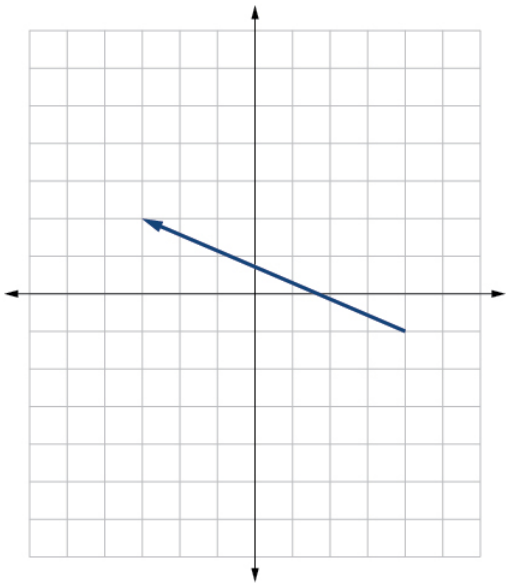
$\langle 4, 1 \rangle$

48) Given initial point $P_1 = (2, 1$ and terminal point $P_2 = (-1, 2)$, write the vector v in terms of i and j, then draw the vector on the graph.

49) Given initial point $P_1 = (4, -1$ and terminal point $P_2 = (-3, 2)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Answer

$v = -7i + 3j$



50) Given initial point $P_1 = (3, 3$ and terminal point $P_2 = (-3, 3)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Extensions

For the exercises 51-54, use the given magnitude and direction in standard position, write the vector in component form.

51) $|v| = 6, \theta = 45^\circ$

Answer

$3\sqrt{2}i + 3\sqrt{2}j$

52) $|v| = 8, \theta = 220^\circ$

53) $|v| = 2, \theta = 300^\circ$

Answer

$i - \sqrt{3}j$

54) $|v| = 5, \theta = 135^\circ$

55) A 60-pound box is resting on a ramp that is inclined 12°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.
b. Find the magnitude of the component of the force that is parallel to the ramp.

Answer

a. 58.7
b. 12.5

56) A 25-pound box is resting on a ramp that is inclined 8°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.
b. Find the magnitude of the component of the force that is parallel to the ramp.

57) Find the magnitude of the horizontal and vertical components of a vector with magnitude 8 pounds pointed in a direction of 27° above the horizontal. Round to the nearest hundredth.

Answer

$x = 7.13$ pounds, $y = 3.63$ pounds

58) Find the magnitude of the horizontal and vertical components of the vector with magnitude 4 pounds pointed in a direction of 127° above the horizontal. Round to the nearest hundredth.

59) Find the magnitude of the horizontal and vertical components of a vector with magnitude 5 pounds pointed in a direction of 55° above the horizontal. Round to the nearest hundredth.

Answer

$x = 2.87$ pounds, $y = 4.10$ pounds

60) Find the magnitude of the horizontal and vertical components of the vector with magnitude 1 pound pointed in a direction of 8° above the horizontal. Round to the nearest hundredth.

Real-World Applications

61) A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

Answer

4.635miles, 17.764° N of E

62) A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

63) A man starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4 miles southwest, and 2 miles east. How far has he walked? If he walked straight home, how far would he have to walk?

Answer

17 miles, 10.318miles

64) A woman starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles south, 5 miles southwest, and 3 miles east. How far has she walked? If she walked straight home, how far would she have to walk?

65) A man starts walking from home and walks 3 miles at 20° north of west, then 5 miles at 10° west of south, then 4 miles at 15° north of east. If he walked straight home, how far would he have to the walk, and in what direction?

Answer

Distance: 2.868, Direction: 86.474° North of West, or 3.526° West of North

66) A woman starts walking from home and walks 6 miles at 40° north of east, then 2 miles at 15° east of south, then 5 miles at 30° south of west. If she walked straight home, how far would she have to walk, and in what direction?

67) An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

Answer

4.924°, 659 km/hr

68) An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

69) An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

Answer

4.424°

70) An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

71) As part of a video game, the point $(5, 7)$ is rotated counterclockwise about the origin through an angle of \(35^{\circ}\). Find the new coordinates of this point.

Answer

$(0.081, 8.602)$

72) As part of a video game, the point $(7, 3)$ is rotated counterclockwise about the origin through an angle of 40°. Find the new coordinates of this point.

73) Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 10 mph relative to the car, and the car is traveling 25 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

Answer

21.801°, relative to the car's forward direction

74) Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 8 mph relative to the car, and the car is traveling 45 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

75) A 50-pound object rests on a ramp that is inclined 19°. Find the magnitude of the components of the force parallel to and perpendicular to (normal) the ramp to the nearest tenth of a pound.

Answer

parallel: 16.28, perpendicular: 47.28pounds

76) Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it upward, and 5 pounds acting on it 45° from the horizontal. What single force is the resultant force acting on the body?

77) Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it -135° from the horizontal, and 5 pounds acting on it directed 150° from the horizontal. What single force is the resultant force acting on the body?

Answer

19.35pounds, 231.54° from the horizontal

78) The condition of equilibrium is when the sum of the forces acting on a body is the zero vector. Suppose a body has a force of 2 pounds acting on it to the right, 5 pounds acting on it upward, and 3 pounds acting on it 45° from the horizontal. What single force is needed to produce a state of equilibrium on the body?

79) Suppose a body has a force of 3 pounds acting on it to the left, 4 pounds acting on it upward, and 2 pounds acting on it \(30^{\circ}\) from the horizontal. What single force is needed to produce a state of equilibrium on the body? Draw the vector.

Answer

5.1583pounds, 75.8° from the horizontal

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5.R: Further Applications of Trigonometry (Review)

8.1 & 8.3: Non-right Triangles: Law of Sines

For the exercises 1-5 assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

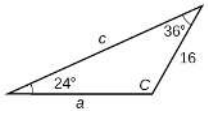
1) $\beta = 50^\circ, a = 105, b = 45$

Answer

Not possible

2) $\alpha = 43.1^\circ, a = 184.2, b = 242.8$

3) Solve the triangle.



Answer

$C = 120^\circ, a = 23.1, c = 34.1$

4) Find the area of the triangle.

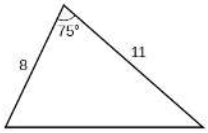
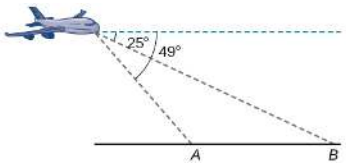


Figure below. Find the distance of the plane from point A and the elevation of the plane.



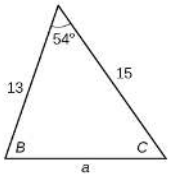
Answer

distance of the plane from point A : 2.2 km, elevation of the plane: 1.6 km

8.2 & 8.3: Non-right Triangles - Law of Cosines

1) Solve the triangle, rounding to the nearest tenth, assuming α is opposite side a, β is opposite side b, and γ s opposite side c : $a = 4, b = 6, c = 8$.

2) Solve the triangle in the Figure below, rounding to the nearest tenth.

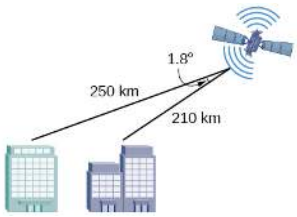


Answer

$B = 71.0^\circ, C = 55.0^\circ, a = 12.8$

3) Find the area of a triangle with sides of length 8.3, 6.6, and 9.1.

4) To find the distance between two cities, a satellite calculates the distances and angle shown in the Figure below (not to scale). Find the distance between the cities. Round answers to the nearest tenth.



Answer

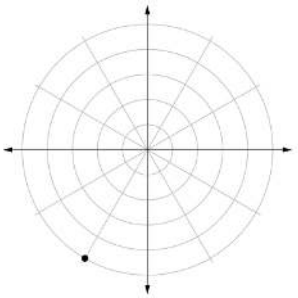
40.6km

8.4: Polar Coordinates

1) Plot the point with polar coordinates $\left(3, \frac{\pi}{6}\right)$.

2) Plot the point with polar coordinates $\left(5, \frac{-2\pi}{3}\right)$.

Answer



3) Convert $\left(6, \frac{-3\pi}{4}\right)$ to rectangular coordinates.

4) Convert $\left(-2, \frac{3\pi}{2}\right)$ to rectangular coordinates.

Answer

$(0, 2)$

5) Convert $(7, -2)$ to polar coordinates.

6) Convert $(-9, -4)$ to polar coordinates.

Answer

$(9.8489, 203.96^\circ)$

For the exercises 7-9, convert the given Cartesian equation to a polar equation.

7) $x = -2$

8) $x^2 + y^2 = 64$

Answer

$r = 8$

9) $x^2 + y^2 = -2y$

For the exercises 10-11, convert the given polar equation to a Cartesian equation.

10) $r = 7\cos\theta$

Answer

$x^2 + y^2 = 7x$

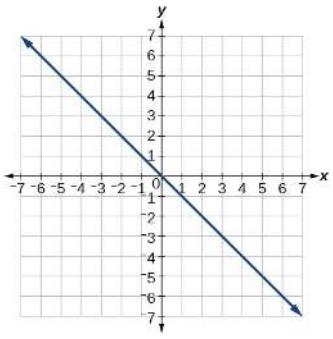
11) $r = \frac{-2}{4\cos\theta + \sin\theta}$

For the exercises 12-13, convert to rectangular form and graph.

12) $\theta = \frac{3\pi}{4}$

Answer

$y = -x$



13) $r = 5\sec\theta$

8.5: Polar Coordinates - Graphs

For the exercises 1-5, test each equation for symmetry.

1) $r = 4 + 4\sin\theta$

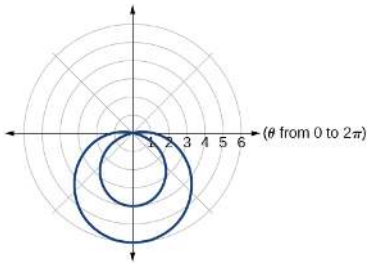
Answer

symmetric with respect to the line $\theta = \frac{\pi}{2}$

2) $r = 7$

3) Sketch a graph of the polar equation $r = 1 - 5\sin\theta$. Label the axis intercepts.

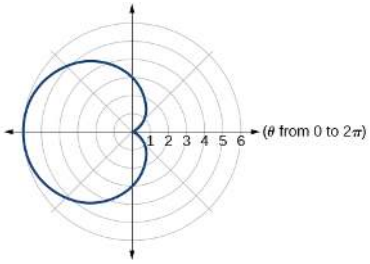
Answer



4) Sketch a graph of the polar equation $r = 5\sin(7\theta)$.

5) Sketch a graph of the polar equation $r = 3 - 3\cos\theta$

Answer



8.6: Polar Form of Complex Numbers

For the exercises 1-2, find the absolute value of each complex number.

1) $-2 + 6i$

2) $4 - 3i$

Answer

5

Write the complex number in polar form.

3) $5 + 9i$

4) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Answer

$\text{cis}\left(-\frac{\pi}{3}\right)$

For the exercises 5-6, convert the complex number from polar to rectangular form.

5) $z = 5\text{cis}\left(\frac{5\pi}{6}\right)$

6) $z = 3\text{cis}(40^\circ)$

Answer

$2.3 + 1.9i$

For the exercises 7-8, find the product $z_1 z_2$ in polar form.

7) $z_1 = 2\text{cis}(89^\circ)$
$z_2 = 5\text{cis}(23^\circ)$

8) $z_1 = 10\text{cis}\left(\frac{\pi}{6}\right)$
$z_2 = 6\text{cis}\left(\frac{\pi}{3}\right)$

Answer

$60\text{cis}\left(\frac{\pi}{2}\right)$

For the exercises 9-10, find the quotient $\frac{z_1}{z_2}$ in polar form.

9) $z_1 = 12\text{cis}(55^\circ)$
$z_2 = 3\text{cis}(18^\circ)$

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10) $z_1 = 27\text{cis}\left(\frac{5\pi}{3}\right)$
$z_2 = 9\text{cis}\left(\frac{\pi}{3}\right)$

Answer

$3\text{cis}\left(\frac{4\pi}{3}\right)$

For the exercises 11-12, find the powers of each complex number in polar form.

11) Find z^4 when $z = 2\text{cis}(70^\circ)$

12) Find z^2 when $z = 5\text{cis}\left(\frac{3\pi}{4}\right)$

Answer

$25\text{cis}\left(\frac{3\pi}{2}\right)$

For the exercises 13-14, evaluate each root.

13) Evaluate the cube root of z when $z = 64\text{cis}(210^\circ)$.

14) Evaluate the square root of z when $z = 25\text{cis}\left(\frac{3\pi}{2}\right)$.

Answer

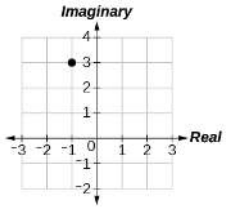
$5\text{cis}\left(\frac{3\pi}{4}\right), 5\text{cis}\left(\frac{7\pi}{4}\right)$

For the exercises 15-16, plot the complex number in the complex plane.

15) $6 - 2i$

16) $-1 + 3i$

Answer



8.7: Parametric Equations

1) $\begin{cases} x(t) = 3t - 1 \\ y(t) = \sqrt{t} \end{cases}$

2) $\begin{cases} x(t) = -\cos t \\ y(t) = 2\sin^2 t \end{cases}$

Answer

$x^2 + \frac{1}{2}y = 1$

3) Parameterize (write a parametric equation for) each Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$ for $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

4) Parameterize the line from $(-2, 3)$ to $(4, 7)$ so that the line is at $(-2, 3)$ at $t = 0$ and $(4, 7)$ at $t = 1$.

Answer

$\begin{cases} x(t) = -2 + 6t \\ y(t) = 3 + 4t \end{cases}$

8.8: Parametric Equations - Graphs

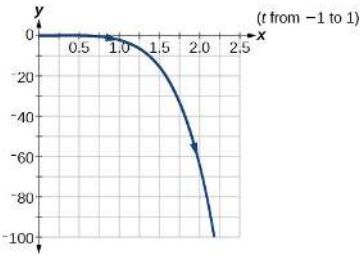
For the exercises 1-, make a table of values for each set of parametric equations, graph the equations, and include an orientation; then write the Cartesian equation.

1) $\begin{cases} x(t) = 3t^2 \\ y(t) = 2t - 1 \end{cases}$

2) $\begin{cases} x(t) = e^t \\ y(t) = -2e^{5t} \end{cases}$

Answer

$y = -2x^5$



3) $\begin{cases} x(t) = 3\cos t \\ y(t) = 2\sin t \end{cases}$

4) A ball is launched with an initial velocity of 80 feet per second at an angle of 40° to the horizontal. The ball is released at a height of 4 feet above the ground.

a. Where is the ball after 3 seconds?

b. How long is the ball in the air?

Answer

a. $\begin{cases} x(t) = (80\cos(40^\circ))t \\ y(t) = -16t^2 + (80\sin(40^\circ))t + 4 \end{cases}$

b. The ball is 14 feet high and 184 feet from where it was launched.

c. 3.3 seconds

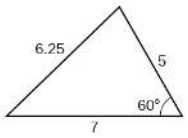
Practice Test

1) Assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve the triangle, if possible, and round each answer to the nearest tenth, given $\beta = 68^\circ, b = 21, c = 16$.

Answer

$\alpha = 67.1^\circ, \gamma = 44.9^\circ, a = 20.9$

2) Find the area of the triangle in the Figure below (from chapter 8). Round each answer to the nearest tenth.



3) A pilot flies in a straight path for 2 hours. He then makes a course correction, heading 15° to the right of his original course, and flies 1 hour in the new direction. If he maintains a constant speed of 575 miles per hour, how far is he from his starting position?

Answer

1712miles

4) Convert $(2, 2)$ to polar coordinates, and then plot the point.

5) Convert $\left(2, \frac{\pi}{3}\right)$ to rectangular coordinates.

Answer

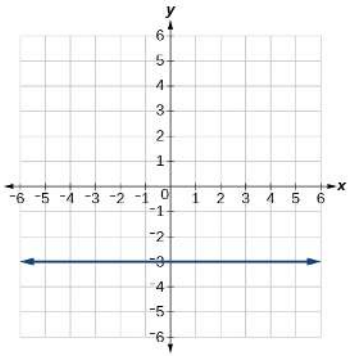
$(1, \sqrt{3})$

6) Convert the polar equation to a Cartesian equation: $x^2 + y^2 = 5y$.

7) Convert to rectangular form and graph:\(r=-3\csc θ\).

Answer

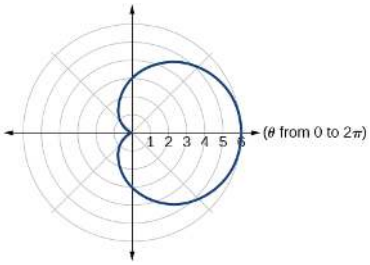
$y = -3$



8) Test the equation for symmetry: $r = -4\sin(2\theta)$.

9) Graph $r = 3 + 3\cos\theta$.

Answer



10) Graph $r = 3 - 5\sin\theta$.

11) Find the absolute value of the complex number $5 - 9i$.

Answer

$\sqrt{106}$

12) Write the complex number in polar form: $4 + i$.

13) Convert the complex number from polar to rectangular form: $z = 5\text{cis}\left(\frac{2\pi}{3}\right)$

Answer

$\frac{-5}{2} + i\frac{5\sqrt{3}}{2}$

14) $z_1 z_2$

15) $\frac{z_1}{z_2}$

Answer

$4\text{cis}(21^\circ)$

16) $(z_2)^3$

17) $\sqrt{z_1}$

Answer

$2\sqrt{2}\text{cis}(18^\circ), 2\sqrt{2}\text{cis}(198^\circ)$

18) Plot the complex number $-5 - i$ in the complex plane.

19) Eliminate the parameter t to rewrite the following parametric equations as a Cartesian equation: $\begin{cases} x(t) = t + 1 \\ y(t) = 2t^2 \end{cases}$

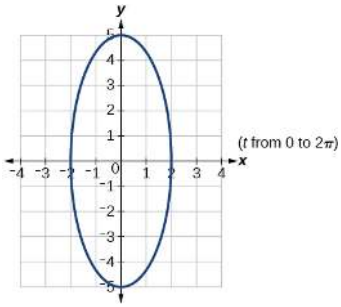
Answer

$y = 2(x-1)^2$

20) Parameterize (write a parametric equation for) the following Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$: $\frac{x^2}{36} + \frac{y^2}{100} = 1$

21) Graph the set of parametric equations and find the Cartesian equation: $\begin{cases} x(t) = -2\sin t \\ y(t) = 5\cos t \end{cases}$

Answer



22) A ball is launched with an initial velocity of 95 feet per second at an angle of 52° to the horizontal. The ball is released at a height of 3.5 feet above the ground.

a. Where is the ball after 2 seconds?
b. How long is the ball in the air?

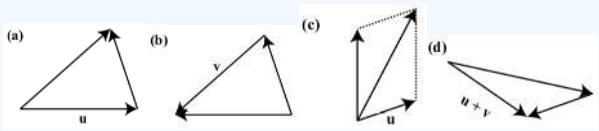
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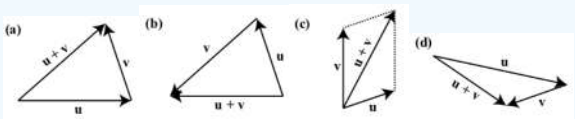
In each of the following diagrams, one of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ is labeled.

NOTE: In each diagram $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ may represent different vectors.

TASK: Label the other two vectors to make the diagram a valid representation of $\mathbf{u}+\mathbf{v}$



Answer



Exercise 5.R.2

On the following diagram, draw the vectors $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $2\mathbf{u}+\mathbf{v}$, and $2\mathbf{u}-\mathbf{v}$.

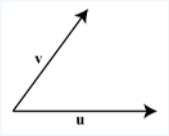


Figure 5.R.11

Answer

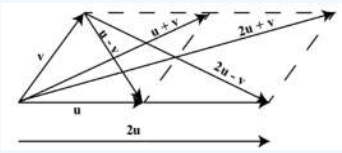


Figure 5.R.12

Exercise 5.R.3

In the following diagram, $|\mathbf{a}|=10$ and $|\mathbf{a}|+|\mathbf{b}|=14$. In addition, the angle θ between the vectors $\mathbf{a}$ and $\mathbf{b}$ is 30°. Determine the magnitude of the vector $\mathbf{b}$ and the angle between the vectors $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$.

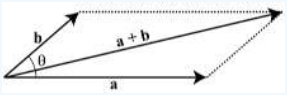


Figure 5.R.13

Answer

The angle between the vectors $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$ is approximately 9.075°. In addition, $|\mathbf{b}|\approx 4.416$

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Exercise $5.R.4$

Suppose that vectors $\mathbf{a}$ and $\mathbf{b}$ have magnitudes of 125 and 180, respectively. Also assume that the angle between these two vectors is 35°. Determine the magnitude of the vector $\mathbf{a}+\mathbf{b}$ and the measure of the angle between the vectors $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$.

Exercise $5.R.5$

A car that weighs 3250 pounds is on an inclined plane that makes an angle of 4.5° with the horizontal. Determine the magnitude of the force of the car on the inclined plane, and determine the magnitude of the force on the car down the plane due to gravity. What is the magnitude of the smallest force necessary to keep the car from rolling down the plane?

Exercise $5.R.6$

An experiment determined that a force of 45 pounds is necessary to keep a 250 pound object from sliding down an inclined plane. Determine the angle the inclined plane makes with the horizontal.

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Exercise 5.R.7

A cable that can withstand a force of 4500 pounds is used to pull an object up an inclined plane that makes an angle of 15 degrees with the horizontal. What is the heaviest object that can be pulled up this plane with the cable? (Assume that friction can be ignored.)

Exercise 5.R.8

Determine the magnitude and the direction angle of each of the following vectors.

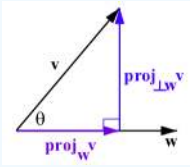


Figure 5.R.14: The Projection of $\mathbf{v}$ onto $\mathbf{w}$

1. $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$ (5.R.1)
2. $\mathbf{w} = \langle -3, 6 \rangle$ (5.R.2)
3. $\mathbf{a} = 4\mathbf{i} - 7\mathbf{j}$ (5.R.3)
4. $\mathbf{u} = \langle -3, -5 \rangle$ (5.R.4)

Answer

1. $|\mathbf{v}| = \sqrt{34}$. The direction angle is approximately 59.036°.
2. $|\mathbf{w}| = \sqrt{45}$. The direction angle is approximately 116.565°.

Exercise 5.R.9

Determine the horizontal and vertical components of each of the following vectors. Write each vector in $\mathbf{i}$, $\mathbf{j}$ form.

The vector $\mathbf{v}$ with magnitude 12 and direction angle 50°?

The vector $\mathbf{u}$ with $\mathbf{u} = \sqrt{20}$ and direction angle 125°.

The vector $\mathbf{w}$ with magnitude 5.25 and direction angle 200°.

Answer

1. $\mathbf{v} = 12\cos(50^\circ) + 12\sin(50^\circ) \approx 7.713\mathbf{i} + 9.193\mathbf{j}$.
2. $\mathbf{u} = \sqrt{20}\cos(125^\circ) + \sqrt{20}\sin(125^\circ) \approx -2.565\mathbf{i} + 3.663\mathbf{j}$

Exercise 5.R.10

Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + 5\mathbf{j}$, and $\mathbf{w} = 4\mathbf{i} - 2\mathbf{j}$. Determine the $\mathbf{i}$, $\mathbf{j}$ form of each of the following:

1. $5\mathbf{u} - \mathbf{v}$ (5.R.5)
2. $2\mathbf{v} + 7\mathbf{w}$ (5.R.6)
3. $\mathbf{u} + \mathbf{v} + \mathbf{w}$ (5.R.7)
4. $3\mathbf{u} + 5\mathbf{w}$ (5.R.8)

Answer

1. $5\mathbf{u} - \mathbf{v} = 11\mathbf{i} + 10\mathbf{j}$
3. $\mathbf{u} + \mathbf{v} + \mathbf{w} = 5\mathbf{i} + 6\mathbf{j}$

Exercise 5.R.11

Determine the value of the dot product for each of the following pairs of vectors.

1. $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} - 2\mathbf{j}$
2. $\mathbf{a}$ and $\mathbf{b}$ where $|\mathbf{a}| = 6$, $|\mathbf{w}| = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is 30°
3. $\mathbf{a}$ and $\mathbf{b}$ where $|\mathbf{a}| = 6$, $|\mathbf{w}| = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is 150°
4. $\mathbf{a}$ and $\mathbf{b}$ where $|\mathbf{a}| = 6$, $|\mathbf{w}| = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is 50°
5. $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j}$

Answer

1. $\mathbf{v} \cdot \mathbf{w} = -4$.
2. $\mathbf{a} \cdot \mathbf{b} = 9\sqrt{3}$.

Exercise $5.R.12$

Determine the angle between each of the following pairs of vectors.

1. $$\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} \tag{5.R.9}$$
   and
   $$\mathbf{w} = 3\mathbf{i} - 2\mathbf{j} \tag{5.R.10}$$
2. $$\mathbf{a} = 5\mathbf{i} - 2\mathbf{j} \tag{5.R.11}$$
   and
   $$\mathbf{b} = 2\mathbf{i} + 5\mathbf{j} \tag{5.R.12}$$
3. $$\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} \tag{5.R.13}$$
   and
   $$\mathbf{w} = -\mathbf{i} + 4\mathbf{j} \tag{5.R.14}$$

Answer

1. The angle between $\mathbf{v}$ and $\mathbf{w}$ is $\cos^{-1}\left(\frac{-4}{\sqrt{29}\sqrt{13}}\right) \approx 101.89^{\circ}$.

Exercise $5.R.13$

For each pair of vectors, determine $\mathbf{proj}_{\mathbf{v}}\mathbf{w}$, $\mathbf{proj}_{\perp \mathbf{v}}\mathbf{w}$ and verify that $\mathbf{w} = \mathbf{proj}_{\perp \mathbf{w}}\mathbf{v} + \mathbf{proj}_{\mathbf{w}}\mathbf{v}$. Draw a picture showing all of the vectors involved in this.

1. $$\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} \tag{5.R.15}$$
   and
   $$\mathbf{w} = 3\mathbf{i} - 2\mathbf{j} \tag{5.R.16}$$
2. $$\mathbf{v} = \langle -2, 3 \rangle \tag{5.R.17}$$
   and

$$\mathbf{w} = \langle 1, 1 \rangle \tag{5.R.18}$$

Answer

1.

$$\mathbf{proj}_v \mathbf{w} = \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{-4}{29}(2\mathbf{i} + 5\mathbf{j}) = -\frac{8}{29}\mathbf{i} - \frac{20}{29}\mathbf{j} \tag{5.R.19}$$

$$\mathbf{proj}_{\perp_v} \mathbf{w} = \mathbf{w} - \mathbf{proj}_v \mathbf{w} = \frac{95}{29}\mathbf{i} - \frac{38}{29}\mathbf{j} \tag{5.R.20}$$

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CHAPTER OVERVIEW

6: Vectors

In this chapter, we will look at vectors in several forms. Once we have a general understanding of notation, we will learn how vectors have a wide range of applications in surveying, work, force and tension. We may also find how helpful these vectors can be for manipulating free-body diagrams (used widely in physics).

6.0: Prelude to Vectors
6.1: Vectors from a Geometric Point of View
6.2: Vectors from an Algebraic Point of View
6.3: Solving Systems of Equations with Augmented Matrices
6.E: Triangles and Vectors (Exercises)

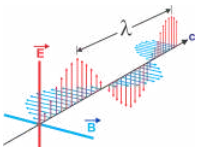
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6.0: Prelude to Vectors

Using vectors to describe magnitude and direction will be the main focus of this chapter. Vectors were independently developed by two different mathematicians, Josiah Gibbs and Oliver Heaviside to describe electromagnetism.



What is particularly unique about vectors is that they are not relegated to the Origin. With only magnitude and direction, vectors prove to have a wide and powerful application range.

A Note On Magnitude

Since magnitude is not always a common word in conversational language it is helpful to remember that it can reference length, speed, weight, mass, force, gravity and much more! It is a measure of some amount based on the context or situation.

A Note on Direction

Direction is a much more common word for everyday conversation, it might be helpful to remember that direction is usually an angle in contexts relating to vectors. The angle can be measured as an angle positively or negatively measured off the horizontal, or a bearing off north. It is important to familiarize yourself with the notation for both nautical navigation and flight navigation, because it will appear throughout the chapter.

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6.1: Vectors from a Geometric Point of View

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is a vector?
- How do we use the geometric form of vectors to find the sum of two vectors?
- How do we use the geometric form of vectors to find a scalar multiple of a vector?
- How do we use the geometric form of vectors to find the difference of two vectors?
- What is the angle between two vectors?
- Why is force a vector and how do we use vectors and triangles to determine forces acting on an object?

We have all had the experience of dropping something and watching it fall to the ground. What is happening, of course, is that the force of gravity is causing the object to fall to the ground. In fact, we experience the force of gravity everyday simply by being on Earth. Each person's weight is a measure of the force of gravity since pounds are a unit of force. So when a person weighs 150 pounds, it means that gravity is exerting a force of 150 pounds straight down on that person. Notice that we described this with a quantity and a direction (straight down). Such a quantity (with magnitude and direction) is called a vector.

Now suppose that person who weighs 150 pounds is standing on a hill. In mathematics, we simplify the situation and say that the person is standing on an inclined plane as shown in Figure 3.20. (By making the hill a straight line, we simplify the mathematics involved.) The diagram in Figure 6.1.1, an object is on the inclined plane at the point P. The inclined plane makes an angle of θ with the horizontal. The vector $\mathbf{w}$ shows the weight of the object (force of gravity, straight down). The diagram also shows two other vectors. The vector $\mathbf{b}$ is perpendicular

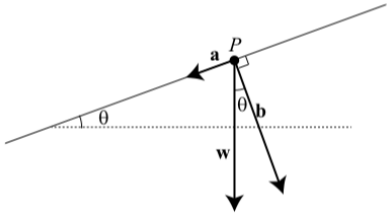


Figure 6.1.1: Inclined Plane

to the plane represents the force that the object exerts on the plane. The vector $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and parallel to the inclined plane. This vector represents the force of gravity along the plane. In this and the next section, we will learn more about these vectors and how to determine the magnitudes of these vectors. We will also see that with our definition of the addition of two vectors that $\mathbf{w} = \mathbf{a} + \mathbf{b}$.

Definitions

There are some quantities that require only a number to describe them. We call this number the magnitude of the quantity. One such example is temperature since we describe this with only a number such as 68 degrees Fahrenheit. Other such quantities are length, area, and mass. These types of quantities are often called **scalar quantities**. However, there are other quantities that require both a magnitude and a direction. One such example is force, and another is velocity. We would describe a velocity with something like 45 miles per hour northwest. Velocity and force are examples of a **vector quantity**. Other examples of vectors are acceleration and displacement.

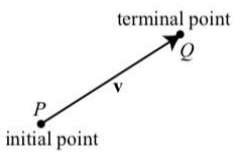
Some vectors are closely associated with scalars. In mathematics and science, we make a distinction between *speed* and *velocity*. Speed is a scalar and we would say something like our speed is 65 miles per hour. However, if we used a velocity, we would say something like 65 miles per hour east. This is different than a velocity of 65 miles per hour north even though in both cases, the speed is 65 miles per hour.

Definitions: Vectors and Scalars

- A **vector** is a quantity that has both magnitude and direction.
- A **scalar** is a quantity that has magnitude only.

Geometric Representation of Vectors

Vectors can be represented geometrically by arrows (directed line segments). The **arrowhead** indicates the direction of the vector, and the **length** of the arrow describes the magnitude of the vector.



A vector with initial point P (the tail of the arrow) and terminal point Q (the tip of the arrowhead) can be represented by

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$$\overrightarrow{PQ}, \mathbf{v}, or\ \vec{v}. \tag{6.1.1}$$

We often write $v = \overrightarrow{PQ}$. In this text, we will use boldface font to designate a vector. When writing with pencil and paper, we always use an arrow above the letter (such as $\vec{v}$) to designate a vector. The **magnitude** (or **norm** or **length**) of the vector $\mathbf{v}$ is designated by $|\mathbf{v}|$. It is important to remember that $|\mathbf{v}|$ is a number that represents the magnitude or length of the vector $\mathbf{v}$.

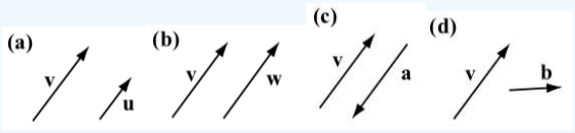
According to our definition, a vector possesses the attributes of length (magnitude) and direction, but position is not mentioned. So we will consider two vectors to be equal if they have the same magnitude and direction. For example, if two different cars are both traveling at 45 miles per hour northwest (but in different locations), they have equal velocity vectors. We make a more formal definition.

Definition: Equal Vectors

Two vectors are **equal** if and only if they have the same magnitude and the same direction. When the vectors $\mathbf{v}$ and $\mathbf{w}$ are equal, we write $\mathbf{v} = \mathbf{w}$.

Exercise 6.1.1

The following diagrams show the vector $\mathbf{v}$ next to four other vectors. Which (if any) of these four vectors are equal to the vector $\mathbf{v}$?



Answer

The vector $\mathbf{w}$ is the only vector that is equal to the vector $\mathbf{v}$. Vector $\mathbf{u}$ has the same direction as $\mathbf{v}$ but a different magnitude. Vector $\mathbf{a}$ has the same magnitude as $\mathbf{v}$ but a different direction (note that the direction of a is the opposite direction of $\mathbf{v}$). Vector $\mathbf{b}$ has a different direction and a different magnitude than $\mathbf{v}$.

Operations on Vectors

Scalar Multiple of a Vector

Doubling a scalar quantity is simply a matter of multiplying its magnitude by 2. For example, if a container has 20 ounces of water and the amount of water is doubled, it will then have 40 ounces of water. What do we mean by doubling a vector? The basic idea is to keep the same direction and multiply the magnitude by 2. So if an object has a velocity of 5 feet per second southeast and a second object has a velocity of twice that, the second object will have a velocity of 10 feet per second in the southeast direction. In this case, we say that we multiplied the vector by the scalar 2. We now make a definition that also takes into account that a scalar can be negative.

Definition

For any vector $\mathbf{v}$ and any scalar c, the vector $c\mathbf{v}$ (called a scalar multiple of the vector $\mathbf{v}$ is a vector whose magnitude is $|c|$ times the magnitude of the vector $\mathbf{v}$.

- If $c > 0$, then the direction of $c\mathbf{v}$ is the same as the direction of $\mathbf{v}$.
- If $c < 0$, then the direction of $c\mathbf{v}$ is the opposite of the direction of $\mathbf{v}$.
- If $c = 0$, then $c\mathbf{v} = 0\mathbf{v} = 0$.

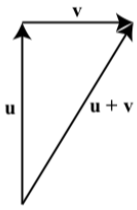
The vector $\mathbf{0}$ is called the **zero vector** and the zero vector has no magnitude and no direction. We sometimes write $\vec{0}$ for the zero vector.

In this definition, $|c|$ is the absolute value of the scalar c. Care must be taken not to confuse this with the notation $|\mathbf{v}|$, which is the magnitude of the vector $\mathbf{v}$. This is one reason it is important to have a notation that clearly indicates when we are working with a vector or a scalar.

Addition of Vectors

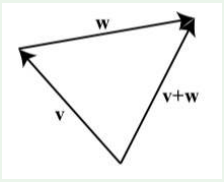
We illustrate how to add vectors with two displacement vectors. As with velocity and speed, there is a distinction between displacement and distance. **Distance** is a scalar. So we might say that we have traveled 2 miles. **Displacement**, on the other hand, is a vector consisting of a distance and a direction. So the vectors 2 miles north and 2 miles east are different displacement vectors.

Now if we travel 3 miles north and then travel 2 miles east, we end at a point that defines a new displacement vector. See the diagram to the right. In this diagram, $\mathbf{u}$ is "3 miles north" and $\mathbf{v}$ is "2 miles east." The vector sum $\mathbf{u} + \mathbf{v}$ goes from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.



Definition: sum of two vectors

The **sum of two vectors** is defined as follows: We position the vectors so that the initial point of $\mathbf{w}$ coincides with the terminal point of $\mathbf{v}$. The vector $\mathbf{v}+\mathbf{w}$ is the vector whose initial point coincides with the initial point of $\mathbf{v}$ and whose terminal point coincides with the terminal point of $\mathbf{w}$.



The vector \(\textbf{v} + \textbf{w}\) is called the sum or resultant of the vectors $\mathbf{v}$ and $\mathbf{w}$.

In the definition, notice that the vectors $\mathbf{v}$, $\mathbf{w}$, and \(\textbf{v} + \textbf{w}\) are placed so that the result is a triangle. The lengths of the sides of that triangle are the magnitudes of these sides $|\mathbf{v}|$, $|\mathbf{w}|$, and $|\mathbf{v}+\mathbf{w}|$. If we place the two vectors $\mathbf{v}$ and $\mathbf{w}$ so that their initial points coincide, we can use a parallelogram to add the two vectors. This is shown in Figure 6.1.2.

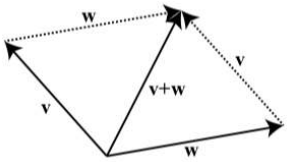


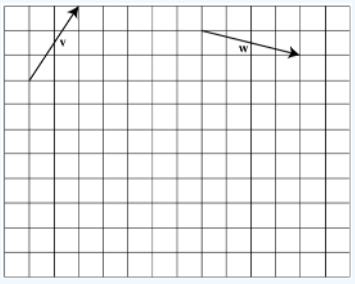
Figure 6.1.2: Sum of Two Vectors Using a Parallelogram

Notice that the vector $\mathbf{v}$ forms a pair of opposite sides of the parallelogram as does the vector $\mathbf{w}$.

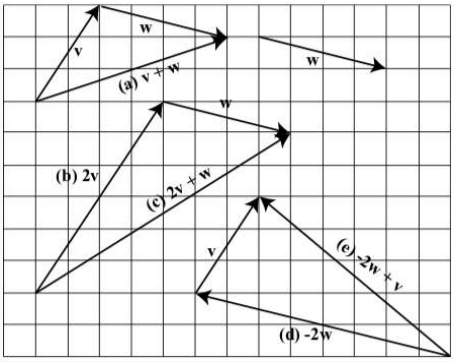
Exercise 6.1.2

The following diagram shows two vectors, $\mathbf{v}$ and $\mathbf{w}$. Draw the following vectors:

1. $\mathbf{v}+\mathbf{w}$
2. $2\mathbf{v}$
3. $2\mathbf{v}+\mathbf{w}$
4. $-2\mathbf{w}$
5. $-2\mathbf{w}+\mathbf{v}$



Answer



Subtraction of Vectors

Before explaining how to subtract vectors, we will first explain what is meant by the "negative of a vector." This works similarly to the negative of a real number. For example, we know that when we add -3 to 3, the result is 0. That is, $3+(-3)=0$

We want something similar for vectors. For a vector $\mathbf{w}$, the idea is to use the scalar multiple $(-1)\mathbf{w}$. The vector $(-1)\mathbf{w}$ has the same magnitude as $\mathbf{w}$ but has the opposite direction of $\mathbf{w}$. We define $-\mathbf{w}$ to be $(-1)\mathbf{w}$. Figure shows that when we add $-\mathbf{w}$ to $\mathbf{w}$, the terminal point of the sum is the same as the initial point of the sum and so the result is the zero vector. That is, $\mathbf{w}+(-\mathbf{w})=\mathbf{0}$.

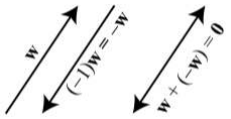


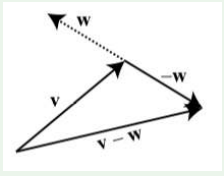
Figure 6.1.3: The Sum of a Vector and Its Negative

We are now in a position to define subtraction of vectors. The idea is much the same as subtraction of real numbers in that for any two real numbers a and b, $a-b=a+(-b)$.

Definition: Difference between vectors

For any two vectors $\mathbf{v}$ and $\mathbf{w}$, the **difference between v and w** is denoted by $\mathbf{v}-\mathbf{w}$ and is defined as follows:

$$\mathbf{v}-\mathbf{w}=\mathbf{v}+(-\mathbf{w}). \tag{6.1.2}$$

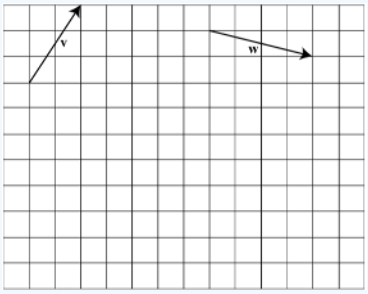


We also say that we are subtracting the vector $\mathbf{w}$ from the vector $\mathbf{v}$.

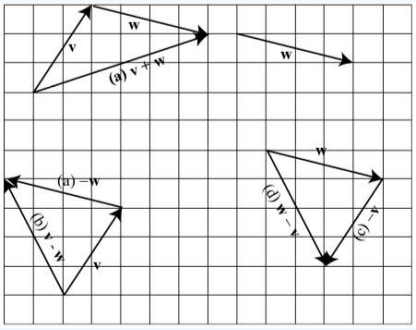
Exercise 6.1.3

The following diagram shows two vectors, $\mathbf{v}$ and $\mathbf{w}$. Draw the following vectors:

1. $-\mathbf{w}$
2. **v - w**
3. $-\mathbf{v}$
4. **w - v**



Answer



The Angle Between Two Vectors

We have seen that we can use triangles to help us add or subtract two vectors. The lengths of the sides of the triangle are the magnitudes of certain vectors. Since we are dealing with triangles, we will also use angles determined by the vectors.

Definition

The angle θ between vectors is the angle formed by these two vectors (with $0^\circ \leq \theta \leq 180^\circ$) when they have the same initial point.

So in the diagram on the left in Figure 6.1.4, the angle θ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$. However, when we want to determine the sum of two angles, we often form the parallelogram determined by the two vectors as shown in the diagram on the right in Figure 6.1.3.

We then will use the angle $180^\circ - \theta$ and the Law of Cosines since we know the two sides of the triangle are the lengths of $\mathbf{v}$ and $\mathbf{w}$ and the included angle is $180^\circ - \theta$.

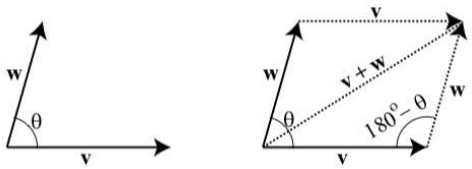


Figure 6.1.3: Angle Between Two Vectors

Exercise 6.1.4

Suppose that the vectors $\mathbf{a}$ and $\mathbf{b}$ have magnitudes of 80 and 60, respectively, and that the angle θ between the two vectors is 53 degrees. In Figure 6.1.4, we have drawn the parallelogram determined by these two vectors and have labeled the vertices for reference.

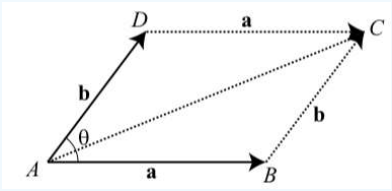


Figure 6.1.4

Remember that a vector is determined by its magnitude and direction. We will determine $|\mathbf{a} + \mathbf{b}|$ and the measure of the angle between $\mathbf{a}$ and $\mathbf{a} + \mathbf{b}$

Determine the measure of $\angle ABC$.

In $\triangle ABC$, the length of side AB is $|\mathbf{a}| = 80$ and the length of side BC is $|\mathbf{b}| = 60$. Use this triangle and the Law of Cosines to determine the length of the third side, which is $|\mathbf{a} + \mathbf{b}|$.

Determine the measure of the angle between $\mathbf{a}$ and $\mathbf{a} + \mathbf{b}$. This is $\angle CAB$ in $\triangle ABC$.

Answer

1. $\angle ABC = 180^\circ - \theta = 127^\circ$.
2. Using the Law of Cosines, we see that

$$|\mathbf{a}+\mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}| \cdot |\mathbf{b}| \cos(\angle ABC) = 80^2 + 60^2 - 2 \cdot 60 \cdot 80 \cos(127^\circ) = 10000 - 9600 \cos(127^\circ) \approx 15777.42422 \quad (6.1.3)$$

so we see that $|\mathbf{a}+\mathbf{b} \approx 125.61|$.

3. The angle between the vectors a and a C b is †CAB. In 4ABC , we know that †ABC D 127ı, and so †CAB must be an acute angle. We will use the Law of Sines to determine this angle.

$$\frac{\sin(\angle CAB)}{|\mathbf{b}|} = \frac{\sin(\angle CAB)}{|\mathbf{a}+\mathbf{b}|} \tag{6.1.4}$$

$$\sin(\angle CAB = \frac{60\sin(127^\circ)}{|\mathbf{a}+\mathbf{b}|} \tag{6.1.5}$$

$$\sin(\angle CAB \approx 0.38148341 \tag{6.1.6}$$

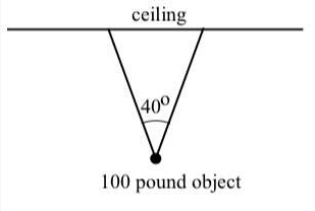
So the angle between the vectors $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$ is approximately 22.43°.

Force

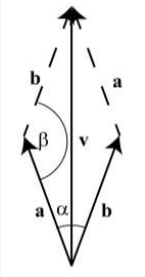
An important vector quantity is that of force. In physics, a force on an object is defined as any interaction that, when left unopposed, will change the motion of the object. So a force will cause an object to change its velocity, that is the object will accelerate. More informally, a force is a push or a pull on an object.

One force that affects our lives is the force of gravity. The magnitude of the force of gravity on a person is that person's weight. The direction of the force of gravity is straight down. So if a person who weighs 150 pounds is standing still on the ground, then the ground is also exerting a force of 150 pounds on the person in the upward direction. The net force on the stationary person is zero. This is an example of what is known as *static equilibrium*. When an object is in static equilibrium, the sum of the forces acting on the object is equal to the zero vector.

Example 6.1.1: Object Suspended from a Ceiling



Suppose a 100 pound object is suspended from the ceiling by two wires that form a 40° angle as shown in the diagram to the right. Because the object is stationary, the two wires must exert a force on the object so that the sum of these two forces is equal to 100 pounds straight up. (The force of gravity is 100 pounds straight down.)



We will assume that the two wires exert an forces of equal magnitudes and that the angle between these forces and the vertical in 20°. So our first step is to draw a picture of these forces, which is shown on the right. The vector $\mathbf{v}$ is a vector of magnitude 100 pounds. The vectors $\mathbf{a}$ and $\mathbf{b}$ are the vectors for the forces exerted by the two wires. (We have $|\mathbf{a}| = |\mathbf{b}|$.) We also know that $\alpha = 20^\circ$ and so by the properties of parallelograms, $\beta = 140^\circ$.

Inclined Planes

At the beginning of this section, we discussed the forces involved when an object is place on an inclined plane. Figure 6.1.5 is the diagram we used, but we now have added labels for some of the angles. Recall that the vector $\mathbf{w}$ shows the weight of the object (force of gravity, straight down), the vector $\mathbf{b}$ is perpendicular to the plane and represents the force that the object exerts on the plane, and the vector $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and parallel to the inclined plane. This vector represents the force of gravity along the plane. Notice that we have also added a second copy of the vector a that begins at the tip of the vector $\mathbf{b}$.

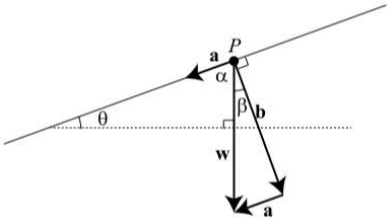


Figure 6.1.5: Inclined Plane

Using the angles shown, we see that $\alpha+\beta=90^\circ$ since they combine to form a right angle, and $\alpha+\theta=90^\circ$ since they are the two acute angles in a right triangle. From this, we conclude that $\beta=\theta$. This gives us the final version of the diagram of the forces on an inclined plane shown in Figure 6.1.6. Notice that the vectors $\mathbf{a}$,

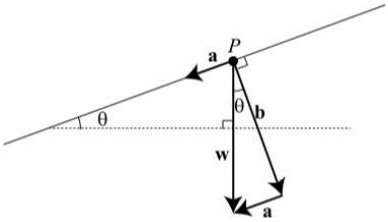


Figure 6.1.6: Inclined Plane

$\mathbf{b}$, and $\mathbf{w}$ form a right triangle, and so we can use right triangle trigonometry for problems dealing with the forces on an inclined plane.

Exercise 6.1.6

An object that weighs 250 pounds is placed on an inclined plane that makes an angle of 12° degrees with the horizontal. Using a diagram like the one in Figure 3.26, determine the magnitude of the force against the plane caused by the object and the magnitude of the force down the plane on the object due to gravity.

Note: The magnitude of the force down the plane will be the force in the direction up the plane that is required to keep the object stationary.

Answer

Using the notation in Figure 6.1.6, we obtain the following:

$$\frac{|\mathbf{b}|}{|\mathbf{w}|}=\cos(12^\circ) \tag{6.1.7}$$

$$\frac{|\mathbf{a}|}{|\mathbf{w}|}=\sin(12^\circ) \tag{6.1.8}$$

$$|\mathbf{b}|=|\mathbf{w}|\cos(12^\circ) \tag{6.1.9}$$

$$|\mathbf{a}|=|\mathbf{w}|\sin(12^\circ) \tag{6.1.10}$$

$$|\mathbf{b}|\approx 244.54 \tag{6.1.11}$$

$$|\mathbf{a}|\approx 51.98 \tag{6.1.12}$$

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The object exerts a force of about 244.54pounds perpendicular to the plane and the force of gravity down the plane on the object is about 51.98 pounds. So in order to keep the object stationary, a force of about 51.98pounds up the plane must be applied to the object.

Summary

In this section, we studied the following important concepts and ideas:

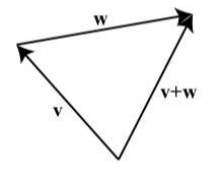
- Vectors and Scalars
  - A **vector** is a quantity that has both magnitude and direction.
  - A **scalar** is a quantity that has magnitude only.
  - Two vectors are **equal** if and only if they have the same magnitude and the same direction.
- Scalar Multiple of a Vector

  For any vector $\mathbf{v}$ and any scalar c, the vector $c\mathbf{v}$ (called a scalar multiple of the vector $\mathbf{v}$ is a vector whose magnitude is $|c|$ times the magnitude of the vector $\mathbf{v}$.

  - If $c > 0$, then the direction of $c\mathbf{v}$ is the same as the direction of $\mathbf{v}$.
  - If $c < 0$, then the direction of $c\mathbf{v}$ is the opposite of the direction of $\mathbf{v}$.
  - If $c = 0$, then $c\mathbf{v} = 0\mathbf{v} = 0$.

  The vector \(**0**\) is called the zero vector and the zero vector has no magnitude and no direction. We sometimes write $\vec{0}$ for the zero vector.
- The Sum of Two Vectors

  The **sum of two vectors** is defined as follows: We position the vectors so that the initial point of $\mathbf{w}$ coincides with the terminal point of $\mathbf{v}$. The vector $\mathbf{v} + \mathbf{w}$ is the vector whose initial point coincides with the initial point of $\mathbf{v}$ and whose terminal point coincides with the terminal point of $\mathbf{w}$.

  The vector \(\textbf{v} + \textbf{w}\) is called the sum or resultant of the vectors $\mathbf{v}$ and $\mathbf{w}$.
- The Angle Between Two Vectors

  The angle θ between vectors is the angle formed by these two vectors (with $0^\circ \leq \theta \leq 180^\circ$) when they have the same initial point.

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6.2: Vectors from an Algebraic Point of View

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we find the component form of a vector?
- How do we find the magnitude and the direction of a vector written in component form?
- How do we add and subtract vectors written in component form and how do we find the scalar product of a vector written in component form?
- What is the dot product of two vectors?
- What does the dot product tell us about the angle between two vectors?
- How do we find the projection of one vector onto another?

Introduction and Terminology

We have seen that a vector is completely determined by magnitude and direction. So two vectors that have the same magnitude and direction are equal. That means that we can position our vector in the plane and identify it in different ways. For one, we can place the tip of a vector $\mathbf{v}$ at the origin and the tail will wind up at some point (v_1, v_2) as illustrated in Figure 6.2.1.

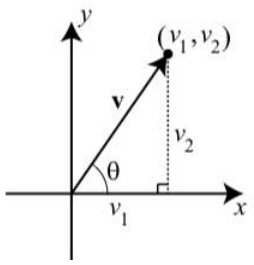


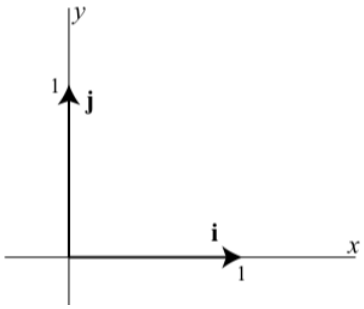
Figure 6.2.1: A Vector in Standard Position

A vector with its initial point at the origin is said to be in **standard position** and is represent by $\mathbf{v} = \langle v_1, v_2 \rangle$. Please note the important distinction in notation between the vector $\mathbf{v} = \langle v_1, v_2 \rangle$ and the point (v_1, v_2). The coordinates of the terminal point (v_1, v_2) are called the **components** of the vector $\mathbf{v}$. We call $\mathbf{v} = \langle v_1, v_2 \rangle$ the **component form** of the vector $\mathbf{v}$. The first coordinate $\mathbf{v}_1$ is called the x-component or the **horizontal component** of the vector $\mathbf{v}$, and the first coordinate $\mathbf{v}_2$ is called the y-component or the **vertical component** of the vector $\mathbf{v}$. The nonnegative angle θ between the vector and the positive x-axis (with $0 \leq \theta \leq 360^\circ$) is called the **direction angle** of the vector. See Figure 6.2.1.

Using Basis Vectors

There is another way to algebraically write a vector if the components of the vector are known. This uses the so-called **standard basis vectors** for vectors in the plane. These two vectors are denoted by $\mathbf{i}$ and $\mathbf{j}$ and are defined as follows and are shown in the diagram below.

$\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$



The diagram in Figure $6.2.2$ shows how to use the vectors $\mathbf{i}$ and $\mathbf{j}$ to represent a vector $\mathbf{v} = \langle a, b \rangle$.

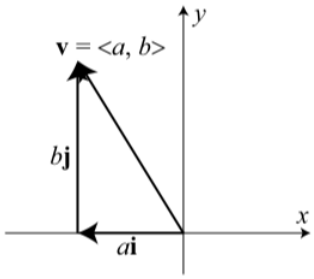


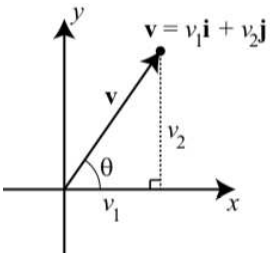
Figure $6.2.2$: Using the Vectors $\mathbf{i}$ and $\mathbf{j}$

The diagram shows that if we place the base of the vector $\mathbf{j}$ at the tip of the vector $\mathbf{i}$, we see that

$$\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j} \tag{6.2.1}$$

This is often called the $\mathbf{i}$, $\mathbf{j}$ **form of a vector**, and the real number a is called the $\mathbf{i}$**-component** of $\mathbf{v}$ and the real number b is called the $\mathbf{j}$**-component** of $\mathbf{v}$

Algebraic Formulas for Geometric Properties of a Vector



Vectors have certain geometric properties such as length and a direction angle. With the use of the component form of a vector, we can write algebraic formulas for these properties. We will use the diagram to the right to help explain these formulas.

- The magnitude (or length) of the vector $\mathbf{v}$ is the distance from the origin to the point (v_1, v_2) and so

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2} \tag{6.2.2}$$

- The direction angle of $\mathbf{v}$ is, where $0 \leq \theta < 360^\circ$, and

$$\cos(\theta) = \frac{v_1}{|\mathbf{v}|} \tag{6.2.3}$$

and

$$\sin(\theta) = \frac{v_2}{|\mathbf{v}|} \tag{6.2.4}$$

- The horizontal and component and vertical component of the vector $\mathbf{v}$ and direction angle θ are

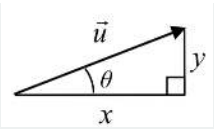
$$v_1 = |\mathbf{v}| \cos(\theta) \tag{6.2.5}$$

and

$$v_2 = |\mathbf{v}| \sin(\theta) \tag{6.2.6}$$

ANOTHER WAY TO LOOK AT THE COMPONENT FORM OF A VECTOR

The component form of a vector represents the vector using two components. $\vec{u} = \langle x, y \rangle$ indicates the vector represents a displacement of x units horizontally and y units vertically.



Notice how we can see the magnitude of the vector as the length of the hypotenuse of a right triangle, or in polar form as the radius, r.

ALTERNATE NOTATION FOR VECTOR COMPONENTS

Sometimes you may see vectors written as the combination of unit vectors $\vec{i}$ and $\vec{j}$, where $\vec{i}$ points the right and $\vec{j}$ points up. In other words, $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle \theta, 1 \rangle$.

In this notation, the vector $\vec{u} = \langle 3, -4 \rangle$ would be written as $\vec{u} = 3\vec{i} - 4\vec{j}$ since both indicate a displacement of 3 units to the right, and 4 units down.

Exercise 6.2.1

1. Suppose the horizontal component of a vector $\mathbf{v}$ is 7 and the vertical component is -3. So we have $\mathbf{v} = 7\mathbf{i} + (-3)\mathbf{j} = 7\mathbf{i} - 3\mathbf{j}$. Determine the magnitude and the direction angle of $\mathbf{v}$.
2. Suppose a vector $\mathbf{w}$ has a magnitude of 20 and a direction angle of -200°. Determine the horizontal and vertical components of $\mathbf{w}$ and write $\mathbf{w}$ in $\mathbf{i}$, $\mathbf{j}$ form.

Answer

1. $\mathbf{v} = 7\mathbf{i} + (-3)\mathbf{j}$. So $|\mathbf{v}| = \sqrt{7^2 + (-3)^2} = \sqrt{58}$. In addition,

$$\cos(\theta) = \frac{7}{\sqrt{58}} \text{ and } \sin(\theta) = \frac{-3}{\sqrt{58}} \tag{6.2.7}$$

So the terminal side of θ is in the fourth quadrant, and we can write

$$\theta = 360^\circ - \arccos(\frac{7}{\sqrt{58}}). \tag{6.2.8}$$

So $\theta \approx 336.80^\circ$.

2. We are given $|\mathbf{w}| = 20$ and the direction angle θ of $\mathbf{w}$ is 200°. So if we write $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}$. Then

$$w_1 = 20\cos(200^\circ) \approx -1.794 \tag{6.2.9}$$

$$w_1 = 20\cos(200^\circ) \approx -6.840 \tag{6.2.10}$$

Operations on Vectors

In pervious sections we earned how to add two vectors and how to multiply a vector by a scalar. At that time, the descriptions of these operations was geometric in nature. We now know about the component form of a vector. So a good question is, “Can we use the component form of vectors to add vectors and multiply a vector by a scalar?”

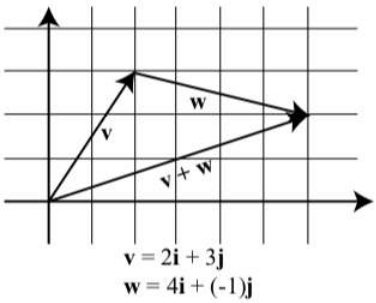


Figure 6.2.3: The Sum of Vector as Compared to Unit Vector Components

To illustrate the idea above in Figure 6.2.3, remember that a vector added to another vector is arranged tip to tail. Then we draw a new vector from the starting point (tail of first vector) to the ending point (tip of last vector). This new vector is the sum of the two. Although we did not use the component forms of these vectors in the previous section, we can now see that

$$\mathbf{v} = \langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j} \tag{6.2.11}$$

and

$$\mathbf{w} = \langle 4, -1 \rangle = 4\mathbf{i} + (-1)\mathbf{j} \tag{6.2.12}$$

The diagram above now shows the vectors in a coordinate plane.

Notice that

$$\mathbf{v} + \mathbf{w} = 6\mathbf{i} + 2\mathbf{j} \tag{6.2.13}$$

$$\mathbf{v} + \mathbf{w} = (2 + 4)\mathbf{i} + (3 + (-1))\mathbf{j} \tag{6.2.14}$$

This means that we can add two vectors by adding their horizontal components and by adding their vertical components. See Figure 6.2.4

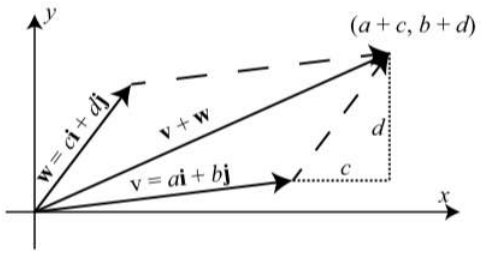


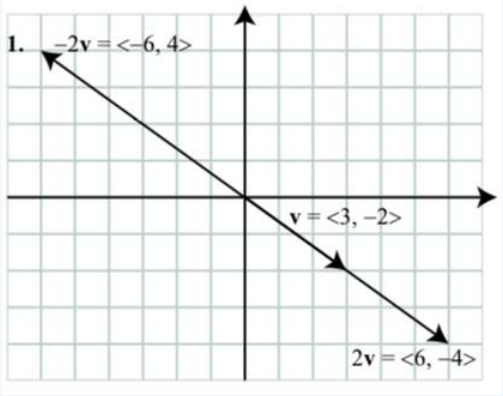
Figure 6.2.4: The Sum of Two Vectors

Exercise 6.2.2

Let $\mathbf{v} = \langle 3, -2 \rangle$. Draw the vector v in standard position and then draw the vectors $2\mathbf{v}$ and $-2\mathbf{v}$ in standard position. What are the component forms of the vectors $2\mathbf{v}$ and $-2\mathbf{v}$?

In general, how do you think a scalar multiple of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ by a scalar c should be defined? Write a formal definition of a scalar multiple of a vector based on your intuition.

Answer



2. For a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ and a scalar c, we define the scalar multiple $c\mathbf{a}$ to be

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle. \tag{6.2.15}$$

Based on the work we have done, we make the following formal definitions.

Definition

For vectors $\mathbf{v} = \langle v_1, v_2 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = \langle w_1, w_2 \rangle = w_1\mathbf{i} + w_2\mathbf{j}$ and scalar c, we make the following definition:

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle \tag{6.2.16}$$

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1)\mathbf{i} + (v_2 + w_2)\mathbf{j} \tag{6.2.17}$$

$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2 \rangle \tag{6.2.18}$$

$$\mathbf{v} - \mathbf{w} = (v_1 - w_1)\mathbf{i} + (v_2 - w_2)\mathbf{j} \tag{6.2.19}$$

$$c\mathbf{v} = \langle cv_1, cv_2 \rangle \tag{6.2.20}$$

$$c\mathbf{v} = (cv_1)\mathbf{i} + (cv_2)\mathbf{j} \tag{6.2.21}$$

Exercise 6.2.3

Let $\mathbf{u} = \langle 1, -2 \rangle$, $\mathbf{u} = \langle 0, 4 \rangle$, and $\mathbf{u} = \langle -5, 7 \rangle$.

1. Determine the component form of the vector $2\mathbf{u} - 3\mathbf{v}$.
2. Determine the magnitude and the direction angle for $2\mathbf{u} - 3\mathbf{v}$.
3. Determine the component form of the vector $u\mathbf{u} + 2\mathbf{v} - 7\mathbf{w}$.

Answer

Let $\mathbf{u} = \langle 1, -2 \rangle$, and $\mathbf{w} = \langle -5, 7 \rangle$.

1. $2\mathbf{u} - 3\mathbf{v} = \langle 2, -4 \rangle - \langle 0, 12 \rangle = \langle 2, -16 \rangle$
2. $|2\mathbf{u} - 3\mathbf{v}| = \sqrt{2^2 + (-16)^2} = \sqrt{260}$. So now let θ be the direction angle of $2\mathbf{u} - 3\mathbf{v}$. Then

$$\cos(\theta) = \frac{2}{\sqrt{260}} \text{ and } \sin(\theta) = \frac{-16}{\sqrt{260}}. \tag{6.2.22}$$

So the terminal side of θ is in the fourth quadfrant. We see that $\arcsin(\frac{-16}{\sqrt{260}}) \approx -82.87^\circ$. Since the direction angle θ must satisfy $a \leq \theta < 360^circ$, we see that $\theta \approx -82.87^\circ + 360^\circ \approx 277.13^\circ$.

3. $\mathbf{u} + 2\mathbf{v} - 7\mathbf{w} = \langle 1, -2 \rangle + \langle 0, 8 \rangle - \langle -35, 49 \rangle = \langle 36, -43 \rangle$.

The Dot Product of Two Vectors

Finding optimal solutions to systems is an important problem in applied mathematics. It is often the case that we cannot find an exact solution that satisfies certain constraints, so we look instead for the "best" solution that satisfies the constraints. An example of this is fitting a least squares curve to a set of data like our calculators do when computing a sine regression curve. The dot product is useful in these situations to find "best" solutions to certain types of problems. Although we won't see it in this course, having collections of perpendicular vectors is very important in that it allows for fast and efficient computations. The dot product of vectors allows us to measure the angle between them and thus determine if the vectors are perpendicular. The dot product has many applications, e.g., finding components of forces acting in different directions in physics and engineering. We introduce and investigate dot products in this section.

We have seen how to add vectors and multiply vectors by scalars, but we have not yet introduced a product of vectors. In general, a product of vectors should give us another vector, but there turns out to be no really useful way to define such a product of vectors. However, there is a dot "product" of vectors whose output is a scalar instead of a vector, and the dot product is a very useful product (even though it isn't a product of vectors in a technical sense).

Recall that the magnitude (or length) of the vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is:

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = \sqrt{u_1 u_1 + u_2 u_2} \tag{6.2.23}$$

The expression under the second square root is an important one and we extend it and give it a special name.

Definition

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors in the plane. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2. \tag{6.2.24}$$

This may seem like a strange number to compute, but it turns out that the dot product of two vectors is useful in determining the angle between two vectors. Recall we can use the Law of Cosines to determine the sum of two vectors and then used the Law of Sines to determine the angle between the sum and one of those vectors. We have now seen how much easier it is to compute the sum of two vectors when the vectors are in component form. The dot product will allow us to determine the cosine of the angle between two vectors in component form. This is due to the following result:

The Dot Product and the Angle between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ ($0^\circ \leq \theta \leq 180^\circ$), then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta) \tag{6.2.25}$$

or

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \tag{6.2.26}$$

Notice that if we have written the vectors $\mathbf{v}$ and $\mathbf{w}$ in component form, then we have formulas to compute $|\mathbf{v}|$, $|\mathbf{w}|$, and $\mathbf{v} \cdot \mathbf{w}$. This result may seem surprising but it is a fairly direct consequence of the Law of Cosines as we will now show. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$ as illustrated in Figure $6.2.5$

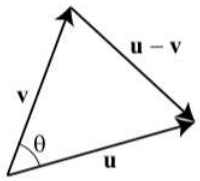


Figure 6.2.5: The angle between $\mathbf{u}$ and $\mathbf{v}$

We can apply the Law of Cosines to determine the angle θ as follows:

$$|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos(\theta) \tag{6.2.27}$$

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos(\theta) \tag{6.2.28}$$

$$(\mathbf{u}\cdot\mathbf{u}) - 2(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos(\theta) \tag{6.2.29}$$

$$|\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos(\theta) \tag{6.2.30}$$

$$-2(\mathbf{u}\cdot\mathbf{u}) = -2|\mathbf{u}||\mathbf{v}|\cos(\theta) \tag{6.2.31}$$

$$\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}||\mathbf{v}|\cos(\theta) \tag{6.2.32}$$

Exercise 6.2.4

1. Determine the angle θ between the vectors $\mathbf{u} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{v} = -5\mathbf{i}+2\mathbf{j}$.

2. Determine all vectors perpendicular to $\mathbf{u} = \langle 1, 3\rangle$. How many such vectors are there? **Hint**: Let $\mathbf{v} = \langle a, b\rangle$. Under what conditions will the angle between $\mathbf{u}$ and $\mathbf{v}$ be 90°?

Answer

1. If θ is the angle between $\mathbf{u} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{v} = -5\mathbf{i}+2\mathbf{j}$, then

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{-13}{\sqrt{10}\sqrt{29}} \tag{6.2.33}$$

$$\theta = \cos^{-1}\left(\frac{-13}{\sqrt{10}\sqrt{29}}\right) \tag{6.2.34}$$

so $\theta \approx 139.764^\circ$.

2. If $\mathbf{v} = \langle a, b\rangle$ is perpendicular to $\mathbf{u} = \langle 1, 3\rangle$, then the angle θ between them is 90° and so $\cos(\theta) = 0$. So we must have $\mathbf{u}\cdot\mathbf{v} = 0$ and this means that $a+3b = 0$. So any vector $\mathbf{v} = \langle a, b\rangle$ where $a = -3b$ will be perpendicular to $\mathbf{v}$, and there are infinitely many such vectors. One vector perpendicular to $\mathbf{u}$ is $\langle -3, 1\rangle$.

One purpose of Progress Check 3.34 was to use the formula

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \tag{6.2.35}$$

to determine when two vectors are perpendicular. Two vectors $\mathbf{u}$ and $\mathbf{v}$ will be perpendicular if and only if the angle θ between them is 90°. Since $\cos(90^\circ) = 0$, we see that this formula implies that $\mathbf{u}$ and $\mathbf{v}$ will be perpendicular if and only if $\mathbf{u}\cdot\mathbf{v} = 0$ (This is because a fraction will be equal to 0 only when the numerator is equal to 0 and the denominator is not zero.) So we have:

Two vectors are perpendicular if and only if their dot product is equal to 0.

> **Note**
>
> When two vectors are perpendicular, we also say that they are **orthogonal**.

Projections

Another useful application of the dot product is in finding the projection of one vector onto another. An example of where such a calculation is useful is the following.

Usain Bolt from Jamaica excited the world of track and field in 2008 with his world record performances on the track. Bolt won the 100 meter race in a world record time of 9.69 seconds. He has since bettered that time with a race of 9.58 seconds with a wind assistance of 0.9 meters per second in Berlin on August 16, 20092. The wind assistance is a measure of the wind speed that is helping push the runners down the track. It is much easier to run a very fast race if the wind is blowing hard in the direction of the race. So that world records aren't dependent on the weather conditions, times are only recorded as record times if the wind aiding the runners is less than or equal to 2 meters per second. Wind speed for a race is recorded by a wind gauge that is set up close to the track. It is important to note, however, that weather is not always as cooperative as we might like. The wind does not always blow exactly in the direction of the track, so the gauge must account for the angle the wind makes with the track.

If the wind is blowing in the direction of the vector $\mathbf{u}$ and the track is in the direction of the vector $\mathbf{v}$ in Figure 6.2.6, then only part of the total wind vector is actually working to help the runners.

