# MATH 141: PRECALCULUS I (OLD EDITION)



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# Precalculus I - MATH 141

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# TABLE OF CONTENTS


Licensing

## 1: Academic Success Skills

- 1.0: The Power to Learn
- 1.1: It's All in the Mindset
- 1.2: Failure Is Not an Option
- 1.3: Active Learning from Lectures
- 1.4: Active Learning from Reading
- 1.5: Taking Notes
- 1.6: How to Manage Time
- 1.7: Enhanced Strategies for Time and Task Management
- 1.8: Goal Setting and Motivation
- 1.9: Test Taking and Preparation
- 1.10: Pre- Mid- and Post-Test-Taking Strategies
- 1.11: Test-Taking Strategy Specifics

## 2: Inequalities and Functions

- 2.0: Prelude to Inequalities and Functions
- 2.1: Equations and Inequalities
  - 2.1.0: Extracting Square Roots and Completing the Square (Review)
  - 2.1.1: Solving Equations Quadratic in Form (Review)
  - 2.1.2: Quadratic Functions and Their Graphs (Review)
  - 2.1.3: Solving Quadratic Inequalities
  - 2.1.4: Solving Polynomial and Rational Inequalities
  - 2.1.E: Solving Equations and Inequalities (Exercises)
- 2.2: Functions and Function Notation
- 2.3: Domain and Range
- 2.4: Rates of Change and Behavior of Graphs
- 2.5: Transformation of Functions
- 2.6: Composition of Functions
- 2.7: Absolute Value Functions
- 2.8: Inverse Functions
- 2.E: Functions (Exercises)
- 2.R: Functions (Review)

## 3: Polynomial and Rational Functions

- 3.0: Prelude to Polynomial and Rational Functions
- 3.1: Complex Numbers Review
- 3.2: Quadratic Functions Review
- 3.3: Power Functions and Polynomial Functions Behavior
- 3.4: Graphs of Polynomial Functions
- 3.5: Dividing Polynomials
- 3.6: Zeros of Polynomial Functions
- 3.7: Rational Functions
- 3.8: Inverses and Radical Functions
- 3.9: Modeling Using Variation (optional)

- 3.E: Polynomial and Rational Functions(Exercises)
- 3.R: Polynomial and Rational Functions(Review)

## 4: Exponential and Logarithmic Functions

- 4.0: Prelude to Exponential and Logarithmic Functions
- 4.1: Exponential Functions
- 4.2: Graphs of Exponential Functions
- 4.3: Logarithmic Functions
- 4.4: Graphs of Logarithmic Functions
- 4.5: Logarithmic Properties
- 4.6: Exponential and Logarithmic Equations
- 4.7: Exponential and Logarithmic Models
- 4.8: Fitting Exponential Models to Data
- 4.9: Trigonometry Preview - Circles
- 4.E: Exponential and Logarithmic Functions (Exercises)
- 4.R: Exponential and Logarithmic Functions (Review)

Index

Glossary

Detailed Licensing

# Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

# CHAPTER OVERVIEW

## 1: Academic Success Skills


1.0: The Power to Learn
1.1: It's All in the Mindset
1.2: Failure Is Not an Option
1.3: Active Learning from Lectures
1.4: Active Learning from Reading
1.5: Taking Notes
1.6: How to Manage Time
1.7: Enhanced Strategies for Time and Task Management
1.8: Goal Setting and Motivation
1.9: Test Taking and Preparation
1.10: Pre- Mid- and Post-Test-Taking Strategies
1.11: Test-Taking Strategy Specifics


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# 1.0: The Power to Learn

**Questions to consider**

- What actually happens to me when I learn something?
- Am I aware of different types of learning?
- Do I approach studying or practicing differently depending on the desired outcome?

Welcome to one of the most empowering chapters in this book! While each chapter focuses on showing you clear paths to success as a student, this one deals specifically with what is at the core of being a student: the act of learning.

We humans have been obsessed with how we learn and understand things since ancient times. Because of this, some of our earliest recorded philosophies have tried to explain how we take in information about the world around us, how we acquire new knowledge, and even how we can be certain what we learn is correct. This obsession has produced a large number of theories, ideas, and research into how we learn. There is a great deal of information out there on the subject—some of it is very good, and some of it, while well intentioned, has been a bit misguided.

Because of this obsession with learning, over the centuries, people have continually come up with new ideas about how we acquire knowledge. The result has been that commonly held “facts” about education have been known to change frequently. Often, what was once thought to be the newest, greatest discovery about learning was debunked later on. One well-known example of this is that of corporal punishment. For most of the time formal education has existed in our society, educators truly believed that beating students when they made a mistake actually helped them learn faster. Thankfully, *birching* (striking someone with a rod made from a birch tree) has fallen out of favor in education circles, and our institutions of learning have adopted different approaches. In this chapter, not only will you learn about current learning theories that are backed by neuroscience (something we did not have back in the days of birching), but you will also learn other learning theories that did not turn out to be as effective or as thoroughly researched as once thought. That does not mean those ideas about learning are useless. Instead, in these cases you find ways to separate the valuable parts from the myths to make good learning choices.

> ***"Research has shown that one of the most influential aids in learning is an understanding about learning itself."***

## What Is the Nature of Learning?

To begin with, it is important to recognize that learning is work. Sometimes it is easy and sometimes it is difficult, but there is always work involved. For many years people made the error of assuming that learning was a passive activity that involved little more than just absorbing information. Learning was thought to be a lot like copying and pasting words in a document; the student’s mind was blank and ready for an instructor to teach them facts that they could quickly take in. As it turns out, learning is much more than that. In fact, at its most rudimentary level, it is an actual process that physically changes our brains. Even something as simple as learning the meaning of a new word requires the physical alteration of neurons and the creation of new paths to receptors. These new electrochemical pathways are formed and strengthened as we utilize, practice, or remember what we have learned. If the new skill or knowledge is used in conjunction with other things we have already learned, completely different sections of the brain, our nerves, or our muscles may be tied in as a part of the process. A good example of this would be studying a painting or drawing that depicts a scene from a story or play you are already familiar with. Adding additional connections, memories, and mental associations to things you already know something about expands your knowledge and understanding in a way that cannot be reversed. In essence, it can be said that every time we learn something new we are no longer the same.

In addition to the physical transformation that takes place during learning, there are also a number of other factors that can influence how easy or how difficult learning something can be. While most people would assume that the ease or difficulty would really depend on what is being learned, there are actually several other factors that play a greater role.

In fact, research has shown that one of the most influential factors in learning is a clear understanding about learning itself. This is not to say that you need to become neuroscientists in order to do well in school, but instead, knowing a thing or two about learning and how we learn in general can have strong, positive results for your own learning. This is called *metacognition* (i.e., thinking about thinking).

Some of the benefits to how we learn can be broken down into different areas such as

- attitude and motivation toward learning,
- types of learning,
- methods of learning, and
- your own preferences for learning.

In this chapter you will explore these different areas to better understand how they may influence your own learning, as well as how to make conscious decisions about your own learning process to maximize positive outcomes.

## All Learning Is Not the Same

The first, fundamental point to understand about learning is that there are several types of learning. Different kinds of knowledge are learned in different ways. Each of these different types of learning can require different processes that may take place in completely different parts of our brain.

For example, simple memorization is a form of learning that does not always require deeper understanding. Children often learn this way when they memorize poems or verses they recite. An interesting example of this can be found in the music industry, where there have been several hit songs sung in English by vocalists who do not speak English. In these cases, the singers did not truly understand what they were singing, but instead they were taught to memorize the sounds of the words in the proper order.



Figure 1.0.2 Learning has many levels and forms. For example, collaborative learning and showing your work require different skills and produce different results than reading or notetaking on your own. (Credit: StartUpStockPhotos / Pexels)

Memorizing sounds is a very different type of learning than, say, acquiring a deep understanding of Einstein's general theory of relativity.

Notice in the comparative examples of music and physics that the different levels of learning are being defined by what they allow you to know or do. When classifying learning in this way, people usually agree on six different levels of learning. In this next section we will take a detailed look at each of these.

In the table below, the cells in the left column each contain one of the main levels of learning, categorized by what the learning allows you to do. To the right of each category are the "skill acquired" and a set of real-world examples of what those skills might be as applied to a specific topic. This set of categories is called Bloom's Taxonomy, and it is often used as a guide for educators when they are determining what students should learn within a course.

Table 2.1

| Category of Learning | Skill acquired | Example 1: Musical ability | Example 2: Historical information on Charles the Bald |
|---|---|---|---|

| Category of Learning | Skill acquired | Example 1: Musical ability | Example 2: Historical information on Charles the Bald |
|---|---|---|---|
| Create | Produce new or original work | Compose a piece of music | Write a paper on Charles that draws a new conclusion about his reign |
| Evaluate | Justify or support an idea or decision | Make critical decisions about the notes that make up a melody—what works, what doesn't, and why | Make arguments that support the idea that Charles was a good ruler |
| Analyze | Draw connections | Play the specific notes that are found in the key of A | Compare and contrast the historical differences between the reign of Charles and his grandfather, Charlemagne |
| Apply | Use information in new ways | Use knowledge to play several notes that sound good together | Use the information to write a historical account on the reign of Charles |
| Understand/ Comprehend | Explain ideas or concepts | Understand the relationship between the musical notes and how to play each on a musical instrument | Explain the historical events that enabled Charles to become Emperor |
| Remember | Recall facts and basic concepts | Memorize notes on a musical scale | Recall that Charles the Bald was Holy Roman Emperor from 875–877 CE |

A review of the above table shows that actions in the left column (or what you will be able to do with the new knowledge) has a direct influence over what needs to be learned and can even dictate the type of learning approach that is best. For example, *remembering* requires a type of learning that allows the person basic memorization. In the case of Charles the Bald and his reign, it is simply a matter of committing the dates to memory. When it comes to *understanding and comprehension*, being able to explain how Charles came to power requires not only the ability to recall several events, but also for the learner to be able to understand the cause and effect of those events and how they worked together to make Charles emperor. Another example would be the ability to *analyze*. In this particular instance the information learned would not only be about Charles, but also about other rulers, such as Charlemagne. The information would have to be of such a depth that the learner could compare the events and facts about each ruler.

When you engage in any learning activity, take the time to understand what you will do with the knowledge once you have attained it. This can help a great deal when it comes to making decisions on how to go about it. Using flashcards to help memorize angles does not really help you solve problems using geometry formulas. Instead, practicing problem-solving with the actual formulas is a much better approach. The key is to make certain the learning activity fits your needs.

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# 1.1: It's All in the Mindset

**Questions to consider**

- What is a growth mindset, and how does it affect my learning?
- What are performance goals versus learning goals?

In the previous sections of this chapter you have focused on a number of concepts and models about learning. One of the things they all have in common is that they utilize different approaches to education by presenting new ways to think about learning. In each of these, the common element has been a better understanding of yourself as a learner and how to apply what you know about yourself to your own learning experience. If you were to distill all that you have learned in this chapter so far down to a single factor, it would be about using your mindset to your best advantage. In this next section, you will examine how all of this works in a broader sense by learning about the significance of certain mindsets and how they can hinder or promote your own learning efforts.



Figure 1.1.7: Many fields of study and work create intersections of growth and fixed mindset. People may feel great ability to grow and learn in some areas, like art and communication, but feel more limited in others, such as planning and financials. Recognizing these intersections will help you approach new topics and tasks. (Credit: mentatdgt / Pexels)

## Performance vs. Learning Goals

As you have discovered in this chapter, much of our ability to learn is governed by our motivations and goals. What has not yet been covered in detail has been how sometimes hidden goals or mindsets can impact the learning process. In truth, we all have goals that we might not be fully aware of, or if we are aware of them, we might not understand how they help or restrict our ability to learn. An illustration of this can be seen in a comparison of a student that has *performance*-based goals with a student that has *learning*-based goals.

If you are a student with strict performance goals, your primary psychological concern might be to appear intelligent to others. At first, this might not seem to be a bad thing for college, but it can truly limit your ability to move forward in your own learning. Instead, you would tend to play it safe without even realizing it. For example, a student who is strictly performance-goal-oriented will often only says things in a classroom discussion when they think it will make them look knowledgeable to the instructor or their classmates. For example, a performance-oriented student might ask a question that she knows is beyond the topic being covered (e.g., asking about the economics of Japanese whaling while discussing the book *Moby Dick* in an American literature course). Rarely will they ask a question in class because they actually do not understand a concept. Instead they will ask questions that make them look intelligent to others or in an effort to “stump the teacher.” When they do finally ask an honest question, it may be because they are more afraid that their lack of understanding will result in a poor performance on an exam rather than simply wanting to learn.

If you are a student who is driven by learning goals, your interactions in classroom discussions are usually quite different. You see the opportunity to share ideas and ask questions as a way to gain knowledge quickly. In a classroom discussion you can ask for

clarification immediately if you don't quite understand what is being discussed. If you are a person guided by learning goals, you are less worried about what others think since you are there to learn and you see that as the most important goal.

Another example where the difference between the two mindsets is clear can be found in assignments and other coursework. If you are a student who is more concerned about performance, you may avoid work that is challenging. You will take the "easy A" route by relying on what you already know. You will not step out of your comfort zone because your psychological goals are based on approval of your performance instead of being motivated by learning.

This is very different from a student with a learning-based psychology. If you are a student who is motivated by learning goals, you may actively seek challenging assignments, and you will put a great deal of effort into using the assignment to expand on what you already know. While getting a good grade is important to you, what is even more important is the learning itself.

If you find that you sometimes lean toward performance-based goals, do not feel discouraged. Many of the best students tend to initially focus on performance until they begin to see the ways it can restrict their learning. The key to switching to learning-based goals is often simply a matter of first recognizing the difference and seeing how making a change can positively impact your own learning.

What follows in this section is a more in-depth look at the difference between performance- and learning-based goals. This is followed by an exercise that will give you the opportunity to identify, analyze, and determine a positive course of action in a situation where you believe you could improve in this area.

## WHAT STUDENTS SAY

1. In the past, did you feel like you had control over your own learning?
    a. No. Someone has always dictated how and what I learned.
    b. Yes. I always look for ways to take control of what and how I learned.
    c. I am uncertain. I never thought about it before.
2. Have you ever heard of learning styles or do you know your own learning style?
    a. No. I have never heard of learning styles.
    b. Yes. I have heard of learning styles and know my own.
    c. Yes. I have heard of learning styles, but I don't think they're accurate or relate to me.
3. Which factors other than intelligence do you think have the greatest influence on learning?
    a. Motivation
    b. Perseverance
    c. Understanding how I learn
    d. Good teachers and support

You can also take the anonymous What Students Say surveys to add your voice to this textbook. Your responses will be included in updates.

Students offered their views on these questions, and the results are displayed in the graphs below.

In the past, did you feel like you had control over your own learning?

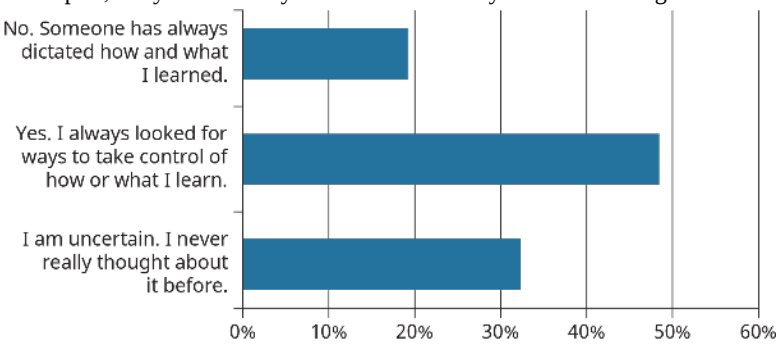


Figure 1.1.8

Have you ever heard of learning styles or do you know your own learning style?

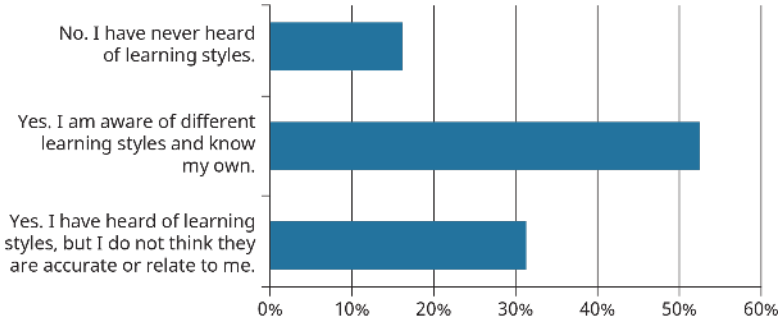


Figure 1.1.9

Which factors other than intelligence do you think have the greatest influence on learning?

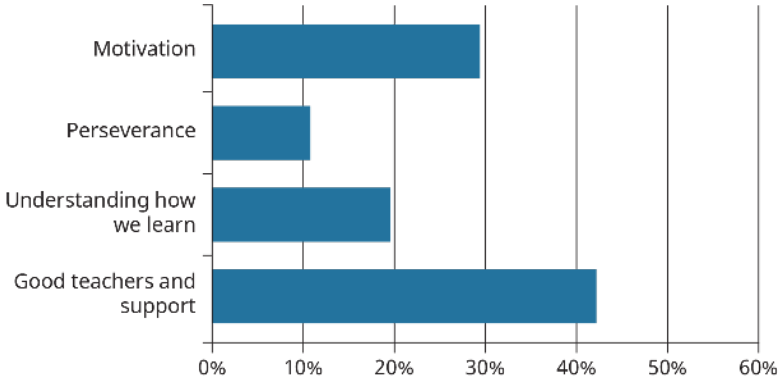


Figure 1.1.10

## Fixed vs. Growth Mindset

The research-based model of these two mindsets and their influence on learning was presented in 1988 by Carol Dweck.[7] In Dr. Dweck's work, she determined that a student's perception about their own learning accompanied by a broader goal of learning had a significant influence on their ability to overcome challenges and grow in knowledge and ability. This has become known as the Fixed vs. Growth Mindset model. In this model, the *performance*-goal-oriented student is represented by the *fixed* mindset, while the *learning*-goal-oriented student is represented by the *growth* mindset.

In the following graphic, based on Dr. Dweck's research, you can see how many of the components associated with learning are impacted by these two mindsets.

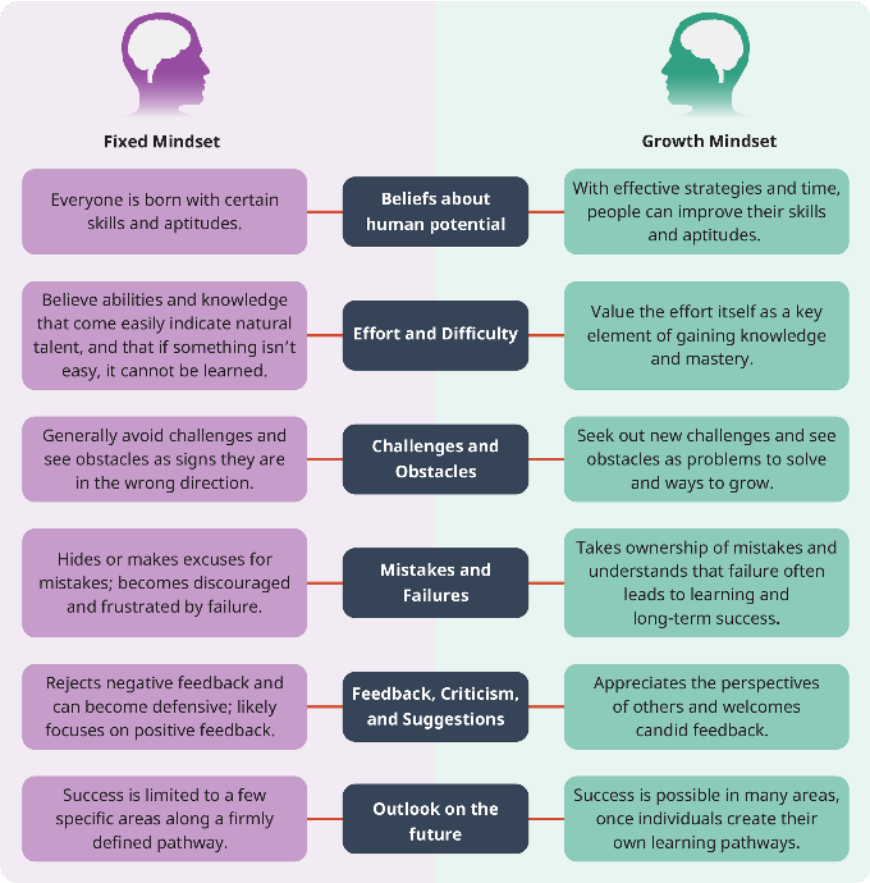



Figure 1.1.11: The differences between fixed and growth mindset are clear when aligned to key elements of learning and personality. (Credit: Based on work by Dr. Carol Dweck)

### The Growth Mindset and Lessons About Failing

Something you may have noticed is that a growth mindset would tend to give a learner grit and persistence. If you had learning as your major goal, you would normally keep trying to attain that goal even if it took you multiple attempts. Not only that, but if you learned a little bit more with each try you would see each attempt as a success, even if you had not achieved complete mastery of whatever it was you were working to learn.

With that in mind, it should come as no surprise that Dr. Dweck found that those people who believed their abilities could change through learning (growth vs. a fixed mindset) readily accepted learning challenges and persisted despite early failures.

### Improving Your Ability to Learn

As strange as it may seem, research into fixed vs. growth mindsets has shown that if you believe you can learn something new, you greatly improve your ability to learn. At first, this may seem like the sort of feel-good advice we often encounter in social media posts or quotes that are intended to inspire or motivate us (e.g., *believe in yourself!*), but in looking at the differences outlined between a fixed and a growth mindset, you can see how each part of the growth mindset path would increase your probability of success when it came to learning.

**ACTIVITY**

Very few people have a strict fixed or growth mindset all of the time. Often we tend to lean one way or another in certain situations. For example, a person trying to improve their ability in a sport they enjoy may exhibit all of the growth mindset traits and characteristics, but they find themselves blocked in a fixed mindset when they try to learn something in another area like computer programming or arithmetic.

In this exercise, do a little self-analysis and think of some areas where you may find yourself hindered by a fixed mindset. Using the outline presented below, in the far right column, write down how you can change your own behavior for each of the parts of the learning process. What will you do to move from a fixed to a growth mindset? For example, say you were trying to

learn to play a musical instrument. In the *Challenges* row, you might pursue a growth path by trying to play increasingly more difficult songs rather than sticking to the easy ones you have already mastered. In the *Criticism* row, you might take someone's comment about a weakness in timing as a motivation for you to practice with a metronome. For *Success of others* you could take inspiration from a famous musician that is considered a master and study their techniques.

Whatever it is that you decide you want to use for your analysis, apply each of the Growth characteristics to determine a course of action to improve.

Table 2.4

| Parts of the learning process | Growth characteristic | What will you do to adopt a growth mindset? |
|---|---|---|
| Challenges | Embraces challenges | |
| Obstacles | Persists despite setbacks | |
| Effort | Sees effort as a path to success | |
| Criticism | Learns from criticism | |
| Success of Others | Finds learning and inspiration in the success of others | |

## Footnotes

- Dweck, C.S. & Leggett, E.L. (1988). A Social-Cognitive Approach to Motivation and Personality

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# 1.2: Failure Is Not an Option

## Failure Is Not an Option

Nathan Wallace

In the movie Apollo 13, Ed Harris portrays NASA flight director Gene Kranz as he successfully guides the crew of a damaged spacecraft to safety. In a famous scene during which Kranz and his staff are attempting to overcome some extremely daunting challenges, Harris shouts, “Failure is not an option!” This singular statement perfectly articulated the determination of Kranz to bring the Apollo astronauts back to Earth.

This “failure is not an option” credo was perfect for the life and death situation that NASA was facing. Failure meant that the astronauts on Apollo 13 would never come home, and that outcome was unacceptable. Attending college, on the other hand, shouldn’t be a life or death experience, though it sometimes might feel like one. Failure, though never the intended outcome, can and sometimes does happen. Sometimes failure manifests itself in election results for a student government post, in a test score, or even in a final grade.

Throughout my life I have had many failures. In high school I drove my parents and teachers crazy because of my lack of academic achievement. I even managed to get an F- in Spanish on my report card. When I told my mom that it was a typo she responded, “So you didn’t get an F?” “No,” I said, “I definitely earned the F, but there’s no such thing as an F-.” To this day I’m not so sure that my reply was accurate. I might have earned that minus after all.

My failures in high school led to only one acceptance from of all the colleges I applied to attend. Furthermore, I was not accepted to the school’s main campus, but to their branch campus. During my first semester there my effort wasn’t much better than in high school, but since my parents were now paying for my education I did enough work to avoid academic probation. It wasn’t until my second semester that I found my niche as a Religious Studies major and started getting good grades, moved to the main campus, and eventually graduated with honors.

Since graduating from college, my career path has taken me into higher education as a Student Affairs administrator. This career has exposed me to many great theories regarding student success, and many of them gave me insight into my own college experience. But it was Stanford psychologist Carol Dweck who appeared to be thinking of me when she wrote the following about fixed mindsets in the introduction to her book titled *Mindset: The New Psychology of Success*:

Believing that your qualities are carved in stone—the fixed mindset—creates an urgency to prove yourself over and over. If you have only a certain amount of intelligence, a certain personality, and a certain moral character—well, then you’d better prove that you have a healthy dose of them. It simply wouldn’t do to look or feel deficient in these most basic characteristics. (Dweck, 2006)

This statement was a revelation to me. I finally understood my problem throughout high school and even in college. I earned good grades because I liked Religious Studies but never really challenged myself inside or outside of the classroom. My problem was that I had a fixed mindset about academic success. I believed that a person is either smart or they’re not, and nothing could be done to significantly change that. I also believed that I was one of the fortunate ones to be “gifted” with an abundance of intelligence.

One might think that having confidence in your intelligence is a whole lot better than thinking that you’re stupid, but the result was the same. My fixed mindset was holding me back because it led to a paralyzing fear of failure. Since as far back as I could remember, my family, friends, and teachers were always telling me how smart I was, and I believed them. But that belief was a double-edged sword. High school and college offered many occasions when self-confidence in my inherent intelligence could be threatened. If I fail on this test or in this course it means that I’m not the smart person I thought I was. If I fail, my family and friends will find out that they were wrong about me.

However, there was a way to avoid all of the risks of academic rigor. I could just not try. If I don’t try I’ll get bad marks on my report card, but those won’t be true indicators of my intelligence. By not putting forth any effort, my intelligence would never be disproven. I would always be able to say to myself and others that, “I could do the work and be a straight A student, but I’m just not interested.” Looking back on this time in my life, it is clear to me that this wasn’t a conscious decision to save face. It was fear, not logic, which was guiding my behavior.

After reading *Mindset* I have made a conscious effort to identify and thwart any remaining fixed mindset thoughts that I continue to hold. Dweck’s book acts as a manual for rooting out fixed mindset thoughts, because she explains that the idea of fixed mindsets is only half of her mindset theory. There is another kind of mindset, and she calls it growth mindset. Dweck writes that, “This growth

mindset is based on the belief that your basic qualities are things you can cultivate through your efforts" (Dweck, 2006). Dweck goes on to explain that we can choose to have a growth mindset about any type of ability, whether it's math, art, athletics, or any other skill that one wishes to cultivate.

I put this theory to the test not long after reading the book. A few years ago I attended a meeting only to find out that it wasn't any ordinary meeting. During this meeting we would be brainstorming solutions to a specific problem. This was going to be a true brainstorming session, led by a facilitator trained in the science of soliciting uninhibited ideas from an audience. As soon as I heard the word *brainstorming* I froze. I have always hated brainstorming. I'm the type of person that likes to think things through two or three times before expressing an opinion. My fear of failing at this task in front of my coworkers paralyzed my mind. I couldn't think.

That's when it hit me. This was fixed mindset thinking. My belief in my brainstorming inadequacies was preventing me from even trying. So I flipped this thinking on its head and decided the best way to improve my brainstorming abilities was to clear my mind and start firing out ideas. I gave it a shot, and though the ideas didn't come out at the prolific rate of some of my colleagues, I had never before had such a positive outcome and experience while brainstorming. Through this experience I found that I really could choose to have a growth mindset, and that this choice produces a greater chance of success. With a greater chance of success comes a smaller chance of failure.

Nevertheless, when it comes to academic success and success in all phases of life, failure is always an option. Though it can be painful, failure can lead to great learning and progress when a specific failure is analyzed through the lens of a growth mindset. By focusing more on effort than on outcomes anyone can learn and grow, regardless of their skill level. Therefore, to make the most of their time in college, students must seek out challenges that will stretch their abilities. These challenges can take many forms and they can occur in a variety of settings, both inside and outside of the classroom. When seeking out challenges there is always the possibility of agonizing defeat, but out of that defeat can be the seeds of great success in the future.

## Reference

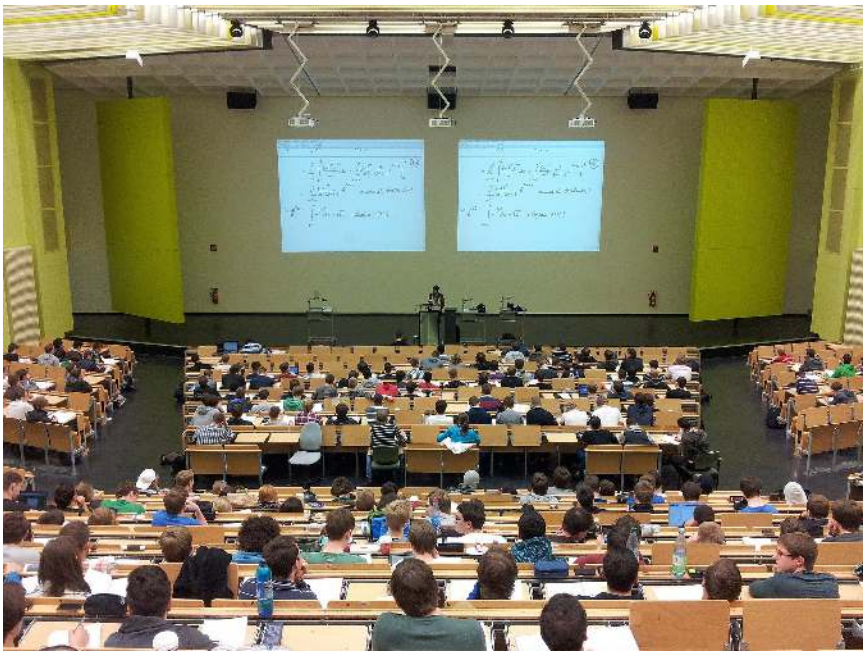
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## 1.3: Active Learning from Lectures

The learning of information and skills presented in class lectures or discussions should be viewed as a process: preparation to take in the new information, the act of taking in the new information, and then reviewing the information so that it is later accessible (recalled from memory) to use for a project, paper, or test.



The activities suggested for “Before Class” in Table 5-2 help students develop a mental framework into which this new information can be “filed” or organized which aids information recall (Bower, Clark, Lesgold, & Winzenz, 1969). It also makes note taking easier because you are familiar with the material and feel less panicked about getting everything noted in full detail.

Attending class and taking the notes is important primarily because the professor may be presenting information that is not presented in the text. It should also be seen as a structured opportunity to engage with the material that will further aid recall when needed. Oftentimes, students can also get clues as to what a professor perceives as important material from attending class that will aid in test preparation.

Table 5-2: Effective Techniques for Learning from Class

| Before Class | During Class | After Class |
|---|---|---|
| Complete assigned readings to prepare for class. Take notes or mark your text for easy reference. | Take notes. Active engagement with the material (like writing down what you are hearing) helps improve memory and concentration and material to later review. Have a positive mindset about taking notes you are writing down answers to questions on the test.” | Review your notes and handouts within 24 hours of taking them. Fill in details that you remember but did not have time to write down. Write questions for the notes that you can use for self-testing later. Write a summary of the day’s notes. |
| Prepare questions that you may have about the reading. | Put a date on your note paper. It helps to also include the text chapter or general topic that might be related to these notes as well. | Compare your notes with a reliable classmate who may help give you ideas on what you missed that might be important or how to better organize the information. Discuss the notes to add another-layer of processing. |

| Before Class | During Class | After Class |
|---|---|---|
| Review notes from the previous class period to get a sense of where you may have left off. | Use active listening techniques. Refer to the informational box called “Ten Tips for Active Listening” for more information. | Follow up on questions that you have written down with your professor, the teaching assistant, or another reliable classmate. |
| Be sure you have your materials ready for class: text, three-ring notebook with paper, writing utensils, highlighters, charged laptop, etc. | Have your textbook accessible in case your faculty member refers to specific information. Write down that reference in your notes as it is probably important information. | Integrate your class notes with your text notes or markings. If you highlighted in your book, add some of that information to your notes for the “whole” picture. |
| Get a good night’s sleep. It is easier to concentrate on the lecture when you are alert. | Write down the main points a speaker is making along with any supple-mental details like examples or experiments. Do not expect to be able to write down every word the speaker says. | Develop tools that swill serve to help you review the material now, as well as later. Create visual diagrams* of concepts, develop ram cards* of concepts, reorganize your notes, etc. *Refer to informational boxes on “Visualize Your Information” and “Cram Cards for Long-Term Review.” |
| Make it a habit to attend every class. Put your classes on your schedule and treat them as a limited opportunity to engage with your professor and this material. | Use abbreviations for commonly used words to increase your speed in note taking. Create some abbreviations of your own. | Self-test your comprehension of the information on an ongoing basis. Do not just “look over” or read what you have written. Use the questions you have created. Cover up the “answer” and see if you can explain what is in the notes. Use the cram cards or visual diagrams to test yourself. |
| If you cannot attend class due to illness or an-other extraordinary reason, contact a classmate to arrange for notes. It is always a good idea to send an e-mail or call your professor as well. | Develop a system of organization for your notes that works for you. Common methods include the Cornell, Outlining, Mapping, Text-Class, and Sentence Methods. Refer to informational box called “Note Taking Methods: What Is Right for You?” | |
| | Leave space between main points so that you can add in new information that you may have missed or from the text. | |
| | If the professor writes it down or puts it on a PowerPoint slide, it is probably important, so write it down. | |
| | If the professor repeats it, it is important, so write it down. | |
| | Listen for cues like “very important,” “in summary,” or sequence words like “first,” “second,” etc. This information provides cues for what is important and possible note organization. | |
| | Ask questions you may have. If you are not comfortable talking in class, write them down and talk with your professor later. | |

Adapted from Dembo Sell (2008) and Downing (2008).

## Ten Tips for Active Listening

Effective note taking in class requires the use of active listening techniques for optimal results.

1. Accept responsibility as a listener.
2. Adopt a positive attitude toward listening. Listening is a choice.
3. Sit somewhere in the classroom that will minimize distractions and allow you to focus on the professor.
4. Maintain eye contact with your professor.
5. Focus on the content being presented rather than the delivery.
6. Ask questions in class.
7. Ask mental questions and search for the answer in the lecture or discussion.
8. Avoid emotional involvement that may impair your ability to concentrate or filter important information.
9. Clue in on non-verbal communication, such as exaggerated movements, excitement, etc., as it may serve as a cue to important information.
10. Monitor your concentration throughout the class period and continually refocus.

Adapted from Kline (2002) and Treuer (2006).



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## 1.4: Active Learning from Reading

As with learning from lectures and discussions, the learning of information and skills presented in readings, texts, and written text from a Web site should be viewed as a process: preparation to take in the new information, the act of taking in the new information, and then reviewing the information so that it is later accessible (recalled from memory) to use for a project, paper, or test.



Students who simply start and finish a reading with no further actions taken are really wasting their time. Reading a text needs to be approached properly in order to ensure comprehension and retention of the information. Planning for an active reading session by engaging in the "before reading" activities facilitates an active mindset to the reading session and discourages "auto-pilot reading" that results in scanning of the words but no real careful thought about what is being said.

Preparing for class by reading assigned texts related to the lecture material is extremely important to optimize the lecture experience. If you have a professor who tends to lecture at a fast pace, you have the foundational information and only need to focus on writing down unfamiliar information. If your professor expects class participation in discussions or utilizes active classroom techniques, such as case studies, small group activities and discussions, simulations, or demonstrations, you will understand what is happening and be able to contribute to the learning experience. Marking or highlighting your text for main ideas while reading not only keeps your mind focused on the reading but also serves to prepare you for easier exam review.

Table 5-3. Effective Techniques for Reading

| Before Reading | During Reading | After Reading |
|---|---|---|
| Preview the reading headings to get a "big picture" of the outline of the reading. Look at the pictures and figures. Check out the bold and italicized words. | As you are reading, seek out the answers to the questions that the reading or you generated rather than passively scanning the words. | Take reading notes from the highlighted material from the text. Utilize visual organizers and summaries to capture information. Revisit the informational box on "Note-Taking Methods: What Is Right for You?" and visit "Visualize Your information and "Cram Cards for Long-term Review." |
| Preview any questions that the chapter may offer so that you can actively seek answers to those questions. | Monitor your concentration and comprehension. After each paragraph, ask yourself, "What was the main point of that paragraph?" After each section, summarize what you have read. | Answer the questions that a the reading may have provided or that you developed. Reread sections in which you remember little information. |
| Turn text headings into questions so that you can actively seek answers to those questions. Jot those questions down in the margin of the text or in your notes if you plan to take reading notes. | Mark your text. Highlight the main idea of a paragraph or write down the main idea of a paragraph in your notes. If you cannot mark your text, take reading notes on the main ideas. Refer to the informational box called "In Search of Main Ideas" for more information | Integrate your reading notes and lecture/discussion notes into one location for easier review for an exam. |
| Plan for a high energy time of the day to read. Consider breaking up a long reading assignment into multiple, shorter reading sessions. | Look up the definitions to words that you do not understand to help build your vocabulary and facilitate understanding of the topic. | Discuss readings with classmates on a regular basis. Take turns explaining to each other sections of the reading. Ask and answer reading questions. Jot down questions that are unclear to the study group to ask the professor in class or during office hours. |
| Be sure you have your materials ready for reading: text, notebook paper, writing utensils, high-lighters, charged laptop, etc. | | |

Adapted from Dembo and Seli (2008) and Downing (2008).

### In Search of Main Ideas

Some students do not actively take notes or mark their readings and texts because they determine it is a lot of extra work to write down so much information. However, the problem is that students are usually writing down too much information and not really cluing in to the main ideas. Here are some tips for identifying the main ideas when reading:

1. The Table of Contents and chapter outlines provide a broad view of the main points that will be covered in a reading. Flesh out the outlines that are already provided for you.
2. Oftentimes in a textbook, the main idea is the first or last sentence of a paragraph. If it is not the first or the last sentence, then look back at the entire paragraph to see what the overall issue seems to be. Look for the overall pat- terns of your textbooks.
3. Titles, headings, and subheadings announce the major subject. Make these headings into questions, and the answers to the questions will likely be the main ideas.
4. Bold and italic words point to a main idea or key concept that you need to understand.
5. Repetition of key words or phrases throughout the text point to a main idea.
6. Chapter questions at the end of the chapter are about the main ideas of the text. Answer those questions and you will identify your main ideas.
7. Summaries presented at the end of the chapter also tend to restate the main ideas briefly. Flesh these ideas out with some supporting ideas, and you would have a good view of the entire chapter.
8. Stop and look at the visuals—pictures, diagrams, tables, etc. Oftentimes, the message depicted in the graph or picture is typically a main idea.
9. Detailed statistics, several examples in a row, and other details often signal that a main idea is being clarified, proven, supported, etc. Track back or ahead to find the main idea they are trying to illustrate.
10. Text that includes bullet points, numbering, or sequences is often a main idea.
11. Look for organizational patterns in the reading that might highlight the main ideas. For instance, are two issues being compared or contrasted? What was the effect of a certain event? Are problems and various solutions being presented? Is there a timeline of events that is important?
12. Be intentional about searching for the main ideas. Ask yourself at the end of each section or paragraph, "What is the point?" or "What is it that the author wants me to know?"
13. If you are reading a narrative, ask yourself questions like, "Who are the main characters?" "Why is this character important to this story?" "Why did the author chose to tell this part of the story?"

Finally, taking the time to think critically about the reading you have done further solidifies that information in your memory and can serve to prepare you for exam review. Summarizing material using visual organizers (refer to informational box "Visualize Your Information" and Figure 5-1) moves your thinking from low levels of thinking to high levels of thinking on Bloom's Taxonomy. Integrating text and lecture notes and discussing readings with your classmates and professor promotes memory storage of that information and thinking at higher levels as well.

## Visualize Your Information

Summarizing main ideas of the readings or lectures can be difficult without the right tools. Visual organizers can help students condense important information into visual diagrams that can be easily reviewed for exams or help organize a writing assignment. Muskingum College's Center for Advancement and Learning (n.d.) hosts a useful Web site that summarizes and provides examples of various visual organizers (a.k.a. Information Organization Strategies) at

**muskingum.edu/~cal/database/g...anization.html (Web site wants username and password now.)**

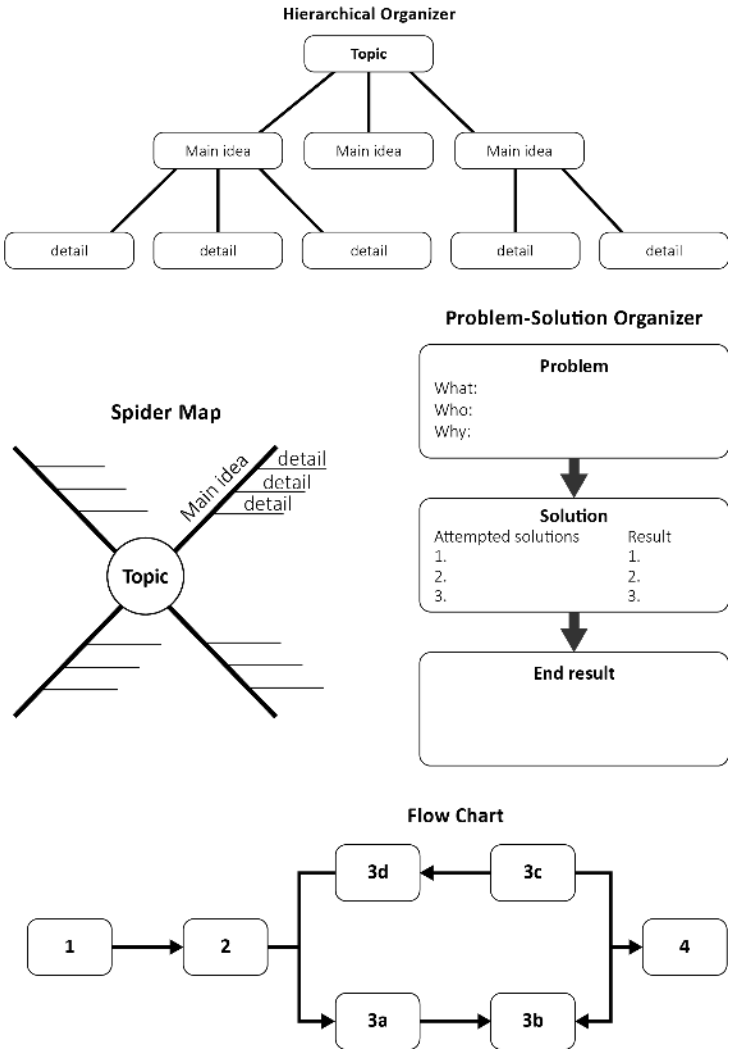
Consider how you might be able to apply several pages of text to a simple flowchart, hierarchical organizer, comparison-contrast organizer, spider map, etc. (See Figure 5-1.)

An additional benefit of using visual organizers is that you are thinking about the material at a deeper level as you identify the proper visual tool and apply the information. While at first it may take more time to determine the proper tool, it will get easier with practice and become second nature.

### Activity 5-2

Utilizing what you have learned from the informational boxes "In Search of Main Ideas" and "Visualize Your Information," select a text chapter or a week of class notes and create some visual organizers that summarize the main ideas of that material. Ask a classmate to share in the same exercise and compare your work in a critical manner. What have you learned from this exercise and each other?

Figure 5-1. Examples of visual organizers



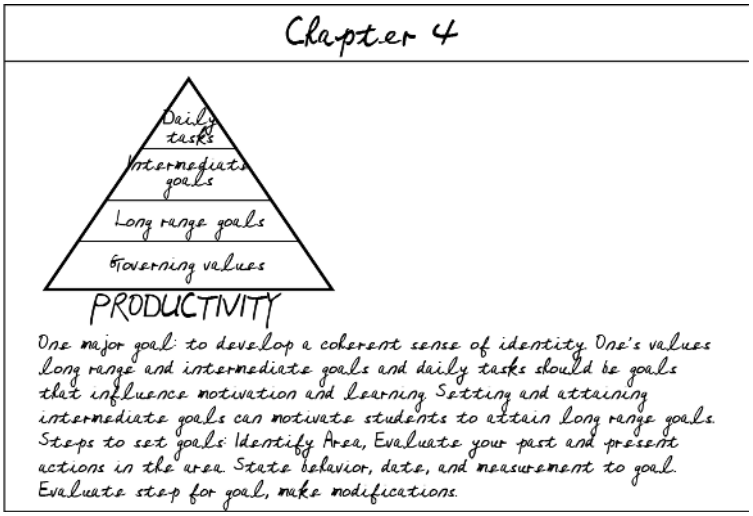

Source: Muskingum College's Center for Advancement and Learning (n.d)

## Cram Cards for Long-term Review

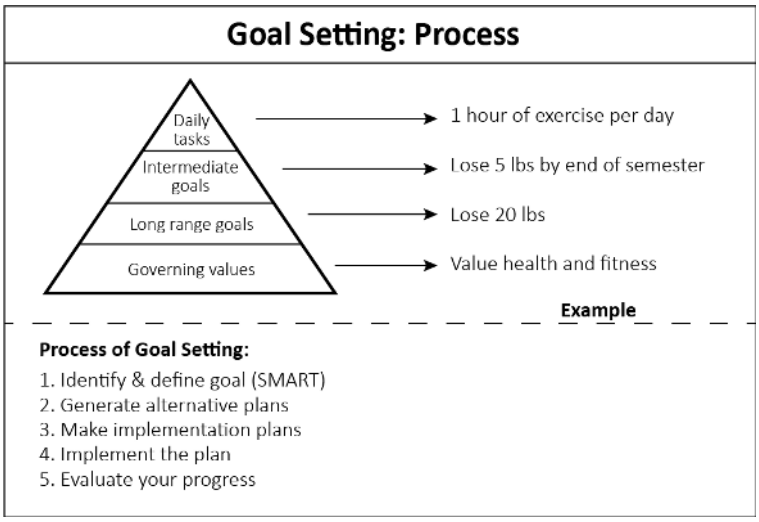
The use of "cram" in cram cards is deceiving. Cram cards are a study tool that promote the active learning of reading or lecture notes and allows for the easy and portable review of material over a longer period of time, therefore promoting later recall. It is a technique that will take some practice to hone your skills in selecting main ideas, but students who use cram cards indicate it is well worth their time.

Tips for Writing Cram Cards:

- Use 4 x 6 index cards (3 x 5 size tends to be too small)—some enthusiasts will even use the colored index cards for different sources of information (i.e., notes versus text reading) or topics (e.g., blue for English and green for Biology).
- Make one cram card per major concept or main idea.
- Do not simply take notes on the card; capture a summary of the main ideas maintaining the proper relationships of the concepts.
- Write the information in your own words.
- Include book examples or provide real-life examples for difficult concepts.
- Include key words.
- Be sure to write legibly.




Example of a Bad Cram Card




Example of a Good Cram Card

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# 1.5: Taking Notes

**Questions to consider**

- How can you prepare to take notes to maximize the effectiveness of the experience?
- What are some specific strategies you can employ for better notetaking?
- Why is annotating your notes after the notetaking session a critical step to follow?

Beyond providing a record of the information you are reading or hearing, notes help you organize the ideas and help you make meaning out of something about which you may not be familiar, so notetaking and reading are two compatible skill sets. Taking notes also helps you stay focused on the question at hand. Nanami often takes notes during presentations or class lectures so she can follow the speaker's main points and condense the material into a more readily usable format. Strong notes build on your prior knowledge of a subject, help you discuss trends or patterns present in the information, and direct you toward areas needing further research or reading.



Figure 1.5.9 Strong notes build on your prior knowledge of a subject, help you discuss trends or patterns present in the information, and direct you toward areas needing further research or reading.

It is not a good habit to transcribe every single word a speaker utters—even if you have an amazing ability to do that. Most of us don't have that court-reporter-esque skill level anyway, and if we try, we would end up missing valuable information. Learn to listen for main ideas and distinguish between these main ideas and details that typically support the ideas. Include examples that explain the main ideas, but do so using understandable abbreviations.

Think of all notes as potential study guides. In fact, if you only take notes without actively working on them after the initial notetaking session, the likelihood of the notes helping you is slim. Research on this topic concludes that without active engagement after taking notes, most students forget 60–75 percent of material over which they took the notes—within two days! That sort of defeats the purpose, don't you think? This information about memory loss was first brought to light by 19th-century German psychologist Hermann Ebbinghaus. Fortunately, you do have the power to thwart what is sometimes called the Ebbinghaus Forgetting Curve by reinforcing what you learned through review at intervals shortly after you take in the material and frequently thereafter.

If you are a musician, you'll understand this phenomenon well. When you first attempt a difficult piece of music, you may not remember the chords and notes well at all, but after frequent practice and review, you generate a certain muscle memory and cognitive recall that allows you to play the music more easily.

Notetaking may not be the most glamorous aspect of your higher-education journey, but it is a study practice you will carry throughout college and into your professional life. Setting yourself up for successful notetaking is almost as important as the actual taking of notes, and what you do after your notetaking session is equally significant. Well-written notes help you organize your thoughts, enhance your memory, and participate in class discussion, and they prepare you to respond successfully on exams. With all that riding on your notes, it would behoove you to learn how to take notes properly and continue to improve your notetaking skills.

**Analysis Question**

Do you currently have a preferred way to take notes? When did you start using it? Has it been effective? What other strategy might work for you?

## Preparing to Take Notes

Preparing to take notes means more than just getting out your laptop or making sure you bring pen and paper to class. You'll do a much better job with your notes if you understand why we take notes, have a strong grasp on your preferred notetaking system, determine your specific priorities depending on your situation, and engage in some version of efficient shorthand.

Like handwriting and fingerprints, we all have unique and fiercely independent notetaking habits. These understandably and reasonably vary from one situation to the next, but you can only improve your skills by learning more about ways to take effective notes and trying different methods to find a good fit.

The very best notes are the ones you take in an organized manner that encourages frequent review and use as you progress through a topic or course of study. For this reason, you need to develop a way to organize all your notes for each class so they remain together and organized. As old-fashioned as it sounds, a clunky three-ring binder is an excellent organizational container for class notes. You can easily add to previous notes, insert handouts you may receive in class, and maintain a running collection of materials for each separate course. If the idea of carrying around a heavy binder has you rolling your eyes, then transfer that same structure into your computer files. If you don't organize your many documents into some semblance of order on your computer, you will waste significant time searching for improperly named or saved files.

You may be interested in relatively new research on what is the more effective notetaking strategy: handwriting versus typing directly into a computer. While individuals have strong personal opinions on this subject, most researchers agree that the format of student notes is less important than what students do with the notes they take afterwards. Both handwriting notes and using a computer for notetaking have pros and cons.

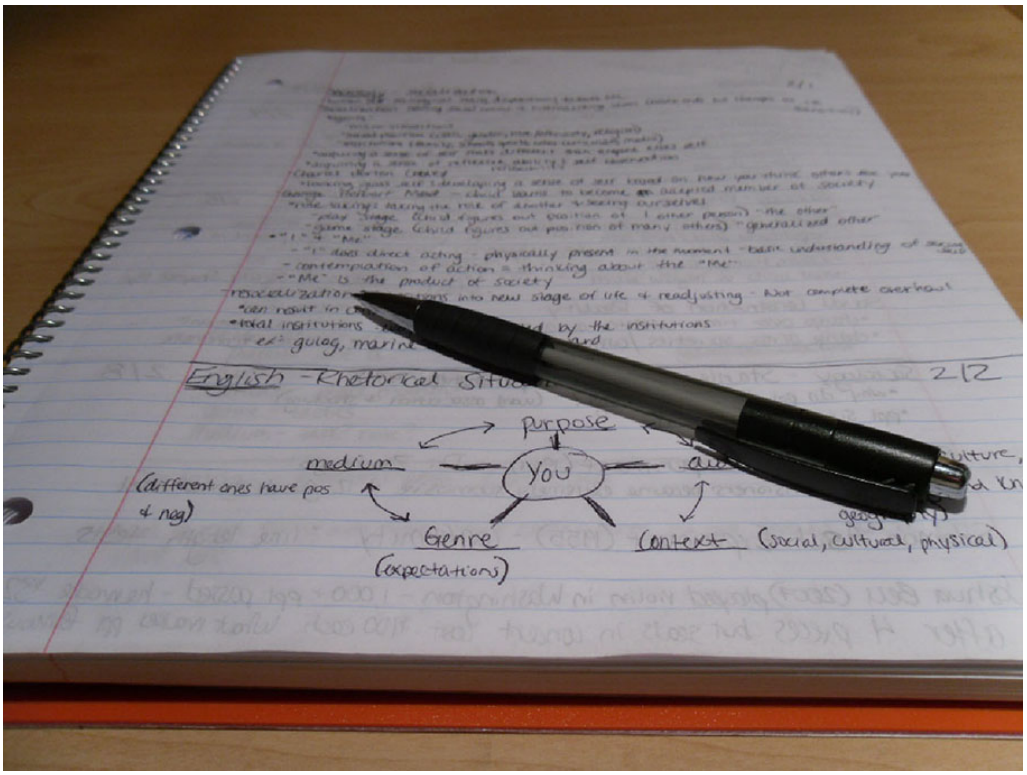


Figure 1.5.10 The best notes are the ones you take in an organized manner. Frequent review and further annotation are important to build a deep and useful understanding of the material. (Credit: English106 / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

### Managing Notetaking Systems (Computer, Paper/Pen, Note Cards, Textbook)

Whichever of the many notetaking systems you choose (and new ones seem to come out almost daily), the very best one is the one that you will use consistently. The skill and art of notetaking is not automatic for anyone; it takes a great deal of practice, patience, and continuous attention to detail. Add to that the fact that you may need to master multiple notetaking techniques for different classes, and you have some work to do. Unless you are specifically directed by your instructor, you are free to combine the best parts of different systems if you are most comfortable with that hybrid system.

Just to keep yourself organized, all your notes should start off with an identifier, including at the very least the date, the course name, the topic of the lecture/presentation, and any other information you think will help you when you return to use the notes for further study, test preparation, or assignment completion. Additional, optional information may be the number of notetaking sessions about this topic or reminders to cross-reference class handouts, textbook pages, or other course materials. It's also always a good idea to leave some blank space in your notes so you can insert additions and questions you may have as you review the material later.

## Notetaking Strategies

You may have a standard way you take all your notes for all your classes. When you were in high school, this one-size-fits-all approach may have worked. Now that you're in college, reading and studying more advanced topics, your general method may still work some of the time, but you should have some different strategies in place if you find that your method isn't working as well with college content. You probably will need to adopt different notetaking strategies for different subjects. The strategies in this section represent various ways to take notes in such a way that you are able to study after the initial notetaking session.

### Cornell Method

One of the most recognizable notetaking systems is called the *Cornell Method*, a relatively simple way to take effective notes devised by Cornell University education professor Dr. Walter Pauk in the 1940s. In this system, you take a standard piece of note paper and divide it into three sections by drawing a horizontal line across your paper about one to two inches from the bottom of the page (the summary area) and then drawing a vertical line to separate the rest of the page above this bottom area, making the left side about two inches (the recall column) and leaving the biggest area to the right of your vertical line (the notes column). You may want to make one page and then copy as many pages as you think you'll need for any particular class, but one advantage of this

system is that you can generate the sections quickly. Because you have divided up your page, you may end up using more paper than you would if you were writing on the entire page, but the point is not to keep your notes to as few pages as possible. The Cornell Method provides you with a well-organized set of notes that will help you study and review your notes as you move through the course. If you are taking notes on your computer, you can still use the Cornell Method in Word or Excel on your own or by using a template someone else created.

| Topic/Objective: | Name: |
|---|---|
| | Class/Period: |
| | Date: |
| Essential Question: | |
| | |
| Questions: | Notes: |
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| | |
| Summary: | |
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| | |

Figure 1.5.11: The Cornell Method provides a straightforward, organized, and flexible approach

Now that you have the notetaking format generated, the beauty of the Cornell Method is its organized simplicity. Just write on one side of the page (the right-hand notes column)—this will help later when you are reviewing and revising your notes. During your notetaking session, use the notes column to record information over the main points and concepts of the lecture; try to put the ideas into your own words, which will help you not transcribe the speaker's words verbatim. Skip lines between each idea in this column. Practice the shortcut abbreviations covered in the next section and avoid writing in complete sentences. Don't make your notes too cryptic, but you can use bullet points or phrases equally well to convey meaning—we do it all the time in conversation. If you know you will need to expand the notes you are taking in class but don't have time, you can put reminders directly in the notes by adding and underlining the word *expand* by the ideas you need to develop more fully.

As soon as possible after your notetaking session, preferably within eight hours but no more than twenty-four hours, read over your notes column and fill in any details you missed in class, including the places where you indicated you wanted to expand your notes. Then in the recall column, write any key ideas from the corresponding notes column—you can't stuff this smaller recall column as if you're explaining or defining key ideas. Just add the one- or two-word main ideas; these words in the recall column serve as cues to help you remember the detailed information you recorded in the notes column.

Once you are satisfied with your notes and recall columns, summarize this page of notes in two or three sentences using the summary area at the bottom of the sheet. This is an excellent time to get with another classmate or a group of students who all heard the same lecture to make sure you all understood the key points. Now, before you move onto something else, cover the large notes column, and quiz yourself over the key ideas you recorded in the recall column. Repeat this step often as you go along, not just immediately before an exam, and you will help your memory make the connections between your notes, your textbook reading, your in-class work, and assignments that you need to succeed on any quizzes and exams.

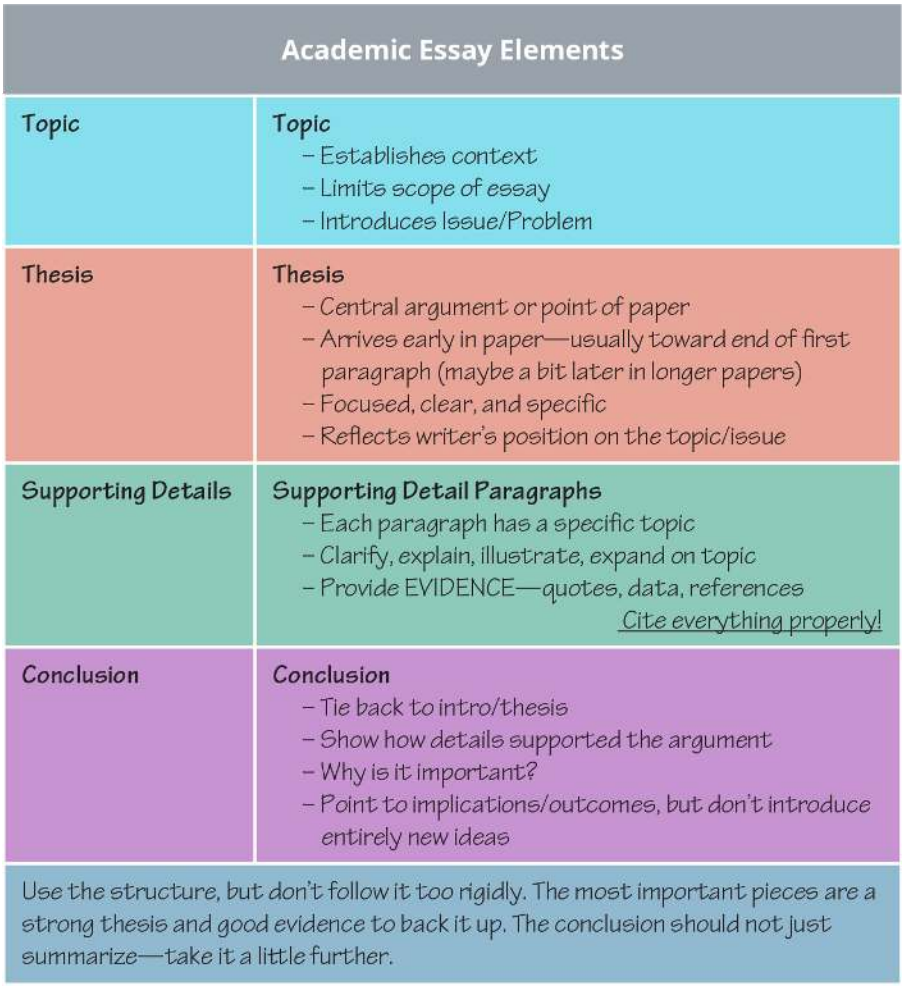



Figure 1.5.12 This sample set of notes in the Cornell Method is designed to make sense of a large amount of information. The process of organizing the notes can help you retain the information more effectively than less consistent methods.

The main advantage of the Cornell Method is that you are setting yourself up to have organized, workable notes. The neat format helps you move into study-mode without needing to re-copy less organized notes or making sense of a large mass of information you aren't sure how to process because you can't remember key ideas or what you meant. If you write notes in your classes without any sort of system and later come across something like "Napoleon—short" in the middle of a glob of notes, what can you do at this point? Is that important? Did it connect with something relevant from the lecture? How would you possibly know? You are your best advocate for setting yourself up for success in college.

## Outlining

Other note organizing systems may help you in different disciplines. You can take notes in a formal outline if you prefer, using Roman numerals for each new topic, moving down a line to capital letters indented a few spaces to the right for concepts related to the previous topic, then adding details to support the concepts indented a few more spaces over and denoted by an Arabic numeral. You can continue to add to a formal outline by following these rules.

You don't absolutely have to use the formal numerals and letter, but you have to then be careful to indent so you can tell when you move from a higher level topic to the related concepts and then to the supporting information. The main benefit of an outline is how

organized it is. You have to be on your toes when you are taking notes in class to ensure you keep up the organizational format of the outline, which can be tricky if the lecture or presentation is moving quickly or covering many diverse topics.

The following formal outline example shows the basic pattern:

I. Dogs *(main topic–usually general)*
   A. German Shepherd *(concept related to main topic)*
      1. Protection *(supporting info about the concept)*
      2. Assertive
      3. Loyal
   B. Weimaraner *(concept related to main topic)*
      1. Family-friendly *(supporting info about the concept)*
      2. Active
      3. Healthy
II. Siamese

You would just continue on with this sort of numbering and indenting format to show the connections between main ideas, concepts, and supporting details. Whatever details you do not capture in your notetaking session, you can add after the lecture as you review your outline.

### Chart or table

Similar to creating an outline, you can develop a chart to compare and contrast main ideas in a notetaking session. Divide your paper into four or five columns with headings that include either the main topics covered in the lecture or categories such as How?, What?, When used?, Advantages/Pros, Disadvantages/Cons, or other divisions of the information. You write your notes into the appropriate columns as that information comes to light in the presentation.

**Example of a Chart to Organize Ideas and Categories**

| | Structure | Types | Functions in Body | Additional Notes |
|---|---|---|---|---|
| Carbohydrates | | | | |
| Lipids | | | | |
| Proteins | | | | |
| Nucleic Acid | | | | |

This format helps you pull out the salient ideas and establishes an organized set of notes to study later. (If you haven't noticed that this *reviewing later* idea is a constant across all notetaking systems, you should…take note of that.) Notes by themselves that you never reference again are little more than scribblings. That would be a bit like compiling an extensive grocery list so you stay on budget when you shop, work all week on it, and then just throw it away before you get to the store. You may be able to recall a few items, but likely won't be as efficient as you could be if you had the notes to reference. Just as you cannot read all the many books, articles, and documents you need to peruse for your college classes, you cannot remember the most important ideas of all the notes you will take as part of your courses, so you must review.

### Concept Mapping and Visual Notetaking

One final notetaking method that appeals to learners who prefer a visual representation of notes is called *mapping* or sometimes *mind mapping* or *concept mapping*, although each of these names can have slightly different uses. Variations of this method abound, so you may want to look for more versions online, but the basic principles are that you are making connections between main ideas through a graphic depiction; some can get rather elaborate with colors and shapes, but a simple version may be more useful at least to begin. Main ideas can be circled or placed in a box with supporting concepts radiating off these ideas shown with a connecting line and possibly details of the support further radiating off the concepts. You can present your main ideas vertically or horizontally, but turning your paper long-ways, or in landscape mode, may prove helpful as you add more main ideas.

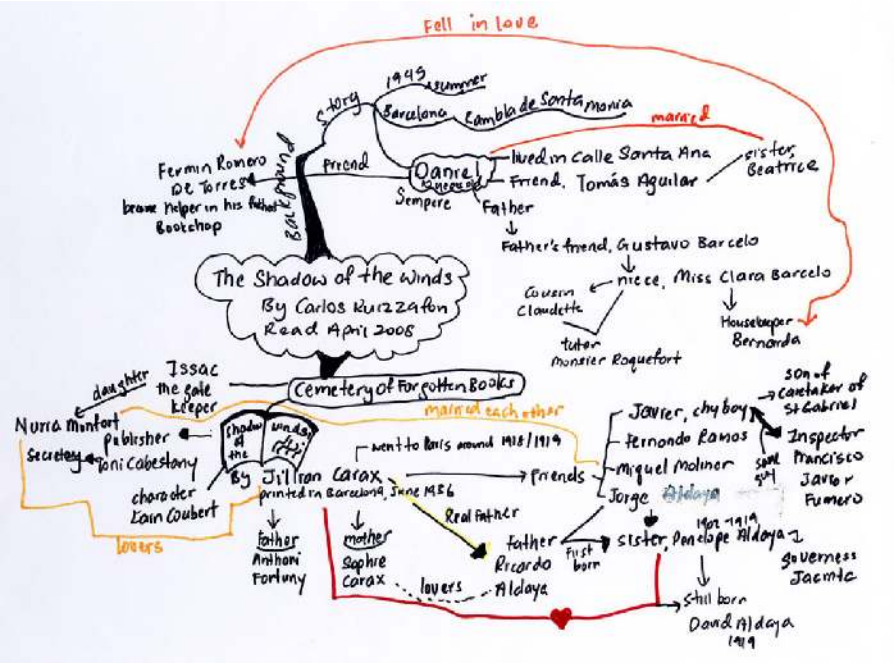



Figure 1.5.13 Concept mapping, sometimes referred to as mind mapping, can be an effective and very personalized approach to capturing information. (Credit: ArtistIvanChew / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

You may be interested in trying visual notetaking or adding pictures to your notes for clarity. Sometimes when you can't come up with the exact wording to explain something or you're trying to add information for complex ideas in your notes, sketching a rough image of the idea can help you remember. According to educator Sherrill Knezel in an article entitled "The Power of Visual Notetaking," this strategy is effective because "When students use images and text in notetaking, it gives them two different ways to pull up the information, doubling their chances of recall." Don't shy away from this creative approach to notetaking just because you believe you aren't an artist; the images don't need to be perfect. You may want to watch Rachel Smith's TEDx Talk called "Drawing in Class" to learn more about visual notetaking.

You can play with different types of notetaking suggestions and find the method(s) you like best, but once you find what works for you, stick with it. You will become more efficient with the method the more you use it, and your notetaking, review, and test prep will become, if not easier, certainly more organized, which can delete decrease your anxiety.

## Practicing Decipherable Shorthand

Most college students don't take a class in shorthand, once the domain of secretaries and executive assistants, but maybe they should. That almost-lost art in the age of computers could come in very handy during intense notetaking sessions. Elaborate shorthand systems do exist, but you would be better served in your college notetaking adventures to hone a more familiar, personalized form of shorthand to help you write more in a shorter amount of time. Seemingly insignificant shortcuts can add up to ease the stress notetaking can induce—especially if you ever encounter an "I'm not going to repeat this" kind of presenter! Become familiar with these useful abbreviations:

| Shortcut symbol | Meaning |
|---|---|
| w/, w/o, w/in | with, without, within |
| & | and |
| # | number |
| b/c | because |
| X, √ | incorrect, correct |
| Diff | different, difference |
| etc. | and so on |

| | |
|---|---|
| ASAP | as soon as possible |
| US, UK | United States, United Kingdom |
| info | information |
| Measurements: ft, in, k, m | foot, inch, thousand, million |
| ¶ | paragraph or new paragraph |
| Math symbols: =, +, >, <, ÷ | equal, plus, greater, less, divided by |
| WWI, WWII | World Wars I and II |
| impt | important |
| ?, !, ** | denote something is very significant; don't over use |

Do you have any other shortcuts or symbols that you use in your notes? Ask your parents if they remember any that you may be able to learn.

## Annotating Notes After Initial Notetaking Session

Annotating notes after the initial notetaking session may be one of the most valuable study skills you can master. Whether you are highlighting, underlining, or adding additional notes, you are reinforcing the material in your mind and memory.

Admit it—who can resist highlighting markers? Gone are the days when yellow was the star of the show, and you had to be very careful not to press too firmly for fear of obliterating the words you were attempting to emphasize. Students now have a veritable rainbow of highlighting options and can color-code notes and text passages to their hearts' content. Technological advances may be important, but highlighter color choice is monumental! Maybe.

The only reason to highlight anything is to draw attention to it, so you can easily pick out that ever-so-important information later for further study or reflection. One problem many students have is not knowing when to stop. If what you need to recall from the passage is a particularly apt and succinct definition of the term important to your discipline, highlighting the entire paragraph is less effective than highlighting just the actual term. And if you don't rein in this tendency to color long passages (possibly in multiple colors) you can end up with a whole page of highlighted text. Ironically, that is no different from a page that is not highlighted at all, so you have wasted your time. Your mantra for highlighting text should be *less is more*. Always read your text selection first before you start highlighting anything. You need to know what the overall message is before you start placing emphasis in the text with highlighting.

Another way to annotate notes after initial notetaking is underlying significant words or passages. Albeit not quite as much fun as its colorful cousin highlighting, underlining provides precision to your emphasis.

Some people think of annotations as only using a colored highlighter to mark certain words or phrases for emphasis. Actually, annotations can refer to anything you do with a text to enhance it for your particular use (either a printed text, handwritten notes, or other sort of document you are using to learn concepts). The annotations may include highlighting passages or vocabulary, defining those unfamiliar terms once you look them up, writing questions in the margin of a book, underlining or circling key terms, or otherwise marking a text for future reference. You can also annotate some electronic texts.

Realistically, you may end up doing all of these types of annotations at different times. We know that repetition in studying and reviewing is critical to learning, so you may come back to the same passage and annotate it separately. These various markings can be invaluable to you as a study guide and as a way to see the evolution of your learning about a topic. If you regularly begin a reading session writing down any questions you may have about the topic of that chapter or section and also write out answers to those questions at the end of the reading selection, you will have a good start to what that chapter covered when you eventually need to study for an exam. At that point, you likely will not have time to reread the entire selection especially if it is a long reading selection, but with strong annotations in conjunction with your class notes, you won't need to do that. With experience in reading discipline-specific texts and writing essays or taking exams in that field, you will know better what sort of questions to ask in your annotations

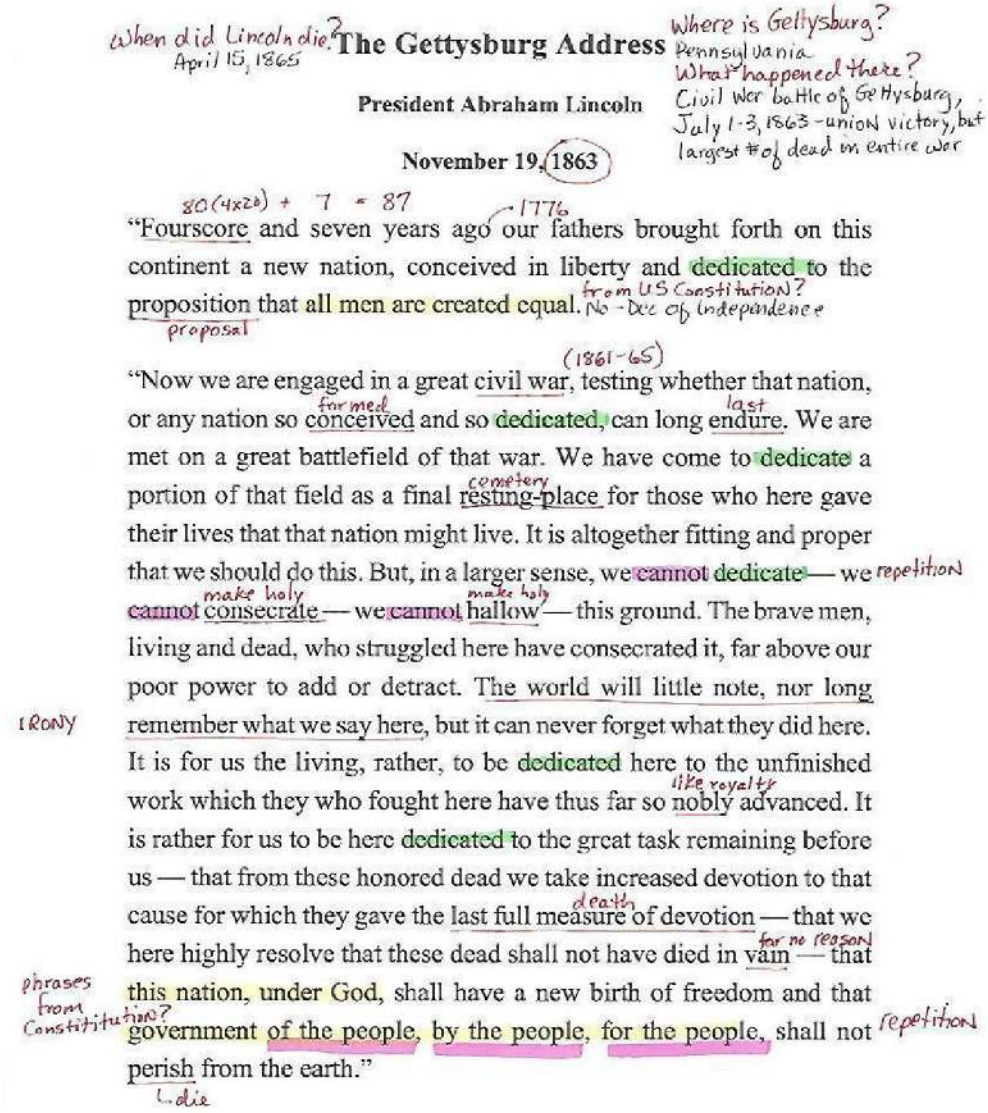



Figure 1.5.14 Annotations may include highlighting important topics, defining unfamiliar terms, writing questions in, underlining or circling key terms, or otherwise marking a text for future reference. Whichever approach you choose, try not to overdo it; neat, organized, and efficient notes are more effective than crowded or overdone notes.

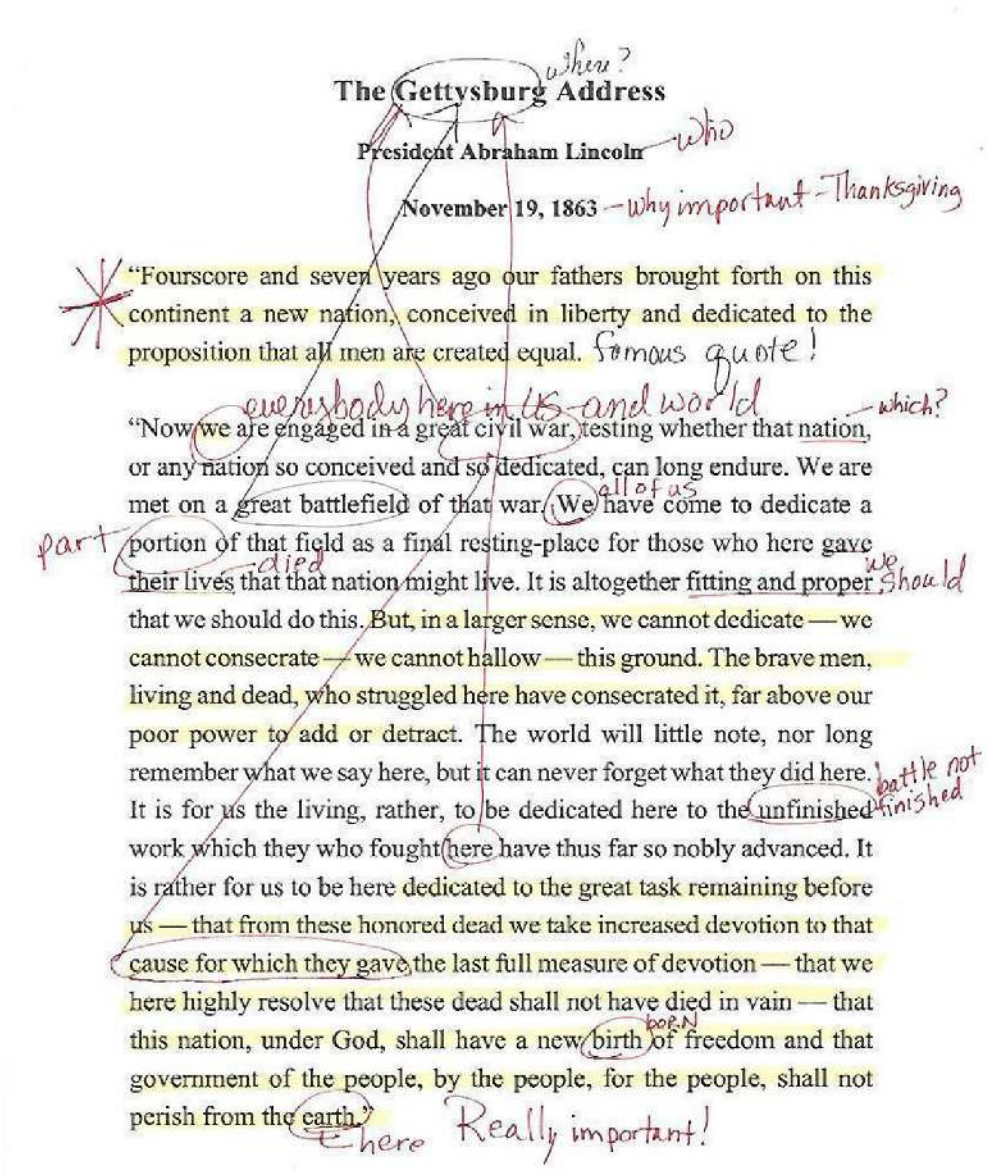



Figure 1.5.15 While these notes may be meaningful to the person who took them, they are neither organized nor consistent. For example, note that some of the more commonly used terms, like "we" and "unfinished," are defined, but less common ones -- "consecrate" and "hallow" -- are not.

What you have to keep in the front of your mind while you are annotating, especially if you are going to conduct multiple annotation sessions, is to not overdo whatever method you use. Be judicious about what you annotate and how you do it on the page, which means you must be neat about it. Otherwise, you end up with a mess of either color or symbols combined with some cryptic notes that probably took you quite a long time to create, but won't be worth as much to you as a study aid as they could be. This is simply a waste of time and effort.

You cannot eat up every smidgen of white space on the page writing out questions or summaries and still have a way to read the original text. If you are lucky enough to have a blank page next to the beginning of the chapter or section you are annotating, use this, but keep in mind that when you start writing notes, you aren't exactly sure how much space you'll need. Use a decipherable shorthand and write only what you need to convey the meaning in very small print. If you are annotating your own notes, you can make a habit of using only one side of the paper in class, so that if you need to add more notes later, you could use the other side. You can also add a blank page to your notes before beginning the next class date in your notebook so you'll end up with extra paper for annotations when you study.

Professional resources may come with annotations that can be helpful to you as you work through the various documentation requirements you'll encounter in college as well. Purdue University's Online Writing Lab (OWL) provides an annotated sample for

how to format a college paper according to guidelines in the Modern Language Association (MLA) manual that you can see, along with other annotations.

### Adding Needed Additional Explanations to Notes

Marlon was totally organized and ready to take notes in a designated course notebook at the beginning of every philosophy class session. He always dated his page and indicated what the topic of discussion was. He had various colored highlighters ready to denote the different note purposes he had defined: vocabulary in pink, confusing concepts in green, and note sections that would need additional explanations later in yellow. He also used his own shorthand and an impressive array of symbols to indicate questions (red question mark), highly probable test material (he used a tiny bomb exploding here), additional reading suggestions, and specific topics he would ask his instructor before the next class. Doing everything so precisely, Marlon's methods seemed like a perfect example of how to take notes for success. Inevitably though, by the end of the hour-and-a-half class session, Marlon was frantically switching between writing tools, near to tears, and scouring his notes as waves of yellow teased him with uncertainty. What went wrong?

As with many of us who try diligently to do everything we know how to do for success or what we think we know because we read books and articles on success in between our course work, Marlon is suffering from trying to do too much simultaneously. It's an honest mistake we can make when we are trying to save a little time or think we can multitask and kill two birds with one stone.

Unfortunately, this particular error in judgement can add to your stress level exponentially if you don't step back and see it for what it is. Marlon attempted to take notes in class as well as annotate his notes to get them ready for his test preparation. It was too much to do at one time, but even if he could have done all those things during class, he's missing one critical point about notetaking.

As much as we may want to hurry and get it over with, notetaking in class is just the beginning. Your instructor likely gave you a pre-class assignment to read or complete before coming to that session. The intention of that preparatory lesson is for you to come in with some level of familiarity for the topic under consideration and questions of your own. Once you're in class, you may also need to participate in a group discussion, work with your classmates, or perform some other sort of lesson-directed activity that would necessarily take you away from taking notes. Does that mean you should ignore taking notes for that day? Most likely not. You may just need to indicate in your notes that you worked on a project or whatever other in-class event you experienced that date.

Very rarely in a college classroom will you engage in an activity that is not directly related to what you are studying in that course. Even if you enjoyed every minute of the class session and it was an unusual format for that course, you still need to take some notes. Maybe your first note could be to ask yourself why you think the instructor used that unique teaching strategy for the class that day. Was it effective? Was it worth using the whole class time? How will that experience enhance what you are learning in that course?

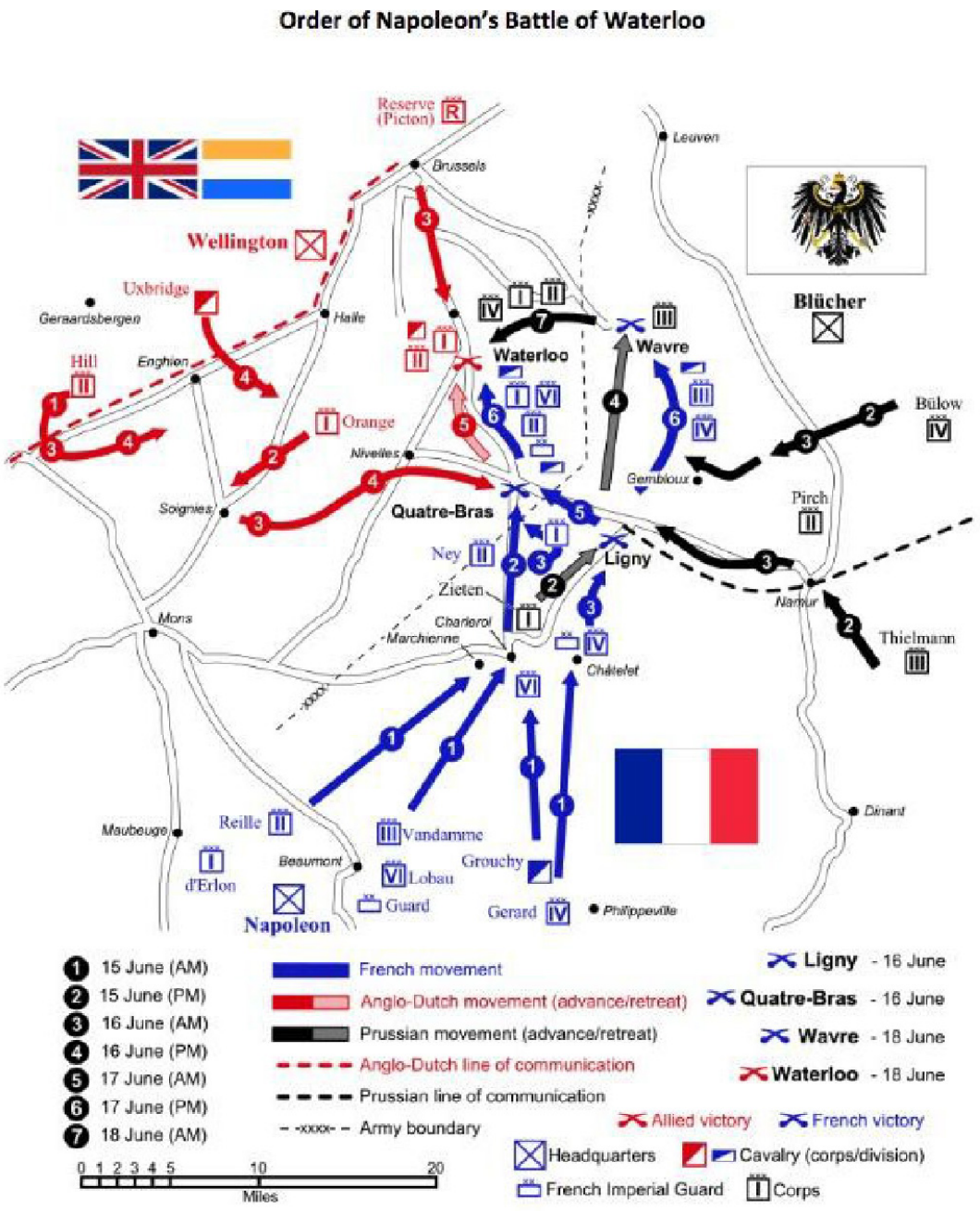
If you use an ereader or ebooks to read texts for class or read articles from the Internet on your laptop or tablet, you can still take effective notes. Depending on the features of your device, you have many choices. Almost all electronic reading platforms allow readers to highlight and underline text. Some devices allow you to add a written text in addition to marking a word or passage that you can collect at the end of your notetaking session. Look into the specific tools for your device and learn how to use the features that allow you to take notes electronically. You can also find apps on devices to help with taking notes, some of which you may automatically have installed when you buy the product. Microsoft's OneNote, Google Keep, and the Notes feature on phones are relatively easy to use, and you may already have free access to those.

### Taking Notes on Non-Text Items (i.e., Tables, Maps, Figures, etc.)

You may also encounter situations as you study and read textbooks, primary sources, and other resources for your classes that are not actually texts. You can still take notes on maps, charts, graphs, images, and tables, and your approach to these non-text features is similar to when you prepare to take notes over a passage of text. For example, if you are looking at the following map, you may immediately come up with several questions. Or it may initially appear overwhelming. Start by asking yourself these questions:

#### What is the main point of this map?

- Who is the intended audience?
- Where is it?
- What time period does it depict?
- What does the map's legend (the explanation of symbols) include?
- What other information do I need to make sense of this map?




Source: Wikipedia Creative Commons, https://en.wikipedia.org/wiki/Order_of_battle_of_the_Waterloo_campaign

Figure 1.5.16 Graphics, charts, graphs, and other visual items are also important to annotate. Not only do they often convey important information, but they may appear on exams or in other situations where you'll need to use or demonstrate knowledge. Credit: "Lpankonin" / Wikipedia Commons / Attribution 3.0 Generic (CC BY 3.0)

You may want to make an extra copy of a graphic or table before you add annotations if you are dealing with a lot of information. Making sense of all the elements will take time, and you don't want to add to the confusion.

## Returning to Your Notes

Later, as soon as possible after the class, you can go back to your notes and add in missing parts. Just as you may generate questions as you're reading new material, you may leave a class session or lecture or activities with many questions. Write those

down in a place where they won't get lost in all your other notes.

The exact timing of when you get back to the notes you take in class or while you are reading an assignment will vary depending on how many other classes you have or what other obligations you have in your daily schedule. A good starting place that is also easy to remember is to make every effort to review your notes within 24 hours of first taking them. Longer than that and you are likely to have forgotten some key features you need to include; must less time than that, and you may not think you need to review the information you so recently wrote down, and you may postpone the task too long.

Use your phone or computer to set reminders for all your note review sessions so that it becomes a habit and you keep on top of the schedule.

Your personal notes play a significant role in your test preparation. They should enhance how you understand the lessons, textbooks, lab sessions, and assignments. All the time and effort you put into first taking the notes and then annotating and organizing the notes will be for naught if you do not formulate an effective and efficient way to use them before sectional exams or comprehensive tests.

The whole cycle of reading, notetaking in class, reviewing and enhancing your notes, and preparing for exams is part of a continuum you ideally will carry into your professional life. Don't try to take short cuts; recognize each step in the cycle as a building block. Learning doesn't end, which shouldn't fill you with dread; it should help you recognize that all this work you're doing in the classroom and during your own study and review sessions is ongoing and cumulative. Practicing effective strategies now will help you be a stronger professional.

**Activity**

What resources can you find about reading and notetaking that will actually help you with these crucial skills? How do you go about deciding what resources are valuable for improving your reading and notetaking skills?

The selection and relative value of study guides and books about notetaking vary dramatically. Ask your instructors for recommendations and see what the library has available on this topic. The following list is not comprehensive, but will give you a starting point for books and articles on notetaking in college.

- *College Rules!: How to Study, Survive, and Succeed in College*, by Sherri Nist-Olejnik and Jodi Patrick Holschuh. More than just notetaking, this book covers many aspects of transitioning into the rigors of college life and studying.
- *Effective Notetaking*, by Fiona McPherson. This small volume has suggestions for using your limited time wisely before, during, and after notetaking sessions.
- *How to Study in College*, by Walter Pauk. This is the book that introduced Pauk's notetaking suggestions we now call the Cornell Method. It is a bit dated (from the 1940s), but still contains some valuable information.
- *Learn to Listen, Listen to Learn 2: Academic Listening and Note-taking*, by Roni S. Lebauer. The main point of this book is to help students get the most from college lectures by watching for clues to lecture organization and adapting this information into strong notes.
- *Study Skills: Do I Really Need this Stuff?*, by Steve Piscitelli. Written in a consistently down-to-earth manner, this book will help you with the foundations of strong study skills, including time management, effective notetaking, and seeing the big picture.
- "What Reading Does for the Mind," by Anne Cunningham and Keith Stanovich, 1998, https://www.aft.org/sites/default/files/periodicals/cunningham.pdf
- Adler, Mortimer J. and Charles Van Doren. *How to Read a Book: The Classic Guide to Intelligent Reading. NY: Simon & Schuster, 1940*.
- Berns, Gregory S., Kristina Blaine, Michael J. Prietula, and Brandon E. Pye. *Brain Connectivity.* Dec 2013.ahead of print http://doi.org/10.1089/brain.2013.0166

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# 1.6: How to Manage Time

**Questions to consider:**

- How can I use time-on-task estimates to improve time management?
- What behaviors can help or hinder when it comes to managing time?

In this next section you will learn about managing time and prioritizing tasks. This is not only a valuable skill for pursuing an education, but it can become an ability that follows you through the rest of your life, especially if your career takes you into a leadership role.




Figure 1.6.7 An online calendar is a very useful tool for keeping track of classes, meetings, and other events. Most learning management systems contain these features, or you can use a calendar application.

## ANALYSIS QUESTION

Read each statement in the brief self-evaluation tool below, and check the answer that best applies to you. There are no right or wrong answers.

Table 3.3

| | Always | Usually | Sometimes | Rarely | Never |
|---|---|---|---|---|---|
| I like to be given strict deadlines for each task. It helps me stay organized and on track. | | | | | |
| I would rather be 15 minutes early than 1 minute late. | | | | | |
| I like to improvise instead of planning everything out ahead of time. | | | | | |

| | Always | Usually | Sometimes | Rarely | Never |
|---|---|---|---|---|---|
| I prefer to be able to manage when and how I do each task. | | | | | |
| I have a difficult time estimating how long a task will take. | | | | | |
| I have more motivation when there is an upcoming deadline. It helps me focus. | | | | | |
| I have difficulty keeping priorities in the most beneficial order. | | | | | |

This exercise is intended to help you recognize some things about your own time management style. The important part is for you to identify any areas where you might be able to improve and to find solutions for them. This chapter will provide some solutions, but there are many others that can be found by researching time management strategies.

After you have decided your best response to each statement, think about what they may mean in regard to potential strengths and/or challenges for you when it comes to time management in college. If you are a person that likes strict deadlines, what would you do if you took a course that only had one large paper due at the end? Would you set yourself a series of mini deadlines that made you more comfortable and that kept things moving along for you? Or, if you have difficulty prioritizing tasks, would it help you to make a list of the tasks to do and order them, so you know which ones must be finished first?

## How to Manage Time

The simplest way to manage your time is to accurately plan for how much time it will take to do each task, and then set aside that amount of time. How you divide the time is up to you. If it is going to take you five hours to study for a final exam, you can plan to spread it over five days, with an hour each night, or you can plan on two hours one night and three hours the next. What you would not want to do is plan on studying only a few hours the night before the exam and find that you fell very short on the time you estimated you would need. If that were to happen, you would have run out of time before finishing, with no way to go back and change your decision. In this kind of situation, you might even be tempted to “pull an all-nighter,” which is a phrase that has been used among college students for decades. In essence it means going without sleep for the entire night and using that time to finish an assignment. While this method of trying to make up for poor planning is common enough to have a name, rarely does it produce the best work.

Table 3.4 Sample Time Estimate Table

| Activity | Estimated Time | Actual Time |
|---|---|---|
| Practice Quiz | 5 minutes | 15 minutes |
| Lab Conclusions | 20 minutes | 35 minutes |
| Food shopping | 45 minutes | 30 minutes |
| Drive to work | 20 minutes | 20 minutes |
| Physical Therapy | 1 hour | 50 minutes |

Of all the parts of time management, accurately predicting how long a task will take is usually the most difficult—and the most elusive. Part of the problem comes from the fact that most of us are not very accurate timekeepers, especially when we are busy applying ourselves to a task. The other issue that makes it so difficult to accurately estimate time on task is that our estimations must also account for things like interruptions or unforeseen problems that cause delays.?

When it comes to academic activities, many tasks can be dependent upon the completion of other things first, or the time a task takes can vary from one instance to another, both of which add to the complexity and difficulty of estimating how much time and effort are required.

For example, if an instructor assigned three chapters of reading, you would not really have any idea how long each chapter might take to read until you looked at them. The first chapter might be 30 pages long while the second is 45. The third chapter could be only 20 pages but made up mostly of charts and graphs for you to compare. By page count, it might seem that the third chapter would take the least amount of time, but actually studying charts and graphs to gather information can take longer than regular reading.?

To make matters even more difficult, when it comes to estimating time on task for something as common as reading, not all reading takes the same amount of time. Fiction, for example, is usually a faster read than a technical manual. But something like the novel *Finnegan's Wake* by James Joyce is considered so difficult that most readers never finish it.

ACTIVITY

To better understand how much time different kinds of material can take to read, try this experiment. You will use two examples of famous texts that are very close to being the same number of words: *The Gettysburg Address* and the opening paragraphs from *A Christmas Carol*. Before you begin, estimate how long it will take you to read each, and predict which you think will take longer. When you do the reading, use a stopwatch function on a device such as a phone or some other timer to see how long it actually takes.

Make certain that you are reading for understanding, not just skimming over words. If you must reread a section to better comprehend what is being said, that is appropriate. The goal here is to compare reading of different texts, not to see how fast you can sight-read the words on a page.

After you have finished *The Gettysburg Address*, read and time *A Christmas Carol* and compare both of your times.

**The Gettysburg Address**
**Abraham Lincoln**
**Gettysburg, Pennsylvania November 19, 1863**
**Word count: 278**

Four score and seven years ago our fathers brought forth on this continent, a new nation, conceived in Liberty, and dedicated to the proposition that all men are created equal.

Now we are engaged in a great civil war, testing whether that nation, or any nation so conceived and so dedicated, can long endure. We are met on a great battle-field of that war. We have come to dedicate a portion of that field, as a final resting place for those who here gave their lives that that nation might live. It is altogether fitting and proper that we should do this.

But, in a larger sense, we can not dedicate -- we can not consecrate -- we can not hallow -- this ground. The brave men, living and dead, who struggled here, have consecrated it, far above our poor power to add or detract. The world will little note, nor long remember what we say here, but it can never forget what they did here. It is for us the living, rather, to be dedicated here to the unfinished work which they who fought here have thus far so nobly advanced. It is rather for us to be here dedicated to the great task remaining before us -- that from these honored dead we take increased devotion to that cause for which they gave the last full measure of devotion -- that we here highly resolve that these dead shall not have died in vain -- that this nation, under God, shall have a new birth of freedom -- and that government of the people, by the people, for the people, shall not perish from the earth.

**A Christmas Carol**
**Charles Dickens**
**Chapman & Hall, 1843**
**Word count: 260**

> Marley was dead: to begin with. There is no doubt whatever about that. The register of his burial was signed by the clergyman, the clerk, the undertaker, and the chief mourner. Scrooge signed it: and Scrooge's name was good upon 'Change, for anything he chose to put his hand to. Old Marley was as dead as a door-nail.
>
> Mind! I don't mean to say that I know, of my own knowledge, what there is particularly dead about a door-nail. I might have been inclined, myself, to regard a coffin-nail as the deadest piece of ironmongery in the trade. But the wisdom of our ancestors is in the simile; and my unhallowed hands shall not disturb it, or the Country's done for. You will therefore permit me to repeat, emphatically, that Marley was as dead as a door-nail.
>
> Scrooge knew he was dead? Of course he did. How could it be otherwise? Scrooge and he were partners for I don't know how many years. Scrooge was his sole executor, his sole administrator, his sole assign, his sole residuary legatee, his sole friend, and sole mourner. And even Scrooge was not so dreadfully cut up by the sad event, but that he was an excellent man of business on the very day of the funeral, and solemnised it with an undoubted bargain.
>
> The mention of Marley's funeral brings me back to the point I started from. There is no doubt that Marley was dead. This must be distinctly understood, or nothing wonderful can come of the story I am going to relate.
>
> In comparing the two, was one or the other easier to understand or faster to read? Was it the piece you predicted you would read faster?
>
> It is important to note that in this case both readings were only three paragraphs long. While there may have only been half a minute or so between the reading of each, that amount of time would multiply greatly over an entire chapter.

## Knowing Yourself

While you can find all sorts of estimates online as to how long a certain task may take, it is important to know these are only averages. People read at different speeds, people write at different speeds, and those numbers even change for each individual depending on the environment.

If you are trying to read in surroundings that have distractions (e.g., conversations, phone calls, etc.), reading 10 pages can take you a lot longer than if you are reading in a quiet area. By the same token, you may be reading in a quiet environment (e.g., in bed after everyone in the house has gone to sleep), but if you are tired, your attention and retention may not be what it would be if you were refreshed.

In essence, the only way you are going to be able to manage your time accurately is to know yourself and to know how long it takes you to do each task. But where to begin?

Below, you will find a table of common college academic activities. This list has been compiled from a large number of different sources, including colleges, publishers, and professional educators, to help students estimate their own time on tasks. The purpose of this table is to both give you a place to begin in your estimates and to illustrate how different factors can impact the actual time spent.

You will notice that beside each task there is a column for the *unit*, followed by the average *time on task*, and a column for notes. The *unit* is whatever is being measured (e.g., pages read, pages written, etc.), and the *time on task* is an average time it takes students to do these tasks. It is important to pay attention to the notes column, because there you will find factors that influence the time on task. These factors can dramatically change the amount of time the activity takes.

Table 3.5 Time on task for common college activities.

| Time on Task | | | |
|---|---|---|---|
| **Activity** | **Unit** | **Time on task** | **Notes** |
| General academic reading (textbook, professional journals) | 1 page | 5–7 minutes | Be aware that your personal reading speed may differ and may change over time. |
| Technical reading (math, charts and data) | 1 page | 10–15 minutes | Be aware that your personal reading speed may differ and may change over time. |

| Time on Task | | | |
|---|---|---|---|
| Simple Quiz or homework question: short answer—oriented toward recall or identification type answers | Per question | 1–2 minutes | Complexity of question will greatly influence the time required. |
| Complex Quiz or homework question: short answer—oriented toward application, evaluation, or synthesis of knowledge | Per question | 2–3 minutes | Complexity of question will greatly influence the time required. |
| Math problem sets, complex | Per question | 15 minutes | For example, algebra, complex equations, financial calculations |
| Writing: short, no research | Per page | 60 minutes | Short essays, single-topic writing assignments, summaries, freewriting assignments, journaling—includes drafting, writing, proofing, and finalizing |
| Writing: research paper | Per page | 105 minutes | Includes research time, drafting, editing, proofing, and finalizing (built into per-page calculation) |
| Study for quiz | Per chapter | 60 minutes | 45–90 minutes per chapter, depending upon complexity of material |
| Study for exam | Per exam | 90 minutes | 1–2 hours, depending upon complexity of material |

Again, these are averages, and it does not mean anything if your times are a little slower or a little faster. There is no "right amount of time," only the time that it takes you to do something so you can accurately plan and manage your time.

There is also another element to look for in the table. These are differentiations in the similar activities that will also affect the time you spend. A good example of this can be found in the first four rows. Each of these activities involves reading, but you can see that depending on the material being read and its complexity, the time spent can vary greatly. Not only do these differences in time account for the different types of materials you might read (as you found in the comparative reading exercise earlier in this chapter), but also they also take into consideration the time needed to think about what you are reading to truly understand and comprehend what it is saying.

GET CONNECTED

Which apps help you best prepare for success when managing your time?

Do you have trouble keeping track of multiple tasks over the course of a term?

Trello lets you organize all your obligations in helpful boards. You can share them with others (project collaborators), set alerts as reminders, and mark tasks off as you complete them.

Do you use a particular app to help you manage your time?

Sticky note apps are available for PC, Mac, and mobile devices. They let you post quick reminders, reorganize them as needed, and view them separately or as a full to-do list.

What do you wish you could improve about your time management skills?

> Toggl helps you keep track of how and where you are spending your time so you can budget better and make time management changes that free you up for the really important stuff.

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# 1.7: Enhanced Strategies for Time and Task Management

**Questions to consider:**

- What strategy helps me prioritize my top tasks?
- How do I make the best use of my time when prioritizing?
- How do I make sure I tackle unpleasant tasks instead of putting them off?
- What's the best way to plan for long-term tasks?
- How do I find time in a busy schedule?

Over the years, people have developed a number of different strategies to manage time and tasks. Some of the strategies have proven to be effective and helpful, while others have been deemed not as useful.

The good news is that the approaches that do not work very well or do not really help in managing time do not get passed along very often. But others, those which people find of value, do. What follows here are three unique strategies that have become staples of time management. While not everyone will find that all three work for them in every situation, enough people have found them beneficial to pass them along with high recommendations.

## Daily Top Three

The idea behind the *daily top three* approach is that you determine which three things are the most important to finish that day, and these become the tasks that you complete. It is a very simple technique that is effective because each day you are finishing tasks and removing them from your list. Even if you took one day off a week and completed no tasks on that particular day, a *daily top three* strategy would have you finishing 18 tasks in the course of a single week. That is a good amount of things crossed off your list.

**analysis question**

Analysis: Think about what would be your top three tasks for today? What would you have on the list tomorrow?

## Pomodoro Technique




Figure 1.7.14 The Pomodoro Technique is named after a type of kitchen timer, but you can use any clock or countdown timer. (Marco Verch /Flickr / Attribution 2.0 Generic (CC BY 2.0))

The Pomodoro Technique was developed by Francesco Cirillo. The basic concept is to use a timer to set work intervals that are followed by a short break. The intervals are usually about 25 minutes long and are called *pomodoros*, which comes from the Italian word for tomato because Cirillo used a tomato-shaped kitchen timer to keep track of the intervals.

In the original technique there are six steps:

1. Decide on the task to be done.
2. Set the timer to the desired interval.
3. Work on the task.
4. When the timer goes off, put a check mark on a piece of paper.
5. If you have fewer than four check marks, take a short break (3–5 minutes), then go to Step 1 or 2 (whichever is appropriate).
6. After four pomodoros, take a longer break (15–30 minutes), reset your check mark count to zero, and then go to Step 1 or 2.

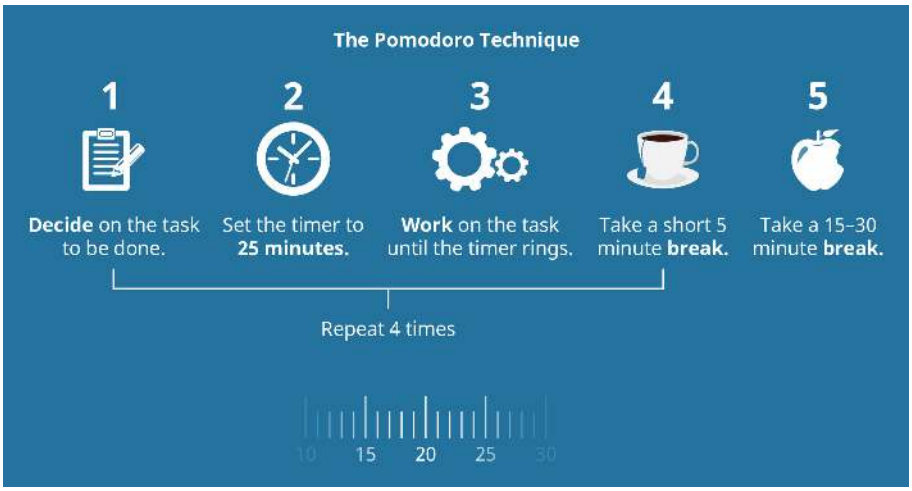


Figure 1.7.15 The Pomodoro Technique contains five defined steps.

There are several reasons this technique is deemed effective for many people. One is the benefit that is derived from quick cycles of work and short breaks. This helps reduce mental fatigue and the lack of productivity caused by it. Another is that it tends to encourage practitioners to break tasks down to things that can be completed in about 25 minutes, which is something that is usually manageable from the perspective of time available. It is much easier to squeeze in three 25-minute sessions of work time during the day than it is to set aside a 75- minute block of time.

## Eat the Frog

Of our three quick strategies, *eat the frog* probably has the strangest name and may not sound the most inviting. The name comes from a famous quote, attributed to Mark Twain: “Eat a live frog first thing in the morning and nothing worse will happen to you the rest of the day.” *Eat the Frog* is also the title of a best-selling book by Brian Tracy that deals with time management and avoiding procrastination.

How this applies to time and task management is based on the concept that if a person takes care of the biggest or most unpleasant task first, everything else will be easier after that.

Although stated in a humorous way, there is a good deal of truth in this. First, we greatly underestimate how much worry can impact our performance. If you are continually distracted by anxiety over a task you are dreading, it can affect the task you are working on at the time. Second, not only will you have a sense of accomplishment and relief when the task you are concerned with is finished and out of the way, but other tasks will seem lighter and not as difficult.

### Application: Try Three Time Management Strategies

Over the next two weeks, try each of these three methods to see which ones might work for you. Is there one you favor over the others? Might each of these three approaches serve you better in different situations or with different tasks? Do you have a creative alternative or possibly a way to use some combination of these techniques?

In addition to these three strategies, you could also develop whole new approaches from suggestions found earlier in this chapter. For example, you could apply some of the strategies for avoiding procrastination or for setting appropriate priorities and see how they work in combination with these techniques or on their own.

The key is to find which system works best for you.

## Breaking Down the Steps and Spreading Them over Shorter Work Periods

Above, you read about several different tried-and-tested strategies for effective time management—approaches that have become staples in the professional world. In this section you will read about two more creative techniques that combine elements from these other methods to handle tasks when time is scarce and long periods of time are a luxury you just do not have.

The concept behind this strategy is to break tasks into smaller, more manageable units that do not require as much time to complete. As an illustration of how this might work, imagine that you are assigned a two-page paper that is to include references. You estimate that to complete the paper—start to finish—would take you between four and a half and five hours. You look at your

calendar over the next week and see that there simply are no open five-hour blocks (unless you decided to only get three hours of sleep one night). Rightly so, you decide that going without sleep is not a good option. While looking at your calendar, you do see that you can squeeze in an hour or so every night. Instead of trying to write the entire paper in one sitting, you break it up into much smaller components as shown in the table below:

Table 3.8: Breaking Down Projects into Manageable-Sized Tasks

| Day/Time | Task | Time |
|---|---|---|
| Monday, 6:00 p.m. | Write outline; look for references. | 60 minutes |
| Tuesday, 6:00 p.m. | Research references to support outline; look for good quotes. | 60 minutes |
| Wednesday, 7:00 p.m. | Write paper introduction and first page draft. | 60 minutes |
| Thursday, 6:00 p.m. | Write second page and closing draft. | 60 minutes |
| Friday, 5:00 p.m. | Rewrite and polish final draft. | 60 minutes |
| *Saturday, 10:00 a.m.* | *Only if needed—finish or polish final draft.* | 60 minutes? |

While this is a simple example, you can see how it would redistribute tasks to fit your available time in a way that would make completing the paper possible. In fact, if your time constraints were even more rigid, it would be possible to break these divided tasks down even further. You could use a variation of the Pomodoro Technique and write for three 20-minute segments each day at different times. The key is to look for ways to break down the entire task into smaller steps and spread them out to fit your schedule.

Table 3.9

| | Monday | Tuesday | Wednesday | Thursday | Friday | Saturday | Sunday |
|---|---|---|---|---|---|---|---|
| 8:00–10:00 | | Work | | Work | | | |
| 10:00–12:00 | Algebra | Work | Algebra | Work | Algebra | 10 a.m.–11 a.m. *Only if needed* | Work |
| 12:00–2:00 | Lunch/study | 1 p.m. English Comp | Lunch/study | 1 p.m. English Comp | Lunch/study | Family picnic | Work |
| 2:00–4:00 | History | English Comp | History | English Comp | History | Family picnic | |
| 4:00–6:00 | Study for Algebra quiz. | Grocery | Study for History exam. | Study for History exam. | 5 p.m.–6 p.m. Rewrite and polish final draft. | Family picnic | Laundry |
| 6:00–7:00 | Write outline; look for references. | Research references to support outline; look for good quotes. | Research presentation project. | Write second page and closing draft | Create presentation. | Meet with Darcy. | Prepare school stuff for next week. |
| 7:00–8:00 | Free time | Free time | Write paper introduction and first page draft. | Research presentation project. | Create presentation. | | Free time |

STUDENT PROFILE

"Time management is probably one of the hardest things I had to pick up when I got to college. For starters, I didn't have anyone to come wake me up if I forgot to set an alarm or to tell me to get out of bed so that I wouldn't be late. I had to start placing my phone far away from my bed; so that way, I would have to get out of bed in order to turn the alarm off. Accomplishing work on time can also be difficult. It's tough to find the fine balance between when you have to stay in and work on assignments and when is acceptable to go out and do leisure activities.

"I learned the 8-8-8 rule. Every day you spend eight hours working on school work or going to class, eight hours of free time to do what you want, and then eight hours to sleep at night so that you will get enough rest. Sleep is crucial for time management. I learned very quickly that you cannot focus or be productive if you are struggling to keep your head from falling over because you are so tired. Basically, I've learned that if you want to be successful in college, then you have to be on top of your game when it comes to time. It's something thing you cannot make up once it's gone."

**—Preston Allen**, University of Central Arkansas

## Analyzing Your Schedule and Creating Time to Work

Of all the strategies covered in this chapter, this one may require the most discipline, but it can also be the most beneficial in time management. The fact is most of us waste time throughout the day. Some of it is due to a lack of awareness, but it can also be caused by the constraints of our current schedules. An example of this is when we have 15 to 20 minutes before we must leave to go somewhere. We don't do anything with that time because we are focused on leaving or where we are going, and we might not be organized enough to accomplish something in that short of a time period. In fact, a good deal of our 24- hour days are spent a few minutes at a time waiting for the next thing scheduled to occur. These small units of time add up to a fair amount each day.

The intent of this strategy is to recapture those lost moments and use them to your advantage. This may take careful observation and consideration on your part, but the results of using this as a method of time management are more than worth it.

The first step is to look for those periods of time that are wasted or that can be repurposed. In order to identify them, you will need to pay attention to what you do throughout the day and how much time you spend doing it. The example of waiting for the next thing in your schedule has already been given, but there are many others. How much time do you spend in activities after you have really finished doing them but are still lingering because you have not begun to do something else (e.g., letting the next episode play while binge-watching, reading social media posts or waiting for someone to reply, surfing the Internet, etc.)? You might be surprised to learn how much time you use up each day by just adding a few unproductive minutes here and there.

If you set a limit on how much time you spend on each activity, you might find that you can recapture time to do other things. An example of this would be limiting yourself to reading news for 30 minutes. Instead of reading the main things that interest you and then spending an additional amount of time just looking at things that you are only casually interested in because that is what you are doing at the moment, you could stop after a certain allotted period and use the extra time you have gained on something else.

After you identify periods of lost time, the next step will be to envision how you might restructure your activities to bring those extra minutes together into useful blocks of time. Using the following scenario as an illustration, we will see how this could be accomplished.



Figure 1.7.16 Sarah has to balance a lot of obligations.

On Tuesday nights, Sarah has a routine: After work, she does her shopping for the week (2 hours driving and shopping) and then prepares and eats dinner (1 hour). After dinner, she spends time on homework (1 hour) and catching up with friends, reading the news, and other Internet activities (1 hour), and then she watches television or reads before going to bed (1 hour). While it may seem that there is very little room for improvement in her schedule without cutting out something she enjoys, limiting the amount of time she spends on each activity and rethinking how she goes about each task can make a significant difference.

In this story, Sarah's Tuesday-night routine includes coming home from work, taking stock of which items in her home she might need to purchase, and then driving to the store. While at the store, she spends time picking out and selecting groceries as she plans for meals she will eat during the rest of the week. Then, after making her purchases, she drives home. Instead, if she took the time to make a list and plan for what she needed at the store before she arrived, she would not spend as much time looking for inspiration in each aisle. Also, if she had a prepared list, not only could she quickly pick up each item, but she could stop at the store on the way home from work, thus cutting out the extra travel time. If purchasing what she needed took 30 minutes less because she was more organized and she cut out an additional 20 minutes of travel time by saving the extra trip to the store from her house, she could recapture a significant amount of her Tuesday evening. If she then limited the time she spent catching up with friends and such to 30 minutes or maybe did some of that while she prepared dinner, she would find that she had added almost an extra hour and a half to the time available to her on that evening, without cutting out anything she needed to do or enjoys. If she decided to spend her time on study or homework, this would more than double the time she previously had available in her schedule for homework.

**Analysis Question: Reflection**

Analysis: Identify areas in the way you spend your day where you may be able to recapture and repurpose time. Are there things you can move around to gain more time? Are there ways you can combine tasks or reduce travel time?

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# 1.8: Goal Setting and Motivation

**Questions to consider:**

- How do I set motivational goals?
- What are SMART goals?
- What's the importance of an action plan?
- How do I keep to my plan?

Motivation often means the difference between success and failure. That applies to school, to specific tasks, and to life in general. One of the most effective ways to keep motivated is to set goals.

Goals can be big or small. A goal can range from *I am going to write one extra page tonight*, to *I am going to work to get an A in this course*, all the way to *I am going to graduate in the top of my class so I can start my career with a really good position*. The great thing about goals is that they can include and influence a number of other things that all work toward a much bigger picture. For example, if your goal is to get an A in a certain course, all the reading, studying, and every assignment you do for that course contributes to the larger goal. You have motivation to do each of those things and to do them well.

Setting goals is something that is frequently talked about, but it is often treated as something abstract. Like time management, goal setting is best done with careful thought and planning. This next section will explain how you can apply tested techniques to goal setting and what the benefits of each can be.

## Set Goals That Motivate You

The first thing to know about goal setting is that a goal is a specific end result you desire. If the goal is not something you are really interested in, there is little motivational drive to achieve it. Think back to when you were much younger and some well-meaning adult set a goal for you—something that didn't really appeal to you at all. How motivated were you to achieve the goal? More than likely, if you were successful at all in meeting the goal, it was because you were motivated by earning the approval of someone or receiving a possible reward, or you were concerned with avoiding something adverse that might happen if you did not do what you were told. From an honest perspective in that situation, your real goal was based on something else, not the meeting of the goal set for you. To get the most from the goals you set, make sure they are things that you are interested in achieving.

That is not to say you shouldn't set goals that are supported by other motivations (e.g., If I finish studying by Friday, I can go out on Saturday), but the idea is to be intellectually honest with your goals.

## Set SMART Goals

Goals should also be SMART. In this case, the word *smart* is not only a clever description of the type of goal, but it is also an acronym that stands for Specific, Measurable, Attainable, Relevant, and Time-bound. The reason these are all desirable traits for your goals is because they not only help you plan how to meet the goal, but they can also contribute to your decision-making processes during the planning stage.

What does it mean to create SMART goals?

- Specific—For a goal to be specific, it must be defined enough to actually determine the goal. A goal of *get a good job when I graduate* is too general. It doesn't define what a good job is. In fact, it doesn't even necessarily include a job in your chosen profession. A more specific goal would be something like *be hired as a nurse in a place of employment where it is enjoyable to work and that has room for promotion.*
- Measurable—The concept of *measurable* is one that is often overlooked when setting goals. What this means is that the goal should have clearly defined outcomes that are detailed enough to measure and can be used for planning of how you will achieve the goal. For example, setting a goal of *doing well in school* is a bit undefined, but making a goal of *graduating with a GPA above 3.0* is measurable and something you can work with. If your goal is measurable, you can know ahead of time how many points you will have to earn on a specific assignment to stay in that range or how many points you will need to make up in the next assignment if you do not do as well as you planned.
- Attainable—*Attainable* or *achievable* goals means they are reasonable and within your ability to accomplish. While a goal of *make an extra one million dollars by the end of the week* is something that would be nice to achieve, the odds that you could make that happen in a single week are not very realistic.

- Relevant—For goal setting, *relevant* means it applies to the situation. In relation to college, a goal of *getting a horse to ride* is not very relevant, but *getting dependable transportation* is something that would contribute to your success in school.
- Time-bound—Time-bound means you set a specific time frame to achieve the goal. *I will get my paper written by Wednesday* is time-bound. You know when you have to meet the goal. *I will get my paper written sometime soon* does not help you plan how and when you will accomplish the goal.

In the following table you can see some examples of goals that do and do not follow the SMART system. As you read each one, think about what elements make them SMART or how you might change those that are not.

Table 3.6

| Goal | Is it SMART? | |
|---|---|---|
| I am going to be rich someday. | No | There is nothing really specific, measurable, or time-bound in this goal. |
| I will graduate with my degree, on time. | Yes | The statement calls out specific, measureable, and time-bound details. The other attributes of attainable and relevant are implied. |
| I am going to save enough money to buy a newer car by June. | Yes | All SMART attributes are covered in this goal. |
| I would like to do well in all my courses next semester. | No | While this is clearly time-bound and meets most of the SMART goal attributes, it is not specific or measurable without defining what “do well” means. |
| I am going to start being a nicer person. | No | While most of the SMART attributes are implied, there is nothing really measurable in this goal. |
| I will earn at least a 3.0 GPA in all my courses next semester. | Yes | All of the SMART attributes are present in this goal. |
| I am going to start being more organized. | No | While most of the SMART attributes are implied, there is nothing really measurable in this goal. |

APPLICATION

Try writing two SMART goals—something with a one-week time frame and something that you will accomplish over the next year. Make certain that you include all the appropriate elements—Specific, Measurable, Attainable, Relevant, and Time-bound.

## Make an Action Plan

Like anything else, making a step-by-step action plan of how you will attain your goals is the best way to make certain you achieve them. It doesn’t matter if it is a smaller goal with immediate results (e.g., finish all your homework due by Friday) or something bigger that takes several years to accomplish (graduate with my degree in the proper amount of time).

The planning techniques you use for time management and achieving goals can be similar. In fact, accurate goal setting is very much a part of time management if you treat the completion of each task as a goal.

What follows is an example of a simple action plan that lists the steps for writing a short paper. You can use something like this or modify it in a way that would better suit your own preferences.

Table 3.7

| Action Plan | | |
|---|---|---|
| **Task** | **Objective** | **When** |

| Action Plan | | |
|---|---|---|
| Choose topic. | Select something interesting. | Needs to be done by Monday! |
| Write outline, look for references. | Create structure of paper and outline each part. | Monday, 6:00 p.m. |
| Research references to support outline, look for good quotes. | Strengthen paper and resources. | Tuesday, 6:00 p.m. |
| Write paper introduction and first page draft. | Get main ideas and thesis statement down. | Wednesday, 7:00 p.m. |
| Write second page and closing draft. | Finish main content and tie it all together. | Thursday, 6:00 p.m. |
| Rewrite and polish final draft. | Clean up for grammar, writing style, and effective communication. | Friday, 5:00 p.m. |

Another useful approach to goal setting is to create SMART goals and then write them down. For most people there is a higher level of commitment when we write something down. If you have your goals written out, you can refer to each component of the SMART acronym and make certain you are on track to achieve it.

## Stick with It!

As with anything else, the key to reaching goals is to keep at it, keep yourself motivated, and overcome any obstacles along the way. In the following graphic you will find seven methods that highly successful people use to accomplish this.




Figure 1.8.13 These seven ways to stay motivated are good suggestions from highly successful people. What other strategies would you suggest?

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# 1.9: Test Taking and Preparation

Students need to develop a mindset that every time they attend class or complete a reading assignment they are preparing for the next exam, because oftentimes so much of their grade rides on those exams. In college, it is not uncommon for your entire course grade to be based on two or three exams with no options for a retake or extra credit. Instead of an exam being over one or two chapters, exams are more likely to be over 5+ chapters of material, and in the case of a comprehensive exam, an entire semester's worth of material. Intentionally working through the process of preparing for, taking, and learning from the testing process will facilitate successful outcomes.



*Table 5-4. Effective Technique for Test Preparation and Taking*

| Before the Test | During the Test | After the Test |
|---|---|---|
| Go to all of your classes and take lecture notes and read all of your assigned reading using techniques described in this chapter. | Mark your exam with the necessary identification. Read all instructions on the exam. Scan the exam for question types, point values, etc. | Conduct an error analysis for items missed on the test. Look up the correct answers. Determine the nature of the errors you made. |
| Develop a study plan and schedule the time to review for an exam over several days. Refer to "Create a Five-Day Study Plan for Exams." | Develop a plan for how you will use your time to complete all questions on the exam. | Talk with your professor about items you missed on the exam. Discuss adjustments in your test preparation methods if necessary. |

| | | |
|---|---|---|
| Schedule study sessions with classmates in advance of the night before the exam. | Read each item carefully and fully before marking an answer. You might find clues to the answer and will be less likely to make a preventable mistake. | Analyze the test questions to see if patterns arise, such as,"Were most of the questions from the lecture, the book, or both?" "What types of questions were asked?" "What levels of thinking were being tested based on Bloom's Taxonomy?" etc. |
| Be sure you have materials you need for the exam ready at least the night before the exam, such as multiple writing utensils, scantrons, blue books, notebook paper, etc. | Skip difficult questions and come back to them if you are pressed for time. Caution: if using a scantron to record answers be very careful that you do not incorrectly record your answers on the wrong line and get off track. | Review the methods that you used to study for the exam and determine what adjustments you need to make in preparing for the next exam. |
| Get a good night's rest the night before the exam. | If time is available at the end of the exam, review questions you marked as "difficult" in order to check for errors. | Determine your overall grade in the course based on the new information that this grade has provided you. Talk with your professor and academic advisor if you have concerns about your overall grade. |
| Eat a good breakfast prior to the exam. Avoid caffeine prior to the exam. | If you have an essay question to write, take the time to briefly outline an answer to ensure that you are answering all aspects of the question and that you provide a well-organized answer for your faculty member to read. | Archive your exam for future reference. |
| To ensure that you are on time to your exam, set two alarm clocks or arrange to have a friend call you to make sure you are awake. | Try not to leave any answers blank. You might be awarded partial credit or even make a correct guess. | |
| | Take a moment to take some deep cleansing breaths (or any other quick relaxation techniques) if you feel stressed by a question or the exam as a whole. | |

Preparation for an exam should begin the first day of class. Repeated exposure to material over a longer period of time (Baddeley, 1982; Bahrick & Hall, 1991) optimizes the recall of information when it is needed. Every assigned reading, every opportunity to take notes and engage in class activities, and a routine of reviewing notes should be considered activities toward exam preparation. The volume of information that students are responsible for on each exam is much larger than experienced in high school. Careful review of the material prior to each exam is required for optimal performance, and student reports tell us that the habit of "cramming" the night before the exam is not going to lead to positive, long-term results. Research of the brain supports this typical finding. Physical and neurochemical changes in the synapses of the brain occur when information is learned, a process called long-term potentiation (Kandel & Schwartz, 1982). Without proper time and repeated stimulation of the neural circuits, long-term learning of this information will not occur efficiently and may not be accessible when needed for a test. Planning for a review period over five days prior to the exam and healthy eating and sleep habits cannot be overemphasized.

### Activity 5-4

Identify when your next substantial exam will be given in a class. Create a five-day study plan using the formula described in "Create a Five-Day Study Plan for Exams" and the "Five-Day Study Plan Worksheet" at the end of this chapter. Once you have created this plan, incorporate the schedule into your time management calendar for implementation.

Oftentimes, students jump right in to taking an exam feeling stressed and rushed with a need to regurgitate information immediately. However, approaching a test in a careful and methodical way can help ensure the best results. You can best utilize your exam time by taking the time to survey the exam directions and point values and planning how to approach sections of the

test. Strategy is particularly important for tests with mixed types of questions (i.e., multiple choice and essay) or tests with multiple essay questions. For ex- ample, if you were to spend too much time on the multiple choice questions and not allocate enough time to complete an essay question, you may have answered the majority of the total number of questions but still do poorly on the exam if the essay question was worth a larger portion of the exam's total points. Plan your time carefully and manage your stress.

## Create a Five-Day Study Plan for Exams

Making a plan that will allow for the careful review of all of the assigned and presented material leads to less stress the day of an exam and the best conditions for recall and performance on an exam. Here is a simple method for creating a study plan:

1. Break the material for the exam into four manageable "chunks." If material can be logically divided by chapters, use that method. If not, make up your own chunks based on the content of the material. Note: It is not a good idea to break up information by the method of delivery (i.e., one chunk equals notes, one chunk equals readings, etc.). You want to include all information on specific topics together in one chunk (e.g., information about the topic of osmosis from the lecture notes and the textbook in a chunk).
2. Plan to spend about two hours studying on each of the five days.
3. You should work with the material on the exam in two ways, by preparing that material for review and by reviewing that material. Preparation of material incorporates study techniques discussed earlier in this chapter such as combining and clarifying lecture and textbook notes, creating visual organizers for topics, developing cram cards, solving practice problems, creating self-testing questions using multiple levels of Bloom's Taxonomy, etc. Once you get the hang of preparing material for review, you could do some of that along the way (for instance creating cram cards for readings and notes) to reduce the amount of time needed for preparation at this time, leaving more time for review.

### Example of the Schedule for a Five-Day Study Plan

**Day One**
Prepare 1st Chunk – 2 hours

**Day Two**
Prepare 2nd Chunk – 2 hours
Review 1st Chunk – 30 minutes

**Day Three**
Prepare 3rd Chunk – 1.5 hours
Review 2nd Chunk – 30 minutes
Review 1st Chunk – 15 minutes

**Day Four**
Prepare 4th Chunk – 1 hour
Review 3rd Chunk – 30 minutes
Review 2nd Chunk – 15 minutes
Review 1st Chunk – 10 minutes

**Day Five**
Review 4th Chunk – 1 hour
Review 3rd Chunk – 30 minutes
Review 2nd Chunk – 30 minutes
Review 1st Chunk – 30 minutes

Taking some time to plan out what material will go into each chunk will ensure that you do not miss anything. Use the "Five-Day Study Plan Worksheet" at the end of this chapter as a guide for your planning.

This method typically works very well for courses that offer two or three tests throughout the semester; however, a similar method could be modified by the number of days or the amount of time spent each day if a course offers more exams with less material on each exam.

## Tips for Better Test Taking

The types of test items on a test are as different as the professors who created the tests. "Objective" test questions refer to items in which students have to recognize the correct answer from a list of provided options (e.g., multiple choice, true-false, matching). In

most cases there is one best answer, though it is important to note that a professor could indicate in the directions that you can select more than one answer. “Subjective” test items are the fill-in-the-blank, short answer, or essay questions in which students must recall and produce the answer.

### Multiple Choice Questions

- Read the question and all of the answer options first.
- Mark out any answers that you know are not correct.
- Once you have selected an answer, do not change it unless you misread the question and know the new answer is correct. Usually, your first answer is the correct one if you are making an educated guess.
- Many times, the correct answer has more information in it.
- If there is no penalty for guessing, do not leave any items blank—make an educated guess.
- If there is an “All of the Above” option and there are two options that are correct, select “All of the Above.”
- In a question with “All of the Above” and “None of the Above,” if you are certain that one of the answers is true, do not select “None of the Above” and, likewise, if one of the answers is false, then do not select “All of the Above.”

### True-False Questions

- Read the statement carefully, but do not read too much into the statement. Base your answer on the information provided.
- With statements that have multiple facts, if any one of the facts is false the entire statement is false.
- Qualifying words like “always,” “never,” and “every” indicate that this would have to be true all of the time. If it is not true all of the time, then you should answer false.
- Qualifying words like “usually,” “sometimes,” and “generally” indicate that it could be true or false depending on the situation. Oftentimes the answer is true.

### Essay Questions

- Most Important Rule: Make sure you understand what the question is asking you to do. An essay question is not an opportunity to simply regurgitate everything you know about a particular topic. Revisit Question Cues listed in the “Table 5-1: “Bloom’s Taxonomy: Cognitive Domain” and be sure you know what you are supposed to accomplish when you see those cues. Underline key words and let those words help guide you. If you are not sure, ask your professor for clarification on what he or she wants you to accomplish in your answer.
- Take a minute or so to plan out your answer and jot down a brief outline of key points to guide your writing before you start. A well-organized answer tends to score more points when graded.
- Formula for Essay Questions:
  - Introductory statements that define terms and describe what you will accomplish in your essay. This section should be brief.
  - State your first main idea, and then give supporting facts, examples, statistics, or details. Follow with your next main idea and continue the pattern until complete. The bulk of your time should be spent developing this section of the essay as it would likely include the details your professor wants to see in grading. Typically, each main idea would have its own paragraph. Use transitional works like “first, second, third…,” “next,” “also,” “however,” etc., to aid your reader.
  - End with a summary or final conclusion. This section is also likely to be brief.
- Budget time to proofread and revise if needed.
- Write legibly. Neater papers tend to receive higher marks.