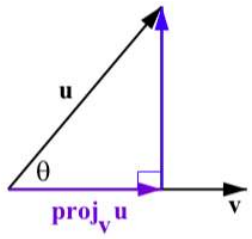


Figure 6.2.6: The Projection of $\mathbf{u}$ onto $\mathbf{v}$

This part is called the **projection of the vector $\mathbf{u}$ onto the vector $\mathbf{v}$** and is denoted $\mathbf{proj}_{\mathbf{v}}\mathbf{u}$.

We can find this projection with a little trigonometry. To do so, we let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$ as shown in Figure 6.2.5. Using right triangle trigonometry, we see that

$$|\mathbf{proj}_{\mathbf{v}}\mathbf{u}| = |\mathbf{u}|\cos(\theta) = |\mathbf{u}|\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} \tag{6.2.36}$$

The quantity we just derived is called the **scalar projection (or component)** of $\mathbf{u}$ onto $\mathbf{v}$ and is denoted by $\mathbf{comp}_{\mathbf{v}}\mathbf{u}$. Thus $\mathbf{comp}_{\mathbf{v}}\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$

This gives us the length of the vector projection. So to determine the vector, we use a scalar multiple of this length times a unit vector in the same direction, which is $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$. So we obtain

$$\mathbf{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{proj}_{\mathbf{v}}\mathbf{u}|\left(\frac{1}{|\mathbf{v}|}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\left(\frac{1}{|\mathbf{v}|}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} \tag{6.2.37}$$

We can also write the projection of $\mathbf{u}$ onto $\mathbf{v}$ as

$$\mathbf{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \tag{6.2.38}$$

The wind component that acts perpendicular to the direction of $\mathbf{v}$ in Figure 6.2.7 is called the **projection of $\mathbf{u}$ orthogonal to $\mathbf{v}$** and is denoted $\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$. Since $\mathbf{u} = \mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} + \mathbf{proj}_{\mathbf{v}}\mathbf{u}$, we have that

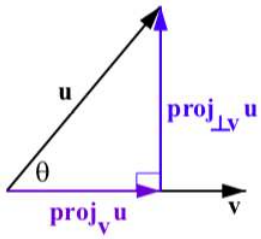


Figure 6.2.7: The Projection of $\mathbf{u}$ onto $\mathbf{v}$

$$\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} = \mathbf{u} - \mathbf{proj}_{\mathbf{v}}\mathbf{u} \tag{6.2.39}$$

Following is a summary of the results we have obtained.

For nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, the **projection of the vector $\mathbf{u}$ onto the vector $\mathbf{v}$** $\mathbf{proj}_{\mathbf{v}}\mathbf{u}$, is given by

$$\mathbf{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \tag{6.2.40}$$

See Figure 6.2.6. The **projection of $\mathbf{u}$ orthogonal to $\mathbf{v}$**, denoted $\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$, is

$$\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} = \mathbf{u} - \mathbf{proj}_{\mathbf{v}}\mathbf{u} \tag{6.2.41}$$

We note that

$$\mathbf{u} = \mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} + \mathbf{proj}_{\mathbf{v}}\mathbf{u} \tag{6.2.42}$$

Exercise 6.2.5

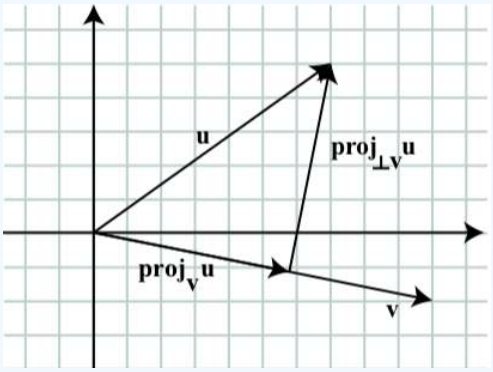
Let $\mathbf{u} = \langle 7, 5\rangle$ and $\mathbf{v} = \langle 10, -2\rangle$. Determine $\mathbf{proj}_{\mathbf{v}}\mathbf{u}$ and $\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$, and verify that $\mathbf{u} = \mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} + \mathbf{proj}_{\mathbf{v}}\mathbf{u}$. Draw a picture showing all of the vectors involved in this.

Answer

Let $\mathbf{u} = \langle 7, 5\rangle$ and $\mathbf{v} = \langle 10, -2\rangle$. Then

$$\mathbf{proj}_{v}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \langle\frac{600}{104}, \frac{-120}{104}\rangle \approx \langle 5.769, -1.154\rangle \tag{6.2.43}$$

$$\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \langle 7, 5\rangle - \langle\frac{600}{104}, \frac{-120}{104}\rangle = \langle\frac{128}{104}, \frac{640}{104}\rangle \approx \langle 1.231, 6.154\rangle \tag{6.2.44}$$



While it can be convenient to think of the vector $\vec{u} = \langle x, y\rangle$ as an arrow from the origin to the point (x, y), be sure to remember that most vectors can be situated anywhere in the plane, and simply indicate a displacement (change in position) rather than a position.

It is common to need to convert from a magnitude and angle to the component form of the vector and vice versa. Happily, this

process is identical to converting from polar coordinates to Cartesian coordinates, or from the polar form of complex numbers to the $a+bi$ form.

Example Applications

Example 6.2.1

Find the component form of a vector with length 7 at an angle of 135 degrees.

Solution

Using the conversion formulas $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we can find the components

$$x = 7\cos(135^\circ) = -\frac{7\sqrt{2}}{2}$$

$$y = 7\sin(135^\circ) = \frac{7\sqrt{2}}{2}$$

The vector can be written in component form as $\langle -\frac{7\sqrt{2}}{2}, \frac{7\sqrt{2}}{2} \rangle$

Example 6.2.2

Find the magnitude and angle θ representing the vector $\vec{u} = \langle 3, -2 \rangle$.

Solution

First we can find the magnitude by remembering the relationship between x, y and r:

$$r^2 = 3^2 + (-2)^2 = 13$$

$$r = \sqrt{13}$$

Second we can find the angle. Using the tangent,

$$\tan(\theta) = \frac{-2}{3}$$

$$\theta = \tan^{-1}(-\frac{2}{3}) \approx -33.69^\circ$$

or written as a coterminal positive angle, 326.31°. This angle is in the 4th quadrant as desired.

Example 6.2.3

Using vector $\vec{v}$ from the first Exercise, the vector that travels from the origin to the point (3, 5), find the components, magnitude and angle θ that represent this vector.

Answer

$$\vec{v} = \langle 3, 5 \rangle$$

$$\text{magnitude} = \sqrt{34}$$

$$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.04^\circ$$

In addition to representing distance movements, vectors are commonly used in physics and engineering to represent any quantity that has both direction and magnitude, including velocities and forces.

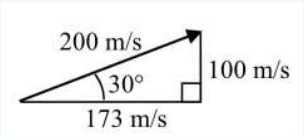
Example 6.2.4

An object is launched with initial velocity 200 meters per second at an angle of 30 degrees. Find the initial horizontal and vertical velocities.

Solution

By viewing the initial velocity as a vector, we can resolve the vector into horizontal and vertical components.

$$x = 200\cos(30^\circ = 200 \cdot \frac{\sqrt{3}}{2} \approx 173.205 \text{ m/sec}$$



$$y = 200\sin(30^\circ = 200 \cdot \frac{1}{2} = 100 \text{ m/sec}$$

This tells us that, absent wind resistance, the object will travel horizontally at about 173 meters each second. Gravity will cause the vertical velocity to change over time – we'll leave a discussion of that to physics or calculus classes.

Adding and Scaling Vectors in Component Form

To add vectors in component form, we can simply add the corresponding components. To scale a vector by a constant, we scale each component by that constant.

COMBINING VECTORS IN COMPONENT FORM

To **add, subtract, or scale vectors** in component form

If $\vec{u} = \langle u_1, u_2 \rangle$, $\vec{v} = \langle v_1, v_2 \rangle$, and c is any constant, then

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \tag{6.2.45}$$

$$\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2 \rangle \tag{6.2.46}$$

$$c\vec{u} = \langle cu_1, cu_2 \rangle \tag{6.2.47}$$

Example 6.2.5

Given $\vec{u} = \langle 3, -2 \rangle$ and $\vec{v} = \langle -1, 4 \rangle$, find a new vector $\vec{w} = 3\vec{u} - 2\vec{v}$

Solution

Using the vectors given,

$$\begin{aligned} \vec{w} &= 3\vec{u} - 2\vec{v} \\ &= 3\langle 3, -2 \rangle - 2\langle -1, 4 \rangle && \text{Scale each vector} \\ &= \langle 9, -6 \rangle - \langle -2, 8 \rangle && \text{Subtract corresponding components} \\ &= \langle 11, -14 \rangle \end{aligned}$$

By representing vectors in component form, we can find the resulting displacement vector after a multitude of movements without needing to draw a lot of complicated non- right triangles. For a simple example, we revisit the problem from the opening of the section. The general procedure we will follow is:

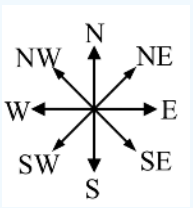
1) Convert vectors to component form
2) Add the components of the vectors
3) Convert back to length and direction if needed to suit the context of the question

Example 6.2.6

A woman leaves home, walks 3 miles north, then 2 miles southeast. How far is she from home, and what direction would she need to walk to return home? How far has she walked by the time she gets home?

Solution

Let's begin by understanding the question in a little more depth. When we use vectors to describe a traveling direction, we often position things so north points in the upward direction, east points to the right, and so on, as pictured here.



Consequently, travelling NW, SW, NE or SE, means we are travelling through the quadrant bordered by the given directions at a 45 degree angle.

With this in mind, we begin by converting each vector to components.

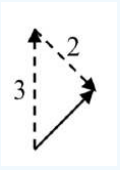
A walk 3 miles north would, in components, be $\langle 0, 3\rangle$.

A walk of 2 miles southeast would be at an angle of 45° South of East. Measuring from standard position the angle would be 315°.

Converting to components, we choose to use the standard position angle so that we do not have to worry about whether the signs are negative or positive; they will work out automatically.

$$\langle 2\cos(315^\circ),\ 2\sin(315^\circ)\rangle = \langle 2\cdot\frac{\sqrt{2}}{2},\ 2\cdot\frac{-\sqrt{2}}{2}\rangle \approx \langle 1.414, -1.414\rangle$$

Adding these vectors gives the sum of the movements in component form



$$\langle 0, 3\rangle + \langle 1.414, -1.414\rangle = \langle 1.4141.586\rangle$$

To find how far she is from home and the direction she would need to walk to return home, we could find the magnitude and angle of this vector.

$$\text{Length} = \sqrt{1.414^2 + 1.586^2} = 2.125$$

To find the angle, we can use the tangent

$$\tan(\theta) = \frac{1.586}{1.414}$$

$$\theta = tan^{-1}(\frac{1.586}{1.414}) = 48.273^\circ \text{ north of east}$$

Of course, this is the angle from her starting point to her ending point. To return home, she would need to head the opposite direction, which we could either describe as $180^\circ + 48.273^\circ = 228.273^\circ$ measured in standard position, or as 48.273° south of west (or 41.727° west of south).

She has walked a total distance of 3 + 2 + 2.125 = 7.125 miles.

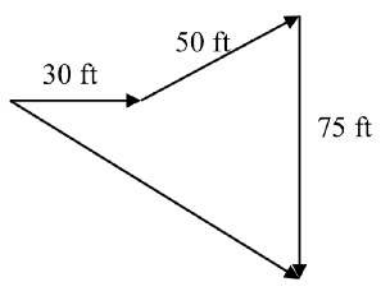
Keep in mind that total distance traveled is not the same as the length of the resulting displacement vector or the "return" vector.

Example 6.2.7

In a scavenger hunt, directions are given to find a buried treasure. From a starting point at a flag pole you must walk 30 feet east, turn 30 degrees to the north and travel 50 feet, and then turn due south and travel 75 feet. Sketch a picture of these

vectors, find their components, and calculate how far and in what direction you must travel to go directly to the treasure from the flag pole without following the map.

Answer



$$\vec{v}_1 = \langle 30, 0\rangle \quad \vec{v}_2 = \langle 50\cos(30^\circ), 50\sin(30^{circ})\rangle \quad \vec{v}_3 = \langle 0, -75\rangle$$

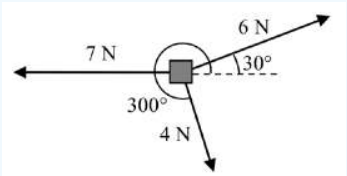
$$\vec{v}_f = \langle 30 + 50\cos(30^\circ), 50\sin(30^\circ) - 75\rangle = \langle 73.301, -50\rangle$$

Magnitude = 88.73 feet at an angle of 34.3° south of east.

While using vectors is not much faster than using law of cosines with only two movements, when combining three or more movements, forces, or other vector quantities, using vectors quickly becomes much more efficient than trying to use triangles.

Example 6.2.8

Three forces are acting on an object as shown below, each measured in Newtons (N). What force must be exerted to keep the object in equilibrium (where the sum of the forces is zero)?



Solution

We start by resolving each vector into components.

The first vector with magnitude 6 Newtons at an angle of 30 degrees will have components

$$\langle 6\cos(30^\circ).\, 6\sin(30^\circ)\rangle = \langle 6\cdot\frac{\sqrt{3}}{2}, 6\cdot\frac{1}{2}\rangle = \langle 3\sqrt{33}\rangle$$

The second vector is only in the horizontal direction, so can be written as $\langle -7, 0\rangle$.

The third vector with magnitude 4 Newtons at an angle of 300 degrees will have components

$$\langle 4\cos(300^\circ), 4\sin(300^\circ)\rangle = \langle 4\cdot\frac{1}{2}, 4\cdot\frac{-\sqrt{3}}{2}\rangle = \langle 2, -2\sqrt{3}\rangle$$

To keep the object in equilibrium, we need to find a force vector x, y so the sum of the four vectors is the zero vector, $\langle 0, 0\rangle$.

$$\langle 3\sqrt{3}, 3\rangle + \langle -7, 0\rangle + \langle 2, -2\sqrt{3}\rangle + \langle x, y\rangle = \langle 0, 0\rangle$$

Add component-wise

$$\langle 3\sqrt{3} - 7 + 2, 3 + 0 - 2\sqrt{3}\rangle + \langle x, y\rangle = \langle 0, 0\rangle$$

Simplify

$$\langle 3\sqrt{3}-5, 3-2\sqrt{3}\rangle + \langle x, y\rangle = \langle 0, 0\rangle$$

Solve

$$\langle x, y\rangle = \langle 0, 0\rangle - \langle 3\sqrt{3}-5, 3-2\sqrt{3}\rangle$$

$$\langle x, y\rangle = \langle -3\sqrt{3}+5, -3+2\sqrt{3}\rangle \approx \langle -0.196, 0.464\rangle$$

This vector gives in components the force that would need to be applied to keep the object in equilibrium. If desired, we could find the magnitude of this force and direction it would need to be applied in.

$$\text{Magnitude} = \sqrt{(-0.196)^2 + 0.464^2} = 0.504 \text{ N}$$

Angle:

$$\tan(\theta) = \frac{0.464}{-0.196}$$

$$\theta = \tan^{-1}(\frac{0.464}{-0.196} = -67.089^\circ$$

This is in the wrong quadrant, so we adjust by finding the next angle with the same tangent value by adding a full period of tangent:

$$\theta = -67.089^\circ + 180^\circ = 112.911^{circ}$$

To keep the object in equilibrium, a force of 0.504 Newtons would need to be applied at an angle of 112.911°

Summary of Section

In this section, we studied the following important concepts and ideas:

The **component form** of a vector $\mathbf{v}$ is written as $\mathbf{v} = \langle v_1, v_2\rangle$ and the$\mathbf{i}, \mathbf{j}$ form of the same vector is $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$. Using this notation, we have

- The magnitude (or length) of the vector $\mathbf{v}$ is $\mathbf{v} = \sqrt{v_1^2 + v_2^2}$.
- The direction angle of $\mathbf{v}$ is θ, where $0^\circ \leq \theta < 360^\circ$, and

$$\cos(\theta) = \frac{v_1}{|\mathbf{v}|} \tag{6.2.48}$$

and

$$\sin(\theta) = \frac{v_2}{|\mathbf{v}|} \tag{6.2.49}$$

- The horizontal and component and vertical component of the vector $\mathbf{v}$ and direction angle θ are

$$v_1 = |\mathbf{v}|\cos(\theta) \tag{6.2.50}$$

and

$$v_2 = |\mathbf{v}|\sin(\theta) \tag{6.2.51}$$

For two vectors $\mathbf{v}$ and $\mathbf{w}$ with $\mathbf{v} = \langle v_1, v_2\rangle = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = \langle w_1, w_2\rangle = w_1\mathbf{i} + w_2\mathbf{j}$ and a scalar c:

- $$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2\rangle = (v_1 + w_1)\mathbf{i} + (v_2 + w_2)\mathbf{j} \tag{6.2.52}$$
- $$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2\rangle = (v_1 - w_1)\mathbf{i} + (v_2 - w_2)\mathbf{j} \tag{6.2.53}$$

- $$c\mathbf{v} = \langle cv_1, cv_2 \rangle = (cv_1)\mathbf{i} + (cv_2)\mathbf{j} \tag{6.2.54}$$
- The **dot product** of $\mathbf{v}$ and $\mathbf{w}$ is $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$.
- If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then

  $$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}||\mathbf{v}| \cos(\theta) \tag{6.2.55}$$

  or

  $$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \tag{6.2.56}$$
- The projection of the vector $\mathbf{v}$ onto the vector $\mathbf{w}$ $\mathbf{proj}_{\mathbf{w}}\mathbf{v}$, is given by

  $$\mathbf{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w} \tag{6.2.57}$$

  The projection of $\mathbf{v}$ orthogonal to $\mathbf{w}$, denoted $\mathbf{proj}_{\perp\mathbf{w}}\mathbf{v}$, is

  $$\mathbf{proj}_{\perp\mathbf{w}}\mathbf{v} = \mathbf{v} - \mathbf{proj}_{\mathbf{w}}\mathbf{v} \tag{6.2.58}$$

  We note that

  $$\mathbf{w} = \mathbf{proj}_{\perp\mathbf{w}}\mathbf{v} + \mathbf{proj}_{\mathbf{w}}\mathbf{v} \tag{6.2.59}$$

  See Figure 6.2.8.

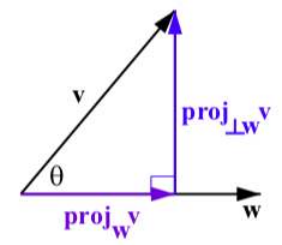


Figure 6.2.8: The Projection of $\mathbf{v}$ onto $\mathbf{w}$

- Scaling of vectors
- Components of vectors
- Vectors as velocity
- Vectors as forces
- Adding & Scaling vectors in component form - Total distance traveled vs. total displacement

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6.3: Solving Systems of Equations with Augmented Matrices

Learning Objectives

- Be able to describe the definition of an augmented matrix.
- Recognize when an augmented matrix would improve the speed at which a system of equations might be solved.
- Be able to correctly enter a system of equations into a calculator and interpret the reduced row echelon form of the matrix.

If you have ever solved a system of equations, you know that it can be time intensive and tedious. Fortunately, there is a process by which a calculator can complete the task for you! This section will go over the basic process by which we can solve a system of equations quickly and effectively! This will be particularly helpful for vector questions with tension in a rope or when a mass is hanging from a cable. In this situation there are two tensions and a system of equations is generated to calculate the tension in each rope/cable, where the components are broken out - creating a system of equations.

What is an Augmented Matrix?

Let's first talk about a matrix. For the purposes of this class we will define a matrix to have rows and columns. The rows of the matrix will be associated with the coefficients of each term in an equation. When working with matrices, we must always place the same terms for each equation in the SAME order; this allows us to assume the variable location and, specifically, which variable we are referencing later in the process without having to write it in every step. Here is an example of a system of equations:

$$3x+8y=11 \tag{6.3.1}$$
$$5x+7y=35 \tag{6.3.2}$$

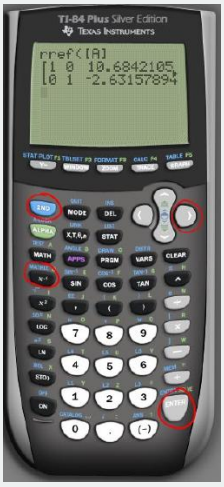
Notice that the x term coefficients are in the first column and the y term coefficients are in the second column. The third column would be considered the constants or the value that balances the equation. Including the constant as the third column makes this an Augmented Matrix as shown below:

$$\begin{bmatrix} 3 & 8 & 11 \\ 5 & 7 & 35 \end{bmatrix}$$

NOTE: Sometimes you will see the augmented matrix represented by a vertical line, separating the coefficients from the constants column as below, which wordlessly implies it is an augmented matrix.

$$\left[\begin{array}{cc|c} 3 & 8 & 11 \\ 5 & 7 & 35 \end{array}\right] \tag{6.3.3}$$

Matrix Application on a Calculator to Solve a System of Equations



- Create a coefficient matrix corresponding to the equation:
  - Select 2nd > MATRIX
  - Arrow to EDIT then press ENTER
  - Enter the number of rows desired then press ENTER
  - Enter the the number of columns that are desired then press ENTER
  - Enter each value for each location in the matrix in the same way you entered the previous values.
- Click 2nd > QUIT
- Press 2nd > MATRIX, MATH, and arrow down to “rref” and press ENTER
- Press 2nd > MATRIX, arrow down to the matrix you want, and press ENTER
- Press ENTER
- The Row Reduced Matrix should be shown in a diagonal of ones and zeros with the solution to the first "1" corresponds to 10.68... and the second row "1" corresponds to -2.63... .

Notice that in this particular image, the keys used to build the matrix are circled in red - the 2nd button in the top left, the arrow right button in the top right, the Matrix button on the middle left and the enter button in the bottom right.

Example 6.3.1

In this scenario a Zipline is VERY loosely attached to two trees. Calculate the tension in the wire supporting the 90.0-kg human. In Figure 6.3.1 the free body diagram is shown with angles of 57 degrees and 38 degrees respectively off the horizontal. Remember that if you calculate these components of x and y you will need to use negatives for the x values to the left and y downwards, or in the case of cosine, you will need to use the difference between 180 degrees and 57 degrees.

The arrow downward represents the weight of the human and is not to scale!

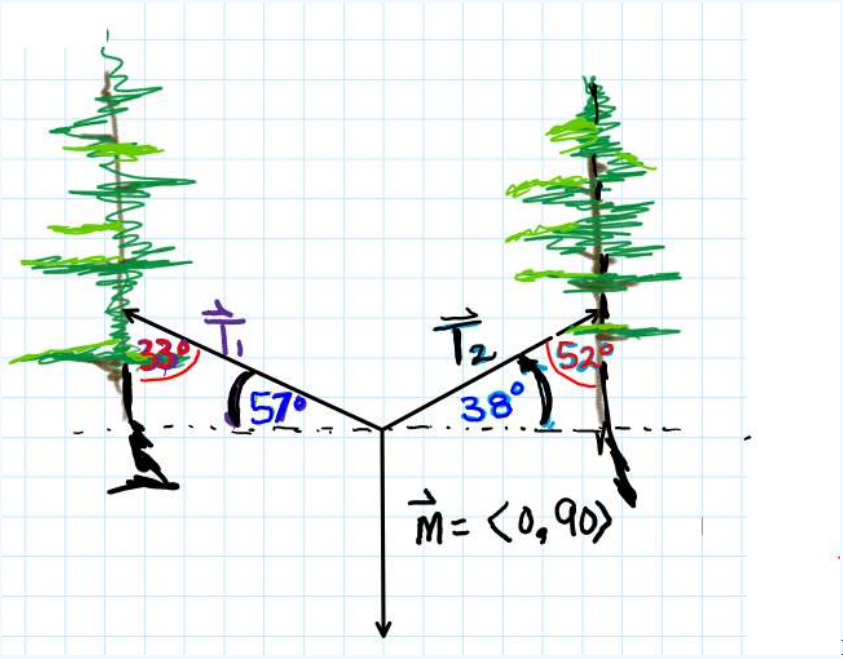


Figure 6.3.1: Free Body Diagram Sketch of Tension in Cable and Mass of Human Downward

Solution

We need to break down the components into the x direction and the y direction separately.

We will list the equation for the x direction components in the first row and the y direction components in the second row:

$$T1\cos(180^o - 57^o) + T2\cos(38^o) = 0 \quad (6.3.4)$$

$$T1\sin(180^o - 57^o) + T2\sin(38^o) - 90 = 0 \quad (6.3.5)$$

As an augmented matrix we would have:

$$\left[\begin{array}{cc|c} \cos(123^o) & \cos(38^o) & 0 \\ \sin(123^o) & \sin(38^o) & 90 \end{array}\right] \quad (6.3.6)$$

As a row reduced echelon form the tension in the ropes are as follows:

$$\left[\begin{array}{cc|c} 1 & 0 & 71.19187 \\ 0 & 1 & 49.20475 \end{array}\right] \quad (6.3.7)$$

Tension $T1 = 71.19187kg$)

Tension $T2 = 49.20475kg$)

Key Concepts

- Augmented matrices are used to quickly solve systems of equations. The columns of the matrix represent the coefficients for each variable present in the system, and the constant on the other side of the equals sign. This implies there will always be one more column than there are variables in the system.

- Practice the process of using a matrix to solve a system of equations a few times. This will help with remembering the steps on your calculator - calculators are different. You might need to search for the specific instructions for your calculator. We covered what it looks like when using a TI-84 Plus Silver Edition.
- Use the number of equations and the number of variables to determine the appropriate size of the matrix.
- When using trig functions within your matrix, be sure to be in the correct mode. If a trig function is negative, be sure to include the sign with the entry.

Glossary

Rows and Columns

a system of equations can be converted into matrix form, where the number of equations is equal to the number of rows and the number of columns is determined by the number of variables plus one more column for the constants on the other side of the equals sign.

Augmented Matrix

a matrix that includes an additional column for the constant term.

Row Reduced Echelon Form

A system of equations in it's simplified or solved form, where diagonals of ones are followed by the corresponding value for each variable in the system's original order.

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6.E: Triangles and Vectors (Exercises)

Vector (Exercises)

Verbal

1) What are the characteristics of the letters that are commonly used to represent vectors?

Answer

lowercase, bold letter, usually u, v, w

2) How is a vector more specific than a line segment?

3) What are i and j, and what do they represent?

Answer

They are unit vectors. They are used to represent the horizontal and vertical components of a vector. They each have a magnitude of 1.

4) What is component form?

5) When a unit vector is expressed as $\langle a, b\rangle$ which letter is the coefficient of the i and which the j?

Answer

The first number always represents the coefficient of the i and the second represents the j.

Algebraic

6) Given a vector with initial point $(5, 2)$ and terminal point $(-1, -3)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b\rangle$.

7) Given a vector with initial point $(-4, 2)$ and terminal point $(3, -3)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b\rangle$.

Answer

$\langle 7, -5\rangle$

8) Given a vector with initial point $(7, -1)$ and terminal point $(-1, -7)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b\rangle$.

For the exercises 9-15, determine whether the two vectors u and v are equal, where u has an initial point P_1 and a terminal point P_2 and v has an initial point P_3 and a terminal point P_4.

9) $P_1 = (5, 1), P_2 = (3, -2), P_3 = (-1, 3), P4 = (9, -4)$

Answer

not equal

10) $P_1 = (2, -3), P_2 = (5, 1), P_3 = (6, -1), P_4 = (9, 3)$

11) $P_1 = (-1, -1), P_2 = (-4, 5), P_3 = (-10, 6), P_4 = (-13, 12)$

Answer

equal

12) $P_1 = (3, 7), P_2 = (2, 1), P_3 = (1, 2), P_4 = (-1, -4)$

13) $P_1 = (8, 3), P_2 = (6, 5), P_3 = (11, 8), P4 = (9, 10)$

Answer

equal

14) Given initial point $P_1 = (-3, 1)$ and terminal point $P_2 = (5, 2)$, write the vector v in terms of i and j.

15) Given initial point $P_1 = (6, 0)$ and terminal point $P_2 = (-1, -3)$, write the vector v in terms of i and j.

Answer

$7i - 3j$

For the exercises 16-17, use the vectors $u = i + 5j, v = -2i - 3j, w = 4i - j$

16) Find $u + (v - w)$

17) Find $4v + 2u$

Answer

$-6i - 2j$

For the exercises 18-21, use the given vectors to compute $u + v, u - v, 2u - 3v$

18) $u = \langle 2, -3\rangle, v = \langle 1, 5\rangle$

19) $u = \langle -3, 4\rangle, v = \langle -2, 1\rangle$

Answer

$u + v = \langle -5, 5\rangle, u - v = \langle -1, 3\rangle, 2u - 3v = \langle 0, 5\rangle$

20) Let $v = -4i + 3j$. Find a vector that is half the length and points in the same direction as v.

21) Let $v = 5i + 2j$. Find a vector that is twice the length and points in the opposite direction as v.

Answer

$-10i-4j$

For the exercises 22-27, find a unit vector in the same direction as the given vector.

22) $a=3i+4j$

23) $b=-2i+5j$

Answer

$-\frac{2\sqrt{29}}{29}i+\frac{5\sqrt{29}}{29}j$

24) $c=10i-j$

25) $d=-\frac{1}{3}i+\frac{5}{2}j$

Answer

$-\frac{2\sqrt{229}}{229}i+\frac{15\sqrt{229}}{229}j$

26) $u=100i+200j$

27) $u=-14i+2j$

Answer

$-\frac{7\sqrt{2}}{10}i+\frac{\sqrt{2}}{10}j$

For the exercises 28-35, find the magnitude and direction of the vector, $0\le\theta<2\pi$.

28) $\langle 0,4\rangle$

29) $\langle 6,5\rangle$

Answer

$|v|=7.810, \theta=39.806^{\circ}$

30) $\langle 2,-5\rangle$

31) $\langle -4,-6\rangle$

Answer

$|v|=7.211, \theta=236.310^{\circ}$

32) Given $u=3i-4j$ and $v=-2i+3j$, calculate $u\cdot v$.

33) Given $u=-i-j$ and $v=i+5j$, calculate $u\cdot v$.

Answer

-6

34) Given $u=\langle -2,4\rangle$ and $v=\langle -3,1\rangle$, calculate $u\cdot v$.

35) Given $u=\langle -1,6\rangle$ and $v=\langle 6,-1\rangle$,calculate $u\cdot v$.

Answer

-12

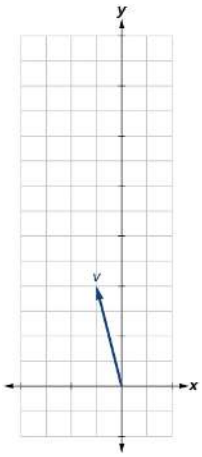
Graphical

For the exercises 36-,38 given v, draw v, $3v$, and $\frac{1}{2}v$.

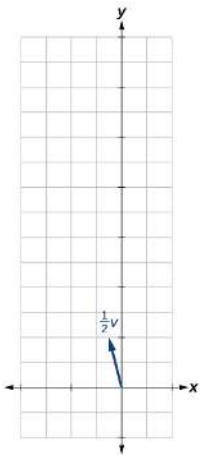
36) $\langle 2,-1\rangle$

37) $\langle -1,4\rangle$

Answer



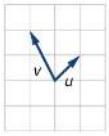




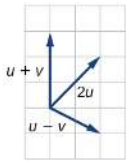
38) $\langle -3, -2 \rangle$

For the exercises 39-41, use the vectors shown to sketch $u + v$, $u - v$, and $2u$.

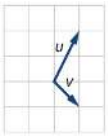
39)



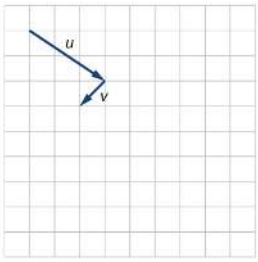
Answer



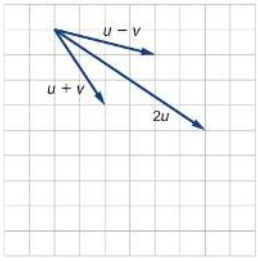
40)



41)

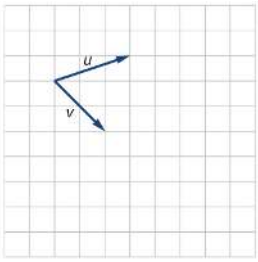


Answer

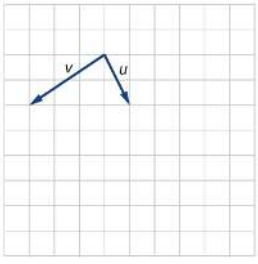


For the exercises 42-43, use the vectors shown to sketch $2u + v$.

42)



43)

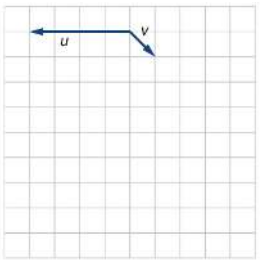


Answer

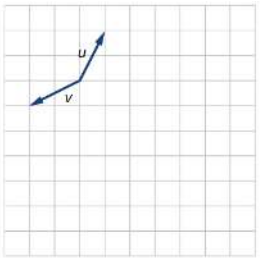


For the exercises 44-45, use the vectors shown to sketch $u - 3v$.

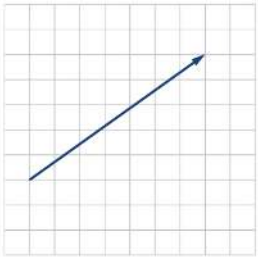
44)



45)



Answer



For the exercises 46-47, write the vector shown in component form.

46)



47)



Answer

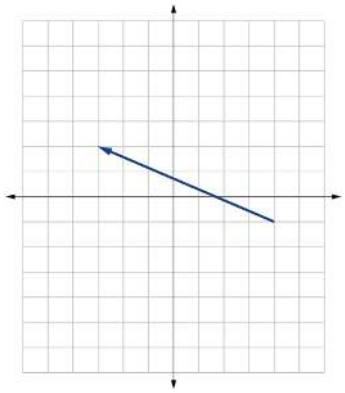
$\langle 4, 1 \rangle$

48) Given initial point $P_1 = (2, 1$ and terminal point $P_2 = (-1, 2)$, write the vector v in terms of i and j, then draw the vector on the graph.

49) Given initial point $P_1 = (4, -1$ and terminal point $P_2 = (-3, 2)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Answer

$v = -7i + 3j$



50) Given initial point $P_1 = (3, 3$ and terminal point $P_2 = (-3, 3)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Extensions

For the exercises 51-54, use the given magnitude and direction in standard position, write the vector in component form.

51) $|v| = 6, \theta = 45^\circ$

Answer

$3\sqrt{2}i + 3\sqrt{2}j$

52) $|v| = 8, \theta = 220^\circ$

53) $|v| = 2, \theta = 300^\circ$

Answer

$i - \sqrt{3}j$

54) $|v| = 5, \theta = 135^\circ$

55) A 60-pound box is resting on a ramp that is inclined 12°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.
b. Find the magnitude of the component of the force that is parallel to the ramp.

Answer

a. 58.7
b. 12.5

56) A 25-pound box is resting on a ramp that is inclined 8°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.
b. Find the magnitude of the component of the force that is parallel to the ramp.

57) Find the magnitude of the horizontal and vertical components of a vector with magnitude 8 pounds pointed in a direction of 27° above the horizontal. Round to the nearest hundredth.

Answer

$x = 7.13$ pounds, $y = 3.63$ pounds

58) Find the magnitude of the horizontal and vertical components of the vector with magnitude 4 pounds pointed in a direction of 127° above the horizontal. Round to the nearest hundredth.

59) Find the magnitude of the horizontal and vertical components of a vector with magnitude 5 pounds pointed in a direction of 55° above the horizontal. Round to the nearest hundredth.

Answer

$x = 2.87$ pounds, $y = 4.10$ pounds

60) Find the magnitude of the horizontal and vertical components of the vector with magnitude 1 pound pointed in a direction of 8° above the horizontal. Round to the nearest hundredth.

Real-World Applications

61) A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

Answer

4.635miles, 17.764° N of E

62) A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

63) A man starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4 miles southwest, and 2 miles east. How far has he walked? If he walked straight home, how far would he

Answer

17 miles, 10.318miles

64) A woman starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles south, 5 miles southwest, and 3 miles east. How far has she walked? If she walked straight home, how far wou

65) A man starts walking from home and walks 3 miles at 20° north of west, then 5 miles at 10° west of south, then 4 miles at 15° north of east. If he walked straight home, how far would he have to the

Answer

Distance: 2.868 Direction: 86.474° North of West, or 3.526° West of North

66) A woman starts walking from home and walks 6 miles at 40° north of east, then 2 miles at 15° east of south, then 5 miles at 30° south of west. If she walked straight home, how far would she have t

67) An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and what is the plan

Answer

4.924°, 659 km/hr

68) An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the plan

69) An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, how many degrees west c

Answer

4.424°

70) An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, how many degrees west o

71) As part of a video game, the point $(5, 7)$ is rotated counterclockwise about the origin through an angle of \(35^{\circ}\). Find the new coordinates of this point.

Answer

$(0.081, 8.602)$

72) As part of a video game, the point $(7, 3)$ is rotated counterclockwise about the origin through an angle of 40°. Find the new coordinates of this point.

73) Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 10 mph relative to the car, and the car is traveling 25 mph down the road. If one child d

Answer

21.801°, relative to the car's forward direction

74) Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 8 mph relative to the car, and the car is traveling 45 mph down the road. If one child do

75) A 50-pound object rests on a ramp that is inclined 19°. Find the magnitude of the components of the force parallel to and perpendicular to (normal) the ramp to the nearest tenth of a pound.

Answer

parallel: 16.28 perpendicular: 47.28pounds

76) Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it upward, and 5 pounds acting on it 45° from the horizontal. What single force is the resultant force acting on t

77) Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it -135° from the horizontal, and 5 pounds acting on it directed 150° from the horizontal. What single force is

Answer

19.35pounds, 231.54° from the horizontal

78) The condition of equilibrium is when the sum of the forces acting on a body is the zero vector. Suppose a body has a force of 2 pounds acting on it to the right, 5 pounds acting on it upward, and 3 p

79) Suppose a body has a force of 3 pounds acting on it to the left, 4 pounds acting on it upward, and 2 pounds acting on it \(30^{\circ}\) from the horizontal. What single force is needed to produce a state of equilibrium on the body? Draw the vector.

Answer

5.1583pounds, 75.8° from the horizontal

More Practice w/Vectors

For the exercises 1-2, determine whether the two vectors, u and v, are equal, where u has an initial point P_1 and a terminal point P_2, and v has an initial point P_3 and a terminal point P_4.

1) $P_1 = (-1, 4), P_2 = (3, 1), P_3 = (5, 5), P_4 = (9, 2)$

2) $P_1 = (6, 11), P_2 = (-2, 8), P_3 = (0, -1), P_4 = (-8, 2)$

Answer

not equal

For the exercises 3-4, use the vectors $u = 2i - j, v = 4i - 3j$, and $w = -2i + 5j$ to evaluate the expression.

3) $u - v$

4) $2v - u + w$

Answer

$4i$

For the exercises 5-6, find a unit vector in the same direction as the given vector.

5) $a = 8i - 6j$

6) $b = -3i - j$

Answer

$$-\frac{3\sqrt{10}}{10}i - \frac{\sqrt{10}}{10}j$$

For the exercises 7-11, calculate $u \cdot v$.

7) $u = -2i + j$ and $v = 3i + 7j$

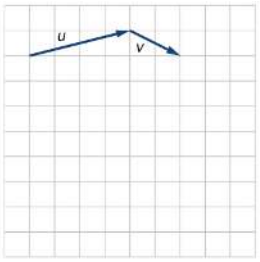
8) $u = i + 4j$ and $v = 4i + 3j$

Answer

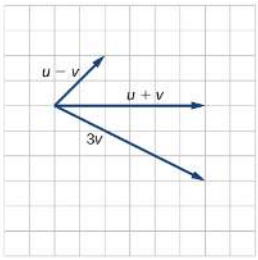
16

9) Given $v = \langle -3, 4 \rangle$ draw v, $2v$, and $\frac{1}{2}v$.

10) Given the vectors shown in the Figure below, sketch $u + v$, $u - v$ and $3v$.



Answer



11) Given initial point $P_1 = (3, 2)$ and terminal point $P_2 = (-5, -1)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

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CHAPTER OVERVIEW

7: Analytic Geometry

In this chapter, we will investigate the two-dimensional figures that are formed when a right circular cone is intersected by a plane. We will begin by studying each of three figures created in this manner. We will develop defining equations for each figure and then learn how to use these equations to solve a variety of problems. Conic sections are formed when a plane intersects two right circular cones aligned tip to tip and extending infinitely far in opposite directions, which we also call a cone. The way in which we slice the cone will determine the type of conic section formed at the intersection. A circle is formed by slicing a cone with a plane perpendicular to the axis of symmetry of the cone. An ellipse is formed by slicing a single cone with a slanted plane not perpendicular to the axis of symmetry.

7.0: Prelude to Analytic Geometry
7.1: The Ellipse
7.2: The Hyperbola
7.3: The Parabola
7.4: Rotation of Axes
7.5: Conic Sections in Polar Coordinates
7.E: Analytic Geometry (Exercises)
7.R: Analytic Geometry (Review)

Thumbnail: Conic sections can also be described by a set of points in the coordinate plane. This section focuses on the four variations of the standard form of the equation for the ellipse. An ellipse is the set of all points $(x,y)(x,y)$ in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus(plural: foci).

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7.0: Prelude to Analytic Geometry

The Greek mathematician Menaechmus (c. 380–c. 320 BCE) is generally credited with discovering the shapes formed by the intersection of a plane and a right circular cone. Depending on how he tilted the plane when it intersected the cone, he formed different shapes at the intersection–beautiful shapes with near-perfect symmetry. It was also said that Aristotle may have had an intuitive understanding of these shapes, as he observed the orbit of the planet to be circular. He presumed that the planets moved in circular orbits around Earth, and for nearly 2000 years this was the commonly held belief.



Figure* 7.0.1*: *(a) Greek philosopher Aristotle (384–322 BCE) (b) German mathematician and astronomer Johannes Kepler (1571–1630)*

It was not until the Renaissance movement that Johannes Kepler noticed that the orbits of the planet were not circular in nature. His published law of planetary motion in the 1600s changed our view of the solar system forever. He claimed that the sun was at one end of the orbits, and the planets revolved around the sun in an oval-shaped path. In this chapter, we will investigate the two-dimensional figures that are formed when a right circular cone is intersected by a plane. We will begin by studying each of three figures created in this manner. We will develop defining equations for each figure and then learn how to use these equations to solve a variety of problems.

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7.1: The Ellipse

Learning Objectives

- Write equations of ellipses in standard form.
- Graph ellipses centered at the origin.
- Graph ellipses not centered at the origin.
- Solve applied problems involving ellipses.

Can you imagine standing at one end of a large room and still being able to hear a whisper from a person standing at the other end? The National Statuary Hall in Washington, D.C., shown in Figure 7.1.1, is such a room. It is an oval-shaped room called a *whispering chamber* because the shape makes it possible for sound to travel along the walls. In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.



Figure 7.1.1: The National Statuary Hall in Washington, D.C. (credit: Greg Palmer, Flickr)

Writing Equations of Ellipses in Standard Form

A conic section, or conic, is a shape resulting from intersecting a right circular cone with a plane. The angle at which the plane intersects the cone determines the shape, as shown in Figure 7.1.2.

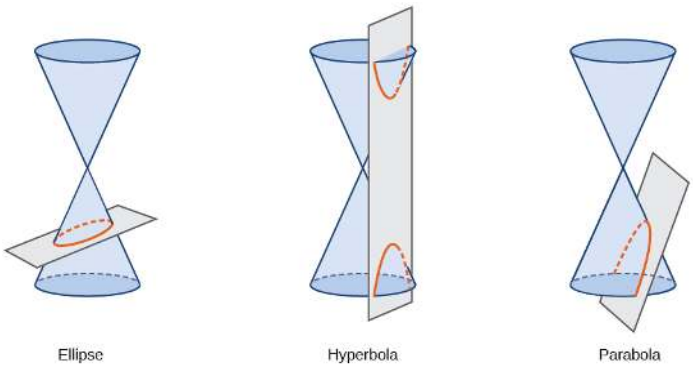


Figure 7.1.2

Conic sections can also be described by a set of points in the coordinate plane. Later in this chapter, we will see that the graph of any quadratic equation in two variables is a conic section. The signs of the equations and the coefficients of the variable terms determine the shape. This section focuses on the four variations of the standard form of the equation for the ellipse. An ellipse is the set of all points (x, y) in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus(plural: foci).

We can draw an ellipse using a piece of cardboard, two thumbtacks, a pencil, and string. Place the thumbtacks in the cardboard to form the foci of the ellipse. Cut a piece of string longer than the distance between the two thumbtacks (the length of the string represents the constant in the definition). Tack each end of the string to the cardboard, and trace a curve with a pencil held taut against the string. The result is an ellipse. See Figure 7.1.3.

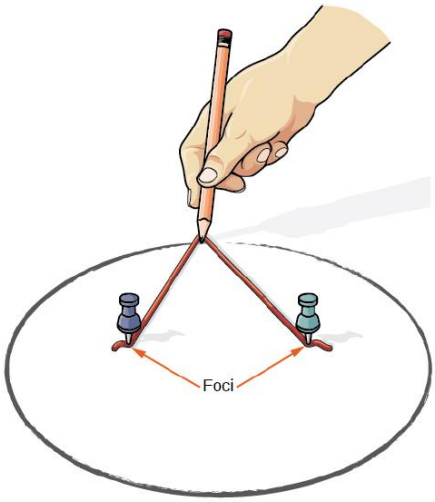


Figure 7.1.3

Every ellipse has two axes of symmetry. The longer axis is called the major axis, and the shorter axis is called the minor axis. Each endpoint of the major axis is the vertex of the ellipse (plural: vertices), and each endpoint of the minor axis is a co-vertex of the ellipse. The center of an ellipse is the midpoint of both the major and minor axes. The axes are perpendicular at the center. The foci always lie on the major axis, and the sum of the distances from the foci to any point on the ellipse (the constant sum) is greater than the distance between the foci (Figure 7.1.4).

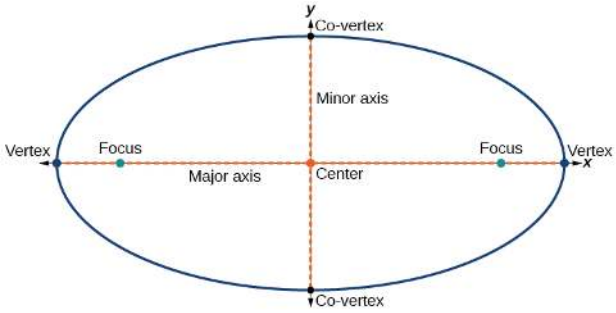


Figure 7.1.4

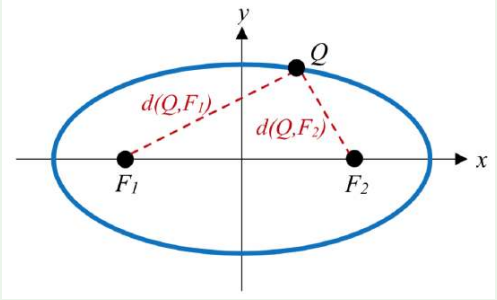
In this section, we restrict ellipses to those that are positioned vertically or horizontally in the coordinate plane. That is, the axes will either lie on or be parallel to the x- and y-axes. Later in the chapter, we will see ellipses that are rotated in the coordinate plane.

To work with horizontal and vertical ellipses in the coordinate plane, we consider two cases: those that are centered at the origin and those that are centered at a point other than the origin. First we will learn to derive the equations of ellipses, and then we will learn how to write the equations of ellipses in standard form. Later we will use what we learn to draw the graphs.

Definition: ELLIPSE Definition AND VOCABULARY

An ellipse is the set of all points $Q(x, y)$ for which the sum of the distance to two fixed points $F_1(x_1, y_1)$ and $F_2(x_2, y_2)$, called the *foci* (plural of focus), is a constant k:

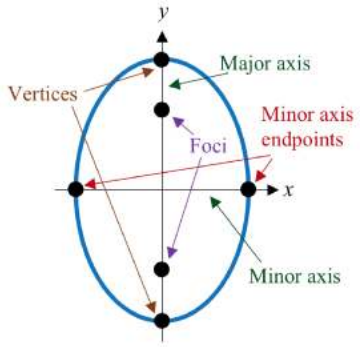
$$d(Q, F_1) + d(Q, F_2) = k \tag{7.1.1}$$



- The **major axis** is the line passing through the foci.
- **Vertices** are the points on the ellipse which intersect the major axis.

- The **major axis length** is the length of the line segment between the vertices.
- The **center** is the midpoint between the vertices (or the midpoint between the foci).
- The **minor axis** is the line perpendicular to the minor axis passing through the center.
- **Minor axis endpoints** are the points on the ellipse which intersect the minor axis.
- The minor axis endpoints are also sometimes called **co-vertices**.
- The **minor axis length** is the length of the line segment between minor axis endpoints.

Note that which axis is major and which is minor will depend on the orientation of the ellipse. In the ellipse shown at right, the foci lie on the y axis, so that is the major axis, and the x axis is the minor axis. Because of this, the vertices are the endpoints of the ellipse on the y axis, and the minor axis endpoints (co-vertices) are the endpoints on the x axis.



Deriving the Equation of an Ellipse Centered at the Origin

To derive the equation of an **ellipse** centered at the origin, we begin with the foci $(-c, 0)$ and $(c, 0)$. The ellipse is the set of all points (x, y) such that the sum of the distances from (x, y) to the foci is constant, as shown in Figure 7.1.5.

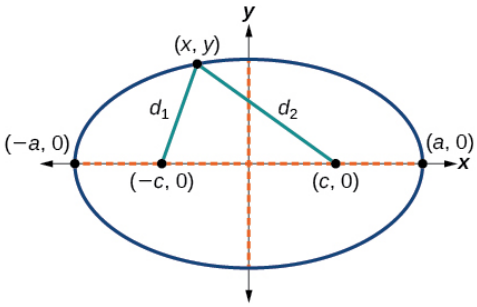


Figure 7.1.5

If $(a, 0)$ is a vertex of the ellipse, the distance from $(-c, 0)$ to $(a, 0)$ is $a-(-c)=a+c$. The distance from $(c, 0)$ to $(a, 0)$ is $a-c$. The sum of the distances from the foci to the vertex is

$$(a+c)+(a-c)=2a$$

If (x, y) is a point on the ellipse, then we can define the following variables:

- $d_1 =$ the distance from $(-c, 0)$ to (x, y)
- $d_2 =$ the distance from $(c, 0)$ to (x, y)

By the definition of an ellipse, d_1+d_2 is constant for any point (x, y) on the ellipse. We know that the sum of these distances is $2a$ for the vertex $(a, 0)$. It follows that $d_1+d_2=2a$ for any point on the ellipse. We will begin the derivation by applying the distance formula. The rest of the derivation is algebraic.

$$
\begin{aligned}
d_1+d_2 &= 2a \\
\sqrt{(x-(-c))^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2} &= 2a \quad \text{Distance formula} \\
\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} &= 2a \quad \text{Simplify expressions.} \\
\sqrt{(x+c)^2+y^2} &= 2a-\sqrt{(x-c)^2+y^2} \quad \text{Move radical to opposite side.} \\
(x+c)^2+y^2 &= \left[2a-\sqrt{(x-c)^2+y^2}\right]^2 \quad \text{Square both sides.} \\
x^2+2cx+c^2+y^2 &= 4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 \quad \text{Expand the squares.} \\
x^2+2cx+c^2+y^2 &= 4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2 \quad \text{Expand remaining squares.} \\
2cx &= 4a^2-4a\sqrt{(x-c)^2+y^2}-2cx \quad \text{Combine like terms.} \\
4cx-4a^2 &= -4a\sqrt{(x-c)^2+y^2} \quad \text{Isolate the radical.} \\
cx-a^2 &= -a\sqrt{(x-c)^2+y^2} \quad \text{Divide by 4.} \\
\left[cx-a^2\right]^2 &= a^2\left[\sqrt{(x-c)^2+y^2}\right]^2 \quad \text{Square both sides.} \\
c^2x^2-2a^2cx+a^4 &= a^2(x^2-2cx+c^2+y^2) \quad \text{Expand the squares.} \\
c^2x^2-2a^2cx+a^4 &= a^2x^2-2a^2cx+a^2c^2+a^2y^2 \quad \text{Distribute } a^2 \\
a^2x^2-c^2x^2+a^2y^2 &= a^4-a^2c^2 \quad \text{Rewrite.} \\
x^2(a^2-c^2)+a^2y^2 &= a^2(a^2-c^2) \quad \text{Factor common terms.} \\
x^2b^2+a^2y^2 &= a^2b^2 \quad \text{Set } b^2=a^2-c^2 \\
\frac{x^2b^2}{a^2b^2}+\frac{a^2y^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} \quad \text{Divide both sides by } a^2b^2 \\
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1 \quad \text{Simplify}
\end{aligned}
$$

Thus, the standard equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ .This equation defines an ellipse centered at the origin. If $a>b$,the ellipse is stretched further in the horizontal direction, and if $b>a$, the ellipse is stretched further in the vertical direction.

Writing Equations of Ellipses Centered at the Origin in Standard Form

Standard forms of equations tell us about key features of graphs. Take a moment to recall some of the standard forms of equations we've worked with in the past: linear, quadratic, cubic, exponential, logarithmic, and so on. By learning to interpret standard forms of equations, we are bridging the relationship between algebraic and geometric representations of mathematical phenomena.

The key features of the **ellipse** are its center, **vertices**, **co-vertices**, **foci**, and lengths and positions of the **major and minor axes**. Just as with other equations, we can identify all of these features just by looking at the standard form of the equation. There are four variations of the standard form of the ellipse. These variations are categorized first by the location of the center (the origin or not the origin), and then by the position (horizontal or vertical). Each is presented along with a description of how the parts of the equation relate to the graph. Interpreting these parts allows us to form a mental picture of the ellipse.

STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER $(0,0)$

The standard form of the equation of an ellipse with center $(0,0)$ and major axis on the *x-axis* is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{7.1.2}$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(\pm a,0)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(0,\pm b)$
- the coordinates of the foci are $(\pm c,0)$, where $c^2=a^2-b^2$. See Figure 7.1.6*a*

The standard form of the equation of an ellipse with center $(0,0)$ and major axis on the *y-axis* is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \tag{7.1.3}$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(0,\pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(\pm b,0)$
- the coordinates of the foci are $(0,\pm c)$, where $c^2=a^2-b^2$. See Figure 7.1.6*b*

Note that the vertices, co-vertices, and foci are related by the equation $c^2=a^2-b^2$. When we are given the coordinates of the foci and vertices of an ellipse, we can use this relationship to find the equation of the ellipse in standard form.

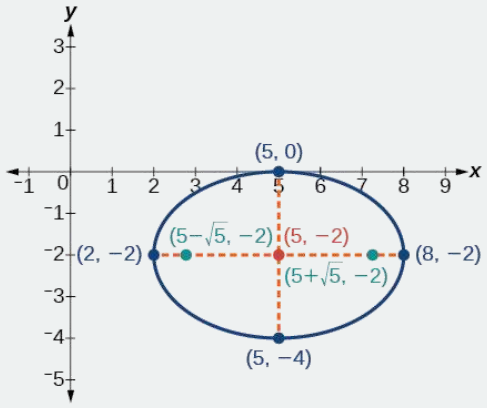


Figure 7.1.6: (a) Horizontal ellipse with center $(0,0)$ (b) Vertical ellipse with center $(0,0)$

How to: Given the vertices and foci of an ellipse centered at the origin, write its equation in standard form

1. Determine whether the major axis lies on the *x*- or *y*-axis.
   - If the given coordinates of the vertices and foci have the form $(\pm a,0)$ and $(\pm c,0)$ respectively, then the major axis is the *x*-axis. Use the standard form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
   - If the given coordinates of the vertices and foci have the form $(0,\pm a)$ and $(\pm c,0)$,respectively, then the major axis is the *y*-axis. Use the standard form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
2. Use the equation $c^2=a^2-b^2$, along with the given coordinates of the vertices and foci, to solve for b^2.
3. Substitute the values for a^2 and b^2 into the standard form of the equation determined in Step 1.

Example 7.1.1: Writing the Equation of an Ellipse Centered at the Origin in Standard Form

What is the standard form equation of the ellipse that has vertices $(\pm 8,0)$ and foci $(\pm 5,0)$?

Solution

The foci are on the x-axis, so the major axis is the x-axis. Thus, the equation will have the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The vertices are $(\pm 8,0)$,so $a=8$ and $a^2=64$.

The foci are $(\pm 5,0)$,so $c=5$ and $c^2=25$.

We know that the vertices and foci are related by the equation $c^2=a^2-b^2$. Solving for b^2, we have:

$$\begin{aligned} c^2 &= a^2-b^2 \\ 25 &= 64-b^2 \qquad \text{Substitute for } c^2 \text{ and } a^2 \\ b^2 &= 39 \qquad \text{Solve for } b^2 \end{aligned}$$

Now we need only substitute $a^2=64$ and $b^2=39$ into the standard form of the equation. The equation of the ellipse is $\frac{x^2}{64}+\frac{y^2}{39}=1$.

Exercise 7.1.1

What is the standard form equation of the ellipse that has vertices $(0, \pm 4)$ and foci $(0, \pm\sqrt{15})$?

Answer

$$x^2 + \frac{y^2}{16} = 1$$

Q&A

Can we write the equation of an ellipse centered at the origin given coordinates of just one focus and vertex?

Yes. Ellipses are symmetrical, so the coordinates of the vertices of an ellipse centered around the origin will always have the form $(\pm a, 0)$ or $(0, \pm a)$. Similarly, the coordinates of the foci will always have the form $(\pm c, 0)$ or $(0, \pm c)$. Knowing this, we can use a and c from the given points, along with the equation $c^2 = a^2 - b^2$, to find b^2.

Writing Equations of Ellipses Not Centered at the Origin

Like the graphs of other equations, the graph of an **ellipse** can be translated. If an ellipse is translated h units horizontally and k units vertically, the center of the ellipse will be (h, k). This **translation** results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and y replaced by $(y - k)$.

STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER (H, K)

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the x-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{7.1.4}$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$. See Figure 7.1.7a.

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the y-axis is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \tag{7.1.5}$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = a^2 - b^2$. See Figure 7.1.7b.

Just as with ellipses centered at the origin, ellipses that are centered at a point (h, k) have vertices, co-vertices, and foci that are related by the equation $c^2 = a^2 - b^2$. We can use this relationship along with the midpoint and distance formulas to find the equation of the ellipse in standard form when the vertices and foci are given.

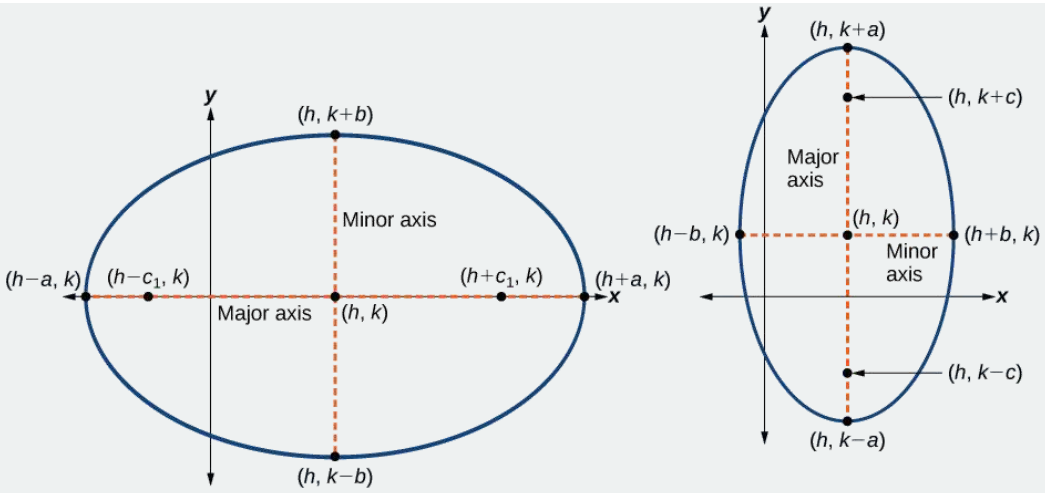


Figure 7.1.7: (a) Horizontal ellipse with center (h, k) (b) Vertical ellipse with center (h, k)

How to: Given the vertices and foci of an ellipse not centered at the origin, write its equation in standard form

1. Determine whether the major axis is parallel to the x- or y-axis.
   - If the y-coordinates of the given vertices and foci are the same, then the major axis is parallel to the x-axis. Use the standard form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$
   - If the x-coordinates of the given vertices and foci are the same, then the major axis is parallel to the y-axis. Use the standard form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$
2. Identify the center of the ellipse (h, k) using the midpoint formula and the given coordinates for the vertices.
3. Find a^2 by solving for the length of the major axis, $2a$, which is the distance between the given vertices.
4. Find c^2 using h and k, found in Step 2, along with the given coordinates for the foci.
5. Solve for b^2 using the equation $c^2=a^2-b^2$.
6. Substitute the values for h, k, a^2, and b^2 into the standard form of the equation determined in Step 1.

Example 7.1.2: Writing the Equation of an Ellipse Centered at a Point Other Than the Origin

What is the standard form equation of the ellipse that has vertices $(-2,-8)$ and $(-2,2)$ and foci $(-2,-7)$ and $(-2,1)$?

Solution

The x-coordinates of the vertices and foci are the same, so the major axis is parallel to the y-axis. Thus, the equation of the ellipse will have the form

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

First, we identify the center, (h, k). The center is halfway between the vertices, $(-2,-8)$ and $(-2,2)$. Applying the midpoint formula, we have:

$$\begin{aligned}(h,k)&=\left(\frac{-2+(-2)}{2},\frac{-8+2}{2}\right)\\&=(-2,-3)\end{aligned}$$

Next, we find a^2. The length of the major axis, $2a$, is bounded by the vertices. We solve for a by finding the distance between the y-coordinates of the vertices.

$$\begin{aligned}2a&=2-(-8)\\2a&=10\\a&=5\end{aligned}$$

So $a^2=25$.

Now we find c^2. The foci are given by $(h, k\pm c)$. So, $(h, k-c)=(-2,-7)$ and $(h, k+c)=(-2,1)$. We substitute $k=-3$ using either of these points to solve for c.

$$\begin{aligned} k+c &= 1 \\ -3+c &= 1 \\ c &= 4 \end{aligned}$$

So $c^2 = 16$.

Next, we solve for b^2 using the equation $c^2 = a^2 - b^2$.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 16 &= 25 - b^2 \\ b^2 &= 9 \end{aligned}$$

Finally, we substitute the values found for h, k, a^2, and b^2 into the standard form equation for an ellipse:

$$\frac{(x+2)^2}{9} + \frac{(y+3)^2}{25} = 1$$

Exercise 7.1.2

What is the standard form equation of the ellipse that has vertices $(-3, 3)$ and $(5, 3)$ and foci $(1-2\sqrt{3}, 3)$ and $(1+2\sqrt{3}, 3)$?

Answer

$$\frac{(x-1)^2}{16} + \frac{(y-3)^2}{4} = 1$$

Graphing Ellipses Centered at the Origin

Just as we can write the equation for an ellipse given its graph, we can graph an ellipse given its equation. To graph ellipses centered at the origin, we use the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b \text{ for horizontal ellipses}$$

and

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b \text{ for vertical ellipses}$$

How to: Given the standard form of an equation for an ellipse centered at $(0, 0)$, sketch the graph.

1. Use the standard forms of the equations of an ellipse to determine the major axis, vertices, co-vertices, and foci.
   - If the equation is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, then
     - the major axis is the x-axis
     - the coordinates of the vertices are $(\pm a, 0)$
     - the coordinates of the co-vertices are $(0, \pm b)$
     - the coordinates of the foci are $(\pm c, 0)$
   - If the equation is in the form $x^2b^2 + y^2a^2 = 1$,where $a > b$, then
     - the major axis is the y-axis
     - the coordinates of the vertices are $(0, \pm a)$
     - the coordinates of the co-vertices are $(\pm b, 0)$
     - the coordinates of the foci are $(0, \pm c)$
2. Solve for c using the equation $c^2 = a^2 - b^2$.
3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

Example 7.1.3: Graphing an Ellipse Centered at the Origin

Graph the ellipse given by the equation, $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Identify and label the center, vertices, co-vertices, and foci.

Solution

First, we determine the position of the major axis. Because $25 > 9$,the major axis is on the y-axis. Therefore, the equation is in the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$,where $b^2 = 9$ and $a^2 = 25$. It follows that:

- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(0, \pm a) = (0, \pm\sqrt{25}) = (0, \pm 5)$
- the coordinates of the co-vertices are $(\pm b, 0) = (\pm 9, 0) = (\pm 3, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = a^2 - b^2$ Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{25 - 9} \\ &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned}$$

Therefore, the coordinates of the foci are $(0, \pm 4)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse. See Figure 7.1.8.

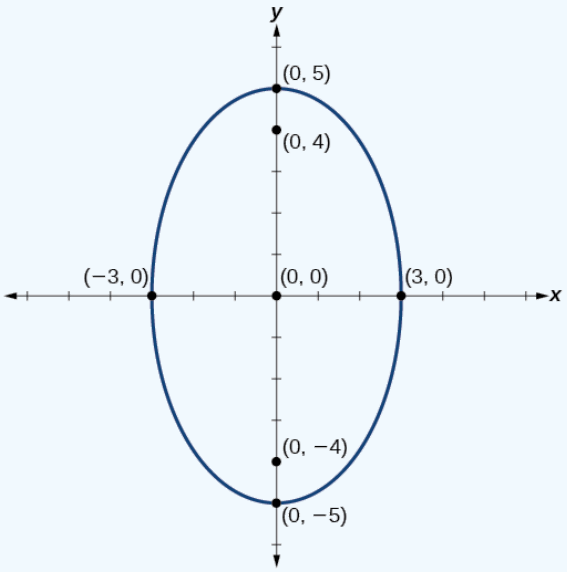


Figure 7.1.8

Exercise 7.1.3

Graph the ellipse given by the equation $\frac{x^2}{36} + \frac{y^2}{4} = 1$. Identify and label the center, vertices, co-vertices, and foci.

Answer

center: $(0, 0)$; vertices: $(\pm 6, 0)$; co-vertices: $(0, \pm 2)$; foci: $(\pm 4\sqrt{2}, 0)$

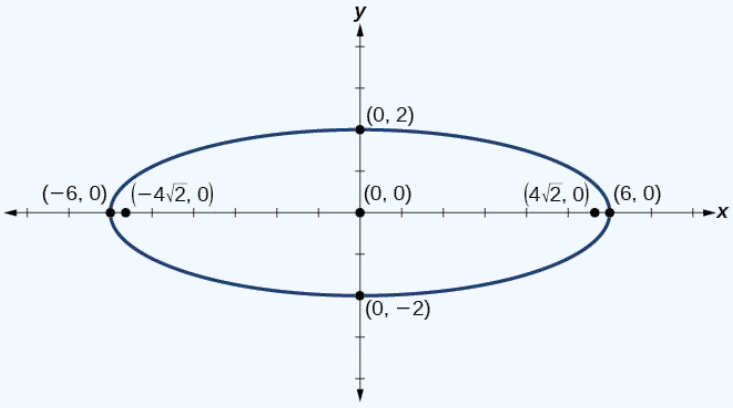


Figure 7.1.9

Example 7.1.4: Graphing an Ellipse Centered at the Origin from an Equation Not in Standard Form

Graph the ellipse given by the equation $4x^2+25y^2=100$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Solution

First, use algebra to rewrite the equation in standard form.

$$\begin{aligned} 4x^2+25y^2 &= 100 \\ \frac{4x^2}{100}+\frac{25y^2}{100} &= \frac{100}{100} \\ \frac{x^2}{25}+\frac{y^2}{4} &= 1 \end{aligned}$$

Next, we determine the position of the major axis. Because $25>4$, the major axis is on the x-axis. Therefore, the equation is in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a^2=25$ and $b^2=4$. It follows that:

- the center of the ellipse is $(0,0)$
- the coordinates of the vertices are $(\pm a,0)=(\pm\sqrt{25},0)=(\pm 5,0)$
- the coordinates of the co-vertices are $(0,\pm b)=(0,\pm\sqrt{4})=(0,\pm 2)$
- the coordinates of the foci are $(\pm c,0)$, where $c^2=a^2-b^2$. Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2-b^2} \\ &= \pm\sqrt{25-4} \\ &= \pm\sqrt{21} \end{aligned}$$

Therefore the coordinates of the foci are $(\pm\sqrt{21},0)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

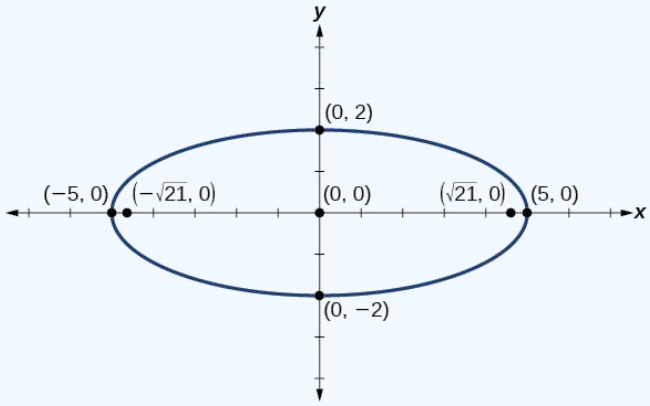


Figure 7.1.10

Exercise 7.1.4

Graph the ellipse given by the equation $49x^2+16y^2=784$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Answer

Standard form: $\frac{x^2}{16}+\frac{y^2}{49}=1$; center: $(0,0)$; vertices: $(0,\pm 7)$; co-vertices: $(\pm 4,0)$; foci: $(0,\pm\sqrt{33})$

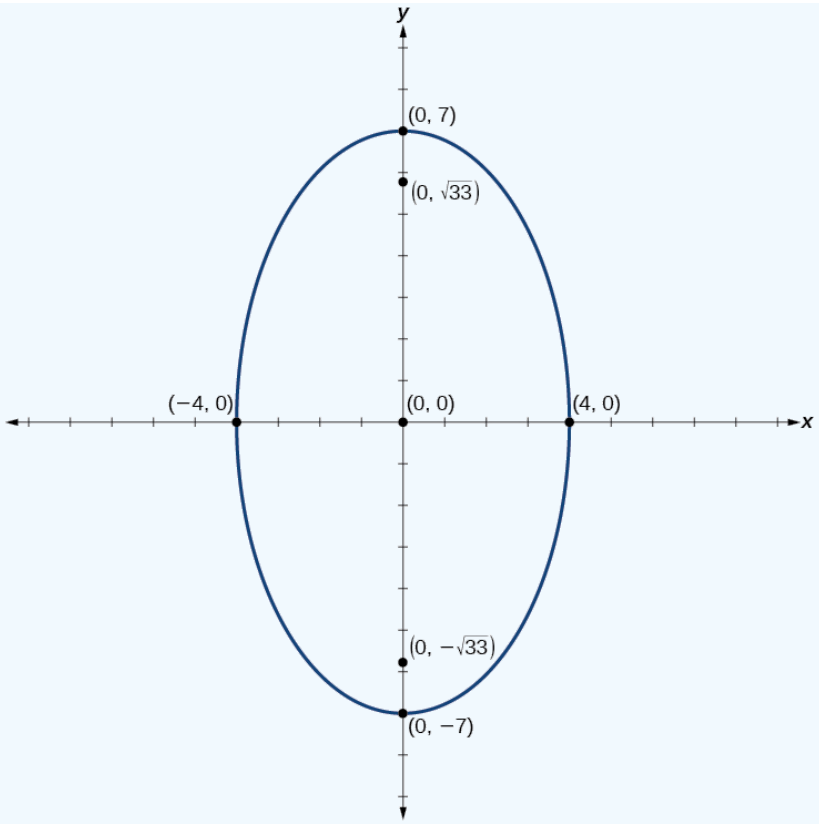


Figure 7.1.11

Graphing Ellipses Not Centered at the Origin

When an **ellipse** is not centered at the origin, we can still use the standard forms to find the key features of the graph. When the ellipse is centered at some point, (h, k),we use the standard forms $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, $a>b$ for horizontal ellipses and $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, $a>b$ for vertical ellipses. From these standard equations, we can easily determine the center, vertices, co-vertices, foci, and positions of the major and minor axes.

How to: Given the standard form of an equation for an ellipse centered at (h, k), sketch the graph.

1. Use the standard forms of the equations of an ellipse to determine the center, position of the major axis, vertices, co-vertices, and foci.
   - If the equation is in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a>b$, then
     - the center is (h, k)
     - the major axis is parallel to the x-axis
     - the coordinates of the vertices are $(h \pm a, k)$
     - the coordinates of the co-vertices are $(h, k \pm b)$
     - the coordinates of the foci are $(h \pm c, k)$
   - If the equation is in the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, where $a>b$, then
     - the center is (h, k)
     - the major axis is parallel to the y-axis
     - the coordinates of the vertices are $(h, k \pm a)$
     - the coordinates of the co-vertices are $(h \pm b, k)$
     - the coordinates of the foci are $(h, k \pm c)$
2. Solve for c using the equation $c^2=a^2-b^2$.
3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

Example 7.1.5: Graphing an Ellipse Centered at (h, k)

Graph the ellipse given by the equation, $\frac{(x+2)^2}{4}+\frac{(y-5)^2}{9}=1$. Identify and label the center, vertices, co-vertices, and foci.

Solution

First, we determine the position of the major axis. Because $9>4$, the major axis is parallel to the y-axis. Therefore, the equation is in the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, where $b^2=4$ and $a^2=9$. It follows that:

- the center of the ellipse is $(h, k)=(-2, 5)$
- the coordinates of the vertices are $(h, k \pm a)=(-2, 5 \pm \sqrt{9})=(-2, 5 \pm 3)$, or $(-2, 2)$ and $(-2, 8)$
- the coordinates of the co-vertices are $(h \pm b, k)=(-2 \pm \sqrt{4}, 5)=(-2 \pm 2, 5)$, or $(-4, 5)$ and $(0, 5)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2=a^2-b^2$. Solving for c,we have:

$$\begin{aligned} c &= \pm\sqrt{a^2-b^2} \\ &= \pm\sqrt{9-4} \\ &= \pm\sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $(-2, 5-\sqrt{5})$ and $(-2, 5+\sqrt{5})$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

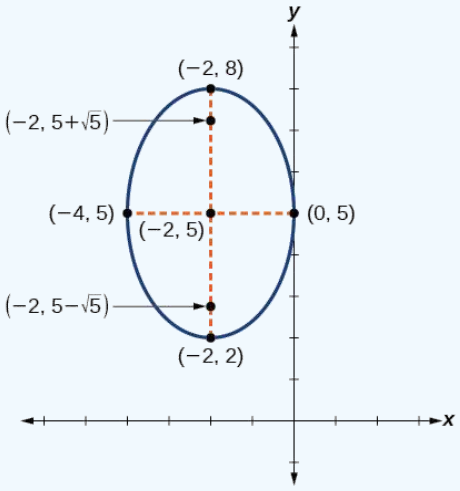


Figure 7.1.12

Exercise 7.1.5

Graph the ellipse given by the equation $\frac{(x-4)^2}{36}+\frac{(y-2)^2}{20}=1$. Identify and label the center, vertices, co-vertices, and foci.

Answer

Center: $(4, 2)$; vertices: $(-2, 2)$ and $(10, 2)$; co-vertices: $(4, 2-2\sqrt{5})$ and $(4, 2+2\sqrt{5})$; foci: $(0, 2)$ and $(8, 2)$

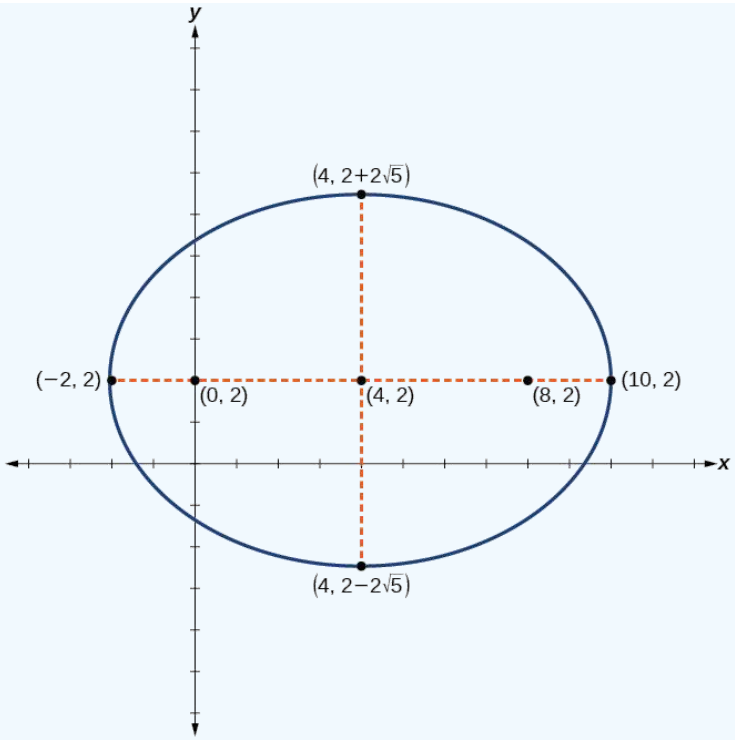


Figure 7.1.13

How to: Given the general form of an equation for an ellipse centered at (h,k), express the equation in standard form.

1. Recognize that an ellipse described by an equation in the form $ax^2+by^2+cx+dy+e=0$ is in general form.
2. Rearrange the equation by grouping terms that contain the same variable. Move the constant term to the opposite side of the equation.
3. Factor out the coefficients of the x^2 and y^2 terms in preparation for completing the square.
4. Complete the square for each variable to rewrite the equation in the form of the sum of multiples of two binomials squared set equal to a constant, $m_1(x-h)^2+m_2(y-k)^2=m_3$, where m_1, m_2,and m_3 are constants.
5. Divide both sides of the equation by the constant term to express the equation in standard form.

Example 7.1.6: Graphing an Ellipse Centered at (h,k) by First Writing It in Standard Form

Graph the ellipse given by the equation $4x^2+9y^2-40x+36y+100=0$. Identify and label the center, vertices, co-vertices, and foci.

Solution

We must begin by rewriting the equation in standard form.

$$4x^2+9y^2-40x+36y+100=0$$

Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$(4x^2-40x)+(9y^2+36y)=-100$$

Factor out the coefficients of the squared terms.

$$4(x^2-10x)+9(y^2+4y)=-100$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$4(x^2-10x+25)+9(y^2+4y+4)=-100+100+36$$

Rewrite as perfect squares.

$$4(x-5)^2+9(y+2)^2=36$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-5)^2}{9}+\frac{(y+2)^2}{4}=1$$

Now that the equation is in standard form, we can determine the position of the major axis. Because $9 > 4$, the major axis is parallel to the x-axis. Therefore, the equation is in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $a^2 = 9$ and $b^2 = 4$. It follows that:

- the center of the ellipse is $(h, k) = (5, -2)$
- the coordinates of the vertices are $(h \pm a, k) = (5 \pm \sqrt{9}, -2) = (5 \pm 3, -2)$, or $(2, -2)$ and $(8, -2)$
- the coordinates of the co-vertices are $(h, k \pm b) = (5, -2 \pm \sqrt{4}) = (5, -2 \pm 2)$, or $(5, -4)$ and $(5, 0)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$. Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{9 - 4} \\ &= \pm\sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $(5 - \sqrt{5}, -2)$ and $(5 + \sqrt{5}, -2)$.

Next we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse as shown in Figure 7.1.14

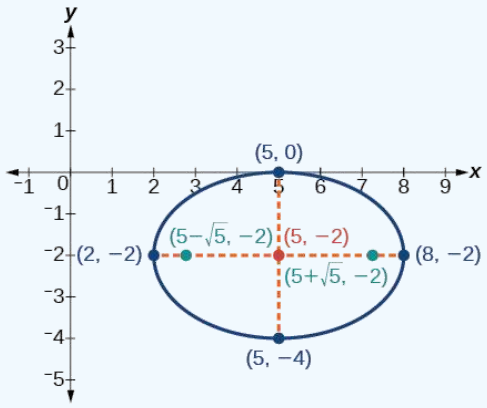


Figure 7.1.14

Exercise 7.1.6

Express the equation of the ellipse given in standard form. Identify the center, vertices, co-vertices, and foci of the ellipse.

$$4x^2 + y^2 - 24x + 2y + 21 = 0$$

Answer

$\frac{(x-3)^2}{4} + \frac{(y+1)^2}{16} = 1$; center: $(3, -1)$; vertices: $(3, -5)$ and $(3, 3)$; co-vertices: $(1, -1)$ and $(5, -1)$; foci: $(3, -1 - 2\sqrt{3})$ and $(3, -1 + 2\sqrt{3})$

Solving Applied Problems Involving Ellipses

Many real-world situations can be represented by ellipses, including orbits of planets, satellites, moons and comets, and shapes of boat keels, rudders, and some airplane wings. A medical device called a lithotripter uses elliptical reflectors to break up kidney stones by generating sound waves. Some buildings, called whispering chambers, are designed with elliptical domes so that a person whispering at one focus can easily be heard by someone standing at the other focus. This occurs because of the acoustic properties of an ellipse. When a sound wave originates at one focus of a whispering chamber, the sound wave will be reflected off the elliptical dome and back to the other focus (Figure 7.1.15). In the whisper chamber at the Museum of Science and Industry in Chicago, two people standing at the foci—about 43 feet apart—can hear each other whisper.

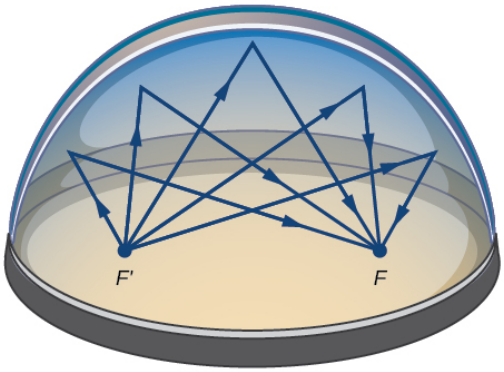


Figure 7.1.15: Sound waves are reflected between foci in an elliptical room, called a whispering chamber.

Example 7.1.7: Locating the Foci of a Whispering Chamber

The Statuary Hall in the Capitol Building in Washington, D.C. is a whispering chamber. Its dimensions are 46 feet wide by 96 feet long as shown in Figure 7.1.16.

a. What is the standard form of the equation of the ellipse representing the outline of the room? ***Hint:*** assume a horizontal ellipse, and let the center of the room be the point $(0, 0)$.
b. If two senators standing at the foci of this room can hear each other whisper, how far apart are the senators? Round to the nearest foot.

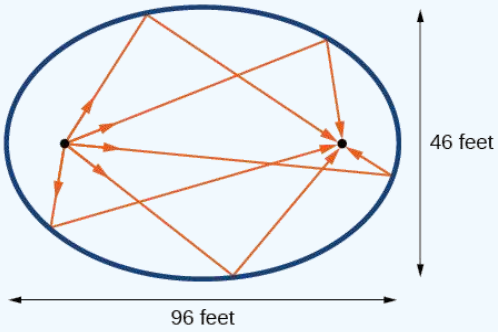


Figure 7.1.16

Solution

a. We are assuming a horizontal ellipse with center $(0, 0)$, so we need to find an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b$. We know that the length of the major axis, $2a$, is longer than the length of the minor axis, $2b$. So the length of the room, 96, is represented by the major axis, and the width of the room, 46, is represented by the minor axis.

   Therefore, the equation of the ellipse is \[(dfrac{x^2}{2304}+\dfrac{y^2}{529}=1\)

   - Solving for a, we have $2a = 96$, so $a = 48$, and $a^2 = 2304$.
   - Solving for b, we have $2b = 46$, so $b = 23$, and $b^2 = 529$.

b. To find the distance between the senators, we must find the distance between the foci, $(\pm c, 0)$, where $c^2 = a^2 - b^2$. Solving for c,we have:

$$\begin{aligned} c^2 &= a^2 - b^2 \\ c^2 &= 2304 - 529 \quad \text{Substitute using the values found in part } (a) \\ c &= \pm\sqrt{2304 - 529} \quad \text{Take the square root of both sides.} \\ c &= \pm\sqrt{1775} \quad \text{Subtract.} \\ c &\approx \pm 42 \quad \text{Round to the nearest foot.} \end{aligned}$$

The points $(\pm 42, 0)$ represent the foci. Thus, the distance between the senators is $2(42) = 84$ feet.

Exercise 7.1.7

Suppose a whispering chamber is 480 feet long and 320 feet wide.

a. What is the standard form of the equation of the ellipse representing the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0, 0)$.
b. If two people are standing at the foci of this room and can hear each other whisper, how far apart are the people? Round to the nearest foot.

Answer a

$$\frac{x^2}{57,600} + \frac{y^2}{25,600} = 1$$

Answer b

The people are standing 358 feet apart.

Media

Access these online resources for additional instruction and practice with ellipses.

- Conic Sections: The Ellipse
- Graph an Ellipse with Center at the Origin
- Graph an Ellipse with Center Not at the Origin

Key Equations

Horizontal ellipse, center at origin	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,\, a > b$
Vertical ellipse, center at origin	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1\,,\, a > b$
Horizontal ellipse, center (h, k)	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1\,,\, a > b$
Vertical ellipse, center (h, k)	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1\,,\, a > b$

Key Concepts

- An ellipse is the set of all points (x, y) in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus (plural: foci).
- When given the coordinates of the foci and vertices of an ellipse, we can write the equation of the ellipse in standard form. See Example $7.1.1$ and Example $7.1.2$.
- When given an equation for an ellipse centered at the origin in standard form, we can identify its vertices, co-vertices, foci, and the lengths and positions of the major and minor axes in order to graph the ellipse. See Example $7.1.3$ and Example $7.1.4$.
- When given the equation for an ellipse centered at some point other than the origin, we can identify its key features and graph the ellipse. See Example $7.1.5$ and Example $7.1.6$.
- Real-world situations can be modeled using the standard equations of ellipses and then evaluated to find key features, such as lengths of axes and distance between foci. See Example $7.1.7$.

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7.2: The Hyperbola

Learning Objectives

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Solve applied problems involving hyperbolas.

What do paths of comets, supersonic booms, ancient Grecian pillars, and natural draft cooling towers have in common? They can all be modeled by the same type of **conic**. For instance, when something moves faster than the speed of sound, a shock wave in the form of a cone is created. A portion of a conic is formed when the wave intersects the ground, resulting in a sonic boom (Figure 7.2.1).

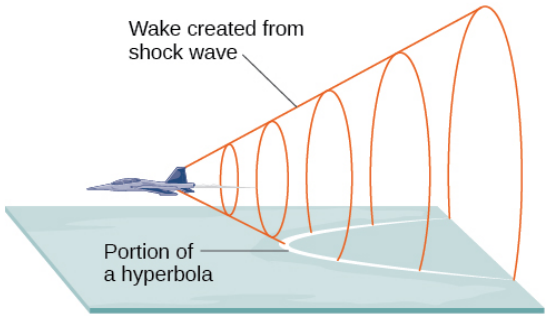


Figure 7.2.1: A shock wave intersecting the ground forms a portion of a conic and results in a sonic boom.

Most people are familiar with the sonic boom created by supersonic aircraft, but humans were breaking the sound barrier long before the first supersonic flight. The crack of a whip occurs because the tip is exceeding the speed of sound. The bullets shot from many firearms also break the sound barrier, although the bang of the gun usually supersedes the sound of the sonic boom.

Locating the Vertices and Foci of a Hyperbola

In analytic geometry, a **hyperbola** is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other (Figure 7.2.2).

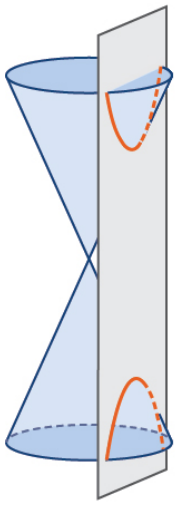


Figure 7.2.2: A hyperbola

Like the ellipse, the hyperbola can also be defined as a set of points in the coordinate plane. A hyperbola is the set of all points (x, y) in a plane such that the difference of the distances between (x, y) and the foci is a positive constant.

Notice that the definition of a hyperbola is very similar to that of an ellipse. The distinction is that the hyperbola is defined in terms of the *difference* of two distances, whereas the ellipse is defined in terms of the *sum* of two distances.

As with the ellipse, every hyperbola has two **axes of symmetry**. The **transverse axis** is a line segment that passes through the center of the hyperbola and has vertices as its endpoints. The foci lie on the line that contains the transverse axis. The **conjugate axis** is perpendicular to

the transverse axis and has the co-vertices as its endpoints. The **center of a hyperbola** is the midpoint of both the transverse and conjugate axes, where they intersect. Every hyperbola also has two **asymptotes** that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes. The **central rectangle** of the hyperbola is centered at the origin with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle (Figure 7.2.3).

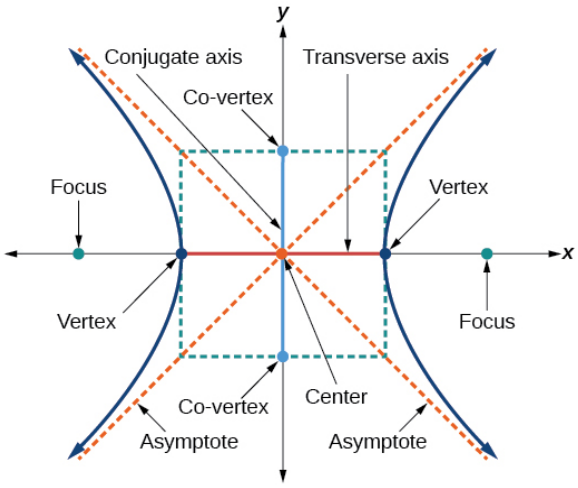


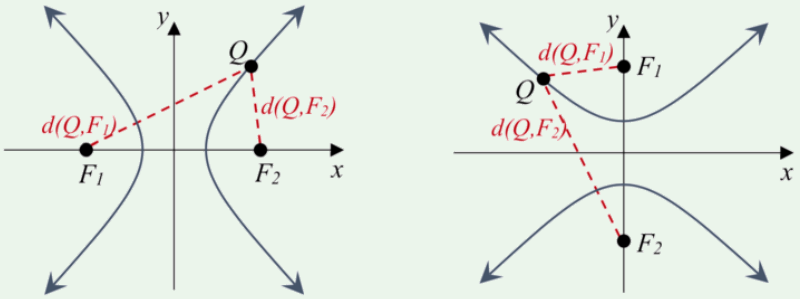
Figure 7.2.3: Key features of the hyperbola

In this section, we will limit our discussion to hyperbolas that are positioned vertically or horizontally in the coordinate plane; the axes will either lie on or be parallel to the x- and y-axes. We will consider two cases: those that are centered at the origin, and those that are centered at a point other than the origin.

Definition: Hyperbola

A hyperbola is the set of all points $Q\ (x, y)$ for which the absolute value of the difference of the distances to two fixed points $F_1\ (x_1, y_1)$ and $F_2\ (x_2, y_2)$ called the foci (plural for focus) is a constant k:

$$|d(Q, F_1) - d(Q, F_2)| = k \tag{7.2.1}$$



- The **transverse axis** is the line passing through the foci.
- **Vertices** are the points on the hyperbola which intersect the transverse axis.
- The **transverse axis length** is the length of the line segment between the vertices.
- The **center** is the midpoint between the vertices (or the midpoint between the foci).
- The other axis of symmetry through the center is the **conjugate axis**.
- The two disjoint pieces of the curve are called **branches**.
- A hyperbola has two **asymptotes**.

Which axis is the transverse axis will depend on the orientation of the hyperbola. As a helpful tool for graphing hyperbolas, it is common to draw a **central rectangle** as a guide. This is a rectangle drawn around the center with sides parallel to the coordinate axes that pass through each vertex and co-vertex. The asymptotes will follow the diagonals of this rectangle.

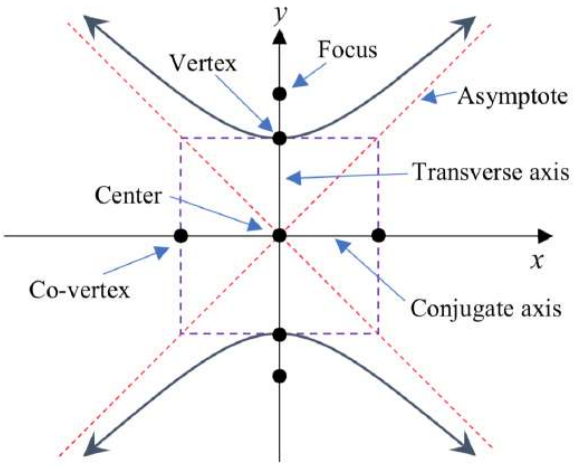


Figure 7.2.4:

Deriving the Equation of an Ellipse Centered at the Origin

Let $(-c,0)$ and $(c,0)$ be the foci of a hyperbola centered at the origin. The hyperbola is the set of all points (x,y) such that the difference of the distances from (x,y) to the foci is constant. See Figure 7.2.5.

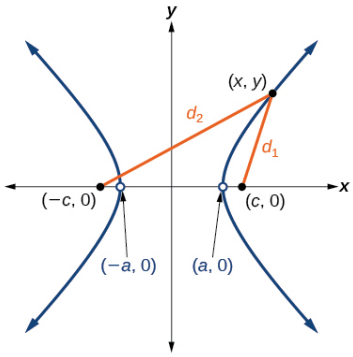


Figure 7.2.5

If $(a,0)$ is a vertex of the hyperbola, the distance from $(-c,0)$ to $(a,0)$ is $a-(-c)=a+c$. The distance from $(c,0)$ to $(a,0)$ is $c-a$. The sum of the distances from the foci to the vertex is

$$(a+c)-(c-a)=2a$$

If (x,y) is a point on the hyperbola, we can define the following variables:

$$d_2 = \text{the distance from } (-c,0) \text{ to } (x,y)$$

$$d_1 = \text{the distance from } (c,0) \text{ to } (x,y)$$

By definition of a hyperbola, d_2-d_1 is constant for any point (x,y) on the hyperbola. We know that the difference of these distances is $2a$ for the vertex $(a,0)$. It follows that $d_2-d_1=2a$ for any point on the hyperbola. As with the derivation of the equation of an ellipse, we will begin by applying the **distance formula**. The rest of the derivation is algebraic. Compare this derivation with the one from the previous section for ellipses.

$$
\begin{aligned}
d_2 - d_1 &= 2a \\
\sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} &= 2a \qquad \text{Distance Formula} \\
\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} &= 2a \qquad \text{Simplify expressions.} \\
\sqrt{(x+c)^2+y^2} &= 2a + \sqrt{(x-c)^2+y^2} \qquad \text{Move radical to opposite side.} \\
(x+c)^2+y^2 &= \left(2a + \sqrt{(x-c)^2+y^2}\right)^2 \qquad \text{Square both sides.} \\
x^2+2cx+c^2+y^2 &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2 \qquad \text{Expand the squares.} \\
x^2+2cx+c^2+y^2 &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2 \qquad \text{Expand remaining square.} \\
2cx &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} - 2cx \qquad \text{Combine like terms.} \\
4cx - 4a^2 &= 4a\sqrt{(x-c)^2+y^2} \qquad \text{Isolate the radical.} \\
cx - a^2 &= a\sqrt{(x-c)^2+y^2} \qquad \text{Divide by 4.} \\
(cx-a^2)^2 &= a^2\left[\sqrt{(x-c)^2+y^2}\right]^2 \qquad \text{Square both sides.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2(x^2 - 2cx + c^2 + y^2) \qquad \text{Expand the squares.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 \qquad \text{Distribute } a^2 \\
a^4 + c^2x^2 &= a^2x^2 + a^2c^2 + a^2y^2 \qquad \text{Combine like terms.} \\
c^2x^2 - a^2x^2 - a^2y^2 &= a^2c^2 - a^4 \qquad \text{Rearrange terms.} \\
x^2(c^2-a^2) - a^2y^2 &= a^2(c^2-a^2) \qquad \text{Factor common terms.} \\
x^2b^2 - a^2y^2 &= a^2b^2 \qquad \text{Set } b^2 = c^2 - a^2 \\
\cdot \frac{x^2b^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} \qquad \text{Divide both sides by } a^2b^2 \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1
\end{aligned}
$$

This equation defines a hyperbola centered at the origin with vertices $(\pm a, 0)$ and co-vertices $(0, \pm b)$.

STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER $(0, 0)$

The standard form of the equation of a hyperbola with center $(0, 0)$ and transverse axis on the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(\pm c, 0)$
- the equations of the asymptotes are $y = \pm \frac{b}{a}x$

See Figure 7.2.6a.

The standard form of the equation of a hyperbola with center $(0, 0)$ and transverse axis on the y-axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(\pm b, 0)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$

- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y = \pm \frac{a}{b}x$

See Figure 7.2.6b

Note that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. When we are given the equation of a hyperbola, we can use this relationship to identify its vertices and foci.

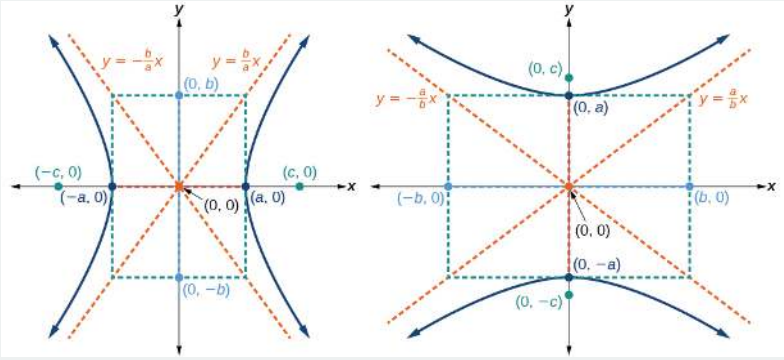


Figure 7.2.6: (a) Horizontal hyperbola with center $(0, 0)$ (b) Vertical hyperbola with center $(0, 0)$

How to: Given the equation of a hyperbola in standard form, locate its vertices and foci

1. Determine whether the transverse axis lies on the x- or y-axis. Notice that a^2 is always under the variable with the positive coefficient. So, if you set the other variable equal to zero, you can easily find the intercepts. In the case where the hyperbola is centered at the origin, the intercepts coincide with the vertices.
   - If the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the transverse axis lies on the x-axis. The vertices are located at $(\pm a, 0)$, and the foci are located at $(\pm c, 0)$.
   - If the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then the transverse axis lies on the y-axis. The vertices are located at $(0, \pm a)$, and the foci are located at $(0, \pm c)$.
2. Solve for a using the equation $a = \sqrt{a^2}$.
3. Solve for c using the equation $c = \sqrt{a^2 + b^2}$.

Example 7.2.1: Locating a Hyperbola's Vertices and Foci

Identify the vertices and foci of the hyperbola with equation $\frac{y^2}{49} - \frac{x^2}{32} = 1$.

Solution

The equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, so the transverse axis lies on the y-axis. The hyperbola is centered at the origin, so the vertices serve as the y-intercepts of the graph. To find the vertices, set $x = 0$, and solve for y.

$$\begin{aligned} 1 &= \frac{y^2}{49} - \frac{x^2}{32} \\ 1 &= \frac{y^2}{49} - \frac{0^2}{32} \\ 1 &= \frac{y^2}{49} \\ y^2 &= 49 \\ y &= \pm\sqrt{49} \\ &= \pm 7 \end{aligned}$$

The foci are located at $(0, \pm c)$. Solving for c,

$$\begin{aligned} c &= \sqrt{a^2+b^2} \\ &= \sqrt{49+32} \\ &= \sqrt{81} \\ &= 9 \end{aligned}$$

Therefore, the vertices are located at $(0, \pm 7)$, and the foci are located at $(0, 9)$.

Exercise 7.2.1

Identify the vertices and foci of the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{25} = 1$.

Answer

Vertices: $(\pm 3, 0)$; Foci: $(\pm\sqrt{34}, 0)$

Writing Equations of Hyperbolas in Standard Form

Just as with ellipses, writing the equation for a hyperbola in standard form allows us to calculate the key features: its center, vertices, co-vertices, foci, asymptotes, and the lengths and positions of the transverse and conjugate axes. Conversely, an equation for a hyperbola can be found given its key features. We begin by finding standard equations for hyperbolas centered at the origin. Then we will turn our attention to finding standard equations for hyperbolas centered at some point other than the origin.

Hyperbolas Centered at the Origin

Reviewing the standard forms given for hyperbolas centered at $(0, 0)$,we see that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. Note that this equation can also be rewritten as $b^2 = c^2 - a^2$. This relationship is used to write the equation for a hyperbola when given the coordinates of its foci and vertices.

How to: Given the vertices and foci of a hyperbola centered at $(0, 0)$, write its equation in standard form

1. Determine whether the transverse axis lies on the x- or y-axis.
   - If the given coordinates of the vertices and foci have the form $(\pm a, 0)$ and $(\pm c, 0)$, respectively, then the transverse axis is the x-axis. Use the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
   - If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(0, \pm c)$, respectively, then the transverse axis is the y-axis. Use the standard form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.
2. Find b^2 using the equation $b^2 = c^2 - a^2$.
3. Substitute the values for a^2 and b^2 into the standard form of the equation determined in Step 1.

Example 7.2.2: Finding the Equation of a Hyperbola Centered at $(0, 0)$ Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices $(\pm 6, 0)$ and foci $(\pm 2\sqrt{10}, 0)$?

Solution

The vertices and foci are on the x-axis. Thus, the equation for the hyperbola will have the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The vertices are $(\pm 6, 0)$, so $a = 6$ and $a^2 = 36$.

The foci are $(\pm 2\sqrt{10}, 0)$, so $c = 2\sqrt{10}$ and $c^2 = 40$.

Solving for b^2, we have

$$\begin{aligned} b^2 &= c^2 - a^2 \\ b^2 &= 40 - 36 \qquad \text{Substitute for } c^2 \text{ and } a^2 \\ b^2 &= 4 \qquad \text{Subtract.} \end{aligned}$$

Finally, we substitute $a^2 = 36$ and $b^2 = 4$ into the standard form of the equation, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The equation of the hyperbola is $\frac{x^2}{36} - \frac{y^2}{4} = 1$, as shown in Figure 7.2.7.

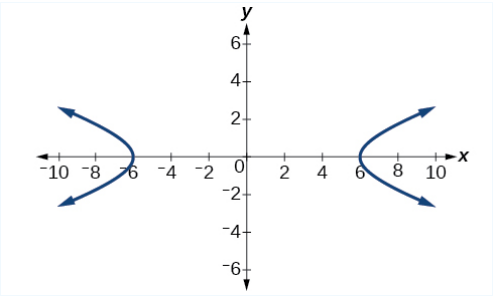


Figure 7.2.7

Exercise 7.2.2

What is the standard form equation of the hyperbola that has vertices $(0, \pm 2)$ and foci $(0, \pm 2\sqrt{5})$?

Answer

$$\frac{y^2}{4} - \frac{x^2}{16} = 1$$

Hyperbolas Not Centered at the Origin

Like the graphs for other equations, the graph of a hyperbola can be translated. If a hyperbola is translated h units horizontally and k units vertically, the center of the hyperbola will be (h, k). This translation results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and y replaced by $(y - k)$.

STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER (H, K)

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the x-axis is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{7.2.2}$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h \pm c, k)$

The asymptotes of the hyperbola coincide with the diagonals of the central rectangle. The length of the rectangle is $2a$ and its width is $2b$. The slopes of the diagonals are $\pm\frac{b}{a}$,and each diagonal passes through the center (h, k). Using the point-slope formula, it is simple to show that the equations of the asymptotes are $y = \pm\frac{b}{a}(x - h) + k$. See Figure 7.2.8a

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the y-axis is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \tag{7.2.3}$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h, k \pm c)$

Using the reasoning above, the equations of the asymptotes are $y = \pm\frac{a}{b}(x - h) + k$. See Figure 7.2.8b

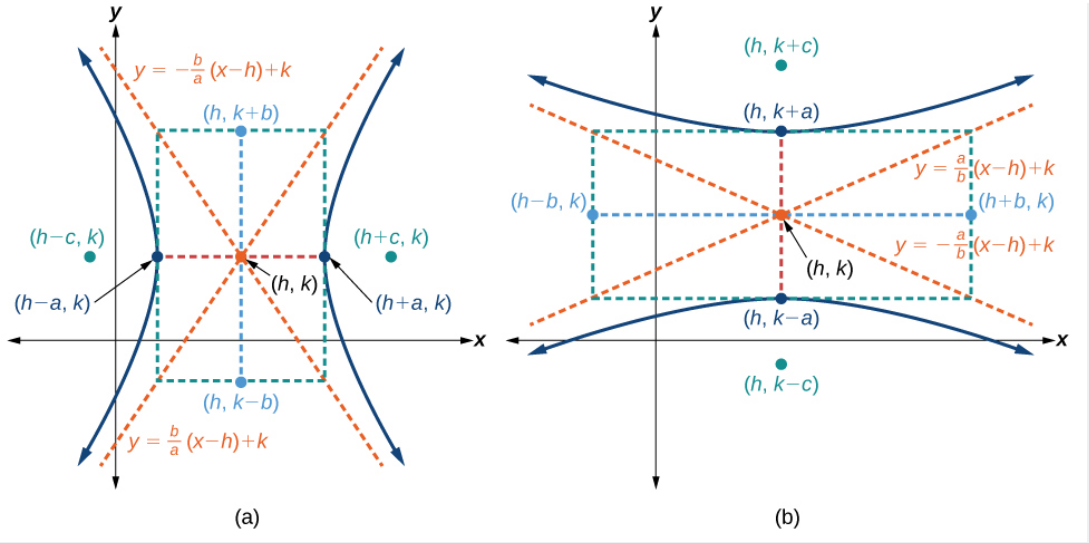


Figure 7.2.8: (a) Horizontal hyperbola with center (h, k) (b) Vertical hyperbola with center (h, k)

Like hyperbolas centered at the origin, hyperbolas centered at a point (h, k) have vertices, co-vertices, and foci that are related by the equation $c^2 = a^2 + b^2$. We can use this relationship along with the midpoint and distance formulas to find the standard equation of a hyperbola when the vertices and foci are given.

How to: Given the vertices and foci of a hyperbola centered at (h, k),write its equation in standard form

1. Determine whether the transverse axis is parallel to the x- or y-axis.
   - If the y-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the x-axis. Use the standard form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.
   - If the x-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the y-axis. Use the standard form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.
2. Identify the center of the hyperbola, (h, k),using the midpoint formula and the given coordinates for the vertices.
3. Find a^2 by solving for the length of the transverse axis, $2a$, which is the distance between the given vertices.
4. Find c^2 using h and k found in Step 2 along with the given coordinates for the foci.
5. Solve for b^2 using the equation $b^2 = c^2 - a^2$.
6. Substitute the values for h, k, a^2, and b^2 into the standard form of the equation determined in Step 1.

Example 7.2.3: Finding the Equation of a Hyperbola Centered at (h, k) Given its Foci and Vertices

What is the standard form equation of the **hyperbola** that has vertices at $(0, -2)$ and $(6, -2)$ and foci at $(-2, -2)$ and $(8, -2)$?

Solution

The y-coordinates of the vertices and foci are the same, so the transverse axis is parallel to the x-axis. Thus, the equation of the hyperbola will have the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

First, we identify the center, (h, k). The center is halfway between the vertices $(0, -2)$ and $(6, -2)$. Applying the midpoint formula, we have

$$(h, k) = \left(\frac{0+6}{2}, \frac{-2+(-2)}{2}\right) = (3, -2)$$

Next, we find a^2. The length of the transverse axis, $2a$,is bounded by the vertices. So, we can find a^2 by finding the distance between the x-coordinates of the vertices.

Now we need to find c^2. The coordinates of the foci are $(h \pm c, k)$. So $(h-c, k) = (-2, -2)$ and $(h+c, k) = (8, -2)$. We can use the x-coordinate from either of these points to solve for c. Using the point $(8, -2)$, and substituting $h = 3$,

$$\begin{aligned} h + c &= 8 \\ 3 + c &= 8 \\ c &= 5 \\ c^2 &= 25 \end{aligned}$$

Next, solve for b^2 using the equation $b^2 = c^2 - a^2$:

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 25 - 9 \\ &= 16 \end{aligned}$$

Finally, substitute the values found for h, k, a^2,and b^2 into the standard form of the equation.

$$\frac{(x-3)^2}{9} - \frac{(y+2)^2}{16} = 1$$

Exercise 7.2.3

What is the standard form equation of the hyperbola that has vertices $(1, -2)$ and $(1, 8)$ and foci $(1, -10)$ and $(1, 16)$?

Answer

$$\frac{(y-3)^2}{25} + \frac{(x-1)^2}{144} = 1$$

Graphing Hyperbolas Centered at the Origin

When we have an equation in standard form for a hyperbola centered at the origin, we can interpret its parts to identify the key features of its graph: the center, vertices, co-vertices, asymptotes, foci, and lengths and positions of the transverse and conjugate axes. To graph hyperbolas centered at the origin, we use the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for horizontal hyperbolas and the standard form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ for vertical hyperbolas.

How to: Given a standard form equation for a hyperbola centered at $(0, 0)$, sketch the graph

1. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the vertices, co-vertices, and foci; and the equations for the asymptotes.
   - If the equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then
     - the transverse axis is on the x-axis
     - the coordinates of the vertices are \((\pm a,0)\0
     - the coordinates of the co-vertices are $(0, \pm b)$
     - the coordinates of the foci are $(\pm c, 0)$
     - the equations of the asymptotes are $y = \pm \frac{b}{a} x$
   - If the equation is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then
     - the transverse axis is on the y-axis
     - the coordinates of the vertices are $(0, \pm a)$
     - the coordinates of the co-vertices are $(\pm b, 0)$
     - the coordinates of the foci are $(0, \pm c)$
     - the equations of the asymptotes are $y = \pm \frac{a}{b} x$
3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2 + b^2}$.
4. Plot the vertices, co-vertices, foci, and asymptotes in the coordinate plane, and draw a smooth curve to form the hyperbola.

Example 7.2.4: Graphing a Hyperbola Centered at $(0,0)$ Given an Equation in Standard Form

Graph the hyperbola given by the equation $\frac{y^2}{64}-\frac{x^2}{36}=1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

Solution

The standard form that applies to the given equation is $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$. Thus, the transverse axis is on the y-axis

The coordinates of the vertices are $(0,\pm a)=(0,\pm\sqrt{64})=(0,\pm 8)$

The coordinates of the co-vertices are $(\pm b,0)=(\pm\sqrt{36},0)=(\pm 6,0)$

The coordinates of the foci are $(0,\pm c)$, where $c=\pm\sqrt{a^2+b^2}$. Solving for c, we have

$$c=\pm\sqrt{a^2+b^2}=\pm\sqrt{64+36}=\pm\sqrt{100}=\pm 10$$

Therefore, the coordinates of the foci are $(0,\pm 10)$

The equations of the asymptotes are $y=\pm\frac{a}{b}x=\pm\frac{8}{6}x=\pm\frac{4}{3}x$

Plot and label the vertices and co-vertices, and then sketch the central rectangle. Sides of the rectangle are parallel to the axes and pass through the vertices and co-vertices. Sketch and extend the diagonals of the central rectangle to show the asymptotes. The central rectangle and asymptotes provide the framework needed to sketch an accurate graph of the hyperbola. Label the foci and asymptotes, and draw a smooth curve to form the hyperbola, as shown in Figure 7.2.9.

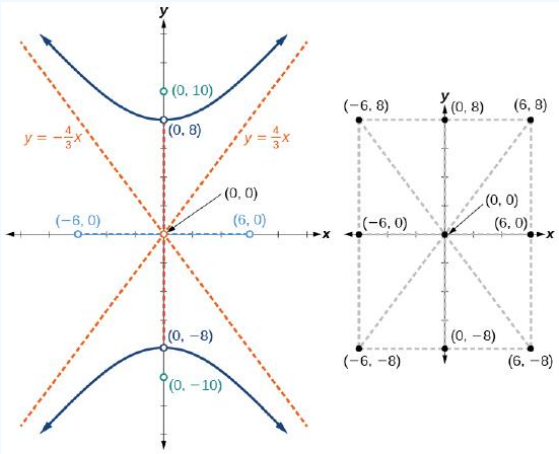


Figure 7.2.9

Exercise 7.2.4

Graph the hyperbola given by the equation $\frac{x^2}{144}-\frac{y^2}{81}=1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

Answer

vertices: $(\pm 12,0)$; co-vertices: $(0,\pm 9)$; foci: $(\pm 15,0)$; asymptotes: $y=\pm\frac{3}{4}x$;

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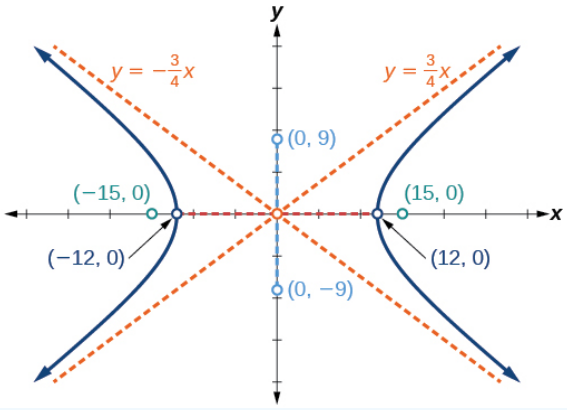


Figure 7.2.10

Graphing Hyperbolas Not Centered at the Origin

Graphing hyperbolas centered at a point (h, k) other than the origin is similar to graphing ellipses centered at a point other than the origin. We use the standard forms $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ for horizontal hyperbolas, and $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ for vertical hyperbolas. From these standard form equations we can easily calculate and plot key features of the graph: the coordinates of its center, vertices, co-vertices, and foci; the equations of its asymptotes; and the positions of the transverse and conjugate axes.

How to: Given a general form for a hyperbola centered at (h, k), sketch the graph

1. Convert the general form to that standard form. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the center, vertices, co-vertices, foci; and equations for the asymptotes.
   1. If the equation is in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, then
      - the transverse axis is parallel to the x-axis
      - the center is (h, k)
      - the coordinates of the vertices are $(h \pm a, k)$
      - the coordinates of the co-vertices are $(h, k \pm b)$
      - the coordinates of the foci are $(h \pm c, k)$
      - the equations of the asymptotes are $y = \pm\frac{b}{a}(x-h) + k$
   2. If the equation is in the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, then
      - the transverse axis is parallel to the y-axis
      - the center is (h, k)
      - the coordinates of the vertices are $(h, k \pm a)$
      - the coordinates of the co-vertices are $(h \pm b, k)$
      - the coordinates of the foci are $(h, k \pm c)$
      - the equations of the asymptotes are $y = \pm\frac{a}{b}(x-h) + k$
3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2 + b^2}$.
4. Plot the center, vertices, co-vertices, foci, and asymptotes in the coordinate plane and draw a smooth curve to form the hyperbola.

Example 7.2.5: Graphing a Hyperbola Centered at (h, k) Given an Equation in General Form

Graph the hyperbola given by the equation $9x^2 - 4y^2 - 36x - 40y - 388 = 0$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Solution

Start by expressing the equation in standard form. Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$(9x^2-36x)-(4y^2+40y)=388$$

Factor the leading coefficient of each expression.

$$9(x^2-4x)-4(y^2+10y)=388$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$9(x^2-4x+4)-4(y^2+10y+25)=388+36-100$$

Rewrite as perfect squares.

$$9(x-2)^2-4(y+5)^2=324$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-2)^2}{36}-\frac{(y+5)^2}{81}=1$$

The standard form that applies to the given equation is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, where $a^2=36$ and $b^2=81$,or $a=6$ and $b=9$. Thus, the transverse axis is parallel to the x-axis. It follows that:

the center of the ellipse is $(h,k)=(2,-5)$

the coordinates of the vertices are $(h\pm a,k)=(2\pm 6,-5)$, or $(-4,-5)$ and $(8,-5)$

the coordinates of the co-vertices are $(h,k\pm b)=(2,-5\pm 9)$, or $(2,-14)$ and $(2,4)$

the coordinates of the foci are $(h\pm c,k)$, where $c=\pm\sqrt{a^2+b^2}$. Solving for c,we have

$$c=\pm\sqrt{36+81}=\pm\sqrt{117}=\pm 3\sqrt{13}$$

Therefore, the coordinates of the foci are $(2-3\sqrt{13},-5)$ and $(2+3\sqrt{13},-5)$.

The equations of the asymptotes are $y=\pm\frac{b}{a}(x-h)+k=\pm\frac{3}{2}(x-2)-5$.

Next, we plot and label the center, vertices, co-vertices, foci, and asymptotes and draw smooth curves to form the hyperbola, as shown in Figure 7.2.11.

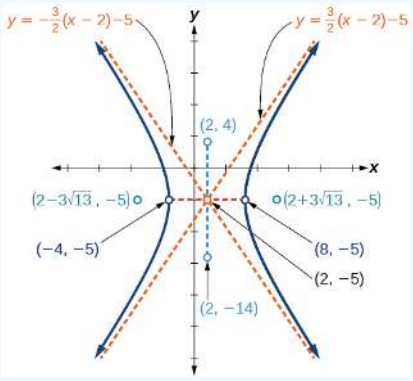


Figure 7.2.11

Exercise 7.2.5

Graph the hyperbola given by the standard form of an equation $\frac{(y+4)^2}{100}-\frac{(x-3)^2}{64}=1$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Answer

center: $(3,-4)$; vertices: $(3,-14)$ and $(3,6)$; co-vertices: $(-5,-4)$; and $(11,-4)$; foci: $(3,-4-2\sqrt{41})$ and $(3,-4+2\sqrt{41})$; asymptotes: $y=\pm\frac{5}{4}(x-3)-4$

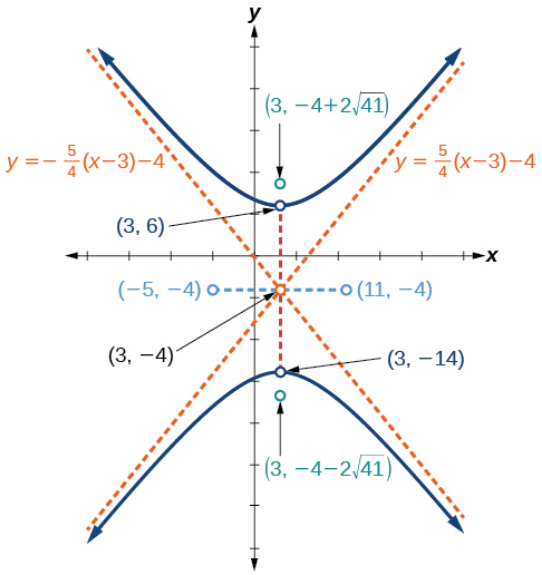


Figure 7.2.12

Solving Applied Problems Involving Hyperbolas

As we discussed at the beginning of this section, hyperbolas have real-world applications in many fields, such as astronomy, physics, engineering, and architecture. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere and are often touted for their ability to generate power efficiently. Because of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength (Figure 7.2.13). For example, a 500-foot tower can be made of a reinforced concrete shell only 6 or 8 inches wide!



Figure 7.2.13: Cooling towers at the Drax power station in North Yorkshire, United Kingdom (credit: Les Haines, Flickr)

The first hyperbolic towers were designed in 1914 and were 35 meters high. Today, the tallest cooling towers are in France, standing a remarkable 170 meters tall. In Example 7.2.6 we will use the design layout of a cooling tower to find a hyperbolic equation that models its sides.

Example 7.2.6: Solving Applied Problems Involving Hyperbolas

The design layout of a cooling tower is shown in Figure 7.2.14 The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart.

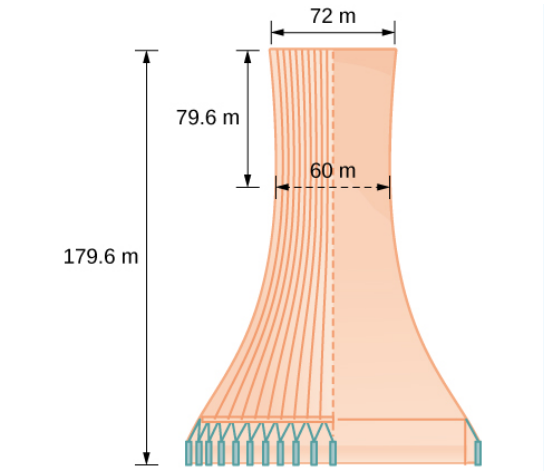


Figure 7.2.14: Project design for a natural draft cooling tower

Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

Solution

We are assuming the center of the tower is at the origin, so we can use the standard form of a horizontal hyperbola centered at the origin: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where the branches of the hyperbola form the sides of the cooling tower. We must find the values of a^2 and b^2 to complete the model.

First, we find a^2. Recall that the length of the transverse axis of a hyperbola is $2a$. This length is represented by the distance where the sides are closest, which is given as 65.3 meters. So, $2a=60$. Therefore, $a=30$ and $a^2=900$.

To solve for b^2,we need to substitute for x and y in our equation using a known point. To do this, we can use the dimensions of the tower to find some point (x,y) that lies on the hyperbola. We will use the top right corner of the tower to represent that point. Since the y-axis bisects the tower, our x-value can be represented by the radius of the top, or 36 meters. The y-value is represented by the distance from the origin to the top, which is given as 79.6 meters. Therefore,

$$\begin{aligned}\frac{x^2}{a^2}-\frac{y^2}{b^2}&=1 && \text{Standard form of horizontal hyperbola.}\\ b^2&=\frac{y^2}{\frac{x^2}{a^2}-1} && \text{Isolate } b^2\\ &=\frac{(79.6)^2}{\frac{(36)^2}{900}-1} && \text{Substitute for } a^2,\ x,\ \text{and } y\\ &\approx 14400.3636 && \text{Round to four decimal places}\end{aligned}$$

The sides of the tower can be modeled by the hyperbolic equation

$$\frac{x^2}{900}-\frac{y^2}{14400.3636}=1 \text{ ,or } \frac{x^2}{30^2}-\frac{y^2}{120.0015^2}=1$$

Exercise 7.2.6

A design for a cooling tower project is shown in Figure 7.2.15. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

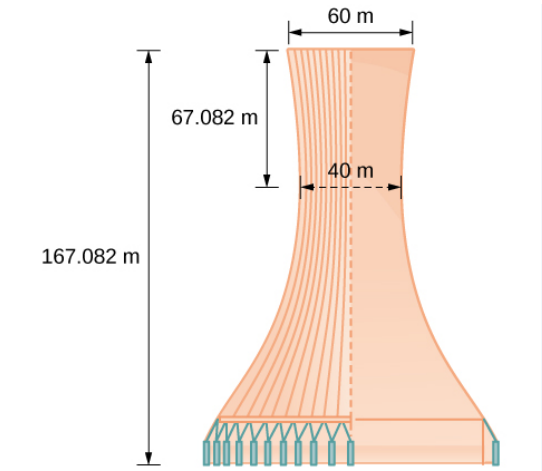


Figure 7.2.15

Answer

The sides of the tower can be modeled by the hyperbolic equation. $\frac{x^2}{400} - \frac{y^2}{3600} = 1$ or $\frac{x^2}{20^2} - \frac{y^2}{60^2} = 1$.

Media

Access these online resources for additional instruction and practice with hyperbolas.

- Conic Sections: The Hyperbola Part 1 of 2
- Conic Sections: The Hyperbola Part 2 of 2
- Graph a Hyperbola with Center at Origin
- Graph a Hyperbola with Center not at Origin

Key Equations

Hyperbola, center at origin, transverse axis on *x*-axis	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Hyperbola, center at origin, transverse axis on *y*-axis	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Hyperbola, center at (h, k),transverse axis parallel to *x*-axis	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
Hyperbola, center at (h, k),transverse axis parallel to *y*-axis	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Key Concepts

- A hyperbola is the set of all points (x, y) in a plane such that the difference of the distances between (x, y) and the foci is a positive constant.
- The standard form of a hyperbola can be used to locate its vertices and foci. See Example $7.2.1$.
- When given the coordinates of the foci and vertices of a hyperbola, we can write the equation of the hyperbola in standard form. See Example $7.2.2$ and Example $7.2.3$.
- When given an equation for a hyperbola, we can identify its vertices, co-vertices, foci, asymptotes, and lengths and positions of the transverse and conjugate axes in order to graph the hyperbola. See Example $7.2.4$ and Example $7.2.5$.
- Real-world situations can be modeled using the standard equations of hyperbolas. For instance, given the dimensions of a natural draft cooling tower, we can find a hyperbolic equation that models its sides. See Example $7.2.6$.

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7.3: The Parabola

Learning Objectives

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Solve applied problems involving parabolas.

Did you know that the Olympic torch is lit several months before the start of the games? The ceremonial method for lighting the flame is the same as in ancient times. The ceremony takes place at the Temple of Hera in Olympia, Greece, and is rooted in Greek mythology, paying tribute to Prometheus, who stole fire from Zeus to give to all humans. One of eleven acting priestesses places the torch at the focus of a parabolic mirror (Figure 7.3.1), which focuses light rays from the sun to ignite the flame.



Figure 7.3.1: The Olympic torch concludes its journey around the world when it is used to light the Olympic cauldron during the opening ceremony. (credit: Ken Hackman, U.S. Air Force)

Parabolic mirrors (or reflectors) are able to capture energy and focus it to a single point. The advantages of this property are evidenced by the vast list of parabolic objects we use every day: satellite dishes, suspension bridges, telescopes, microphones, spotlights, and car headlights, to name a few. Parabolic reflectors are also used in alternative energy devices, such as solar cookers and water heaters, because they are inexpensive to manufacture and need little maintenance. In this section we will explore the parabola and its uses, including low-cost, energy-efficient solar designs.

Graphing Parabolas with Vertices at the Origin

Previously, we saw that an ellipse is formed when a plane cuts through a right circular cone. If the plane is parallel to the edge of the cone, an unbounded curve is formed. This curve is a *parabola* (Figure 7.3.2).

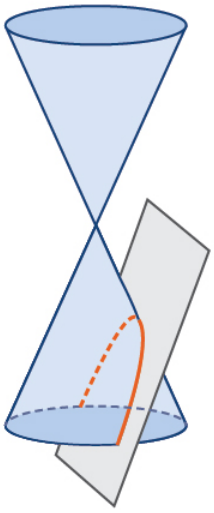


Figure 7.3.2: Parabola

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points (x, y) in a plane that are the same distance from a fixed line, called the *directrix*, and a fixed point (the *focus*) not on the directrix.

Previously, we learned about a parabola's vertex and axis of symmetry. Now we extend the discussion to include other key features of the parabola (Figure 7.3.3). Notice that the axis of symmetry passes through the focus and vertex and is perpendicular to the directrix. The vertex is the midpoint between the directrix and the focus. The line segment that passes through the focus and is parallel to the directrix is called the *latus rectum*. The endpoints of the latus rectum lie on the curve. By definition, the distanced d from the focus to any point P on the parabola is equal to the distance from P to the directrix.

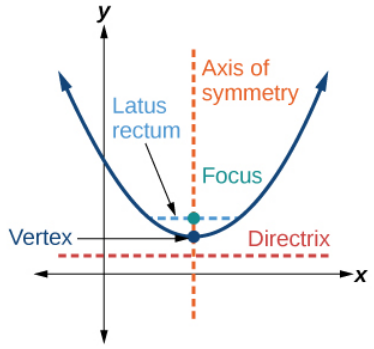


Figure 7.3.3: Key features of the parabola

To work with parabolas in the *coordinate plane*, we consider two cases: those with a vertex at the origin and those with a *vertex* at a point other than the origin. We begin with the former.

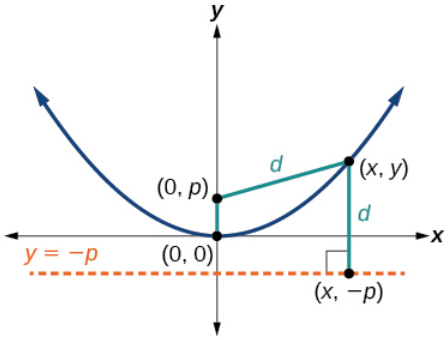
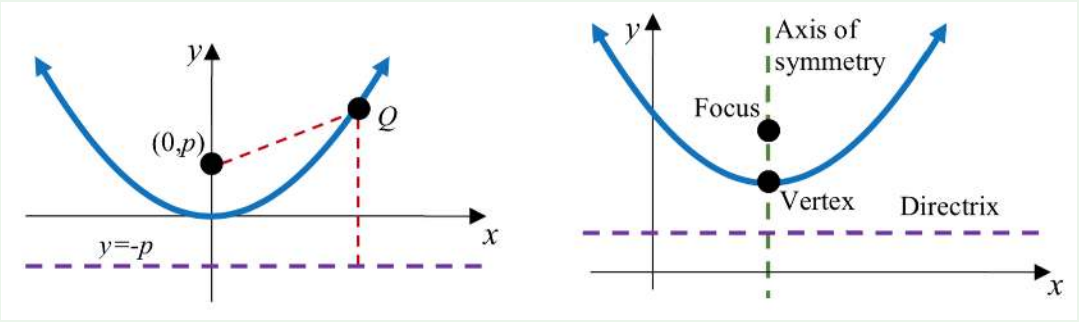


Figure 7.3.4

Definition: PARABOLA Definition AND VOCABULARY

A parabola with vertex at the origin can be defined by placing a fixed point at $F\,(0, p)$ called the **focus**, and drawing a line at $y = -p$, called the **directrix**. The parabola is the set of all points $Q\,(x, y)$ that are an equal distance between the fixed point and the directrix.



For general parabolas,
The **axis of symmetry** is the line passing through the foci, perpendicular to the directrix.

The **vertex** is the point where the parabola crosses the axis of symmetry.
The distance from the vertex to the focus, p, is the **focal length**.

Deriving the Equation for a Parabola with Vertex, Focus and Directrix

Let (x, y) be a point on the parabola with vertex $(0, 0)$, focus $(0, p)$,and directrix $y = -p$ as shown in Figure 7.3.4. The distanced d from point (x, y) to point $(x, -p)$ on the directrix is the difference of the y-values: $d = y + p$. The distance from the focus $(0, p)$ to the point (x, y) is also equal to d and can be expressed using the distance formula.

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-p)^2} \\ &= \sqrt{x^2 + (y-p)^2} \end{aligned}$$

Set the two expressions for d equal to each other and solve for y to derive the equation of the parabola. We do this because the distance from (x, y) to $(0, p)$ equals the distance from (x, y) to $(x, -p)$.

$$\sqrt{x^2 + (y-p)^2} = y + p \tag{7.3.1}$$

We then square both sides of the equation, expand the squared terms, and simplify by combining like terms.

$$\begin{aligned} x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 - 2py &= 2py \\ x^2 &= 4py \end{aligned}$$

The equations of parabolas with vertex $(0, 0)$ are $y^2 = 4px$ when the x-axis is the axis of symmetry and $x^2 = 4py$ when the y-axis is the axis of symmetry. These standard forms are given below, along with their general graphs and key features.

STANDARD FORMS OF PARABOLAS WITH VERTEX $(0, 0)$

Table 7.3.1 and Figure 7.3.5 summarize the standard features of parabolas with a vertex at the origin.

Table 7.3.1

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
x-axis	$y^2 = 4px$	$(p, 0)$	$x = -p$	$(p, \pm 2p)$
y-axis	$x^2 = 4py$	$(0, p)$	$y = -p$	$(\pm 2p, p)$

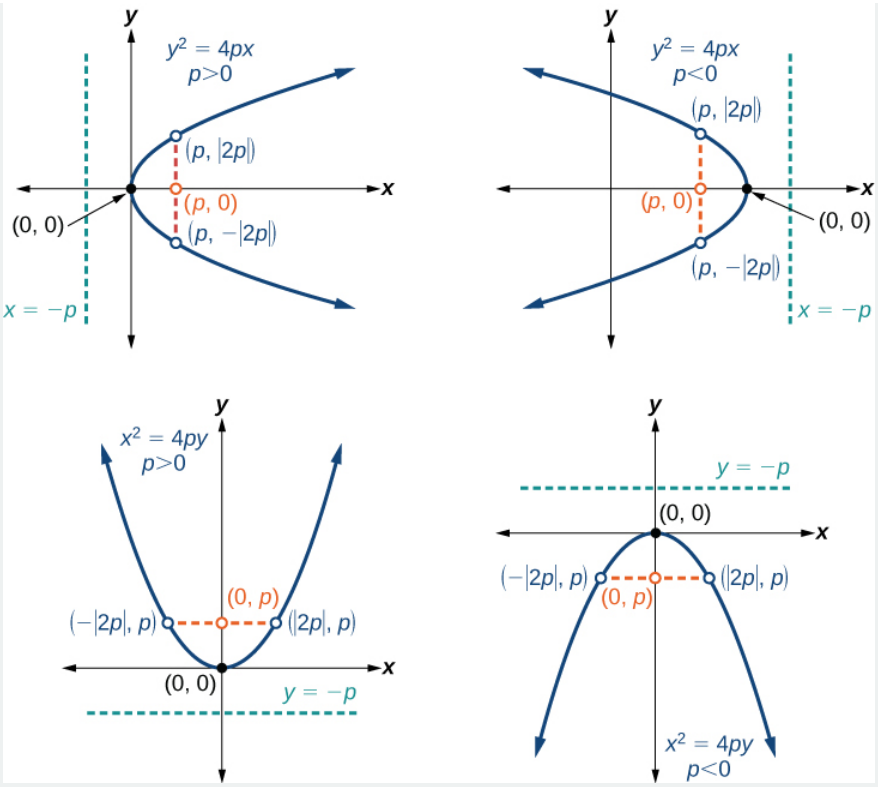


Figure 7.3.5: (a) When $p > 0$ and the axis of symmetry is the x-axis, the parabola opens right. (b) When $p < 0$ and the axis of symmetry is the x-axis, the parabola opens left. (c) When $p < 0$ and the axis of symmetry is the y-axis, the parabola opens up. (d) When $p < 0$ and the axis of symmetry is the y-axis, the parabola opens down.

The key features of a parabola are its vertex, axis of symmetry, focus, directrix, and *latus rectum* (Figure 7.3.5). When given a standard equation for a parabola centered at the origin, we can easily identify the key features to graph the parabola. A line is said to be tangent to a curve if it intersects the curve at exactly one point. If we sketch lines tangent to the parabola at the endpoints of the *latus rectum*, these lines intersect on the axis of symmetry, as shown in Figure 7.3.6.

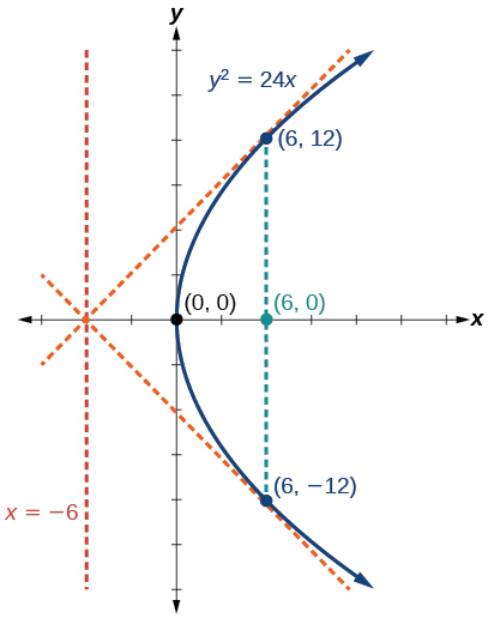


Figure 7.3.6

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How to: Given a standard form equation for a parabola centered at $(0,0)$, sketch the graph

1. Determine which of the standard forms applies to the given equation: $y^2=4px$ or $x^2=4py$.
2. Use the standard form identified in Step 1 to determine the axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   - If the equation is in the form $y^2=4px$, then
     - the axis of symmetry is the x-axis, $y=0$
     - set $4p$ equal to the coefficient of x in the given equation to solve for p. If $p>0$, the parabola opens right. If $p<0$, the parabola opens left.
     - use p to find the coordinates of the focus, $(p,0)$
     - use p to find the equation of the directrix, $x=-p$
     - use p to find the endpoints of the latus rectum, $(p,\pm 2p)$. Alternately, substitute $x=p$ into the original equation.
   - If the equation is in the form $x^2=4py$,then
     - the axis of symmetry is the y-axis, $x=0$
     - set $4p$ equal to the coefficient of y in the given equation to solve for p. If $p>0$, the parabola opens up. If $p<0$, the parabola opens down.
     - use p to find the coordinates of the focus, $(0,p)$
     - use p to find equation of the directrix, $y=-p$
     - use p to find the endpoints of the latus rectum, $(\pm 2p,p)$
3. Plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

-axis as the Axis of Symmetry

Graph $y^2=24x$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

Solution

The standard form that applies to the given equation is $y^2=4px$. Thus, the axis of symmetry is the *x*-axis. It follows that:

- $24=4p$, so $p=6$. Since $p>0$, the parabola opens right
- the coordinates of the focus are $(p,0)=(6,0)$
- the equation of the directrix is $x=-p=-6$
- the endpoints of the latus rectum have the same *x*-coordinate at the focus. To find the endpoints, substitute $x=6$ into the original equation: $(6,\pm 12)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 7.3.7).

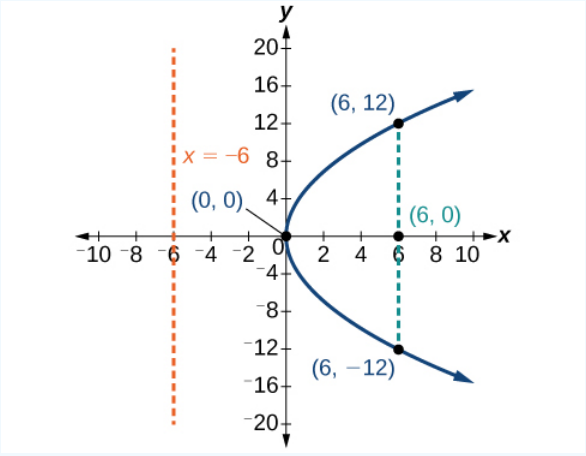


Figure 7.3.7

Exercise 7.3.1

Graph $y^2 = -16x$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

Answer

- Focus: $(-4, 0)$
- Directrix: $x = 4$
- Endpoints of the latus rectum: $(-4, \pm 8)$

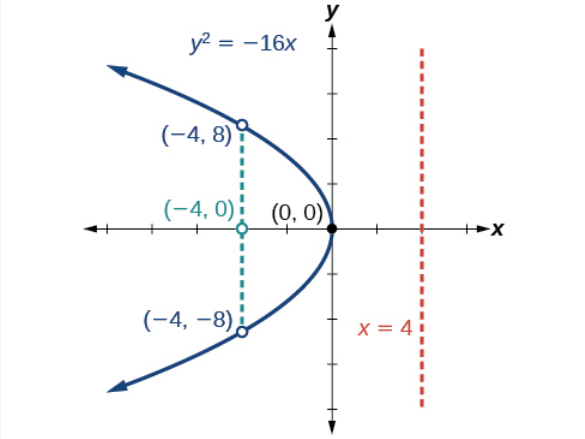


Figure 7.3.8

-axis as the Axis of Symmetry

Graph $x^2 = -6y$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

Solution

The standard form that applies to the given equation is $x^2 = 4py$. Thus, the axis of symmetry is the y-axis. It follows that:

- $-6 = 4p$,so $p = -\frac{3}{2}$. Since $p < 0$, the parabola opens down.
- the coordinates of the focus are $(0, p) = (0, -\frac{3}{2})$
- the equation of the directrix is $y = -p = \frac{3}{2}$
- the endpoints of the latus rectum can be found by substituting $y = \frac{3}{2}$ into the original equation, $(\pm 3, -\frac{3}{2})$

Next we plot the focus, directrix, and *latus rectum*, and draw a smooth curve to form the parabola.

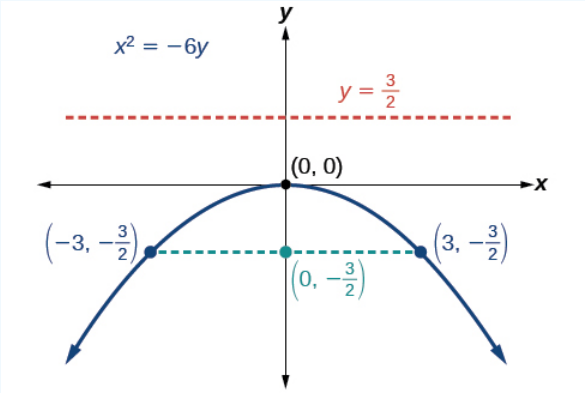


Figure 7.3.9

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Exercise 7.3.2

Graph $x^2 = 8y$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

Answer

- Focus: $(0, 2)$
- Directrix: $y = -2$
- Endpoints of the latus rectum: $(\pm 4, 2)$.

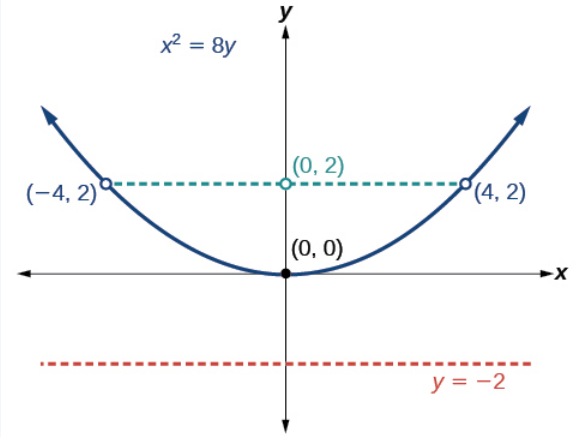


Figure 7.3.10

Writing Equations of Parabolas in Standard Form

In the previous examples, we used the standard form equation of a parabola to calculate the locations of its key features. We can also use the calculations in reverse to write an equation for a parabola when given its key features.