Adapted from Dembo and Seli (2008) and TestTakingTips.com (2003–2009).

Once the exam has been completed, the learning experience is not over. Look over the test after you get it back from your professor and continue your review of the material that may be important for a comprehensive final or information that serves as a foundation for material to be learned in other courses. Even if only a scantron answer sheet is returned to you, professors are willing to give students access to review the test results during their office hours. Comprehensive post-test review not only includes reviewing your correct and incorrect answers to the question but also looking at where questions came from (i.e., lecture notes, readings, or both). Also, consider the study strategies you used and how they worked for you in order to help you prepare for future exams. Overall, it is important to emphasize the benefit of talking with your professor after each exam to gain insights into the information presented in the class as well as your learning and test strategies.

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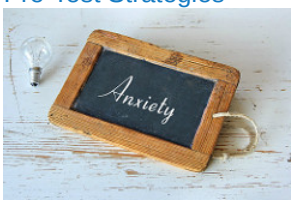
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# 1.10: Pre- Mid- and Post-Test-Taking Strategies

Phyllis Nissila

## Pre-Test Strategies




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Q: When should you start preparing for the first test? Circle…

1. The night before.
2. The week prior.
3. The first day of classes.

If you answered "3. The first day of classes," you are correct. If you circled all three, you are also correct. Preparing to pass tests is something that begins when learning begins and continues all the way through to the final exam.

Many students, however, don't start thinking about test taking, whether weekly exams, mid-terms, or finals, until the day before when they engage in an all-nighter, or cramming. From the previous unit on memory, you might recall that the brain can only process an average of 5-7 new pieces of information at a time. Additionally, unless memory devices are used to aid memory and to cement information into long term memory (or at least until the test is over tomorrow!) chances are slim that students who cram will effectively learn and remember the information.

Additionally, a lot of students are unaware of the many strategies available to help with the test-taking experience before, during, and after. For starters, take a look at what has helped you so far.

**EXERCISE 30-1**

Pre-Test Taking Strategies

PART A:

Put a check mark next to the pre-test strategies you already employ.

____ Organize your notebook and other class materials the first week of classes.

____ Maintain your organized materials throughout the term.

____ Take notes on key points from lectures and other materials.

____ Make sure you understand the information as you go along.

____ Access your instructor's help and the help of a study group, as needed.

____ Organize a study group, if desired.

____ Create study tools such as flashcards, graphic organizers, etc. as study aids.

____ Complete all homework assignments on time.

____ Review likely test items several times beforehand.

____ Ask your instructor what items are likely to be covered on the test.

____ Ask your instructor if she or he can provide a study guide or practice test.

____ Ask your instructor if he/she gives partial credit for test items such as essays.

____ Maintain an active learner attitude.

____ Schedule extra study time in the days just prior to the test.

____ Gather all notes, handouts, and other materials needed before studying.

____ Review all notes, handouts, and other materials.

____ Organize your study area for maximum concentration and efficiency.

____ Create and use mnemonic devices to aid memory.

____ Put key terms, formulas, etc., on a single study sheet that can be quickly reviewed.

____ Schedule study times short enough (1-2 hours) so you do not get burned out.

____ Get plenty of sleep the night before.

____ Set a back-up alarm in case the first alarm doesn't sound or you sleep through it.

____ Have a good breakfast with complex carbs and protein to see you through.

____ Show up 5-10 minutes early to get completely settled before the test begins.

____ Use the restroom beforehand to minimize distractions.

PART B

By reviewing the pre-test strategies, above, you have likely discovered new ideas to add to what you already use. Make a list of them.

## Mid-Test Strategies

Here is a list of the most common–and useful–strategies to survive this ubiquitous college experience.

- Scan the test, first, to get the big picture of how many test items there are, what types there are (multiple choice, matching, essay, etc.), and the point values of each item or group of items.
- Determine which way you want to approach the test: Some students start with the easy questions first, that is, the ones they immediately know the answers to, saving the difficult ones for later, knowing they can spend the remaining time on them. Some students begin with the biggest-point items first, to make sure they get the most points.
- Determine a schedule that takes into consideration how long you have to test, and the types of questions on the test. Essay questions, for example, will require more time than multiple choice, or matching questions.
- Keep your eye on the clock.
- If you can mark on the test, put a check mark next to items you are not sure of just yet. It is easy to go back and find them to answer later on. You might just find help in other test questions covering similar information.
- Sit where you are most comfortable. That said, sitting near the front has a couple of advantages: You may be less distracted by other students. If a classmate comes up with a question for the instructor and there is an important clarification given, you will be better able to hear it and apply it, if needed.
- Wear ear plugs, if noise distracts you.
- You do NOT have to start with #1! It you are unsure of it, mark it to come back to later on.
- Bring water…this helps calm the nerves, for one, and water is also needed for optimum brain function.
- If permitted, get up and stretch (or stretch in your chair) from time to time to relieve tension and assist the blood to the brain!
- Remember to employ strategies to reduce test-taking anxiety (covered in the next lesson)
- If despite all of your best efforts to prepare for a test you just cannot remember the answer to a given item for multiple choice, matching, and/or true/false questions, employ one or more of the following educated guessing (also known as "educated selection") techniques. By using these techniques, you have a *better* chance of selecting the correct answer. It is usually best to avoid selecting an extreme or all-inclusive answer (also known as 100% modifiers) such as "always," and "never". Choose, instead, words such as "usually," "sometimes," etc. (also known as in-between modifiers). If the answers are numbers, choose one of the middle numbers. If you have options such as "all of the above," or "both A and B," make sure each item is true before selecting those options. Choose the longest, or most inclusive, answer. Make sure to match the grammar of question and answer. For example, if the question indicates a plural answer, look for the plural answer. Regarding matching tests: count both sides to be matched. If there are more questions than answers, ask if you can use an answer more than once. Pay close attention to items that ask you to choose the "best" answer. This means one answer is better or more inclusive than a similar answer. Read all of the response options.

## Post-Test Strategies

In addition to sighing that big sigh of relief, here are a few suggestions to help with future tests.

- If you don't understand why you did not get an item right, ask the instructor. This is especially useful for quizzes that contain information that may be incorporated into more inclusive exams such as mid-terms and finals.
- Analyze your results to help you in the future. For example, see if most of your incorrect answers were small things such as failing to include the last step in a math item, or neglecting to double-check for simple errors in a short-answer or essay item. See where in the test you made the most errors: beginning, middle, or end. Also analyze which type of questions, true/false, multiple choice, essay, etc. And which topics were missed. This will help you pay closer attention to those sections in the future.

**EXERCISE 30-2**

Write a letter of advice to Chen incorporating 10 test-taking tips and strategies you think will help him.

Chen believes he is good at organization, and he usually is–for about the first two weeks of classes. He then becomes overwhelmed with all of the handouts and materials and tends to start slipping in the organization department. When it comes to tests, he worries that his notes might not cover all of the right topics and that he will not be able to remember all of the key terms and points–especially for his math class. During tests, he sometimes gets stuck on an item and tends to spend too much time there. He also sometimes changes answers but finds out later that his original selection was correct. Chen is also easily distracted by other students and noises which makes it hard for him to concentrate sometimes, and, unfortunately, he does admit to occasionally "cramming" the night before.

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Adaptions: Changed formatting, removed one exercise.

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# 1.11: Test-Taking Strategy Specifics

Dave Dillon

*"By failing to prepare, you are preparing to fail."*

– Benjamin Franklin

## Discipline, Preparation and Execution

From what I have seen in my own educational experiences, along with coaching and teaching at the college and university level for over 20 years, test taking (with few exceptions) comes down to discipline, preparation and execution. Students wanting to be successful have to have the self-discipline to schedule time to study well in advance of the exam. They have to actually do the work: the preparation needed in order to have the best opportunity for success on the exam. Then they must execute: they have to be able to apply their preparation accordingly and perform well on the exam.

Preparation for an exam is not glamorous. It's easy to find other things to do that are more interesting and fun. Students need to keep themselves motivated with their "eyes on the prize." Think of it like this: if the most important event of your life was coming up and you wanted to perform to the best of your ability in that event, you would likely spend some time preparing for it, rehearsing for it, practicing it, etc. A student may argue that an exam they will be taking would not be the most important event of their life, but think back to the chapter on Passion. If you're already spending the time, effort, energy and money to attend college, why not do it to the best of your ability?

It would be beneficial to spread this preparation and practice out over time and prepare periodically rather than to wait until the last minute and binge study or cram. Your preparation would not be the same and this may affect your test score. Binge studying and cramming also are not healthy. Staying up late puts stress on our brain and body, and not getting adequate sleep places our bodies at risk for getting sick.

*"One of the most important keys to success is having the discipline to do what you know you should do, even when you don't feel like doing it."*

– Unknown

*"The will to succeed is important, but what's more important is the will to prepare."*

– Bobby Knight

Everyone wants to be successful. When the exam is passed out, everyone wants to perform well. But what often separates successful students and less successful students is the preparation time put in.

Studying the right thing is a process and a skill. As you gain more experience, you will learn how to become better at knowing what to study. It can be very frustrating to spend a lot of time preparing and studying and then finding out that what you studied was not on the exam. You will see a lot of variance with exams due to different instructors, classes and types of tests. The better you become at predicting what will be on the exam and study accordingly, the better you will perform on your exams. Try placing yourself in your instructor's shoes and design questions you think your instructor would ask. It's often an eye-opening experience for students and a great study strategy.

## Preparation for Exam Strategies

### Find out as much about the exam in advance as you can.

Some professors will tell you how many questions there will be, what format the exam will be in, how much time you will have, etc., and others will not. I encourage students to ask questions about the exam if there is not information given. I also encourage students to ask those questions before class, after class, in professors' office hours or via e-mail rather than during class.

### Know the test

If you know how many questions, what the format is, and/or how much time you will have, you can start to mentally prepare for the exam much more so than if you are coming in with no information. There are two more important aspects that you may or may not know: a) what will be covered or asked on the exam; b) how the exam will be scored. Obviously, the more you know about what will be covered, the easier it is for you to be able to prepare for the exam. Most exam scoring is standardized, but not always.

Look for opportunities where some areas of the exam are worth more points than others. For example: An exam consists of 21 questions, with 10 being True/False, 10 being multiple choice, and one essay question. The T/F questions are worth 1 point each (10 points), the multiple-choice questions are worth 2 points each (20 points), and the essay question is worth 30 points. We know that the essay question is the most valuable (it is worth half of the value of the exam). And we should allocate our time for it accordingly. I would advise starting with the essay question. Do a quick analysis of time to be able to spend your time on the exam wisely. You want to spend some time with the exam question since it is so valuable, without sacrificing adequate time to ensure the T/F and multiple-choice questions are answered.

Often, the order of the exam in this scenario will be: T/F first, multiple choice second and essay third. Most students will go in the chronological order of the exam, but a savvy student would start with the essay. If an exam were to last for 30 minutes with this format of questions, I might recommend a student spent 15 minutes on the essay question, ten minutes on the multiple choice, three minutes on the T/F and two minutes reviewing their answer.

Also, look for situations where exams penalize students for incorrectly answering a question. This does not occur very often, but is the case with some exams. With the SAT for example, students are awarded one point for a correct answer and ¼ of a point subtracted for an incorrect answer. Points are not awarded nor subtracted for leaving a question blank. Thus, the strategy for a multiple-choice question is: if you can narrow down the potentially correct answer to two rather than four or five, it is statistically advantageous to answer the question and guess between the two answers; however, if a student had no idea if any of the answers were correct or incorrect, it would be best to leave the answer blank. Remember, this is rare, but it is important to understand the strategy when students take these exams.

In conclusion, the more information you have about the exam, the better you can prepare for content, allocation of time spent on aspects of the exam, and the more confident you will be in knowing how and when to attempt to answer questions.

### Take care of your body

Before the exam, it is important to prepare your brain and body for optimal performance for your exam. Do not cram the night before. Get a good night's sleep. Make sure you eat (nutritiously) before the exam. I recommend exercising the day before and if possible a few hours before the exam.

## Strategies for Specific Exam Formats

### True or False Questions

Look for qualifiers. A qualifier is a word that is absolute. Examples are: all, never, no, always, none, every, only, entirely. They are often seen in false statements. This is because it is more difficult to create a true statement using a qualifier like never, no, always, etc. For example, "All cats chase mice." Cats may be known for chasing mice, but not all of them do so. The answer here is false and the qualifier "all" gave us a tip. Qualifiers such as: sometimes, many, some, most, often, and usually are commonly found in true statements. For example: "Most cats chase mice." This is true and the qualifier "most" gave us a tip.

Make sure to read the entire statement. All parts of a sentence must be true if the whole statement is to be true. If one part of it is false, the whole sentence is false. Long sentences are often false for this reason.

Students should guess on True or False questions they do not know the answer to unless there is a penalty for an incorrect answer.

### Multiple Choice Questions

Think of multiple choice questions as four (or five) true or false statements in one. One of the statements is true (the correct answer) and the others will be false. Apply the same strategy toward qualifiers. If you see an absolute qualifier in one of the answer choices, it is probably false and not the correct answer. Try to identify the true statement. If you can do this, you have the answer as there is only one. If you cannot do this at first, try eliminating answers you know to be false.

If there is no penalty for incorrect answers, my suggestion is to guess if you are not certain of the answer. If there is a penalty for incorrect answers, common logic is to guess if you can eliminate two of the answers as incorrect (pending what the penalty is). If there's a penalty and you cannot narrow down the answers, it's best to leave it blank. You may wish to ask your instructor for clarification.

Answers that are strange and unrelated to the question are usually false. If two answers have a word that looks or sounds similar, one of those is usually correct. For example: abductor/ adductor. If you see these as two of the four or five choices, one of them is usually correct. Also look for answers that are grammatically incorrect. These are usually incorrect answers. If you have to completely guess, choose B or C. It is statistically proven to be correct more than 25 percent of the time. If there are four answers

for each question, and an exam had standardized the answers, each answer on the exam A, B, C and D would be equal. But most instructors do not standardize their answers, and more correct answers are found in the middle (B and C then the extremes A and D or E). “People writing isolated four-choice questions hide the correct answer in the two middle positions about 70% of the time.”[1] This is 20 percent more correct answers found in B or C than a standardized exam with equal correct answers for each letter.

### Matching Questions

Although less common than the other types of exams, you will likely see some matching exams during your time in college. First, read the instructions and take a look at both lists to determine what the items are and their relationship. It is especially important to determine if both lists have the same number of items and if all items are to be used, and used only once.

Matching exams become much more difficult if one list has more items than the other or if items either might not be used or could be used more than once. If your exam instructions do not discern this, you may wish to ask your instructor for further clarification. I advise students to take a look at the whole list before selecting an answer because a more correct answer may be found further into the list. Mark items when you are sure you have a match (pending the number of items in the list this may eliminate answers for the future). Guessing (if needed) should take place once you have selected answers you are certain about.

### Short-Answer Questions

Read all of the instructions first. Budget your time and then read all of the questions. Answer the ones you know best or feel the most confident with. Then go back to the other ones. If you do not know the answer and there is no penalty for incorrect answers, guess. Use common sense. Sometimes instructors will award partial credit for a logical answer that is related even if it is not the correct answer.

### Essay Questions

Keep in mind that knowing the format of the exam can help you determine how to study. If I know that I am taking a True-False exam, I know that I will need to discern whether a statement is True or False. I will need to know subject content for the course. But if I am studying for short answer and especially for essay questions, I must know a lot more. For essay questions, I must have much greater content knowledge and be able to make a coherent argument that answers the question using information from textbooks, lectures or other course materials. I have to place a lot more time and thought into studying for an essay exam than for True-False or Multiple-Choice exams.

Read the essay question(s) and the instructions first. Plan your time wisely and organize your answer before you start to write. Address the answer to the question in your first or second sentence. It may help to restate the original question. Write clearly and legibly. Instructors have difficulty grading essays that they cannot read. Save some time for review when you have finished writing to check spelling, grammar and coherent thought in your answer. Make sure you have addressed all parts of the essay question.

During the Exam:

Always read the directions first. Read them thoroughly.

Preview the exam to help you allocate proper time for each area.

Skip questions if you do not know the answer but make a mark somewhere to ensure you are able to go back to those questions (you may need to reallocate your prepared time for this depending on how many there are).

Allocate some time to review your answers before submitting your exam or the exam time expiring.

One of the biggest mistakes that students make after they take an exam in a course is that they do not use the exam for the future. The exam contains a lot of information that can be helpful in studying for future exams. Students that perform well on an exam often put it away thinking they do not need it anymore. Students who do poorly on an exam often put it away, not wanting to think about it any further.

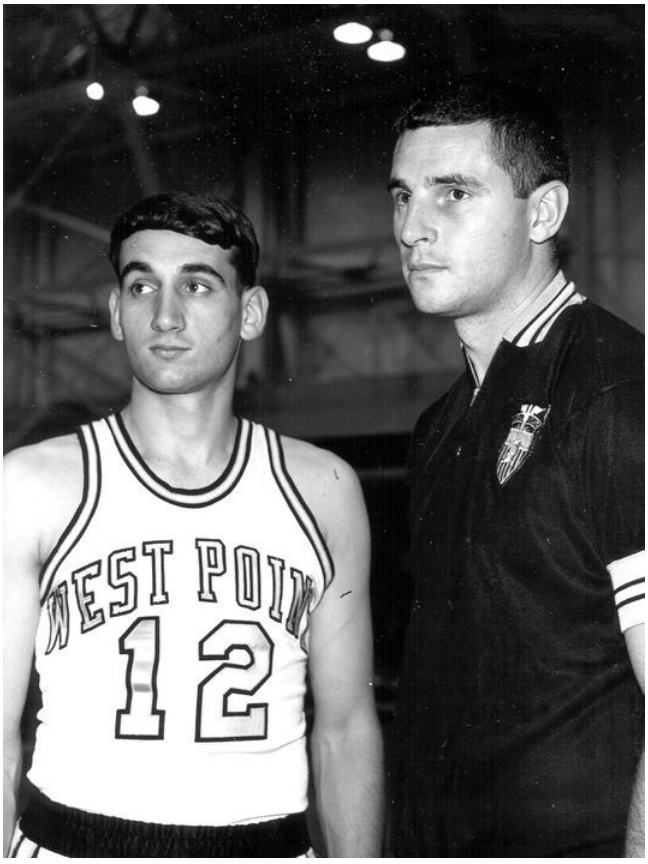
In both cases, students are missing out on the value of reviewing their exams. It is wise to review exams for three reasons: 1) students should review the answers that were correct because they may see those questions on future exams and it is important to reinforce learning; 2) students should review the answers that were incorrect in order to learn what the correct answer was and why. These questions also may appear on a future exam. In addition, occasionally an answer is marked incorrect, when it should have been marked correct. The student would never know this if they didn’t review their exam; and 3) there is value in reviewing the exam to try to predict what questions or what format will be used by a professor for a future exam in the same course.

## Author's Story

What does it mean to be a "poor test taker?" Think about that. Does it mean that the student has put effort into studying but has difficulty under pressure? Does it mean that a student studies the wrong material? Is the student prepared but does not execute well? Could the student have a learning disability? Are they missing key strategies for taking tests? It could mean any of those things. And while I believe that it may be true that a student may be a "poor test taker," it does not by any means mean it is permanent. Students willing to work hard and learn can improve their test taking skills and raise their confidence.

In high school chemistry, I earned a D grade. In microeconomics at UC Santa Cruz, I struggled and barely passed. Both of these classes had grading systems that were heavily weighted by exams. I was an above average student in my other classes in both high school and college, and so I was told (and believed for a long time) that I was not a "good test taker." I no longer believe this. I will not make excuses. The responsibility for the grade I earned rests solely with me. I mention this because many students come to my office and complain that their professor grades in a way that is unfair to them because so much emphasis is on exams and that they understand the material but are "not good test takers." It may be the case that my grade or another students' grade would have been better had the course grade been determined with less weight levied to exams. That to me is irrelevant. It does not absolve me from being responsible for knowing what the grading metrics were at the beginning of the course and choosing to continue in the course after learning this knowledge. In the end, I realized I was not a poor test taker and that I needed to spend more time studying and preparing to perform well on exams. I changed my attitude from being afraid of exams to one of believing I was going to do well regardless of the class or the format of the exam. Once I had confidence that I had the necessary tools and was willing to work hard, it changed my entire perspective and experience with exams.

Note: if you think you may have a learning disability and would like to get assessed, contact your college to see what steps to follow to begin the assessment process.



Mike Krzyzewski, point guard at Army with Bob Knight, coach at Army.

Thinking about excuses reminds me of a story about Mike Krzyzewski, the successful Duke University men's basketball coach. Before Krzyzewski coached, he was a student at the United States Military Academy at West Point. One day in his first year, Krzyzweski and his roommate were walking and his roommate stepped in a puddle. Mud splashed onto Krzyzewski's uniform. Immediately, an upperclassman screamed at Krzyzewski about not knowing the rules of wearing a clean uniform. Plebes (first-year

students at military academies) are allowed three answers when asked a question: “Yes sir!” “No sir!” And “No excuse sir!” Krzyzewski repeated himself, “No excuse sir!” Despite wanting to explain what had happened and that it was not his fault, he realized that there was no excuse. It was his responsibility to keep a clean uniform.

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1. Yigal Attali and Maya Bar-Hillel, “Guess Where: The Position of Correct Answers in Multiple-Choice Test Items as a Psychometric Variable,” *Journal of Educational Measurement* 40 no. 2 (2003):109-128. ↵

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# CHAPTER OVERVIEW

## 2: Inequalities and Functions


2.0: Prelude to Inequalities and Functions
2.1: Equations and Inequalities
2.1.0: Extracting Square Roots and Completing the Square (Review)
2.1.1: Solving Equations Quadratic in Form (Review)
2.1.2: Quadratic Functions and Their Graphs (Review)
2.1.3: Solving Quadratic Inequalities
2.1.4: Solving Polynomial and Rational Inequalities
2.1.E: Solving Equations and Inequalities (Exercises)
2.2: Functions and Function Notation
2.3: Domain and Range
2.4: Rates of Change and Behavior of Graphs
2.5: Transformation of Functions
2.6: Composition of Functions
2.7: Absolute Value Functions
2.8: Inverse Functions
2.E: Functions (Exercises)
2.R: Functions (Review)


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# 2.0: Prelude to Inequalities and Functions

Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. The *Standard and Poor's Index* tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1100 by the beginning of 2000. That five-year period became known as the "dot-com bubble" because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

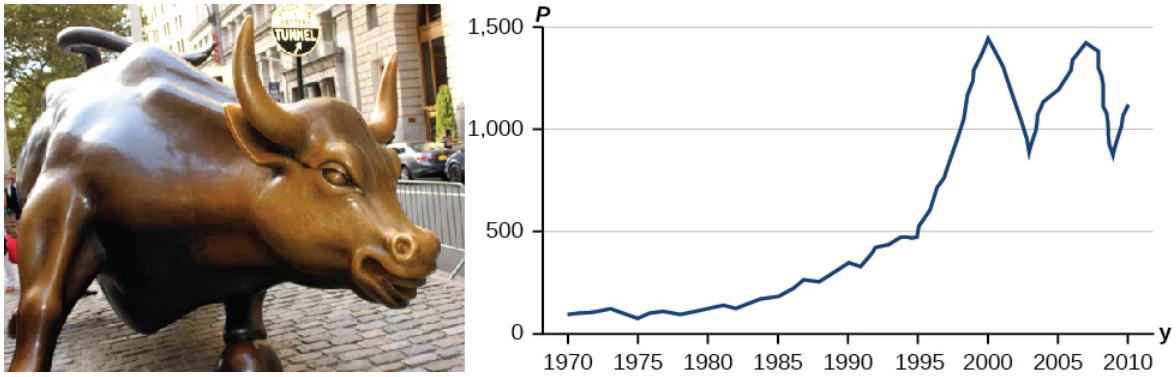



Figure 2.0.1: Figure of a Bull and a graph of market prices. Standard and Poor's Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

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# SECTION OVERVIEW

## 2.1: Equations and Inequalities


2.1.0: Extracting Square Roots and Completing the Square (Review)

2.1.1: Solving Equations Quadratic in Form (Review)

2.1.2: Quadratic Functions and Their Graphs (Review)

2.1.3: Solving Quadratic Inequalities

2.1.4: Solving Polynomial and Rational Inequalities

2.1.E: Solving Equations and Inequalities (Exercises)


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# 2.1.0: Extracting Square Roots and Completing the Square (Review)

**Learning Objectives**

- Solve certain quadratic equations by extracting square roots.
- Solve any quadratic equation by completing the square.

## Extracting Square Roots

Recall that a quadratic equation is in **standard form**[1] if it is equal to 0:

$$ax^2+bx+c=0$$

where $a, b,$ and $c$ are real numbers and $a \neq 0$. A solution to such an equation is a root of the quadratic function defined by $f(x)=ax^2+bx+c$ . Quadratic equations can have two real solutions, one real solution, or no real solution—in which case there will be two complex solutions. If the quadratic expression factors, then we can solve the equation by factoring. For example, we can solve $4x^2-9=0$ by factoring as follows:

$$\begin{aligned} 4x^2-9 &= 0 \\ (2x+3)(2x-3) &= 0 \\ 2x+3=0 \quad &\text{or} \quad 2x-3=0 \\ 2x=-3 \quad &\quad 2x=3 \\ x=-\tfrac{3}{2} \quad &\quad x=\tfrac{3}{2} \end{aligned}$$

The two solutions are $\pm\frac{3}{2}$. Here we use $\pm$ to write the two solutions in a more compact form. The goal in this section is to develop an alternative method that can be used to easily solve equations where $b=0$, giving the form

$$ax^2+c=0$$

The equation $4x^2-9=0$ is in this form and can be solved by first isolating $x^2$.

$$\begin{aligned} 4x^2-9 &= 0 \\ 4x^2 &= 9 \\ x^2 &= \frac{9}{4} \end{aligned}$$

If we take the square root of both sides of this equation, we obtain the following:

$$\begin{aligned} \sqrt{x^2} &= \sqrt{\frac{9}{4}} \\ |x| &= \frac{3}{2} \end{aligned}$$

Here we see that $x=\pm\frac{3}{2}$ are solutions to the resulting equation. In general, this describes the **square root property**[2]; for any real number $k$,

$$\text{If} \quad x^2=k, \quad \text{then} \quad x=\pm\sqrt{k}$$

Applying the square root property as a means of solving a quadratic equation is called **extracting the root**[3]. This method allows us to solve equations that do not factor.

**Example 2.1.0.1:**

Solve: $9x^2-8=0$ .

**Solution**

Notice that the quadratic expression on the left does not factor. However, it is in the form $ax^2+c=0$ and so we can solve it by extracting the roots. Begin by isolating $x^2$.

$$\begin{aligned} 9x^2 - 8 &= 0 \\ 9x^2 &= 8 \\ x^2 &= \frac{8}{9} \end{aligned}$$

Next, apply the square root property. Remember to include the $\pm$ and simplify.

$$\begin{aligned} x &= \pm\sqrt{\frac{8}{9}} \\ &= \pm\frac{2\sqrt{2}}{3} \end{aligned}$$

For completeness, check that these two real solutions solve the original quadratic equation.

Table 2.1.0.1

| Check $x = -\frac{2\sqrt{2}}{3}$ | Check $x = \frac{2\sqrt{2}}{3}$ |
|---|---|
| $9x^2 - 8 = 0$<br>$9\left(-\frac{2\sqrt{2}}{3}\right)^2 - 8 = 0$<br>$9\left(\frac{4.2}{9}\right) - 8 = 0$<br>$8 - 8 = 0$<br>$0 = 0$ ✓ | $9x^2 - 8 = 0$<br>$9\left(\frac{2\sqrt{2}}{3}\right)^2 - 8 = 0$<br>$9\left(\frac{4.2}{9}\right) - 8 = 0$<br>$8 - 8 = 0$<br>$0 = 0$ ✓ |

**Answer**:

Two real solutions, $\pm\frac{2\sqrt{2}}{3}$

Sometimes quadratic equations have no real solution. In this case, the solutions will be complex numbers.

**Example 2.1.0.2:**

Solve: $x^2 + 25 = 0$ .

**Solution**

Begin by isolating $x^2$ and then apply the square root property.

$$\begin{aligned} x^2 + 25 &= 0 \\ x^2 &= -25 \\ x &= \pm\sqrt{-25} \end{aligned}$$

After applying the square root property, we are left with the square root of a negative number. Therefore, there is no real solution to this equation; the solutions are complex. We can write these solutions in terms of the imaginary unit $i = \sqrt{-1}$.

$$\begin{aligned} x &= \pm\sqrt{-25} \\ &= \pm\sqrt{-1 \cdot 25} \\ &= \pm i \cdot 5 \\ &= \pm 5i \end{aligned}$$

Table 2.1.0.2

| Check $x = -5i$ | Check $x = 5i$ |
|---|---|

$$\begin{aligned} x^2+25&=0\\ (-5i)^2+25&=0\\ 25i^2+25&=0\\ 25(-1)+25&=0\\ -25+25&=0\\ 0&=0\ \checkmark \end{aligned}$$

$$\begin{aligned} x^2+25&=0\\ (-5i)^2+25&=0\\ 25i^2+25&=0\\ 25(-1)+25&=0\\ -25+25&=0\\ 0&=0\ \checkmark \end{aligned}$$

**Answer**:

Two complex solutions, $\pm 5i$.

**Exercise 2.1.0.1**

Solve: $2x^2-3=0$ .

**Answer**

The solutions are $\pm\frac{\sqrt{6}}{2}$ .

www.youtube.com/v/9ff7QGhFytQ

Consider solving the following equation:

$$(x+5)^2=9$$

To solve this equation by factoring, first square $x+5$ and then put the equation in standard form, equal to zero, by subtracting 9 from both sides.

$$\begin{aligned} (x+5)^2&=9\\ x^2+10x+25&=9\\ x^2+10x+16&=0 \end{aligned}$$

Factor and then apply the zero-product property.

$$\begin{aligned} x^2+10x+16&=0\\ (x+8)(x+2)&=0\\ x+8=0 \quad &\text{or} \quad x+2=0\\ x=-8 \quad &\quad\ \ x=-2 \end{aligned}$$

The two solutions are $-8$ and $-2$. When an equation is in this form, we can obtain the solutions in fewer steps by extracting the roots.

**Example 2.1.0.3:**

Solve by extracting roots: $(x+5)^2=9$ .

**Solution**

The term with the square factor is isolated so we begin by applying the square root property.

$$\begin{aligned} (x+5)^2&=9 && \textit{Apply the square root property.}\\ x+5&=\pm\sqrt{9} && \textit{Simplify.}\\ x+5&=\pm\sqrt{9}\\ x&=-5\pm 3 \end{aligned}$$

At this point, separate the “plus or minus” into two equations and solve each individually.

$$\begin{aligned} x=-5+3 \quad &\text{or} \quad x=-5-3\\ x=-2 \quad &\quad\ \ x=-8 \end{aligned}$$

**Answer**:

The solutions are $-2$ and $-8$.

In addition to fewer steps, this method allows us to solve equations that do not factor.

**Example 2.1.0.4:**

Solve: $2(x-2)^2-5=0$ .

**Solution**

Begin by isolating the term with the square factor.

$$\begin{aligned} 2(x-2)^2-5 &= 0 \\ 2(x-2)^2 &= 5 \\ (x-2)^2 &= \frac{5}{2} \end{aligned}$$

Next, extract the roots, solve for $x$, and then simplify.

$$\begin{aligned} x-2 &= \pm\sqrt{\frac{5}{2}} \qquad \textit{Rationalize the denominator.} \\ x &= 2 \pm \frac{\sqrt{5}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ x &= 2 \pm \frac{\sqrt{10}}{2} \\ x &= \frac{4 \pm \sqrt{10}}{2} \end{aligned}$$

**Answer**:

The solutions are $\frac{4-\sqrt{10}}{2}$ and $\frac{4+\sqrt{10}}{2}$.

**Exercise 2.1.0.2**

Solve: $2(3x-1)^2+9=0$ .

**Answer**

The solutions are $\frac{1}{3} \pm \frac{\sqrt{2}}{2}i$ .

www.youtube.com/v/RQrrtff3rw0

## Completing the Square

In this section, we will devise a method for rewriting any quadratic equation of the form

$$ax^2+bx+c=0$$

as an equation of the form

$$(x-p)^2=q$$

This process is called **completing the square**[4]. As we have seen, quadratic equations in this form can be easily solved by extracting roots. We begin by examining perfect square trinomials:

$$\begin{array}{rcccc} (x+3)^2 = x^2 + & 6x & + & 9 \\ & \downarrow & & \uparrow \\ & \left(\frac{6}{2}\right)^2 = (3)^2 = & & 9 \end{array}$$

The last term, 9, is the square of one-half of the coefficient of $x$. In general, this is true for any perfect square trinomial of the form $x^2+bx+c$ .

$$\left(x+\frac{b}{2}\right)^2 = x^2 + 2\cdot\frac{b}{2}x + \left(\frac{b}{2}\right)^2$$
$$= x^2 + bx + \left(\frac{b}{2}\right)^2$$

In other words, any trinomial of the form $x^2+bx+c$ will be a perfect square trinomial if

$$c = \left(\tfrac{b}{2}\right)^2$$

**Note**

It is important to point out that the leading coefficient must be equal to 1 for this to be true.

**Example 2.1.0.5:**

Complete the square: $x^2-6x+?=(x+?)^2$ .

**Solution**

In this example, the coefficient $b$ of the middle term is $-6$. Find the value that completes the square as follows:

$$\left(\tfrac{b}{2}\right)^2 = \left(\tfrac{-6}{2}\right)^2 = (-3)^2 = 9$$

The value that completes the square is $9$.

$$x^2-6x+9 = (x-3)(x-3)$$
$$= (x-3)^2$$

**Answer**:

$x^2-6x+9=(x-3)^2$

**Example 2.1.0.6:**

Complete the square: $x^2+x+?=(x+?)^2$ .

**Solution**

Here $b=1$. Find the value that will complete the square as follows:

$$\left(\tfrac{b}{2}\right)^2 = \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

The value $\frac{1}{4}$ completes the square:

$$x^2+x+\frac{1}{4} = \left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)$$
$$= \left(x+\frac{1}{2}\right)^2$$

**Answer**:

$x^2+x+\frac{1}{4} = \left(x+\frac{1}{2}\right)^2$

We can use this technique to solve quadratic equations. The idea is to take any quadratic equation in standard form and complete the square so that we can solve it by extracting roots. The following are general steps for solving a quadratic equation with leading coefficient 1 in standard form by completing the square.

**Example 2.1.0.7:**

Solve by completing the square: $x^2-8x-2=0$ .

**Solution**

It is important to notice that the leading coefficient is 1.

**Step 1**: Add or subtract the constant term to obtain an equation of the form $x^2+bx=c$. Here we add 2 to both sides of the equation.

$$\begin{aligned} x^2-8x-2 &= 0 \\ x^2-8x &= 2 \end{aligned}$$

**Step 2**: Use $\left(\frac{b}{2}\right)^2$ to determint the value that completes the square. In this case, $b=-8$.

$$\left(\tfrac{b}{2}\right)^2=\left(\tfrac{-8}{2}\right)^2=(-4)^2=16$$

**Step 3**: Add $\left(\frac{b}{2}\right)^2$ to both sides of the equation and complete the square.

$$\begin{aligned} x^2-8x &= 2 \\ x^2-8x+16 &= 2+16 \\ (x-4)(x-4) &= 18 \\ (x-4)^2 &= 18 \end{aligned}$$

**Step 4**: Solve by extracting roots.

$$\begin{aligned} (x-4)^2 &= 18 \\ x-4 &= \pm\sqrt{18} \\ x &= 4\pm\sqrt{9\cdot 2} \\ x &= 4\pm 3\sqrt{2} \end{aligned}$$

**Answer**:

The solutions are $4-3\sqrt{2}$ and $4+3\sqrt{2}$. The check is left to the reader.

## Example 2.1.0.8:

Solve by completing the square: $x^2+2x-48=0$ .

**Solution**

Begin by adding 48 to both sides

$$\begin{aligned} x^2+2x-48 &= 0 \\ x^2+2x &= 48 \end{aligned}$$

Next, find the value that completes the square using $b=2$.

$$\left(\tfrac{b}{2}\right)^2=\left(\tfrac{2}{2}\right)^2=(1)^2=1$$

To complete the square, add 1 to both sides, complete the square, and then solve by extracting the roots.

$$\begin{aligned} x^2+2x &= 48 && \textit{Complete the square.} \\ x^2+2x+1 &= 48+1 \\ (x+1)(x+1) &= 49 \\ (x+1)^2 &= 49 && \textit{Extract the roots.} \\ x+1 &= \pm\sqrt{49} \\ x+1 &= \pm 7 \\ x &= -1\pm 7 \end{aligned}$$

At this point, separate the “plus or minus” into two equations and solve each individually.

$$\begin{aligned} x=-1-7 \text{ or } x=-1+7 \\ x=-8 \qquad x=6 \end{aligned}$$

**Answer**:

The solutions are $-8$ and 6.

**Note**

In the previous example the solutions are integers. If this is the case, then the original equation will factor.

$$x^2+2x-48=0$$
$$(x-6)(x+8)=0$$

If an equation factors, we can solve it by factoring. However, not all quadratic equations will factor. Furthermore, equations often have complex solutions.

**Example 2.1.0.9:**

Solve by completing the square: $x^2-10x+26=0$ .

**Solution**

Begin by subtracting $26$ from both sides of the equation.

$$\begin{aligned} x^2-10x+26 &= 0 \\ x^2-10x &= -26 \end{aligned}$$

Here $b=-10$, and we determine the value that completes the square as follows:

$$\left(\tfrac{b}{2}\right)^2=\left(\tfrac{-10}{2}\right)^2=(-5)^2=25$$

To complete the square, add $25$ to both sides of the equation.

$$\begin{aligned} x^2-10x &= -26 \\ x^2-10x+25 &= -26+25 \\ x^2-10x+25 &= -1 \end{aligned}$$

Factor and then solve by extracting roots.

$$\begin{aligned} x^2-10x+25 &= -1 \\ (x-5)(x-5) &= -1 \\ (x-5)^2 &= -1 \\ x-5 &= \pm\sqrt{-1} \\ x-5 &= \pm i \\ x &= 5\pm i \end{aligned}$$

**Answer**:

The solutions are $5\pm i$ .

**Exercise 2.1.0.3**

Solve by completing the square: $x^2-2x-17=0$ .

**Answer**

The solutions are $x=1\pm 3\sqrt{2}$ .

www.youtube.com/v/i8WVWpe-Ct0

The coefficient of $x$ is not always divisible by $2$.

**Example 2.1.0.10:**

Solve by completing the square: $x^2+3x+4=0$ .

**Solution**

Begin by subtracting $4$ from both sides.

$$x^2+3x+4 = 0$$
$$x^2+3x = -4$$

Use $b=3$ to find the value that completes the square:

$$\left(\tfrac{b}{2}\right)^2 = \left(\tfrac{3}{2}\right)^2 = \tfrac{9}{4}$$

To complete the square, add $\frac{9}{4}$ to both sides of the equation.

$$x^2+3x = -4$$
$$x^2+3x+\frac{9}{4} = -4+\frac{9}{4}$$
$$\left(x+\frac{3}{2}\right)\left(x+\frac{3}{2}\right) = \frac{-16}{4}+\frac{9}{4}$$
$$\left(x+\frac{3}{2}\right)^2 = \frac{-7}{4}$$

Solve by extracting roots.

$$\left(x+\frac{3}{2}\right)^2 = -\frac{7}{4}$$
$$x+\frac{3}{2} = \pm\sqrt{\frac{-1\cdot 7}{4}}$$
$$x+\frac{3}{2} = \pm\frac{i\sqrt{7}}{2}$$
$$x = -\frac{3}{2}\pm\frac{\sqrt{7}}{2}i$$

**Answer**:

The solutions are $-\frac{3}{2}\pm\frac{\sqrt{7}}{2}i$.

So far, all of the examples have had a leading coefficient of 1. The formula $\left(\frac{b}{2}\right)^2$ determines the value that completes the square only if the leading coefficient is 1. If this is not the case, then simply divide both sides by the leading coefficient before beginning the steps outlined for completing the square.

**Example 2.1.0.11:**

Solve by completing the square: $2x^2+5x-1=0$ .

**Solution**

Notice that the leading coefficient is 2. Therefore, divide both sides by 2 before beginning the steps required to solve by completing the square.

$$\tfrac{2x^2+5x-1}{2} = \tfrac{0}{2}$$
$$\tfrac{2x^2}{2}+\tfrac{5x}{2}-\tfrac{1}{2} = 0$$
$$x^2+\tfrac{5}{2}x-\tfrac{1}{2} = 0$$

Add $\frac{1}{2}$ to both sides of the equation.

$$x^2+\tfrac{5}{2}x-\tfrac{1}{2} = 0$$
$$x^2+\tfrac{5}{2}x = \tfrac{1}{2}$$

Here $b=\frac{5}{2}$, and we can find the value that completes the square as follows:

$$\left(\tfrac{b}{2}\right)^2 = \left(\tfrac{5/2}{2}\right)^2 = \left(\tfrac{5}{2}\cdot\tfrac{1}{2}\right)^2 = \left(\tfrac{5}{4}\right)^2 = \tfrac{25}{16}$$

To complete the square, add $\frac{25}{16}$ to both sides of the equation.

$$\begin{aligned} x^2+\frac{5}{2}x &= \frac{1}{2} \\ x^2+\frac{5}{2}x+\frac{25}{16} &= \frac{1}{2}+\frac{25}{16} \\ \left(x+\frac{5}{4}\right)\left(x+\frac{5}{4}\right) &= \frac{8}{16}+\frac{25}{16} \\ \left(x+\frac{5}{4}\right)^2 &= \frac{33}{16} \end{aligned}$$

Next, solve by extracting roots.

$$\begin{aligned} \left(x+\frac{5}{4}\right)^2 &= \frac{33}{16} \\ x+\frac{5}{4} &= \pm\sqrt{\frac{33}{16}} \\ x+\frac{5}{4} &= \pm\frac{\sqrt{33}}{4} \\ x &= -\frac{5}{4}\pm\frac{\sqrt{33}}{4} \\ x &= \frac{-5\pm\sqrt{33}}{4} \end{aligned}$$

**Answer**:

The solutions are $\frac{-5\pm\sqrt{33}}{4}$.

**Exercise 2.1.0.4**

Solve by completing the square: $3x^2-2x+1=0$ .

**Answer**

The solutions are $x=\frac{1}{3}\pm\frac{\sqrt{2}}{3}i$

www.youtube.com/v/A-i6LKqNmtY

## Key Takeaways

- Solve equations of the form $ax^2+c=0$ by extracting the roots.
- Extracting roots involves isolating the square and then applying the square root property. Remember to include "$\pm$" when taking the square root of both sides.
- After applying the square root property, solve each of the resulting equations. Be sure to simplify all radical expressions and rationalize the denominator if necessary.
- Solve any quadratic equation by completing the square.
- You can apply the square root property to solve an equation if you can first convert the equation to the form $(x-p)^2=q$ .
- To complete the square, first make sure the equation is in the form $x^2+bx=c$ . The leading coefficient must be 1. Then add the value $\left(\frac{b}{2}\right)^2$ to both sides and factor.
- The process for completing the square always works, but it may lead to some tedious calculations with fractions. This is the case when the middle term, $b$, is not divisible by 2.

**Exercise 2.1.0.5**

Solve by factoring and then solve by extracting roots. Check answers.

1. $x^2-16=0$
2. $x^2-36=0$
3. $9y^2-1=0$

4. $4y^2 - 25 = 0$
5. $(x-2)^2 - 1 = 0$
6. $(x+1)^2 - 4 = 0$
7. $4(y-2)^2 - 9 = 0$
8. $9(y+1)^2 - 4 = 0$
9. $(u-5)^2 - 25 = 0$
10. $(u+2)^2 - 4 = 0$

**Answer**

1. $-4, 4$
3. $-\frac{1}{3}, \frac{1}{3}$
5. $1, 3$
7. $\frac{1}{2}, \frac{7}{2}$
9. \(0,10)

## Exercise 2.1.0.6

Solve by extracting the roots.

1. $x^2 = 81$
2. $x^2 = 1$
3. $y^2 = \frac{1}{9}$
4. $y^2 = \frac{1}{16}$
5. $x^2 = 12$
6. $x^2 = 18$
7. $16x^2 = 9$
8. $4x^2 = 25$
9. $2t^2 = 1$
10. $3t^2 = 2$
11. $x^2 - 40 = 0$
12. $x^2 - 24 = 0$
13. $x^2 + 1 = 0$
14. $x^2 + 100 = 0$
15. $5x^2 - 1 = 0$
16. $6x^2 - 5 = 0$
17. $8x^2 + 1 = 0$
18. $12x^2 + 5 = 0$
19. $y^2 + 4 = 0$
20. $y^2 + 1 = 0$
21. $x^2 - \frac{4}{9} = 0$
22. $x^2 - \frac{9}{25} = 0$
23. $x^2 - 8 = 0$
24. $t^2 - 18 = 0$
25. $x^2 + 8 = 0$
26. $x^2 + 125 = 0$
27. $5y^2 - 2 = 0$
28. $3x^2 - 1 = 0$
29. $(x+7)^2 - 4 = 0$
30. $(x+9)^2 - 36 = 0$
31. $(x-5)^2 - 20 = 0$
32. $(x+1)^2 - 28 = 0$

33. $(3t+2)^2+6=0$
34. $(3t-5)^2+10=0$
35. $4(3x+1)^2-27=0$
36. $9(2x-3)^2-8=0$
37. $2(3x-1)^2+3=0$
38. $5(2x-1)^2+2=0$
39. $3\left(y-\frac{2}{3}\right)^2-\frac{3}{2}=0$
40. $2\left(3y-\frac{1}{3}\right)^2-\frac{5}{2}=0$
41. $-3(t-1)^2+12=0$
42. $-2(t+1)^2+8=0$
43. Solve for $x: px^2-q=0, p, q>0$
44. Solve for $x (x-p)^2-q=0, p, q>0$
45. The diagonal of a square measures $3$ centimeters. Find the length of each side.
46. The length of a rectangle is twice its width. If the diagonal of the rectangle measures $10$ meters, then find the dimensions of the rectangle.
47. If a circle has an area of $50\pi$ square centimeters, then find its radius.
48. If a square has an area of $27$ square centimeters, then find the length of each side.
49. The height in feet of an object dropped from an $18$-foot stepladder is given by $h(t)=-16t^2+18$, where $t$ represents the time in seconds after the object is dropped. How long does it take the object to hit the ground? (Hint: The height is $0$ when the object hits the ground. Round to the nearest hundredth of a second.)
50. The height in feet of an object dropped from a $50$-foot platform is given by $h(t)=-16t^2+50$, where $t$ represents the time in seconds after the object is dropped. How long does it take the object to hit the ground? (Round to the nearest hundredth of a second.)
51. How high does a $22$-foot ladder reach if its base is $6$ feet from the building on which it leans? Round to the nearest tenth of a foot.
52. The height of a triangle is $\frac{1}{2}$ the length of its base. If the area of the triangle is $72$ square meters, find the exact length of the triangle's base.

**Answer**

1. $\pm 9$
3. $\pm\frac{1}{3}$
5. $\pm 2\sqrt{3}$
7. $\pm\frac{3}{4}$
9. $\pm\frac{\sqrt{2}}{2}$
11. $\pm 2\sqrt{10}$
13. $\pm i$
15. $\pm\frac{\sqrt{5}}{5}$
17. $\pm\frac{\sqrt{2}}{4}i$
19. $\pm 2i$
21. $\pm\frac{2}{3}$
23. $\pm 2\sqrt{2}$
25. $\pm 2i\sqrt{2}$
27. $\pm\frac{\sqrt{10}}{5}$
29. $-9, -5$

31. $5 \pm 2\sqrt{5}$
33. $-\frac{2}{3} \pm \frac{\sqrt{6}}{3} i$
35. $\frac{-2 \pm 3\sqrt{3}}{6}$
37. $\frac{1}{3} \pm \frac{\sqrt{6}}{6} i$
39. $\frac{4 \pm 3\sqrt{2}}{6}$
41. $-1, -3$
43. $x = \pm \frac{\sqrt{pq}}{p}$
45. $\frac{3\sqrt{2}}{2}$ centimeters
47. $5\sqrt{2}$ centimeters
49. $1.06$seconds
51. $21.2$feet

## Exercise 2.1.0.7

Complete the square.

1. $x^2 - 2x + ? = (x - ?)^2$
2. $x^2 - 4x + ? = (x - ?)^2$
3. $x^2 + 10x + ? = (x + ?)^2$
4. $x^2 + 12x + ? = (x + ?)^2$
5. $x^2 + 7x + ? = (x + ?)^2$
6. $x^2 + 5x + ? = (x + ?)^2$
7. $x^2 - x + ? = (x - ?)^2$
8. $x^2 - \frac{1}{2}x + ? = (x - ?)^2$
9. $x^2 + \frac{2}{3}x + ? = (x + ?)^2$
10. $x^2 + \frac{4}{5}x + ? = (x + ?)^2$

**Answer**

1. $x^2 - 2x + 1 = (x - 1)^2$
3. $x^2 + 10x + 25 = (x + 5)^2$
5. $x^2 + 7x + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$
7. $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$
9. $x^2 + \frac{2}{3}x + \frac{1}{9} = \left(x + \frac{1}{3}\right)^2$

## Exercise 2.1.0.8

Solve by factoring and then solve by completing the square. Check answers.

1. $x^2 + 2x - 8 = 0$
2. $x^2 - 8x + 15 = 0$
3. $y^2 + 2y - 24 = 0$
4. $y^2 - 12y + 11 = 0$
5. $t^2 + 3t - 28 = 0$
6. $t^2 - 7t + 10 = 0$

7. $2x^2+3x-2=0$
8. $3x^2-x-2=0$
9. $2y^2-y-1=0$
10. $2y^2+7y-4=0$

**Answer**

1. $-4, 2$
3. $-6, 4$
5. $-7, 4$
7. $-2, \frac{1}{2}$
9. $-\frac{1}{2}, 1$

## Exercise 2.1.0.9

Solve by completing the square.

1. $x^2+6x-1=0$
2. $x^2+8x+10=0$
3. $x^2-2x-7=0$
4. $x^2-6x-3=0$
5. $y^2-2y+4=0$
6. $y^2-4y+9=0$
7. $t^2+10t-75=0$
8. $t^2+12t-108=0$
9. $u^2-\frac{2}{3}u-\frac{1}{3}=0$
10. $u^2-\frac{4}{5}u-\frac{1}{5}=0$
11. $x^2+x-1=0$
12. $x^2+x-3=0$
13. $y^2+3y-2=0$
14. $y^2+5y-3=0$
15. $x^2+3x+5=0$
16. $x^2+x+1=0$
17. $x^2-7x+\frac{11}{2}=0$
18. $x^2-9x+\frac{3}{2}=0$
19. $t^2-\frac{1}{2}t-1=0$
20. $t^2-\frac{1}{3}t-2=0$
21. $4x^2-8x-1=0$
22. $2x^2-4x-3=0$
23. $3x^2+6x+1=0$
24. $5x^2+10x+2=0$
25. $3x^2+2x-3=0$
26. $5x^2+2x-5=0$
27. $4x^2-12x-15=0$
28. $2x^2+4x-43=0$
29. $2x^2-4x+10=0$
30. $6x^2-24x+42=0$
31. $2x^2-x-2=0$
32. $2x^2+3x-1=0$
33. $3u^2+2u-2=0$
34. $3u^2-u-1=0$

35. $x^2-4x-1=15$
36. $x^2-12x+8=-10$
37. $x(x+1)-11(x-2)=0$
38. $(x+1)(x+7)-4(3x+2)=0$
39. $y^2=(2y+3)(y-1)-2(y-1)$
40. $(2y+5)(y-5)-y(y-8)=-24$
41. $(t+2)^2=3(3t+1)$
42. $(3t+2)(t-4)-(t-8)=1-10t$

**Answer**

1. $-3\pm\sqrt{10}$
3. $1\pm2\sqrt{2}$
5. $1\pm i\sqrt{3}$
7. $-15, 5$
9. $-\frac{1}{3}, 1$
11. $\frac{-1\pm\sqrt{5}}{2}$
13. $\frac{-3\pm\sqrt{17}}{2}$
15. $-\frac{3}{2}\pm\frac{\sqrt{11}}{2}i$
17. $\frac{7\pm3\sqrt{3}}{2}$
19. $\frac{1\pm\sqrt{17}}{4}$
21. $\frac{2\pm\sqrt{5}}{2}$
23. $\frac{-3\pm\sqrt{6}}{3}$
25. $\frac{-1\pm\sqrt{10}}{3}$
27. $\frac{3\pm2\sqrt{6}}{2}$
29. $1\pm2i$
31. $\frac{1\pm\sqrt{17}}{4}$
33. $\frac{-1\pm\sqrt{7}}{3}$
35. $2\pm2\sqrt{5}$
37. $5\pm\sqrt{3}$
39. $\frac{1\pm\sqrt{5}}{2}$
41. $\frac{5\pm\sqrt{21}}{2}$

## Exercise 2.1.0.10

Solve by completing the square and round the solutions to the nearest hundredth.

1. $(2x-1)^2=2x$
2. $(3x-2)^2=5-15x$
3. $(2x+1)(3x+1)=9x+4$
4. $(3x+1)(4x-1)=17x-4$
5. $9x(x-1)-2(2x-1)=-4x$

6. $(6x+1)^2-6(6x+1)=0$

**Answer**

1. $0.19, 1.31$
3. $-0.45, 1.12$
5. $0.33, 0.67$

**Exercise 2.1.0.11**

1. Create an equation of your own that can be solved by extracting the roots. Share it, along with the solution, on the discussion board.
2. Explain why the technique of extracting roots greatly expands our ability to solve quadratic equations.
3. Explain why the technique for completing the square described in this section requires that the leading coefficient be equal to 1.
4. Derive a formula for the diagonal of a square in terms of its sides.

**Answer**

1. Answer may vary
3. Answer may vary

## Footnotes

[1]Any quadratic equation in the form $ax^2+bx+c=0$ , where $a$, $b$, and $c$ are real numbers and $a\neq 0$.

[2]For any real number $k$, if $x^2=k$, then $x=\pm\sqrt{k}$.

[3]Applying the square root property as a means of solving a quadratic equation.

[4]The process of rewriting a quadratic equation to be in the form $(x-p)^2=q$.

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# 2.1.1: Solving Equations Quadratic in Form (Review)

**Learning Objectives**

- Develop a general strategy for solving quadratic equations.
- Solve equations that are quadratic in form.

Use the coefficients of a quadratic equation to help decide which method is most appropriate for solving it. While the quadratic formula always works, it is sometimes not the most efficient method. If given any quadratic equation in standard form,

$$ax^2+bx+c=0$$

where $c=0$, then it is best to factor out the GCF and solve by factoring.

**Example 2.1.1.1:**

Solve: $12x^2-3x=0$

**Solution**

In this case, $c=0$ and we can solve by factoring out the GCF $3x$.

$$\begin{aligned} 12x^2-3x&=0\\ 3x(4x-1)&=0 \end{aligned}$$

Then apply the zero-product property and set each factor equal to zero.

$$\begin{aligned} 3x&=0 \quad \text{or} \quad & 4x-1&=0\\ x&=0 & 4x&=1\\ & & x&=\frac{1}{4} \end{aligned}$$

**Answer**

The solutions are $0$ and $\frac{1}{4}$.

If $b=0$, then we can solve by extracting the roots.

**Example 2.1.1.2:**

Solve $5x^2+8=0$

**Solution**

In this case, $b=0$ and we can solve by extracting the roots. Begin by isolating the square.

$$\begin{aligned} 5x^2+8&=0\\ 5x^2&=-8\\ x^2&=-\frac{8}{5} \end{aligned}$$

Next, apply the square root property. Remember to include the $\pm$.

$$\begin{aligned} x&=\pm\sqrt{-\frac{8}{5}} && \textit{Rationalize the denominator.}\\ &=\pm\frac{\sqrt{-4\cdot 2}}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} && \textit{Simplify.}\\ &=\pm\frac{2i\sqrt{2}\cdot\sqrt{5}}{5}\\ &=\pm\frac{2i\sqrt{10}}{5} \end{aligned}$$

**Answer**

The solutions are $\pm\frac{2i\sqrt{10}}{5}$.

**Howto: General Guidelines for Solving Quadratic Equations**

When given a quadratic equation in standard form where $a$, $b$, and $c$ are all nonzero, determine the value for the discriminant using the formula $b^2-4ac$.

1. If the discriminant is a perfect square, then solve by factoring.
2. If the discriminant is not a perfect square, then solve using the quadratic formula.

Recall that if the discriminant is not a perfect square and positive, the quadratic equation will have two irrational solutions. And if the discriminant is negative, the quadratic equation will have two complex conjugate solutions.

**Example 2.1.1.3:**

Solve $(3x+5)(3x+7)=6x+10$.

**Solution**

Begin by rewriting the quadratic equation in standard form.

$$\begin{aligned}(3x+5)(3x+7) &= 6x+10\\ 9x^2+21x+15x+35 &= 6x+10\\ 9x^2+36x+35 &= 6x+10\\ 9x^2+30x+25 &= 0\end{aligned}$$

Substitute, $a=9, b=30$, and $c=25$ into the discriminant formula.

$$\begin{aligned}b^2-4ac &= (30)^2-4(9)(25)\\ &= 900-900\\ &= 0\end{aligned}$$

Since the discriminant is 0, solve by factoring and expect one real solution, a double root.

$$\begin{aligned}9x^2+30x+25 &= 0\\ (3x+5)(3x+5) &= 0\end{aligned}$$

$$\begin{aligned}3x+5=0 \quad &\text{or} \quad 3x+5=0\\ 3x=-5 \quad &\qquad 3x=-5\\ x=-\frac{5}{3} \quad &\qquad x=-\frac{5}{3}\end{aligned}$$

**Answer**

The solution is $-\frac{5}{3}$.

It is good to know that the quadratic formula will work to find the solutions to all of the examples in this section. However, it is not always the best solution. If the equation can be solved by factoring or by extracting the roots, you should use that method.

## Solving Equations Quadratic in Form

In this section we outline an algebraic technique that is used extensively in mathematics to transform equations into familiar forms. We begin by defining **quadratic form**[7],

$$au^2+bu+c=0$$

Here $u$ represents an algebraic expression. Some examples follow:

$$\left(\frac{t+2}{t}\right)^2+8\left(\frac{t+2}{t}\right)+7=0 \overset{u=\frac{t+2}{t}}{\Longrightarrow} u^2+8u+7=0$$

$$x^{2/3}-3x^{1/3}-10=0 \overset{u=x^{1/3}}{\Longrightarrow} u^2-3u-10=0$$

$$3y^{-2}+7y^{-1}-6=0 \overset{u=y^{-1}}{\Longrightarrow} 3u^2+7u-6=0$$

If we can express an equation in quadratic form, then we can use any of the techniques used to solve quadratic equations. For example, consider the following fourth-degree polynomial equation,

$$x^4 - 4x^2 - 32 = 0$$

If we let $u = x^2$ then $u^2 = \left(x^2\right)^2 = x^4$ and we can write

$$\begin{array}{rcl} x^4 - 4x^2 - 32 = 0 \Rightarrow \left(x^2\right)^2 - 4\left(x^2\right) - 32 = 0 \\ \downarrow \quad \downarrow \\ u^2 \quad - \quad 4u \quad - 32 = 0 \end{array}$$

This substitution transforms the equation into a familiar quadratic equation in terms of $u$ which, in this case, can be solved by factoring.

$$\begin{array}{rl} u^2 - 4u - 32 & = 0 \\ (u-8)(u+4) & = 0 \\ u = 8 \quad \text{or} \quad u & = -4 \end{array}$$

Since $u = x^2$ we can back substitute and then solve for $x$.

$$\begin{array}{ll} u = 8 \quad \text{or} & u = -4 \\ \downarrow & \downarrow \\ x^2 = 8 & x^2 = -4 \\ x = \pm\sqrt{8} & x = \pm\sqrt{-4} \\ x = \pm 2\sqrt{2} & x = \pm 2i \end{array}$$

Therefore, the equation $x^4 - 4x^2 - 32 = 0$ has four solutions $\{\pm 2\sqrt{2}, \pm 2i\}$, two real and two complex. This technique, often called a **u-substitution**[8], can also be used to solve some non-polynomial equations.

**Example 2.1.1.4:**

Solve: $x - 2\sqrt{x} - 8 = 0$ .

**Solution**

This is a radical equation that can be written in quadratic form. If we let $u = \sqrt{x}$ then $u^2 = (\sqrt{x})^2 = x$ and we can write

$$\begin{array}{rl} x - 2\sqrt{x} - 8 & = 0 \\ \downarrow \quad \downarrow & \\ u^2 \quad - 2u - 8 & = 0 \end{array}$$

Solve for $u$.

$$\begin{array}{c} u^2 - 2u - 8 = 0 \\ (u-4)(u+2) = 0 \\ u = 4 \text{ or } u = -2 \end{array}$$

Back substitute $u = \sqrt{x}$ and solve for $x$.

$$\begin{array}{ll} \sqrt{x} = 4 \quad \text{or} & \sqrt{x} = -2 \\ (\sqrt{x})^2 = (4)^2 & (\sqrt{x})^2 = (-2)^2 \\ x = 16 & x = 4 \end{array}$$

Recall that squaring both sides of an equation introduces the possibility of extraneous solutions. Therefore we must check our potential solutions.

| $Check\ x = 16$ | $Check\ x = 4$ |
|---|---|
| $x - 2\sqrt{x} - 8 = 0$ | $x - 2\sqrt{x} - 8 = 0$ |
| $16 - 2\sqrt{16} - 18 = 0$ | $4 - 2\sqrt{4} - 8 = 0$ |
| $16 - 2 \cdot 4 - 8 = 0$ | $4 - 2 \cdot 2 - 8 = 0$ |
| $16 - 8 - 8 = 0$ | $4 - 4 - 8 = 0$ |
| $0 = 0$ ✓ | $-8 = 0$ ✗ |

Because $x = 4$ is extraneous, there is only one solution, $x = 16$.

**Answer**

The solution is 16.

**Example 2.1.1.5:**

Solve: $x^{2/3} - 3x^{1/3} - 10 = 0$ .

**Solution**

If we let $u = x^{1/3}$ then $u^2 = \left(x^{1/3}\right)^2 = x^{2/3}$ and we can write

$$\begin{array}{c} x^{2/3} - 3x^{1/3} - 10 = 0 \\ \downarrow \quad \downarrow \\ u^2 \quad - 3u - 10 = 0 \end{array}$$

Solve for $u$.

$$\begin{array}{c} u^2 - 3u - 10 = 0 \\ (u-5)(u+2) = 0 \\ u = 5 \quad \text{or} \quad u = -2 \end{array}$$

Back substitute $u = x^{1/3}$ and solve for $x$.

$$\begin{array}{cc} x^{1/3} = 5 & \text{or} \quad x^{1/3} = -2 \\ \left(x^{1/3}\right)^3 = (5)^3 & \left(x^{1/3}\right)^3 = (-2)^3 \\ x = 125 & x = -8 \end{array}$$

**Check**

| $Check\ x = 125$ | $Check\ x = -8$ |
|---|---|
| $x^{2/3} - 3x^{1/3} - 10 = 0$ | $x^{2/3} - 3x^{1/3} - 10 = 0$ |
| $(125)^{2/3} - 3(125)^{1/3} - 10 = 0$ | $(-8)^{2/3} - 3(-8)^{1/3} - 10 = 0$ |
| $\left(5^3\right)^{2/3} - 3\left(5^3\right)^{1/3} - 10 = 0$ | $\left[(-2)^3\right]^{2/3} - 3\left[(-2)^3\right]^{1/3} - 10 = 0$ |
| $5^2 - 3 \cdot 5 - 10 = 0$ | $(-2)^2 - 3 \cdot (-2) - 10 = 0$ |
| $25 - 15 - 10 = 0$ | $4 + 6 - 10 = 0$ |
| $0 = 0$✓ | $0 = 0$✓ |

**Answer**

The solutions are $-8, 125$.

**Example 2.1.1.6:**

Solve: $3y^{-2} + 7y^{-1} - 6 = 0$

**Solution**

If we let $u = y^{-1}$, then $u^2 = \left(y^{-1}\right)^2 = y^{-2}$ and we can write

$$\begin{array}{c} 3y^{-2} + 7y^{-1} - 6 = 0 \\ \downarrow \quad \downarrow \\ 3u^2 + 7u - 6 = 0 \end{array}$$

Solve for $u$.

$$\begin{array}{c} 3u^2 + 7u - 6 = 0 \\ (3u-2)(u+3) = 0 \\ u = \frac{2}{3} \quad \text{or} \quad u = -3 \end{array}$$

Back substitute $u = y^{-1}$ and solve for $y$.

$$y^{-1} = \frac{2}{3} \text{ or } y^{-1} = -3$$
$$\frac{1}{y} = \frac{2}{3} \qquad \frac{1}{y} = -3$$
$$y = \frac{3}{2} \qquad y = -\frac{1}{3}$$

The original equation is actually a rational equation where $y \neq 0$. In this case, the solutions are not restrictions; they solve the original equation.

**Answer**

The solutions are $-\frac{1}{3}, \frac{3}{2}$.

**Example 2.1.1.7:**

Solve: $\left(\frac{t+2}{t}\right)^2 + 8\left(\frac{t+2}{t}\right) + 7 = 0$

**Solution**

If we let $u = \frac{t+2}{t}$, then $u^2 = \left(\frac{t+2}{t}\right)^2$ and we can write

$$\left(\frac{t+2}{t}\right)^2 + 8\left(\frac{t+2}{t}\right) + 7 = 0$$
$$\downarrow \qquad \downarrow$$
$$u^2 + \quad 8u \quad + 7 = 0$$

Solve for $u$.

$$u^2 + 8u + 7 = 0$$
$$(u+1)(u+7) = 0$$
$$u = -1 \quad \text{or} \quad u = -7$$

Back substitute $u = \frac{t+2}{t}$, and solve for $t$.

$$\frac{t+2}{t} = -1 \text{ or } \frac{t+2}{t} = -7$$
$$t + 2 = -t \qquad t + 2 = -7t$$
$$2t = -2 \qquad 8t = -2$$
$$t = -1 \qquad t = -\frac{1}{4}$$

**Answer**

The solutions are $-1, -\frac{1}{4}$. The check is left to the reader.

**Exercise 2.1.1.1**

Solve: $12x^{-2} - 16x^{-1} + 5 = 0$

**Answer**

The solutions are $\frac{6}{5}, 2$.

www.youtube.com/v/pAuR53nLNDo

So far all of the examples were of equations that factor. As we know, not all quadratic equations factor. If this is the case, we use the quadratic formula.

**Example 2.1.1.8:**

Solve $x^4 - 10x^2 + 23 = 0$ . Approximate to the nearest hundredth.

**Solution**

If we let $u = x^2$, then $u^2 = \left(x^2\right)^2 = x^4$ and we can write

$$\begin{array}{c} x^4 - 10x^2 + 23 = 0 \\ \downarrow \quad \downarrow \\ u^2 - 10u + 23 = 0 \end{array}$$

This equation does not factor; therefore, use the quadratic formula to find the solutions for $u$. Here $a = 1, b = -10$, and $c = 23$.

$$\begin{aligned} u &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(23)}}{2(1)} \\ &= \frac{10 \pm \sqrt{8}}{2} \\ &= \frac{10 \pm 2\sqrt{2}}{2} \\ &= 5 \pm \sqrt{2} \end{aligned}$$

Therefore, $u = 5 \pm \sqrt{2}$ . Now back substitute $u = x^2$ and solve for $x$.

$$\begin{array}{ccc} u = 5 - \sqrt{2} & \text{or} & u = 5 + \sqrt{2} \\ \downarrow & & \downarrow \\ x^2 = 5 - \sqrt{2} & & x^2 = 5 + \sqrt{2} \\ x = \pm\sqrt{5 - \sqrt{2}} & & x = \pm\sqrt{5 + \sqrt{2}} \end{array}$$

Round the four solutions as follows.

$$\begin{array}{cc} x = -\sqrt{5 - \sqrt{2}} \approx -1.89 & x = -\sqrt{5 + \sqrt{2}} \approx -2.53 \\ x = \sqrt{5 - \sqrt{2}} \approx 1.89 & x = \sqrt{5 + \sqrt{2}} \approx 2.53 \end{array}$$

**Answer**

The solutions are approximately $\pm 1.89, \pm 2.53$

If multiple roots and complex roots are counted, then the **fundamental theorem of algebra**[9] implies that every polynomial with one variable will have as many roots as its degree. For example, we expect $f(x) = x^3 - 8$ to have three roots. In other words, the equation

$$x^3 - 8 = 0$$

should have three solutions. To find them one might first think of trying to extract the cube roots just as we did with square roots,

$$\begin{aligned} x^3 - 8 &= 0 \\ x^3 &= 8 \\ x &= \sqrt[3]{8} \\ x &= 2 \end{aligned}$$

As you can see, this leads to one solution, the real cube root. There should be two others; let's try to find them.

**Example 2.1.1.9:**

Find the set of all roots: $f(x) = x^3 - 8$ .

**Solution**

Notice that the expression $x^3-8$ is a difference of cubes and recall that $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$ . Here $a=x$ and $b=2$ and we can write

$$\begin{aligned} x^3-8 &= 0 \\ (x-2)\left(x^2+2x+4\right) &= 0 \end{aligned}$$

Next apply the zero-product property and set each factor equal to zero. After setting the factors equal to zero we can then solve the resulting equation using the appropriate methods.

$$\begin{aligned} x-2=0 \qquad \text{or} \qquad & x^2+2x+4=0 \\ x=2 \qquad & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ & =\frac{-(2)\pm\sqrt{b^2-4ac}}{2a} \\ & =\frac{-2\pm\sqrt{-12}}{2} \\ & =\frac{-2\pm 2i\sqrt{3}}{2} \\ & =-1\pm i\sqrt{3} \end{aligned}$$

Using this method, we were able to obtain the set of all three roots $\{2,-1\pm i\sqrt{3}\}$, one real and two complex.

**Answer**

$\{2,-1\pm i\sqrt{3}\}$

Sometimes the roots of a function will occur multiple times. For example, $g(x)=(x-2)^3$ has degree three where the roots can be found as follows:

$$\begin{aligned} (x-2)^3 &= 0 \\ (x-2)(x-2)(x-2) &= 0 \\ x-2=0 \text{ or } x-2=0 \text{ or } & x-2=0 \\ x=2 \qquad x=2 \qquad & x=2 \end{aligned}$$

Even though $g$ is of degree 3 there is only one real root $\{2\}$; it occurs 3 times.

## Key Takeaways

- The quadratic formula can solve any quadratic equation. However, it is sometimes not the most efficient method.
- If a quadratic equation can be solved by factoring or by extracting square roots you should use that method.
- We can sometimes transform equations into equations that are quadratic in form by making an appropriate $u$-substitution. After solving the equivalent equation, back substitute and solve for the original variable.
- Counting multiple and complex roots, the fundamental theorem of algebra guarantees as many roots as the degree of a polynomial equation with one variable.

### Exercise 2.1.1.2

Solve

1. $x^2-9x=0$
2. $x^2+10x=0$
3. $15x^2+6x=0$
4. $36x^2-18x=0$
5. $x^2-90=0$
6. $x^2+48=0$
7. $2x^2+1=0$
8. $7x^2-1=0$
9. $6x^2-11x+4=0$
10. $9x^2+12x-5=0$

11. $x^2+x+6=0$
12. $x^2+2x+8=0$
13. $4t^2+28t+49=0$
14. $25t^2-20t+4=0$
15. $u^2-4u-1=0$
16. $u^2-2u-11=0$
17. $2(x+2)^2=11+4x-2x^2$
18. $(2x+1)(x-3)+2x^2=3(x-1)$
19. $(3x+2)^2=6(2x+1)$
20. $(2x-3)^2+5x^2=4(2-3x)$
21. $4(3x-1)^2-5=0$
22. $9(2x+3)^2-2=0$

**Answer**

1. $0, 9$
3. $-\frac{2}{5}, 0$
5. $\pm 3\sqrt{10}$
7. $\pm\frac{\sqrt{2}}{2}i$
9. $\frac{1}{2}, \frac{4}{3}$
11. $-\frac{1}{2}\pm\frac{\sqrt{23}}{2}i$
13. $-\frac{7}{2}$
15. $2\pm\sqrt{5}$
17. $-\frac{3}{2}, \frac{1}{2}$
19. $\pm\frac{\sqrt{2}}{3}$
21. $\frac{2\pm\sqrt{5}}{6}$

## Exercise 2.1.1.3

Find all solutions.

1. $x^4+x^2-72=0$
2. $x^4-17x^2-18=0$
3. $x^4-13x^2+36=0$
4. $4x^4-17x^2+4=0$
5. $x+2\sqrt{x}-3=0$
6. $x-\sqrt{x}-2=0$
7. $x-5\sqrt{x}+6=0$
8. $x-6\sqrt{x}+5=0$
9. $x^{2/3}+5x^{1/3}+6=0$
10. $x^{2/3}-2x^{1/3}-35=0$
11. $4x^{2/3}-4x^{1/3}+1=0$
12. $3x^{2/3}-2x^{1/3}-1=0$
13. $5x^{-2}+9x^{-1}-2=0$
14. $3x^{-2}+8x^{-1}-3=0$
15. $8x^{-2}+14x^{-1}-15=0$
16. $9x^{-2}-24x^{-1}+16=0$

17. $\left(\frac{x-3}{x}\right)^2-2\left(\frac{x-3}{x}\right)-24=0$
18. $\left(\frac{2x+1}{x}\right)^2+9\left(\frac{2x+1}{x}\right)-36=0$
19. $2\left(\frac{x}{x+1}\right)^2-5\left(\frac{x}{x+1}\right)-3=0$
20. $3\left(\frac{x}{3x-1}\right)^2+13\left(\frac{x}{3x-1}\right)-10=0$
21. $4y^{-2}-9=0$
22. $16y^{-2}+4y^{-1}=0$
23. $30y^{2/3}-15y^{1/3}=0$
24. $y^{2/3}-9=0$
25. $81y^4-1=0$
26. $5\left(\frac{1}{x+2}\right)^2-3\left(\frac{1}{x+2}\right)-2=0$
27. $12\left(\frac{x}{2x-3}\right)^2-11\left(\frac{x}{2x-3}\right)+2=0$
28. $10x^{-2}-19x^{-1}-2=0$
29. $x^{1/2}-3x^{1/4}+2=0$
30. $x+5\sqrt{x}-50=0$
31. $8x^{2/3}+7x^{1/3}-1=0$
32. $x^{4/3}-13x^{2/3}+36=0$
33. $y^4-14y^2+46=0$
34. $x^{4/3}-2x^{2/3}+1=0$
35. $2y^{-2}-y^{-1}-1=0$
36. $2x^{-2/3}-3x^{-1/3}-2=0$
37. $4x^{-1}-17x^{-1/2}+4=0$
38. $3x^{-1}-8x^{-1/2}+4=0$
39. $2x^{1/3}-3x^{1/6}+1=0$
40. $x^{1/3}-x^{1/6}-2=0$

**Answer**

1. $\pm 2\sqrt{2}, \pm 3i$
3. $\pm 2, \pm 3$
5. $1$
7. $4, 9$
9. $-27, -8$
11. $\frac{1}{8}$
13. $-\frac{1}{2}, 5$
15. $-\frac{2}{5}, \frac{4}{3}$
17. $\pm\frac{3}{5}$
19. $-\frac{3}{2}, -\frac{1}{3}$
21. $\pm\frac{2}{3}$
23. $0, \frac{1}{8}$
25. $\pm\frac{1}{3}, \pm\frac{i}{3}$
27. $-\frac{3}{2}, 6$
29. $1, 16$

31. $-1, \frac{1}{512}$
33. $\pm\sqrt{7-\sqrt{3}}, \pm\sqrt{7+\sqrt{3}}$
35. $-2, 1$
37. $\frac{1}{16}, 16$
39. $\frac{1}{64}, 1$

## Exercise 2.1.1.4

Find all solutions. Round your answers to the nearest hundredth.

1. $x^4-6x^2+7=0$
2. $x^4-6x^2+6=0$
3. $x^4-8x^2+14=0$
4. $x^4-12x^2+31=0$
5. $4x^4-16x^2+13=0$
6. $9x^4-30x^2+1=0$

**Answer**

1. $\pm 1.26, \pm 2.10$
3. $\pm 1.61, \pm 2.33$
5. $\pm 1.06, \pm 1.69$

## Exercise 2.1.1.5

Find the set of all roots.

1. $f(x)=x^3-1$
2. $g(x)=x^3+1$
3. $f(x)=x^3-27$
4. $g(x)=x^4-16$
5. $h(x)=x^4-1$
6. $h(x)=x^6-1$
7. $f(x)=(2x-1)^3$
8. $g(x)=x^2(x-4)^2$
9. $f(x)=x^3-q^3, q>0$
10. $f(x)=x^3+q^3, q>0$

**Answer**

1. $\left\{1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right\}$
3. $\left\{3, -\frac{3}{2} \pm \frac{3\sqrt{3}}{2} i\right\}$
5. $\{\pm 1, \pm i\}$
7. $\left\{\frac{1}{2}\right\}$
9. $\left\{q, -\frac{q}{2} \pm \frac{q\sqrt{3}}{2} i\right\}$

## Exercise 2.1.1.6

Find all solutions.

1. $x^6+7x^3-8=0$
2. $x^6-7x^3-8=0$
3. $x^6+28x^3+27=0$
4. $x^6+16x^3+64=0$
5. $\left|x^2+2x-5\right|=1$
6. $\left|x^2-2x-3\right|=3$
7. $\left|2x^2-5\right|=4$
8. $\left|3x^2-9x\right|=6$

**Answer**

1. $-2, 1, 1\pm i\sqrt{3}, -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$
3. $-3, -1, \frac{3}{2}\pm\frac{3\sqrt{3}}{2}i, \frac{1}{2}\pm\frac{\sqrt{3}}{2}i$
5. $-1\pm\sqrt{7}, -1\pm\sqrt{5}$
7. $\pm\frac{\sqrt{2}}{2}, \pm\frac{3\sqrt{2}}{2}$

## Exercise 2.1.1.7

Find a quadratic function with integer coefficients and the given set of roots. (Hint: if $r_1$ and $r_2$ are roots, then $(x-r_1)(x-r_2)=0$ .)

1. $\{\pm 3i\}$
2. $\{\pm i\sqrt{5}\}$
3. $\{\pm\sqrt{3}\}$
4. $\{\pm 2\sqrt{6}\}$
5. $\{1\pm\sqrt{3}\}$
6. $\{2\pm 3\sqrt{2}\}$
7. $\{1\pm 6i\}$
8. $\{2\pm 3i\}$

**Answer**

1. $f(x)=x^2+9$
3. $f(x)=x^2-3$
5. $f(x)=x^2-2x-2$
7. $f(x)=x^2-2x+37$

## Exercise 2.1.1.8

1. On a note card, write out your strategy for solving a quadratic equation. Share your strategy on the discussion board.
2. Make up your own equation that is quadratic in form. Share it and the solution on the discussion board

**Answer**

1. Answer may vary

## Footnotes

[7]An equation of the form $au^2 + bu + c = 0$ where $a, b$ and $c$ are real numbers and u represents an algebraic expression.

[8]A technique in algebra using substitution to transform equations into familiar forms.

[9]If multiple roots and complex roots are counted, then every polynomial with one variable will have as many roots as its degree.

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# 2.1.2: Quadratic Functions and Their Graphs (Review)

**Learning Objectives**

- Graph a parabola.
- Find the intercepts and vertex of a parabola.
- Find the maximum and minimum $y$-value.
- Find the vertex of a parabola by completing the square.

## The Graph of a Quadratic Function

A **quadratic function** is a polynomial function of degree 2 which can be written in the general form,

$$f(x) = ax^2 + bx + c$$

Here $a, b$ and $c$ represent real numbers where $a \neq 0$ .The squaring function $f(x) = x^2$ is a quadratic function whose graph follows.

$f(x) = x^2$

| $x$ | $f(x)$ |
|---|---|
| $-2$ | 4 |
| $-1$ | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |

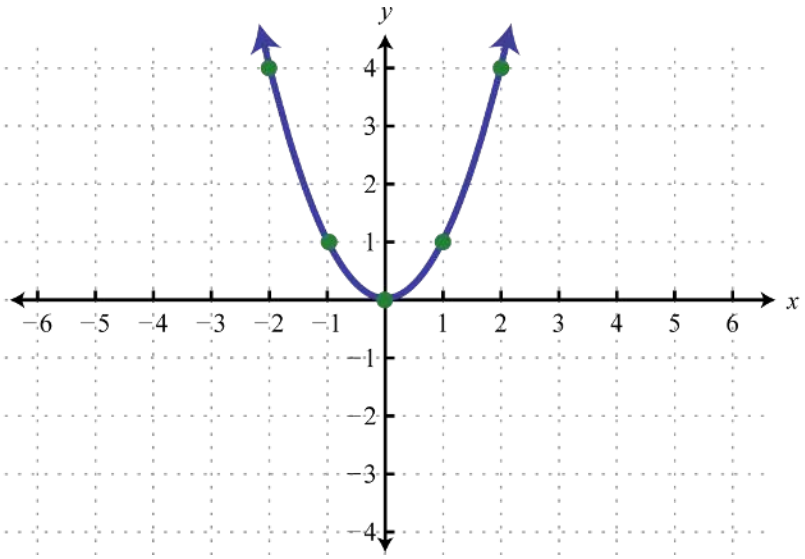



Figure 2.1.2.1

This general curved shape is called a **parabola**[10] and is shared by the graphs of all quadratic functions. Note that the graph is indeed a function as it passes the vertical line test. Furthermore, the domain of this function consists of the set of all real numbers $(-\infty, \infty)$ and the range consists of the set of nonnegative numbers $[0, \infty)$.

When graphing parabolas, we want to include certain special points in the graph. The $y$-intercept is the point where the graph intersects the $y$-axis. The $x$-intercepts are the points where the graph intersects the $x$-axis. The **vertex**[11] is the point that defines the minimum or maximum of the graph. Lastly, the **line of symmetry**[12] (also called the **axis of symmetry**[13]) is the vertical line through the vertex, about which the parabola is symmetric.

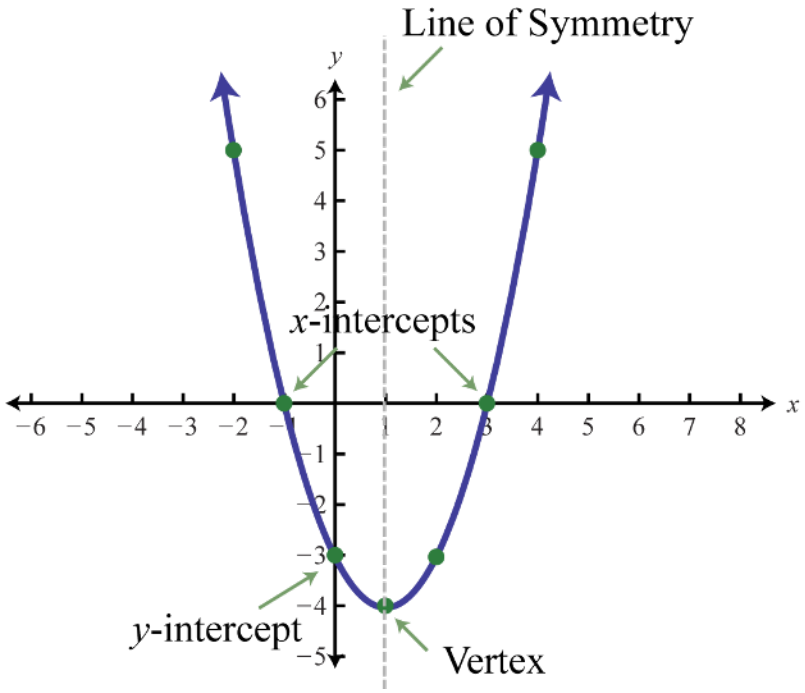



Figure 2.1.2.2

For any parabola, we will find the vertex and $y$-intercept. In addition, if the $x$-intercepts exist, then we will want to determine those as well. Guessing at the $x$-values of these special points is not practical; therefore, we will develop techniques that will facilitate finding them. Many of these techniques will be used extensively as we progress in our study of algebra.

Given a quadratic function $f(x)=ax^2+bx+c$ , find the $y$-intercept by evaluating the function where $x=0$. In general, $f(0)=a(0)^2+b(0)+c=c$ , and we have

$$\begin{array}{c} y-intercept \\ (0,c) \end{array}$$

Next, recall that the $x$-intercepts, if they exist, can be found by setting $f(x)=0$. Doing this, we have $a^2+bx+c=0$ , which has general solutions given by the quadratic formula, \(x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\).Therefore, the $x$-intercepts have this general form:

$$\begin{array}{c} x-intercepts \\ \left(\frac{-b-\sqrt{b^2-4ac}}{2a},0\right) \text{ and } \left(\frac{-b+\sqrt{b^2-4ac}}{2a},0\right) \end{array}$$

Using the fact that a parabola is symmetric, we can determine the vertical line of symmetry using the $x$-intercepts. To do this, we find the $x$-value midway between the $x$-intercepts by taking an average as follows:

$$\begin{aligned} x &= \left(\frac{-b-\sqrt{b^2-4ac}}{2a}+\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\div 2 \\ &= \left(\frac{-b-\cancel{\sqrt{b^2-4ac}}-b+\cancel{\sqrt{b^2-4ac}}}{2a}\right)\div\left(\frac{2}{1}\right) \\ &= \frac{-2b}{2a}\cdot\frac{1}{2} \\ &= -\frac{b}{2a} \end{aligned}$$

Therefore, the line of symmetry is the vertical line $x=-\frac{b}{2a}$ . We can use the line of symmetry to find the the vertex.

$$\begin{array}{cc} Line\ of\ symmetry & Vertex \\ x=-\frac{b}{2a} & \left(-\frac{b}{2a},f\left(-\frac{b}{2a}\right)\right) \end{array}$$

Generally three points determine a parabola. However, in this section we will find five points so that we can get a better approximation of the general shape. The steps for graphing a parabola are outlined in the following example.

**Example 2.1.2.1:**

Graph: $f(x) = -x^2 - 2x + 3$ .

**Solution**

**Step 1**: Determine the $y$-intercept. To do this, set $x$=0 and find $f(0)$.

$$\begin{aligned} f(x) &= -x^2 - 2x + 3 \\ f(0) &= -(0)^2 - 2(0) + 3 \\ &= 3 \end{aligned}$$

The $y$-intercept is $(0, 3)$.

**Step 2**: Determine the $x$-intercepts if any. To do this, set $f(x) = 0$ and solve for $x$.

$$\begin{aligned} f(x) &= -x^2 - 2x + 3 && \textit{Set } f(x) = 0. \\ 0 &= -x^2 - 2x + 3 && \textit{Multiply both sides by } -1. \\ 0 &= x^2 + 2x - 3 && \textit{Factor.} \\ 0 &= (x + 3)(x - 1) && \textit{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x + 3 = 0 \quad &\text{or } x - 1 = 0 \\ x = -3 \quad &\quad x = 1 \end{aligned}$$

Here where $f(x) = 0$, we obtain two solutions. Hence, there are two $x$-intercepts, $(-3, 0)$ and $(1, 0)$.

**Step 3**: Determine the vertex. One way to do this is to first use $x = -\frac{b}{2a}$ to find the $x$-value of the vertex and then substitute this value in the function to find the corresponding $y$-value. In this example, $a = -1$ and $b = -2$.

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-2)}{2(-1)} \\ &= \frac{2}{-2} \\ &= -1 \end{aligned}$$

Substitute $-1$ into the original function to find the corresponding $y$-value.

$$\begin{aligned} f(x) &= -x^2 - 2x + 3 \\ f(-1) &= -(-1)^2 - 2(-1) + 3 \\ &= -1 + 2 + 3 \\ &= 4 \end{aligned}$$

The vertex is $(-1, 4)$.

**Step 4**: Determine extra points so that we have at least five points to plot. Ensure a good sampling on either side of the line of symmetry. In this example, one other point will suffice. Choose $x = -2$ and find the corresponding $y$-value.

Table 2.1.2.1

| $x$ | $y$ | | Point |
|---|---|---|---|
| $-2$ | 3 | $f(-2) = -(-2)^2 - 2(-2) + 3$ | $= -4 + 4 + 3$ $(-2, 3)$ |

Our fifth point is $(-2, 3)$.

**Step 5**: Plot the points and sketch the graph. To recap, the points that we have found are

$y$-intercept: $(0, 3)$

$x$-intercepts: $(-3, 0)$ and $(1, 0)$

Vertex: $(-1, 4)$

Extra point: $(-2, 3)$

**Answer**:

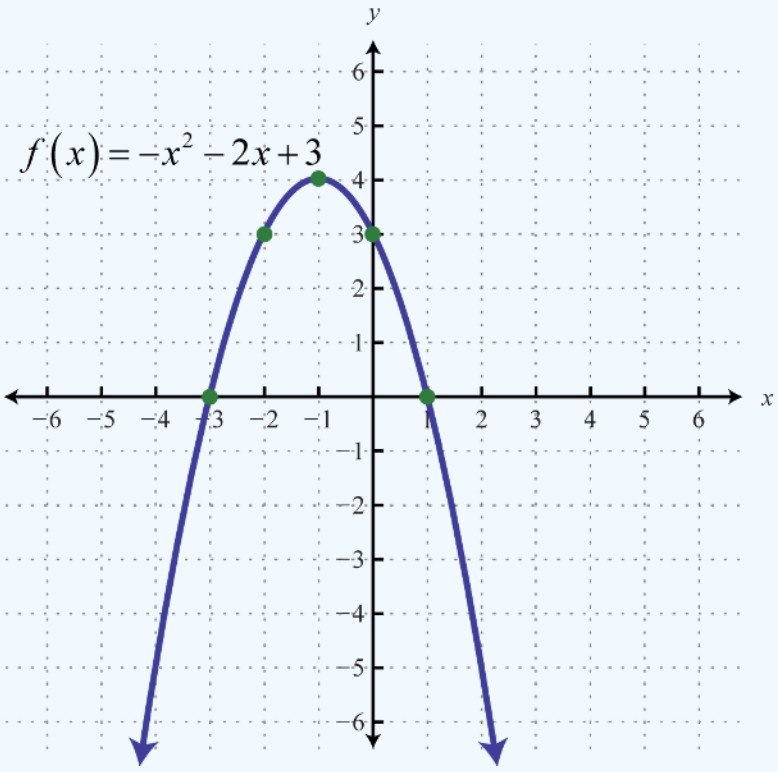



Figure 2.1.2.3

The parabola opens downward. In general, use the leading coefficient to determine if the parabola opens upward or downward. If the leading coefficient is negative, as in the previous example, then the parabola opens downward. If the leading coefficient is positive, then the parabola opens upward.

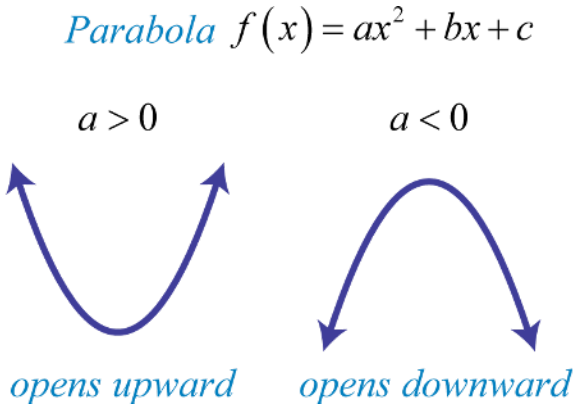



Figure 2.1.2.4

All quadratic functions of the form $f(x) = ax^2 + bx + c$ have parabolic graphs with $y$-intercept $(0, c)$. However, not all parabolas have $x$-intercepts.

**Example 2.1.2.2:**

Graph: $f(x) = 2x^2 + 4x + 5$ .

**Solution**

Because the leading coefficient 2 is positive, we note that the parabola opens upward. Here $c = 5$ and the $y$-intercept is $(0, 5)$. To find the $x$-intercepts, set $f(x) = 0$.

$$\begin{aligned} f(x) &= 2x^2 + 4x + 5 \\ 0 &= 2x^2 + 4x + 5 \end{aligned}$$

In this case, $a = 2, b = 4$, and $c = 5$. Use the discriminant to determine the number and type of solutions.

$$\begin{aligned} b^2 - 4ac &= (4)^2 - 4(2)(5) \\ &= 16 - 40 \\ &= -24 \end{aligned}$$

Since the discriminant is negative, we conclude that there are no real solutions. Because there are no real solutions, there are no $x$-intercepts. Next, we determine the $x$-value of the vertex.

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(4)}{2(2)} \\ &= \frac{-4}{4} \\ &= -1 \end{aligned}$$

Given that the $x$-value of the vertex is $-1$, substitute $-1$ into the original equation to find the corresponding $y$-value.

$$\begin{aligned} f(x) &= 2x^2 + 4x + 5 \\ f(-1) &= 2(-1)^2 + 4(-1) + 5 \\ &= 2 - 4 + 5 \\ &= 3 \end{aligned}$$

The vertex is $(-1, 3)$. So far, we have only two points. To determine three more, choose some $x$-values on either side of the line of symmetry, $x = -1$. Here we choose $x$-values $-3, -2$, and $1$.

Table 2.1.2.2

| $x$ | $y$ | | Points |
|---|---|---|---|
| $-3$ | 11 | $f(-3) = 2(-3)^2 + 4(-3) + 5 = 18 - 12 + 5$ | $(-3, 11)$ |
| $-2$ | 5 | $f(-2) = 2(-2)^2 + 4(-2) + 5 = 8 - 8 + 5$ | $(-2, 5)$ |
| 1 | 11 | $f(1) = 2(1)^2 + 4(1) + 5 = 2 + 4 + 5 = 11$ | $(1, 11)$ |

To summarize, we have

$y$-intercept: $(0, 5)$

$x$-intercepts: None

Vertex: $(-1, 3)$

Extra points: $(-3, 11), (-2, 5), (1, 11)$

Plot the points and sketch the graph.

**Answer**:

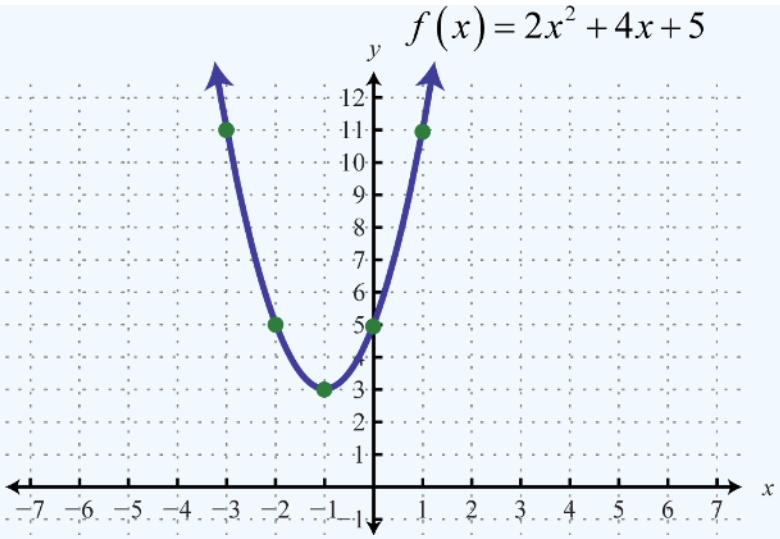



Figure 2.1.2.5

**Example 2.1.2.3:**

Graph: $f(x) = x^2 - 2x - 1$ .

**Solution**

Since $a = 1$, the parabola opens upward. Furthermore, $c = -1$, so the $y$-intercept is $(0, -1)$. To find the $x$-intercepts, set $f(x) = 0$.

$$\begin{aligned} f(x) &= x^2 - 2x - 1 \\ 0 &= x^2 - 2x - 1 \end{aligned}$$

In this case, solve using the quadratic formula with $a = 1, b = -2$ , and $c = -1$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{2 \pm \sqrt{8}}{2} \\ &= \frac{2 \pm 2\sqrt{2}}{2} \\ &= \frac{2(1 \pm \sqrt{2})}{2} \\ &= 1 \pm \sqrt{2} \end{aligned}$$

Here we obtain two real solutions for $x$, and thus there are two $x$-intercepts:

$$(1 - \sqrt{2}, 0) \text{ and } (1 + \sqrt{2}, 0) \quad \textit{Exact values}$$
$$(-0.41, 0) \qquad (2.41, 0) \qquad \textit{Approximate values}$$

Approximating the $x$-intercepts using a calculator will help us plot the points. However, we will present the exact $x$-intercepts on the graph. Next, find the vertex.

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-2)}{2(1)} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

Given that the $x$-value of the vertex is 1, substitute into the original equation to find the corresponding $y$-value.

$$\begin{aligned} y &= x^2 - 2x - 1 \\ &= (1)^2 - 2(1) - 1 \\ &= 1 - 2 - 1 \\ &= -2 \end{aligned}$$

The vertex is $(1, -2)$. We need one more point.

Table 2.1.2.3

| $x$ | $y$ | | Point |
|---|---|---|---|
| 2 | $-1$ | $f(2) = (2)^2 - 2(2) - 1 = 4 - 4 - 1 = -1$ | $(2, -1)$ |

To summarize, we have

$y$-intercepts: $(0, 1)$

$x$-intercepts: $(1 - \sqrt{2}, 0)$ and $(1 + \sqrt{2}, 0)$

Vertex: $(1, -2)$

Extra point: $(2, -1)$

Plot the points and sketch the graph.

**Answer**:

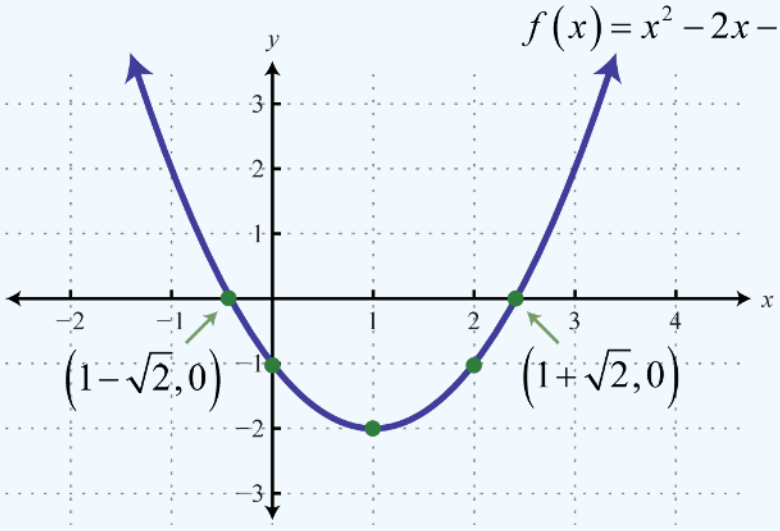



Figure 2.1.2.6

## Exercise 2.1.2.1

Graph: $g(x) = -4x^2 + 12x - 9$ .

**Answer**

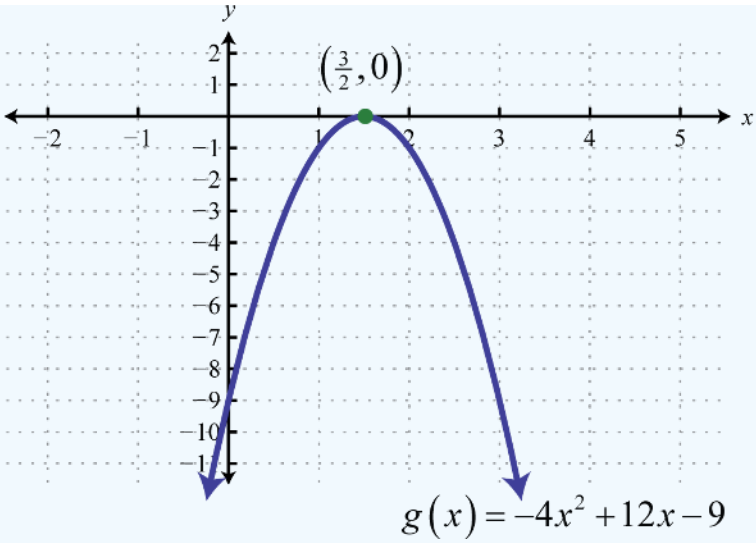



Figure 2.1.2.7

www.youtube.com/v/tQiDXNhn7ik

## Finding the Maximum or Minimum

It is often useful to find the maximum and/or minimum values of functions that model real-life applications. To find these important values given a quadratic function, we use the vertex. If the leading coefficient $a$ is positive, then the parabola opens upward and there will be a minimum $y$-value. If the leading coefficient $a$ is negative, then the parabola opens downward and there will be a maximum $y$-value.

**Example 2.1.2.4:**

Determine the maximum or minimum: $y=-4x^2+24x-35$ .

**Solution**

Since $a=-4$ , we know that the parabola opens downward and there will be a maximum $y$-value. To find it, first find the $x$-value of the vertex.

$$\begin{aligned} x &= -\frac{b}{2a} && x-value\ of\ the\ vertex. \\ &= -\frac{24}{2(-4)} && Substitute\ a=-4\ and\ b-24. \\ &= -\frac{24}{-8} && Simplify. \\ &= 3 \end{aligned}$$

The $x$-value of the vertex is $3$. Substitute this value into the original equation to find the corresponding $y$-value.

$$\begin{aligned} y &= -4x^2+24x-35 && Substitute\ x=3. \\ &= -4(3)^2+24(3)-35 && Simplify. \\ &= -36+72-35 \\ &= 1 \end{aligned}$$

The vertex is $(3,1)$. Therefore, the maximum $y$-value is $1$, which occurs where $x=3$, as illustrated below:

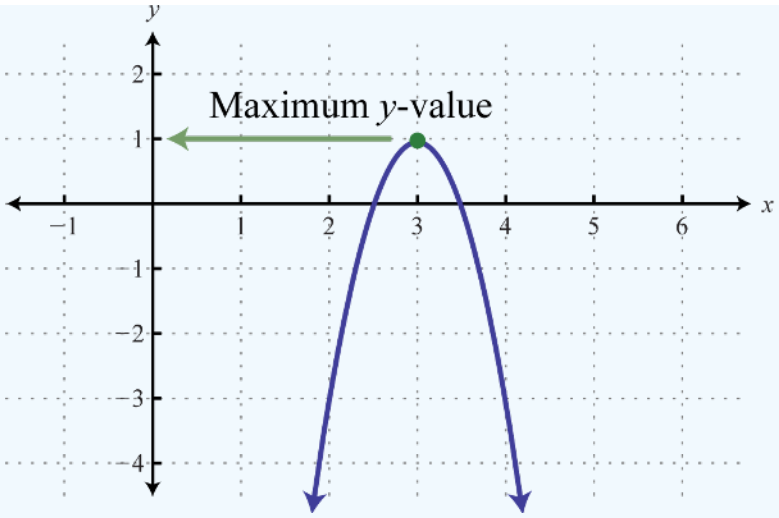



Figure 2.1.2.8

**Note**: The graph is not required to answer this question.

**Answer**:

The maximum is 1.

## Example 2.1.2.5:

Determine the maximum or minimum: $y = 4x^2 - 32x + 62$ .

**Solution**

Since $a = 4$, the parabola opens upward and there is a minimum $y$-value. Begin by finding the $x$-value of the vertex.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-32}{2(4)} \quad \textit{Substitute } a = 4 \textit{ and } b = -32. \\ &= -\frac{-32}{8} \quad \textit{Simplify.} \\ &= 4 \end{aligned}$$

Substitute $x = 4$ into the original equation to find the corresponding $y$-value.

$$\begin{aligned} y &= 4x^2 - 32x + 62 \\ &= 4(4)^2 - 32(4) + 62 \\ &= 64 - 128 + 62 \\ &= -2 \end{aligned}$$

The vertex is $(4, -2)$. Therefore, the minimum $y$-value of $-2$ occurs where $x = 4$, as illustrated below:

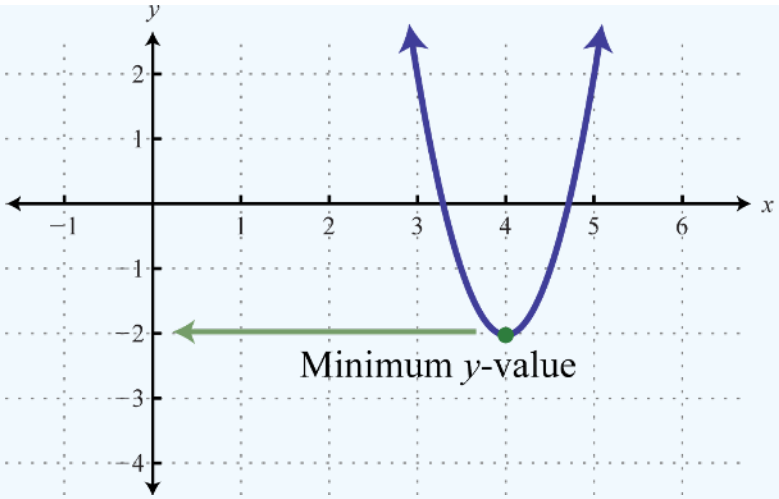



Figure 2.1.2.9

**Answer**:

The minimum is $-2$.

**Example 2.1.2.6:**

The height in feet of a projectile is given by the function $h(t) = -16t^2 + 72t$, where $t$ represents the time in seconds after launch. What is the maximum height reached by the projectile?

**Solution**

Here $a = -16$, and the parabola opens downward. Therefore, the $y$-value of the vertex determines the maximum height. Begin by finding the time at which the vertex occurs.

$$t = -\frac{b}{2a} = -\frac{72}{2(-16)} = \frac{72}{32} = \frac{9}{4}$$

The maximum height will occur in $\frac{9}{4}$ seconds (or $2\frac{1}{4}$ seconds). Substitute this time into the function to determine the maximum height attained.

$$\begin{aligned} h\left(\frac{9}{4}\right) &= -16\left(\frac{9}{4}\right)^2 + 72\left(\frac{9}{4}\right) \\ &= -16\left(\frac{81}{16}\right) + 72\left(\frac{9}{4}\right) \\ &= -81 + 162 \\ &= 81 \end{aligned}$$

**Answer**:

The maximum height of the projectile is 81 feet.

## Finding the Vertex by Completing the Square

In this section, we demonstrate an alternate approach for finding the vertex. Any quadratic function $f(x) = ax^2 + bx + c$ can be rewritten in **vertex form**[14],

$$f(x) = a(x-h)^2 + k$$

In this form, the vertex is $(h, k)$. To see that this is the case, consider graphing $f(x) = (x-2)^2 + 3$ using the transformations.

$$\begin{aligned} y &= x^2 && \textit{Basic squaring function} \\ y &= (x-2)^2 && \textit{Horizontal shift right 2 units} \\ y &= (x-2)^2 + 3 && \textit{Vertical shift up 3 units} \end{aligned}$$

Use these translations to sketch the graph,

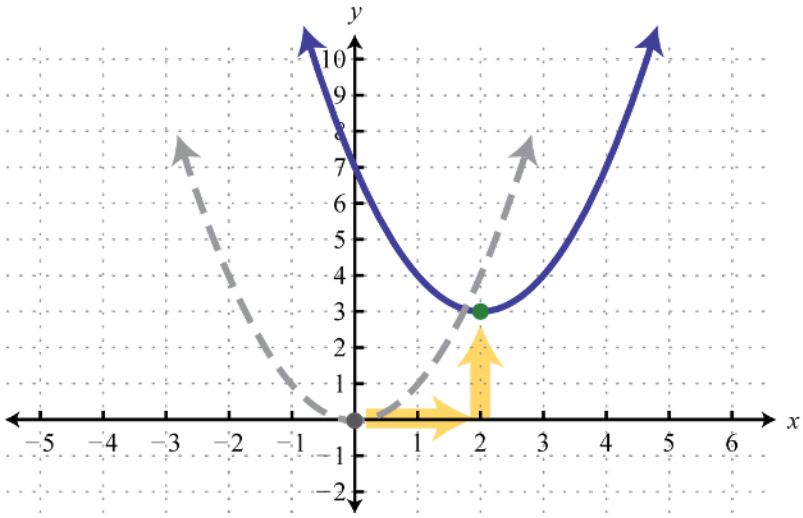


Figure 2.1.2.10

Here we can see that the vertex is $(2, 3)$.

$$f(x) = a(x-h)^2 + k$$
$$\downarrow \quad \downarrow$$
$$f(x) = (x-2)^2 + 3$$

When the equation is in this form, we can read the vertex directly from it.

**Example 2.1.2.7:**

Determine the vertex: $f(x) = 2(x+3)^2 - 2$ .

**Solution**

Rewrite the equation as follows before determining $h$ and $k$.

$$f(x) = \ a\,(\,x\,-\,h\,)^2\ +\ k$$
$$\downarrow \quad \downarrow$$
$$f(x) = 2[x-(-3)]^2 + (-2)$$

Here $h = -3$ and $k = -2$ .

**Answer**:

The vertex is $(-3, -2)$.

Often the equation is not given in vertex form. To obtain this form, complete the square.

**Example 2.1.2.8:**

Rewrite in vertex form and determine the vertex: $f(x) = x^2 + 4x + 9$ .

**Solution**

Begin by making room for the constant term that completes the square.

$$f(x) = x^2 + 4x + 9$$
$$= x^2 + 4x + \_\_\_ + 9 - \_\_\_$$

The idea is to add and subtract the value that completes the square, $\left(\frac{b}{2}\right)^2$, and then factor. In this case, add and subtract $\left(\frac{4}{2}\right)^2 = (2)^2 = 4$ .

$$\begin{aligned} f(x) &= x^2+4x+9 && \textit{Add and subtract } 4. \\ &= x^2+4x+4+9-4 && \textit{Factor.} \\ &= \left(x^2+4x+4\right)+5 \\ &= (x+3)(x+2)+5 \\ &= (x+2)^2+5 \end{aligned}$$

Adding and subtracting the same value within an expression does not change it. Doing so is equivalent to adding 0. Once the equation is in this form, we can easily determine the vertex.

$$\begin{aligned} f(x) &= a(x-h)^2 \ + \ k \\ &\quad\ \downarrow \quad\ \downarrow \\ f(x) &= (x-(-2))^2+5 \end{aligned}$$

Here $h=-2$ and $k=5$.

**Answer**:

The vertex is $(-2,5)$.

If there is a leading coefficient other than 1, then we must first factor out the leading coefficient from the first two terms of the trinomial.

**Example 2.1.2.9:**

Rewrite in vertex form and determine the vertex: $f(x)=2x^2-4x+8$.

**Solution**

Since $a=2$, factor this out of the first two terms in order to complete the square. Leave room inside the parentheses to add and subtract the value that completes the square.

$$\begin{aligned} f(x) &= 2x^2-4x+8 \\ &= 2\left(x^2-2x\right)+8 \end{aligned}$$

Now use $-2$ to determine the value that completes the square. In this case, $\left(\frac{-2}{2}\right)^2=(-1)^2=1$. Add and subtract 1 and factor as follows:

$$\begin{aligned} f(x) &= 2x^2-4x+8 \\ &= 2\left(x^2-2x+\_\_\_-\_\_\_\right)+8 && \textit{Add and subtract } 1. \\ &= 2\left(x^2-2x+1-1\right)+8 && \textit{Factor.} \\ &= 2[(x-1)(x-1)-1]+8 \\ &= 2\left[(x-1)^2-1\right]+8 && \textit{Distribute the } 2. \\ &= 2(x-1)^2-2+8 \\ &= 2(x-1)^2+6 \end{aligned}$$

In this form, we can easily determine the vertex.

$$\begin{aligned} f(x) &= a(x-h)^2+k \\ &\quad\ \downarrow \quad\ \downarrow \\ f(x) &= 2(x-1)^2+6 \end{aligned}$$

Here $h=1$ and $k=6$.

**Answer**:

The vertex is $(1,6)$.

**Exercise 2.1.2.1**

Rewrite in vertex form and determine the vertex: $f(x) = -2x^2 - 12x + 3$ .

**Answer**

$f(x) = -2(x+3)^2 + 21$ ; vertex: $(-3, 21)$

www.youtube.com/v/cmwMn6oZ2dI

## Key Takeaways

- The graph of any quadratic function $f(x) = ax^2 + bx + c$ , where $a, b$, and $c$ are real numbers and $a \neq 0$, is called a parabola.
- When graphing a parabola always find the vertex and the $y$-intercept. If the $x$-intercepts exist, find those as well. Also, be sure to find ordered pair solutions on either side of the line of symmetry, $x = -\frac{b}{2a}$ .
- Use the leading coefficient, $a$, to determine if a parabola opens upward or downward. If $a$ is positive, then it opens upward. If $a$ is negative, then it opens downward.
- The vertex of any parabola has an $x$-value equal to $-\frac{b}{2a}$. After finding the $x$-value of the vertex, substitute it into the original equation to find the corresponding $y$-value. This $y$-value is a maximum if the parabola opens downward, and it is a minimum if the parabola opens upward.
- The domain of a parabola opening upward or downward consists of all real numbers. The range is bounded by the $y$-value of the vertex.
- An alternate approach to finding the vertex is to rewrite the quadratic function in the form $f(x) = a(x-h)^2 + k$ . When in this form, the vertex is $(h, k)$ and can be read directly from the equation. To obtain this form, take $f(x) = ax^2 + bx + c$ and complete the square.

**Exercise 2.1.2.3**

Does the parabola open upward or downward? Explain.

1. $y = x^2 - 9x + 20$
2. $y = x^2 - 12x + 32$
3. $y = -2x^2 + 5x + 12$
4. $y = -6x^2 + 13x - 6$
5. $y = 64 - x^2$
6. $y = -3x + 9x^2$

**Answer**

1. Upward
3. Downward
5. Downward

**Exercise 2.1.2.4**

Determine the $x$- and $y$-intercepts.

1. $y = x^2 + 4x - 12$
2. $y = x^2 - 13x + 12$
3. $y = 2x^2 + 5x - 3$
4. $y = 3x^2 - 4x - 4$
5. $y = -5x^2 - 3x + 2$
6. $y = -6x^2 + 11x - 4$
7. $y = 4x^2 - 27$
8. $y = 9x^2 - 50$
9. $y = x^2 - x + 1$

10. $y = x^2 - 6x + 4$

**Answer**

1. $x$-intercepts: $(-6, 0), (2, 0)$; $y$-intercept: $(0, -12)$
3. $x$-intercepts: $(-3, 0), \left(\frac{1}{2}, 0\right)$; $y$-intercept: $(0, -3)$
5. $x$-intercepts: $(-1, 0), \left(\frac{2}{5}, 0\right)$; $y$-intercept: $(0, 2)$
7. $x$-intercepts: $\left(-\frac{3\sqrt{3}}{2}, 0\right), \left(\frac{3\sqrt{3}}{2}, 0\right)$; $y$-intercept: $(0, -27)$
9. $x$-intercepts: none; $y$-intercept: $(0, 1)$

## Exercise 2.1.2.5

Find the vertex and the line of symmetry.

1. $y = -x^2 + 10x - 34$
2. $y = -x^2 - 6x + 1$
3. $y = -4x^2 + 12x - 7$
4. $y = -9x^2 + 6x + 2$
5. $y = 4x^2 - 1$
6. $y = x^2 - 16$

**Answer**

1. Vertex: $(5, -9)$; line of symmetry: $x = 5$
3. Vertex: $\left(\frac{3}{2}, 2\right)$; line of symmetry: $x = \frac{3}{2}$
5. Vertex: $(0, -1)$; line of symmetry: $x = 0$

## Exercise 2.1.2.6

Graph. Find the vertex and the $y$-intercept. In addition, find the $x$-intercepts if they exist.

1. $f(x) = x^2 - 2x - 8$
2. $f(x) = x^2 - 4x - 5$
3. $f(x) = -x^2 + 4x + 12$
4. $f(x) = -x^2 - 2x + 15$
5. $f(x) = x^2 - 10x$
6. $f(x) = x^2 + 8x$
7. $f(x) = x^2 - 9$
8. $f(x) = x^2 - 25$
9. $f(x) = 1 - x^2$
10. $f(x) = 4 - x^2$
11. $f(x) = x^2 - 2x + 1$
12. $f(x) = x^2 + 4x + 4$
13. $f(x) = -4x^2 + 12x - 9$
14. $f(x) = -4x^2 - 4x + 3$
15. $f(x) = x^2 - 2$
16. $f(x) = x^2 - 3$
17. $f(x) = -4x^2 + 4x - 3$
18. $f(x) = 4x^2 + 4x + 3$
19. $f(x) = x^2 - 2x - 2$
20. $f(x) = x^2 - 6x + 6$
21. $f(x) = -2x^2 + 6x - 3$

22. $f(x) = -4x^2 + 4x + 1$
23. $f(x) = x^2 + 3x + 4$
24. $f(x) = -x^2 + 3x - 4$
25. $f(x) = -2x^2 + 3$
26. $f(x) = -2x^2 - 1$
27. $f(x) = 2x^2 + 4x - 3$
28. $f(x) = 3x^2 + 2x - 2$

**Answer**

1.

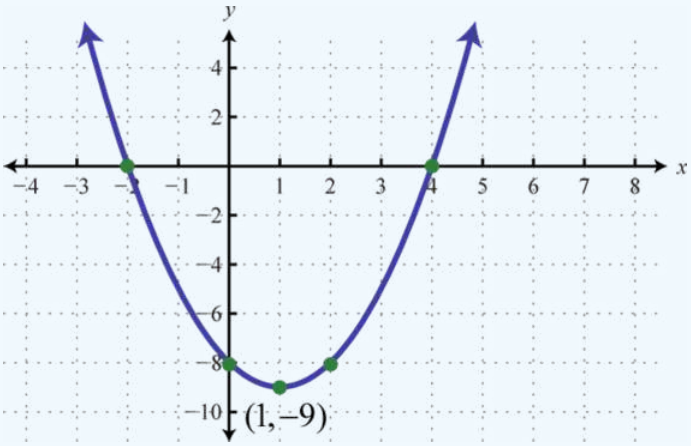



Figure 2.1.2.11

3.

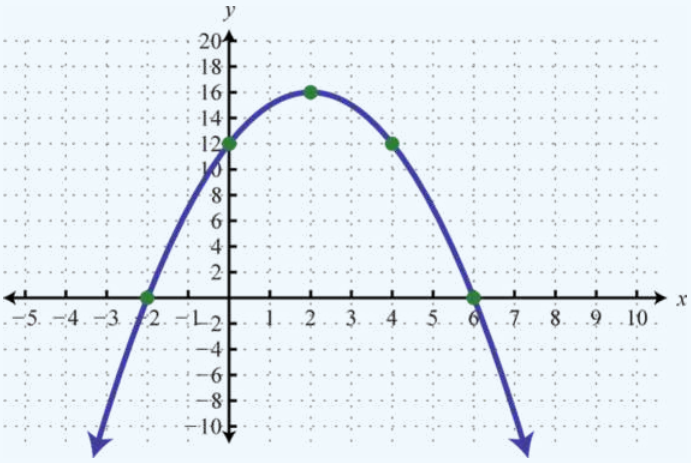



Figure 2.1.2.12

5.

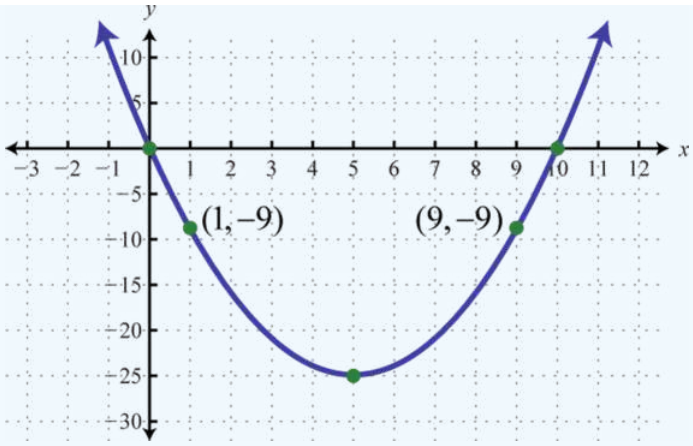



Figure 2.1.2.13

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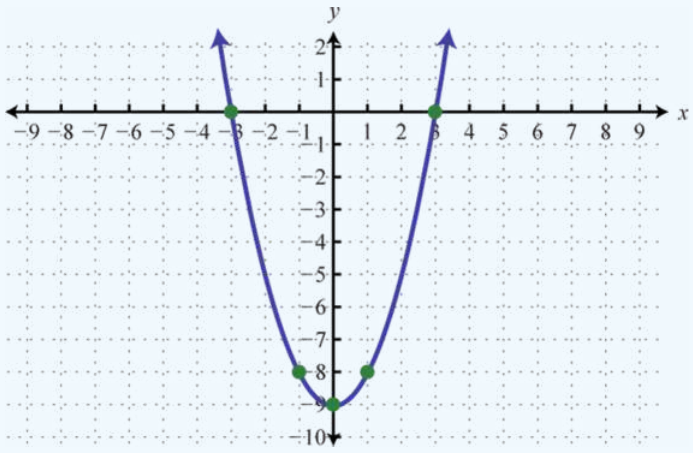



Figure 2.1.2.14

9.

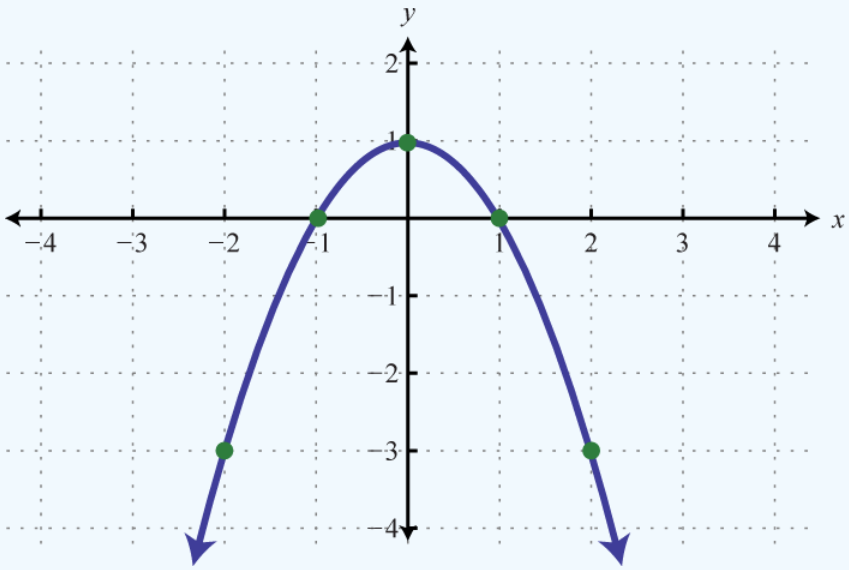



Figure 2.1.2.15

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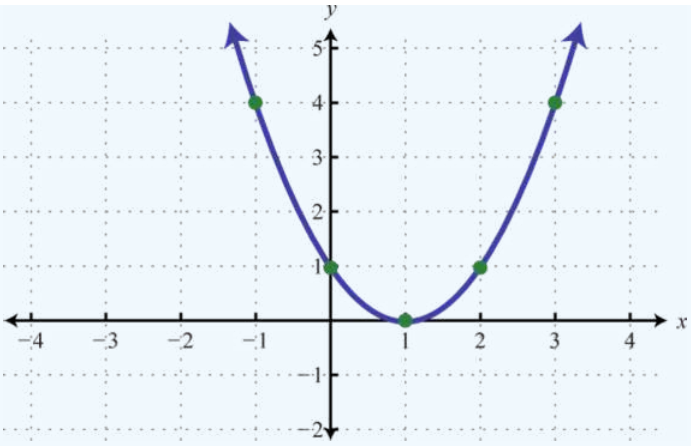



Figure 2.1.2.16

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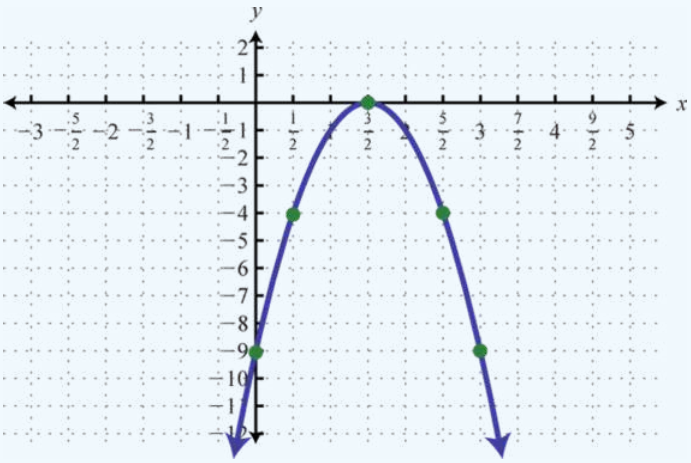



Figure 2.1.2.17

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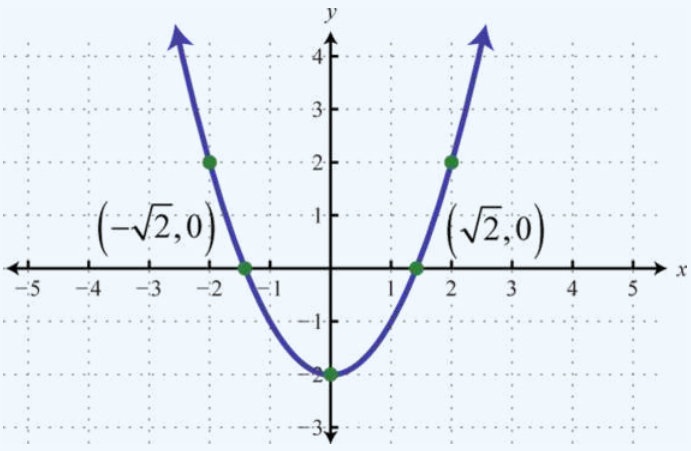



Figure 2.1.2.18

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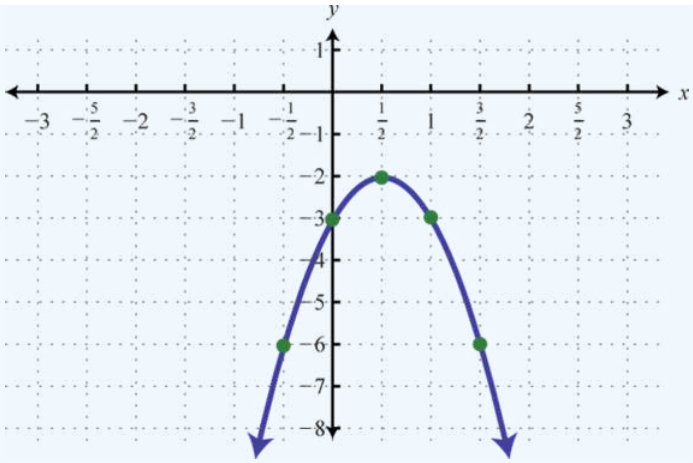


Figure 2.1.2.19

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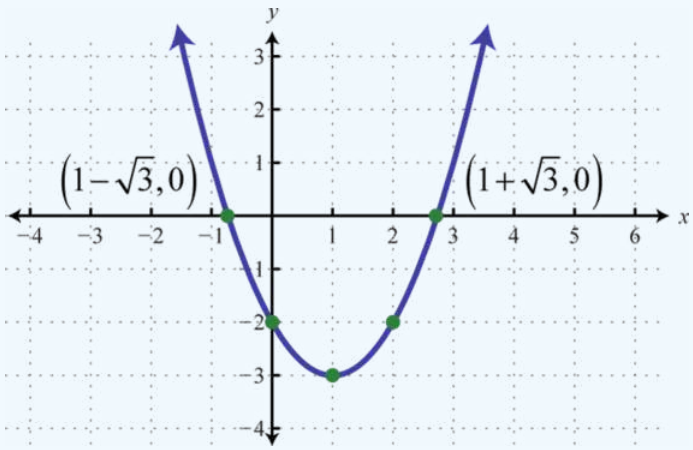



Figure 2.1.2.20

21.

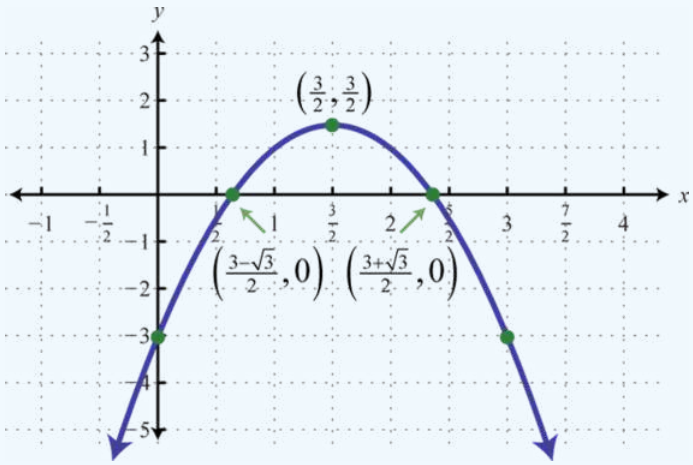



Figure 2.1.2.21

23.