How to: Given its focus and directrix, write the equation for a parabola in standard form

1. Determine whether the axis of symmetry is the x- or y-axis.
   i. If the given coordinates of the focus have the form $(p, 0)$, then the axis of symmetry is the x-axis. Use the standard form $y^2 = 4px$.
   ii. If the given coordinates of the focus have the form $(0, p)$, then the axis of symmetry is the y-axis. Use the standard form $x^2 = 4py$.
2. Multiply $4p$.
3. Substitute the value from Step 2 into the equation determined in Step 1.

Example 7.3.3: Writing the Equation of a Parabola in Standard Form Given its Focus and Directrix

What is the equation for the parabola with focus $(-\frac{1}{2}, 0)$ and directrix $x = \frac{1}{2}$?

Solution

The focus has the form $(p, 0)$, so the equation will have the form $y^2 = 4px$.

- Multiplying $4p$, we have $4p = 4(-\frac{1}{2}) = -2$.
- Substituting for $4p$, we have $y^2 = 4px = -2x$.=

Therefore, the equation for the parabola is $y^2 = -2x$.

Exercise 7.3.3

What is the equation for the parabola with focus $\left(0, \frac{7}{2}\right)$ and directrix $y = -\frac{7}{2}$?

Answer

$x^2 = 14y$.

Graphing Parabolas with Vertices Not at the Origin

Like other graphs we've worked with, the graph of a parabola can be translated. If a parabola is translated h units horizontally and k units vertically, the vertex will be (h, k). This translation results in the standard form of the equation we saw previously with x replaced by $(x - h)$ and y replaced by $(y - k)$.

To graph parabolas with a vertex (h, k) other than the origin, we use the standard form $(y - k)^2 = 4p(x - h)$ for parabolas that have an axis of symmetry parallel to the x-axis, and $(x - h)^2 = 4p(y - k)$ for parabolas that have an axis of symmetry parallel to the y-axis. These standard forms are given below, along with their general graphs and key features.

STANDARD FORMS OF PARABOLAS WITH VERTEX (H, K)

Table 7.3.2 and Figure 7.3.11 summarize the standard features of parabolas with a vertex at a point (h, k).

Table 7.3.2

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
$y = k$	$(y - k)^2 = 4p(x - h)$	$(h + p, k)$	$x = h - p$	$(h + p, k \pm 2p)$
$x = h$	$(x - h)^2 = 4p(y - k)$	$(h, k + p)$	$y = k - p$	$(h \pm 2p, k + p)$

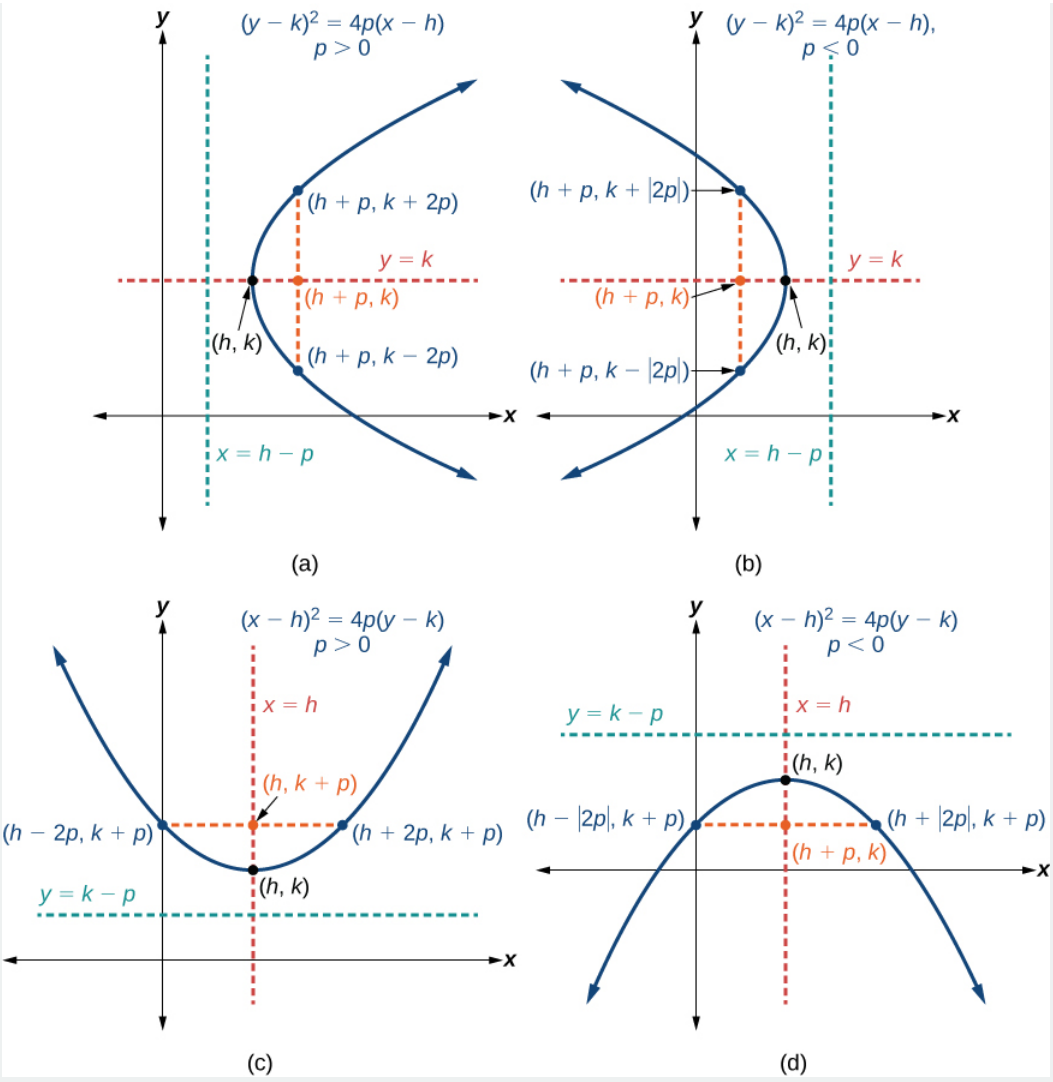


Figure 7.3.11: (a) When $p > 0$, the parabola opens right. (b) When $p < 0$, the parabola opens left. (c) When $p > 0$,the parabola opens up. (d) When $p < 0$, the parabola opens down.

How to: Given a standard form equation for a parabola centered at (h, k), sketch the graph

1. Determine which of the standard forms applies to the given equation: $(y-k)^2 = 4p(x-h)$ or $(x-h)^2 = 4p(y-k)$.
2. Use the standard form identified in Step 1 to determine the vertex, axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   - If the equation is in the form $(y-k)^2 = 4p(x-h)$,then:
     - use the given equation to identify h and k for the vertex, (h, k)
     - use the value of k to determine the axis of symmetry, $y = k$
     - set $4p$ equal to the coefficient of $(x-h)$ in the given equation to solve for p. If $p > 0$,the parabola opens right. If $p < 0$, the parabola opens left.
     - use h, k, and p to find the coordinates of the focus, $(h+p, k)$
     - use h andp p to find the equation of the directrix, $x = h-p$
     - use h, k, and p to find the endpoints of the latus rectum, $(h+p, k \pm 2p)$
   - If the equation is in the form $(x-h)^2 = 4p(y-k)$,then:
     - use the given equation to identify h and k for the vertex, (h, k)
     - use the value of h to determine the axis of symmetry, $x = h$

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- set $4p$ equal to the coefficient of $(y-k)$ in the given equation to solve for p. If $p>0$, the parabola opens up. If $p<0$, the parabola opens down.
- use h, k, and p to find the coordinates of the focus, $(h, k+p)$
- use k and p to find the equation of the directrix, $y=k-p$
- use h, k, and p to find the endpoints of the latus rectum, $(h \pm 2p, k+p)$

3. Plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

Example 7.3.4: Graphing a Parabola with Vertex (h, k) and Axis of Symmetry Parallel to the x-axis

Graph $(y-1)^2=-16(x+3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solution

The standard form that applies to the given equation is $(y-k)^2=4p(x-h)$. Thus, the axis of symmetry is parallel to the x-axis. It follows that:

- the vertex is $(h, k)=(-3, 1)$
- the axis of symmetry is $y=k=1$
- $-16=4p$, so $p=-4$. Since $p<0$, the parabola opens left.
- the coordinates of the focus are $(h+p, k)=(-3+(-4), 1)=(-7, 1)$
- the equation of the directrix is $x=h-p=-3-(-4)=1$
- the endpoints of the latus rectum are $(h+p, k \pm 2p)=(-3+(-4), 1 \pm 2(-4))$, or $(-7, -7)$ and $(-7, 9)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 7.3.10).

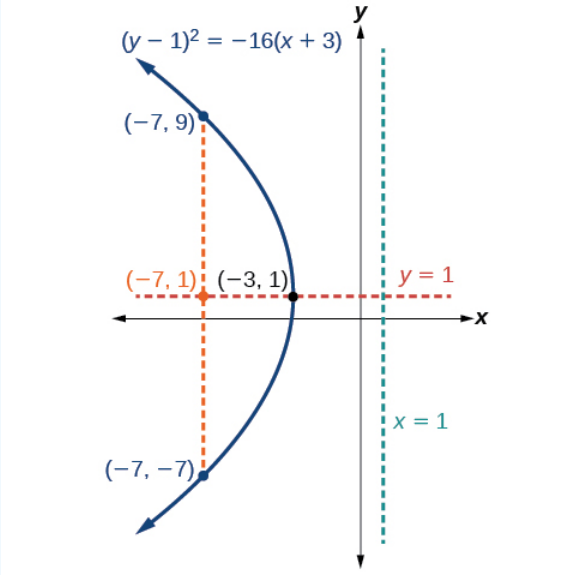


Figure 7.3.12

Exercise 7.3.4

Graph $(y+1)^2=4(x-8)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

Answer

- Vertex: $(8, -1)$
- Axis of symmetry: $y=-1$
- Focus: $(9, -1)$
- Directrix: $x=7$

- Endpoints of the *latus rectum*: $(9, -3)$ and $(9, 1)$.

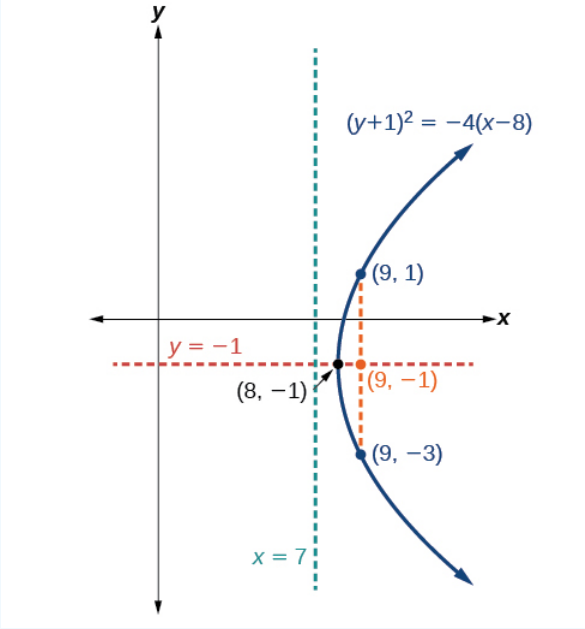


Figure 7.3.13

Example 7.3.5: Graphing a Parabola from an Equation Given in General Form

Graph $x^2 - 8x - 28y - 208 = 0$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

Solution

Start by writing the equation of the parabola in standard form. The standard form that applies to the given equation is $(x-h)^2 = 4p(y-k)$. Thus, the axis of symmetry is parallel to the y-axis. To express the equation of the parabola in this form, we begin by isolating the terms that contain the variable x in order to complete the square.

$$\begin{aligned} x^2 - 8x - 28y - 208 &= 0 \\ x^2 - 8x &= 28y + 208 \\ x^2 - 8x + 16 &= 28y + 208 + 16 \\ (x-4)^2 &= 28y + 224 \\ (x-4)^2 &= 28(y+8) \\ (x-4)^2 &= 4 \cdot 7 \cdot (y+8) \end{aligned}$$

It follows that:

- the vertex is $(h, k) = (4, -8)$
- the axis of symmetry is $x = h = 4$
- since $p = 7$, $p > 0$ and so the parabola opens up
- the coordinates of the focus are $(h, k+p) = (4, -8+7) = (4, -1)$
- the equation of the directrix is $y = k - p = -8 - 7 = -15$
- the endpoints of the latus rectum are $(h \pm 2p, k+p) = (4 \pm 2(7), -8+7)$, or $(-10, -1)$ and $(18, -1)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 7.3.14).

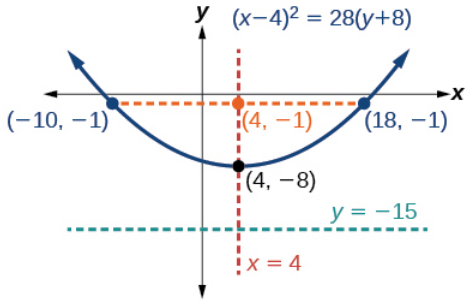


Figure 7.3.14

Exercise 7.3.5

Graph $(x+2)^2 = -20(y-3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

Answer

- Vertex: $(-2, 3)$
- Axis of symmetry: $x = -2$
- Focus: $(-2, -2)$
- Directrix: $y = 8$
- Endpoints of the *latus rectum*: $(-12, -2)$ and $(8, -2)$.

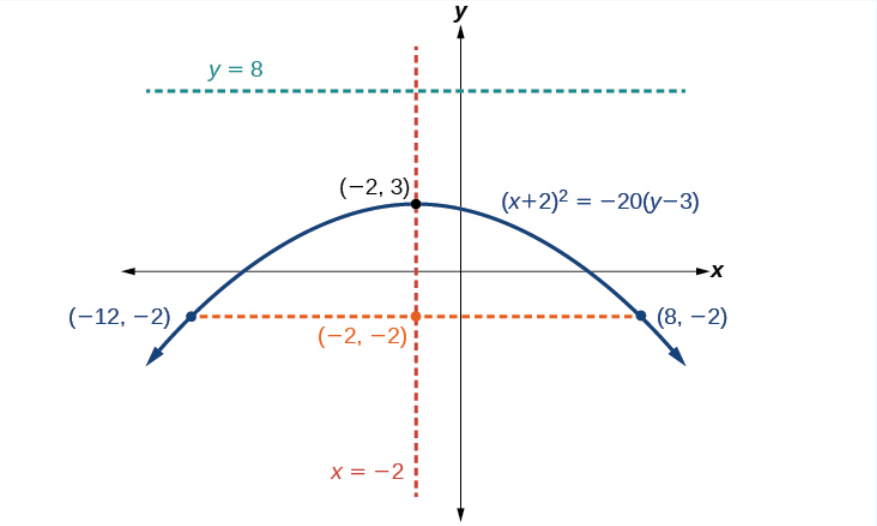


Figure 7.3.15

Solving Applied Problems Involving Parabolas

As we mentioned at the beginning of the section, parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones, and radar equipment. Parabolic mirrors, such as the one used to light the Olympic torch, have a very unique reflecting property. When rays of light parallel to the parabola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus (Figure 7.3.16). This is why the Olympic torch is ignited when it is held at the focus of the parabolic mirror.

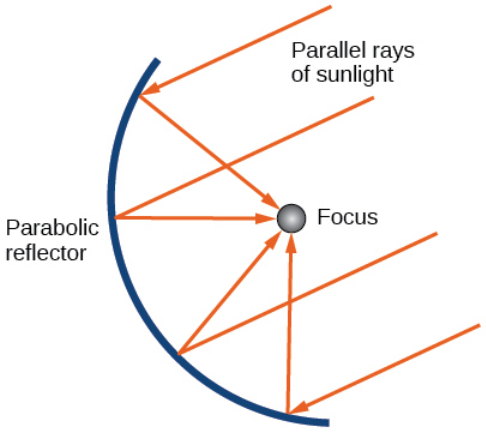


Figure 7.3.16: Reflecting property of parabolas

Parabolic mirrors have the ability to focus the sun's energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus, parabolic mirrors are featured in many low-cost, energy efficient solar products, such as solar cookers, solar heaters, and even travel-sized fire starters.

Example 7.3.6: Solving Applied Problems Involving Parabolas

A cross-section of a design for a travel-sized solar fire starter is shown in Figure 7.3.17. The sun's rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds.

a. Find the equation of the parabola that models the fire starter. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane.
b. Use the equation found in part (a) to find the depth of the fire starter.

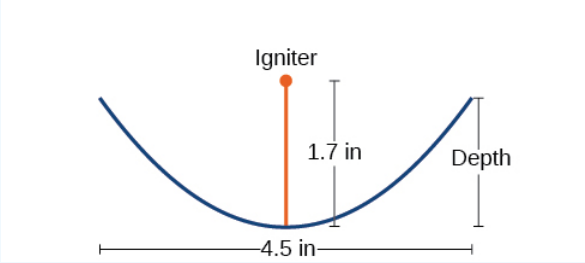


Figure 7.3.17 Cross-section of a travel-sized solar fire starter

Solution

a. The vertex of the dish is the origin of the coordinate plane, so the parabola will take the standard form $x^2 = 4py$,where $p > 0$. The igniter, which is the focus, is 1.7 inches above the vertex of the dish. Thus we have $p = 1.7$.

$$\begin{aligned} x^2 &= 4py && \text{Standard form of upward-facing parabola with vertex } (0,0) \\ x^2 &= 4(1.7)y && \text{Substitute } 1.7 \text{ for } p \\ x^2 &= 6.8y && \text{Multiply.} \end{aligned}$$

b. The dish extends $\frac{4.5}{2} = 2.25$ inches on either side of the origin. We can substitute 2.25 for x in the equation from part (a) to find the depth of the dish.

$$\begin{aligned} x^2 &= 6.8y && \text{Equation found in part } (a) \\ (2.25)^2 &= 6.8y && \text{Substitute } 2.25 \text{ for } x \\ y &\approx 0.74 && \text{Solve for } y \end{aligned}$$

The dish is about 0.74 inches deep.

Exercise 7.3.6

Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of 1600 mm. The sun's rays reflect off the parabolic mirror toward the "cooker," which is placed 320 mm from the base.

a. Find an equation that models a cross-section of the solar cooker. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane, and that the parabola opens to the right (i.e., has the *x*-axis as its axis of symmetry).
b. Use the equation found in part (a) to find the depth of the cooker.

Answer a

$y^2 = 1280x$

Answer b

The depth of the cooker is 500 mm

Key Equations

Parabola, vertex at origin, axis of symmetry on *x*-axis	$y^2 = 4px$
Parabola, vertex at origin, axis of symmetry on *y*-axis	$x^2 = 4py$
Parabola, vertex at (h, k),axis of symmetry on *x*-axis	$(y-k)^2 = 4p(x-h)$
Parabola, vertex at (h, k),axis of symmetry on *y*-axis	$(x-h)^2 = 4p(y-k)$

Key Concepts

- A parabola is the set of all points (x, y) in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix.
- The standard form of a parabola with vertex $(0, 0)$ and the *x*-axis as its axis of symmetry can be used to graph the parabola. If $p > 0$, the parabola opens right. If $p < 0$, the parabola opens left. See Example 7.3.1.
- The standard form of a parabola with vertex $(0, 0)$ and the *y*-axis as its axis of symmetry can be used to graph the parabola. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down. See Example 7.3.2.
- When given the focus and directrix of a parabola, we can write its equation in standard form. See Example 7.3.3.
- The standard form of a parabola with vertex (h, k) and axis of symmetry parallel to the x-axis can be used to graph the parabola. If $p > 0$, the parabola opens right. If $p < 0$, the parabola opens left. See Example 7.3.4.
- The standard form of a parabola with vertex (h, k) and axis of symmetry parallel to the y-axis can be used to graph the parabola. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down. See Example 7.3.5.
- Real-world situations can be modeled using the standard equations of parabolas. For instance, given the diameter and focus of a cross-section of a parabolic reflector, we can find an equation that models its sides. See Example 7.3.6.

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7.4: Rotation of Axes

Learning Objectives

- Identify nondegenerate conic sections given their general form equations.
- Use rotation of axes formulas.
- Write equations of rotated conics in standard form.
- Identify conics without rotating axes.

As we have seen, conic sections are formed when a plane intersects two right circular cones aligned tip to tip and extending infinitely far in opposite directions, which we also call a *cone*. The way in which we slice the cone will determine the type of conic section formed at the intersection. A circle is formed by slicing a cone with a plane perpendicular to the axis of symmetry of the cone. An ellipse is formed by slicing a single cone with a slanted plane not perpendicular to the axis of symmetry. A parabola is formed by slicing the plane through the top or bottom of the double-cone, whereas a hyperbola is formed when the plane slices both the top and bottom of the cone (Figure 7.4.1).

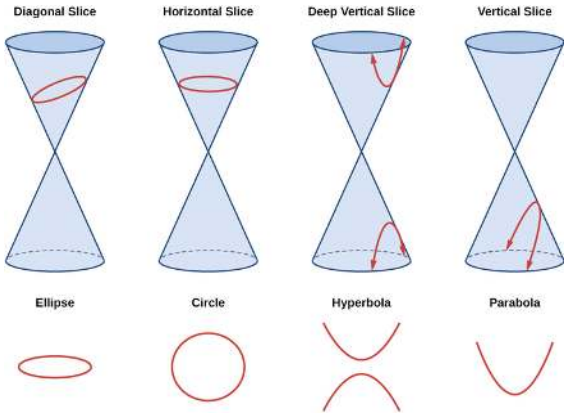


***Figure* 7.4.1:** *The nondegenerate conic sections*

Ellipses, circles, hyperbolas, and parabolas are sometimes called the nondegenerate conic sections, in contrast to the degenerate conic sections, which are shown in Figure 7.4.2. A degenerate conic results when a plane intersects the double cone and passes through the apex. Depending on the angle of the plane, three types of degenerate conic sections are possible: a point, a line, or two intersecting lines.

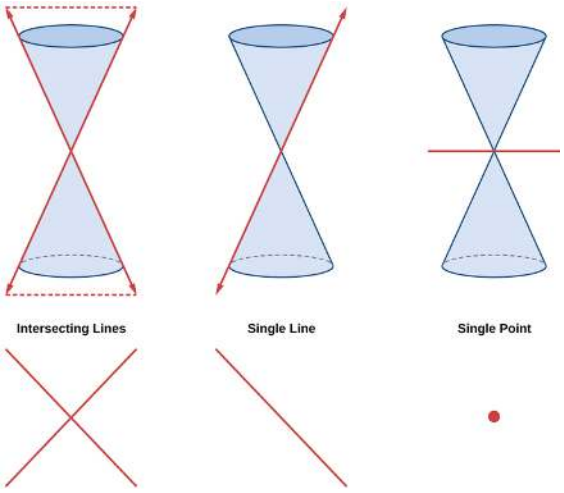


***Figure* 7.4.2:** *Degenerate conic sections*

Identifying Nondegenerate Conics in General Form

In previous sections of this chapter, we have focused on the standard form equations for nondegenerate conic sections. In this section, we will shift our focus to the general form equation, which can be used for any conic. The general form is set equal to zero, and the terms and coefficients are given in a particular order, as shown below.

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

where A, B, and C are not all zero. We can use the values of the coefficients to identify which type conic is represented by a given equation.

You may notice that the general form equation has an xy term that we have not seen in any of the standard form equations. As we will discuss later, the xy term rotates the conic whenever B is not equal to zero.

Table 7.4.1

Conic Sections	Example
ellipse	$4x^2+9y^2=1$
circle	$4x^2+4y^2=1$
hyperbola	$4x^2-9y^2=1$
parabola	$4x^2=9y$ or $4y^2=9x$

Conic Sections	Example
one line	$4x+9y=1$
intersecting lines	$(x-4)(y+4)=0$
parallel lines	$(x-4)(x-9)=0$
a point	$4x^2+4y^2=0$
no graph	$4x^2+4y^2=-1$

GENERAL FORM OF CONIC SECTIONS

A *conic section* has the general form

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \tag{7.4.1}$$

where A, B, and C are not all zero. Table 7.4.2 summarizes the different conic sections where $B=0$, and A and C are nonzero real numbers. This indicates that the conic has not been rotated.

Table 7.4.2

ellipse	$Ax^2+Cy^2+Dx+Ey+F=0$, $A\neq C$ and $AC>0$
circle	$Ax^2+Cy^2+Dx+Ey+F=0$, $A=C$
hyperbola	$Ax^2-Cy^2+Dx+Ey+F=0$ or $-Ax^2+Cy^2+Dx+Ey+F=0$, where A and C are positive
parabola	$Ax^2+Dx+Ey+F=0$ or $Cy^2+Dx+Ey+F=0$

How to: Given the equation of a conic, identify the type of conic

1. Rewrite the equation in the general form (Equation 7.4.1), $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$
2. Identify the values of A and C from the general form.
   - If A and C are nonzero, have the same sign, and are not equal to each other, then the graph may be an ellipse.
   - If A and C are equal and nonzero and have the same sign, then the graph may be a circle.
   - If A and C are nonzero and have opposite signs, then the graph may be a hyperbola.
   - If either A or C is zero, then the graph may be a parabola.

If $B=0$, the conic section will have a vertical and/or horizontal axes. If B does not equal 0, as shown below, the conic section is rotated. Notice the phrase "may be" in the definitions. That is because the equation may not represent a conic section at all, depending on the values of A, B, C, D, E, and F. For example, the degenerate case of a circle or an ellipse is a point:

$$Ax^2+By^2=0, \tag{7.4.2}$$

when A and B have the same sign.

The degenerate case of a hyperbola is two intersecting straight lines: $Ax^2+By^2=0$, when A and B have opposite signs.

On the other hand, the equation, $Ax^2+By^2+1=0$, when A and B are positive does not represent a graph at all, since there are no real ordered pairs which satisfy it.

Example 7.4.1: Identifying a Conic from Its General Form

Identify the graph of each of the following nondegenerate conic sections.

a. $4x^2-9y^2+36x+36y-125=0$
b. $9y^2+16x+36y-10=0$
c. $3x^2+3y^2-2x-6y-4=0$
d. $-25x^2-4y^2+100x+16y+20=0$

Solution

1. Rewriting the general form (Equation 7.4.1), we have

$$\begin{aligned} Ax^2+Bxy+Cy^2+Dx+Ey+F &= 0 \\ 4x^2+0xy+(-9)y^2+36x+36y+(-125) &= 0 \end{aligned}$$

   with $A=4$ and $C=-9$, so we observe that A and C have opposite signs. The graph of this equation is a hyperbola.
2. Rewriting the general form (Equation 7.4.1), we have

$$\begin{aligned} Ax^2+Bxy+Cy^2+Dx+Ey+F &= 0 \\ 0x^2+0xy+9y^2+16x+36y+(-10) &= 0 \end{aligned}$$

   with $A=0$ and $C=9$. We can determine that the equation is a parabola, since A is zero.
3. Rewriting the general form (Equation 7.4.1), we have

$$\begin{aligned} Ax^2+Bxy+Cy^2+Dx+Ey+F &= 0 \\ 3x^2+0xy+3y^2+(-2)x+(-6)y+(-4) &= 0 \end{aligned}$$

   with $A=3$ and $C=3$. Because $A=C$, the graph of this equation is a circle.
4. Rewriting the general form (Equation 7.4.1), we have

$$\begin{aligned} Ax^2+Bxy+Cy^2+Dx+Ey+F &= 0 \\ (-25)x^2+0xy+(-4)y^2+100x+16y+20 &= 0 \end{aligned}$$

   with $A=-25$ and $C=-4$. Because $AC>0$ and $A\neq C$, the graph of this equation is an ellipse.

Exercise 7.4.1

Identify the graph of each of the following nondegenerate conic sections.

a. $16y^2-x^2+x-4y-9=0$
b. $16x^2+4y^2+16x+49y-81=0$

Answer a

hyperbola

Answer b

ellipse

Finding a New Representation of the Given Equation after Rotating through a Given Angle

Until now, we have looked at equations of conic sections without an xy term, which aligns the graphs with the *x*- and *y*-axes. When we add an xy term, we are rotating the conic about the origin. If the *x*- and *y*-axes are rotated through an angle, say θ,then every point on the plane may be thought of as having two representations: (x, y) on the Cartesian plane with the original *x*-axis and *y*-axis, and (x', y') on the new plane defined by the new, rotated axes, called the *x*′-axis and *y*′-axis (Figure 7.4.3).

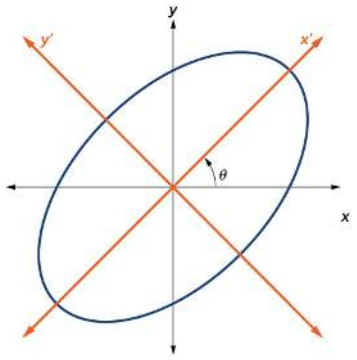


Figure 7.4.3: *The graph of the rotated ellipse* $x^2 + y^2 - xy - 15 = 0$

We will find the relationships between x and y on the Cartesian plane with x' and y' on the new rotated plane (Figure 7.4.4).

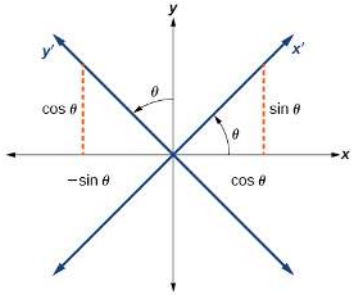


Figure 7.4.4: *The Cartesian plane with* x*- and* y*-axes and the resulting* $x'-$ *and* $y'-$*axes formed by a rotation by an angle* θ*.*

The original coordinate *x*- and *y*-axes have unit vectors $\hat{i}$ and $\hat{j}$. The rotated coordinate axes have unit vectors $\hat{i}'$ and $\hat{j}'$.The angle θ is known as the angle of rotation (Figure 7.4.5). We may write the new unit vectors in terms of the original ones.

$$\hat{i}' = \cos\theta\hat{i} + \sin\theta\hat{j} \tag{7.4.3}$$

$$\hat{j}' = -\sin\theta\hat{i} + \cos\theta\hat{j} \tag{7.4.4}$$

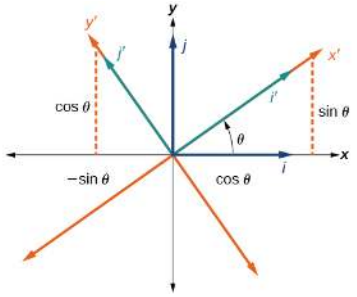


Figure 7.4.5: *Relationship between the old and new coordinate planes.*

Consider a vector $\vec{u}$ in the new coordinate plane. It may be represented in terms of its coordinate axes.

$$\begin{aligned}
\vec{u} &= x'i' + y'j' \\
&= x'(i\cos\theta + j\sin\theta) + y'(-i\sin\theta + j\cos\theta) && \text{Substitute.} \\
&= ix'\cos\theta + jx'\sin\theta - iy'\sin\theta + jy'\cos\theta && \text{Distribute.} \\
&= ix'\cos\theta - iy'\sin\theta + jx'\sin\theta + jy'\cos\theta && \text{Apply commutative property.} \\
&= (x'\cos\theta - y'\sin\theta)i + (x'\sin\theta + y'\cos\theta)j && \text{Factor by grouping.}
\end{aligned}$$

Because $\vec{u} = x'i' + y'j'$, we have representations of x and y in terms of the new coordinate system.

$$x = x'\cos\theta - y'\sin\theta$$

and

$$y = x'\sin\theta + y'\cos\theta$$

EQUATIONS OF ROTATION

If a point (x, y) on the Cartesian plane is represented on a new coordinate plane where the axes of rotation are formed by rotating an angle θ from the positive x-axis, then the coordinates of the point with respect to the new axes are (x', y'). We can use the following equations of rotation to define the relationship between (x, y) and (x', y'):

$$x = x' \cos\theta - y' \sin\theta \tag{7.4.5}$$

and

$$y = x' \sin\theta + y' \cos\theta \tag{7.4.6}$$

How to: Given the equation of a conic, find a new representation after rotating through an angle

1. Find x and y where $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$.
2. Substitute the expression for x and y into in the given equation, then simplify.
3. Write the equations with x' and y' in standard form.

Example 7.4.2: Finding a New Representation of an Equation after Rotating through a Given Angle

Find a new representation of the equation $2x^2 - xy + 2y^2 - 30 = 0$ after rotating through an angle of $\theta = 45^\circ$.

Solution

Find x and y, where $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$.

Because $\theta = 45^\circ$,

$$\begin{aligned} x &= x' \cos(45^\circ) - y' \sin(45^\circ) \\ x &= x' \left(\frac{1}{\sqrt{2}}\right) - y' \left(\frac{1}{\sqrt{2}}\right) \\ x &= \frac{x' - y'}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} y &= x' \sin(45^\circ) + y' \cos(45^\circ) \\ y &= x' \left(\frac{1}{\sqrt{2}}\right) + y' \left(\frac{1}{\sqrt{2}}\right) \\ y &= \frac{x' + y'}{\sqrt{2}} \end{aligned}$$

Substitute $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$ into $2x^2 - xy + 2y^2 - 30 = 0$.

$$2\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - \left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 2\left(\frac{x' + y'}{\sqrt{2}}\right)^2 - 30 = 0$$

Simplify.

$$2\frac{(x' - y')(x' - y')}{2} - \frac{(x' - y')(x' + y')}{2} + 2\frac{(x' + y')(x' + y')}{2} - 30 = 0 \quad \text{FOIL method}$$

$$x'^2 - 2x'y' + y'^2 - \frac{(x'^2 - y'^2)}{2} + x'^2 + 2x'y' + y'^2 - 30 = 0 \quad \text{Combine like terms.}$$

$$2x'^2 + 2y'^2 - \frac{(x'^2 - y'^2)}{2} = 30 \quad \text{Combine like terms.}$$

$$2(2x'^2 + 2y'^2 - \frac{(x'^2 - y'^2)}{2}) = 2(30) \quad \text{Multiply both sides by 2.}$$

$$4x'^2 + 4y'^2 - (x'^2 - y'^2) = 60 \quad \text{Simplify.}$$

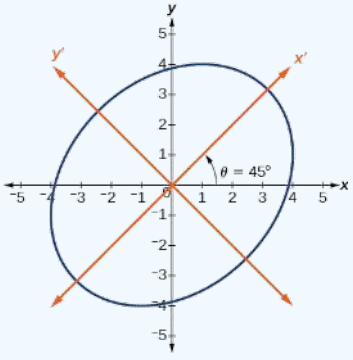
$$4x'^2 + 4y'^2 - x'^2 + y'2 = 60 \quad \text{Distribute.}$$

$$\frac{3x'^2}{60} + \frac{5y'^2}{60} = \frac{60}{60} \quad \text{Set equal to 1.}$$

Write the equations with x' and y' in the standard form.

$$\frac{x'^2}{20} + \frac{y'^2}{12} = 1$$

This equation is an ellipse. Figure 7.4.6 shows the graph.



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Figure **7.4.6**

Writing Equations of Rotated Conics in Standard Form

Now that we can find the standard form of a conic when we are given an angle of rotation, we will learn how to transform the equation of a conic given in the form $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ into standard form by rotating the axes. To do so, we will rewrite the general form as an equation in the x' and y' coordinate system without the $x'y'$ term, by rotating the axes by a measure of θ that satisfies

$$\cot(2\theta)=\frac{A-C}{B} \tag{7.4.7}$$

We have learned already that any conic may be represented by the second degree equation

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

where A, B,and C are not all zero. However, if $B\neq 0$, then we have an xy term that prevents us from rewriting the equation in standard form. To eliminate it, we can rotate the axes by an acute angle θ where $\cot(2\theta)=\dfrac{A-C}{B}$.

- If $\cot(2\theta)>0$, then 2θ is in the first quadrant, and θ is between $(0^\circ, 45^\circ)$.
- If $\cot(2\theta)<0$, then 2θ is in the second quadrant, and θ is between $(45^\circ, 90^\circ)$.
- If $A=C$, then $\theta=45^\circ$.

How to: Given an equation for a conic in the $x'y'$ system, rewrite the equation without the $x'y'$ term in terms of x' and y',where the x' and y' axes are rotations of the standard axes by θ degrees

1. Find $\cot(2\theta)$.
2. Find $\sin\theta$ and $\cos\theta$.
3. Substitute $\sin\theta$ and $\cos\theta$ into $x=x'\cos\theta-y'\sin\theta$ and $y=x'\sin\theta+y'\cos\theta$.
4. Substitute the expression for x and y into in the given equation, and then simplify.
5. Write the equations with x' and y' in the standard form with respect to the rotated axes.

Example 7.4.3: Rewriting an Equation with respect to the x' and y' axes without the $x'y'$ Term

Rewrite the equation $8x^2-12xy+17y^2=20$ in the $x'y'$ system without an $x'y'$ term.

Solution

First, we find $\cot(2\theta)$.

$$8x^2-12xy+17y^2=20 \rightarrow A=8,\ B=-12 \text{ and } C=17$$

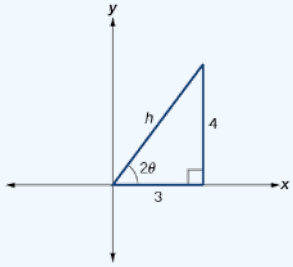


Figure **7.4.7**

From Figure 7.4.7:

$$\cot(2\theta)=\frac{A-C}{B}=\frac{8-17}{-12}$$
$$=\frac{-9}{-12}=\frac{3}{4}$$
$$\cot(2\theta)=\frac{3}{4}=\frac{\text{adjacent}}{\text{opposite}}$$

So the hypotenuse is

$$3^2+4^2=h^2$$
$$9+16=h^2$$
$$25=h^2$$
$$h=5$$

Next, we find $\sin\theta$ and $\cos\theta$.

$$\sin\theta=\sqrt{\frac{1-\cos(2\theta)}{2}}=\sqrt{\frac{1-\frac{3}{5}}{2}}=\sqrt{\frac{\frac{5}{5}-\frac{3}{5}}{2}}=\sqrt{\frac{5-3}{5}\cdot\frac{1}{2}}=\sqrt{\frac{2}{10}}=\sqrt{\frac{1}{5}}$$
$$\sin\theta=\frac{1}{\sqrt{5}}$$
$$\cos\theta=\sqrt{\frac{1+\cos(2\theta)}{2}}=\sqrt{\frac{1+\frac{3}{5}}{2}}=\sqrt{\frac{\frac{5}{5}+\frac{3}{5}}{2}}=\sqrt{\frac{5+3}{5}\cdot\frac{1}{2}}=\sqrt{\frac{8}{10}}=\sqrt{\frac{4}{5}}$$
$$\cos\theta=\frac{2}{\sqrt{5}}$$

Substitute the values of $\sin\theta$ and $\cos\theta$ into $x=x'\cos\theta-y'\sin\theta$ and $y=x'\sin\theta+y'\cos\theta$.

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ &= x'\left(\frac{2}{\sqrt{5}}\right) - y'\left(\frac{1}{\sqrt{5}}\right) \\ &= \frac{2x'-y'}{\sqrt{5}} \end{aligned}$$

and

$$\begin{aligned} y &= x'\sin\theta + y'\cos\theta \\ &= x'\left(\frac{1}{\sqrt{5}}\right) + y'\left(\frac{2}{\sqrt{5}}\right) \\ &= \frac{x'+2y'}{\sqrt{5}} \end{aligned}$$

Substitute the expressions for x and y into in the given equation, and then simplify.

$$8\left(\frac{2x'-y'}{\sqrt{5}}\right)^2 - 12\left(\frac{2x'-y'}{\sqrt{5}}\right)\left(\frac{x'+2y'}{\sqrt{5}}\right) + 17\left(\frac{x'+2y'}{\sqrt{5}}\right)^2 = 20$$

$$8\left(\frac{(2x'-y')(2x'-y')}{5}\right) - 12\left(\frac{(2x'-y')(x'+2y')}{5}\right) + 17\left(\frac{(x'+2y')(x'+2y')}{5}\right) = 20$$

$$8(4x'^2 - 4x'y' + y'^2) - 12(2x'^2 + 3x'y' - 2y'^2) + 17(x'^2 + 4x'y' + 4y'^2) = 100$$

$$32x'^2 - 32x'y' + 8y'^2 - 24x'^2 - 36x'y' + 24y'^2 + 17x'^2 + 68x'y' + 68y'^2 = 100$$

$$25x'^2 + 100y'^2 = 100$$

$$\frac{25}{100}x'^2 + \frac{100}{100}y'^2 = \frac{100}{100}$$

Write the equations with x' and y' in the standard form with respect to the new coordinate system.

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1$$

Figure 7.4.8 shows the graph of the ellipse.

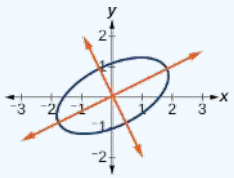


***Figure* 7.4.8**

? Exercise 7.4.2

Rewrite the $13x^2 - 6\sqrt{3}xy + 7y^2 = 16$ in the $x'y'$ system without the $x'y'$ term.

Answer

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1$$

✓ Example 7.4.4 :Graphing an Equation That Has No $x'y'$ Terms

Graph the following equation relative to the $x'y'$ system:

$$x^2 + 12xy - 4y^2 = 30$$

Solution

First, we find $\cot(2\theta)$.

$$x^2 + 12xy - 4y^2 = 20 \rightarrow A = 1\ ,\ B = 12, \text{and } C = -4$$

$$\cot(2\theta) = \frac{A-C}{B}$$

$$\cot(2\theta) = \frac{1-(-4)}{12}$$

$$\cot(2\theta) = \frac{5}{12}$$

Because $\cot(2\theta) = \frac{5}{12}$, we can draw a reference triangle as in Figure 7.4.9.

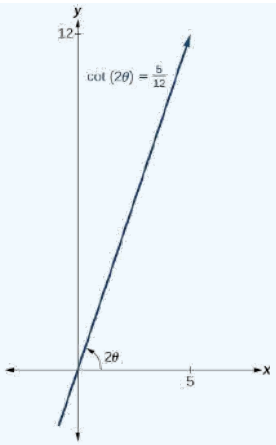


***Figure* 7.4.9**

$$\cot(2\theta) = \frac{5}{12} = \frac{adjacent}{opposite}$$

Thus, the hypotenuse is

$$\begin{aligned} 5^2 + 12^2 &= h^2 \\ 25 + 144 &= h^2 \\ 169 &= h^2 \\ h &= 13 \end{aligned}$$

Next, we find $\sin\theta$ and $\cos\theta$. We will use half-angle identities.

$$\sin\theta = \sqrt{\frac{1-\cos(2\theta)}{2}} = \sqrt{\frac{1-\frac{5}{13}}{2}} = \sqrt{\frac{\frac{13}{13}-\frac{5}{13}}{2}} = \sqrt{\frac{8}{13}\cdot\frac{1}{2}} = \frac{2}{\sqrt{13}}$$

$$\cos\theta = \sqrt{\frac{1+\cos(2\theta)}{2}} = \sqrt{\frac{1+\frac{5}{13}}{2}} = \sqrt{\frac{\frac{13}{13}+\frac{5}{13}}{2}} = \sqrt{\frac{18}{13}\cdot\frac{1}{2}} = \frac{3}{\sqrt{13}}$$

Now we find x and y.

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ x &= x'\left(\frac{3}{\sqrt{13}}\right) - y'\left(\frac{2}{\sqrt{13}}\right) \\ x &= \frac{3x'-2y'}{\sqrt{13}} \end{aligned}$$

and

$$\begin{aligned} y &= x'\sin\theta + y'\cos\theta \\ y &= x'\left(\frac{2}{\sqrt{13}}\right) + y'\left(\frac{3}{\sqrt{13}}\right) \\ y &= \frac{2x'+3y'}{\sqrt{13}} \end{aligned}$$

Now we substitute $x = \frac{3x'-2y'}{\sqrt{13}}$ and $y = \frac{2x'+3y'}{\sqrt{13}}$ into $x^2 + 12xy - 4y^2 = 30$.

$$\left(\frac{3x'-2y'}{\sqrt{13}}\right)^2 + 12\left(\frac{3x'-2y'}{\sqrt{13}}\right)\left(\frac{2x'+3y'}{\sqrt{13}}\right) - 4\left(\frac{2x'+3y'}{\sqrt{13}}\right)^2 = 30$$

$$\left(\frac{1}{13}\right)\left[(3x'-2y')^2 + 12(3x'-2y')(2x'+3y') - 4(2x'+3y')^2\right] = 30 \quad \text{Factor.}$$

$$\left(\frac{1}{13}\right)\left[9x'^2 - 12x'y' + 4y'^2 + 12(6x'^2 + 5x'y' - 6y'^2) - 4(4x'^2 + 12x'y' + 9y'^2)\right] = 30 \quad \text{Multiply.}$$

$$\left(\frac{1}{13}\right)\left[9x'^2 - 12x'y' + 4y'^2 + 72x'^2 + 60x'y' - 72y'^2 - 16x'^2 - 48x'y' - 36y'^2\right] = 30 \quad \text{Distribute.}$$

$$\left(\frac{1}{13}\right)\left[65x'^2 - 104y'^2\right] = 30 \quad \text{Combine like terms.}$$

$$65x'^2 - 104y'^2 = 390 \quad \text{Multiply.}$$

$$\frac{x'^2}{6} - \frac{4y'^2}{15} = 1 \quad \text{Divide by 390.}$$

Figure 7.4.10 shows the graph of the hyperbola $\frac{x'^2}{6} - \frac{4y'^2}{15} = 1$

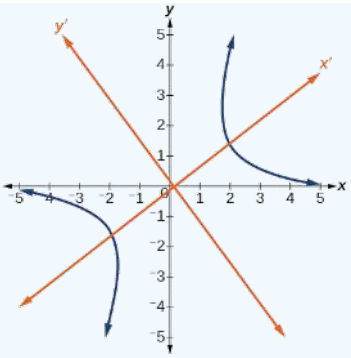


***Figure* 7.4.10**

Identifying Conics without Rotating Axes

Now we have come full circle. How do we identify the type of conic described by an equation? What happens when the axes are rotated? Recall, the general form of a conic is

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

If we apply the rotation formulas to this equation we get the form

$$A'x'^2+B'x'y'+C'y'^2+D'x'+E'y'+F'=0$$

It may be shown that

$$B^2-4AC=B'^2-4A'C'$$

The expression does not vary after rotation, so we call the expression *invariant*. The discriminant, B^2-4AC, is invariant and remains unchanged after rotation. Because the discriminant remains unchanged, observing the discriminant enables us to identify the conic section.

HOWTO: USING THE DISCRIMINANT TO IDENTIFY A CONIC

If the equation

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \tag{7.4.8}$$

is transformed by rotating axes into the equation

$$A'x'^2+B'x'y'+C'y'^2+D'x'+E'y'+F'=0 \tag{7.4.9}$$

then

$$B^2-4AC=B'^2-4A'C' \tag{7.4.10}$$

The equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ is an ellipse, a parabola, or a hyperbola, or a degenerate case of one of these. If the discriminant, B^2-4AC, is

- <0, the conic section is an ellipse
- $=0$, the conic section is a parabola
- >0, the conic section is a hyperbola

Example 7.4.5: Identifying the Conic without Rotating Axes

Identify the conic for each of the following without rotating axes.

a. $5x^2+2\sqrt{3}xy+2y^2-5=0$
b. $5x^2+2\sqrt{3}xy+12y^2-5=0$

Solution

a. Let's begin by determining A, B, and C.

$$\underbrace{5}_{A}x^2+\underbrace{2\sqrt{3}}_{B}xy+\underbrace{2}_{C}y^2-5=0$$

Now, we find the discriminant.

$$\begin{aligned}B^2-4AC&=(2\sqrt{3})^2-4(5)(2)\\&=4(3)-40\\&=12-40\\&=-28<0\end{aligned}$$

Therefore, $5x^2+2\sqrt{3}xy+2y^2-5=0$ represents an ellipse.

b. Again, let's begin by determining A,B, and C.

$$\underbrace{5}_{A}x^2+\underbrace{2\sqrt{3}}_{B}xy+\underbrace{12}_{C}y^2-5=0$$

Now, we find the discriminant.

$$\begin{aligned}B^2-4AC&=(2\sqrt{3})^2-4(5)(12)\\&=4(3)-240\\&=12-240\\&=-228<0\end{aligned}$$

Therefore, $5x^2+2\sqrt{3}xy+12y^2-5=0$ represents an ellipse.

? Exercise 7.4.3

Identify the conic for each of the following without rotating axes.

a. $x^2 - 9xy + 3y^2 - 12 = 0$
b. $10x^2 - 9xy + 4y^2 - 4 = 0$

Answer a

hyperbola

Answer b

ellipse

Key Equations

General Form equation of a conic section	$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
Rotation of a conic section	$x = x' \cos\theta - y' \sin\theta$ $y = x' \sin\theta + y' \cos\theta$
Angle of rotation	θ, where $\cot(2\theta) = \dfrac{A - C}{B}$

Key Concepts

- Four basic shapes can result from the intersection of a plane with a pair of right circular cones connected tail to tail. They include an ellipse, a circle, a hyperbola, and a parabola.
- A nondegenerate conic section has the general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where A, B and C are not all zero. The values of A, B, and C determine the type of conic. See Example 7.4.1.
- Equations of conic sections with an xy term have been rotated about the origin. See Example 7.4.2.
- The general form can be transformed into an equation in the x' and y' coordinate system without the $x'y'$ term. See Example 7.4.3 and Example 7.4.4.
- An expression is described as invariant if it remains unchanged after rotating. Because the discriminant is invariant, observing it enables us to identify the conic section. See Example 7.4.5.

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7.5: Conic Sections in Polar Coordinates

Learning Objectives

- Identify a conic in polar form.
- Graph the polar equations of conics.
- Define conics in terms of a focus and a directrix.

Most of us are familiar with orbital motion, such as the motion of a planet around the sun or an electron around an atomic nucleus. Within the planetary system, orbits of planets, asteroids, and comets around a larger celestial body are often elliptical. Comets, however, may take on a parabolic or hyperbolic orbit instead. And, in reality, the characteristics of the planets' orbits may vary over time. Each orbit is tied to the location of the celestial body being orbited and the distance and direction of the planet or other object from that body. As a result, we tend to use polar coordinates to represent these orbits.

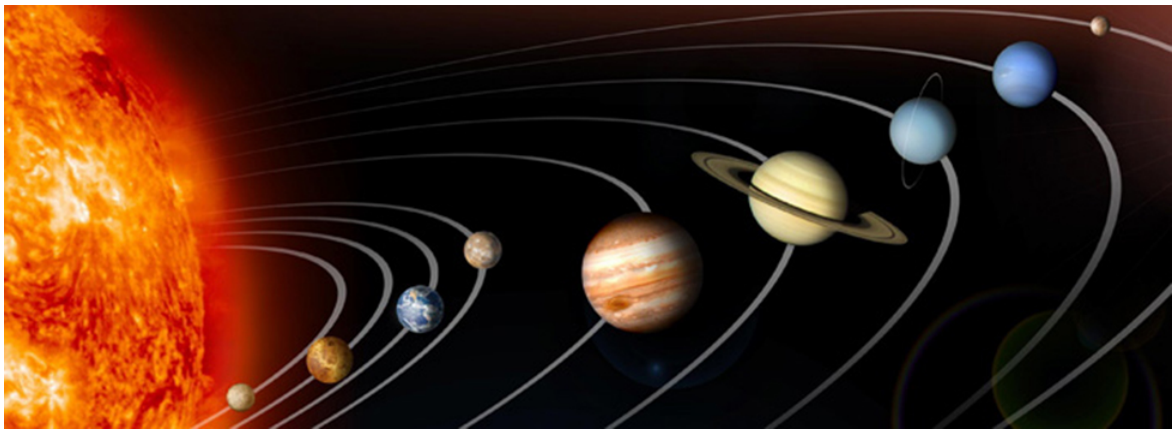


Figure 7.5.1: Planets orbiting the sun follow elliptical paths. (credit: NASA Blueshift, Flickr)

In an elliptical orbit, the **periapsis** is the point at which the two objects are closest, and the **apoapsis** is the point at which they are farthest apart. Generally, the velocity of the orbiting body tends to increase as it approaches the periapsis and decrease as it approaches the apoapsis. Some objects reach an escape velocity, which results in an infinite orbit. These bodies exhibit either a parabolic or a hyperbolic orbit about a body; the orbiting body breaks free of the celestial body's gravitational pull and fires off into space. Each of these orbits can be modeled by a conic section in the polar coordinate system.

Identifying a Conic in Polar Form

Any conic may be determined by three characteristics: a single **focus**, a fixed line called the **directrix**, and the ratio of the distances of each to a point on the graph. Consider the **parabola** $x = 2 + y^2$ shown in Figure 7.5.2.

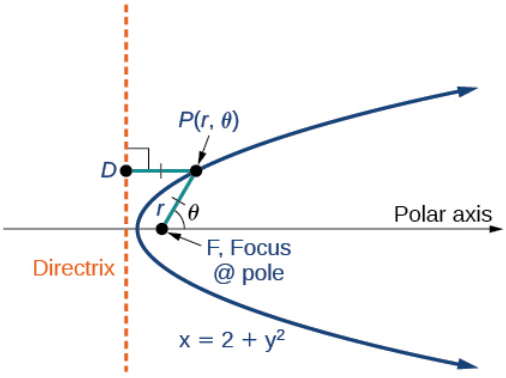


Figure 7.5.2

We previously learned how a parabola is defined by the focus (a fixed point) and the directrix (a fixed line). In this section, we will learn how to define any conic in the polar coordinate system in terms of a fixed point, the focus $P(r, \theta)$ at the pole, and a line, the directrix, which is perpendicular to the polar axis.

If F is a fixed point, the focus, and D is a fixed line, the directrix, then we can let e be a fixed positive number, called the **eccentricity**, which we can define as the ratio of the distances from a point on the graph to the focus and the point on the graph to the directrix. Then the set of all points P such that $e = \dfrac{PF}{PD}$ is a conic. In other words, we can define a conic as the set of all points P with the property that the ratio of the distance from P to F to the distance from P to D is equal to the constant e.

For a conic with eccentricity e,

- if $0 \le e < 1$, the conic is an ellipse
- if $e = 1$, the conic is a parabola
- if $e > 1$, the conic is an hyperbola

With this definition, we may now define a conic in terms of the directrix, $x = \pm p$, the eccentricity e, and the angle θ. Thus, each conic may be written as a **polar equation**, an equation written in terms of r and θ.

THE POLAR EQUATION FOR A CONIC

For a conic with a focus at the origin, if the directrix is $x = \pm p$, where p is a positive real number, and the eccentricity is a positive real number e, the conic has a polar equation

$$r = \frac{ep}{1 \pm e\cos\theta} \tag{7.5.1}$$

For a conic with a focus at the origin, if the directrix is $y = \pm p$, where p is a positive real number, and the eccentricity is a positive real number e, the conic has a polar equation

$$r = \frac{ep}{1 \pm e\sin\theta} \tag{7.5.2}$$

How to: Given the polar equation for a conic, identify the type of conic, the directrix, and the eccentricity.

1. Multiply the numerator and denominator by the reciprocal of the constant in the denominator to rewrite the equation in standard form.
2. Identify the eccentricity e as the coefficient of the trigonometric function in the denominator.
3. Compare e with 1 to determine the shape of the conic.
4. Determine the directrix as $x = p$ if cosine is in the denominator and $y = p$ if sine is in the denominator. Set ep equal to the numerator in standard form to solve for x or y.

Example 7.5.1: Identifying a Conic Given the Polar Form

For each of the following equations, identify the conic with focus at the origin, the directrix, and the eccentricity.

a. $r = \dfrac{6}{3 + 2\sin\theta}$

b. $r = \dfrac{12}{4 + 5\cos\theta}$

c. $r = \dfrac{7}{2 - 2\sin\theta}$

Solution

For each of the three conics, we will rewrite the equation in standard form. Standard form has a 1 as the constant in the denominator. Therefore, in all three parts, the first step will be to multiply the numerator and denominator by the reciprocal of the constant of the original equation, $\dfrac{1}{c}$, where c is that constant.

a. Multiply the numerator and denominator by $\dfrac{1}{3}$.

$$r=\frac{6}{3+2\sin\theta}\cdot\frac{\left(\frac{1}{3}\right)}{\left(\frac{1}{3}\right)}=\frac{6\left(\frac{1}{3}\right)}{3\left(\frac{1}{3}\right)+2\left(\frac{1}{3}\right)\sin\theta}=\frac{2}{1+\frac{2}{3}\sin\theta}$$

Because $\sin\theta$ is in the denominator, the directrix is $y=p$. Comparing to standard form, note that $e=\frac{2}{3}$ .Therefore, from the numerator,

$$\begin{aligned}2&=ep\\2&=\frac{2}{3}p\\\left(\frac{3}{2}\right)2&=\left(\frac{3}{2}\right)\frac{2}{3}p\\3&=p\end{aligned}$$

Since $e<1$, the conic is an **ellipse**. The eccentricity is $e=\frac{2}{3}$ and the directrix is $y=3$.

b. Multiply the numerator and denominator by $\frac{1}{4}$.

$$\begin{aligned}r&=\frac{12}{4+5\cos\theta}\cdot\frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{4}\right)}\\r&=\frac{12\left(\frac{1}{4}\right)}{4\left(\frac{1}{4}\right)+5\left(\frac{1}{4}\right)\cos\theta}\\r&=\frac{3}{1+\frac{5}{4}\cos\theta}\end{aligned}$$

Because $\cos\theta$ is in the denominator, the directrix is $x=p$. Comparing to standard form, $e=\frac{5}{4}$. Therefore, from the numerator,

$$\begin{aligned}3&=ep\\3&=\frac{5}{4}p\\\left(\frac{4}{5}\right)3&=\left(\frac{4}{5}\right)\frac{5}{4}p\\\frac{12}{5}&=p\end{aligned}$$

Since $e>1$, the conic is a **hyperbola**. The eccentricity is $e=\frac{5}{4}$ and the directrix is $x=\frac{12}{5}=2.4$.

c. Multiply the numerator and denominator by $\frac{1}{2}$.

$$r = \frac{7}{2-2\sin\theta} \cdot \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}$$

$$r = \frac{7\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)\sin\theta}$$

$$r = \frac{\frac{7}{2}}{1-\sin\theta}$$

Because sine is in the denominator, the directrix is $y = -p$. Comparing to standard form, $e = 1$. Therefore, from the numerator,

$$\frac{7}{2} = ep$$
$$\frac{7}{2} = (1)p$$
$$\frac{7}{2} = p$$

Because $e = 1$, the conic is a parabola. The eccentricity is $e = 1$ and the directrix is $y = -\frac{7}{2} = -3.5$.

? Exercise 7.5.1

Identify the conic with focus at the origin, the directrix, and the eccentricity for $r = \frac{2}{3-\cos\theta}$.

Answer

ellipse; $e = \frac{1}{3}$; $x = -2$

Graphing the Polar Equations of Conics

When graphing in Cartesian coordinates, each conic section has a unique equation. This is not the case when graphing in polar coordinates. We must use the eccentricity of a conic section to determine which type of curve to graph, and then determine its specific characteristics. The first step is to rewrite the conic in standard form as we have done in the previous example. In other words, we need to rewrite the equation so that the denominator begins with 1. This enables us to determine e and, therefore, the shape of the curve. The next step is to substitute values for θ and solve for r to plot a few key points. Setting θ equal to 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ provides the vertices so we can create a rough sketch of the graph.

✓ Example 7.5.2A: Graphing a Parabola in Polar Form

Graph $r = \frac{5}{3+3\cos\theta}$.

Solution

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 3, which is $\frac{1}{3}$.

$$r=\frac{5}{3+3\cos\theta}=\frac{5\left(\frac{1}{3}\right)}{3\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)\cos\theta}$$

$$r=\frac{\frac{5}{3}}{1+\cos\theta}$$

Because $e=1$,we will graph a **parabola** with a focus at the origin. The function has a $\cos\theta$, and there is an addition sign in the denominator, so the directrix is $x=p$.

$$\frac{5}{3}=ep$$
$$\frac{5}{3}=(1)p$$
$$\frac{5}{3}=p$$

The directrix is $x=\frac{5}{3}$.

Plotting a few key points as in Table 7.5.1 will enable us to see the vertices. See Figure 7.5.3.

Table 7.5.1

	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r=\frac{5}{3+3\cos\theta}$	$\frac{5}{6}\approx 0.83$	$\frac{5}{3}\approx 1.67$	undefined	$\frac{5}{3}\approx 1.67$

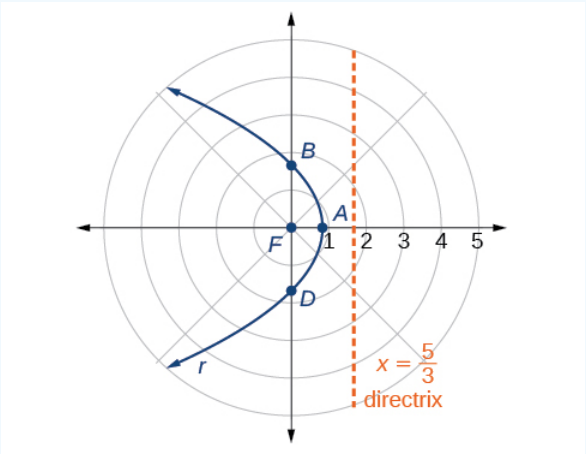


Figure 7.5.3

We can check our result with a graphing utility. See Figure 7.5.4.

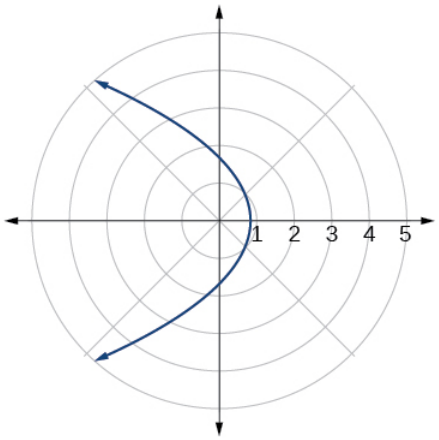


Figure 7.5.4

Example 7.5.2B: Graphing a Hyperbola in Polar Form

Graph $r = \dfrac{8}{2 - 3\sin\theta}$.

Solution

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 2, which is $\frac{1}{2}$.

$$r = \frac{8}{2 - 3\sin\theta} = \frac{8\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}\right) - 3\left(\frac{1}{2}\right)\sin\theta}$$

$$r = \frac{4}{1 - \frac{3}{2}\sin\theta}$$

Because $e = \frac{3}{2}$, $e > 1$, so we will graph a hyperbola with a focus at the origin. The function has a $\sin\theta$ term and there is a subtraction sign in the denominator, so the directrix is $y = -p$.

$$\begin{aligned} 4 &= ep \\ 4 &= \left(\frac{3}{2}\right)p \\ 4\left(\frac{2}{3}\right) &= p \\ \frac{8}{3} &= p \end{aligned}$$

The directrix is $y = -\frac{8}{3}$.

Plotting a few key points as in Table 7.5.2 will enable us to see the vertices. See Figure 7.5.5.

Table 7.5.2

	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{8}{2 - 3\sin\theta}$	4	-8	4	$\frac{8}{5} = 1.6$

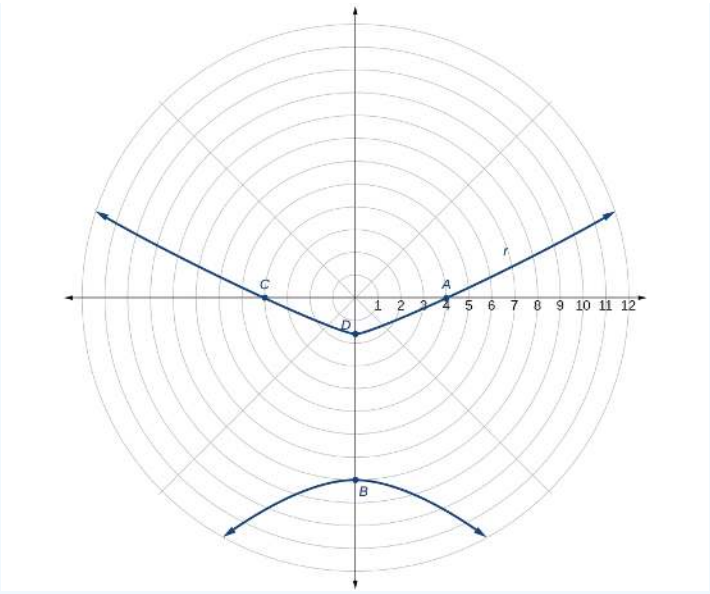


Figure 7.5.5

Example 7.5.2C: Graphing an Ellipse in Polar Form

Graph $r = \dfrac{10}{5 - 4\cos\theta}$.

Solution

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 5, which is $\frac{1}{5}$.

$$r = \frac{10}{5 - 4\cos\theta} = \frac{10\left(\frac{1}{5}\right)}{5\left(\frac{1}{5}\right) - 4\left(\frac{1}{5}\right)\cos\theta}$$

$$r = \frac{2}{1 - \frac{4}{5}\cos\theta}$$

Because $e = \frac{4}{5}$, $e < 1$, so we will graph an **ellipse** with a **focus** at the origin. The function has a $\cos\theta$, and there is a subtraction sign in the denominator, so the **directrix** is $x = -p$.

$$2 = ep$$
$$2 = \left(\frac{4}{5}\right)p$$
$$2\left(\frac{5}{4}\right) = p$$
$$\frac{5}{2} = p$$

The directrix is $x = -\frac{5}{2}$.

Plotting a few key points as in Table 7.5.3 will enable us to see the vertices. See Figure 7.5.6.

Table 7.5.3

	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

	A	B	C	D
$r = \dfrac{10}{5 - 4\cos\theta}$	10	2	$\dfrac{10}{9} \approx 1.1$	2

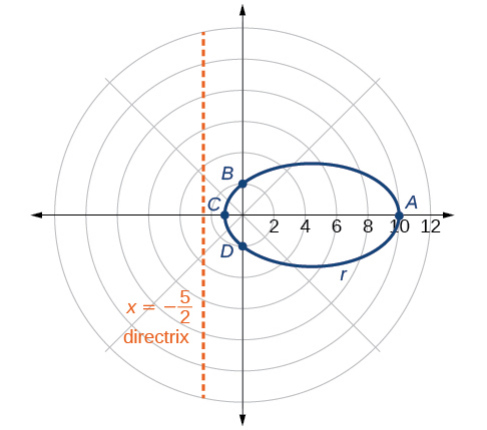


Figure 7.5.6

Analysis

We can check our result using a graphing utility. See Figure 7.5.7.

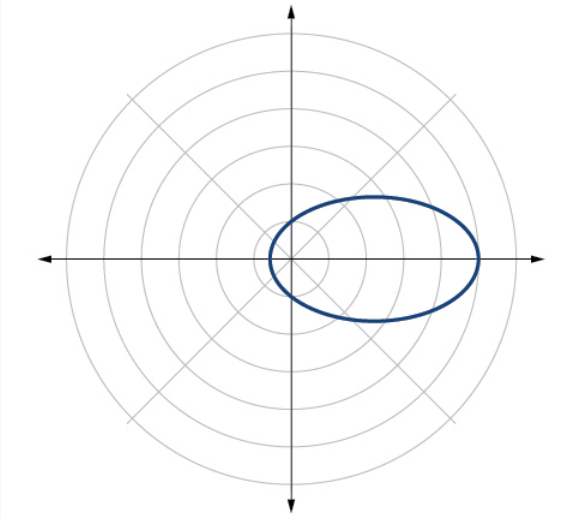


Figure 7.5.6: $r = \dfrac{10}{5 - 4\cos\theta}$ graphed on a viewing window of $[-3, 12, 1]$ by $[-4, 4, 1]$, $\theta min = 0$ and $\theta max = 2\pi$.

? Exercise 7.5.2

Graph $r = \dfrac{2}{4 - \cos\theta}$.

Answer

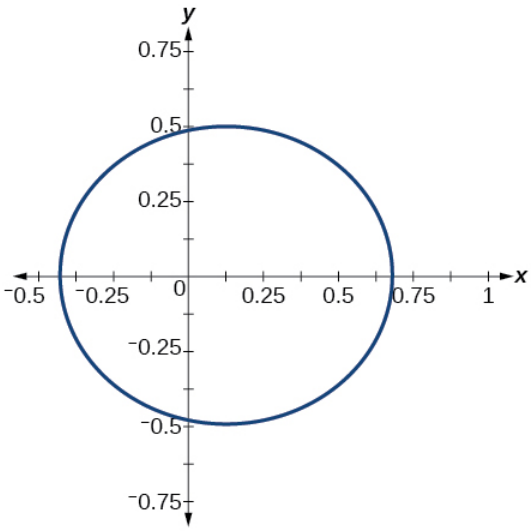


Figure 7.5.7

Defining Conics in Terms of a Focus and a Directrix

So far we have been using polar equations of conics to describe and graph the curve. Now we will work in reverse; we will use information about the origin, eccentricity, and directrix to determine the polar equation.

How to: Given the focus, eccentricity, and directrix of a conic, determine the polar equation

1. Determine whether the directrix is horizontal or vertical. If the directrix is given in terms of y, we use the general polar form in terms of sine. If the directrix is given in terms of x, we use the general polar form in terms of cosine.
2. Determine the sign in the denominator. If $p < 0$, use subtraction. If $p > 0$, use addition.
3. Write the coefficient of the trigonometric function as the given eccentricity.
4. Write the absolute value of p in the numerator, and simplify the equation.

Example 7.5.3A: Finding the Polar Form of a Vertical Conic Given a Focus at the Origin and the Eccentricity and Directrix

Find the polar form of the conic given a focus at the origin, $e = 3$ and directrix $y = -2$.

Solution

The directrix is $y = -p$, so we know the trigonometric function in the denominator is sine.

Because $y = -2, -2 < 0$, so we know there is a subtraction sign in the denominator. We use the standard form of

$$r = \frac{ep}{1 - e\sin\theta}$$

and $e = 3$ and $|-2| = 2 = p$.

Therefore,

$$r = \frac{(3)(2)}{1 - 3\sin\theta}$$
$$r = \frac{6}{1 - 3\sin\theta}$$

Example 7.5.3B: Finding the Polar Form of a Horizontal Conic Given a Focus at the Origin and the Eccentricity and Directrix

Find the polar form of a conic given a focus at the origin, $e = \frac{3}{5}$, and directrix $x = 4$.

Solution

Because the directrix is $x = p$, we know the function in the denominator is cosine. Because $x = 4$, $4 > 0$, so we know there is an addition sign in the denominator. We use the standard form of

$$r = \frac{ep}{1 + e\cos\theta}$$

and $e = \frac{3}{5}$ and $|4| = 4 = p$.

Therefore,

$$r = \frac{\left(\frac{3}{5}\right)(4)}{1 + \frac{3}{5}\cos\theta}$$

$$r = \frac{\frac{12}{5}}{1 + \frac{3}{5}\cos\theta}$$

$$r = \frac{\frac{12}{5}}{1\left(\frac{5}{5}\right) + \frac{3}{5}\cos\theta}$$

$$r = \frac{\frac{12}{5}}{\frac{5}{5} + \frac{3}{5}\cos\theta}$$

$$r = \frac{12}{5} \cdot \frac{5}{5 + 3\cos\theta}$$

$$r = \frac{12}{5 + 3\cos\theta}$$

Exercise 7.5.3

Find the polar form of the conic given a focus at the origin, $e = 1$, and directrix $x = -1$.

Answer

$$r = \frac{1}{1 - \cos\theta}$$

Example 7.5.4: Converting a Conic in Polar Form to Rectangular Form

Convert the conic $r = \frac{1}{5 - 5\sin\theta}$ to rectangular form.

Solution

We will rearrange the formula to use the identities $r = \sqrt{x^2 + y^2}$, $x = r\cos\theta$,and $y = r\sin\theta$.

$$
\begin{aligned}
r &= \frac{1}{5-5\sin\theta} \\
r\cdot(5-5\sin\theta) &= \frac{1}{5-5\sin\theta}\cdot(5-5\sin\theta) && \text{Eliminate the fraction.} \\
5r-5r\sin\theta &= 1 && \text{Distribute.} \\
5r &= 1+5r\sin\theta && \text{Isolate } 5r. \\
25r^2 &= (1+5r\sin\theta)^2 && \text{Square both sides.} \\
25(x^2+y^2) &= (1+5y)^2 && \text{Substitute } r=\sqrt{x^2+y^2} \text{ and } y=r\sin\theta. \\
25x^2+25y^2 &= 1+10y+25y^2 && \text{Distribute and use FOIL.} \\
25x^2-10y &= 1 && \text{Rearrange terms and set equal to 1.}
\end{aligned}
$$

Exercise 7.5.4

Convert the conic $r=\dfrac{2}{1+2\cos\theta}$ to rectangular form.

Answer

$4-8x+3x^2-y^2=0$

Media

Access these online resources for additional instruction and practice with conics in polar coordinates.

- Polar Equations of Conic Sections
- Graphing Polar Equations of Conics - 1
- Graphing Polar Equations of Conics - 2

Visit this website for additional practice questions from Learningpod.

Key Concepts

- Any conic may be determined by a single focus, the corresponding eccentricity, and the directrix. We can also define a conic in terms of a fixed point, the focus $P(r,\theta)$ at the pole, and a line, the directrix, which is perpendicular to the polar axis.
- A conic is the set of all points $e=\dfrac{PF}{PD}$, where eccentricity e is a positive real number. Each conic may be written in terms of its polar equation. See Example 7.5.1.
- The polar equations of conics can be graphed. See Example 7.5.2, Example 7.5.3, and Example 7.5.4.
- Conics can be defined in terms of a focus, a directrix, and eccentricity. See Example 7.5.5 and Example 7.5.6.
- We can use the identities $r=\sqrt{x^2+y^2}$, $x=r\cos\theta$, and $y=r\sin\theta$ to convert the equation for a conic from polar to rectangular form. See Example 7.5.7.

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7.E: Analytic Geometry (Exercises)

10.1: The Ellipse

In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.

Verbal

1) Define an ellipse in terms of its foci.

Answer

An ellipse is the set of all points in the plane the sum of whose distances from two fixed points, called the foci, is a constant.

2) Where must the foci of an ellipse lie?

3) What special case of the ellipse do we have when the major and minor axis are of the same length?

Answer

This special case would be a circle.

4) For the special case mentioned above, what would be true about the foci of that ellipse?

5) What can be said about the symmetry of the graph of an ellipse with center at the origin and foci along the y-axis?

Answer

It is symmetric about the x-axis, y-axis, and the origin.

Algebraic

For the exercises 6-10, determine whether the given equations represent ellipses. If yes, write in standard form.

6) $2x^2+y=4$

7) $4x^2+9y^2=36$

Answer

yes; $\dfrac{x^2}{3^2}+\dfrac{y^2}{2^2}=1$

8) $4x^2-y^2=4$

9) $4x^2+9y^2=1$

Answer

yes; $\dfrac{x^2}{\left(\frac{1}{2}\right)^2}+\dfrac{y^2}{\left(\frac{1}{2}\right)^2}=1$

10) $4x^2-8x+9y^2-72y+112=0$

For the exercises 11-26, write the equation of an ellipse in standard form, and identify the end points of the major and minor axes as well as the foci.

11) $\dfrac{x^2}{4}+\dfrac{y^2}{49}=1$

Answer

$\frac{x^2}{2^2}+\frac{y^2}{7^2}=1$; Endpoints of major axis $(0,7)$ and $(0,-7)$. Endpoints of minor axis $(2,0)$ and $(-2,0)$. Foci at $(0,3\sqrt{5})$, $(0,-3\sqrt{5})$.

12) $\frac{x^2}{100}+\frac{y^2}{64}=1$

13) $x^2+9y^2=1$

Answer

$\frac{x^2}{(1)^2}+\frac{y^2}{\left(\frac{1}{3}\right)^2}=1$; Endpoints of major axis $(1,0)$ and $(-1,0)$. Endpoints of minor axis $\left(0,\frac{1}{3}\right)$, $\left(0,-\frac{1}{3}\right)$. Foci at $\left(\frac{2\sqrt{2}}{3},0\right)$, $\left(-\frac{2\sqrt{2}}{3},0\right)$.

14) $4x^2+16y^2=1$

15) $\frac{(x-2)^2}{49}+\frac{(y-4)^2}{25}=1$

Answer

$\frac{(x-2)^2}{7^2}+\frac{(y-4)^2}{5^2}=1$; Endpoints of major axis $(9,4)$, $(-5,4)$. Endpoints of minor axis $(2,9)$, $(2,-1)$. Foci at $(2+2\sqrt{6},4)$, $(2-2\sqrt{6},4)$

16) $\frac{(x-2)^2}{81}+\frac{(y+1)^2}{16}=1$

17) $\frac{(x+5)^2}{4}+\frac{(y-7)^2}{9}=1$

Answer

$\frac{(x+5)^2}{2^2}+\frac{(y-7)^2}{3^2}=1$; Endpoints of major axis $(-5,10)$, $(-5,4)$. Endpoints of minor axis $(-3,7)$, $(-7,7)$. Foci at $(-5,7+\sqrt{5})$, $(-5,7-\sqrt{5})$

18) $\frac{(x-7)^2}{49}+\frac{(y-7)^2}{49}=1$

19) $4x^2-8x+9y^2-72y+112=0$

Answer

$\frac{(x-1)^2}{3^2}+\frac{(y-4)^2}{2^2}=1$; Endpoints of major axis $(4,4)$, $(-2,4)$. Endpoints of minor axis $(1,6)$, $(1,2)$. Foci at $(1+\sqrt{5},4)$, $(1-\sqrt{5},4)$

20) $9x^2-54x+9y^2-54y+81=0$

21) $4x^2-24x+36y^2-360y+864=0$

Answer

$\frac{(x-3)^2}{(3\sqrt{2})^2}+\frac{(y-5)^2}{\sqrt{2}^2}=1$; Endpoints of major axis $(3+3\sqrt{2},5)$, $(3-3\sqrt{2},5)$. Endpoints of minor axis $(3,5+\sqrt{2})$, $(3,5-\sqrt{2})$. Foci at $(7,5)$, $(-1,5)$

22) $4x^2+24x+16y^2-128y+228=0$

23) $4x^2+40x+25y^2-100y+100=0$

Answer

$\frac{(x+5)^2}{(5)^2}+\frac{(y-2)^2}{(2)^2}=1$; Endpoints of major axis $(0,2)$, $(-10,2)$. Endpoints of minor axis $(-5,4)$, $(-5,0)$. Foci at $(-5+\sqrt{21},2)$, $(-5-\sqrt{21},2)$

24) $x^2+2x+100y^2-1000y+2401=0$

25) $4x^2+24x+25y^2+200y+336=0$

Answer

$\frac{(x+3)^2}{(5)^2}+\frac{(y+4)^2}{(2)^2}=1$; Endpoints of major axis $(2,-4)$, $(-8,-4)$. Endpoints of minor axis $(-3,-2)$, $(-3,-6)$. Foci at $(-3+\sqrt{21},-4)$, $(-3-\sqrt{21},-4)$

26) $9x^2+72x+16y^2+16y+4=0$

For the exercises 27-31, find the foci for the given ellipses.

27) $\frac{(x+3)^2}{25}+\frac{(y+1)^2}{36}=1$

Answer

Foci $(-3,-1+\sqrt{11})$, $(-3,-1-\sqrt{11})$

28) $\frac{(x+1)^2}{100}+\frac{(y-2)^2}{4}=1$

29) $x^2+y^2=1$

Answer

Focus $(0,0)$

30) $x^2+4y^2+4x+8y=1$

31) $10x^2+y^2+200x=0$

Answer

Foci $(-10,30)$, $(-10,-30)$

Graphical

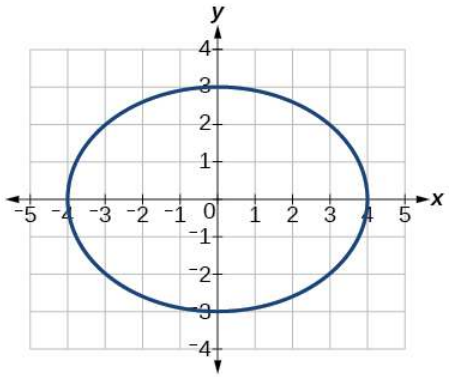
For the exercises 32-45, graph the given ellipses, noting center, vertices, and foci.

32) $\frac{x^2}{25}+\frac{y^2}{36}=1$

33) $\frac{x^2}{16}+\frac{y^2}{9}=1$

Answer

Center $(0,0)$, Vertices $(4,0)$, $(-4,0)$, $(0,3)$, $(0,-3)$. Foci $(\sqrt{7},0)$, $(-\sqrt{7},0)$

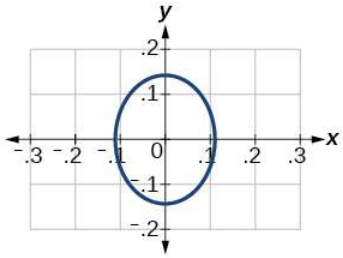


34) $4x^2+9y^2=1$

35) $81x^2+49y^2=1$

Answer

Center $(0,0)$, Vertices $\left(\frac{1}{9},0\right)$, $\left(-\frac{1}{9},0\right)$, $\left(0,\frac{1}{7}\right)$, $\left(0,\frac{1}{7}\right)$. Foci $\left(0,\frac{4\sqrt{2}}{63}\right)$, $\left(0,-\frac{4\sqrt{2}}{63}\right)$



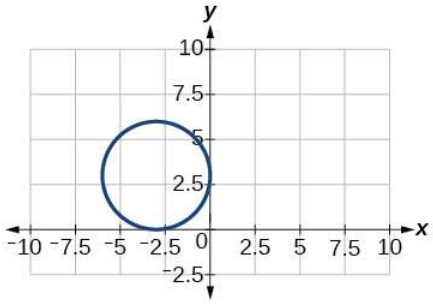
36) $\frac{(x-2)^2}{64}+\frac{(y-4)^2}{16}=1$

37) $\frac{(x+3)^2}{9}+\frac{(y-3)^2}{9}=1$

Answer

Center $(-3,3)$, Vertices $(0,3)$, $(-6,3)$, $(-3,0)$, $(-3,6)$. Focus $(-3,3)$

Note that this ellipse is a circle. The circle has only one focus, which coincides with the center.

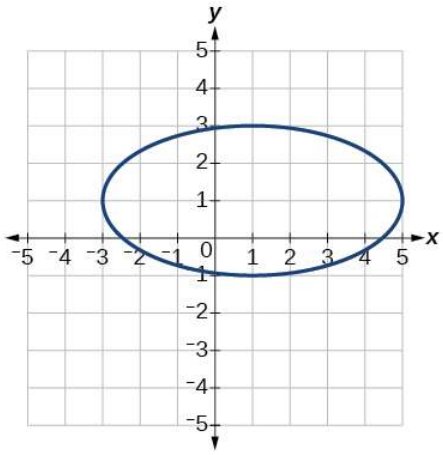


38) $\frac{x^2}{2}+\frac{(y+1)^2}{5}=1$

39) $4x^2-8x+16y^2-32y-44=0$

Answer

Center $(1,1)$, Vertices $(5,1)$, $(-3,1)$, $(1,3)$, $(1,-1)$. Foci $(1,1+4\sqrt{3})$, $(1,1-4\sqrt{3})$

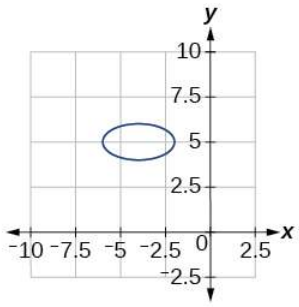


40) $x^2-8x+25y^2-100y+91=0$

41) $x^2+8x+4y^2-40y+112=0$

Answer

Center $(-4,5)$, Vertices $(-2,5)$, $(-6,4)$, $(-4,6)$, $(-4,4)$. Foci $(-4+\sqrt{3},5)$, $(-4-4\sqrt{3},5)$

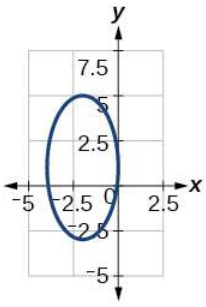


42) $64x^2+128x+9y^2-72y-368=0$

43) $16x^2+64x+4y^2-8y+4=0$

Answer

Center $(-2,1)$, Vertices $(0,1)$, $(-4,1)$, $(-2,5)$, $(-2,-3)$. Foci $(-2,1+2\sqrt{3})$, $(-2,1-2\sqrt{3})$

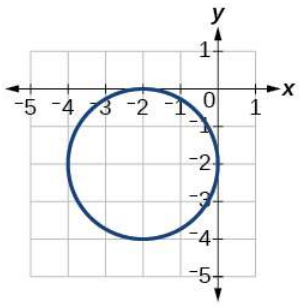


44) $100x^2+1000x+y^2-10y+2425=0$

45) $4x^2+16x+4y^2+16y+16=0$

Answer

Center $(-2,2)$, Vertices $(0,-2)$, $(-4,-2)$, $(-2,0)$, $(-2,-4)$. Focus $(-2,-2)$



For the exercises 46-51, use the given information about the graph of each ellipse to determine its equation.

46) Center at the origin, symmetric with respect to the x- and y-axes, focus at $(4, 0)$ and point on graph $(0, 3)$.

47) Center at the origin, symmetric with respect to the x- and y-axes, focus at $(0, -2)$ and point on graph $(5, 0)$.

Answer

$$\frac{x^2}{25} + \frac{y^2}{29} = 1$$

48) Center at the origin, symmetric with respect to the x- and y-axes, focus at $(3, 0)$, and major axis is twice as long as minor axis.

49) Center $(4, 2)$; vertex $(9, 2)$; one focus: $(4 + 2\sqrt{6}, 2)$

Answer

$$\frac{(x-4)^2}{25} + \frac{(y-2)^2}{1} = 1$$

50) Center $(3, 5)$; vertex $(3, 11)$; one focus: $(3, 5 + 4\sqrt{2})$

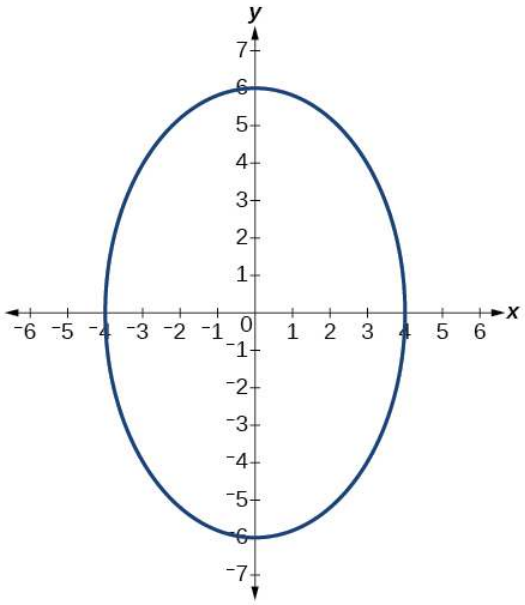
51) Center $(-3, 4)$; vertex $(1, 4)$; one focus: $(-3 + 2\sqrt{3}, 4)$

Answer

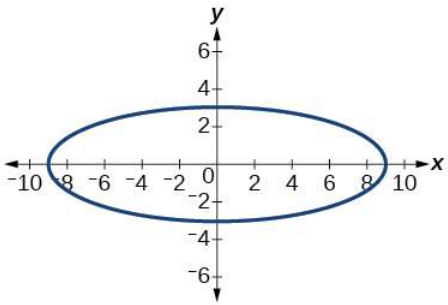
$$\frac{(x+3)^2}{16} + \frac{(y-4)^2}{4} = 1$$

For the exercises 52-56, given the graph of the ellipse, determine its equation.

52)



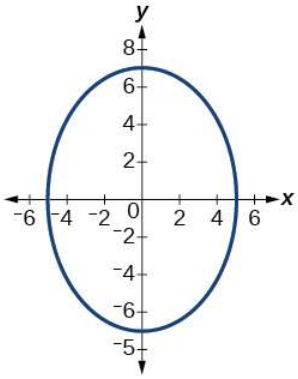
53)



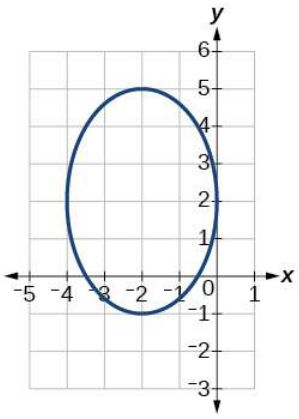
Answer

$$\frac{x^2}{81}+\frac{y^2}{9}=1$$

54)



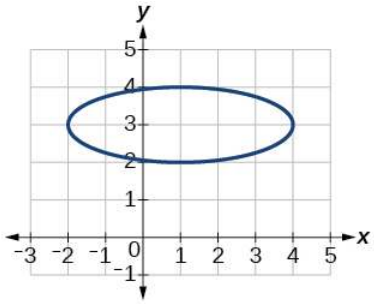
55)



Answer

$$\frac{(x+2)^2}{4}+\frac{(y-2)^2}{9}=1$$

56)



Extensions

For the exercises 57-61, find the area of the ellipse. The area of an ellipse is given by the formula $\text{Area}=a\cdot b\cdot\pi$

57) $\frac{(x-3)^2}{9}+\frac{(y-3)^2}{16}=1$

Answer

$\text{Area}=12\pi$ square units

58) $\frac{(x+6)^2}{16}+\frac{(y-6)^2}{36}=1$

59) $\frac{(x+1)^2}{4}+\frac{(y-2)^2}{5}=1$

Answer

$\text{Area}=2\sqrt{5}\pi$ square units

60) $4x^2-8x+9y^2-72y+112=0$

61) $9x^2-54x+9y^2-54y+81=0$

Answer

$\text{Area}=9\pi$ square units

Real-World Applications

62) Find the equation of the ellipse that will just fit inside a box that is 8 units wide and 4 units high.

63) Find the equation of the ellipse that will just fit inside a box that is four times as wide as it is high. Express in terms of h, the height.

Answer

$$\frac{x^2}{4h^2}+\frac{y^2}{\frac{1}{4}h^2}=1$$

64) An arch has the shape of a semi-ellipse (the top half of an ellipse). The arch has a height of 8 feet and a span of 20 feet. Find an equation for the ellipse, and use that to find the height to the nearest 0.01 foot of the arch at a distance of 4 feet from the center.

65) An arch has the shape of a semi-ellipse. The arch has a height of 12 feet and a span of 40 feet. Find an equation for the ellipse, and use that to find the distance from the center to a point at which the height is 6 feet. Round to the nearest hundredth.

Answer

$\frac{x^2}{400}+\frac{y^2}{144}=1$. Distance = 17.32feet

66) A bridge is to be built in the shape of a semi-elliptical arch and is to have a span of 120 feet. The height of the arch at a distance of 40 feet from the center is to be 8 feet. Find the height of the arch at its center.

67) A person in a whispering gallery standing at one focus of the ellipse can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet, and the foci are located 30 feet from the center, find the height of the ceiling at the center.

Answer

Approximately 51.96feet

68) A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus, and the other focus is 80 feet away, what is the length and height at the center of the gallery?

10.2: The Hyperbola

In analytic geometry, a hyperbola is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other.

Verbal

1) Define a hyperbola in terms of its foci.

Answer

A hyperbola is the set of points in a plane the difference of whose distances from two fixed points (foci) is a positive constant.

2) What can we conclude about a hyperbola if its asymptotes intersect at the origin?

3) What must be true of the foci of a hyperbola?

Answer

The foci must lie on the transverse axis and be in the interior of the hyperbola.

4) If the transverse axis of a hyperbola is vertical, what do we know about the graph?

5) Where must the center of hyperbola be relative to its foci?

Answer

The center must be the midpoint of the line segment joining the foci.

Algebraic

For the exercises 6-10, determine whether the following equations represent hyperbolas. If so, write in standard form.

6) $3y^2+2x=6$

7) $\frac{x^2}{36}-\frac{y^2}{9}=1$

Answer

yes $\frac{x^2}{6^2}-\frac{y^2}{3^2}=1$

8) $5y^2+4x^2=6x$

9) $25x^2-16y^2=400$

Answer

yes $\frac{x^2}{4^2}-\frac{y^2}{5^2}=1$

10) $-9x^2+18x+y^2+4y-14=0$

For the exercises 11-25, write the equation for the hyperbola in standard form if it is not already, and identify the vertices and foci, and write equations of asymptotes.

11) $\frac{x^2}{25}-\frac{y^2}{36}=1$

Answer

$\frac{x^2}{5^2}-\frac{y^2}{6^2}=1$; vertices: $(5,0)$, $(-5,0)$; foci: $(\sqrt{61},0)$, $(-\sqrt{61},0)$; asymptotes: $y=\frac{6}{5}x$, $y=-\frac{6}{5}x$

12) $\frac{x^2}{100}-\frac{y^2}{9}=1$

13) $\frac{y^2}{4}-\frac{x^2}{81}=1$

Answer

$\frac{y^2}{2^2}-\frac{x^2}{9^2}=1$; vertices: $(0,2)$, $(0,-2)$; foci: $(0,\sqrt{85})$, $(0,-\sqrt{85})$; asymptotes: $y=\frac{2}{9}x$, $y=-\frac{2}{9}x$

14) $9y^2-4x^2=1$

15) $\frac{(x-1)^2}{9}-\frac{(y-2)^2}{16}=1$

Answer

$\frac{(x-1)^2}{3^2}-\frac{(y-2)^2}{4^2}=1$; vertices: $(4,2)$, $(-2,2)$; foci: $(6,2)$, $(-4,2)$; asymptotes: $y=\frac{4}{3}(x-1)+2$, $y=-\frac{4}{3}(x-1)+2$

16) $\frac{(y-6)^2}{36}-\frac{(x+1)^2}{16}=1$

17) $\frac{(x-2)^2}{49}-\frac{(y+7)^2}{49}=1$

Answer

$\frac{(x-2)^2}{7^2}-\frac{(y+7)^2}{7^2}=1$; vertices: $(9,-7),\ (-5,-7)$; foci: $(2+7\sqrt{2},-7),\ (2-7\sqrt{2},-7)$; asymptotes: $y=x-9$, $y=-x-5$

18) $4x^2-8x-9y^2-72y+112=0$

19) $-9x^2-54x+9y^2-54y+81=0$

Answer

$\frac{(x+3)^2}{3^2}-\frac{(y-3)^2}{3^2}=1$; vertices: $(0,3),\ (-6,3)$; foci: $(-3+3\sqrt{2},1),\ (-3-3\sqrt{2},1)$; asymptotes: $y=x+6$, $y=-x$

20) $4x^2-24x-36y^2-360y+864=0$

21) $-4x^2+24x+16y^2-128y+156=0$

Answer

$\frac{(y-4)^2}{2^2}-\frac{(x-3)^2}{4^2}=1$; vertices: $(3,6),\ (3,2)$; foci: $(3,4+2\sqrt{5}),\ (3,4-2\sqrt{5})$; asymptotes: $y=\frac{1}{2}(x-3)+4$, $y=-\frac{1}{2}(x-3)+4$

22) $-4x^2+40x+25y^2-100y+100=0$

23) $x^2+2x-100y^2-1000y+2401=0$

Answer

$\frac{(y+5)^2}{7^2}-\frac{(x+1)^2}{70^2}=1$; vertices: $(-1,2),\ (-1,-12)$; foci: $(-1,-5+7\sqrt{101}),\ (-1,-5-7\sqrt{101})$; asymptotes: $y=\frac{1}{10}(x+1)-5$, $y=-\frac{1}{10}(x+1)-5$

24) $-9x^2+72x+16y^2+16y+4=0$

25) $4x^2+24x-25y^2+200y-464=0$

Answer

$\frac{(x+3)^2}{5^2}-\frac{(y-4)^2}{2^2}=1$; vertices: $(2,4),\ (-8,4)$; foci: $(-3+\sqrt{29},4),\ (-3-\sqrt{29},4)$; asymptotes: $y=\frac{2}{5}(x+3)+4$, $y=-\frac{2}{5}(x+3)+4$

For the exercises 26-30, find the equations of the asymptotes for each hyperbola.

26) $\frac{y^2}{3^2}-\frac{x^2}{3^2}=1$

27) $\frac{(x-3)^2}{5^2}-\frac{(y+4)^2}{2^2}=1$

Answer

$y=\frac{2}{5}(x-3)-4$, $y=-\frac{2}{5}(x-3)-4$

28) $\frac{(y-3)^2}{3^2}-\frac{(x+5)^2}{6^2}=1$

29) $9x^2-18x-16y^2+32y-151=0$

Answer

$$y=\frac{3}{4}(x-1)+1 \ , y=-\frac{3}{4}(x-1)+1$$

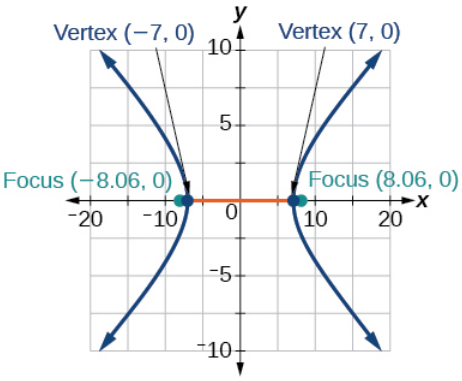
30) $16y^2+96y-4x^2+16x+112=0$

Graphical

For the exercises 31-44, sketch a graph of the hyperbola, labeling vertices and foci.

31) $\frac{x^2}{49}-\frac{y^2}{16}=1$

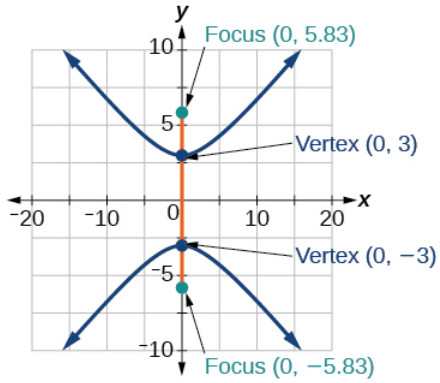
Answer



32) $\frac{x^2}{64}-\frac{y^2}{4}=1$

33) $\frac{y^2}{9}-\frac{x^2}{25}=1$

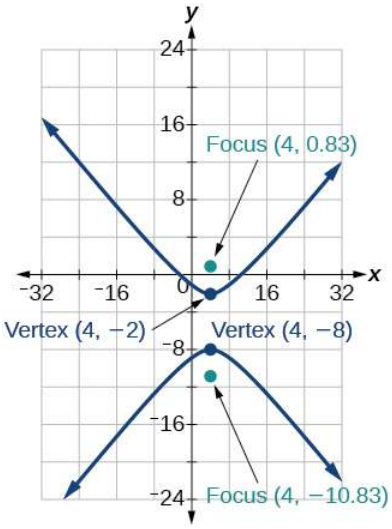
Answer



34) $81x^2-9y^2=1$

35) $\frac{(y+5)^2}{9}-\frac{(x-4)^2}{25}=1$

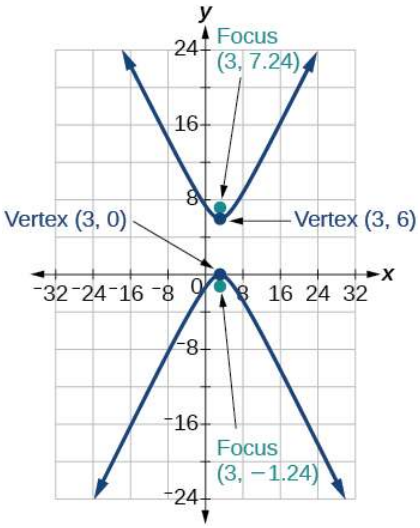
Answer



36) $\frac{(x-2)^2}{8} - \frac{(y+3)^2}{27} = 1$

37) $\frac{(y-3)^2}{9} - \frac{(x-3)^2}{9} = 1$

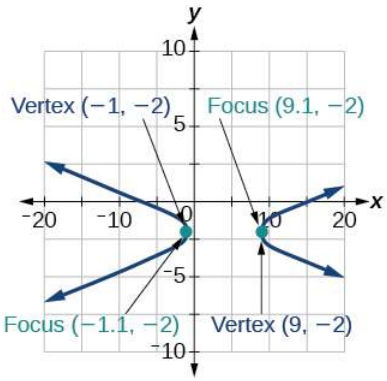
Answer



38) $-4x^2 - 8x + 16y^2 - 32y - 52 = 0$

39) $x^2 - 8x - 25y^2 - 100y - 109 = 0$

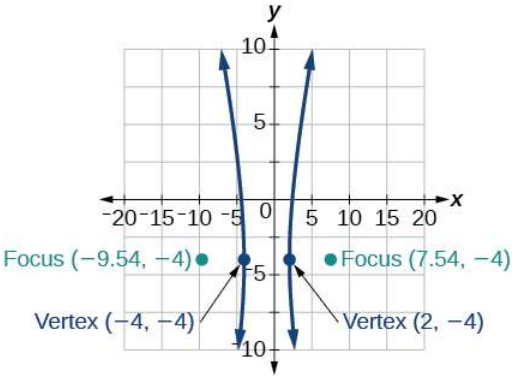
Answer



40) $-x^2+8x+4y^2-40y+88=0$

41) $64x^2+128x-9y^2-72y-656=0$

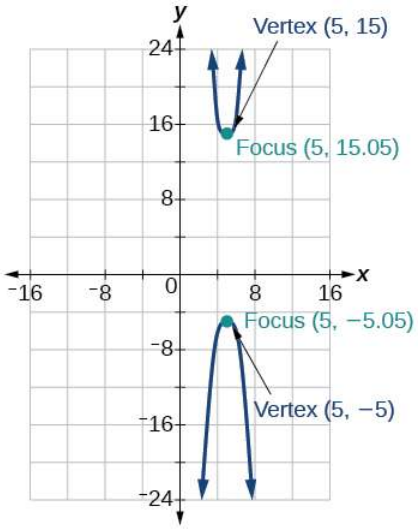
Answer



42) $16x^2+64x-4y^2-8y-4=0$

43) $-100x^2+1000x+y^2-10y-2575=0$

Answer



44) $4x^2+16x-4y^2+16y+16=0$

For the exercises 45-50, given information about the graph of the hyperbola, find its equation.

45) Vertices at $(3, 0)$ and $(-3, 0)$ and one focus at $(5, 0)$.

Answer

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

46) Vertices at $(0, 6)$ and $(0, -6)$ and one focus at $(0, -8)$.

47) Vertices at $(1, 1)$ and $(11, 1)$ and one focus at $(12, 1)$.

Answer

$$\frac{(x-6)^2}{25} - \frac{(y-1)^2}{11} = 1$$

48) Center: $(0, 0)$; vertex: $(0, -13)$; one focus: $(0, \sqrt{313})$.

49) Center: $(4, 2)$; vertex: $(9, 2)$; one focus: $(4 + \sqrt{26}, 2)$.

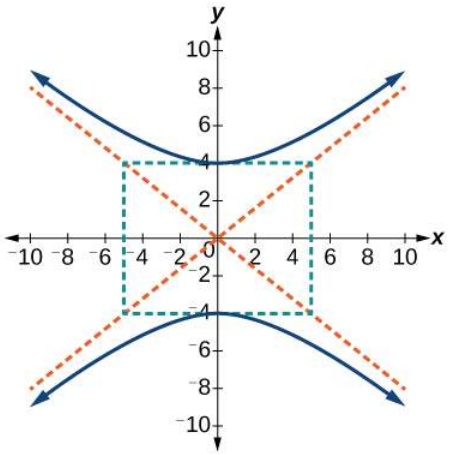
Answer

$$\frac{(x-4)^2}{25} - \frac{(y-2)^2}{1} = 1$$

50) Center: $(3, 5)$; vertex: $(3, 11)$; one focus: $(3, 5 + 2\sqrt{10})$.

For the exercises 51-,55 given the graph of the hyperbola, find its equation.

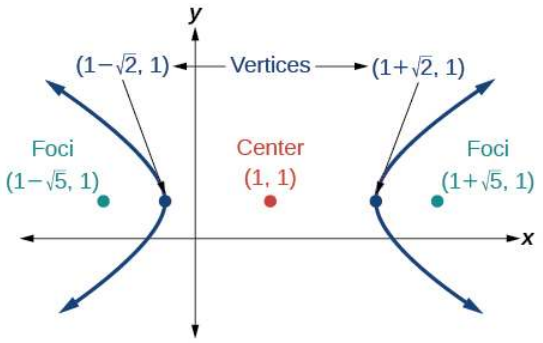
51)



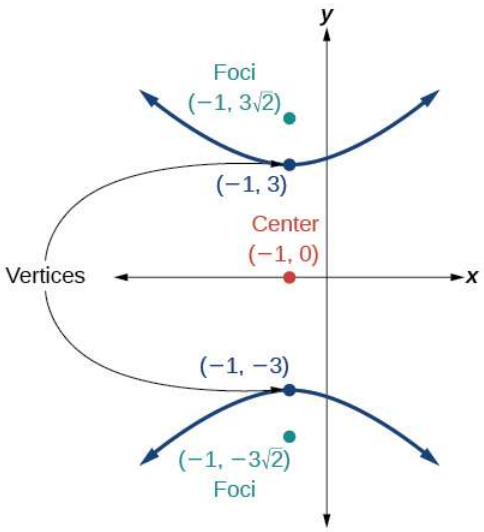
Answer

$$\frac{y^2}{16} - \frac{x^2}{25} = 1$$

52)



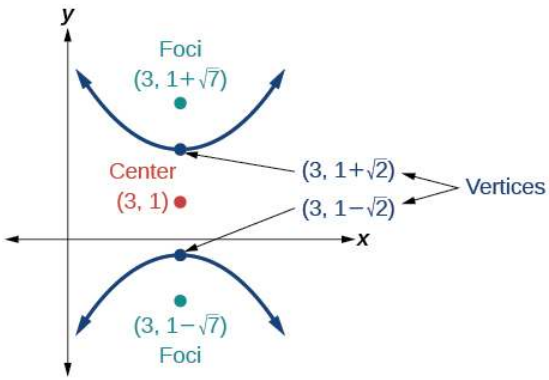
53)



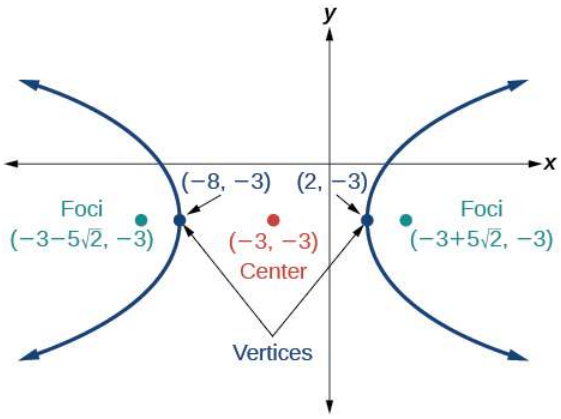
Answer

$$\frac{y^2}{9}-\frac{(x+1)^2}{9}=1$$

54)



55)



Answer

$$\frac{(x+3)^2}{25}-\frac{(y+3)^2}{25}=1$$

Extensions

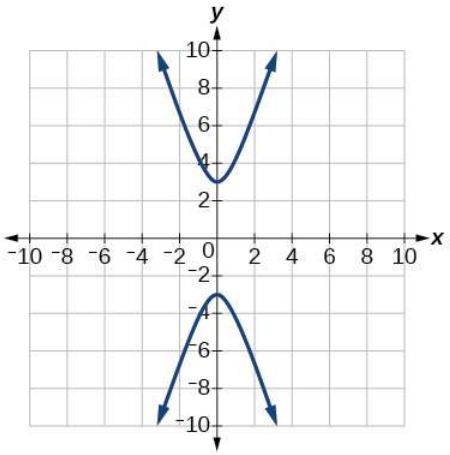
For the exercises 56-60, express the equation for the hyperbola as two functions, with y as a function of x. Express as simply as possible. Use a graphing calculator to sketch the graph of the two functions on the same axes.

56) $\frac{x^2}{4}-\frac{y^2}{9}=1$

57) $\frac{y^2}{9}-\frac{x^2}{1}=1$

Answer

$y(x)=3\sqrt{x^2+1}$, $y(x)=-3\sqrt{x^2+1}$

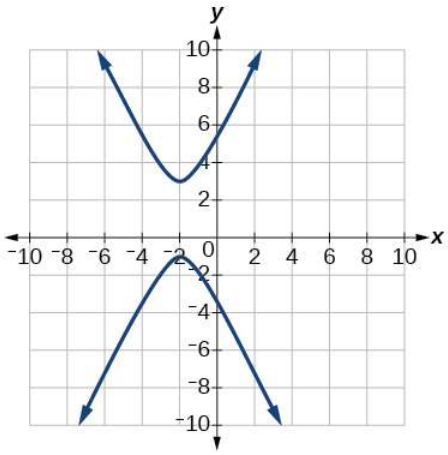


58) $\frac{(x-2)^2}{16}-\frac{(y+3)^2}{25}=1$

59) $-4x^2-16x+y^2-2y-19=0$

Answer

$y(x)=1+2\sqrt{x^2+4x+5}$, $y(x)=1-2\sqrt{x^2+4x+5}$



60) $4x^2 - 24x - y^2 - 4y + 16 = 0$

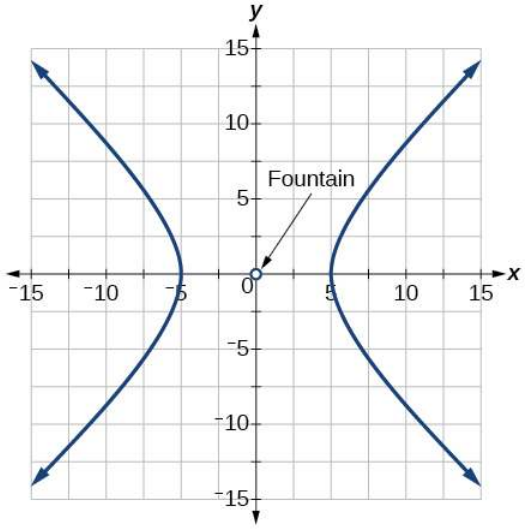
Real-World Applications

For the exercises 61-65, a hedge is to be constructed in the shape of a hyperbola near a fountain at the center of the yard. Find the equation of the hyperbola and sketch the graph.

61) The hedge will follow the asymptotes $y = x$ and $y = -x$, and its closest distance to the center fountain is 5 yards.

Answer

$$\frac{x^2}{25} - \frac{y^2}{25} = 1$$

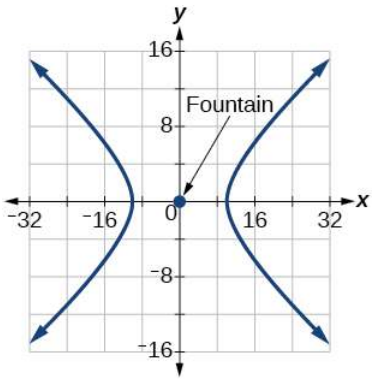


62) The hedge will follow the asymptotes $y = 2x$ and $y = -2x$, and its closest distance to the center fountain is 6 yards.

63) The hedge will follow the asymptotes $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$, and its closest distance to the center fountain is 10 yards.

Answer

$$\frac{x^2}{100} - \frac{y^2}{25} = 1$$

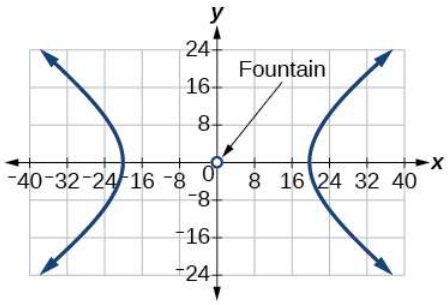


64) The hedge will follow the asymptotes $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$, and its closest distance to the center fountain is 12 yards.

65) The hedge will follow the asymptotes $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$, and its closest distance to the center fountain is 20 yards.

Answer

$$\frac{x^2}{400} - \frac{y^2}{225} = 1$$



For the exercises 66-70, assume an object enters our solar system and we want to graph its path on a coordinate system with the sun at the origin and the x-axis as the axis of symmetry for the object's path. Give the equation of the flight path of each object using the given information.

66) The object enters along a path approximated by the line $y = x - 2$ and passes within 1 au (astronomical unit) of the sun at its closest approach, so that the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -x + 2$.

67) The object enters along a path approximated by the line $y = 2x - 2$ and passes within 0.5 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -2x + 2$.

Answer

$$\frac{(x-1)^2}{0.25} - \frac{y^2}{0.75} = 1$$

68) The object enters along a path approximated by the line $y = 0.5x + 2$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -0.5x - 2$.

69) The object enters along a path approximated by the line $y = \frac{1}{3}x - 1$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -\frac{1}{3}x + 1$.

Answer

$$(x-3)^2 - 9y^2 = 4$$

70) The object enters along a path approximated by the line $y = 3x - 9$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -3x + 9$.

10.3: The Parabola

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix.

Verbal

1) Define a parabola in terms of its focus and directrix.

Answer

A parabola is the set of points in the plane that lie equidistant from a fixed point, the focus, and a fixed line, the directrix.

2) If the equation of a parabola is written in standard form and p is positive and the directrix is a vertical line, then what can we conclude about its graph?

3) If the equation of a parabola is written in standard form and p is negative and the directrix is a horizontal line, then what can we conclude about its graph?

Answer

The graph will open down.

4) What is the effect on the graph of a parabola if its equation in standard form has increasing values of p?

5) As the graph of a parabola becomes wider, what will happen to the distance between the focus and directrix?

Answer

The distance between the focus and directrix will increase.

Algebraic

For the exercises 6-10, determine whether the given equation is a parabola. If so, rewrite the equation in standard form.

6) $y^2 = 4 - x^2$

7) $y = 4x^2$

Answer

yes $y = 4(1)x^2$

8) $3x^2 - 6y^2 = 12$

9) $(y-3)^2 = 8(x-2)$

Answer

yes $(y-3)^2 = 4(2)(x-2)$

10) $y^2 + 12x - 6y - 51 = 0$

For the exercises 11-30, rewrite the given equation in standard form, and then determine the vertex (V), focus (F), and directrix (d) of the parabola.

11) $x = 8y^2$

Answer

$y^2 = \frac{1}{8}x$, $V : (0,0)$, $F : \left(\frac{1}{32}, 0\right)$, $d : x = -\frac{1}{32}$

12) $y = \frac{1}{4}x^2$

13) $y = -4x^2$

Answer

$x^2 = -\frac{1}{4}y$, $V : (0,0)$, $F : \left(0, -\frac{1}{16}\right)$, $d : y = -\frac{1}{16}$

14) $x = \frac{1}{8}y^2$

15) $x = 36y^2$

Answer

$y^2 = \frac{1}{36}x$, $V : (0,0)$, $F : \left(\frac{1}{144}, 0\right)$, $d : x = -\frac{1}{144}$

16) $x = \frac{1}{36}y^2$

17) $(x-1)^2 = 4(y-1)$

Answer

$(x-1)^2 = 4(y-1)$, $V : (1,1)$, $F : (1,2)$, $d : y = 0$

18) $(y-2)^2 = \frac{4}{5}(x+4)$

19) $(y-4)^2 = 2(x+3)$

Answer

$(y-4)^2 = 2(x+3)$, $V : (-3,4)$, $F : \left(-\frac{5}{2}, 4\right)$, $d : x = -\frac{7}{2}$

20) $(x+1)^2 = 2(y+4)$

21) $(x+4)^2 = 24(y+1)$

Answer

$(x+4)^2 = 24(y+1)$, $V : (-4,-1)$, $F : (-4,5)$, $d : y = -7$

22) $(y+4)^2 = 16(x+4)$

23) $y^2 + 12x - 6y + 21 = 0$

Answer

$(y-3)^2 = -12(x+1)$, $V : (-1,3)$, $F : (-4,3)$, $d : x = 2$

24) $x^2 - 4x - 24y + 28 = 0$

25) $5x^2 - 50x - 4y + 113 = 0$

Answer

$(x-5)^2 = \frac{4}{5}(y+3)$, $V : (5,-3)$, $F : \left(5, -\frac{14}{5}\right)$, $d : y = -\frac{16}{5}$

26) $y^2 - 24x + 4y - 68 = 0$

27) $x^2 - 4x + 2y - 6 = 0$

Answer

$$(x-2)^2=-2(y-5)\ ,V:(2,5),F:\left(2,\frac{9}{2}\right),d:y=\frac{11}{2}$$

28) $y^2-6y+12x-3=0$

29) $3y^2-4x-6y+23=0$

Answer

$$(y-1)^2=\frac{4}{3}(x-5)\ ,V:(5,1),F:\left(\frac{16}{3},1\right),d:x=\frac{14}{3}$$

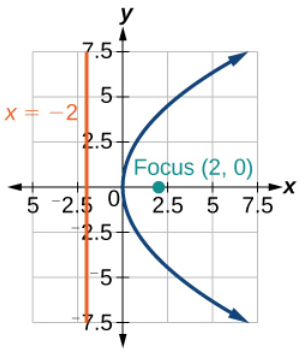
30) $x^2+4x+8y-4=0$

Graphical

For the exercises 31-44, graph the parabola, labeling the focus and the directrix.

31) $x=\frac{1}{8}y^2$

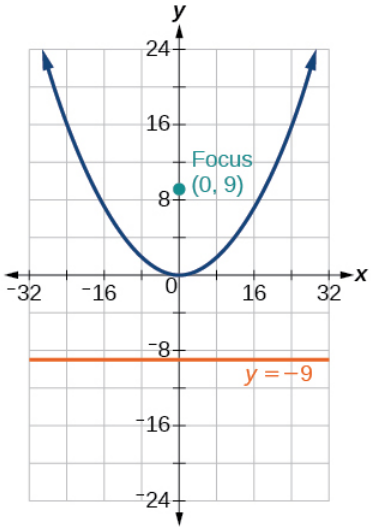
Answer



32) $y=36x^2$

33) $y=\frac{1}{36}x^2$

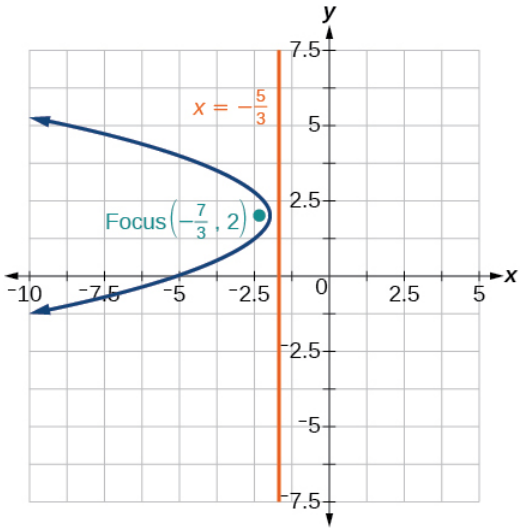
Answer



34) $y = -9x^2$

35) $(y-2)^2 = -\frac{4}{3}(x+2)$

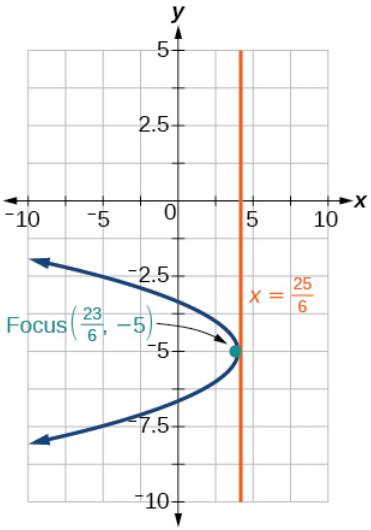
Answer



36) $-5(x+5)^2 = 4(y+5)$

37) $-6(y+5)^2 = 4(x-4)$

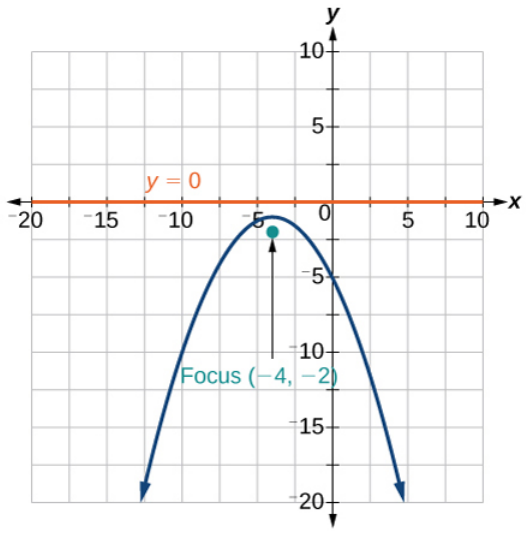
Answer



38) $y^2 - 6y - 8x + 1 = 0$

39) $x^2 + 8x + 4y + 20 = 0$

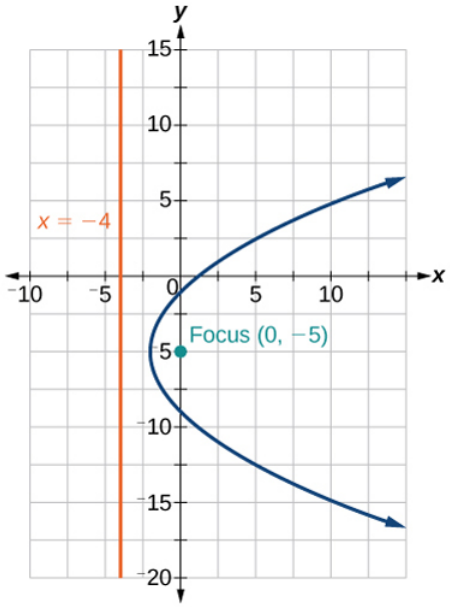
Answer



40) $3x^2+30x-4y+95=0$

41) $y^2-8x+10y+9=0$

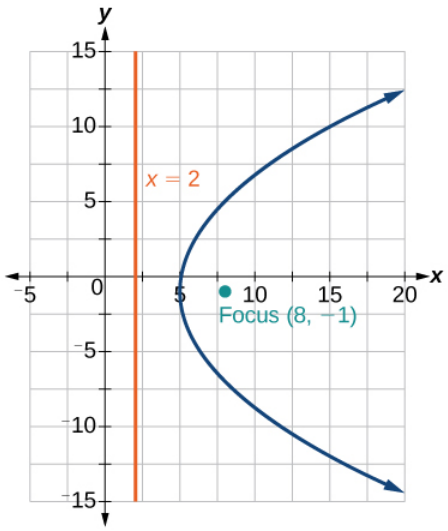
Answer



42) $x^2+4x+2y+2=0$

43) $y^2+2y-12x+61=0$

Answer



44) $-2x^2+8x-4y-24=0$

For the exercises 45-50, find the equation of the parabola given information about its graph.

45) Vertex is $(0,0)$; directrix is $y=4$, focus is $(0,-4)$.

Answer

$x^2=-16y$

46) Vertex is $(0,0)$; directrix is $x=4$, focus is $(-4,0)$.

47) Vertex is $(2,2)$; directrix is $x=2-\sqrt{2}$, focus is $(2+\sqrt{2},2)$.

Answer

$(y-2)^2=4\sqrt{2}(x-2)$

48) Vertex is $(-2,3)$; directrix is $x=-\frac{7}{2}$, focus is $\left(-\frac{1}{2},3\right)$.

49) Vertex is $(\sqrt{2},-\sqrt{3})$; directrix is $x=2\sqrt{2}$, focus is $(0,-\sqrt{3})$.

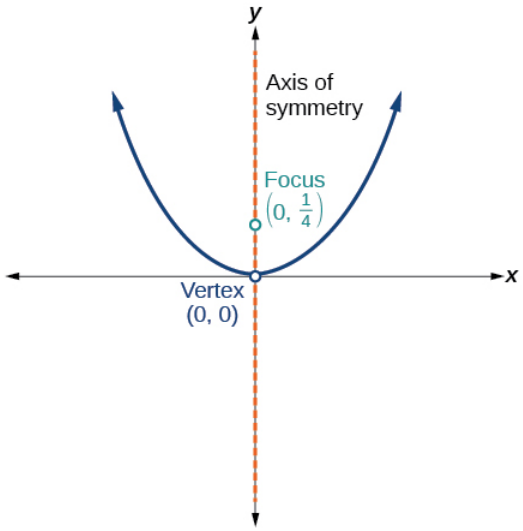
Answer

$(y+\sqrt{3})^2=-4\sqrt{2}(x-\sqrt{2})$

50) Vertex is $(-,21)$; directrix is $y=\frac{11}{3}$, focus is $\left(1,\frac{1}{3}\right)$.

For the exercises 51-55, determine the equation for the parabola from its graph.

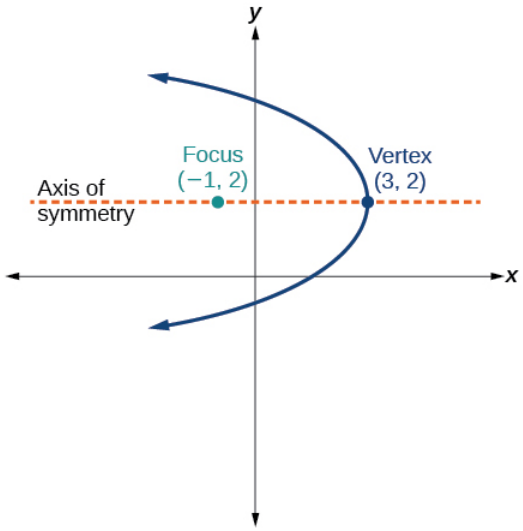
51)



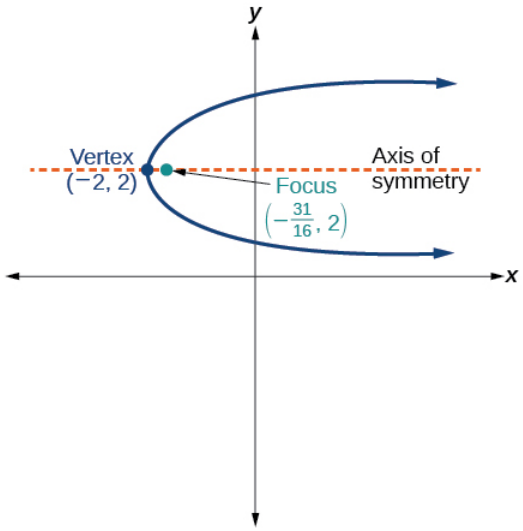
Answer

$x^2 = y$

52)



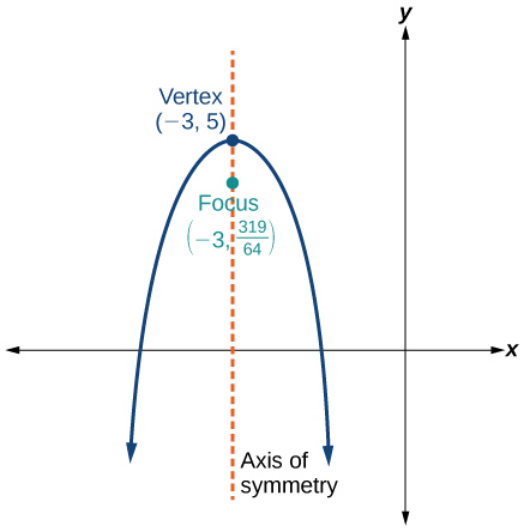
53)



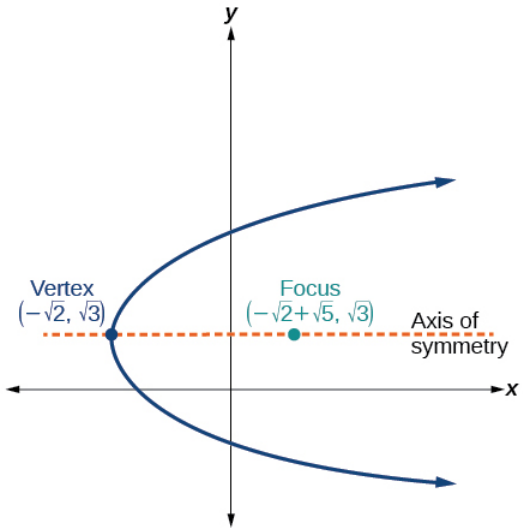
Answer

$$(y-2)^2 = \frac{1}{4}(x+2)$$

54)



55)



Answer

$(y-\sqrt{2})^2=4\sqrt{5}(x+\sqrt{2})$

Extensions

For the exercises 56-60, the vertex and endpoints of the latus rectum of a parabola are given. Find the equation.

56) $V(0,0)$, Endpoints $(2,1),(-2,1)$

57) $V(0,0)$, Endpoints $(-2,4),(-2,-4)$

Answer

$y^2=-8x$

58) $V(1,2)$, Endpoints $(-5,5),(7,5)$

59) $V(-3,-1)$, Endpoints $(0,5),(0,-7)$

Answer

$(y+1)^2=12(x+3)$

60) $V(4,-3)$, Endpoints $\left(5,-\frac{7}{2}\right),\left(3,-\frac{7}{2}\right)$

Real-World Applications

61) The mirror in an automobile headlight has a parabolic cross-section with the light bulb at the focus. On a schematic, the equation of the parabola is given as $x^2=4y$. At what coordinates should you place the light bulb?

Answer

$(0,1)$

62) If we want to construct the mirror from the previous exercise such that the focus is located at $(0,0.25)$ what should the equation of the parabola be?

63) A satellite dish is shaped like a paraboloid of revolution. This means that it can be formed by rotating a parabola around its axis of symmetry. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

Answer

At the point 2.25 feet above the vertex.

64) Consider the satellite dish from the previous exercise. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?

65) A searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

Answer

0.5625 feet

66) If the searchlight from the previous exercise has the light source located 6 inches from the base along the axis of symmetry and the opening is 4 feet, find the depth.

67) An arch is in the shape of a parabola. It has a span of 100 feet and a maximum height of 20 feet. Find the equation of the parabola, and determine the height of the arch 40 feet from the center.

Answer

$x^2 = -125(y - 20)$, height is 7.2 feet

68) If the arch from the previous exercise has a span of 160 feet and a maximum height of 40 feet, find the equation of the parabola, and determine the distance from the center at which the height is 20 feet.

69) An object is projected so as to follow a parabolic path given by $y = -x^2 + 96x$ where x is the horizontal distance traveled in feet and y is the height. Determine the maximum height the object reaches.

Answer

2304 feet

70) For the object from the previous exercise, assume the path followed is given by $y = -0.5x^2 + 80x$. Determine how far along the horizontal the object traveled to reach maximum height.

10.4: Rotation of Axes

In previous sections of this chapter, we have focused on the standard form equations for nondegenerate conic sections. In this section, we will shift our focus to the general form equation, which can be used for any conic. The general form is set equal to zero, and the terms and coefficients are given in a particular order, as shown below.

Verbal

1) What effect does the xy term have on the graph of a conic section?

Answer

The xy term causes a rotation of the graph to occur.

2) If the equation of a conic section is written in the form $Ax^2 + By^2 + Cx + Dy + E = 0$ and $AB = 0$ what can we conclude?

3) If the equation of a conic section is written in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, and $B^2 - 4AC > 0$, what can we conclude?

Answer

The conic section is a hyperbola.

4) Given the equation $ax^2 + 4x + 3y^2 - 12 = 0$, what can we conclude if $a > 0$?

5) For the equation \(Ax^2+Bxy+Cy^2+Dx+Ey+F=0\) the value of θ that satisfies $\cot(2\theta) = \dfrac{A - C}{B}$ gives us what information?

Answer

It gives the angle of rotation of the axes in order to eliminate the xy term.

Algebraic

For the exercises 6-17, determine which conic section is represented based on the given equation.

6) $9x^2+4y^2+72x+36y-500=0$

7) $x^2-10x+4y-10=0$

Answer

$AB=0$, parabola

8) $2x^2-2y^2+4x-6y-2=0$

9) $4x^2-y^2+8x-1=0$

Answer

$AB=-4<0$, hyperbola

10) $4y^2-5x+9y+1=0$

11) $2x^2+3y^2-8x-12y+2=0$

Answer

$AB=6>0$, ellipse

12) $4x^2+9xy+4y^2-36y-125=0$

13) $3x^2+6xy+3y^2-36y-125=0$

Answer

$B^2-4AC=0$, parabola

14) $-3x^2+3\sqrt{3}xy-4y^2+9=0$

15) $2x^2+4\sqrt{3}xy+6y^2-6x-3=0$

Answer

$B^2-4AC=0$, parabola

16) $-x^2+4\sqrt{2}xy+2y^2-2y+1=0$

17) $8x^2+4\sqrt{2}xy+4y^2-10x+1=0$

Answer

$B^2-4AC=-96<0$, ellipse

For the exercises 18-22, find a new representation of the given equation after rotating through the given angle.

18) $3x^2+xy+3y^2-5=0, \theta=45^\circ$

19) $4x^2-xy+4y^2-2=0, \theta=45^\circ$

Answer

$7x'^2+9y'^2-4=0$

20) $2x^2+8xy-1=0, \theta=30^\circ$

21) $-2x^2+8xy+1=0, \theta=45^\circ$

Answer

$3x'^2+2x'y'-5y'^2+1=0$

22) $4x^2+\sqrt{2}xy+4y^2+y+2=0, \theta=45^\circ$

For the exercises 23-30, determine the angle θ that will eliminate the xy term and write the corresponding equation without the xy term.

23) $x^2+3\sqrt{3}xy+4y^2+y-2=0$

Answer

$\theta=60^\circ, 11x'^2-y'2+\sqrt{3}x'+y'-4=0$

24) $4x^2+2\sqrt{3}xy+6y^2+y-2=0$

25) $9x^2-3\sqrt{3}xy+6y^2+4y-3=0$

Answer

$\theta=150^\circ, 21x'^2+9y'^2+4x'-4\sqrt{3}y'-6=0$

26) $-3x^2-\sqrt{3}xy-2y^2-x=0$

27) $16x^2+24xy+9y^2+6x-6y+2=0$

Answer

$\theta\approx 36.9^\circ, 125x'^2+6x'-42y'+10=0$

28) $x^2+4xy+4y^2+3x-2=0$

29) $x^2+4xy+y^2-2x+1=0$

Answer

$\theta=45^\circ, 3x'^2-y'^2-\sqrt{2}x'+\sqrt{2}y'+1=0$

30) $4x^2-2\sqrt{3}xy+6y^2-1=0$

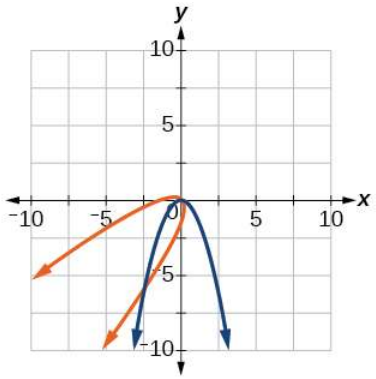
Graphical

For the exercises 31-38, rotate through the given angle based on the given equation. Give the new equation and graph the original and rotated equation.

31) $y=-x^2, \theta=-45^\circ$

Answer

$\frac{\sqrt{2}}{2}(x'+y')=\frac{1}{2}(x'-y')^2$

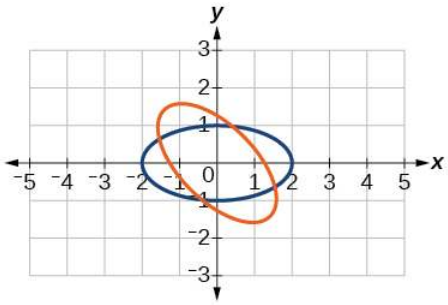


32) $x=y^2, \theta=45^\circ$

33) $\frac{x^2}{4}+\frac{y^2}{1}=1, \theta=45^\circ$

Answer

$$\frac{(x'-y')^2}{8}+\frac{(x'+y')^2}{2}=1$$

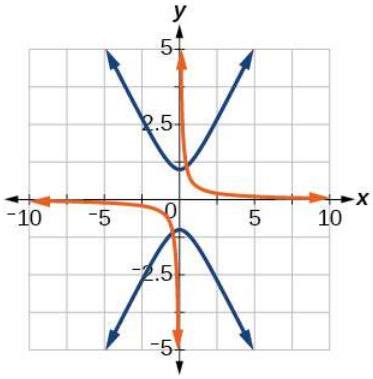


34) $\frac{y^2}{16}+\frac{x^2}{9}=1, \theta=45^\circ$

35) $y^2-x^2=1, \theta=45^\circ$

Answer

$$\frac{(x'+y')^2}{2}-\frac{(x'-y')^2}{2}=1$$

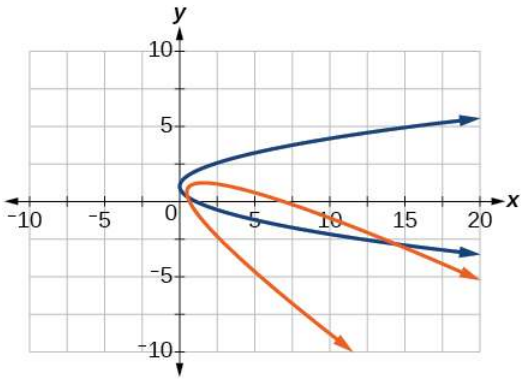


36) $y=\frac{x^2}{2}, \theta=30^\circ$

37) $x=(y-1)^2, \theta=30^\circ$

Answer

$$\frac{\sqrt{3}}{2}x'-\frac{1}{2}y'=\left(\frac{1}{2}x'+\frac{\sqrt{3}}{2}x'-1\right)^2$$

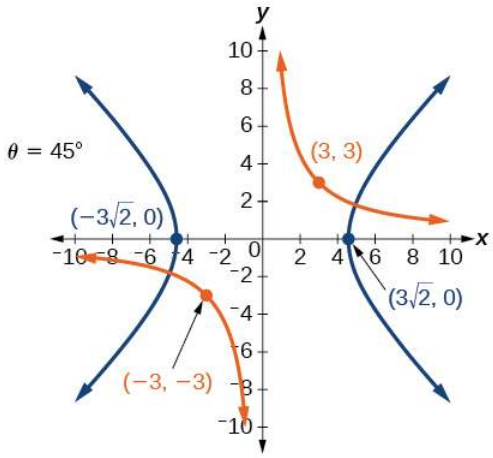


38) $\frac{x^2}{9}+\frac{y^2}{4}=1, \theta=30^\circ$

For the exercises 39-49, graph the equation relative to the $x'y'$ system in which the equation has no $x'y'$ term.

39) $xy=9$

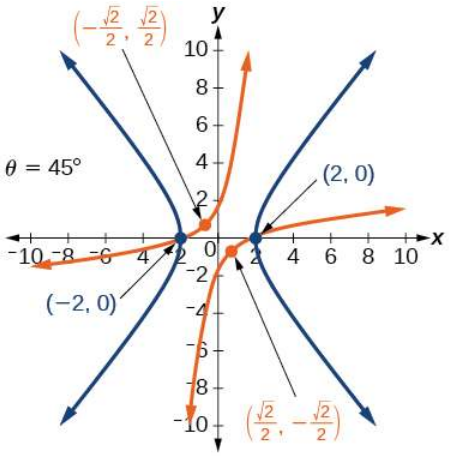
Answer



40) $x^2+10xy+y^2-6=0$

41) $x^2-10xy+y^2-24=0$

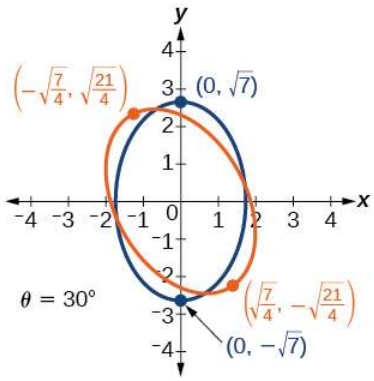
Answer



42) $4x^2 - 3\sqrt{3}xy + y^2 - 22 = 0$

43) $6x^2 + 2\sqrt{3}xy + 4y^2 - 21 = 0$

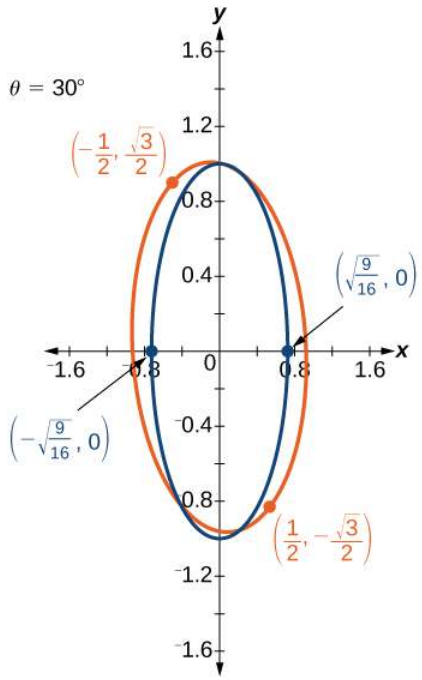
Answer



44) $11x^2 + 10\sqrt{3}xy + y^2 - 64 = 0$

45) $21x^2 + 2\sqrt{3}xy + 19y^2 - 18 = 0$

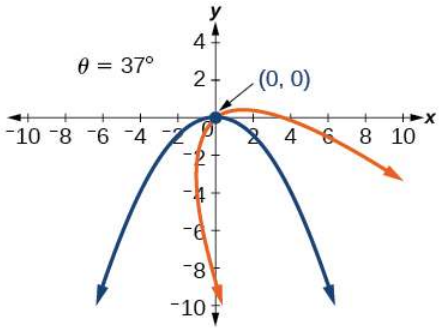
Answer



46) $16x^2 + 24xy + 9y^2 - 130x + 90y = 0$

47) $16x^2 + 24xy + 9y^2 - 60x + 80y = 0$

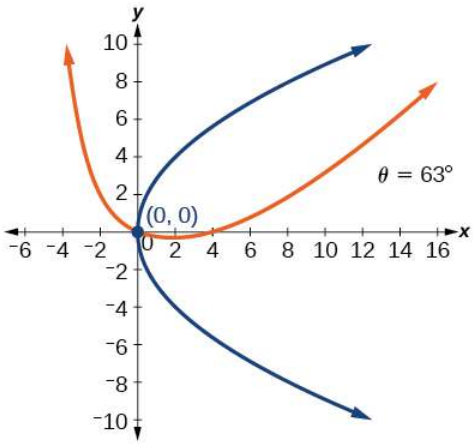
Answer



48) $13x^2 - 6\sqrt{3}xy + 7y^2 - 16 = 0$

49) $4x^2 - 4xy + y^2 - 8\sqrt{5}x - 16\sqrt{5}y = 0$

Answer



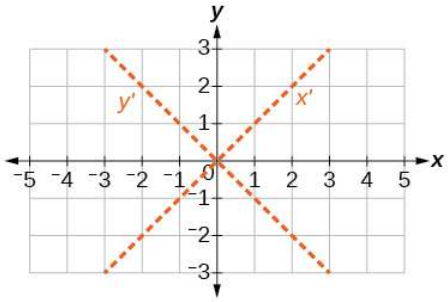
For the exercises 50-55, determine the angle of rotation in order to eliminate the xy term. Then graph the new set of axes.