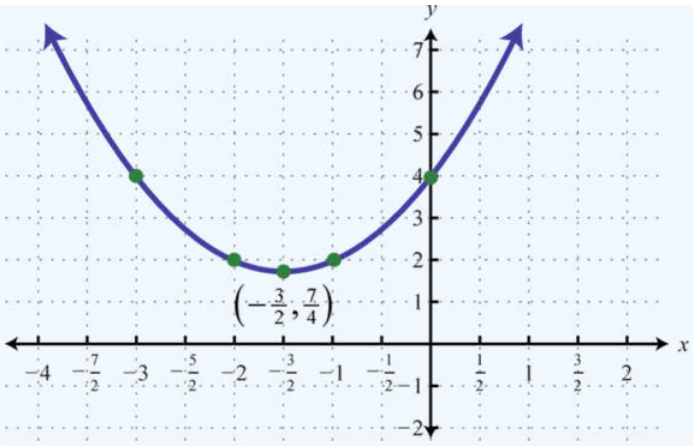



Figure 2.1.2.22

25.

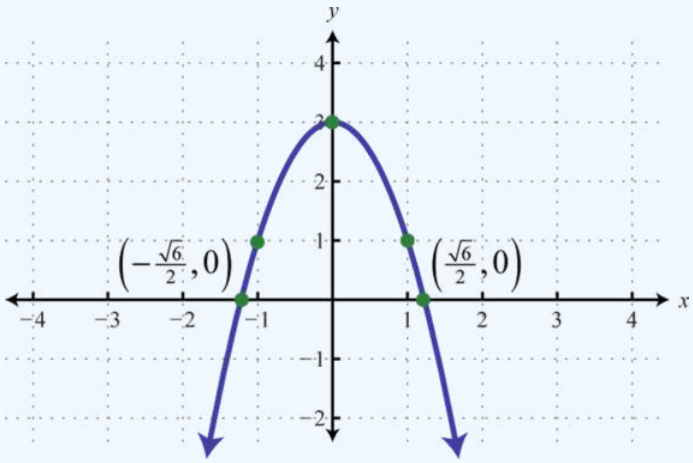



Figure 2.1.2.23

27.

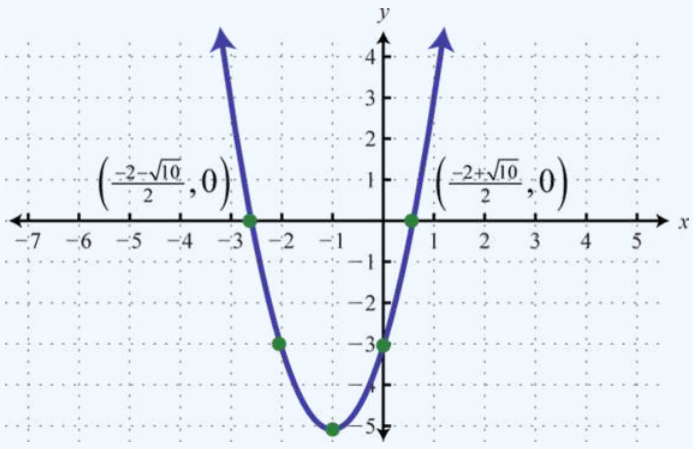



Figure 2.1.2.24

## Exercise 2.1.2.7

Determine the maximum or minimum $y$-value.

1. $y = -x^2 - 6x + 1$
2. $y = -x^2 - 4x + 8$

3. $y = 25x^2 - 10x + 5$
4. $y = 16x^2 - 24x + 7$
5. $y = -x^2$
6. $y = 1 - 9x^2$
7. $y = 20x - 10x^2$
8. $y = 12x + 4x^2$
9. $y = 3x^2 - 4x - 2$
10. $y = 6x^2 - 8x + 5$
11. $y = x^2 - 5x + 1$
12. $y = 1 - x - x^2$

**Answer**

1. Maximum: $y = 10$
3. Minimum: $y = \frac{4}{5}$
5. Maximum: $y = 0$
7. Maximum: $y = 10$
9. Minimum: $y = -\frac{10}{3}$
11. Minimum: $y = -\frac{21}{4}$

## Exercise 2.1.2.8

Given the following quadratic functions, determine the domain and range.

1. $f(x) = 3x^2 + 30x + 50$
2. $f(x) = 5x^2 - 10x + 1$
3. $g(x) = -2x^2 + 4x + 1$
4. $g(x) = -7x^2 - 14x - 9$
5. $f(x) = x^2 + x - 1$
6. $f(x) = -x^2 + 3x - 2$
7. The height in feet reached by a baseball tossed upward at a speed of 48 feet per second from the ground is given by the function $h(t) = -16t^2 + 48t$ , where $t$ represents the time in seconds after the ball is thrown. What is the baseball's maximum height and how long does it take to attain that height?
8. The height in feet of a projectile launched straight up from a mound is given by the function $h(t) = -16t^2 + 96t + 4$ , where $t$ represents seconds after launch. What is the maximum height?
9. The profit in dollars generated by producing and selling $x$ custom lamps is given by the function $P(x) = -10x^2 + 800x - 12,000$. What is the maximum profit?
10. The profit in dollars generated from producing and selling a particular item is modeled by the formula $P(x) = 100x - 0.0025x^2$, where $x$ represents the number of units produced and sold. What number of units must be produced and sold to maximize revenue?
11. The average number of hits to a radio station Web site is modeled by the formula $f(x) = 450t^2 - 3,600t + 8,000$, where $t$ represents the number of hours since 8 : 00 a.m. At what hour of the day is the number of hits to the Web site at a minimum?
12. The value in dollars of a new car is modeled by the formula $V(t) = 125t^2 - 3,000t + 22,000$, where $t$ represents the number of years since it was purchased. Determine the minimum value of the car.
13. The daily production cost in dollars of a textile manufacturing company producing custom uniforms is modeled by the formula $C(x) = 0.02x^2 - 20x + 10,000$, where $x$ represents the number of uniforms produced.
    1. How many uniforms should be produced to minimize the daily production costs?
    2. What is the minimum daily production cost?
14. The area in square feet of a certain rectangular pen is given by the formula $A = 14w - w^2$ , where $w$ represents the width in feet. Determine the width that produces the maximum area.

**Answer**

1. Domain: $(-\infty, \infty)$ ; range:$[-25, \infty)$
3. Domain: $(-\infty, \infty)$ ; range: $(-\infty, 3]$
5. Domain: $(-\infty, \infty)$ ; range: $[-\frac{5}{4}, \infty)$
7. The maximum height of $36$ feet occurs after $1.5$ seconds.
9. $\$4,000$
11. $12:00$ p.m.
13. (1) $500$ uniforms (2) $\$5,000$

## Exercise 2.1.2.9

Determine the vertex.

1. $y=-(x-5)^2+3$
2. $y=-2(x-1)^2+7$
3. $y=5(x+1)^2+6$
4. $y=3(x+4)^2+10$
5. $y=-5(x+8)^2-1$
6. $y=(x+2)^2-5$

**Answer**

1. $(5,3)$
3. $(-1,6)$
5. $(-8,-1)$

## Exercise 2.1.2.10

Rewrite in vertex form $y=a(x-h)^2+k$ and determine the vertex.

1. $y=x^2-14x+24$
2. $y=x^2-12x+40$
3. $y=x^2+4x-12$
4. $y=x^2+6x-1$
5. $y=2x^2-12x-3$
6. $y=3x^2-6x+5$
7. $y=-x^2+16x+17$
8. $y=-x^2+10x$

**Answer**

1. $y=(x-7)^2-25$ ; vertex: $(7,-25)$
3. $y=(x+2)^2-16$ ; vertex: $(-2,-16)$
5. $y=2(x-3)^2-21$ ; vertex: $(3,-21)$
7. $y=-(x-8)^2+81$ ; vertex: $(8,81)$

## Exercise 2.1.2.11

Graph. Find the vertex and the $y$-intercept. In addition, find the $x$-intercepts if they exist.

1. $f(x)=x^2-1$
2. $f(x)=x^2+1$
3. $f(x)=(x-1)^2$
4. $f(x)=(x+1)^2$
5. $f(x)=(x-4)^2-9$
6. $f(x)=(x-1)^2-4$
7. $f(x)=-2(x+1)^2+8$
8. $f(x)=-3(x+2)^2+12$
9. $f(x)=-5(x-1)^2$
10. $f(x)=-(x+2)^2$
11. $f(x)=-4(x-1)^2-2$
12. $f(x)=9(x+1)^2+2$
13. $f(x)=(x+5)^2-15$
14. $f(x)=2(x-5)^2-3$
15. $f(x)=-2(x-4)^2+22$
16. $f(x)=2(x+3)^2-13$

**Answer**

1.

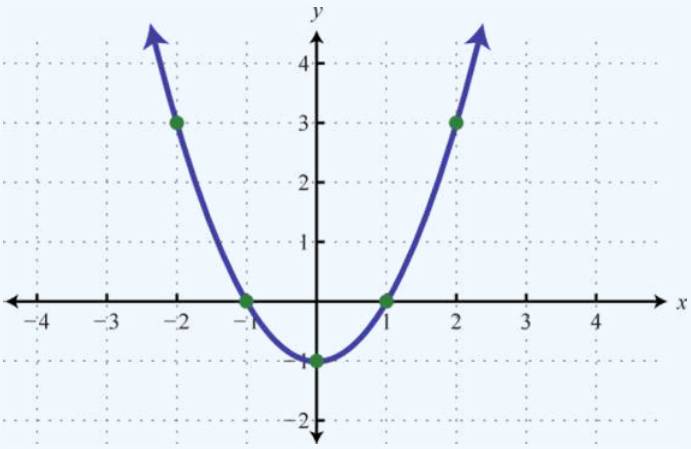



Figure 2.1.2.25

3.

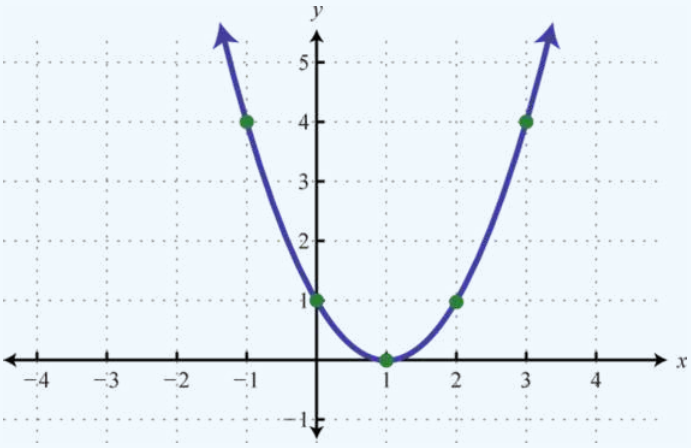



Figure 2.1.2.26

5.

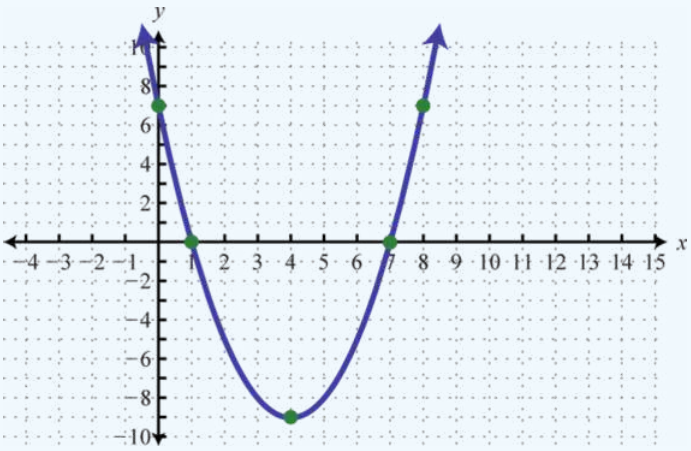



Figure 2.1.2.27

7.

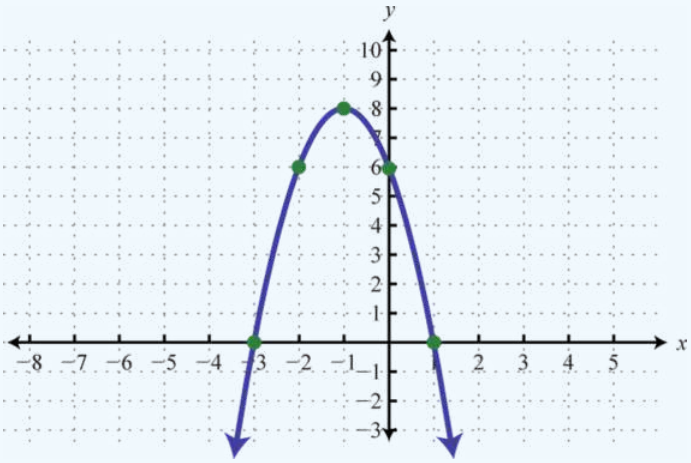



Figure 2.1.2.28

9.

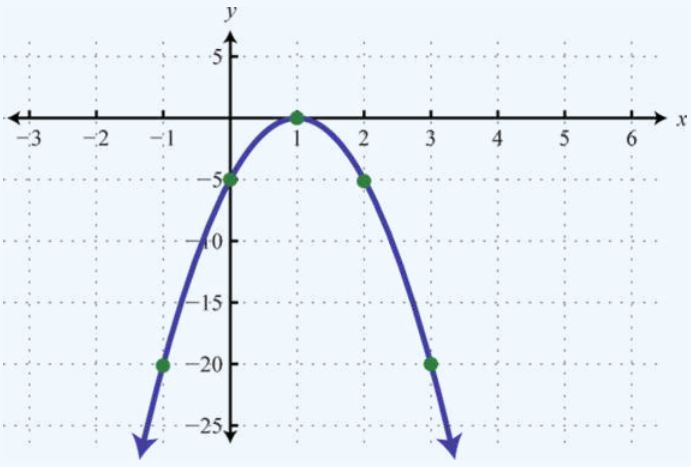



Figure 2.1.2.29

11.

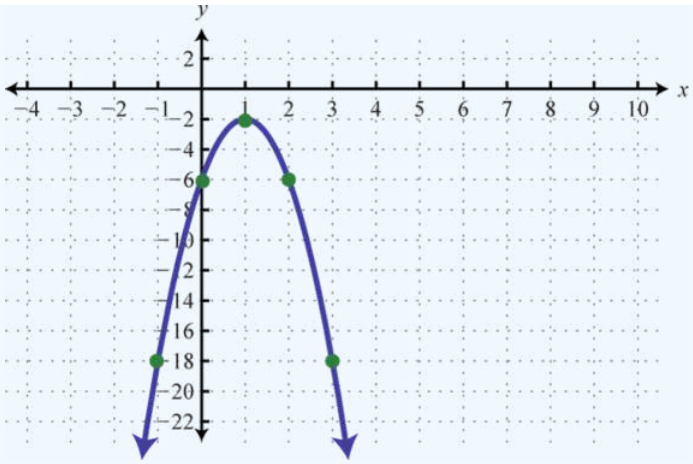


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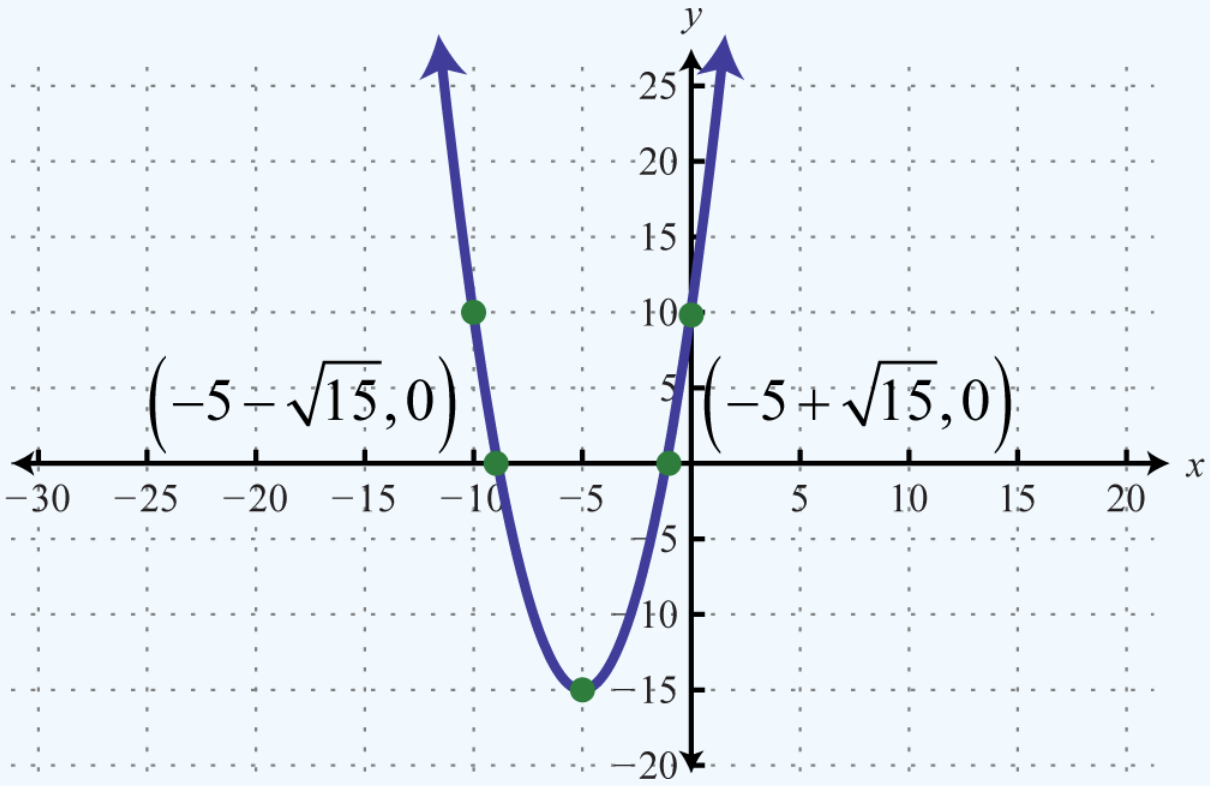



Figure 2.1.2.31

15.


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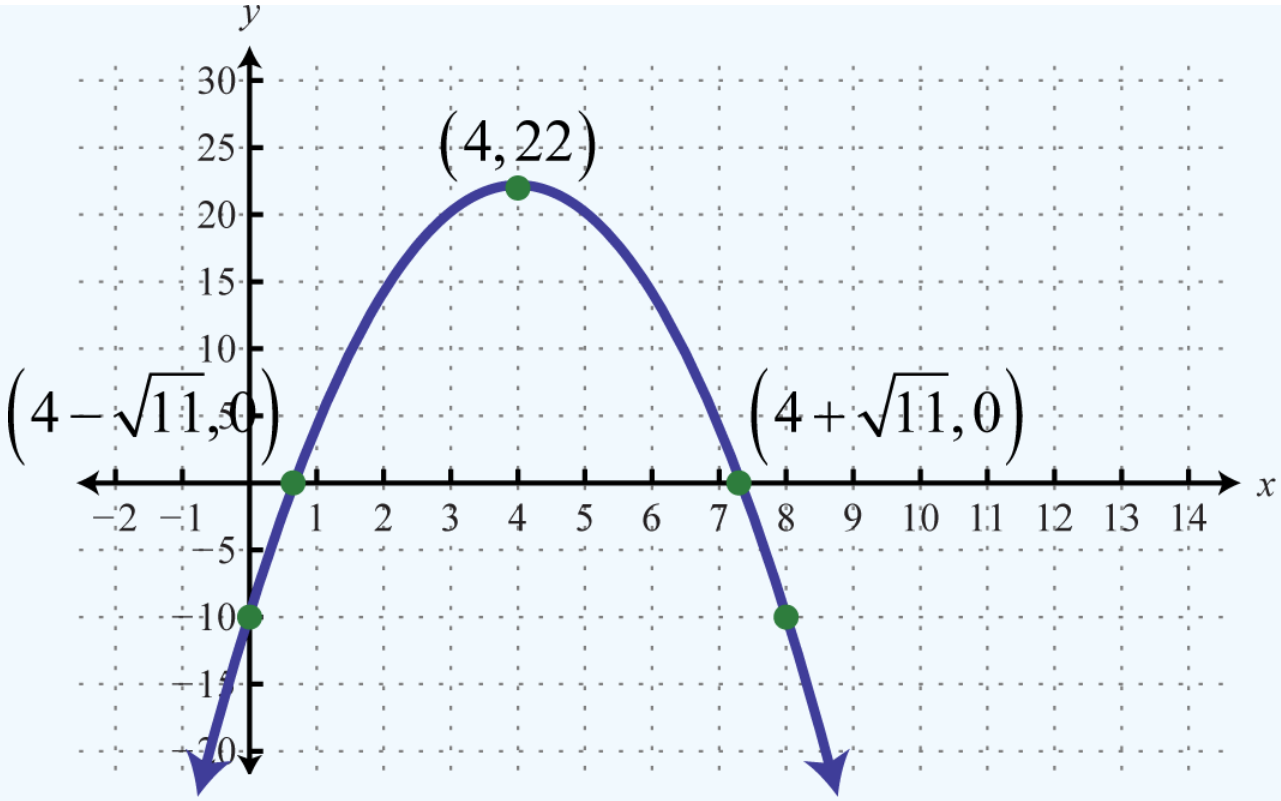



Figure 2.1.2.32

**Exercise 2.1.2.12**

1. Write down your plan for graphing a parabola on an exam. What will you be looking for and how will you present your answer? Share your plan on the discussion board.
2. Why is any parabola that opens upward or downward a function? Explain to a classmate how to determine the domain and range.
3. Research and discuss ways of finding a quadratic function that has a graph passing through any three given points. Share a list of steps as well as an example of how to do this.

**Answer**

1. Answer may vary
3. Answer may vary

## Footnotes

[10]The U-shaped graph of any quadratic function defined by $f(x) = ax^2 + bx + c$ , where $a, b$, and $c$ are real numbers and $a \neq 0$.

[11]The point that defines the minimum or maximum of a parabola.

[12]The vertical line through the vertex, $x = -\frac{b}{2a}$, about which the parabola is symmetric.

[13]A term used when referencing the line of symmetry.

[14]A quadratic function written in the form $f(x) = a(x - h)^2 + k$ .

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# 2.1.3: Solving Quadratic Inequalities

**Learning Objectives**

- Check solutions to quadratic inequalities with one variable.
- Understand the geometric relationship between solutions to quadratic inequalities and their graphs.
- Solve quadratic inequalities.

## Solutions to Quadratic Inequalities

A **quadratic inequality**[15] is a mathematical statement that relates a quadratic expression as either less than or greater than another. Some examples of quadratic inequalities solved in this section follow.

Table 2.1.3.1

| | | |
|---|---|---|
| | | $9-x^2>0$ |

A solution to a quadratic inequality is a real number that will produce a true statement when substituted for the variable.

**Example 2.1.3.1:**

Are $-3$, $-2$, and $-1$ solutions to $x^2-x-6\le 0$ ?

**Solution**

Substitute the given value in for $x$ and simplify.

| | | |
|---|---|---|
| $x^2-x-6\le 0$ | $x^2-x-6\le 0$ | $x^2-x-6\le 0$ |
| $(-3)^2-(-3)-6\le 0$ | $(-2)^2-(-2)-6\le 0$ | $(-1)^2-(-1)-6\le 0$ |
| $9+3-6\le 0$ | $4+2-6\le 0$ | $1+1-6\le 0$ |
| $6\le 0$ ✗ | $0\le 0$ ✓ | $-4\le 0$ ✓ |

**Answer**:

$-2$ and $-1$ are solutions and $-3$ is not.

Quadratic inequalities can have infinitely many solutions, one solution, or no solution. If there are infinitely many solutions, graph the solution set on a number line and/or express the solution using interval notation. Graphing the function defined by $f(x)=x^2-x-6$ found in the previous example we have

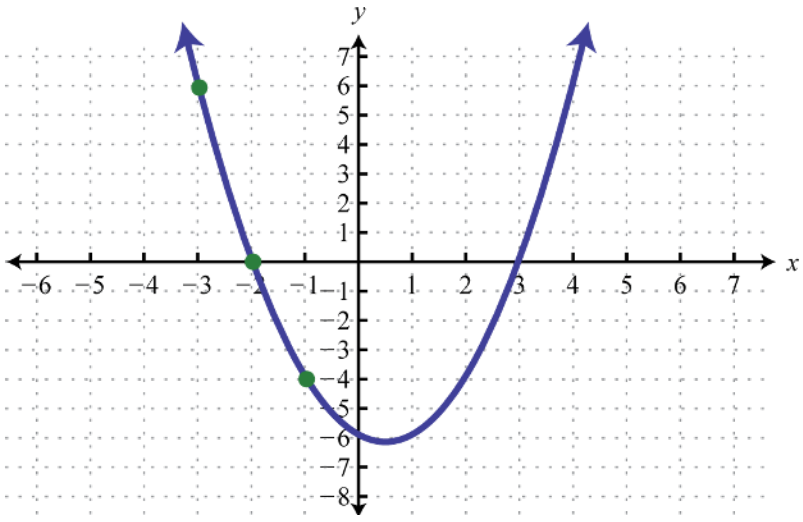



Figure 2.1.3.1

The result of evaluating for any $x$-value will be negative, zero, or positive.

The values in the domain of a function that separate regions that produce positive or negative results are called **critical numbers**[16]. In the case of a quadratic function, the critical numbers are the roots, sometimes called the zeros. For example, $f(x) = x^2 - x - 6 = (x+2)(x-3)$ has roots $-2$ and $3$. These values bound the regions where the function is positive (above the $x$-axis) or negative (below the $x$-axis).

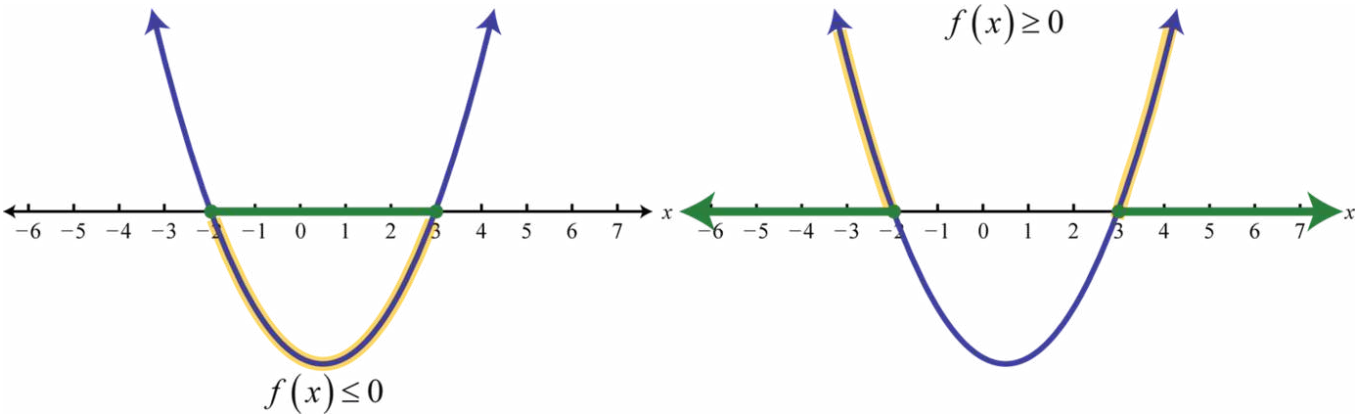



Figure 2.1.3.2

Therefore $x^2 - x - 6 \leq 0$ has solutions where $-2 \leq x \leq 3$, using interval notation $[-2, 3]$. Furthermore, $x^2 - x - 6 \geq 0$ has solutions where $x \leq -2$ or $x \geq 3$, using interval notation $(-\infty, -2] \cup [-3, \infty)$.

**Example 2.1.3.2:**

Given the graph of $f$ determine the solutions to $f(x) > 0$:

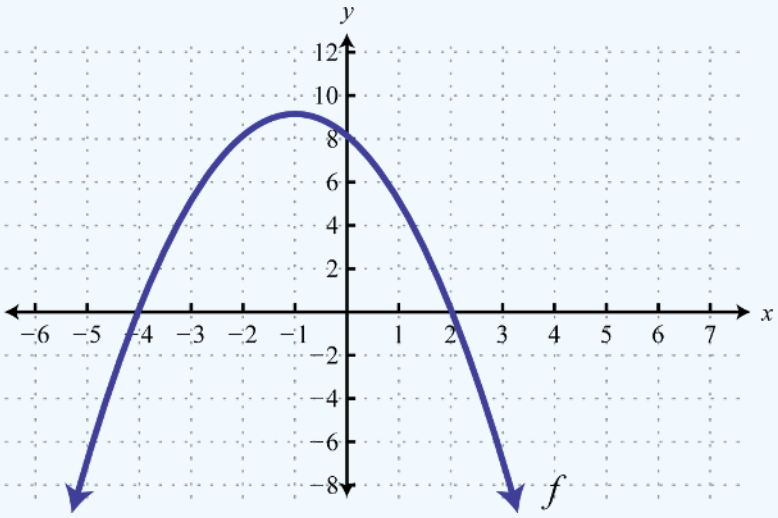



Figure 2.1.3.4

Because of the strict inequality, the solution set is shaded with an open dot on each of the boundaries. This indicates that these critical numbers are not actually included in the solution set. This solution set can be expressed two ways,

$$\{x \mid -4 < x < 2\} \qquad \textit{Set Notation}$$
$$(-4,2) \qquad \textit{Interval Notation}$$

In this textbook, we will continue to present answers in interval notation.

**Answer**:

$(-4, 2)$

**Exercise 2.1.3.1**

Given the graph of $f$ determine the solutions to $f(x) < 0$:

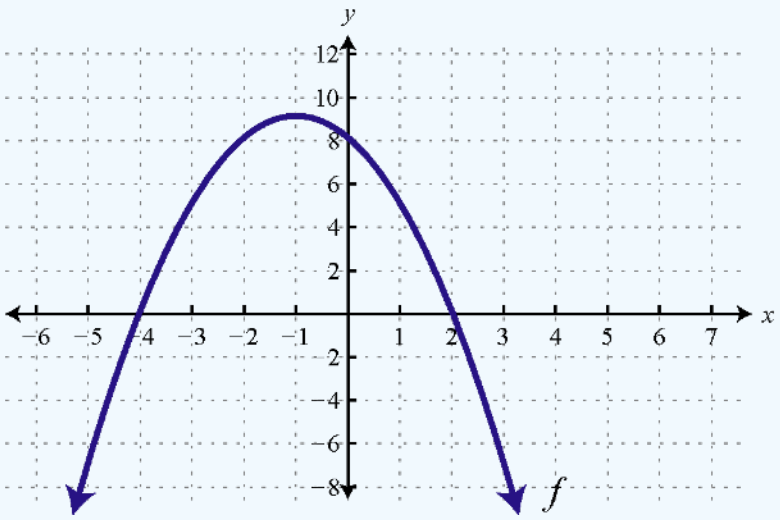



Figure 2.1.3.5

**Answer**

$(-\infty, -4) \cup (2, \infty)$

www.youtube.com/v/kkBaoM8buBo

## Solving Quadratic Inequalities

Next we outline a technique used to solve quadratic inequalities without graphing the parabola. To do this we make use of a **sign chart**[17] which models a function using a number line that represents the $x$-axis and signs $(+ \text{ or } -)$ to indicate where the function is positive or negative. For example,

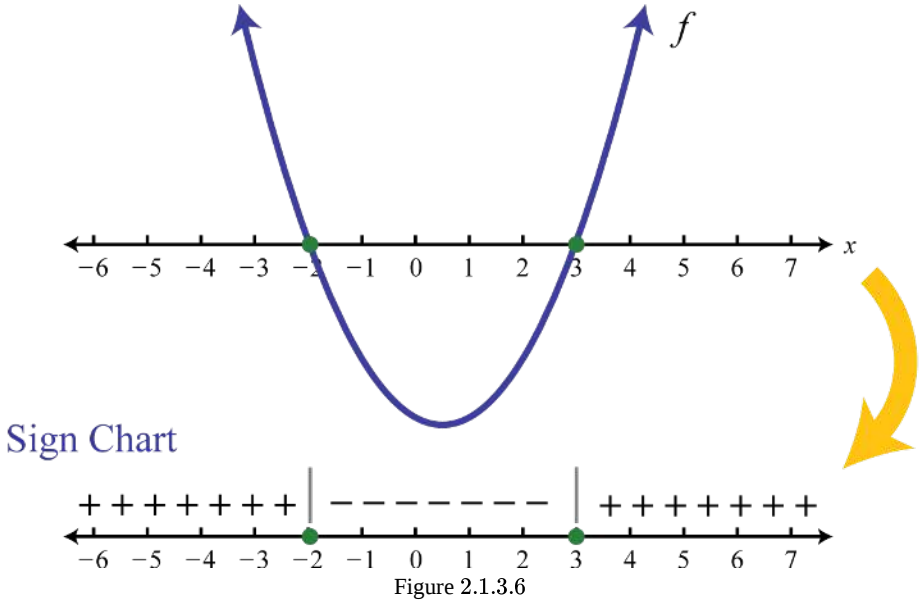



Figure 2.1.3.6

The plus signs indicate that the function is positive on the region. The negative signs indicate that the function is negative on the region. The boundaries are the critical numbers, $-2$ and $3$ in this case. Sign charts are useful when a detailed picture of the graph is not needed and are used extensively in higher level mathematics. The steps for solving a quadratic inequality with one variable are outlined in the following example.

Example 2.1.3.3:

Solve: $-x^2+6x+7 \geq 0$ .

**Solution**

It is important to note that this quadratic inequality is in standard form, with zero on one side of the inequality.

**Step 1**: Determine the critical numbers. For a quadratic inequality in standard form, the critical numbers are the roots. Therefore, set the function equal to zero and solve.

$$\begin{aligned} -x^2+6x+7 &= 0 \\ -\left(x^2-6x-7\right) &= 0 \\ -(x+1)(x-7) &= 0 \end{aligned}$$

The critical numbers are $-1$ and $7$.

**Step 2**: Create a sign chart. Since the critical numbers bound the regions where the function is positive or negative, we need only test a single value in each region. In this case the critical numbers partition the number line into three regions and we choose test values $x=-3$, $x=0$, and $x=10$.



Figure 2.1.3.7

Test values may vary. In fact, we need only determine the sign $(+$ or $-)$ of the result when evaluating $f(x)=-x^2+6x+7=-(x+1)(x-7)$ . Here we evaluate using the factored form.

$$\begin{aligned} f(-3)=-(-3+1)(-3-7)=-(-2)(-10) &= -Negative \\ f(0)=-(0+1)(0-7)=-(1)(-7) &= +Positive \\ f(10)=-(10+1)(10-7)=-(11)(3) &= -Negative \end{aligned}$$

Since the result of evaluating for $-3$ was negative, we place negative signs above the first region. The result of evaluating for $0$ was positive, so we place positive signs above the middle region. Finally, the result of evaluating for $10$ was negative, so we place negative signs above the last region, and the sign chart is complete.

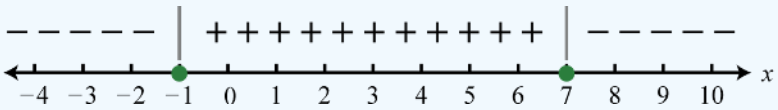



Figure 2.1.3.8

**Step 3**: Use the sign chart to answer the question. In this case, we are asked to determine where $f(x)\geq 0$, or where the function is positive or zero. From the sign chart we see this occurs when $x$-values are inclusively between $-1$ and $7$.

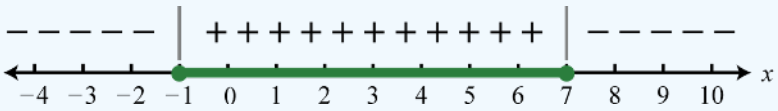



Figure 2.1.3.9

Using interval notation, the shaded region is expressed as $[-1,7]$. The graph is not required; however, for the sake of completeness it is provided below.

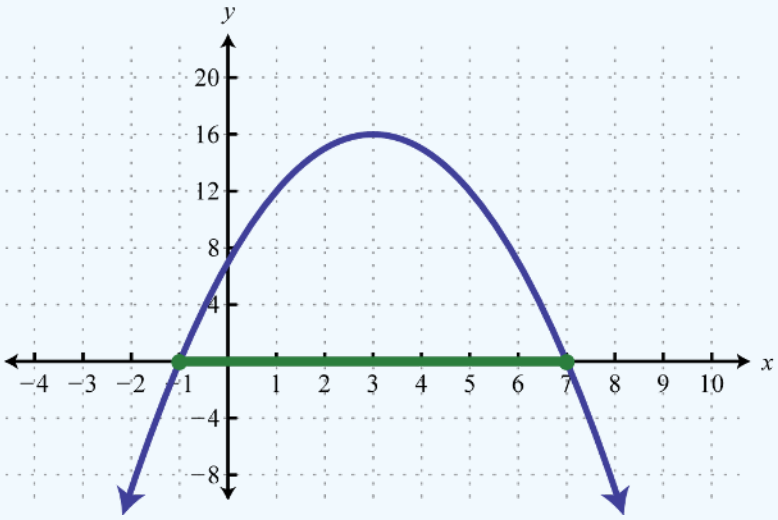



Figure 2.1.3.10

Indeed the function is greater than or equal to zero, above or on the $x$-axis, for $x$-values in the specified interval.

**Answer**:

$[-1,7]$

**Example 2.1.3.4:**

Solve: $2x^2-7x+3>0$ .

**Solution**

Begin by finding the critical numbers, in this case, the roots of $f(x)=2x^2-7x+3$ .

The critical numbers are $\frac{1}{2}$ and $3$. Because of the strict inequality > we will use open dots.

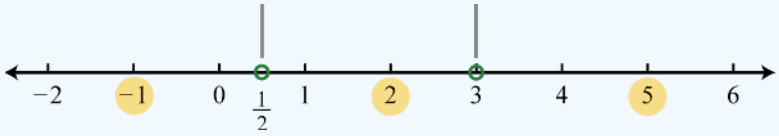



Figure 2.1.3.11

Next choose a test value in each region and determine the sign after evaluating $f(x)=2x^2-7x+3=(2x-1)(x-3)$ . Here we choose test values $-1, 2$, and $5$.

$$\begin{aligned} f(-1)&=[2(-1)-1](-1-3) &&=(-)(-)=+\\ f(2)&=[2(2)-1](2-3) &&=(+)(-)=-\\ f(5)&=[2(5)-1](5-3) &&=(+)(+)=+ \end{aligned}$$

And we can complete the sign chart.

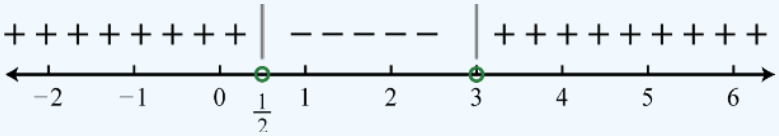



Figure 2.1.3.12

The question asks us to find the $x$-values that produce positive results (greater than zero). Therefore, shade in the regions with a $+$ over them. This is the solution set.

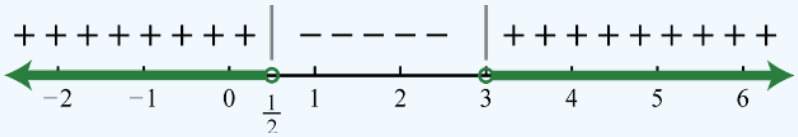



Figure 2.1.3.13

**Answer**:

$\left(-\infty, \frac{1}{2}\right) \cup (3, \infty)$

Sometimes the quadratic function does not factor. In this case we can make use of the quadratic formula.

**Example 2.1.3.5:**

Solve: $x^2-2x-11 \leq 0$

**Solution**

Find the critical numbers.

Identify $a, b$, and $c$ for use in the quadratic formula. Here $a=1, b=-2$ , and $c=-11$. Substitute the appropriate values into the quadratic formula and then simplify.

Therefore the critical numbers are $1-2\sqrt{3} \approx -2.5$ and $1+2\sqrt{3} \approx 4.5$ . Use a closed dot on the number to indicate that these values will be included in the solution set.

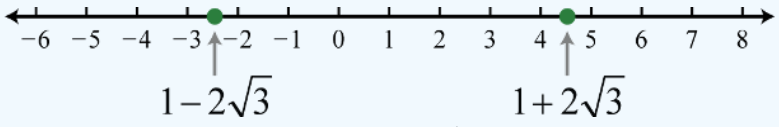



Figure 2.1.3.14

Here we will use test values $-5, 0$, and $7$.

After completing the sign chart shade in the values where the function is negative as indicated by the question $(f(x) \leq 0)$.



Figure 2.1.3.15

**Answer**:

$$[1-2\sqrt{3}, 1+2\sqrt{3}]$$

**Exercise 2.1.3.2**

Solve: $9-x^2>0$ .

**Answer**

$(-3,3)$

www.youtube.com/v/7gMJ8gUvASw

It may be the case that there are no critical numbers.

**Example 2.1.3.6:**

Solve: $x^2-2x+3>0$ .

**Solution**

To find the critical numbers solve,

Substitute $a=1, b=-2$ , and $c=3$ into the quadratic formula and then simplify.

Because the solutions are not real, we conclude there are no real roots; hence there are no critical numbers. When this is the case, the graph has no $x$-intercepts and is completely above or below the $x$-axis. We can test any value to create a sign chart. Here we choose $x=0$.

$$f(0)=(0)^2-2(0)+3=+$$

Because the test value produced a positive result the sign chart looks as follows:

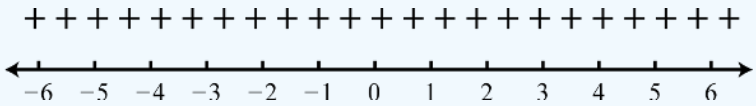



Figure 2.1.3.16

We are looking for the values where $f(x)>0$; the sign chart implies that any real number for $x$ will satisfy this condition.

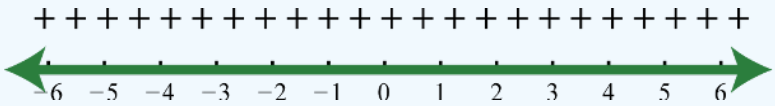



Figure 2.1.3.17

**Answer**:

$(-\infty, \infty)$

The function in the previous example is graphed below.

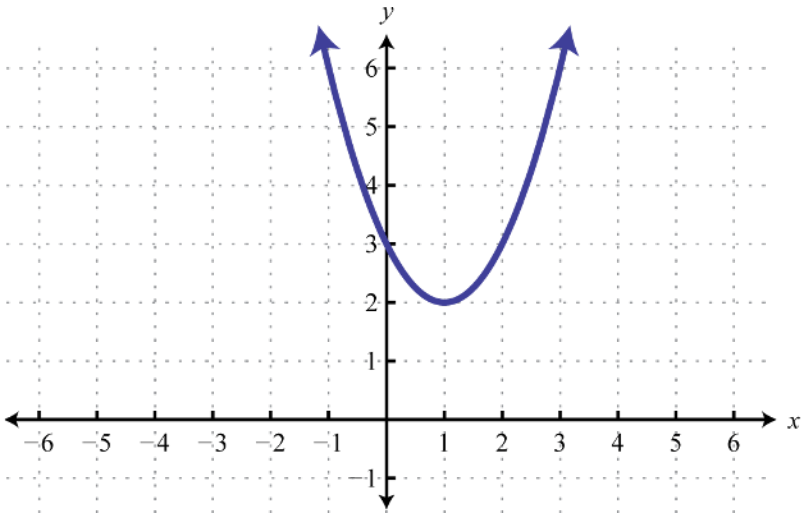


Figure 2.1.3.18

We can see that it has no $x$-intercepts and is always above the $x$-axis (positive). If the question was to solve $x^2-2x+3<0$ , then the answer would have been no solution. The function is never negative.

**Exercise 2.1.3.3**

Solve: $9x^2-12x+4\leq 0$

**Answer**

One solution, $\frac{2}{3}$.

www.youtube.com/v/E7VcOYVV_Ds

**Example 2.1.3.7:**

Find the domain: $f(x)=\sqrt{x^2-4}$.

**Solution**

Recall that the argument of a square root function must be nonnegative. Therefore, the domain consists of all real numbers for $x$ such that $x^2-4$ is greater than or equal to zero.

It should be clear that $x^2-4=0$ has two solutions $x=\pm 2$; these are the critical values. Choose test values in each interval and evaluate $f(x)=x^2-4$.

$f(-3)=(-3)^2-4=9-4=+$

$f(0)=(0)^2-4=0-4=-$

$f(3)=(3)^2-4=9-4=+$

Shade in the $x$-values that produce positive results.

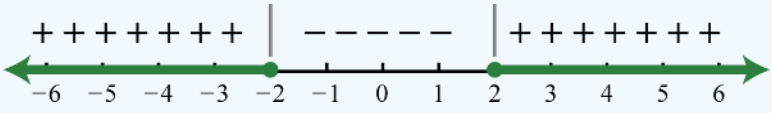


Figure 2.1.3.19

**Answer**:

Domain: $(-\infty,-2]\cup[2,\infty)$


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## Key Takeaways

- Quadratic inequalities can have infinitely many solutions, one solution or no solution.
- We can solve quadratic inequalities graphically by first rewriting the inequality in standard form, with zero on one side. Graph the quadratic function and determine where it is above or below the $x$-axis. If the inequality involves "less than," then determine the $x$-values where the function is below the $x$-axis. If the inequality involves "greater than," then determine the $x$-values where the function is above the $x$-axis.
- We can streamline the process of solving quadratic inequalities by making use of a sign chart. A sign chart gives us a visual reference that indicates where the function is above the $x$-axis using positive signs or below the $x$-axis using negative signs. Shade in the appropriate x-values depending on the original inequality.
- To make a sign chart, use the function and test values in each region bounded by the roots. We are only concerned if the function is positive or negative and thus a complete calculation is not necessary.

### Exercise 2.1.3.4

Determine whether or not the given value is a solution.

1. $x^2 - x + 1 < 0; x = -1$
2. $x^2 + x - 1 > 0; x = -2$
3. $4x^2 - 12x + 9 \leq 0; x = \frac{3}{2}$
4. $5x^2 - 8x - 4 < 0; x = -\frac{2}{5}$
5. $3x^2 - x - 2 \geq 0; x = 0$
6. $4x^2 - x + 3 \leq 0; x = -1$
7. $2 - 4x - x^2 < 0; x = \frac{1}{2}$
8. $5 - 2x - x^2 > 0; x = 0$
9. $-x^2 - x - 9 < 0; x = -3$
10. $-x^2 + x - 6 \geq 0; x = 6$

**Answer**

1. No
3. Yes
5. No
7. Yes
9. Yes

### Exercise 2.1.3.5

Given the graph of $f$ determine the solution set.

1. $f(x) \leq 0;$

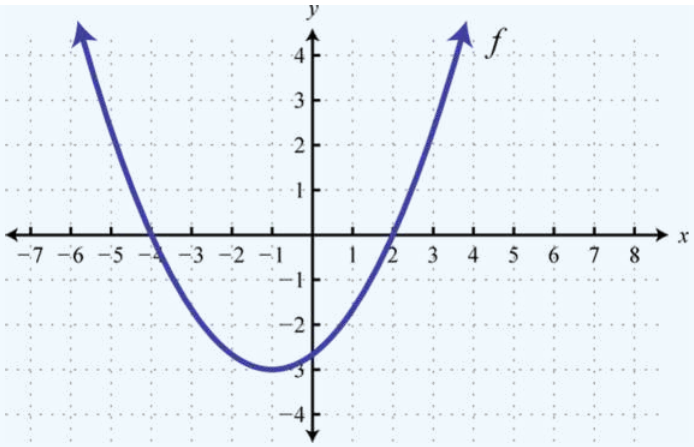



Figure 2.1.3.20

2. $f(x) \geq 0$;

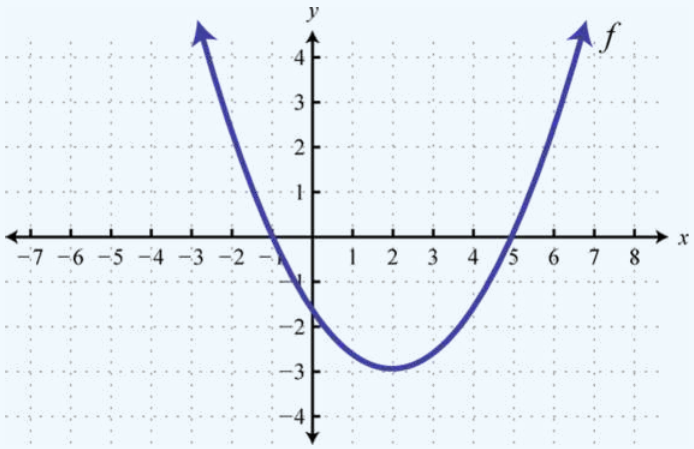



Figure 2.1.3.21

3. $f(x) \geq 0$;

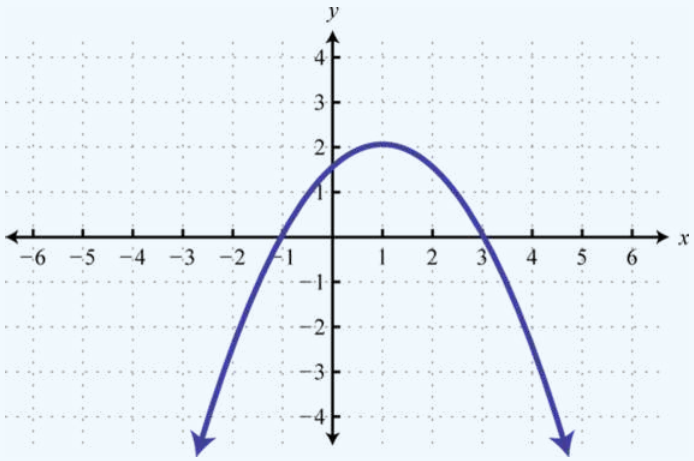



Figure 2.1.3.22

4. $f(x) \leq 0$;

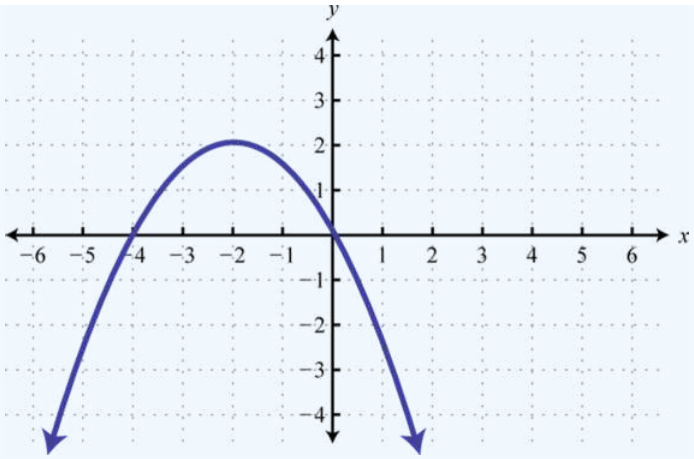



Figure 2.1.3.23

5. $f(x) > 0$;

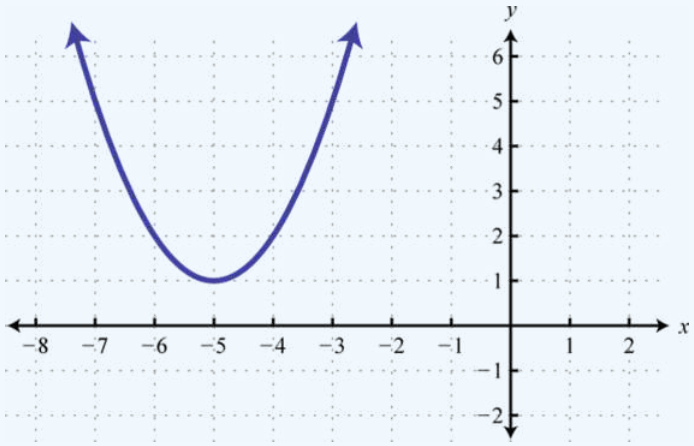



Figure 2.1.3.24

6. $f(x) < 0$;

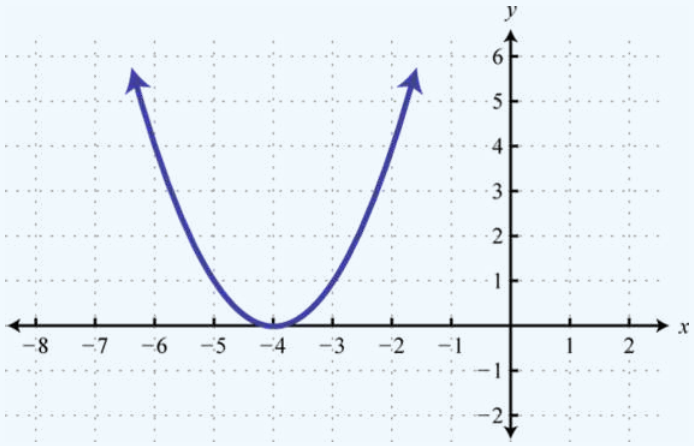



Figure 2.1.3.25

7. $f(x) > 0$;


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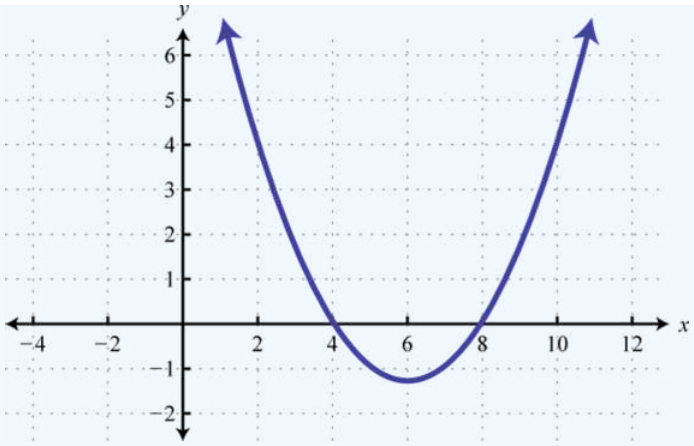



Figure 2.1.3.26

8. $f(x) < 0$;

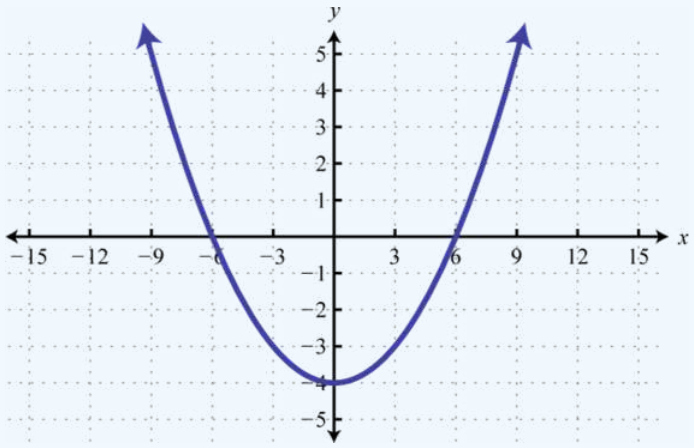



Figure 2.1.3.27

9. $f(x) \geq 0$;

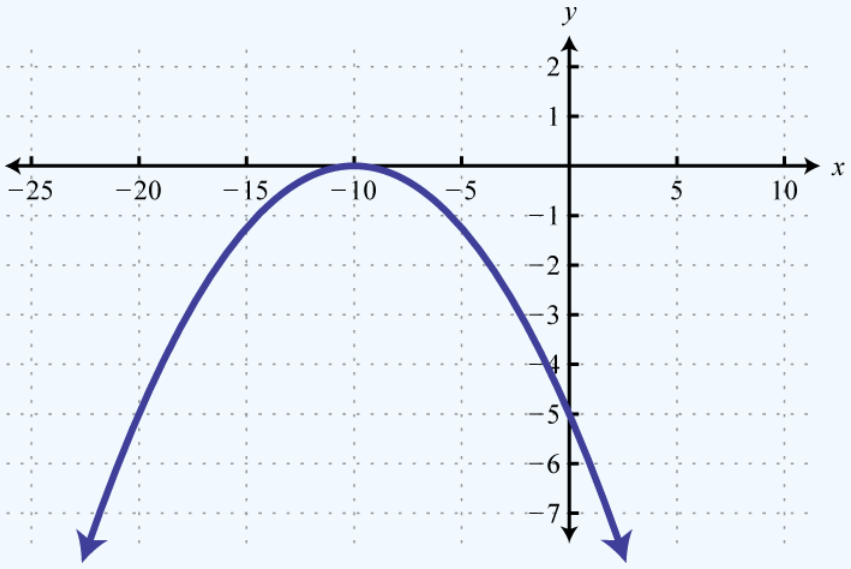



Figure 2.1.3.28

10. $f(x) < 0$;


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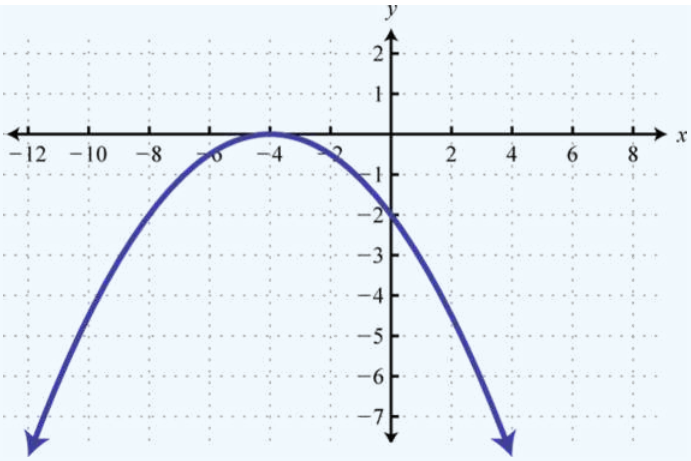



Figure 2.1.3.29

**Answer**

1. $[-4, 2]$
3. $[-1, 3]$
5. $(-\infty, \infty)$
7. $(-\infty, 4) \cup (8, \infty)$
9. $\{-10\}$

## Exercise 2.1.3.6

Use the transformations to graph the following and then determine the solution set.

1. $x^2 - 1 > 0$
2. $x^2 + 2 > 0$
3. $(x-1)^2 > 0$
4. $(x+2)^2 \leq 0$
5. $(x+2)^2 - 1 \leq 0$
6. $(x+3)^2 - 4 > 0$
7. $-x^2 + 4 \geq 0$
8. $-(x+2)^2 > 0$
9. $-(x+3)^2 + 1 < 0$
10. $-(x-4)^2 + 9 > 0$

**Answer**

1. $(-\infty, -1) \cup (1, \infty)$

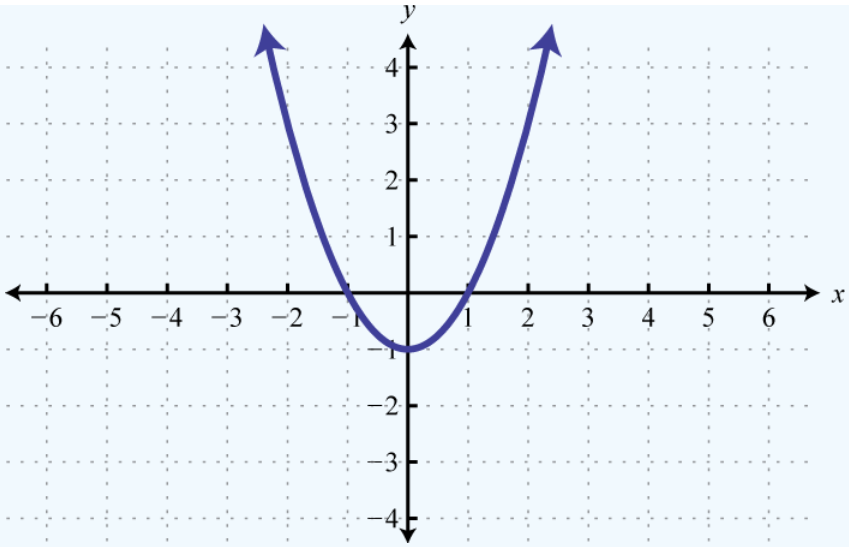



Figure 2.1.3.30

3. $(-\infty, 1) \cup (1, \infty)$

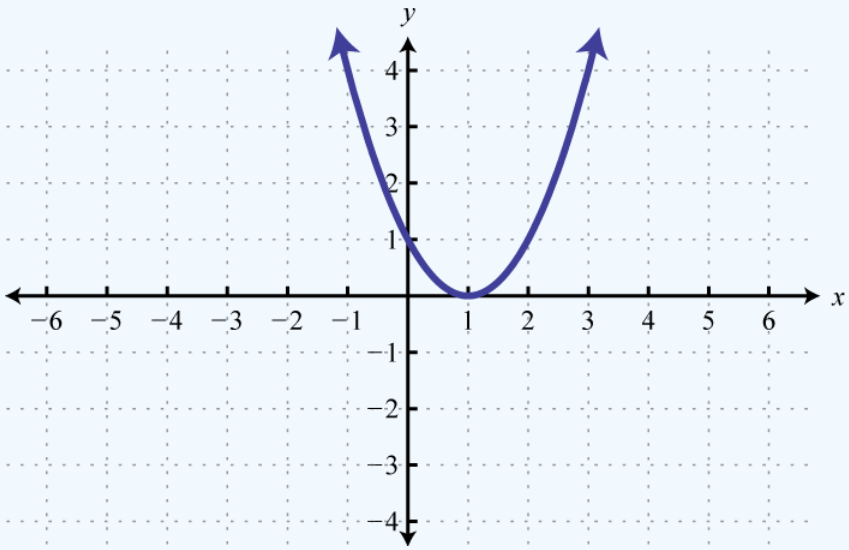



Figure 2.1.3.31

5. $[-3, -1]$

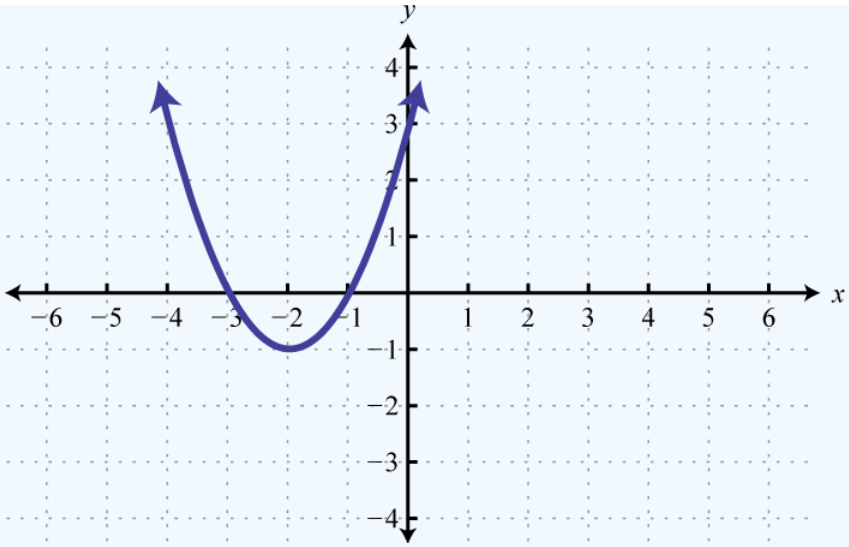



Figure 2.1.3.32

7. $[-2, 2]$

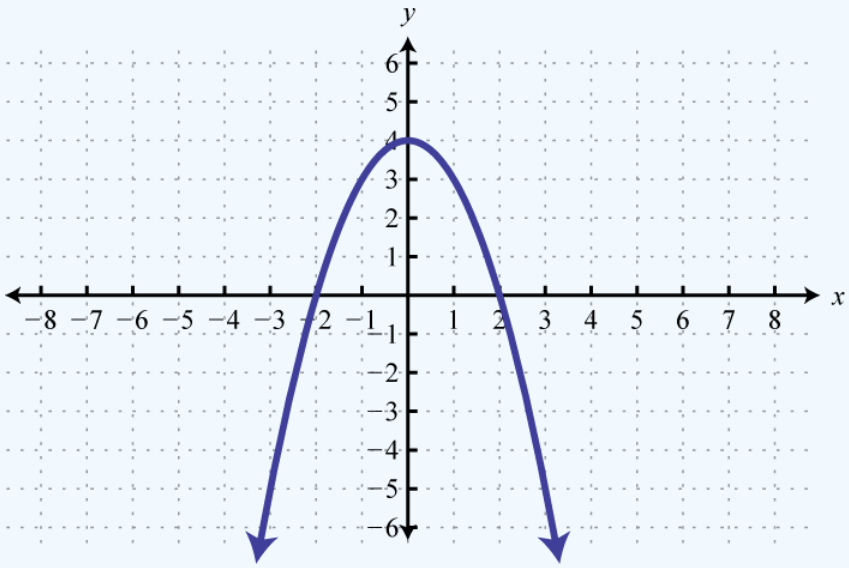



Figure 2.1.3.33

9. $(-\infty, -4) \cup (-2, \infty)$

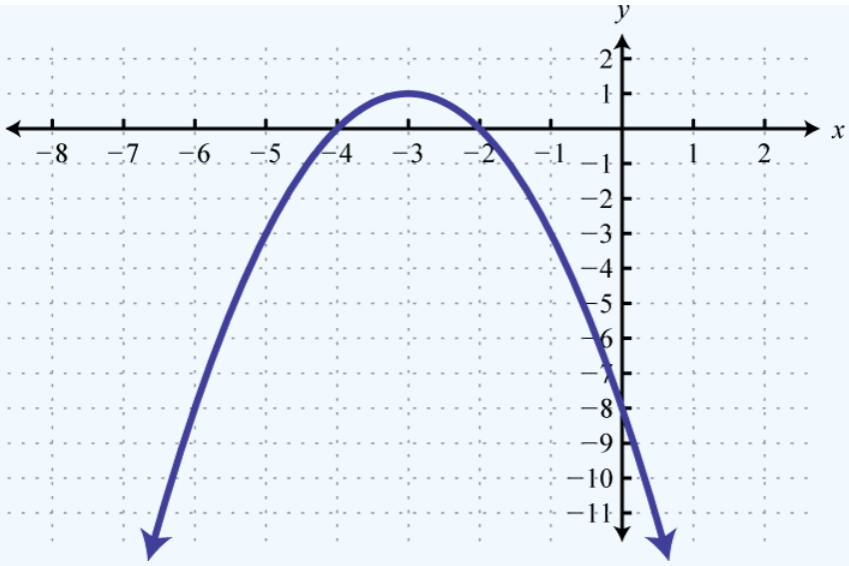



Figure 2.1.3.34

## Exercise 2.1.3.7

Use a sign chart to solve and graph the solution set. Present answers using interval notation.

1. $x^2 - x - 12 > 0$
2. $x^2 - 10x + 16 > 0$
3. $x^2 + 2x - 24 < 0$
4. $x^2 + 15x + 54 < 0$
5. $x^2 - 23x - 24 \leq 0$
6. $x^2 - 12x + 20 \leq 0$
7. $2x^2 - 11x - 6 \geq 0$
8. $3x^2 + 17x - 6 \geq 0$
9. $8x^2 - 18x - 5 < 0$
10. $10x^2 + 17x + 6 > 0$
11. $9x^2 + 30x + 25 \leq 0$
12. $16x^2 - 40x + 25 \leq 0$
13. $4x^2 - 4x + 1 > 0$
14. $9x^2 + 12x + 4 > 0$
15. $-x^2 - x + 30 \geq 0$
16. $-x^2 - 6x + 27 \leq 0$
17. $x^2 - 64 < 0$
18. $x^2 - 81 \geq 0$
19. $4x^2 - 9 \geq 0$
20. $16x^2 - 25 < 0$
21. $25 - 4x^2 \geq 0$
22. $1 - 49x^2 < 0$
23. $x^2 - 8 > 0$
24. $x^2 - 75 \leq 0$
25. $2x^2 + 1 > 0$
26. $4x^2 + 3 < 0$
27. $x - x^2 > 0$
28. $3x - x^2 \leq 0$
29. $x^2 - x + 1 < 0$
30. $x^2 + x - 1 > 0$


https://math.libretexts.org/@go/page/69070

31. $4x^2 - 12x + 9 \leq 0$
32. $5x^2 - 8x - 4 < 0$
33. $3x^2 - x - 2 \geq 0$
34. $4x^2 - x + 3 \leq 0$
35. $2 - 4x - x^2 < 0$
36. $5 - 2x - x^2 > 0$
37. $-x^2 - x - 9 < 0$
38. $-x^2 + x - 6 \geq 0$
39. $-2x^2 + 4x - 1 \geq 0$
40. $-3x^2 - x + 1 \leq 0$

**Answer**

1. $(-\infty, -3) \cup (4, \infty)$
3. $(-6, 4)$
5. $[-1, 24]$
7. $\left(-\infty, -\frac{1}{2}\right] \cup [6, \infty)$
9. $\left(-\frac{1}{4}, \frac{5}{2}\right)$
11. $-\frac{5}{3}$
13. $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$
15. $[-6, 5]$
17. $(-8, 8)$
19. $\left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{3}{2}, \infty\right)$
21. $\left[-\frac{5}{2}, \frac{5}{2}\right]$
23. $(-\infty, -2\sqrt{2}) \cup (2\sqrt{2}, \infty)$
25. $(-\infty, \infty)$
27. $(0, 1)$
29. $\emptyset$
31. $\frac{3}{2}$
33. $\left(-\infty, -\frac{2}{3}\right] \cup [1, \infty)$
35. $(-\infty, -2 - \sqrt{6}) \cup (-2 + \sqrt{6}, \infty)$
37. $(-\infty, \infty)$
39. $\left[\frac{2-\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}\right]$

## Exercise 2.1.3.8

Find the domain of the function.

1. $f(x) = \sqrt{x^2 - 25}$
2. $f(x) = \sqrt{x^2 + 3x}$
3. $g(x) = \sqrt{3x^2 - x - 2}$
4. $g(x) = \sqrt{12x^2 - 9x - 3}$
5. $h(x) = \sqrt{16 - x^2}$
6. $h(x) = \sqrt{3 - 2x - x^2}$

7. $f(x)=\sqrt{x^2+10}$
8. $f(x)=\sqrt{9+x^2}$
9. A robotics manufacturing company has determined that its weekly profit in thousands of dollars is modeled by $P(n)=-n^2+30n-200$ where $n$ represents the number of units it produces and sells. How many units must the company produce and sell to maintain profitability. (Hint: Profitability occurs when profit is greater than zero.)
10. The height in feet of a projectile shot straight into the air is given by $h(t)=-16t^2+400t$ where $t$ represents the time in seconds after it is fired. In what time intervals is the projectile under $1,000$ feet? Round to the nearest tenth of a second.

**Answer**

1. $(-\infty,-5]\cup[5,\infty)$
3. $\left(-\infty,-\frac{2}{3}\right]\cup[1,\infty)$
5. $[-4,4]$
7. $(-\infty,\infty)$
9. The company must produce and sell more than $10$ units and fewer than $20$ units each week.

**Exercise 2.1.3.9**

1. Does the sign chart for any given quadratic function always alternate? Explain and illustrate your answer with some examples.
2. Research and discuss other methods for solving a quadratic inequality.
3. Explain the difference between a quadratic equation and a quadratic inequality. How can we identify and solve each? What is the geometric interpretation of each?

**Answer**

1. Answer may vary
3. Answer may vary

## Footnotes

[15] A mathematical statement that relates a quadratic expression as either less than or greater than another.

[16] The values in the domain of a function that separate regions that produce positive or negative results.

[17] A model of a function using a number line and signs $(+$ or $-)$ to indicate regions in the domain where the function is positive or negative.

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# 2.1.4: Solving Polynomial and Rational Inequalities

**Learning Objectives**

- Solve polynomial inequalities.
- Solve rational inequalities.

## Solving Polynomial Inequalities

A **polynomial inequality**[18] is a mathematical statement that relates a polynomial expression as either less than or greater than another. We can use sign charts to solve polynomial inequalities with one variable.

**Example 2.1.4.1**

Solve $x(x+3)^2(x-4)<0$.

**Solution**

Begin by finding the critical numbers. For a polynomial inequality in standard form, with zero on one side, the critical numbers are the roots. Because $f(x)=x(x+3)^2(x-4)$ is given in its factored form the roots are apparent. Here the roots are: $0, -3$, and $4$. Because of the strict inequality, plot them using open dots on a number line.

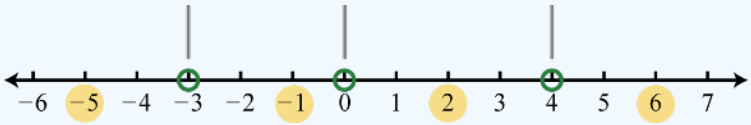



Figure 2.1.4.1

In this case, the critical numbers partition the number line into four regions. Test values in each region to determine if f is positive or negative. Here we choose test values $-5, -1, 2$, and $6$. Remember that we are only concerned with the sign $(+$ or $-)$ of the result.

$$\begin{aligned} f(-5)&=(-5)(-5+3)^2(-5-4)=(-)(-)^2(-) &&=+Positive\\ f(-1)&=(-1)(-1+3)^2(-1-4)=(-)(+)^2(-) &&=+Positive\\ f(2)&=(2)(2+3)^2(2-4)=(+)(+)^2(-) &&=-Negative\\ f(6)&=(6)(6+3)^2(6-4)=(+)(+)^2(+) &&=+Positive \end{aligned}$$

After testing values we can complete a sign chart.

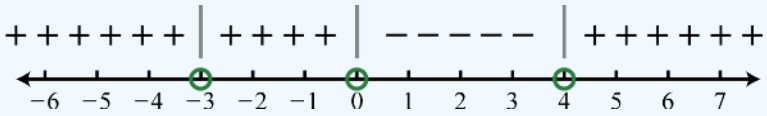



Figure 2.1.4.2

The question asks us to find the values where $f(x)<0$, or where the function is negative. From the sign chart we can see that the function is negative for $x$-values in between 0 and 4.

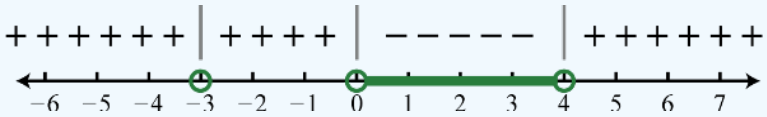



Figure 2.1.4.3

We can express this solution set in two ways:

$$\begin{aligned} \{x|0<x<4\} &\qquad Set\ notation\\ (0,4) &\qquad Interval\ notation \end{aligned}$$

In this textbook we will continue to present solution sets using interval notation.

**Answer**:

$(0,4)$

Graphing polynomials such as the one in the previous example is beyond the scope of this textbook. However, the graph of this function is provided below. Compare the graph to its corresponding sign chart.

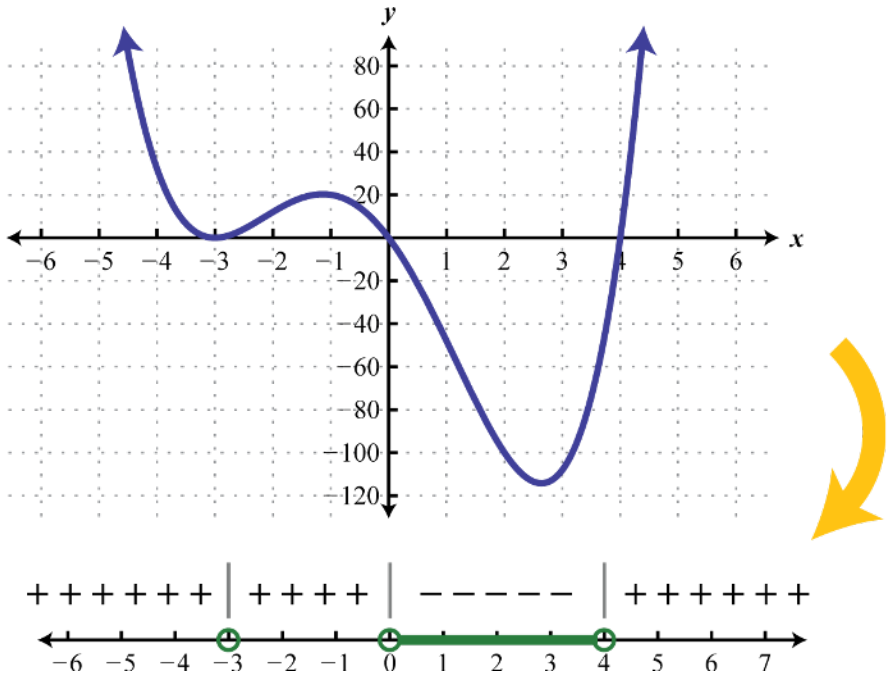



Figure 2.1.4.4

Certainly it may not be the case that the polynomial is factored nor that it has zero on one side of the inequality. To model a function using a sign chart, all of the terms should be on one side and zero on the other. The general steps for solving a polynomial inequality are listed in the following example.

**Example 2.1.4.2:**

Solve: $2x^4 > 3x^3 + 9x^2$.

**Solution**

**Step 1**: Obtain zero on one side of the inequality. In this case, subtract to obtain a polynomial on the left side in standard from.

$$0\end{aligned}\)$$

**Step 2**: Find the critical numbers. Here we can find the zeros by factoring.

$$x^2\left(2x^2-3x-9\right)=0$$
$$x^2(2x+3)(x-3)=0$$

There are three solutions, hence, three critical numbers $-\frac{3}{2}, 0$, and 3. The strict inequality indicates that we should use open dots.

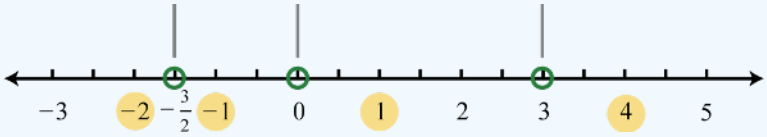



Figure 2.1.4.5

**Step 3**: Create a sign chart. In this case use $f(x)=x^2(2x+3)(x-3)$ and test values $-2, -1, 1$, and 4 to determine the sign of the function in each interval.

$$\begin{aligned} f(-2)&=(-2)^2[2(-2)+3](-2-3) &&=(-)^2(-)(-)=+\\ f(-1)&=(-1)^2[2(-1)+3](-1-3) &&=(-)^2(+)(-)=-\\ f(1)&=(1)^2[2(1)+3](1-3) &&=(+)^2(+)(-)=-\\ f(4)&=(4)^2[2(4)+3](4-3) &&=(+)^2(+)(+)=+ \end{aligned}$$

With this information we can complete the sign chart.




Figure 2.1.4.6

**Step 4**: Use the sign chart to answer the question. Here the solution consists of all values for which $f(x) > 0$. Shade in the values that produce positive results and then express this set in interval notation.




Figure 2.1.4.7

**Answer**:

$\left(-\infty, -\frac{3}{2}\right) \cup (3, \infty)$

## Example 2.1.4.3

Solve: $x^3 + x^2 \leq 4(x+1)$ .

**Solution**

Begin by rewriting the inequality in standard form, with zero on one side.

Next find the critical numbers of $f(x) = x^3 + x^2 - 4x - 4$ :

The critical numbers are $-2$, $-1$, and 2. Because of the inclusive inequality $(\leq)$ we will plot them using closed dots.

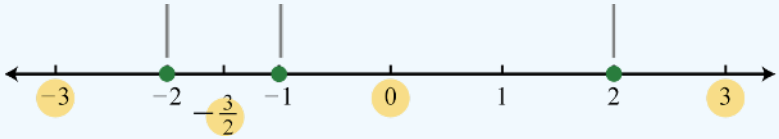



Figure 2.1.4.8

Use test values, $-3$, $-\frac{3}{2}$, 0, and 3 to create a sign chart.

And we have



Figure 2.1.4.9

Use the sign chart to shade in the values that have negative results $(f(x) \leq 0)$.

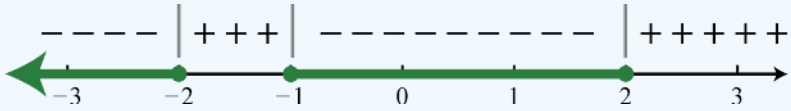



Figure 2.1.4.10

**Answer**

$(-\infty, -2] \cup [-1, 2]$

## Exercise 2.1.4.1

Solve $-3x^4 + 12x^3 - 9x^2 > 0$ .

**Answer**

$(1, 3)$

www.youtube.com/v/EXpe_0LzbSY

## Solving Rational Inequalities

A **rational inequality**[19] is a mathematical statement that relates a rational expression as either less than or greater than another. Because rational functions have restrictions to the domain we must take care when solving rational inequalities. In addition to the zeros, we will include the restrictions to the domain of the function in the set of critical numbers.

### Example 2.1.4.4

Solve: $\frac{(x-4)(x+2)}{(x-1)} \geq 0$

**Solution**

The zeros of a rational function occur when the numerator is zero and the values that produce zero in the denominator are the restrictions. In this case,

| Roots (Numerator) | Restriction(Denominator) |
|---|---|
| $x-4=0$ or $x+2=0$ | $x-1=0$ |
| $x=4 \quad x=-2$ | $x=1$ |

Therefore the critical numbers are $-2, 1$, and $4$. Because of the inclusive inequality $(\geq)$ use a closed dot for the roots $-2, 4$ and always use an open dot for restrictions $1$. Restrictions are never included in the solution set.

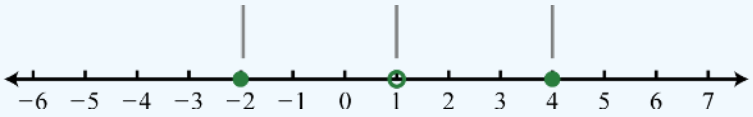



Figure 2.1.4.11

Use test values $x=-4, 0, 2, 6$.

$$f(-4)=\frac{(-4-4)(-4+2)}{(-4-1)} = \frac{(-)(-)}{(-)} = -$$
$$f(0)=\frac{(0-4)(0+2)}{(0-1)} = \frac{(-)(+)}{(-)} = +$$
$$f(2)=\frac{(2-4)(2+2)}{(2-1)} = \frac{(-)(+)}{(+)} = -$$
$$f(6)=\frac{(6-4)(6+2)}{(6-1)} = \frac{(+)(+)}{(+)} = +$$

And then complete the sign chart.

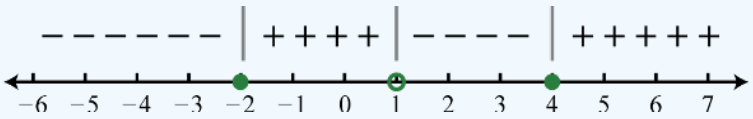



Figure 2.1.4.12

The question asks us to find the values for which $f(x) \geq 0$, in other words, positive or zero. Shade in the appropriate regions and present the solution set in interval notation.

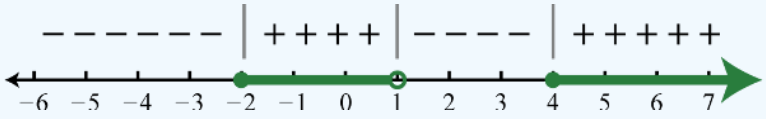



Figure 2.1.4.13

**Answer**

$[-2, 1) \cup [4, \infty)$

Graphing such rational functions like the one in the previous example is beyond the scope of this textbook. However, the graph of this function is provided below. Compare the graph to its corresponding sign chart.

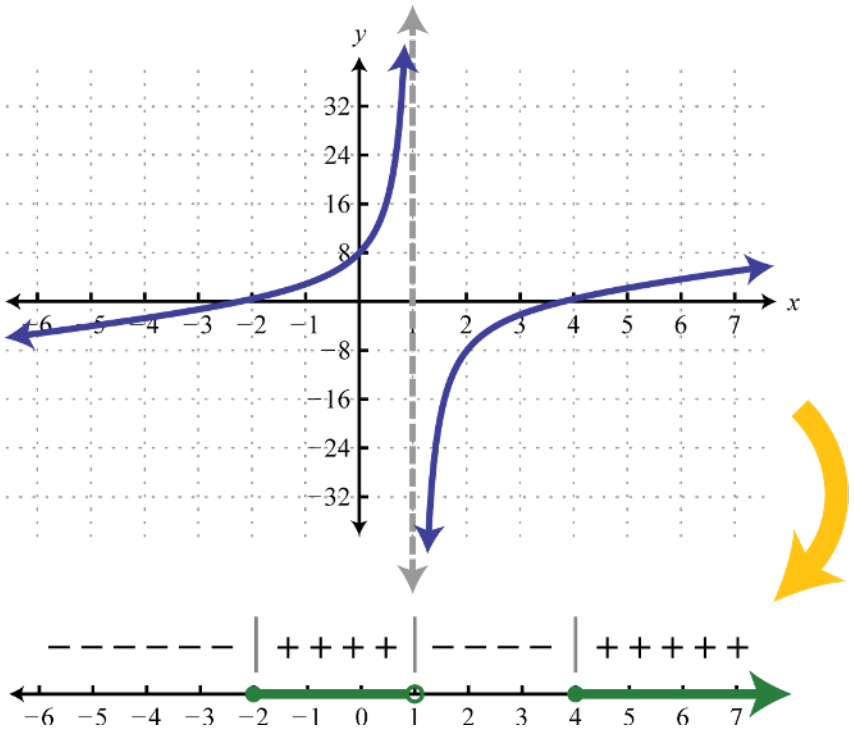



Figure 2.1.4.14

Notice that the restriction $x = 1$ corresponds to a vertical asymptote which bounds regions where the function changes from positive to negative. While not included in the solution set, the restriction is a critical number. Before creating a sign chart we must ensure the inequality has a zero on one side. The general steps for solving a rational inequality are outlined in the following example.

**Example 2.1.4.5:**

Solve $\frac{7}{x+3} < 2$.

**Solution**

**Step 1**: Begin by obtaining zero on the right side.

$$\frac{7}{x+3} < 2$$
$$\frac{7}{x+3} - 2 < 0$$

**Step 2**: Determine the critical numbers. The critical numbers are the zeros and restrictions. Begin by simplifying to a single algebraic fraction.

Next find the critical numbers. Set the numerator and denominator equal to zero and solve.

| Root | Restriction |
|---|---|
| $-2x+1=0$ | |
| $-2x=-1$ | $x+3=0$ |
| $x=\frac{1}{2}$ | $x=-3$ |

In this case, the strict inequality indicates that we should use an open dot for the root.

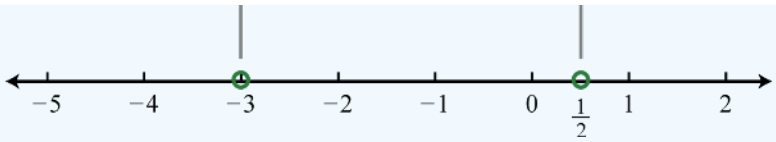



Figure 2.1.4.15

**Step 3**: Create a sign chart. Choose test values $-4, 0$, and $1$.

$$f(-4) = \frac{-2(-4)+1}{-4+3} = \frac{+}{-} = -$$
$$f(0) = \frac{-2(0)+1}{0+3} = \frac{+}{+} = +$$
$$f(1) = \frac{-2(1)+1}{1+3} = \frac{-}{+} = -$$

And we have

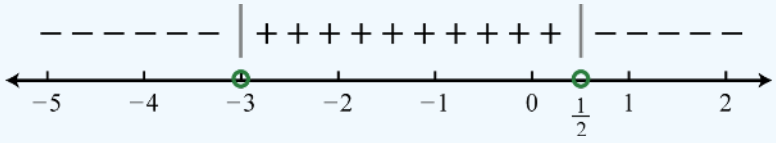



Figure 2.1.4.16

**Step 4**: Use the sign chart to answer the question. In this example we are looking for the values for which the function is negative, $f(x) < 0$. Shade the appropriate values and then present your answer using interval notation.

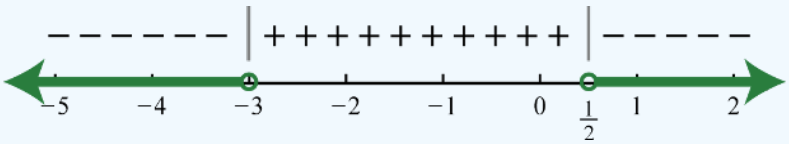



Figure 2.1.4.17

**Answer**:

$(-\infty, -3) \cup \left(\frac{1}{2}, \infty\right)$

## Example 2.1.4.6:

Solve: $\frac{1}{x^2-4} \leq \frac{1}{2-x}$.

**Solution**

Begin by obtaining zero on the right side.

$$\frac{1}{x^2-4} \leq \frac{1}{2-x}$$
$$\frac{1}{x^2-4} - \frac{1}{2-x} \leq 0$$

Next simplify the left side to a single algebraic fraction.

$$\frac{1}{x^2-4} - \frac{1}{2-x} \leq 0$$
$$\frac{1}{(x+2)(x-2)} - \frac{1}{-(x-2)} \leq 0$$
$$\frac{1}{(x+2)(x-2)} + \frac{1(x+2)}{(x-2)(x+2)} \leq 0$$
$$\frac{1+x+2}{(x+2)(x-2)} \leq 0$$
$$\frac{x+3}{(x+2)(x-2)} \leq 0$$

The critical numbers are $-3, -2$, and $2$. Note that $\pm 2$ are restrictions and thus we will use open dots when plotting them on a number line. Because of the inclusive inequality we will use a closed dot at the root $-3$.

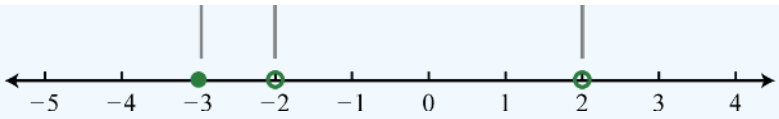


Figure 2.1.4.18

Choose test values $-4, -2\frac{1}{2} = -\frac{5}{2}, 0$, and $3$.

Construct a sign chart.


Figure 2.1.4.19

Answer the question; in this case, find $x$ where $f(x) \leq 0$.

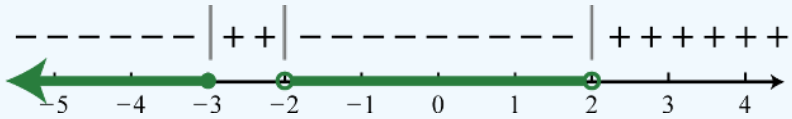


Figure 2.1.4.20

**Answer**

$$(-\infty, -3] \cup (-2, 2)$$

## Exercise 2.1.4.2

Solve: $\frac{2x^2}{2x^2+7x-4} \geq \frac{x}{x+4}$.

**Answer**

$(-4, 0] \cup \left(\frac{1}{2}, \infty\right)$

www.youtube.com/v/3ljcHJqExBY

## Key Takeaways

- When a polynomial inequality is in standard form, with zero on one side, the roots of the polynomial are the critical numbers. Create a sign chart that models the function and then use it to answer the question.
- When a rational inequality is written as a single algebraic fraction, with zero on one side, the roots as well as the restrictions are the critical numbers. The values that produce zero in the numerator are the roots, and the values that produce zero in the denominator are the restrictions. Always use open dots for restrictions, regardless of the given inequality, because restrictions are not part of the domain. Create a sign chart that models the function and then use it to answer the question.

## Exercise 2.1.4.3

Solve. Present answers using interval notation.

1. $x(x+1)(x-3) > 0$
2. $x(x-1)(x+4) < 0$
3. $(x+2)(x-5)^2 < 0$
4. $(x-4)(x+1)^2 \geq 0$
5. $(2x-1)(x+3)(x+2) \leq 0$
6. $(3x+2)(x-4)(x-5) \geq 0$
7. $x(x+2)(x-5)^2 < 0$
8. $x(2x-5)(x-1)^2 > 0$
9. $x(4x+3)(x-1)^2 \geq 0$
10. $(x-1)(x+1)(x-4)^2 < 0$
11. $(x+5)(x-10)(x-5)^2 \geq 0$
12. $(3x-1)(x-2)(x+2)^2 \leq 0$

13. $-4x(4x+9)(x-8)^2>0$
14. $-x(x-10)(x+7)^2>0$

**Answer**

1. $(-1,0)\cup(3,\infty)$
3. $(-\infty,-2)$
5. $(-\infty,-3]\cup\left[-2,\frac{1}{2}\right]$
7. $(-2,0)$
9. $\left(-\infty,-\frac{3}{4}\right]\cup[0,\infty)$
11. $(-\infty,-5]\cup[5,5]\cup[10,\infty)$
13. $\left(-\frac{9}{4},0\right)$

## Exercise 2.1.4.4

Solve.

1. $x^3+2x^2-24x\geq 0$
2. $x^3-3x^2-18x\leq 0$
3. $4x^3-22x^2-12x<0$
4. $9x^3+30x^2-24x>0$
5. $12x^4+44x^3>80x^2$
6. $6x^4+12x^3<48x^2$
7. $x\left(x^2+25\right)<10x^2$
8. $x^3>12x(x-3)$
9. $x^4-5x^2+4\leq 0$
10. $x^4-13x^2+36\geq 0$
11. $x^4>3x^2+4$
12. $4x^4<3-11x^2$
13. $9x^3-3x^2-81x+27\leq 0$
14. $2x^3+x^2-50x-25\geq 0$
15. $x^3-3x^2+9x-27>0$
16. $3x^3+5x^2+12x+20<0$

**Answer**

1. $[-6,0]\cup[4,\infty)$
3. $\left(-\infty,-\frac{1}{2}\right)\cup(0,6)$
5. $(-\infty,-5)\cup\left(\frac{4}{3},\infty\right)$
7. $(-\infty,0)$
9. $[-2,-1]\cup[1,2]$
11. $(-\infty,-2)\cup(2,\infty)$
13. $(-\infty,-3]\cup\left[\frac{1}{3},3\right]$
15. $(3,\infty)$

## Exercise 2.1.4.5

Solve.

1. $\frac{x}{x-3} > 0$
2. $\frac{x-5}{x} > 0$
3. $\frac{(x-3)(x+1)}{x} < 0$
4. $\frac{(x+5)(x+4)}{(x-2)} < 0$
5. $\frac{(2x+1)(x+5)}{(x-3)(x-5)} \leq 0$
6. $\frac{(3x-1)(x+6)}{(x-1)(x+9)} \geq 0$
7. $\frac{(x-8)(x+8)}{-2x(x-2)} \geq 0$
8. $\frac{(2x+7)(x+4)}{x(x+5)} \leq 0$
9. $\frac{x^2}{(2x+3)(2x-3)} \leq 0$
10. $\frac{(x-4)^2}{-x(x+1)} > 0$
11. $\frac{-5x(x-2)^-}{(x+5)(x-6)} \geq 0$
12. $\frac{(3x-4)(x+5)}{x(x-4)^2} \geq 0$
13. $\frac{1}{(x-5)^4} > 0$
14. $\frac{1}{(x-5)^4} < 0$

**Answer**

1. $(-\infty, -0) \cup (3, \infty)$
3. $(-\infty, -1) \cup (0, 3)$
5. $\left[-5, -\frac{1}{2}\right] \cup (3, 5)$
7. $[-8, 0) \cup (2, 8]$
9. $\left(-\frac{3}{2}, \frac{3}{2}\right)$
11. $(-\infty, -5) \cup [0, 6)$
13. $(-\infty, 5) \cup (5, \infty)$

## Exercise 2.1.4.6

Solve.

1. $\frac{x^2-11x-12}{x+4} < 0$
2. $\frac{x^2-10x+24}{x-2} > 0$
3. $\frac{x^2+x-30}{2x+1} \geq 0$
4. $\frac{2x^2+x-3}{x-3} \leq 0$
5. $\frac{3x^2-4x+1}{x^2-9} \leq 0$
6. $\frac{x^2-16}{2x^2-3x-2} \geq 0$
7. $\frac{x^2-12x+20}{x^2-10x+25} > 0$
8. $\frac{x^2+15x+36}{x^2-8x+16} < 0$
9. $\frac{8x^2-2x-1}{2x^2-3x-14} \leq 0$
10. $\frac{4x^2-4x-15}{x^2+4x-5} \geq 0$
11. $\frac{1}{x+5} + \frac{5}{x-1} > 0$

12. $\frac{5}{x+4}-\frac{1}{x-4}<0$
13. $\frac{1}{x+7}>1$
14. $\frac{1}{x-1}<-5$
15. $x\geq\frac{30}{x-1}$
16. $x\leq\frac{1-2x}{x-2}$
17. $\frac{1}{x-1}\leq\frac{2}{x}$
18. $\frac{3}{x+1}>-\frac{1}{x}$
19. $\frac{4}{x-3}\leq\frac{1}{x+3}$
20. $\frac{2x-9}{x}+\frac{49}{x-8}<0$
21. $\frac{x}{2(x+2)}-\frac{1}{x+2}\leq\frac{12}{x(x+2)}$
22. $\frac{1}{2x+1}-\frac{9}{2x-1}>2$
23. $\frac{3x}{x^2-4}-\frac{2}{x-2}<0$
24. $\frac{x}{2x+1}+\frac{4}{2x^2-7x-4}<0$
25. $\frac{x+1}{2x^2+5x-3}\geq\frac{x}{4x^2-1}$
26. $\frac{x^2-14}{2x^2-7x-4}\leq\frac{5}{1+2x}$

**Answer**

1. $(-\infty,-4)\cup(-1,12)$
3. $\left[-6,-\frac{1}{2}\right)\cup[5,\infty)$
5. $\left(-3,\frac{1}{3}\right]\cup[1,3)$
7. $(-\infty,2)\cup(10,\infty)$
9. $\left(-2,-\frac{1}{4}\right]\cup\left[\frac{1}{2},\frac{7}{2}\right)$
11. $(-5,-4)\cup(1,\infty)$
13. $(-7,-6)$
15. $[-5,1)\cup[6,\infty)$
17. $(0,1)\cup[2,\infty)$
19. $(-\infty,5]\cup(-3,3)$
21. $[-4,-2)\cup(0,6]$
23. $(-\infty,-2)\cup(2,4)$
25. $\left(-3,-\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$

## Exercise 2.1.4.7

1. Does the sign chart for any given polynomial or rational function always alternate? Explain and illustrate your answer with some examples.
2. Write down your own steps for solving a rational inequality and illustrate them with an example. Do your steps also work for a polynomial inequality? Explain.

**Answer**

1. Answer may vary

## Footnotes

[18]A mathematical statement that relates a polynomial expression as either less than or greater than another.

[19]A mathematical statement that relates a rational expression as either less than or greater than another.

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# 2.1.E: Solving Equations and Inequalities (Exercises)

## Exercise 2.1.$E$.1

Solve by extracting the roots.

1. $x^2-81=0$
2. $y^2-\frac{1}{4}=0$
3. $9x^2-8=0$
4. $5x^2-12=0$
5. $2y^2-7=0$
6. $3y^2-6=0$
7. $(2x-3)^2-16=0$
8. $4(x-1)^2-5=0$
9. $9(x-3)^2+4=0$
10. $5(2x+1)^2+1=0$
11. $2x^2+10=0$
12. $x^2+64=0$
13. The height in feet of an object dropped from a 20-foot stepladder is given by $h(t)=-16t^2+20$ where $t$ represents the time in seconds after the object has been dropped. How long does it take the object to hit the ground after it has been dropped? Round to the nearest tenth of a second.
14. A 20-foot ladder, leaning against a building, reaches a height of 19 feet. How far is the base of the ladder from the wall? Round to the nearest tenth of a foot.

**Answer**

1. $\pm 9$
3. $\pm\frac{2\sqrt{2}}{3}$
5. $\pm\frac{\sqrt{14}}{2}$
7. $-\frac{1}{2},\frac{7}{2}$
9. $3\pm\frac{2}{3}i$
11. $\pm i\sqrt{5}$
13. 1.1 seconds

## Exercise 2.1.$E$.2

Solve by completing the square.

1. $x^2+4x-5=0$
2. $x^2+2x-17=0$
3. $x^2-4x+1=0$
4. $x^2-6x-2=0$
5. $x^2-3x-1=0$
6. $x^2+5x-6=0$
7. $x^2+x-2=0$
8. $x^2-x-4=0$
9. $5x^2-10x+1=0$
10. $4x^2+8x-3=0$
11. $2x^2-6x+1=0$
12. $3x^2+10x+6=0$
13. $x^2-x+3=0$
14. $2x^2+6x+5=0$

15. $x(x+9)+10=5x+2$
16. $(2x+5)(x+2)=8x+7$

**Answer**

1. $-5, 1$
3. $2 \pm \sqrt{3}$
5. $\frac{3 \pm \sqrt{13}}{2}$
7. $-2, 1$
9. $\frac{5 \pm 2\sqrt{5}}{5}$
11. $\frac{3 \pm \sqrt{7}}{2}$
13. $\frac{1}{2} \pm \frac{\sqrt{11}}{2} i$
15. $-2 \pm 2i$

## Exercise 2.1.$E$.3

Solve using the quadratic formula.

1. $2x^2 - x - 6 = 0$
2. $3x^2 + x - 4 = 0$
3. $9x^2 + 12x + 2 = 0$
4. $25x^2 - 10x - 1 = 0$
5. $-x^2 + 8x - 2 = 0$
6. $-x^2 - x + 1 = 0$
7. $5 - 2x - x^2 = 0$
8. $2 + 4x - 3x^2 = 0$
9. $3x^2 - 2x + 4 = 0$
10. $7x^2 - x + 1 = 0$
11. $-x^2 + 2x - 6 = 0$
12. $-3x^2 + 4x - 2 = 0$
13. $36x^2 + 60x + 25 = 0$
14. $72x^2 + 54x - 35 = 0$
15. $1.3x^2 - 2.8x - 4.2 = 0$
16. $5.5x^2 - 4.1x + 2.2 = 0$
17. $(x+2)^2 - 3x = 4$
18. $(3x+1)^2 - 6 = 6x - 3$
19. The height in feet of a baseball tossed upward at a speed of 48 feet per second from the ground is given by the function, $h(t) = -16t^2 + 48t$ , where $t$ represents the time in seconds after the ball is tossed. At what time does the baseball reach a height of 18 feet? Round off to the nearest hundredth of a second.
20. The height in feet reached by a model rocket launched from a 3-foot platform is given by the function $h(t) = -16t^2 + 256t + 3$ where $t$ represents time in seconds after launch. At what times will the rocket reach $1,000$ feet? Round off to the nearest tenth of a second.

**Answer**

1. $-\frac{3}{2}, 2$
3. $\frac{-2 \pm \sqrt{2}}{3}$
5. $4 \pm \sqrt{14}$
7. $-1 \pm \sqrt{6}$


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9. $\frac{1}{3} \pm \frac{\sqrt{11}}{3} i$
11. $1 \pm i\sqrt{5}$
13. $-\frac{5}{6}$
15. $x \approx -1.0, x \approx 3.2$
17. $-1, 0$
19. The ball will reach $18$ feet at $0.44$seconds and again at $2.56$seconds.

## Exercise 2.1.$E$.4

Use the discriminant to determine the number and type of solutions.

1. $-x^2 + 6x + 1 = 0$
2. $-x^2 + x - 3 = 0$
3. $4x^2 - 4x + 1 = 0$
4. $16x^2 - 9 = 0$

**Answer**

1. Two irrational solutions
3. One rational solution

## Exercise 2.1.$E$.5

Solve using any method.

1. $x^2 - 4x - 96 = 0$
2. $25x^2 + x = 0$
3. $25t^2 - 1 = 0$
4. $t^2 + 25 = 0$
5. $y^2 - y - 7 = 0$
6. $5y^2 - 25y = 0$
7. $2x^2 - 9 = 0$
8. $25x^2 - 10x + 1 = 0$
9. $(2x + 5)^2 - 9 = 0$
10. $(x - 2)(x - 5) = 5$
11. The length of a rectangle is $3$ inches less than twice the width. If the area of the rectangle measures $30$ square inches, then find the dimensions of the rectangle. Round off to the nearest hundredth of an inch.
12. The value in dollars of a new car is modeled by the function $V(t) = 125t^2 - 2,500t + 18,000$ where $t$ represents the number of years since it was purchased. Determine the age of the car when its value is $\$18,000$

**Answer**

1. $-8, 12$
3. $\pm\frac{1}{5}$
5. $\frac{1 \pm \sqrt{29}}{2}$
7. $\pm\frac{3\sqrt{2}}{2}$
9. $-4, -1$
11. Length: $6.38$inches; width: $4.69$inches

## Exercise 2.1.$E$.6

Find all solutions.

1. $x^4 - 16x^2 + 48 = 0$
2. $x^{2/3} - x^{1/3} - 20 = 0$
3. $x^{-2} - 5x^{-1} - 50 = 0$
4. $\left(\frac{t+3}{t}\right)^2 + 11\left(\frac{t+3}{t}\right) - 12 = 0$
5. $x + 2\sqrt{x} - 24 = 0$
6. $2x^{1/2} - 3x^{1/4} + 1 = 0$
7. $4\left(\frac{1}{x+1}\right)^2 - 4\left(\frac{1}{x+1}\right) - 3 = 0$
8. $5t^{-2} - 27t^{-1} - 18 = 0$
9. $3x^{2/3} - 5x^{1/3} + 2 = 0$
10. $4x + 4\sqrt{x} + 1 = 0$
11. $16y^4 - 25 = 0$
12. $x^{-2} - 64 = 0$

**Answer**

1. $\pm 2, \pm 2\sqrt{3}$
3. $-\frac{1}{5}, \frac{1}{10}$
5. $16$
7. $-3, -\frac{1}{3}$
9. $1, \frac{8}{27}$
11. $\pm\frac{\sqrt{5}}{2}, \pm\frac{\sqrt{5}}{2}i$

## Exercise 2.1.$E$.7

Find the set of all roots.

1. $f(x) = x^2 - 50$
2. $f(x) = x^3 - 64$
3. $f(x) = x^4 - 81$
4. $f(x) = x^4 + 8x$

**Answer**

1. $\{\pm 5\sqrt{2}\}$
3. $\{\pm 3, \pm 3i\}$

## Exercise 2.1.$E$.8

Find a quadratic equation with integer coefficients and the given set of solutions.

1. $\left\{\frac{4}{3}, -\frac{1}{2}\right\}$
2. $\{\pm\sqrt{5}\}$
3. $\{\pm 4\sqrt{2}\}$
4. $\{\pm 6i\}$
5. $\{2 \pm i\}$
6. $\{3 \pm \sqrt{5}\}$

**Answer**

1. $6x^2 - 5x - 4 = 0$
3. $x^2 - 32 = 0$

5. $x^2 - 4x + 5 = 0$

## Exercise 2.1.$E$.9

Determine the $x$- and $y$-intercepts.

1. $y = 2x^2 + 5x - 12$
2. $y = x^2 - 18$
3. $y = x^2 + 4x + 7$
4. $y = -9x^2 + 12x - 4$

**Answer**

1. $x$-intercepts: $(-4, 0), (\frac{3}{2}, 0)$; $y$-intercept: $(0, -12)$
3. $x$-intercepts: none; $y$-intercept: $(0, 7)$

## Exercise 2.1.$E$.10

Find the vertex and the line of symmetry.

1. $y = x^2 - 4x - 12$
2. $y = -x^2 + 8x - 1$
3. $y = x^2 + 3x - 1$
4. $y = 4x^2 - 1$

**Answer**

1. Vertex: $(2, -16)$; line of symmetry: $x = 2$
3. Vertex: $(-\frac{3}{2}, -\frac{13}{4})$; line of symmetry: $x = -\frac{3}{2}$

## Exercise 2.1.$E$.11

Graph. Find the vertex and the $y$-intercept. In addition, find the $x$-intercepts if they exist.

1. $y = x^2 + 8x + 12$
2. $y = -x^2 - 6x + 7$
3. $y = -2x^2 - 4$
4. $y = x^2 + 4x$
5. $y = 4x^2 - 4x + 1$
6. $y = -2x^2$
7. $y = -2x^2 + 8x - 7$
8. $y = 3x^2 - 1$

**Answer**

1.

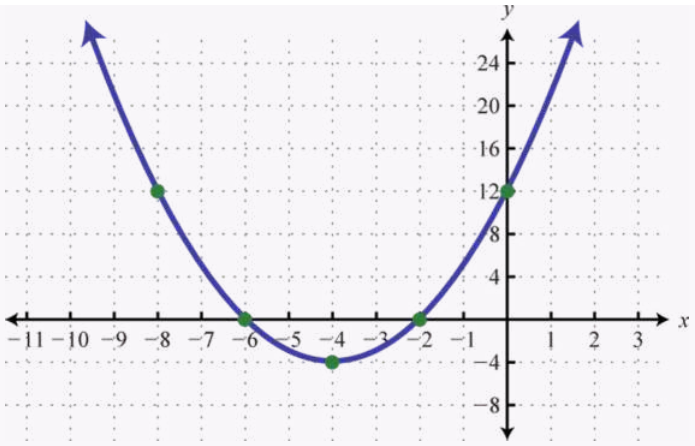



*Figure 6.E.1*

3.

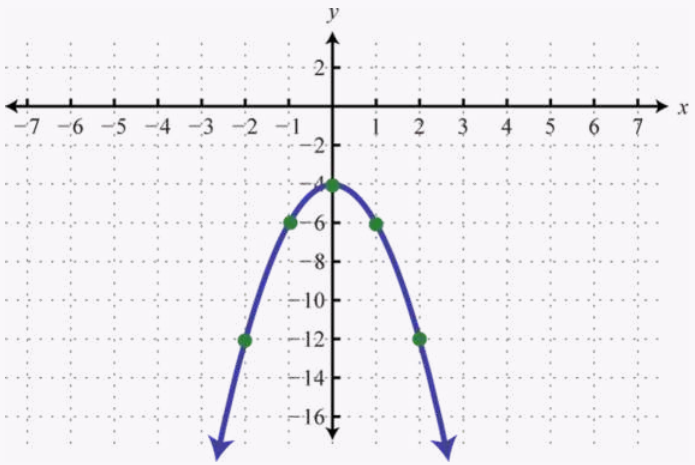



*Figure 6.E.2*

5.

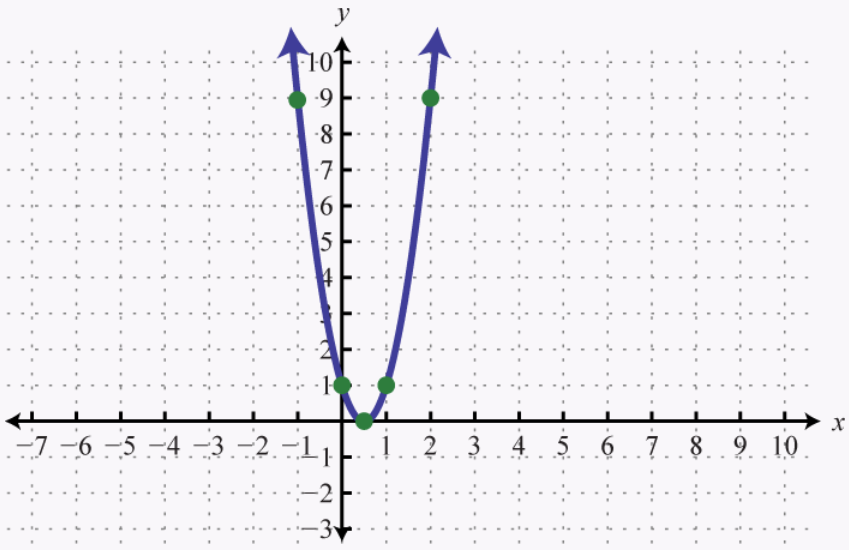



*Figure 6.E.3*

7.


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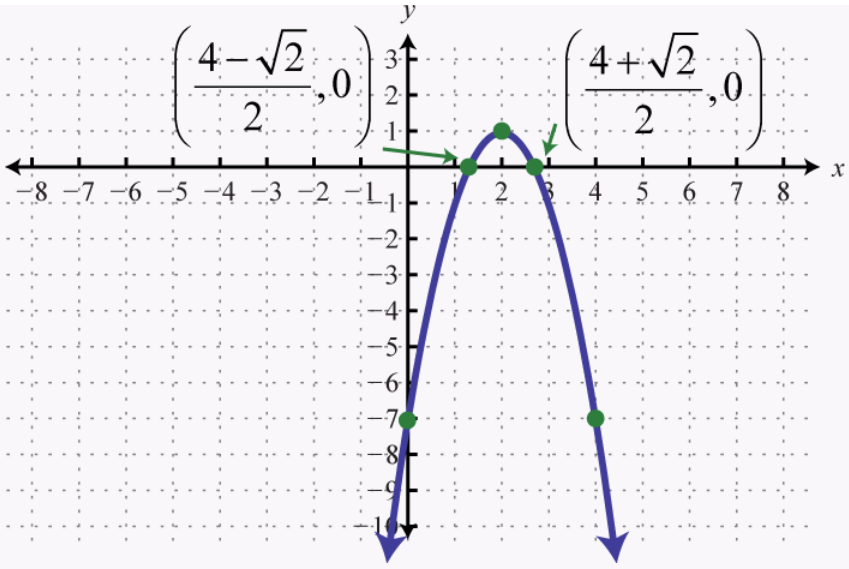



***Figure 6.E.4***

## Exercise 2.1.$E$.12

Determine the maximum or minimum $y$-value.

1. $y = x^2 - 10x + 1$
2. $y = -x^2 + 10x - 1$
3. $y = -3x^2 + 2x - 1$
4. $y = 2x^2 - x + 2$

**Answer**

1. Minimum: $y = -24$
3. Maximum: $y = -\frac{2}{3}$

## Exercise 2.1.$E$.13

Rewrite in vertex form $y = a(x-h)^2 + k$ and determine the vertex.

1. $y = x^2 - 6x + 13$
2. $y = x^2 + 10x + 24$
3. $y = 2x^2 - 4x - 1$
4. $y = -x^2 - 8x - 11$

**Answer**

1. $y = (x-3)^2 + 4$ ; vertex: $(3, 4)$
3. $y = 2(x-1)^2 - 3$ ; vertex: $(1, -3)$

## Exercise 2.1.$E$.14

Graph. Find the vertex and the $y$-intercept. In addition, find the $x$-intercepts if they exist.

1. $f(x) = (x-4)^2 - 2$
2. $f(x) = -(x+6)^2 + 4$
3. $f(x) = -x^2 + 10$
4. $f(x) = (x+10)^2 - 20$
5. $f(x) = 2(x-1)^2 - 3$

6. $f(x) = -3(x+1)^2 - 2$
7. The value in dollars of a new car is modeled by $V(t) = 125t^2 - 3,000t + 22,000$ where $t$ represents the number of years since it was purchased. Determine the age of the car when its value is at a minimum.
8. The height in feet of a baseball tossed upward at a speed of 48 feet per second from the ground is given by the function, $h(t) = -16t^2 + 32t$ , where $t$ represents the time in seconds after it is tossed. What is the maximum height of the baseball?
9. The rectangular area in square feet that can be enclosed with 200 feet of fencing is given by $A(w) = w(100 - w)$ where $w$ represents the width of the rectangular area in feet. What dimensions will maximize the area that can be enclosed?
10. A manufacturing company has found that production costs in thousands of dollars are modeled by $C(x) = 0.4x^2 - 72x + 8,050$ where $x$ represents the number of employees. Determine the number of employees that will minimize production costs.

**Answer**

1.

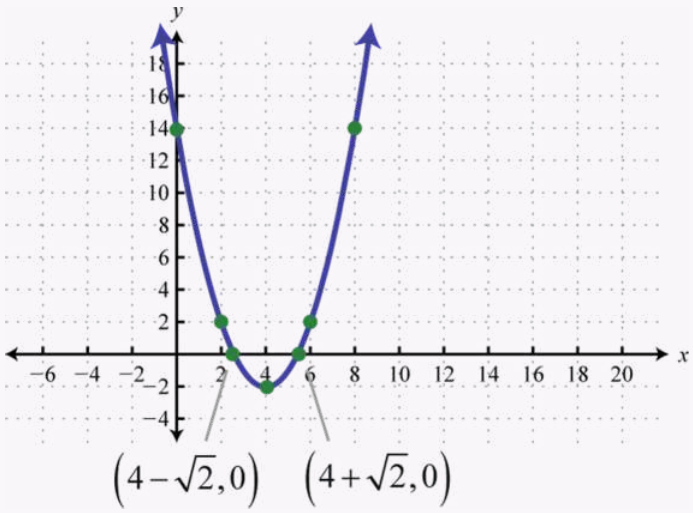



***Figure 6.E.5***

3.

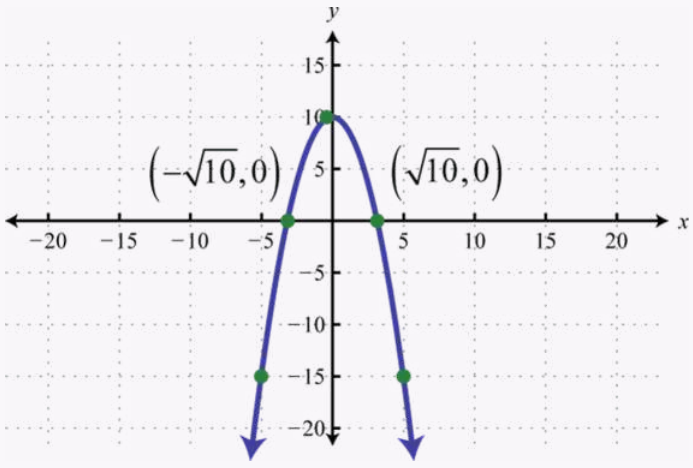



***Figure 6.E.6***

5.

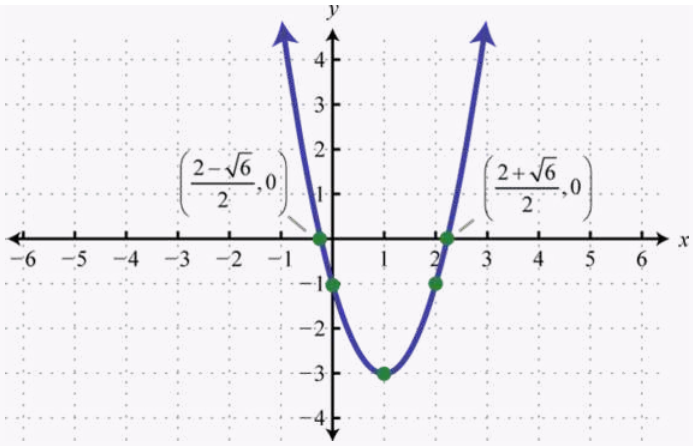



***Figure 6.E.7***

7. The car will have minimum value 12 years after it is purchased.

9. Length: 50 feet; width: 50 feet

## Exercise 2.1.$E$.15

Solve. Present answers using interval notation.

1. $-2(x-1)(x+3)<0$
2. $x^2+2x-35<0$
3. $x^2-6x-16\leq 0$
4. $x^2+14x+40\geq 0$
5. $x^2-10x-24>0$
6. $36-x^2>0$
7. $1-9x^2<0$
8. $8x-12x^2\leq 0$
9. $5x^2+3\leq 0$
10. $x^2-28\geq 0$
11. $9x^2-30x+25\leq 0$
12. $x^2-8x+18>0$
13. $x^2-2x-4<0$
14. $-x^2+3x+18>0$

**Answer**

1. $(-\infty,-3)\cup(1,\infty)$
3. $[-2,8]$
5. $(-\infty,-2)\cup(12,\infty)$
7. $\left(-\infty,-\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$
9. $\varnothing$
11. $\frac{5}{3}$
13. $(1-\sqrt{5},1+\sqrt{5})$

## Exercise 2.1.$E$.16

Find the domain of the function.

1. $f(x)=\sqrt{x^2-100}$

2. $f(x)=\sqrt{3x-6x^2}$
3. $g(x)=\sqrt{3x^2+9}$
4. $g(x)=\sqrt{8+2x-x^2}$

**Answer**

1. $(-\infty,-10]\cup[10,\infty)$
3. $(-\infty,\infty)$

### Exercise 2.1.$E$.17

Solve. Present answers using interval notation.

1. $x(x-5)(x+2)>0$
2. $(x+4)^2(x-3)<0$
3. $x^2(x+3)\geq 0$
4. $x(x-1)^2\leq 0$
5. $x^3+4x^2-9x-36>0$
6. $2x(4x-1)\geq 3$
7. $4x^3-12x^2+9x<0$
8. $x^3-9x^2+20x\geq 0$
9. $x^3-2x^2-x+2<0$
10. $6x(x+1)+5x\leq 35$
11. $\frac{(x-2)(2x+1)}{x(x-1)}\leq 0$
12. $\frac{x(x-3)^2}{x-4}\leq 0$
13. $\frac{x^2+4x+4}{4x^2-1}<0$
14. $\frac{x^2-10x+24}{x^2+10x+25}>0$
15. $\frac{1}{x-2}+\frac{3}{x}\geq 0$
16. $\frac{2}{x-1}-\frac{1}{x+1}\leq 0$
17. $\frac{3(x+1)}{x^2+2x-3}\leq\frac{2}{x-1}$
18. $\frac{x-4}{x+5}\geq\frac{x-2}{x-5}$

**Answer**

1. $(-2,0)\cup(5,\infty)$
3. $[-3,\infty)$
5. $(-4,-3)\cup(3,\infty)$
7. $(-\infty,0)$
9. $(-\infty,-1)\cup(1,2)$
11. $\left[-\frac{1}{2},0\right)\cup(1,2]$
13. $\left(-\frac{1}{2},\frac{1}{2}\right)$
15. $\left(0,\frac{3}{2}\right]\cup(2,\infty)$
17. $(-\infty,-3)\cup(1,3]$

## Sample Exam

### Exercise 2.1.$E$.18

1. Solve by extracting the roots: $2x^2-5=0$ .
2. Solve by completing the square: $x^2-16x+1=0$ .


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**Answer**

1. $\pm\frac{\sqrt{10}}{2}$

## Exercise 2.1.$E$.19

Solve using the quadratic formula.

1. $x^2+x+1=0$
2. $2x^2-x-4=0$
3. $-4x^2+2x-1=0$
4. $(x-4)(x-2)=6$

**Answer**

1. $-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$
3. $\frac{1}{4}\pm\frac{\sqrt{3}}{4}i$

## Exercise 2.1.$E$.20

1. Find a quadratic equation with integer coefficients and solutions $\{\pm\sqrt{5}\}$.
2. The area of a rectangle is 22 square centimeters. If the length is 5 centimeters less than twice the width, then find the dimensions of the rectangle. Round off to the nearest tenth of a centimeter.
3. Assuming dry road conditions and average reaction times, the safe stopping distance in feet of a certain car is given by $d(x)=\frac{1}{20}x^2+x$ where $x$ represents the speed of the car in miles per hour. Determine the safe speed of the car if you expect to stop in 100 feet. Round off to the nearest mile per hour.

**Answer**

1. $x^2-5=0$

9. 36 miles per hour

## Exercise 2.1.$E$.21

Find all solutions.

1. $x^4+x^2-12=0$
2. $3x^{-2}-5x^{-1}-2=0$
3. $2x^{2/3}+3x^{1/3}-2=0$
4. $x-3\sqrt{x}-4=0$
5. $\left(\frac{t}{t+1}\right)^2+4\left(\frac{t}{t+1}\right)-12=0$

**Answer**

2. $-3,\ \frac{1}{2}$

4. 16

## Exercise 2.1.$E$.22

Graph. Find the vertex and the $y$-intercept. In addition, find the $x$-intercepts if they exist.

1. $f(x)=x^2+4x-12$
2. $f(x)=-x^2+2x+3$

**Answer**

1.

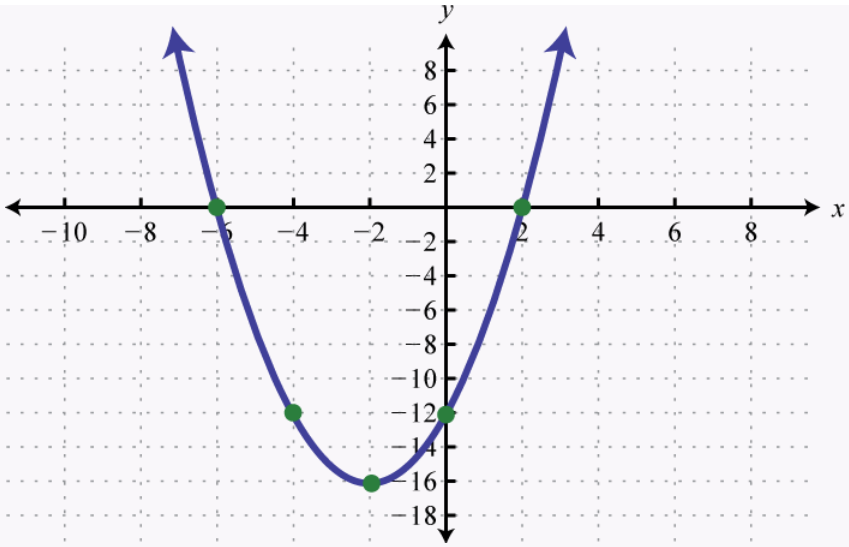



***Figure 6.E.8***

## Exercise 2.1.$E$.23

1. Given the function defined by $y = 3x^2 - 6x - 5$ :
    1. Does the function have a minimum or maximum? Explain.
    2. Find the minimum or maximum $y$-value.
2. The height in feet of a water rocket launched from the ground is given by the function $h(t) = -16t^2 + 96t$ where $t$ represents the number of seconds after launch. What is the maximum height attained by the rocket?

**Answer**

1. (1) Minimum (2) $y = -8$

## Exercise 2.1.$E$.24

Sketch the graph and use it to solve the given inequality.

1. Graph $f(x) = (x+1)^2 - 4$ and find $x$ where $f(x) \geq 0$.
2. Graph $f(x) = -x^2 + 4$ and find $x$ where $f(x) \geq 0$.

**Answer**

1. $(-\infty, -3] \cup [1, \infty)$

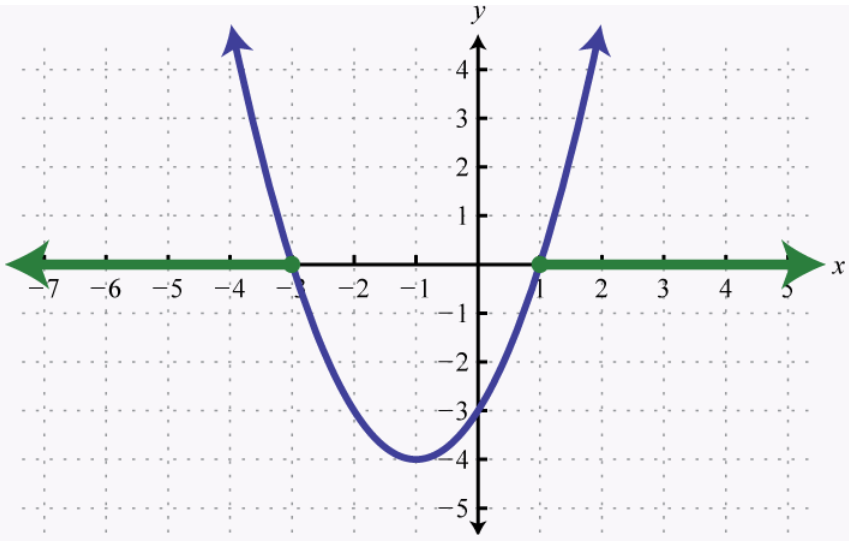



***Figure 6.E.9***

## Exercise 2.1.$E$.25

Solve. Present answers using interval notation.

1. $x^2-2x-15<0$
2. $x(2x-1)>10$
3. $x(x+3)(x-2)^2\leq 0$
4. $\frac{x^2-10x+25}{x+1}\geq 0$
5. $\frac{x^2-5x+4}{x^2+x}\leq 0$

**Answer**

1. $(-3,5)$
3. $[-3,0]\cup\{2\}$
5. $(-1,0)\cup[1,4]$

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# 2.2: Functions and Function Notation

Page ID 69073

**Learning Objectives**

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

## Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\} \tag{1.1.1}$$

The domain is $\{1,2,3,4,5\}$ The range is $\{2,4,6,8,10\}$

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter $x$. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter $y$.

A function $f$ is a relation that assigns a single value in the range to each value in the domain. In other words, no $x$-values are repeated. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\} \tag{1.1.2}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$ For example, the term "odd" corresponds to three values from the range, $\{1,3,5\}$,and the term "even" corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure 2.2.1 compares relations that are functions and not functions.

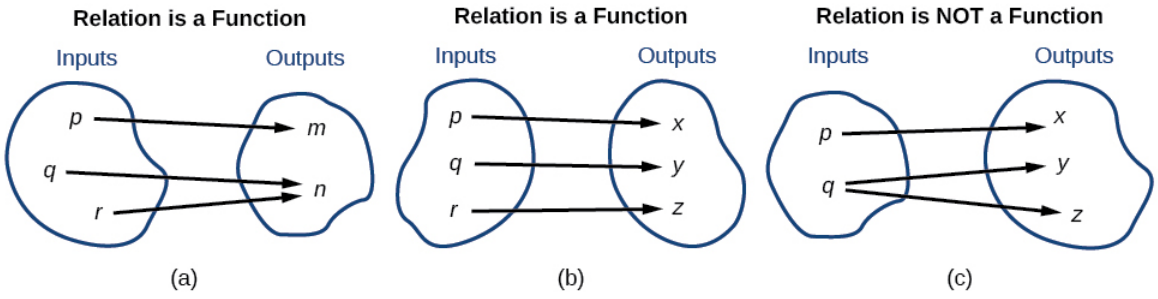



Figure 2.2.1: (a) This relationship is a function because each input is associated with a single output. Note that input $q$ and $r$ both give output $n$. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input $q$ is associated with two different outputs.

## Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.

## How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

## Example 2.2.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure 2.2.2 consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

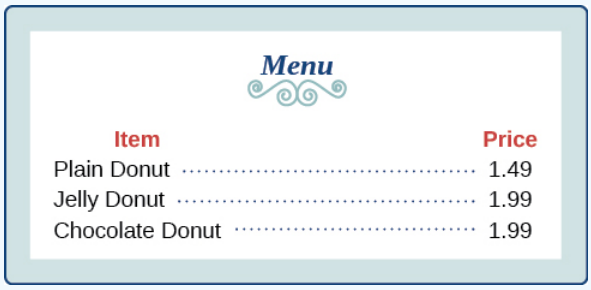



Figure 2.2.2: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

**Solution**

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure 2.2.3.




Figure 2.2.3: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure 2.2.4.