50) $6x^2 - 5\sqrt{3}xy + y^2 + 10x - 12y = 0$

51) $6x^2 - 5xy + 6y^2 + 20x - y = 0$

Answer

$\theta = 45°$

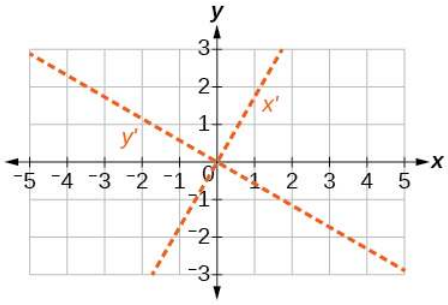


52) $6x^2 - 8\sqrt{3}xy + 14y^2 + 10x - 3y = 0$

53) $4x^2 + 6\sqrt{3}xy + 10y^2 + 20x - 40y = 0$

Answer

$\theta = 60°$

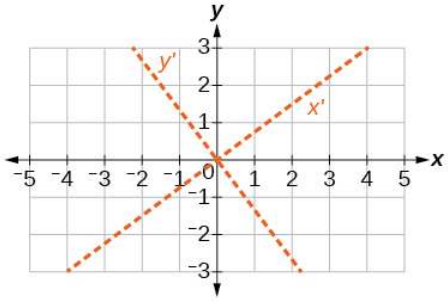


54) $8x^2+3xy+4y^2+2x-4=0$

55) $16x^2+24xy+9y^2+20x-44y=0$

Answer

$\theta \approx 36.9^\circ$



For the exercises 56-60, determine the value of k based on the given equation.

56) Given $4x^2+kxy+16y^2+8x+24y-48=0$, find k for the graph to be a parabola.

57) Given $2x^2+kxy+12y^2+10x-16y+28=0$, find k for the graph to be an ellipse.

Answer

$-4\sqrt{6}<k<4\sqrt{6}$

58) Given $3x^2+kxy+4y^2-6x+20y+128=0$, find k for the graph to be a hyperbola.

59) Given $kx^2+8xy+8y^2-12x+16y+18=0$ find k for the graph to be a parabola.

Answer

$k=2$

60) Given $6x^2+12xy+ky^2+16x+10y+4=0$ find k for the graph to be an ellipse.

10.5: Conic Sections in Polar Coordinates

In this section, we will learn how to define any conic in the polar coordinate system in terms of a fixed point, the focus at the pole, and a line, the directrix, which is perpendicular to the polar axis.

Verbal

1) Explain how eccentricity determines which conic section is given.

Answer

If eccentricity is less than 1, it is an ellipse. If eccentricity is equal to 1, it is a parabola. If eccentricity is greater than 1, it is a hyperbola.

2) If a conic section is written as a polar equation, what must be true of the denominator?

3) If a conic section is written as a polar equation, and the denominator involves $\sin\theta$ what conclusion can be drawn about the directrix?

Answer

The directrix will be parallel to the polar axis.

4) If the directrix of a conic section is perpendicular to the polar axis, what do we know about the equation of the graph?

5) What do we know about the focus/foci of a conic section if it is written as a polar equation?

Answer

One of the foci will be located at the origin.

Algebraic

For the exercises 6-17, identify the conic with a focus at the origin, and then give the directrix and eccentricity.

6) $r=\dfrac{6}{1-2\cos\theta}$

7) $r=\dfrac{3}{4-4\sin\theta}$

Answer

Parabola with $e=1$ and directrix $\dfrac{3}{4}$ units below the pole.

8) $r=\dfrac{8}{4-3\cos\theta}$

9) $r=\dfrac{5}{1+2\sin\theta}$

Answer

Hyperbola with $e=2$ and directrix $\dfrac{5}{2}$ units above the pole.

10) $r=\dfrac{15}{4+3\cos\theta}$

11) $r=\dfrac{3}{10+10\cos\theta}$

Answer

Parabola with $e=1$ and directrix $\dfrac{3}{10}$ units to the right of the pole.

12) $r=\dfrac{2}{1-\cos\theta}$

13) $r=\dfrac{4}{7+2\cos\theta}$

Answer

Ellipse with $e=\dfrac{2}{7}$ and directrix 2 units to the right of the pole.

14) $r(1-\cos\theta)=3$

15) $r(3+5\sin\theta)=11$

Answer

Hyperbola with $e = \frac{5}{3}$ and directrix $\frac{11}{5}$ units above the pole.

16) $r(4 - 5\sin\theta) = 1$

17) $r(7 + 8\sin\theta) = 7$

Answer

Hyperbola with $e = \frac{8}{7}$ and directrix $\frac{7}{8}$ units to the right of the pole.

For the exercises 18-30, convert the polar equation of a conic section to a rectangular equation.

18) $r = \frac{4}{1 + 3\sin\theta}$

19) $r = \frac{2}{5 - 3\sin\theta}$

Answer

$25x^2 + 16y^2 - 12y - 4 = 0$

20) $r = \frac{8}{3 - 2\cos\theta}$

21) $r = \frac{3}{2 + 5\cos\theta}$

Answer

$21x^2 - 4y^2 - 30x + 9 = 0$

22) $r = \frac{4}{2 + 2\sin\theta}$

23) $r = \frac{3}{8 - 8\cos\theta}$

Answer

$64y^2 = 48x + 9$

24) $r = \frac{2}{6 + 7\cos\theta}$

25) $r = \frac{5}{5 - 11\sin\theta}$

Answer

$96y^2 - 25x^2 + 110y + 25 = 0$

26) $r(5 + 2\cos\theta) = 6$

27) $r(2 - \cos\theta) = 1$

Answer

$3x^2 + 4y^2 - 2x - 1 = 0$

28) $r(2.5 - 2.5\sin\theta) = 5$

29) $r = \frac{6\sec\theta}{-2 + 3\sec\theta}$

Answer

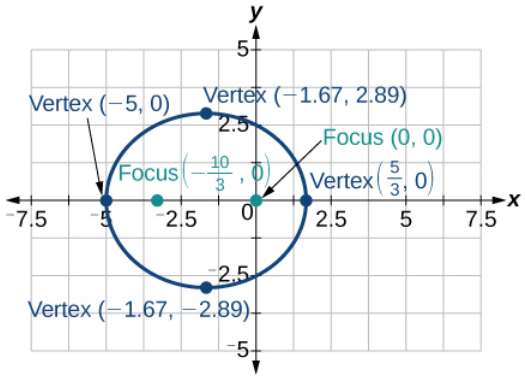
$$5x^2+9y^2-24x-36=0$$

30) $r=\dfrac{6\csc\theta}{3+2\csc\theta}$

For the exercises 31-42, graph the given conic section. If it is a parabola, label the vertex, focus, and directrix. If it is an ellipse, label the vertices and foci. If it is a hyperbola, label the vertices and foci.

31) $r=\dfrac{5}{2+\cos\theta}$

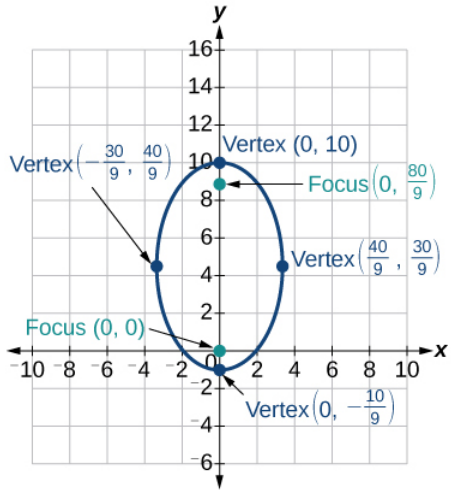
Answer



32) $r=\dfrac{2}{3+3\sin\theta}$

33) $r=\dfrac{10}{5-4\sin\theta}$

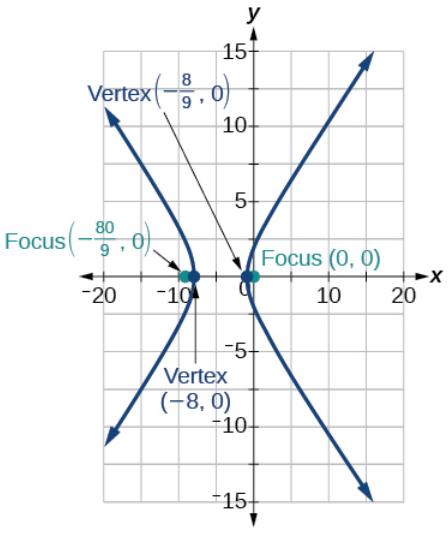
Answer



34) $r=\dfrac{3}{1+2\cos\theta}$

35) $r=\dfrac{8}{4-5\cos\theta}$

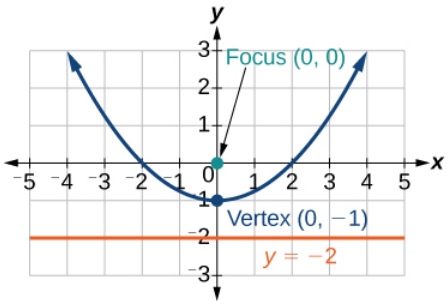
Answer



36) $r = \dfrac{3}{4 - 4\cos\theta}$

37) $r = \dfrac{2}{1 - \sin\theta}$

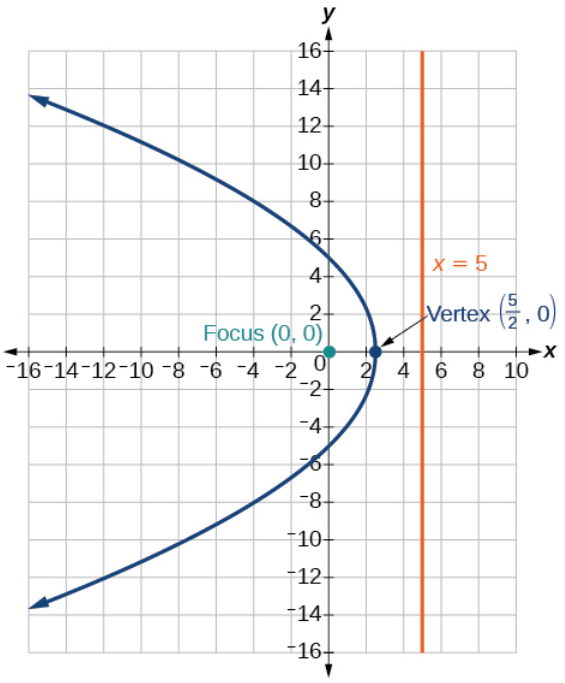
Answer



38) $r = \dfrac{6}{3 + 2\sin\theta}$

39) $r(1 + \cos\theta) = 5$

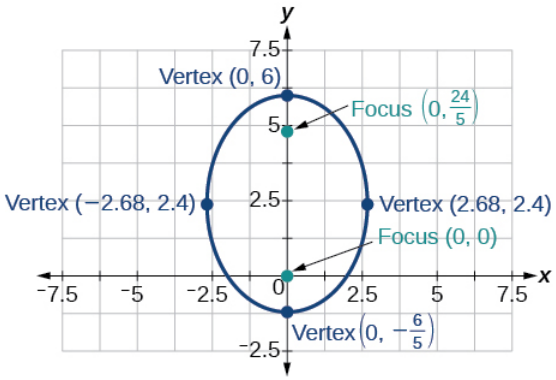
Answer



40) $r(3-4\sin\theta)=9$

41) $r(3-2\sin\theta)=6$

Answer



42) $r(4-6\cos\theta)=5$

For the exercises 43-, find the polar equation of the conic with focus at the origin and the given eccentricity and directrix.

43) Directrix: $x=4$; $e=\dfrac{1}{5}$

Answer

$$r=\frac{4}{5+\cos\theta}$$

44) Directrix: $x=-4$; $e=5$

45) Directrix: $y=2$; $e=2$

Answer

$$r=\frac{4}{1+2\sin\theta}$$

46) Directrix: $y=-2$; $e=\frac{1}{2}$

47) Directrix: $x=1$; $e=1$

Answer

$$r=\frac{1}{1+\cos\theta}$$

48) Directrix: $x=-1$; $e=1$

49) Directrix: $x=-\frac{1}{4}$; $e=\frac{7}{2}$

Answer

$$r=\frac{7}{8-28\cos\theta}$$

50) Directrix: $y=-\frac{2}{5}$; $e=\frac{7}{2}$

51) Directrix: $y=4$; $e=\frac{3}{2}$

Answer

$$r=\frac{12}{2+3\sin\theta}$$

52) Directrix: $x=-2$; $e=\frac{8}{3}$

53) Directrix: $x=-5$; $e=\frac{3}{4}$

Answer

$$r=\frac{15}{4-3\cos\theta}$$

54) Directrix: $y=2$; $e=2.5$

55) Directrix: $x=-3$; $e=\frac{1}{3}$

Answer

$$r=\frac{3}{3-3\cos\theta}$$

Extensions

Recall from Rotation of Axes that equations of conics with an xy term have rotated graphs. For the following exercises, express each equation in polar form with r as a function of θ.

56) $xy=2$

57) $x^2+xy+y^2=4$

Answer

$$r=\pm\frac{2}{\sqrt{1+\sin\theta\cos\theta}}$$

58) $2x^2+4xy+2y^2=9$

59) $16x^2+24xy+9y^2=4$

Answer

$$r = \pm \frac{2}{4\cos\theta + 3\sin\theta}$$

60) $2xy + y = 1$

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7.R: Analytic Geometry (Review)

10.1: The Ellipse

In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.

For the exercises 1-4, write the equation of the ellipse in standard form. Then identify the center, vertices, and foci.

1) $\frac{x^2}{25}+\frac{y^2}{64}=1$

Answer

$\frac{x^2}{5^2}+\frac{y^2}{8^2}=1$; center: $(0,0)$; Vertices: $(5,0)$, $(-5,0)$, $(0,8)$, $(0,-8)$; foci: $(0,\sqrt{39})$, $(0,-\sqrt{39})$

2) $\frac{(x-2)^2}{100}+\frac{(y+3)^2}{36}=1$

3) $9x^2+y^2+54x-4y+76=0$

Answer

$\frac{(x+3)^2}{1^2}+\frac{(y-2)^2}{3^2}=1$; center: $(-3,2)$; Vertices: $(-2,2)$, $(-4,2)$, $(-3,5)$, $(-3,-1)$; foci: $(-3,2+2\sqrt{2})$, $(-3,2-2\sqrt{2})$

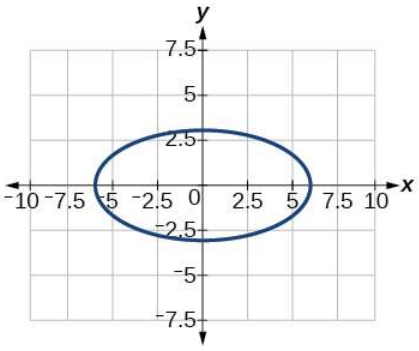
4) $9x^2+36y^2-36x+72y+36=0$

For the exercises 5-8, graph the ellipse, noting center, vertices, and foci.

5) $\frac{x^2}{36}+\frac{y^2}{9}=1$

Answer

center: $(0,0)$; Vertices: $(6,0$, $(-6,0)$, $(0,3)$, $(0,-3)$; foci: $(3\sqrt{3},0)$, $(-3\sqrt{3},0)$

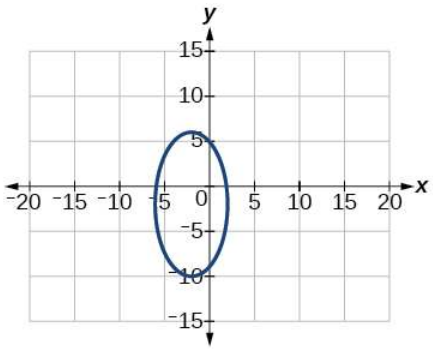


6) $\frac{(x-4)^2}{25}+\frac{(y+3)^2}{49}=1$

7) $4x^2+y^2+16x+4y-44=0$

Answer

center: $(-2,-2)$; Vertices: $(2,-2)$, $(-6,-2)$, $(-2,6)$, $(-2,-10)$; foci: $(-2,-2+4\sqrt{3})$, $(-2,-2-4\sqrt{3})$



8) $2x^2+3y^2-20x+12y+38=0$

For the exercises 9-11, use the given information to find the equation for the ellipse.

9) Center at $(0,0)$, focus at $(3,0)$, vertex at $(-5,0)$

Answer

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

10) Center at $(2,-2)$, vertex at $(7,-2)$, focus at $(4,-2)$

11) A whispering gallery is to be constructed such that the foci are located 35 feet from the center. If the length of the gallery is to be 100 feet, what should the height of the ceiling be?

Answer

Approximately 35.71feet

10.2: The Hyperbola

In analytic geometry, a hyperbola is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other.

For the exercises 1-4, write the equation of the hyperbola in standard form. Then give the center, vertices, and foci.

1) $\frac{x^2}{81}-\frac{y^2}{9}=1$

2) $\frac{(y+1)^2}{16}-\frac{(x-4)^2}{36}=1$

Answer

$\frac{(y+1)^2}{4^2}-\frac{(x-4)^2}{6^2}=1$; center: $(4,-1)$; Vertices: $(4,3)$, $(4,-5)$; foci: $(4,-1+2\sqrt{13})$, $(4,-1-2\sqrt{13})$

3) $9y^2-4x^2+54y-16x+29=0$

4) $3x^2-y^2-12x-6y-9=0$

Answer

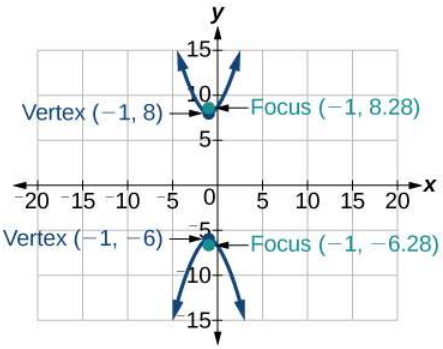
$\frac{(x-2)^2}{2^2}-\frac{(y+3)^2}{(2\sqrt{3})^2}=1$; center: $(2,-3)$; Vertices: $(4,-3)$, $(0,-3)$; foci: $(6,-3)$, $(-2,-3)$

For the exercises 5-8, graph the hyperbola, labeling vertices and foci.

5) $\frac{x^2}{9}-\frac{y^2}{16}=1$

6) $\frac{(y-1)^2}{49} - \frac{(x+1)^2}{4} = 1$

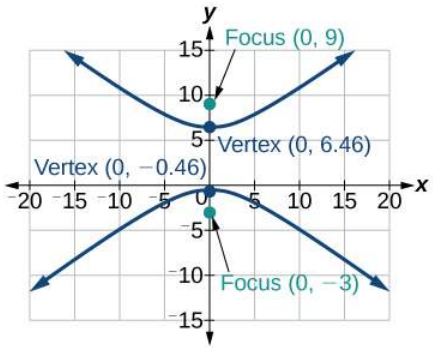
Answer



7) $x^2 - 4y^2 + 6x + 32y - 91 = 0$

8) $2y^2 - x^2 - 12y - 6 = 0$

Answer



For the exercises 9-10, find the equation of the hyperbola.

9) Center at $(0, 0)$, vertex at $(0, 4)$, focus at $(0, -6)$

10) Foci at $(3, 7)$ and $(7, 7)$, vertex at $(6, 7$

Answer

$\frac{(x-5)^2}{1} - \frac{(y-7)^2}{3} = 1$

10.3: The Parabola

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix.

For the exercises 1-4, write the equation of the parabola in standard form. Then give the vertex, focus, and directrix.

1) $y^2 = 12x$

2) $(x+2)^2 = \frac{1}{2}(y-1)$

Answer

$(x+2)^2 = \frac{1}{2}(y-1)$; vertex: $(-2, 1)$; focus: $\left(-2, \frac{9}{8}\right)$; directrix: $y = \frac{7}{8}$

3) $y^2 - 6y - 6x - 3 = 0$

4) $x^2 + 10x - y + 23 = 0$

Answer

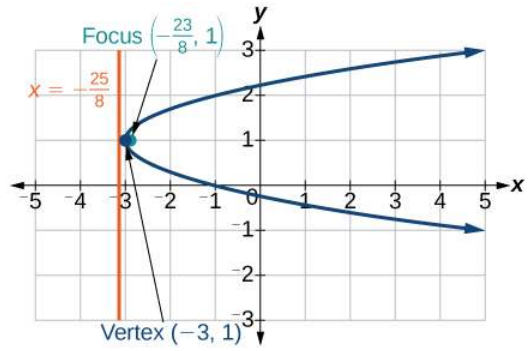
$(x+5)^2 = (y+2)$; vertex: $(-5, -2)$; focus: $\left(-5, -\frac{7}{4}\right)$; directrix: $y = -\frac{9}{4}$

For the exercises 5-8, graph the parabola, labeling vertex, focus, and directrix.

5) $x^2 + 4y = 0$

6) $(y-1)^2 = \frac{1}{2}(x+3)$

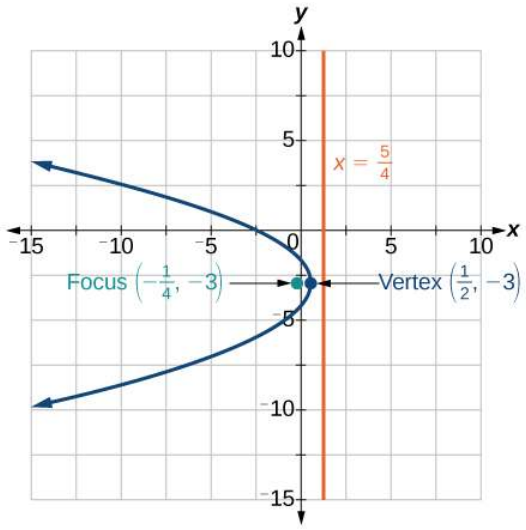
Answer



7) $x^2 - 8x - 10y + 46 = 0$

8) $2y^2 + 12y + 6x + 15 = 0$

Answer



For the exercises 9-11, write the equation of the parabola using the given information.

9) Focus at $(-4, 0)$; directrix is $x = 4$

10) Focus at $\left(2, \frac{9}{8}\right)$; directrix is $y = \frac{7}{8}$

Answer

$$(x-2)^2=\left(\frac{1}{2}\right)(y-1)$$

11) A cable TV receiving dish is the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is 5 feet across at its opening and 1.5 feet deep.

10.4: Rotation of Axes

In previous sections of this chapter, we have focused on the standard form equations for nondegenerate conic sections. In this section, we will shift our focus to the general form equation, which can be used for any conic. The general form is set equal to zero, and the terms and coefficients are given in a particular order, as shown below.

For the exercises 1-3, determine which of the conic sections is represented.

1) $16x^2+24xy+9y^2+24x-60y-60=0$

Answer

$B^2-4AC=0$, parabola

2) $4x^2+14xy+5y^2+18x-6y+30=0$

3) $4x^2+xy+2y^2+8x-26y+9=0$

Answer

$B^2-4AC=-31<0$, ellipse

For the exercises 4-5, determine the angle θ that will eliminate the xy term, and write the corresponding equation without the xy term.

4) $x^2+4xy-2y^2-6=0$

5) $x^2-xy+y^2-6=0$

Answer

$\theta=45^\circ, x'^2+3y'^2-12=0$

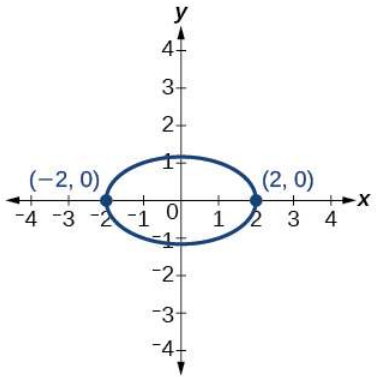
For the exercises 6-8, graph the equation relative to the $x'y'$ system in which the equation has no $x'y'$ term.

6) $9x^2-24xy+16y^2-80x-60y+100=0$

7) $x^2-xy+y^2-2=0$

Answer

$\theta=45^\circ$



8) $6x^2+24xy-y^2-12x+26y+11=0$

10.5: Conic Sections in Polar Coordinates

In this section, we will learn how to define any conic in the polar coordinate system in terms of a fixed point, the focus at the pole, and a line, the directrix, which is perpendicular to the polar axis.

For the exercises 1-4, given the polar equation of the conic with focus at the origin, identify the eccentricity and directrix.

1) $r = \dfrac{10}{1 - 5\cos\theta}$

Answer

Hyperbola with $e = 5$ and directrix 2 units to the left of the pole.

2) $r = \dfrac{6}{3 + 2\cos\theta}$

3) $r = \dfrac{1}{4 + 3\sin\theta}$

Answer

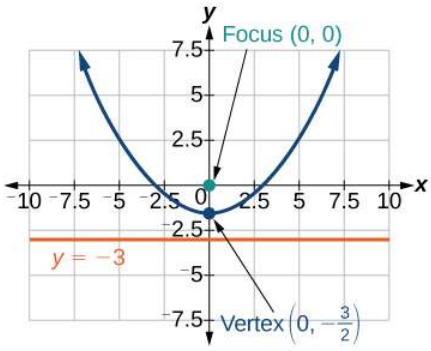
Ellipse with $e = \dfrac{3}{4}$ and directrix $\dfrac{1}{3}$ unit above the pole.

4) $r = \dfrac{3}{5 - 5\sin\theta}$

For the exercises 5-8, graph the conic given in polar form. If it is a parabola, label the vertex, focus, and directrix. If it is an ellipse or a hyperbola, label the vertices and foci.

5) $r = \dfrac{3}{1 - \sin\theta}$

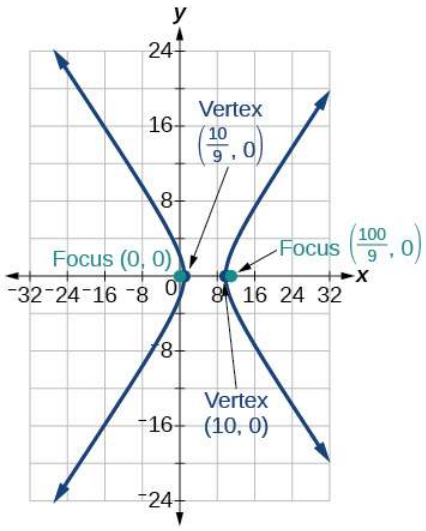
Answer



6) $r = \dfrac{8}{4 + 3\sin\theta}$

7) $r = \dfrac{10}{4 + 5\cos\theta}$

Answer



8) $r = \dfrac{9}{3 - 6\cos\theta}$

For the exercises 9-10, given information about the graph of a conic with focus at the origin, find the equation in polar form.

9) Directrix is $x = 3$ and eccentricity $e = 1$

Answer

$$r = \frac{3}{1 + \cos\theta}$$

10) Directrix is $y = -2$ and eccentricity $e = 4$

Practice Test

For the exercises 1-2, write the equation in standard form and state the center, vertices, and foci.

1) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Answer

$\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$; center: $(0, 0)$; vertices: $(3, 0)$, $(-3, 0)$, $(0, 2)$, $(0, -2)$; foci: $(\sqrt{5}, 0)$, $(-\sqrt{5}, 0)$

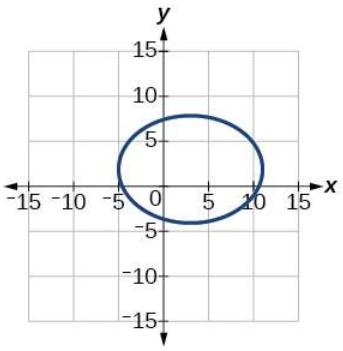
2) $9y^2 + 16x^2 - 36y + 32x - 92 = 0$

For the exercises 3-6, sketch the graph, identifying the center, vertices, and foci.

3) $\dfrac{(x-3)^2}{64} + \dfrac{(y-2)^2}{36} = 1$

Answer

center: $(3, 2)$; vertices: $(11, 2)$, $(-5, 2)$, $(3, 8)$, $(3, -4)$; foci: $(3 + 2\sqrt{7}, 2)$, $(3 - 2\sqrt{7}, 2)$



4) $2x^2+y^2+8x-6y-7=0$

5) Write the standard form equation of an ellipse with a center at $(1,2)$, vertex at $(7,2)$, and focus at $(4,2)$.

Answer

$$\frac{(x-1)^2}{36}+\frac{(y-2)^2}{27}=1$$

6) A whispering gallery is to be constructed with a length of 150 feet. If the foci are to be located 20 feet away from the wall, how high should the ceiling be?

For the exercises 7-8, write the equation of the hyperbola in standard form, and give the center, vertices, foci, and asymptotes.

7) $\frac{x^2}{49}-\frac{y^2}{81}=1$

Answer

$\frac{x^2}{7^2}-\frac{y^2}{9^2}=1$; center: $(0,0)$; vertices: $(7,0)$, $(-7,0)$; foci: $(\sqrt{130},0)$, $(-\sqrt{130},0)$; asymptotes: $y=\pm\frac{9}{7}x$

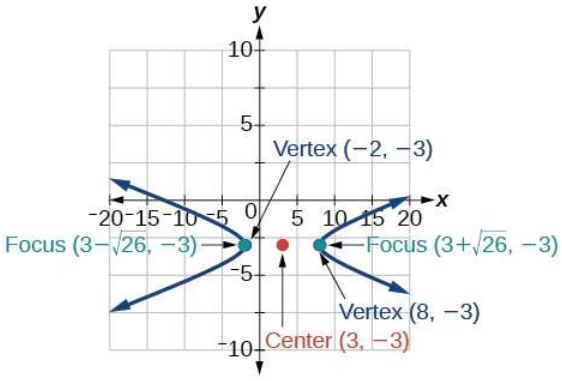
8) $16y^2-9x^2+128y+112=0$

For the exercises 9-11, graph the hyperbola, noting its center, vertices, and foci. State the equations of the asymptotes.

9) $\frac{(x-3)^2}{25}-\frac{(y+3)^2}{1}=1$

Answer

center: $(3,-3)$; vertices: $(8,-3)$, $(-2,-3)$; foci: $(3+\sqrt{26},-3)$, $(3-\sqrt{26},-3)$; asymptotes: $y=\pm\frac{1}{5}(x-3)-3$



10) $y^2-x^2+4y-4x-18=0$

11) Write the standard form equation of a hyperbola with foci at $(1,0)$, and $(1,6)$, and a vertex at $(1,2)$.

Answer

$$\frac{(y-3)^2}{1}-\frac{(x-1)^2}{8}=1$$

For the exercises 12-13, write the equation of the parabola in standard form, and give the vertex, focus, and equation of the directrix.

12) $y^2+10x=0$

13) $3x^2-12x-y+11=0$

Answer

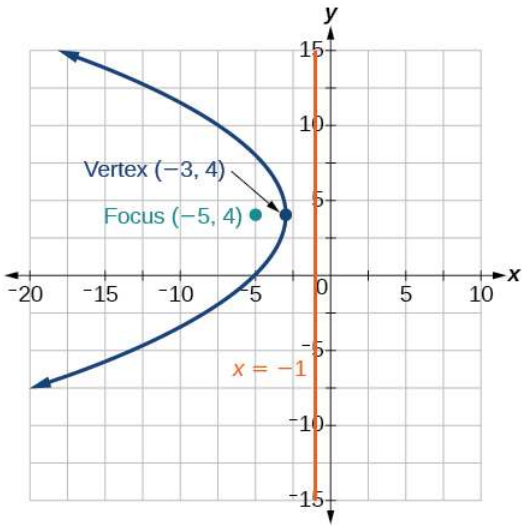
$(x-2)^2=\frac{1}{3}(y+1)$; vertex: $(2,-1)$; focus: $(2,-\frac{11}{12})$; directrix: $y=-\frac{13}{12}$

For the exercises 14-17, graph the parabola, labeling the vertex, focus, and directrix.

14) $(x-1)^2=-4(y+3)$

15) $y^2+8x-8y+40=0$

Answer



16) Write the equation of a parabola with a focus at $(2,3)$ and directrix $y=-1$.

17) A searchlight is shaped like a paraboloid of revolution. If the light source is located 1.5 feet from the base along the axis of symmetry, and the depth of the searchlight is 3 feet, what should the width of the opening be?

Answer

Approximately 8.49feet

For the exercises 18-19, determine which conic section is represented by the given equation, and then determine the angle θ that will eliminate the xy term.

18) $3x^2-2xy+3y^2=4$

19) $x^2+4xy+4y^2+6x-8y=0$

Answer

parabola; $\theta\approx 63.4^\circ$

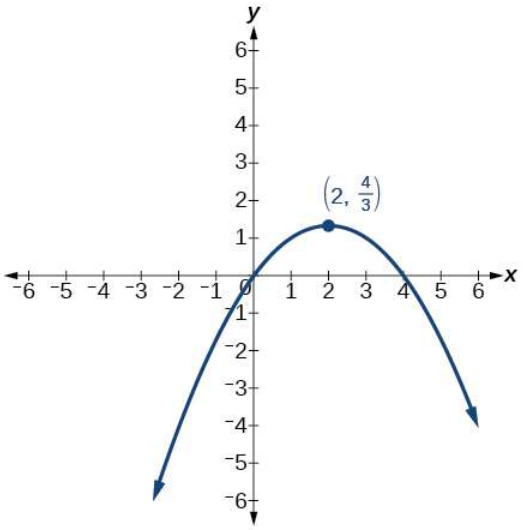
For the exercises 20-21, rewrite in the $x'y'$ system without the $x'y'$ term, and graph the rotated graph.

20) $11x^2+10\sqrt{3}xy+y^2=4$

21) $16x^2+24xy+9y^2-125x=0$

Answer

$x'^2-4x'+3y'=0$



For the exercises 22-23, identify the conic with focus at the origin, and then give the directrix and eccentricity.

22) $r=\dfrac{3}{2-\sin\theta}$

23) $r=\dfrac{5}{4+6\cos\theta}$

Answer

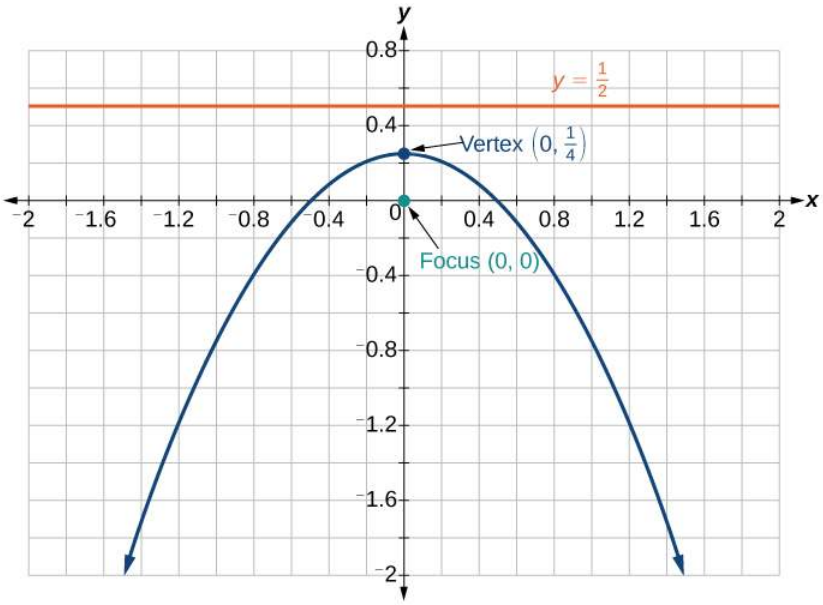
Hyperbola with $e=\dfrac{3}{2}$ and directrix $\dfrac{5}{6}$ units to the right of the pole.

For the exercises 24-26, graph the given conic section. If it is a parabola, label vertex, focus, and directrix. If it is an ellipse or a hyperbola, label vertices and foci.

24) $r=\dfrac{12}{4-8\sin\theta}$

25) $r=\dfrac{2}{4+4\sin\theta}$

Answer



26) Find a polar equation of the conic with focus at the origin, eccentricity of $e = 2$, and directrix: $x = 3$.

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CHAPTER OVERVIEW

8: Introduction to Calculus

Calculus is the broad area of mathematics dealing with such topics as instantaneous rates of change, areas under curves, and sequences and series. Underlying all of these topics is the concept of a limit, which consists of analyzing the behavior of a function at points ever closer to a particular point, but without ever actually reaching that point. Calculus has two basic applications: differential calculus and integral calculus.

8.0: Prelude to Calculus
8.1: Finding Limits - Numerical and Graphical Approaches
8.2: Finding Limits - Properties of Limits
8.3: Continuity
8.4: Derivatives
8.E: Introduction to Calculus (Exercises)
8.R: Introduction to Calculus (Review)

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8.0: Prelude to Calculus

The eight-time world champion and winner of six Olympic gold medals in sprinting, Usain Bolt has truly earned his nickname as the “fastest man on Earth.” Also known as the “lightning bolt,” he set the track on fire by running at a top speed of 27.79 mph—the fastest time ever recorded by a human runner.



Like the fastest land animal, a cheetah, Bolt does not run at his top speed at every instant. How then, do we approximate his speed at any given instant? We will find the answer to this and many related questions in this chapter.

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8.1: Finding Limits - Numerical and Graphical Approaches

Intuitively, we know what a **limit** is. A car can go only so fast and no faster. A trash can might hold 33 gallons and no more. It is natural for measured amounts to have limits. What, for instance, is the limit to the height of a woman? The tallest woman on record was Jinlian Zeng from China, who was 8 ft 1 in.[1] Is this the limit of the height to which women can grow? Perhaps not, but there is likely a limit that we might describe in inches if we were able to determine what it was.

To put it mathematically, the function whose input is a woman and whose output is a measured height in inches has a limit. In this section, we will examine numerical and graphical approaches to identifying limits.

Understanding Limit Notation

We have seen how a **sequence** can have a limit, a value that the sequence of terms moves toward as the nu mber of terms increases. For example, the terms of the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$$

gets closer and closer to 0. A sequence is one type of function, but functions that are not sequences can also have limits. We can describe the behavior of the function as the input values get close to a specific value. If the limit of a function $f(x) = L$, then as the input x gets closer and closer to a, the output y-coordinate gets closer and closer to L. We say that the output "approaches" L.

Figure 8.1.1 provides a visual representation of the mathematical concept of limit. As the input value x approaches a, the output value $f(x)$ approaches L.

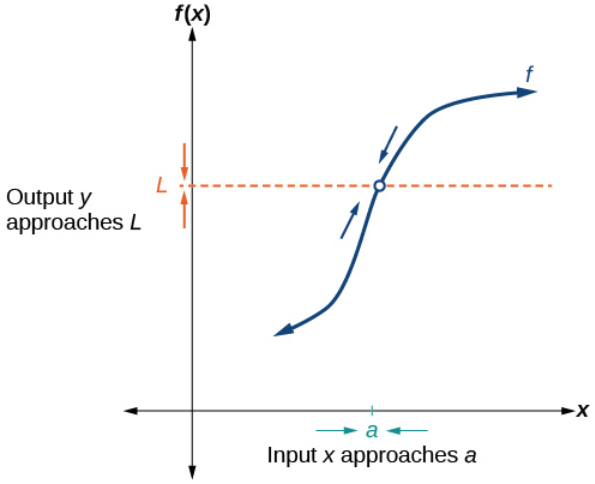


Figure 8.1.1: The output (y--coordinate) approaches L as the input (x-coordinate) approaches a.

We write the equation of a limit as

$$\lim_{x \to a} f(x) = L.$$

This notation indicates that as x approaches a both from the left of $x = a$ and the right of $x = a$, the output value approaches L.

Consider the function

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}. \tag{8.1.1}$$

We can factor the function in Equation 8.1.1 as shown.

$$f(x) = \frac{\cancel{(x-7)}(x+1)}{\cancel{x-7}} \qquad \text{Cancel like factors in numerator and denominator.}$$

$$f(x) = x + 1, x \neq 7 \qquad \text{Simplify.}$$

Notice that x cannot be 7, or we would be dividing by 0, so 7 is not in the domain of the function in Equation 8.1.1. To avoid changing the function when we simplify, we set the same condition, $x \neq 7$, for the simplified function. We can represent the function graphically as shown in Figure 8.1.2.

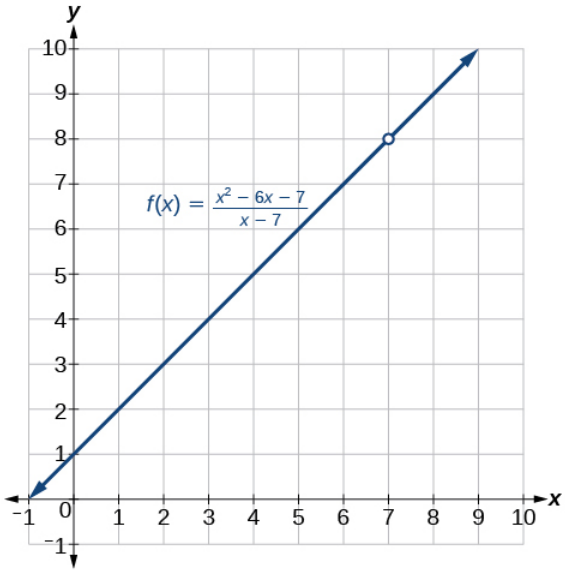


Figure 8.1.2: Because 7 is not allowed as an input, there is no point at $x = 7$.

What happens at $x = 7$ is completely different from what happens at points close to $x = 7$ on either side. The notation

$$\lim_{x \to 7} f(x) = 8$$

indicates that as the input x approaches 7 from either the left or the right, the output approaches 8. The output can get as close to 8 as we like if the input is sufficiently near 7.

What happens at $x = 7$? When $x = 7$, there is no corresponding output. We write this as

$$f(7) \text{does not exist.}$$

This notation indicates that 7 is not in the domain of the function. We had already indicated this when we wrote the function as

$$f(x) = x + 1, x \neq 7.$$

Notice that the limit of a function can exist even when $f(x)$ is not defined at $x = a$. Much of our subsequent work will be determining limits of functions as x nears a, even though the output at $x = a$ does not exist.

THE LIMIT OF A FUNCTION

A quantity L is the limit of a function $f(x)$ as x approaches a if, as the input values of x approach a (but do not equal a),the corresponding output values of $f(x)$ get closer to L. Note that the value of the limit is not affected by the output value of $f(x)$ at a. Both a and L must be real numbers. We write it as

$$\lim_{x \to a} f(x) = L$$

Example 8.1.1: Understanding the Limit of a Function

For the following limit, define a, $f(x)$, and L.

$$\lim_{x \to 2}(3x + 5) = 11$$

Solution

First, we recognize the notation of a limit. If the limit exists, as x approaches a, we write

$$\lim_{x \to a} f(x) = L.$$

We are given

$$\lim_{x \to 2}(3x + 5) = 11.$$

This means that $a = 2, f(x) = 3x + 5,$ and $L = 11.$

Analysis

Recall that $y = 3x + 5$ is a line with no breaks. As the input values approach 2, the output values will get close to 11. This may be phrased with the equation $\lim_{x \to 2}(3x + 5) = 11,$ which means that as x nears 2 (but is not exactly 2), the output of the function $f(x) = 3x + 5$ gets as close as we want to $3(2) + 5,$ or 11, which is the limit L, as we take values of x sufficiently near 2 but not at $x = 2.$

Exercise 8.1.1

For the following limit, define a, $f(x)$, and L.

$$\lim_{x \to 5}(2x^2 - 4) = 46$$

Solution

$a = 5, f(x) = 2x^2 - 4,$ and $L = 46.$

Understanding Left-Hand Limits and Right-Hand Limits

We can approach the input of a function from either side of a value—from the left or the right. Figure 8.1.3 shows the values of

$$f(x) = x + 1, x \neq 7$$

as described earlier and depicted in Figure 8.1.3.

	Values of x approach 7 from the left ($x < 7$)			$x = 7$	Values of x approach 7 from the right ($x > 7$)		
x	6.9	6.99	6.999	7	7.001	7.01	7.1
$f(x)$	7.9	7.99	7.999	Undefined	8.001	8.01	8.1
	Values of output approach the limit, 8				Values of output approach the limit, 8		

Figure 8.1.3 are $6.9, 6.99,$ and 6.999. The corresponding outputs are $7.9, 7.99,$ and 7.999. These values are getting closer to 8. The limit of values of $f(x)$ as x approaches from the left is known as the left-hand limit. For this function, 8 is the left-hand limit of the function $f(x) = x + 1, x \neq 7$ as x approaches 7.

Values described as "from the right" are greater than the input value 7 and would therefore appear to the right of the value on a number line. The input values that approach 7 from the right in Figure 8.1.3 are $7.1, 7.01,$and 7.001. The corresponding outputs are $8.1, 8.01,$and 8.001. These values are getting closer to 8. The limit of values of $f(x)$ as x approaches from the right is known as the right-hand limit. For this function, 8 is also the right-hand limit of the function $f(x) = x + 1, x \neq 7$ as x approaches 7.

Figure 8.1.3 shows that we can get the output of the function within a distance of 0.1 from 8 by using an input within a distance of 0.1 from 7. In other words, we need an input x within the interval $6.9 < x < 7.1$ to produce an output value of $f(x)$ within the interval $7.9 < f(x) < 8.1$.

We also see that we can get output values of $f(x)$ successively closer to 8 by selecting input values closer to 7. In fact, we can obtain output values within any specified interval if we choose appropriate input values.

Figure 8.1.4 provides a visual representation of the left- and right-hand limits of the function. From the graph of $f(x)$, we observe the output can get infinitesimally close to $L = 8$ as x approaches 7 from the left and as x approaches 7 from the right.

To indicate the left-hand limit, we write

$$\lim_{x \to 7^-} f(x) = 8.$$

To indicate the right-hand limit, we write

$$\lim_{x \to 7^+} f(x) = 8.$$

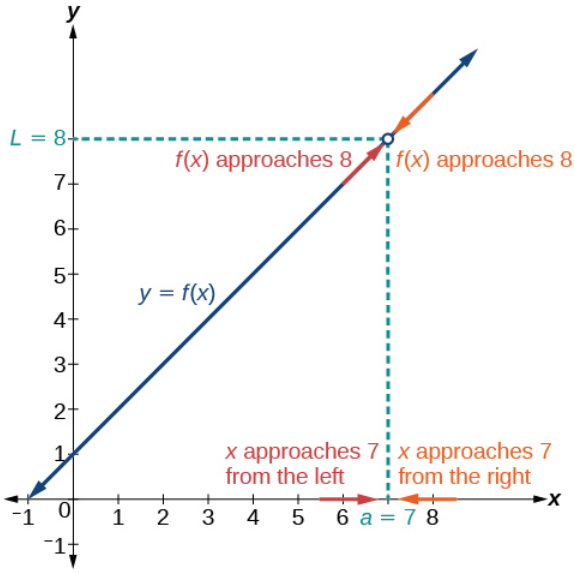


Figure 8.1.4: The left- and right-hand limits are the same for this function.

LEFT- AND RIGHT-HAND LIMITS

The left-hand limit of a function $f(x)$ as x approaches a from the left is equal to L, denoted by

$$\lim_{x \to a^-} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a such that $x < a$ and $x \neq a$.

The right-hand limit of a function $f(x)$, as x approaches a from the right, is equal to L, denoted by

$$\lim_{x \to a^+} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a but greater than a. Both a and L are real numbers.

Understanding Two-Sided Limits

In the previous example, the left-hand limit and right-hand limit as x approaches a are equal. If the left- and right-hand limits are equal, we say that the function $f(x)$ has a two-sided limit as x approaches a. More commonly, we simply refer to a **two-sided limit** as a limit. If the left-hand limit does not equal the right-hand limit, or if one of them does not exist, we say the limit does not exist.

A

The limit of a function $f(x)$, as x approaches a, is equal to L, that is,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

In other words, the left-hand limit of a function $f(x)$ as x approaches a is equal to the right-hand limit of the same function as x approaches a. If such a limit exists, we refer to the limit as a two-sided limit. Otherwise we say the limit does not exist.

Finding a Limit Using a Graph

To visually determine if a limit exists as x approaches a, we observe the graph of the function when x is very near to $x = a$. In Figure $8.1.5$ we observe the behavior of the graph on both sides of a.

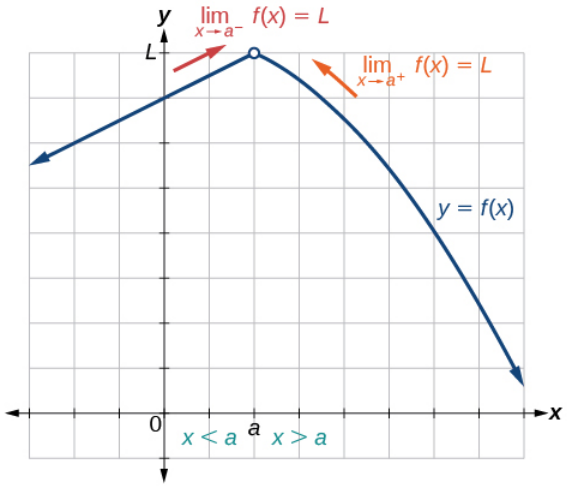


Figure 8.1.5

To determine if a left-hand limit exists, we observe the branch of the graph to the left of $x = a$, but near $x = a$. This is where $x < a$. We see that the outputs are getting close to some real number L so there is a left-hand limit.

To determine if a right-hand limit exists, observe the branch of the graph to the right of $x = a$, but near $x = a$. This is where $x > a$. We see that the outputs are getting close to some real number L, so there is a right-hand limit.

If the left-hand limit and the right-hand limit are the same, as they are in Figure $8.1.5$, then we know that the function has a two-sided limit. Normally, when we refer to a "limit," we mean a two-sided limit, unless we call it a one-sided limit.

Finally, we can look for an output value for the function $f(x)$ when the input value x is equal to a. The coordinate pair of the point would be $(a, f(a))$. If such a point exists, then $f(a)$ has a value. If the point does not exist, as in Figure $8.1.5$, then we say that $f(a)$ does not exist.

HOW TO: Given a function $f(x)$, use a graph to find the limits and a function value as x approaches a.

1. Examine the graph to determine whether a left-hand limit exists.
2. Examine the graph to determine whether a right-hand limit exists.
3. If the two one-sided limits exist and are equal, then there is a two-sided limit—what we normally call a "limit."
4. If there is a point at $x = a$, then $f(a)$ is the corresponding function value.

Example 8.1.2: Finding a Limit Using a Graph

Determine the following limits and function value for the function f shown in Figure $8.1.6$.

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$
c. $\lim_{x \to 2} f(x)$
d. $f(2)$

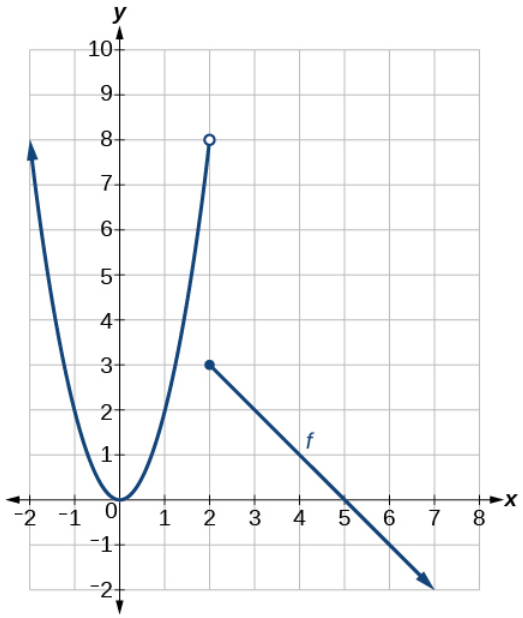


Figure 8.1.6

Determine the following limits and function value for the function f f shown in Figure.

i. $\lim_{x\to 2^-} f(x)$
ii. $\lim_{x\to 2^+} f(x)$
iii. $\lim_{x\to 2} f(x)$
iv. $f(2)$

a. Looking at Figure:
   i. $\lim_{x\to 2^-} f(x) = 8$; when $x < 2$,but infinitesimally close to 2, the output values get close to $y = 8$.
   ii. $\lim_{x\to 2^+} f(x) = 3$; when $x > 2$,but infinitesimally close to 2, the output values approach $y = 3$.
   iii. $\lim_{x\to 2} f(x)$ does not exist because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$; the left and right-hand limits are not equal.
   iv. $f(2) = 3$ because the graph of the function f passes through the point $(2, f(2))$ or $(2, 3)$.

b. Looking at Figure:
   i. $\lim_{x\to 2^-} f(x) = 8$; when $x < 2$ but infinitesimally close to 2, the output values approach $y = 8$.
   ii. $\lim_{x\to 2^+} f(x) = 8$; when $x > 2$ but infinitesimally close to 2, the output values approach $y = 8$.
   iii. $\lim_{x\to 2} f(x) = 8$; because $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 8$; the left and right-hand limits are equal.
   iv. $f(2) = 4$ because the graph of the function f passes through the point $(2, f(2))$ or $(2, 4)$.

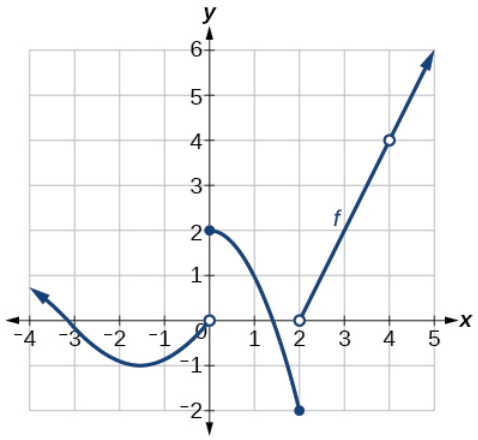
Exercise 8.1.2:

Using the graph of the function y=f(x) y=f(x) shown in Figure, estimate the following limits.

a. $\lim_{x \to 0^-} f(x)$ b. $\lim_{x \to 0^+} f(x)$ c. $\lim_{x \to 0} f(x)$

d. $\lim_{x \to 2^-} f(x)$ e. $\lim_{x \to 2^+} f(x)$ f. $\lim_{x \to 2} f(x)$

g. $\lim_{x \to 4^-} f(x)$ h. $\lim_{x \to 4^+} f(x)$ i. $\lim_{x \to 4} f(x)$



Solution

a. 0; b. 2; c. does not exist; d.−2; −2; e. 0; f. does not exist; g. 4; h. 4; i. 4

Finding a Limit Using a Table

Creating a table is a way to determine limits using numeric information. We create a table of values in which the input values of x approach a from both sides. Then we determine if the output values get closer and closer to some real value, the limit L.

Let's consider an example using the following function:

$$\lim_{x \to 5} \left(\frac{x^3 - 125}{x - 5} \right)$$

To create the table, we evaluate the function at values close to $x = 5$. We use some input values less than 5 and some values greater than 5 as in Figure. The table values show that when $x > 5$ but nearing 5, the corresponding output gets close to 75. When $x > 5$ but nearing 5, the corresponding output also gets close to 75.

x	4.9	4.99	4.999	5	5.001	5.01	5.1
f(x)	73.51	74.8501	74.985001	Undefined	75.015001	75.1501	76.51

$\lim_{x \to 5^-} f(x) = 75$ $\lim_{x \to 5^+} f(x) = 75$

Because

$$\lim_{x \to 5^-} f(x) = 75 = \lim_{x \to 5^+} f(x),$$

then

$$\lim_{x \to 5} f(x) = 75.$$

Remember that $f(5)$ does not exist.

How to: Given a function f,use a table to find the limit as x approaches a and the value of $f(a)$, if it exists.

1. Choose several input values that approach a a from both the left and right. Record them in a table.
2. Evaluate the function at each input value. Record them in the table.
3. Determine if the table values indicate a left-hand limit and a right-hand limit.
4. If the left-hand and right-hand limits exist and are equal, there is a two-sided limit.
5. Replace x with a to find the value of $f(a)$.

Example 8.1.3: Finding a Limit Using a Table

Numerically estimate the limit of the following expression by setting up a table of values on both sides of the limit.

$$\lim_{x\to 0}\left(\frac{5\sin(x)}{3x}\right)$$

Solution

We can estimate the value of a limit, if it exists, by evaluating the function at values near $x=0$. We cannot find a function value for $x=0$ directly because the result would have a denominator equal to 0, and thus would be undefined.

$$f(x)=\frac{5\sin(x)}{3x}$$

We create Figure by choosing several input values close to $x=0$, with half of them less than $x=0$ and half of them greater than $x=0$. Note that we need to be sure we are using radian mode. We evaluate the function at each input value to complete the table.

The table values indicate that when $x<0$ but approaching 0, the corresponding output nears $\frac{5}{3}$.

When $x>0$ but approaching 0, the corresponding output also nears $\frac{5}{3}$.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	1.66389	1.666639	1.666666	Undefined	1.666666	1.666639	1.66389

$\lim_{x\to 0^-} f(x)=\frac{5}{3}$ $\lim_{x\to 0^+} f(x)=\frac{5}{3}$

Because

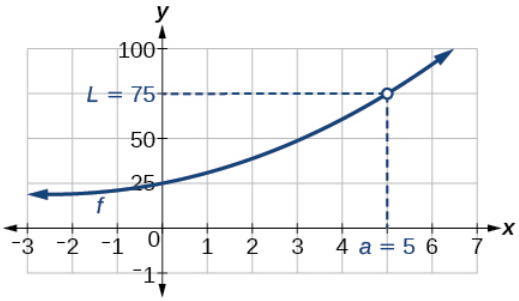
$$\lim_{x\to 0^-} f(x)=\frac{5}{3}=\lim_{x\to 0^+} f(x),$$

then

$$\lim_{x\to 0} f(x)=\frac{5}{3}.$$

Q & A: Is it possible to check our answer using a graphing utility?

Yes. We previously used a table to find a limit of 75 for the function $f(x)=\frac{x^3-125}{x-5}$ as x approaches 5. To check, we graph the function on a viewing window as shown in Figure. A graphical check shows both branches of the graph of the function get close to the output 75 as x nears 5. Furthermore, we can use the 'trace' feature of a graphing calculator. By appraoching $x=5$ we may numerically observe the corresponding outputs getting close to 75.



Exercise 8.1.3:

Numerically estimate the limit of the following function by making a table:

$$\lim_{x \to 0} \left(\frac{20 \sin(x)}{4x} \right)$$

Solution

$$\lim_{x \to 0} \left(\frac{20 \sin(x)}{4x} \right) = 5$$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	4.9916708	4.9999167	4.9999992	Error	4.9999992	4.9999167	4.9916708

$\lim_{x \to 0^-} \left(\frac{20 \sin(x)}{4x} \right) \longrightarrow 5 \qquad 5 \longleftarrow \lim_{x \to 0^+} \left(\frac{20 \sin(x)}{4x} \right)$

Q & A

Is one method for determining a limit better than the other?

No. Both methods have advantages. Graphing allows for quick inspection. Tables can be used when graphical utilities aren't available, and they can be calculated to a higher precision than could be seen with an unaided eye inspecting a graph.

Example 8.1.4: Using a Graphing Utility to Determine a Limit

With the use of a graphing utility, if possible, determine the left- and right-hand limits of the following function as x approaches 0. If the function has a limit as x approaches 0, state it. If not, discuss why there is no limit.

$$f(x) = 3 \sin(\frac{\pi}{x})$$

Solution

We can use a graphing utility to investigate the behavior of the graph close to $x = 0$. Centering around $x = 0$, we choose two viewing windows such that the second one is zoomed in closer to $x = 0$ than the first one. The result would resemble Figure for $[-2, 2]$ by $[-3, 3]$.

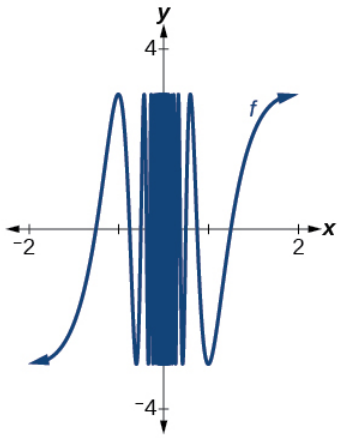


Figure_12_01_012">Figure for$[-0.1, 0.1]$ by $[-3, 3]$.

Even closer to zero, we are even less able to distinguish any limits.

The closer we get to 0, the greater the swings in the output values are. That is not the behavior of a function with either a left-hand limit or a right-hand limit. And if there is no left-hand limit or right-hand limit, there certainly is no limit to the function $f(x)$ as x approaches 0.

We write

$$\lim_{x \to 0^-} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

$$\lim_{x \to 0^+} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

$$\lim_{x \to 0} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

Exercise 8.1.4:

Numerically estimate the following limit: $\lim_{x \to 0}(\sin(\frac{2}{x}))$.

Solution

does not exist

Access these online resources for additional instruction and practice with finding limits.

- Introduction to Limits
- Formal Definition of a Limit

Key Concepts

- A function has a limit if the output values approach some value L as the input values approach some quantity a. a. See Example.
- A shorthand notation is used to describe the limit of a function according to the form $\lim_{x \to a} f(x) = L$, which indicates that as x approaches a, both from the left of $x = a$ and the right of $x = a$, the output value gets close to L.
- A function has a left-hand limit if $f(x)$ approaches L as x approaches a a where $x < a$. A function has a right-hand limit if $f(x)$ approaches L as x approaches a where $x > a$.
- A two-sided limit exists if the left-hand limit and the right-hand limit of a function are the same. A function is said to have a limit if it has a two-sided limit.
- A graph provides a visual method of determining the limit of a function.
- If the function has a limit as x approaches a, the branches of the graph will approach the same $y-$ coordinate near $x = a$ from the left and the right. See Example.

- A table can be used to determine if a function has a limit. The table should show input values that approach a from both directions so that the resulting output values can be evaluated. If the output values approach some number, the function has a limit. See Example.
- A graphing utility can also be used to find a limit. See Example.

Footnotes

- 1 https://en.Wikipedia.org/wiki/Human_height and http://en.Wikipedia.org/wiki/List_of_tallest_people

Glossary

left-hand limit

the limit of values of $f(x)$ as x approaches from a the left, denoted

$$\lim_{x \to a^-} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a a such that $x < a$ and $x \neq a$.. Both a and L are real numbers.

limit

when it exists, the value, L,that the output of a function $f(x)$ approaches as the input x gets closer and closer to a but does not equal a. The value of the output, $f(x)$, can get as close to L as we choose to make it by using input values of x sufficiently near to $x = a$,but not necessarily at $x = a$. Both a and L are real numbers, and L is denoted

$$\lim_{x \to a} f(x) = L.$$

right-hand limit

the limit of values of $f(x)$ as x approaches a from the right, denoted

$$\lim_{x \to a^+} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a where $x > a,$ and $x \neq a.$ Both a and L are real numbers.

two-sided limit

the limit of a function

$$f(x),$$

as x approaches a,is equal to L,that is,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

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8.2: Finding Limits - Properties of Limits

Consider the *rational function*

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

The function can be factored as follows:

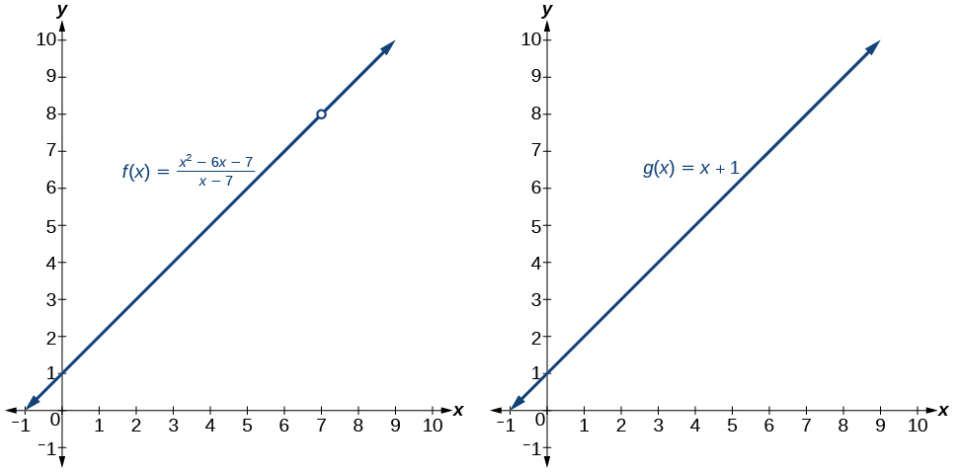
$$f(x) = \frac{\cancel{(x-7)}(x+1)}{\cancel{x-7}}$$

which gives us

$$f(x) = x + 1, x \neq 7.$$

Does this mean the function $f(x)$ is the same as the function $g(x) = x + 1$?

The answer is no. Function $f(x)$ does not have $x = 7$ in its domain, but $g(x)$ does. Graphically, we observe there is a hole in the graph of $f(x)$ at $x = 7$, as shown in Figure and no such hole in the graph of $g(x)$, as shown in Figure.



(left) The graph of function f contains a break at $x = 7$ and is therefore not continuous at $x = 7$. (Right)The graph of function g is continuous.

So, do these two different functions also have different limits as x approaches 7? Not necessarily. Remember, in determining a limit of a function as x approaches a, what matters is whether the output approaches a real number as we get close to $x = a$. The existence of a limit does not depend on what happens when x equals a.

Look again at Figure and Figure. Notice that in both graphs, as x approaches 7, the output values approach 8. This means

$$\lim_{x \to 7} f(x) = \lim_{x \to 7} g(x).$$

Remember that when determining a limit, the concern is what occurs near $x = a$, not at $x = a$. In this section, we will use a variety of methods, such as rewriting functions by factoring, to evaluate the limit. These methods will give us formal verification for what we formerly accomplished by intuition.

Finding the Limit of a Sum, a Difference, and a Product

Graphing a function or exploring a table of values to determine a limit can be cumbersome and time-consuming. When possible, it is more efficient to use the **properties of limits**, which is a collection of theorems for finding limits.

Knowing the properties of limits allows us to compute limits directly. We can add, subtract, multiply, and divide the limits of functions as if we were performing the operations on the functions themselves to find the limit of the result. Similarly, we can find the limit of a function raised to a power by raising the limit to that power. We can also find the limit of the root of a function by taking the root of the limit. Using these operations on limits, we can find the limits of more complex functions by finding the limits of their simpler component functions.

properties of limits

Let a, k, A, and B represent real numbers, and f and g be functions, such that $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$. For limits that exist and are finite, the properties of limits are summarized in Table

Constant, k	$\lim_{x \to a} k = k$
Constant times a function	$\lim_{x \to a}[k \cdot f(x)] = k \lim_{x \to a} f(x) = kA$
Sum of functions	$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{xtoa} g(x) = A + B$

Difference of functions	$\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)=A-B$
Product of functions	$\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)=A\cdot B$
Quotient of functions	$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{A}{B}, B\neq 0$
Function raised to an exponent	$\lim_{x\to a}[f(x)]^n=[\lim_{x\to\infty}f(x)]^n=A^n$, where n is a positive integer
nth root of a function, where n is a positive integer	$\lim_{x\to a}f(x)\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a}[f(x)]}=\sqrt[n]{A}$
Polynomial function	$\lim_{x\to a}p(x)=p(a)$

Example 8.2.1: Evaluating the Limit of a Function Algebraically

Evaluate

$$\lim_{x\to 3}(2x+5).$$

Solution

$$\begin{aligned}\lim_{x\to 3}(2x+5) &= \lim_{x\to 3}(2x)+\lim_{x\to 3}(5) && \text{Sum of functions property} && (8.2.1)\\ &= 2\lim_{x\to 3}(x)+\lim_{x\to 3}(5) && \text{Constant times a function property} && (8.2.2)\\ &= 2(3)+5 && \text{Evaluate} && (8.2.3)\\ &= 11 && && (8.2.4)\end{aligned}$$

Exercise 8.2.1:

Evaluate the following limit:

$$\lim_{x\to -12}(-2x+2).$$

Solution

26

Finding the Limit of a Polynomial

Not all functions or their limits involve simple addition, subtraction, or multiplication. Some may include polynomials. Recall that a polynomial is an expression consisting of the sum of two or more terms, each of which consists of a constant and a variable raised to a nonnegative integral power. To find the limit of a polynomial function, we can find the limits of the individual terms of the function, and then add them together. Also, the limit of a polynomial function as x approaches a is equivalent to simply evaluating the function for a.

how to: Given a function containing a polynomial, find its limit

1. Use the properties of limits to break up the polynomial into individual terms.
2. Find the limits of the individual terms.
3. Add the limits together.
4. Alternatively, evaluate the function for a.

Example 8.2.1: Evaluating the Limit of a Function Algebraically

Evaluate

$$\lim_{x\to 3}(5x^2).$$

Solution

$$\begin{aligned}\lim_{x\to 3}(5x^2) &= 5\lim_{x\to 3}(x^2) && \text{Constant times a function property} && (8.2.5)\\ &= 5(3^2) && \text{Function raised to an exponent property} && (8.2.6)\\ &= 45 && && (8.2.7)\end{aligned}$$

Exercise 8.2.1:

Evaluate

$$\lim_{x\to 4}(x^3-5).$$

Solution

59

Example 8.2.2: Evaluating the Limit of a Polynomial Algebraically

Evaluate

$$\lim_{x\to 5}(2x^3-3x+1).$$

Solution

$$\begin{aligned}\lim_{x\to 5}(2x^3-3x+1) &= \lim_{x\to 5}(2x3)-\lim_{x\to 5}(3x)+\lim_{x\to 5}(1) && \text{Sum of functions} && (8.2.8)\\ &= 2\lim_{x\to 5}(x^3)-3\lim_{x\to 5}(x)+\lim_{x\to 5}(1) && \text{Constant times a function} && (8.2.9)\\ &= 2(5^3)-3(5)+1 && \text{Function raised to an exponent} && (8.2.10)\\ &= 236 && \text{Evaluate} && (8.2.11)\end{aligned}$$

Exercise 8.2.2:

Evaluate the following limit:

$$\lim_{x\to -1}(x^4-4x^3+5).$$

Solution

10

Finding the Limit of a Power or a Root

When a limit includes a power or a root, we need another property to help us evaluate it. The square of the limit of a function equals the limit of the square of the function; the same goes for higher powers. Likewise, the square root of the limit of a function equals the limit of the square root of the function; the same holds true for higher roots.

Example 8.2.3: Evaluating a Limit of a Power

Evaluate

$$\lim_{x\to 2}(3x+1)^5.$$

Solution

We will take the limit of the function as x approaches 2 and raise the result to the 5th power.

$$\begin{aligned}\lim_{x\to 2}(3x+1)^5 &= (\lim_{x\to 2}(3x+1))^5 && (8.2.12)\\ &= (3(2)+1)^5 && (8.2.13)\\ &= 7^5 && (8.2.14)\\ &= 16,807 && (8.2.15)\end{aligned}$$

Exercise 8.2.3:

Evaluate the following limit: $\lim_{x\to -4}(10x+36)^3$.

Solution

−64

Q & A: If we can't directly apply the properties of a limit, for example in $\lim_{x\to 2}(\frac{x^2+6x+8}{x-2})$, can we still determine the limit of the function as x approaches a?