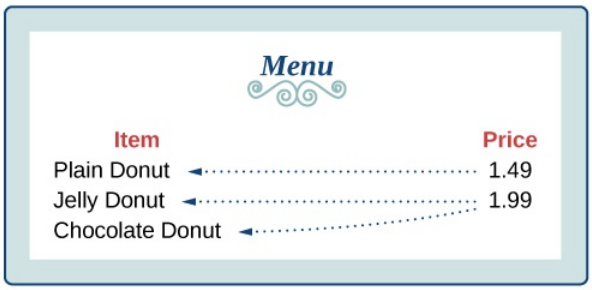



Figure 2.2.4: Association of the prices to the donuts.

Therefore, the item is a not a function of price.

### Example 2.2.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 2.2.1 shows a possible rule for assigning grade points.

Table 2.2.1: Class grade points.

| **Percent grade** | 0–56 | 57–61 | 62–66 | 67–71 | 72–77 | 78–86 | 87–91 | 92–100 |
|---|---|---|---|---|---|---|---|---|
| **Grade point average** | 0.0 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |

**Solution**

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

### Exercise 2.2.2

Table 2.2.2 lists the five greatest baseball players of all time in order of rank.

Table 2.2.2: Five greatest baseball players.

| Player | Rank |
|---|---|
| Babe Ruth | 1 |
| Willie Mays | 2 |
| Ty Cobb | 3 |
| Walter Johnson | 4 |
| Hank Aaron | 5 |

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

**Answer a**

Yes

**Answer b**

yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

## Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables $h$ for height and $a$ for age. The letters $f$, $g$,and $h$ are often used to represent functions just as we use $x$, $y$,and $z$ to represent numbers and $A$, $B$, and $C$ to represent sets.

$$\begin{aligned} &h \text{ is } f \text{ of } a && \text{We name the function } f\text{; height is a function of age.} \\ &h = f(a) && \text{We use parentheses to indicate the function input.} \\ &f(a) && \text{We name the function } f \text{ ; the expression is read as "} f \text{ of } a\text{."} \end{aligned} \tag{2.2.1}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that $h$ depends on $a$. The value $a$ must be put into the function $h$ to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add $a$ and $b$, and the result is the input for the function $f$." The operations must be performed in this order to obtain the correct result.

### Function Notation

The notation $y = f(x)$ defines a function named $f$. This is read as "$y$ is a function of $x$." The letter $x$ represents the input value, or independent variable. The letter $y$, or $f(x)$, represents the output value, or dependent variable.

### Example 2.2.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

**Solution**

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function $f$, we write days $= f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

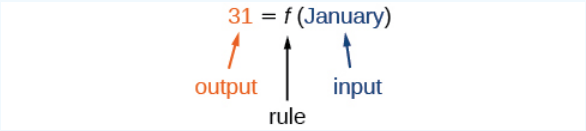



Figure 2.2.5: The function $31 = f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March}) = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, $d$ (the output), is dependent on the name of the month, $m$ (the input).

**Analysis**

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

**Example 2.2.3B: Interpreting Function Notation**

A function $N = f(y)$ gives the number of police officers, $N$, in a town in year $y$. What does $f(2005) = 300$ represent?

**Solution**

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers $(N)$, is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

**Exercise 2.2.3**

Use function notation to express the weight of a pig in pounds as a function of its age in days $d$.

**Answer**

$w = f(d)$

**Q&A**

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "$y$ is a function of $x$?"

*Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as $f$, which is a rule or procedure, and the output $y$ we get by applying $f$ to a particular input $x$. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.*

## Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 2.2.3 lists the input number of each month ($\text{January} = 1$, $\text{February} = 2$, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function $f$ where $D = f(m)$ identifies months by an integer rather than by name.

Table 2.2.3: Months and number of days per month.

| **Month number, $m$ (input)** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Days in month, $D$ (output)** | 31 | 28 | 31 | 30 | 31 | 30 | 31 | 31 | 30 | 31 | 30 | 31 |

Table 2.2.4 defines a function $Q = g(n)$ Remember, this notation tells us that $g$ is the name of the function that takes the input $n$ and gives the output $Q$.

Table 2.2.4: Function $Q = g(n)$

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $Q$ | 8 | 6 | 7 | 6 | 8 |

Table 2.2.5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table 2.2.5: Age of children and their corresponding heights.

| **Age in years, $a$ (input)** | 5 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| **Height in inches, $h$ (output)** | 40 | 42 | 44 | 47 | 50 | 52 | 54 |

**How To: Given a table of input and output values, determine whether the table represents a function**

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

**Example 2.2.5: Identifying Tables that Represent Functions**

Which table, Table 2.2.6, Table 2.2.7, or Table 2.2.8, represents a function (if any)?

Table 2.2.6

| Input | Output |
|---|---|
| 2 | 1 |
| 5 | 3 |
| 8 | 6 |

Table 2.2.7

| Input | Output |
|---|---|
| -3 | 5 |
| 0 | 1 |
| 4 | 5 |

Table 2.2.8

| Input | Output |
|---|---|
| 1 | 0 |
| 5 | 2 |
| 5 | 4 |

**Solution**

Table 2.2.6 and Table 2.2.7 define functions. In both, each input value corresponds to exactly one output value. Table 2.2.8 does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 2.2.6 can be represented by writing

$$f(2) = 1,\ f(5) = 3, \text{ and } f(8) = 6$$

Similarly, the statements

$$g(-3) = 5,\ g(0) = 1, \text{ and } g(4) = 5$$

represent the function in Table 2.2.7.

Table 2.2.8 cannot be expressed in a similar way because it does not represent a function.

**Exercise 2.2.5**

Does Table 2.2.9 represent a function?

Table 2.2.9

| Input | Output |
|---|---|
| 1 | 10 |
| 2 | 100 |
| 3 | 1000 |

**Answer**

yes

## Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

### Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

**How To: Given the formula for a function, evaluate.**

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

**Example 2.2.6A: Evaluating Functions at Specific Values**

1. Evaluate $f(x) = x^2 + 3x - 4$ at
   a. $2$
   b. $a$
   c. $a + h$
   d. Evaluate $\frac{f(a+h) - f(a)}{h}$

**Solution**

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a+h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} \\ &= \frac{(2ah + h^2 + 3h)}{h} \\ &= \frac{h(2a + h + 3)}{h} && \text{Factor out h.} \\ &= 2a + h + 3 && \text{Simplify.} \end{aligned}$$

## Example 2.2.6$B$: Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

**Solution**

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p \\ h(4) &= (4)^2 + 2(4) \\ &= 16 + 8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

## Exercise 2.2.6

Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

**Answer**

$g(5) = 1$

## Example 2.2.7: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

**Solution**

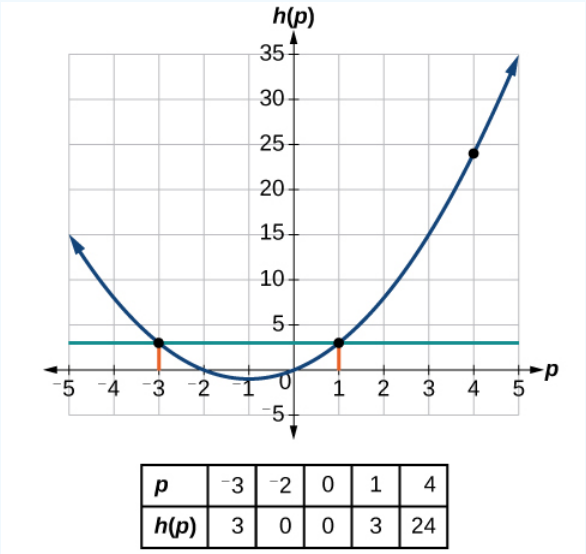
$$\begin{aligned} h(p) &= 3 \\ p^2+2p &= 3 \quad \text{Substitute the original function} \\ p^2+2p-3 &= 0 \quad \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 \quad \text{Factor.} \end{aligned}$$

If $(p+3)(p-1)=0$ , either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0). We will set each factor equal to 0 and solve for $p$ in each case.

$$(p+3)=0,\ p=-3$$

$$(p-1)=0,\ p=1$$

This gives us two solutions. The output $h(p)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing as in Figure 2.2.6. The graph verifies that $h(1)=h(-3)=3$ and $h(4)=24$.




| p | −3 | −2 | 0 | 1 | 4 |
|---|---|---|---|---|---|
| h(p) | 3 | 0 | 0 | 3 | 24 |

Figure 2.2.6: Graph of $h(p)=p^2+2p$

**Exercise 2.2.7**

Given the function $g(m)=\sqrt{m-4}$ , solve $g(m)=2$.

**Answer**

$m=8$

## Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in **equation** form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between $n$ and $p$. We can rewrite it to decide if $p$ is a function of $n$.

**How to: Given a function in equation form, write its algebraic formula.**

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

### Example 2.2.8$A$: Finding an Equation of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

**Solution**

To express the relationship in this form, we need to be able to write the relationship where $p$ is a function of $n$, which means writing it as $p=[\text{expression involving } n]$.

$$\begin{aligned} 2n+6p &= 12 \\ 6p &= 12-2n && \text{Subtract 2n from both sides.} \\ p &= \frac{12-2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6}-\frac{2n}{6} \\ p &= 2-\frac{1}{3}n \end{aligned}$$

Therefore, $p$ as a function of $n$ is written as

$$p=f(n)=2-\frac{1}{3}n$$

**Analysis**

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

### Example 2.2.8$B$: Expressing the Equation of a Circle as a Function

Does the equation $x^2+y^2=1$ represent a function with $x$ as input and $y$ as output? If so, express the relationship as a function $y=f(x)$.

**Solution**

First we subtract $x^2$ from both sides.

$$y^2=1-x^2$$

We now try to solve for $y$ in this equation.

$$y=\pm\sqrt{1-x^2}$$

$$\text{so, } y=\sqrt{1-x^2} \text{ and } y=-\sqrt{1-x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y=f(x)$.

### Exercise 2.2.8

If $x-8y^3=0$, express $y$ as a function of $x$.

**Answer**

$$y=f(x)=\frac{\sqrt[3]{x}}{2}$$

### Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x=y+2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only $x$ that equals $y$. However, each $x$ does determine a unique value for $y$, and there are

mathematical procedures by which $y$ can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for $y$ as a function of $x$, even though the formula cannot be written explicitly.

## Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table (Table 2.2.10).

Table 2.2.10

| Pet Memory | span in hours |
|---|---|
| Puppy | 0.008 |
| Adult Dog | 0.083 |
| Cat | 3 |
| Goldfish | 2160 |
| Beta Fish | 3600 |

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function $P$. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function $P$ at the input value of "goldfish." We would write $P(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

### How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

### Example 2.2.9: Evaluating and Solving a Tabular Function

Using Table 2.2.11,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

Table 2.2.11

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $g(n)$ | 8 | 6 | 7 | 6 | 8 |

**Solution**

a. Evaluating $g(3)$ means determining the output value of the function $g$ for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. Table 2.2.12 shows two solutions: 2 and 4.

Table 2.2.12

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $g(n)$ | 8 | 6 | 7 | 6 | 8 |

When we input 2 into the function $g$, our output is 6. When we input 4 into the function $g$, our output is also 6.

**Exercise 2.2.1**

Using Table 2.2.12, evaluate $g(1)$.

**Answer**

$g(1) = 8$

## Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

**Example 2.2.10: Reading Function Values from a Graph**

Given the graph in Figure 2.2.7,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

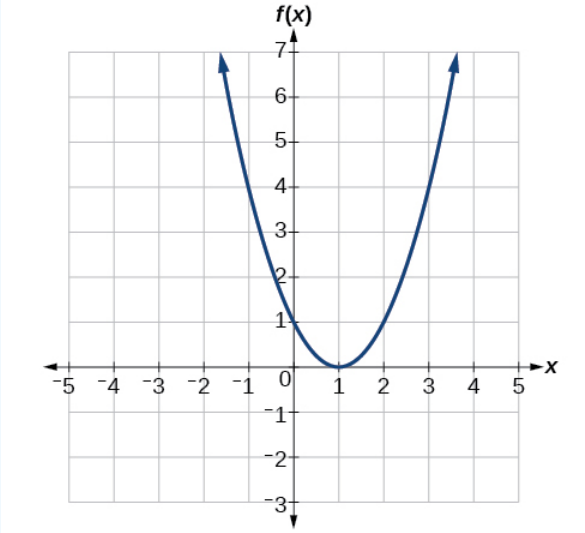
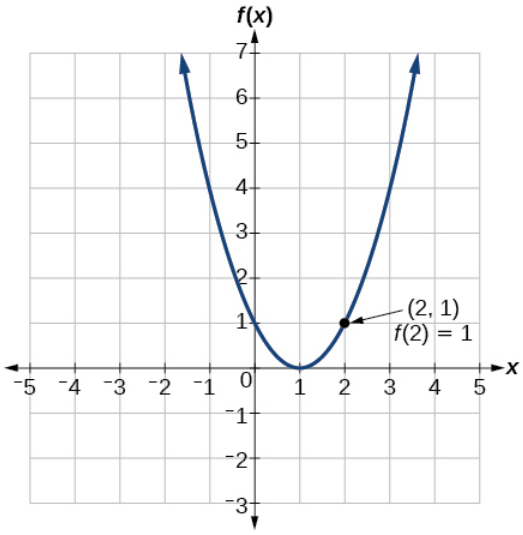



Figure 2.2.7: Graph of a positive parabola centered at $(1, 0)$.

**Solution**

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 2.2.8.




2.2.8: Graph of a positive parabola centered at $(1,0)$ with the labeled point $(2,1)$ where $f(2)=1$.

To solve $f(x)=4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y=4$, we locate two points of the curve with output value 4: $(-1,4)$ and $(3,4)$. These points represent the two solutions to $f(x)=4$: $-1$ or 3. This means $f(-1)=4$ and $f(3)=4$, or when the input is $-1$ or 3, the output is 4. See Figure 2.2.9.

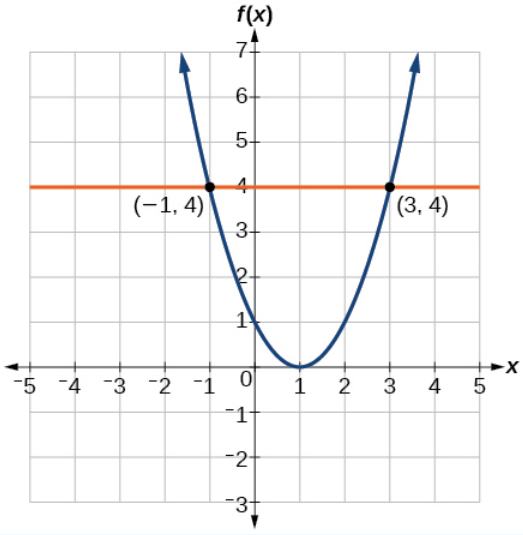



Figure 2.2.9: Graph of an upward-facing parabola with a vertex at $(0,1)$ and labeled points at $(-1,4)$ and $(3,4)$. A line at $y=4$ intersects the parabola at the labeled points.

**Exercise 2.2.10**

Given the graph in Figure 2.2.7, solve $f(x)=1$.

**Answer**

$x=0$ or $x=2$

## Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the Figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 2.2.13

Table 2.2.13: Letter grades and decimal equivalents.

| Letter Grade | Grade Point Average |
|---|---|
| A | 4.0 |
| B | 3.0 |
| C | 2.0 |
| D | 1.0 |

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figures 2.2.1*a* and 2.2.1*b*. The function in part (a) shows a relationship that is not a one-to-one function because inputs $q$ and $r$ both give output $n$. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

### One-to-One Functions

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

### Example 2.2.11: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

**Solution**

A circle of radius $r$ has a unique area measure given by $A = \pi r^2$, so for any input, $r$, there is only one output, $A$. The area is a function of radius$r$.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure $A$ is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

### Exercise 2.2.11$A$

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

**Answer**

a. yes, because each bank account has a single balance at any given time;

b. no, because several bank account numbers may have the same balance;

c. no, because the same output may correspond to more than one input.

### Exercise 2.2.11$B$

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

**Answer**

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

## Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value $x$ and the output $y$, and we say $y$ is a function of $x$, or $y = f(x)$ when the function is named $f$. The graph of the function is the set of all points $(x, y)$ in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 2.2.10 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points $(x, y)$ satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

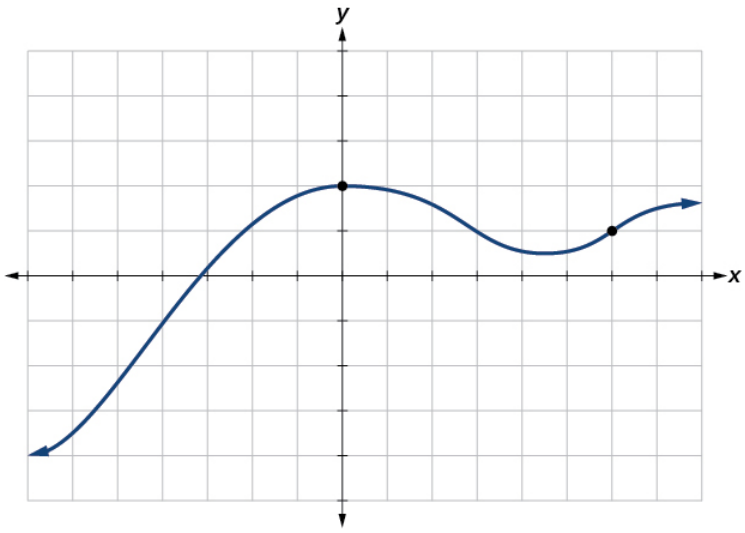



Figure 2.2.10: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure 2.2.11.

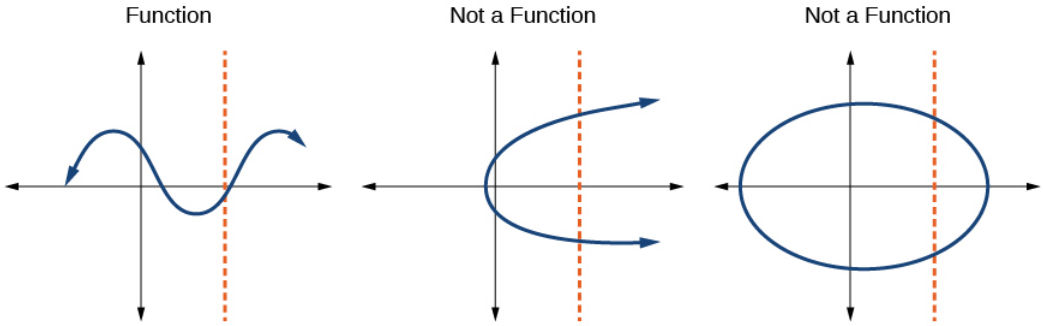



Figure 2.2.11: Three graphs visually showing what is and is not a function.

**Howto: Given a graph, use the vertical line test to determine if the graph represents a function**

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

## Example 2.2.12: Applying the Vertical Line Test

Which of the graphs in Figure 2.2.12 represent(s) a function $y = f(x)$?

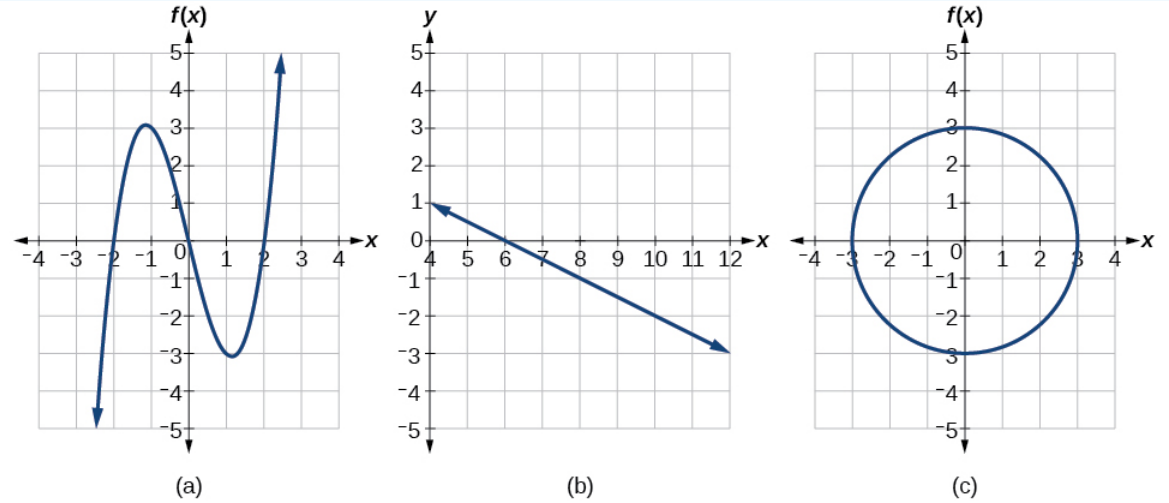



Figure 2.2.12: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

**Solution**

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 2.2.12 From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure 2.2.13

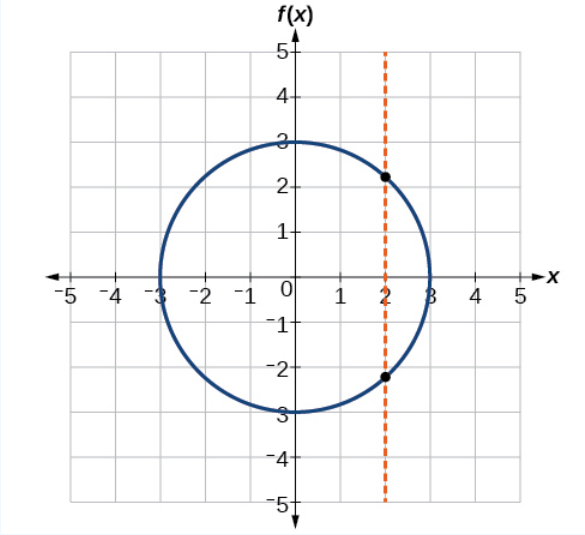



Figure 2.2.13: Graph of a circle.

## Exercise 2.2.12

Does the graph in Figure 2.2.14 represent a function?


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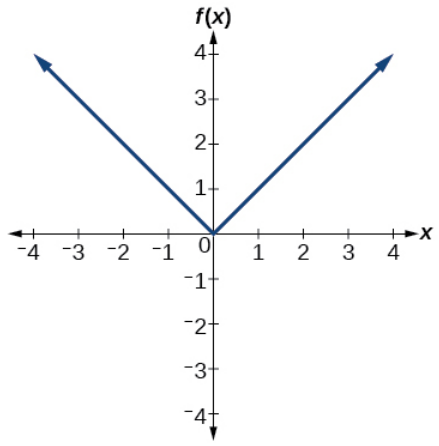



Figure 2.2.14: Graph of absolute value function.

**Answer**

yes

## Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

### Howto: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

### Example 2.2.13: Applying the Horizontal Line Test

Consider the functions shown in Figure $2.2.12a$ and Figure $2.2.12b$ Are either of the functions one-to-one?

**Solution**

The function in Figure $2.2.12a$ is not one-to-one. The horizontal line shown in Figure $2.2.15$ intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

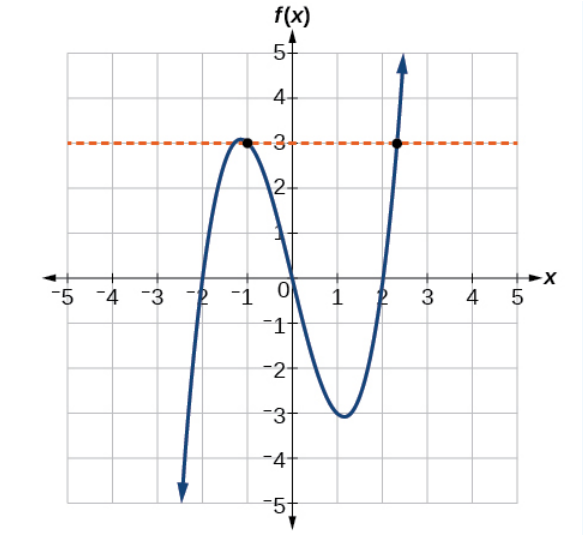



Figure 2.2.15: Graph of a polynomial with a horizontal line crossing through 2 points

The function in Figure $2.2.12b$is one-to-one. Any horizontal line will intersect a diagonal line at most once.

**Exercise 2.2.13**

Is the graph shown in Figure $2.2.13$one-to-one?

**Answer**

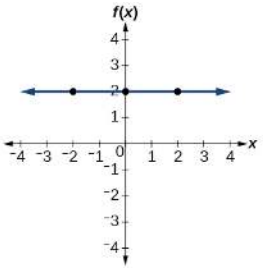
No, because it does not pass the horizontal line test.

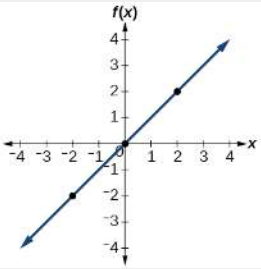
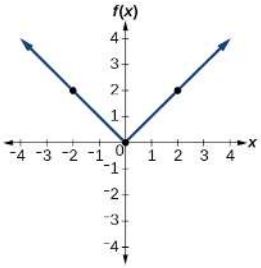
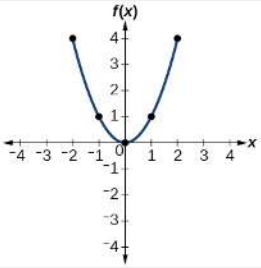
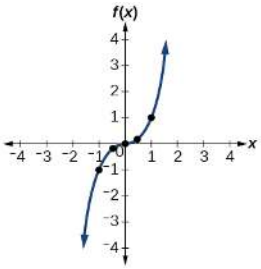
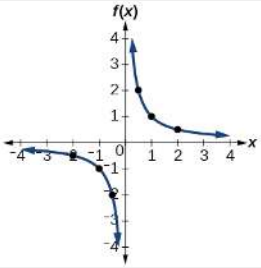
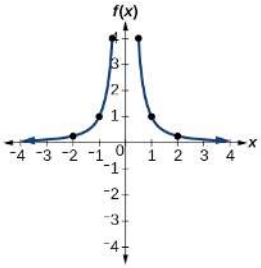
## Identifying Basic Toolkit Functions

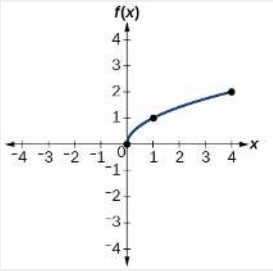
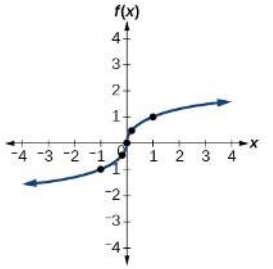
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table $2.2.14$

Table 2.2.14: Toolkit Functions

| Name | Function | Graph |
|---|---|---|
| Constant | $f(x) = c$ where $c$ is a constant |   <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table> |

| Name | Function | Graph |
|---|---|---|
| Identity | $f(x) = x$ |   <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table> |
| Absolute Value | $f(x) = \lvert x \rvert$ |   <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table> |
| Quadratic | $f(x) = x^2$ |   <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>4</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table> |
| Cubic | $f(x) = x^3$ |   <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−0.125</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.5</td><td>0.125</td></tr><tr><td>1</td><td>1</td></tr></table> |
| reciprocal | $f(x) = \frac{1}{x}$ |   <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−0.5</td></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−2</td></tr><tr><td>0.5</td><td>2</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.5</td></tr></table> |
| Reciprocal squared | $f(x) = \frac{1}{x^2}$ |   <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>0.25</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>−0.5</td><td>4</td></tr><tr><td>0.5</td><td>4</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.25</td></tr></table> |

| Name | Function | Graph |
|---|---|---|
| Square root | $f(x) = \sqrt{x}$ |   <table><tr><th>x</th><th>f(x)</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>4</td><td>2</td></tr></table> |
| Cube root | $f(x) = \sqrt[3]{x}$ |   <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.125</td><td>−0.5</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.125</td><td>0.5</td></tr><tr><td>1</td><td>1</td></tr></table> |

## Key Equations

- Constant function $f(x) = c$, where $c$ is a constant
- Identity function $f(x) = x$
- Absolute value function $f(x) = |x|$
- Quadratic function $f(x) = x^2$
- Cubic function $f(x) = x^3$
- Reciprocal function $f(x) = \dfrac{1}{x}$
- Reciprocal squared function $f(x) = \frac{1}{x^2}$
- Square root function $f(x) = \sqrt{x}$
- Cube root function $f(x) = 3\sqrt{x}$

## Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$.
- In tabular form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- The graph of a one-to-one function passes the horizontal line test.

## Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.
2 www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

## Glossary

**dependent variable**
an output variable

**domain**
the set of all possible input values for a relation

**function**
a relation in which each input value yields a unique output value

**horizontal line test**
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

**independent variable**
an input variable

**input**
each object or value in a domain that relates to another object or value by a relationship known as a function

**one-to-one function**
a function for which each value of the output is associated with a unique input value

**output**
each object or value in the range that is produced when an input value is entered into a function

**range**
the set of output values that result from the input values in a relation

**relation**
a set of ordered pairs

**vertical line test**
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

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# 2.3: Domain and Range

**Learning Objectives**

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—I am Legend, Hannibal, The Ring, The Grudge, and The Conjuring. Figure 2.3.1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

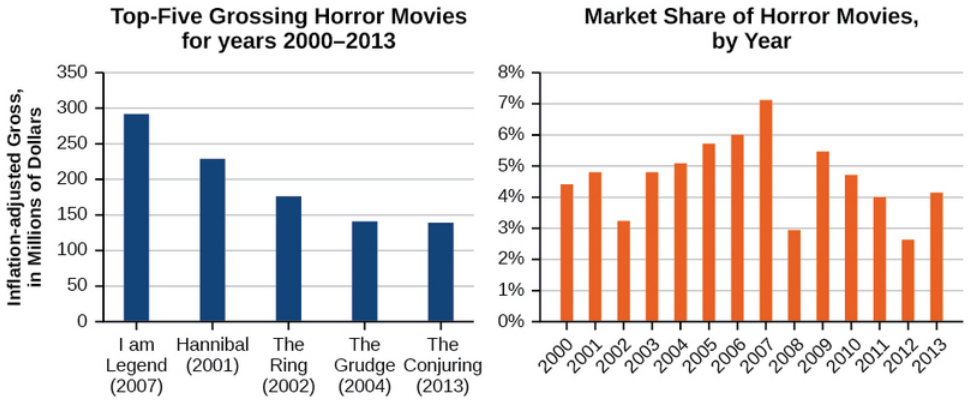



Figure 2.3.1: Graph of the Top-Five Grossing Horror Movies for years 2000-2003, and a Graph of the Market Share of Horror Movies by Year. Based on data compiled by www.the-numbers.com.

## Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (Figure 2.3.2).

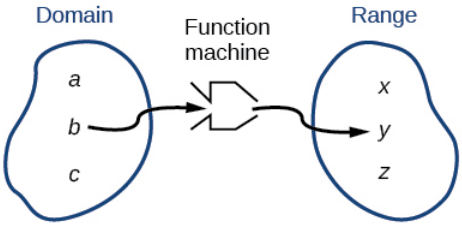



Figure 2.3.2: Diagram of how a function relates two relations

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [ when the set includes the endpoint and a parenthesis ( to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, ( or ), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [ or ], are used to indicate that an endpoint is included, called inclusive.

See Figure 2.3.3 for a summary of interval notation.

| Inequality | Interval Notation | Graph on Number Line | Description |
|---|---|---|---|
| $x > a$ | $(a, \infty)$ |   | $x$ is greater than $a$ |
| $x < a$ | $(-\infty, a)$ |   | $x$ is less than $a$ |
| $x \geq a$ | $[a, \infty)$ |   | $x$ is greater than or equal to $a$ |
| $x \leq a$ | $(-\infty, a]$ |   | $x$ is less than or equal to $a$ |
| $a < x < b$ | $(a, b)$ |   | $x$ is strictly between $a$ and $b$ |
| $a \leq x < b$ | $[a, b)$ |   | $x$ is between $a$ and $b$, to include $a$ |
| $a < x \leq b$ | $(a, b]$ |   | $x$ is between $a$ and $b$, to include $b$ |
| $a \leq x \leq b$ | $[a, b]$ |   | $x$ is between $a$ and $b$, to include $a$ and $b$ |

Figure 2.3.3: Summary of interval notation.

### Example 2.3.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$

**Solution**

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

### Exercse 2.3.1

Find the domain of the function:

$$\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$$

**Answer**

$\{-5, 0, 5, 10, 15\}$

### How To: Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

### Example 2.3.2: Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$ .

**Solution**

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

### Exercse 2.3.2

Find the domain of the function:

$$f(x) = 5 - x + x^3$$

**Answer**

$(-\infty, \infty)$

### Howto: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

### Example 2.3.3: Finding the Domain of a Function Involving a Denominator

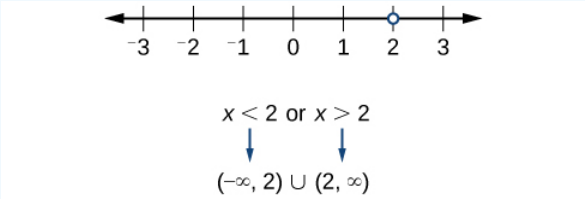
Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

**Solution**

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$2-x=0$$
$$-x=-2$$
$$x=2$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x<2$ or $x>2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty,2)\cup(2,\infty)$.




In interval form, the domain of f is $(-\infty,2)\cup(2,\infty)$.

## Exercse 2.3.3

Find the domain of the function:

$$f(x)=\frac{1+4x}{2x-1}$$

**Answer**

$$(-\infty,\frac{1}{2})\cup(\frac{1}{2},\infty)$$

### How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

## Example 2.3.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x)=\sqrt{7-x}.$$

**Solution**

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$7-x\geq 0$$
$$-x\geq -7$$
$$x\leq 7$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty,7]$.

## Exercse 2.3.4

Find the domain of the function

$$f(x)=\sqrt{5+2x}.$$

**Answer**

$$[-2.5, \infty)$$

**Q&A: Can there be functions in which the domain and range do not intersect at all?**

Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

### Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure $2.3.4$ compares inequality notation, set-builder notation, and interval notation.

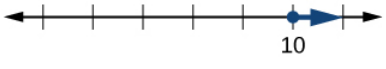
| | Inequality Notation | Set-builder Notation | Interval Notation |
|---|---|---|---|
|  | $5 < h \leq 10$ | $\{h \mid 5 < h \leq 10\}$ | $(5, 10]$ |
|  | $5 \leq h < 10$ | $\{h \mid 5 \leq h < 10\}$ | $[5, 10)$ |
|  | $5 < h < 10$ | $\{h \mid 5 < h < 10\}$ | $(5, 10)$ |
|  | $h < 10$ | $\{h \mid h < 10\}$ | $(-\infty, 10)$ |
|  | $h \geq 10$ | $\{h \mid h \geq 10\}$ | $[10, \infty)$ |
|  | All real numbers | $\mathbb{R}$ | $(-\infty, \infty)$ |

Figure 2.3.4: Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3\} = (-\infty, -3] \cup [3, \infty) \tag{2.3.1}$$

## Set-Builder Notation and Interval Notation

*Set-builder notation* is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|$ statement about x$\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \leq 12\}$$

*Interval notation* is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

**Given a line graph, describe the set of values using interval notation.**

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [ with each end value to be included in the set (solid dot) or ( for each excluded end value (open dot).
3. At the right end of each interval, use ] with each end value to be included in the set (filled dot) or ) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

## Example 2.3.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure 2.3.5 using inequality notation, set-builder notation, and interval notation.




Figure 2.3.5: Line graph of $1 \leq x \leq 3$ or $x > 5$ .

**Solution**

To describe the values, $x$, included in the intervals shown, we would say, "$x$ is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x|1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

## Exercse 2.3.5

Given Figure 2.3.6, specify the graphed set in

a. words
b. set-builder notation
c. interval notation




Figure 2.3.6: Line graph of $-2 \leq x$, $-1 \leq x < 3$.leq}x\), $-1 \leq x < 3$.


https://math.libretexts.org/@go/page/69074

**Answer a**

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

**Answer b**

$\{x|x \leq -2 or -1 \leq x < 3\}$

**Answer c**

$(-\infty, -2] \cup [-1, 3)$

## Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 2.3.7.

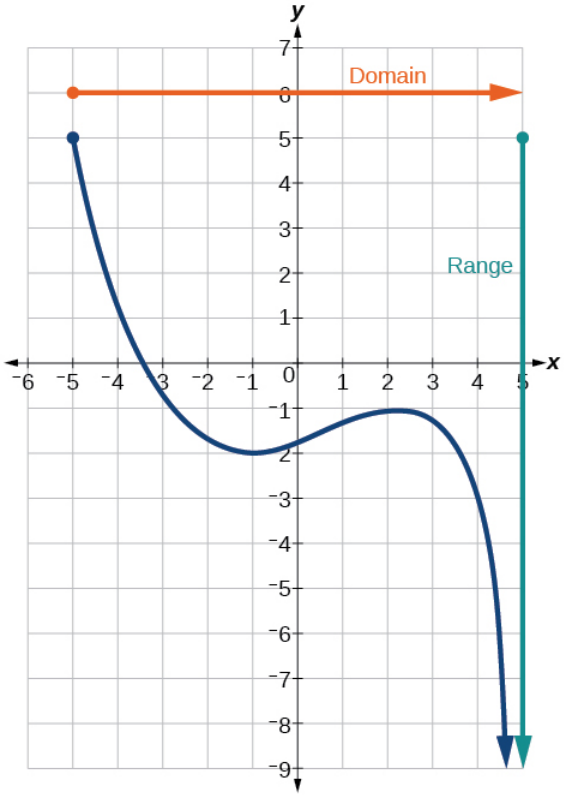



Figure 2.3.7: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from −5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

### Example 2.3.6A: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

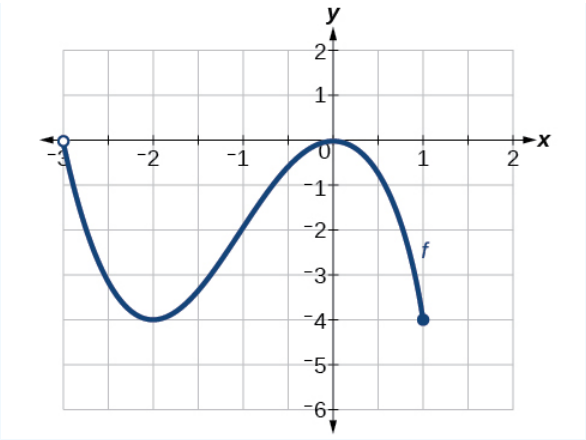



Figure 2.3.8: Graph of a function from (-3, 1].

**Solution**

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See Figure 2.3.9.

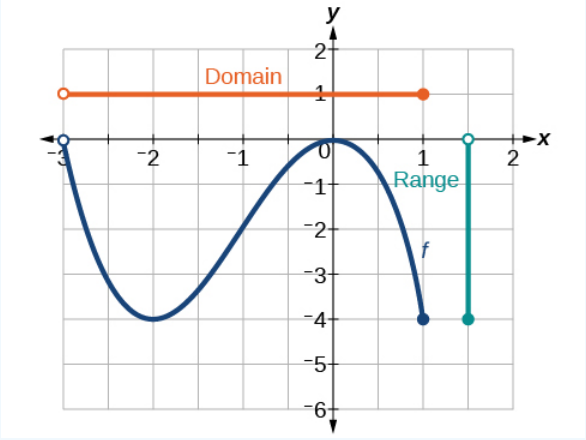



Figure 2.3.9: Graph of the previous function shows the domain and range

## Example 2.3.6B: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure 2.3.10.

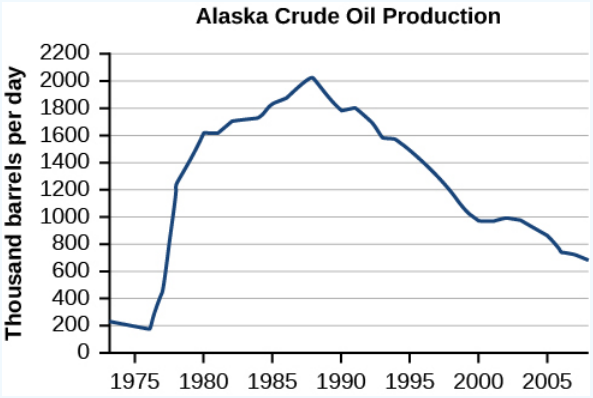



Figure 2.3.10: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

**Solution**

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is $[1973, 2008]$ and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

**Exercse 2.3.6**

Given Figure 2.3.11, identify the domain and range using interval notation.

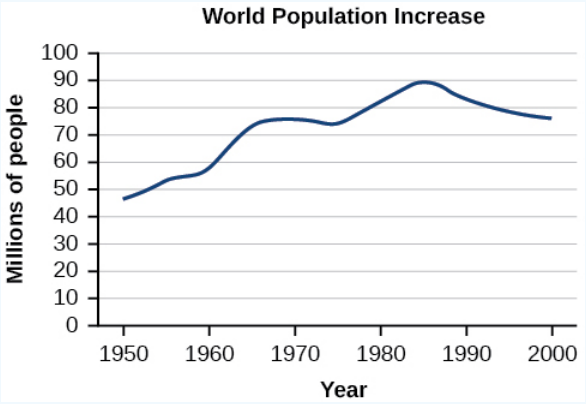



Figure 2.3.11: Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.

**Answer**

domain =$[1950, 2002]$

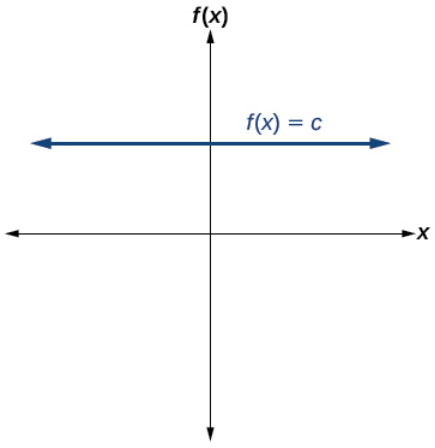
range = $[47,000,000, 89,000,000]$

**Can a function’s domain and range be the same?**

*Yes. For example, the domain and range of the cube root function are both the set of all real numbers.*

## Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.




Domain: $(-\infty, \infty)$
Range: $[c,c]$

Figure 2.3.12: Constant function $f(x) = c$.

For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant $c$, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with $c$.

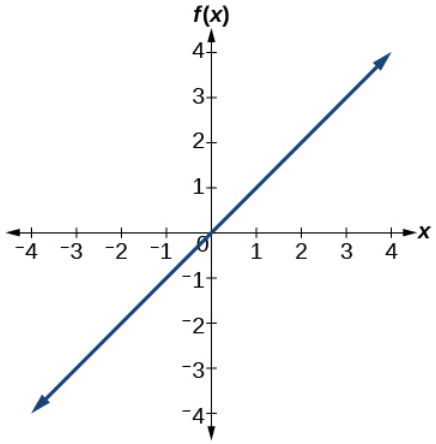



***Figure* 2.3.13:** *Identity function f(x)=x.*

For the **identity function** $f(x) = x$, there is no restriction on $x$. Both the domain and range are the set of all real numbers.

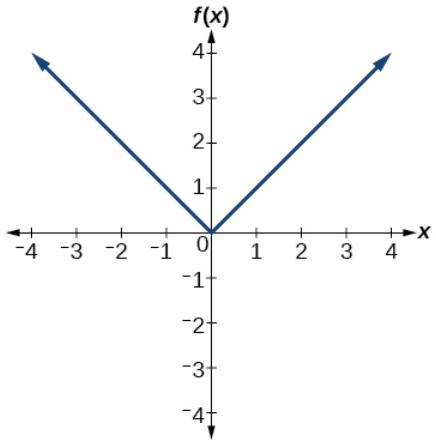
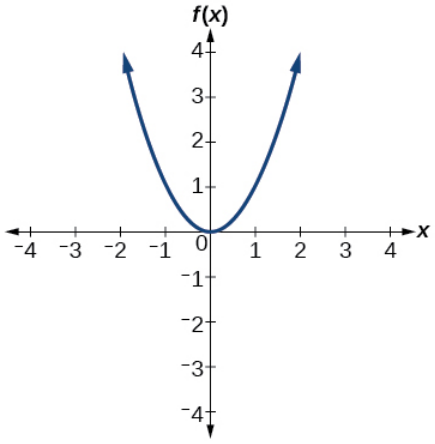



Figure 2.3.14: Absolute function $f(x) = |x|$.

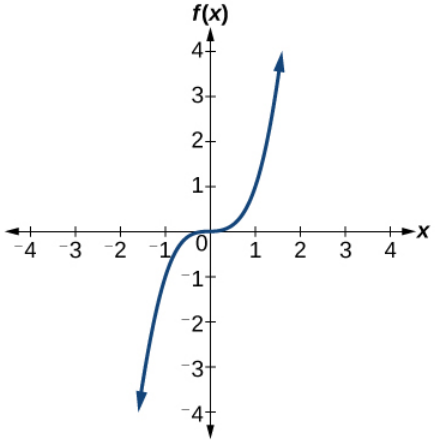
For the **absolute value function** $f(x) = |x|$, there is no restriction on $x$. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.




Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.3.15: Quadratic function $f(x) = x^2$.

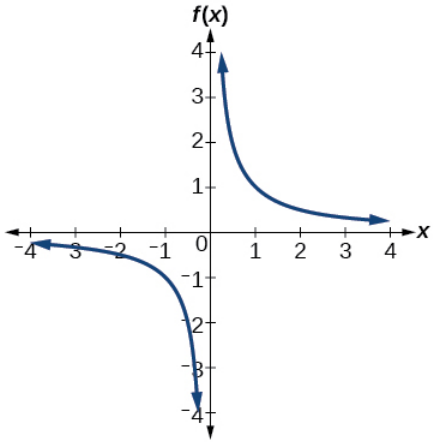
For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.




Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.3.16: Cubic function $f(x) - x^3$.

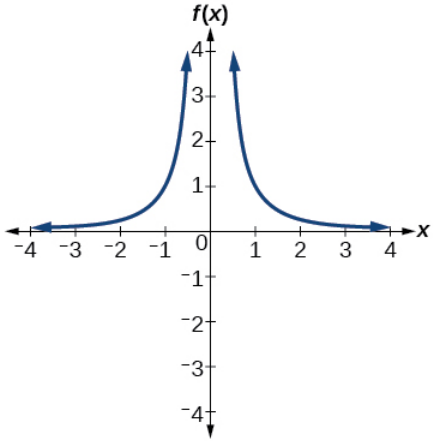
For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.




Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 2.3.17: Reciprocal function $f(x) = \frac{1}{x}$.

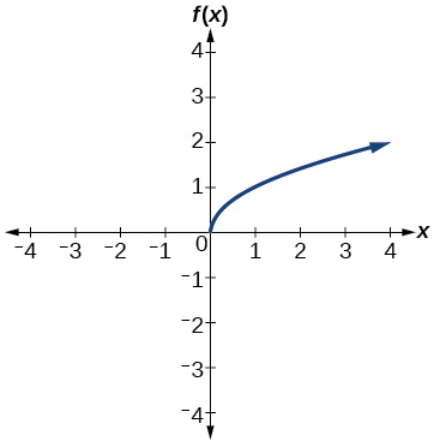
For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x | x \neq 0\}$,the set of all real numbers that are not zero.




Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 2.3.18: Reciprocal squared function $f(x) = \frac{1}{x^2}$

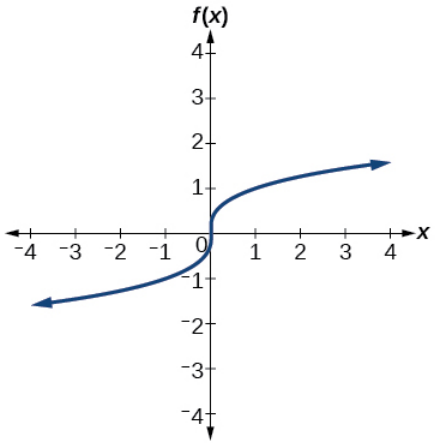
For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$ ,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.




Domain: [0, ∞)
Range: [0, ∞)

***Figure* 2.3.19:** *Square root function* $f(x)=\sqrt{(x)}$.

For the **square root function** $f(x)=\sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number $x$ is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us $x$.




Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.3.20: Cube root function $f(x)=\sqrt[3]{x}$.

For the **cube root function** $f(x)=\sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

**How To: Given the formula for a function, determine the domain and range.**

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

### Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x)=2x^3-x$ .

**Solution**

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

### Example 2.3.7$B$: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

**Solution**

We cannot evaluate the function at $-1$ because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

### Example 2.3.7$C$: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

**Solution**

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as $x$ increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

**Analysis**

Figure 2.3.19represents the function $f$.

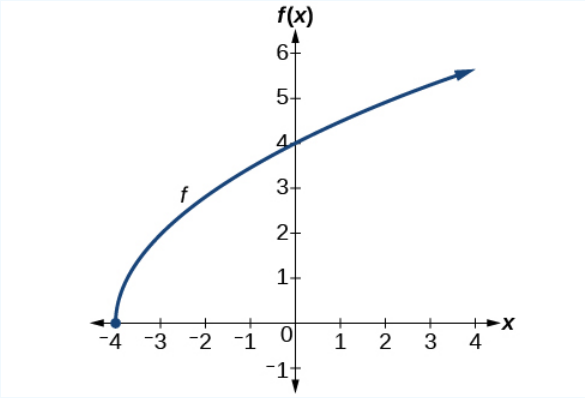



Figure 2.3.19: Graph of a square root function at $(-4, 0)$.

### Exercise 2.3.7

Find the domain and range of

$f(x) = \sqrt{-2-x}$.

**Answer**

domain: $(-\infty, -2]$

range: $[0, \infty)$

## Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, **absolute value** can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to $10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

## Piecewise Function

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1} & \text{if x is in domain 1} \\ \text{formula 2} & \text{if x is in domain 2} \\ \text{formula 3} & \text{if x is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

**How To: Given a piecewise function, write the formula and identify the domain for each interval.**

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

### Example 2.3.8A: Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a **function** relating the number of people, $n$, to the cost, $C$.

**Solution**

Two different formulas will be needed. For $n$-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

**Analysis**

The function is represented in Figure 2.3.20. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

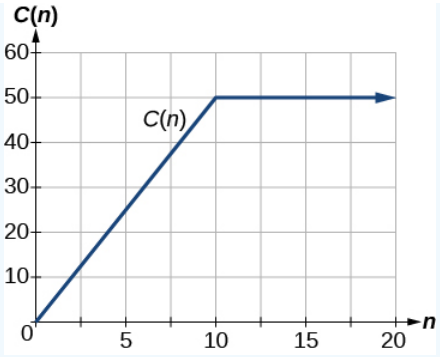


Figure 2.3.20

## Example 2.3.8$B$: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g-2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

**Soltuion**

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, C(4), we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4-2) = \$45$$

**Analysis**

The function is represented in Figure 2.3.21. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

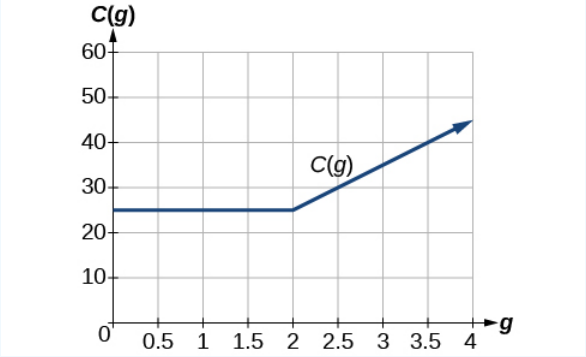


Figure 2.3.21

**How To: Given a piecewise function, sketch a graph.**

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

## Example 2.3.8C: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 & \text{if } 1 < x \leq 2 \\ x & \text{if } x > 2 \end{cases}$$

**Solution**

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 2.3.20 shows the three components of the piecewise function graphed on separate coordinate systems.

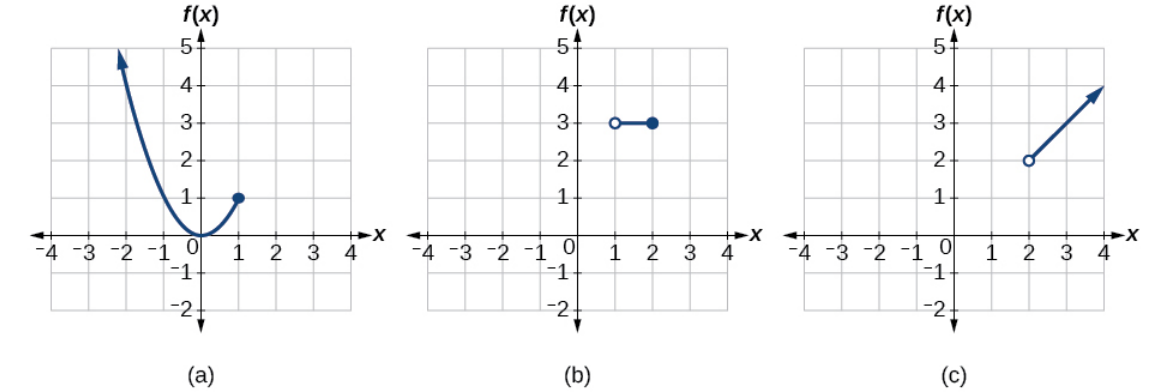



***Figure 2.3.20:*** *Graph of each part of the piece-wise function f(x)*

(a) $f(x) = x^2$ if $x \leq 1$; (b) $f(x) = 3$ if $1 < x \leq 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure 2.3.21.

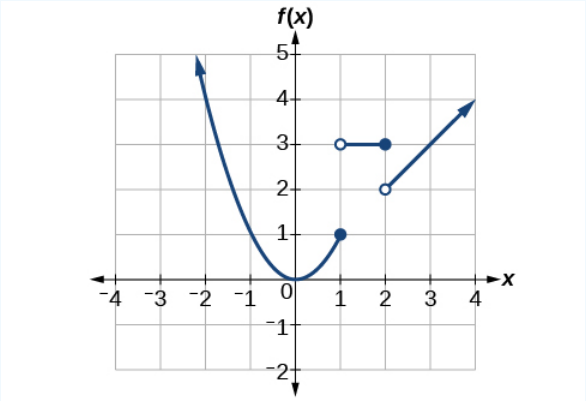



Figure 2.3.21: Graph of the entire function.

**Analysis**

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

## Exercise 2.3.8

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

**Answer**

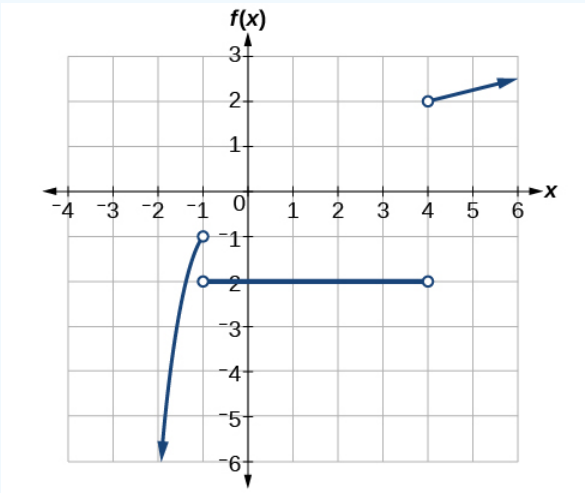



Figure 2.3.22

**Can more than one formula from a piecewise function be applied to a value in the domain?**

*No. Each value corresponds to one equation in a piecewise formula.*

## Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.
- A piecewise function is described by more than one formula.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain.

## Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014

2 www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

## Glossary

**interval notation**

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

**piecewise function**

a function in which more than one formula is used to define the output

**set-builder notation**

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

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# 2.4: Rates of Change and Behavior of Graphs

**Learning Objectives**

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

Gasoline costs have experienced some wild fluctuations over the last several decades. Table 2.4.1 lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

Table 2.4.1

| $y$ | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 |
|---|---|---|---|---|---|---|---|---|
| $C(y)$ | 2.31 | 2.62 | 2.84 | 3.30 | 2.41 | 2.84 | 3.58 | 3.68 |

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from \$2.31 to \$3.68, an increase of \$1.37. While this is interesting, it might be more useful to look at how much the price changed per year. In this section, we will investigate changes such as these.

## Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 2.4.1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter $\Delta$ (delta) signifies the change in a quantity; we read the ratio as "delta-$y$ over delta-$x$" or "the change in $y$ divided by the change in $x$." Occasionally we write $\Delta f$ instead of $\Delta y$, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by \$1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196 \text{ dollars per year.} \tag{2.4.1}$$

On average, the price of gas increased by about 19.6¢ each year. Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by \$4,000 per quarter

**Definition: Rate of Change**

A **rate of change** describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{2.4.2}$$

**How To...**

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values $x_1$ and $x_2$.

1. Calculate the difference $y_2 - y_1 = \Delta y$ .
2. Calculate the difference $x_2 - x_1 = \Delta x$ .
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

**Example 2.4.1: Computing an Average Rate of Change**

Using the data in Table 2.4.1, find the average rate of change of the price of gasoline between 2007 and 2009.

**Solution**

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} \\ &= \frac{-\$0.43}{2 \text{ years}} \\ &= -\$0.22 \text{ per year} \end{aligned}$$

**Analysis**

Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

**Exercise 2.4.1**

Using the data in Table 2.4.1, find the average rate of change between 2005 and 2010.

**Solution**

$$\frac{\$2.84 - \$2.315}{5 \text{ years}} = \frac{\$0.535}{5 \text{ years}} = \$0.106 \text{per year.}$$

**Example 2.4.2: Computing Average Rate of Change from a Graph**

Given the function $g(t)$ shown in Figure 2.4.1, find the average rate of change on the interval $[-1, 2]$.

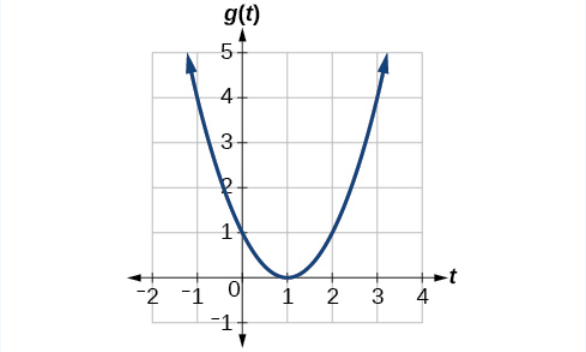



Figure 2.4.1: Graph of a parabola.

**Solution**

At $t=-1$, Figure 2.4.2 shows $g(-1)=4$. At $t=2$,the graph shows $g(2)=1$.

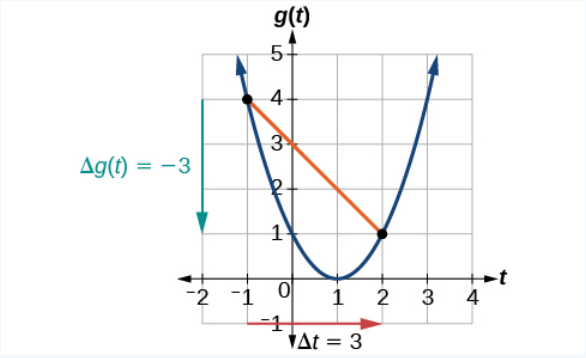



Figure 2.4.2: Graph of a parabola with a line from points (-1, 4) and (2, 1) to show the changes for g(t) and t.

The horizontal change $\Delta t=3$ is shown by the red arrow, and the vertical change $\Delta g(t)=-3$ is shown by the turquoise arrow. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)}=\frac{-3}{3}=-1 \tag{2.4.3}$$

**Analysis**

Note that the order we choose is very important. If, for example, we use $\frac{y_2-y_1}{x_1-x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as $(x_1,y_1)$ and $(x_2,y_2)$.

## Example 2.4.3: Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table 2.4.2. Find her average speed over the first 6 hours.

Table 2.4.2

| $t$ (hours) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $D(t)$ (miles) | 10 | 55 | 90 | 153 | 214 | 240 | 292 | 300 |

**Solution**

Here, the average speed is the average rate of change. She traveled 292 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0}=\frac{282}{6}$$
$$=47$$

The average speed is about 47 miles per hour.

**Analysis**

Because the speed is not constant, the average speed depends on the interval chosen. For the interval $[2,3]$, the average speed is 63 miles per hour.

## Example 2.4.4: Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x)=x^2-\frac{1}{x}$ on the interval $[2,4]$.

**Solution**

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2-\frac{1}{2} & f(4) &= 4^2-\frac{1}{4} \\ &= 4-\frac{1}{2} & &= 16-\frac{1}{4} \\ &= 72 & &= \frac{63}{4} \end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned} \text{Average rate of change} &= \frac{f(4)-f(2)}{4-2} \\ &= \frac{\frac{63}{4}-\frac{7}{2}}{4-2} \\ &= \frac{\frac{49}{4}}{2} \\ &= \frac{49}{8} \end{aligned}$$

## Exercise 2.4.2

Find the average rate of change of $f(x)=x-2\sqrt{x}$ on the interval $[1,9]$.

**Solution**

$\frac{1}{2}$

## Example 2.4.5: Finding the Average Rate of Change of a Force

The electrostatic force $F$, measured in newtons, between two charged particles can be related to the distance between the particles $d$,in centimeters, by the formula $F(d)=\frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

**Solution**

We are computing the average rate of change of $F(d)=\dfrac{2}{d^2}$ on the interval $[2,6]$.

$$\begin{aligned}
\text{Average rate of change} &= \frac{F(6)-F(2)}{6-2} \\
&= \frac{\frac{2}{6^2}-\frac{2}{2^2}}{6-2} && \text{Simplify} \\
&= \frac{\frac{2}{36}-\frac{2}{4}}{4} \\
&= \frac{-\frac{16}{36}}{4} && \text{Combine numerator terms.} \\
&= -\frac{1}{9} && \text{Simplify}
\end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

### Example 2.4.6: Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t)=t^2+3t+1$ on the interval $[0,a]$. The answer will be an expression involving $a$.

**Solution**

We use the average rate of change formula.

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a)-g(0)}{a-0} && \text{Evaluate.} \\
&= \frac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0} && \text{Simplify.} \\
&= \frac{a^2+3a+1-1}{a} && \text{Simplify and factor.} \\
&= \frac{a(a+3)}{a} && \text{Divide by the common factor a.} \\
&= a+3
\end{aligned}$$

This result tells us the average rate of change in terms of a between $t=0$ and any other point $t=a$. For example, on the interval $[0,5]$, the average rate of change would be $5+3=8$.

### Exercise 2.4.3

Find the average rate of change of $f(x)=x^2+2x-8$ on the interval $[5,a]$.

**Solution**

$a+7$

## Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. Figure 2.4.3 shows examples of increasing and decreasing intervals on a function.

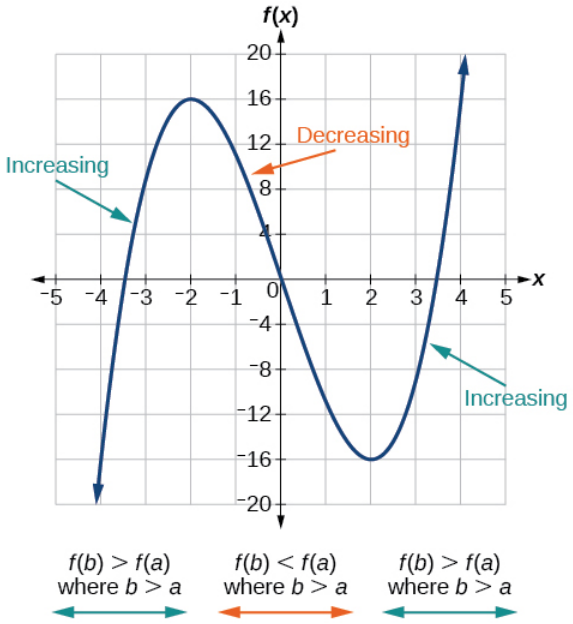



Figure 2.4.3: The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is "local minima." Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is "extremum.") Often, the term local is replaced by the term relative. In this text, we will use the term local.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of local extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in Figure 2.4.4, the local maximum is 16, and it occurs at $x = -2$. The local minimum is $-16$ and it occurs at $x = 2$.

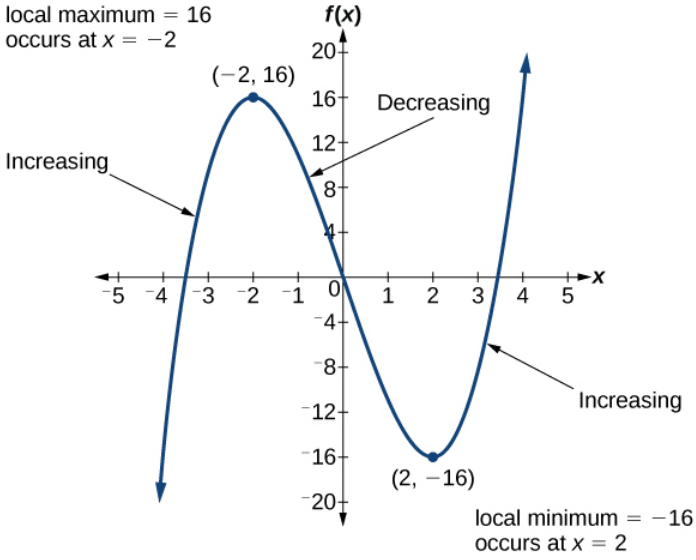



Figure 2.4.4: Graph of a polynomial that shows the increasing and decreasing intervals and local maximum.maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a

local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure 2.4.5 illustrates these ideas for a local maximum.

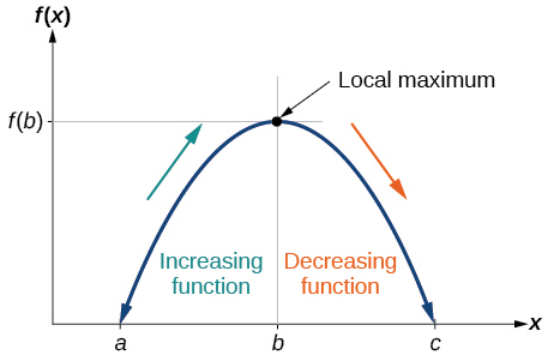



Figure 2.4.5: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

### Local Minima and Local Maxima

- A function $f$ is an **increasing function** on an open interval if $f(b) > f(a)$ for every $a$, $b$ interval where $b > a$.
- A function $f$ is a **decreasing function** on an open interval if $f(b) < f(a)$ for every $a$, $b$ interval where $b > a$.

A function $f$ has a local maximum at a point $b$ in an open interval $(a, c)$ if $f(b)$ is greater than or equal to $f(x)$ for every point $x$ ($x$ does not equal $b$) in the interval. Likewise, $f$ has a local minimum at a point $b$ in $(a, c)$ if $f(b)$ is less than or equal to $f(x)$ for every $x$ ($x$ does not equal $b$) in the interval.

### Example 2.4.7 Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in Figure 2.4.6, identify the intervals on which the function appears to be increasing.

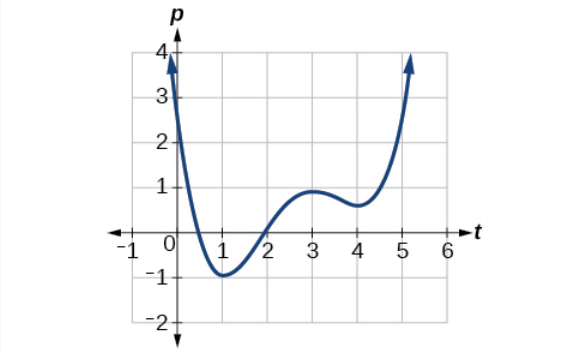



Figure 2.4.6: Graph of a polynomial.

**Solution**

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

**Analysis**

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

## Example 2.4.8: Finding Local Extrema from a Graph

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

**Solution**

Using technology, we find that the graph of the function looks like that in Figure 2.4.7. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$

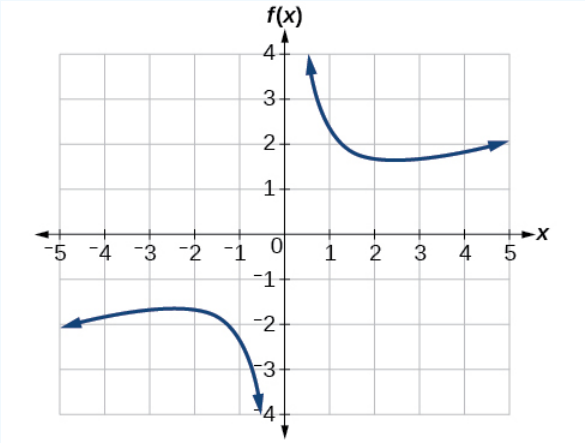



Figure 2.4.7: Graph of a reciprocal function.

**Analysis**

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Figure 2.4.8 provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

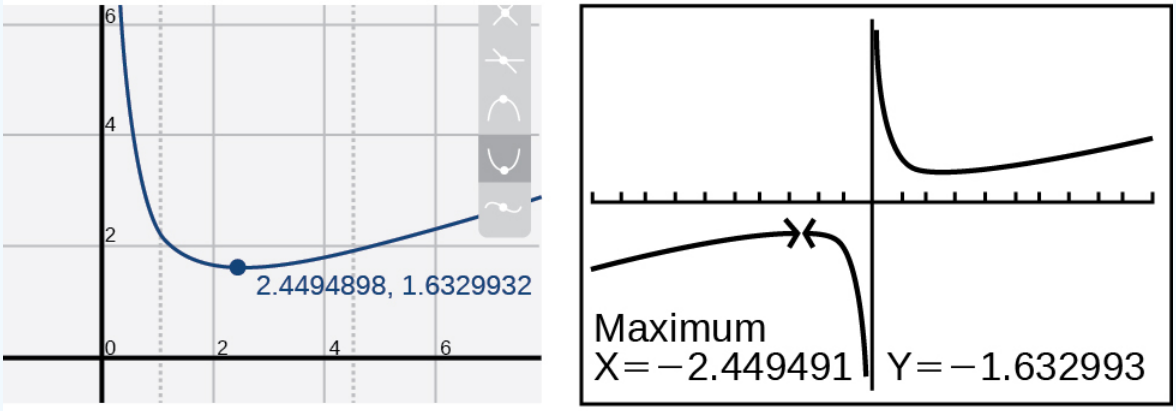



Figure 2.4.8: Graph of the reciprocal function on a graphing calculator.

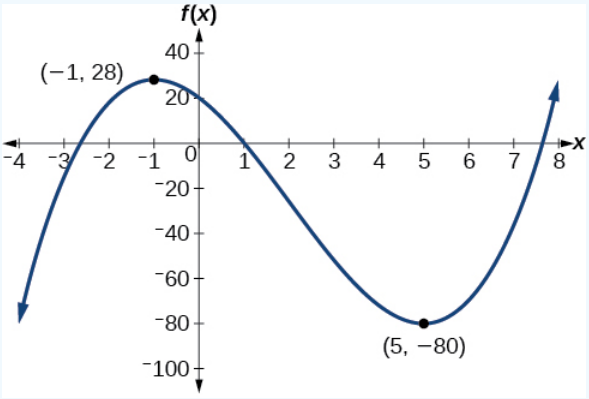
Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

## Exercise 2.4.8

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

**Solution**

The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.




*Graph of a polynomial with a local maximum at (-1, 28) and local minimum at (5, -80).*

**Example 2.4.9: Finding Local Maxima and Minima from a Graph**

For the function f whose graph is shown in Figure 2.4.9, find all local maxima and minima.

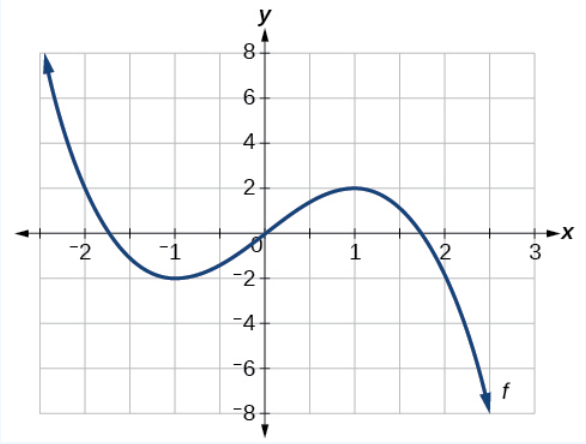



Figure 2.4.9: Graph of a polynomial.

**Solution**

Observe the graph of $f$. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is −2.

## Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in Figure 2.4.10, Figure 2.4.11, and Figure 2.4.12

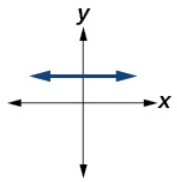
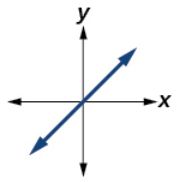
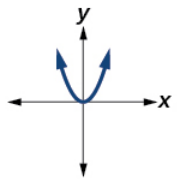
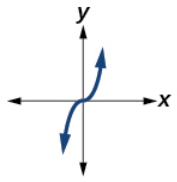
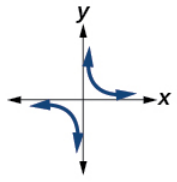
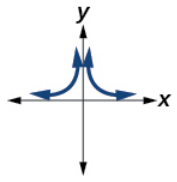
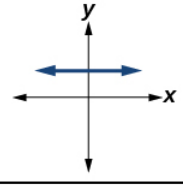
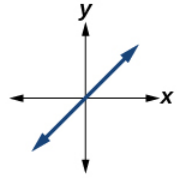
| Function | Increasing/Decreasing | Example |
|---|---|---|
| Constant Function<br>$f(x) = c$ | Neither increasing nor decreasing |  |
| Identity Function<br>$f(x) = x$ | Increasing |  |
| Quadratic Function<br>$f(x) = x^2$ | Increasing on $(0, \infty)$<br>Decreasing on $(-\infty, 0)$<br>Minimum at $x = 0$ |  |

Figure 2.4.10

| Function | Increasing/Decreasing | Example |
|---|---|---|
| Cubic Function<br>$f(x) = x^3$ | Increasing |  |
| Reciprocal<br>$f(x) = \frac{1}{x}$ | Decreasing $(-\infty, 0) \cup (0, \infty)$ |  |
| Reciprocal Squared<br>$f(x) = \frac{1}{x^2}$ | Increasing on $(-\infty, 0)$<br>Decreasing on $(0, \infty)$ |  |

.

Figure 2.4.11

| Function | Increasing/Decreasing | Example |
|---|---|---|
| Constant Function<br>$f(x) = c$ | Neither increasing nor decreasing |  |
| Identity Function<br>$f(x) = x$ | Increasing |  |

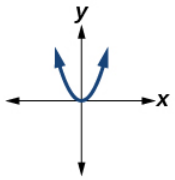
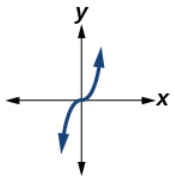
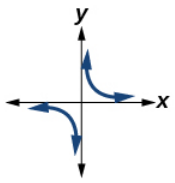
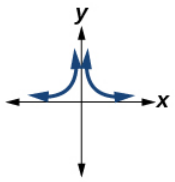
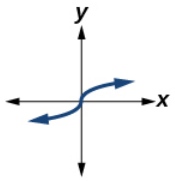
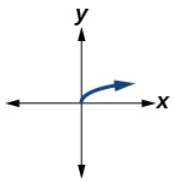
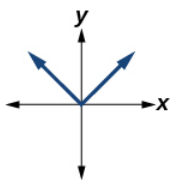
| | | |
|---|---|---|
| Quadratic Function<br>$f(x) = x^2$ | Increasing on $(0, \infty)$<br>Decreasing on $(-\infty, 0)$<br>Minimum at $x = 0$ |  |
| Cubic Function<br>$f(x) = x^3$ | Increasing |  |
| Reciprocal<br>$f(x) = \frac{1}{x}$ | Decreasing $(-\infty, 0) \cup (0, \infty)$ |  |
| Reciprocal Squared<br>$f(x) = \frac{1}{x^2}$ | Increasing on $(-\infty, 0)$<br>Decreasing on $(0, \infty)$ |  |
| Cube Root<br>$f(x) = \sqrt[3]{x}$ | Increasing |  |
| Square Root<br>$f(x) = \sqrt{x}$ | Increasing on $(0, \infty)$ |  |
| Absolute Value<br>$f(x) = \lvert x \rvert$ | Increasing on $(0, \infty)$<br>Decreasing on $(\infty, 0)$ |  |

***Figure* 2.4.12**

## Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively. To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function (Figure 2.4.13).

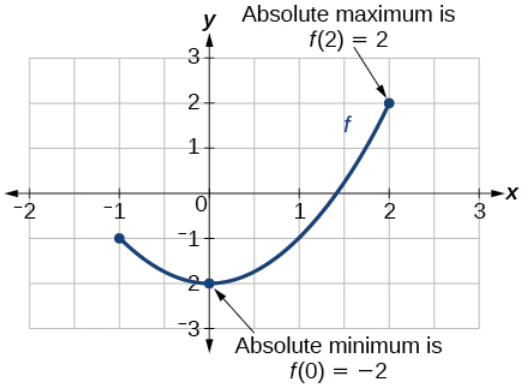



Figure 2.4.13: Graph of a segment of a parabola with an absolute minimum at (0, -2) and absolute maximum at (2, 2).

Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

### Absolute Maxima and Minima

- The **absolute maximum** of $f$ at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all $x$ in the domain of $f$.
- The **absolute minimum** of $f$ at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all $x$ in the domain of $f$.

### Example 2.4.10: Finding Absolute Maxima and Minima from a Graph

For the function f shown in Figure 2.4.14, find all absolute maxima and minima.

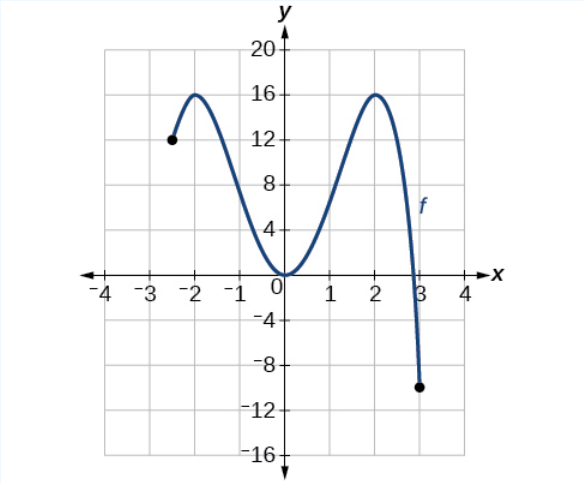



Figure 2.4.14: Graph of a polynomial.

**Solution**

Observe the graph of $f$. The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at x=3, because it is the lowest point on the domain of the function's graph. The absolute minimum is the y-coordinate at x=3,which is−10.

## Key Equations

- Average rate of change: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

## Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See Example.

- Identifying points that mark the interval on a graph can be used to find the average rate of change. See Example.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See Example.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See Example and Example.
- The average rate of change can sometimes be determined as an expression. See Example.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See Example.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See Example and Example.
- The highest and lowest points on a graph indicate the maxima and minima. See Example.

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# 2.5: Transformation of Functions

**Learning Objectives**

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.



Figure 2.5.1: (credit: "Misko"/Flickr)

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

## Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically $k$ units. See Figure 2.5.2 for an example.

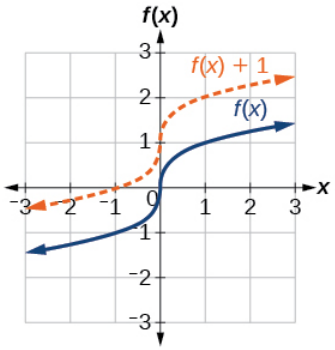



Figure 2.5.2: Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of $y$ is replaced by $y + k$, so the $y$-value increases or decreases depending on the value of $k$. The result is a shift upward or downward.

**Definition: Vertical Shift**

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where $k$ is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by $k$ units. If $k$ is positive, the graph will shift up. If $k$ is negative, the graph will shift down.

**Example 2.5.1: Adding a Constant to a Function**

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 2.5.3 shows the area of open vents $V$ (in square feet) throughout the day in hours after midnight, $t$. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

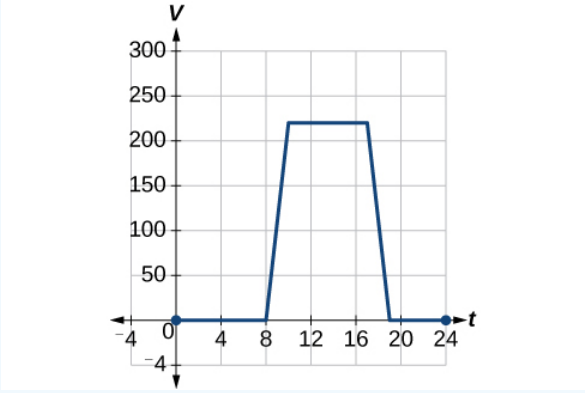



Figure 2.5.3

**Solution**

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 2.5.4.

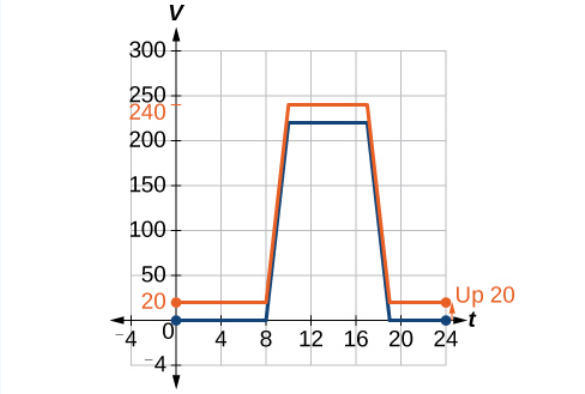



Figure 2.5.4

Notice that in Figure 2.5.4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20 \tag{2.5.1}$$

This notation tells us that, for any value of $t$,$S(t)$ can be found by evaluating the function $V$ at the same input and then adding 20 to the result. This defines $S$ as a transformation of the function $V$, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 2.5.1.

Table 2.5.1

| $t$ | 0 | 8 | 10 | 17 | 19 | 24 |
|---|---|---|---|---|---|---|
| $V(t)$ | 0 | 0 | 220 | 220 | 0 | 0 |

| | | | | | | |
|---|---|---|---|---|---|---|
| $S(t)$ | 20 | 20 | 240 | 240 | 20 | 20 |

**How To...**

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

**Example 2.5.2: Shifting a Tabular Function Vertically**

A function $f(x)$ is given in Table 2.5.2. Create a table for the function $g(x) = f(x) - 3$.

Table 2.5.2

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

**Solution**

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of $g$ by subtracting 3 from the output values of $f$. For example:

$$\begin{aligned} f(x) &= 1 && \text{Given} \\ g(x) &= f(x) - 3 && \text{Given Transformation} \\ g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 2.5.3.

Table 2.5.3

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |
| $g(x)$ | -2 | 0 | 4 | 8 |

**Analysis**

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

**Exercise 2.5.1**

The function $h(t) = -4.9t^2 + 30t$ gives the height $h$ of a ball (in meters) thrown upward from the ground after $t$ seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

**Answer**

$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

## Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function

left or right in what is known as a **horizontal shift**, shown in Figure 2.5.4.

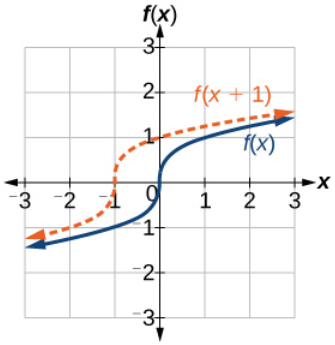



Figure 2.5.4: Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = -1$ shifts the graph to the left, that is, towards negative values of $x$.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in $f$.

**Definition: Horizontal Shift**

Given a function $f$, a new function $g(x) = f(x-h)$, where $h$ is a constant, is a **horizontal shift** of the function $f$. If $h$ is positive, the graph will shift right. If $h$ is negative, the graph will shift left.

**Example 2.5.4: Adding a Constant to an Input**

Returning to our building airflow example from Figure 2.5.2, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

**Solution**

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function $V$ was 2 hours later. For example, in the original function $V$, the airflow starts to change at 8 a.m., whereas for the function $F$, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 2.5.5. Notice also that the vents first opened to $220\text{ft}^2$ at 10 a.m. under the original plan, while under the new plan the vents reach $220\text{ft}^2$ at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$ .

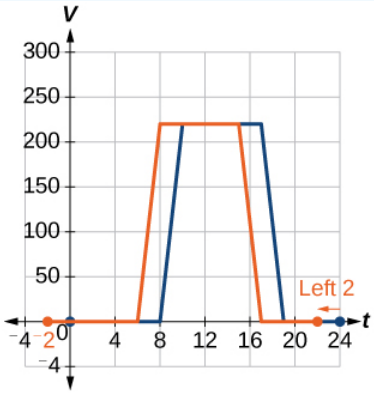



Figure 2.5.5

**Analysis**

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of $V$ to find the corresponding output for $F$ : $F(t)=V(t+2)$ .

How To...

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

**Example 2.5.5: Shifting a Tabular Function Horizontally**

A function $f(x)$ is given in Table 2.5.4. Create a table for the function $g(x)=f(x-3)$.

Table 2.5.4

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

**Solution**

The formula $g(x)=f(x-3)$ tells us that the output values of $g$ are the same as the output value of $f$ when the input value is 3 less than the original value. For example, we know that $f(2)=1$. To get the same output from the function $g$, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function $f$.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 2.5.5.

Table 2.5.5

| $x$ | 5 | 7 | 9 | 11 |
|---|---|---|---|---|
| $x-3$ | 2 | 4 | 6 | 8 |
| $f(x)$ | 1 | 3 | 7 | 11 |
| $g(x)$ | 1 | 3 | 7 | 11 |

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, $x$, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

**Analysis**

Figure 2.5.6 represents both of the functions. We can see the horizontal shift in each point.

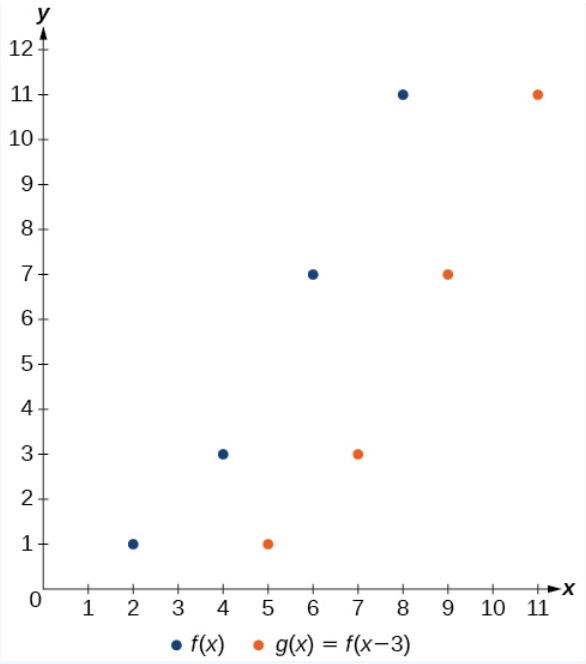



Figure 2.5.6: Graph of the points from the previous table for $f(x)$ and $g(x) = f(x-3)$

## Example 2.5.6: Identifying a Horizontal Shift of a Toolkit Function

Figure 2.5.7 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

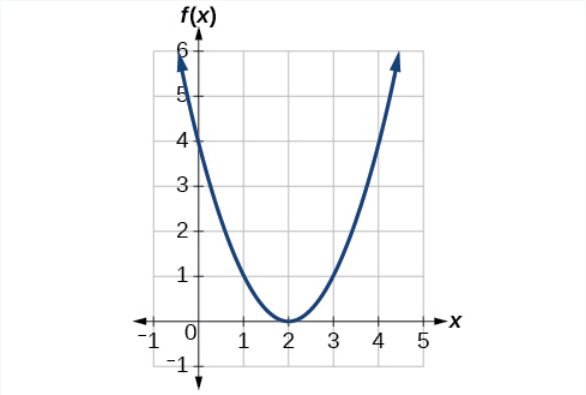



Figure 2.5.7: Graph of a parabola

**Solution**

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the $x$-values are shifted to the right 2 units. The vertex used to be at $(0, 0)$, but now the vertex is at $(2, 0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x-2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the $x$-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x-2) \\ g(x) &= f(x-2) = (x-2)^2 \end{aligned}$$

**Analysis**

To determine whether the shift is $+2$ or $-2$, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0)=0$. In our shifted function, $g(2)=0$. To obtain the output value of 0 from the function $f$, we need to decide whether a plus or a minus sign will work to satisfy $g(2)=f(x-2)=f(0)=0$ . For this to work, we will need to subtract 2 units from our input values.

### Example 2.5.7: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive $m$ miles. Interpret $G(m)+10$ and $G(m+10)$

**Solution**

$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive $m$ miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than $m$ miles. The graph would indicate a horizontal shift.

### Exercise 2.5.7

Given the function $f(x)=\sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x)=f(x+2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

**Answer**

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

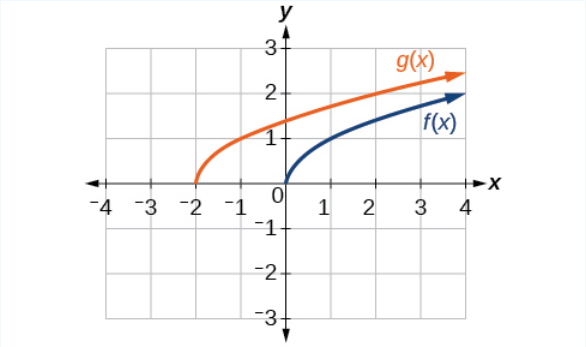



Figure 2.5.8

## Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output $(y-)$ axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input $(x-)$ axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

### How To...

**Given a function and both a vertical and a horizontal shift, sketch the graph.**

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

## Example 2.5.8: Graphing Combined Vertical and Horizontal Shifts

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x+1)-3$ .

**Solution**

The function $f$ is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of $h$ has transformed $f$ in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 2.5.9.

Let us follow one point of the graph of $f(x)=|x|$.

- The point $(0,0)$ is transformed first by shifting left 1 unit:$(0,0)\rightarrow(-1,0)$
- The point $(-1,0)$ is transformed next by shifting down 3 units:$(-1,0)\rightarrow(-1,-3)$

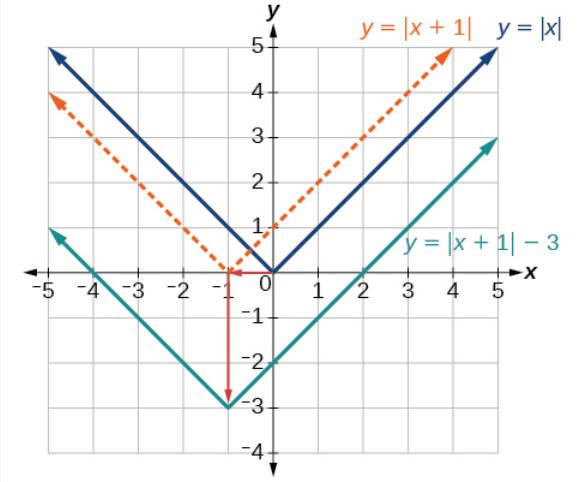



Figure 2.5.9: Graph of an absolute function, $y=|x|$ , and how it was transformed to $y=|x+1|-3$

Figure 2.5.10shows the graph of $h$.

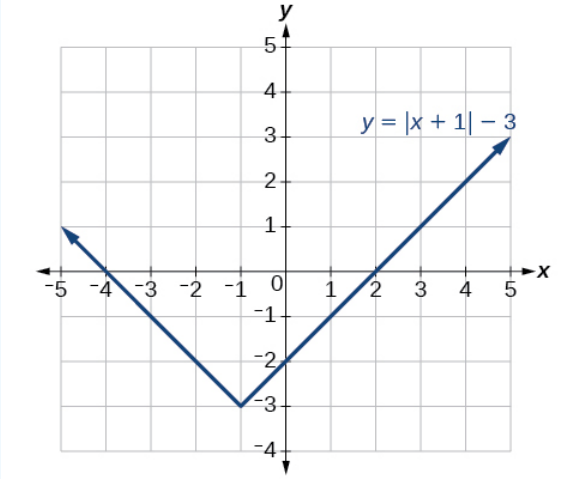



Figure 2.5.10: The final function $y=|x+1|-3$ .

## Exercise 2.5.8

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x-2)+4$ .

**Answer**

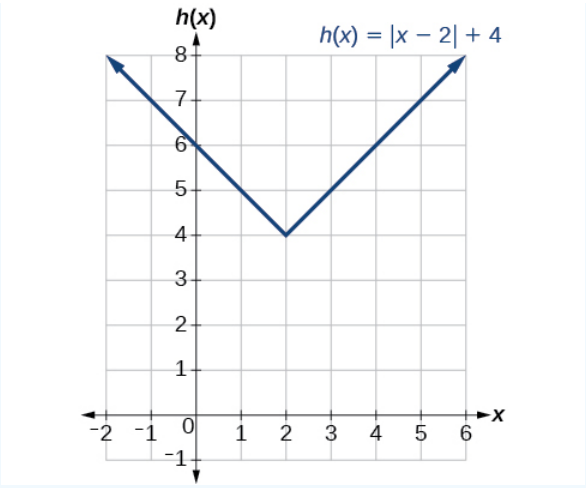



Figure 2.5.11

## Example 2.5.9: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 2.5.12, which is a transformation of the toolkit square root function.

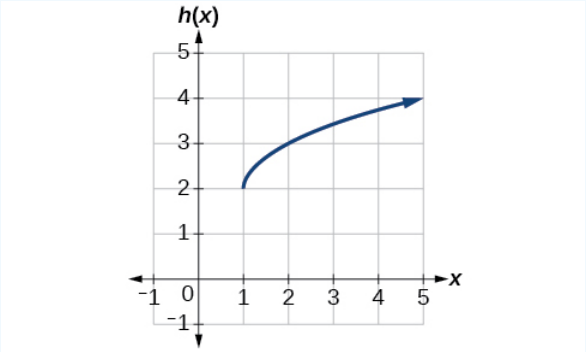



Figure 2.5.12: Graph of a square root function transposed right one unit and up 2.

**Solution**

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1)+2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1}+2$$

**Analysis**

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

## Exercise 2.5.9

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

**Answer**

$$g(x) = \frac{1}{x-1}+1$$

## Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 2.5.13.

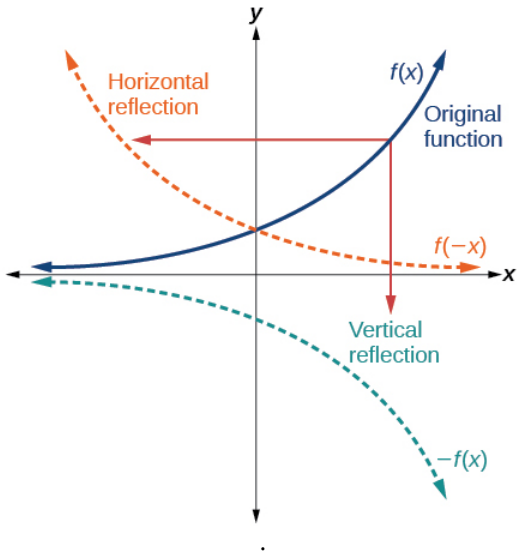



Figure 2.5.13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

**Definitions: Reflections**

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

**How To...**

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

**Example 2.5.10: Reflecting a Graph Horizontally and Vertically**

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

**Solution**

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 2.5.14

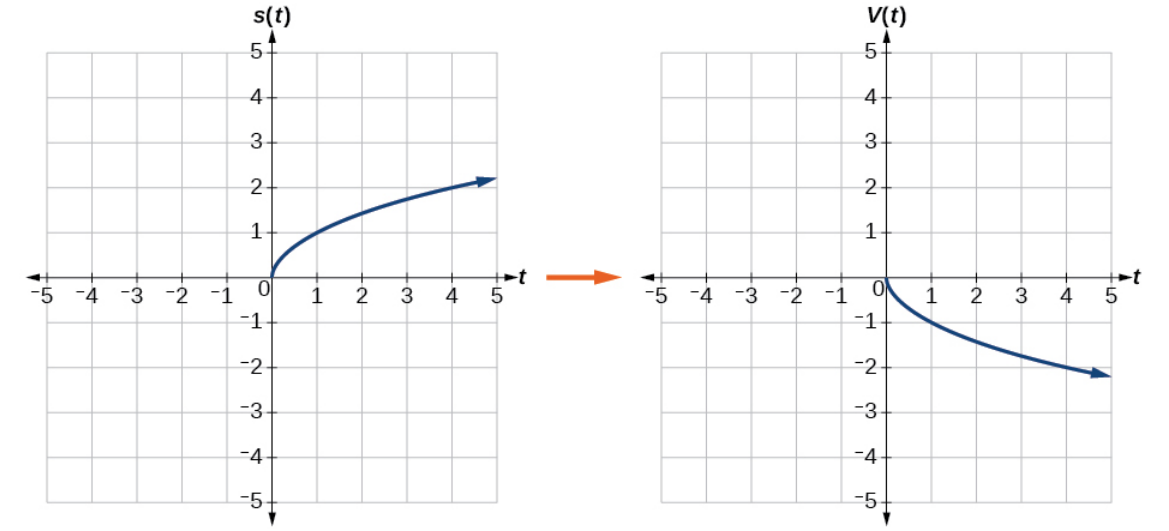



Figure 2.5.14: Graph of the vertical reflection of the square root function.

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 2.5.15.

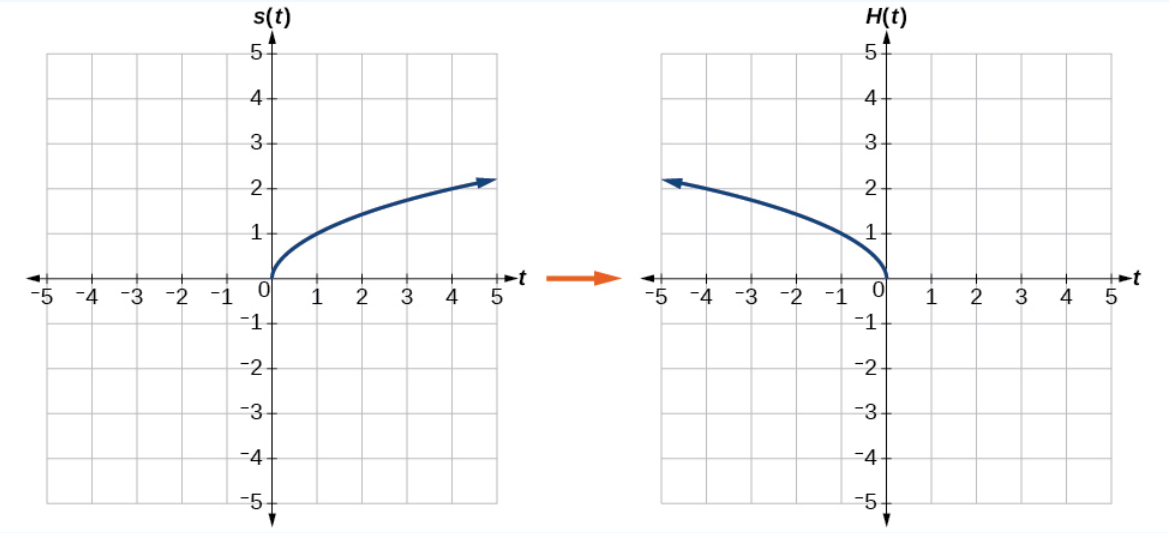



Figure 2.5.15: Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

## Exercise 2.5.5

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

**Answer**

a.

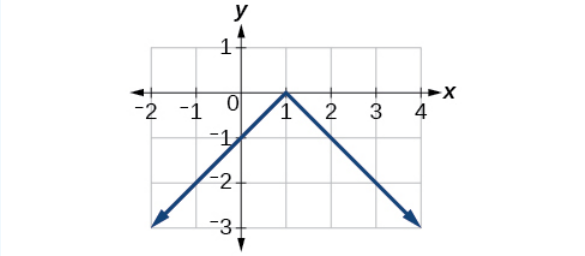



Figure 2.5.16: Graph of a vertically reflected absolute function.

b.

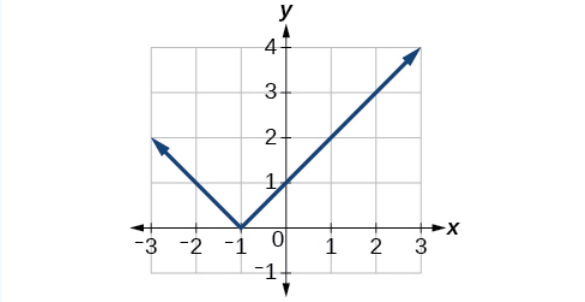



Figure 2.5.17: Graph of an absolute function translated one unit left.

## Example 2.5.11: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 2.5.6. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 2.5.6

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 2.5.7.

Table 2.5.7

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x)$ | -1 | -3 | -7 | -11 |

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 2.5.8.

Table 2.5.8

| $x$ | -2 | -4 | -6 | -8 |
|---|---|---|---|---|
| $h(x)$ | 1 | 3 | 7 | 11 |

## Exercise 2.5.6

A function $f(x)$ is given as Table 2.5.9. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 2.5.9

| $x$ | -2 | 0 | 2 | 4 |
|---|---|---|---|---|
| $f(x)$ | 5 | 10 | 15 | 20 |

**Answer**

**a.** $g(x) = -f(x)$

Table 2.5.10

| $x$ | -2 | 0 | 2 | 4 |
|---|---|---|---|---|
| $g(x)$ | -5 | -10 | -15 | -20 |

**b.** $h(x) = f(-x)$

Table 2.5.11

| $x$ | -2 | 0 | 2 | -4 |
|---|---|---|---|---|
| $h(x)$ | 15 | 10 | 5 | 20 |

## Example 2.5.12: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$ , where $k$ is the percentage of mastery that can be achieved after $t$ practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 2.5.18. Sketch a graph of $k(t)$.

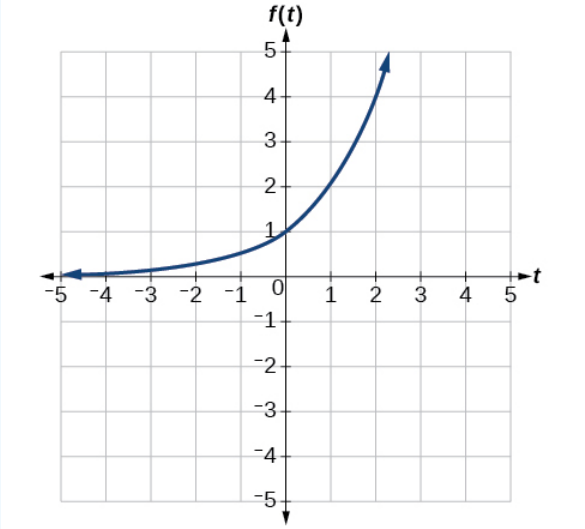



Figure 2.5.18: Graph of $k(t)$

**Solution**

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$

- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t)+1 = -2^{-t}+1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

- First, we apply a horizontal reflection: $(0, 1)$ $(-1, 2)$.
- Then, we apply a vertical reflection: $(0, -1)$ $(-1, -2)$.
- Finally, we apply a vertical shift: $(0, 0)$ $(-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 2.5.19, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

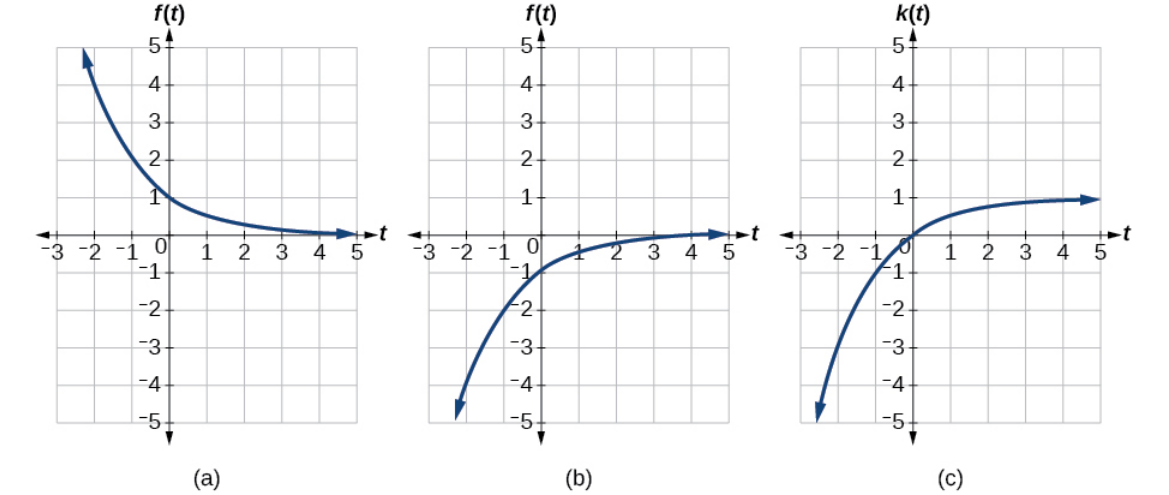



Figure 2.5.19: Graphs of all the transformations.

**Analysis**

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

## Exercise 2.5.7

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

**Answer**

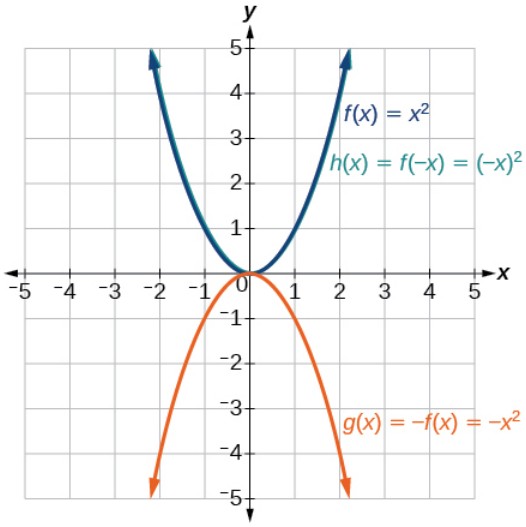



Figure 2.5.20: Graph of $x^2$ and its reflections.

Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

## Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure 2.5.21.

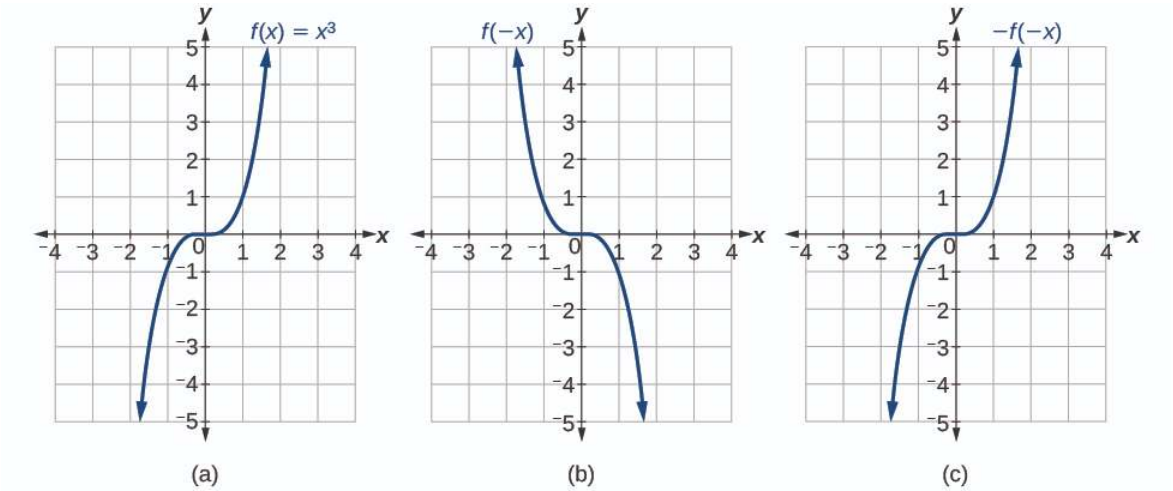



Figure 2.5.21: (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

### Definitions: Even and Odd Functions

A function is called an **even function** if for every input $x$

$f(x) = f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input $x$

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

### How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

### Example 2.5.13: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

**Solution**

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

**Analysis**

Consider the graph of $f$ in Figure 2.5.22 Notice that the graph is symmetric about the origin. For every point $(x, y)$ on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of $f$, and the corresponding point $(-1, -3)$ is also on the graph.

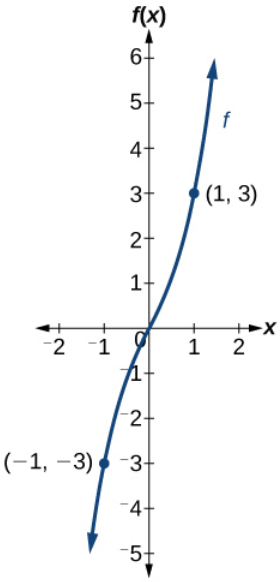



Figure 2.5.22: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

**Exercise 2.5.8**

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

**Answer**

even

## Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

### Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 2.5.23 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

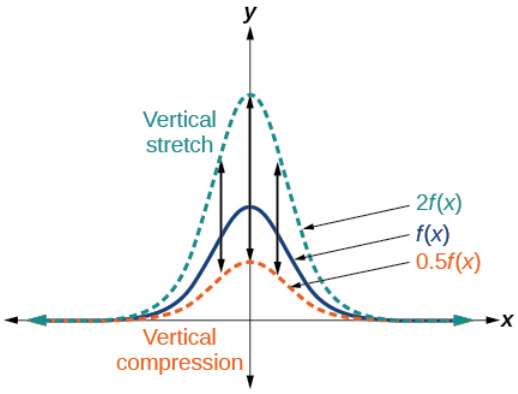



Figure 2.5.23: Vertical stretch and compression

**Definitions: Vertical Stretches and Compressions**

Given a function $f(x)$, a new function $g(x) = af(x)$, where $a$ is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

**How To...**

Given a function, graph its vertical stretch.

a. Identify the value of $a$.
b. Multiply all range values by $a$
c. If $a > 1$, the graph is stretched by a factor of $a$.
d. If $0 < a < 1$, the graph is compressed by a factor of $a$.
e. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

**Example 1.5.14: Graphing a Vertical Stretch**

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure $2.5.24$

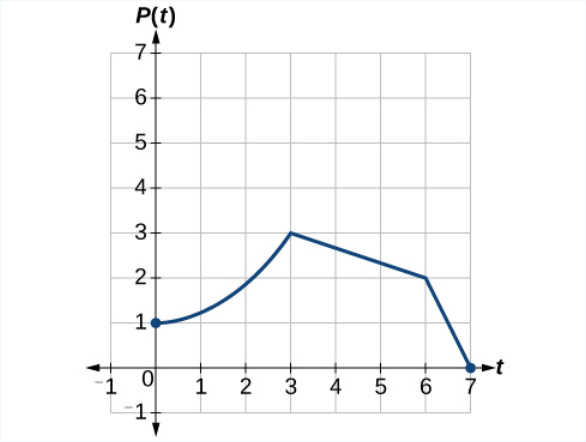



Figure 2.5.24: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, $Q$, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

**Solution**

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure $2.5.25$

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2) \tag{2.5.2}$$
$$(3, 3) \to (3, 6) \tag{2.5.3}$$
$$(6, 2) \to (6, 4) \tag{2.5.4}$$
$$(7, 0) \to (7, 0) \tag{2.5.5}$$

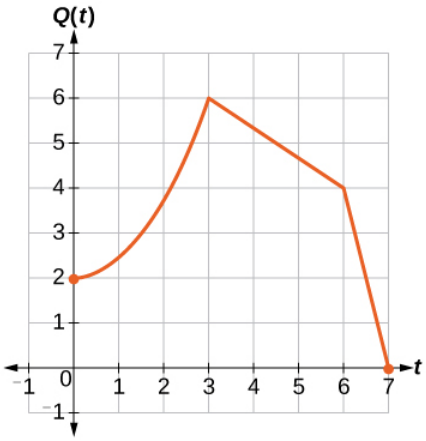



Figure 2.5.25: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input $t$, the value of the function $Q$ is twice the value of the function $P$. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, $t$, stay the same while the output values are twice as large as before.

**How To...**

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of $a$.
2. Multiply all of the output values by $a$.

**Example 2.5.15: Finding a Vertical Compression of a Tabular Function**

A function $f$ is given as Table 2.5.12 Create a table for the function $g(x) = \frac{1}{2}f(x)$.

Table 2.5.12

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

**Solution**

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of $g$ are half of the output values of $f$ with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 2.5.13

Table 2.5.13

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x)$ | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{7}{2}$ | $\frac{11}{2}$ |

**Analysis**

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

## Exercise 2.5.9

A function $f$ is given as Table 2.5.14 Create a table for the function $g(x) = \frac{3}{4}f(x)$.

Table 2.5.14

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 12 | 16 | 20 | 0 |

**Answer**

Table 2.5.15

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x)$ | 9 | 12 | 15 | 0 |

## Example 2.5.16: Recognizing a Vertical Stretch

The graph in Figure 2.5.26 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

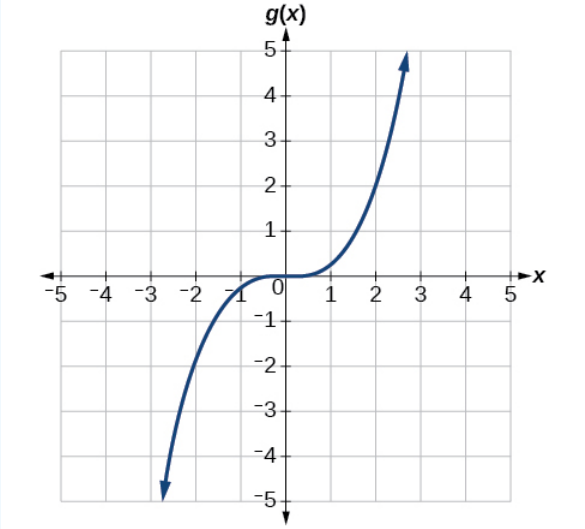



Figure 2.5.26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of $g$ are $\frac{1}{4}$ the outputs of the function $f$ because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for $g$ by using the definition of the function $f$.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3. \tag{2.5.6}$$

**Exercise 2.5.1**

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

**Answer**

$g(x) = 3x - 2$

## Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

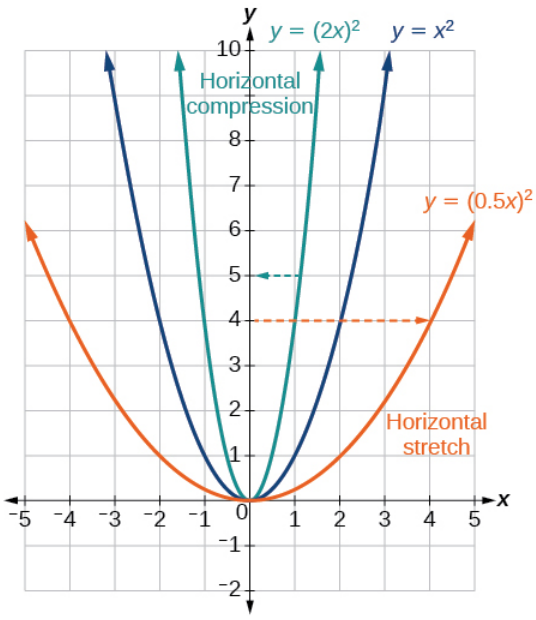



Figure 2.5.27: Graph of the vertical stretch and compression of $x^2$.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 2.5.27. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

**Definitions: Horizontal Stretches and Compressions**

Given a function $f(x)$, a new function $g(x) = f(bx)$, where $b$ is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

**How To...**

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

## Example 2.5.17: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, $R$, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

**Solution**

Symbolically, we could write

$$R(1) = P(2), \tag{2.5.7}$$
$$R(2) = P(4), \quad \text{and in general,} \tag{2.5.8}$$
$$R(t) = P(2t). \tag{2.5.9}$$

See Figure 2.5.28 for a graphical comparison of the original population and the compressed population.

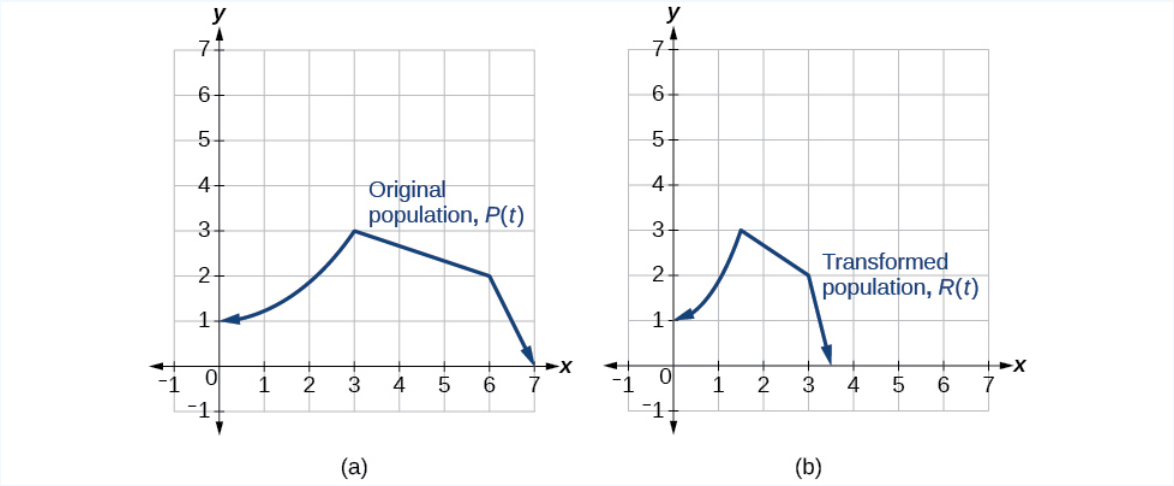



Figure 2.5.28: (a) Original population graph (b) Compressed population graph

**Analysis**

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

## Example 2.5.18: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 2.5.16. Create a table for the function $g(x) = f(\frac{1}{2}x)$.

Table 2.5.16

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for $g$ are the same as the output values for the function $f$ at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$ , and we do not have a value for $f(1)$ in our table. Our input values to $g$ will need to be twice as large to get inputs for $f$ that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1 \tag{2.5.10}$$

We do the same for the other values to produce Table 2.5.17.

Table 2.5.17

| $x$ | 4 | 8 | 12 | 16 |
|---|---|---|---|---|

| $g(x)$ | 1 | 3 | 7 | 11 |
|---|---|---|---|---|

Figure 2.5.29 shows the graphs of both of these sets of points.

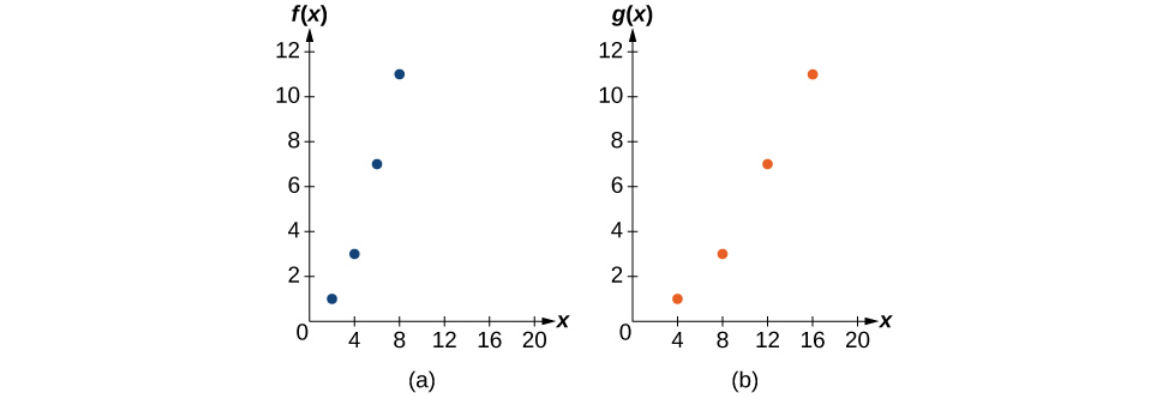



Figure 2.5.29: Graph of the previous table.

**Analysis**

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

## Example 2.5.19: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 2.5.30.

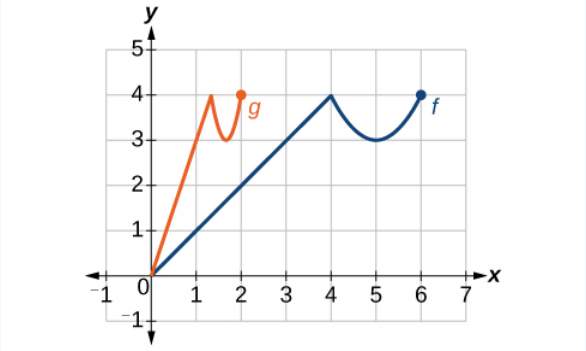



Figure 2.5.30: Graph of $f(x)$ being vertically compressed to $g(x)$.

**Solution**

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

**Analysis**

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

## Exercise 2.5.11

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

**Answer**

$g(x) = f(\frac{1}{3}x)$ , so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

## Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x+3)$, for example, we have to think about how the inputs to the function $g$ relate to the inputs to the function $f$. Suppose we know $f(7) = 12$. What input to $g$ would produce that output? In other words, what value of $x$ will allow $g(x) = f(2x+3) = 12$? We would need $2x+3 = 7$. To solve for $x$, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p) = f(b(x+\frac{p}{b}))$$

Let's work through an example.

$$f(x) = (2x+4)^2$$

We can factor out a 2.

$$f(x) = (2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

### Combining Transformations

- When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by $a$ and then vertically shift by $k$.
- When combining horizontal transformations written in the form $f(bx+h)$, first horizontally shift by $h$ and then horizontally stretch by $\frac{1}{b}$.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by $h$.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

### Example 2.5.20: Finding a Triple Transformation of a Tabular Function

Given Table 2.5.18 for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x)+1$.

Table 2.5.18

| $x$ | 6 | 12 | 18 | 24 |
|---|---|---|---|---|
| $f(x)$ | 10 | 14 | 15 | 17 |

**Solution**

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each $x$-value by $\frac{1}{3}$.See Table 2.5.19

Table 2.5.19

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(3x)$ | 10 | 14 | 15 | 17 |

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 2.5.20

Table 2.5.20

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $2f(3x)$ | 20 | 28 | 30 | 34 |

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 2.5.21

Table 2.5.21

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x)=2f(3x)+1+1$ | 21 | 29 | 31 | 35 |

## Example 2.5.21: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 2.5.31 to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$ .

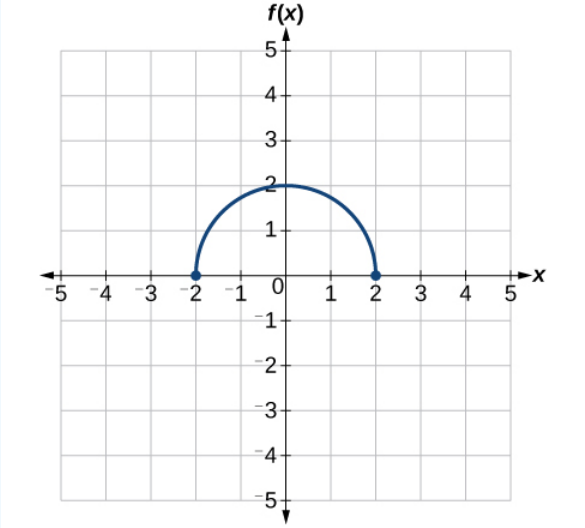



Figure 2.5.31: Graph of a half-circle.

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3 \tag{2.5.11}$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure 2.5.32

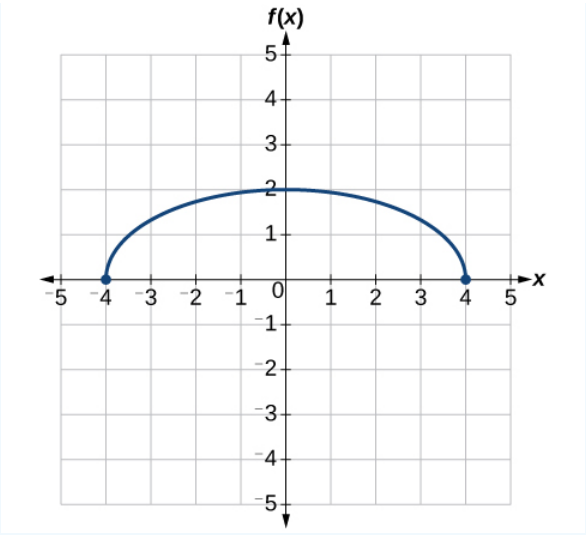



Figure 2.5.32: Graph of a vertically stretch half-circle.

Next, we horizontally shift left by 2 units, as indicated by $x+2$. See Figure 2.5.33

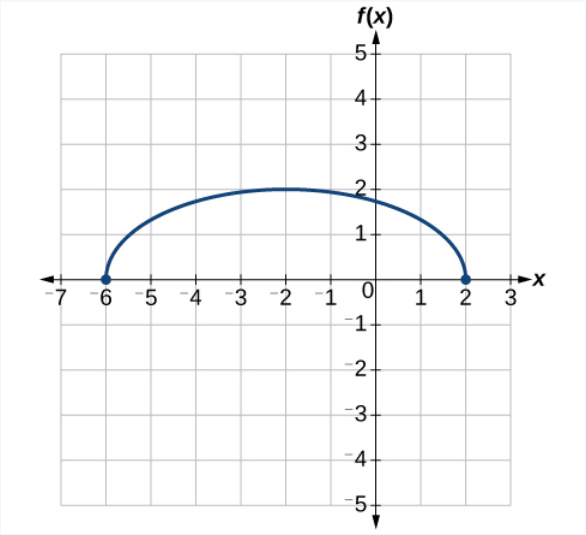



Figure 2.5.33: Graph of a vertically stretch and translated half-circle.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the $-3$ on the outside of the function. See Figure 2.5.34

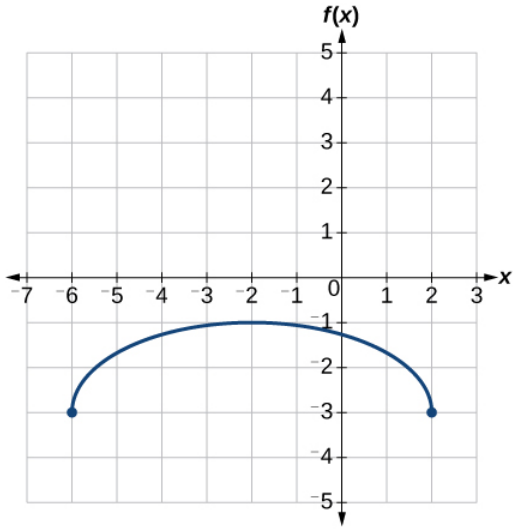



Figure 2.5.34: Graph of a vertically stretch and translated half-circle.

## Key Equations

- Vertical shift $g(x) = f(x) + k$ (up for $k > 0$)
- Horizontal shift $g(x) = f(x - h)$ (right) for $h > 0$
- Vertical reflection $g(x) = -f(x)$
- Horizontal reflection $g(x) = f(-x)$
- Vertical stretch $g(x) = af(x)$ (a>0 )
- Vertical compression $g(x) = af(x)$ (0<a<1)
- Horizontal stretch $g(x) = f(bx)(0 < b < 1)$
- Horizontal compression $g(x) = f(bx)$ (b>1)

## Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

## Glossary

**even function**

a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

**horizontal compression**
a transformation that compresses a function's graph horizontally, by multiplying the input by a constant b>1

**horizontal reflection**
a transformation that reflects a function's graph across the y-axis by multiplying the input by −1

**horizontal shift**
a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

**horizontal stretch**
a transformation that stretches a function's graph horizontally by multiplying the input by a constant 0<b<1

**odd function**
a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

**vertical compression**
a function transformation that compresses the function's graph vertically by multiplying the output by a constant 0<a<1

**vertical reflection**
a transformation that reflects a function's graph across the x-axis by multiplying the output by −1

**vertical shift**
a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output

**vertical stretch**
a transformation that stretches a function's graph vertically by multiplying the output by a constant a>1

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# 2.6: Composition of Functions

**Learning Objectives**

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost $C$ of heating a house for a given average daily temperature in $T$ degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $Cost = C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the **cost function** at that temperature. We would write $C(T(5))$.

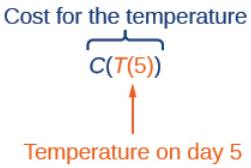



Figure 2.6.1: Explanation of $C(T(5))$, which is the cost for the temperature and $T(5)$ is the temperature on day 5.

By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

## Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year $y$, and we want $T$ to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, $fg$, and $\frac{f}{g}$ by the relations.

$$(f+g)(x) = f(x)+g(x)$$
$$(f-g)(x) = f(x)-g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

### Example 2.6.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x)=x-1$ and $g(x)=x^2-1$. Are they the same function?

**Solution**

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ &= x^2-x\\ &= x(x-1)\end{aligned}$$

$$\left(\frac{g}{f}\right)(x) = g(x)f(x)$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ &= \frac{(x+1)(x-1)}{x-1}\\ &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

### Exercise 2.6.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

$$f(x) = x-1$$

and

$$g(x) = x^2-1$$

Are they the same function?

**Answer**

$(fg)(x) = f(x)g(x) = (x-1)(x2-1) = x^3-x^2-x+1$

$(f-g)(x) = f(x)-g(x) = (x-1)-(x^2-1) = x-x^2$

No, the functions are not the same.

## Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f \circ g(x) = f(g(x)) \tag{2.6.1}$$

We read the left-hand side as"$f$ composed with $g$ at $x$," and the right-hand side as"$f$ of $g$ of $x$."The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function $g$ takes the input $x$ first and yields an output $g(x)$. Then the function $f$ takes $g(x)$ as an input and yields an output $f(g(x))$.

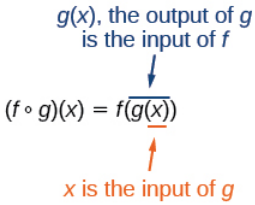



Figure 2.6.2: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all $x$. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

### Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input $x$ and functions $f$ and $g$, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \tag{2.6.2}$$

The domain of the composite function $f \circ g$ is all $x$ such that $x$ is in the domain of $g$ and $g(x)$ is in the domain of $f$.

It is important to realize that the product of functions $fg$ is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

### Example 2.6.2: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x)=2x+1 \quad g(x)=3-x$$

**Solution**

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3-x)+1 \\ &= 6-2x+1 \\ &= 7-2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= 2-2x \end{aligned}$$

We find that $g(f(x))\neq f(g(x))$, so the operation of function composition is not commutative.

### Example 2.6.3: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing $s$ sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in $t$ minutes. Interpret $c(s(3))$.

**Solution**

The inside expression in the composition is $s(3)$. Because the input to the $s$-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

### Example 2.6.4: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in $x$ hours and $g(y)$ gives the gallons of gas used in driving $y$ miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

**Solution**

The function $y=f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, $g$, driving a certain number of miles, $f(x)$, in $x$ hours.

Question/Answer

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order

Exercise 2.6.2

The gravitational force on a planet a distance $r$ from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force $F$ is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

**Answer**

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance $r$ from the Sun (due to gravity), but $G(a(F))$ does not make sense.

## Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

### Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 2.6.5: Using a Table to Evaluate a Composite Function

Using Table 2.6.1, evaluate $f(g(3))$ and $g(f(3))$.

Table 2.6.1

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 1 | 6 | 3 |
| 2 | 8 | 5 |
| 3 | 3 | 2 |
| 4 | 1 | 7 |

**Solution**

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function $g$ : $g(3) = 2$ . We can then use that result as the input to the function $f$, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function $f$, we find that $f(2) = 8$.

$$g(3) = 2$$

$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for $g$, we can evaluate

$$g(f(3)) = g(3) = 2$$

Table $2.6.2$ shows the composite functions $f \circ g$ and $g \circ f$ as tables.

Table 2.6.2

| $x$ | $g(x)$ | $f(g(x))$ | $f(x)$ | $g(f(x))$ |
|---|---|---|---|---|
| 3 | 2 | 8 | 3 | 2 |

**Exercise 2.6.3**

Using Table $2.6.1$, evaluate $f(g(1))$ and $g(f(4))$.

**Answer**

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

## Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

**How To ...**

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

**Example 2.6.6: Using a Graph to Evaluate a Composite Function**

Using Figure $2.6.3$, evaluate $f(g(1))$.

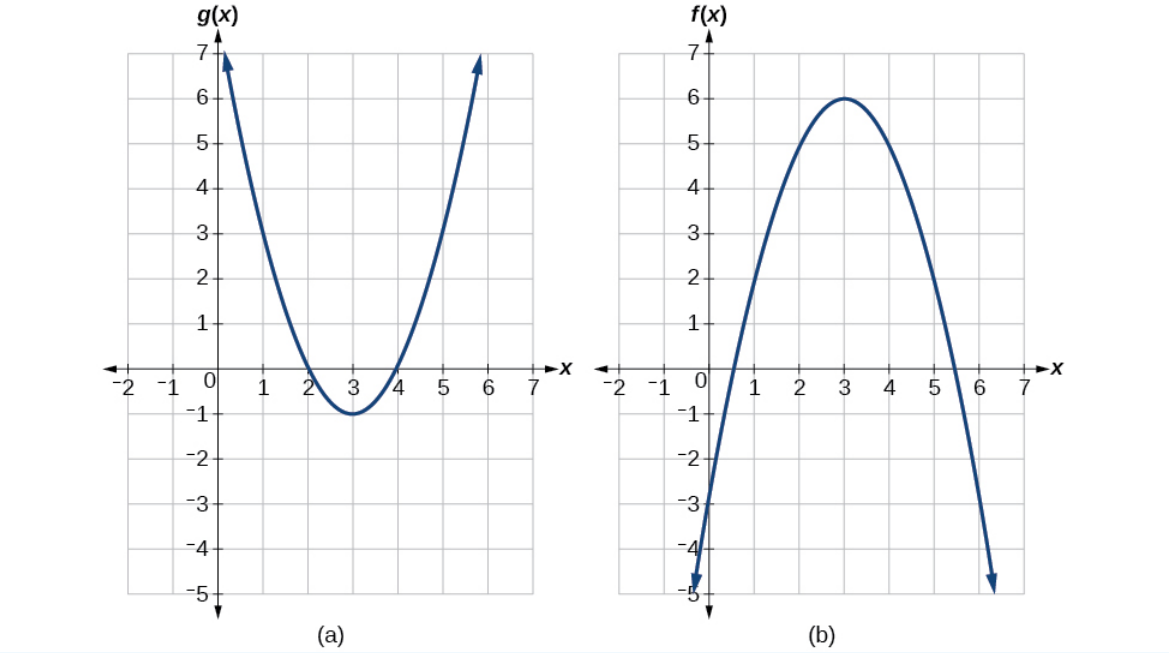



Figure 2.6.3: Two graphs of a positive and negative parabola.

**Solution**

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure $2.6.4$.

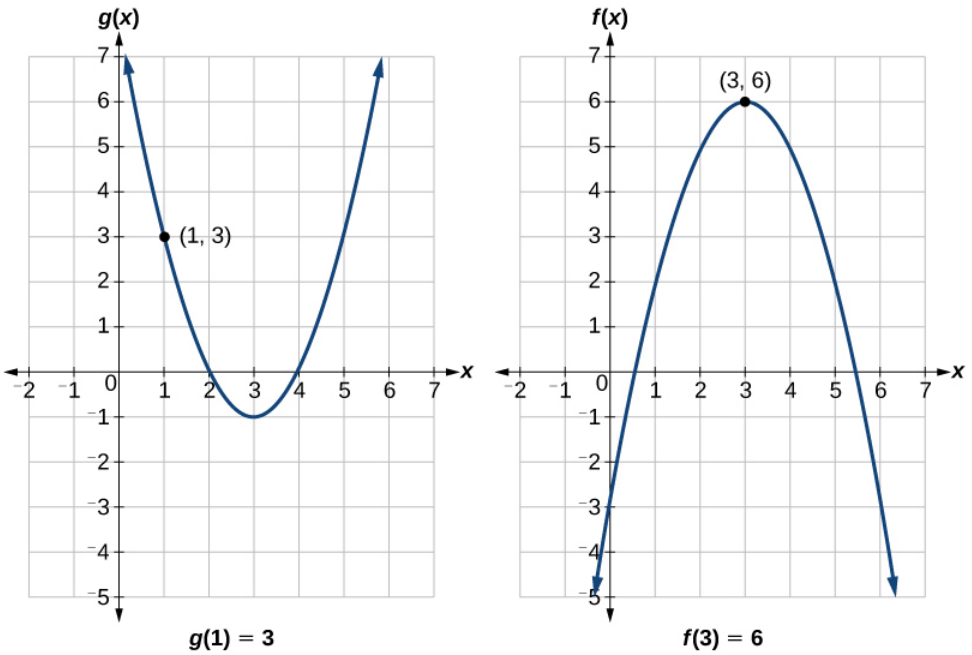



Figure 2.6.4: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function $f$.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

**Analysis**

Figure 2.6.5 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

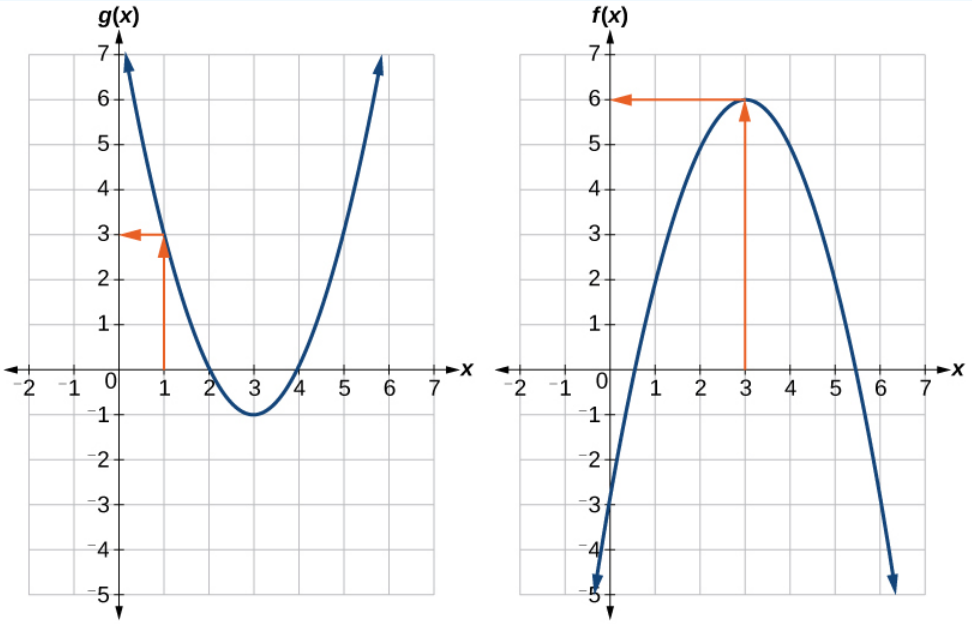



Figure 2.6.5: Two graphs of a positive and negative parabola.

**Exercise 2.6.4**

Using Figure 2.6.3, evaluate $g(f(2))$.

**Answer**

$g(f(2)) = g(5) = 3$

## Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$ , we substitute the value inside the parentheses into the formula wherever we see the input variable.

**How To...**

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

**Example 2.6.7: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input**

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

**Solution**

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$h(1) = 3(1) + 2$$
$$h(1) = 5$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$f(h(1)) = f(5)$$
$$f(h(1)) = 5^2 - 5$$
$$f(h(1)) = 20$$

**Analysis**

It makes no difference what the input variables $t$ and $x$ were called in this problem because we evaluated for specific numerical values.

**Exercise 2.6.5**

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

**Answer a**

8

**Answer b**

20

## Finding the Domain of a Composite Function

As we discussed previously, the **domain of a composite function** such as $f \circ g$ is dependent on the domain of $g$ and the domain of $f$. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions $f$ and $g$ separately. If we write the composite function for an input $x$ as $f(g(x))$, we can see right away that $x$ must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of $f$, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of $g$ that produce outputs from $g$ belonging to the domain of $f$. Note that the domain of $f$ composed with $g$ is the set of all $x$ such that $x$ is in the domain of $g$ and g(x)\) is in the domain of $f$.

**Definition: Domain of a Composite Function**

The **domain of a composite function** $f(g(x))$ is the set of those inputs $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$.

**How To...**

Given a function composition $f(g(x))$, determine its domain.

1. Find the domain of $g$.
2. Find the domain of $f$.
3. Find those inputs $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$. That is, exclude those inputs $x$ from the domain of $g$ for which $g(x)$ is not in the domain of $f$. The resulting set is the domain of $f \circ g$.

**Example 2.6.8A: Finding the Domain of a Composite Function**

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

**Solution**

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of $f$ consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of $x$ for which $g(x) = 1$.

$$\begin{aligned} \frac{4}{3x-2} &= 1 \\ 4 &= 3x-2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

### Example 2.6.8$B$: Finding the Domain of a Composite Function Involving Radicals

Find the domain of

$$(f\circ g)(x) \text{ where } f(x)=\sqrt{x+2} \text{ and } g(x)=\sqrt{3-x}$$

**Solution**

Because we cannot take the square root of a negative number, the domain of $g$ is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f\circ g)(x)=\sqrt{\sqrt{3-x}+2}$$

For $(f\circ g)(x)=\sqrt{\sqrt{3-x}+2}$, $\sqrt{3-x}+2\geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $\sqrt{3-x}\geq 0$,or, $3-x\geq 0$, which gives a domain of $(-\infty, 3]$.

**Analysis**

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f\circ g$ can contain values that are not in the domain of $f$, though they must be in the domain of $g$.

### Exercise 2.6.6

Find the domain of

$$(f\circ g)(x) \text{ where } f(x)=\frac{1}{x-2} \text{ and } g(x)=\sqrt{x+4}$$

**Answer**

$[-4, 0)\cup(0, \infty)$

## Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to **decompose a composite function**, so we may choose the decomposition that appears to be most expedient.

### Example 2.6.9: Decomposing a Function

Write $f(x)=\sqrt{5-x^2}$ as the composition of two functions.

**Solution**

We are looking for two functions, $g$ and $h$, so $f(x)=g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5-x^2$ is the inside of the square root. We could then decompose the function as

$$h(x)=5-x^2 \text{ and } g(x)=\sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x))=g(5-x^2)=\sqrt{5-x^2}$$

### Exercise 2.6.7

Write $f(x)=\dfrac{4}{3-\sqrt{4+x^2}}$ as the composition of two functions.

**Answer**

Possible answers:

$g(x) = \sqrt{4 + x^2}$

$h(x) = \dfrac{4}{3 - x}$

$f = h \circ g$

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions (http://openstaxcollege.org/l/compfunction)
- Composite Function Notation Application (http://openstaxcollege.org/l/compfuncnot)
- Composite Functions Using Graphs (http://openstaxcollege.org/l/compfuncgraph)
- Decompose Functions (http://openstaxcollege.org/l/decompfunction)
- Composite Function Values (http://openstaxcollege.org/l/compfuncvalue)

## Key Equation

- Composite function $(f \circ g)(x) = f(g(x))$

## Key Concepts

- We can perform algebraic operations on functions. See Example.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example and Example.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example.
- A composite function can be evaluated from a graph. See Example.
- A composite function can be evaluated from a formula. See Example.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Example and Example.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See Example.

## Glossary

**composite function**

the new function formed by function composition, when the output of one function is used as the input of another

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# 2.7: Absolute Value Functions

**Learning Objectives**

- Graph an absolute value function.
- Solve an absolute value equation.

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate **absolute value functions**.



Figure 2.7.1: Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

## Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the **absolute value** function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

**Absolute Value Function**

The absolute value function can be defined as a piecewise function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{2.7.1}$$

**Example 2.7.1: Determine a Number within a Prescribed Distance**

Describe all values $x$ within or including a distance of 4 from the number 5.

**Solution**

We want the distance between $x$ and 5 to be less than or equal to 4. We can draw a number line, such as the one in , to represent the condition to be satisfied.

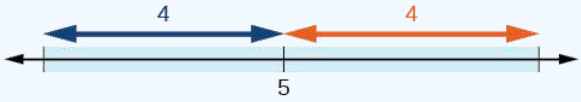



Figure 2.7.2: Number line describing the difference of the distance of 4 away from 5.

The distance from $x$ to 5 can be represented using the absolute value as $|x-5|$. We want the values of $x$ that satisfy the condition $|x-5| \leq 4$.

**Analysis**

Note that

$$-4 \leq x-5 \qquad x-5 \leq 4$$
$$1 \leq x \qquad\qquad x \leq 9$$

So $|x-5| \leq 4$ is equivalent to $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

**Exercise 2.7.1**

Describe all values $x$ within a distance of 3 from the number 2.

**Answer**

$|x-2| \leq 3$

**Example 2.7.2: Resistance of a Resistor**

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often ±1%, ±5%, or ±10%.

Suppose we have a resistor rated at 680 ohms, ±5%. Use the absolute value function to express the range of possible values of the actual resistance.

**Solution**

5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance $R$ in ohms,

$$|R-680| \leq 34$$

**Exercise 2.7.2**

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

**Answer**

Using the variable $p$ for passing, $|p-80| \leq 20$

## Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 2.7.3.

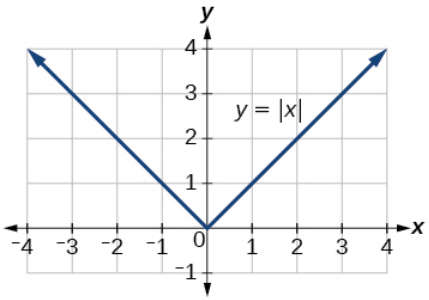



Figure 2.7.3: Graph of an absolute function.

Figure 2.7.3 shows the graph of $y = 2|x-3|+4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

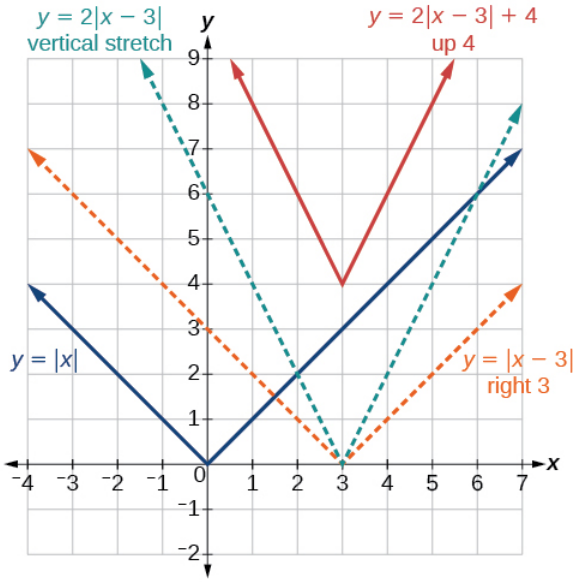



Figure 2.7.4: Graph of the different types of transformations for an absolute function.

## Example 2.7.3: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 2.7.5.

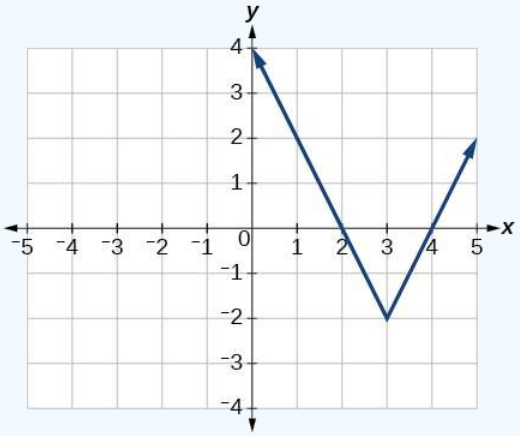



Figure 2.7.5: Graph of an absolute function.

**Solution**

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 2.7.6.

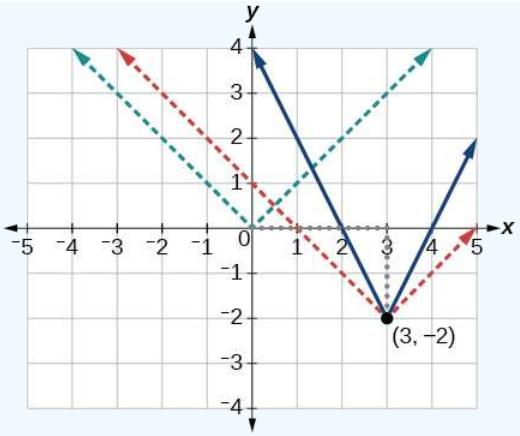



Figure 2.7.6: Graph of two transformations for an absolute function at $(3, -2)$.

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 2.7.7.

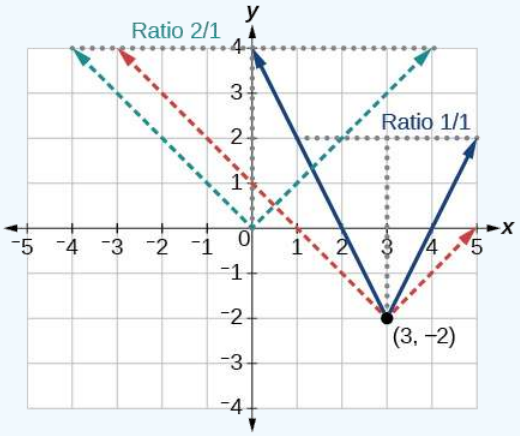



Figure 2.7.7: Graph of two transformations for an absolute function at $(3, -2)$ and the ratios between the two different transformations.

From this information we can write the equation

$$\begin{aligned} f(x) &= 2|x-3|-2, && \text{treating the stretch as a vertial stretch, or} \\ f(x) &= |2(x-3)|-2, && \text{treating the stretch as a horizontal compression.} \end{aligned}$$

**Analysis**

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

Q & A

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

**Answer**

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for $x$ and $f(x)$.

$$f(x) = a|x-3|-2$$

Now substituting in the point $(1, 2)$

$$2 = a|1-3| - 2$$
$$4 = 2a$$
$$a = 2$$

**Exercise 2.7.3**

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

**Answer**

$f(x) = -|x+2| + 3$

**Q & A**

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

**Answer**

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (Figure 2.7.8).

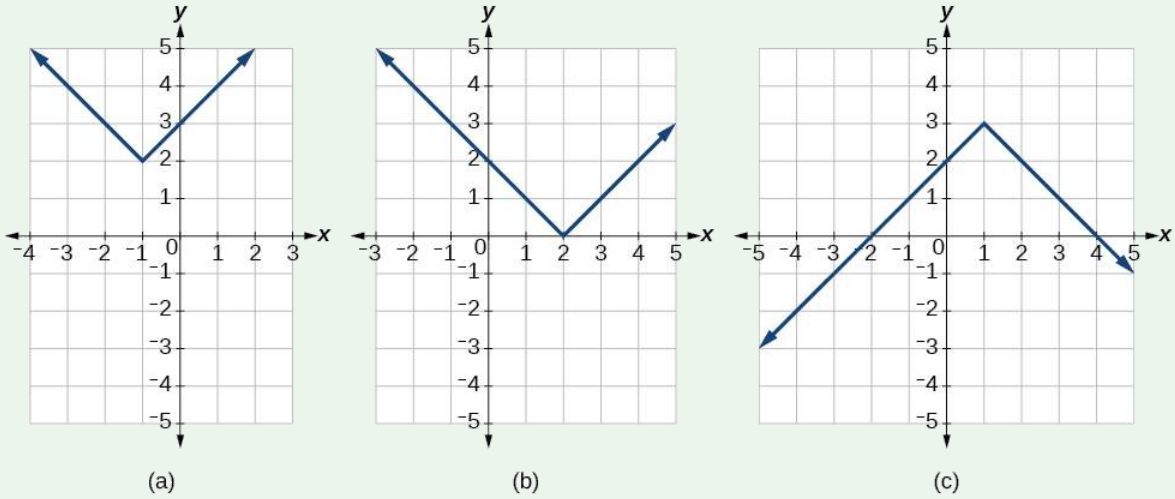



Figure 2.7.8: (a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

## Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$2x - 6 = 8 \quad \text{or} \quad 2x - 6 = -8$$
$$2x = 14 \qquad 2x = -2$$
$$x = 7 \qquad x = -1$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An **absolute value equation** is an equation in which the unknown variable appears in absolute value bars. For example,

$$|x| = 4,$$
$$|2x-1| = 3,$$
$$|5x+2| - 4 = 9.$$

### Solutions to Absolute Value Equations

For real numbers $A$ and $B$, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

### How To ...

Given the formula for an absolute value function, find the horizontal intercepts of its graph.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $-A = B$, assuming $B > 0$.
3. Solve for $x$.

### Example 2.7.4: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x+1| - 7$, find the values of $x$ such that $f(x) = 0$.

**Solution**

| | | |
|---|---|---|
| $0 = \|4x+1\| - 7$ | | Substitute 0 for f(x). |
| $7 = \|4x+1\|$ | | Isolate the absolute value on one side of the equation. |
| $7 = 4x+1$ | or $-7 = 4x+1$ | Break into two separate equations and solve. |
| $6 = 4x$ | $-8 = 4x$ | |
| $x = \frac{6}{4} = 1.5$ | $x = \frac{-8}{4} = -2$ | |

The function outputs 0 when $x = 1.5$ or $x = -2$ (Figure 2.7.9).

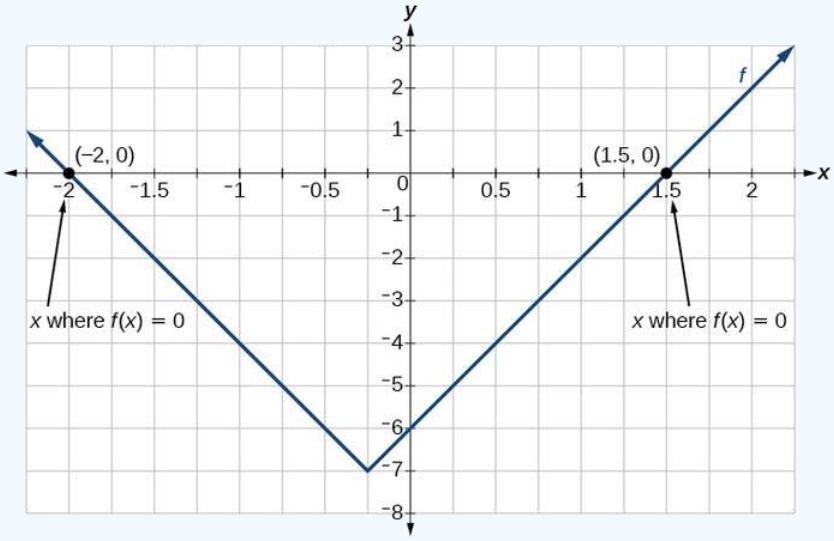



Figure 2.7.9: Graph of an absolute function with x-intercepts at -2 and 1.5.

### Exercise 2.7.4

For the function $f(x) = |2x-1| - 3$, find the values of $x$ such that $f(x) = 0$.

**Solution**

$x = -1$ or $x = 2$

**Q & A**

Should we always expect two answers when solving $|A| = B$?

**Answer**

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x - 5| = 1$.

**How To ...**

Given an absolute value equation, solve it.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $A = -B$.
3. Solve for $x$.

**Example 2.7.5: Solving an Absolute Value Equation**

Solve $1 = 4|x - 2| + 2$.

**Solution**

Isolating the absolute value on one side of the equation gives the following.

$$\begin{aligned} 1 &= 4|x-2|+2 \\ -1 &= 4|x-2| \\ -\frac{1}{4} &= |x-2| \end{aligned}$$

The absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value. At this point, we notice that this equation has no solutions.

**Q & A**

In Example 2.7.5, if $f(x) = 1$ and $g(x) = 4|x - 2| + 2$ were graphed on the same set of axes, would the graphs intersect?

**Answer**

No. The graphs of $f$ and $g$ would not intersect, as shown in Figure 2.7.10. This confirms, graphically, that the equation $1 = 4|x - 2| + 2$ has no solution.

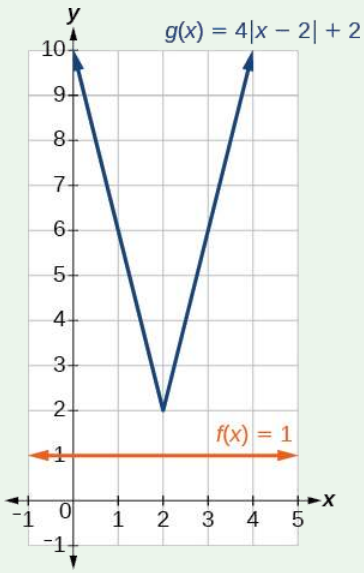



Figure 2.7.10: Graph of $g(x) = 4|x - 2| + 2$ and $f(x) = 1$.


https://math.libretexts.org/@go/page/69078

Find where the graph of the function $f(x) = -|x+2|+3$ intersects the horizontal and vertical axes.

$f(0) = 1$, so the graph intersects the vertical axis at $(0, 1)$. $f(x) = 0$ when $x = -5$ and $x = 1$ so the graph intersects the horizontal axis at $(-5, 0)$ and $(1, 0)$.

## Solving an Absolute Value Inequality

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an **absolute value inequality** to solve such an equation. An absolute value inequality is an equation of the form

$$|A| < B,\ |A| \leq B, |A| > B,$$

or

$$|A| \geq B,$$

where an expression $A$ (and possibly but not usually $B$) depends on a variable $x$. Solving the inequality means finding the set of all $x$ that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as

$$|x| < 200$$

or

$$-200 < x < 200$$

Suppose we want to know all possible returns on an investment if we could earn some amount of money within \$200 of \$600. We can solve algebraically for the set of values $x$ such that the distance between $x$ and 600 is less than 200. We represent the distance between $x$ and 600 as $|x - 600|$.

$$|x - 600| < 200 \tag{2.7.2}$$

or

$$-200 < x - 600 < 200 \tag{2.7.3}$$

$$-200 + 600 < x - 600 + 600 < 200 + 600$$

$$400 < x < 800$$

This means our returns would be between \$400 and \$800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

**How To ...**

Given an absolute value inequality of the form $|x - A| \leq B$ for real numbers $a$ and $b$ where $b$ is positive, solve the absolute value inequality algebraically.

1. Find boundary points by solving $|x - A| = B$.
2. Test intervals created by the boundary points to determine where $|x - A| \leq B$.
3. Write the interval or union of intervals satisfying the inequality in interval, inequality, or set-builder notation.

**Example 2.7.6: Solving an Absolute Value Inequality**

Solve $|x - 5| \leq 4$.

**Solution**

With both approaches, we will need to know first where the corresponding equality is true. In this case we first will find where $|x-5|=4$. We do this because the absolute value is a function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x-5|=4$.

$$\begin{aligned} x-5 &= 4 \quad \text{or} \quad x=9 \\ x-5 &= -4 \quad\quad\;\; x=1 \end{aligned}$$

After determining that the absolute value is equal to 4 at $x=1$ and $x=9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:

$$x<1,\ 1<x<9, \text{ and } x>9.$$

To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table 2.7.1.

Table 2.7.1

| Interval test $x$ | $f(x)$ | $<4$ or $>4$ | |
|---|---|---|---|
| $x<1$ | 0 | $\|0-5\|=5$ | Greater than |
| $1<x<9$ | 6 | $\|6-5\|=1$ | Less than |
| $x>9$ | 11 | $\|11-5\|=6$ | Greater than |

Because $1\leq x\leq 9$ is the only interval in which the output at the test value is less than 4, we can conclude that the solution to $|x-5|\leq 4$ is $1\leq x\leq 9$, or $[1,9]$.

To use a graph, we can sketch the function $f(x)=|x-5|$. To help us see where the outputs are 4, the line $g(x)=4$ could also be sketched as in Figure 2.7.11.

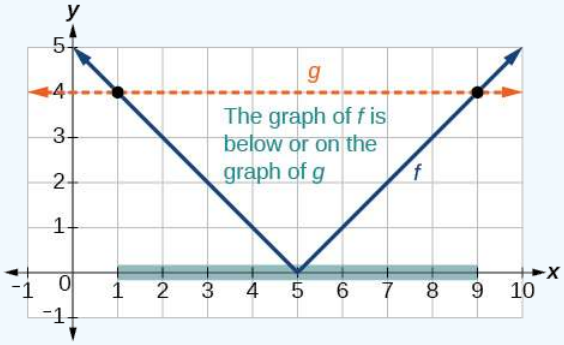



Figure 2.7.11: Graph to find the points satisfying an absolute value inequality.

We can see the following:

- The output values of the absolute value are equal to 4 at $x=1$ and $x=9$.
- The graph of $f$ is below the graph of $g$ on $1<x<9$. This means the output values of $f(x)$ are less than the output values of $g(x)$.
- The absolute value is less than or equal to 4 between these two points, when $1\leq x\leq 9$. In interval notation, this would be the interval $[1,9]$.

**Analysis**

For absolute value inequalities,

$$\begin{aligned} |x-A|<C, &\qquad |x-A|>C, \\ -C<x-A<C, &\quad x-A<-C \text{ or } x-A>C. \end{aligned}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for this example, we could use this alternative approach.

$$
\begin{aligned}
|x-5| &\leq 4 \\
-4 \leq x-5 &\leq 4 && \text{Rewrite by removing the absolute value bars.} \\
-4+5 \leq x-5+5 &\leq 4+5 && \text{Isolate the x.} \\
1 \leq x &\leq 9
\end{aligned}
$$

**Exercise 2.7.5**

Solve $|x+2| \leq 6$.

**Answer**

$-8 \leq x \leq 4$

**How To ...**

Given an absolute value function, solve for the set of inputs where the output is positive (or negative).

1. Set the function equal to zero, and solve for the boundary points of the solution set.
2. Use test points or a graph to determine where the function's output is positive or negative.

**Example 2.7.7: Using a Graphical Approach to Solve Absolute Value Inequalities**

Given the function $f(x) = -\frac{1}{2}|4x-5|+3$, determine the $x$-values for which the function values are negative.

**Solution**

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x-5|+3 < 0$. We begin by isolating the absolute value.

$$
\begin{aligned}
-\frac{1}{2}|4x-5| &< -3 && \text{Multiply both sides by } -2, \text{ and reverse the inequality.} \\
|4x-5| &> 6
\end{aligned}
$$

Next we solve for the equality $|4x-5| = 6$.

$$
\begin{aligned}
4x-5 &= 6 \qquad 4x-5 = -6 \\
4x-6 &= 6
\end{aligned}
$$

or

$$
\begin{aligned}
4x &= -1 \\
x &= \frac{11}{4} \qquad x = -\frac{1}{4}
\end{aligned}
$$

Now, we can examine the graph of $f$ to observe where the output is negative. We will observe where the branches are below the $x$-axis. Notice that it is not even important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$ and that the graph has been reflected vertically. See Figure 2.7.12

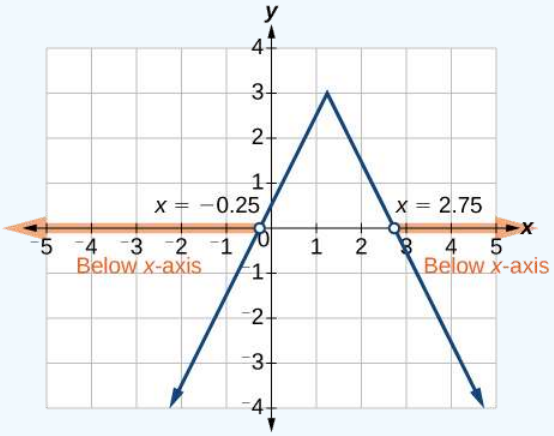



Figure 2.7.12: Graph of an absolute function with x-intercepts at -0.25 and 2.75.]

We observe that the graph of the function is below the $x$-axis left of $x = -\frac{1}{4}$ and right of $x = \frac{11}{4}$. This means the function values are negative to the left of the first horizontal intercept at $x = -\frac{1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality.

$$x < -\frac{1}{4} \text{ or } x > 1\frac{1}{4}$$

In interval notation, this would be $(-\infty, -0.25) \cup (2.75, \infty)$.

**Exercise 2.7.6**

Solve $-2|k-4| \leq -6$.

**Answer**

$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$

## Key Concepts

- The absolute value function is commonly used to measure distances between points.
- Applied problems, such as ranges of possible values, can also be solved using the absolute value function.
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable.
- An absolute value equation may have one solution, two solutions, or no solutions.
- An absolute value inequality is similar to an absolute value equation but takes the form | A |<B, | A |≤B, | A |>B, or | A |≥B.It can be solved by determining the boundaries of the solution set and then testing which segments are in the set.
- Absolute value inequalities can also be solved graphically.

## Glossary

**absolute value equation**
an equation of the form $|A| = B$, with $B \geq 0$; it will have solutions when $A = B$ or $A = -B$

**absolute value inequality**
a relationship in the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$

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# 2.8: Inverse Functions

**Learning Objectives**

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure 2.8.1 provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

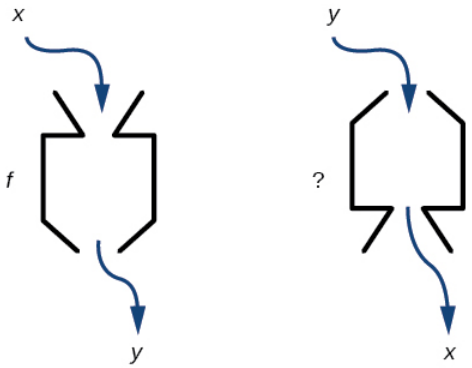



***Figure 2.8.1:*** *Can a function "machine" operate in reverse?*

## Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the **Celsius** scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees **Fahrenheit** to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32) \tag{2.8.1}$$

and substitutes 75 for $F$ to calculate

$$\frac{5}{9}(75 - 32) \approx 24^{\circ} \tag{2.8.2}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure 2.8.2 for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

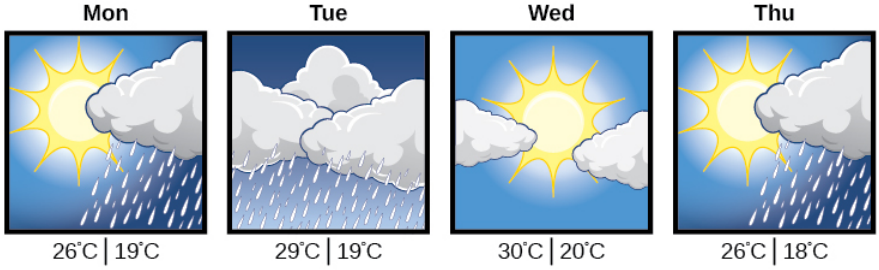



Figure 2.8.2: A forecast of Monday's through Thursday's weather.

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for $F$ after substituting a value for $C$. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F-32) \tag{2.8.3}$$

$$26 \cdot \frac{9}{5} = F-32 \tag{2.8.4}$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79 \tag{2.8.5}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "$f$ inverse of $x$." The raised $-1$ is part of the notation. It is not an exponent; it does not imply a power of $-1$ . In other words, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of $f$ and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number $a$, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x \tag{2.8.6}$$

This holds for all $x$ in the domain of $f$. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a **reciprocal**, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x \tag{2.8.7}$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x \tag{2.8.8}$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2,-8)$, $(0,0)$, and $(2,8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8,-2)$, $(0,0)$, and $(8,2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

**Definition: Inverse Function**

For any **one-to-one function** $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of $f$ if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all $x$ in the domain of $f$. It also follows that $f(f^{-1}(x)) = x$ for all $x$ in the domain of $f^{-1}$ if $f^{-1}$ is the inverse of $f$.

The notation $f^{-1}$ is read "$f$ inverse." Like any other function, we can use any variable name as the input for $f^{-1}$, so we will often write $f^{-1}(x)$, which we read as "$f$ inverse of $x$." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)} \tag{2.8.9}$$

and not all functions have inverses.

### Example 2.8.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2)=4$ and $f(5)=12$, what are the corresponding input and output values for the inverse function?

**Solution**

The inverse function reverses the input and output quantities, so if

$$\begin{aligned} f(2)=4, \text{ then } f^{-1}(4)=2; \\ f(5)=12, \text{ then } f^{-1}(12)=5 \end{aligned} \tag{2.8.10}$$

.

Alternatively, if we want to name the inverse function $g$, then $g(4)=2$ and $g(12)=5$.

**Analysis**

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table 2.8.1.

Table 2.8.1

| $(x, f(x))$ | $(x, g(x))$ |
|---|---|
| $(2, 4)$ | $(4, 2)$ |
| $(5, 12)$ | $(12, 5)$ |

### Exercise 2.8.1

Given that $h^{-1}(6)=2$, what are the corresponding input and output values of the original function $h$?

**Answer**

$h(2)=6$

### How To: Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x))=x$ or $g(f(x))=x$.
2. If both statements are true, then $g=f^{-1}$ and $f=g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

### Example 2.8.2: Testing Inverse Relationships Algebraically

If $f(x)=\frac{1}{x+2}$ and $g(x)=\frac{1}{x}-2$, is $g=f^{-1}$?

**Solution**

$$g(f(x)) = \frac{1}{(\frac{1}{x+2})-2} \tag{2.8.11}$$

$$= x+2-2 \tag{2.8.12}$$

$$= x \tag{2.8.13}$$

so

$$g=f^{-1} \text{ and } f=g^{-1} \tag{2.8.14}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$f(g(x)) = \frac{1}{\frac{1}{x} - 2 + 2} \tag{2.8.15}$$

$$= \frac{1}{\frac{1}{x}} \tag{2.8.16}$$

$$= x \tag{2.8.17}$$

**Analysis**

Notice the inverse operations are in reverse order of the operations from the original function.

**Exercise 2.8.2**

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

**Answer**

Yes

**Example 2.8.3: Determining Inverse Relationships for Power Functions**

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

**Solution**

$$f(g(x)) = \frac{x^3}{27} \neq x \tag{2.8.18}$$

No, the functions are not inverses.

**Analysis**

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

**Exercise 2.8.3**

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

**Answer**

Yes

## Finding Domain and Range of Inverse Functions

The outputs of the function $f$ are the inputs to $f^{-1}$, so the range of $f$ is also the domain of $f^{-1}$. Likewise, because the inputs to $f$ are the outputs of $f^{-1}$, the domain of $f$ is the range of $f^{-1}$. We can visualize the situation as in Figure 2.8.3.

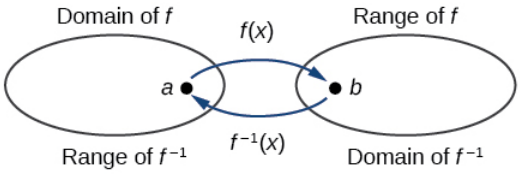



***Figure 2.8.3:*** *Domain and range of a function and its inverse.*

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs

3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its range limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$ .

- The domain of $f$ = range of $f^{-1} = [1, \infty)$.
- The domain of $f^{-1}$ = range of $f = [0, \infty)$.

**Is it possible for a function to have more than one inverse?**

*No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function $f$, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.*

**Note: Domain and Range of Inverse Functions**

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

**How To: Given a function, find the domain and range of its inverse.**

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

**Example 2.8.4: Finding the Inverses of Toolkit Functions**

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 2.8.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table 2.8.2

| Constant | Identity | Quadratic | Cubic | Reciprocal |
|---|---|---|---|---|
| $f(x) = c$ | $f(x) = x$ | $f(x) = x^2$ | $f(x) = x^3$ | $f(x) = \frac{1}{x}$ |
| Reciprocal squared | Cube Root | Square Root | Absolute Value | |
| $f(x) = \frac{1}{x^2}$ | $f(x) = \sqrt[3]{x}$ | $f(x) = \sqrt{x}$ | $f(x) = \lvert x \rvert$ | |

**Solution**

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

**Analysis**

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 2.8.4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal

line test, then in that restricted domain, it can have an inverse.

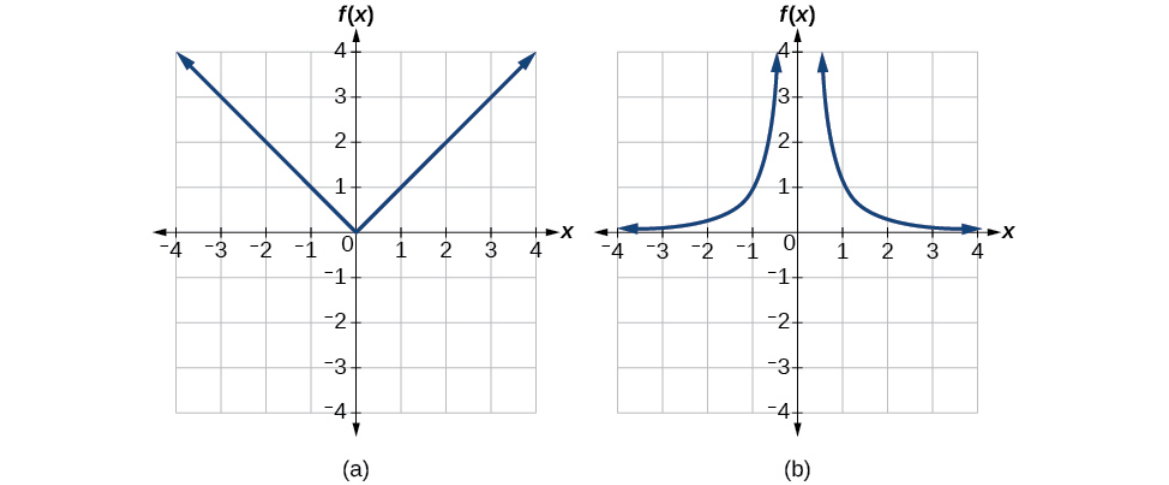



***Figure* 2.8.4:** *(a) Absolute value (b) Reciprocal squared*

**2.8.4:** The domain of function $f$ is $(1, \infty)$ and the range of function $f$ is $(-\infty, -2)$. Find the domain and range of the inverse function.

**Solution**

The domain of function $f^{-1}$ is $(-\infty, -2)$ and the range of function $f^{-1}$ is $(1, \infty)$.

## Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

### Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

**Example 2.8.5: Interpreting the Inverse of a Tabular Function**

A function $f(t)$ is given in Table 2.8.3, showing distance in miles that a car has traveled in $t$ minutes. Find and interpret $f^{-1}(70)$

Table 2.8.3

| $t$ (minutes) | 30 | 50 | 70 | 90 |
|---|---|---|---|---|
| $f(t)$ (miles) | 20 | 40 | 60 | 70 |

The inverse function takes an output of $f$ and returns an input for $f$. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function $f$, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value $a$ so that $f(a) = 70$. In this case, we are looking for a $t$ so that $f(t) = 70$, which is when $t = 90$.

## Exercise 2.8.5

Using Table 2.8.4, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

Table 2.8.4

| $t$ (minutes) | 30 | 50 | 60 | 70 | 90 |
|---|---|---|---|---|---|
| $f(t)$ (miles) | 20 | 40 | 50 | 60 | 70 |

**Answer**

$f(60) = 50$. In 60 minutes, 50 miles are traveled.
$f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

### Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

**How To: Given the graph of a function, evaluate its inverse at specific points.**

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

## Example 2.8.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 2.8.5. Find $g(3)$ and $g^{-1}(3)$.
.

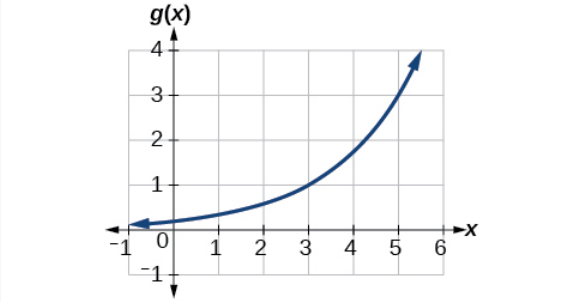



Figure 2.8.5: Graph of $g(x)$

**Solution**

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of $x$ for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 2.8.6.

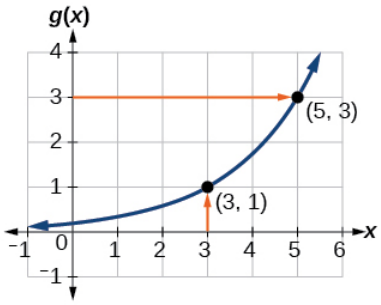



Figure 2.8.6: Graph of $g(x)$.

**Exercise 2.8.6**

Using the graph in Figure 2.8.6, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

**Answer a**

3

**Answer b**

5.6

## Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, $y$ as a function of $x$— we can often find the inverse function by solving to obtain $x$ as a function of $y$.

**How To: Given a function represented by a formula, find the inverse.**

1. Make sure $f$ is a one-to-one function.
2. Solve for $x$
3. Interchange $x$ and $y$.

**Example 2.8.7: Inverting the Fahrenheit-to-Celsius Function**

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32) \tag{2.8.19}$$

**Solution**

$$C = \frac{5}{9}(F - 32) \tag{2.8.20}$$

$$C \cdot \frac{9}{5} = F - 32 \tag{2.8.21}$$

$$F = \frac{9}{5}C + 32 \tag{2.8.22}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32) \tag{2.8.23}$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32. \tag{2.8.24}$$

In this case, we introduced a function $h$ to represent the conversion because the input and output variables are descriptive, and writing $C^{-1}$ could get confusing.

### Exercise 2.8.7

Solve for $x$ in terms of $y$ given $y = \frac{1}{3}(x-5)$

**Answer**

$x = 3y+5$

### Example 2.8.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3}+4$.

**Solution**

$$y = \frac{2}{x-3+4} \quad \text{Set up an equation.} \tag{2.8.25}$$

$$y-4 = \frac{2}{x-3} \quad \text{Subtract 4 from both sides.} \tag{2.8.26}$$

$$x-3 = \frac{2}{y-4} \quad \text{Multiply both sides by x−3 and divide by y−4.} \tag{2.8.27}$$

$$x = \frac{2}{y-4}+3 \quad \text{Add 3 to both sides.} \tag{2.8.28}$$

So $f^{-1}(y) = \frac{2}{y-4}+3$ or $f^{-1}(x) = \frac{2}{x-4}+3$.

**Analysis**

The domain and range of $f$ exclude the values 3 and 4, respectively. $f$ and $f^{-1}$ are equal at two points but are not the same function, as we can see by creating Table 2.8.5.

Table 2.8.5

| $x$ | 1 | 2 | 5 | $f^{-1}(y)$ |
|---|---|---|---|---|
| $f(x)$ | 3 | 2 | 5 | $y$ |

### Example 2.8.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2+\sqrt{x-4}$.

**Solution**

$$y = 2+\sqrt{x-4} \tag{2.8.29}$$

$$(y-2)^2 = x-4 \tag{2.8.30}$$

$$x = (y-2)^2+4 \tag{2.8.31}$$

So $f^{-1}(x) = (x-2)^2+4$.

The domain of $f$ is $[4,\infty)$. Notice that the range of $f$ is $[2,\infty)$, so this means that the domain of the inverse function $f^{-1}$ is also $[2,\infty)$

**Analysis**

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real $x$. However, $f^{-1}$ itself must have an inverse (namely, $f$) so we have to restrict the domain of $f^{-1}$ to $[2,\infty)$ in order to make $f^{-1}$ a one-to-one function. This domain of $f^{-1}$ is exactly the range of $f$.

**Exercise 2.8.8**

What is the inverse of the function $f(x) = 2 - \sqrt{x}$ ? State the domains of both the function and the inverse function.

**Answer**

$f^{-1}(x) = (2-x)^2$ ; domain of $f$: $[0, \infty)$; domain of $f^{-1}$: $(-\infty, 2]$

## Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure 2.8.7.

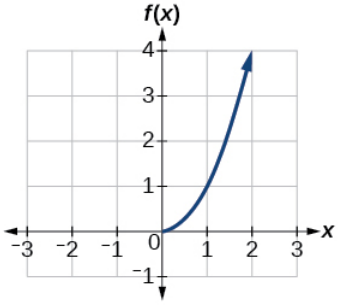



***Figure 2.8.7:*** *Quadratic function with domain restricted to* $[0, \infty)$.

**Restricting the domain** to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both $f$ and $f^{-1}$ on the same set of axes, using the x-axis for the input to both $f$ and $f^{-1}$?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 2.8.8.

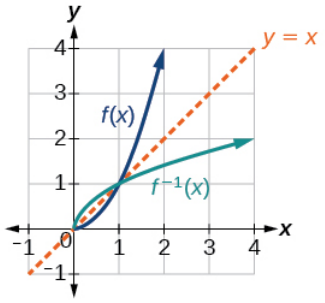



***Figure 2.8.8:*** *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

**Example 2.8.10: Finding the Inverse of a Function Using Reflection about the Identity Line**

Given the graph of $f(x)$ in Figure 2.8.9, sketch a graph of $f^{-1}(x)$.

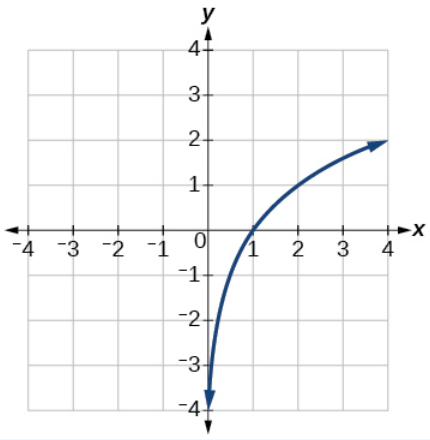



Figure 2.8.9: Graph of $f^(-1)(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 2.8.10.

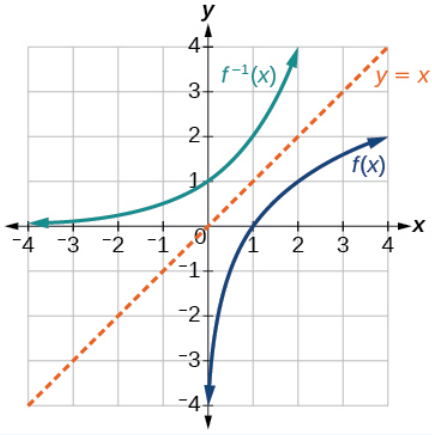



Figure 2.8.10: The function and its inverse, showing reflection about the identity line

## Exercise 2.8.1

Draw graphs of the functions $f$ and $f^{-1}$ from Example 2.8.8.

**Answer**

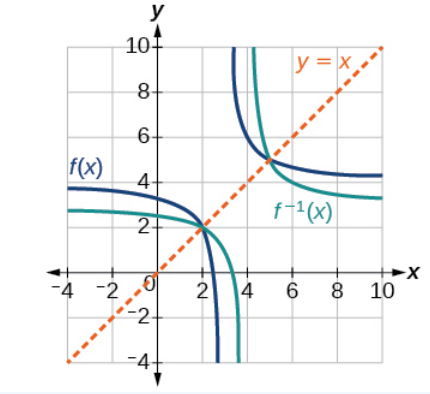



Figure 2.8.11: Graph of $f(x)$ and $f^(-1)(x)$.

**Is there any function that is equal to its own inverse?**

*Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because*

$$\frac{1}{\frac{1}{x}} = x \tag{2.8.32}$$

*Any function $f(x) = c - x$, where $c$ is a constant, is also equal to its own inverse.*

## Key Concepts

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$.
- Each of the toolkit functions has an inverse.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse.
- The inverse of a function can be determined at specific points on its graph.
- To find the inverse of a formula, solve the equation $y = f(x)$ for $x$ as a function of $y$. Then exchange the labels $x$ and $y$.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$.

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# 2.E: Functions (Exercises)

This page is a draft and is under active development.

## 1.1: Functions and Function Notation

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

### Verbal

1) What is the difference between a relation and a function?

**Answer**

A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

2) What is the difference between the input and the output of a function?

3) Why does the vertical line test tell us whether the graph of a relation represents a function?

**Answer**

When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

4) How can you determine if a relation is a one-to-one function?

5) Why does the horizontal line test tell us whether the graph of a function is one-to-one?

**Answer**

When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

### Algebraic

**For the exercises 6-7, determine whether the relation represents a function.**

6) $\{(a,b),(c,d),(a,c)\}$

7) $\{(a,b),(b,c),(c,c)\}$

**Answer**

function

**For the exercises 8-26, determine whether the relation represents $y$ as a function of $x$.**

8) $5x+2y=10$

9) $y=x^2$

**Answer**

function

10) $x=y^2$

11) $3x^2+y=14$

**Answer**

function

12) $2x+y^2=6$

13) $y=-2x^2+40x$

**Answer**

function

14) $y=\dfrac{1}{x}$

15) $x=\dfrac{3y+5}{7y-1}$

**Answer**

function

16) $x=\sqrt{1-y^2}$

17) $y=\dfrac{3x+5}{7x-1}$

**Answer**

function

18) $x^2+y^2=9$

19) $2xy=1$

**Answer**

function

20) $x=y^3$

21) $y=x^3$

**Answer**

function

22) $y=\sqrt{1-x^2}$

23) $x=\pm\sqrt{1-y}$

**Answer**

function

24) $y=\pm\sqrt{1-x}$

25) $y^2=x^2$

**Answer**

not a function

26) $y^3=x^2$

**For the exercises 27-39, evaluate the function $f$ at the indicated values $f(-3), f(2), f(-a), -f(a), f(a+h)$.**

27) $f(x)=2x-5$

**Answer**

$f(-3)=-11$; $f(2)=-1$; $f(-a)=-2a-5$ ; $-f(a)=-2a+5$ ; $f(a+h)=2a+2h-5$

28) $f(x)=-5x^2+2x-1$

29) $f(x)=\sqrt{2-x}+5$

**Answer**

$f(-3)=\sqrt{5+5}$; $f(2)=5$; $f(-a)=\sqrt{2+a+5}$ ; $-f(a)=-\sqrt{2-a}-5$ ; $f(a+h)=\sqrt{2-a-h}+5$

30) $f(x)=\dfrac{6x-1}{5x+2}$

31) $f(x)=|x-1|-|x+1|$

**Answer**

$f(-3)=2$; $f(2)=1-3=-2$ ; $f(-a)=|-a-1|-|-a+1|$ ; $-f(a)=-|a-1|+|a+1|$ ;
$f(a+h)=|a+h-1|-|a+h+1|$

32) Given the function $g(x)=5-x^2$ , evaluate $g(x+h)-g(x)h$ , $h\neq 0$.

33) Given the function $g(x)=x^2+2x$ , evaluate $\dfrac{g(x)-g(a)}{x-a}$, $x\neq a$.

**Answer**

$\dfrac{g(x)-g(a)}{x-a}=x+a+2$ , $x\neq a$

34) Given the function $k(t)=2t-1$ :

a. Evaluate $k(2)$.
b. Solve $k(t)=7$.

35) Given the function $f(x)=8-3x$:

a. Evaluate $f(-2)$.
b. Solve $f(x)=-1$.

**Answer**

a. $f(-2)=14$; b. $x=3$

36) Given the function $p(c)=c^2+c$:

a. Evaluate $p(-3)$.
b. Solve $p(c)=2$.

37) Given the function $f(x)=x^2-3x$:

a. Evaluate $f(5)$.
b. Solve $f(x)=4$.

**Answer**

a. $f(5)=10$; b. $x=-1$ or $x=4$

38) Given the function $f(x)=\sqrt{x+2}$:

a. Evaluate $f(7)$.
b. Solve $f(x)=4$.

39) Consider the relationship $3r+2t=18$.

a. Write the relationship as a function $r=f(t)$.
b. Evaluate $f(-3)$.

c. Solve $f(t) = 2$.

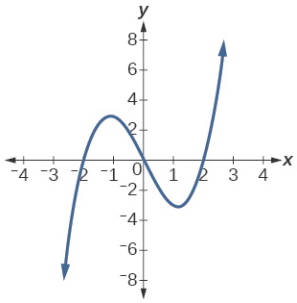
**Answer**

a. $f(t) = 6 - \frac{2}{3}t$ ; b. $f(-3) = 8$; c. $t = 6$

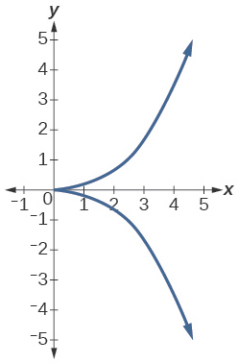
## Graphical

**For the exercises 40-54, use the vertical line test to determine which graphs show relations that are functions.**

40)



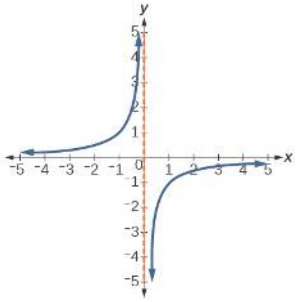

41)



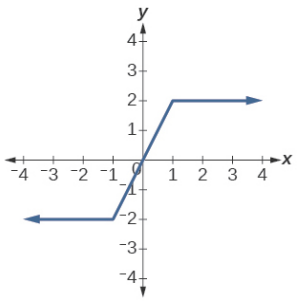

**Answer**

not a function

42)



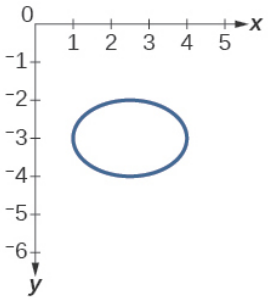

43)



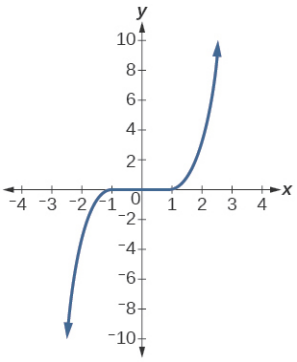

**Answer**

function

44)



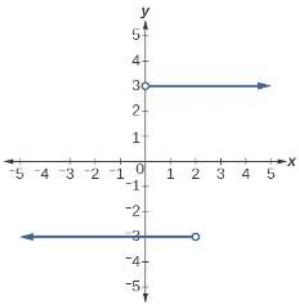

45)



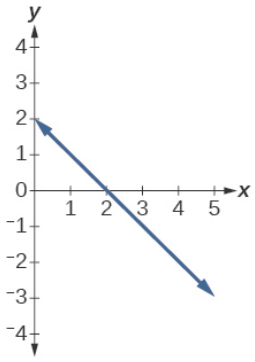

**Answer**

function

46)



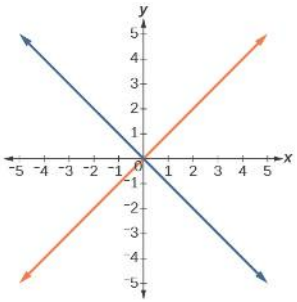

47)



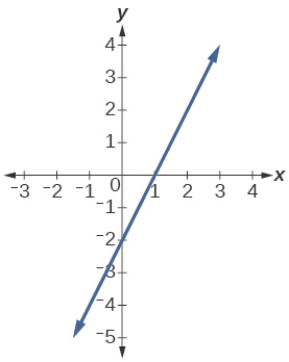

**Answer**

function

48)



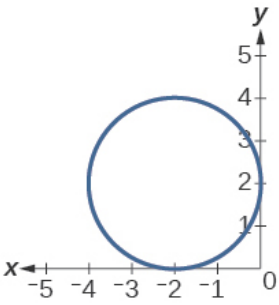

49)



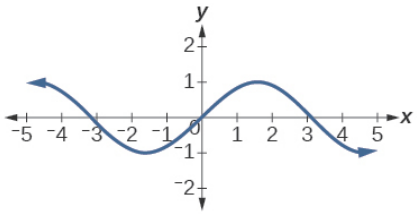

**Answer**

function

50)



51)

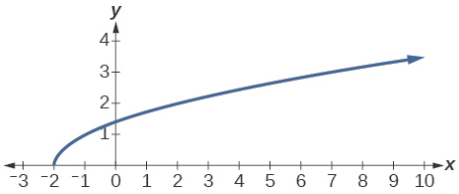



**Answer**

function

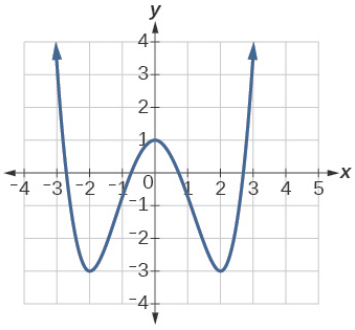
52) Given the following graph,

- Evaluate $f(-1)$.
- Solve for $f(x) = 3$.




53) Given the following graph,

- Evaluate $f(0)$.
- Solve for $f(x) = -3$.

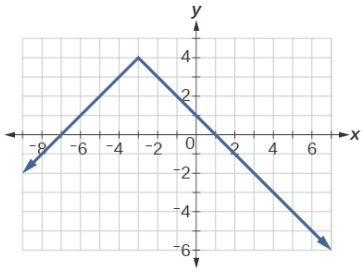



**Answer**

a.$f(0) = 1$; b. $f(x) = -3$, $x = -2$ or $x = 2$

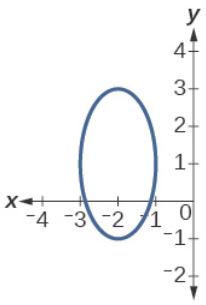
54) Given the following graph,

- Evaluate $f(4)$.
- Solve for $f(x) = 1$.




**For the exercises 55-59, determine if the given graph is a one-to-one function.**

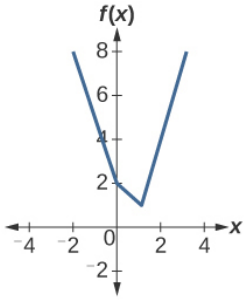
55)



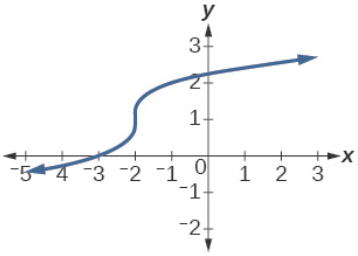

**Answer**

not a function so it is also not a one-to-one function

56)



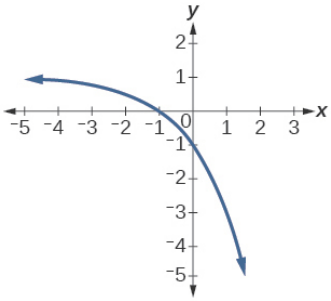

57)



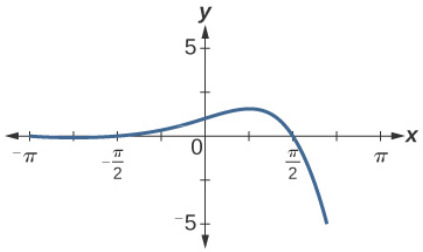

**Answer**

one-to-one function

58)




59)




**Answer**

function, but not one-to-one

## Numeric

**For the exercises 60-62, determine whether the relation represents a function.**

60) $\{(-1,-1),(-2,-2),(-3,-3)\}$

61) $\{(3,4),(4,5),(5,6)\}$

**Answer**

function

62) $(2,5),(7,11),(15,8),(7,9)\}$

**For the exercises 63-65, determine if the relation represented in table form represents $y$ as a function of $x$.**

63)

| $x$ | 5 | 10 | 15 |
|---|---|---|---|
| $y$ | 3 | 8 | 14 |

**Answer**

function

64)

| $x$ | 5 | 10 | 15 |
|---|---|---|---|
| $y$ | 3 | 8 | 8 |

65)

| $x$ | 5 | 10 | 10 |
|---|---|---|---|
| $y$ | 3 | 8 | 14 |

**Answer**

not a function

**For the exercises 66-67, use the function $f$ represented in the Table below**

| $x$ | $f(x)$ |
|---|---|
| 0 | 74 |
| 1 | 28 |
| 2 | 1 |

| $x$ | $f(x)$ |
|---|---|
| 3 | 53 |
| 4 | 56 |
| 5 | 3 |
| 6 | 36 |
| 7 | 45 |
| 8 | 14 |
| 9 | 47 |

66) Evaluate $f(3)$.

67) Solve $f(x) = 1$.

**Answer**

$f(x) = 1$ , $x = 2$

**For the exercises 68-73, evaluate the function $f$ at the values $f(-2)$, $f(-1)$, $f(0)$, $f(1)$,and $f(2)$.**

68) $f(x) = 4 - 2x$

69) $f(x) = 8 - 3x$

**Answer**

$f(-2) = 14; f(-1) = 11; f(0) = 8; f(1) = 5; f(2) = 2$

70) $f(x) = 8x^2 - 7x + 3$

71) $f(x) = 3 + \sqrt{x+3}$

**Answer**

$f(-2) = 4$; $f(-1) = 4.414$; $f(0) = 4.732$; $f(1) = 4.5$; $f(2) = 5.236$

72) $f(x) = \dfrac{x-2}{x+3}$

73) $f(x) = 3^x$

**Answer**

$f(-2) = \frac{1}{9}$ ; $f(-1) = \frac{1}{3}$ ; $f(0) = 1$; $f(1) = 3$; $f(2) = 9$

**For the exercises 74-75, evaluate the expressions, given functions $f$, $g$, and $h$:**

- $f(x) = 3x - 2$
- $g(x) = 5 - x^2$
- $h(x) = -2x^2 + 3x - 1$

74) $3f(1) - 4g(-2)$

75) $f\left(\dfrac{7}{3}\right) - h(-2)$

**Answer**

20

## Technology

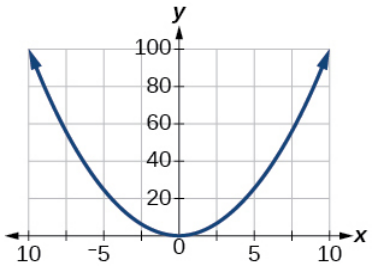
**For the exercises 76-78, graph $y = x^2$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.**

76) $[-0.1, 0.1]$

77) $[-10, 10]$

**Answer**

$[0, 100]$



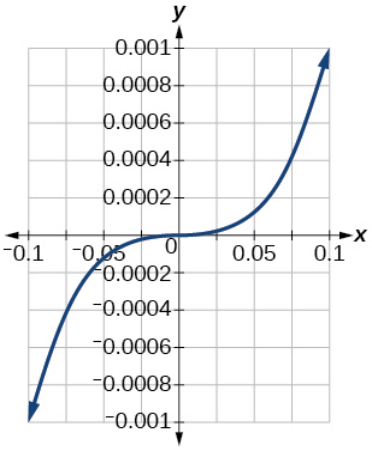

78) $[-100, 100]$

**For the exercises 79-81, graph $y = x^3$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.**

79) $[-0.1, 0.1]$

**Answer**

$[-0.001, 0.001]$



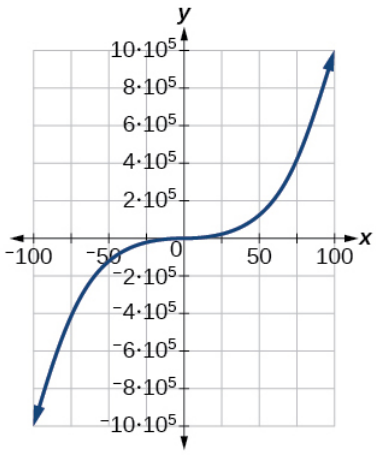

80) $[-10, 10]$

81) $[-100, 100]$

**Answer**

$[-1,000,000, 1,000,000]$


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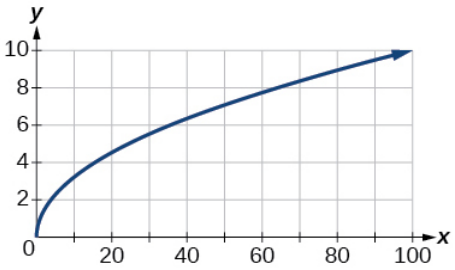

**For the exercises 82-84, graph $y = \sqrt{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.**

82) $[0, 0.01]$

83) $[0, 100]$

**Answer**

$[0, 10]$




84) $[0, 10,000]$

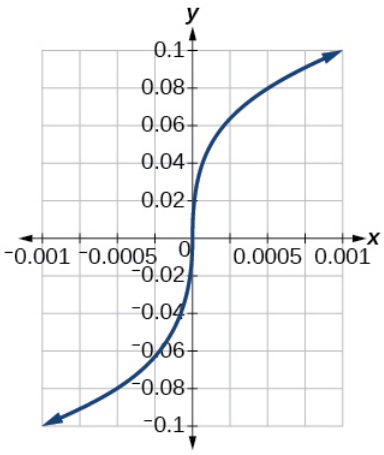
**For the exercises 85-87, graph $y = \sqrt[3]{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.**

85) $[-0.001, 0.001]$

**Answer**

$[-0.1, 0.1]$


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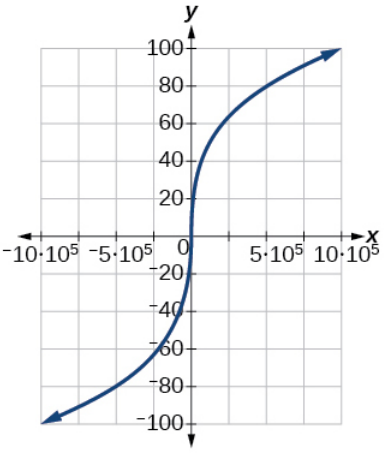



86) $[-1000, 1000]$

87) $[-1,000,000, 1,000,000]$

**Answer**

$[-100, 100]$




## Real-World Applications

88) The amount of garbage, $G$, produced by a city with population $p$ is given by $G = f(p)$. $G$ is measured in tons per week, and $p$ is measured in thousands of people.

The town of Tola has a population of $40,000$ and produces $13$ tons of garbage each week. Express this information in terms of the function $f$.Tola has a population of $40,000$ and produces $13$ tons of garbage each week. Express this information in terms of the function $f$.
Explain the meaning of the statement $f(5) = 2$.

89) The number of cubic yards of dirt, $D$, needed to cover a garden with area a square feet is given by $D = g(a)$.

a. A garden with area $5000\ ft^2$ requires $50\ yd^3$ of dirt. Express this information in terms of the function $g$.
b. Explain the meaning of the statement $g(100) = 1$.

**Answer**

a. $g(5000) = 50$; b. The number of cubic yards of dirt required for a garden of $100$ square feet is $1$.

90) Let $f(t)$ be the number of ducks in a lake $t$ years after 1990. Explain the meaning of each statement:

a. f(5)=30
b. f(10)=40

91) Let $h(t)$ be the height above ground, in feet, of a rocket $t$ seconds after launching. Explain the meaning of each statement:

a. $h(1) = 200$
b. $h(2) = 350$

**Answer**

a. The height of a rocket above ground after $1$ second is $200$ ft. b. the height of a rocket above ground after $2$ seconds is $350$ ft.

92) Show that the function $f(x) = 3(x-5)^2 + 7$ is not one-to-one.

## 1.2: Domain and Range

In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions.

### Verbal

1) Why does the domain differ for different functions?

**Answer**

The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

2) How do we determine the domain of a function defined by an equation?

3) Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.

**Answer**

There is no restriction on x for $f(x) = \sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x-values are restricted for $f(x) = \sqrt{x}$ to nonnegative numbers and the domain is $[0, \infty)$.

4) When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?

5) How do you graph a piecewise function?

**Answer**

Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints with an open circle. Use an arrow to indicate $-\infty$ or $\infty$. Combine the graphs to find the graph of the piecewise function.

### Algebraic

**For the exercises 6-25, find the domain of each function using interval notation.**

6) $f(x) = -2x(x-1)(x-2)$

7) $f(x) = 5 - 2x^2$

**Answer**

$(-\infty, \infty)$

8) $f(x) = 3\sqrt{x-2}$

9) $f(x) = 3 - \sqrt{6-2x}$

**Answer**

$(-\infty, 3]$

10) $f(x) = \sqrt{4-3x}$

11) $f(x) = \sqrt{x^2+4}$

**Answer**

$(-\infty, \infty)$

12) $f(x) = \sqrt[3]{1-2x}$

13) $f(x) = \sqrt[3]{x-1}$

**Answer**

$(-\infty, \infty)$

14) $f(x) = \dfrac{9}{x-6}$

15) $f(x) = \dfrac{3x+1}{4x+2}$

**Answer**

$\left(-\infty, -\dfrac{1}{2}\right) \cup \left(-\dfrac{1}{2}, \infty\right)$

16) $f(x) = \dfrac{\sqrt{x+4}}{x-4}$

17) $f(x) = \dfrac{x-3}{x^2+9x-22}$

**Answer**

$(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$

18) $f(x) = \dfrac{1}{x^2-x-6}$

19) $f(x) = \dfrac{2x^3-250}{x^2-2x-15}$

**Answer**

$(-\infty, -3) \cup (-3, 5) \cup (5, \infty)$

20) $\dfrac{5}{\sqrt{x-3}}$

21) $\dfrac{2x+1}{\sqrt{5-x}}$

**Answer**

$(-\infty, 5)$

22) $\dfrac{\sqrt{x-4}}{\sqrt{x-6}}$

23) $\dfrac{\sqrt{x-6}}{\sqrt{x-4}}$

**Answer**

$[6, \infty)$

24) $f(x) = \dfrac{x}{x}$

25) $f(x) = \dfrac{x^2 - 9x}{x^2 - 81}$

**Answer**

$(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$

26) Find the domain of the function $f(x) = \sqrt{2x^3 - 50x}$ by:

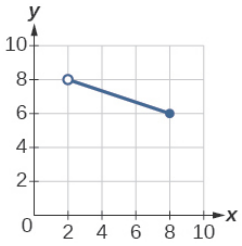
a. using algebra

b. graphing the function in the radicand and determining intervals on the x-axis for which the radicand is nonnegative.

## Graphical

**For the exercises 27-37, write the domain and range of each function using interval notation.**

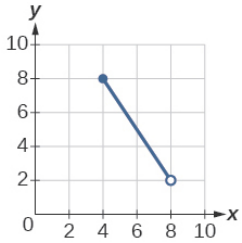
27)



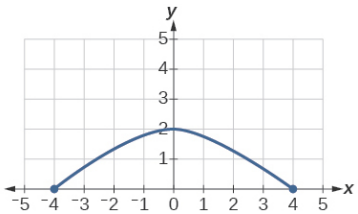

**Answer**

domain: $(2, 8]$, range $[6, 8)$

28)




29)

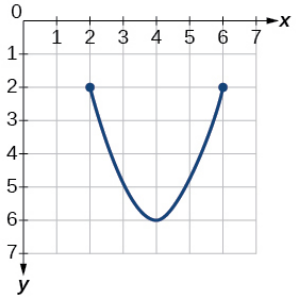



**Answer**

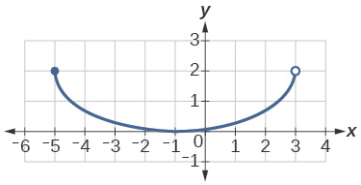
domain: $[-4, 4]$, range: $[0, 2]$

30)


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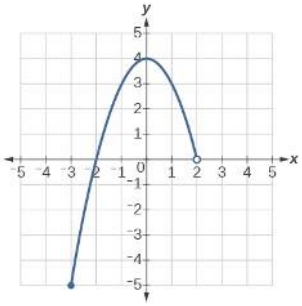

31)



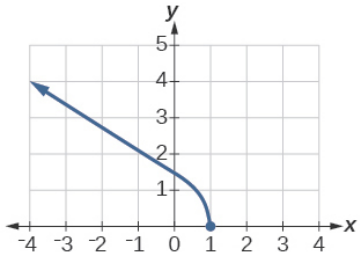

**Answer**

domain: $[-5, 3)$, range: $[0, 2]$

32)




33)

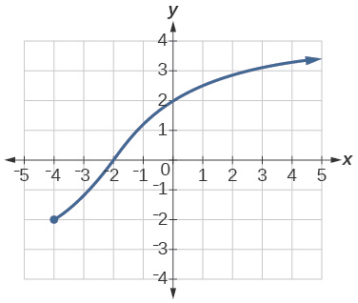



**Answer**

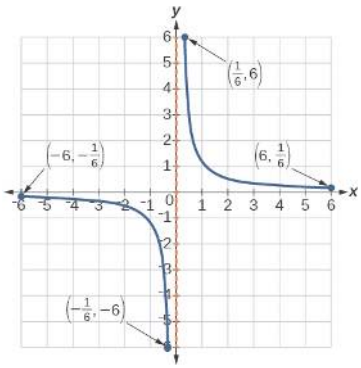
domain: $(-\infty, 1]$, range: $[0, \infty)$

34)


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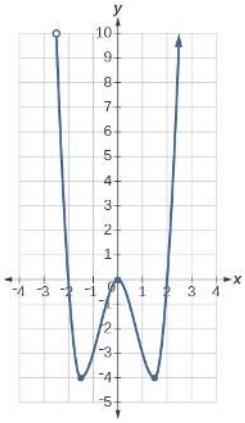
35)



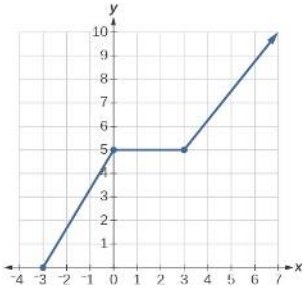

**Answer**

domain: $[-6, -16] \cup [16, 6]$; range: $[-6, -16] \cup [16, 6]$

36)



37)



**Answer**

domain: $[-3, \infty)$; range: $[0, \infty)$

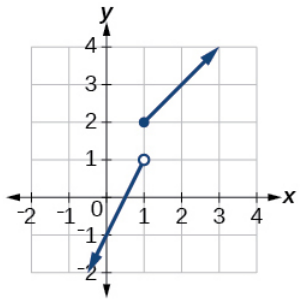
**For the exercises 38-45, sketch a graph of the piecewise function. Write the domain in interval notation.**

38) $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

39) $f(x) = \begin{cases} 2x-1 & \text{if } x < 1 \\ 1+x & \text{if } x \geq 1 \end{cases}$

**Answer**

domain: $(-\infty, \infty)$

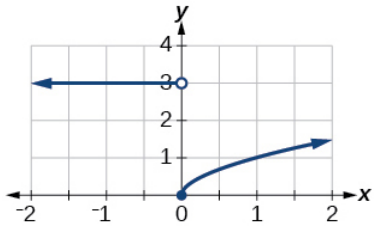



40) $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x-1 & \text{if } x > 0 \end{cases}$

41) $f(x) = \begin{cases} 3 & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$

**Answer**

domain: $(-\infty, \infty)$

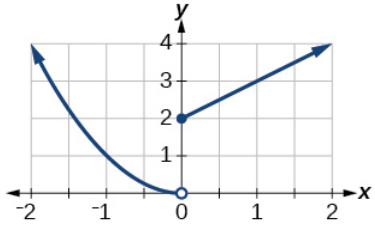



42) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1-x & \text{if } x > 0 \end{cases}$

43) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x+2 & \text{if } x \geq 0 \end{cases}$

**Answer**

domain: $(-\infty, \infty)$





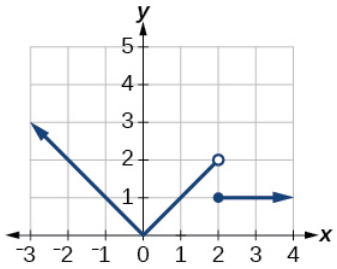
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44) $f(x)=\begin{cases} x+1 & \text{if } x<1 \\ x^3 & \text{if } x\geq 1 \end{cases}$

45) $f(x)=\begin{cases} |x| & \text{if } x<2 \\ 1 & \text{if } x\geq 2 \end{cases}$

**Answer**

domain: $(-\infty,\infty)$




## Numeric

**For the exercises 46-48, given each function $f$,evaluate $f(-3)$, $f(-2)$, $f(-1)$, and $f(0)$.**

46) $f(x)=\begin{cases} x+1 & \text{if } x<-2 \\ -2x-3 & \text{if } x\geq -2 \end{cases}$

47) $f(x)=\begin{cases} 1 & \text{if } x\leq -3 \\ 0 & \text{if } x>-3 \end{cases}$

**Answer**

$f(-3)=1;\ f(-2)=0;\ f(-1)=0;\ f(0)=0$

48) $f(x)=\begin{cases} -2x^2+3 & \text{if } x\leq -1 \\ 5x-7 & \text{if } x>-1 \end{cases}$

**For the exercises 49-51, given each function $f$, evaluate $f(-1)$, $f(0)$, $f(2)$, and $f(4)$.**

49) $f(x)=\begin{cases} 7x+3 & \text{if } x<0 \\ 7x+6 & \text{if } x\geq 0 \end{cases}$

**Answer**

$f(-1)=-4;\ f(0)=6;\ f(2)=20;\ f(4)=34$

50) $f(x)=\begin{cases} x^2-2 & \text{if } x<2 \\ 4+|x-5| & \text{if } x\geq 2 \end{cases}$

51) $f(x)=\begin{cases} 5x & \text{if } x<0 \\ 3 & \text{if } 0\geq x\leq 2 \\ x^2 & \text{if } x>3 \end{cases}$

**Answer**

$f(-1)=-5;\ f(0)=3;\ f(2)=3;\ f(4)=16$

**For the exercises 52-54, write the domain for the piecewise function in interval notation.**

52) $f(x)=\begin{cases} x+1 & \text{if } x<-2 \\ -2x-3 & \text{if } x\geq -2 \end{cases}$

53) $f(x)=\begin{cases} x^2-2 & \text{if } x<1 \\ -x^2+2 & \text{if } x>1 \end{cases}$

**Answer**

domain: $(-\infty, 1) \cup (1, \infty)$

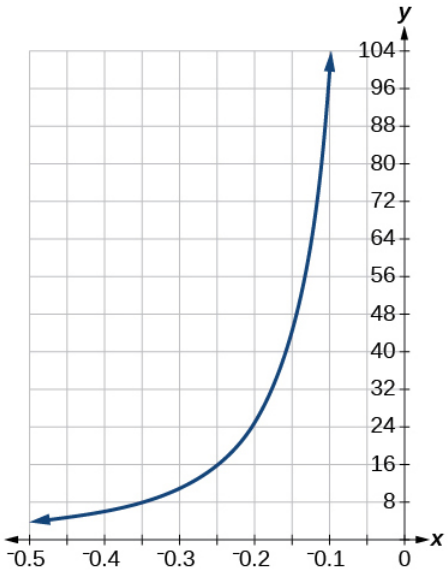
54) $f(x) = \begin{cases} x^2 - 3 & \text{if } x < 0 \\ -3x^2 & \text{if } x \geq 2 \end{cases}$

## Technology

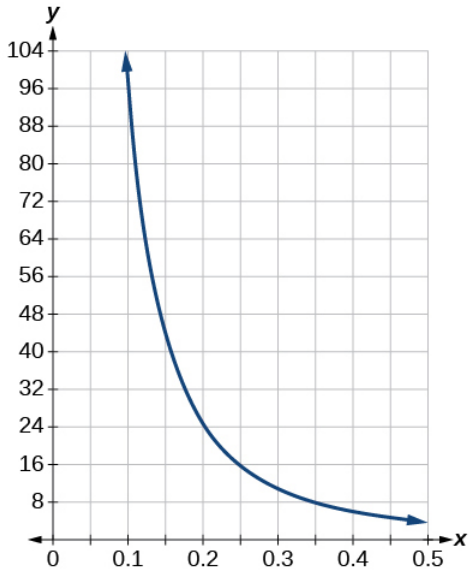
55) Graph $y = \dfrac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

**Answer**

window: $[-0.5, -0.1]$; range: $[4, 100]$




window: $[0.1, 0.5]$; range: $[4, 100]$




56) Graph $y = \dfrac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

### Extension

57) Suppose the range of a function $f$ is $[-5, 8]$. What is the range of $|f(x)|$?

**Answer**

$[0, 8]$

58) Create a function in which the range is all nonnegative real numbers.

59) Create a function in which the domain is $x > 2$.

**Answer**

Many answers. One function is $f(x) = \dfrac{1}{\sqrt{x-2}}$.

### Real-World Applications

60) The height $h$ of a projectile is a function of the time $t$ it is in the air. The height in feet for $t$ seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

**Answer**

The domain is $[0, 6]$; it takes 6 seconds for the projectile to leave the ground and return to the ground

61) The cost in dollars of making $x$ items is given by the function $C(x) = 10x + 500$.

a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.

b. What is the cost of making $25$ items?

c. Suppose the maximum cost allowed is $\$1500$. What are the domain and range of the cost function, $C(x)$?

## 1.3: Rates of Change and Behavior of Graphs

In this section, we will investigate changes in functions. For example, a rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. Identifying points that mark the interval on a graph can be used to find the average rate of change. Comparing pairs of input and output values in a table can also be used to find the average rate of change.

### Verbal

1) Can the average rate of change of a function be constant?

**Answer**

Yes, the average rate of change of all linear functions is constant.

2) If a function $f$ is increasing on $(a, b)$ and decreasing on $(b, c)$, then what can be said about the local extremum of $f$ on $(a, c)$?

3) How are the absolute maximum and minimum similar to and different from the local extrema?

**Answer**

The absolute maximum and minimum relate to the entire graph, whereas the local extrema relate only to a specific region around an open interval.

4) How does the graph of the absolute value function compare to the graph of the quadratic function, $y = x^2$, in terms of increasing and decreasing intervals?

### Algebraic

**For the exercises 5-15, find the average rate of change of each function on the interval specified for real numbers $b$ or $h$.**

5) $f(x) = 4x^2 - 7$ on $[1, b]$


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**Answer**

$4(b+1)$

6) $g(x)=2x^2-9$ on $[4,b]$

7) $p(x)=3x+4$ on $[2,2+h]$

**Answer**

$3$

8) $k(x)=4x-2$ on $[3,3+h]$

9) $f(x)=2x^2+1$ on $[x,x+h]$

**Answer**

$4x+2h$

10) $g(x)=3x^2-2$ on $[x,x+h]$

11) $a(t)=\dfrac{1}{t+4}$ on $[9,9+h]$

**Answer**

$\dfrac{-1}{13(13+h)}$

12) $b(x)=1x+3$ on $[1,1+h]$

13) $j(x)=3x^3$ on $[1,1+h]$

**Answer**

$3h^2+9h+9$

14) $r(t)=4t^3$ on $[2,2+h]$

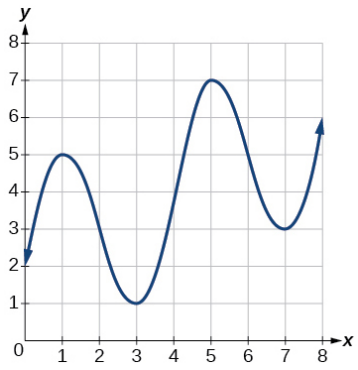
15) $\dfrac{f(x+h)-f(x)}{h}$ given $f(x)=2x^2-3x$ on $[x,x+h]$

**Answer**

$4x+2h-3$

## Graphical

**For the exercises 16-17, consider the graph of $f$ shown in the Figure below**




*Graph of a polynomial.*

16) Estimate the average rate of change from $x=1$ to $x=4$.

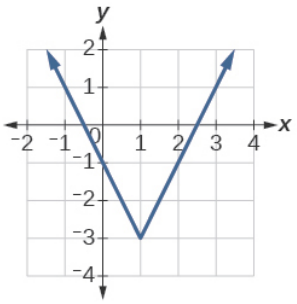
17) Estimate the average rate of change from $x = 2$ to $x = 5$.

**Answer**

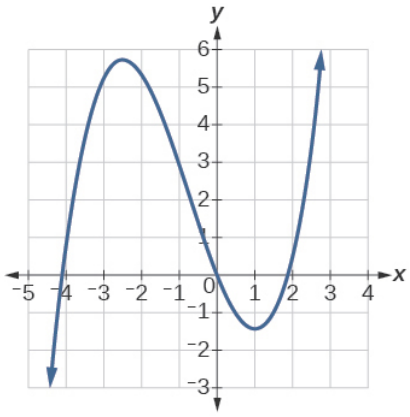
$\frac{4}{3}$

**For the following exercises, use the graph of each function to estimate the intervals on which the function is increasing or decreasing.**

18)



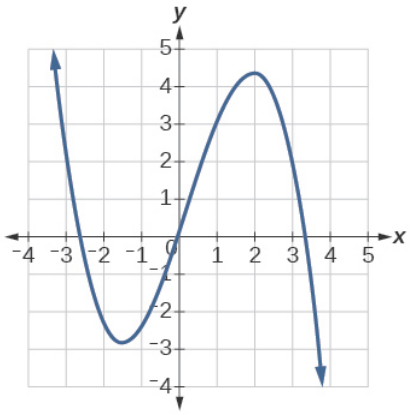

19)



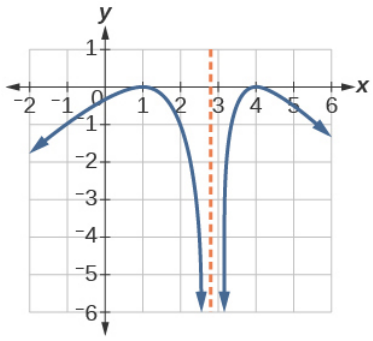

**Answer**

increasing on $(-\infty, -2.5) \cup (1, \infty)$, decreasing on $(-2.5, 1)$

20)



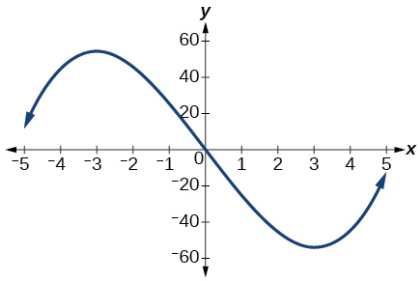

21)




**Answer**

increasing on $(-\infty, 1) \cup (3, 4)$, decreasing on $(1, 3) \cup (4, \infty)$

**For the exercises 22-23, consider the graph shown in the Figure below.**




*Graph of a cubic function.*

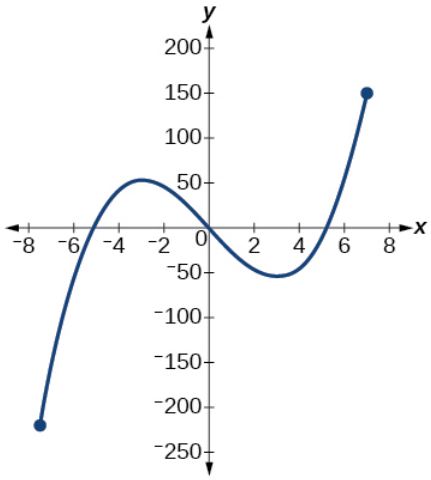
22) Estimate the intervals where the function is increasing or decreasing.

23) Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

**Answer**

local maximum: $(-3, 60)$, local minimum: $(3, -60)$

**For the exercises 24-25, consider the graph in the Figure below.**




*Graph of a cubic function.*

24) If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.

25) If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

**Answer**

absolute maximum at approximately $(7, 150)$, absolute minimum at approximately $(-7.5, -220)$

### Numeric

26) The Table below gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (a) between 2001 and 2002, and (b) between 2001 and 2004?

| Year | Sales (millions of dollars) |
|---|---|
| 1998 | 201 |
| 1999 | 219 |
| 2000 | 233 |
| 2001 | 243 |
| 2002 | 249 |
| 2003 | 251 |
| 2004 | 249 |
| 2005 | 243 |
| 2006 | 233 |

27) The Table below gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of population (a) between 2002 and 2004, and (b) between 2002 and 2006?

| Year | Population (thousands) |
|---|---|
| 2000 | 87 |
| 2001 | 84 |
| 2002 | 83 |
| 2003 | 80 |
| 2004 | 77 |
| 2005 | 76 |
| 2006 | 78 |
| 2007 | 81 |
| 2008 | 85 |

**Answer**

a. $-3000$; b. $-1250$

**For the exercises 28-34, find the average rate of change of each function on the interval specified.**

28) $f(x) = x^2$ on $[1, 5]$

29) $h(x) = 5 - 2x^2$ on $[-2, 4]$

**Answer**

$-4$

30) $q(x) = x^3$ on $[-4, 2]$

31) $g(x) = 3x^3 - 1$ on $[-3, 3]$

**Answer**

$27$

32) $y = \dfrac{1}{x}$ on $[1, 3]$

33) $p(t) = \dfrac{(t^2 - 4)(t + 1)}{t^2 + 3}$ on $[-3, 1]$

**Answer**

$-0.167$

34) $k(t) = 6t^2 + \dfrac{4}{t^3}$ on $[-1, 3]$

## Technology

**For the exercises 35-40, use a graphing utility to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.**

35) $f(x) = x^4 - 4x^3 + 5$

**Answer**

Local minimum at $(3, -22)$, decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$

36) $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

37) $g(t) = t\sqrt{t + 3}$

**Answer**

Local minimum at $(-2, -2)$, decreasing on $(-3, -2)$, increasing on $(-2, \infty)$

38) $k(t) = 3t^{\frac{2}{3}} - t$

39) $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

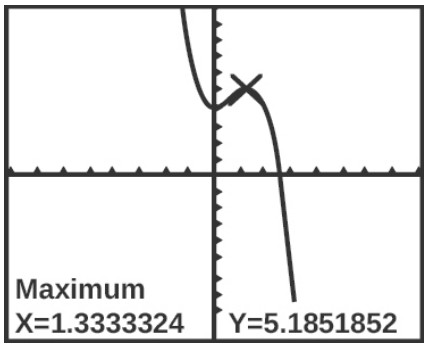
**Answer**

Local maximum at $(-0.5, 6)$, local minima at $(-3.25, -47)$ and $(2.1, -32)$, decreasing on $(-\infty, -3.25)$ and $(-0.5, 2.1)$, increasing on $(-3.25, -0.5)$and $(2.1, \infty)$

40) $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

## Extension

41) The graph of the function $f$ is shown in the Figure below:



*Graph of f(x) on a graphing calculator.*

Based on the calculator screen shot, the point $(1.333, 5.185)$is which of the following?

a. a relative (local) maximum of the function
b. the vertex of the function
c. the absolute maximum of the function
d. a zero of the function

**Answer**

a

42) Let $f(x) = \frac{1}{x}$. Find a number $c$ such that the average rate of change of the function f on the interval $(1, c)$ is $-\frac{1}{4}$.

43) Let $f(x) = \frac{1}{x}$. Find the number $b$ such that the average rate of change of f on the interval $(2, b)$ is $-\frac{1}{10}$.

**Answer**

$b = 5$

## Real-World Applications

44) At the start of a trip, the odometer on a car read $21,395$. At the end of the trip, $13.5$ hours later, the odometer read $22,125$. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?

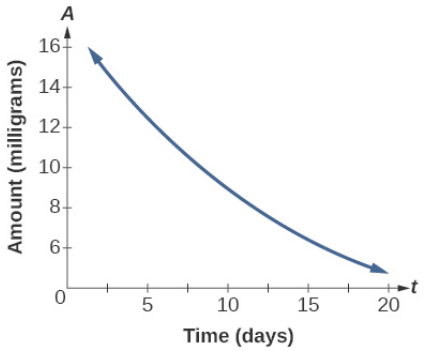
45) A driver of a car stopped at a gas station to fill up his gas tank. He looked at his watch, and the time read exactly 3:40 p.m. At this time, he started pumping gas into the tank. At exactly 3:44, the tank was full and he noticed that he had pumped $10.7$ gallons. What is the average rate of flow of the gasoline into the gas tank?

**Answer**

$2.7$ gallons per minute

46) Near the surface of the moon, the distance that an object falls is a function of time. It is given by $d(t) = 2.6667t^2$, where $t$ is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.

47) The graph in the Figure below illustrates the decay of a radioactive substance over $t$ days.




*Graph of an exponential function.*

Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

**Answer**

approximately $-0.6$ milligrams per day


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## 1.4: Composition of Functions

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. The cost depends on the temperature, and the temperature depends on the day. By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

### Verbal

1) How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

**Answer**

Find the numbers that make the function in the denominator $g$ equal to zero, and check for any other domain restrictions on $f$ and $g$, such as an even-indexed root or zeros in the denominator.

2) What is the composition of two functions, $f \circ g$?

3) If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

**Answer**

Yes. Sample answer: Let $f(x) = x+1$ and $g(x) = x-1$. Then $f(g(x)) = f(x-1) = (x-1)+1 = x$ and $g(f(x)) = g(x+1) = (x+1)-1 = x$. So $f \circ g = g \circ f$.

4) How do you find the domain for the composition of two functions, $f \circ g$?

### Algebraic

5) Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$ ,find $f+g$, $f-g$, $fg$, and $\frac{f}{g}$.Determine the domain for each function in interval notation.

**Answer**

$(f+g)(x) = 2x+6$ , domain: $(-\infty, \infty)$

$(f-g)(x) = 2x^2 + 2x - 6$ , domain: $(-\infty, \infty)$

$(fg)(x) = -x^4 - 2x^3 + 6x^2 + 12x$ , domain: $(-\infty, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{x^2+2x}{6-x^2}$, domain: $(-\infty, -\sqrt{6}) \cup (\sqrt{6}, \sqrt{6}) \cup (\sqrt{6}, \infty)$

6) Given $f(x) = -3x^2 + x$ and $g(x) = 5$, find $f+g$, $f-g$, $fg$, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

7) Given $f(x) = 2x^2 + 4x$ and $g(x) = \frac{1}{2x}$ ,find $f+g$, $f-g$, $fg$, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

**Answer**

$(f+g)(x) = \frac{4x^3 + 8x^2 + 1}{2x}$ , domain: $(-\infty, 0) \cup (0, \infty)$

$(f-g)(x) = \frac{4x3 + 8x2 - 1}{2x}$ , domain: $(-\infty, 0) \cup (0, \infty)$

$(fg)(x) = x + 2$ , domain: $(-\infty, 0) \cup (0, \infty)$

$(fg)(x) = 4x^3 + 8x^2$ , domain: $(-\infty, 0) \cup (0, \infty)$

8) Given $f(x)=\dfrac{1}{x-4}$ and $g(x)=\dfrac{1}{6-x}$, find$f+g$, $f-g$, $fg$, and $\dfrac{f}{g}$. Determine the domain for each function in interval notation.

9) Given $f(x)=3x^2$ and $g(x)=\sqrt{x-5}$, find $f+g$, $f-g$, $fg$, and $\dfrac{f}{g}$. Determine the domain for each function in interval notation.

**Answer**

$(f+g)(x)=3x^2+\sqrt{x-5}$ , domain: $[5,\infty)$

$(f-g)(x)=3x^2-\sqrt{x-5}$ , domain: $[5,\infty)$

$(fg)(x)=3x^2\sqrt{x-5}$ , domain: $[5,\infty)$

$\left(\dfrac{f}{g}\right)(x)=\dfrac{3x^2}{\sqrt{x-5}}$ , domain: $(5,\infty)$

10) Given $f(x)=\sqrt{x}$ and $g(x)=|x-3|$, find $\dfrac{g}{f}$. Determine the domain of the function in interval notation.

11) Given $f(x)=2x^2+1$ and $g(x)=3x-5$, find the following:

a. $f(g(2))$
b. $f(g(x))$
c. $g(f(x))$
d. $(g\circ g)(x)$
e. $(f\circ f)(-2)$

**Answer**

a. $3$
b. $f(g(x))=2(3x-5)^2+1$
c. $f(g(x))=6x^2-2$
d. $(g\circ g)(x)=3(3x-5)-5=9x-20$
e. $(f\circ f)(-2)=163$

**For the exercises 12-17, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.**

12) $f(x)=x^2+1$, $g(x)=\sqrt{x+2}$

13) $f(x)=\sqrt{x}+2$, $g(x)=x^2+3$

**Answer**

$f(g(x))=\sqrt{x+3}$ , $g(f(x))=x+4\sqrt{x}+7$

14) $f(x)=|x|$, $g(x)=5x+1$

15) $f(x)=\sqrt[3]{x}$, $g(x)=\dfrac{x+1}{x^3}$

**Answer**

$f(g(x))=\sqrt[3]{\dfrac{x+1}{x^3}}=\dfrac{\sqrt[3]{x+1}}{x}$ , $g(f(x))=\dfrac{\sqrt[3]{x}+1}{x}$

16) $f(x)=\dfrac{1}{x-6}$, $g(x)=\dfrac{7}{x}+6$

17) $f(x)=\dfrac{1}{x-4}$, $g(x)=\dfrac{2}{x}+4$

**Answer**

$(f\circ g)(x)=\dfrac{1}{\frac{2}{x}+4-4}=\dfrac{x}{2}$ , $(g\circ f)(x)=2x-4$

**For the exercises 18-25, use each set of functions to find f(g(h(x))). Simplify your answers.**

18) $f(x)=x^4+6$ , $g(x)=x-6$ , and $h(x)=\sqrt{x}$

19) $f(x)=x^2+1$ , $g(x)=\dfrac{1}{x}$ , and $h(x)=x+3$

**Answer**

$f(g(h(x)))=\left(\dfrac{1}{x+3}\right)^2+1$

20) Given $f(x)=\dfrac{1}{x}$ and $g(x)=x-3$ , find the following:

a. $(f\circ g)(x)$
b. the domain of $(f\circ g)(x)$ in interval notation
c. $(g\circ f)(x)$
d. the domain of $(g\circ f)(x)$
e. $\left(\dfrac{f}{g}\right)x$

21) Given $f(x)=\sqrt{2-4x}$ and $g(x)=-\dfrac{3}{x}$ , find the following:

a. $(g\circ f)(x)$
b. the domain of $(g\circ f)(x)$ in interval notation

**Answer**

a. $(g\circ f)(x)=-\dfrac{3}{\sqrt{2-4x}}$
b. $\left(-\infty,\dfrac{1}{2}\right)$

22) Given the functions $f(x)=\dfrac{1-x}{x}$ and $g(x)=\dfrac{1}{1+x^2}$ , find the following:

a. $(g\circ f)(x)$
b. $(g\circ f)(2)$

23) Given functions $p(x)=\dfrac{1}{\sqrt{x}}$ and $m(x)=x^2-4$ , state the domain of each of the following functions using interval notation:

a. $\dfrac{p(x)}{m(x)}$
b. $p(m(x))$
c. $m(p(x))$

**Answer**

a. $(0,2)\cup(2,\infty)$
b. $(-\infty,-2)\cup(2,\infty)$
c. $(0,\infty)$

24) Given functions $q(x)=\dfrac{1}{\sqrt{x}}$ and $h(x)=x^2-9$ , state the domain of each of the following functions using interval notation.

a. $\dfrac{q(x)}{h(x)}$

b. $q(h(x))$
c. $h(q(x))$

25) For $f(x)=\dfrac{1}{x}$ and $g(x)=\sqrt{x-1}$, write the domain of $(f\circ g)(x)$ in interval notation.

**Answer**

$(1,\infty)$

**For the exercises 26-41, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x)=f(g(x))$.**

26) $h(x)=(x+2)^2$

27) $h(x)=(x-5)^3$

**Answer**

sample: $f(x)=x^3$

$g(x)=x-5$

28) $h(x)=\dfrac{3}{x-5}$

29) $h(x)=\dfrac{4}{(x+2)^2}$

**Answer**

sample: $f(x)=\dfrac{4}{x}$

$g(x)=(x+2)^2$

30) $h(x)=4+\sqrt[3]{x}$

31) $h(x)=\sqrt[3]{\dfrac{1}{2x-3}}$

**Answer**

sample: $f(x)=\sqrt[3]{x}$

$g(x)=\dfrac{1}{2x-3}$

32) $h(x)=\dfrac{1}{(3x^2-4)^{-3}}$

33) $h(x)=\sqrt[4]{\dfrac{3x-2}{x+5}}$

**Answer**

sample: $f(x)=\sqrt[4]{x}$

$g(x)=\dfrac{3x-2}{x+5}$

34) $h(x)=(\dfrac{8+x^3}{8-x^3})^4$

35) $h(x)=\sqrt{2x+6}$

**Answer**

sample: $f(x)=\sqrt{x}$


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$g(x) = 2x + 6$

36) $h(x) = (5x - 1)^3$

37) $h(x) = \sqrt[3]{x - 1}$

**Answer**

sample: $f(x) = \sqrt[3]{x}$
$g(x) = (x - 1)$

38) $h(x) = |x^2 + 7|$

39) $h(x) = \dfrac{1}{(x - 2)^3}$

**Answer**

sample: $f(x) = x^3$
$g(x) = \dfrac{1}{x - 2}$

40) $h(x) = \left(\dfrac{1}{2x - 3}\right)^2$

41) $h(x) = \sqrt{\dfrac{2x - 1}{3x + 4}}$

**Answer**

sample: $f(x) = \sqrt{x}$
$g(x) = \dfrac{2x - 1}{3x + 4}$

## Graphical

**For the exercises 42-49, use the graphs of $f$,shown in Figure 1, and $g$,shown in Figure 2, to evaluate the expressions.**

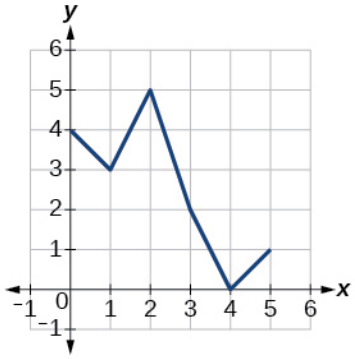



Figure 1: Graph of a function.


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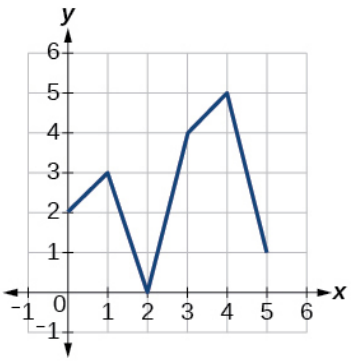



***Figure 2:*** *Graph of a function.*

42) $f(g(3))$

43) $f(g(1))$

**Answer**

2

44) $g(f(1))$

45) $g(f(0))$

**Answer**

5

46) $f(f(5))$

47) $f(f(4))$

**Answer**

4

48) $g(g(2))$

49) $g(g(0))$

**Answer**

0

**For the exercises 50-57, use graphs of $f(x)$,shown in Figure 3, $g(x)$,shown in Figure 4, and $h(x)$,shown in Figure 5, to evaluate the expressions.**

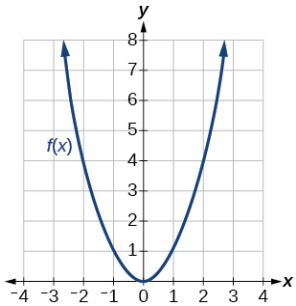



Figure 3: Graph of a parabola.


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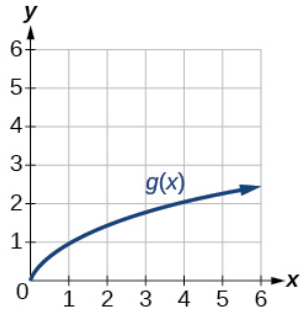



***Figure 4:*** *Graph of a square root function.*

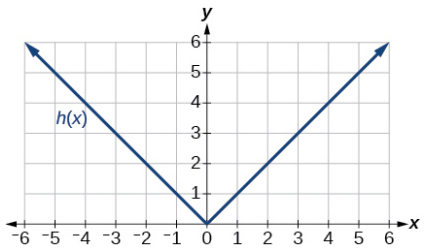



Figure 5: Graph of an absolute value function.

50) $g(f(1))$

51) $g(f(2))$

**Answer**

2

52) $f(g(4))$

53) $f(g(1))$

**Answer**

1

54) $f(h(2))$

55) $h(f(2))$

**Answer**

4

56) $f(g(h(4)))$

57) $f(g(f(-2)))$

**Answer**

4

## Numeric

**For the exercises 58-65, use the function values for $f$ and $g$ shown in the Table below to evaluate each expression.**

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 0 | 7 | 9 |
| 1 | 6 | 5 |
| 2 | 5 | 6 |

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 3 | 8 | 2 |
| 4 | 4 | 1 |
| 5 | 0 | 8 |
| 6 | 2 | 7 |
| 7 | 1 | 3 |
| 8 | 9 | 4 |
| 9 | 3 | 0 |

58) $f(g(8))$

59) $f(g(5))$

**Answer**

9

60) $g(f(5))$

61) $g(f(3))$

**Answer**

4

62) $f(f(4))$

63) $f(f(1))$

**Answer**

2

64) $g(g(2))$

65) $g(g(6))$

**Answer**

3

**For the exercises 66-71, use the function values for $f$ and $g$ shown in the Table below to evaluate the expressions.**

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| -3 | 11 | -8 |
| -2 | 9 | -3 |
| -1 | 7 | 0 |
| 0 | 5 | 1 |
| 1 | 3 | 0 |
| 2 | 1 | -3 |
| 3 | -1 | -8 |

66) $(f \circ g)(1)$

67) $(f\circ g)(2)$

**Answer**

11

68) $(g\circ f)(2)$

69) $(g\circ f)(3)$

**Answer**

0

70) $(g\circ g)(1)$

71) $(f\circ f)(3)$

**Answer**

7

**For the exercises 72-75, use each pair of functions to find $f(g(0))$ and $g(f(0))$.**

72) $f(x)=4x+8$, $g(x)=7-x^2$

73) $f(x)=5x+7$, $g(x)=4-2x^2$

**Answer**

$f(g(0))=27$, $g(f(0))=-94$

74) $f(x)=\sqrt{x+4}$, $g(x)=12-x^3$

75) $f(x)=\frac{1}{x+2}$, $g(x)=4x+3$

**Answer**

$f(g(0))=\frac{1}{5}$, $g(f(0))=5$

**For the exercises 76-79, use the functions $f(x)=2x^2+1$ and $g(x)=3x+5$ to evaluate or find the composite function as indicated.**

76) $f(g(2))$

77) $f(g(x))$

**Answer**

$18x^2+60x+51$

78) $g(f(-3))$

79) $(g\circ g)(x)$

**Answer**

$g\circ g(x)=9x+20$

## Extensions

**For the exercises 80-84, use $f(x)=x^3+1$ and $g(x)=\sqrt[3]{x-1}$.**

80) Find $(f\circ g)(x)$ and $(g\circ f)(x)$. Compare the two answers.

81) Find $(f\circ g)(2)$ and $(g\circ f)(2)$.

**Answer**

2

82) What is the domain of $(g \circ f)(x)$?

83) What is the domain of $(f \circ g)(x)$?

**Answer**

$(-\infty, \infty)$

84) Let $f(x) = \frac{1}{x}$.

a. Find $(f \circ f)(x)$.
b. Is $(f \circ f)(x)$ for any function $f$ the same result as the answer to part (a) for any function? Explain.

**For the exercises 85-87, let $F(x) = (x+1)^5$, $f(x) = x^5$, and $g(x) = x + 1$.**

85) True or False: $(g \circ f)(x) = F(x)$.

**Answer**

False

86) True or False: $(f \circ g)(x) = F(x)$.

**For the exercises 87-89, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x-2}$.**

87) $(f \circ g)(6)$; $(g \circ f)(6)$

**Answer**

$(f \circ g)(6) = 6$; $(g \circ f)(6) = 6$

88) $(g \circ f)(a)$; $(f \circ g)(a)$

89) $(f \circ g)(11)$; $(g \circ f)(11)$

**Answer**

$(f \circ g)(11) = 11$, $(g \circ f)(11) = 11$

## Real-World Applications

90) The function $D(p)$ gives the number of items that will be demanded when the price is $p$. The production cost $C(x)$ is the cost of producing $x$ items. To determine the cost of production when the price is \$6, you would do which of the following?

a. Evaluate $D(C(6))$
b. Evaluate $C(D(6))$
c. Solve $D(C(x)) = 6$
d. Solve $C(D(p)) = 6$

91) The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with $d$ milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after $t$ minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?

a. Evaluate $A(m(4))$
b. Evaluate $m(A(4))$
c. Solve $A(m(t)) = 4$
d. Solve $m(A(d)) = 4$

**Answer**

c

92) A store offers customers a $30\%$ discount on the price $x$ of selected items. Then, the store takes off an additional $15\%$ at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price $x$. (Hint: Use

function composition to find your answer.)

93) A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.

**Answer**

$A(t) = \pi(25\sqrt{t+2})^2$ and $A(2) = \pi(25\sqrt{4})^2 = 2500\pi$ square inches

94) A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t) = 2t + 1$, express the area burned as a function of time, $t$ (minutes).

95) Use the function you found in the previous exercise to find the total area burned after 5 minutes.

**Answer**

$A(5) = \pi(2(5)+1)2 = 121\pi$ square units

96) The radius $r$, in inches, of a spherical balloon is related to the volume, $V$, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after $t$ seconds is given by $V(t) = 10 + 20t$.

a. Find the composite function $r(V(t))$.
b. Find the exact time when the radius reaches 10 inches.

97) The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where $T$ is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where $t$ is the time in hours.

a. Find the composite function $N(T(t))$.
b. Find the time (round to two decimal places) when the bacteria count reaches 6752.

**Answer**

a. $N(T(t)) = 23(5t+1.5)^2 - 56(5t+1.5) + 1$
b. $3.38$hours

## 1.5: Transformation of Functions

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

### Verbal

1) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?

**Answer**

A horizontal shift results when a constant is added to or subtracted from the input. A vertical shifts results when a constant is added to or subtracted from the output.

2) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?

3) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?

**Answer**

A horizontal compression results when a constant greater than 1 is multiplied by the input. A vertical compression results when a constant between 0 and 1 is multiplied by the output.

4) When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the $x$-axis from a reflection with respect to the $y$-axis?

5) How can you determine whether a function is odd or even from the formula of the function?

**Answer**

For a function $f$, substitute (−x) for (x) in $f(x)$. Simplify. If the resulting function is the same as the original function, $f(-x)=f(x)$, then the function is even. If the resulting function is the opposite of the original function, $f(-x)=-f(x)$, then the original function is odd. If the function is not the same or the opposite, then the function is neither odd nor even.

### Algebraic

6) Write a formula for the function obtained when the graph of $f(x)=\sqrt{x}$ is shifted up 1 unit and to the left 2 units.

7) Write a formula for the function obtained when the graph of $f(x)=|x|$ is shifted down 3 units and to the right 1 unit.

**Answer**

$g(x)=|x-1|-3$

8) Write a formula for the function obtained when the graph of $f(x)=\dfrac{1}{x}$ is shifted down 4 units and to the right 3 units.

9) Write a formula for the function obtained when the graph of $f(x)=\dfrac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

**Answer**

$g(x)=\dfrac{1}{(x+4)^2}+2$

**For the exercises 10-19, describe how the graph of the function is a transformation of the graph of the original function $f$.**

10) $y=f(x-49)$

11) $y=f(x+43)$

**Answer**

The graph of $f(x+43)$ is a horizontal shift to the left 43 units of the graph of $f$.

12) $y=f(x+3)$

13) $y=f(x-4)$

**Answer**

The graph of $f(x-4)$ is a horizontal shift to the right 4 units of the graph of $f$.

14) $y=f(x)+5$

15) $y=f(x)+8$

**Answer**

The graph of $f(x)+8$ is a vertical shift up 8 units of the graph of $f$.

16) $y=f(x)-2$

17) $y=f(x)-7$

**Answer**

The graph of $f(x)-7$ is a vertical shift down 7 units of the graph of $f$.

18) $y=f(x-2)+3$

19) $y = f(x+4) - 1$

**Answer**

The graph of $f(x+4)-1$ is a horizontal shift to the left 4 units and a vertical shift down 1 unit of the graph of $f$.

**For the exercises 20-23, determine the interval(s) on which the function is increasing and decreasing.**

20) $f(x) = 4(x+1)^2 - 5$

21) $g(x) = 5(x+3)^2 - 2$

**Answer**

decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$

22) $a(x) = \sqrt{-x+4}$

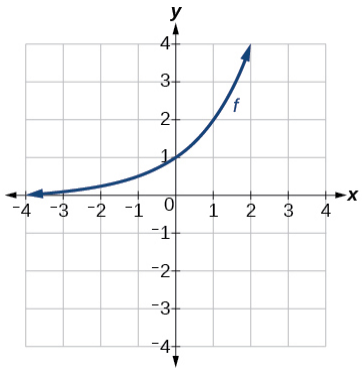
23) $k(x) = -3\sqrt{x} - 1$

**Answer**

decreasing on $(0, \infty)$

## Graphical

**For the exercises 24-26, use the graph of $f(x) = 2^x$ shown in the Figure below to sketch a graph of each transformation of $f(x)$.**



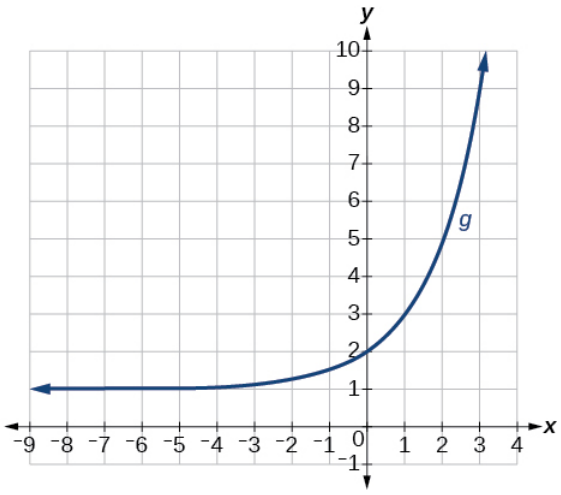

Graph of $f(x)$.

24) $h(x) = 2^x - 3$

25) $g(x) = 2^x + 1$

**Answer**


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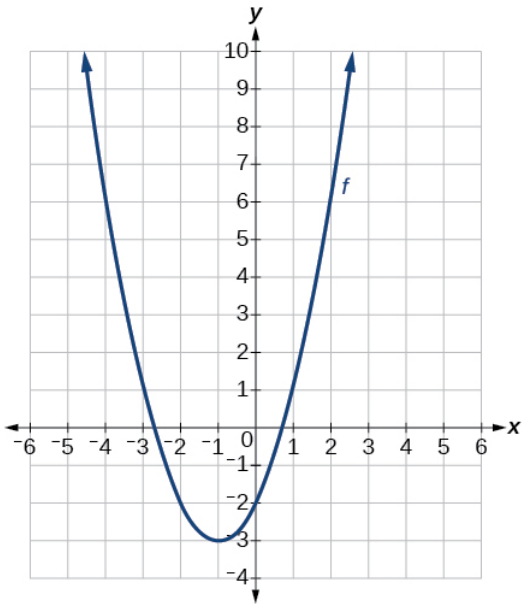

Graph of $g(x)$

26) $w(x) = 2^{x-1}$

**For the exercises 27-30, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.**

27) $f(t) = (t+1)^2 - 3$

**Answer**



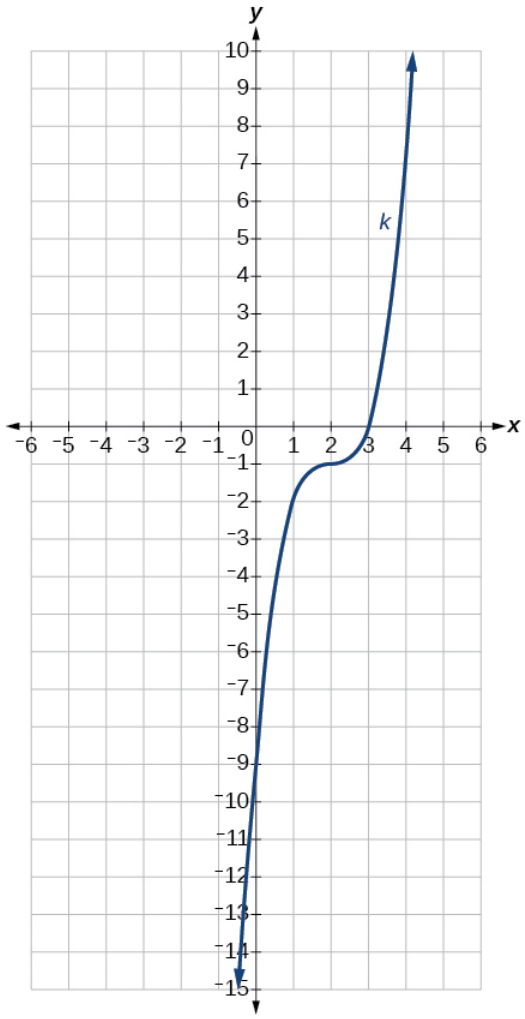

Graph of $f(t)$.

28) $h(x) = |x-1| + 4$

29) $k(x) = (x-2)^3 - 1$

**Answer**


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Graph of $k(x)$.

30) $m(t) = 3 + \sqrt{t+2}$

## Numeric

31) Tabular representations for the functions $f$, $g$, and $h$ are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | -2 | -1 | -3 | 1 | 2 |

| $x$ | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $g(x)$ | -2 | -1 | -3 | 1 | 2 |

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $h(x)$ | -1 | 0 | -2 | 2 | 3 |

**Answer**

$g(x) = f(x-1)$ , $h(x) = f(x)+1$

32) Tabular representations for the functions $f$, $g$, and $h$ are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

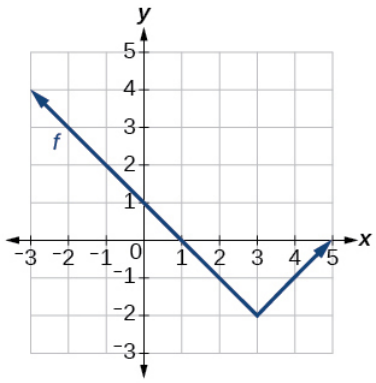
| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | -1 | -3 | 4 | 2 | 1 |

| $x$ | -3 | -2 | -1 | 0 | 1 |
|---|---|---|---|---|---|
| $g(x)$ | -1 | -3 | 4 | 2 | 1 |

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $h(x)$ | -2 | -4 | 3 | 1 | 0 |

**For the exercises 33-40, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.**

33)

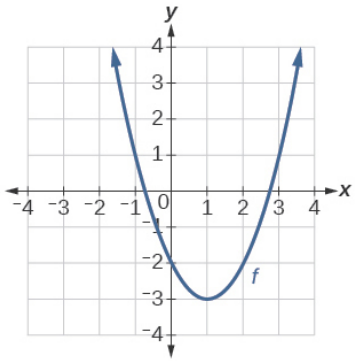



*Graph of an absolute function.*

**Answer**

$f(x) = |x - 3| - 2$

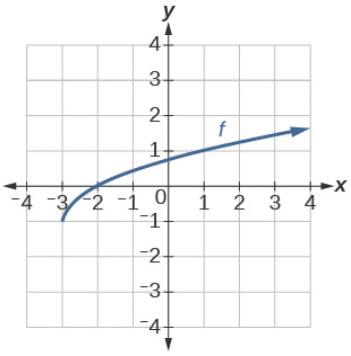
34)




*Graph of a parabola.*

35)


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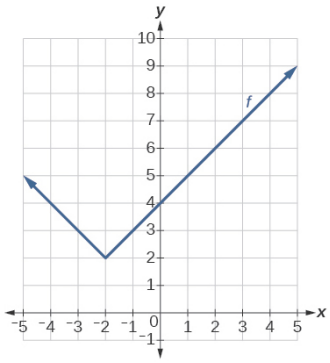



*Graph of a square root function.*

**Answer**

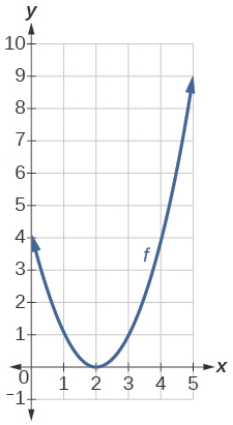
$f(x) = \sqrt{x+3} - 1$

36)




*Graph of an absolute function.*

37)



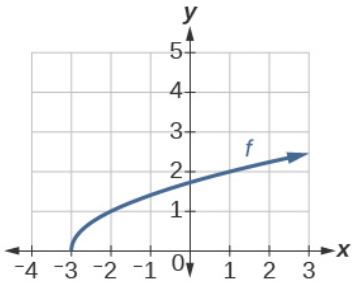

*Graph of a parabola*

**Answer**

$f(x) = (x-2)^2$

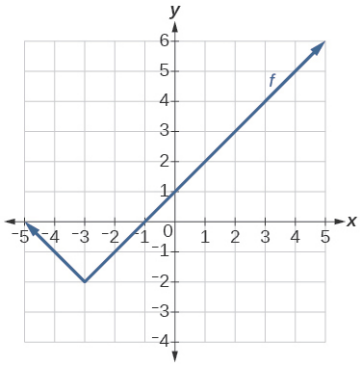
38)


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*Graph of a square root function.*

39)

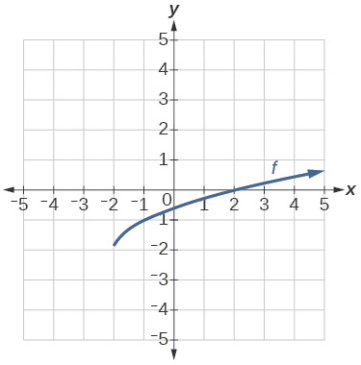



*Graph of an absolute function.*

**Answer**

$f(x) = |x + 3| - 2$

40)

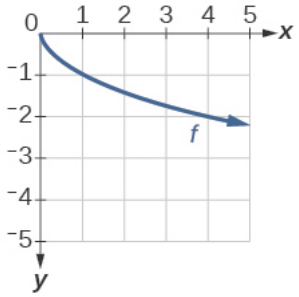



*Graph of a square root function.*

**For the exercises 41-42, use the graphs of transformations of the square root function to find a formula for each of the functions.**

41)


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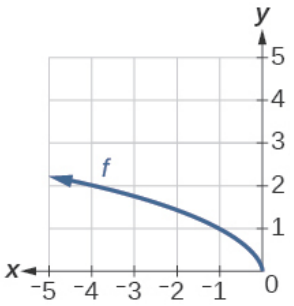



*Graph of a square root function.*

**Answer**

$f(x) = -\sqrt{x}$

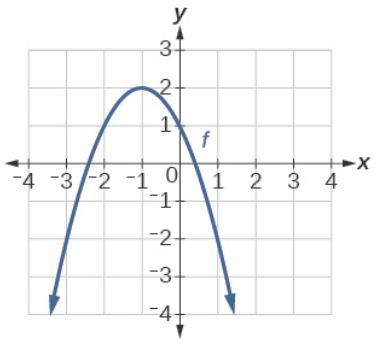
42)




*Graph of a square root function.*

**For the exercises 43-46, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.**

43)



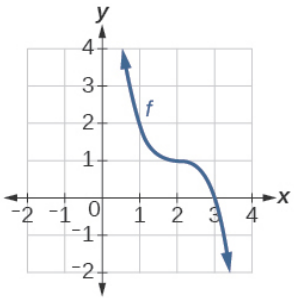

*Graph of a parabola.*

**Answer**

$f(x) = -(x+1)^2 + 2$

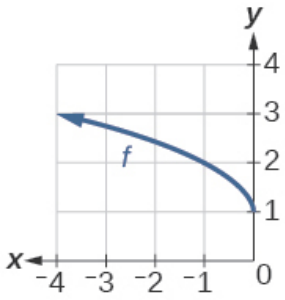
44)


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*Graph of a cubic function.*

45)

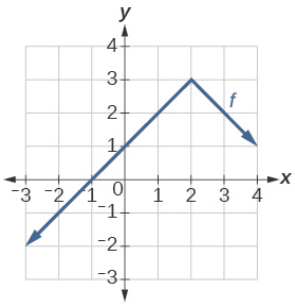



*Graph of a square root function.*

**Answer**

$f(x) = \sqrt{-x} + 1$

46)




*Graph of an absolute function.*

**For the exercises 47-52, determine whether the function is odd, even, or neither.**

47) $f(x) = 3x^4$

**Answer**

even

48) $g(x) = \sqrt{x}$

49) $h(x) = \dfrac{1}{x} + 3x$

**Answer**

odd


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50) $f(x) = (x-2)^2$

51) $g(x) = 2x^4$

**Answer**

even

52) $h(x) = 2x - x^3$

**For the exercises 53-62, describe how the graph of each function is a transformation of the graph of the original function $f$.**

53) $g(x) = -f(x)$

**Answer**

The graph of $g$ is a vertical reflection (across the x-axis) of the graph of $f$.

54) $g(x) = f(-x)$

55) $g(x) = 4f(x)$

**Answer**

The graph of $g$ is a vertical stretch by a factor of 4 of the graph of $f$.

56) $g(x) = 6f(x)$

57) $g(x) = f(5x)$

**Answer**

The graph of $g$ is a horizontal compression by a factor of 15 of the graph of $f$.

58) $g(x) = f(2x)$

59) $g(x) = f\left(\frac{1}{3}x\right)$

**Answer**

The graph of $g$ is a horizontal stretch by a factor of 3 of the graph of $f$.