Yes. Some functions may be algebraically rearranged so that one can evaluate the limit of a simplified equivalent form of the function.

Finding the Limit of a Quotient

Finding the limit of a function expressed as a quotient can be more complicated. We often need to rewrite the function algebraically before applying the properties of a limit. If the denominator evaluates to 0 when we apply the properties of a limit directly, we must rewrite the quotient in a different form. One approach is to write the quotient in factored form and simplify.

Example 8.2.4: Evaluating the Limit of a Quotient by Factoring

Evaluate

$$\lim_{x\to 2}\left(\frac{x^2-6x+8}{x-2}\right).$$

Solution

Factor where possible, and simplify.

$$\lim_{x\to 2}\left(\frac{x^2-6x+8}{x-2}\right)=\lim_{x\to 2}\left(\frac{(x-2)(x-4)}{x-2}\right) \quad \text{Factor the numerator.} \tag{8.2.16}$$

$$=\lim_{x\to 2}\left(\frac{\cancel{(x-2)}(x-4)}{\cancel{x-2}}\right) \quad \text{Cancel the common factors.} \tag{8.2.17}$$

$$=\lim_{x\to 2}(x-4) \quad \text{Evaluate.} \tag{8.2.18}$$

$$=2-4=-2 \tag{8.2.19}$$

Analysis

When the limit of a rational function cannot be evaluated directly, factored forms of the numerator and denominator may simplify to a result that can be evaluated.

Notice, the function

$$f(x)=\frac{x^2-6x+8}{x-2}$$

is equivalent to the function

$$f(x)=x-4, x\neq 2.$$

Notice that the limit exists even though the function is not defined at $x=2$.

Exercise 8.2.4

Evaluate the following limit:

$$\lim_{x\to 7}\left(\frac{x^2-11x+28}{7-x}\right).$$

Solution

-3

Example 8.2.5: Evaluating the Limit of a Quotient by Finding the LCD

Evaluate

$$\lim_{x\to 5}\left(\frac{\frac{1}{x}-\frac{1}{5}}{x-5}\right).$$

Solution

Find the LCD for the denominators of the two terms in the numerator, and convert both fractions to have the LCD as their denominator.

Analysis

When determining the limit of a rational function that has terms added or subtracted in either the numerator or denominator, the first step is to find the common denominator of the added or subtracted terms; then, convert both terms to have that denominator, or simplify the rational function by multiplying numerator and denominator by the least common denominator. Then check to see if the resulting numerator and denominator have any common factors.

Exercise 8.2.5:

Evaluate

$$\lim_{x\to -5}\left(\frac{\frac{1}{5}+\frac{1}{x}}{10+2x}\right).$$

Solution

$-\frac{1}{50}$

how to: Given a limit of a function containing a root, use a conjugate to evaluate

1. If the quotient as given is not in indeterminate $(\frac{0}{0})$ form, evaluate directly.
2. Otherwise, rewrite the sum (or difference) of two quotients as a single quotient, using the **least common denominator (LCD).**
3. If the numerator includes a root, rationalize the numerator; multiply the numerator and denominator by the **conjugate** of the numerator. Recall that $a \pm \sqrt{b}$ are conjugates.
4. Simplify.
5. Evaluate the resulting limit.

Example 8.2.6: Evaluating a Limit Containing a Root Using a Conjugate

Evaluate

$$\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right).$$

Solution

$$\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right)=\lim_{x\to 0}\left(\frac{(\sqrt{25-x}-5)}{x}\cdot\frac{(\sqrt{25-x}+5)}{(\sqrt{25-x}+5)}\right) \quad \text{Multiply numerator and denominator by the conjugate.} \quad (8.2.20)$$

$$=\lim_{x\to 0}\left(\frac{(25-x)-25}{x(\sqrt{25-x}+5)}\right) \quad \text{Multiply: } (\sqrt{25-x}-5)\cdot(\sqrt{25-x}+5)=(25-x)-25. \quad (8.2.21)$$

$$=\lim_{x\to 0}\left(\frac{-\cancel{x}}{\cancel{x}(25-x+5)}\right) \quad \text{Combine like terms.} \quad (8.2.22)$$

$$=\lim_{x\to 0}\left(\frac{-\cancel{x}}{\cancel{x}(\sqrt{25-x}+5)}\right) \quad \text{Simplify } \frac{-x}{x}=-1. \quad (8.2.23)$$

$$=\frac{-1}{\sqrt{25-0}+5} \quad \text{Evaluate.} \quad (8.2.24)$$

$$=\frac{-1}{5+5}=-\frac{1}{10} \quad (8.2.25)$$

Analysis

When determining a **limit** of a function with a root as one of two terms where we cannot evaluate directly, think about multiplying the numerator and denominator by the conjugate of the terms.

Exercise 8.2.6

Evaluate the following limit: $\lim_{h\to 0}\left(\frac{\sqrt{16-h}-4}{h}\right)$.

Solution

$-\frac{1}{8}$

Example 8.2.7: Evaluating the Limit of a Quotient of a Function by Factoring

Evaluate

$$\lim_{x\to 4}\left(\frac{4-x}{\sqrt{x}-2}\right).$$

Solution

$$\lim_{x\to 4}\left(\frac{4-x}{\sqrt{x}-2}\right)=\lim_{x\to 4}\left(\frac{(2+\sqrt{x})(2-x)}{\sqrt{x}-2}\right) \quad \text{Factor.} \quad (8.2.26)$$

$$=\lim_{x\to 4}\left(\frac{(2+\sqrt{x})(\,2-\cancel{\sqrt{x}})}{-\,(2-\cancel{\sqrt{x}})}\right) \quad \text{Factor } -1 \text{ out of the denominator. Simplify.} \quad (8.2.27)$$

$$=\lim_{x\to 4}-(2+x) \quad \text{Evaluate.} \quad (8.2.28)$$

$$=-(2+\sqrt{4}) \quad (8.2.29)$$

$$=-4 \quad (8.2.30)$$

Analysis

Multiplying by a conjugate would expand the numerator; look instead for factors in the numerator. Four is a perfect square so that the numerator is in the form

$$a^2-b^2$$

and may be factored as

$$(a+b)(a-b).$$

Exercise 8.2.7

Evaluate the following limit:

$$\lim_{x\to 3}\left(\frac{x-3}{\sqrt{x}-\sqrt{3}}\right).$$

Solution

$2\sqrt{3}$

how to: Given a quotient with absolute values, evaluate its limit

1. Try factoring or finding the LCD.
2. If the **limit** cannot be found, choose several values close to and on either side of the input where the function is undefined.
3. Use the numeric evidence to estimate the limits on both sides.

Example 8.2.8: Evaluating the Limit of a Quotient with Absolute Values

Evaluate

$$\lim_{x\to 7}\frac{|x-7|}{x-7}.$$

Solution

The function is undefined at $x=7$, so we will try values close to 7 from the left and the right.

Left-hand limit:

$$\frac{|6.9-7|}{6.9-7}=\frac{|6.99-7|}{6.99-7}=\frac{|6.999-7|}{6.999-7}=-1$$

Right-hand limit:

$$\frac{|7.1-7|}{7.1-7}=\frac{|7.01-7|}{7.01-7}=\frac{|7.001-7|}{7.001-7}=1$$

Since the left- and right-hand limits are not equal, there is no limit.

Exercise 8.2.8

Evaluate

$$\lim_{x\to 6^+}\frac{6-x}{|x-6|}.$$

Solution

Key Concepts

- The properties of limits can be used to perform operations on the limits of functions rather than the functions themselves. See Example.
- The limit of a polynomial function can be found by finding the sum of the limits of the individual terms. See Example and Example.
- The limit of a function that has been raised to a power equals the same power of the limit of the function. Another method is direct substitution. See Example.
- The limit of the root of a function equals the corresponding root of the limit of the function.
- One way to find the limit of a function expressed as a quotient is to write the quotient in factored form and simplify. See Example.
- Another method of finding the limit of a complex fraction is to find the LCD. See Example.
- A limit containing a function containing a root may be evaluated using a conjugate. See Example.
- The limits of some functions expressed as quotients can be found by factoring. See Example.
- One way to evaluate the limit of a quotient containing absolute values is by using numeric evidence. Setting it up piecewise can also be useful. See Example.

Glossary

properties of limits

a collection of theorems for finding limits of functions by performing mathematical operations on the limits

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8.3: Continuity

Arizona is known for its dry heat. On a particular day, the temperature might rise as high as $118^\circ F$ and drop down only to a brisk $95^\circ F$. Figure $8.3.1$ shows the function T, where the output of $T(x)$ is the temperature in Fahrenheit degrees and the input x is the time of day, using a 24-hour clock on a particular summer day.

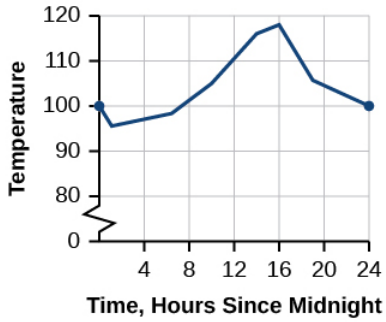


***Figure* 8.3.1**: *Temperature as a function of time forms a continuous function.*

When we analyze this graph, we notice a specific characteristic. There are no breaks in the graph. We could trace the graph without picking up our pencil. This single observation tells us a great deal about the function. In this section, we will investigate functions with and without breaks.

Determining Whether a Function Is Continuous at a Number

Let's consider a specific example of temperature in terms of date and location, such as June 27, 2013, in Phoenix, AZ. The graph in Figure $8.3.1$ indicates that, at 2 a.m., the temperature was $96^\circ F$. By 2 p.m. the temperature had risen to $116^\circ F$, and by 4 p.m. it was $118^\circ F$. Sometime between 2 a.m. and 4 p.m., the temperature outside must have been exactly $110.5^\circ F$. In fact, any temperature between $96^\circ F$ and $118^\circ F$ occurred at some point that day. This means all real numbers in the output between $96^\circ F$ and $118^\circ F$ are generated at some point by the function according to the intermediate value theorem,

Look again at Figure $8.3.1$. There are no breaks in the function's graph for this 24-hour period. At no point did the temperature cease to exist, nor was there a point at which the temperature jumped instantaneously by several degrees. A function that has no holes or breaks in its graph is known as a continuous function. Temperature as a function of time is an example of a continuous function.

If temperature represents a continuous function, what kind of function would not be continuous? Consider an example of dollars expressed as a function of hours of parking. Let's create the function D, where $D(x)$ is the output representing cost in dollars for parking x number of hours (Figure $8.3.2$).

Suppose a parking garage charges $4.00 per hour or fraction of an hour, with a $25 per day maximum charge. Park for two hours and five minutes and the charge is $12. Park an additional hour and the charge is $16. We can never be charged $13, $14, or $15. There are real numbers between 12 and 16 that the function never outputs. There are breaks in the function's graph for this 24-hour period, points at which the price of parking jumps instantaneously by several dollars.

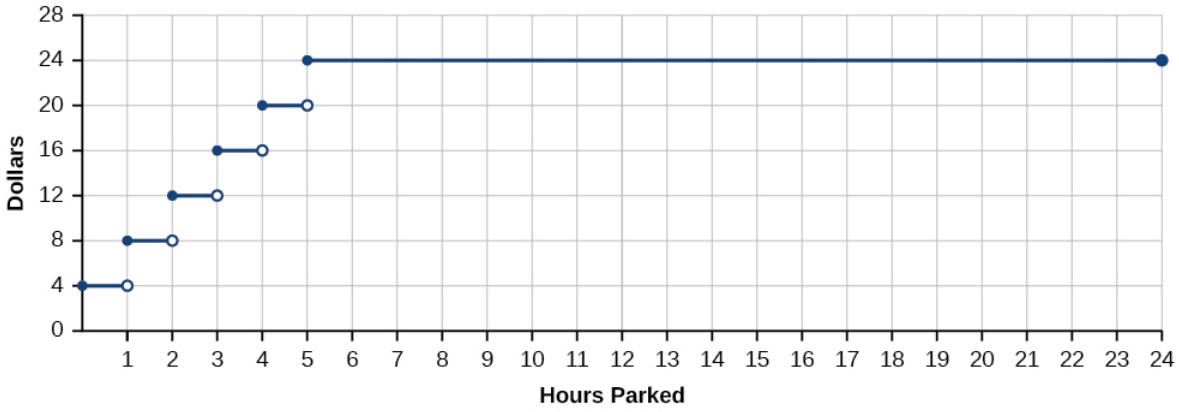


Figure 8.3.2: Parking-garage charges form a discontinuous function.

A function that remains level for an interval and then jumps instantaneously to a higher value is called a **stepwise** function. This function is an example.

A function that has any hole or break in its graph is known as a **discontinuous function**. A stepwise function, such as parking-garage charges as a function of hours parked, is an example of a discontinuous function.

So how can we decide if a function is continuous at a particular number? We can check three different conditions. Let's use the function $y = f(x)$ represented in Figure as an example.

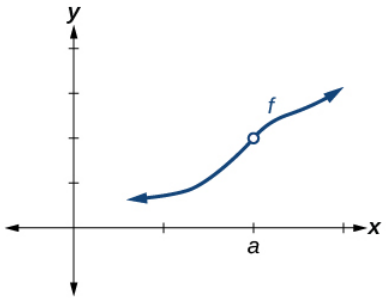


Figure_12_03_004">Figure.

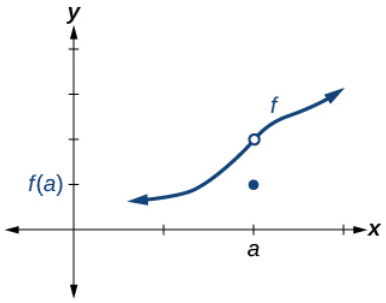


Figure_12_03_003">Figure approaches $x = a$ from the left and right, the same *y*-coordinate is approached. Therefore, Condition 2 is satisfied. However, there could still be a hole in the graph at $x = a$.

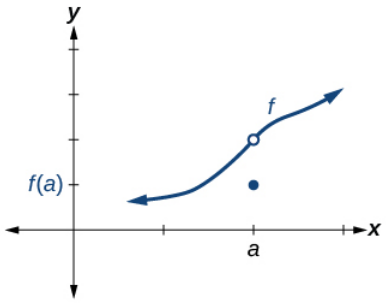
Condition 3 According to Condition 3, the corresponding y y coordinate at $x = a$ fills in the hole in the graph of f. This is written $\lim_{x \to a} f(x) = f(a)$.

Satisfying all three conditions means that the function is continuous. All three conditions are satisfied for the function represented in Figure so the function is continuous as $x = a$.

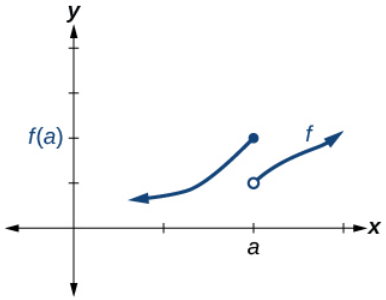
All three conditions are satisfied. The function is continuous at $x = a$.

Figure through Figure provide several examples of graphs of functions that are not continuous at $x = a$ and the condition or conditions that fail.

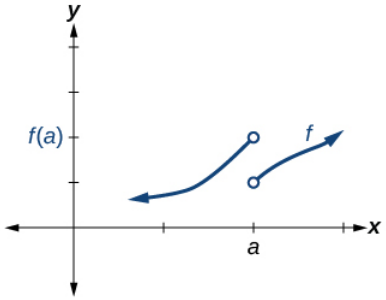
Condition 2 is satisfied. Conditions 1 and 3 both fail.



Conditions 1 and 2 are both satisfied. Condition 3 fails.



Condition 1 is satisfied. Conditions 2 and 3 fail.



Conditions 1, 2, and 3 all fail.

Definition of continuity

A function $f(x)$ is **continuous** at $x = a$ provided all three of the following conditions hold true:

- Condition 1: $f(a)$ exists.
- Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
- Condition 3: $\lim_{x \to a} f(x) = f(a)$

If a function $f(x)$ is not continuous at $x = a$,the function is **discontinuous** at $x = a$.

Identifying a Jump Discontinuity

Discontinuity can occur in different ways. We saw in the previous section that a function could have a left-hand limit and a right-hand limit even if they are not equal. If the left- and right-hand limits exist but are different, the graph "jumps" at $x = a$. The function is said to have a jump discontinuity.

As an example, look at the graph of the function $y = f(x)$ in Figure. Notice as x approaches a how the output approaches different values from the left and from the right.

Graph of a function with a jump discontinuity.

JUMP DISCONTINUITY

A function $f(x)$ has a jump discontinuity at $x=a$ if the left- and right-hand limits both exist but are not equal: $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$

Identifying Removable Discontinuity

Some functions have a discontinuity, but it is possible to redefine the function at that point to make it continuous. This type of function is said to have a removable discontinuity. Let's look at the function $y=f(x)$ represented by the graph in Figure. The function has a limit. However, there is a hole at $x=a$. The hole can be filled by extending the domain to include the input $x=a$ and defining the corresponding output of the function at that value as the limit of the function at $x=a$.

Graph of function f with a removable discontinuity at $x=a$.

removable discontinuity

A function f(x) f(x) has a removable discontinuity at $x=a$ if the limit, $\lim\limits_{x\to a} f(x)$, exists, but either

1. $f(a)$ does not exist *or*
2. $f(a)$, the value of the function at $x=a$ does not equal the limit, $f(a) \neq \lim\limits_{x\to a} f(x)$.

Example 8.3.1: Identifying Discontinuities

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x)=\frac{x^2-2x-15}{x-5}$

b. $$g(x)=\begin{cases} x+1, & x<2 \\ -x, & x\geq 2 \end{cases} \tag{8.3.1}$$

---

a. Notice that the function is defined everywhere except at $x=5$.

Thus, $f(5)$ does not exist, Condition 2 is not satisfied. Since Condition 1 is satisfied, the limit as x approaches 5 is 8, and Condition 2 is not satisfied.This means there is a removable discontinuity at $x=5$.

b. Condition 2 is satisfied because $g(2)=-2$.

Notice that the function is a **piecewise function**, and for each piece, the function is defined everywhere on its domain. Let's examine Condition 1 by determining the left- and right-hand limits as x approaches 2.

Left-hand limit: $\lim\limits_{x\to 2^-}(x+1)=2+1=3$. The left-hand limit exists.

Right-hand limit: $\lim\limits_{x\to 2^+}(-x)=-2$. The right-hand limit exists. But

$$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x). \tag{8.3.2}$$

So, $\lim\limits_{x\to 2} f(x)$ does not exist, and Condition 2 fails: There is no removable discontinuity. However, since both left- and right-hand limits exist but are not equal, the conditions are satisfied for a jump discontinuity at $x=2$.

Exercise 8.3.1:

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x)=\frac{x^2-6x}{x-6}$

b. $g(x)=\begin{cases} \sqrt{x}, & 0\leq x<4 \\ 2x, & x\geq 4 \end{cases}$

---

a. removable discontinuity at $x = 6$;
b. jump discontinuity at $x = 4$

Recognizing Continuous and Discontinuous Real-Number Functions

Many of the functions we have encountered in earlier chapters are continuous everywhere. They never have a hole in them, and they never jump from one value to the next. For all of these functions, the limit of $f(x)$ as x approaches a a is the same as the value of $f(x)$ when $x = a$. So $\lim_{x \to a} f(x) = f(a)$. There are some functions that are continuous everywhere and some that are only continuous where they are defined on their domain because they are not defined for all real numbers.

EXAMPLES OF CONTINUOUS FUNCTIONS

The following functions are continuous everywhere:

Polynomial functions	Ex: $f(x) = x^4 - 9x^2$
Exponential functions	Ex: $f(x) = 4^{x+2} - 5$
Sine functions	Ex: $f(x) = \sin(2x) - 4$
Cosine functions	Ex: $f(x) = -\cos(x + \frac{\pi}{3})$

The following functions are continuous everywhere they are defined on their domain:

Logarithmic functions	Ex: $f(x) = 2\ln(x), x > 0$
Tangent functions	Ex: $f(x) = \tan(x) + 2, x \neq \frac{\pi}{2} + k\pi, k$ is an integer
Rational functions	Ex: $f(x) = \frac{x^2 - 25}{x - 7}, x \neq 7$

how to:Given a function $f(x)$, determine if the function is continuous at $x = a$.

1. Check Condition 1: $f(a)$ exists.
2. Check Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
3. Check Condition 3: $\lim_{x \to a} f(x) = f(a)$.
4. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions is not satisfied, the function is not continuous at $x = a$.

Example 8.3.2: Determining Whether a Piecewise Function is Continuous at a Given Number

Determine whether the function $f(x) = \begin{cases} 4x, & x \leq 3 \\ 8 + x, & x > 3 \end{cases}$ is continuous at

a. $x = 3$
b. $x = \frac{8}{3}$

To determine if the function f is continuous at $x = a$, we will determine if the three conditions of continuity are satisfied at $x = a$.

a. Condition 1: Does $f(a)$ exist?

$$f(3) = 4(3) = 12 \tag{8.3.3}$$
$$\Rightarrow \text{Condition 1 is satisfied.} \tag{8.3.4}$$

Condition 2: Does $\lim_{x \to 3} f(x)$ exist?

To the left of $x=3$, $f(x)=4x$; to the right of $x=3$, $f(x)=8+x$. We need to evaluate the left- and right-hand limits as x approaches 1.

- Left-hand limit: $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-} 4(3)=12$
- Right-hand limit: $\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+} (8+x)=8+3=11$

Because $\lim\limits_{x\to 1^-} f(x)\neq \lim\limits_{x\to 1^+} f(x)$, $\lim\limits_{x\to 1} f(x)$ does not exist.

$$\Rightarrow \text{Condition 2 fails.} \tag{8.3.5}$$

There is no need to proceed further. Condition 2 fails at $x=3$. If any of the conditions of continuity are not satisfied at $x=3$, the function $f(x)$ is not continuous at $x=3$.

b. $x=\frac{8}{3}$

Condition 1: Does $f(\frac{8}{3})$ exist?

$$f(\frac{8}{3})=4(\frac{8}{3})=\frac{32}{3} \tag{8.3.6}$$

$$\Rightarrow \text{Condition 1 is satisfied.} \tag{8.3.7}$$

Condition 2: Does $\lim\limits_{x\to \frac{8}{3}} f(x)$ exist?

To the left of $x=\frac{8}{3}$, $f(x)=4x$; to the right of $x=\frac{8}{3}$, $f(x)=8+x$. We need to evaluate the left- and right-hand limits as x approaches $\frac{8}{3}$.

- Left-hand limit: $\lim\limits_{x\to \frac{8}{3}^-} f(x)=\lim\limits_{x\to \frac{8}{3}^-} 4(\frac{8}{3})=\frac{32}{3}$
- Right-hand limit: $\lim\limits_{x\to \frac{8}{3}^+} f(x)=\lim\limits_{x\to \frac{8}{3}^+} (8+x)=8+\frac{8}{3}=\frac{32}{3}$

Because $\lim\limits_{x\to \frac{8}{3}} f(x)$ exists,

$$\Rightarrow \text{Condition 2 is satisfied.} \tag{8.3.8}$$

Condition 3: Is $f(\frac{8}{3})=\lim\limits_{x\to \frac{8}{3}} f(x)$?

$$f(\frac{32}{3})=\frac{32}{3}=\lim_{x\to \frac{8}{3}} f(x) \tag{8.3.9}$$

$$\Rightarrow \text{Condition 3 is satisfied.} \tag{8.3.10}$$

Because all three conditions of continuity are satisfied at $x=\frac{8}{3}$, the function $f(x)$ is continuous at $x=\frac{8}{3}$.

Exercise 8.3.2:

Determine whether the function $f(x)=\begin{cases} \frac{1}{x}, & x\leq 2 \\ 9x-11.5, & x>2 \end{cases}$ is continuous at $x=2$.

---

yes

Example 8.3.3: Determining Whether a Rational Function is Continuous at a Given Number

Determine whether the function $f(x)=\frac{x^2-25}{x-5}$ is continuous at $x=5$.

To determine if the function f is continuous at $x=5$, we will determine if the three conditions of continuity are satisfied at $x=5$.

Condition 1:

$$f(5) \text{ does not exist.} \tag{8.3.11}$$
$$\Rightarrow \text{Condition 1 fails.} \tag{8.3.12}$$

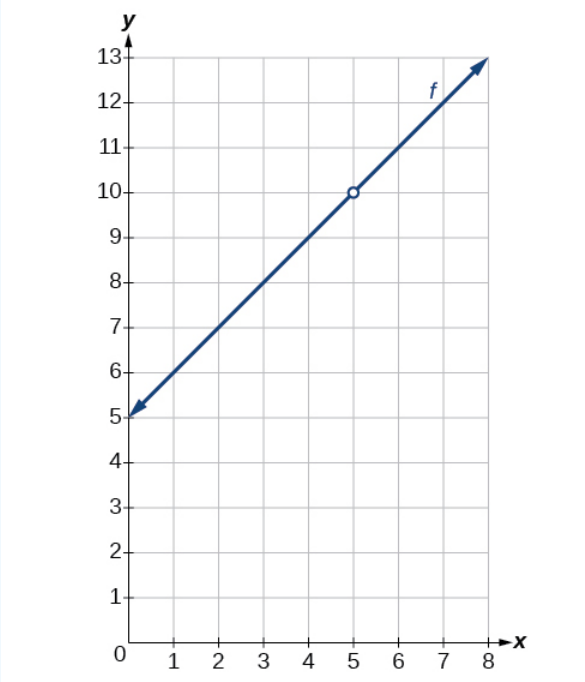
There is no need to proceed further. Condition 2 fails at $x = 5$. If any of the conditions of continuity are not satisfied at $x = 5$,the function f f is not continuous at $x = 5$.

Analysis

See Figure. Notice that for Condition 2 we have

$$\lim_{x\to 5} \frac{x^2 - 25}{x - 5} = \lim_{x\to 3} \frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}} \tag{8.3.13}$$
$$= \lim_{x\to 5}(x+5) \tag{8.3.14}$$
$$= 5 + 5 = 10 \tag{8.3.15}$$
$$\Rightarrow \text{Condition 2 is satisfied.} \tag{8.3.16}$$

At x=5, x=5,there exists a removable discontinuity. See Figure.



Exercise 8.3.3:

Determine whether the function $f(x) = \frac{9-x^2}{x^2-3x}$ is continuous at $x = 3$. If not, state the type of discontinuity.

No, the function is not continuous at $x = 3$. There exists a removable discontinuity at $x = 3$.

Determining the Input Values for Which a Function Is Discontinuous

Now that we can identify continuous functions, jump discontinuities, and removable discontinuities, we will look at more complex functions to find discontinuities. Here, we will analyze a piecewise function to determine if any real numbers exist where the function is not continuous. A **piecewise function** may have discontinuities at the boundary points of the function as well as within the functions that make it up.

To determine the real numbers for which a piecewise function composed of **polynomial** functions is not continuous, recall that polynomial functions themselves are continuous on the set of real numbers. Any discontinuity would be at the boundary points. So we need to explore the three conditions of continuity at the boundary points of the piecewise function.

how to: Given a piecewise function, determine whether it is continuous at the boundary points

1. For each boundary point a of the piecewise function, determine the left- and right-hand limits as x approaches a, as well as the function value at a.
2. Check each condition for each value to determine if all three conditions are satisfied.
3. Determine whether each value satisfies condition 1: $f(a)$ exists.
4. Determine whether each value satisfies condition 2: $\lim_{x\to a} f(x)$ exists.
5. Determine whether each value satisfies condition 3: $\lim_{x\to a} f(x) = f(a)$.
6. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions fails, the function is not continuous at $x = a$.

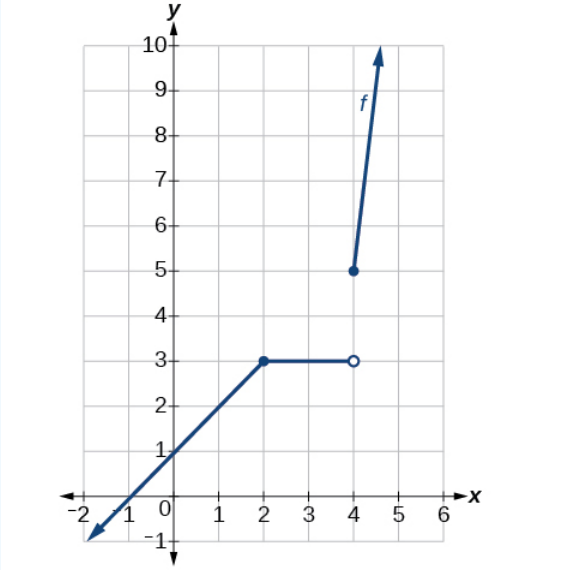
Example 8.3.4: Determining the Input Values for Which a Piecewise Function Is Discontinuous

Determine whether the function f f is discontinuous for any real numbers.

$$fx = \begin{cases} x+1, & x < 2 \\ 3, & 2 \le x < 4 \\ x^2 - 11, & x \ge 4 \end{cases} \tag{8.3.17}$$

Analysis

See Figure. At $x = 4$,there exists a jump discontinuity. Notice that the function is continuous at $x = 2$.



Graph is continuous at $x = 2$ but shows a jump discontinuity at $x = 4$.

Exercise 8.3.4:

Determine where the function $f(x) = \begin{cases} \frac{\pi x}{4}, & x < 2 \\ \frac{\pi}{x}, & 2 \le x \le 6 \\ 2\pi x, & x > 6 \end{cases}$ is discontinuous.

$x = 6$

Determining Whether a Function Is Continuous

To determine whether a **piecewise function** is continuous or discontinuous, in addition to checking the boundary points, we must also check whether each of the functions that make up the piecewise function is continuous.

how to: Given a piecewise function, determine whether it is continuous.

1. Determine whether each component function of the piecewise function is continuous. If there are discontinuities, do they occur within the domain where that component function is applied?
2. For each boundary point $x = a$ of the piecewise function, determine if each of the three conditions hold.

Example 8.3.5: Determining Whether a Piecewise Function Is Continuous

Determine whether the function below is continuous. If it is not, state the location and type of each discontinuity.

$$fx = \begin{cases} \sin(x), & x < 0 \\ x^3, & x > 0 \end{cases} \tag{8.3.18}$$

The two functions composing this piecewise function are $f(x) = \sin(x)$ on $x < 0$ and $f(x) = x^3$ on $x > 0$. The sine function and all polynomial functions are continuous everywhere. Any discontinuities would be at the boundary point,

At $x = 0$,let us check the three conditions of continuity.

Condition 1:

$$f(0) \text{ does not exist.} \tag{8.3.19}$$
$$\Rightarrow \text{Condition 1 fails.} \tag{8.3.20}$$

Because all three conditions are not satisfied at $x = 0$, the function $f(x)$ is discontinuous at $x = 0$.

Analysis

See Figure. There exists a removable discontinuity at $x = 0$; $\lim_{x \to 0} f(x) = 0$, thus the limit exists and is finite, but $f(a)$ does not exist.

Function has removable discontinuity at 0.

Media

Access these online resources for additional instruction and practice with continuity.

- Continuity at a Point
- Continuity at a Point: Concept Check

Key Concepts

- A continuous function can be represented by a graph without holes or breaks.
- A function whose graph has holes is a discontinuous function.
- A function is continuous at a particular number if three conditions are met:
  - Condition 1: $f(a)$ exists.
  - Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
  - Condition 3: $\lim_{x \to a} f(x) = f(a)$.
- A function has a jump discontinuity if the left- and right-hand limits are different, causing the graph to “jump.”
- A function has a removable discontinuity if it can be redefined at its discontinuous point to make it continuous. See Example.
- Some functions, such as polynomial functions, are continuous everywhere. Other functions, such as logarithmic functions, are continuous on their domain. See Example and Example.

- For a piecewise function to be continuous each piece must be continuous on its part of the domain and the function as a whole must be continuous at the boundaries. See Example and Example.

Glossary

continuous function

a function that has no holes or breaks in its graph

discontinuous function

a function that is not continuous at $x = a$

jump discontinuity

a point of discontinuity in a function $f(x)$ at $x = a$ where both the left and right-hand limits exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

removable discontinuity

a point of discontinuity in a function $f(x)$ where the function is discontinuous, but can be redefined to make it continuous

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8.4: Derivatives

The average teen in the United States opens a refrigerator door an estimated 25 times per day. Supposedly, this average is up from 10 years ago when the average teenager opened a refrigerator door 20 times per day[1]. It is estimated that a television is on in a home 6.75 hours per day, whereas parents spend an estimated 5.5 minutes per day having a meaningful conversation with their children. These averages, too, are not the same as they were 10 years ago, when the television was on an estimated 6 hours per day in the typical household, and parents spent 12 minutes per day in meaningful conversation with their kids. What do these scenarios have in common? The functions representing them have changed over time. In this section, we will consider methods of computing such changes over time.

Finding the Average Rate of Change of a Function

The functions describing the examples above involve a change over time. Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

A **tangent line** to a curve is a line that intersects the curve at only a single point but does not cross it there. (The tangent line may intersect the curve at another point away from the point of interest.) If we zoom in on a curve at that point, the curve appears linear, and the **slope of the curve** at that point is close to the slope of the tangent line at that point.

Figure 8.4.1 represents the function $f(x) = x^3 - 4x$. We can see the slope at various points along the curve.

- slope at $x = -2$ is 8
- slope at $x = -1$ is –1
- slope at $x = 2$ is 8

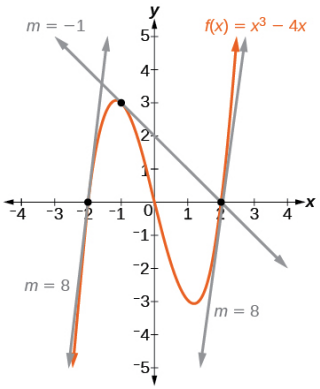


Figure 8.4.1: Graph showing tangents to curve at –2, –1, and 2.

Let's imagine a point on the curve of function f at $x = a$ as shown in Figure 8.4.1. The coordinates of the point are $(a, f(a))$. Connect this point with a second point on the curve a little to the right of $x = a$, with an x-value increased by some small real number h. The coordinates of this second point are $(a+h, f(a+h))$ for some positive-value h.

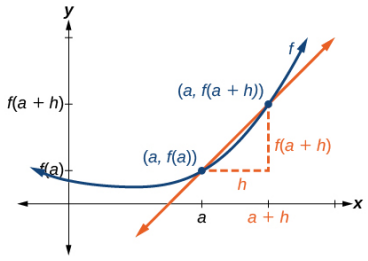


Figure 8.4.2: Connecting point a with a point just beyond allows us to measure a slope close to that of a tangent line at $x = a$.

We can calculate the slope of the line connecting the two points $(a, f(a))$ and $(a+h, f(a+h))$, called a **secant line**, by applying the slope formula,

$$\text{slope} = \frac{\text{change in y}}{\text{change in x}} \tag{8.4.1}$$

We use the notation m_{sec} to represent the slope of the secant line connecting two points.

$$m_{sec} = \frac{f(a+h) - f(a)}{(a+h) - (a)} \tag{8.4.2}$$

$$= \frac{f(a+h) - f(a)}{\cancel{a}+h-\cancel{a}} \tag{8.4.3}$$

The slope m_{sec} equals the *average rate of change* between two points $(a, f(a))$ and $(a+h, f(a+h))$.

$$m_{sec} = \frac{f(a+h) - f(a)}{h} \tag{8.4.4}$$

the AVERAGE RATE OF CHANGE BETWEEN TWO POINTS ON A CURVE

The **average rate of change** (AROC) between two points $(a, f(a))$ and $(a+h, f(a+h))$ on the curve of f is the slope of the line connecting the two points and is given by

$$\text{AROC} = \frac{f(a+h) - f(a)}{h} \tag{8.4.5}$$

Example 8.4.1: Finding the Average Rate of Change

Find the average rate of change connecting the points $(2, -6)$ and $(-1, 5)$.

Solution

We know the average rate of change connecting two points may be given by

$$\text{AROC} = \frac{f(a+h) - f(a)}{h} \tag{8.4.6}$$

If one point is $(2, -6)$, or $(2, f(2))$, then $f(2) = -6$.

The value h is the displacement from 2 to -1, which equals $-1 - 2 = -3$.

For the other point, $f(a+h)$ is the y-coordinate at $a+h$, which is $2+(-3)$ or -1, so $f(a+h) = f(-1) = 5$.

$$\text{AROC} = \frac{f(a+h) - f(a)}{h} \tag{8.4.7}$$

$$= \frac{5 - (-6)}{-3} \tag{8.4.8}$$

$$= \frac{11}{-3} \tag{8.4.9}$$

$$= -\frac{11}{3} \tag{8.4.10}$$

Exercise 8.4.1

Find the average rate of change connecting the points $(-5, 1.5)$ and $(-2.5, 9)$

Solution

3

Understanding the Instantaneous Rate of Change

Now that we can find the average rate of change, suppose we make h in Figure 8.4.2 smaller and smaller. Then $a+h$ will approach a as h gets smaller, getting closer and closer to 0. Likewise, the second point $(a+h, f(a+h))$ will approach the first point, $(a, f(a))$. As a consequence, the connecting line between the two points, called the secant line, will get closer and closer to being a tangent to the function at $x = a$, and the slope of the secant line will get closer and closer to the slope of the tangent at $x = a$ (Figure 8.4.3).

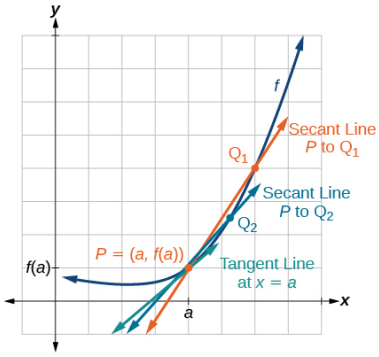


Figure 8.4.3: The connecting line between two points moves closer to being a tangent line at $x = a$.

Because we are looking for the slope of the tangent at $x = a$, we can think of the measure of the slope of the curve of a function f at a given point as the rate of change at a particular instant. We call this slope the **instantaneous rate of change**, or the **derivative** of the function at $x = a$. Both can be found by finding the limit of the slope of a line connecting the point at $x = a$ with a second point infinitesimally close along the curve. For a function f both the instantaneous rate of change of the function and the derivative of the function at $x = a$ are written as $f'(a)$, and we can define them as a two-sided limit that has the same value whether approached from the left or the right.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{8.4.11}$$

The expression by which the limit is found is known as the **difference quotient**.

DEFINITION OF INSTANTANEOUS RATE OF CHANGE AND DERIVATIVE

The derivative, or instantaneous rate of change, of a function f at $x = a$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{8.4.12}$$

The expression $\frac{f(a+h)-f(a)}{h}$ is called the difference quotient.

We use the difference quotient to evaluate the limit of the rate of change of the function as h approaches 0.

Derivatives: Interpretations and Notation

The derivative of a function can be interpreted in different ways. It can be observed as the behavior of a graph of the function or calculated as a numerical rate of change of the function.

- The derivative of a function $f(x)$ at a point $x = a$ is the slope of the tangent line to the curve $f(x)$ at $x = a$. The derivative of $f(x)$ at $x = a$ is written $f'(a)$.
- The derivative $f'(a)$ measures how the curve changes at the point $(a, f(a))$.

- The derivative $f'(a)$ may be thought of as the instantaneous rate of change of the function $f(x)$ at $x=a$.
- If a function measures distance as a function of time, then the derivative measures the instantaneous **velocity** at time $t=a$.

NOTATIONS FOR THE DERIVATIVE

The equation of the derivative of a function $f(x)$ is written as $y'=f'(x)$, where $y=f(x)$. The notation $f'(x)$ is read as "f prime of x." Alternate notations for the derivative include the following:

$$f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\frac{d}{dx}f(x)=Df(x) \tag{8.4.13}$$

The expression $f'(x)$ is now a function of x; this function gives the slope of the curve $y=f(x)$ at any value of x. The derivative of a function $f(x)$ at a point $x=a$ is denoted $f'(a)$.

how to: Given a function f, find the derivative by applying the definition of the derivative.

1. Calculate $f(a+h)$.
2. Calculate $f(a)$.
3. Substitute and simplify $\frac{f(a+h)-f(a)}{h}$.
4. Evaluate the limit if it exists: $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

Example 8.4.1: Finding the Derivative of a Polynomial Function

Find the derivative of the function $f(x)=x^2-3x+5$ at $x=a$.

Solution

We have:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad \text{Definition of a derivative} \tag{8.4.14}$$

Substitute $f(a+h)=(a+h)^2-3(a+h)+5$ and $f(a)=a^2-3a+5$.

$$f'(a)=\lim_{h\to 0}\frac{(a+h)(a+h)-3(a+h)+5-(a^2-3a+5)}{h} \tag{8.4.15}$$

$$=\lim_{h\to 0}\frac{a^2+2ah+h^2-3a-3h+5-a^2+3a-5}{h} \qquad \text{Evaluate to remove parentheses.} \tag{8.4.16}$$

$$=\lim_{h\to 0}\frac{\cancel{a^2}+2ah+h^2-\cancel{3a}-3h+\cancel{5}-\cancel{a^2}+\cancel{3a}-\cancel{5}}{h} \qquad \text{Simplify.} \tag{8.4.17}$$

$$=\lim_{h\to 0}\frac{2ah+h^2-3h}{h} \qquad \text{Factor out an } h. \tag{8.4.18}$$

$$=2a+0-3 \qquad \text{Evaluate the limit.} \tag{8.4.19}$$

$$=2a-3 \tag{8.4.20}$$

Exercise 8.4.1

Find the derivative of the function $f(x)=3x^2+7x$ at $x=a$

Solution

$f'(a)=6a+7$

Finding Derivatives of Rational Functions

To find the derivative of a rational function, we will sometimes simplify the expression using algebraic techniques we have already learned.

Example 8.4.1: Finding the Derivative of a Rational Function

Find the derivative of the function $f(x)=\frac{3+x}{2-x}$ at $x=a$.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \tag{8.4.21}$$

$$=\lim_{h\to 0}\frac{\frac{3+(a+h)}{2-(a+h)}-(\frac{3+a}{2-a})}{h} \quad \text{Substitute } f(a+h) \text{ and } f(a) \tag{8.4.22}$$

$$=\lim_{h\to 0}\frac{(2-(a+h))(2-a)[\frac{3+(a+h)}{2-(a+h)}-(\frac{3+a}{2-a})]}{(2-(a+h))(2-a)(h)} \quad \text{Multiply numerator and denominator by } (2-(a+h))(2-a) \tag{8.4.23}$$

$$=\lim_{h\to 0}\frac{(\cancel{2-(a+h)})(2-a)(\frac{3+(a+h)}{\cancel{(2-(a+h))}})-(2-(a+h))(\cancel{2-a})(\frac{3+a}{\cancel{2-a}})}{(2-(a+h))(2-a)(h)} \quad \text{Distribute} \tag{8.4.24}$$

$$=\lim_{h\to 0}\frac{6-3a+2a-a^2+2h-ah-6+3a+3h-2a+a^2+ah}{(2-(a+h))(2-a)(h)} \quad \text{Multiply} \tag{8.4.25}$$

$$=\lim_{h\to 0}\frac{5\cancel{h}}{(2-(a+h))(2-a)(\cancel{h})} \quad \text{Combine like terms} \tag{8.4.26}$$

$$=\lim_{h\to 0}\frac{5}{(2-(a+h))(2-a)} \quad \text{Cancel like factors} \tag{8.4.27}$$

$$=\frac{5}{(2-(a+0))(2-a)}=\frac{5}{(2-a)(2-a)}=\frac{5}{(2-a)^2} \quad \text{Evaluate the limit} \tag{8.4.28}$$

Exercise 8.4.1:

Find the derivative of the function $f(x)=\frac{10x+11}{5x+4}$ at $x=a$.

—

$$f'(a)=\frac{-15}{(5a+4)^2} \tag{8.4.29}$$

Finding Derivatives of Functions with Roots

To find derivatives of functions with roots, we use the methods we have learned to find limits of functions with roots, including multiplying by a conjugate.

Example 8.4.1: Finding the Derivative of a Function with a Root

Find the derivative of the function $f(x)=4\sqrt{x}$ at $x=36$.

—

We have

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \tag{8.4.30}$$

$$=\lim_{h\to 0}\frac{4\sqrt{a+h}-4\sqrt{a}}{h} \quad \text{Substitute } f(a+h) \text{ and } f(a) \tag{8.4.31}$$

Multiply the numerator and denominator by the conjugate: $\frac{4\sqrt{a+h}+4\sqrt{a}}{4\sqrt{a+h}+4\sqrt{a}}$.

$$f'(a)=\lim_{h\to 0}\left(\frac{4\sqrt{a+h}-4\sqrt{a}}{h}\right)\cdot\left(\frac{4\sqrt{a+h}+4\sqrt{a}}{4\sqrt{a+h}+4\sqrt{a}}\right) \tag{8.4.32}$$

$$=\lim_{h\to 0}\left(\frac{16(a+h)-16a}{h4(\sqrt{a+h}+4\sqrt{a})}\right) \quad \text{Multiply.} \tag{8.4.33}$$

$$=\lim_{h\to 0}\left(\frac{\cancel{16a}+16h-\cancel{16a}}{h4(\sqrt{a+h}+4\sqrt{a})}\right) \quad \text{Distribute and combine like terms.} \tag{8.4.34}$$

$$=\lim_{h\to 0}\left(\frac{16\cancel{h}}{\cancel{h}(4\sqrt{a+h}+4\sqrt{a})}\right) \quad \text{Simplify.} \tag{8.4.35}$$

$$=\lim_{h\to 0}\left(\frac{16}{4\sqrt{a+h}+4\sqrt{a}}\right) \quad \text{Evaluate the limit by letting } h=0. \tag{8.4.36}$$

$$=\frac{16}{8\sqrt{a}}=\frac{2}{\sqrt{a}} \tag{8.4.37}$$

$$f'(36)=\frac{2}{\sqrt{36}} \quad \text{Evaluate the derivative at } x=36. \tag{8.4.38}$$

$$=\frac{2}{6} \tag{8.4.39}$$

$$=\frac{1}{3} \tag{8.4.40}$$

Exercise 8.4.1:

Find the derivative of the function $f(x)=9\sqrt{x}$ at $x=9$.

—

$\frac{3}{2}$

Finding Instantaneous Rates of Change

Many applications of the derivative involve determining the rate of change at a given instant of a function with the independent variable time—which is why the term *instantaneous* is used. Consider the height of a ball tossed upward with an initial velocity of 64 feet per second, given by $s(t) = -16t^2 + 64t + 6$, where t is measured in seconds and $s(t)$ is measured in feet. We know the path is that of a parabola. The derivative will tell us how the height is changing at any given point in time. The height of the ball is shown in Figure as a function of time. In physics, we call this the "*s*-*t* graph."

Example 8.4.1: Finding the Instantaneous Rate of Change

Using the function above, $s(t) = -16t^2 + 64t + 6$,what is the instantaneous velocity of the ball at 1 second and 3 seconds into its flight?

The velocity at $t = 1$ and $t = 3$ is the instantaneous rate of change of distance per time, or velocity. Notice that the initial height is 6 feet. To find the instantaneous velocity, we find the **derivative** and evaluate it at $t = 1$ and $t = 3$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{8.4.41}$$

$$= \lim_{h \to 0} \frac{-16(t+h)^2 + 64(t+h) + 6 - (-16t^2 + 64t + 6)}{h} \quad \text{Substitute } s(t+h) \text{ and } s(t). \tag{8.4.42}$$

$$= \lim_{h \to 0} \frac{-16t^2 - 32ht - h^2 + 64t + 64h + 6 + 16t^2 - 64t - 6}{h} \quad \text{Distribute} \tag{8.4.43}$$

$$= \lim_{h \to 0} \frac{-32ht - h^2 + 64h}{h} \quad \text{Simplify.} \tag{8.4.44}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-32t - h + 64)}{\cancel{h}} \quad \text{Factor the numerator.} \tag{8.4.45}$$

$$= \lim_{h \to 0} -32t - h + 64 \quad \text{Cancel out the common factor } h. \tag{8.4.46}$$

$$s'(t) = -32t + 64 \quad \text{Evaluate the limit by letting } h = 0. \tag{8.4.47}$$

For any value of t, $s'(t)$ tells us the velocity at that value of t.

Evaluate $t = 1$ and $t = 3$.

$$s'(1) = -32(1) + 64 = 32 \tag{8.4.48}$$

$$s'(3) = -32(3) + 64 = -32 \tag{8.4.49}$$

The velocity of the ball after 1 second is 32 feet per second, as it is on the way up.

The velocity of the ball after 3 seconds is −32 feet per second, as it is on the way down.

Exercise 8.4.1:

The position of the ball is given by$s(t) = -16t^2 + 64t + 6$. What is its velocity 2 seconds into flight?

0

Using Graphs to Find Instantaneous Rates of Change

We can estimate an instantaneous rate of change at $x = a$ by observing the slope of the curve of the function $f(x)$ at $x = a$. We do this by drawing a line tangent to the function at $x = a$ and finding its slope.

how to: Given a graph of a function $f(x)$, find the instantaneous rate of change of the function at $x = a$.

1. Locate $x = a$ on the graph of the function $f(x)$.
2. Draw a tangent line, a line that goes through $x = a$ at a and at no other point in that section of the curve. Extend the line far enough to calculate its slope as

$$\frac{\text{change in } y}{\text{change in } x}. \tag{8.4.50}$$

Example 8.4.1: Estimating the Derivative at a Point on the Graph of a Function

From the graph of the function $y = f(x)$ presented in Figure, estimate each of the following:

$f(0); f(2); f'(0); f'(2)$

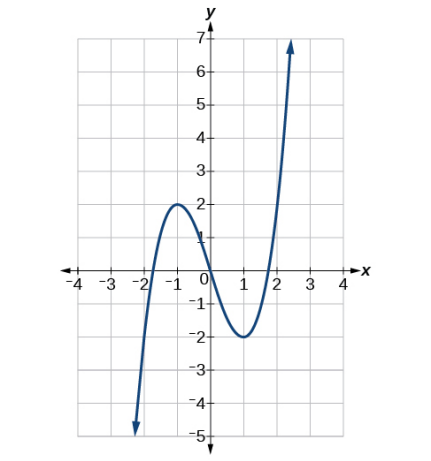
To find the functional value, $f(a)$, find the *y*-coordinate at $x = a$.

To find the **derivative** at $x = a$, $f'(a)$, draw a tangent line at $x = a$, and estimate the slope of that tangent line. See Figure.

- $f(0)$ is the *y*-coordinate at $x = 0$. The point has coordinates $(0, 1)$, thus $f(0) = 1$.
- $f(2)$ is the *y*-coordinate at $x = 2$. The point has coordinates $(2, 1)$, thus $f(2) = 1$.
- $f'(0)$ is found by estimating the slope of the tangent line to the curve at $x = 0$. The tangent line to the curve at $x = 0$ appears horizontal. Horizontal lines have a slope of 0, thus $f'(0) = 0$.
- $f'(2)$ is found by estimating the slope of the tangent line to the curve at $x = 2$. Observe the path of the tangent line to the curve at $x = 2$. As the x value moves one unit to the right, the y value moves up four units to another point on the line. Thus, the slope is 4, so $f'(2) = 4$.

Exercise 8.4.1:

Using the graph of the function $f(x) = x^3 - 3x$ shown in Figure, estimate: $f(1)$, $f'(1)$, $f(0)$and $f'(0)$.



−2,−2,0, 0, −3

Using Instantaneous Rates of Change to Solve Real-World Problems

Another way to interpret an instantaneous rate of change at $x = a$ is to observe the function in a real-world context. The unit for the derivative of a function $f(x)$ is

$$\frac{\text{output units}}{\text{input unit}} \tag{8.4.51}$$

Such a unit shows by how many units the output changes for each one-unit change of input. The instantaneous rate of change at a given instant shows the same thing: the units of change of output per one-unit change of input.

One example of an instantaneous rate of change is a marginal cost. For example, suppose the production cost for a company to produce x items is given by $C(x)$, in thousands of dollars. The derivative function tells us how the cost is changing for any value of x in the domain of the function. In other words, $C'(x)$ is interpreted as a **marginal cost**, the additional cost in thousands of dollars of producing one more item when x items have been produced. For example, $C'(11)$ is the approximate additional cost in thousands of dollars of producing the 12th item after 11 items have been produced. $C'(11) = 2.50$ means that when 11 items have been produced, producing the 12th item would increase the total cost by approximately $2,500.00.

Example 8.4.1: Finding a Marginal Cost

The cost in dollars of producing x laptop computers in dollars is $f(x) = x^2 - 100x$. At the point where 200 computers have been produced, what is the approximate cost of producing the 201stunit?

—

If $f(x) = x^2 - 100x$ describes the cost of producing x computers, $f'(x)$ will describe the marginal cost. We need to find the derivative. For purposes of calculating the derivative, we can use the following functions:

$$f(a+b) = (x+h)^2 - 100(x+h) \tag{8.4.52}$$

$$f(a) = a^2 - 100a \tag{8.4.53}$$

$$f'(x) = \frac{f(a+h) - f(a)}{h} \quad \text{Formula for a derivative} \tag{8.4.54}$$

$$= \frac{(x+h)^2 - 100(x+h) - (x^2 - 100x)}{h} \tag{8.4.55}$$

Substitute $f(a+h)$ and $f(a)$. (8.4.56)

$$= \frac{x^2 + 2xh + h^2 - 100x - 100h - x^2 + 100x}{h} \quad \text{Multiply polynomials, distribute.} \tag{8.4.57}$$

$$= 2\text{xh+h\^{}2-100h}h \quad \text{Collect like terms.} \tag{8.4.58}$$

$$= \frac{h\not{(}2x + h - 100)}{h\not{}} \quad \text{Factor and cancel like terms.} \tag{8.4.59}$$

$$= 2x + h - 100 \quad \text{Simplify.} \tag{8.4.60}$$

$$= 2x - 100 \quad \text{Evaluate when } h = 0. \tag{8.4.61}$$

$$f'(x) = 2x - 100 \quad \text{Formula for marginal cost} \tag{8.4.62}$$

$$f'(200) = 2(200) - 100 = 300 \quad \text{Evaluate for 200 units.} \tag{8.4.63}$$

The marginal cost of producing the 201st unit will be approximately $300.

Example 8.4.1:Interpreting a Derivative in Context

A car leaves an intersection. The distance it travels in miles is given by the function $f(t)$, where t represents hours. Explain the following notations:

$f(0) = 0 f'(1) = 60 f(1) = 70 f(2.5) = 150$

—

First we need to evaluate the function $f(t)$ and the derivative of the function $f'(t)$, and distinguish between the two. When we evaluate the function $f(t)$, we are finding the distance the car has traveled in t hours. When we evaluate the derivative f'(t), f'(t), we are finding the speed of the car after t hours.

a. $f(0) = 0$ means that in zero hours, the car has traveled zero miles.
b. $f'(1) = 60$ means that one hour into the trip, the car is traveling 60 miles per hour.
c. $f(1) = 70$ means that one hour into the trip, the car has traveled 70 miles. At some point during the first hour, then, the car must have been traveling faster than it was at the 1-hour mark.
d. $f(2.5) = 150$ means that two hours and thirty minutes into the trip, the car has traveled 150 miles.

Exercise 8.4.1

A runner runs along a straight east-west road. The function $f(t)$ gives how many feet eastward of her starting point she is after t seconds. Interpret each of the following as it relates to the runner.

$f(0) = 0; f(10) = 150; f'(10) = 15; f'(20) = -10; f(40) = -100$

a. After zero seconds, she has traveled 0 feet.
b. After 10 seconds, she has traveled 150 feet east.
c. After 10 seconds, she is moving eastward at a rate of 15 ft/sec.
d. After 20 seconds, she is moving westward at a rate of 10 ft/sec.
e. After 40 seconds, she is 100 feet westward of her starting point.

Finding Points Where a Function's Derivative Does Not Exist

To understand where a function's derivative does not exist, we need to recall what normally happens when a function $f(x)$ has a derivative at $x = a$. Suppose we use a graphing utility to zoom in on $x = a$. If the function $f(x)$ is differentiable, that is, if it is a function that can be differentiated, then the closer one zooms in, the more closely the graph approaches a straight line. This characteristic is called *linearity*.

Look at the graph in Figure. The closer we zoom in on the point, the more linear the curve appears.

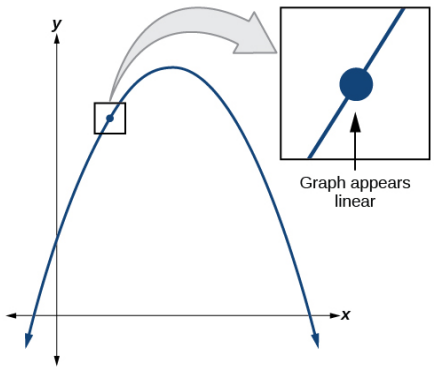


Figure_12_04_009">Figure, the graph does not approach a straight line. No matter how close we zoom in, the graph maintains its sharp corner.

Graph of the function $f(x) = |x|$,with *x*-axis from –0.1 to 0.1 and *y*-axis from –0.1 to 0.1.

What are the characteristics of a graph that is not differentiable at a point? Here are some examples in which function $f(x)$ is not differentiable at $x = a$.

In Figure, we see the graph of

$$f(x) = \begin{cases} x^2, & x \leq 2 \\ 8 - x, & x > 2. \end{cases} . \tag{8.4.64}$$

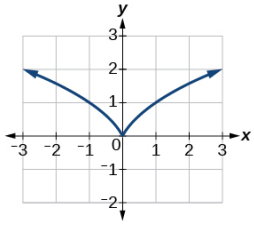
Notice that, as x approaches 2 from the left, the left-hand limit may be observed to be 4, while as x approaches 2 from the right, the right-hand limit may be observed to be 6. We see that it has a discontinuity at $x = 2$.

The graph of $f(x)$ has a discontinuity at $x = 2$.

In Figure, we see the graph of $f(x) = |x|$. We see that the graph has a corner point at $x = 0$.

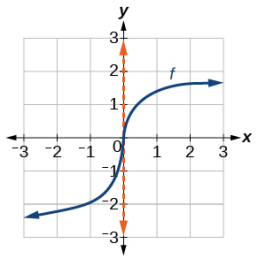
The graph of $f(x) = |x|$ has a corner point at $x = 0$.

In Figure, we see that the graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$. A cusp has a unique feature. Moving away from the cusp, both the left-hand and right-hand limits approach either infinity or negative infinity. Notice the tangent lines as x approaches 0 from both the left and the right appear to get increasingly steeper, but one has a negative slope, the other has a positive slope.



The graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$.

In Figure, we see that the graph of $f(x) = x^{frac13}$ has a vertical tangent at $x = 0$. Recall that vertical tangents are vertical lines, so where a vertical tangent exists, the slope of the line is undefined. This is why the derivative, which measures the slope, does not exist there.



The graph of $f(x) = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$.

differentiability

A function $f(x)$ is differentiable at $x = a$ if the derivative exists at $x = a$,which means that $f'(a)$ exists.

There are four cases for which a function $f(x)$ is not differentiable at a point $x = a$.

1. When there is a discontinuity at $x = a$.
2. When there is a corner point at $x = a$.
3. When there is a cusp at $x = a$.
4. Any other time when there is a vertical tangent at $x = a$.

Example 8.4.1: Determining Where a Function Is Continuous and Differentiable from a Graph

Using Figure, determine where the function is

a. continuous
b. discontinuous
c. differentiable
d. not differentiable

At the points where the graph is discontinuous or not differentiable, state why.

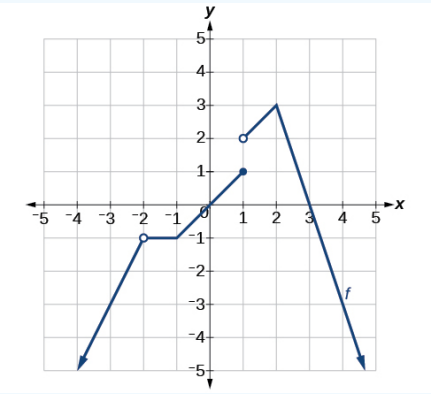


Figure_12_04_016">Figure.

Three intervals where the function is continuous

The graph of is differentiable on $(-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, 2) \cup (2, \infty)$. The graph of $f(x)$ is not differentiable at $x = -2$ because it is a point of discontinuity, at $x = -1$ because of a sharp corner, at $x = 1$ because it is a point of discontinuity, and at $x = 2$ because of a sharp corner. See Figure.

Five intervals where the function is differentiable

Exercise 8.4.1:

Determine where the function $y = f(x)$ shown in Figure is continuous and differentiable from the graph.

The graph of f is continuous on $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$. The graph of f f is discontinuous at $x = 1$ and $x = 3$. The graph of f is differentiable on $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$. The graph of f is not differentiable at $x = 1$ and $x = 3$.

Finding an Equation of a Line Tangent to the Graph of a Function

The equation of a tangent line to a curve of the function $f(x)$ at $x = a$ is derived from the point-slope form of a line, $y = m(x - x_1) + y_1$. The slope of the line is the slope of the curve at $x = a$ and is therefore equal to $f'(a)$, the derivative of $f(x)$ at $x = a$. The coordinate pair of the point on the line at $x = a$ is $(a, f(a))$.

If we substitute into the point-slope form, we have

The equation of the tangent line is

$$y = f'(a)(x - a) + f(a) \tag{8.4.65}$$

F

The equation of a line tangent to the curve of a function f at a point $x=a$ is

$$y=f'(a)(x-a)+f(a) \tag{8.4.66}$$

how to: Given a function f, find the equation of a line tangent to the function at $x=a$.

1. Find the derivative of $f(x)$ at $x=a$ using $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.
2. Evaluate the function at $x=a$. This is $f(a)$.
3. Substitute $(a,f(a))$ and $f'(a)$ into $y=f'(a)(x-a)+f(a)$.
4. Write the equation of the tangent line in the form $y=mx+b$.

Example 8.4.1: Finding the Equation of a Line Tangent to a Function at a Point

Find the equation of a line tangent to the curve $f(x)=x^2-4x$ at $x=3$.

—

Using:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \tag{8.4.67}$$

Substitute $f(a+h)=(a+h)^2-4(a+h)$ and $f(a)=a^2-4a$.

$$f'(a)=\lim_{h\to 0}\frac{(a+h)(a+h)-4(a+h)-(a2-4a)}{h} \tag{8.4.68}$$

$$=\lim_{h\to 0}\frac{a^2+2ah+h^2-4a-4h-a^2+4a}{h} \quad \text{Remove parentheses.} \tag{8.4.69}$$

$$=\lim_{h\to 0}\frac{\cancel{a^2}+2ah+h^2-\cancel{4a}-4h-\cancel{a^2}+\cancel{4a}}{h} \quad \text{Combine like terms.} \tag{8.4.70}$$

$$=\lim_{h\to 0}\frac{2ah+h^2-4h}{h} \tag{8.4.71}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2a+h-4)}{h} \quad \text{Factor out } h. \tag{8.4.72}$$

$$=2a+0-4 \tag{8.4.73}$$

$$f'(a)=2a-4 \quad \text{Evaluate the limit.} \tag{8.4.74}$$

$$f'(3)=2(3)-4=2 \tag{8.4.75}$$

Equation of tangent line at $x=3$:

$$y=f'(a)(x-a)+f(a) \tag{8.4.76}$$

$$y=f'(3)(x-3)+f(3) \tag{8.4.77}$$

$$y=2(x-3)+(-3) \tag{8.4.78}$$

$$y=2x-9 \tag{8.4.79}$$

Analysis

We can use a graphing utility to graph the function and the tangent line. In so doing, we can observe the point of tangency at $x=3$ as shown in Figure.

Graph confirms the point of tangency at $x=3$.

Exercise 8.4.1:

Find the equation of a tangent line to the curve of the function $f(x)=5x^2-x+4$ at $x=2$.

—

$y=19x-16$

Finding the Instantaneous Speed of a Particle

If a function measures position versus time, the derivative measures displacement versus time, or the speed of the object. A change in speed or direction relative to a change in time is known as ***velocity***. The velocity at a given instant is known as **instantaneous velocity**.

In trying to find the speed or velocity of an object at a given instant, we seem to encounter a contradiction. We normally define speed as the distance traveled divided by the elapsed time. But in an instant, no distance is traveled, and no time elapses. How will we divide zero by zero? The use of a derivative solves this problem. A derivative allows us to say that even while the object's velocity is constantly changing, it has a certain velocity at a given instant. That means that if the object traveled at that exact velocity for a unit of time, it would travel the specified distance.

INSTANTANEOUS VELOCITY

Let the function $s(t)$ represent the position of an object at time t. The **instantaneous velocity** or velocity of the object at time $t=a$ is given by

$$s'(a)=\lim_{h\to 0}\frac{s(a+h)-s(a)}{h} \tag{8.4.80}$$

Example 8.4.1: Finding the Instantaneous Velocity

A ball is tossed upward from a height of 200 feet with an initial velocity of 36 ft/sec. If the height of the ball in feet after t seconds is given by $s(t) = -16t^2 + 36t + 200$, find the instantaneous velocity of the ball at $t = 2$.

First, we must find the derivative $s'(t)$. Then we evaluate the derivative at $t = 2$, using $s(a+h) = -16(a+h)^2 + 36(a+h) + 200$ and $s(a) = -16a^2 + 36a + 200$.

$$s'(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} \tag{8.4.81}$$

$$= \lim_{h \to 0} \frac{-16(a+h)^2 + 36(a+h) + 200 - (-16a^2 + 36a + 200)}{h} \tag{8.4.82}$$

$$= \lim_{h \to 0} \frac{-16(a^2 + 2ah + h^2) + 36(a+h) + 200 - (-16a^2 + 36a + 200)}{h} \tag{8.4.83}$$

$$= \lim_{h \to 0} \frac{-16a^2 - 32ah - 16h^2 + 36a + 36h + 200 + 16a^2 - 36a - 200}{h} \tag{8.4.84}$$

$$= \lim_{h \to 0} \frac{\cancel{-16a^2} - 32ah - 16h^2 + \cancel{36a} + 36h + \cancel{200} + \cancel{16a^2} - \cancel{36a} - \cancel{200}}{h} \tag{8.4.85}$$

$$= \lim_{h \to 0} \frac{-32ah - 16h^2 + 36h}{h} \tag{8.4.86}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-32a - 16h + 36)}{\cancel{h}} \tag{8.4.87}$$

$$= \lim_{h \to 0} (-32a - 16h + 36) \tag{8.4.88}$$

$$= -32a - 16 \cdot 0 + 36 \tag{8.4.89}$$

$$s'(a) = -32a + 36 \tag{8.4.90}$$

$$s'(2) = -32(2) + 36 \tag{8.4.91}$$

$$= -28 \tag{8.4.92}$$

Analysis

This result means that at time $t = 2$ seconds, the ball is dropping at a rate of 28 ft/sec.

Exercise 8.4.1:

A fireworks rocket is shot upward out of a pit 12 ft below the ground at a velocity of 60 ft/sec. Its height in feet after t seconds is given by $s = -16t^2 + 60t - 12$. What is its instantaneous velocity after 4 seconds?

–68 ft/sec, it is dropping back to Earth at a rate of 68 ft/s.

Access these online resources for additional instruction and practice with derivatives.

- Estimate the Derivative
- Estimate the Derivative Ex. 4

Visit this website for additional practice questions from Learningpod.

Key Equations

average rate of change	$\text{AROC} = \frac{f(a+h)-f(a)}{h}$
derivative of a function	$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Key Concepts

- The slope of the secant line connecting two points is the average rate of change of the function between those points. See Example.
- The derivative, or instantaneous rate of change, is a measure of the slope of the curve of a function at a given point, or the slope of the line tangent to the curve at that point. See Example, Example, and Example.
- The difference quotient is the quotient in the formula for the instantaneous rate of change:

  $\frac{f(a+h)-f(a)}{h}$
- Instantaneous rates of change can be used to find solutions to many real-world problems. See Example.
- The instantaneous rate of change can be found by observing the slope of a function at a point on a graph by drawing a line tangent to the function at that point. See Example.
- Instantaneous rates of change can be interpreted to describe real-world situations. See Example and Example.
- Some functions are not differentiable at a point or points. See Example.
- The point-slope form of a line can be used to find the equation of a line tangent to the curve of a function. See Example.
- Velocity is a change in position relative to time. Instantaneous velocity describes the velocity of an object at a given instant. Average velocity describes the velocity maintained over an interval of time.
- Using the derivative makes it possible to calculate instantaneous velocity even though there is no elapsed time. See Example.

Section Exercises

Verbal

How is the slope of a linear function similar to the derivative?

The slope of a linear function stays the same. The derivative of a general function varies according to x. Both the slope of a line and the derivative at a point measure the rate of change of the function.

What is the difference between the average rate of change of a function on the interval $[x, x+h]$ and the derivative of the function at x?

A car traveled 110 miles during the time period from 2:00 P.M. to 4:00 P.M. What was the car's average velocity? At exactly 2:30 P.M., the speed of the car registered exactly 62 miles per hour. What is another name for the speed of the car at 2:30 P.M.? Why does this speed differ from the average velocity?

Average velocity is 55 miles per hour. The instantaneous velocity at 2:30 p.m. is 62 miles per hour. The instantaneous velocity measures the velocity of the car at an instant of time whereas the average velocity gives the velocity of the car over an interval.

Explain the concept of the slope of a curve at point x.

Suppose water is flowing into a tank at an average rate of 45 gallons per minute. Translate this statement into the language of mathematics.

The average rate of change of the amount of water in the tank is 45 gallons per minute. If $f(x)$ is the function giving the amount of water in the tank at any time t,then the average rate of change of $f(x)$ between $t=a$ and $t=b$ is $f(a)+45(b-a)$.

Algebraic

For the following exercises, use the definition of derivative $\lim_{hto0} \frac{f(x+h)-f(x)}{h}$ to calculate the derivative of each function.

$f(x)=3x-4$

$f(x)=-2x+1$

$f'(x)=-2$

$f(x)=x^2-2x+1$

\(f(x)=2x^2+x−3

$f'(x)=4x+1$

$f(x)=2x^2+5$

$f(x)=\frac{-1}{x-2}$

$f'(x)=\frac{1}{(x-2)^2}$

$f(x)=\frac{2+x}{1-x}$

$f(x)=\frac{5-2x}{3+2x}$

$\frac{-16(}{3+2x)^2}$

$f(x)=\sqrt{1+3x}$

$f(x)=3x^3-x^2+2x+5$

$f'(x)=9x^2-2x+2$

$f(x)=5$

$f(x)=5\pi$

$f'(x)=0$

For the following exercises, find the average rate of change between the two points.

$(-2,0)$ and $(-4,5)$

$(4,-3)$ and $(-2,-1)$

$-\frac{1}{3}$

$(0,5)$ and $(6,5)$

$(7,-2)$ and $(7,10)$

---

undefined

For the following polynomial functions, find the derivatives.

$f(x)=x^3+1$

$f(x)=-3x^2-7x=6$

$f'(x) = -6x - 7$

$f(x) = 7x^2$

$f(x) = 3x^3 + 2x^2 + x - 26$

$f'(x) = 9x^2 + 4x + 1$

For the following functions, find the equation of the tangent line to the curve at the given point x on the curve.