60) $g(x) = f\left(\frac{1}{5}x\right)$

61) $g(x) = 3f(-x)$

**Answer**

The graph of $g$ is a horizontal reflection across the y-axis and a vertical stretch by a factor of 3 of the graph of $f$.

62) $g(x) = -f(3x)$

**For the exercises 63-68, write a formula for the function $g$ that results when the graph of a given toolkit function is transformed as described.**

63) The graph of $f(x) = |x|$ is reflected over the $y$-axis and horizontally compressed by a factor of $\frac{1}{4}$.

**Answer**

$g(x) = |-4x|$

64) The graph of $f(x) = \sqrt{x}$ is reflected over the $x$-axis and horizontally stretched by a factor of 2.

65) The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.

**Answer**

$$g(x) = \frac{1}{3(x+2)^2} - 3$$

66) The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of $8$, then shifted to the right $4$ units and up $2$ units.

67) The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right $5$ units and up $1$ unit.

**Answer**

$$g(x) = \frac{1}{2}(x-5)^2 + 1$$

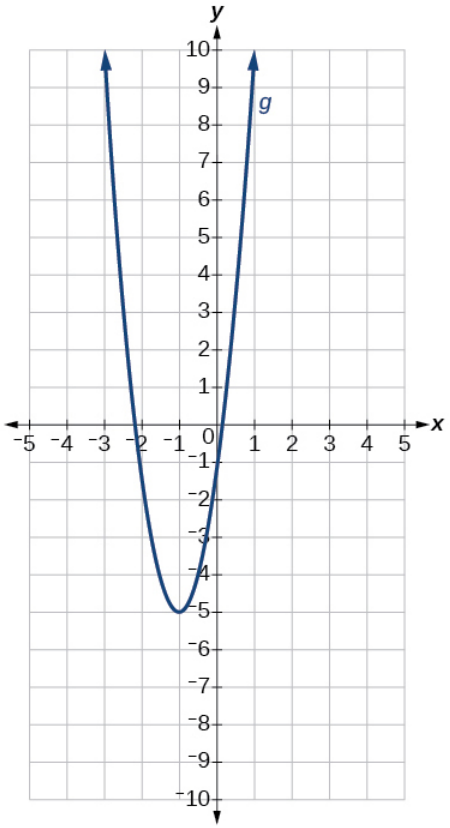
68) The graph of $f(x) = x^2$ is horizontally stretched by a factor of $3$, then shifted to the left $4$ units and down $3$ units.

**For the exercises 69-77, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.**

69) $g(x) = 4(x+1)^2 - 5$

**Answer**

The graph of the function $f(x) = x^2$ is shifted to the left $1$ unit, stretched vertically by a factor of $4$, and shifted down $5$ units.




Graph of a parabola.

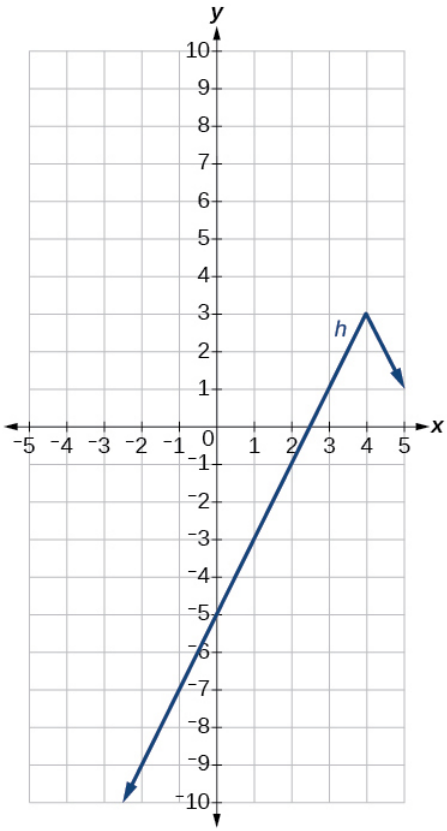
70) $g(x) = 5(x+3)^2 - 2$

71) $h(x) = -2|x-4| + 3$

**Answer**


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The graph of $f(x) = |x|$ is stretched vertically by a factor of 2, shifted horizontally 4 units to the right, reflected across the horizontal axis, and then shifted vertically 3 units up.




Graph of an absolute value function.

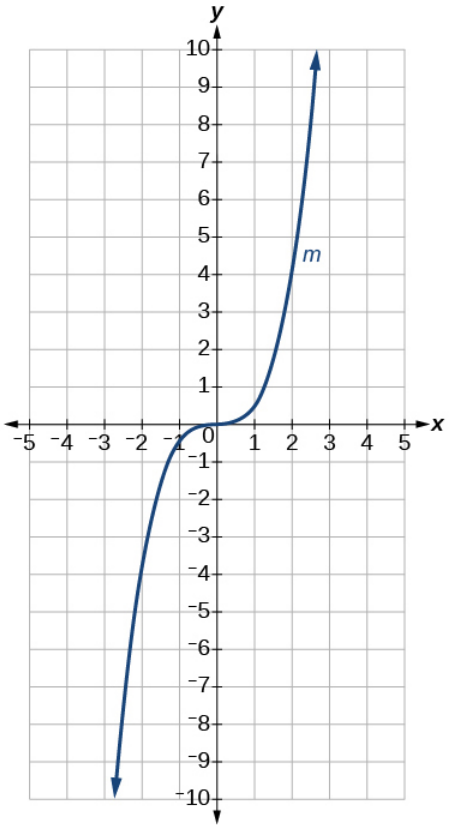
72) $k(x) = -3\sqrt{x} - 1$

73) $m(x) = \frac{1}{2}x^3$

**Answer**

The graph of the function $f(x) = x^3$ is compressed vertically by a factor of $\frac{1}{2}$.


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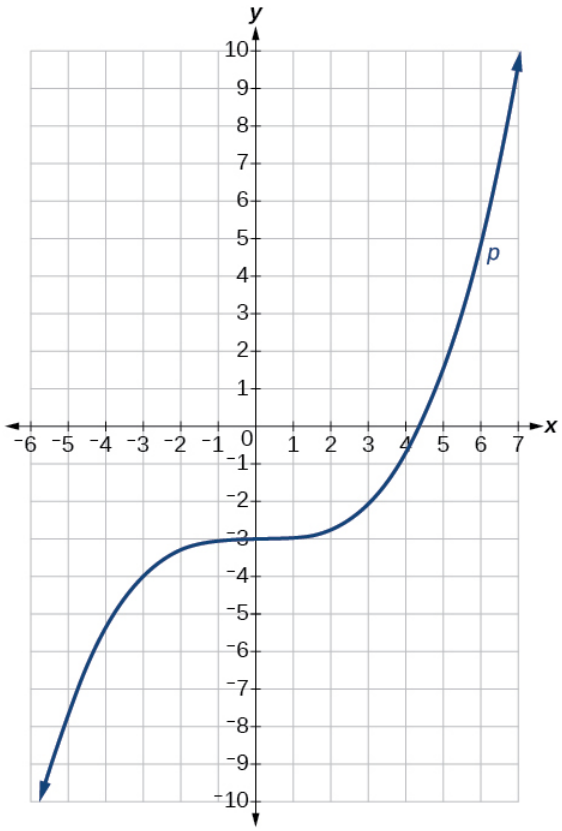

Graph of a cubic function.

74) $n(x) = \frac{1}{3}|x-2|$

75) $p(x) = (\frac{1}{3}x)^3 - 3$

**Answer**

The graph of the function is stretched horizontally by a factor of 3 and then shifted vertically downward by 3 units.




Graph of a cubic function.

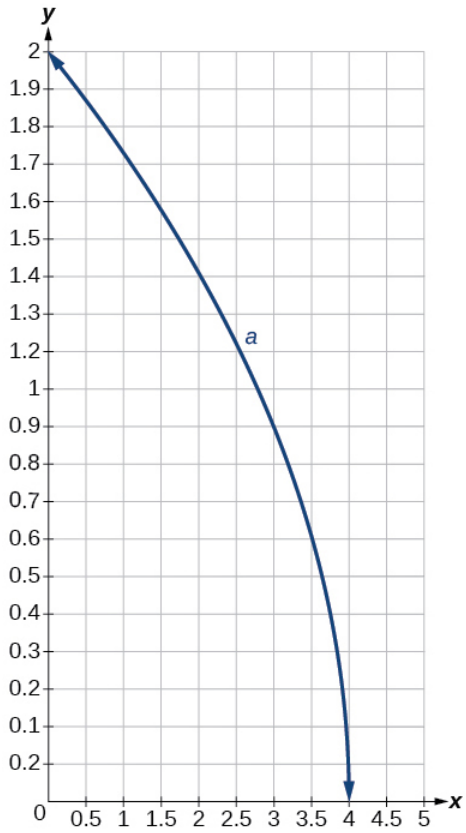
76) $q(x) = (\frac{1}{4}x)^3 + 1$

77) $a(x) = \sqrt{-x+4}$

**Answer**

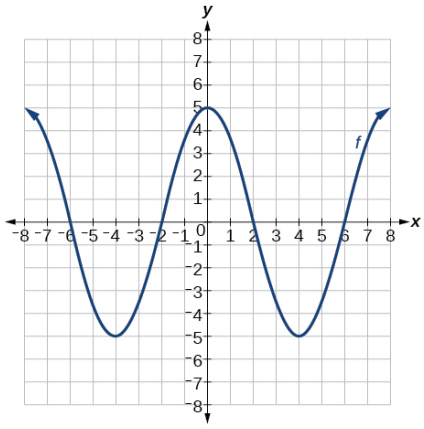
The graph of $f(x) = \sqrt{x}$ is shifted right 4 units and then reflected across the vertical line $x = 4$.


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Graph of a square root function.

**For the exercises 78-81, use the graph in the Figure below to sketch the given transformations.**



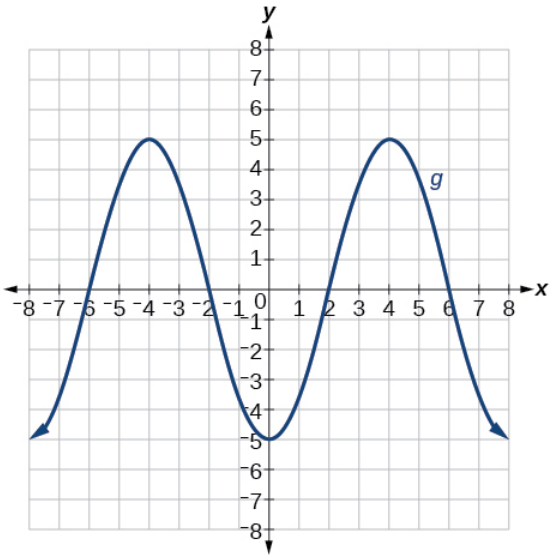

*Graph of a polynomial.*

78) $g(x) = f(x) - 2$

79) $g(x) = -f(x)$

**Answer**


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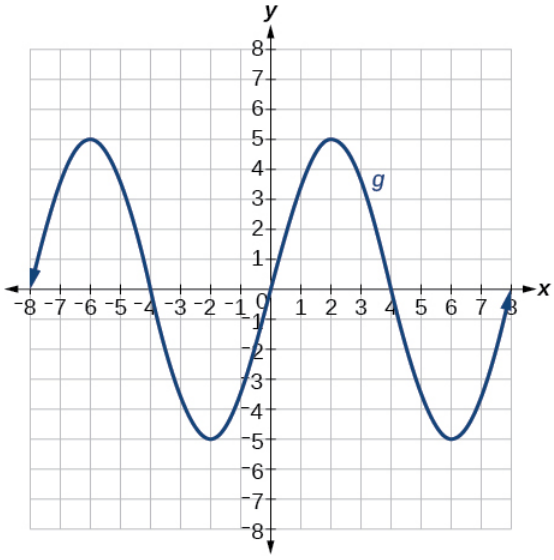



*Graph of a polynomial.*

80) $g(x) = f(x+1)$

81) $g(x) = f(x-2)$

**Answer**




Graph of a polynomial.

## 1.6: Absolute Value Functions

Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate absolute value functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

### Verbal

1) How do you solve an absolute value equation?

**Answer**

Isolate the absolute value term so that the equation is of the form $|A| = B$. Form one equation by setting the expression inside the absolute value symbol, $A$, equal to the expression on the other side of the equation, $B$. Form a second equation by setting $A$ equal to the opposite of the expression on the other side of the equation, $-B$. Solve each equation for the variable.

2) How can you tell whether an absolute value function has two $x$-intercepts without graphing the function?

3) When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?

**Answer**

The graph of the absolute value function does not cross the $x$-axis, so the graph is either completely above or completely below the $x$-axis.

4) How can you use the graph of an absolute value function to determine the $x$-values for which the function values are negative?

5) How do you solve an absolute value inequality algebraically?

**Answer**

First determine the boundary points by finding the solution(s) of the equation. Use the boundary points to form possible solution intervals. Choose a test value in each interval to determine which values satisfy the inequality.

## Algebraic

6) Describe all numbers $x$ that are at a distance of $4$ from the number $8$. Express this using absolute value notation.

7) Describe all numbers $x$ that are at a distance of $\frac{1}{2}$ from the number $-4$. Express this using absolute value notation.

**Answer**

$|x+4| = \frac{1}{2}$

8) Describe the situation in which the distance that point $x$ is from $10$ is at least $15$ units. Express this using absolute value notation.

9) Find all function values $f(x)$ such that the distance from $f(x)$ to the value $8$ is less than $0.03$ units. Express this using absolute value notation.

**Answer**

$|f(x) - 8| < 0.03$

**For the exercises 10-24, solve the equations below and express the answer using set notation.**

10) $|x+3| = 9$

11) $|6-x| = 5$

**Answer**

$1, 11$

12) $|5x-2| = 11$

13) $|4x-2| = 11$

**Answer**

$\left\{\frac{9}{4}, \frac{13}{4}\right\}$

14) $2|4-x| = 7$

15) $3|5-x|=5$

**Answer**

$\left\{\frac{10}{3},\frac{20}{3}\right\}$

16) $3|x+1|-4=5$

17) $5|x-4|-7=2$

**Answer**

$\left\{\frac{11}{5},\frac{29}{5}\right\}$

18) $0=-|x-3|+2$

19) $2|x-3|+1=2$

**Answer**

$\left\{\frac{5}{2},\frac{7}{2}\right\}$

20) $|3x-2|=7$

21) $|3x-2|=-7$

**Answer**

No solution

22) $\left|\frac{1}{2}x-5\right|=11$

23) $\left|\frac{1}{3}x+5\right|=14$

**Answer**

$\{-57,27\}$

24) $-\left|\frac{1}{3}x+5\right|+14=0$

**For the exercises 25-28, find the $x$- and $y$-intercepts of the graphs of each function.**

25) $f(x)=2|x+1|-10$

**Answer**

$(0,-8)$; $(-6,0)$, $(4,0)$

26) $f(x)=4|x-3|+4$

27) $f(x)=-3|x-2|-1$

**Answer**

$(0,-7)$; no $x$-intercepts

28) $f(x)=-2|x+1|+6$

**For the exercises 29-36, solve each inequality and write the solution in interval notation.**

29) $|x-2|>10$

**Answer**

$(-\infty,-8)\cup(12,\infty)$

30) $2|v-7|-4\geq 42$

31) $|3x-4|\geq 8$

**Answer**

$-\frac{4}{3}\leq x\leq 4$

32) $|x-4|\geq 8$

33) $|3x-5|\geq -13$

**Answer**

$\left(-\infty,-\frac{8}{3}\right]\cup[6,\infty)$

34) $|3x-5|\geq -13$

35) $|\frac{3}{4}x-5|\geq 7$

**Answer**

$\left(-\infty,-\frac{8}{3}\right]\cup[16,\infty)$

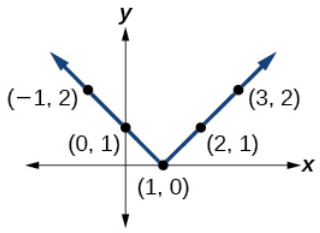
36) $\left|\frac{3}{4}x-5\right|+1\leq 16$

## Graphical

**For the exercises 37-39, graph the absolute value function. Plot at least five points by hand for each graph.**

37) $y=|x-1|$

**Answer**

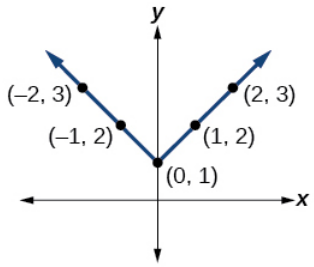



38) $y=|x+1|$

39) $y=|x|+1$

**Answer**


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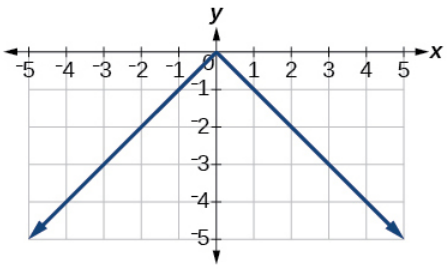



**For the exercises 40-52, graph the given functions by hand.**

40) $y = |x| - 2$

41) $y = -|x|$

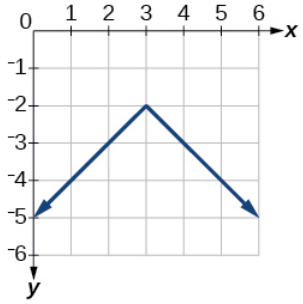
**Answer**




42) $y = -|x| - 2$

43) $y = -|x - 3| - 2$

**Answer**

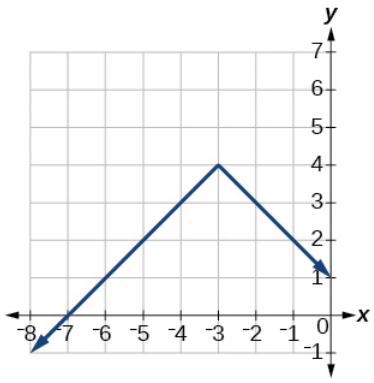



44) $f(x) = -|x - 1| - 2$

45) $f(x) = -|x + 3| + 4$

**Answer**

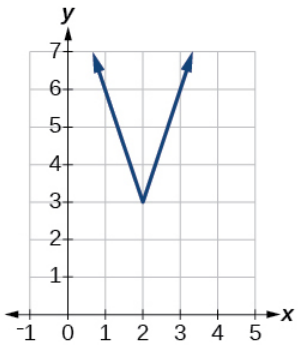

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46) $f(x) = 2|x+3|+1$

47) $f(x) = 3|x-2|+3$

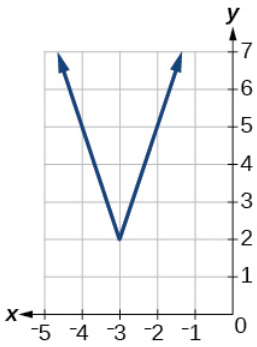
**Answer**




48) $f(x) = |2x-4|-3$

49) $f(x) = |3x+9|+2$

**Answer**

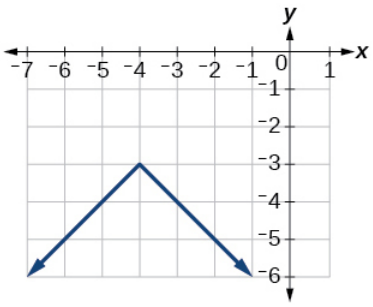



50) $f(x) = -|x-1|-3$

51) $f(x) = -|x+4|-3$

**Answer**


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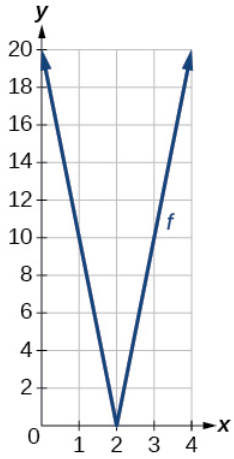

52) $f(x) = \frac{1}{2}|x+4| - 3$

## Technology

53) Use a graphing utility to graph $f(x) = 10|x-2|$ on the viewing window $[0, 4]$. Identify the corresponding range. Show the graph.

**Answer**

range: $[0, 20]$



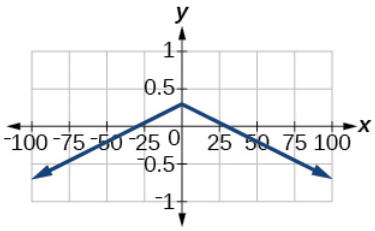

54) Use a graphing utility to graph $f(x) = -100|x| + 100$ on the viewing window $[-5, 5]$. Identify the corresponding range. Show the graph.

**For the exercises 55-56, graph each function using a graphing utility. Specify the viewing window.**

55) $f(x) = -0.1|0.1(0.2 - x)| + 0.3$

**Answer**

$x$-intercepts:




56) $f(x) = 4 \times 10^9|x - (5 \times 10^9)| + 2 \times 10^9$

### Extensions

**For the exercises 57-59, solve the inequality.**

57) $\left|-2x-\frac{2}{3}(x+1)\right|+3>-1$

**Answer**

$(-\infty,\infty)$

58) If possible, find all values of a such that there are no $x$-intercepts for $f(x)=2|x+1|+a$.

59) If possible, find all values of a such that there are no $y$-intercepts for $f(x)=2|x+1|+a$.

**Answer**

There is no solution for a that will keep the function from having a $y$-intercept. The absolute value function always crosses the $y$-intercept when $x=0$.

### Real-World Applications

60) Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least $100$ miles and $x$ represents the distance from city B to city A, express this using absolute value notation.

61) The true proportion $p$ of people who give a favorable rating to Congress is $8\%$ with a margin of error of $1.5\%$. Describe this statement using an absolute value equation.

**Answer**

$|p-0.08|\leq 0.015$

62) Students who score within $18$ points of the number $82$ will pass a particular test. Write this statement using absolute value notation and use the variable $x$ for the score.

63) A machinist must produce a bearing that is within $0.01$ inches of the correct diameter of $5.0$ inches. Using $x$ as the diameter of the bearing, write this statement using absolute value notation.

**Answer**

$|x-5.0|\leq 0.01$

64) The tolerance for a ball bearing is $0.01$. If the true diameter of the bearing is to be $2.0$ inches and the measured value of the diameter is $x$ inches, express the tolerance using absolute value notation.

## 1.7: Inverse Functions

If some physical machines can run in two directions, we might ask whether some of the function “machines” we have been studying can also run backwards. In this section, we will consider the reverse nature of functions.

### Verbal

1) Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

**Answer**

Each output of a function must have exactly one output for the function to be one-to-one. If any horizontal line crosses the graph of a function more than once, that means that $y$-values repeat and the function is not one-to-one. If no horizontal line crosses the graph of the function more than once, then no $y$-values repeat and the function is one-to-one.

2) Why do we restrict the domain of the function $f(x)=x^2$ to find the function’s inverse?

3) Can a function be its own inverse? Explain.

**Answer**

Yes. For example, $f(x)=\dfrac{1}{x}$ is its own inverse.

4) Are one-to-one functions either always increasing or always decreasing? Why or why not?

5) How do you find the inverse of a function algebraically?

**Answer**

Given a function $y=f(x)$, solve for $x$ in terms of $y$. Interchange the $x$ and $y$. Solve the new equation for $y$. The expression for $y$ is the inverse, $y=f^{-1}(x)$.

### Algebraic

6) Show that the function $f(x)=a-x$ is its own inverse for all real numbers $a$.

**For the exercises 7-12, find $f^{-1}(x)$ for each function.**

7) $f(x)=x+3$

**Answer**

$f^{-1}(x)=x-3$

8) $f(x)=x+5$

9) $f(x)=2-x$

**Answer**

$f^{-1}(x)=2-x$

10) $f(x)=3-x$

11) $f(x)=\dfrac{x}{x+2}$

**Answer**

$f^{-1}(x)=\dfrac{-2x}{x-1}$

12) $f(x)=\dfrac{2x+3}{5x+4}$

**For the exercises 13-16, find a domain on which each function $f$ is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of $f$ restricted to that domain.**

13) $f(x)=(x+7)^2$

**Answer**

domain of $f(x)$: $[-7,\infty)$; $f^{-1}(x)=\sqrt{x}-7$

14) $f(x)=(x-6)^2$

15) $f(x)=x^2-5$

**Answer**

domain of $f(x)$: $[0,\infty)$; $f^{-1}(x)=\sqrt{x+5}$

16) Given $f(x)=\dfrac{x}{2+x}$ and $g(x)=\dfrac{2x}{1-x}$:

a. Find $f(g(x))$ and $g(f(x))$.

b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

**Answer**

a. $f(g(x)) = x$ and $g(f(x)) = x$

b. This tells us that $f$ and $g$ are inverse functions

**For the exercises 17-18, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.**

17) $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$

**Answer**

$f(g(x)) = x$ , $g(f(x)) = x$

18) $f(x) = -3x + 5$ and $g(x) = \dfrac{x-5}{-3}$

## Graphical

**For the exercises 19-22, use a graphing utility to determine whether each function is one-to-one.**

19) $f(x) = \sqrt{x}$

**Answer**

one-to-one

20) $f(x) = \sqrt[3]{3x+1}$

21) $f(x) = -5x + 1$

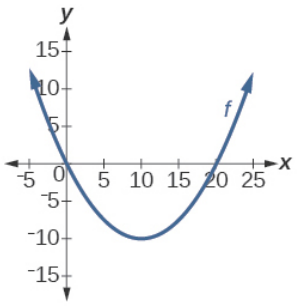
**Answer**

one-to-one

22) $f(x) = x^3 - 27$

**For the exercises 23-24, determine whether the graph represents a one-to-one function.**

23)




**Answer**

not one-to-one

24)


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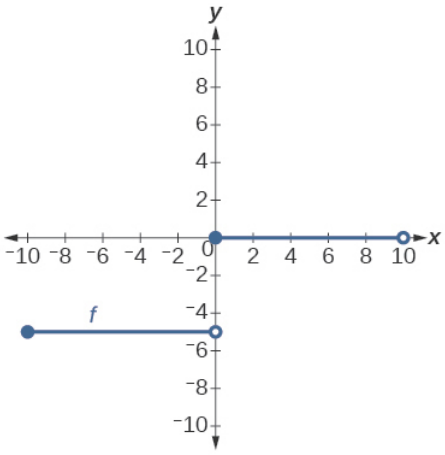
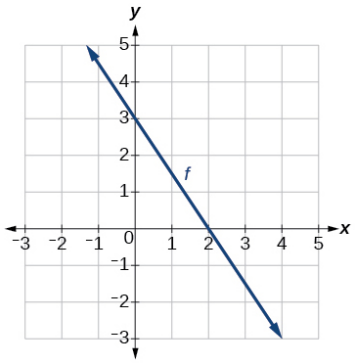



Figure below.




*Graph of a line*

25) Find $f(0)$.

**Answer**

3

26) Solve $f(x) = 0$.

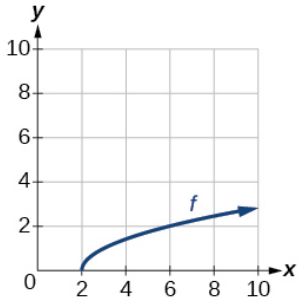
27) Find $f^{-1}(0)$.

**Answer**

2

28) Solve $f^{-1}(x) = 0$.

**For the exercises 29-32, use the graph of the one-to-one function shown in the Figure below**

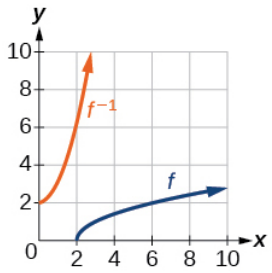



*Graph of a square root function.*

29) Sketch the graph of $f^{-1}$.


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**Answer**




30) Find $f(6)$ and $f^{-1}(2)$.

31) If the complete graph of $f$ is shown, find the domain of $f$.

**Answer**

$[2, 10]$

32) If the complete graph of $f$ is shown, find the range of $f$

## Numeric

**For the exercises 33-36, evaluate or solve, assuming that the function $f$ is one-to-one.**

33) If $f(6) = 7$, find $f^{-1}(7)$.

**Answer**

6

34) If $f(3) = 2$, find $f^{-1}(2)$.

35) If $f^{-1}(-4) = -8$, find $f(-8)$.

**Answer**

$-4$

36) If $f^{-1}(-2)$=$-1$, find $f(-1)$.

**For the exercises 37-40, use the values listed in the Table below to evaluate or solve.**

| $x$ | $f(x)$ |
|---|---|
| 0 | 8 |
| 1 | 0 |
| 2 | 7 |
| 3 | 4 |
| 4 | 2 |
| 5 | 6 |
| 6 | 5 |
| 7 | 3 |
| 8 | 9 |
| 9 | 1 |

37) Find $f(1)$.


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**Answer**

0

38) Solve $f(x) = 3$.

39) Find $f^{-1}(0)$.

**Answer**

1

40) Solve $f^{-1}(x) = 7$.

41) Use the tabular representation of $f$ in the Table below to create a table for $f^{-1}(x)$.

| $x$ | 3 | 6 | 9 | 13 | 14 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 4 | 7 | 12 | 16 |

**Answer**

| $x$ | 1 | 4 | 7 | 12 | 16 |
|---|---|---|---|---|---|
| $f^{-1}(x)$ | 3 | 6 | 9 | 13 | 14 |

## Technology

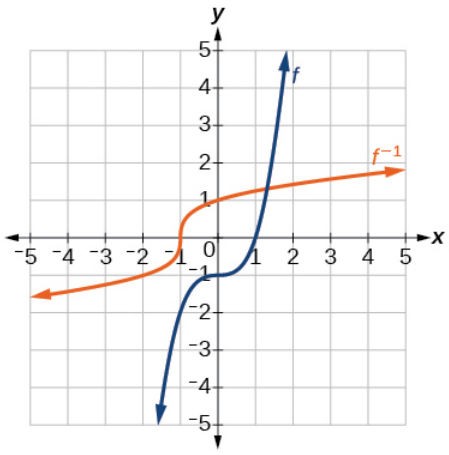
**For the exercises 42-44, find the inverse function. Then, graph the function and its inverse.**

42) $f(x) = \dfrac{3}{x-2}$

43) $f(x) = x^3 - 1$

**Answer**

$f^{-1}(x) = (1+x)^{1/3}$




44) Find the inverse function of $f(x) = \dfrac{1}{x-1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

## Real-World Applications

45) To convert from $x$ degrees Celsius to $y$ degrees Fahrenheit, we use the formula $f(x) = \dfrac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.

**Answer**

$f^{-1}(x) = \frac{5}{9}(x - 32)$ . Given the Fahrenheit temperature, $x$, this formula allows you to calculate the Celsius temperature.

46) The circumference $C$ of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.

47) A car travels at a constant speed of $50$ miles per hour. The distance the car travels in miles is a function of time, $t$, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

**Answer**

$t(d) = \frac{d}{50}$ , $t(180) = \frac{180}{50}$. The time for the car to travel $180$ miles is $3.6$ hours.

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# 2.R: Functions (Review)

This page is a draft and is under active development.

## 1.1: Functions and Function Notation

**For the exercises 1-4, determine whether the relation is a function.**

1) $\{(a,b),(c,d),(e,d)\}$

**Answer**

function

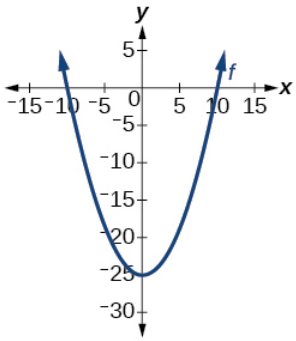
2) $\{(5,2),(6,1),(6,2),(4,8)\}$

3) $y^2+4=x$ , for $x$ the independent variable and $y$ the dependent variable

**Answer**

not a function

4) Is the graph in the Figure below a function?




**For the exercises 5-6, evaluate the function at the indicated values: $f(-3); f(2); f(-a); -f(a); f(a+h)$**

5) $f(x)=-2x^2+3x$

**Answer**

$f(-3)=-27; f(2)=-2; f(-a)=-2a^2-3a; -f(a)=2a^2-3a; f(a+h)=-2a^2+3a-4ah+3h-2h^2$

6) $f(x)=2|3x-1|$

**For the exercises 7-8, determine whether the functions are one-to-one.**

7) $f(x)=-3x+5$

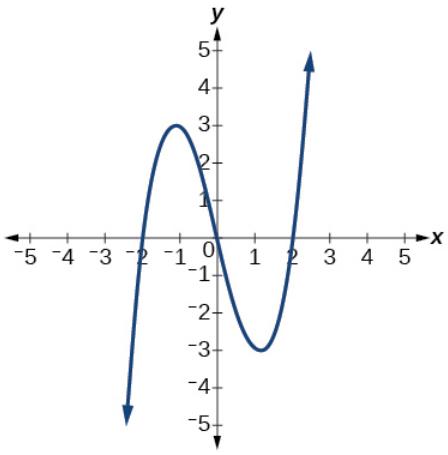
**Answer**

one-to-one

8) $f(x)=|x-3|$

**For the exercises 9-11, use the vertical line test to determine if the relation whose graph is provided is a function.**

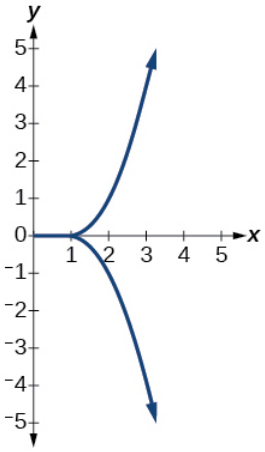
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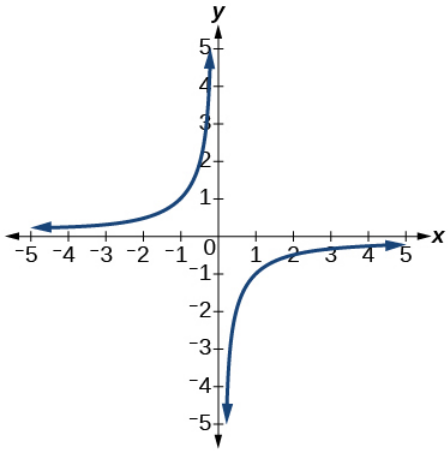

**Answer**

function

10)




11)




**Answer**

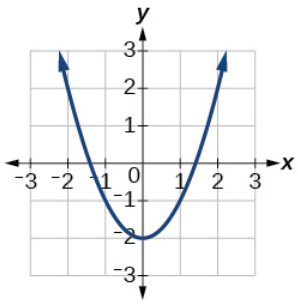
function

**For the exercises 12-13, graph the functions.**

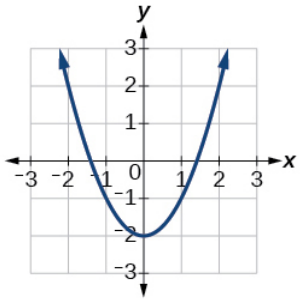
12) $f(x) = |x+1|$

13) $f(x) = x^2 - 2$

**Answer**




**For the exercises 14-17, use the Figure below to approximate the values.**




14) $f(2)$

15) $f(-2)$

**Answer**

2

16) If $f(x) = -2$, then solve for $x$

17) If $f(x) = 1$, then solve for $x$

**Answer**

$x = -1.8$ or $x = 1.8$

**For the exercises 18-19, use the function $h(t) = -16t^2 + 80t$ to find the values.**

18) $\dfrac{h(2) - h(1)}{2 - 1}$

19) $\dfrac{h(a) - h(1)}{a - 1}$

**Answer**

$$\frac{-64 + 80a - 16a^2}{-1 + a} = -16a + 64$$

## 1.2: Domain and Range

**For the exercises 1-4, find the domain of each function, expressing answers using interval notation.**

1) $f(x) = \dfrac{2}{3x + 2}$

2) $f(x) = \frac{x-3}{x^2-4x-12}$

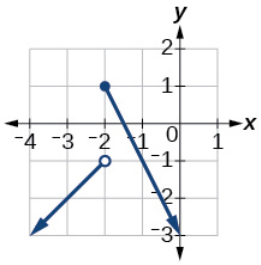
**Answer**

$(-\infty, -2) \cup (-2, 6) \cup (6, \infty)$

3)

4) Graph this piecewise function: $f(x) = \begin{cases} x+1 & x < -2 \\ -2x-3 & x \geq -2 \end{cases}$

**Answer**




## 1.3: Rates of Change and Behavior of Graphs

**For the exercises 1-3, find the average rate of change of the functions from $x = 1$ to $x = 2$**

1) $f(x) = 4x - 3$

2) $f(x) = 10x^2 + x$

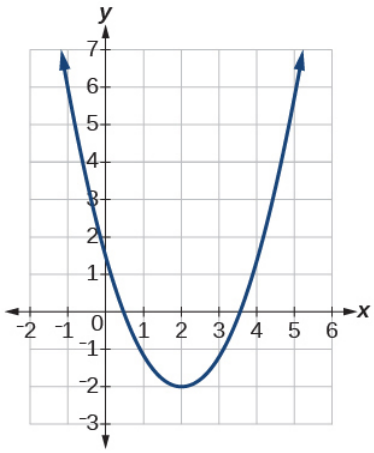
**Answer**

31

3) $f(x) = -\frac{2}{x^2}$

**For the exercises 4-6, use the graphs to determine the intervals on which the functions are increasing, decreasing, or constant.**

4)

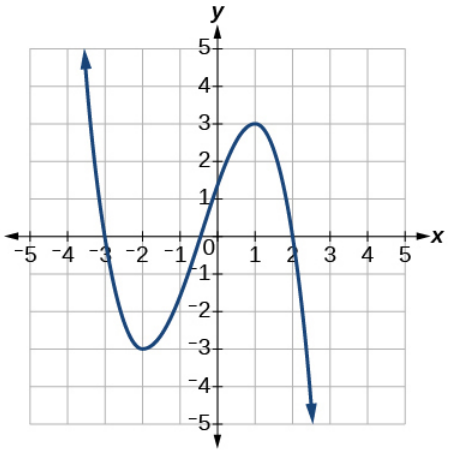



**Answer**

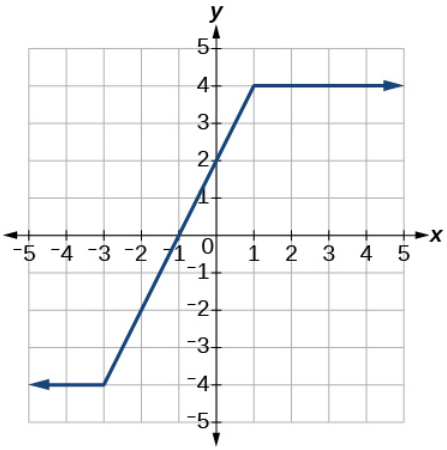
increasing $(2, \infty)$; decreasing $(-\infty, 2)$

5)


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6)




**Answer**

increasing $(-3, 1)$; constant $(-\infty, -3) \cup (1, \infty)$

7) Find the local minimum of the function graphed in Exercise 4.

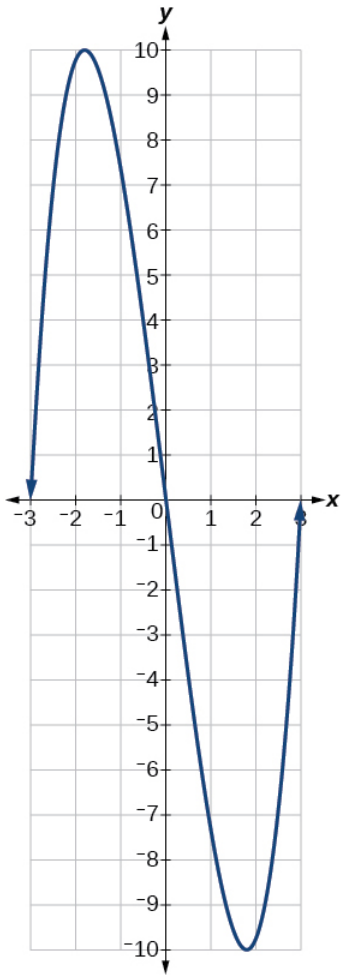
8) Find the local extrema for the function graphed in Exercise 5.

**Answer**

local minimum $(-2, -3)$; local maximum $(1, 3)$

9) For the graph in the Figure in Exercise 10, the domain of the function is $[-3, 3]$. The range is $[-10, 10]$. Find the absolute minimum of the function on this interval.

10) Find the absolute maximum of the function graphed in the Figure below.




**Answer**

$(-1.8, 10)$

## 1.4: Composition of Functions

**For the exercises 1-5, find $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions.**

1) $f(x) = 4 - x, g(x) = -4x$

2) $f(x) = 3x + 2, g(x) = 5 - 6x$

**Answer**

$(f \circ g)(x) = 17 - 18x; (g \circ f)(x) = -7 - 18x$

3) $f(x) = x^2 + 2x, g(x) = 5x + 1$

4) $f(x) = \sqrt{x+2}, g(x) = \dfrac{1}{x}$

**Answer**

$(f \circ g)(x) = \sqrt{\dfrac{1}{x} + 2}; (g \circ f)(x) = \dfrac{1}{\sqrt{x+2}}$

5) $f(x) = \dfrac{x+3}{2}, g(x) = \sqrt{1-x}$

**For the exercises 6-9, find $(f \circ g)$ and the domain for $(f \circ g)(x)$ for each pair of functions.**

6) $f(x)=\frac{x+1}{x+4}, g(x)=\frac{1}{x}$

**Answer**

$(f\circ g)(x)=\dfrac{1+x}{1+4x}, x\neq 0, x\neq -\dfrac{1}{4}$

7) $f(x)=\dfrac{1}{x+3}, g(x)=\dfrac{1}{x-9}$

8) $f(x)=\dfrac{1}{x}, g(x)=\sqrt{x}$

**Answer**

$(f\circ g)(x)=\frac{1}{\sqrt{x}}, x>0$

9) $f(x)=\frac{1}{x^2-1}, g(x)=\sqrt{x+1}$

**For the exercises 10-11, express each function $H$ as a composition of two functions $f$ and $g$ where $H(x)=(f\circ g)(x)$**

10) $H(x)=\sqrt{\frac{2x-1}{3x+4}}$

**Answer**

sample: $g(x)=\dfrac{2x-1}{3x+4}; f(x)=\sqrt{x}$

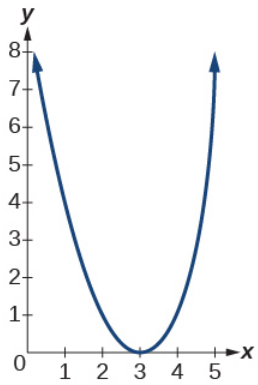
11) $H(x)=\dfrac{1}{(3x^2-4)^{-3}}$

## 1.5: Transformation of Functions

**For the exercises 1-8, sketch a graph of the given function.**

1) $f(x)=(x-3)^2$

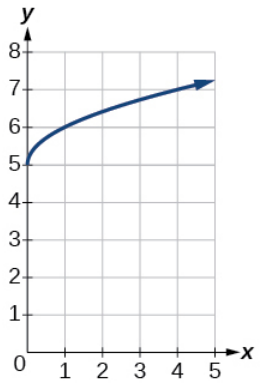
**Answer**




2) $f(x)=(x+4)^3$

3) $f(x)=\sqrt{x}+5$

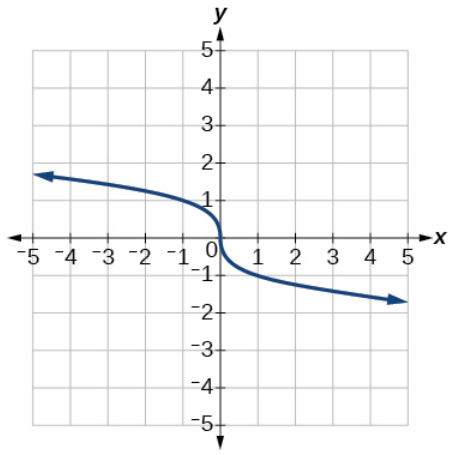
**Answer**




4) $f(x) = -x^3$

5) $f(x) = \sqrt[3]{-x}$

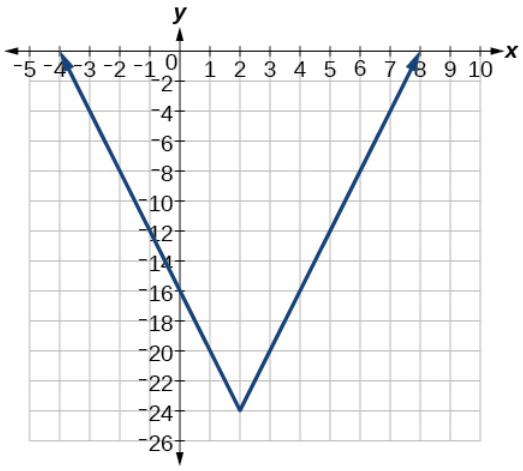
**Answer**




6) $f(x) = 5\sqrt{-x} - 4$

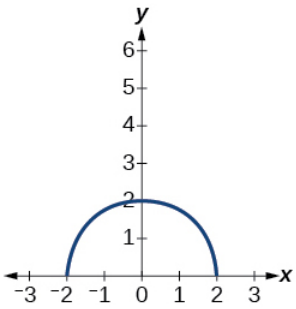
7) $f(x) = 4[|x-2| - 6]$

**Answer**



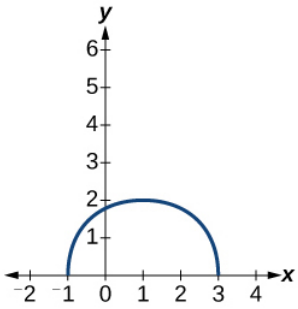

8) $f(x) = -(x+2)^2 - 1$

**For the exercises 9-10, sketch the graph of the function $g$ if the graph of the function $f$ is shown in the Figure below.**




9) $g(x) = f(x-1)$

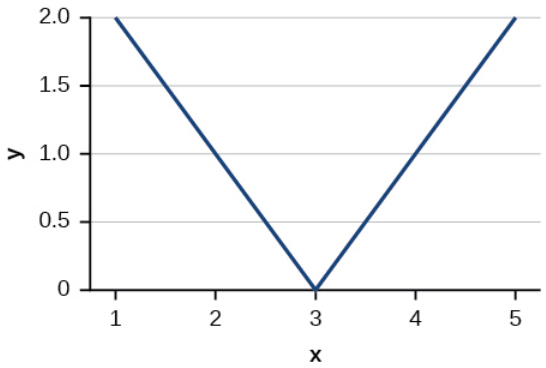
**Answer**




10) $g(x) = 3f(x)$

**For the exercises 11-12, write the equation for the standard function represented by each of the graphs below.**

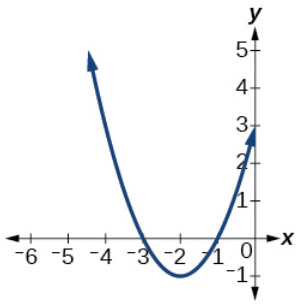
11)




**Answer**

$f(x) = |x-3|$

12)



**For the exercises 13-15, determine whether each function below is even, odd, or neither.**

13) $f(x) = 3x^4$

**Answer**

even

14) $g(x) = \sqrt{x}$

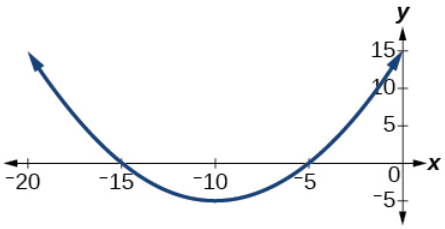
15) $h(x) = \frac{1}{x} + 3x$

**Answer**

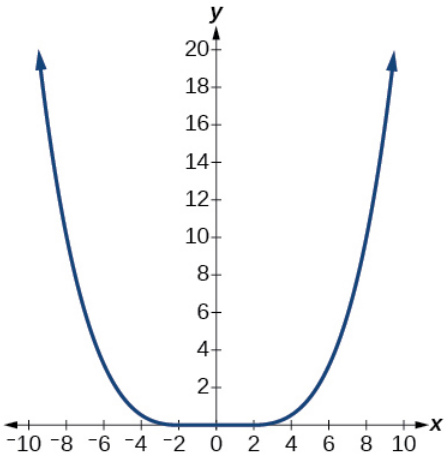
odd

**For the exercises 16-18, analyze the graph and determine whether the graphed function is even, odd, or neither.**

16)




17)

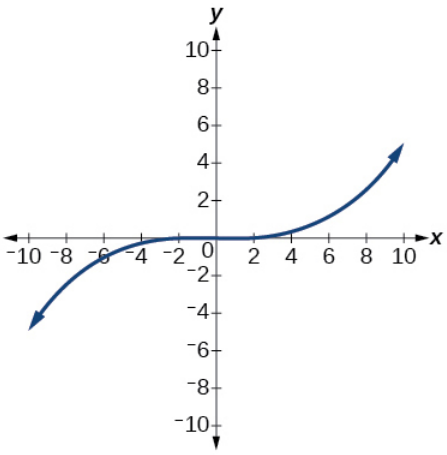



**Answer**

even

18)

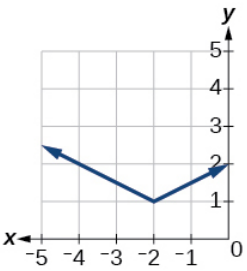

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## 1.6: Absolute Value Functions

**For the exercises 1-3, write an equation for the transformation of $f(x) = |x|$.**

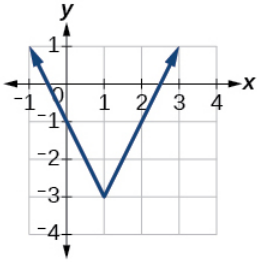
1)



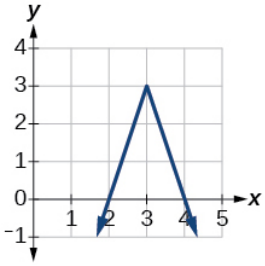

**Answer**

$$f(x) = \frac{1}{2}|x+2|+1$$

2)




3)




**Answer**

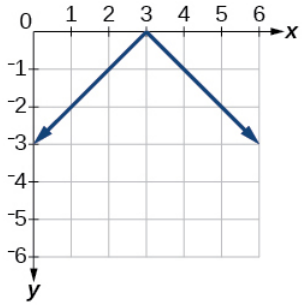
$$f(x) = -3|x-3|+3$$

**For the exercises 4-6, graph the absolute value function.**

4) $f(x) = |x-5|$

5) $f(x) = -|x-3|$

**Answer**




6) $f(x) = |2x-4|$

**For the exercises 7-8, solve the absolute value equation.**

7) $|x+4| = 18$

**Answer**

$$x = -22, x = 14$$

8) $\left|\frac{1}{3}x+5\right| = \left|\frac{3}{4}x-2\right|$

**For the exercises 9-10, solve the inequality and express the solution using interval notation.**

9) $|3x-2| < 7$

**Answer**

$$\left(-\frac{5}{3}, 3\right)$$

10) $\left|\frac{1}{3}x-2\right| \leq 7$

## 1.7: Inverse Functions

**For the exercises 1-2, find $f^{-1}(x)$ for each function.**

1) $f(x) = 9+10x$

2) $f(x) = \dfrac{x}{x+2}$

**Answer**

$$f^{-1}(x) = \frac{-2x}{x-1}$$

3) For the following exercise, find a domain on which the function $f$ is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of $f$ restricted to that domain.

$$f(x) = x^2+1 \tag{2.R.1}$$

4) Given $f(x) = x^3-5$ and $g(x) = \sqrt[3]{x+5}$ :

a. Find $f(g(x))$ and $g(f(x))$.
b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

**Answer**

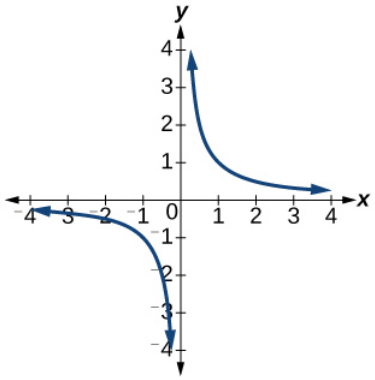
a. $f(g(x)) = x$ and $g(f(x)) = x$
b. This tells us that $f$ and $g$ are inverse functions

**For the exercises 5-8, use a graphing utility to determine whether each function is one-to-one.**

5) $f(x) = \dfrac{1}{x}$

**Answer**

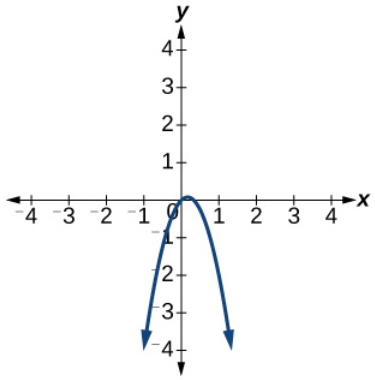
The function is one-to-one.




6) $f(x) = -3x^2 + x$

**Answer**

The function is not one-to-one.




7) If $f(5) = 2$, find $f^{-1}(2)$

**Answer**

5

8) If $f(1) = 4$, find $f^{-1}(4)$

## Practice Test

**For the exercises 1-2, determine whether each of the following relations is a function.**

1) $y = 2x + 8$

**Answer**

The relation is a function.

2) $\{(2,1),(3,2),(-1,1),(0,-2)\}$

**For the exercises 3-4, evaluate the function $f(x)=-3x^2+2x$ at the given input.**

3) $f(-2)$

**Answer**

$-16$

4) $f(a)$

5) Show that the function $f(x)=-2(x-1)^2+3$ is not one-to-one.

**Answer**

The graph is a parabola and the graph fails the horizontal line test.

6) Write the domain of the function $f(x)=\sqrt{3-x}$ in interval notation.

7) Given $f(x)=2x^2-5x$, find $f(a+1)-f(1)$

**Answer**

$2a^2-a$

8) Graph the function $f(x)=\begin{cases} x+1 & \text{if} \quad -2<x<3 \\ -x & \text{if} \quad x\geq 3 \end{cases}$

9) Find the average rate of change of the function $f(x)=3-2x^2+x$ by finding $\dfrac{f(b)-f(a)}{b-a}$

**Answer**

$-2(a+b)+1$

**For the exercises 10-11, use the functions $f(x)=3-2x^2+x$ and $g(x)=\sqrt{x}$ to find the composite functions.**

10) $(g\circ f)(x)$

11) $(g\circ f)(1)$

**Answer**

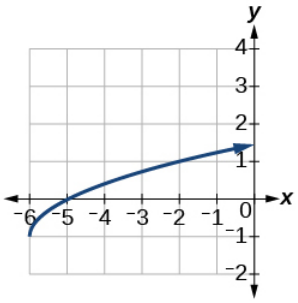
$\sqrt{2}$

12) Express $H(x)=\sqrt[3]{5x^2-3x}$ a composition of two functions, $f$ and $g$, where $(f\circ g)(x)=H(x)$

**For the exercises 13-14, graph the functions by translating, stretching, and/or compressing a toolkit function.**

13) $f(x)=\sqrt{x+6}-1$

**Answer**




14) $f(x) = \dfrac{1}{x+2} - 1$

**For the exercises 15-17, determine whether the functions are even, odd, or neither.**

15) $f(x) = -\dfrac{5}{x^2} + 9x^6$

**Answer**

even

16) $f(x) = -\dfrac{5}{x^3} + 9x^5$

17) $f(x) = \dfrac{1}{x}$

**Answer**

odd

18) Graph the absolute value function $f(x) = -2|x-1| + 3$.

19) Solve $|2x-3| = 17$.

**Answer**

$x = -7$ and $x = 10$

20) Solve $-\left|\dfrac{1}{3}x - 3\right| \geq 17$. Express the solution in interval notation.

**For the exercises 21-22, find the inverse of the function.**

21) $f(x) = 3x - 5$

**Answer**

$f^{-1}(x) = \dfrac{x+5}{3}$

22) $f(x) = \dfrac{4}{x+7}$

**For the exercises 23-26, use the graph of *g* shown in the Figure below.**

23) On what intervals is the function increasing?

**Answer**

$(-\infty, -1.1)$ and $(1.1, \infty)$

24) On what intervals is the function decreasing?

25) Approximate the local minimum of the function. Express the answer as an ordered pair.

**Answer**

$(1.1, -0.9)$

26) Approximate the local maximum of the function. Express the answer as an ordered pair.

**For the exercises 27-29, use the graph of the piecewise function shown in the Figure below.**

27) Find $f(2)$.

**Answer**

$f(2) = 2$

28) Find $f(-2)$.

29) Write an equation for the piecewise function.

**Answer**

$$f(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$$

**For the exercises 30-35, use the values listed in the Table below.**

| $x$ | $F(x)$ |
|---|---|
| 0 | 1 |
| 1 | 3 |
| 2 | 5 |
| 3 | 7 |
| 4 | 9 |
| 5 | 11 |
| 6 | 13 |
| 7 | 15 |
| 8 | 17 |

30) Find $F(6)$.

31) Solve the equation $F(x) = 5$

**Answer**

$x = 2$

32) Is the graph increasing or decreasing on its domain?

33) Is the function represented by the graph one-to-one?

**Answer**

yes

34) Find $F^{-1}(15)$.

35) Given $f(x) = -2x + 11$, find $f^{-1}(x)$.

**Answer**

$$f^{-1}(x) = -\frac{x-11}{2}$$

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# CHAPTER OVERVIEW

## 3: Polynomial and Rational Functions

In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.


3.0: Prelude to Polynomial and Rational Functions
3.1: Complex Numbers Review
3.2: Quadratic Functions Review
3.3: Power Functions and Polynomial Functions Behavior
3.4: Graphs of Polynomial Functions
3.5: Dividing Polynomials
3.6: Zeros of Polynomial Functions
3.7: Rational Functions
3.8: Inverses and Radical Functions
3.9: Modeling Using Variation (optional)
3.E: Polynomial and Rational Functions(Exercises)
3.R: Polynomial and Rational Functions(Review)


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# 3.0: Prelude to Polynomial and Rational Functions

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

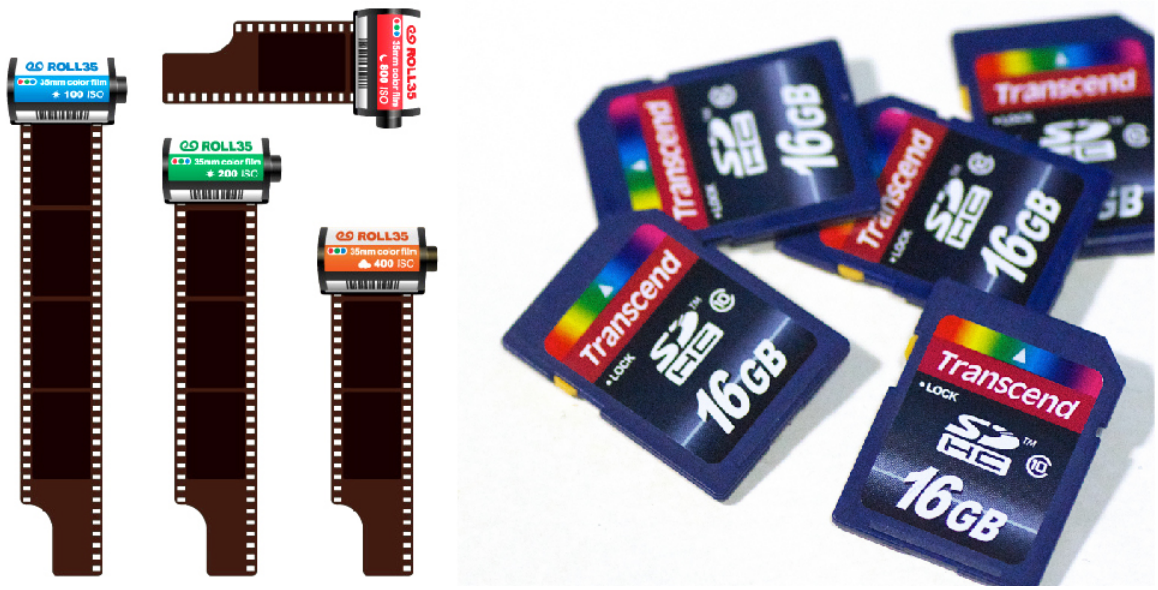



***Figure* 3.0.1*:*** *35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit "film": modification of work by Horia Varlan; credit "memory cards": modification of work by Paul Hudson)*

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# 3.1: Complex Numbers Review

**Learning Objectives**

- Express square roots of negative numbers as multiples of $i$.
- Plot complex numbers on the complex plane.
- Add and subtract complex numbers.
- Multiply and divide complex numbers.

The study of mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. For example, we still have no solution to equations such as

$$x^2+4=0 \tag{3.1.1}$$

Our best guesses might be +2 or –2. But if we test +2 in this equation, it does not work. If we test –2, it does not work. If we want to have a solution for this equation, we will have to go farther than we have so far. After all, to this point we have described the square root of a negative number as undefined. Fortunately, there is another system of numbers that provides solutions to problems such as these. In this section, we will explore this number system and how to work within it.

## Expressing Square Roots of Negative Numbers as Multiples of *i*

We know how to find the square root of any positive real number. In a similar way, we can find the square root of a negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number. The imaginary number i is defined as the square root of negative 1.

$$\sqrt{-1}=i \tag{3.1.2}$$

So, using properties of radicals,

$$i^2=(\sqrt{-1})^2=-1 \tag{3.1.3}$$

We can write the square root of any negative number as a multiple of i. Consider the square root of –25.

$$\sqrt{-25}=\sqrt{25\cdot(-1)} \tag{3.1.4}$$

$$=\sqrt{25}\sqrt{-1} \tag{3.1.5}$$

$$=5i \tag{3.1.6}$$

We use $5i$ and not $-5i$ because the principal root of 25 is the positive root.

A **complex number** is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a+bi$ where $a$ is the real part and $bi$ is the imaginary part. For example, $5+2i$ is a complex number. So, too, is $3+4\sqrt{3}i$.

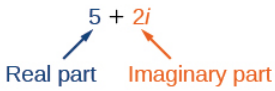



Figure 3.1.1

Imaginary numbers are distinguished from real numbers because a squared imaginary number produces a negative real number. Recall, when a positive real number is squared, the result is a positive real number and when a negative real number is squared, again, the result is a positive real number. Complex numbers are a combination of real and imaginary numbers.

**Imaginary and Complex Numbers**

A complex number is a number of the form $a+bi$ where

- $a$ is the real part of the complex number.
- $bi$ is the imaginary part of the complex number.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and $b$ is not equal to 0, the complex number is called an **imaginary number**. An imaginary number is an even root of a negative number.

### Standard Form

Given an imaginary number, express it in standard form.

1. Write $\sqrt{-a}$ as $\sqrt{a}\sqrt{-1}$.
2. Express $\sqrt{-1}$ as $i$.
3. Write $\sqrt{a} \cdot i$ in simplest form.

### Example 3.1.1: Expressing an Imaginary Number in Standard Form

Express $\sqrt{-9}$ in standard form.

**Solution**

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$$

In standard form, this is $0 + 3i$.

### Exercise 3.1.1

Express $\sqrt{-24}$ in standard form.

**Answer**

$\sqrt{-24} = 0 + 2i\sqrt{6}$

## Plotting a Complex Number on the Complex Plane

We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number we need to address the two components of the number. We use the complex plane, which is a coordinate system in which the horizontal axis represents the real component and the vertical axis represents the imaginary component. Complex numbers are the points on the plane, expressed as ordered pairs $(a, b)$, where $a$ represents the coordinate for the horizontal axis and $b$ represents the coordinate for the vertical axis.

Let's consider the number $-2 + 3i$. The real part of the complex number is$-2$ and the imaginary part is $3i$. We plot the ordered pair $(-2, 3)$ to represent the complex number $-2 + 3i$ as shown in Figure 3.1.2

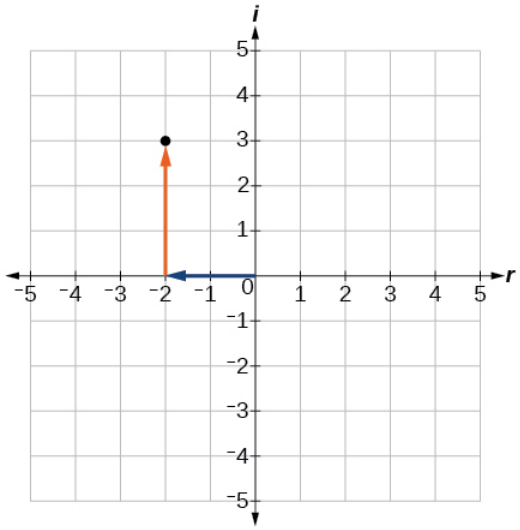



Figure 3.1.2: Plot of a complex number, $-2 + 3i$. Note that the real part $(-2)$ is plotted on the x-axis and the imaginary part $(3i)$ is plotted on the y-axis.

## Complex Plane

In the complex plane, the horizontal axis is the real axis, and the vertical axis is the imaginary axis as shown in Figure 3.1.3.

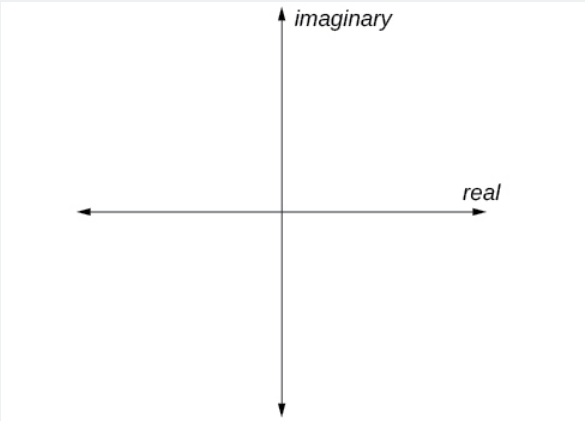



Figure 3.1.3: The complex plane showing that the horizontal axis (in the real plane, the x-axis) is known as the real axis and the vertical axis (in the real plane, the y-axis) is known as the imaginary axis.

## How To ...

Given a complex number, represent its components on the complex plane.

1. Determine the real part and the imaginary part of the complex number.
2. Move along the horizontal axis to show the real part of the number.
3. Move parallel to the vertical axis to show the imaginary part of the number.
4. Plot the point.

## Example 3.1.2: Plotting a Complex Number on the Complex Plane

Plot the complex number $3-4i$ on the complex plane.

**Solution**

The real part of the complex number is 3, and the imaginary part is $-4i$. We plot the ordered pair $(3,-4)$ as shown in Figure 3.1.4.

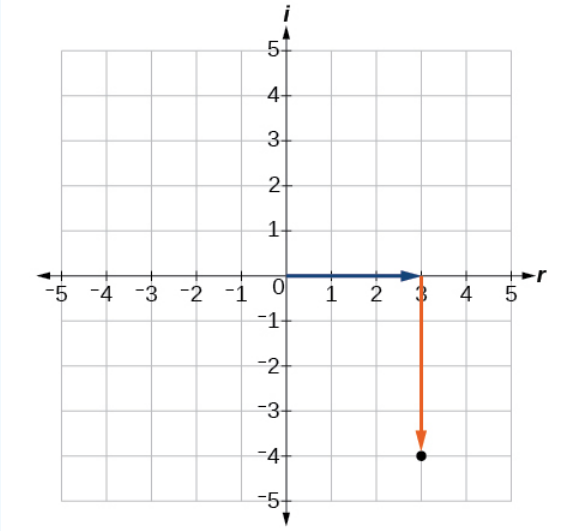



Figure 3.1.4: Plot of a complex number, $3-4i$. Note that the real part $(3)$ is plotted on the x-axis and the imaginary part $(-4i)$ is plotted on the y-axis.

**Exercise 3.1.1**

Plot the complex number $-4-i$ on the complex plane.

**Answer**

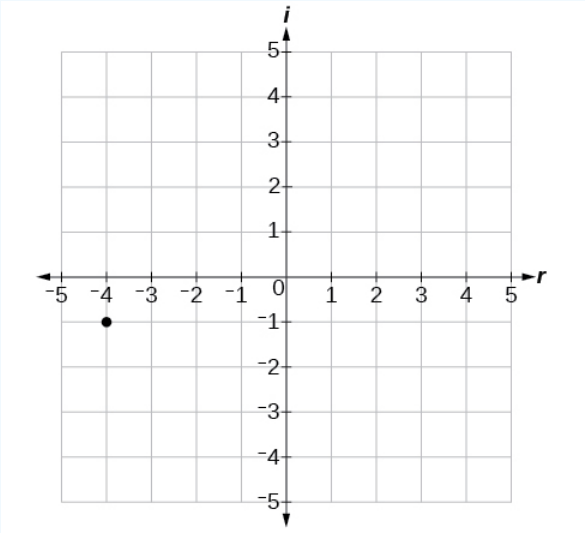



Figure 3.1.5

## Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and combine the imaginary parts.

**Complex Numbers: Addition and Subtraction**

Adding complex numbers:

$$(a+bi)+(c+di)=(a+c)+(b+d)i \tag{3.1.7}$$

Subtracting complex numbers:

$$(a+bi)-(c+di)=(a-c)+(b-d)i \tag{3.1.8}$$

**How To...**

Given two complex numbers, find the sum or difference.

1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

**Example 3.1.3: Adding Complex Numbers**

Add $3-4i$ and $2+5i$.

**Solution**

We add the real parts and add the imaginary parts.

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i\\ (3-4i)+(2+5i) &= (3+2)+(-4+5)i\\ &= 5+i\end{aligned}$$

**Exercise 3.1.3**

Subtract $2+5i$ from $3-4i$.

**Answer**

$(3-4i)-(2+5i)=1-9i$

## Multiplying Complex Numbers

Multiplying complex numbers is much like multiplying binomials. The major difference is that we work with the real and imaginary parts separately.

### Multiplying a Complex Numbers by a Real Number

Let's begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. So, for example,

$$\begin{aligned} 3(6+2i) &= (3\cdot 6)+(3\cdot 2i) && \text{Distribute.}\\ &= 18+6i && \text{Simplify.}\end{aligned}$$

Figure 3.1.6

**How To...**

Given a complex number and a real number, multiply to find the product.

1. Use the distributive property.
2. Simplify.

**Example 3.1.4: Multiplying a Complex Number by a Real Number**

Find the product $4(2+5i)$.

**Solution**

Distribute the 4.

$$\begin{aligned} 4(2+5i) &= (4\cdot 2)+(4\cdot 5i)\\ &= 8+20i\end{aligned}$$

**Exercise 3.1.4**

Find the product $-4(2+6i)$.

**Answer**

$-8-24i$

### Multiplying Complex Numbers Together

Now, let's multiply two complex numbers. We can use either the distributive property or the FOIL method. Recall that FOIL is an acronym for multiplying First, Outer, Inner, and Last terms together. Using either the distributive property or the FOIL method, we get

$$(a+bi)(c+di)=ac+adi+bci+bdi^2$$

Because $i^2=-1$, we have

$$(a+bi)(c+di)=ac+adi+bci-bd$$

To simplify, we combine the real parts, and we combine the imaginary parts.

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$$

**How To...**

Given two complex numbers, multiply to find the product.

1. Use the distributive property or the FOIL method.
2. Simplify.

**Example 3.1.5: Multiplying a Complex Number by a Complex Number**

Multiply $(4+3i)(2-5i)$.

**Solution**

Use $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$

$$\begin{aligned}(4+3i)(2-5i) &= (4\cdot 2-3\cdot(-5))+(4\cdot(-5)+3\cdot 2)i\\ &= (8+15)+(-20+6)i\\ &= 23-14i\end{aligned}$$

**Exercise 3.1.5**

Multiply $(3-4i)(2+3i)$.

**Answer**

$18+i$

## Dividing Complex Numbers

Division of two complex numbers is more complicated than addition, subtraction, and multiplication because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator. We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the **complex conjugate** of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of $a+bi$ is $a-bi$.

Note that complex conjugates have a reciprocal relationship: The complex conjugate of $a+bi$ is $a-bi$, and the complex conjugate of $a-bi$ is $a+bi$. Further, when a quadratic equation with real coefficients has complex solutions, the solutions are always complex conjugates of one another.

Suppose we want to divide $c+di$ by $a+bi$, where neither a nor $b$ equals zero. We first write the division as a fraction, then find the complex conjugate of the denominator, and multiply.

$$\frac{c+di}{a+bi} \text{ where } a\neq 0 \text{ and } b\neq 0$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(c+di)}{(a+bi)}\cdot\frac{(a-bi)}{(a-bi)}=\frac{(c+di)(a-bi)}{(a+bi)(a-bi)}$$

Apply the distributive property.

$$=\frac{ca-cbi+adi-bdi^2}{a^2-abi+abi-b^2i^2}$$

Simplify, remembering that $i^2=-1$.

$$= \frac{ca - cbi + adi - bd(-1)}{a^2 - abi + abi - b^2(-1)}$$
$$= \frac{(ca + bd) + (ad - cb)i}{a^2 + b^2}$$

**Definition: The Complex Conjugate**

The *complex conjugate* of a complex number $a + bi$ is $a - bi$. It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

**Example 3.1.6: Finding Complex Conjugates**

Find the complex conjugate of each number.

a. $2 + i\sqrt{5}$
b. $-\frac{1}{2}i$

**Solution**

a. The number is already in the form $a + bi$. The complex conjugate is $a - bi$, or $2 - i\sqrt{5}$.
b. We can rewrite this number in the form $a + bi$ as $0 - \frac{1}{2}i$. The complex conjugate is $a - bi$, or $0 + \frac{1}{2}i$. This can be written simply as $\frac{1}{2}i$.

**Analysis**

Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. To obtain a real number from an imaginary number, we can simply multiply by $i$.

**How To...**

Given two complex numbers, divide one by the other.

1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

**Example 3.1.7: Dividing Complex Numbers**

Divide $(2 + 5i)$ by $(4 - i)$.

**Solution**

We begin by writing the problem as a fraction.

$$\frac{(2 + 5i)}{(4 - i)}$$

Then we multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(2 + 5i)}{(4 - i)} \cdot \frac{(4 + i)}{(4 + i)}$$

To multiply two complex numbers, we expand the product as we would with polynomials (the process commonly called FOIL).

$$\frac{(2+5i)}{(4-i)}\cdot\frac{(4+i)}{(4+i)}=\frac{8+2i+20i+5i^2}{16+4i-4i-i^2} \tag{3.1.9}$$

$$=\frac{8+2i+20i+5(-1)}{16+4i-4i-(-1)} \quad \text{because } i^2=-1 \tag{3.1.10}$$

$$=\frac{3+22i}{17} \tag{3.1.11}$$

$$=\frac{3}{17}+\frac{22}{17}i \quad \text{Separate real and imaginary parts.} \tag{3.1.12}$$

Note that this expresses the quotient in standard form.

### Example 3.1.8: Substituting a Complex Number into a Polynomial Function

Let $f(x)=x^2-5x+2$ . Evaluate $f(3+i)$.

**Solution**

Substitute $x=3+i$ into the function $f(x)=x^2-5x+2$ and simplify.

$$\begin{aligned} f(3+i) &= (3+i)^2-5(3+i)+2 && \text{Substitute 3+i for x} \\ &= (3+6i+i^2)-(15+5i)+2 && \text{Multiply} \\ &= 9+6i+(-1)-15-5i+2 && \text{Subsitute -1 for } i^2 \\ &= -5+i && \text{Combine like terms} \end{aligned}$$

**Analysis**

We write $f(3+i)=-5+i$ . Notice that the input is $3+i$ and the output is $-5+i$.

### Exercise 3.1.8

Let $f(x)=2x^2-3x$. Evaluate $f(8-i)$.

**Answer**

$102-29i$

### Example 3.1.9: Substituting an Imaginary Number in a Rational Function

Let $f(x)=\frac{2+x}{x+3}$. Evaluate $f(10i)$.

**Solution**

Substitute $x=10i$ and simplify.

$$\frac{2+10i}{10i+3} \quad \text{Substitute } 10i \text{ for } x.$$

$$\frac{2+10i}{3+10i} \quad \text{Rewrite the denominator in standard form.}$$

$$\frac{2+10i}{3+10i}\cdot\frac{3-10i}{3-10i} \quad \text{Multiply the numerator and denominator by the complex conjugate of the denominator.}$$

$$\frac{6-20i+30i-100i^2}{9-30i+30i-100i^2} \quad \text{Multiply using the distributive property or the FOIL method.}$$

$$\frac{6-20i+30i-100(-1)}{9-30i+30i-100(-1)} \quad \text{Substitute } -1 \text{ for } i^2.$$

$$\frac{106+10i}{109} \quad \text{Simplify.}$$

$$\frac{106}{109}+\frac{10}{109} \quad \text{Separate the real and imaginary parts.}$$

**Exercise 3.1.9**

Let $f(x)=\frac{x+1}{x-4}$. Evaluate $f(-i)$.

**Answer**

$-\frac{3}{17}+\frac{5i}{17}$

## Simplifying Powers of $i$

The powers of $i$ are cyclic. Let's look at what happens when we raise $i$ to increasing powers.

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= i^2 \cdot i = -1 \cdot i = -i \\ i^4 &= i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1 \\ i^5 &= i^4 \cdot i = 1 \cdot i = i \end{aligned}$$

We can see that when we get to the fifth power of $i$, it is equal to the first power. As we continue to multiply $i$ by itself for increasing powers, we will see a cycle of 4. Let's examine the next 4 powers of $i$.

$$\begin{aligned} i^6 &= i^5 \cdot i = i \cdot i = i^2 = -1 \\ i^7 &= i^6 \cdot i = i^2 \cdot i = i^3 = -i \\ i^8 &= i^7 \cdot i = i^3 \cdot i = i^4 = 1 \\ i^9 &= i^8 \cdot i = i^4 \cdot i = i^5 = i \end{aligned}$$

**Example 3.1.10: Simplifying Powers of $i$**

Evaluate $i^{35}$.

**Solution**

Since $i^4=1$, we can simplify the problem by factoring out as many factors of i^4 as possible. To do so, first determine how many times 4 goes into 35: $35=4\cdot 8+3$.

$$i^{35}=i^{4\cdot 8+3}=i^{4\cdot 8}\cdot i^3=(i^4)^8\cdot i^3=1^8\cdot i^3=i^3=-i$$

**Q & A**

Can we write $i^35$ in other helpful ways?

> As we saw in Example 3.1.10, we reduced $i^{35}$ to $i^3$ by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of $i^{35}$ may be more useful. Table 3.1.1 shows some other possible factorizations.

Table 3.1.1: Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.

| Factorization of $i^{35}$ | $i^{34}\cdot i$ | $i^{33}\cdot i^2$ | $i^{31}\cdot i^4$ | $i^{19}\cdot i^{16}$ |
|---|---|---|---|---|
| Reduced form | $(i^2)^{17}\cdot i$ | $i^{33}\cdot(-1)$ | $i^{31}\cdot 1$ | $i^{19}\cdot(i^4)^4$ |
| Simplified form | $(-1)^{17}\cdot i$ | $-i^{33}$ | $i^{31}$ | $i^{19}$ |

## Key Concepts

- The square root of any negative number can be written as a multiple of $i$.
- To plot a complex number, we use two number lines, crossed to form the complex plane. The horizontal axis is the real axis, and the vertical axis is the imaginary axis.
- Complex numbers can be added and subtracted by combining the real parts and combining the imaginary parts.
- Complex numbers can be multiplied and divided.
- To multiply complex numbers, distribute just as with polynomials.
- To divide complex numbers, multiply both the numerator and denominator by the complex conjugate of the denominator to eliminate the complex number from the denominator.
- The powers of $i$ are cyclic, repeating every fourth one.

## Glossary

**complex conjugate**
the complex number in which the sign of the imaginary part is changed and the real part of the number is left unchanged; when added to or multiplied by the original complex number, the result is a real number

**complex number**
the sum of a real number and an imaginary number, written in the standard form $a+bi$, where $a$ is the real part, and $bi$ is the imaginary part

**complex plane**
a coordinate system in which the horizontal axis is used to represent the real part of a complex number and the vertical axis is used to represent the imaginary part of a complex number

**imaginary number**
a number in the form bi where $i=\sqrt{-1}$

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# 3.2: Quadratic Functions Review

**Learning Objectives**

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.

Curved antennas, such as the ones shown in Figure 3.2.1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

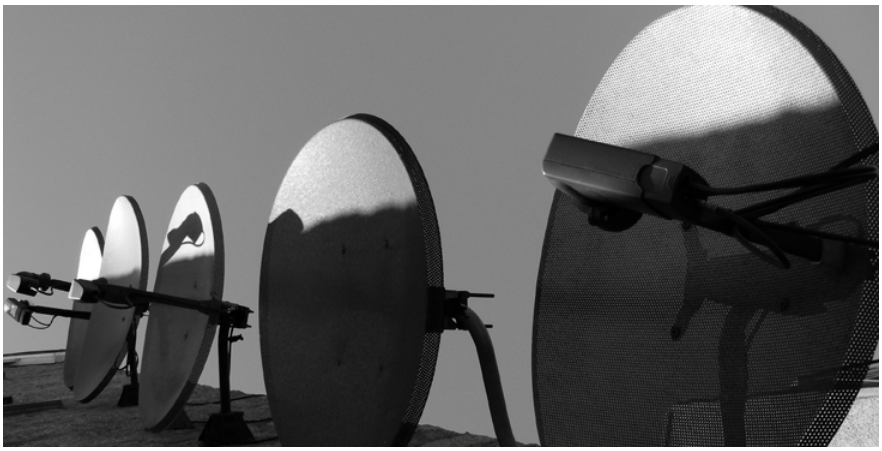


***Figure* 3.2.1*:*** *An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)*

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

## Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 3.2.2.

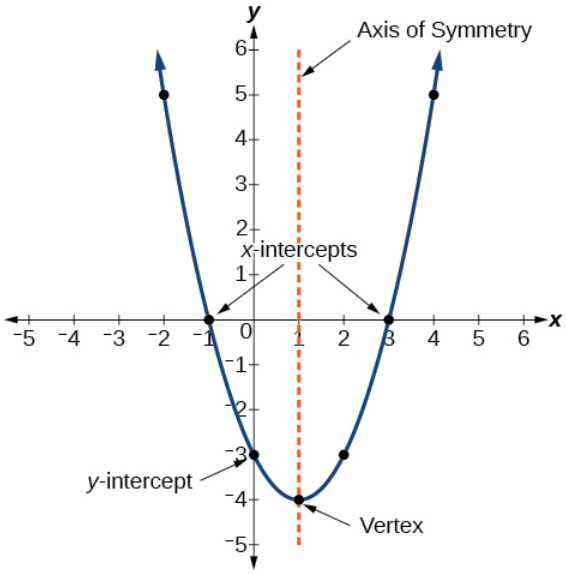



Figure 3.2.2: Graph of a parabola showing where the $x$ and $y$ intercepts, vertex, and axis of symmetry are.

The y-intercept is the point at which the parabola crosses the $y$-axis. The x-intercepts are the points at which the parabola crosses the $x$-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of $x$ at which $y = 0$.

### Example 3.2.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 3.2.3.

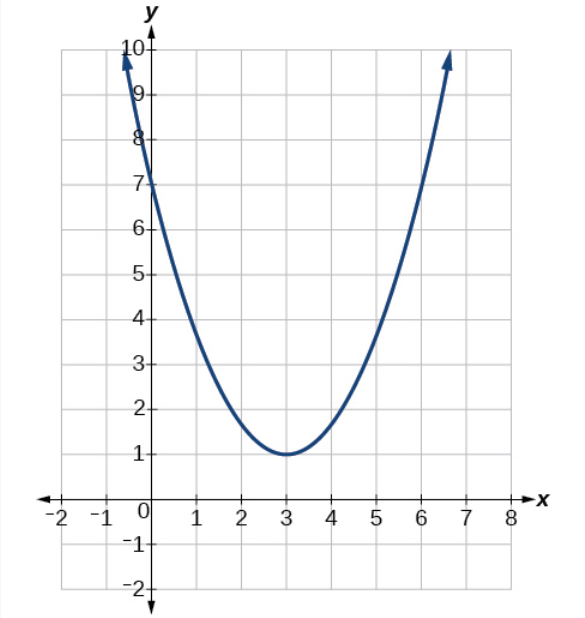



Figure 3.2.3.

**Solution**

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the $y$-axis at $(0, 7)$ so this is the y-intercept.

## Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c \tag{3.2.1}$$

where $a$, $b$, and $c$ are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of $x$ halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 3.2.4 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

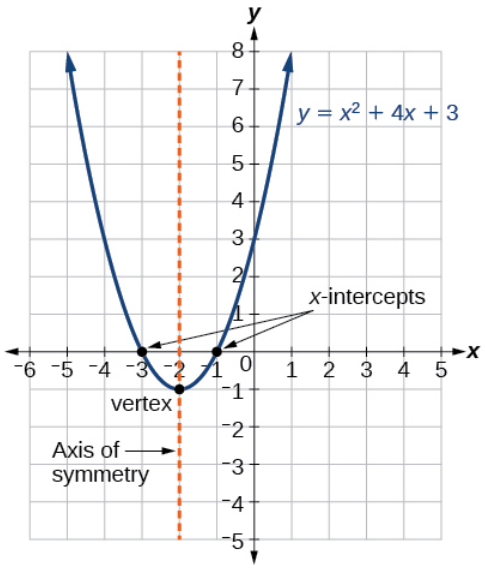



Figure 3.2.4: Graph of a parabola showing where the $x$ and $y$ intercepts, vertex, and axis of symmetry are for the function $y = x^2 + 4x + 3$.

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k \tag{3.2.2}$$

where $(h, k)$ is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure 3.2.5 represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

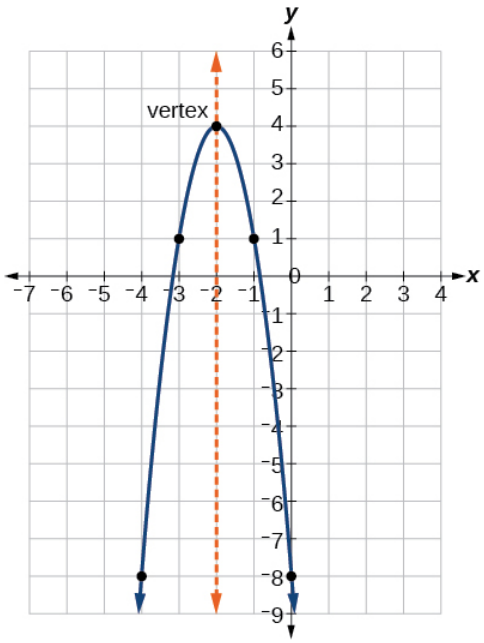



Figure 3.2.5: Graph of a parabola showing where the $x$ and $y$ intercepts, vertex, and axis of symmetry are for the function $y = -3(x+2)^2 + 4$ .

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure 3.2.6 is the graph of this basic function.

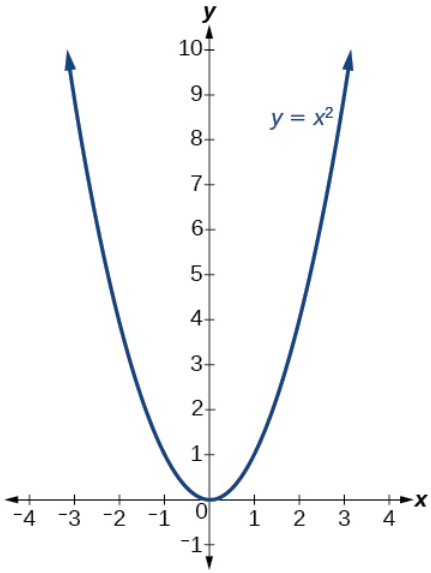



Figure 3.2.6: Graph of $y = x^2$.

If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 3.2.5, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 3.2.5, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of $a$ indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure 3.2.5, $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$\begin{aligned} a(x-h)^2+k &= ax^2+bx+c \\ ax^2-2ahx+(ah^2+k) &= ax^2+bx+c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah=b, \text{ so } h=-\frac{b}{2a}.$$

This is the **axis of symmetry** we defined earlier. Setting the constant terms equal:

$$\begin{aligned} ah^2+k &= c \\ k &= c-ah^2 \\ &= c-a\cdot\left(-\frac{b}{2a}\right)^2 \\ &= c-\frac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that $k$ is the output value of the function when the input is $h$, so $f(h)=k$.

**Definitions: Forms of Quadratic Functions**

A quadratic function is a function of degree two. The graph of a *quadratic function* is a parabola.

- The *general form* of a quadratic function is $f(x)=ax^2+bx+c$ where $a$, $b$, and $c$ are real numbers and $a\neq 0$.
- The standard form of a quadratic function is $f(x)=a(x-h)^2+k$ .
- The vertex $(h,k)$ is located at

$$h=-\frac{b}{2a},\ k=f(h)=f(\frac{-b}{2a}). \tag{3.2.3}$$

**HOWTO: Write a quadratic function in a general form**

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is $h$. Identify the vertical shift of the parabola; this value is $k$.
2. Substitute the values of the horizontal and vertical shift for $h$ and $k$. in the function $f(x)=a(x-h)^2+k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for $x$ and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. If the parabola opens up, $a>0$. If the parabola opens down, $a<0$ since this means the graph was reflected about the x-axis.
6. Expand and simplify to write in general form.

**Example 3.2.2: Writing the Equation of a Quadratic Function from the Graph**

Write an equation for the quadratic function $g$ in Figure 3.2.7 as a transformation of $f(x)=x^2$, and then expand the formula, and simplify terms to write the equation in general form.

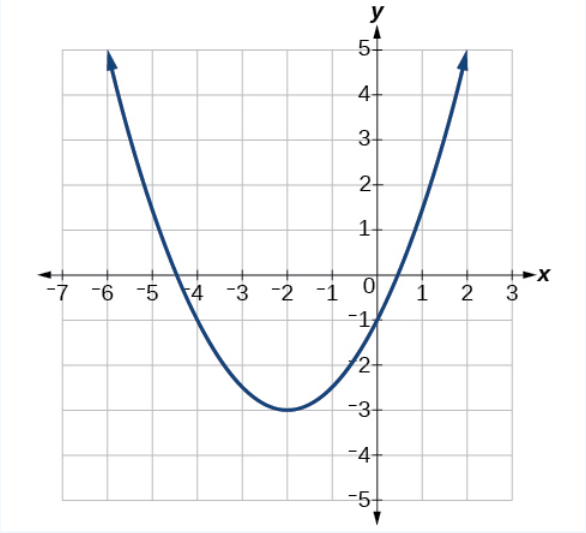



Figure 3.2.7: Graph of a parabola with its vertex at $(-2, -3)$.

**Solution**

We can see the graph of $g$ is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$-1 = a(0+2)^2 - 3 \tag{3.2.4}$$
$$2 = 4a \tag{3.2.5}$$
$$a = \frac{1}{2} \tag{3.2.6}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$g(x) = \frac{1}{2}(x+2)^2 - 3 \tag{3.2.7}$$
$$= \frac{1}{2}(x+2)(x+2) - 3 \tag{3.2.8}$$
$$= \frac{1}{2}(x^2 + 4x + 4) - 3 \tag{3.2.9}$$
$$= \frac{1}{2}x^2 + 2x + 2 - 3 \tag{3.2.10}$$
$$= \frac{1}{2}x^2 + 2x - 1 \tag{3.2.11}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

**Analysis**

We can check our work using the table feature on a graphing utility. First enter $\mathrm{Y1} = \frac{1}{2}(\mathrm{x}+2)^2 - 3$ . Next, select $\mathrm{TBLSET}$, then use $\mathrm{TblStart} = -6$ and $\Delta\mathrm{Tbl} = 2$, and select $\mathrm{TABLE}$. See Table 3.2.1

Table 3.2.1

| $x$ | -6 | -4 | -2 | 0 | 2 |
|---|---|---|---|---|---|
| $y$ | -5 | -1 | -3 | -1 | 5 |

The ordered pairs in the table correspond to points on the graph.

## Exercise 3.2.2

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 3.2.8. Find an equation for the path of the ball. Does the shooter make the basket?

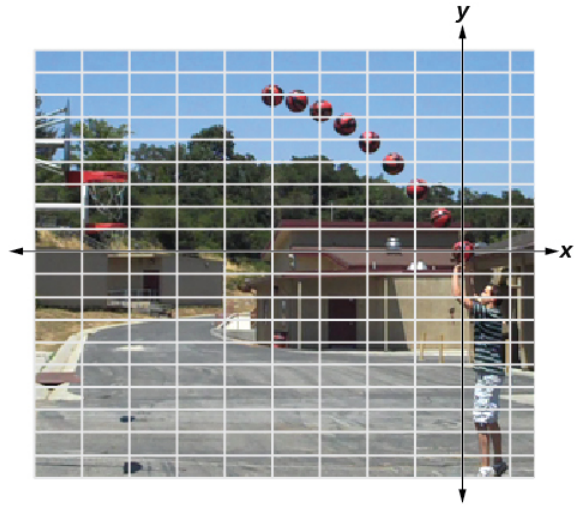



***Figure 3.2.8:*** *Stop motioned picture of a boy throwing a basketball into a hoop to show the parabolic curve it makes. (credit: modification of work by Dan Meyer)*

**Answer**

The path passes through the origin and has vertex at $(-4, 7)$, so $h(x) = -\frac{7}{16}(x+4)^2 + 7$ . To make the shot, $h(-7.5)$ would need to be about 4 but $h(-7.5) \approx 1.64$; he doesn't make it.

**How To: Given a quadratic function in general form, find the vertex of the parabola.**

1. Identify $a$, $b$, and $c$.
2. Find $h$, the x-coordinate of the vertex, by substituting $a$ and $b$ into $h = -\frac{b}{2a}$.
3. Find $k$, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$ .

## Example 3.2.3: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

**Solution**

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} \tag{3.2.12}$$
$$= -\frac{-6}{2(2)} \tag{3.2.13}$$
$$= \frac{6}{4} \tag{3.2.14}$$
$$= \frac{3}{2} \tag{3.2.15}$$

The vertical coordinate of the vertex will be at

$$k = f(h) \tag{3.2.16}$$
$$= f\left(\frac{3}{2}\right) \tag{3.2.17}$$
$$= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \tag{3.2.18}$$
$$= \frac{5}{2} \tag{3.2.19}$$

Rewriting into standard form, the stretch factor will be the same as the $a$ in the original quadratic.

$$f(x) = ax^2 + bx + c \tag{3.2.20}$$
$$f(x) = 2x^2 - 6x + 7$$

Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} \tag{3.2.21}$$

**Analysis**

One reason we may want to identify the vertex of the parabola is that this point will inform us what the maximum or minimum value of the function is, $(k)$,and where it occurs, $(h)$.

**Exercise 3.2.3**

Given the equation $g(x) = 13 + x^2 - 6x$ , write the equation in general form and then in standard form.

**Answer**

$g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x-3)^2 + 4$ in standard form.

## Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

**Definition: Domain and Range of a Quadratic Function**

The domain of any **quadratic function** is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive $a$ value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $[f(-\frac{b}{2a}), \infty)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f(-\frac{b}{2a})$, or $(-\infty, f(-\frac{b}{2a})]$.

The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive $a$ value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative $a$ value is $f(x) \leq k$.

**How To: Given a quadratic function, find the domain and range.**

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether $a$ is positive or negative. If $a$ is positive, the parabola has a minimum. If $a$ is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, $k$.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

### Example 3.2.4: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$ .

**Solution**

As with any quadratic function, the domain is all real numbers.

Because $a$ is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} \tag{3.2.22}$$
$$= -\frac{9}{2(-5)} \tag{3.2.23}$$
$$= \frac{9}{10} \tag{3.2.24}$$

The maximum value is given by $f(h)$.

$$f(\frac{9}{10}) = 5(\frac{9}{10})^2 + 9(\frac{9}{10}) - 1 \tag{3.2.25}$$
$$= \frac{61}{20} \tag{3.2.26}$$

The range is $f(x) \leq \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

### Exercise 3.2.4

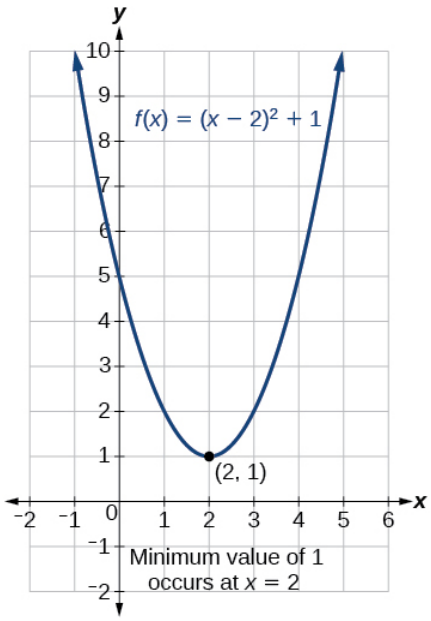
Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$ .

**Answer**

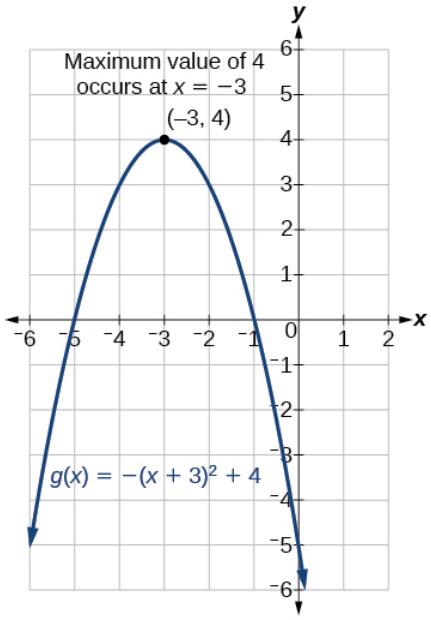
The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$ , or $\left[\frac{8}{11}, \infty\right)$.

## Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the **parabola**. We can see the maximum and minimum values in Figure 3.2.9.




(a)




(b)

Figure 3.2.9: Minimum and maximum of two quadratic functions.

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

### Example 3.2.5: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length $L$.
b. What dimensions should she make her garden to maximize the enclosed area?

**Solution**

Let's use a diagram such as Figure 3.2.10 to record the given information. It is also helpful to introduce a temporary variable, $W$, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

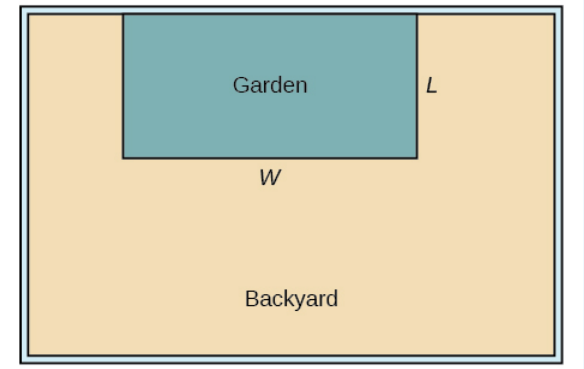



Figure 3.2.10: Diagram of the garden and the backyard.

a. We know we have only 80 feet of fence available, and $L+W+L=80$ , or more simply, $2L+W=80$. This allows us to represent the width, $W$, in terms of $L$.

$$W=80-2L \tag{3.2.27}$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80-2L) \tag{3.2.28}$$
$$A(L) = 80L-2L^2 \tag{3.2.29}$$

This formula represents the area of the fence in terms of the variable length $L$. The function, written in general form, is

$$A(L)=-2L^2+80L \tag{3.2.30}$$

.

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since $a$ is the coefficient of the squared term, $a=-2$ , $b=80$, and $c=0$.

To find the vertex:

$$h=-\frac{80}{2(-2)} \qquad k=A(20) \tag{3.2.31}$$
$$=20 \qquad \text{and} \qquad =80(20)-2(20)^2 \tag{3.2.32}$$
$$=800 \tag{3.2.33}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L=20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

**Analysis**

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 3.2.11.

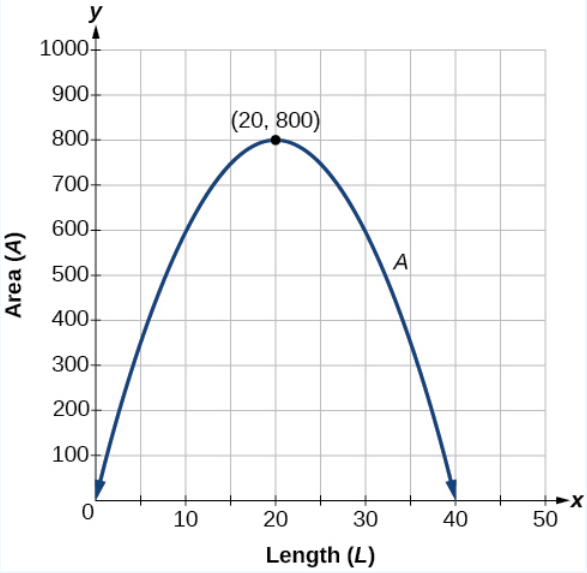



Figure 3.2.11: Graph of the parabolic function $A(L) = -2L^2 + 80L$

**How To: Given an application involving revenue, use a quadratic equation to find the maximum.**

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

### Example 3.2.6: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

**Solution**

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, $p$ for price per subscription and $Q$ for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} \tag{3.2.34}$$

$$= -\frac{5,000}{2} \tag{3.2.35}$$

$$= -2,500 \tag{3.2.36}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$Q = -2500p + b \quad \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \tag{3.2.37}$$

$$84,000 = -2500(30) + b \quad \text{Solve for } b \tag{3.2.38}$$

$$b = 159,000 \tag{3.2.39}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = pQ \tag{3.2.40}$$
$$\text{Revenue} = p(-2,500p + 159,000) \tag{3.2.41}$$
$$\text{Revenue} = -2,500p^2 + 159,000p \tag{3.2.42}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)} \tag{3.2.43}$$
$$= 31.8 \tag{3.2.44}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{maximum revenue} = -2,500(31.8)^2 + 159,000(31.8) \tag{3.2.45}$$
$$= 2,528,100 \tag{3.2.46}$$

**Analysis**

This could also be solved by graphing the quadratic as in Figure 3.2.12 We can see the maximum revenue on a graph of the quadratic function.

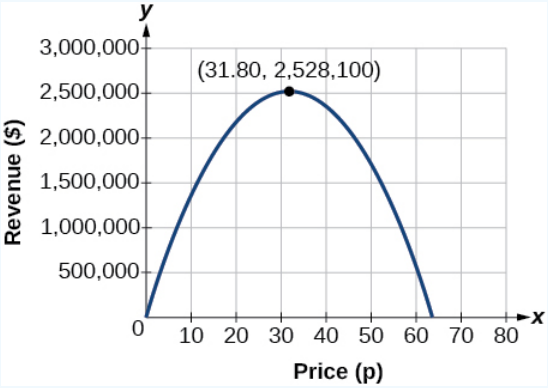



Figure 3.2.12: Graph of the parabolic function

## Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure 3.2.13 that the number of x-intercepts can vary depending upon the location of the graph.

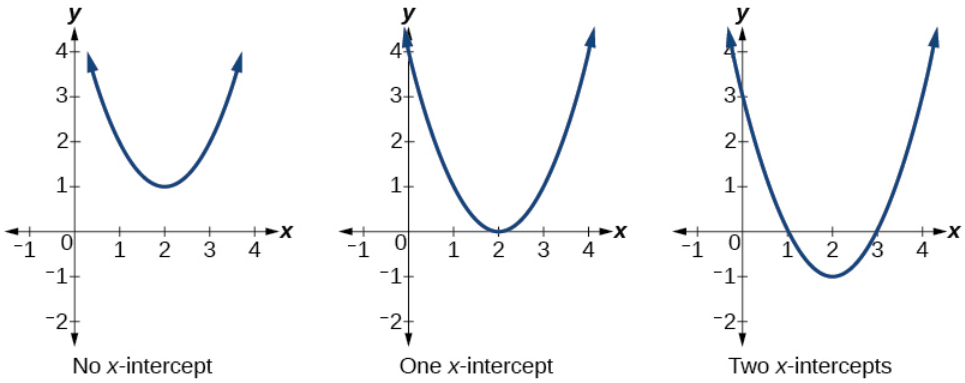



Figure 3.2.13: Number of x-intercepts of a parabola.

**Given a quadratic function $f(x)$, find the y- and x-intercepts.**

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x)=0$ to find the x-intercepts.

### Example 3.2.7: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x)=3x^2+5x-2$ .

**Solution**

We find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2+5(0)-2 \tag{3.2.47}$$
$$= -2 \tag{3.2.48}$$

So the y-intercept is at $(0,-2)$.

For the x-intercepts, we find all solutions of $f(x)=0$.

$$0=3x^2+5x-2 \tag{3.2.49}$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0=(3x-1)(x+2) \tag{3.2.50}$$
$$0=3x-1 \qquad\qquad 0=x+2 \tag{3.2.51}$$
$$x=\frac{1}{3} \qquad \text{or} \qquad x=-2 \tag{3.2.52}$$

So the x-intercepts are at $(\frac{1}{3},0)$ and $(-2,0)$.

**Analysis**

By graphing the function, we can confirm that the graph crosses the $y$-axis at $(0,-2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3},0\right)$ and $(-2,0)$. See Figure 3.2.14

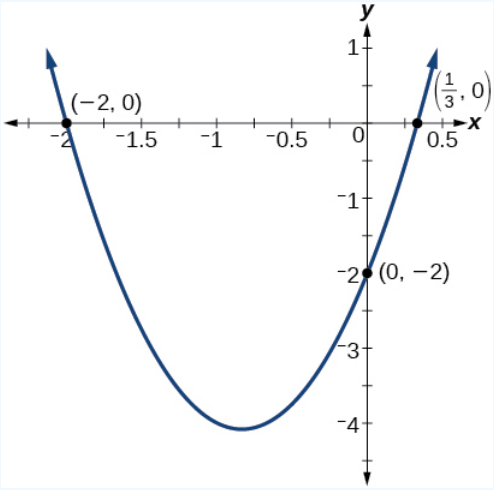



Figure 3.2.14: Graph of a parabola.

## Rewriting Quadratics in Standard Form

In Example 3.2.7, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

**How To: Given a quadratic function, find the x-intercepts by rewriting in standard form.**

1. Substitute a and $b$ into $h=-\frac{b}{2a}$ .
2. Substitute $x=h$ into the general form of the quadratic function to find $k$.
3. Rewrite the quadratic in standard form using $h$ and $k$.

4. Solve for when the output of the function will be zero to find the x-intercepts.

### Example 3.2.8: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$ .

**Solution**

We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x-h)^2 + k$$

We know that $a = 2$. Then we solve for $h$ and $k$.

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x+1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$.

**Analysis**

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See Figure 3.2.15

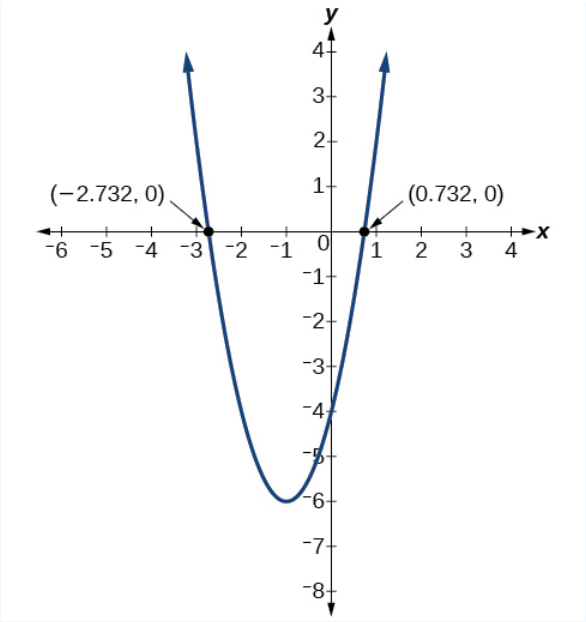



Figure 3.2.15: Graph of a parabola which has the following x-intercepts: $(-2.732, 0)$ and $(0.732, 0)$.

## Exercise 3.2.1

In Try It 3.2.1, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$ . Now find the y- and x-intercepts (if any).

**Answer**

y-intercept at $(0, 13)$, No x-intercepts

## Example 3.2.9: Solving a Quadratic Equation with the Quadratic Formula

Solve $x^2 + x + 2 = 0$ .

**Solution**

Let's begin by writing the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

When applying the quadratic formula, we identify the coefficients $a$, $b$ and $c$. For the equation $x^2 + x + 2 = 0$ , we have $a = 1$ , $b = 1$ , and $c = 2$ . Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (2)}}{2 \cdot 1} \\ &= \frac{-1 \pm \sqrt{1 - 8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $x = \frac{-1 + i\sqrt{7}}{2}$ and $x = \frac{-1 - i\sqrt{7}}{2}$ or $x = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $x = \frac{-1}{2} - \frac{i\sqrt{7}}{2}$ .

## Example 3.2.10: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$ .

When does the ball reach the maximum height?
What is the maximum height of the ball?
When does the ball hit the ground?

The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{80}{2(-16)} \tag{3.2.53}$$
$$= \frac{80}{32} \tag{3.2.54}$$
$$= \frac{5}{2} \tag{3.2.55}$$
$$= 2.5 \tag{3.2.56}$$

The ball reaches a maximum height after 2.5 seconds.

To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$k = H(-\frac{b}{2a}) \tag{3.2.57}$$
$$= H(2.5) \tag{3.2.58}$$
$$= -16(2.5)^2 + 80(2.5) + 40 \tag{3.2.59}$$
$$= 140 \tag{3.2.60}$$

The ball reaches a maximum height of 140 feet.

To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$.

We use the quadratic formula.

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \tag{3.2.61}$$
$$= \frac{-80 \pm \sqrt{8960}}{-32} \tag{3.2.62}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458 \tag{3.2.63}$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 3.2.16.

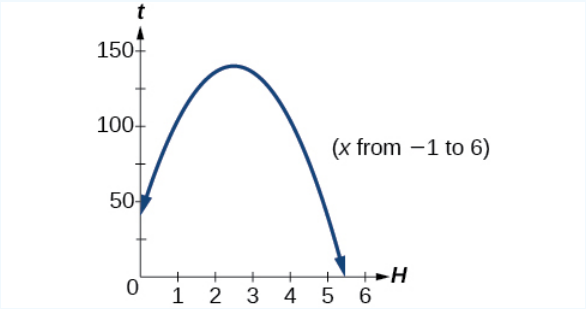



Figure 3.2.16


https://math.libretexts.org/@go/page/69085

**3.2.5:** A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$ .

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

**Solution**

a. 3 seconds b. 256 feet c. 7 seconds

## Key Equations

- general form of a quadratic function: $f(x) = ax^2 + bx + c$
- the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- standard form of a quadratic function: $f(x) = a(x - h)^2 + k$

## Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the $y$-axis.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph.
- The vertex can be found from an equation representing a quadratic function. .
- The domain of a quadratic function is all real numbers. The range varies with the function.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue.
- Some quadratic equations must be solved by using the quadratic formula.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems.

## Glossary

**axis of symmetry**
a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

**general form of a quadratic function**
the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$ , where $a, b$, and $c$ are real numbers and a≠0.

**standard form of a quadratic function**
the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$ , where $(h, k)$ is the vertex.

**vertex**
the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

**vertex form of a quadratic function**
another name for the standard form of a quadratic function

**zeros**
in a given function, the values of $x$ at which $y = 0$, also called roots

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# 3.3: Power Functions and Polynomial Functions Behavior

**Learning Objectives**

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table 3.3.1.

Table 3.3.1

| Year | 2009 | 2010 | 2011 | 2012 | 2013 |
|---|---|---|---|---|---|
| Bird Population | 800 | 897 | 992 | 1,083 | 1,169 |

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$ , where $P(t)$ represents the bird population on the island $t$ years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.



Figure 3.3.1: (credit: Jason Bay, Flickr)

## Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius $r$ is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius $r$ is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, $\pi$ or $\frac{4}{3}\pi$, multiplied by a variable $r$ raised to a power.

**Definition: Power Function**

A *power function* is a function that can be represented in the form

$$f(x) = kx^p \tag{3.3.1}$$

where $k$ and $p$ are real numbers, and $k$ is known as the *coefficient*.

**Q&A: Is $f(x) = 2^x$ a power function?**

*No. A power function contains a variable base raised to a fixed power (Equation 3.3.1). This function has a constant base raised to a variable power. This is called an exponential function, not a power function. This function will be discussed later.*

**Example 3.3.1: Identifying Power Functions**

Which of the following functions are power functions?

| | |
|---|---|
| $f(x) = 1$ | Constant function |
| $f(x) = x$ | Identify function |
| $f(x) = x^2$ | Quadratic function |
| $f(x) = x^3$ | Cubic function |
| $f(x) = \frac{1}{x}$ | Reciprocal function |
| $f(x) = \frac{1}{x^2}$ | Reciprocal squared function |
| $f(x) = \sqrt{x}$ | Square root function |
| $f(x) = \sqrt[3]{x}$ | Cube root function |

**Solution**

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

**Exercise 3.3.1**

Which functions are power functions?

- $f(x) = 2x^2 \cdot 4x^3$
- $g(x) = -x^5 + 5x^3 - 4x$
- $h(x) = \frac{2x^5 - 1}{3x^2 + 4}$

**Answer**

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

## Identifying End Behavior of Power Functions

Figure 3.3.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

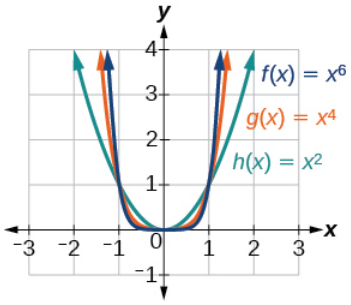



Figure 3.3.2: Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol $\infty$ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that $x$ is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as $x$ approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty,\ f(x) \to \infty$$

Figure 3.3.3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

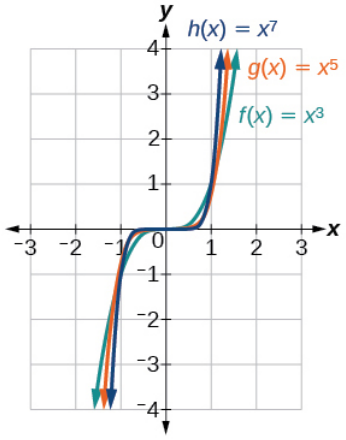



Figure 3.3.3: Odd-power function

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 3.3.2 we see that even functions of the form $f(x) = x^n$, $n$ even, are symmetric about the $y$-axis. In Figure 3.3.3 we see that odd functions of the form $f(x) = x^n$, $n$ odd, are symmetric about the origin.

For these odd power functions, as $x$ approaches negative infinity, $f(x)$ decreases without bound. As $x$ approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$
$$\text{as } x \to \infty,\ f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small $(x \to -\infty)$ and get very large $x \to \infty$ is referred to as the *end behavior* of the function. We can use words or symbols to describe end behavior.

Figure 3.3.4 shows the end behavior of power functions in the form $f(x) = kx^n$ where $n$ is a non-negative integer depending on the power and the constant.

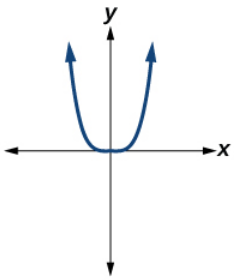
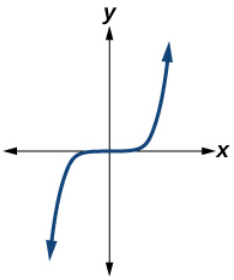
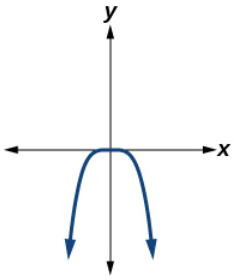
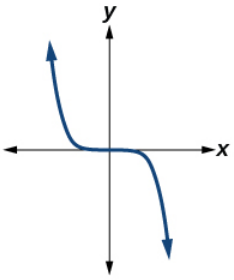
| | Even power | Odd power |
|---|---|---|
| Positive constant $k > 0$ |  $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to \infty$ |  $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to \infty$ |
| Negative constant $k < 0$ |  $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to -\infty$ |  $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to -\infty$ |

Figure 3.3.4

**How To: Given a power function $f(x) = kx^n$ where $n$ is a non-negative integer, identify the end behavior.**

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure $3.3.4$ to identify the end behavior.

**Example 3.3.2: Identifying the End Behavior of a Power Function**

Describe the end behavior of the graph of $f(x) = x^8$.

**Solution**

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As $x$ approaches infinity, the output (value of $f(x)$ ) increases without bound. We write as $x \to \infty$, $f(x) \to \infty$. As $x$ approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty$, $f(x) \to \infty$. We can graphically represent the function as shown in Figure $3.3.5$.

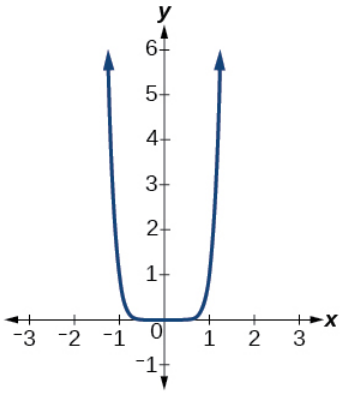



Figure 3.3.5: Graph of $f(x) = x^8$.

## Example 3.3.3: Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

**Solution**

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the $x$-axis of the graph of $f(x) = x^9$. Figure 3.3.6 shows that as $x$ approaches infinity, the output decreases without bound. As $x$ approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty$$
$$\text{as } x \to \infty,\ f(x) \to -\infty$$

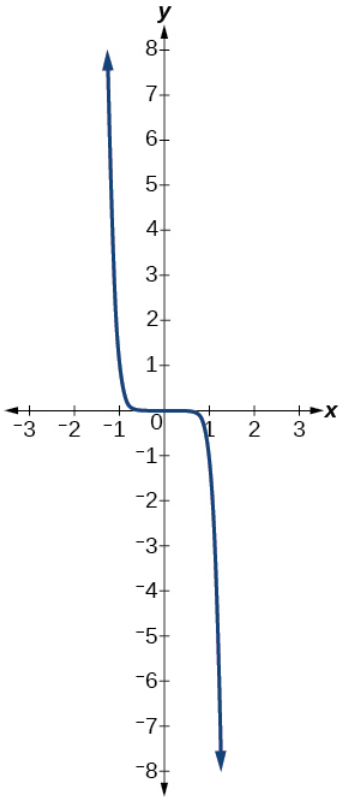



Figure 3.3.6: Graph of $f(x) = -x^9$.

**Analysis**

We can check our work by using the table feature on a graphing utility.

Table 3.3.2

| $x$ | $f(x)$ |
|---|---|
| -10 | 1,000,000,000 |
| -5 | 1,953,125 |
| 0 | 0 |
| 5 | -1,953,125 |
| 10 | -1,000,000,000 |

We can see from Table $3.3.2$ that, when we substitute very small values for $x$, the output is very large, and when we substitute very large values for $x$, the output is very small (meaning that it is a very large negative value).

**Exercise 3.3.2**

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

**Answer**

As $x$ approaches positive or negative infinity, $f(x)$ decreases without bound: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the negative coefficient.

## Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius $r$ of the spill depends on the number of weeks $w$ that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

**Definition: Polynomial Functions**

Let $n$ be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{3.3.2}$$

This is called the general form of a polynomial function. Each $a_i$ is a coefficient and can be any real number. Each product $a_i x^i$ is a *term of a polynomial function.*

### Example 3.3.4: Identifying Polynomial Functions

Which of the following are polynomial functions?

- $f(x) = 2x^3 \cdot 3x + 4$
- $g(x) = -x(x^2 - 4)$
- $h(x) = 5\sqrt{x} + 2$

**Solution**

The first two functions are examples of polynomial functions because they can be written in the form of Equation 3.3.2, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$ .
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

## Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

### Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.

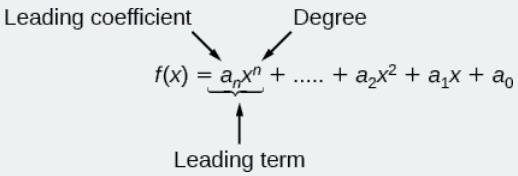



Figure 3.3.7

When a polynomial is written in this way, we say that it is in *general form*.

### How To: Given a polynomial function, identify the degree and leading coefficient

1. Find the highest power of $x$ to determine the degree function.
2. Identify the term containing the highest power of $x$ to find the leading term.
3. Identify the coefficient of the leading term.

### Example 3.3.5: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x) = 3 + 2x^2 - 4x^3$$
$$g(t) = 5t^5 - 2t^3 + 7t$$
$$h(p) = 6p - p^3 - 2$$

**Solution**

For the function $f(x)$, the highest power of $x$ is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, −4.

For the function $g(t)$, the highest power of $t$ is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of $p$ is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, $-1$.

**Exercise 3.3.3**

Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$ .

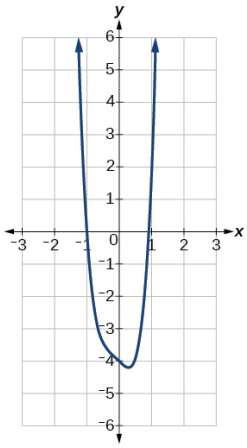
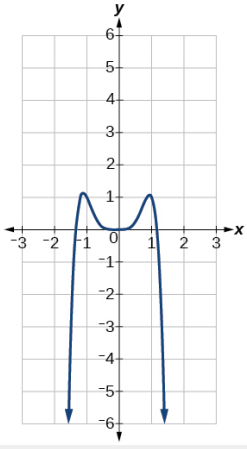
**Answer**

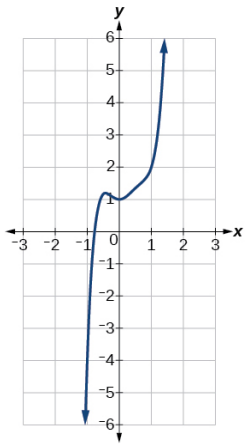
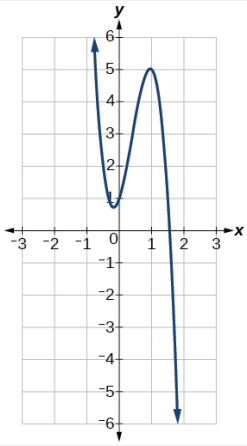
The degree is $6$. The leading term is $-x^6$. The leading coefficient is $-1$.

## Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as $x$ gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree (Table $3.3.3$).

Table 3.3.3

| Polynomial Function | Leading Term | Graph of Polynomial Function |
|---|---|---|
| $f(x) = 5x^4 + 2x^3 - x - 4$ | $5x^4$ |  |
| $f(x) = -2x^6 - x^5 + 3x^4 + x^3$ | $-2x^6$ |  |

| Polynomial Function | Leading Term | Graph of Polynomial Function |
|---|---|---|
| $f(x) = 3x^5 - 4x^4 + 2x^2 + 1$ | $3x^5$ |  |
| $f(x) = -6x^3 + 7x^2 + 3x + 1$ | $-6x^3$ |  |

**Example 3.3.6: Identifying End Behavior and Degree of a Polynomial Function**

Describe the end behavior and determine a possible degree of the polynomial function in Figure 3.3.8.

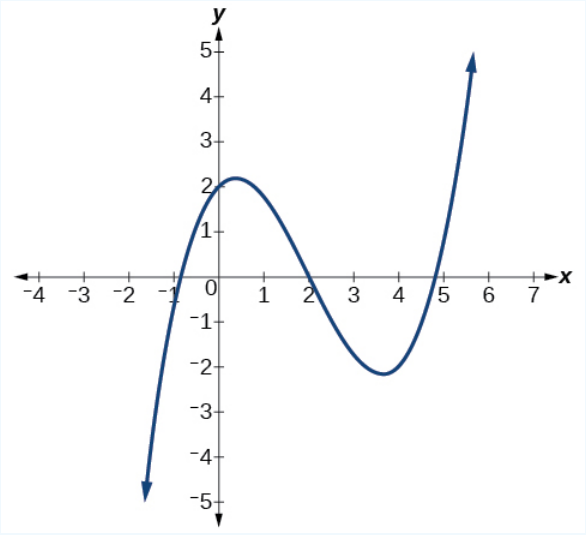


Figure 3.3.8.

**Solution**

As the input values $x$ get very large, the output values $f(x)$ increase without bound. As the input values $x$ get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x\to\infty,\ f(x)\to\infty$$

$$\text{as } x\to-\infty,\ f(x)\to-\infty$$

In words, we could say that as $x$ values approach infinity, the function values approach infinity, and as $x$ values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

### Exercise 3.3.1

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 3.3.9.

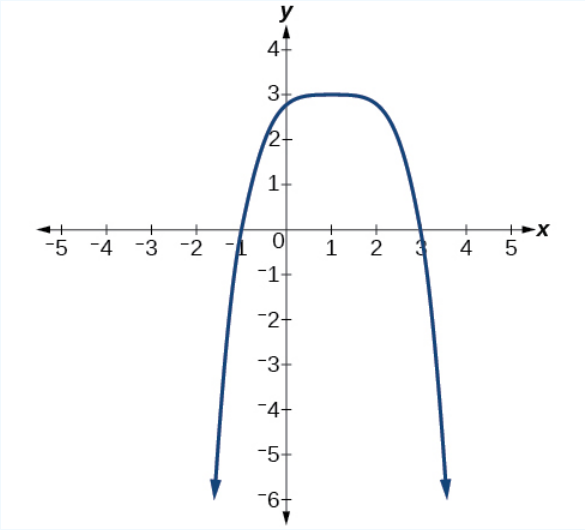



Figure 3.3.9

**Answer**

As $x\to\infty$, $f(x)\to-\infty$ ; as $x\to-\infty$ , $f(x)\to-\infty$ . It has the shape of an even degree power function with a negative coefficient.

### Example 3.3.7: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x)=-3x^2(x-1)(x+4)$ , express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

**Solution**

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The general form is $f(x)=-3x^4-9x^3+12x^2$ . The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x\to-\infty,\ f(x)\to-\infty$$

$$\text{as } x\to\infty,\ f(x)\to-\infty$$

**Exercise 3.3.7**

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

**Answer**

The leading term is $0.2x^3$, so it is a degree 3 polynomial. As $x$ approaches positive infinity, $f(x)$ increases without bound; as $x$ approaches negative infinity, $f(x)$ decreases without bound.

## Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the $y$-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one $y$-intercept $(0, a_0)$. The $x$-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one $x$-intercept. See Figure 3.3.10

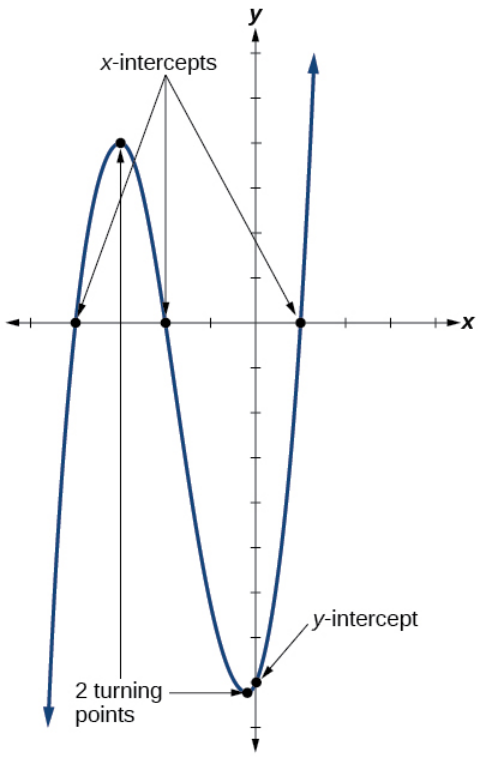



Figure 3.3.10

**Defintion: Intercepts and Turning Points of Polynomial Functions**

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The $y$-intercept is the point at which the function has an input value of zero. The $x$-intercepts are the points at which the output value is zero.

**How To: Given a polynomial function, determine the intercepts.**

1. Determine the $y$-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the $x$-intercepts by solving for the input values that yield an output value of zero.

## Example 3.3.8: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$ , written in factored form for your convenience, determine the $y$- and $x$-intercepts.

**Solution**

The $y$-intercept occurs when the input is zero, so substitute 0 for $x$.

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The $y$-intercept is $(0, 8)$.

The $x$-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{aligned} x-2 &= 0 \quad \text{or} \quad x+1 = 0 \quad \text{or} \quad x-4 = 0 \\ x &= 2 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 4 \end{aligned}$$

The $x$-intercepts are $(2, 0)$,$(-1, 0)$, and $(4, 0)$.

We can see these intercepts on the graph of the function shown in Figure 3.3.11.

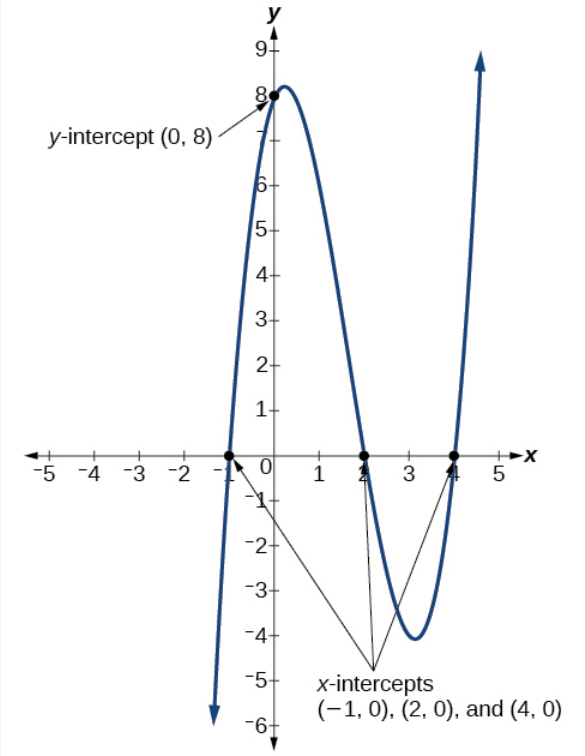



Figure 3.3.11: Graph of $f(x) = (x-2)(x+1)(x-4)$.

## Example 3.3.9: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$ , determine the $y$- and $x$-intercepts.

**Solution**

The $y$-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^2 - 45 \\ &= -45 \end{aligned}$$

The $y$-intercept is $(0, -45)$.

The $x$-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4 - 4x^2 - 45 \\ &= (x^2 - 9)(x^2 + 5) \\ &= (x-3)(x+3)(x^2+5) \end{aligned}$$

$$0 = (x-3)(x+3)(x^2+5)$$

$$\begin{array}{lllll} x-3=0 & \text{or} & x+3=0 & \text{or} & x^2+5=0 \\ x=3 & \text{or} & x=-3 & \text{or} & \text{(no real solution)} \end{array}$$

The $x$-intercepts are $(3, 0)$ and $(-3, 0)$.

We can see these intercepts on the graph of the function shown in Figure 3.3.12 We can see that the function is even because $f(x) = f(-x)$.

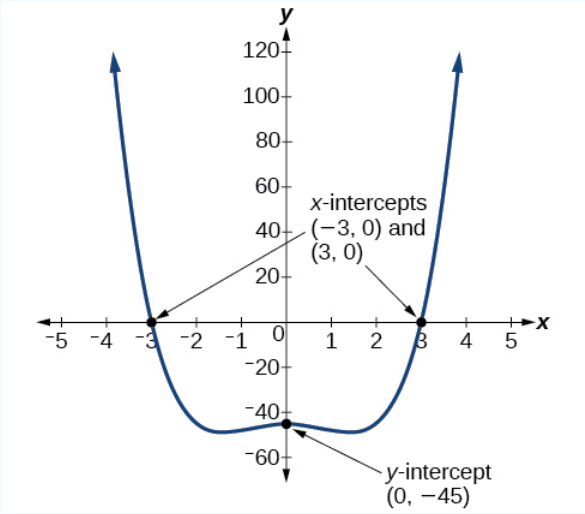



Figure 3.3.12: Graph of $f(x) = x^4 - 4x^2 - 45$ .

**Exercise 3.3.5**

**3.3.5:** Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$ , determine the $y$- and $x$-intercepts.

**Solution**

$y$-intercept $(0, 0)$; $x$-intercepts $(0, 0)$,$(-2, 0)$, and $(5, 0)$

## Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of $x$-intercepts and the number of turning points. A polynomial function of $n^{\text{th}}$ degree is the product of $n$ factors, so it will have at most $n$ roots or zeros, or $x$-intercepts. The graph of the polynomial function of degree $n$ must have at most $n-1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A smooth curve is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

### Intercepts and Turning Points of Polynomials

A polynomial of degree $n$ will have, at most, $n$ $x$-intercepts and $n-1$ turning points.

### Example 3.3.10: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of $x$-intercepts and turning points for $f(x)=-3x^{10}+4x^7-x^4+2x^3$ .

**Solution**

The polynomial has a degree of **10**, so there are at most $n$ $x$-intercepts and at most $n-1$ turning points.

### Exercise 3.3.6

Without graphing the function, determine the maximum number of $x$-intercepts and turning points for $f(x)=108-13x^9-8x^4+14x^{12}+2x^3$

**Answer**

There are at most 12 $x$-intercepts and at most 11 turning points.

### Example 3.3.11: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 3.3.12 based on its intercepts and turning points?

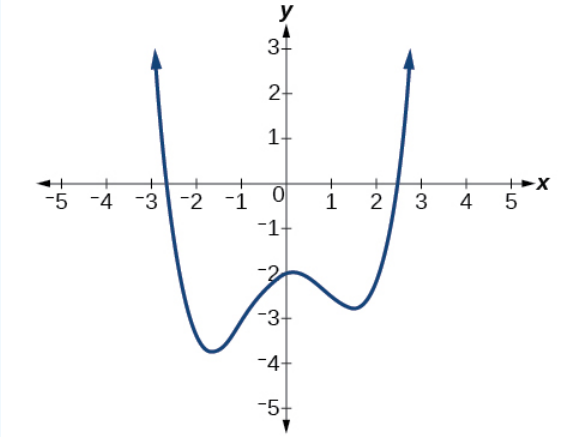



Figure 3.3.13.

**Solution**

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure 3.3.14


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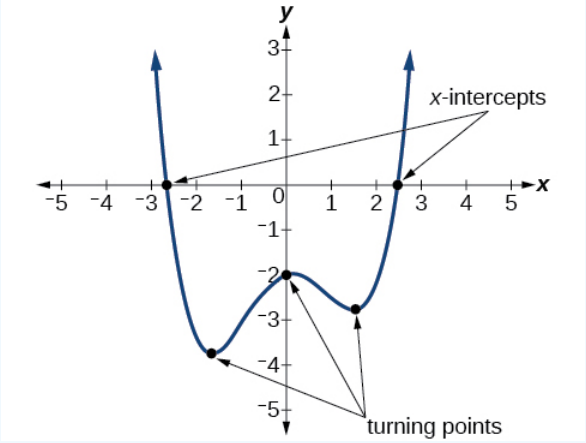



Figure 3.3.14: Graph of an even-degree polynomial.

The graph has 2 $x$-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

## Exercise 3.3.7

What can we conclude about the polynomial represented by the graph shown in Figure 3.3.15 based on its intercepts and turning points?

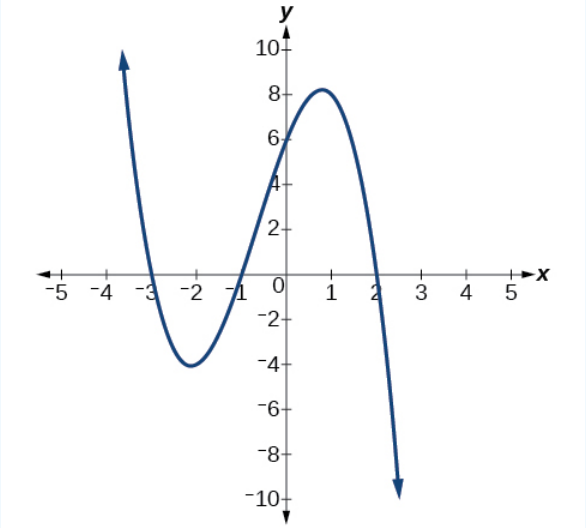



**Figure 3.3.15.**

**Answer**

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**Solution**

The end behavior indicates an odd-degree polynomial function; there are 3 $x$-intercepts and 2 turning points, so the degree is odd and at least 3. Because of the end behavior, we know that the lead coefficient must be negative.

### Example 3.3.12: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

**Solution**

The $y$-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The $y$-intercept is $(0, 0)$.

The $x$-intercepts are found by determining the zeros of the function.

$$\begin{aligned} 0 &= -4x(x+3)(x-4) \\ x &= 0 \quad \text{or} \quad x+3 = 0 \quad \text{or} \quad x-4 = 0 \\ x &= 0 \quad \text{or} \quad x = -3 \quad \text{or} \quad x = 4 \end{aligned}$$

The $x$-intercepts are $(0, 0)$,$(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

### Exercise 3.3.8

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

**Answer**

The $x$-intercepts are $(2, 0)$, $(-1, 0)$, and $(5, 0)$, the $y$-intercept is $(0, 2)$, and the graph has at most 2 turning points.

## Key Equations

- general form of a polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

## Key Concepts

- A power function is a variable base raised to a number power.
- The behavior of a graph as the input decreases without bound and increases without bound is called the end behavior.
- The end behavior depends on whether the power is even or odd.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function.
- A polynomial of degree $n$ will have at most $n$ $x$-intercepts and at most $n-1$ turning points.

## Glossary

**coefficient**

a nonzero real number that is multiplied by a variable raised to an exponent (only the number factor is the coefficient)

**continuous function**

a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

**degree**

the highest power of the variable that occurs in a polynomial

**end behavior**

the behavior of the graph of a function as the input decreases without bound and increases without bound

**leading coefficient**

the coefficient of the leading term

**leading term**

the term containing the highest power of the variable

**polynomial function**

a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

**power function**

a function that can be represented in the form $f(x) = kx^p$ where $k$ is a constant, the base is a variable, and the exponent, $p$, is a constant

**smooth curve**

a graph with no sharp corners

**term of a polynomial function**

any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

**turning point**

the location at which the graph of a function changes direction

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# 3.4: Graphs of Polynomial Functions

**Learning Objectives**

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in Table 3.4.1.

Table 3.4.1

| **Year** | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | 2013 |
|---|---|---|---|---|---|---|---|---|
| **Revenues** | 52.4 | 52.8 | 51.2 | 49.5 | 48.6 | 48.6 | 48.7 | 47.1 |

The revenue can be modeled by the polynomial function

$$R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332 \tag{3.4.1}$$

where $R$ represents the revenue in millions of dollars and $t$ represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing? These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

## Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure 3.4.1 shows a graph that represents a **polynomial function** and a graph that represents a function that is not a polynomial.

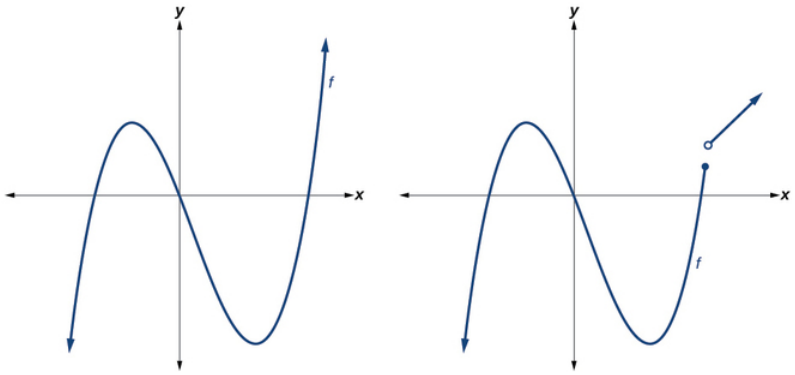



Figure 3.4.1: Graph of $f(x) = x^3 - 0.01x$.

**Example 3.4.1: Recognizing Polynomial Functions**

Which of the graphs in Figure 3.4.2 represents a polynomial function?

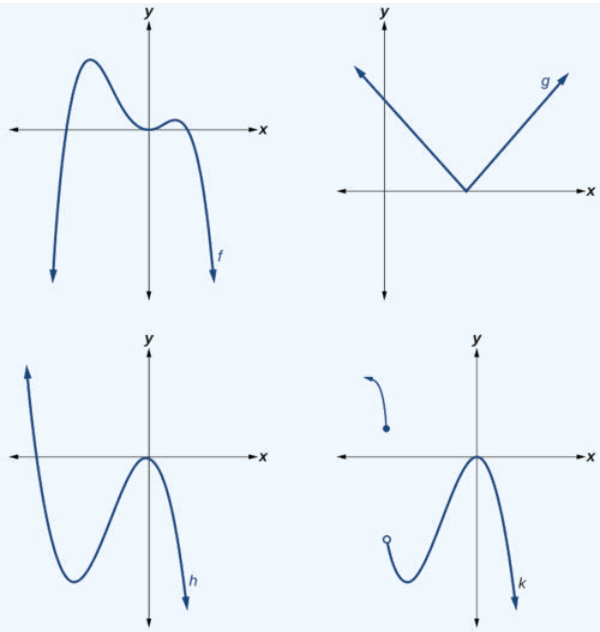



***Figure* 3.4.2**

**Solution**

- The graphs of $f$ and $h$ are graphs of polynomial functions. They are smooth and **continuous**.
- The graphs of $g$ and $k$ are graphs of functions that are not polynomials. The graph of function $g$ has a sharp corner. The graph of function $k$ is not continuous.

Q&A

Do all polynomial functions have as their domain all real numbers?

- Yes. Any real number is a valid input for a polynomial function.

## Using Factoring to Find Zeros of Polynomial Functions

Recall that if $f$ is a polynomial function, the values of $x$ for which $f(x) = 0$ are called **zeros** of $f$. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
The polynomial is given in factored form.
Technology is used to determine the intercepts.

HOwTO: Given a polynomial function $f$, find the x-intercepts by factoring

1. Set $f(x) = 0$.
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

## Example 3.4.2: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$ .

**Solution**

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

$$\begin{aligned} x^6 - 3x^4 + 2x^2 &= 0 && \text{Factor out the greatest common factor.} \\ x^2(x^4 - 3x^2 + 2) &= 0 && \text{Factor the trinomial.} \\ x^2(x^2 - 1)(x^2 - 2) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x^2 &= 0 \quad & (x^2 - 1) &= 0 \quad & (x^2 - 2) &= 0 \\ x^2 &= 0 \quad \text{or} \quad & x^2 &= 1 \quad \text{or} \quad & x^2 &= 2 \\ x &= 0 & x &= \pm 1 & x &= \pm\sqrt{2} \end{aligned}$$

.

This gives us five x-intercepts: $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(\sqrt{2}, 0)$,and $(-\sqrt{2}, 0)$ (Figure 3.4.3). We can see that this is an even function.

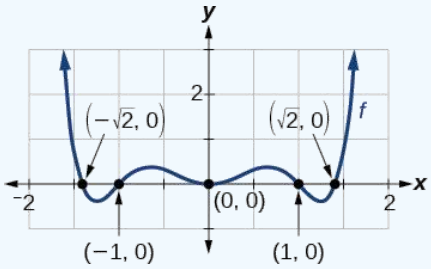



Figure 3.4.3.

## Example 3.4.3: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$ .

**Solution**

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 && \text{Factor by grouping.} \\ x^2(x - 5) - (x - 5) &= 0 && \text{Factor out the common factor.} \\ (x^2 - 1)(x - 5) &= 0 && \text{Factor the difference of squares.} \\ (x + 1)(x - 1)(x - 5) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x + 1 &= 0 \quad & x - 1 &= 0 \quad & x - 5 &= 0 \\ x &= -1 \quad \text{or} \quad & x &= 1 \quad \text{or} \quad & x &= 5 \end{aligned}$$

There are three x-intercepts: $(-1, 0)$, $(1, 0)$, and $(5, 0)$ (Figure 3.4.4).

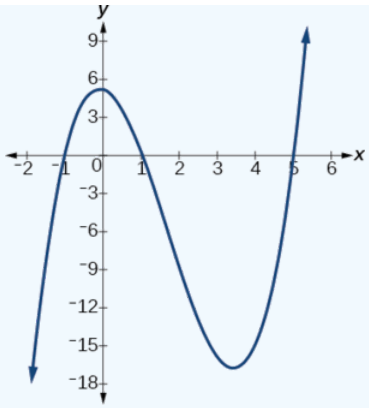



Figure 3.4.4: Graph of $f(x)$.

## Example 3.4.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x-2)^2(2x+3)$ .

**Solution**

The y-intercept can be found by evaluating $g(0)$.

$$\begin{aligned} g(0) &= (0-2)^2(2(0)+3) \\ &= 12 \end{aligned}$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0 \tag{3.4.2}$$

$$\begin{aligned} (x-2)^2 &= 0 \qquad\qquad (2x+3) = 0 \\ x-2 &= 0 \quad \text{or} \quad x = -\frac{3}{2} \\ x &= 2 \end{aligned}$$

So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

**Analysis**

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in Figure 3.4.5.

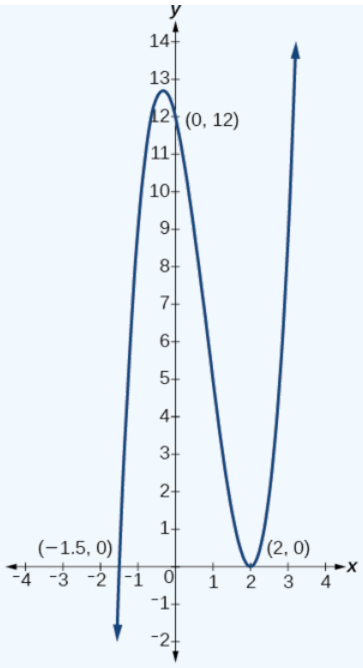



Figure 3.4.5: Graph of $g(x)$.

## Example 3.4.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$ .

**Solution**

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 3.4.6, it appears that there are x-intercepts at $x = -3, -2, \text{ and } 1$.

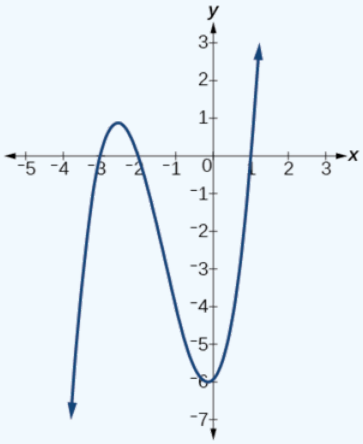



Figure 3.4.6: Graph of $h(x)$.

We can check whether these are correct by substituting these values for $x$ and verifying that

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6$ , we have:

$$\begin{aligned} h(-3) &= (-3)^3+4(-3)^2+(-3)-6=-27+36-3-6=0 \\ h(-2) &= (-2)^3+4(-2)^2+(-2)-6=-8+16-2-6=0 \\ h(1) &= (1)^3+4(1)^2+(1)-6=1+4+1-6=0 \end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$\begin{aligned} h(x) &= x^3+4x^2+x-6 \\ &= (x+3)(x+2)(x-1) \end{aligned}$$

**Exercise 3.4.1**

Find the y-and x-intercepts of the function $f(x)=x^4-19x^2+30x$ .

**Answer**

- y-intercept $(0,0)$;
- x-intercepts $(0,0)$, $(-5,0)$, $(2,0)$, and $(3,0)$

## Identifying Zeros and Their Multiplicities

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off. Suppose, for example, we graph the function

$$f(x)=(x+3)(x-2)^2(x+1)^3. \tag{3.4.3}$$

Notice in Figure 3.4.7 that the behavior of the function at each of the x-intercepts is different.

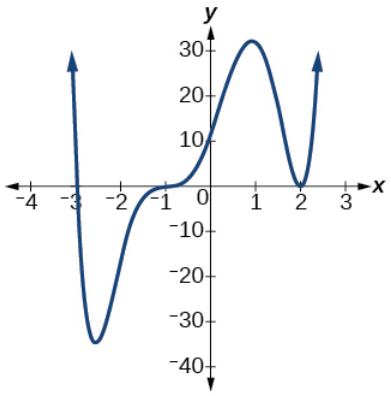



Figure 3.4.7: Identifying the behavior of the graph at an x-intercept by examining the multiplicity of the zero.

The x-intercept −3 is the solution of equation $(x+3)=0$ . The graph passes directly through thex-intercept at $x=-3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The x-intercept 2 is the repeated solution of equation $(x-2)^2=0$ . The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.

$$(x-2)^2=(x-2)(x-2) \tag{3.4.4}$$

The factor is repeated, that is, the factor $(x-2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x=2$, has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept −1 is the repeated solution of factor $(x+1)^3=0$ .The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near

the intercept as the toolkit function $f(x) = x^3$. We call this a triple zero, or a zero with multiplicity 3.

For **zeros** with even multiplicities, the graphs touch or are tangent to the x-axis. For zeros with odd multiplicities, the graphs cross or intersect the x-axis. See Figure 3.4.8 for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

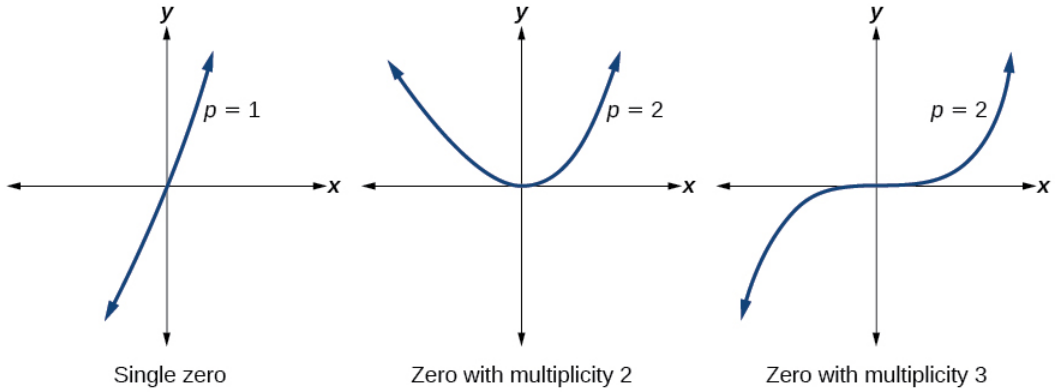



Figure 3.4.8: Three graphs showing three different polynomial functions with multiplicity 1, 2, and 3.

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

### Graphical Behavior of Polynomials at x-Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercepth is determined by the power $p$. We say that $x = h$ is a zero of **multiplicity** $p$.

The graph of a polynomial function will touch the x-axis at zeros with even multiplicities. The graph will cross the x-axis at zeros with odd multiplicities.

The sum of the multiplicities is the degree of the polynomial function.

### HOWTO: Given a graph of a polynomial function of degree $n$, identify the zeros and their multiplicities

1. If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is $n$.

### Example 3.4.6: Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure 3.4.9 to identify the zeros of the function and their possible multiplicities.

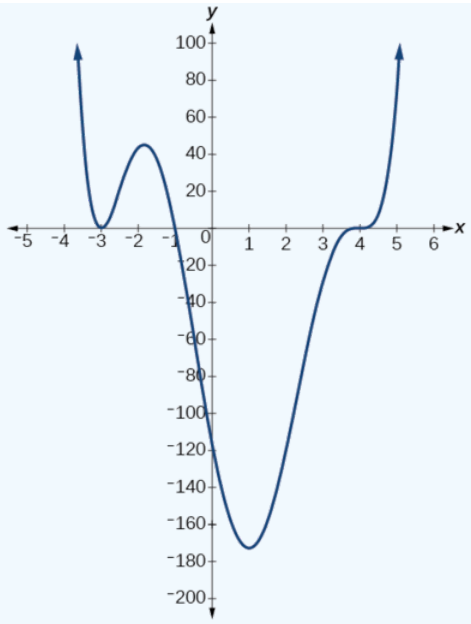



Figure 3.4.9: Graph of a polynomial function with degree 5.

**Solution**

The polynomial function is of degree $n$. The sum of the multiplicities must be $n$.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of −3 has multiplicity 2.

The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 4$.The graph crosses the x-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3 and that the sum of the multiplicities is likely 6.

**Exercise 3.4.2**

Use the graph of the function of degree 5 in Figure 3.4.10to identify the zeros of the function and their multiplicities.

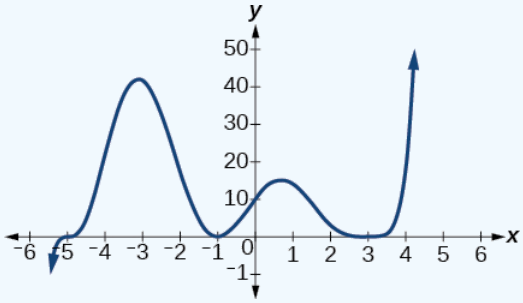



***Figure* 3.4.10*:*** *Graph of a polynomial function with degree 5.*

**Answer**

The graph has a zero of –5 with multiplicity 1, a zero of –1 with multiplicity 2, and a zero of 3 with even multiplicity.

## Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{3.4.5}$$

will either ultimately rise or fall as $x$ increases without bound and will either rise or fall as $x$ decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as $x$ increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as $x$ decreases without bound, $f(x)$ also decreases without bound; as $x$ increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure $3.4.11$ summarizes all four cases.

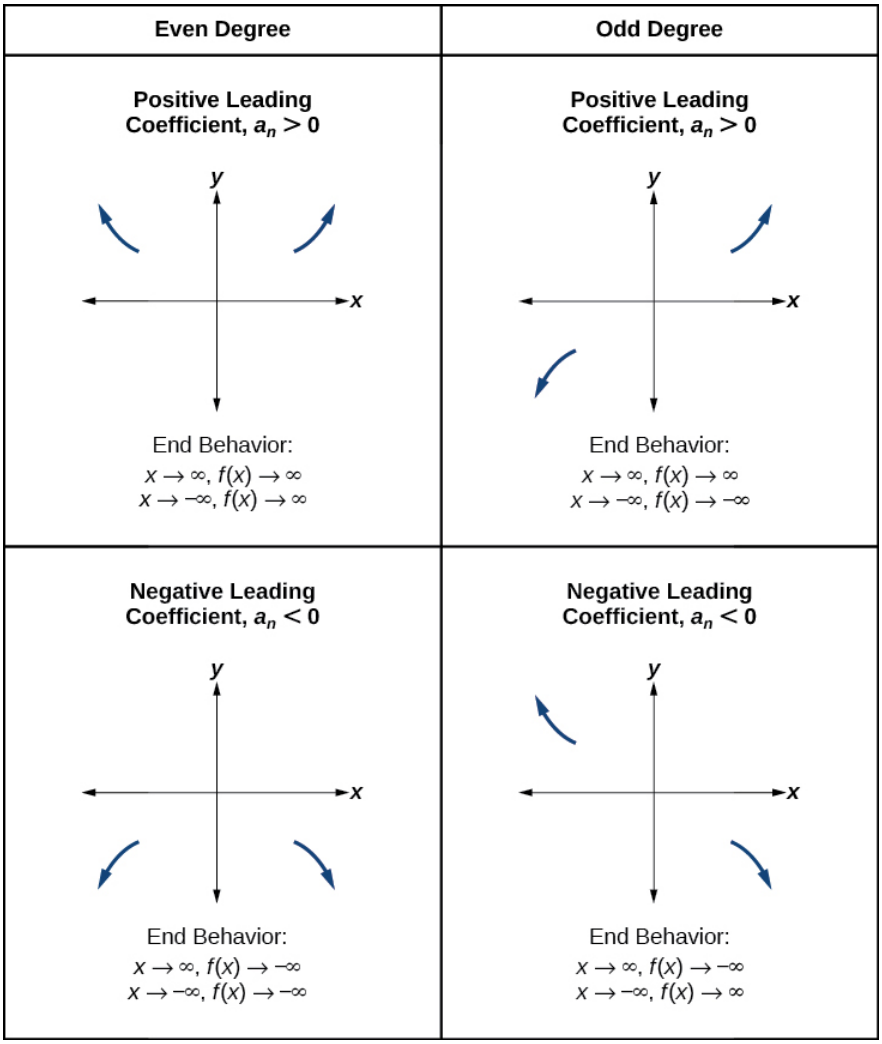



Figure 3.4.11.

## Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure $3.4.12$ The graph has three turning points.

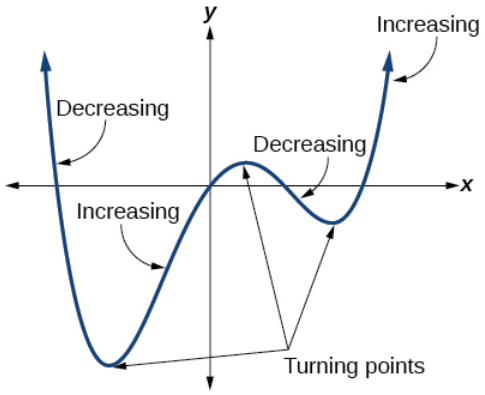



Figure 3.4.12: Graph of $f(x) = x^4 - x^3 - 4x^2 + 4x$

This function $f$ is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

**Definition: Interpreting Turning Points**

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). A polynomial of degree $n$ will have at most $n-1$ turning points.

**Example 3.4.7: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function**

Find the maximum number of turning points of each polynomial function.

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
b. $f(x) = -(x-1)^2(1+2x^2)$

**Solution**

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$

First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum number of turning points is $5 - 1 = 4$.

b. $f(x) = -(x-1)^2(1+2x^2)$

First, identify the leading term of the polynomial function if the function were expanded.

$$f(x) = -(x-1)^2\ (1+2x^2)$$
$$a_n = -(x^2)\ (2x^2) - 2x^4$$

Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4 - 1 = 3$.

## Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

**Howto: Given a polynomial function, sketch the graph**

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.

4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

## Example 3.4.8: Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

**Solution**

This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$,the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The y-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure 3.4.13

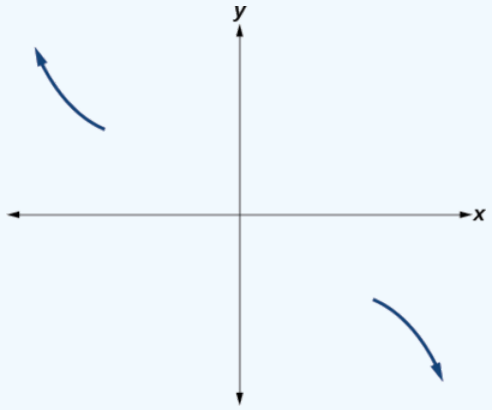



Figure 3.4.13: Showing the distribution for the leading term.

To sketch this, we consider that:

- As $x \to -\infty$ the function $f(x) \to \infty$,so we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of thex-axis, so the function must start increasing.
- At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure 3.4.14

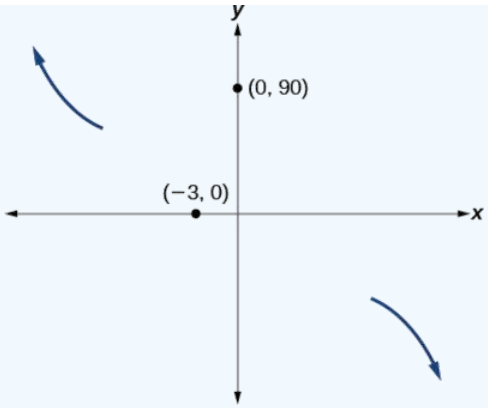



Figure 3.4.14: Graph of the end behavior and intercepts, $(-3,0)$ and $(0,90)$, for the function $f(x)=-2(x+3)^2(x-5)$.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5,0)$. See Figure 3.4.15.

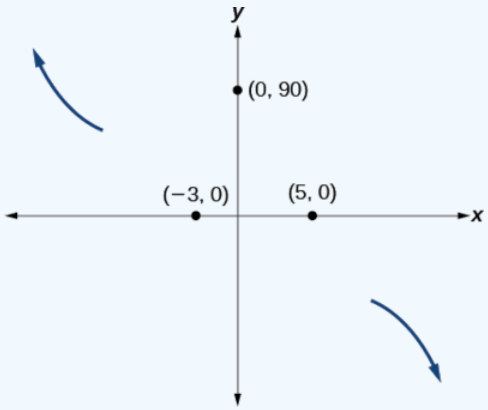



Figure 3.4.15: Graph of the end behavior and intercepts, $(-3,0)$, $(0,90)$ and $(5,0)$, for the function $f(x)=-2(x+3)^2(x-5)$.

As $x\to\infty$ the function $f(x)\to-\infty$,

so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure 3.4.16, and verify that the resulting graph looks like our sketch in Figure 3.4.15.

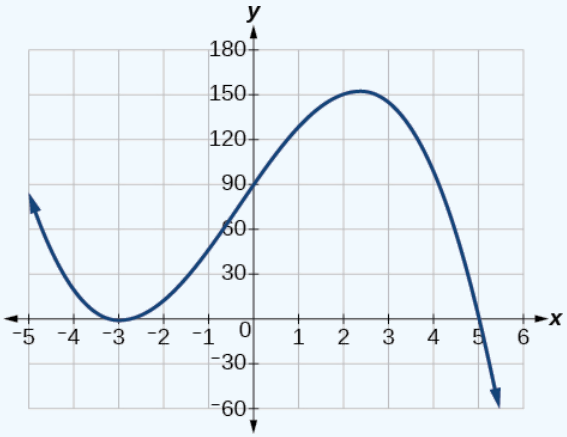



***Figure 3.4.16:*** *The complete graph of the polynomial function* $f(x)=-2(x+3)^2(x-5)$.


https://math.libretexts.org/@go/page/69087

**Exercise 3.4.8**

Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

**Answer**

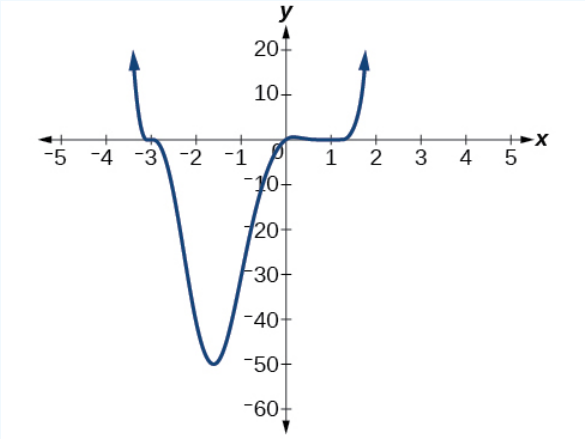



Figure 3.4.17: Graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$

## Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function $f$ whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers $a$ and $b$ in the domain of $f$,if $a < b$ and $f(a) \neq f(b)$,then the function $f$ takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function $f$ at $x = a$ lies above the x-axis and another point at $x = b$ lies below thex-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. Call this point $(c, f(c))$.This means that we are assured there is a solution $c$ where $f(c) = 0$.

In other words, the **Intermediate Value Theorem** tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. Figure 3.4.18shows that there is a zero between $a$ and $b$.

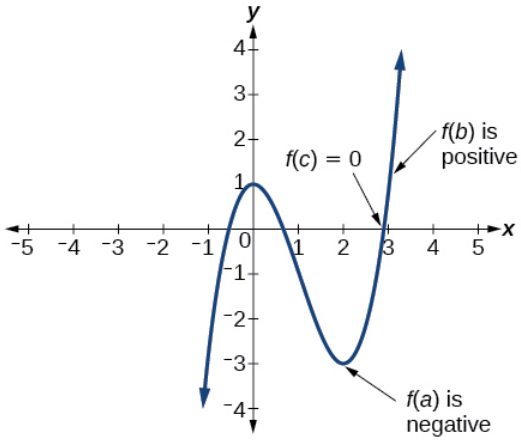



Figure 3.4.18: Using the Intermediate Value Theorem to show there exists a zero.

**Definition: Intermediate Value Theorem**

Let $f$ be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value $c$ between $a$ and $b$ for which $f(c) = 0$.

## Example 3.4.9: Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

**Solution**

As a start, evaluate $f(x)$ at the integer values $x = 1,\ 2,\ 3,$ and $4$ (Table 3.4.2).

Table 3.4.2

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 5 | 0 | -3 | 2 |

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.

We have shown that there are at least two real zeros between $x = 1$ and $x = 4$.

**Analysis**

We can also see on the graph of the function in Figure 3.4.19 that there are two real zeros between $x = 1$ and $x = 4$.

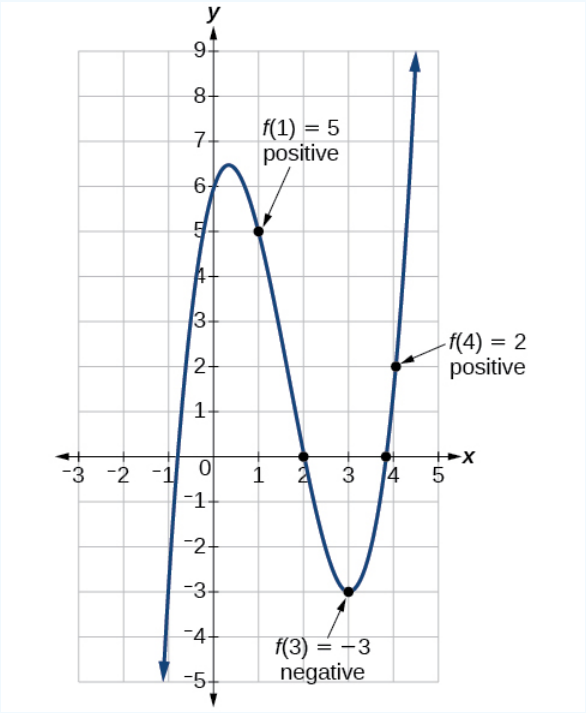



Figure 3.4.19.

## Exercise 3.4.4

Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

**Answer**

Because $f$ is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x = 1$ and $x = 2$.

## Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a **polynomial function** written in factored form will have an x-intercept where each factor is equal to zero, we can form a function

that will pass through a set of x-intercepts by introducing a corresponding set of factors.

**Note: Factored Form of Polynomials**

If a polynomial of lowest degree $p$ has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form: $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$ where the powers $p_i$ on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor $a$ can be determined given a value of the function other than the x-intercept.

**How To: Given a graph of a polynomial function, write a formula for the function.**

1. Identify the x-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the x-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

**Example 3.4.10: Writing a Formula for a Polynomial Function from the Graph**

Write a formula for the polynomial function shown in Figure 3.4.20.

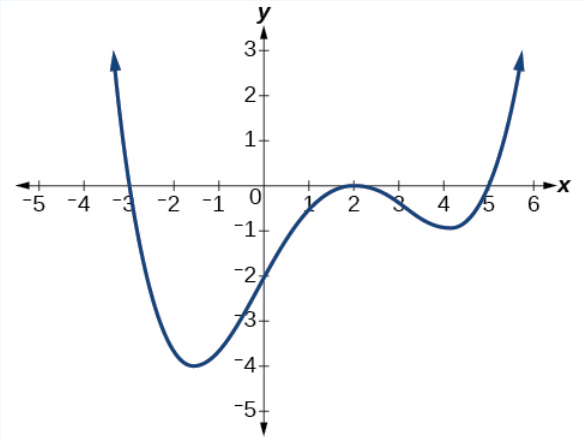



Figure 3.4.20.

**Solution**

This graph has three x-intercepts: $x = -3,\ 2,$ and $5$. The y-intercept is located at $(0, 2)$.At $x = -3$ and $x = 5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x) = a(x+3)(x-2)^2(x-5) \tag{3.4.6}$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, -2)$, to solve for $a$.

$$\begin{aligned} f(0) &= a(0+3)(0-2)^2(0-5) \\ -2 &= a(0+3)(0-2)^2(0-5) \\ -2 &= -60a \\ a &= \frac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$ .

**Exercise 3.4.5**

Given the graph shown in Figure 3.4.21, write a formula for the function shown.

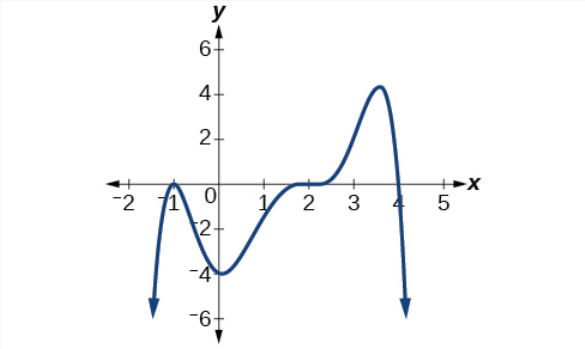



Figure 3.4.21.

**Answer**

$f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

## Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

**Note: Local and Global Extrema**

A **local maximum** or **local minimum** at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$.If a function has a local maximum at $a$, then $f(a) \geq f(x)$for all $x$ in an open interval around $x = a$. If a function has a local minimum at $a$, then $f(a) \leq f(x)$for all $x$ in an open interval around $x = a$.

A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at $a$, then $f(a) \geq f(x)$ for all $x$. If a function has a global minimum at $a$, then $f(a) \leq f(x)$ for all $x$.

We can see the difference between local and global extrema in Figure 3.4.22

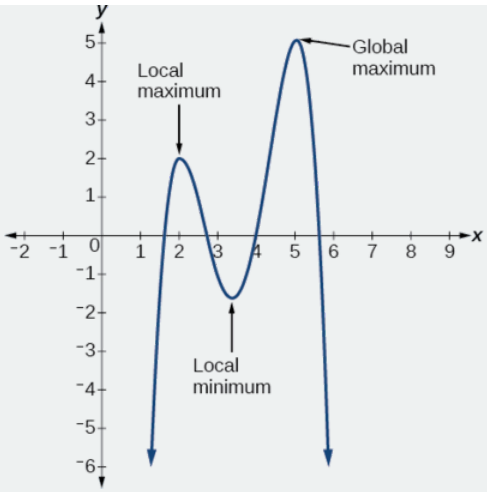



Figure 3.4.22: Graph of an even-degree polynomial that denotes the local maximum and minimum and the global maximum.

**Do all polynomial functions have a global minimum or maximum?**

*No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.*

## Example 3.4.11: Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

**Solution**

We will start this problem by drawing a picture like that in Figure 3.4.23, labeling the width of the cut-out squares with a variable, $w$.

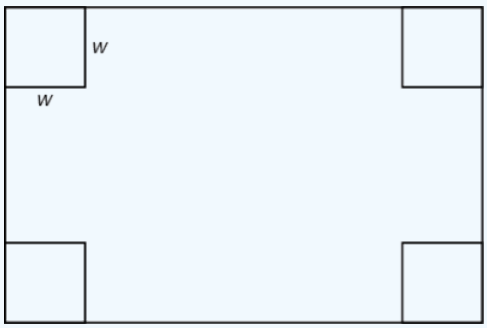



Figure 3.4.23: Diagram of a rectangle with four squares at the corners.

Notice that after a square is cut out from each end, it leaves $a(14-2w)$ cm by $(20-2w)$ cm rectangle for the base of the box, and the box will be $w$ cm tall. This gives the volume

$$\begin{aligned} V(w) &= (20-2w)(14-2w)w \\ &= 280w - 68w^2 + 4w^3 \end{aligned}$$

Notice, since the factors are $20-2w$, $14-2w$, and $w$, the three zeros are $10$, $7$, and $0$, $)respectively. Becauseaheightof\(0$ cm is not reasonable, we consider the only the zeros $10$ and $7$. The shortest side is $14$ and we are cutting off two squares, so values $w$ may take on are greater than zero or less than $7$. This means we will restrict the domain of this function to $0 < w < 7$.Using technology to sketch the graph of $V(w)$ on this reasonable domain, we get a graph like that in Figure 3.4.24 We can use this graph to estimate the maximum value for the volume, restricted to values for $w$ that are reasonable for this problem—values from 0 to 7.

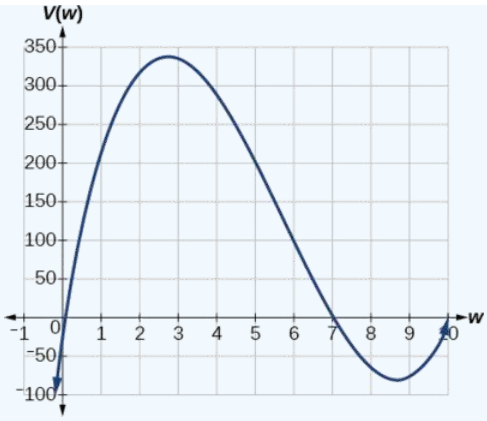



Figure 3.4.24: Graph of $V(w) = (20 - 2w)(14 - 2w)w$

From this graph, we turn our focus to only the portion on the reasonable domain, $[0, 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce Figure 3.4.25

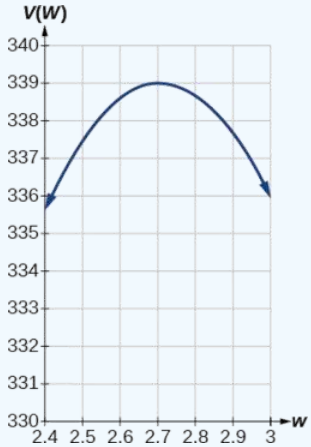



Figure 3.4.25: Graph of $V(w) = (20 - 2w)(14 - 2w)w$

From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

**Exercise 3.4.1**

Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x - 2)^3(x + 1)^2(x - 4)$.

**Answer**

The minimum occurs at approximately the point $(0, -6.5)$,
and the maximum occurs at approximately the point $(3.5, 7)$.

## Key Concepts

- Polynomial functions of degree 2 or more are smooth, continuous functions.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero.
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis.
- The multiplicity of a zero determines how the graph behaves at the x-intercepts.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.

- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree $n$ has at most $n-1$ turning points.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points.
- Graphing a polynomial function helps to estimate local and global extremas.
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value $c$ between $a$ and $b$ for which $f(c)=0$.

## Glossary

**global maximum**
highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all $x$.

**global minimum**
lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all $x$.

**Intermediate Value Theorem**
for two numbers $a$ and $b$ in the domain of $f$, if $a<b$ and $f(a) \neq f(b)$, then the functionf takes on every value between $f(a)$ and $f(b)$; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis

**multiplicity**
the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x-h)^p$, $x=h$ is a zero of multiplicity $p$.

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# 3.5: Dividing Polynomials

**Learning Objectives**

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length $61.5$ meters (m), width $40$ m, and height $30$ m.[1]

We can easily find the volume using elementary geometry.

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 61.5 \cdot 40 \cdot 30 \\ &= 73,800 \end{aligned}$$

So the volume is $73,800$ cubic meters ($m^3$).



Figure 3.5.1: Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

Suppose we knew the volume, length, and width. We could divide to find the height.

$$\begin{aligned} h &= \frac{V}{l \cdot w} \\ &= \frac{73,800}{61.5 \cdot 40} \\ &= 30 \end{aligned}$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$ . The length of the solid is given by $3x$; the width is given by $x - 2$.

To find the height of the solid, we can use polynomial division, which is the focus of this section.

## Using Long Division to Divide Polynomials

We are familiar with the **long division** algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

**Long Division**

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ \underline{-15}\downarrow \\ 28 \\ \underline{-27} \\ 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned}\text{dividend} &= (\text{divisor}\cdot\text{quotient})+\text{remainder}\\ 178 &= (3\cdot 59)+1\\ &= 177+1\\ &= 178\end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm, it would look like this:

$$\begin{array}{rl}
x+2\overline{)2x^3-3x^2+4x+5} & \text{Set up the division problem.}\\[2ex]
\phantom{x+2\overline{)}}2x^2 & 2x^3 \text{ divided by } x \text{ is } 2x^2.\\
x+2\overline{)2x^3-3x^2+4x+5} & \\[2ex]
\phantom{x+2\overline{)}}2x^2 & \\
x+2\overline{)2x^3-3x^2+4x+5} & \\
\underline{-\left(2x^3+4x^2\right)\phantom{+5}} & \text{Multiply } x+2 \text{ by } 2x^2 \text{ and subtract.}\\
\phantom{-(}-7x^2+4x & \text{Bring down the next term.}\\[2ex]
\phantom{x+2\overline{)}}2x^2-7x & -7x^2 \text{ divided by } x \text{ is } -7x.\\
x+2\overline{)2x^3-3x^2+4x+5} & \\
\underline{-\left(2x^3+4x^2\right)\phantom{+5}} & \\
\phantom{-(}-7x^2+4x & \\
\underline{\phantom{-(}-\left(-7x^2-14x\right)} & \text{Multiply } x+2 \text{ by } -7x.\\
\phantom{-(-}18x+5 & \text{Subtract and bring down the next term.}\\[2ex]
\phantom{x+2\overline{)}}2x^2-7x+18 & 18x \text{ divided by } x \text{ is } 18.\\
x+2\overline{)2x^3-3x^2+4x+5} & \\
\underline{-\left(2x^3+4x^2\right)\phantom{+5}} & \\
\phantom{-(}-7x^2+4x & \\
\underline{\phantom{-(}-\left(-7x^2-14x\right)} & \\
\phantom{-(-}18x+\ \ 5 & \\
\underline{\phantom{-(-}-(18x+36)} & \text{Multiply } x+2 \text{ by } 18.\\
\phantom{-(-}-31 & \text{Subtract.}
\end{array}$$

We have found

$$\frac{2x^3-3x^2+4x+5}{x+2}=2x^2-7x+18-\frac{31}{x+2}$$

or

$$2x^3-3x^2+4x+5=(x+2)(2x^2-7x+18)-31$$

We can identify the dividend, the **divisor**, the **quotient**, and the **remainder**.

Writing the result in this manner illustrates the Division Algorithm.

## The Division Algorithm

The Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x) \tag{3.5.1}$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

If $r(x) = 0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

## Given a polynomial and a binomial, use long division to divide the polynomial by the binomial

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom **binomial** from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

## Example 3.5.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

**Solution**

| | |
|---|---|
| $x+1\ \overline{)\,5x^2+3x-2}$ | Set up division problem. |
| $5x$<br>$x+1\ \overline{)\,5x^2+3x-2}$ | $5x^2$ divided by $x$ is $5x$. |
| $5x$<br>$x+1\ \overline{)\,5x^2+3x-2}$ | Multiply $x+1$ by $5x$. |
| $5x$<br>$x+1\ \overline{)\,5x^2+3x-2}$<br>$-(5x^2+5x)$ | Subtract. |
| $5x$<br>$x+1\ \overline{)\,5x^2+3x-2}$<br>$-(5x^2+5x)$<br>$-2x-2$ | Bring down the next term.<br>$-2x$ divided by $x$ is $-2$. |
| $5x-2$<br>$x+1\ \overline{)\,5x^2+3x-2}$<br>$-(5x^2+5x)$<br>$-2x-2$ | |
| $5x-2$<br>$x+1\ \overline{)\,5x^2+3x-2}$<br>$-(5x^2+5x)$<br>$-2x-2$<br>$-(-2x-2)$ | Multiply $x+1$ by $-2$. |
| $5x-2$<br>$x+1\ \overline{)\,5x^2+3x-2}$<br>$-(5x^2+5x)$<br>$-2x-2$<br>$-(-2x-2)$<br>$0$ | Subtract. |

The quotient is $5x - 2$. The remainder is 0. We write the result as

$$\frac{5x^2+3x-2}{x+1} = 5x - 2$$

or

$$5x^2 + 3x - 2 = (x+1)(5x-2)$$

**Analysis**

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

## Example 3.5.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

**Solution**

$$
\begin{array}{rl}
2x^2+\ 5x-\ 7 & 6x^3 \text{ divided by } 3x \text{ is } 2x^2. \\
3x-2\overline{)6x^3+11x^2-31x+15} & \\
\underline{-\left(6x^3-4x^2\right)} & \text{Multiply } 3x-2 \text{ by } 2x^2. \\
15x^2-31x & \text{Subtract. Bring down next term. } 15x^2 \text{ divided by } 3x \text{ is } 5x. \\
\underline{-\left(15x^2-10x\right)} & \text{Multiply } 3x-2 \text{ by } 5x. \\
-21x+15 & \text{Subtract. Bring down next term. } -21x \text{ divided by } 3x \text{ is } -7. \\
\underline{-(-21x+14)} & \text{Multiply } 3x-2 \text{ by } -7. \\
1 & \text{Subtract. The remainder is 1.}
\end{array}
$$

There is a remainder of 1. We can express the result as:

$$\frac{6x^3+11x^2-31x+15}{3x-2}=2x^2+5x-7+\frac{1}{3x-2}$$

**Analysis**

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x-2)(2x^2+5x-7)+1=6x^3+11x^2-31x+15$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is $1$

**Try It! 3.5.2**

Divide $16x^3-12x^2+20x-3$ by $4x+5$.

**Solution**

$$4x^2-8x+15-\frac{78}{4x+5}$$

## Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm.

The final form of the process looked like this:

$$
\begin{array}{r}
2x^2-7x+18 \\
x+2\overline{)2x^3-3x^2+4x+5} \\
\underline{-\left(2x^3+4x^2\right)} \\
-7x^2+4x \\
\underline{-\left(-7x^2-14x\right)} \\
18x+\ 5 \\
\underline{-(18x+36)} \\
-31
\end{array}
$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$
\begin{array}{r|rrrr}
2 & 2 & -3 & 4 & 5 \\
 & -2 & -4 & & \\
\hline
 & & -7 & 14 & \\
 & & & 18 & -36 \\
\hline
 & & & & -31
\end{array}
$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$
\begin{array}{r|rrrr}
-2 & 2 & -3 & 4 & 5 \\
 & & -4 & 14 & -36 \\
\hline
 & 2 & -7 & 18 & -31
\end{array}
$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2-7x+18$ and the remainder is $-31$. The process will be made more clear in Example $3.5.3$.

### Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x-k$. In *synthetic division*, only the coefficients are used in the division process.

### Given two polynomials, use synthetic division to divide

1. Write $k$ for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by $k$. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by $k$. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

### Example $3.5.3$: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2-3x-36$ by $x-3$.

**Solution**

Begin by setting up the synthetic division. Write $k$ and the coefficients.

$$
\begin{array}{r|rrr}
3 & 5 & -3 & -36 \\
\hline
 & & &
\end{array}
$$

Bring down the lead coefficient. Multiply the lead coefficient by $k$.

$$
\begin{array}{r|rrr}
3 & 5 & -3 & -36 \\
 & & 15 & \\
\hline
 & 5 & &
\end{array}
$$

Continue by adding the numbers in the second column. Multiply the resulting number by $k$.Write the result in the next column. Then add the numbers in the third column.

$$
\begin{array}{r|rrr}
3 & 5 & -3 & -36 \\
 & & 15 & 36 \\
\hline
 & 5 & 12 & 0
\end{array}
$$

The result is $5x+12$. The remainder is 0. So $x-3$ is a factor of the original polynomial.

**Analysis**

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x-3)(5x+12)+0=5x^2-3x-36$$

### Example 3.5.4: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3+10x^2-6x-20$ by $x+2$.

**Solution**

The binomial divisor is $x+2$ so $k=-2$. Add each column, multiply the result by –2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -2 & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The result is $4x^2+2x-10$.

The remainder is 0. Thus, $x+2$ is a factor of $4x^3+10x^2-6x-20$.

**Analysis**

The graph of the polynomial function $f(x)=4x^3+10x^2-6x-20$ in Figure 3.5.2 shows a zero at $x=k=-2$. This confirms that $x+2$ is a factor of $4x^3+10x^2-6x-20$.

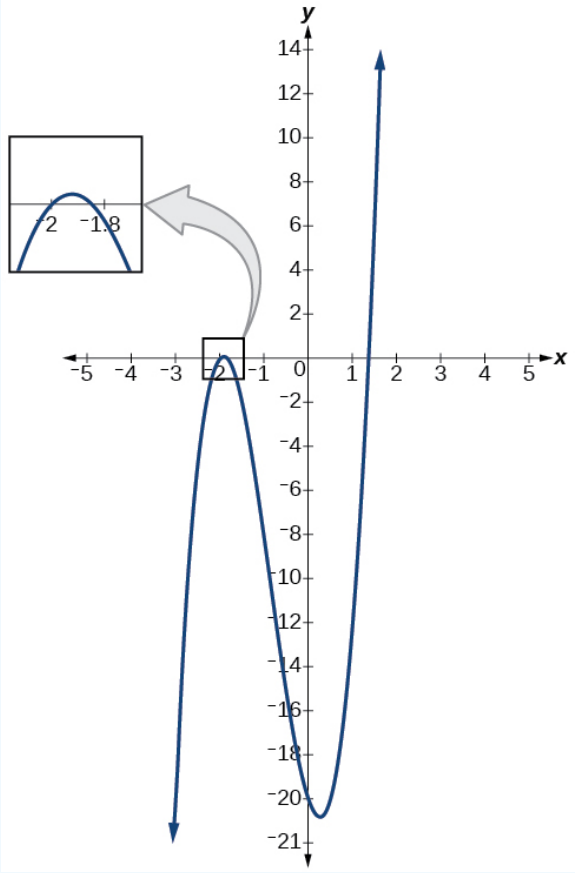



Figure 3.5.2

### Example 3.5.5: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4+10x^3+7x^2-6$ by $x-1$.

**Solution**

Notice there is no $x$-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3+x^2+8x+8+\dfrac{2}{x-1}$ .

### Try It! 3.5.3

Use synthetic division to divide $3x^4+18x^3-3x+40$ by $x+7$.

**Solution**

$$3x^3-3x^2+21x-150+\frac{1090}{x+7}$$

## Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

### Example 3.5.6: Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4-3x^3-33x^2+54x$ . The length of the solid is given by $3x$ and the width is given by $x-2$. Find the height $h$ of the solid.

**Solution**

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in Figure 3.5.3. Let $h$ equal the height of the box.

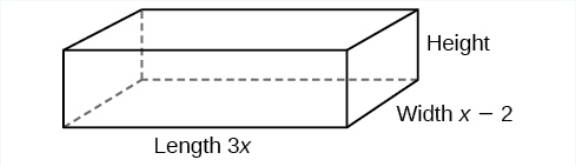



Figure 3.5.3

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$\begin{aligned} V &= l\cdot w\cdot h \\ 3x^4-3x^3-33x^2+54x &= 3x\cdot(x-2)\cdot h \end{aligned}$$

To solve for $h$, first divide both sides by $3x$.

$$\frac{3x\cdot(x-2)\cdot h}{3x}=\frac{3x^4-3x^3-33x^2+54x}{3x}$$

$$(x-2)h=\frac{x^3-x^2-11x+18}{x-2}$$

Now solve for $h$ using synthetic division.

$$h = \frac{x^3 - x^2 - 11x + 18}{x - 2}$$

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The quotient is $x^2 + x - 9$ and the remainder is $0$. The height of the solid is $x^2 + x - 9$ .

**Try It! 3.5.4**

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$ . The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

**Solution**

$3x^2 - 4x + 1$

## Key Equations

Division Algorithm $f(x) = d(x)q(x) + r(x)$ where $q(x) \neq 0$

## Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x - k$.
- Polynomial division can be used to solve application problems, including area and volume.

## Footnotes

[1] National Park Service. "Lincoln Memorial Building Statistics." www.nps.gov/linc/historycultu...statistics.htm. Accessed 4/3/2014

## Glossary

**Division Algorithm**

given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

**synthetic division**

a shortcut method that can be used to divide a polynomial by a binomial of the form $x - k$

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# 3.6: Zeros of Polynomial Functions

**Learning Objectives**

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zero Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs.
- Solve real-world applications of polynomial equations

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

## Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x-k$, the remainder may be found quickly by evaluating the polynomial function at $k$, that is, $f(k)$. Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$,there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x)=d(x)q(x)+r(x)$$

If the divisor, $d(x)$, is $x-k$, this takes the form

$$f(x)=(x-k)q(x)+r$$

Since the divisor $x-k$

is linear, the remainder will be a constant, $r$. And, if we evaluate this for $x=k$, we have

$$\begin{aligned} f(k) &= (k-k)q(k)+r \\ &= 0\cdot q(k)+r \\ &= r \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$by $x-k$.

**The Remainder Theorem**

If a polynomial $f(x)$ is divided by $x-k$,then the remainder is the value $f(k)$.

How To: Given a polynomial function $f$, evaluate $f(x)$ at $x=k$ using the Remainder Theorem.

1. Use synthetic division to divide the polynomial by $x-k$.
2. The remainder is the value $f(k)$.

**Example 3.6.1: Using the Remainder Theorem to Evaluate a Polynomial**

Use the Remainder Theorem to evaluate $f(x)=6x^4-x^3-15x^2+2x-7$ at $x=2$.

**Solution**

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x-2$.

$$\begin{array}{r|rrrrr} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array} \tag{3.6.1}$$

The remainder is 25. Therefore, $f(2)=25$.

**Analysis**

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4-x^3-15x^2+2x-7 \\ f(2) &= 6(2)^4-(2)^3-15(2)^2+2(2)-7 \\ &= 25 \end{aligned}$$

**Exercise 3.6.1**

se the Remainder Theorem to evaluate $f(x)=2x^5-3x^4-9x^3+8x^2+2$ at $x=-3$.

**Answer**

$f(-3)=-412$

## Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm.

$$f(x)=(x-k)q(x)+r \tag{3.6.2}$$

If $k$ is a zero, then the remainder $r$ is $f(k)=0$ and $f(x)=(x-k)q(x)+0$ or $f(x)=(x-k)q(x)$.

Notice, written in this form, $x-k$ is a factor of $f(x)$. We can conclude if $k$ is a zero of $f(x)$, then $x-k$ is a factor of $f(x)$.

Similarly, if $x-k$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x)=(x-k)q(x)+r$ is 0. This tells us that $k$ is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree $n$ in the complex number system will have $n$ zeros. We can use the Factor Theorem to completely factor a polynomial into the product of $n$ factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

**THE FACTOR THEOREM**

According to the Factor Theorem, $k$ is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$.

**How to: Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial**

1. Use synthetic division to divide the polynomial by $(x-k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x-k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

**Example 3.6.2: Using the Factor Theorem to Solve a Polynomial Equation**

Show that $(x+2)$ is a factor of $x^3-6x^2-x+30$ . Find the remaining factors. Use the factors to determine the zeros of the **polynomial**.

**Solution**

We can use synthetic division to show that $(x+2)$ is a factor of the polynomial.

$$\begin{array}{r|cccc} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array} \tag{3.6.3}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x+2)(x^2-8x+15) \tag{3.6.4}$$

We can factor the quadratic factor to write the polynomial as

$$(x+2)(x-3)(x-5) \tag{3.6.5}$$

By the Factor Theorem, the zeros of $x^3-6x^2-x+30$ are –2, 3, and 5.

**Exercise 3.6.2**

Use the Factor Theorem to find the zeros of $f(x)=x^3+4x^2-4x-16$ given that $(x-2)$ is a factor of the polynomial.

**Answer**

The zeros are 2, –2, and –4.

## Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading **coefficient** of the polynomial

Consider a quadratic function with two zeros, $x=\frac{2}{5}$ and $x=\frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.

| | |
|---|---|
| $x-\frac{2}{5}=0$ or $x-\frac{3}{4}=0$ | Set each factor equal to 0. |
| $5x-2=0$ or $4x-3=0$ | Multiply both sides of the equation to eliminate fractions. |
| $f(x)=(5x-2)(4x-3)$ | Create the quadratic function, multiplying the factors. |
| $f(x)=20x^2-23x+6$ | Expand the polynomial. |
| $f(x)=(5\cdot 4)x^2-23x+(2\cdot 3)$ | |

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

**THE RATIONAL ZERO THEOREM**

The Rational Zero Theorem states that, if the polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where $p$ is a factor of the constant term $a_0$ and $q$ is a factor of the leading coefficient $a_n$.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

**How to: Given a polynomial function $f(x)$, use the Rational Zero Theorem to find rational zeros.**

1. Determine all factors of the constant term and all factors of the leading coefficient.

2. Determine all possible values of $\frac{p}{q}$, where $p$ is a factor of the constant term and $q$ is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f(\frac{p}{q})$.

### Example 3.6.3: Listing All Possible Rational Zeros

List all possible rational zeros of $f(x)=2x^4-5x^3+x^2-4$ .

**Solution**

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, –4, and the factors of the leading coefficient, 2.

The constant term is –4; the factors of –4 are $p=\pm1,\pm2,\pm4$.

The leading coefficient is 2; the factors of 2 are $q=\pm1,\pm2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of –4 divided by one of the factors of 2.

$$\frac{p}{q}=\pm\frac{1}{1},\pm\frac{1}{2} \qquad \frac{p}{q}=\pm\frac{2}{1},\pm\frac{2}{2} \qquad \frac{p}{q}=\pm\frac{4}{1},\pm\frac{4}{2}$$

Note that $\frac{2}{2}=1$ and $\frac{4}{2}=2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q}=\frac{\text{Factors of the last}}{\text{Factors of the first}}=\pm1,\pm2,\pm4,\pm\frac{1}{2}$$

### Example 3.6.4: Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x)=2x^3+x^2-4x+1$ .

**Solution**

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then $p$ is a factor of 1 and $q$ is a factor of 2.

$$\frac{p}{q}=\frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$

$$=\frac{factor\ of\ 1}{factor\ of\ 2}$$

The factors of 1 are ±1 and the factors of 2 are ±1 and ±2. The possible values for $\frac{p}{q}$ are ±1 and $\pm\frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for $x$ in $f(x)$.

$$f(-1)=2(-1)^3+(-1)^2-4(-1)+1=4 \tag{3.6.6}$$

$$f(1)=1(1)^3+(1)^2-4(1)+1=0 \tag{3.6.7}$$

$$f(-\frac{1}{2})=2(-\frac{1}{2})^3+(-\frac{1}{2})^2-4(-\frac{1}{2})+1=3 \tag{3.6.8}$$

$$f(\frac{1}{2})=2(\frac{1}{2})^3+(\frac{1}{2})^2-4(\frac{1}{2})+1=-\frac{1}{2} \tag{3.6.9}$$

Of those, $-1,-\frac{1}{2}$, and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

**Exercise 3.6.3**

Use the Rational Zero Theorem to find the rational zeros of $f(x) = x^3 - 5x^2 + 2x + 1$ .

**Answer**

There are no rational zeros.

## Finding the Zeros of Polynomial Functions

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use **synthetic division** repeatedly to determine all of the **zeros** of a polynomial function.

**How to: Given a polynomial function $f$, use synthetic division to find its zeros.**

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.
4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

**Example 3.6.5: Finding the Zeros of a Polynomial Function with Repeated Real Zeros**

Find the zeros of $f(x) = 4x^3 - 3x - 1$ .

**Solution**

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then $p$ is a factor of –1 and $q$ is a factor of 4.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$= \frac{factor\ of\ -1}{factor\ of\ 4}$$

The factors of –1 are ±1 and the factors of 4 are ±1,±2, and ±4. The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}$, and $\pm\frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x-1)(4x^2+4x+1)$$

The quadratic is a perfect square. $f(x)$ can be written as

$$(x-1)(2x+1)^2$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$2x+1=0$$
$$x=-\frac{1}{2}$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

**Analysis**

Look at the graph of the function $f$ in Figure 3.6.1. Notice, at $x = -0.5$, the graph bounces off the x-axis, indicating the even multiplicity (2,4,6…) for the zero −0.5. At $x = 1$, the graph crosses the x-axis, indicating the odd multiplicity (1,3,5…) for the zero $x = 1$.

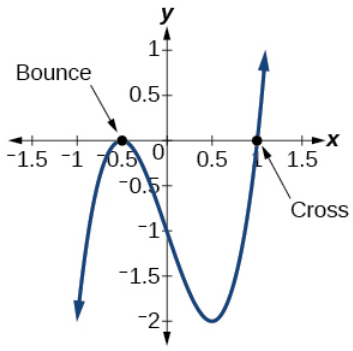



Figure 3.6.1.

## Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose $f$ is a polynomial function of degree four, and $f(x) = 0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it $c_1$. By the Factor Theorem, we can write $f(x)$ as a product of $x - c_1$ and a polynomial quotient. Since $x - c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it $c_2$. So we can write the polynomial quotient as a product of $x - c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

### THE FUNDAMENTAL THEOREM OF ALGEBRA

The *Fundamental Theorem of Algebra* states that, if $f(x)$ is a polynomial of degree $n > 0$, then $f(x)$ has at least one complex zero.

We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n > 0$, and a is a non-zero real number, then $f(x)$ has exactly $n$ linear factors

$$f(x) = a(x - c_1)(x - c_2)\ldots(x - c_n) \tag{3.6.10}$$

where $c_1, c_2,...,c_n$ are complex numbers. Therefore, $f(x)$ has $n$ roots if we allow for multiplicities.

### Q&A: Does every polynomial have at least one imaginary zero?

No. Real numbers are a subset of complex numbers, but not the other way around. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

### Example 3.6.6: Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of $f(x) = 3x^3 + 9x^2 + x + 3$ .

**Solution**

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then $p$ is a factor of 3 and $q$ is a factor of 3.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$

$$= \frac{factor\ of\ 3}{factor\ of\ 3}$$

The factors of 3 are ±1 and ±3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are ±3,±1, and $\pm\frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –3.

Dividing by $(x+3)$ gives a remainder of 0, so –3 is a zero of the function. The polynomial can be written as

$$(x+3)(3x^2+1)$$

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$\begin{aligned} 3x^2+1 &= 0 \\ x^2 &= -\frac{1}{3} \\ x &= \pm -\sqrt{\frac{1}{3}} \\ &= \pm\frac{i\sqrt{3}}{3} \end{aligned}$$

The zeros of $f(x)$ are $-3$ and $\pm\frac{i\sqrt{3}}{3}$.

**Analysis**

Look at the graph of the function $f$ in Figure 3.6.2. Notice that, at $x=-3$, the graph crosses the x-axis, indicating an odd multiplicity (1) for the zero $x=-3$. Also note the presence of the two turning points. This means that, since there is a $3^{rd}$ degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the x-intercepts for the function are shown. So either the multiplicity of $x=-3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x=-3$ is three. Either way, our result is correct.

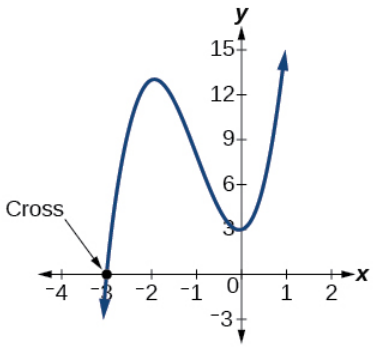



Figure 3.6.2.

**3.6.4**

Find the zeros of $f(x)=2x^3+5x^2-11x+4$ .

**Solution**

The zeros are $-4$, $\frac{1}{2}$, and $1$.

## Using the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the **Fundamental Theorem of Algebra**, as we stated above, is that a polynomial function of degree n will have $n$ zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into $n$ factors. The **Linear Factorization Theorem** tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x-c)$, where c is a complex number.

Let $f$ be a polynomial function with real coefficients, and suppose $a+bi$, $b \neq 0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x-(a+bi)$ is a factor of $f(x)$. For $f$ to have real coefficients, $x-(a-bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x-(a-bi)$, when multiplied by $x-(a+bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function $f$ with real coefficients has a complex zero $a+bi$, then the complex conjugate $a-bi$ must also be a zero of $f(x)$. This is called the **Complex Conjugate Theorem**.

### COMPLEX CONJUGATE THEOREM

According to the Linear Factorization Theorem, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where $c$ is a complex number.

If the polynomial function $f$ has real coefficients and a complex zero in the form $a+bi$, then the complex conjugate of the zero, $a-bi$, is also a zero.

### How to

**Given the zeros of a polynomial function $f$ and a point $(c, f(c))$ on the graph of $f$, use the Linear Factorization Theorem to find the polynomial function.**

1. Use the zeros to construct the linear factors of the polynomial.
2. Multiply the linear factors to expand the polynomial.
3. Substitute $(c, f(c))$ into the function to determine the leading coefficient.
4. Simplify.

### Example 3.6.7: Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of $-3$, $2$, $i$, such that $f(-2)=100$.

**Solution**

Because $x=i$ is a zero, by the Complex Conjugate Theorem $x=-i$ is also a zero. The polynomial must have factors of $(x+3)$, $(x-2)$, $(x-i)$, and $(x+i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$f(x) = a(x+3)(x-2)(x-i)(x+i) \tag{3.6.11}$$
$$f(x) = a(x^2+x-6)(x^2+1) \tag{3.6.12}$$
$$f(x) = a(x^4+x^3-5x^2+x-6) \tag{3.6.13}$$

We need to find $a$ to ensure $f(-2)=100$. Substitute $x=-2$ and $f(-2)=100$ into $f(x)$.

$$100 = a((-2)^4+(-2)^3-5(-2)^2+(-2)-6) \tag{3.6.14}$$
$$100 = a(-20) \tag{3.6.15}$$
$$-5 = a \tag{3.6.16}$$

So the polynomial function is

$$f(x) = -5(x^4+x^3-5x^2+x-6) \tag{3.6.17}$$

or

$$f(x) = -5x^4-5x^3+25x^2-5x+30 \tag{3.6.18}$$

Analysis

We found that both $i$ and $-i$ were zeros, but only one of these zeros needed to be given. If $i$ is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of $i$.

Q&A

**If $2+3i$ were given as a zero of a polynomial with real coefficients, would $2-3i$ also need to be a zero?**

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

3.6.5

Find a third degree polynomial with real coefficients that has zeros of 5 and $-2i$ such that $f(1)=10$.

**Solution**

$f(x)=-\frac{1}{2}x^3+\frac{5}{2}x^2-2x+10$

## Using Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x) = x^4 + x^3 + x^2 + x - 1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x) = (-x)^4 + (-x)^3 + (-x)^2 + (-x) - 1$$
$$f(-x) = +x^4 - x^3 + x^2 - x - 1$$

In this case, $f(-x)$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

DESCARTES' RULE OF SIGNS

According to *Descartes' Rule of Signs*, if we let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.

Example 3.6.8: Using Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x)=-x^4-3x^3+6x^2-4x-12$ .

**Solution**

Begin by determining the number of sign changes.

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine $f(-x)$ to determine the number of negative real roots.

$$f(-x) = -(-x)^4-3(-x)^3+6(-x)^2-4(-x)-12 \quad (3.6.19)$$
$$f(-x) = -x^4+3x^3+6x^2+4x-12 \quad (3.6.20)$$

$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$

Again, there are two sign changes, so there are either 2 or 0 negative real roots.

There are four possibilities, as we can see in Table 3.6.1.

Table 3.6.1

| Positive Real Zeros | Negative Real Zeros | Complex Zeros | Total Zeros |
|---|---|---|---|
| 2 | 2 | 0 | 4 |
| 2 | 0 | 2 | 4 |
| 0 | 2 | 2 | 4 |
| 0 | 0 | 4 | 4 |

**Analysis**

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See Figure 3.6.3. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

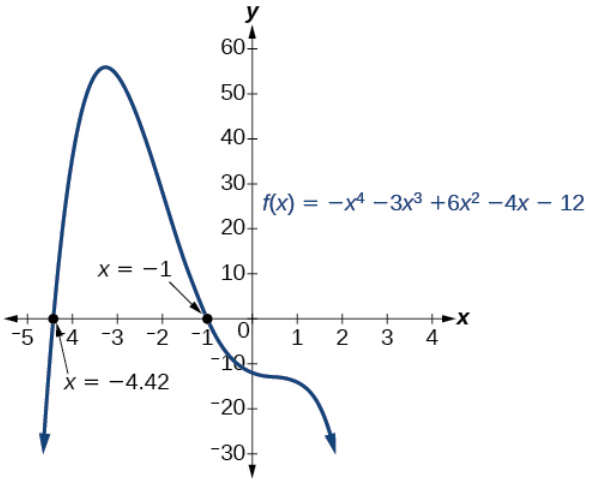



Figure 3.6.3.

**3.6.6**

Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12$ . Use a graph to verify the numbers of positive and negative real zeros for the function.

**Solution**

There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

## Solving Real-World Applications

We have now introduced a variety of tools for solving polynomial equations. Let's use these tools to solve the bakery problem from the beginning of the section.

**Example 3.6.9**

Solving Polynomial Equations

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

**Solution**

Begin by writing an equation for the volume of the cake. The volume of a rectangular solid is given by $V = lwh$. We were given that the length must be four inches longer than the width, so we can express the length of the cake as $l = w + 4$. We were given that the height of the cake is one-third of the width, so we can express the height of the cake as $h = \frac{1}{3}w$. Let's write the volume of the cake in terms of width of the cake.

$$V = (w+4)(w)(\frac{1}{3}w) \tag{3.6.21}$$

$$V = \frac{1}{3}w^3 + \frac{4}{3}w^2 \tag{3.6.22}$$

Substitute the given volume into this equation.

$351 = 13w^3 + 43w^2$ Substitute 351 for V. $1053 = w^3 + 4w^2$ Multiply both sides by 3. $0 = w^3 + 7w^2 - 1053$ Subtract 1053 from both sides.

Descartes' rule of signs tells us there is one positive solution. The Rational Zero Theorem tells us that the possible rational zeros are $\pm 1, \pm 3, \pm 9, \pm 13, \pm 27, \pm 39, \pm 81, \pm 117, \pm 351$, and $\pm 1053$. We can use synthetic division to test these possible zeros. Only positive numbers make sense as dimensions for a cake, so we need not test any negative values. Let's begin by testing values that make the most sense as dimensions for a small sheet cake. Use synthetic division to check $x = 1$.

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 4 | 0 | −1053 |
| | | 1 | 5 | 5 |
| | 1 | 5 | 5 | −1048 |

Since 1 is not a solution, we will check $x = 3$.

| | | | | |
|---|---|---|---|---|
| 3 | 1 | 4 | 0 | −1053 |
| | | 3 | 21 | 63 |
| | 1 | 7 | 21 | −990 |

Since 3 is not a solution either, we will test $x = 9$.

| | | | | |
|---|---|---|---|---|
| 9 | 1 | 4 | 0 | −1053 |
| | | 9 | 117 | 1053 |
| | 1 | 13 | 117 | 0 |

Synthetic division gives a remainder of 0, so 9 is a solution to the equation. We can use the relationships between the width and the other dimensions to determine the length and height of the sheet cake pan.

$l = w + 4 = 9 + 4 = 13$ and $h = \frac{1}{3}w = \frac{1}{3}(9) = 3$

The sheet cake pan should have dimensions 13 inches by 9 inches by 3 inches

**3.6.7**

A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?

**Solution**

3 meters by 4 meters by 7 meters

**Media**

Access these online resources for additional instruction and practice with zeros of polynomial functions.

- Real Zeros, Factors, and Graphs of Polynomial Functions
- Complex Factorization Theorem
- Find the Zeros of a Polynomial Function

- Find the Zeros of a Polynomial Function 2
- Find the Zeros of a Polynomial Function 3

## Key Concepts

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x-k$. This is known as the Remainder Theorem. See Example 3.6.1.
- According to the Factor Theorem, $k$ is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$. See Example3.6.2.
- According to the Rational Zero Theorem, each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See Example 3.6.3 and Example 3.6.4.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See Example 3.6.5.
- According to the Fundamental Theorem, every polynomial function with degree greater than 0 has at least one complex zero. See Example 3.6.6.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x-c)$, where $c$ is a complex number. See Example 3.6.7.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less than the number of sign changes by an even integer. See Example 3.6.8.
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division. See Example 3.6.9.

## Glossary

**Descartes' Rule of Signs**

a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

**Factor Theorem**

$k$ is a zero of polynomial function $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$

**Fundamental Theorem of Algebra**

a polynomial function with degree greater than 0 has at least one complex zero

**Linear Factorization Theorem**

allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where $c$ is a complex number

**Rational Zero Theorem**

the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where $p$ is a factor of the constant term and $q$ is a factor of the leading coefficient.

**Remainder Theorem**

if a polynomial $f(x)$ is divided by $x-k$,then the remainder is equal to the value $f(k)$

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# 3.7: Rational Functions

**Learning Objectives**

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, $x$, produced. This is given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing $x$ items, we would divide the cost function by the number of items, $x$. The average cost function, which yields the average cost per item for $x$ items produced, is

$$f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}$$

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

## Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure 3.7.1, and notice some of their features.

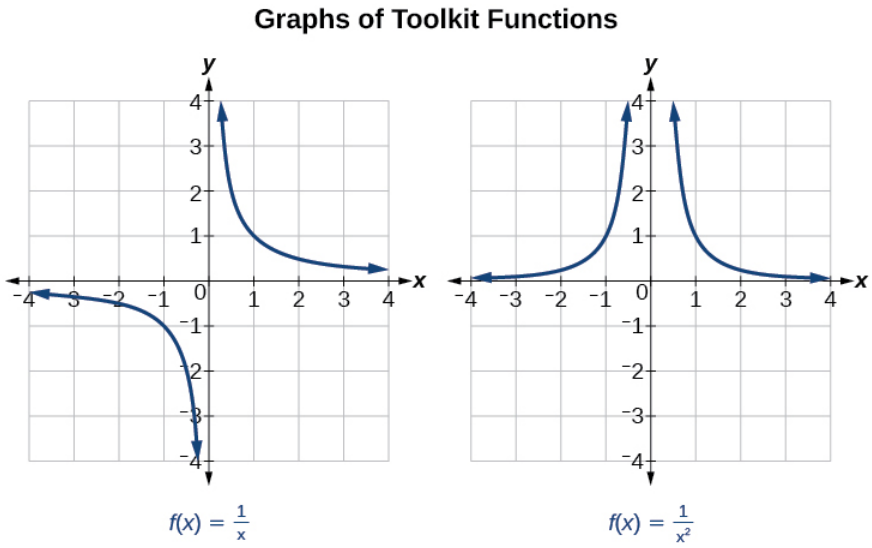



Figure 3.7.1

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

- On the left branch of the graph, the curve approaches the $x$-axis $(y = 0)$ as $x \to -\infty$.
- As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
- Finally, on the right branch of the graph, the curves approaches the $x$-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that $x$ or $f(x)$ is approaching a particular value (Table 3.7.1).

Table 3.7.1

| Symbol | Meaning |
|---|---|

| Symbol | Meaning |
|---|---|
| $x \to a^-$ | $x$ approaches a from the left ($x < a$ but close to $a$ ) |
| $x \to a^+$ | $x$ approaches a from the right ($x > a$ but close to $a$ ) |
| $x \to \infty$ | $x$ approaches infinity ($x$ increases without bound) |
| $x \to -\infty$ | $x$ approaches negative infinity ($x$ decreases without bound) |
| $f(x) \to \infty$ | the output approaches infinity (the output increases without bound) |
| $f(x) \to -\infty$ | the output approaches negative infinity (the output decreases without bound) |
| $f(x) \to a$ | the output approaches $a$ |

## Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 3.7.2.

Table 3.7.2

| $x$ | –0.1 | –0.01 | –0.001 | –0.0001 |
|---|---|---|---|---|
| $f(x) = \frac{1}{x}$ | –10 | –100 | –1000 | –10,000 |

We write in arrow notation

$$\text{as } x \to 0^-, f(x) \to -\infty$$

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 3.7.3.

Table 3.7.3

| $x$ | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|
| $f(x) = \frac{1}{x}$ | 10 | 100 | 1000 | 10,000 |

We write in arrow notation

$$\text{As } x \to 0^+, f(x) \to \infty \text{ .}$$

See Figure 3.7.2.

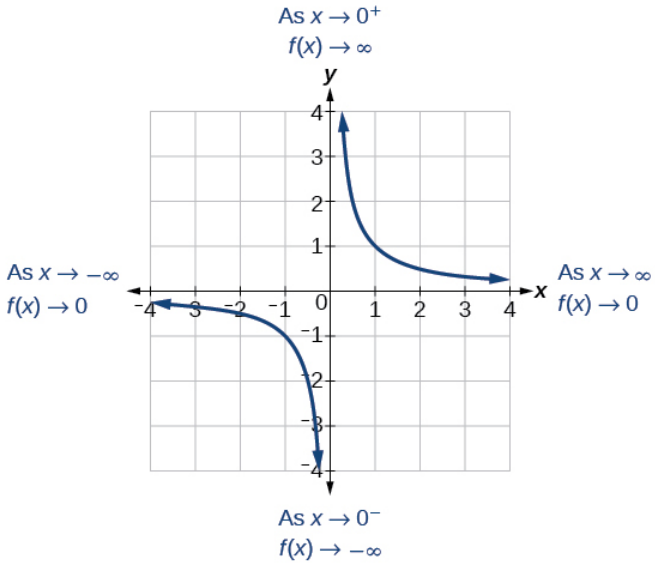



Figure 3.7.2.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero (Figure 3.7.3).

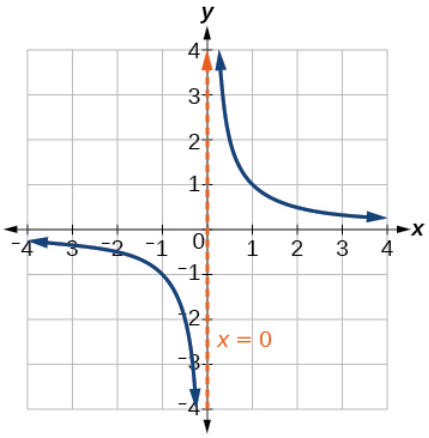



Figure 3.7.3.

**Definition: VERTICAL ASYMPTOTE**

A vertical asymptote of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach $a$. We write

As $x \to a$, $f(x) \to \infty$, or as $x \to a$, $f(x) \to -\infty$.

## End Behavior of $f(x) = \frac{1}{x}$

As the values of $x$ approach infinity, the function values approach 0. As the values of $x$ approach negative infinity, the function values approach 0 (Figure 3.7.4). Symbolically, using arrow notation

As $x \to \infty$, $f(x) \to 0$,and as $x \to -\infty$, $f(x) \to 0$.

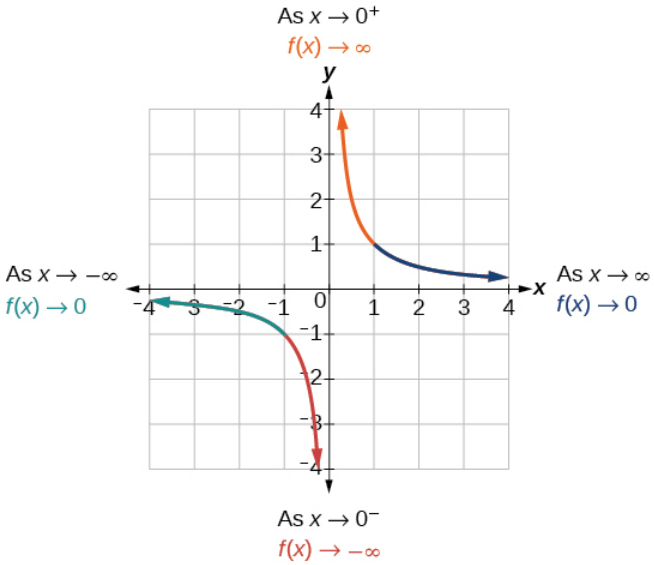



Figure 3.7.4.

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure 3.7.5.

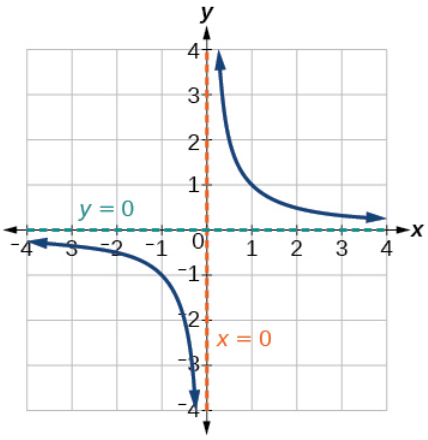



Figure 3.7.5.

**Definition: HORIZONTAL ASYMPTOTE**

A *horizontal asymptote* of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty \text{ , } f(x) \to b.$$

**Example 3.7.1: Using Arrow Notation.**

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 3.7.6.

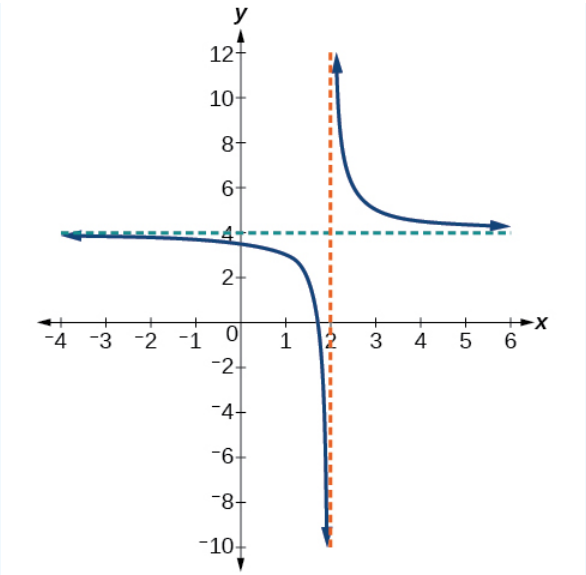



Figure 3.7.6.

**Solution**

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

$$\text{As } x \to 2^-,\ f(x) \to -\infty, \text{ and as } x \to 2^+,\ f(x) \to \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph levels off at 4.

$$\text{As } x \to \infty,\ f(x) \to 4 \text{ and as } x \to -\infty,\ f(x) \to 4.$$

## Exercises 3.7.1

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

**Answer**

End behavior: as $x \to \pm\infty$, $f(x) \to 0$;

Local behavior: as $x \to 0$, $f(x) \to \infty$ (there are no x- or y-intercepts)

## Example 3.7.2: Using Transformations to Graph a Rational Function.

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

**Solution**

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3 \tag{3.7.1}$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2} \tag{3.7.2}$$

The graph of the shifted function is displayed in Figure 3.7.7.

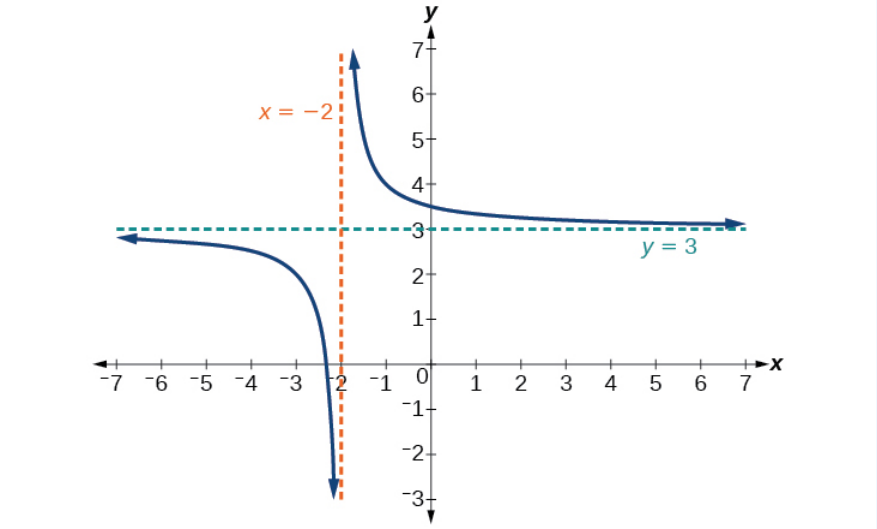



Figure 3.7.7.

Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \to -2^{-}\text{, } f(x) \to -\infty\text{, and as } x \to -2^{+}\text{, } f(x) \to \infty\text{.}$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \to \pm\infty\text{, } f(x) \to 3\text{.}$$

**Analysis**

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

## Exercise 3.7.2

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

**Solution**

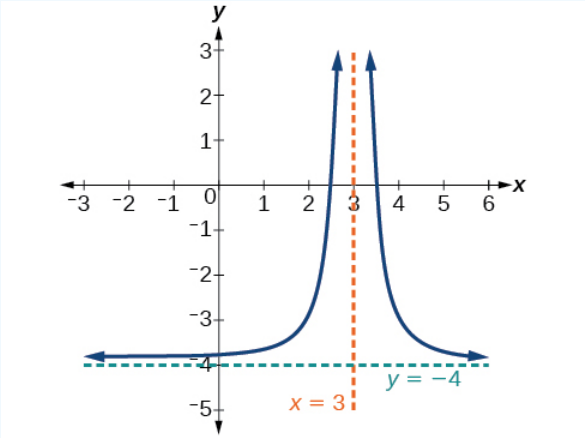



Figure 3.7.8.

The function and the asymptotes are shifted 3 units right and 4 units down. As $x \to 3$, $f(x) \to \infty$, and as $x \to \pm\infty$, $f(x) \to -4$.

The function is $f(x) = \frac{1}{(x-3)^2} - 4$.

## Solving Applied Problems Involving Rational Functions

In Example 3.7.2, we shifted a toolkit function in a way that resulted in the function $f(x)=\frac{3x+7}{x+2}$. This is an example of a rational function. A *rational function* is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

**Definition: RATIONAL FUNCTION**

A ***rational function*** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x)=\frac{P(x)}{Q(x)}=\frac{a_p x^p+a_{p-1}x^{p-1}+\ldots+a_1 x+a_0}{b_q x^q+b_{q-1}x^{q-1}+\ldots+b_1 x+b_0},\ Q(x)\neq 0 \tag{3.7.3}$$

**Example 3.7.3: Solving an Applied Problem Involving a Rational Function**

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

**Solution**

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\begin{array}{c}\text{water: } W(t)=100+10t \text{ in gallons}\\ \text{sugar: } S(t)=5+1t \text{ in pounds}\end{array}$$

The concentration, $C$, will be the ratio of pounds of sugar to gallons of water

$$C(t)=\frac{5+t}{100+10t} \tag{3.7.4}$$

The concentration after 12 minutes is given by evaluating $C(t)$ at $t=12$.

$$C(12)=\frac{5+12}{100+10(12)} \tag{3.7.5}$$

$$=\frac{17}{220} \tag{3.7.6}$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

$$C(0)=\frac{5+0}{100+10(0)} \tag{3.7.7}$$

$$=\frac{1}{20} \tag{3.7.8}$$

Since $\frac{17}{220}\approx 0.08>\frac{1}{20}=0.05$, the concentration is greater after 12 minutes than at the beginning.

**Analysis**

To find the horizontal asymptote, divide the leading coefficient in the numerator by the leading coefficient in the denominator:

$$\frac{1}{10}=0.1 \tag{3.7.9}$$

Notice the horizontal asymptote is $y=0.1$. This means the concentration, $C$, the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

**Exercise 3.7.3**

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

**Answer**

$\frac{12}{11}$

## Finding the Domains of Rational Functions

A *vertical asymptote* represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

**Definition: DOMAIN OF A RATIONAL FUNCTION**

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

**How To: Given a rational function, find the domain.**

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

**Example 3.7.4: Finding the Domain of a Rational Function**

Find the domain of $f(x) = \dfrac{x+3}{x^2-9}$.

**Solution**

Begin by setting the denominator equal to zero and solving.

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

**Analysis**

A graph of this function, as shown in Figure 3.7.9, confirms that the function is not defined when $x = \pm 3$.

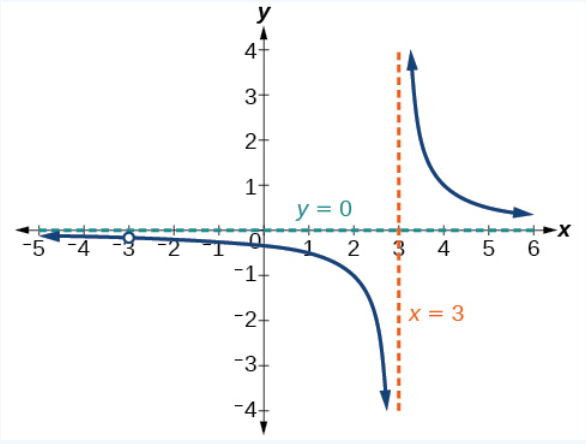



Figure 3.7.9.

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

**Exercise 3.7.4**

Find the domain of $f(x) = \dfrac{4x}{5(x-1)(x-5)}$.

**Answer**

The domain is all real numbers except $x = 1$ and $x = 5$.

## Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

### Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

**How To: Given a rational function, identify any vertical asymptotes of its graph**

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or "holes."

**Example 3.7.5: Identifying Vertical Asymptotes**

Find the vertical asymptotes of the graph of $k(x) = \dfrac{5+2x^2}{2-x-x^2}$.

**Solution**

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$
$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$
$$x = -2,\ x = 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 3.7.10 confirms the location of the two vertical asymptotes.

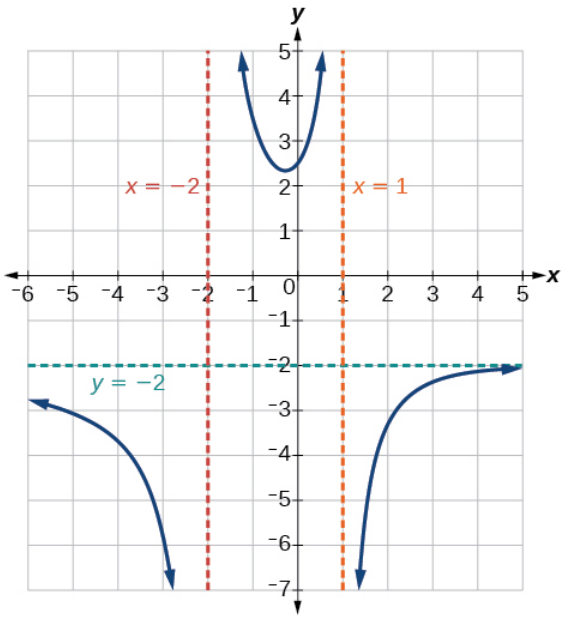



***Figure* 3.7.10.**

## Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $(x-3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x = 3$, is the vertical asymptote. See Figure 3.7.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

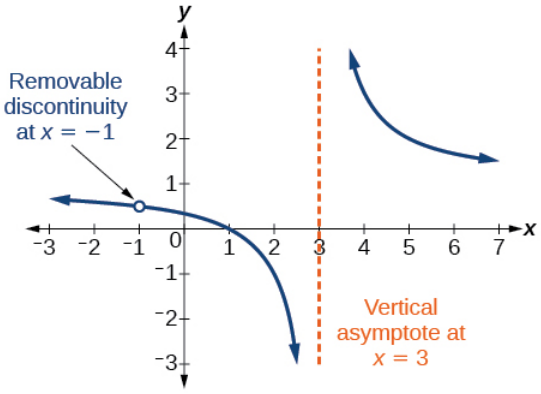



Figure 3.7.11.

### REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x = a$ if $a$ is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the

multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

### Example 3.7.6: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \dfrac{x-2}{x^2-4}$.

**Solution**

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x-2$. The zero for this factor is $x=2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x=-2$. The vertical asymptote is $x=-2$. See Figure 3.7.12

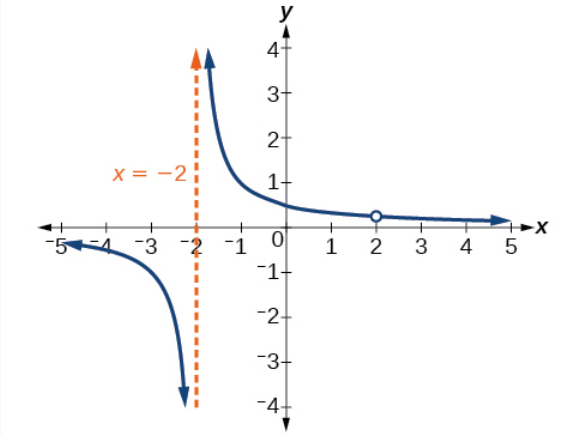



Figure 3.7.12.

The graph of this function will have the vertical asymptote at $x=-2$, but at $x=2$ the graph will have a hole.

### Exercise 3.7.5

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \dfrac{x^2-25}{x^3-6x^2+5x}$.

**Answer**

Removable discontinuity at $x=5$.

Vertical asymptotes: $x=0$, $x=1$.

## Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

**Case 1**: If the degree of the denominator > degree of the numerator, there is a **horizontal asymptote** at $y=0$.

**Example**: $f(x)=\dfrac{4x+2}{x^2+4x-5}$

In this case, the end behavior is $f(x)\approx\dfrac{4x}{x^2}=\dfrac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x)=\dfrac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y=0$. See **Figure** 3.7.13. Note that this graph crosses the horizontal asymptote.

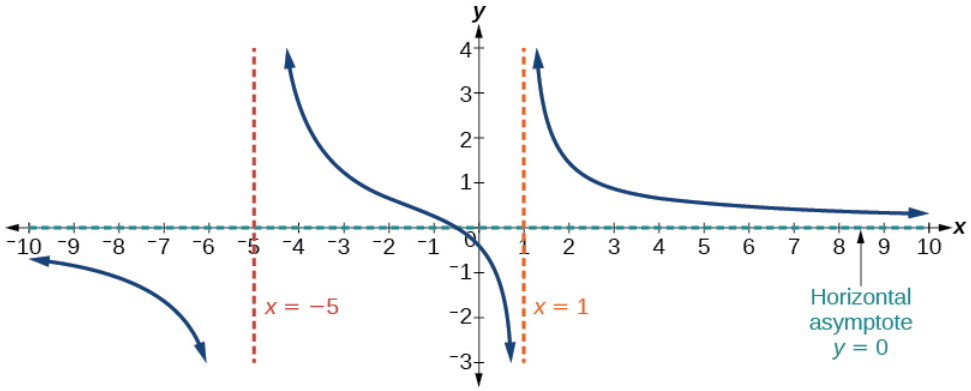



Figure 3.7.13: Horizontal asymptote $y=0$ when $f(x)=\dfrac{p(x)}{q(x)}$, $q(x)\neq 0$ where degree of $p$ < degree of $q$.

**Case 2**: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

**Example**: $f(x)=\dfrac{3x^2-2x+1}{x-1}$

In this case, the end behavior is $f(x)\approx\dfrac{3x^2}{x}=3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x)=3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x)=3x$ looks like a diagonal line, and since $f$ will behave similarly to $g$, it will approach a line close to $y=3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\dfrac{3x^2-2x+1}{x-1}$. The quotient is $3x+1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x)=3x+1$. See Figure 3.7.14

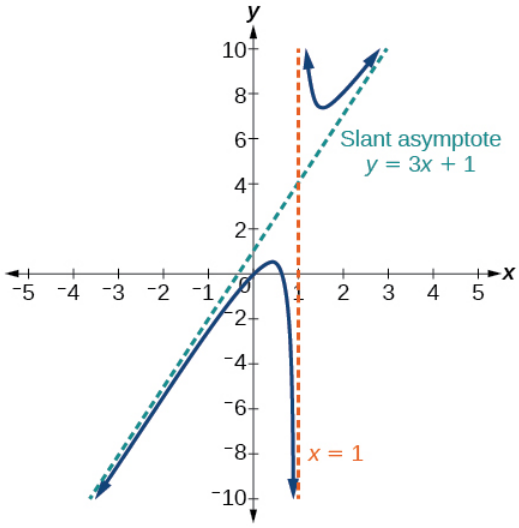



Figure 3.7.14: Slant asymptote when $f(x)=\dfrac{p(x)}{q(x)}$, $q(x)\neq 0$ where degree of $p$ >degree of $q$ by 1.

**Case 3**: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where $a_n$ and $b_n$ are respectively the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$.

**Example**: $f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See Figure 3.7.15. Note that this graph crosses the horizontal asymptote.

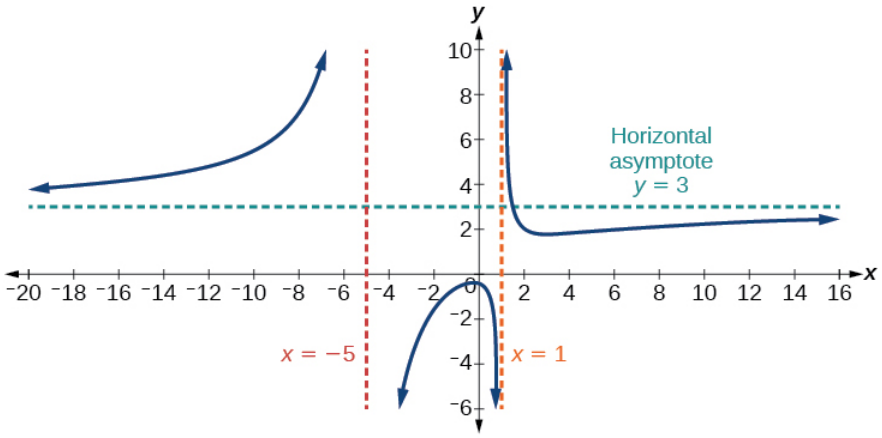



Figure 3.7.15: Horizontal asymptote when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p$ = degree of $q$.

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, f(x) \to \infty$$

### HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

## Example 3.7.7: Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$
3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$

**Solution**

For these solutions, we will use $f(x) = \dfrac{p(x)}{q(x)},\ q(x) \neq 0$.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of $p =$ degree of $q = 3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$: The degree of $p = 2$ and degree of $q = 1$. Since $p > q$ by 1, there is a slant asymptote found at $\dfrac{x^2 - 4x + 1}{x + 2}$.

$$\begin{array}{r|rr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$ : The degree of $p = 2 <$ degree of $q = 3$, so there is a horizontal asymptote $y = 0$.

## Example 3.7.8 Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \dfrac{5 + t}{100 + 10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

**Solution**

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is $t$, with coefficient 1. In the denominator, the leading term is 10t, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty,\ C(t) \to \tfrac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of $t$ increase, the values of $C$ will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

## Example 3.7.9: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function $f(x) = \dfrac{(x - 2)(x + 3)}{(x - 1)(x + 2)(x - 5)}$

**Solution**

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2,$ and $5$, indicating vertical asymptotes at these values.

The numerator has degree $2$, while the denominator has degree $3$. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure $3.7.16$.

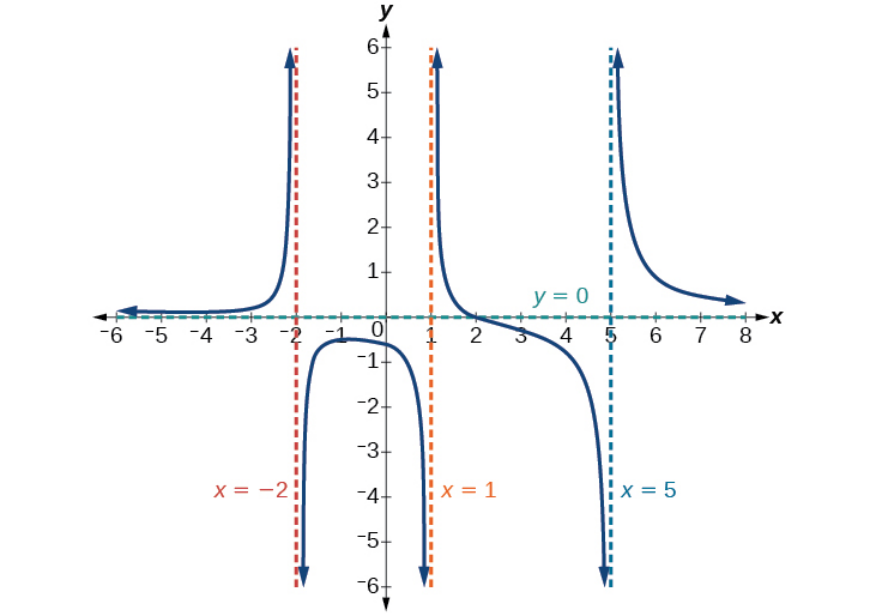



Figure 3.7.16.

## Exercise 3.7.6

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

**Answer**

Vertical asymptotes at $x = 2$ and $x = -3$

horizontal asymptote at $y = 4$.

## INTERCEPTS OF RATIONAL FUNCTIONS

A *rational function* will have a $y$-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a $y$-intercept if the function is not defined at zero.

Likewise, a rational function will have $x$-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

## Example 3.7.10: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

**Solution**

We can find the y-intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)}$$

$$= -\frac{6}{10}$$
$$= -\frac{3}{5}$$
$$= -0.6$$

The x-intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \text{ This is zero when the numerator is zero.}$$

$$0 = (x-2)(x+3)$$

$$x = 2, x = -3$$

The y-intercept is $(0, -0.6)$, the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 3.7.17.

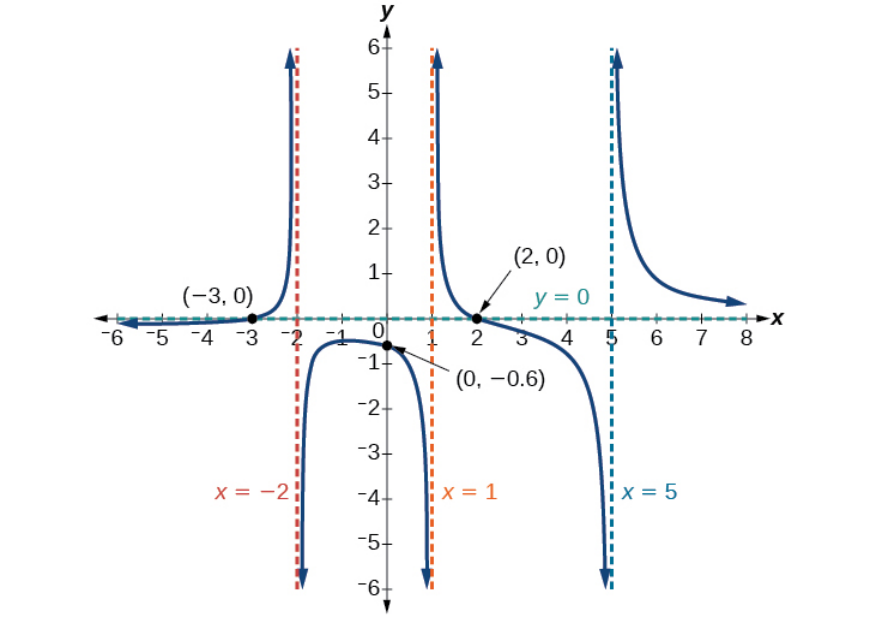



Figure 3.7.17.

## Exercise 3.7.7

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

**Answer**

For the transformed reciprocal squared function, we find the rational form.

$$f(x) = \frac{1}{(x-3)^2} - 4 = \frac{1-4(x-3)^2}{(x-3)^2} = \frac{1-4(x^2-6x+9)}{(x-3)(x-3)} = \frac{-4x^2+24x-35}{x^2-6x+9}$$

Because the numerator is the same degree as the denominator we know that as $x \to \pm\infty$, $f(x) \to -4$; so $y = -4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x = 3$, because as

$x \to 3$, $f(x) \to \infty$. We then set the numerator equal to 0 and find the x-intercepts are at $(2.5, 0)$ and $(3.5, 0)$. Finally, we evaluate the function at 0 and find the y-intercept to be at $(0, -\frac{35}{9})$.

## Graphing Rational Functions

In Example 3.7.10, we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 3.7.18

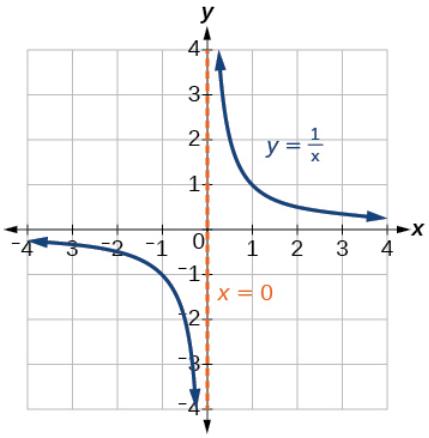



Figure 3.7.18.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 3.7.19

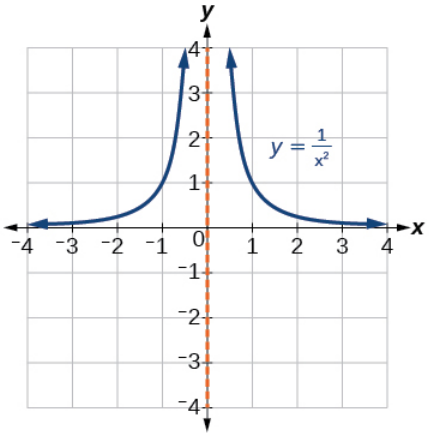



Figure 3.7.19.

For example, the graph of $f(x) = \dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 3.7.20

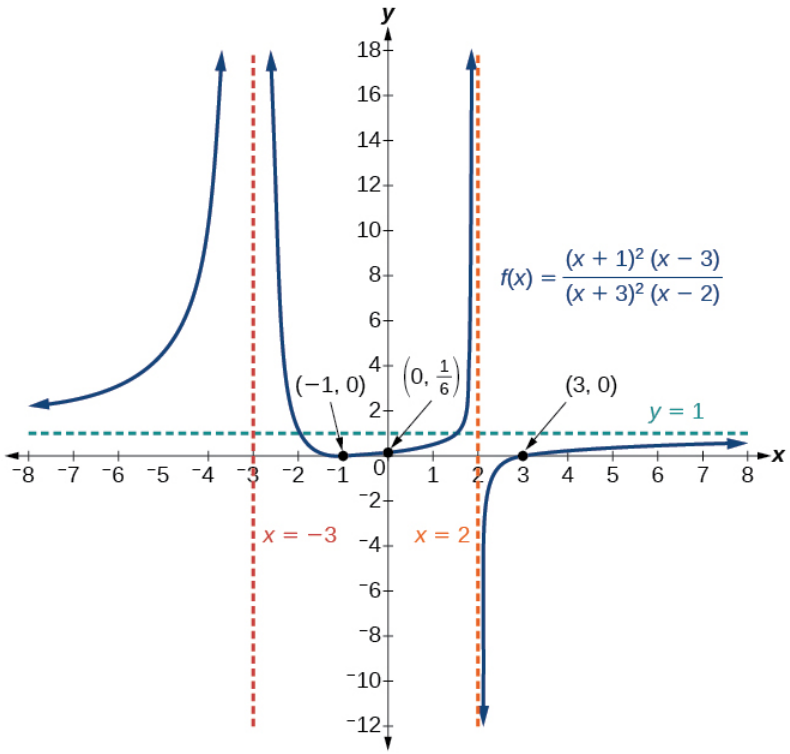



Figure 3.7.20.

- At the x-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the x-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x) = \frac{1}{x}$.

**Howto: Given a rational function, sketch a graph.**

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

**Example 3.7.11: Graphing a Rational Function**

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

**Solution**

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0)=\frac{(0+2)(0-3)}{(0+1)^2(0-2)}=3$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x=-2$ and $x=3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0,3)$ and x-intercepts at $(-2,0)$ and $(3,0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x+1=0$ and when $x-2=0$, giving us vertical asymptotes at $x=-1$ and $x=2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y=0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 3.7.21.

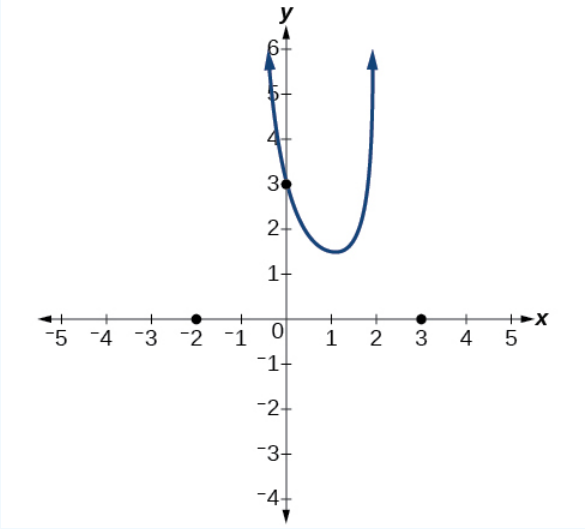



Figure 3.7.21.

The factor associated with the vertical asymptote at $x=-1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x=2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 3.7.22 After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

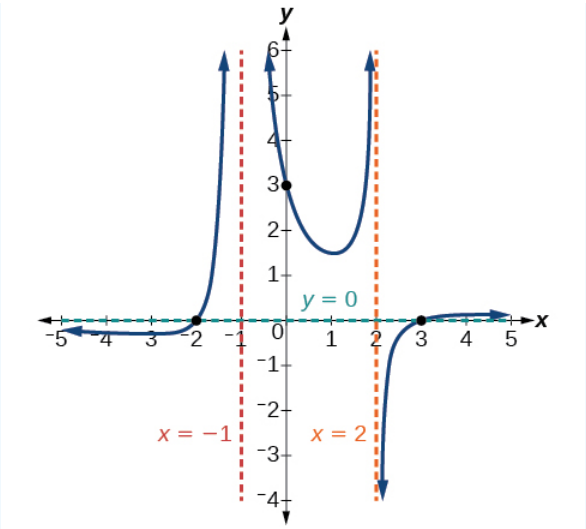



Figure 3.7.22.

## Exercise 3.7.8

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

**Answer**

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $(0, \frac{4}{3})$.

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.

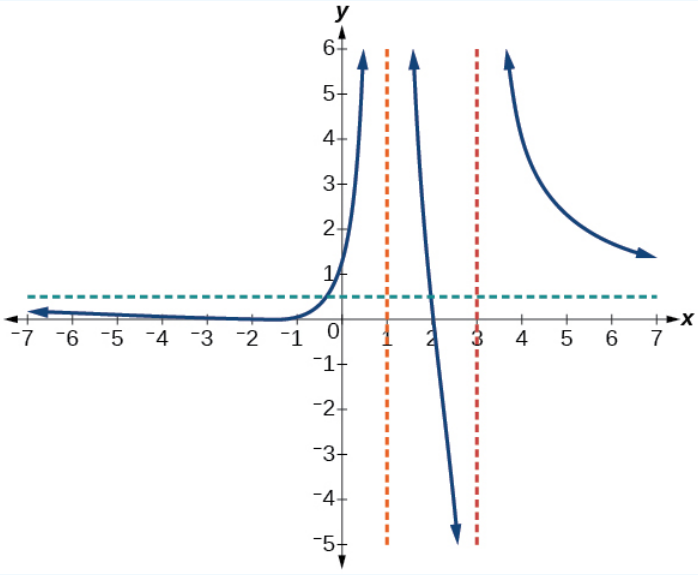



Figure 3.7.23.

## Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an *x*-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a

numerator of a function whose graph will pass through a set of *x*-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

### WRITING RATIONAL FUNCTIONS FROM INTERCEPTS AND ASYMPTOTES

If a **rational function** has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any $v_j$, then the function can be written in the form:

$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$$

where the powers $p_i$ or $q_i$ on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor $a$ can be determined given a value of the function other than the x-intercept or by the horizontal asymptote if it is nonzero.

### Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the *x*-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the "simplest" function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

### Example 3.7.12: Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 3.7.24

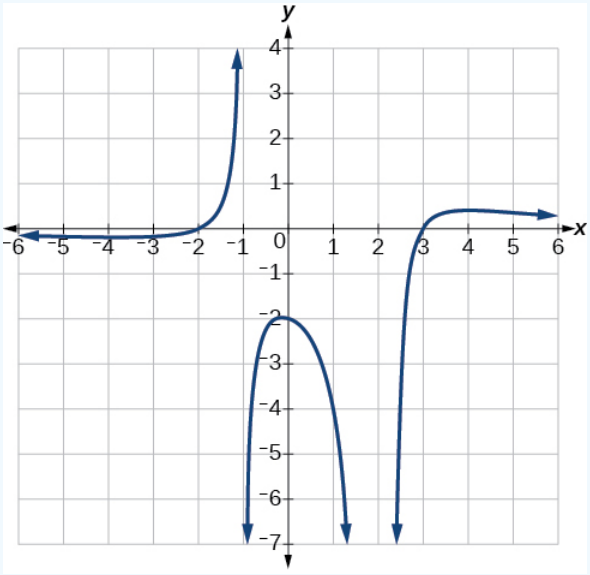



Figure 3.7.24.

**Solution**

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure 3.7.25

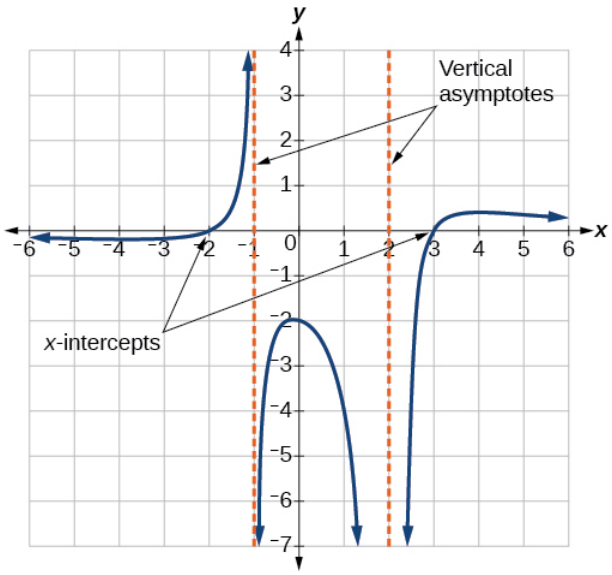



Figure 3.7.25.

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0,-2)$.

$$-2 = a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$
$$-2 = a\frac{-6}{4}$$
$$a = \frac{-8}{-6} = \frac{4}{3}$$

This gives us a final function of $f(x) = \dfrac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

## Media

Access these online resources for additional instruction and practice with rational functions.

- Graphing Rational Functions
- Find the Equation of a Rational Function
- Determining Vertical and Horizontal Asymptotes
- Find the Intercepts, Asymptotes, and Hole of a Rational Function

## Key Equations

Rational Function $\quad f(x) = \dfrac{P(x)}{Q(x)} = \dfrac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0},\ Q(x) \neq 0$

## Key Concepts

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See Example 3.7.1.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See Example.

- Application problems involving rates and concentrations often involve rational functions. See Example.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See Example.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See Example.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See Example.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See Example, Example, Example, and Example.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See Example.
- If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any $v_j$, then the function can be written in the form

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# 3.8: Inverses and Radical Functions

**Learning Objectives**

In this section, you will:

- Find the inverse of an invertible polynomial function.
- Restrict the domain to find the inverse of a polynomial function.

A mound of gravel is in the shape of a cone with the height equal to twice the radius.

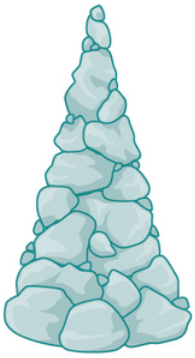


Figure 3.8.1

The volume is found using a formula from elementary geometry.

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi r^2 (2r) \\ &= \frac{2}{3}\pi r^3 \end{aligned}$$

We have written the volume $V$ in terms of the radius $r$. However, in some cases, we may start out with the volume and want to find the radius. For example: A customer purchases 100 cubic feet of gravel to construct a cone shape mound with a height twice the radius. What are the radius and height of the new cone? To answer this question, we use the formula

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This function is the inverse of the formula for $V$ in terms of $r$.

In this section, we will explore the inverses of polynomial and rational functions and in particular the radical functions we encounter in the process.

## Finding the Inverse of a Polynomial Function

Two functions $f$ and $g$ are inverse functions if for every coordinate pair in $f$, $(a, b)$, there exists a corresponding coordinate pair in the inverse function, $g$, $(b, a)$. In other words, the coordinate pairs of the inverse functions have the input and output interchanged. Only one-to-one functions have inverses. Recall that a one-to-one function has a unique output value for each input value and passes the horizontal line test.

For example, suppose a water runoff collector is built in the shape of a parabolic trough as shown in Figure 3.8.2. We can use the information in the figure to find the surface area of the water in the trough as a function of the depth of the water.

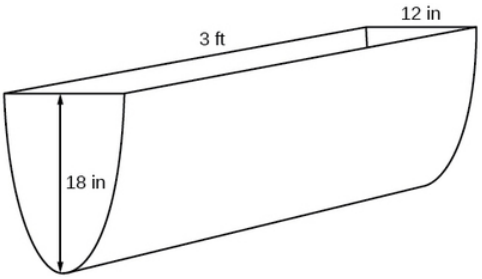



Figure 3.8.2

Because it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with $x$ measured horizontally and $y$ measured vertically, with the origin at the vertex of the parabola (Figure 3.8.3).

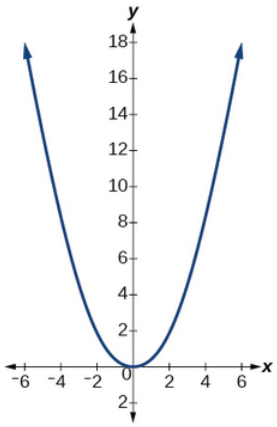



Figure 3.8.3

From this we find an equation for the parabolic shape. We placed the origin at the vertex of the parabola, so we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point $(6, 18)$, from which we can solve for the stretch factor $a$.

$$\begin{aligned} 18 &= a6^2 \\ a &= \frac{18}{36} \\ &= \frac{1}{2} \end{aligned}$$

Our parabolic cross section has the equation

$$y(x) = \frac{1}{2}x^2$$

We are interested in the *surface area* of the water, so we must determine the width at the top of the water as a function of the water depth. For any depth $y$, the width will be given by $2x$, so we need to solve the equation above for $x$ and find the inverse function. However, notice that the original function is not one-to-one, and indeed, given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive $x$ values. On this domain, we can find an inverse by solving for the input variable:

$$\begin{aligned} y &= \frac{1}{2}x^2 \\ 2y &= x^2 \\ x &= \pm\sqrt{2y} \end{aligned}$$

This is not a function as written. We are limiting ourselves to positive $x$ values, so we eliminate the negative solution, giving us the inverse function we're looking for.

$$y = \frac{x^2}{2}, x > 0$$

Because $x$ is the distance from the center of the parabola to either side, the entire width of the water at the top will be $2x$. The trough is 3 feet (36 inches) long, so the surface area will then be:

$$\begin{aligned} \text{Area } &= l \cdot w \\ &= 36 \cdot 2x \\ &= 72x \\ &= 72\sqrt{2y} \end{aligned}$$

This example illustrates two important points:

1. When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2. The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called **radical functions**. While it is not possible to find an inverse of most polynomial functions, some basic polynomials do have inverses. Such functions are called **invertible functions**, and we use the notation $f^{-1}(x)$.

Warning: $f^{-1}(x)$ is not the same as the reciprocal of the function $f(x)$. This use of "–1" is reserved to denote inverse functions. To denote the reciprocal of a function $f(x)$, we would need to write:

$$(f(x))^{-1} = \frac{1}{f(x)}. \tag{3.8.1}$$

An important relationship between inverse functions is that they "undo" each other. If $f^{-1}$ is the inverse of a function $f$, then $f$ is the inverse of the function $f^{-1}$. In other words, whatever the function $f$ does to $x$, $f^{-1}$ undoes it—and vice-versa.

$$f^{-1}(f(x)) = x, \text{ for all } x \text{ in the domain of } f$$

and

$$f(f^{-1}(x)) = x, \text{ for all } x \text{ in the domain of } f^{-1}$$

Note that the inverse switches the domain and range of the original function.

### VERIFYING TWO FUNCTIONS ARE INVERSES OF ONE ANOTHER

Two functions, $f$ and $g$, are inverses of one another if for all $x$ in the domain of $f$ and $g$,

$$g(f(x)) = f(g(x)) = x$$

### Howto: Given a polynomial function, find the inverse of the function by restricting the domain in such a way that the new function is one-to-one

1. Replace $f(x)$ with $y$.
2. Interchange $x$ and $y$.
3. Solve for $y$, and rename the function $f^{-1}(x)$.

### Example 3.8.1: Verifying Inverse Functions

Show that $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses, for $x \neq 0, -1$.

**Solution**

We must show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x+1}\right)$$
$$= \frac{1}{\frac{1}{x+1}} - 1$$
$$= (x+1) - 1$$
$$= x$$

and

$$f(f^{-1}(x)) = f(\frac{1}{x-1})$$
$$= \frac{1}{\left(\frac{1}{x-1}\right)+1}$$
$$= \frac{1}{\frac{1}{x}}$$
$$= x$$

Therefore, $f(x) = \dfrac{1}{x+1}$ and $f^{-1}(x) = \dfrac{1}{x} - 1$ are inverses.

### Exercise 3.8.1

Show that $f(x) = \frac{x+5}{3}$ and $f^{-1}(x) = 3x - 5$ are inverses.

**Answer a**

$f^{-1}(f(x)) = f^{-1}(\frac{x+5}{3}) = 3(\frac{x+5}{3}) - 5 = (x-5) + 5 = x$

**Answer b**

$f(f^{-1}(x)) = f(3x-5) = \frac{(3x-5)+5}{3} = \frac{3x}{3} = x$

### Example 3.8.2: Finding the Inverse of a Cubic Function

Find the inverse of the function $f(x) = 5x^3 + 1$.

**Solution**

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for $x$.

$$y = 5x^3 + 1$$
$$x = 5y^3 + 1$$
$$x - 1 = 5y^3$$
$$\frac{x-1}{5} = y^3$$
$$f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$$

**Analysis**

Look at the graph of $f$ and $f^{-1}$. Notice that one graph is the reflection of the other about the line $y = x$. This is always the case when graphing a function and its inverse function.

Also, since the method involved interchanging $x$ and $y$, notice corresponding points. If $(a, b)$ is on the graph of $f$,then $(b, a)$ is on the graph of $f^{-1}$. Since $(0, 1)$ is on the graph of $f$, then $(1, 0)$ is on the graph of $f^{-1}$. Similarly, since $(1, 6)$ is on the graph of $f$,then $(6, 1)$ is on the graph of $f^{-1}$ (Figure 3.8.9).

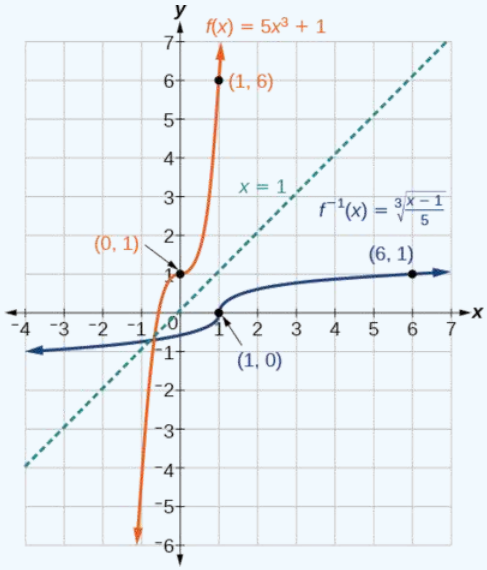



Figure 3.8.4

**Exercise 3.8.2**

Find the inverse function of $f(x) = \sqrt[3]{x+4}$.

**Answer**

$f^{-1}(x) = x^3 - 4$

## Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of **cubic functions** without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an **inverse function**. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

**RESTRICTING THE DOMAIN**

If a function is not one-to-one, it cannot have an inverse. If we restrict the domain of the function so that it becomes one-to-one, thus creating a new function, this new function will have an inverse.

**How to: Given a polynomial function, restrict the domain of a function that is not one-to-one and then find the inverse**

1. Restrict the domain by determining a domain on which the original function is one-to-one.
2. Replace $f(x)$ with $y$.
3. Interchange $x$ and $y$.
4. Solve for $y$, and rename the function or pair of function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

## Example 3.8.3: Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of $f$:

1. $f(x) = (x-4)^2$, $x \geq 4$
2. $f(x) = (x-4)^2$, $x \leq 4$

**Solution**

The original function $f(x) = (x-4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ or $x \leq 4$ on which it is one-to-one (Figure 3.8.6).

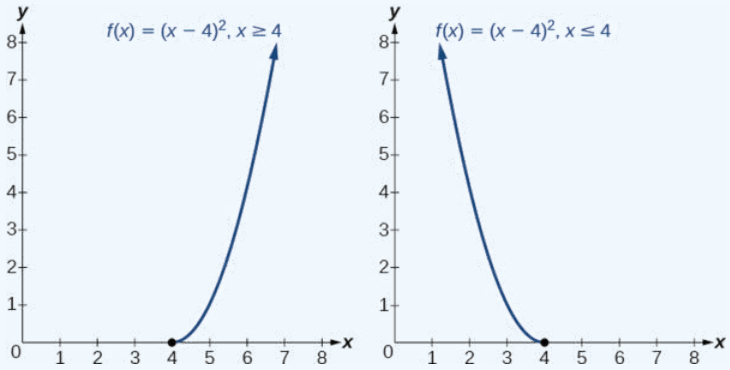



Figure 3.8.5

To find the inverse, start by replacing $f(x)$ with the simple variable $y$.

$$y = (x-4)^2 \text{ Interchange } x \text{ and } y.$$

$$x = (y-4)^2 \text{ Take the square root.}$$

$$\pm\sqrt{x} = y - 4 \text{ Add 4 to both sides.}$$

$$4 \pm \sqrt{x} = y$$

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse. Since we reversed the roles of $x$ and $y$ for the original $f(x)$, we looked at the domain: the values $x$ could assume. When we reversed the roles of $x$ and $y$, this gave us the values $y$ could assume. For this function, $x \geq 4$, so for the inverse, we should have $y \geq 4$, which is what our inverse function gives.