$f(x) = 2x^2 - 3x \quad x = 3$

$f(x) = x^3 + 1 \quad x = 2$

$y = 12x - 15$

$f(x) = \sqrt{x} \quad x = 9$

For the following exercise, find k such that the given line is tangent to the graph of the function.

$f(x) = x^2 - kx, \quad y = 4x - 9$

$k = -10$ or $k = 2$

Graphical

For the following exercises, consider the graph of the function f and determine where the function is continuous/discontinuous and differentiable/not differentiable.

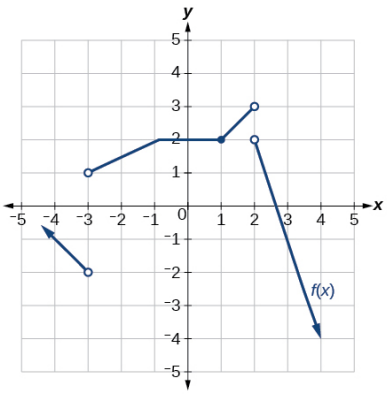


Figure_12_04_202.jpg" style="width: 487px; height: 457px;" width="487px" height="457px" src="/@api/deki/files/7709/CNX_Precalc_Figure_12_04_202.jpg" />

Discontinuous at $x = -2$ and $x = 0$. Not differentiable at –2, 0, 2.

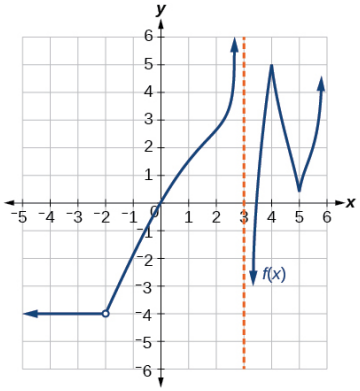
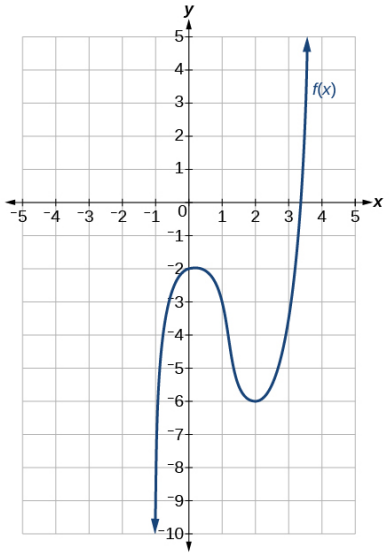


Figure_12_04_204.jpg" style="width: 402px; height: 374px;" width="402px" height="374px" src="/@api/deki/files/7711/CNX_Precalc_Figure_12_04_204.jpg" />

Discontinuous at $x = 5$. Not differentiable at -4, –2, 0, 1, 3, 4, 5.

For the following exercises, use Figure to estimate either the function at a given value of x or the derivative at a given value of x, as indicated.



$f(-1)$

$f(0)$

$f(0) = -2$

$f(1)$

$f(2)$

\(f(2)=−6f(2)=−6

$f(3)$

$f'(-1)$

$f'(-1) = 9$

$f'(0)$

$f'(1)$

$f'(1) = -3$

$f'(2)$

$f'(3)$

$f'(3) = 9$

Sketch the function based on the information below:

$f'(x) = 2x, f(2) = 4$

Technology

Numerically evaluate the derivative. Explore the behavior of the graph of $f(x) = x^2$ around $x = 1$ by graphing the function on the following domains: $[0.9, 1.1], [0.99, 1.01], [0.999, 1.001]$ and $[0.9999, 1.0001]$ We can use the feature on our calculator that automatically sets Ymin and Ymax to the Xmin and Xmax values we preset. (On some of the commonly used graphing calculators, this feature may be called ZOOM FIT or ZOOM AUTO). By examining the corresponding range values for this viewing window, approximate how the curve changes at $x = 1$, that is, approximate the derivative at $x = 1$.

Answers vary. The slope of the tangent line near $x = 1$ is 2.

Real-World Applications

For the following exercises, explain the notation in words. The volume $f(t)$ of a tank of gasoline, in gallons, t minutes after noon.

$f(0) = 600$

$f'(30) = -20$

At 12:30 p.m., the rate of change of the number of gallons in the tank is –20 gallons per minute. That is, the tank is losing 20 gallons per minute.

$f(30) = 0$

$f'(200) = 30$

At 200 minutes after noon, the volume of gallons in the tank is changing at the rate of 30 gallons per minute.

$f(240) = 500$

For the following exercises, explain the functions in words. The height, s, of a projectile after t seconds is given by $s(t) = -16t^2 + 80t$.

$s(2) = 96$

The height of the projectile after 2 seconds is 96 feet.

$s'(2) = 16$

$s(3) = 96$

The height of the projectile at $t = 3$ seconds is 96 feet.

$s'(3) = -16$

$s(0) = 0, s(5) = 0.$

The height of the projectile is zero at $t = 0$ and again at $t = 5$. In other words, the projectile starts on the ground and falls to earth again after 5 seconds.

For the following exercises, the volume V of a sphere with respect to its radius r is given by $V = \frac{4}{3}\pi r^3$.

Find the average rate of change of V as r changes from 1 cm to 2 cm.

Find the instantaneous rate of change of V when r=3 cm. r=3 cm.

36π

For the following exercises, the revenue generated by selling x items is given by $R(x) = 2x^2 + 10x$.

Find the average change of the revenue function as x changes from $x = 10$ to $x = 20$.

Find $R'(10)$ and interpret.

$50.00 per unit, which is the instantaneous rate of change of revenue when exactly 10 units are sold.

Find $R'(15)$ and interpret. Compare $R'(15)$ to $R'(10)$, and explain the difference.

For the following exercises, the cost of producing x cellphones is described by the function $C(x) = x^2 - 4x + 1000$.

Find the average rate of change in the total cost as x changes from $x = 10$ to $x = 15$.

$21 per unit

Find the approximate marginal cost, when 15 cellphones have been produced, of producing the 16th cellphone.

Find the approximate marginal cost, when 20 cellphones have been produced, of producing the 21st cellphone.

$36

Extension

For the following exercises, use the definition for the derivative at a point $x = a$, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, to find the derivative of the functions.

$f(x) = \frac{1}{x^2}$

$f(x) = 5x^2 - x + 4$

$f'(x) = 10a - 1$

$f(x) = -x^2 + 4x + 7$

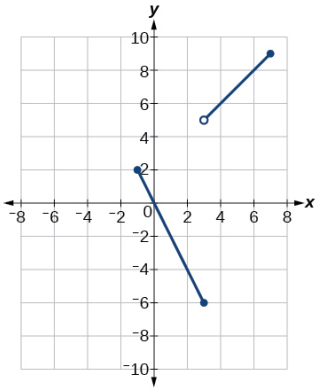
$f(x) = \frac{-4}{3 - x^2}$

$\frac{4}{(3-x)^2}$

Chapter Review Exercises

Finding Limits: A Numerical and Graphical Approach

For the following exercises, use Figure.



$\lim_{x \to -1^+} f(x)$

2

$\lim_{x \to -1^-} f(x)$

$\lim_{x \to -1} f(x)$

---

does not exist

$\lim_{x \to 3} f(x)$

At what values of x is the function discontinuous? What condition of continuity is violated?

Discontinuous at $x = -1$ ($\lim_{x \to a} f(x)$ does not exist), $x = 3$ (jump discontinuity),and $x = 7$ ((\lim \limits_{x \to a} f(x)\) does not exist).

Using Table, estimate $\lim_{x \to 0} f(x)$.

x	$F(x)$
−0.1	2.875
−0.01	2.92
−0.001	2.998
0	Undefined
0.001	2.9987
0.01	2.865
0.1	2.78145
0.15	2.678

3

For the following exercises, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has limit as x approaches a, state it. If not, discuss why there is no limit.

$f(x) = \begin{cases} |x| - 1, & \text{if } x \neq 1 \\ x^3, & \text{if } x = 1 \end{cases} \quad a = 1$

$f(x) = \begin{cases} \frac{1}{x+1}, & \text{if } x = -2 \\ (x+1)^2, & \text{if } x \neq -2 \end{cases} \quad a = -2$

$\lim_{x \to -2} f(x) = 1$

$f(x) = \begin{cases} \sqrt{x+3} & \text{if } x < 1 \\ -\sqrt[3]{x} & \text{if } x > 1 \end{cases} \quad a = 1$

Finding Limits: Properties of Limits

For the following exercises, find the limits if $\lim_{x \to c} f(x) = -3$ and $\lim_{x \to c} g(x) = 5$.

$\lim_{x \to c} (f(x) + g(x))$

2

$\lim_{x \to c} \frac{f(x)}{g(x)}$

$\lim_{x to c}(f(x) \cdot g(x))$

-15

$\lim_{x \to 0^+} f(x), f(x) = \begin{cases} 3x^2 + 2x + 1 & x > 0 \\ 5x + 3 & x < 0 \end{cases}$

$\lim_{x \to 0^-} f(x), f(x) = \begin{cases} 3x^2 + 2x + 1 & x > 0 \\ 5x + 3 & x < 0 \end{cases}$

3

$\lim_{x \to 3^+} (3x - [\![x]\!])$

For the following exercises, evaluate the limits using algebraic techniques.

$\lim_{h \to 0}(\frac{(h+6)^2 - 36}{h})$

12

$\lim_{x \to 25}(\frac{x^2 - 625}{\sqrt{x} - 5})$

$\lim_{x \to 1}(\frac{-x^2 - 9x}{x})$

-10

$\lim_{x \to 4} \frac{7 - \sqrt{12x+1}}{x-4}$

$\lim_{x \to -3}(\frac{\frac{1}{3} + \frac{1}{x}}{3+x})$

$-\frac{1}{9}$

Continuity

For the following exercises, use numerical evidence to determine whether the limit exists at $x = a$. If not, describe the behavior of the graph of the function at $x = a$.

$f(x) = \frac{-2}{x-4}; a = 4$

$f(x) = \frac{-2}{(x-4)^2}; a = 4$

At $x = 4$,the function has a vertical asymptote.

$f(x) = \frac{-x}{x^2 - x - 6}; a = 3$

$f(x) = \frac{6x^2 + 23x + 20}{4x^2 - 25}; a = -\frac{5}{2}$

---

removable discontinuity at $a = -\frac{5}{2}$

$f(x) = \frac{\sqrt{x} - 3}{9 - x}; a = 9$

For the following exercises, determine where the given function $f(x)$ is continuous. Where it is not continuous, state which conditions fail, and classify any discontinuities.

$f(x) = x^2 - 2x - 15$

---

continuous on $(-\infty, \infty)$

$f(x) = \frac{x^2 - 2x - 15}{x-5}$

$f(x) = \frac{x^2 - 2x}{x^2 - 4x + 4}$

---

removable discontinuity at $x = 2. f(2)$ is not defined, but limits exist.

$f(x) = \frac{x^3 - 125}{2x^2 - 12x + 10}$

$f(x) = \frac{x^2 - \frac{1}{x}}{2 - x}$

---

discontinuity at $x = 0$ and $x = 2$. Both $f(0)$ and $f(2)$ are not defined.

$f(x) = \frac{x+2}{x^2-3x-10}$

$f(x) = \frac{x+2}{x^3+8}$

---

removable discontinuity at $x = -2. f(-2)$ is not defined.

Derivatives

For the following exercises, find the average rate of change $\frac{f(x+h)-f(x)}{h}$.

$f(x) = 3x + 2$

$f(x) = 5$

0

$f(x) = \frac{1}{x+1}$

$f(x) = \ln(x)$

$\frac{\ln(x+h)-\ln(x)}{h}$

$f(x) = e^{2x}$

For the following exercises, find the derivative of the function.

$f(x) = 4x - 6$

$= 4$

$f(x) = 5x^2 - 3x$

Find the equation of the tangent line to the graph of $f(x)$ at the indicated x value.

$f(x) = -x^3 + 4x; x = 2.$

$y = -8x + 16$

For the following exercises, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

$f(x) = \frac{x}{|x|}$

Given that the volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$ and that a given cone has a fixed height of 9 cm and variable radius length, find the instantaneous rate of change of volume with respect to radius length when the radius is 2 cm. Give an exact answer in terms of π

12π

Practice Test

For the following exercises, use the graph of f in Figure.

$f(1)$

3

$\lim_{x \to -1^+} f(x)$

$\lim_{x \to -1^-} f(x)$

0

$\lim_{x \to -1} f(x)$

$\lim_{x \to -2} f(x)$

-1

At what values of x is f discontinuous? What property of continuity is violated?

For the following exercises, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has a limit as x approaches a,state it. If not, discuss why there is no limit

$f(x) = \begin{cases} \frac{1}{x} - 3, & \text{if } x \leq 2 \\ x^3 + 1, & \text{if } x > 2 \end{cases} \quad a = 2$

$\lim_{x\to 2^-} f(x) = -\frac{5}{2}a$ and $\lim_{x\to 2^+} f(x) = 9$ Thus, the limit of the function as x approaches 2 does not exist.

$$f(x) = \begin{cases} x^3+1, & \text{if } x < 1 \\ 3x^2-1, & \text{if } x = 1 \ a = 1 \\ -\sqrt{x+3}+4, & \text{if } x > 1 \end{cases}$$

For the following exercises, evaluate each limit using algebraic techniques.

$\lim_{x\to -5}(\frac{\frac{1}{5}+\frac{1}{x}}{10+2x})$

$-\frac{1}{50}$

$\lim_{h\to 0}(\frac{\sqrt{h^2+25}-5}{h^2})$

$\lim_{h\to 0}(\frac{1}{h}-\frac{1}{h^2+h})$

1

For the following exercises, determine whether or not the given function f is continuous. If it is continuous, show why. If it is not continuous, state which conditions fail.

$f(x) = \sqrt{x^2-4}$

$f(x) = \frac{x^3-4x^2-9x+36}{x^3-3x^2+2x-6}$

---

removable discontinuity at $x = 3$

For the following exercises, use the definition of a derivative to find the derivative of the given function at $x = a$.

$f(x) = \frac{3}{5+2x}$

$f(x) = \frac{3}{\sqrt{x}}$

$f'(x) = -\frac{3}{2a^{\frac{3}{2}}}$

$f(x) = 2x^2+9x$

discontinuous at –2,0, not differentiable at –2,0, 2.

For the following exercises, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

$f(x) = |x-2|-|x+2|$

$f(x) = \frac{2}{1+e^{\frac{2}{x}}}$

---

not differentiable at $x = 0$ (no limit)

For the following exercises, explain the notation in words when the height of a projectile in feet, s, is a function of time t t in seconds after launch and is given by the function $s(t)$.

$s(0)$

$s(2)$

---

the height of the projectile at $t = 2$ seconds

$s'(2)$

$\frac{s(2)-s(1)}{2-1}$

---

the average velocity from $t = 1$ to $t = 2$

$s(t) = 0$

For the following exercises, use technology to evaluate the limit.

$\lim_{x\to 0}\frac{\sin(x)}{3x}$

$\frac{1}{3}$

$\lim_{x\to 0}\frac{\tan^2(x)}{2x}$

$\lim_{x\to 0}\frac{\sin(x)(1-\cos(x))}{2x^2}$

0

Evaluate the limit by hand.

$\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 4x-7 & x \neq 1 \\ x^2-4 & x = 1 \end{cases}$

At what value(s) of x is the function below discontinuous?

$f(x) = \begin{cases} 4x-7 & x \neq 1 \\ x^2-4 & x = 1 \end{cases}$

For the following exercises, consider the function whose graph appears in Figure.

Find the average rate of change of the function from $x = 1$ to $x = 3$.

2

Find all values of x at which $f'(x) = 0$.

$x = 1$

Find all values of x at which $f'(x)$ does not exist.

Find an equation of the tangent line to the graph of f the indicated point: $f(x) = 3x^2 - 2x - 6, x = -2$

$y = -14x - 18$

For the following exercises, use the function $f(x) = x(1-x)^{\frac{2}{5}}$.

Graph the function $f(x) = x(1-x)^{\frac{2}{5}}$ by entering $f(x) = x((1-x)^2)^{\frac{1}{5}}$ and then by entering $f(x) = x((1-x)^{\frac{1}{5}})^2$.

Explore the behavior of the graph of $f(x)$ around $x = 1$ by graphing the function on the following domains, [0.9, 1.1], [0.99, 1.01], [0.999, 1.001], and [0.9999, 1.0001]. Use this information to determine whether the function appears to be differentiable at $x = 1$.

The graph is not differentiable at $x = 1$ (cusp).

For the following exercises, find the derivative of each of the functions using the definition: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f(x) = 2x - 8$

$f(x) = 4x^2 - 7$

$f'(x) = 8x$

$f(x) = x - \frac{1}{2}x^2$

$f(x) = \frac{1}{x+2}$

$f'(x) = -\frac{1}{(2+x)^2}$

$f(x) = \frac{3}{x-1}$

$f(x) = -x^3 + 1$

$f'(x) = -3^x 2$

$f(x) = x^2 + x^3$

$f(x) = \sqrt{x-1}$

$f'(x) = \frac{1}{2\sqrt{x-1}}$

Footnotes

- 1 www.csun.edu/science/health/d...tv&health.html Source provided.

Glossary

average rate of change

the slope of the line connecting the two points $(a, f(a))$ and $(a+h, f(a+h))$ on the curve of $f(x)$; it is given by

$$\text{AROC} = \frac{f(a+h)-f(a)}{h}. \tag{8.4.93}$$

derivative

the slope of a function at a given point; denoted $f'(a)$,at a point $x = a$ it is $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$,providing the limit exists.

differentiable

a function $f(x)$ for which the derivative exists at $x = a$. In other words, if f'(a) f'(a) exists.

instantaneous rate of change

the slope of a function at a given point; at $x = a$ it is given by $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

instantaneous velocity

the change in speed or direction at a given instant; a function $s(t)$ represents the position of an object at time t,and the instantaneous velocity or velocity of the object at time $t = a$ is given by $s'(a) = \lim_{h \to 0} \frac{s(a+h)-s(a)}{h}$.

secant line

a line that intersects two points on a curve

tangent line

a line that intersects a curve at a single point

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8.E: Introduction to Calculus (Exercises)

12.1: Finding Limits - Numerical and Graphical Approaches

In this section, we will examine numerical and graphical approaches to identifying limits.

Verbal

1) Explain the difference between a value at $x = a$ and the limit as x approaches a.

Answer

The value of the function, the output, at $x = a$ is $f(a)$. When the $\lim_{x \to a} f(x)$ is taken, the values of x get infinitely close to a but never equal a. As the values of x approach a from the left and right, the limit is the value that the function is approaching.

2) Explain why we say a function does not have a limit as x approaches a if, as x approaches a, the left-hand limit is not equal to the right-hand limit.

Graphical

For the exercises 3-14, estimate the functional values and the limits from the graph of the function f provided in the Figure below.

3) $\lim_{x \to -2^-} f(x)$

Answer

-4

4) $\lim_{x \to -2^+} f(x)$

5) $\lim_{x \to -2} f(x)$

Answer

-4

6) $f(-2)$

7) $\lim_{x \to -1^-} f(x)$

Answer

2

8) $\lim_{x \to 1^+} f(x)$

9) $\lim_{x \to 1} f(x)$

Answer

does not exist

10) $f(1)$

11) $\lim_{x \to 4^-} f(x)$

Answer

4

12) $\lim\limits_{x\to 4^+} f(x)$

13) $\lim\limits_{x\to 4} f(x)$

Answer

does not exist

14) $f(4)$

For the exercises 15-21, draw the graph of a function from the functional values and limits provided.

15) $\lim\limits_{x\to 0^-} f(x)=2,\ \lim\limits_{x\to 0^+} f(x)=-3, \lim\limits_{x\to 2} f(x)=2, f(0)=4, f(2)=-1, f(-3)$ does not exist.

Answer

Answers will vary.

16) $\lim\limits_{x\to 2^-} f(x)=0,\ \lim\limits_{x\to 2^+} =-2, \lim\limits_{x\to 0} f(x)=3, f(2)=5, f(0)$

Answer

Answers will vary.

17) $\lim\limits_{x\to 2^-} f(x)=2,\ \lim\limits_{x\to 2^+} f(x)=-3, \lim\limits_{x\to 0} f(x)=5, f(0)=1, f(1)=0$

Answer

Answers will vary.

18) $\lim\limits_{x\to 3^-} f(x)=0,\ \lim\limits_{x\to 3^+} f(x)=5, \lim\limits_{x\to 5} f(x)=0, f(5)=4, f(3)$ does not exist.

Answer

Answers will vary.

19) $\lim\limits_{x\to 4} f(x)=6,\ \lim\limits_{x\to 6^+} f(x)=-1, \lim\limits_{x\to 0} f(x)=5, f(4)=6, f(2)=6$

Answer

Answers will vary.

20) $\lim\limits_{x\to -3} f(x)=2,\ \lim\limits_{x\to 1^+} f(x)=-2, \lim\limits_{x\to 3} f(x)=-4, f(-3)=0, f(0)=0$

Answer

Answers will vary.

21) $\lim\limits_{x\to \pi} f(x)=\pi^2,\ \lim\limits_{x\to -\pi} f(x)=\dfrac{\pi}{2},\ \lim\limits_{x\to 1^-} f(x)=0, f(\pi)=\sqrt{2}, f(0)$ does not exist.

Answer

Answers will vary.

For the exercises 22-26, use a graphing calculator to determine the limit to 5 decimal places as x approaches 0.

22) $f(x)=(1+x)^{\frac{1}{x}}$

23) $g(x)=(1+x)^{\frac{2}{x}}$

Answer

7.38906

24) $h(x) = (1+x)^{\frac{3}{x}}$

25) $i(x) = (1+x)^{\frac{4}{x}}$

Answer

54.59815

26) $j(x) = (1+x)^{\frac{5}{x}}$

27) Based on the pattern you observed in the exercises above, make a conjecture as to the limit of $f(x) = (1+x)^{\frac{6}{x}}$, $g(x) = (1+x)^{\frac{7}{x}}$, and $h(x) = (1+x)^{\frac{n}{x}}$.

Answer

$e^6 \approx 403.428794, e^7 \approx 1096.633158, e^n$

For the exercises 28-29, use a graphing utility to find graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has a limit as x approaches a,state it. If not, discuss why there is no limit.

28) $(x) = \begin{cases} |x|-1, & \text{if } x \neq 1 \\ x^3, & \text{if } x = 1 \end{cases} \quad a = 1$

29) $(x) = \begin{cases} \frac{1}{x+1}, & \text{if } x = -2 \\ (x+1)^2, & \text{if } x \neq -2 \end{cases} \quad a = -2$

Answer

$\lim\limits_{x \to -2} f(x) = 1$

Numeric

For the exercises 30-38, use numerical evidence to determine whether the limit exists at $x = a$. If not, describe the behavior of the graph of the function near $x = a$. Round answers to two decimal places.

30) $f(x) = \dfrac{x^2 - 4x}{16 - x^2}; a = 4$

31) $f(x) = \dfrac{x^2 - x - 6}{x^2 - 9}; a = 3$

Answer

$\lim\limits_{x \to 3} \left(\dfrac{x^2 - x - 6}{x^2 - 9} \right) = \dfrac{5}{6} \approx 0.83$

32) $f(x) = \dfrac{x^2 - 6x - 7}{x^2 - 7x}; a = 7$

33) $f(x) = \dfrac{x^2 - 1}{x^2 - 3x + 2}; a = 1$

Answer

$\lim\limits_{x \to 1} \left(\dfrac{x^2 - 1}{x^2 - 3x + 2} \right) = -2.00$

34) $f(x) = \dfrac{1 - x^2}{x^2 - 3x + 2}; a = 1$

35) $f(x) = \dfrac{10 - 10x^2}{x^2 - 3x + 2}; a = 1$

Answer

$$\lim_{x\to 1}\left(\frac{10-10x^2}{x^2-3x+2}\right)=20.00$$

36) $f(x)=\dfrac{x}{6x^2-5x-6};a=\dfrac{3}{2}$

37) $f(x)=\dfrac{x}{4x^2+4x+1};a=-\dfrac{1}{2}$

Answer

$\lim\limits_{x\to\frac{-1}{2}}\left(\dfrac{x}{4x^2+4x+1}\right)$ does not exist. Function values decrease without bound as x approaches -0.5 from either left or right.

38) $f(x)=\frac{2}{x-4};a=4$

For the exercises 39-41, use a calculator to estimate the limit by preparing a table of values. If there is no limit, describe the behavior of the function as x approaches the given value.

39) $\lim\limits_{x\to 0}\dfrac{7\tan x}{3x}$

Answer

$$\lim_{x\to 0}\frac{7\tan x}{3x}=\frac{7}{3}$$

x	$f(x)$
-0.1	2.34114234
-0.01	2.33341114
-0.001	2.33333411
0	Error
0.001	2.33333411
0.01	2.33341114
0.1	2.34114234

$\lim\limits_{x\to 0^-}\dfrac{7\tan x}{3x}\downarrow\ \dfrac{7}{3}$

$\dfrac{7}{3}\uparrow\lim\limits_{x\to 0^+}\dfrac{7\tan x}{3x}$

40) $\lim\limits_{x\to 4}\dfrac{x^2}{x-4}$

Answer

x	$f(x)$
3.9	-152.1
3.99	-1592.01
3.999	-15992.001
4	Error
4.001	16008.001
4.01	1608.1
4.1	168.1

$\lim_{x\to 4^-} \frac{x^2}{x-4}$ $\downarrow$ $-\infty$

$+\infty$ $\uparrow$ $\lim_{x\to 4^+} \frac{x^2}{x-4}$

41) $\lim_{x\to 0} \frac{2\sin x}{4\tan x}$

Answer

$\lim_{x\to 0} \frac{2\sin x}{4\tan x} = \frac{1}{2}$

x	$f(x)$
-0.1	0.49750208
-0.01	0.49997500
-0.001	0.49999975
0	Error
0.001	0.49999975
0.01	0.49997500
0.1	0.49750208

$\lim_{x\to 0^-} \frac{2\sin x}{4\tan x}$ $\downarrow$ $\frac{1}{2}$

$\frac{1}{2}$ $\uparrow$ $\lim_{x\to 0^+} \frac{2\sin x}{4\tan x}$

For the exercises 42-49, use a graphing utility to find numerical or graphical evidence to determine the left and right-hand limits of the function given as x approaches a. If the function has a limit as x approaches a, state it. If not, discuss why there is no limit.

42) $\lim_{x\to 0} e^{e^{\frac{1}{x}}}$

43) $\lim_{x\to 0} e^{e^{-\frac{1}{x^2}}}$

Answer

$\lim_{x\to 0} e^{e^{-\frac{1}{x^2}}} = 1.0$

44) $\lim_{x\to 0} \frac{|x|}{x}$

45) $\lim\limits_{x\to -1}\dfrac{|x+1|}{x+1}$

Answer

$\lim\limits_{x\to -1^-}\dfrac{|x+1|}{x+1}=\dfrac{-(x+1)}{(x+1)}=-1$ and $\lim\limits_{x\to -1^+}\dfrac{|x+1|}{x+1}=\dfrac{(x+1)}{(x+1)}=1$; since the right-hand limit does not equal the left-hand limit, $\lim\limits_{x\to -1}\dfrac{|x+1|}{x+1}$ does not exist.

46) $\lim\limits_{x\to 5}\dfrac{|x-5|}{5-x}$

47) $\lim\limits_{x\to -1}\dfrac{1}{(x+1)^2}$

Answer

$\lim\limits_{x\to -1}\dfrac{1}{(x+1)^2}$ does not exist. The function increases without bound as x approaches -1 from either side.

48) $\lim\limits_{x\to 1}\dfrac{1}{(x-1)^3}$

49) $\lim\limits_{x\to 0}\dfrac{5}{1-e^{\frac{2}{x}}}$

Answer

$\lim\limits_{x\to 0}\dfrac{5}{1-e^{\frac{2}{x}}}$ does not exist. Function values approach 5 from the left and approach 0 from the right.

50) Use numerical and graphical evidence to compare and contrast the limits of two functions whose formulas appear similar: $f(x)=\left|\dfrac{1-x}{x}\right|$ and $g(x)=\left|\dfrac{1+x}{x}\right|$ as x approaches 0. Use a graphing utility, if possible, to determine the left- and right-hand limits of the functions $f(x)$ and $g(x)$ as x approaches 0. If the functions have a limit as x approaches 0, state it. If not, discuss why there is no limit.

Extensions

51) According to the Theory of Relativity, the mass m m of a particle depends on its velocity v. That is

$$m=\frac{m_o}{\sqrt{1-(v^2/c^2)}}$$

where m_o is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass, m, as v approaches c^-.

Answer

Through examination of the postulates and an understanding of relativistic physics, as $v\to c, m\to\infty$. Take this one step further to the solution,

$$\lim_{v\to c^-} m=\lim_{v\to c^-}\frac{m_o}{\sqrt{1-(v^2/c^2)}}=\infty$$

52) Allow the speed of light, c, to be equal to 1.0. If the mass, m, is 1, what occurs to m as $v\to c$? Using the values listed in the Table below, make a conjecture as to what the mass is as v approaches 1.00.

v	m
0.5	1.15
0.9	2.29

v	m
0.95	3.20
0.99	7.09
0.999	22.36
0.99999	223.61

12.2: Finding Limits - Properties of Limits

Graphing a function or exploring a table of values to determine a limit can be cumbersome and time-consuming. When possible, it is more efficient to use the properties of limits, which is a collection of theorems for finding limits. Knowing the properties of limits allows us to compute limits directly.

Verbal

1) Give an example of a type of function f whose limit, as x approaches a, is $f(a)$.

Answer

If f is a polynomial function, the limit of a polynomial function as x approaches a will always be $f(a)$.

2) When direct substitution is used to evaluate the limit of a rational function as x approaches a and the result is $f(a)=\frac{0}{0}$,does this mean that the limit of f does not exist?

3) What does it mean to say the limit of $f(x)$, as x approaches c, is undefined?

Answer

It could mean either (1) the values of the function increase or decrease without bound as x approaches c, or (2) the left and right-hand limits are not equal.

Algebraic

For the exercises 4-30, evaluate the limits algebraically.

4) $\lim\limits_{x\to 0}(3)$

5) $\lim\limits_{x\to 2}\left(\frac{-5x}{x^2-1}\right)$

Answer

$\frac{-10}{3}$

6) $\lim\limits_{x\to 2}\left(\frac{x^2-5x+6}{x+2}\right)$

7) $\lim\limits_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$

Answer

6

8) $\lim\limits_{x\to -1}\left(\frac{x^2-2x-3}{x+1}\right)$

9) $\lim\limits_{x\to \frac{3}{2}}\left(\frac{6x^2-17x+12}{2x-3}\right)$

Answer

$\frac{1}{2}$

10) $\lim_{x\to-\frac{7}{2}}\left(\frac{8x^2+18x-35}{2x+7}\right)$

11) $\lim_{x\to3}\left(\frac{x^2-9}{x-5x+6}\right)$

Answer

6

12) $\lim_{x\to-3}\left(\frac{-7x^4-21x^3}{-12x^4+108x^2}\right)$

13) $\lim_{x\to3}\left(\frac{x^2+2x-3}{x-3}\right)$

Answer

does not exist

14) $\lim_{h\to0}\left(\frac{(3+h)^3-27}{h}\right)$

15) $\lim_{h\to0}\left(\frac{(2-h)^3-8}{h}\right)$

Answer

-12

16) $\lim_{h\to0}\left(\frac{(h+3)^2-9}{h}\right)$

17) $\lim_{h\to0}\left(\frac{\sqrt{5-h}-\sqrt{5}}{h}\right)$

Answer

$-\frac{\sqrt{5}}{10}$

18) $\lim_{x\to0}\left(\frac{\sqrt{3-x}-\sqrt{3}}{x}\right)$

19) $\lim_{x\to9}\left(\frac{x^2-81}{3-x}\right)$

Answer

-108

20) $\lim_{x\to1}\left(\frac{\sqrt{x}-x^2}{1-\sqrt{x}}\right)$

21) $\lim_{x\to0}\left(\frac{x}{\sqrt{1+2x}-1}\right)$

Answer

1

22) $\lim_{x\to\frac{1}{2}}\left(\frac{x^2-\frac{1}{4}}{2x-1}\right)$

23) $\lim_{x\to 4}\left(\frac{x^3-64}{x^2-16}\right)$

Answer

6

24) $\lim_{x\to 2^-}\left(\frac{|x-2|}{x-2}\right)$

25) $\lim_{x\to 2^+}\left(\frac{|x-2|}{x-2}\right)$

Answer

1

26) $\lim_{x\to 2}\left(\frac{|x-2|}{x-2}\right)$

27) $\lim_{x\to 4^-}\left(\frac{|x-4|}{4-x}\right)$

Answer

1

28) $\lim_{x\to 4^+}\left(\frac{|x-4|}{4-x}\right)$

29) $\lim_{x\to 4}\left(\frac{|x-4|}{4-x}\right)$

Answer

does not exist

30) $\lim_{x\to 2}\left(\frac{-8+6x-x^2}{x-2}\right)$

For the exercises 31-33, use the given information to evaluate the limits: $\lim_{x\to c} f(x)=3, \lim_{x\to c} g(x)=5$

31) $\lim_{x\to c}[2f(x)+\sqrt{g(x)}]$

Answer

$6+\sqrt{5}$

32) $\lim_{x\to c}[3f(x)+\sqrt{g(x)}]$

33) $\lim_{x\to c}\frac{f(x)}{g(x)}$

Answer

$\frac{3}{5}$

For the exercises 34-43, evaluate the following limits.

34) $\lim_{x\to 2}\cos(\pi x)$

35) $\lim_{x\to 2}\sin(\pi x)$

Answer

0

36) $\lim_{x\to 2}\sin\left(\frac{\pi}{x}\right)$

37) $f(x)=\begin{cases}2x^2+2x+1, & x\le 0\\ x-3, & x>0;\end{cases}\lim_{x\to 0^+} f(x)$

Answer

-3

38) $f(x)=\begin{cases}2x^2+2x+1, & x\le 0\\ x-3, & x>0;\end{cases}\lim_{x\to 0^-} f(x)$

39) $f(x)=\begin{cases}2x^2+2x+1, & x\le 0\\ x-3, & x>0;\end{cases}\lim_{x\to 0} f(x)$

Answer

does not exist; right-hand limit is not the same as the left-hand limit.

40) $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}$

41) $\lim_{x\to 2^+}(2x-[\![\ x]\!]\ )$

Answer

2

42) $\lim_{x\to 2}\frac{\sqrt{x+7}-3}{x^2-x-2}$

43) $\lim_{x\to 3^+}\frac{x^2}{x^2-9}$

Answer

Limit does not exist; limit approaches infinity.

For the exercises 44-53, find the average rate of change $\frac{f(x+h)-f(x)}{h}$.

44) $f(x)=x+1$

45) $f(x)=2x^2-1$

Answer

$4x+2h$

46) $f(x)=x^2+3x+4$

47) $f(x)=x^2+4x-100$

Answer

$2x+h+4$

48) $f(x)=3x^2+1$

49) $f(x)=\cos(x)$

Answer

$$\frac{\cos(x+h)-\cos(x)}{h}$$

50) $f(x)=2x^3-4x$

51) $f(x)=\frac{1}{x}$

Answer

$$\frac{-1}{x(x+h)}$$

52) $f(x)=\frac{1}{x^2}$

53) $f(x)=\sqrt{x}$

Answer

$$\frac{-1}{\sqrt{x+h}+\sqrt{x}}$$

Graphical

54) Find an equation that could be represented by the Figure below.

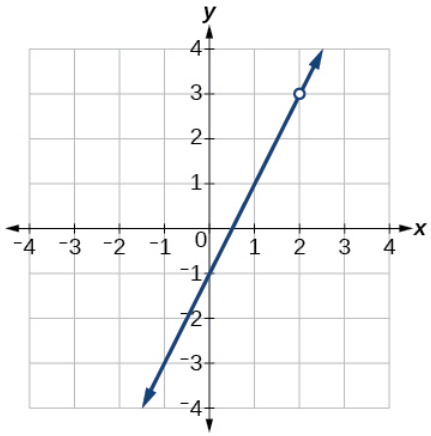


Figure below.

Answer

$$f(x)=\frac{x^2+5x+6}{x+3}$$

For the exercises 56-57, refer to the Figure below.

56) What is the right-hand limit of the function as x approaches 0?

57) What is the left-hand limit of the function as x approaches 0?

Answer

does not exist

Real-World Applications

58) The position function $s(t) = -16t^2 + 144t$ gives the position of a projectile as a function of time. Find the average velocity (average rate of change) on the interval $[1, 2]$.

59) The height of a projectile is given by $s(t) = -64t^2 + 192t$ Find the average rate of change of the height from $t = 1$ second to $t = 1.5$ seconds.

Answer

52

60) The amount of money in an account after t years compounded continuously at 4.25% interest is given by the formula $A = A_0 e^{0.0425t}$,where A_0 is the initial amount invested. Find the average rate of change of the balance of the account from $t = 1$ year to $t = 2$ years if the initial amount invested is $\$1,000.00$.

12.3: Continuity

A function that remains level for an interval and then jumps instantaneously to a higher value is called a stepwise function. This function is an example. A function that has any hole or break in its graph is known as a discontinuous function. A stepwise function, such as parking-garage charges as a function of hours parked, is an example of a discontinuous function. We can check three different conditions to decide if a function is continuous at a particular number.

Verbal

1) State in your own words what it means for a function f to be continuous at $x = c$.

Answer

Informally, if a function is continuous at $x = c$, then there is no break in the graph of the function at $f(c)$, and $f(c)$ is defined.

2) State in your own words what it means for a function to be continuous on the interval (a, b).

Algebraic

For the exercises 3-22, determine why the function f is discontinuous at a given point a on the graph. State which condition fails.

3) $f(x) = \ln|x + 3|, a = -3$

Answer

discontinuous at $a = -3$; $f(-3)$ does not exist

4) $f(x) = \ln|5x - 2|, a = \dfrac{2}{5}$

5) $f(x) = \dfrac{x^2 - 16}{x + 4}, a = -4$

Answer

removable discontinuity at $a = -4$; $f(-4)$ is not defined

6) $f(x) = \dfrac{x^2 - 16x}{x}, a = 0$

7) $f(x) = \begin{cases} x, & x \neq 3 \\ 2x, & x = 3 \end{cases}$ $a = 3$

Answer

Discontinuous at $a = 3; \lim\limits_{x \to 3} f(x) = 3,$ but $f(3) = 6,$ which is not equal to the limit.

8) $f(x)=\begin{cases}5, & x\neq 0\\ 3, & x=0\end{cases}\ a=0$

9) $f(x)=\begin{cases}\dfrac{1}{2-x}, & x\neq 2\\ 3, & x=2\end{cases}\ a=2$

Answer

$\lim\limits_{x\to 2} f(x)$ does not exist.

10) $f(x)=\begin{cases}\dfrac{1}{x+6}, & x=-6\\ x^2, & x\neq -6\end{cases}\ a=-6$

11) $f(x)=\begin{cases}3+x, & x<1\\ x, & x=1\\ x^2, & x>1\end{cases}\ a=1$

Answer

$\lim\limits_{x\to 1^-} f(x)=4$; $\lim\limits_{x\to 1^+} f(x)=1$. Therefore, $\lim\limits_{x\to 1} f(x)$ does not exist.

12) $f(x)=\begin{cases}3-x, & x<1\\ x, & x=1\\ 2x^2, & x>1\end{cases}\ a=1$

13) $f(x)=\begin{cases}3+2x, & x<1\\ x, & x=1\\ -x^2, & x>1\end{cases}\ a=1$

Answer

$\lim\limits_{x\to 1^-} f(x)=5\neq \lim\limits_{x\to 1^+} f(x)=-1$. Thus $\lim\limits_{x\to 1} f(x)$ does not exist.

14) $f(x)=\begin{cases}x^2, & x<-2\\ 2x+1, & x=-2\\ x^3, & x>-2\end{cases}\ a=-2$

15) $f(x)=\begin{cases}\dfrac{x^2-9}{x+3}, & x<-3\\ x-9, & x=-3\\ \dfrac{1}{x}, & x>-3\end{cases}\ a=-3$

Answer

$\lim\limits_{xto-3^+} f(x)=-\dfrac{1}{3}$

Therefore, $\lim\limits_{x\to -3} f(x)$ does not exist.

16) $f(x)=\begin{cases}\dfrac{x^2-9}{x+3}, & x<-3\\ x-9, & x=-3\\ -6, & x>-3\end{cases}\ a=3$

17) $f(x)=\dfrac{x^2-4}{x-2}, a=2$

Answer

$f(2)$ is not defined.

18) $f(x)=\dfrac{25-x^2}{x^2-10x+25}, a=5$

19) $f(x)=\dfrac{x^3-9x}{x^2+11x+24}, a=-3$

Answer

$f(-3)$ is not defined.

20) $f(x)=\dfrac{x^3-27}{x^2-3x}, a=3$

21) $f(x)=\dfrac{x}{|x|}, a=0$

Answer

$f(0)$ is not defined.

22) $f(x)=\dfrac{2|x+2|}{x+2}, a=-2$

For the exercises 23-35, determine whether or not the given function f is continuous everywhere. If it is continuous everywhere it is defined, state for what range it is continuous. If it is discontinuous, state where it is discontinuous.

23) $f(x)=x^3-2x-15$

Answer

Continuous on $(-\infty,\infty)$

24) $f(x)=\dfrac{x^2-2x-15}{x-5}$

25) $f(x)=2\cdot 3^{x+4}$

Answer

Continuous on $(-\infty,\infty)$

26) $f(x)=-\sin(3x)$

27) $f(x)=\dfrac{|x-2|}{x^2-2x}$

Answer

Discontinuous at $x=0$ and$x=2$

28) $f(x)=\tan(x)+2$

29) $f(x)=2x+\dfrac{5}{x}$

Answer

Discontinuous at $x=0$

30) $f(x)=\log_2(x)$

31) $f(x)=\ln x^2$

Answer

Continuous on $(0,\infty)$

32) $f(x) = e^{2x}$

33) $f(x) = \sqrt{x-4}$

Answer

Continuous on $[4, \infty)$

34) $f(x) = \sec(x) - 3$

35) $f(x) = x^2 + \sin(x)$

Answer

Continuous on $(-\infty, \infty)$.

36) Determine the values of b and c such that the following function is continuous on the entire real number line.

$$f(x) = \begin{cases} x+1, & 1 < x < 3 \\ x^2 + bx + c, & |x-2| \geq 1 \end{cases}$$

Graphical

For the exercises 37-39, refer to the Figure below. Each square represents one square unit. For each value of a, determine which of the three conditions of continuity are satisfied at $x = a$ and which are not.

37) $x = -3$

Answer

1, but not 2 or 3

38) $x = 2$

39) $x = 4$

Answer

1 and 2, but not 3

For the exercises 40-43, use a graphing utility to graph the function $f(x) = \sin\left(\frac{12\pi}{x}\right)$ as in Figure. Set the x-axis a short distance before and after 0 to illustrate the point of discontinuity.

40) Which conditions for continuity fail at the point of discontinuity?

41) Evaluate $f(0)$.

Answer

$f(0)$ is undefined.

42) Solve for x if $f(x) = 0$.

43) What is the domain of $f(x)$?

Answer

$(-\infty, 0) \cup (0, \infty)$

For the exercises 44-45, consider the function shown in the Figure below.

44) At what x-coordinates is the function discontinuous?

45) What condition of continuity is violated at these points?

Answer

At $x = -1$, the limit does not exist. At $x = 1$, $f(1)$ does not exist.

At $x = 2$, there appears to be a vertical asymptote, and the limit does not exist.

46) Consider the function shown in the Figure below. At what x-coordinates is the function discontinuous? What condition(s) of continuity were violated?

47) Construct a function that passes through the origin with a constant slope of 1, with removable discontinuities at $x = -7$ and $x = 1$.

Answer

$$\frac{x^3 + 6x^2 - 7x}{(x+7)(x-1)}$$

48) The function $f(x) = \dfrac{x^3 - 1}{x - 1}$ is graphed in the Figure below. It appears to be continuous on the interval $[-3, 3]$, but there is an x-value on that interval at which the function is discontinuous. Determine the value of x at which the function is discontinuous, and explain the pitfall of utilizing technology when considering continuity of a function by examining its graph.

49) Find the limit $\lim\limits_{x \to 1} f(x)$ and determine if the following function is continuous at $x = 1$:

$$fx = \begin{cases} x^2 + 4 & x \neq 1 \\ 2 & x = 1 \end{cases}$$

Answer

The function is discontinuous at $x = 1$ because the limit as x approaches 1 is 5 and $f(1) = 2$.

50) The graph of $f(x) = \dfrac{\sin(2x)}{x}$ is shown in the Figure below. Is the function $f(x)$ continuous at $x = 0$? Why or why not?

12.4: Derivatives

Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

Verbal

1) How is the slope of a linear function similar to the derivative?

Answer

The slope of a linear function stays the same. The derivative of a general function varies according to x. Both the slope of a line and the derivative at a point measure the rate of change of the function.

2) What is the difference between the average rate of change of a function on the interval $[x, x + h]$ and the derivative of the function at x?

3) A car traveled 110 miles during the time period from 2:00 P.M. to 4:00 P.M. What was the car's average velocity? At exactly 2:30 P.M., the speed of the car registered exactly 62 miles per hour. What is another name for the speed of the car at 2:30 P.M.? Why does this speed differ from the average velocity?

Answer

Average velocity is 55 miles per hour. The instantaneous velocity at 2:30 p.m. is 62 miles per hour. The instantaneous velocity measures the velocity of the car at an instant of time whereas the average velocity gives the velocity of the car over an interval.

4) Explain the concept of the slope of a curve at point x.

5) Suppose water is flowing into a tank at an average rate of 45 gallons per minute. Translate this statement into the language of mathematics.

Answer

The average rate of change of the amount of water in the tank is 45 gallons per minute. If $f(x)$ is the function giving the amount of water in the tank at any time t, then the average rate of change of $f(x)$ between $t=a$ and $t=b$ is $f(a)+45(b-a)$.

Algebraic

For the exercises 6-17, use the definition of derivative $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to calculate the derivative of each function.

6) $f(x)=3x-4$

7) $f(x)=-2x+1$

Answer

$f'(x)=-2$

8) $f(x)=x^2-2x+1$

9) $f(x)=2x^2+x-3$

Answer

$f'(x)=4x+1$

10) $f(x)=2x^2+5$

11) $f(x)=\frac{-1}{x-2}$

Answer

$f'(x)=\frac{1}{(x-2)^2}$

12) $f(x)=\frac{2+x}{1-x}$

13) $f(x)=\frac{5-2x}{3+2x}$

Answer

$\frac{-16}{(3+2x)^2}$

14) $f(x)=\sqrt{1+3x}$

15) $f(x)=3x^3-x^2+2x+5$

Answer

$f'(x)=9x^2-2x+2$

16) $f(x)=5$

17) $f(x)=5\pi$

Answer

$f'(x)=0$

For the exercises 18-21, find the average rate of change between the two points.

18) $(-2,0)$ and $(-4,5)$

19) $(4, -3)$ and $(-2, -1)$

Answer

$$-\frac{1}{3}$$

20) $(0, 5)$ and $(6, 5)$

21) $(7, -2)$ and $(7, 10)$

Answer

undefined

For the polynomial functions 22-25, find the derivatives.

22) $f(x) = x^3 + 1$

23) $f(x) = -3x^2 - 7x = 6$

Answer

$f'(x) = -6x - 7$

24) $f(x) = 7x^2$

25) $f(x) = 3x^3 + 2x^2 + x - 26$

Answer

$f'(x) = 9x^2 + 4x + 1$

For the functions 26-28, find the equation of the tangent line to the curve at the given point x on the curve.

26) $f(x) = 2x^2 - 3x \quad x = 3$

27) $f(x) = x^2 + 1 \quad x = 2$

Answer

$y = 12x - 15$

28) $f(x) = \sqrt{x} \quad x = 9$

29) For the following exercise, find k such that the given line is tangent to the graph of the function.

$$f(x) = x^2 - kx \quad y = 4x - 9$$

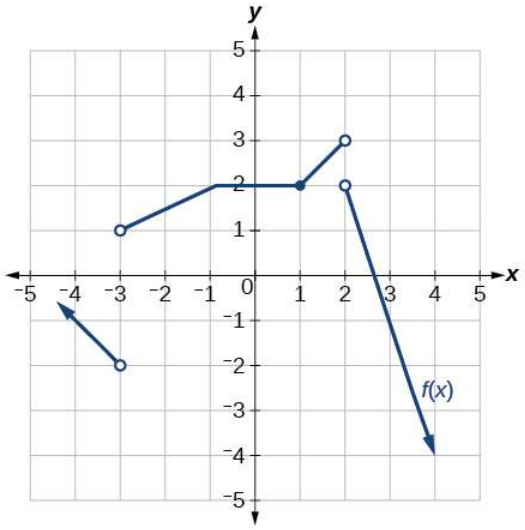
Answer

$k = -10$ or $k = 2$

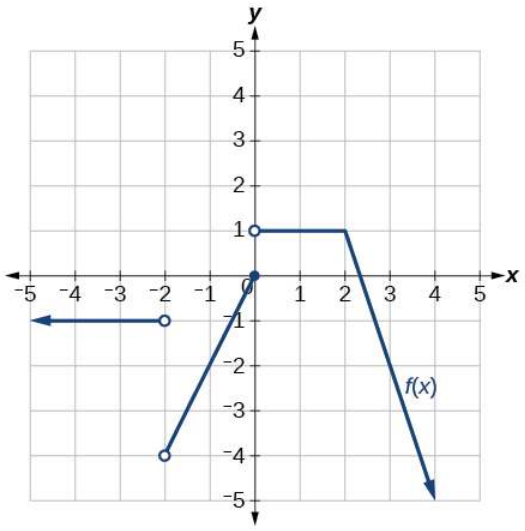
Graphical

For the exercises 30-33, consider the graph of the function f and determine where the function is continuous/discontinuous and differentiable/not differentiable.

30)



31)

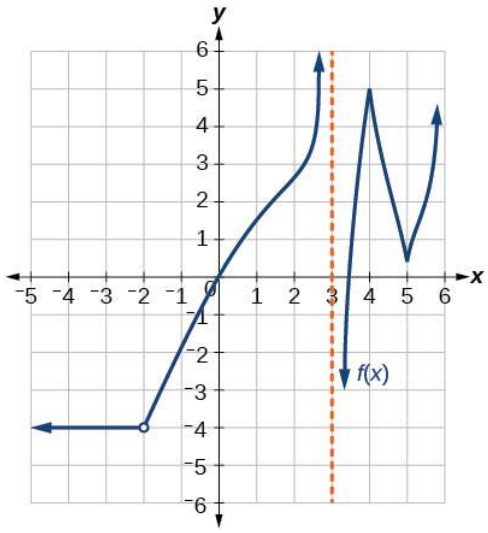


Answer

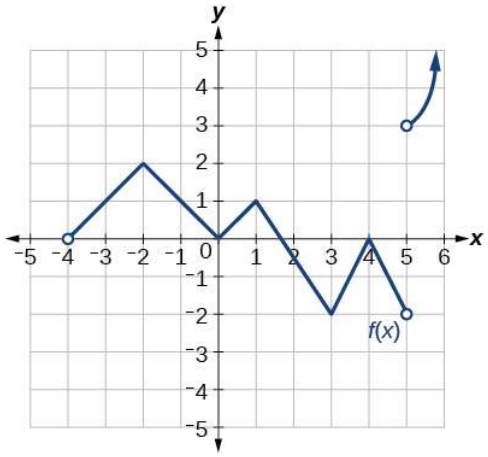
Discontinuous at $x = -2$ and $x = 0$. Not differentiable at $-2, 0, 2$.

32)

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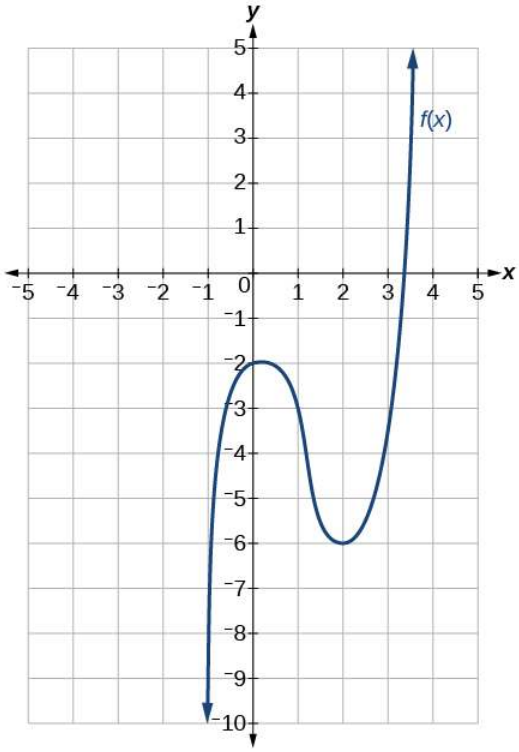
33)



Answer

Discontinuous at $x = 5$. Not differentiable at $-4, -2, 0, 1, 3, 4, 5$

For the exercises 34-43, use the Figure below to estimate either the function at a given value of x or the derivative at a given value of x, as indicated.



34) $f(-1)$

35) $f(0)$

Answer

$f(0) = -2$

36) $f(1)$

37) $f(2)$

Answer

$f(2) = -6$

38) $f(3)$

39) $f'(-1)$

Answer

$f'(-1) = 9$

40) $f'(0)$

41) $f'(1)$

Answer

$f'(1) = -3$

42) $f'(2)$

43) $f'(3)$

Answer

$f'(3) = 9$

44) Sketch the function based on the information below:

$$f'(x) = 2x, f(2) = 4$$

Technology

45) Numerically evaluate the derivative. Explore the behavior of the graph of $f(x) = x^2$ around $x = 1$ by graphing the function on the following domains: $[0.9, 1.1]$, $[0.99, 1.01]$, $[0.999, 1.001]$, $[0.9999, 1.0001]$. We can use the feature on our calculator that automatically sets Ymin and Ymax to the Xmin and Xmax values we preset. (On some of the commonly used graphing calculators, this feature may be called ZOOM FIT or ZOOM AUTO). By examining the corresponding range values for this viewing window, approximate how the curve changes at $x = 1$, that is, approximate the derivative at $x = 1$.

Answer

Answers vary. The slope of the tangent line near $x = 1$ is 2.

Real-World Applications

For the exercises 46-50, explain the notation in words. The volume $f(t)$ of a tank of gasoline, in gallons, t minutes after noon.

46) $f(0) = 600$

47) $f'(30) = -20$

Answer

At 12:30 p.m., the rate of change of the number of gallons in the tank is -20 gallons per minute. That is, the tank is losing 20 gallons per minute.

48) $f(30) = 0$

49) $f'(200) = 30$

Answer

At 200 minutes after noon, the volume of gallons in the tank is changing at the rate of 30 gallons per minute.

50) $f(240) = 500$

For the exercises 51-55, explain the functions in words. The height, s, of a projectile after t seconds is given by $s(t) = -16t^2 + 80t$.

51) $s(2) = 96$

Answer

The height of the projectile after 2 seconds is 96 feet.

52) $s'(2) = 16$

53) $s(3) = 96$

Answer

The height of the projectile at $t = 3$ seconds is 96 feet.

54) $s'(3) = -16$

55) $s(0) = 0, s(5) = 0$

Answer

The height of the projectile is zero at $t = 0$ and again at $t = 5$. In other words, the projectile starts on the ground and falls to earth again after 5 seconds.

For the exercises 56-57, the volume V of a sphere with respect to its radius r is given by $V = \frac{4}{3}\pi r^3$.

56) Find the average rate of change of V as r changes from 1 cm to 2 cm.

57) Find the instantaneous rate of change of V when $r = 3$ cm.

Answer

36π

For the exercises 58-60, the revenue generated by selling x items is given by $R(x) = 2x^2 + 10x$.

58) Find the average change of the revenue function as x changes from $x = 10$ to $x = 20$.

59) Find $R'(10)$ and interpret.

Answer

$\$50.00$per unit, which is the instantaneous rate of change of revenue when exactly 10 units are sold.

60) Find $R'(15)$ and interpret. Compare $R'(15)$ to $R'(10)$, and explain the difference.

For the exercises 61-63, the cost of producing x cellphones is described by the function $C(x) = x^2 - 4x + 1000$.

61) Find the average rate of change in the total cost as x changes from $x = 10$ to $x = 15$.

Answer

$\$21$per unit

62) Find the approximate marginal cost, when 15 cellphones have been produced, of producing the 16th cellphone.

63) Find the approximate marginal cost, when 20 cellphones have been produced, of producing the 21st cellphone.

Answer

$\$36$

Extension

For the exercises 64-67, use the definition for the derivative at a point $x = a$, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, to find the derivative of the functions.

64) $f(x) = \frac{1}{x^2}$

65) $f(x) = 5x^2 - x + 4$

Answer

$f'(x) = 10a - 1$

66) $f(x) = -x^2 + 4x + 7$

67) $f(x) = \frac{-4}{3 - x^2}$

Answer

$\frac{4}{(3 - x)^2}$

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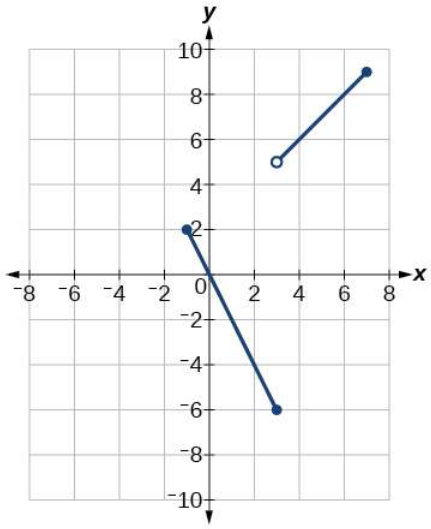
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8.R: Introduction to Calculus (Review)

12.1: Finding Limits - Numerical and Graphical Approaches

For the exercises 1-6, use the Figure below.



1) $\lim_{x \to -1^+} f(x)$

Answer

2

2) $\lim_{x \to -1^-} f(x)$

3) $\lim_{x \to -1} f(x)$

Answer

does not exist

4) $\lim_{x \to 3} f(x)$

5) At what values of x is the function discontinuous? What condition of continuity is violated?

Answer

Discontinuous at $x = -1$ $\left(\lim_{x \to a} f(x) \text{ does not exist}\right)$, $x = 3$ (jump discontinuity) , and $x = 7$ $\left(\lim_{x \to a} f(x) \text{ does not exist}\right)$.

6) Using the Table below, estimate $\lim_{x \to 0} f(x)$.

x	$F(x)$
−0.1	2.875
−0.01	2.92
−0.001	2.998
0	Undefined
0.001	2.9987

x	$F(x)$
0.01	2.865
0.1	2.78145
0.15	2.678

Answer

3

For the exercises 7-9, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has limit as x approaches a, state it. If not, discuss why there is no limit.

7) $f(x)=\begin{cases}|x|-1 & \text{if } x\neq 1\\ x^3 & \text{if } x=1\end{cases}\quad a=1$

8) $f(x)=\begin{cases}\dfrac{1}{x+1} & \text{if } x=-2\\ (x+1)^2 & \text{if } x\neq -2\end{cases}\quad a=-2$

Answer

$\lim\limits_{x\to -2} f(x)=1$

9) $f(x)=\begin{cases}\sqrt{x+3} & \text{if } x<1\\ -\sqrt[3]{x} & \text{if } x>1\end{cases}\quad a=1$

12.2: Finding Limits - Properties of Limits

For the exercises 1-6, find the limits if $\lim\limits_{x\to c} f(x)=-3$ and $\lim\limits_{x\to c} g(x)=5$.

1) $\lim\limits_{x\to c}(f(x)+g(x))$

Answer

2

2) $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$

3) $\lim\limits_{x\to c}(f(x)\cdot g(x))$

Answer

-15

4) $\lim\limits_{x\to 0^+} f(x), f(x)=\begin{cases}3x^2+2x+1 & x>0\\ 5x+3 & x<0\end{cases}$

5) $\lim\limits_{x\to 0^-} f(x), f(x)=\begin{cases}3x^2+2x+1 & x>0\\ 5x+3 & x<0\end{cases}$

Answer

3

6) $\lim\limits_{x\to 3^+}(3x-[\![\ x]\!]\ )$

For the exercises 7-11, evaluate the limits using algebraic techniques.

7) $\lim_{h\to 0}\left(\dfrac{(h+6)^2-36}{h}\right)$

Answer

12

8) $\lim_{x\to 25}\left(\dfrac{x^2-625}{\sqrt{x}-5}\right)$

9) $\lim_{x\to 1}\left(\dfrac{-x^2-9x}{x}\right)$

Answer

-10

10) $\lim_{x\to 4}\left(\dfrac{7-\sqrt{12x+1}}{x-4}\right)$

11) $\lim_{x\to 3}\left(\dfrac{\frac{1}{3}+\frac{1}{x}}{3+x}\right)$

Answer

$-\dfrac{1}{9}$

12.3: Continuity

For the exercises 1-5, use numerical evidence to determine whether the limit exists at $x=a$. If not, describe the behavior of the graph of the function at $x=a$.

1) $f(x)=\dfrac{-2}{x-4};\ a=4$

2) $f(x)=\dfrac{-2}{(x-4)^2};\ a=4$

Answer

At $x=4$, the function has a vertical asymptote.

3) $f(x)=\dfrac{-x}{x^2-x-6};\ a=3$

4) $f(x)=\dfrac{6x^2+23x+20}{4x^2-25};\ a=-\dfrac{5}{2}$

Answer

removable discontinuity at $a=-\dfrac{5}{2}$

5) $f(x)=\dfrac{\sqrt{x}-3}{9-x};\ a=9$

For the exercises 6-12, determine where the given function $f(x)$ is continuous. Where it is not continuous, state which conditions fail, and classify any discontinuities.

6) $f(x)=x^2-2x-15$

Answer

continuous on $(-\infty,\infty)$

7) $f(x)=\dfrac{x^2-2x-15}{x-5}$

8) $f(x)=\dfrac{x^2-2x}{x^2-4x+4}$

Answer

removable discontinuity at $x=2$. $f(2)$ is not defined, but limits exist.

9) $f(x)=\dfrac{x^3-125}{2x^2-12x+10}$

10) $f(x)=\dfrac{x^2-\frac{1}{x}}{2-x}$

Answer

discontinuity at $x=0$ and $x=2$. Both $f(0)$ and $f(2)$ are not defined.

11) $f(x)=\dfrac{x+2}{x^2-3x-10}$

12) $f(x)=\dfrac{x+2}{x^3+8}$

Answer

removable discontinuity at $x=-2$. $f(-2)$ is not defined.

12.4: Derivatives

For the exercises 1-5, find the average rate of change $f(x)=\dfrac{f(x+h)-f(x)}{h}$.

1) $f(x)=3x+2$

2) $f(x)=5$

Answer

0

3) $f(x)=\dfrac{1}{x+1}$

4) $f(x)=\ln(x)$

Answer

$f(x)=\dfrac{\ln(x+h)-\ln(x)}{h}$

5) $f(x)=e^{2x}$

For the exercises 6-7, find the derivative of the function.

6) $f(x)=4x-6$

Answer

4

7) $f(x)=5x^2-3x$

8) Find the equation of the tangent line to the graph of $f(x)$ at the indicated x value.

$$f(x)=-x^3+4x;\ x=2$$

Answer

$y = -8x + 16$

9) For the following exercise, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

$$f(x) = \frac{x}{|x|}$$

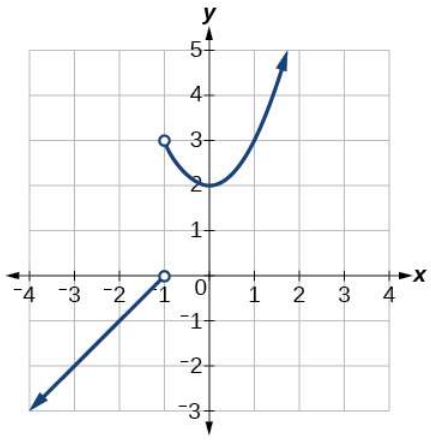
10) Given that the volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$ and that a given cone has a fixed height of 9 cm and variable radius length, find the instantaneous rate of change of volume with respect to radius length when the radius is 2 cm. Give an exact answer in terms of π.

Answer

12π

Practice Test

For the exercises 1-6, use the graph of f in the Figure below.



1) $f(1)$

Answer

3

2) $\lim\limits_{x \to -1^+} f(x)$

3) $\lim\limits_{x \to -1^-} f(x)$

Answer

0

4) $\lim\limits_{x \to -1} f(x)$

5) $\lim\limits_{x \to -2} f(x)$

Answer

-1

6) At what values of x is f discontinuous? What property of continuity is violated?

7) $f(x)=\begin{cases}\frac{1}{3}-3 & \text{if } x\le 2\\ x^3+1 & \text{if } x>2\end{cases}$ $a=2$

Answer

$\lim\limits_{x\to 2^-} f(x)=-\frac{5}{2}a$ and $\lim\limits_{x\to 2^+} f(x)=9$

Thus, the limit of the function as x approaches 2 does not exist.

8) $f(x)=\begin{cases}x^3+1 & \text{if } x<1\\ 3x^2-1 & \text{if } x=1\\ -\sqrt{x+3}+4 & \text{if } x>1\end{cases}$ $a=1$

For the exercises 9-11, evaluate each limit using algebraic techniques.

9) $\lim\limits_{x\to -5}\left(\frac{\frac{1}{5}+\frac{1}{x}}{10+2x}\right)$

Answer

$-\frac{1}{50}$

10) $\lim\limits_{h\to 0}\left(\frac{\sqrt{h^2+25}-5}{h^2}\right)$

11) $\lim\limits_{h\to 0}\left(\frac{1}{h}-\frac{1}{h^2+h}\right)$

Answer

1

For the exercises 12-13, determine whether or not the given function f is continuous. If it is continuous, show why. If it is not continuous, state which conditions fail.

12) $f(x)=\sqrt{x^2-4}$

13) $f(x)=\frac{x^3-4x^2-9x+36}{x^3-3x^2+2x-6}$

Answer

removable discontinuity at $x=3$

For the exercises 14-16, use the definition of a derivative to find the derivative of the given function at $x=a$.

14) $f(x)=\frac{3}{5+2x}$

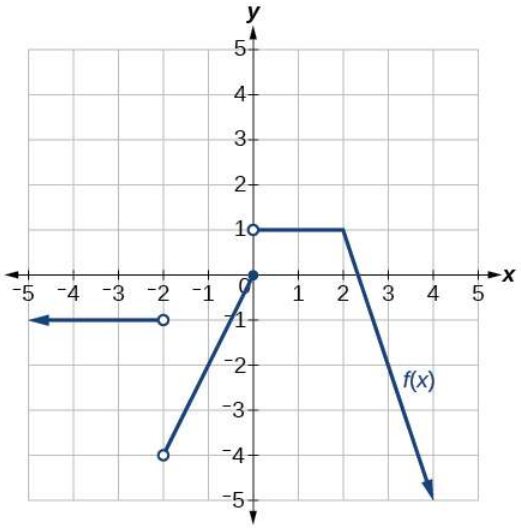
15) $f(x)=\frac{3}{\sqrt{x}}$

Answer

$f'(x)=-\frac{3}{2a^{\frac{3}{2}}}$

16) $f(x)=2x^2+9x$

17) For the graph in the Figure below, determine where the function is continuous/discontinuous and differentiable/not differentiable.



Answer

discontinuous at $-2, 0$, not differentiable at $-2, 0, 2$.

For the exercises 18-19, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

18) $f(x) = |x-2| - |x+2|$

19) $f(x) = \dfrac{2}{1+e^{\frac{2}{x}}}$

Answer

not differentiable at $x = 0$ (no limit)

For the exercises 20-24, explain the notation in words when the height of a projectile in feet, s, is a function of time t in seconds after launch and is given by the function $s(t)$.

20) $s(0)$

21) $s(2)$

Answer

the height of the projectile at $t = 2$ seconds

22) $s'(2)$

23) $\dfrac{s(2) - s(1)}{2-1}$

Answer

the average velocity from $t = 1$ to $t = 2$

24) $s(t) = 0$

For the exercises 25-28, use technology to evaluate the limit.

25) $\lim\limits_{x \to 0} \dfrac{\sin(x)}{3x}$

Answer

$\frac{1}{3}$

26) $\lim_{x \to 0} \frac{\tan^2(x)}{2x}$

27) $\lim_{x \to 0} \frac{\sin(x)(1-\cos(x))}{2x^2}$

Answer

0

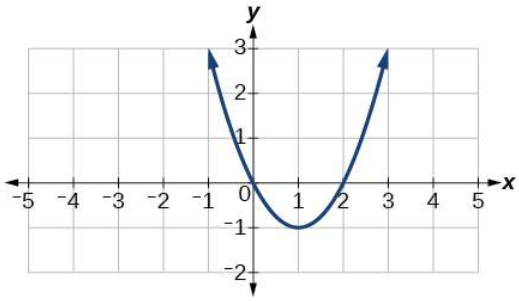
28) Evaluate the limit by hand.

$$\lim_{x \to 1} f(x), \text{ where } f(x) = \begin{cases} 4x-7 & x \neq 1 \\ x^2-4 & x = 1 \end{cases}$$

At what value(s) of x is the function below discontinuous?

$$f(x) = \begin{cases} 4x-7 & x \neq 1 \\ x^2-4 & x = 1 \end{cases}$$

For the exercises 29-32, consider the function whose graph appears in Figure.



29) Find the average rate of change of the function from $x = 1$ to $x = 3$.

Answer

2

30) Find all values of x at which $f'(x) = 0$.

Answer

$x = 1$

31) Find all values of x at which $f'(x)$ does not exist.

32) Find an equation of the tangent line to the graph of f the indicated point: $f(x) = 3x^2 - 2x - 6, \ x = -2$

Answer

$y = -14x - 18$

For the exercises 33-34, use the function $f(x) = x(1-x)^{\frac{2}{5}}$

33) Graph the function $f(x) = x(1-x)^{\frac{2}{5}}$ by entering $f(x) = x\left((1-x)^2\right)^{\frac{1}{5}}$ and then by entering $f(x) = x\left((1-x)^{\frac{1}{5}}\right)^2$.

34) Explore the behavior of the graph of $f(x)$ around $x = 1$ by graphing the function on the following domains, $[0.9, 1.1]$, $[0.99, 1.01]$, $[0.999, 1.001]$, and $[0.9999, 1.0001]$. Use this information to determine whether the function appears to be differentiable at $x = 1$.

Answer

The graph is not differentiable at $x = 1$ (cusp).

For the exercises 35-42, find the derivative of each of the functions using the definition: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

35) $f(x) = 2x - 8$

36) $f(x) = 4x^2 - 7$

Answer

$f'(x) = 8x$

37) $f(x) = x - \frac{1}{2}x^2$

38) $f(x) = \frac{1}{x+2}$

Answer

$f'(x) = -\frac{1}{(2+x)^2}$

39) $f(x) = \frac{3}{x-1}$

40) $f(x) = -x^3 + 1$

Answer

$f'(x) = -3x^2$

41) $f(x) = x^2 + x^3$

42) $f(x) = \sqrt{x-1}$

Answer

$f'(x) = -\frac{1}{2\sqrt{x-1}}$

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