1. The domain of the original function was restricted to $x \geq 4$, so the outputs of the inverse need to be the same, $f(x) \geq 4$, and we must use the + case:
$$f^{-1}(x) = 4 + \sqrt{x}$$
2. The domain of the original function was restricted to $x \leq 4$, so the outputs of the inverse need to be the same, $f(x) \leq 4$, and we must use the – case:
$$f^{-1}(x) = 4 - \sqrt{x}$$

**Analysis**

On the graphs in Figure 3.8.6, we see the original function graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y = x$. The coordinate pair $(4, 0)$ is on the graph off f and the coordinate pair $(0, 4)$ is on the graph of $f^{-1}$. For any coordinate pair, if $(a, b)$ is on the graph of $f$, then $(b, a)$ is on the graph of $f^{-1}$. Finally, observe that the graph of $f$ intersects the graph of $f^{-1}$ on the line $y = x$. Points of intersection for the graphs of $f$ and $f^{-1}$ will always lie on the line $y = x$.

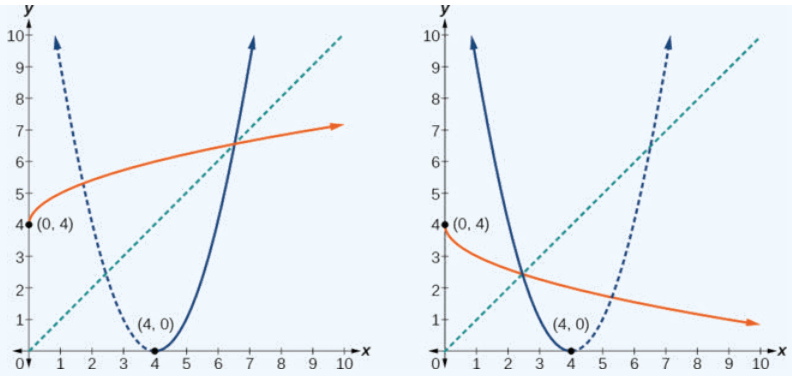


Figure 3.8.6

## Example 3.8.4: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of

$$f(x) = (x-2)^2 - 3\,.$$

**Solution**

We can see this is a parabola with vertex at $(2,-3)$ that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we will use the vertex form of the quadratic. We start by replacing $f(x)$ with a simple variable, $y$, then solve for $x$.

$$y = (x-2)^2 - 3 \text{ Interchange } x \text{ and } y.$$

$$x = (y-2)^2 - 3 \text{ Add 3 to both sides.}$$

$$x+3 = (y-2)^2 \text{ Take the square root.}$$

$$\pm\sqrt{x+3} = y-2 \text{ Add 2 to both sides.}$$

$$2 \pm \sqrt{x+3} = y \text{ Rename the function.}$$

$$f^{-1}(x) = 2 \pm \sqrt{x+3}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the + case

$$f^{-1}(x) = 2 + \sqrt{x+3}$$

If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

Analysis

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just as easily have opted to restrict the domain to $x \leq 2$, in which case $f^{-1}(x) = 2 - \sqrt{x+3}$ . Observe the original function graphed on the same set of axes as its inverse function in Figure 3.8.7. Notice that both graphs show symmetry about the line $y = x$. The coordinate pair $(2,-3)$ is on the graph of $f$ and the coordinate pair $(-3,2)$ is on the graph of $f^{-1}$. Observe from the graph of both functions on the same set of axes that

$$\text{domain of } f = \text{range of } f^{-1} = [2,\infty)$$

and

$$\text{domain of } f^{-1} = \text{range of } f = [-3,\infty).$$

Finally, observe that the graph of $f$ intersects the graph of $f^{-1}$ along the line $y = x$.

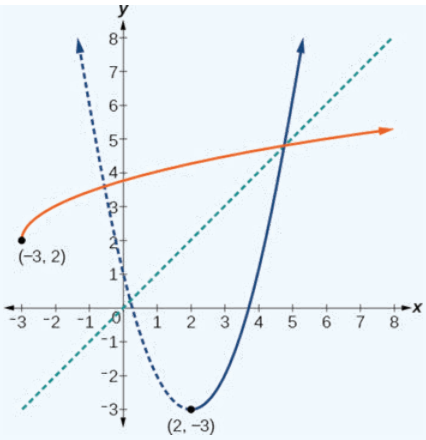



Figure 3.8.7

**Exercise 3.8.4**

Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

**Answer**

$f^{-1}(x) = \sqrt{x-1}$

## Finding Inverses

Notice that the functions from previous examples were all polynomials, and their inverses were radical functions. If we want to find the inverse of a radical function, we will need to restrict the domain of the answer because the range of the original function is limited.

**How to: Given a radical function, find the inverse**

1. Determine the range of the original function.
2. Replace $f(x)$ with $y$, then solve for $x$.
3. If necessary, restrict the domain of the inverse function to the range of the original function.

**Example 3.8.5: Finding the Inverse of a Radical Function**

Restrict the domain of the function $f(x) = \sqrt{x-4}$ and then find the inverse.

**Solution**

Note that the original function has range $f(x) \geq 0$. Replace $f(x)$ with $y$, then solve for $x$.

$$y = \sqrt{x-4} \text{ Replace } f(x) \text{ with } y.$$
$$x = \sqrt{y-4} \text{ Interchange } x \text{ and } y.$$
$$x = \sqrt{y-4} \text{ Square each side.}$$
$$x^2 = y - 4 \text{ Add 4.}$$
$$x^2 + 4 = y \text{ Rename the function } f^{-1}(x).$$
$$f^{-1}(x) = x^2 + 4$$

Recall that the domain of this function must be limited to the range of the original function.

$$f^{-1}(x) = x^2 + 4, \; x \geq 0$$

**Analysis**

Notice in Figure 3.8.8 that the inverse is a reflection of the original function over the line $y = x$. Because the original function has only positive outputs, the inverse function has only nonnegative inputs.

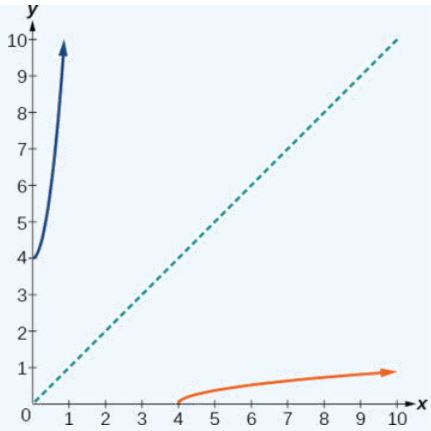


Figure 3.8.8

**Exercise 3.8.5**

Restrict the domain and then find the inverse of the function $f(x) = \sqrt{2x+3}$.

**Answer**

$f^{-1}(x) = \frac{x^2-3}{2}$, $x \geq 0$

## Solving Applications of Radical Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

**Example 3.8.6: Solving an Application with a Cubic Function**

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3$$

Find the inverse of the function $V = \frac{2}{3}\pi r^3$ that determines the volume $V$ of a cone and is a function of the radius $r$. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi = 3.14$.

**Solution**

Start with the given function for $V$. Notice that the meaningful domain for the function is $r > 0$ since negative radii would not make sense in this context nor would a radius of 0. Also note the range of the function (hence, the domain of the inverse function) is $V > 0$. Solve for $r$ in terms of $V$, using the method outlined previously. Note that in real-world applications, we do not swap the variables when finding inverses. Instead, we change which variable is considered to be the independent variable.

$$V = \frac{2}{3}\pi r^3$$

Solve for $r^3$.

$$r^3 = \frac{3V}{2\pi}$$

Solve for $r$.

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi = 3.14$.

$$
\begin{aligned}
r &= \sqrt[3]{\frac{3V}{2\pi}} \\
&= \sqrt[3]{\frac{3 \cdot 100}{2 \cdot 3.14}} \\
&\approx \sqrt[3]{47.7707} \\
&\approx 3.63
\end{aligned}
$$

Therefore, the radius is about 3.63 ft.

### Determining the Domain of a Radical Function Composed with Other Functions

When radical functions are composed with other functions, determining domain can become more complicated.

**Example 3.8.7: Finding the Domain of a Radical Function Composed with a Rational Function**

Find the domain of the function:

$$f(x) = \sqrt{\frac{(x+2)(x-3)}{(x-1)}}.$$

**Solution**

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where

$$\frac{(x+2)(x-3)}{(x-1)} \geq 0.$$

The output of a rational function can change signs (change from positive to negative or vice versa) at *x*-intercepts and at vertical asymptotes. For this equation, the graph could change signs at $x = -2$, 1, and 3.

To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, for this example we will use a graph as shown in Figure 3.8.9.

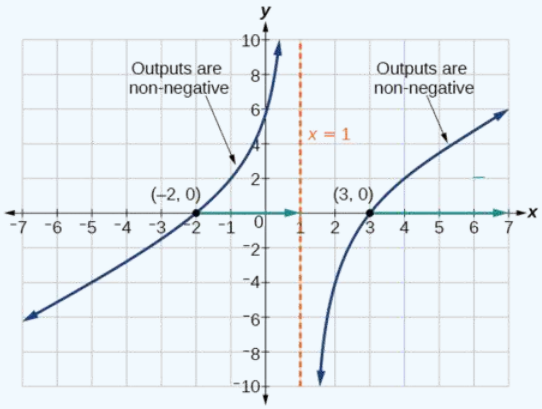



Figure 3.8.9

This function has two *x*-intercepts, both of which exhibit linear behavior near the *x*-intercepts. There is one vertical asymptote, corresponding to a linear factor; this behavior is similar to the basic reciprocal toolkit function, and there is no horizontal asymptote because the degree of the numerator is larger than the degree of the denominator. There is a *y*-intercept at $(0, \sqrt{6})$.

From the *y*-intercept and *x*-intercept at $x = -2$, we can sketch the left side of the graph. From the behavior at the asymptote, we can sketch the right side of the graph.

From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-2 \leq x < 1$ or $x \geq 3$, or in interval notation, $[-2, 1) \cup [3, \infty)$.

## Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the *inverse of a rational function*, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

### Example 3.8.8: Finding the Inverse of a Rational Function

The function

$$C = \frac{20 + 0.4n}{100 + n} \tag{3.8.2}$$

represents the concentration $C$ of an acid solution after $n$ mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for $n$ in terms of $C$. Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

**Solution**

We first want the inverse of the function in order to determine how many mL we need for a given concentration. We will solve for $n$ in terms of $C$.

$$\begin{aligned} C &= \frac{20 + 0.4n}{100 + n} \\ C(100 + n) &= 20 + 0.4n \\ 100C + Cn &= 20 + 0.4n \\ 100C - 20 &= 0.4n - Cn \\ 100C - 20 &= (0.4 - C)n \\ n &= \frac{100C - 20}{0.4 - C} \end{aligned}$$

Now evaluate this function at 35%, which is $C = 0.35$.

$$\begin{aligned} n &= \frac{100(0.35) - 20}{0.4 - 0.35} \\ &= \frac{15}{0.05} \\ &= 300 \end{aligned}$$

We can conclude that 300 mL of the 40% solution should be added.

### Exercise 3.8.8

Find the inverse of the function $f(x) = \frac{x+3}{x-2}$.

**Answer**

$f^{-1}(x) = \frac{2x+3}{x-1}$

### Media

Access these online resources for additional instruction and practice with inverses and radical functions.

- Graphing the Basic Square Root Function
- Find the Inverse of a Square Root Function
- Find the Inverse of a Rational Function
- Find the Inverse of a Rational Function and an Inverse Function Value
- Inverse Functions

## Key Concepts

- The inverse of a quadratic function is a square root function.
- If $f^{-1}$ is the inverse of a function $f$, then $f$ is the inverse of the function $f^{-1}$. See Example $3.8.1$.
- While it is not possible to find an inverse of most polynomial functions, some basic polynomials are invertible. See Example $3.8.2$.
- To find the inverse of certain functions, we must restrict the function to a domain on which it will be one-to-one. See Examples $3.8.3$ and $3.8.4$
- When finding the inverse of a radical function, we need a restriction on the domain of the answer. See Example $3.8.5$ and $3.8.7$.
- Inverse and radical and functions can be used to solve application problems. See Examples $3.8.6$ and $3.8.8$.

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# 3.9: Modeling Using Variation (optional)

**Learning Objectives**

In this section, you will:

- Solve direct variation problems.
- Solve inverse variation problems.
- Solve problems involving joint variation.

A used-car company has just offered their best candidate, Nicole, a position in sales. The position offers 16% commission on her sales. Her earnings depend on the amount of her sales. For instance, if she sells a vehicle for $4,600, she will earn $736. She wants to evaluate the offer, but she is not sure how. In this section, we will look at relationships, such as this one, between earnings, sales, and commission rate.

## Solving Direct Variation Problems

In the example above, Nicole's earnings can be found by multiplying her sales by her commission. The formula $e = 0.16s$ tells us her earnings, $e$, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See **Table 5.8.1**.

| $s$, sales price | $e = 0.16s$ | Interpretation |
|---|---|---|
| $9,200 | $e = 0.16(9,200) = 1,472$ | A sale of a $9,200 vehicle results in $1472 earnings. |
| $4,600 | $e = 0.16(4,600) = 736$ | A sale of a $4,600 vehicle results in $736 earnings. |
| $18,400 | $e = 0.16(18,400) = 2,944$ | A sale of a $18,400 vehicle results in $2944 earnings. |

**Table 5.8.1**

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from $4,600 to $9,200, and we double the earnings from $736 to $1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship **varies directly** with the other.

Figure 5.8.1 represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the car. The formula $y = kx^n$ is used for direct variation. The value $k$ is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$ and $n = 1$. We saw functions like this one when we discussed power functions.

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Figure 5.8.1

**A General Note: DIRECT VARIATION**

If $x$ and $y$ are related by an equation of the form

$$y = kx^n$$

then we say that the relationship is **direct variation** and $y$ **varies directly** with, or is proportional to, the $n$th power of $x$. In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x^n}$, where $k$ is called the **constant of variation**, which help defines the relationship between the variables.

## Given a description of a direct variation problem, solve for an unknown.

1. Identify the input, $x$,and the output, $y$.
2. Determine the constant of variation. You may need to divide $y$ by the specified power of $x$ to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

## Example

Solving a Direct Variation Problem

The quantity $y$ varies directly with the cube of $x$. If $y = 25$ when $x = 2$, find $y$ when $x$ is 6.

**Solution**

The general formula for direct variation with a cube is $y = kx^3$ . The constant can be found by dividing $y$ by the cube of $x$.

$$\begin{aligned} k &= \frac{y}{x^3} \\ &= \frac{25}{2^3} \\ &= \frac{25}{8} \end{aligned}$$

Now use the constant to write an equation that represents this relationship.

$$y = \frac{25}{8}x^3$$

Substitute $x = 6$ and solve for $y$.

$$\begin{aligned} y &= \frac{25}{8}(6)^3 \\ &= 675 \end{aligned}$$

Analysis

The graph of this equation is a simple cubic, as shown in **Figure 5.8.2**.

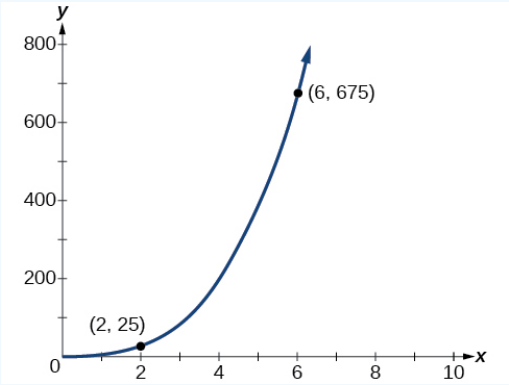



Figure 5.8.2

## Q&A

**Do the graphs of all direct variation equations look like Example?**

*No. Direct variation equations are power functions—they may be linear, quadratic, cubic, quartic, radical, etc. But all of the graphs pass through* $(0, 0)$.

**Exercise**

The quantity $y$ varies directly with the square of $x$. If $y = 24$ when $x = 3$, find $y$ when $x$ is 4.

**Solution**

$\frac{128}{3}$

## Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water's depth. The formula $T = \frac{14{,}000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers 22% of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F.

If we create **Table 5.8.2**, we observe that, as the depth increases, the water temperature decreases.

| $d$, depth | $T = \frac{14{,}000}{d}$ | Interpretation |
|---|---|---|
| 500 ft | $\frac{14{,}000}{500} = 28$ | At a depth of 500 ft, the water temperature is 28° F. |
| 1000 ft | $\frac{14{,}000}{1000} = 14$ | At a depth of 1,000 ft, the water temperature is 14° F. |
| 2000 ft | $\frac{14{,}000}{2000} = 7$ | At a depth of 2,000 ft, the water temperature is 7° F. |

**Table 5.8.2**

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be **inversely proportional** and each term **varies inversely** with the other. Inversely proportional relationships are also called **inverse variations**.

For our example, **Figure 5.8.3** depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula $y = \frac{k}{x}$ for inverse variation in this case uses $k = 14,000$.

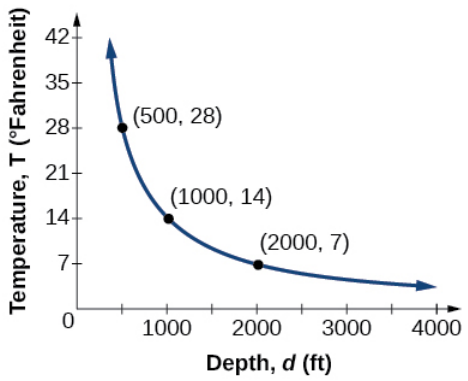



Figure 5.8.3

**A General Note: INVERSE VARIATION**

If $x$ and $y$ are related by an equation of the form

$$y = \frac{k}{x^n}$$

where $k$ is a nonzero constant, then we say that $y$ **varies inversely** with the $n$th power of $x$. In **inversely proportional** relationships, or **inverse variations**, there is a constant multiple $k = x^n y$.

**Example**

Writing a Formula for an Inversely Proportional Relationship

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

**Solution**

Recall that multiplying speed by time gives distance. If we let $t$ represent the drive time in hours, and $v$ represent the velocity (speed or rate) at which the tourist drives, then $vt =$ distance. Because the distance is fixed at 100 miles, $vt = 100$ so $t = \frac{100}{v}$. Because time is a function of velocity, we can write $t(v)$.

$$t(v) = \frac{100}{v}$$
$$= 100v^{-1}$$

We can see that the constant of variation is 100 and, although we can write the relationship using the negative exponent, it is more common to see it written as a fraction. We say that time varies inversely with velocity.

**Given a description of an indirect variation problem, solve for an unknown.**

1. Identify the input, $x$, and the output, $y$.
2. Determine the constant of variation. You may need to multiply $y$ by the specified power of $x$ to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

**Example**

Solving an Inverse Variation Problem

A quantity $y$ varies inversely with the cube of $x$. If $y = 25$ when $x = 2$, find $y$ when $x$ is 6.

**Solution**

The general formula for inverse variation with a cube is $y = \frac{k}{x^3}$. The constant can be found by multiplying $y$ by the cube of $x$.

$$k = x^3 y$$
$$= 2^3 \cdot 25$$
$$= 200$$

Now we use the constant to write an equation that represents this relationship.

$$y = \frac{k}{x^3}, k = 200$$
$$y = \frac{200}{x^3}$$

Substitute $x = 6$ and solve for $y$.

$$y = \frac{200}{6^3}$$
$$= \frac{25}{27}$$

Analysis

The graph of this equation is a rational function, as shown in **Figure 5.8.4**.

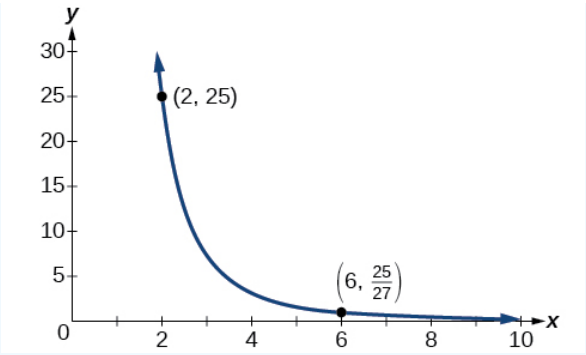



Figure 5.8.4

**Exercise**

A quantity $y$ varies inversely with the square of $x$. If $y = 8$ when $x = 3$, find $y$ when $x$ is 4.

**Solution**

$\frac{9}{2}$

## Solving Problems Involving Joint Variation

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called **joint variation**. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable $c$,cost, varies jointly with the number of students, $n$,and the distance, $d$.

**A General Note: JOINT VARIATION**

Joint variation occurs when a variable varies directly or inversely with multiple variables.

For instance, if $x$ varies directly with both $y$ and $z$, we have $x = kyz$. If $x$ varies directly with $y$ and inversely with $z$,we have $x = \frac{ky}{z}$. Notice that we only use one constant in a joint variation equation.

**Example**

Solving Problems Involving Joint Variation

A quantity $x$ varies directly with the square of $y$ and inversely with the cube root of $z$. If $x = 6$ when $y = 2$ and $z = 8$, find $x$ when $y = 1$ and $z = 27$.

**Solution**

Begin by writing an equation to show the relationship between the variables.

$$x = \frac{ky^2}{\sqrt[3]{z}}$$

Substitute $x = 6$, $y = 2$, and $z = 8$ to find the value of the constant $k$.

$$6 = \frac{k2^2}{\sqrt[3]{8}}$$

$$6 = \frac{4k}{2}$$

$$3 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$x = \frac{3y^2}{\sqrt[3]{z}}$$

To find $x$ when $y = 1$ and $z = 27$, we will substitute values for $y$ and $z$ into our equation.

$$x = \frac{3(1)^2}{\sqrt[3]{27}}$$
$$= 1$$

**Exercise**

A quantity $x$ varies directly with the square of $y$ and inversely with $z$. If $x = 40$ when $y = 4$ and $z = 2$, find $x$ when $y = 10$ and $z = 25$.

**Solution**

$x = 20$

**Media**

Access these online resources for additional instruction and practice with direct and inverse variation.

- Direct Variation
- Inverse Variation
- Direct and Inverse Variation

Visit this website for additional practice questions from Learningpod.

## Key Equations

| | |
|---|---|
| Direct variation | $y = kx^n$, $k$ is a nonzero constant. |
| Inverse variation | $y = \frac{k}{x^n}$, $k$ is a nonzero constant. |

## Key Concepts

- A relationship where one quantity is a constant multiplied by another quantity is called direct variation. See Example.
- Two variables that are directly proportional to one another will have a constant ratio.
- A relationship where one quantity is a constant divided by another quantity is called inverse variation. See Example.
- Two variables that are inversely proportional to one another will have a constant multiple. See Example.
- In many problems, a variable varies directly or inversely with multiple variables. We call this type of relationship joint variation. See Example.

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# 3.E: Polynomial and Rational Functions(Exercises)

## 3.1 Complex Numbers

### Verbal

1) Explain how to add complex numbers.

**Answer**

Add the real parts together and the imaginary parts together.

2) What is the basic principle in multiplication of complex numbers?

3) Give an example to show the product of two imaginary numbers is not always imaginary.

**Answer**

$i$ times $i$ equals $-1$, which is not imaginary (answers vary)

4) What is a characteristic of the plot of a real number in the complex plane?

### Algebraic

**For the exercises 5-10, evaluate the algebraic expressions.**

5) If $f(x)=x^2+x-4$ , evaluate $f(2i)$.

**Answer**

$-8+2i$

6) If $f(x)=x^3-2$ , evaluate $f(i)$.

7) If $f(x)=x^2+3x+5$ ,evaluate $f(2+i)$.

**Answer**

$14+7i$

8) If $f(x)=2x^2+x-3$ , evaluate $f(2-3i)$.

9) If $f(x)=\dfrac{x+1}{2-x}$ , evaluate $f(5i)$.

**Answer**

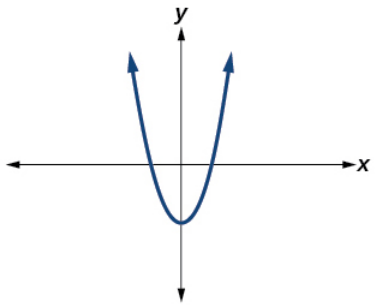
$-\dfrac{23}{29}+\dfrac{15}{29}i$

10) If $f(x)=\dfrac{1+2x}{x+3}$, evaluate $f(4i)$.

### Graphical

**For the exercises 11-12, determine the number of real and nonreal solutions for each quadratic function shown.**

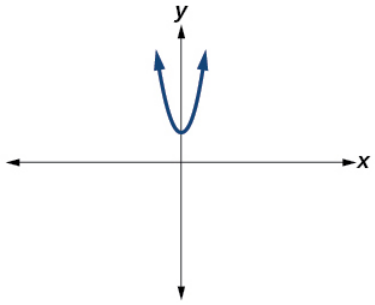
11)




**Answer**

2 real and 0 nonreal

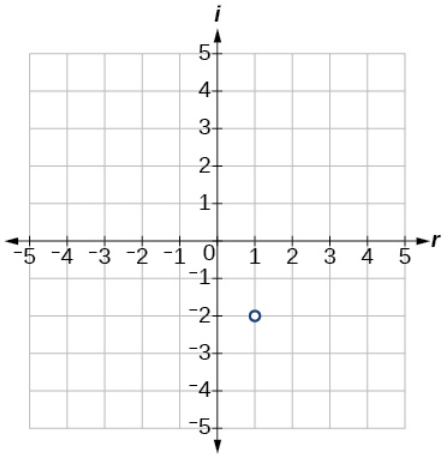
12)




**For the exercises 13-16, plot the complex numbers on the complex plane.**

13) $1-2i$

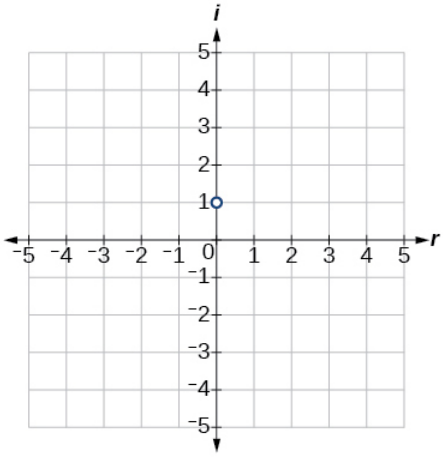
**Answer**




14) $-2+3i$

15) $i$

**Answer**




16) $-3-4i$

## Numeric

**For the exercises 17-43, perform the indicated operation and express the result as a simplified complex number.**

17) $(3+2i)+(5-3i)$

**Answer**

$8-i$

18) $(-2-4i)+(1+6i)$

19) $(-5+3i)-(6-i)$

**Answer**

$-11+4i$

20) $(2-3i)-(3+2i)$

21) $(-4+4i)-(-6+9i)$

**Answer**

$2-5i$

22) $(2+3i)(4i)$

23) $(5-2i)(3i)$

**Answer**

$6+15i$

24) $(6-2i)(5)$

25) $(-2+4i)(8)$

**Answer**

$-16+32i$

26) $(2+3i)(4-i)$

27) $(-1+2i)(-2+3i)$

**Answer**

$-4-7i$

28) $(4-2i)(4+2i)$

29) $(3+4i)(3-4i)$

**Answer**

$25$

30) $\dfrac{3+4i}{2}$

31) $\dfrac{6-2i}{3}$

**Answer**

$2-\dfrac{2}{3}i$

32) $\dfrac{-5+3i}{2i}$

33) $\dfrac{6+4i}{i}$

**Answer**

$4-6i$

34) $\dfrac{2-3i}{4+3i}$

35) $\dfrac{3+4i}{2-i}$

**Answer**

$\dfrac{2}{5}+\dfrac{11}{5}i$

36) $\dfrac{2+3i}{2-3i}$

37) $\sqrt{-9}+3\sqrt{-16}$

**Answer**

$15i$

38) $-\sqrt{-4}-4\sqrt{-25}$

39) $\dfrac{2+\sqrt{-12}}{2}$

**Answer**

$1+i\sqrt{3}$

40) $\dfrac{4+\sqrt{-20}}{2}$

41) $i^8$

**Answer**

$1$

42) $i^{15}$

43) $i^{22}$

**Answer**

$-1$

## Technology

**For the exercises 44-48, use a calculator to help answer the questions.**

44) Evaluate $(1+i)^k$ for $k=4, 8,$ and 12.Predict the value if $k=16$.

45) Evaluate $(1-i)^k$ for $k=2, 6,$ and 10.Predict the value if $k=14$.

**Answer**

$128i$

46) Evaluate $(1+i)^k-(1-i)^k$ for $k=4$, 8, and 12. Predict the value for $k=16$.

47) Show that a solution of $x^6+1=0$ is $\frac{\sqrt{3}}{2}+\frac{1}{2}i$.

**Answer**

$(\frac{\sqrt{3}}{2}+\frac{1}{2}i)^6=-1$

48) Show that a solution of $x^8-1=0$ is $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

## Extensions

**For the exercises 49-58, evaluate the expressions, writing the result as a simplified complex number.**

49) $\frac{1}{i}+\frac{4}{i^3}$

**Answer**

$3i$

50) $\frac{1}{i^{11}}-\frac{1}{i^{21}}$

51) $i^7(1+i^2)$

**Answer**

$0$

52) $i^{-3}+5i^7$

53) $\frac{(2+i)(4-2i)}{(1+i)}$

**Answer**

$5-5i$

54) $\frac{(1+3i)(2-4i)}{(1+2i)}$

55) $\frac{(3+i)^2}{(1+2i)^2}$

**Answer**

$-2i$

56) $\frac{3+2i}{2+i}+(4+3i)$

57) $\frac{4+i}{i}+\frac{3-4i}{1-i}$

**Answer**

$\frac{9}{2}-\frac{9}{2}i$

58) $\frac{3+2i}{1+2i}-\frac{2-3i}{3+i}$

## 3.2 Quadratic Functions

### Verbal

1) Explain the advantage of writing a quadratic function in standard form.

**Answer**

When written in that form, the vertex can be easily identified.

2) How can the vertex of a parabola be used in solving real world problems?

3) Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

**Answer**

If $a = 0$ then the function becomes a linear function.

4) What is another name for the standard form of a quadratic function?

5) What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

**Answer**

If possible, we can use factoring. Otherwise, we can use the quadratic formula.

### Algebraic

**For the exercises 6-13, rewrite the quadratic functions in standard form and give the vertex.**

6) $f(x) = x^2 - 12x + 32$

7) $g(x) = x^2 + 2x - 3$

**Answer**

$g(x) = (x+1)^2 - 4$ , Vertex $(-1, -4)$

8) $f(x) = x^2 - x$

9) $f(x) = x^2 + 5x - 2$

**Answer**

$f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$ , Vertex $\left(-\frac{5}{2}, -\frac{33}{4}\right)$

10) $h(x) = 2x^2 + 8x - 10$

11) $k(x) = 3x^2 - 6x - 9$

**Answer**

$k(x) = 3(x-1)^2 - 12$ , Vertex $(1, -12)$

12) $f(x) = 2x^2 - 6x$

13) $f(x) = 3x^2 - 5x - 1$

**Answer**

$f(x) = 3\left(x - \frac{5}{6}\right)^2 - \frac{37}{12}$ , Vertex $\left(\frac{5}{6}, -\frac{37}{12}\right)$

**For the exercises 14-20, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.**

14) $y(x) = 2x^2 + 10x + 12$

15) $f(x) = 2x^2 - 10x + 4$

**Answer**

Minimum is $-\frac{17}{2}$ and occurs at $\frac{5}{2}$. Axis of symmetry is $x = \frac{5}{2}$.

16) $f(x) = -x^2 + 4x + 3$

17) $f(x) = 4x^2 + x - 1$

**Answer**

Minimum is $-\frac{17}{16}$ and occurs at $-\frac{1}{8}$. Axis of symmetry is $x = -\frac{1}{8}$.

18) $h(t) = -4t^2 + 6t - 1$

19) $f(x) = \frac{1}{2}x^2 + 3x + 1$

**Answer**

Minimum is $-\frac{7}{2}$ and occurs at $-3$. Axis of symmetry is $x = -3$.

20) $f(x) = -\frac{1}{3}x^2 - 2x + 3$

**For the exercises 21-25, determine the domain and range of the quadratic function.**

21) $f(x) = (x-3)^2 + 2$

**Answer**

Domain is $(-\infty, \infty)$. Range is $[2, \infty)$.

22) $f(x) = -2(x+3)^2 - 6$

23) $f(x) = x^2 + 6x + 4$

**Answer**

Domain is $(-\infty, \infty)$. Range is $[-5, \infty)$.

24) $f(x) = 2x^2 - 4x + 2$

25) $k(x) = 3x^2 - 6x - 9$

**Answer**

Domain is $(-\infty, \infty)$ . Range is $[-12, \infty)$.

**For the exercises 26-44, solve the equations over the complex numbers.**

26) $x^2 = -25$

27) $x^2 = -8$

**Answer**

$2i\sqrt{2}, -2i\sqrt{2}$

28) $x^2 + 36 = 0$

29) $x^2 + 27 = 0$

**Answer**

$3i\sqrt{3}, -3i\sqrt{3}$

30) $x^2 + 2x + 5 = 0$

31) $x^2 - 4x + 5 = 0$

**Answer**

$2 + i, 2 - i$

32) $x^2 + 8x + 25 = 0$

33) $x^2 - 4x + 13 = 0$

**Answer**

$2 + 3i, 2 - 3i$

34) $x^2 + 6x + 25 = 0$

35) $x^2 - 10x + 26 = 0$

**Answer**

$5 + i, 5 - i$

36) $x^2 - 6x + 10 = 0$

37) $x(x - 4) = 20$

**Answer**

$2 + 2\sqrt{6}, 2 - 2\sqrt{6}$

38) $x(x - 2) = 10$

39) $2x^2 + 2x + 5 = 0$

**Answer**

$\left\{-\frac{1}{2} + \frac{3}{2}i, -\frac{1}{2} - \frac{3}{2}i\right\}$

40) $5x^2 - 8x + 5 = 0$

41) $5x^2 + 6x + 2 = 0$

**Answer**

$$\left\{-\frac{3}{5}+\frac{1}{5}i,-\frac{3}{5}-\frac{1}{5}i\right\}$$

42) $2x^2-6x+5=0$

43) $x^2+x+2=0$

**Answer**

$$\left\{-\frac{1}{2}+\frac{1}{2}i\sqrt{7},-\frac{1}{2}-\frac{1}{2}i\sqrt{7}\right\}$$

44) $x^2-2x+4=0$

**For the exercises 45-52, use the vertex $(h,k)$ and a point on the graph $(x,y)$ to find the general form of the equation of the quadratic function.**

45) $(h,k)=(2,0),(x,y)=(4,4)$

**Answer**

$f(x)=x^2-4x+4$

46) $(h,k)=(-2,-1),(x,y)=(-4,3)$

47) $(h,k)=(0,1),(x,y)=(2,5)$

**Answer**

$f(x)=x^2+1$

48) $(h,k)=(2,3),(x,y)=(5,12)$

49) $(h,k)=(-5,3),(x,y)=(2,9)$

**Answer**

$$f(x)=\frac{6}{49}x^2+\frac{60}{49}x+\frac{297}{49}$$

50) $(h,k)=(3,2),(x,y)=(10,1)$

51) $(h,k)=(0,1),(x,y)=(1,0)$

**Answer**

$f(x)=-x^2+1$

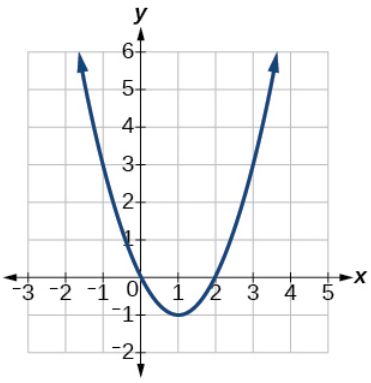
52) $(h,k)=(1,0),(x,y)=(0,1)$

### Graphical

**For the exercises 53-58, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.**

53) $f(x)=x^2-2x$

**Answer**

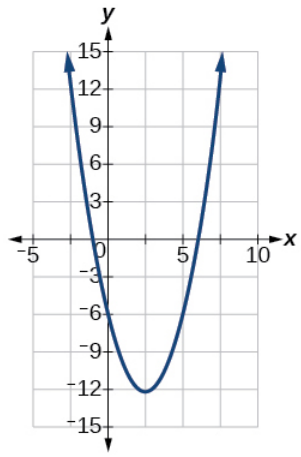



Vertex $(1, -1)$, Axis of symmetry is $x = 1$. Intercepts are $(0, 0), (2, 0)$.

54) $f(x) = x^2 - 6x - 1$

55) $f(x) = x^2 - 5x - 6$

**Answer**

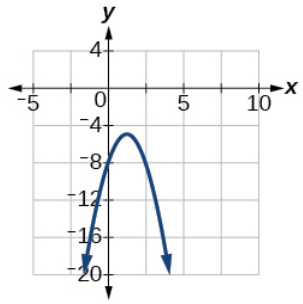



Vertex $\left(\frac{5}{2}, \frac{-49}{4}\right)$, Axis of symmetry is $x = \frac{5}{2}$. Intercepts are $(0, -6), (-1, 0), (6, 0)$.

56) $f(x) = x^2 - 7x + 3$

57) $f(x) = -2x^2 + 5x - 8$

**Answer**

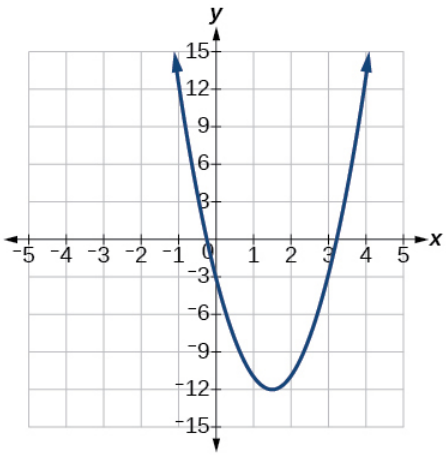



Vertex $\left(\frac{5}{4}, -\frac{39}{8}\right)$, Axis of symmetry is $x = \frac{5}{4}$. Intercepts are $(0, -8)$.

58) $f(x) = 4x^2 - 12x - 3$

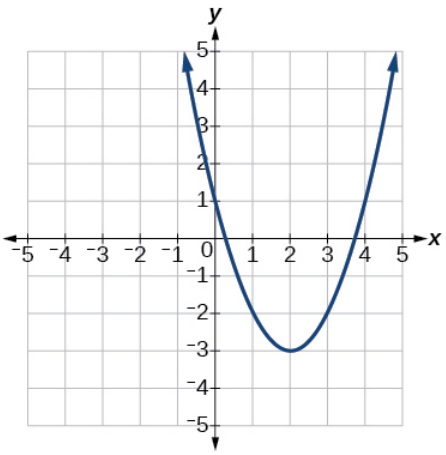

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**Answer**




**For the exercises 59-64, write the equation for the graphed function.**

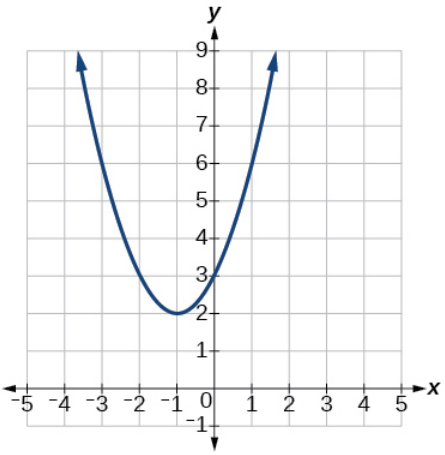
59)



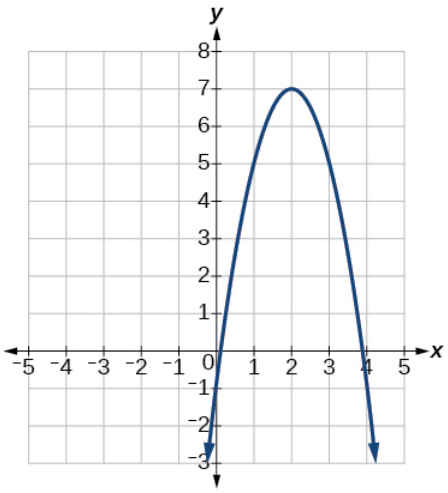

**Answer**

$f(x) = x^2 - 4x + 1$

60)



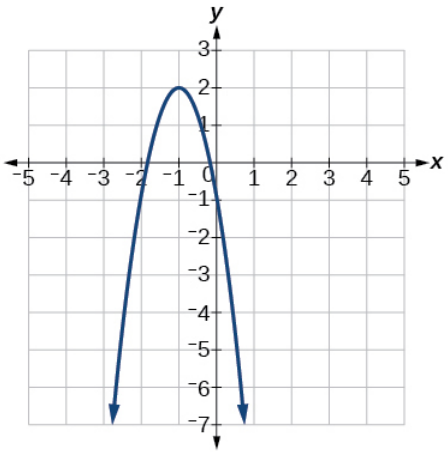

61)



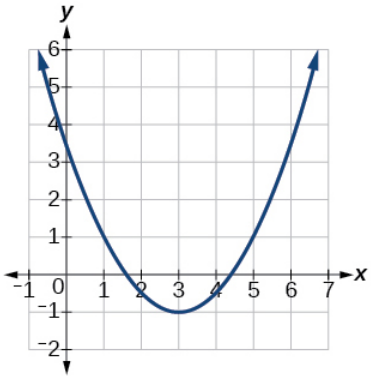

**Answer**

$f(x) = -2x^2 + 8x - 1$

62)



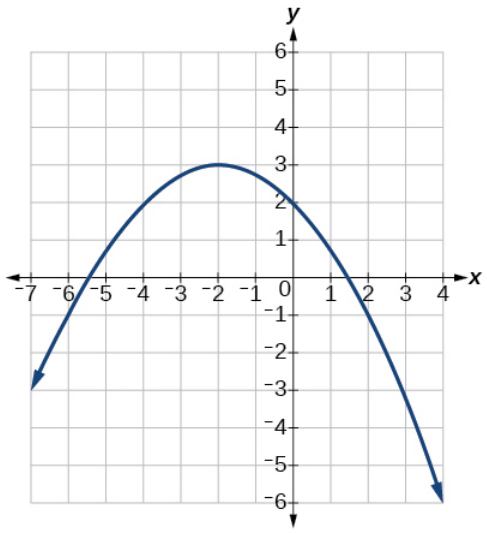

63)




**Answer**

$f(x) = \frac{1}{2}x^2 - 3x + \frac{7}{2}$

64)




## Numeric

**For the exercises 65-69, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.**

65)

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 5 | 2 | 1 | 2 | 5 |

**Answer**

$f(x) = x^2 + 1$

66)

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 1 | 0 | 1 | 4 | 9 |

67)

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | –2 | 1 | 2 | 1 | –2 |

**Answer**

$f(x) = 2 - x^2$

68)

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | –8 | –3 | 0 | 1 | 0 |

69)

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 8 | 2 | 0 | 2 | 8 |

**Answer**

$f(x) = 2x^2$

## Technology

**For the exercises 70-74, use a calculator to find the answer.**

70) Graph on the same set of axes the functions $f(x) = x^2, f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$. What appears to be the effect of changing the coefficient?

71) Graph on the same set of axes $f(x) = x^2, f(x) = x^2 + 2$ and $f(x) = x^2, f(x) = x^2 + 5$ and $f(x) = x^2 - 3$. What appears to be the effect of adding a constant?

**Answer**

The graph is shifted up or down (a vertical shift).

72) Graph on the same set of axes $f(x) = x^2, f(x) = (x-2)^2, f(x-3)2,$ and $f(x) = (x+4)^2$. What appears to be the effect of adding or subtracting those numbers?

73) The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x) = -\frac{32}{(80)^2}x^2 + x$ where $x$ is the horizontal distance traveled and $h(x)$ is the height in feet. Use the TRACE feature of your calculator to determine the height of the object when it has traveled 100 feet away horizontally.

**Answer**

50 feet

74) A suspension bridge can be modeled by the quadratic function $h(x) = .0001x^2$ with $-2000 \leq x \leq 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use the TRACE feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

## Extensions

**For the exercises 75-78, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.**

75) Vertex $(1, -2)$, opens up.

**Answer**

Domain is $(-\infty, \infty)$. Range is $[-2, \infty)$.

76) Vertex $(-1, 2)$ opens down.

77) Vertex $(-5, 11)$, opens down.

**Answer**

Domain is $(-\infty, \infty)$ Range is $(-\infty, 11]$.

78) Vertex $(-100, 100)$, opens up.

**For the exercises 79-84, write the equation of the quadratic function that contains the given point and has the same shape as the given function.**

79) Contains $(1, 1)$ and has shape of $f(x) = 2x^2$. Vertex is on the $y$-axis.

**Answer**

$f(x) = 2x^2 - 1$

80) Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the $y$-axis.

81) Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the $y$-axis.

**Answer**

$f(x) = 3x^2 - 9$

82) Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the $y$-axis.

83) Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the $y$-axis.

**Answer**

$f(x) = 5x^2 - 77$

84) Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has $x$-coordinate of $-1$.

## Real-World Applications

85) Find the dimensions of the rectangular corral producing the greatest enclosed area given $200$ feet of fencing.

**Answer**

$50$ feet by $50$ feet. Maximize $f(x) = -x^2 + 100x$ .

86) Find the dimensions of the rectangular corral split into $2$ pens of the same size producing the greatest possible enclosed area given $300$ feet of fencing.

87) Find the dimensions of the rectangular corral producing the greatest enclosed area split into $3$ pens of the same size given $500$ feet of fencing.

**Answer**

$125$ feet by $62.5$ feet. Maximize $f(x) = -2x^2 + 250x$ .

88) Among all of the pairs of numbers whose sum is $6$, find the pair with the largest product. What is the product?

89) Among all of the pairs of numbers whose difference is $12$, find the pair with the smallest product. What is the product?

**Answer**

$6$ and $-6$; product is $-36$; maximize $f(x) = x^2 + 12x$ .

90) Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x$, where $x$ is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.

91) A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$ . Find the maximum height the rocket attains.

**Answer**

$2909.56$ meters

92) A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$ . How long does it take to reach maximum height?

93) A soccer stadium holds $62,000$ spectators. With a ticket price of $\$11$, the average attendance has been $26,000$. When the price dropped to $\$9$, the average attendance rose to $31,000$. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

**Answer**

$\$10.70$

94) A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

## 3.3 Power Functions and Polynomial Functions

### Verbal

1) Explain the difference between the coefficient of a power function and its degree.

**Answer**

The coefficient of the power function is the real number that is multiplied by the variable raised to a power. The degree is the highest power appearing in the function.

2) If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?

3) In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.

**Answer**

As $x$ decreases without bound, so does $f(x)$. As $x$ increases without bound, so does $f(x)$.

4) What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?

5) What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty, f(x) \to -\infty$ and as $x \to \infty, f(x) \to -\infty$.

**Answer**

The polynomial function is of even degree and leading coefficient is negative.

### Algebraic

**For the exercises 6-11, identify the function as a power function, a polynomial function, or neither.**

6) $f(x) = x^5$

7) $f(x) = (x^2)^3$

**Answer**

Power function

8) $f(x) = x - x^4$

9) $f(x) = \dfrac{x^2}{x^2 - 1}$

**Answer**

Neither

10) $f(x) = 2x(x+2)(x-1)^2$

11) $f(x) = 3^{x+1}$

**Answer**

Neither

**For the exercises 12-16, find the degree and leading coefficient for the given polynomial.**

12) $-3x^4$

13) $7 - 2x^2$

**Answer**

Degree = 2, Coefficient = $-2$

14) $-2x^2 - 3x^5 + x - 6$

15) $x(4 - x^2)(2x + 1)$

**Answer**

Degree =4, Coefficient = $-2$

16) $x^2(2x - 3)^2$

**For the exercises 17-24, determine the end behavior of the functions.**

17) $f(x) = x^4$

**Answer**

As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to \infty$

18) $f(x) = x^3$

19) $f(x) = -x^4$

**Answer**

As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$

20) $f(x) = -x^9$

21) $f(x) = -2x^4 - 3x^2 + x - 1$

**Answer**

As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$

22) $f(x) = 3x^2 + x - 2$

23) $f(x) = x^2(2x^3 - x + 1)$

**Answer**

As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to -\infty$

24) $f(x) = (2 - x)^7$

**For the exercises 25-30, find the intercepts of the functions.**

25) $f(t) = 2(t-1)(t+2)(t-3)$

**Answer**

$y$-intercept is $(0, 12)$, $t$-intercepts are $(1, 0)$; $(-2, 0)$;and $(3, 0)$.

26) $g(n) = -2(3n-1)(2n+1)$

27) $f(x) = x^4 - 16$

**Answer**

$y$-intercept is $(0, -16)$, $x$-intercepts are $(2, 0)$ and $(-2, 0)$.

28) $f(x) = x^3 + 27$

29) $f(x) = x(x^2 - 2x - 8)$

**Answer**

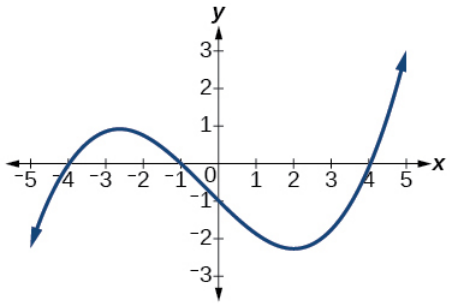
$y$-intercept is $(0, 0)$, $x$-intercepts are $(0, 0)$, $(4, 0)$,and $(-2, 0)$.

30) $f(x) = (x+3)(4x^2 - 1)$

## Graphical

**For the exercises 31-38, determine the least possible degree of the polynomial function shown.**

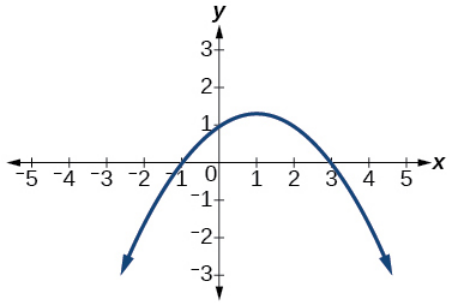
31)



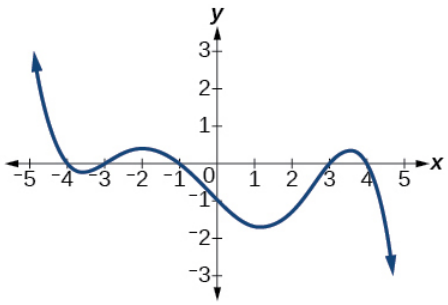

**Answer**

3

32)



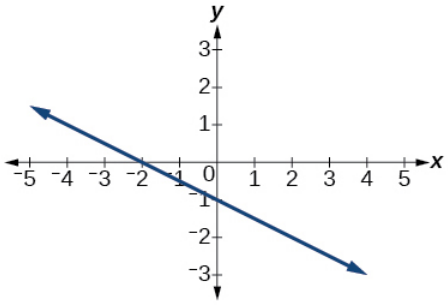
33)



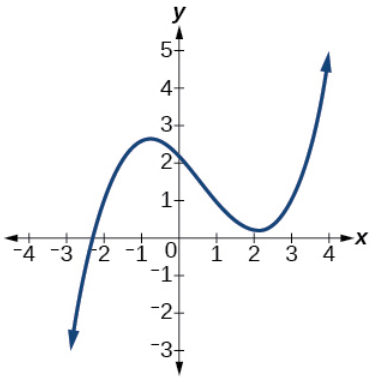

**Answer**

5

34)



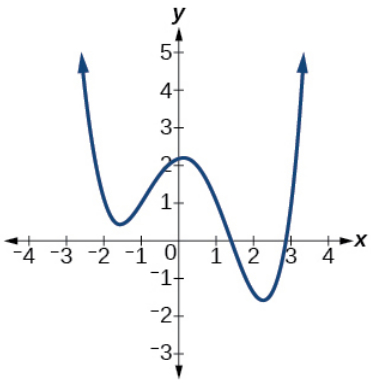

35)



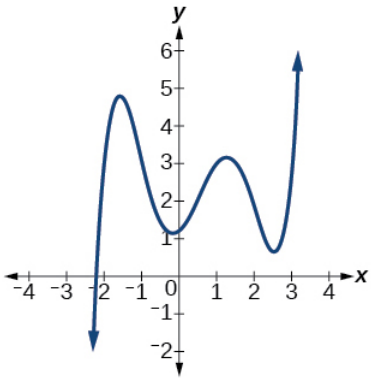

**Answer**

3

36)



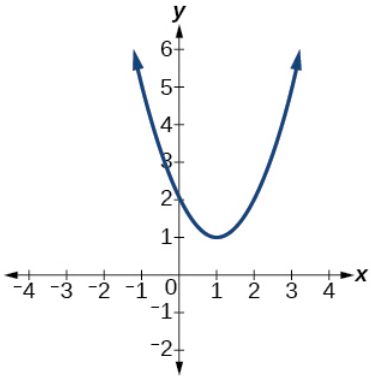

37)




**Answer**

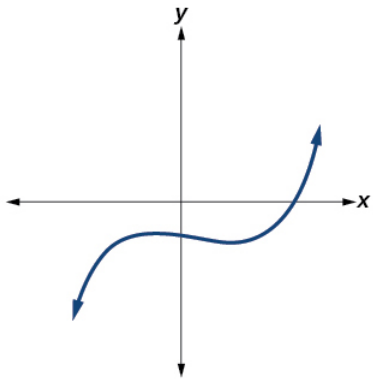
5

38)




**For the exercises 39-45, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.**

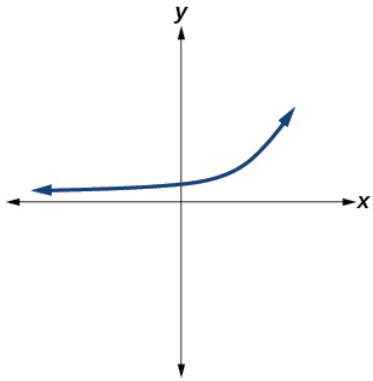
39)



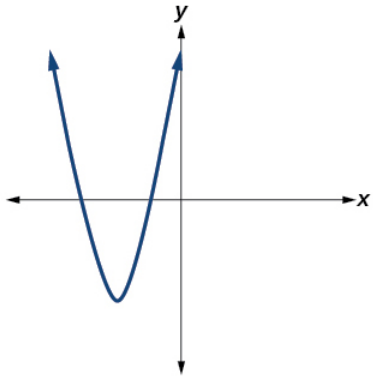

**Answer**

Yes. Number of turning points is 2. Least possible degree is 3.

40)



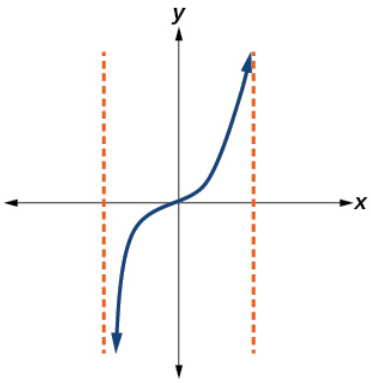

41)



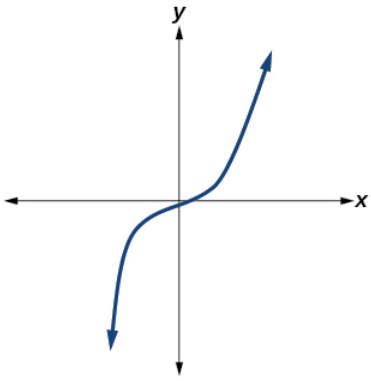

**Answer**

Yes. Number of turning points is 1. Least possible degree is 2.

42)



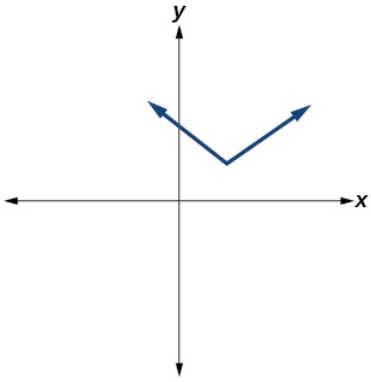

43)




**Answer**

Yes. Number of turning points is 0. Least possible degree is 1.

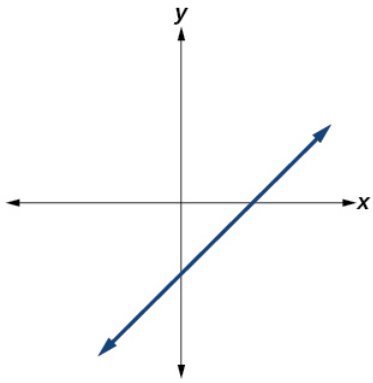
44)




**Answer**

No.

45)




**Answer**

Yes. Number of turning points is 0. Least possible degree is 1.

Numeric

**For the exercises 46-50, make a table to confirm the end behavior of the function.**

46) $f(x) = -x^3$

47) $f(x) = x^4 - 5x^2$

**Answer**

| $x$ | $f(x)$ |
|---|---|
| 10 | 9,500 |
| 100 | 99,950,000 |
| –10 | 9,500 |
| –100 | 99,950,000 |

as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$

48) $f(x) = x^2(1-x)^2$

49) $f(x) = (x-1)(x-2)(3-x)$

**Answer**

| $x$ | $f(x)$ |
|---|---|
| 10 | –504 |
| 100 | –941,094 |
| –10 | 1,716 |
| –100 | 1,061,106 |

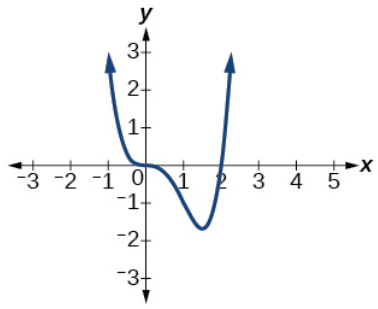
as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to -\infty$

50) $f(x) = \dfrac{x^5}{10} - x^4$

**For the exercises 51-60, graph the polynomial functions using a calculator. Based on the graph, determine the intercepts and the end behavior.**

51) $f(x) = x^3(x-2)$

**Answer**

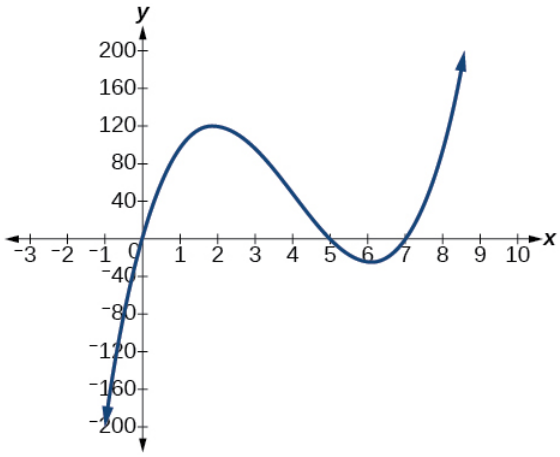



The $y$-intercept is $(0,0)$. The x-intercepts are $(0,0),(2,0)$. As $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$

52) $f(x) = x(x-3)(x+3)$

53) $f(x) = x(14-2x)(10-2x)$

**Answer**

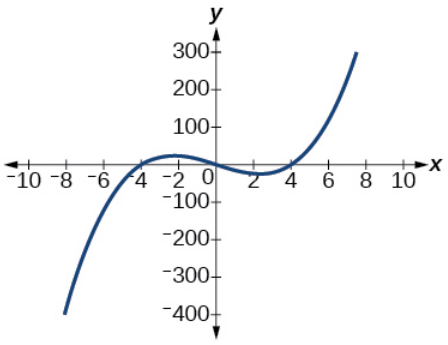



The $y$-intercept is $(0,0)$. The x-intercepts are $(0,0),(5,0),(7,0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$

54) $f(x) = x(14-2x)(10-2x)^2$

55) $f(x) = x^3 - 16x$

**Answer**

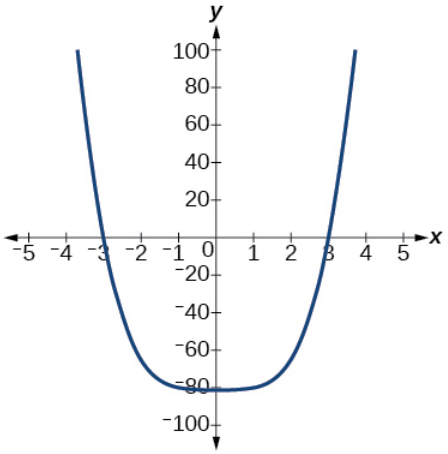



The $y$-intercept is $(0,0)$. The x-intercept is $(-4,0),(0,0),(4,0)$. As $x \to -\infty$ , $f(x) \to -\infty$ , as $x \to \infty,\ f(x) \to \infty$

56) $f(x) = x^3 - 27$

57) $f(x) = x^4 - 81$

**Answer**

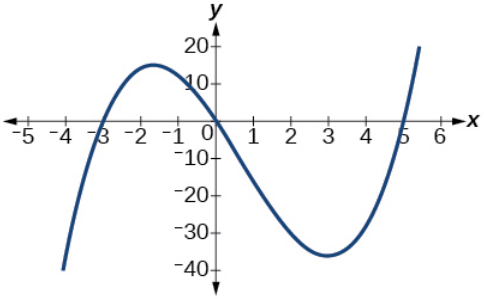



The $y$-intercept is $(0,-81)$. The $x$-intercept are $(3,0),(-3,0)$. As $x \to -\infty,\ f(x) \to \infty$ , as $x \to \infty,\ f(x) \to \infty$

58) $f(x) = -x^3 + x^2 + 2x$

59) $f(x) = x^3 - 2x^2 - 15x$

**Answer**




The $y$-intercept is $(0,0)$. The x-intercepts are $(-3,0),(0,0),(5,0)$.As $x \to -\infty$ , $f(x) \to -\infty$ , as $x \to \infty,\ f(x) \to \infty$

60) $f(x) = x^3 - 0.01x$

## Extensions

**For the exercises 61-65, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is $1$ or $-1$. There may be more than one correct answer.**

61) The $y$-intercept is $(0,-4)$. The $x$-intercepts are $(-2,0),(2,0)$. Degree is $2$.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$.

**Answer**

$f(x) = x^2 - 4$

62) The $y$-intercept is $(0,9)$. The $x$-intercepts are $(-3,0),(3,0)$. Degree is $2$.

End behavior: as $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to -\infty$.

63) The $y$-intercept is $(0,0)$. The $x$-intercepts are $(0,0),(2,0)$. Degree is $3$.

End behavior: as $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$.

**Answer**

$f(x) = x^3 - 4x^2 + 4x$

64) The $y$-intercept is $(0,1)$. The x-intercept is $(1,0)$. Degree is $3$.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to -\infty$.

65) The $y$-intercept is $(0,1)$. There is no $x$-intercept. Degree is $4$.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$.

**Answer**

$f(x) = x^4 + 1$

## Real-World Applications

**For the exercises 66-70, use the written statements to construct a polynomial function that represents the required information.**

66) An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of $20$ meters per day. Express the area of the circle as a function of $d$, the number of days elapsed.

67) A cube has an edge of $3$ feet. The edge is increasing at the rate of $2$ feet per minute. Express the volume of the cube as a function of $m$, the number of minutes elapsed.

**Answer**

$V(m) = 8m^3 + 36m^2 + 54m + 27$

68) A rectangle has a length of $10$ inches and a width of $6$ inches. If the length is increased by $x$ inches and the width increased by twice that amount, express the area of the rectangle as a function of $x$.

69) An open box is to be constructed by cutting out square corners of $x$-inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of $x$.

**Answer**

$V(x) = 4x^3 - 32x^2 + 64x$

70) A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width ( $x$).

## 3.4 Graphs of Polynomial Functions

### Verbal

1) What is the difference between an $x$-intercept and a zero of a polynomial function $f$?

**Answer**

The $x$-intercept is where the graph of the function crosses the $x$-axis, and the zero of the function is the input value for which $f(x) = 0$.

2) If a polynomial function of degree $n$ has $n$ distinct zeros, what do you know about the graph of the function?

3) Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.

**Answer**

If we evaluate the function at $a$ and at $b$ and the sign of the function value changes, then we know a zero exists between $a$ and $b$.

4) Explain how the factored form of the polynomial helps us in graphing it.

5) If the graph of a polynomial just touches the $x$-axis and then changes direction, what can we conclude about the factored form of the polynomial?

**Answer**

There will be a factor raised to an even power.

### Algebraic

**For the exercises 6-23, find the $x$-or $t$-intercepts of the polynomial functions.**

6) $C(t) = 2(t-4)(t+1)(t-6)$

7) $C(t) = 3(t+2)(t-3)(t+5)$

**Answer**

$(-2, 0), (3, 0), (-5, 0)$

8) $C(t) = 4t(t-2)^2(t+1)$

9) $C(t) = 2t(t-3)(t+1)^2$

**Answer**

$(3,0), (-1,0), (0,0)$

10) $C(t) = 2t^4 - 8t^3 + 6t^2$

11) $C(t) = 4t^4 + 12t^3 - 40t^2$

**Answer**

$(0,0), (-5,0), (2,0)$

12) $f(x) = x^4 - x^2$

13) $f(x) = x^3 + x^2 - 20x$

**Answer**

$(0,0), (-5,0), (4,0)$

14) $f(x) = x^3 + 6x^2 - 7x$

15) $f(x) = x^3 + x^2 - 4x - 4$

**Answer**

$(2,0), (-2,0), (-1,0)$

16) $f(x) = x^3 + 2x^2 - 9x - 18$

17) $f(x) = 2x^3 - x^2 - 8x + 4$

**Answer**

$(-2,0), (2,0), \left(\frac{1}{2}, 0\right)$

18) $f(x) = x^6 - 7x^3 - 8$

19) $f(x) = 2x^4 + 6x^2 - 8$

**Answer**

$(1,0), (-1,0)$

20) $f(x) = x^3 - 3x^2 - x + 3$

21) $f(x) = x^6 - 2x^4 - 3x^2$

**Answer**

$(0,0), (\sqrt{3},0), (-\sqrt{3},0)$

22) $f(x) = x^6 - 3x^4 - 4x^2$

23) $f(x) = x^5 - 5x^3 + 4x$

**Answer**

$(0,0),(1,0),(-1,0),(2,0),(-2,0)$

**For the exercises 24-29, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.**

24) $f(x)=x^3-9x$ , between $x=-4$ and $x=-2$.

25) $f(x)=x^3-9x$ , between $x=2$ and $x=4$.

**Answer**

$f(2)=-10$ and $f(4)=28$. Sign change confirms.

26) $f(x)=x^5-2x$ , between $x=1$ and $x=2$.

27) $f(x)=-x^4+4$, between $x=1$ and $x=3$.

**Answer**

$f(1)=3$ and $f(3)=-77$. Sign change confirms.

28) $f(x)=-2x^3-x$ , between $x=-1$ and $x=1$.

29) $f(x)=x^3-100x+2$ , between $x=0.01$ and $x=0.1$

**Answer**

$f(0.01)=1.000001$ and $f(0.1)=-7.999$ Sign change confirms.

**For the exercises 30-41, find the zeros and give the multiplicity of each.**

30) $f(x)=(x+2)^3(x-3)^2$

31) $f(x)=x^2(2x+3)^5(x-4)^2$

**Answer**

0 with multiplicity 2, $-\frac{3}{2}$ with multiplicity 5, 4 with multiplicity 2

32) $f(x)=x^3(x-1)^3(x+2)$

33) $f(x)=x^2(x^2+4x+4)$

**Answer**

0 with multiplicity 2, $-2$ with multiplicity 2

34) $f(x)=(2x+1)^3(9x^2-6x+1)$

35) $f(x)=(3x+2)^5(x^2-10x+25)$

**Answer**

$-\frac{2}{3}$ with multiplicity 5, 5 with multiplicity 2

36) $f(x)=x(4x^2-12x+9)(x^2+8x+16)$

37) $f(x)=x^6-x^5-2x^4$

**Answer**

0 with multiplicity 4, 2 with multiplicity 1, $-1$ with multiplicity 1

38) $f(x) = 3x^4 + 6x^3 + 3x^2$

39) $f(x) = 4x^5 - 12x^4 + 9x^3$

**Answer**

$\frac{3}{2}$ with multiplicity 2, 0 with multiplicity 3

40) $f(x) = 2x^4(x^3 - 4x^2 + 4x)$

41) $f(x) = 4x^4(9x^4 - 12x^3 + 4x^2)$

**Answer**

0 with multiplicity 6, $\frac{2}{3}$ with multiplicity 2

## Graphical

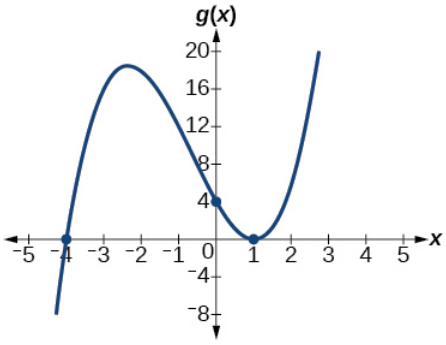
**For the exercises 42-47, graph the polynomial functions. Note $x$- and $y$- intercepts, multiplicity, and end behavior.**

42) $f(x) = (x + 3)^2(x - 2)$

43) $g(x) = (x + 4)(x - 1)^2$

**Answer**

$x$-intercepts, $(1, 0)$ with multiplicity 2, $(-4, 0)$ with multiplicity 1, $y$- intercept $(0, 4)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.

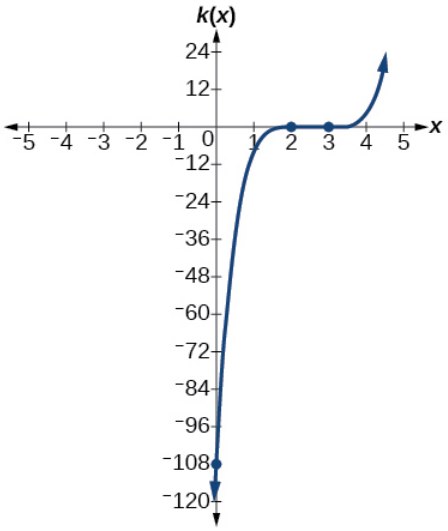



44) $h(x) = (x - 1)^3(x + 3)^2$

45) $k(x) = (x - 3)^3(x - 2)^2$

**Answer**

$x$-intercepts $(3, 0)$ with multiplicity 3, $(2, 0)$ with multiplicity 2, $y$- intercept $(0, -108)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.

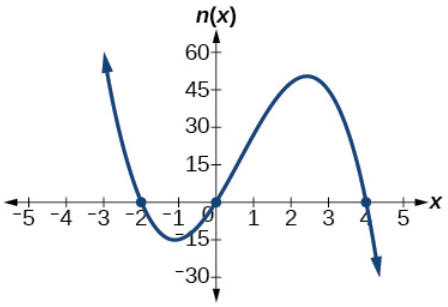



46) $m(x) = -2x(x-1)(x+3)$

47) $n(x) = -3x(x+2)(x-4)$

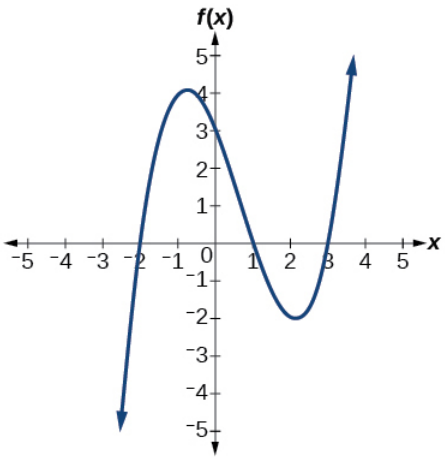
**Answer**

$x$-intercepts $(0,0), (-2,0), (4,0)$with multiplicity 1, $y$-intercept (0, 0). As $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to -\infty$.

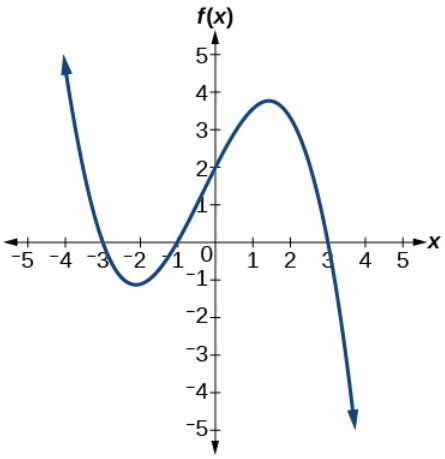



**For the exercises 48-52, use the graphs to write the formula for a polynomial function of least degree.**

48)



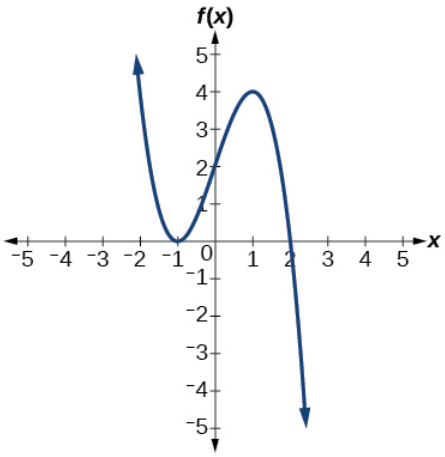
49)



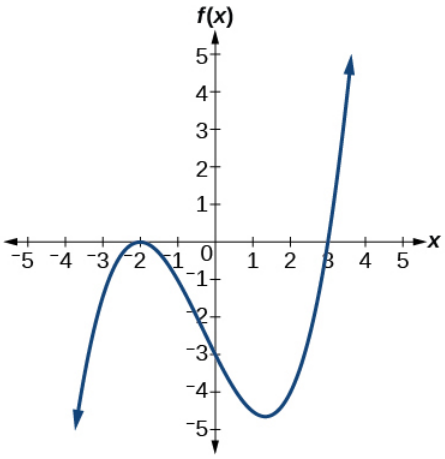

**Answer**

$f(x) = -\frac{2}{9}(x-3)(x+1)(x+3)$

50)



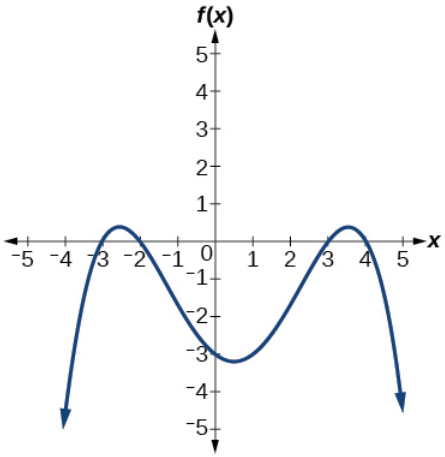

51)



**Answer**

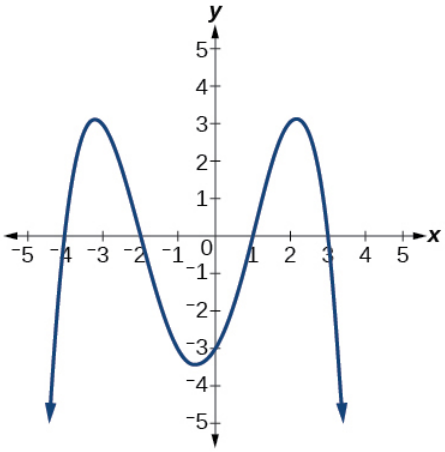
$$f(x) = \frac{1}{4}(x+2)^2(x-3)$$

52)




**For the exercises 53-56, use the graph to identify zeros and multiplicity.**

53)

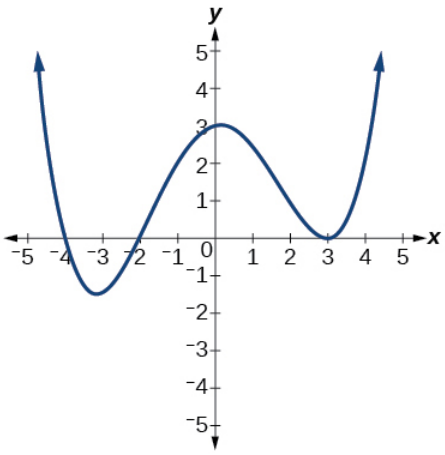



**Answer**

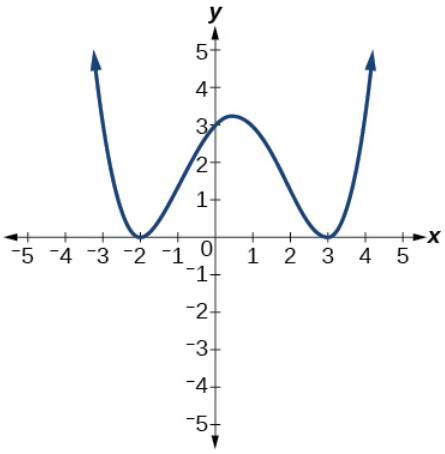
$-4, -2, 1, 3$with multiplicity 1

54)


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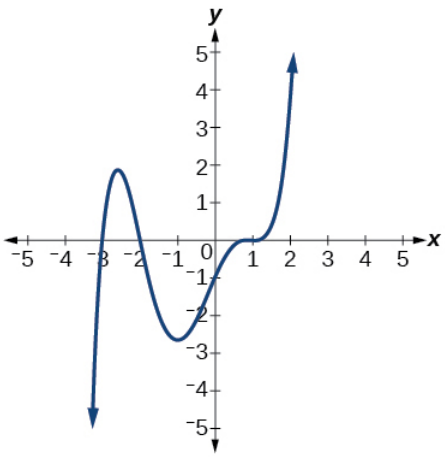

55)




**Answer**

$-2, 3$ each with multiplicity 2

56)




**For the exercises 57-66, use the given information about the polynomial graph to write the equation.**

57) Degree 3. Zeros at $x=-2, x=1$, and $x=3$. $y$-intercept at $(0,-4)$

**Answer**

$f(x)=-\frac{2}{3}(x+2)(x-1)(x-3)$

58) Degree 3. Zeros at $x=-5$, $x=-2$, and $x=1$. $y$-intercept at $(0,6)$

59) Degree 5. Roots of multiplicity 2 at $x=3$ and $x=1$, and a root of multiplicity 1 at $x=-3$. $y$-intercept at $(0,9)$

**Answer**

$f(x)=\frac{1}{3}(x-3)^2(x-1)^2(x+3)$

60) Degree 4. Root of multiplicity 2 at $x=4$, and a roots of multiplicity 1 at $x=1$ and $x=-2$. $y$-intercept at $(0,-3)$

61) Degree 5. Double zero at $x=1$, and triple zero at $x=3$. Passes through the point $(2,15)$

**Answer**

$f(x)=-15(x-1)^2(x-3)^3$

62) Degree 3. Zeros at $x=4$, $x=3$, and $x=2$. $y$-intercept at $(0,-24)$

63) Degree 3. Zeros at $x=-3$, $x=-2$ and $x=1$. $y$-intercept at $(0,12)$

**Answer**

$f(x)=-2(x+3)(x+2)(x-1)$

64) Degree 5. Roots of multiplicity 2 at $x=-3$ and $x=2$ and a root of multiplicity 1 at $x=-2$. $y$-intercept at $(0,4)$.

65) Degree 4. Roots of multiplicity 2 at $x=\frac{1}{2}$ and roots of multiplicity 1 at $x=6$ and $x=-2$. $y$-intercept at $(0,18)$

**Answer**

$f(x)=-\frac{3}{2}(2x-1)^2(x-6)(x+2)$

66) Double zero at $x=-3$ and triple zero at $x=0$. Passes through the point $(1,32)$.

### Technology

**For the exercises 67-71, use a calculator to approximate local minima and maxima or the global minimum and maximum.**

67) $f(x)=x^3-x-1$

**Answer**

local max $(-.58,-.62)$ local min $(.58,-1.38)$

68) $f(x) = 2x^3 - 3x - 1$

69) $f(x) = x^4 + x$

**Answer**

global min $(-.63, -.47)$

70) $f(x) = -x^4 + 3x - 2$

71) $f(x) = x^4 - x^3 + 1$

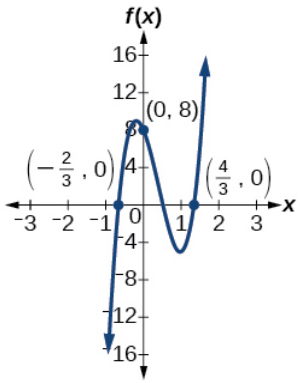
**Answer**

global min $(.75, .89)$

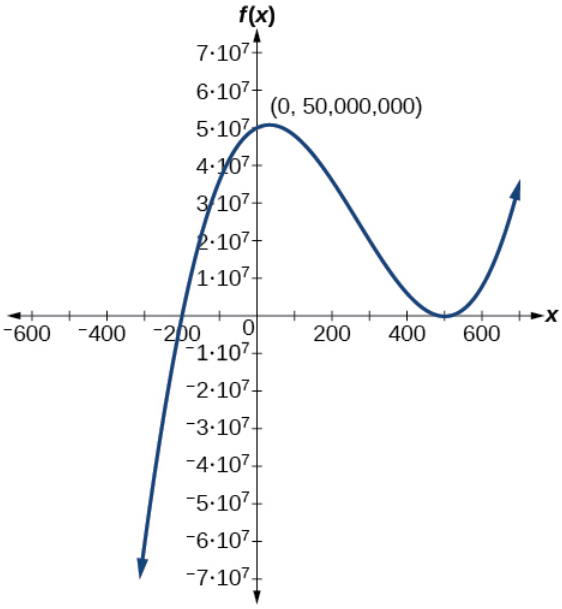
## Extensions

**For the exercises 72-74, use the graphs to write a polynomial function of least degree.**

72)




73)



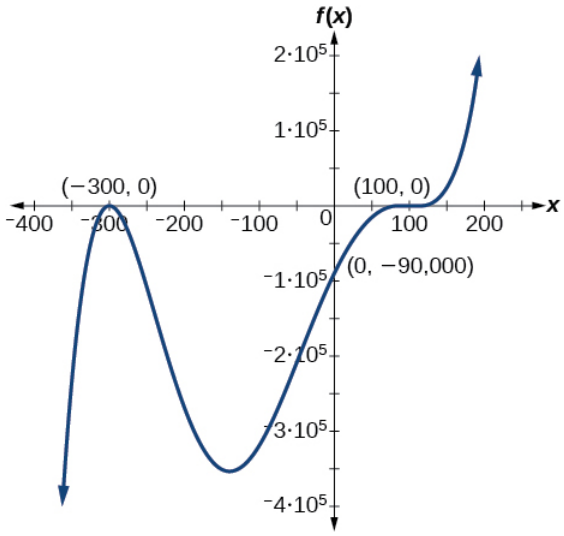


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**Answer**

$f(x) = (x - 500)^2(x + 200)$

74)




## Real-World Applications

**For the exercises 75-, write the polynomial function that models the given situation.**

75) A rectangle has a length of 10 units and a width of 8 units. Squares of $x$ by $x$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of $x$.

**Answer**

$f(x) = 4x^3 - 36x^2 + 80x$

76) Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of $x$.

77) A square has sides of 12 units. Squares $x + 1$ by $x + 1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of $x$.

**Answer**

$f(x) = 4x^3 - 36x^2 + 60x + 100$

78) A cylinder has a radius of $x + 2$ units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.

79) A right circular cone has a radius of $3x + 6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ for radius $r$ and height $h$.

**Answer**

$f(x) = \pi(9x^3 + 45x^2 + 72x + 36)$

## 3.5 Dividing Polynomials

### Verbal

1) If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?

**Answer**

The binomial is a factor of the polynomial.

2) If a polynomial of degree $n$ is divided by a binomial of degree 1, what is the degree of the quotient?

### Algebraic

**For the exercises 3-13, use long division to divide. Specify the quotient and the remainder.**

3) $(x^2+5x-1)\div(x-1)$

**Answer**

$\mathrm{x}+6+\frac{5}{\mathrm{x}-1}$, quotient : $\mathrm{x}+6$, remainder : $5$

4) $(2x^2-9x-5)\div(x-5)$

5) $(3x^2+23x+14)\div(x+7)$

**Answer**

$3\mathrm{x}+2$, quotient : $3\mathrm{x}+2$, remainder : $0$

6) $(4x^2-10x+6)\div(4x+2)$

7) $(6x^2-25x-25)\div(6x+5)$

**Answer**

$\mathrm{x}-5$, quotient : $\mathrm{x}-5$, remainder : $0$

8) $(-x^2-1)\div(x+1)$

9) $(2x^2-3x+2)\div(x+2)$

**Answer**

$2\mathrm{x}-7+\frac{16}{\mathrm{x}+2}$, quotient : $2\mathrm{x}-7$, remainder : $16$

10) $(x^3-126)\div(x-5)$

11) $(3x^2-5x+4)\div(3x+1)$

**Answer**

$\mathrm{x}-2+\frac{6}{3\mathrm{x}+1}$, quotient : $\mathrm{x}-2$, remainder : $6$

12) $(x^3-3x^2+5x-6)\div(x-2)$

13) $(2x^3 + 3x^2 - 4x + 15) \div (x + 3)$

**Answer**

$2x^2 - 3x + 5$, quotient : $2x^2 - 3x + 5$, remainder : $0$

**For the exercises 14-37, use synthetic division to find the quotient.**

14) $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

15) $(2x^3 - 6x^2 - 7x + 6) \div (x - 4)$

**Answer**

$2x^2 + 2x + 1 + \dfrac{10}{x - 4}$

16) $(6x^3 - 10x^2 - 7x - 15) \div (x + 1)$

17) $(4x^3 - 12x^2 - 5x - 1) \div (2x + 1)$

**Answer**

$2x^2 - 7x + 1 - \dfrac{2}{2x + 1}$

18) $(9x^3 - 9x^2 + 18x + 5) \div (3x - 1)$

19) $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

**Answer**

$3x^2 - 11x + 34 - \dfrac{106}{x + 3}$

20) $(-6x^3 + x^2 - 4) \div (2x - 3)$

21) $(2x^3 + 7x^2 - 13x - 3) \div (2x - 3)$

**Answer**

$x^2 + 5x + 1$

22) $(3x^3 - 5x^2 + 2x + 3) \div (x + 2)$

23) $(4x^3 - 5x^2 + 13) \div (x + 4)$

**Answer**

$4x^2 - 21x + 84 - \dfrac{323}{x + 4}$

24) $(x^3 - 3x + 2) \div (x + 2)$

25) $(x^3 - 21x^2 + 147x - 343) \div (x - 7)$

**Answer**

$x^2 - 14x + 49$

26) $(x^3 - 15x^2 + 75x - 125) \div (x - 5)$

27) $(9x^3 - x + 2) \div (3x - 1)$

**Answer**

$3x^2 + x + \dfrac{2}{3x-1}$

28) $(6x^3 - x^2 + 5x + 2) \div (3x + 1)$

29) $(x^4 + x^3 - 3x^2 - 2x + 1) \div (x + 1)$

**Answer**

$x^3 - 3x + 1$

30) $(x^4 - 3x^2 + 1) \div (x - 1)$

31) $(x^4 + 2x^3 - 3x^2 + 2x + 6) \div (x + 3)$

**Answer**

$x^3 - x^2 + 2$

32) $(x^4 - 10x^3 + 37x^2 - 60x + 36) \div (x - 2)$

33) $(x^4 - 8x^3 + 24x^2 - 32x + 16) \div (x - 2)$

**Answer**

$x^3 - 6x^2 + 12x - 8$

34) $(x^4 + 5x^3 - 3x^2 - 13x + 10) \div (x + 5)$

35) $(x^4 - 12x^3 + 54x^2 - 108x + 81) \div (x - 3)$

**Answer**

$x^3 - 9x^2 + 27x - 27$

36) $(4x^4 - 2x^3 - 4x + 2) \div (2x - 1)$

37) $(4x^4 + 2x^3 - 4x^2 + 2x + 2) \div (2x + 1)$

**Answer**

$2x^3 - 2x + 2$

**For the exercises 38-43, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.**

38) $x - 2, 4x^3 - 3x^2 - 8x + 4$

39) $x - 2, 3x^4 - 6x^3 - 5x + 10$

**Answer**

Yes $(x - 2)(3x^3 - 5)$

40) $x + 3, -4x^3 + 5x^2 + 8$

41) $x-2, 4x^4-15x^2-4$

**Answer**

Yes $(x-2)(4x^3+8x^2+x+2)$

42) $x-\frac{1}{2}, 2x^4-x^3+2x-1$

43) $x+\frac{1}{3}, 3x^4+x^3-3x+1$

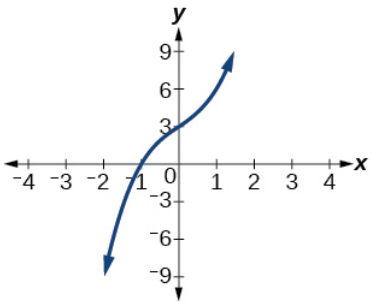
**Answer**

No

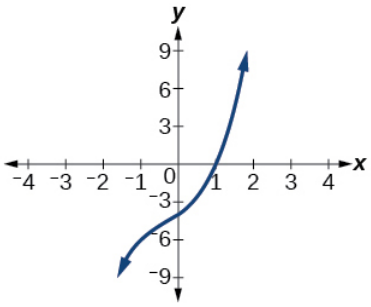
## Graphical

**For the exercises 44-48, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.**

44) Factor is $x^2-x+3$



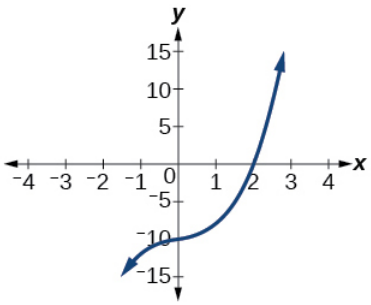

45) Factor is $(x^2+2x+4)$



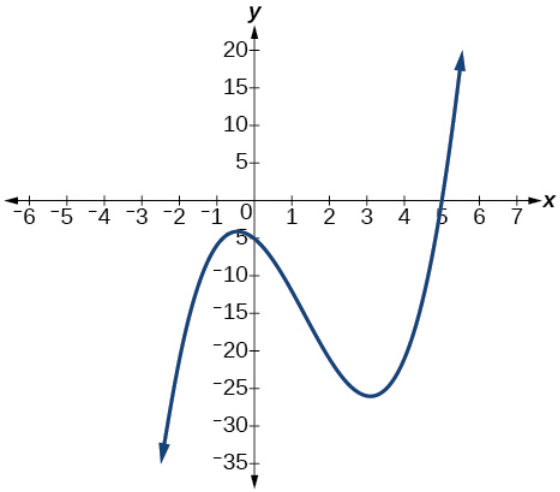

**Answer**

$(x-1)(x^2+2x+4)$

46) Factor is $x^2+2x+5$



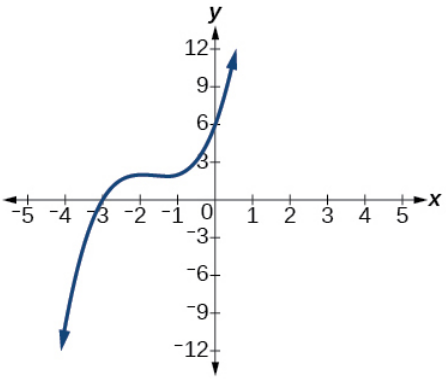

47) Factor is $x^2 + x + 1$




**Answer**

$(x-5)(x^2+x+1)$

48) Factor is $x^2 + 2x + 2$




**For the exercises 49-53, use synthetic division to find the quotient and remainder.**

49) $\dfrac{4x^3 - 33}{x - 2}$

**Answer**

Quotient : $4x^2 + 8x + 16$, remainder : $-1$


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50) $\dfrac{2x^3+25}{x+3}$

51) $\dfrac{3x^3+2x-5}{x-1}$

**Answer**

Quotient : $3x^2+3x+5$, remainder : $0$

52) $\dfrac{-4x^3-x^2-12}{x+4}$

53) $\dfrac{x^4-22}{x+2}$

**Answer**

Quotient : $x^3-2x^2+4x-8$, remainder : $-6$

### Technology

**For the exercises 54-58, use a calculator with CAS to answer the questions.**

54) Consider $\dfrac{x^k-1}{x-1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

55) Consider $\dfrac{x^k+1}{x+1}$ for $k=1,3,5$. What do you expect the result to be if $k=7$?

**Answer**

$x^6-x^5+x^4-x^3+x^2-x+1$

56) Consider $\dfrac{x^4-k^4}{x-k}$ for $k=1,2,3$. What do you expect the result to be if $k=4$?

57) Consider $\dfrac{x^k}{x+1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

**Answer**

$x^3-x^2+x-1+\dfrac{1}{x+1}$

58) Consider $\dfrac{x^k}{x-1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

### Extensions

**For the exercises 59-63, use synthetic division to determine the quotient involving a complex number.**

59) $\dfrac{x+1}{x-i}$

**Answer**

$1+\dfrac{1+i}{x-i}$

60) $\dfrac{x^2+1}{x-i}$

61) $\dfrac{x+1}{x+i}$

**Answer**

$1+\dfrac{1-i}{x+i}$

62) $\dfrac{x^2+1}{x+i}$

63) $\dfrac{x^3+1}{x-i}$

**Answer**

$x^2-ix-1+\dfrac{1-i}{x-i}$

## Real-World Applications

**For the exercises 64-66, use the given length and area of a rectangle to express the width algebraically.**

64) Length is $x+5$, area is $2x^2+9x-5$.

65) Length is $2x+5$, area is $4x^3+10x^2+6x+15$.

**Answer**

$2x^2+3$

66) Length is $3x-4$, area is $6x^4-8x^3+9x^2-9x-4$.

**For the exercises 67-70, use the given volume of a box and its length and width to express the height of the box algebraically.**

67) Volume is $12x^3+20x^2-21x-36$, length is $2x+3$, width is $3x-4$.

**Answer**

$2x+3$

68) Volume is $18x^3-21x^2-40x+48$, length is $3x-4$, width is $3x-4$.

69) Volume is $10x^3+27x^2+2x-24$, length is $5x-4$, width is $2x+3$.

**Answer**

$x+2$

70) Volume is $10x^3+30x^2-8x-24$, length is $2$, width is $x+3$.

**For the exercises 71-73, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.**

71) Volume is $\pi(25x^3 - 65x^2 - 29x - 3)$ , radius is $5x + 1$.

**Answer**

$x - 3$

72) Volume is $\pi(4x^3 + 12x^2 - 15x - 50)$ , radius is $2x + 5$.

73) Volume is $\pi(3x^4 + 24x^3 + 46x^2 - 16x - 32)$ , radius is $x + 4$.

**Answer**

$3x^2 - 2$

## 3.6 Zeros of Polynomial Functions

### Verbal

1) Describe a use for the Remainder Theorem.

**Answer**

The theorem can be used to evaluate a polynomial.

2) Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.

3) What is the difference between rational and real zeros?

**Answer**

Rational zeros can be expressed as fractions whereas real zeros include irrational numbers.

4) If Descartes' Rule of Signs reveals a no change of signs or one sign of changes, what specific conclusion can be drawn?

5) If synthetic division reveals a zero, why should we try that value again as a possible solution?

**Answer**

Polynomial functions can have repeated zeros, so the fact that number is a zero doesn't preclude it being a zero again.

### Algebraic

**For the exercises 6-13, use the Remainder Theorem to find the remainder.**

6) $(x^4 - 9x^2 + 14) \div (x - 2)$

7) $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

**Answer**

$-106$

8) $(x^4 + 5x^3 - 4x - 17) \div (x + 1)$

9) $(-3x^2+6x+24) \div (x-4)$

**Answer**

$0$

10) $(5x^5-4x^4+3x^3-2x^2+x-1) \div (x+6)$

11) $(x^4-1) \div (x-4)$

**Answer**

$255$

12) $(3x^3+4x^2-8x+2) \div (x-3)$

13) $(4x^3+5x^2-2x+7) \div (x+2)$

**Answer**

$-1$

**For the exercises 14-21, use the Factor Theorem to find all real zeros for the given polynomial function and one factor.**

14) $f(x)=2x^3-9x^2+13x-6; x-1$

15) $f(x)=2x^3+x^2-5x+2; x+2$

**Answer**

$-2, 1, \frac{1}{2}$

16) $f(x)=3x^3+x^2-20x+12; x+3$

17) $f(x)=2x^3+3x^2+x+6; x+2$

**Answer**

$-2$

18) $f(x)=-5x^3+16x^2-9; x-3$

19) $x^3+3x^2+4x+12; x+3$

**Answer**

$-3$

20) $4x^3-7x+3; x-1$

21) $2x^3+5x^2-12x-30, 2x+5$

**Answer**

$-\frac{5}{2}, \sqrt{6}, -\sqrt{6}$

**For the exercises 22-39, use the Rational Zero Theorem to find all real zeros.**

22) $x^3 - 3x^2 - 10x + 24 = 0$

23) $2x^3 + 7x^2 - 10x - 24 = 0$

**Answer**

$2, -4, -\frac{3}{2}$

24) $x^3 + 2x^2 - 9x - 18 = 0$

25) $x^3 + 5x^2 - 16x - 80 = 0$

**Answer**

$4, -4, -5$

26) $x^3 - 3x^2 - 25x + 75 = 0$

27) $2x^3 - 3x^2 - 32x - 15 = 0$

**Answer**

$5, -3, -\frac{1}{2}$

28) $2x^3 + x^2 - 7x - 6 = 0$

29) $2x^3 - 3x^2 - x + 1 = 0$

**Answer**

$\frac{1}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

30) $3x^3 - x^2 - 11x - 6 = 0$

31) $2x^3 - 5x^2 + 9x - 9 = 0$

**Answer**

$\frac{3}{2}$

32) $2x^3 - 3x^2 + 4x + 3 = 0$

33) $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$

**Answer**

$2, 3, -1, -2$

34) $x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$

35) $4x^4 + 4x^3 - 25x^2 - x + 6 = 0$

**Answer**

$\frac{1}{2}, -\frac{1}{2}, 2, -3$

36) $2x^4 - 3x^3 - 15x^2 + 32x - 12 = 0$

37) $x^4 + 2x^3 - 4x^2 - 10x - 5 = 0$

**Answer**

$-1, -1, \sqrt{5}, -\sqrt{5}$

38) $4x^3 - 3x + 1 = 0$

39) $8x^4 + 26x^3 + 39x^2 + 26x + 6$

**Answer**

$-\frac{3}{4}, -\frac{1}{2}$

**For the exercises 40-45, find all complex solutions (real and non-real).**

40) $x^3 + x^2 + x + 1 = 0$

41) $x^3 - 8x^2 + 25x - 26 = 0$

**Answer**

$2, 3 + 2i, 3 - 2i$

42) $x^3 + 13x^2 + 57x + 85 = 0$

43) $3x^3 - 4x^2 + 11x + 10 = 0$

**Answer**

$-\frac{2}{3}, 1 + 2i, 1 - 2i$

44) $x^4 + 2x^3 + 22x^2 + 50x - 75 = 0$

45) $2x^3 - 3x^2 + 32x + 17 = 0$

**Answer**

$-\frac{1}{2}, 1 + 4i, 1 - 4i$

## Graphical

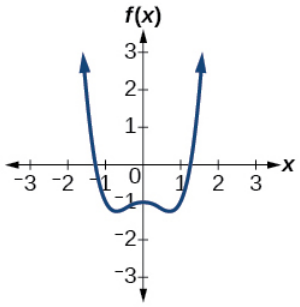
**For the exercises 46-55, use Descartes' Rule to determine the possible number of positive and negative solutions. Confirm with the given graph.**

46) $f(x) = x^3 - 1$

47) $f(x) = x^4 - x^2 - 1$

**Answer**

1 positive, 1 negative

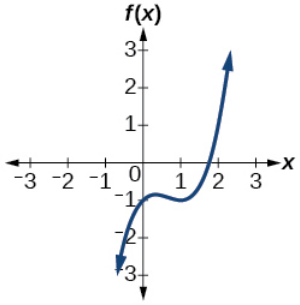



48) $f(x) = x^3 - 2x^2 - 5x + 6$

49) $f(x) = x^3 - 2x^2 + x - 1$

**Answer**

3 or 1 positive, 0 negative

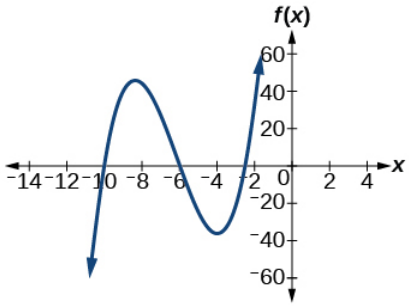



50) $f(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

51) $f(x) = 2x^3 + 37x^2 + 200x + 300$

**Answer**

0 positive, 3 or 1 negative



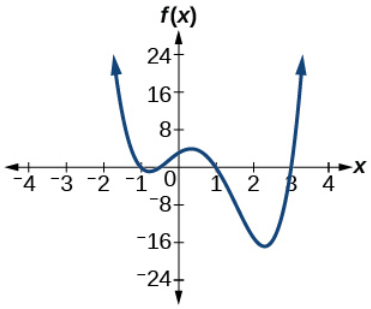

52) $f(x) = x^3 - 2x^2 - 16x + 32$

53) $f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$

**Answer**

2 or 0 positive, 2 or 0 negative


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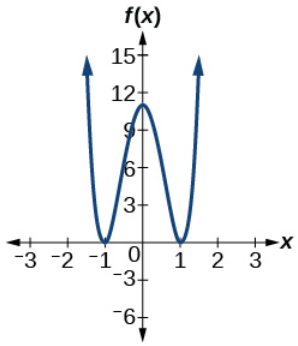



54) $f(x) = 2x^4 - 5x^3 - 14x^2 + 20x + 8$

55) $f(x) = 10x^4 - 21x^2 + 11$

**Answer**

2 or 0 positive, 2 or 0 negative




Numeric

**For the exercises 56-60, list all possible rational zeros for the functions.**

56) $f(x) = x^4 + 3x^3 - 4x + 4$

57) $f(x) = 2x^3 + 3x^2 - 8x + 5$

**Answer**

$\pm 5, \pm 1, \pm \frac{5}{2}$

58) $f(x) = 3x^3 + 5x^2 - 5x + 4$

59) $f(x) = 6x^4 - 10x^2 + 13x + 1$

**Answer**

$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}$

60) $f(x) = 4x^5 - 10x^4 + 8x^3 + x^2 - 8$


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### Technology

**For the exercises 61-65, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.**

61) $f(x) = 6x^3 - 7x^2 + 1$

**Answer**

$1, \frac{1}{2}, -\frac{1}{3}$

62) $f(x) = 4x^3 - 4x^2 - 13x - 5$

63) $f(x) = 8x^3 - 6x^2 - 23x + 6$

**Answer**

$2, \frac{1}{4}, -\frac{3}{2}$

64) $f(x) = 12x^4 + 55x^3 + 12x^2 - 117x + 54$

65) $f(x) = 16x^4 - 24x^3 + x^2 - 15x + 25$

**Answer**

$\frac{5}{4}$

### Extensions

**For the exercises 66-70, construct a polynomial function of least degree possible using the given information.**

66) Real roots: $-1, 1, 3$ and $(2, f(2)) = (2, 4)$

67) Real roots: $-1$ (with multiplicity 2 and 1) and $(2, f(2)) = (2, 4)$

**Answer**

$f(x) = \frac{4}{9}(x^3 + x^2 - x - 1)$

68) Real roots: $-2, \frac{1}{2}$ (with multiplicity 2) and $(-3, f(-3)) = (-3, 5)$

69) Real roots: $-\frac{1}{2}, 0, \frac{1}{2}$ and $(-2, f(-2)) = (-2, 6)$

**Answer**

$f(x) = -\frac{1}{5}(4x^3 - x)$

70) Real roots: $-4, -1, 1, 4$ and $(-2, f(-2)) = (-2, 10)$

### Real-World Applications

**For the exercises 71-75, find the dimensions of the box described.**

71) The length is twice as long as the width. The height is 2 inches greater than the width. The volume is 192 cubic inches.

**Answer**

8 by 4 by 6 inches

72) The length, width, and height are consecutive whole numbers. The volume is 120cubic inches.

73) The length is one inch more than the width, which is one inch more than the height. The volume is 86.625 cubic inches.

**Answer**

5.5 by 4.5 by 3.5 inches

74) The length is three times the height and the height is one inch less than the width. The volume is 108 cubic inches.

75) The length is 3 inches more than the width. The width is 2 inches more than the height. The volume is 120 cubic inches.

**Answer**

8 by 5 by 3 inches

**For the exercises 76-80, find the dimensions of the right circular cylinder described.**

76) The radius is 3 inches more than the height. The volume is $16\pi$ cubic meters.

77) The height is one less than one half the radius. The volume is $72\pi$ cubic meters.

**Answer**

Radius = 6 meters, Height = 2 meters

78) The radius and height differ by one meter. The radius is larger and the volume is $48\pi$ cubic meters.

79) The radius and height differ by two meters. The height is greater and the volume is $28.125\pi$ cubic meters.

**Answer**

Radius = 2.5 meters, Height = 4.5 meters

80) The radius is $\frac{1}{3}$ meter greater than the height. The volume is $\frac{98}{9}\pi$ cubic meters.

## 3.7 Rational Functions

### Verbal

1) What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?

**Answer**

The rational function will be represented by a quotient of polynomial functions.

2) What is the fundamental difference in the graphs of polynomial functions and rational functions?

3) If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?

**Answer**

The numerator and denominator must have a common factor.

4) Can a graph of a rational function have no vertical asymptote? If so, how?

5) Can a graph of a rational function have no x-intercepts? If so, how?

**Answer**

Yes. The numerator of the formula of the functions would have only complex roots and/or factors common to both the numerator and denominator.

## Algebraic

**For the exercises 6-9, find the domain of the rational functions.**

6) $f(x) = \dfrac{x-1}{x+2}$

7) $f(x) = \dfrac{x+1}{x^2-1}$

**Answer**

All reals $x \neq -1, 1$

8) $f(x) = \dfrac{x^2+4}{x^2-2x-8}$

9) $f(x) = \dfrac{x^2+4x-3}{x^4-5x^2+4}$

**Answer**

All reals $x \neq -1, -2, 1, 2$

**For the exercises 10-19, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.**

10) $f(x) = \dfrac{4}{x-1}$

11) $f(x) = \dfrac{2}{5x+2}$

**Answer**

V.A. at $x = -\dfrac{2}{5}$; H.A. at $y = 0$; Domain is all reals $x \neq -\dfrac{2}{5}$

12) $f(x) = \dfrac{x}{x^2-9}$

13) $f(x) = \dfrac{x}{x^2+5x-36}$

**Answer**

V.A. at $x = 4, -9$; H.A. at $y = 0$; Domain is all reals $x \neq 4, -9$

14) $f(x)=\dfrac{3+x}{x^3-27}$

15) $f(x)=\dfrac{3x-4}{x^3-16x}$

**Answer**

V.A. at $x=0,4,-4$; H.A. at $y=0$; Domain is all reals $x\neq 0,4,-4$

16) $f(x)=\dfrac{x^2-1}{x^3+9x^2+14x}$

17) $f(x)=\dfrac{x+5}{x^2-25}$

**Answer**

V.A. at $x=-5$; H.A. at $y=0$; Domain is all reals $x\neq 5,-5$

18) $f(x)=\dfrac{x-4}{x-6}$

19) $f(x)=\dfrac{4-2x}{3x-1}$

**Answer**

V.A. at $x=\dfrac{1}{3}$; H.A. at $y=-\dfrac{2}{3}$; Domain is all reals $x\neq\dfrac{1}{3}$

**For the exercises 20-24, find the $x$- and $y$-intercepts for the functions.**

20) $f(x)=\dfrac{x+5}{x^2+4}$

21) $f(x)=\dfrac{x}{x^2-x}$

**Answer**

none

22) $f(x)=\dfrac{x^2+8x+7}{x^2+11x+30}$

23) $f(x)=\dfrac{x^2+x+6}{x^2-10x+24}$

**Answer**

$x$ -intercepts none, $y$ -intercept $\left(0,\dfrac{1}{4}\right)$

24) $f(x)=\dfrac{94-2x^2}{3x^2-12}$

**For the exercises 25-29, describe the local and end behavior of the functions.**

25) $f(x)=\dfrac{x}{2x+1}$

**Answer**

Local behavior: $x\to-\dfrac{1}{2}^{+}, f(x)\to-\infty, x\to-\dfrac{1}{2}^{-}, f(x)\to\infty$

End behavior: $x\to\pm\infty, f(x)\to\dfrac{1}{2}$

26) $f(x)=\dfrac{2x}{x-6}$

27) $f(x)=\dfrac{-2x}{x-6}$

**Answer**

Local behavior: $x\to 6^{+}, f(x)\to -\infty, x\to 6^{-}, f(x)\to \infty$

End behavior: $x\to \pm\infty, f(x)\to -2$

28) $f(x)=\dfrac{x^2-4x+3}{x^2-4x-5}$

29) $f(x)=\dfrac{2x^2-32}{6x^2+13x-5}$

**Answer**

Local behavior: $x\to -\dfrac{1}{3}, f(x)\to \infty, x\to -\dfrac{1}{3}, f(x)\to -\infty, x\to \dfrac{5}{2}, f(x)\to \infty, x\to \dfrac{5}{2}+f(x)\to -\infty$

End behavior: $x\to \pm\infty, f(x)\to \dfrac{1}{3}$

**For the exercises 30-34, find the slant asymptote of the functions.**

30) $f(x)=\dfrac{24x^2+6x}{2x+1}$

31) $f(x)=\dfrac{4x^2-10}{2x-4}$

**Answer**

$y=2x+4$

32) $f(x)=\dfrac{81x^2-18}{3x-2}$

33) $f(x)=\dfrac{6x^3-5x}{3x^2+4}$

**Answer**

$y=2x$

34) $f(x)=\dfrac{x^2+5x+4}{x-1}$

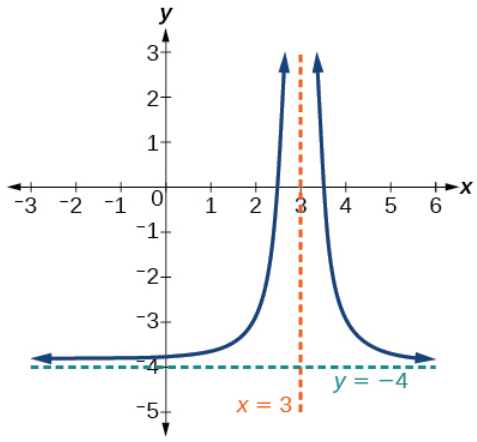
## Graphical

**For the exercises 35-38, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.**

35) The reciprocal function shifted up two units.

**Answer**

V.A. $x=0$, H.A. $y=2$

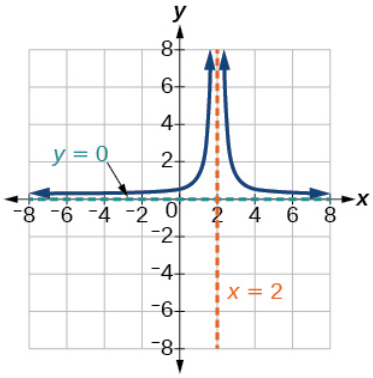



36) The reciprocal function shifted down one unit and left three units.

37) The reciprocal squared function shifted to the right 2 units.

**Answer**

V.A. $x = 2$, H.A. $y = 0$




38) The reciprocal squared function shifted down 2 units and right 1 unit.

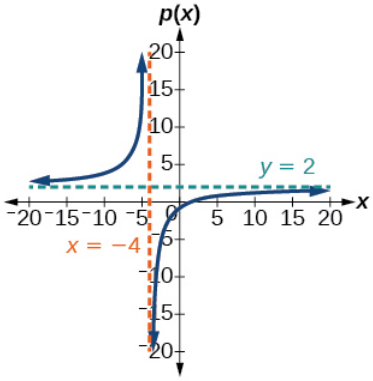
**For the exercises 39-50, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.**

39) $p(x) = \dfrac{2x-3}{x+4}$

**Answer**

V.A. $x = -4$, H.A. $y = 2$; $\left(\dfrac{3}{2}, 0\right)$; $\left(0, -\dfrac{3}{4}\right)$


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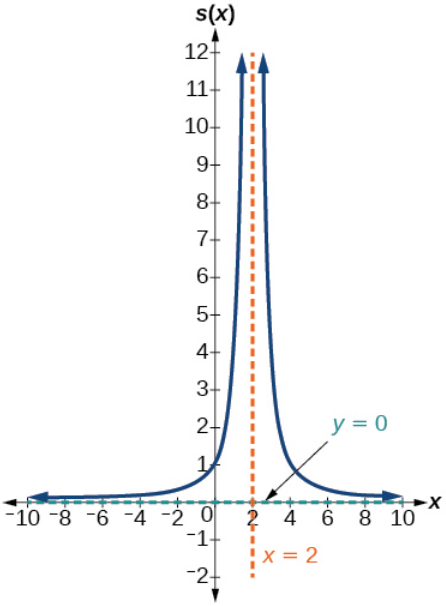



40) $q(x) = \dfrac{x-5}{3x-1}$

41) $s(x) = \dfrac{4}{(x-2)^2}$

**Answer**

V.A. $x = 2$, H.A. $y = 0$, $(0, 1)$



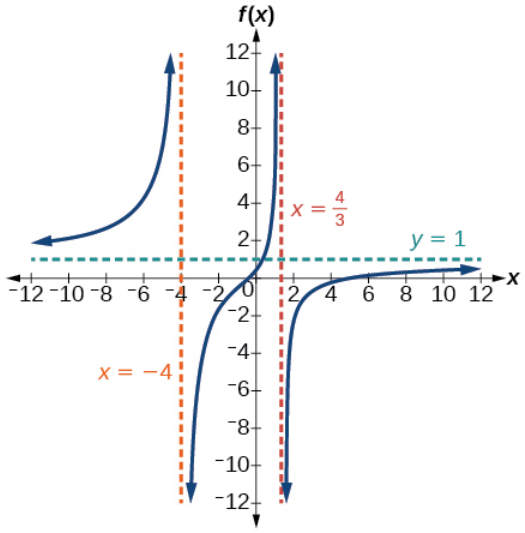

42) $r(x) = \dfrac{5}{(x+1)^2}$

43) $f(x) = \dfrac{3x^2-14x-5}{3x^2+8x-16}$

**Answer**

V.A. $x = -4$, $x = \dfrac{4}{3}$, H.A. $y = 1$; $(5, 0)$; $\left(-\dfrac{1}{3}, 0\right)$; $\left(0, \dfrac{5}{16}\right)$


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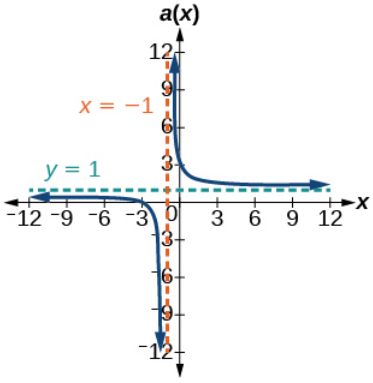



44) $g(x) = \dfrac{2x^2+7x-15}{3x^2-14+15}$

45) $a(x) = \dfrac{x^2+2x-3}{x^2-1}$

**Answer**

V.A. $x=-1$, H.A. $y=1$; $(-3,0)$; $(0,3)$



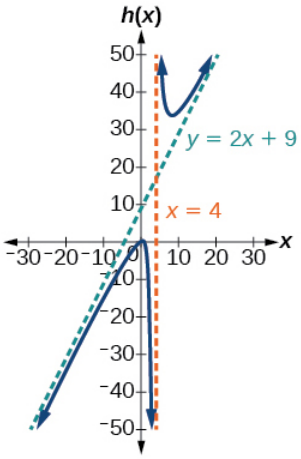

46) $b(x) = \dfrac{x^2-x-6}{x^2-4}$

47) $h(x) = \dfrac{2x^2+x-1}{x-4}$

**Answer**

V.A. $x=4$, S.A. $y=2x+9$; $(-1,0)$; $\left(\dfrac{1}{2},0\right)$; $\left(0,\dfrac{1}{4}\right)$


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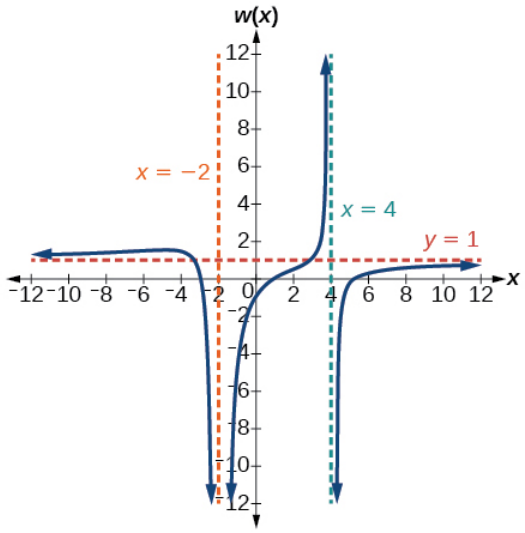



48) $k(x) = \dfrac{2x^2 - 3x - 20}{x - 5}$

49) $w(x) = \dfrac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

**Answer**

V.A. $x = -2$, $x = 4$, H.A. $y = 1$, $(1, 0)$; $(5, 0)$; $(-3, 0)$; $\left(0, -\dfrac{15}{16}\right)$




50) $z(x) = \dfrac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

**For the exercises 51-56, write an equation for a rational function with the given characteristics.**

51) Vertical asymptotes at $x = 5$, $x$ -intercepts at $(2, 0)$ and $(-1, 0)$, $y$-intercept at $(0, 4)$

**Answer**

$y = 50\dfrac{x^2 - x - 2}{x^2 - 25}$

52) Vertical asymptotes at $x = -4$ and $x = -1$, $x$ -intercepts at $(1, 0)$ and $(5, 0)$, $y$-intercept at $(0, 7)$

53) Vertical asymptotes at $x = -4$ and $x = -5$, $x$ -intercepts at $(4, 0)$ and $(-6, 0)$, Horizontal asymptote at $y = 7$


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**Answer**

$$y = 7\frac{x^2+2x-24}{x^2+9x+20}$$

54) Vertical asymptotes at $x=-3$ and $x=6$, $x$ -intercepts at $(-2,0)$ and $(1,0)$, Horizontal asymptote at $y=-2$

55) Vertical asymptote at $x=-1$, Double zero at $x=2$, $y$ -intercept at $(0,2)$

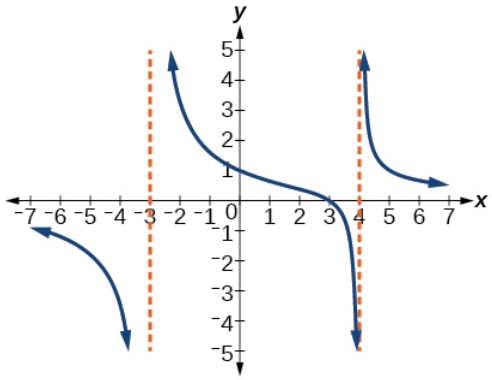
**Answer**

$$y = \frac{1}{2}\frac{x^2-4x+4}{x+1}$$

56) Vertical asymptote at $x=3$, Double zero at $x=1$, $y$ -intercept at $(0,4)$

**For the exercises 57-,64 use the graphs to write an equation for the function.**

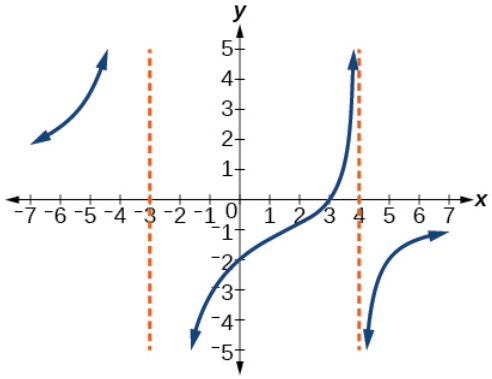
57)




**Answer**

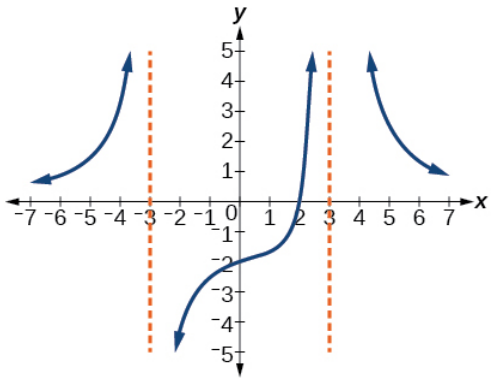
$$y = 4\frac{x-3}{x^2-x-12}$$

58)




59)

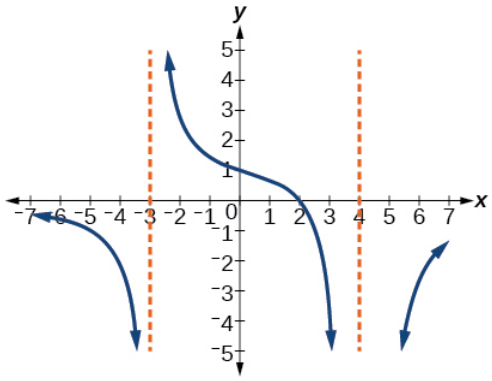

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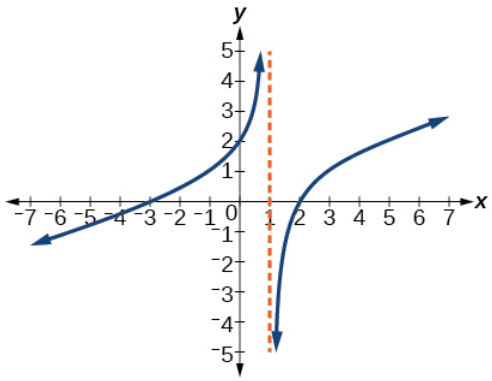

**Answer**

$$y = -9\frac{x-2}{x^2-9}$$

60)




61)

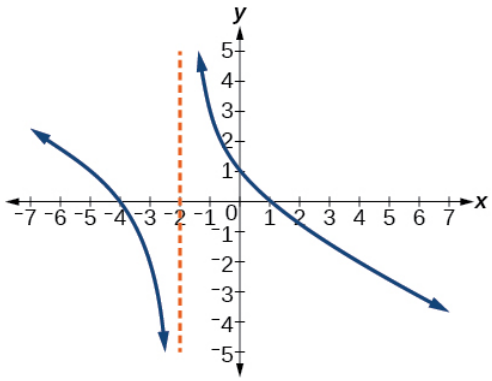



**Answer**

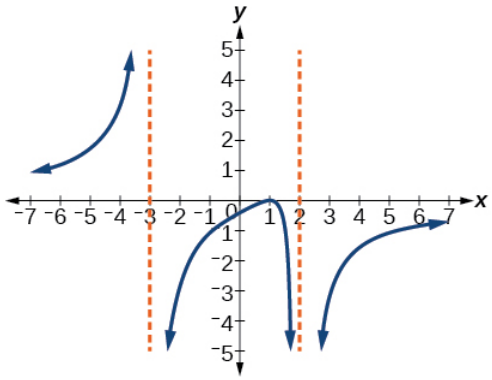
$$y = \frac{1}{3}\frac{x^2+x-6}{x-1}$$

62)


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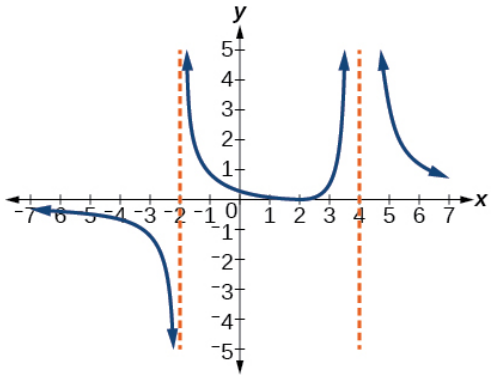

63)




**Answer**

$$y = -6\frac{(x-1)^2}{(x+3)(x-2)^2}$$

64)




## Numeric

**For the exercises 65-69, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote**

65) $f(x) = \dfrac{1}{x-2}$

**Answer**

| $x$ | 2.01 | 2.001 | 2.0001 | 1.99 | 1.999 |
|---|---|---|---|---|---|
| $y$ | 100 | 1,000 | 10,000 | –100 | –1,000 |

| $x$ | 10 | 100 | 1,000 | 10,000 | 100,000 |
|---|---|---|---|---|---|
| $y$ | .125 | .0102 | .001 | .0001 | .00001 |

Vertical asymptote $x = 2$, Horizontal asymptote $y = 0$

66) $f(x) = \dfrac{x}{x-3}$

67) $f(x) = \dfrac{2x}{x+4}$

**Answer**

| $x$ | –4.1 | –4.01 | –4.001 | –3.99 | –3.999 |
|---|---|---|---|---|---|
| $y$ | 82 | 802 | 8,002 | –798 | –7998 |

| $x$ | 10 | 100 | 1,000 | 10,000 | 100,000 |
|---|---|---|---|---|---|
| $y$ | 1.4286 | 1.9331 | 1.992 | 1.9992 | 1.999992 |

Vertical asymptote $x = -4$, Horizontal asymptote $y = 2$

68) $f(x) = \dfrac{2x}{(x-3)^2}$

69) $f(x) = \dfrac{x^2}{x^2+2x+1}$

**Answer**

| $x$ | –.9 | –.99 | –.999 | –1.1 | –1.01 |
|---|---|---|---|---|---|
| $y$ | 81 | 9,801 | 998,001 | 121 | 10,201 |

| $x$ | 10 | 100 | 1,000 | 10,000 | 100,000 |
|---|---|---|---|---|---|
| $y$ | .82645 | .9803 | .998 | .9998 | |

Vertical asymptote $x = -1$, Horizontal asymptote $y = 1$

## Technology

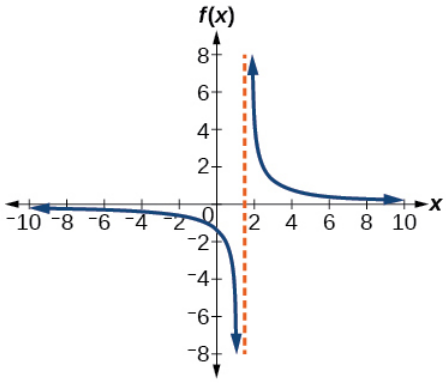
**For the exercises 70-74, use a calculator to graph $f(x)$. Use the graph to solve $f(x) > 0$.**

70) $f(x) = \dfrac{2}{x+1}$

71) $f(x) = \dfrac{4}{2x-3}$

**Answer**

$\left(\dfrac{3}{2}, \infty\right)$

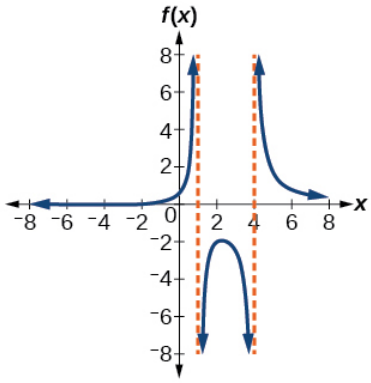



72) $f(x) = \dfrac{2}{(x-1)(x+2)}$

73) $f(x) = \dfrac{x+2}{(x-1)(x-4)}$

**Answer**

$(-2, 1) \cup (4, \infty)$




74) $f(x) = \dfrac{(x+3)^2}{(x-1)^2(x+1)}$

## Extensions

**For the exercises 75-79, identify the removable discontinuity.**

75) $f(x) = \dfrac{x^2-4}{x-2}$

**Answer**

$(2, 4)$

76) $f(x) = \dfrac{x^3+1}{x+1}$

77) $f(x) = \dfrac{x^2+x-6}{x-2}$

**Answer**

$(2, 5)$


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78) $f(x)=\dfrac{2x^2+5x-3}{x+3}$

79) $f(x)=\dfrac{x^3+x^2}{x+1}$

**Answer**

$(-1,1)$

## Real-World Applications

**For the exercises 80-81, express a rational function that describes the situation.**

80) A large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after $t$ minutes.

81) A large mixing tank currently contains 300 gallons of water, into which 8 pounds of sugar have been mixed. A tap will open, pouring 20 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after $t$ minutes.

**Answer**

$C(t)=\dfrac{8+2t}{300+20t}$

**For the exercises 82-83, use the given rational function to answer the question.**

82) The concentration $C$ of a drug in a patient's bloodstream $t$ hours after injection in given by $C(t)=\dfrac{2t}{3+t^2}$. What happens to the concentration of the drug as $t$ increases?

83) The concentration $C$ of a drug in a patients bloodstream, hours after injection is given by $C(t)=\dfrac{100t}{2t^2+75}$. Use a calculator to approximate the time when the concentration is highest.

**Answer**

After about $6.12$hours.

**For the exercises 84-88, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.**

84) An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let $x$ = length of the side of the base.

85) A rectangular box with a square base is to have a volume of 20 cubic feet. The material for the base costs 30 cents/ square foot. The material for the sides costs 10 cents/square foot. The material for the top costs 20 cents/square foot. Determine the dimensions that will yield minimum cost. Let $x$ = length of the side of the base.

**Answer**

$A(x)=50x^2+\dfrac{800}{x}\cdot 2$ by 2 by 5 feet.

86) A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let $x$ = radius.

87) A right circular cylinder with no top has a volume of 50 cubic meters. Find the radius that will yield minimum surface area. Let $x$ = radius.

**Answer**

$A(x)=\pi x^2+\dfrac{100}{x}$ . Radius = $2.52$meters.

88) A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents/square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let $x$ = radius.

## 3.8 Inverses and Radical Functions

### Verbal

1) Explain why we cannot find inverse functions for all polynomial functions.

**Answer**

It can be too difficult or impossible to solve for in terms of $y$.

2) Why must we restrict the domain of a quadratic function when finding its inverse?

3) When finding the inverse of a radical function, what restriction will we need to make?

**Answer**

We will need a restriction on the domain of the answer.

4) The inverse of a quadratic function will always take what form?

### Algebraic

**For the exercises 5-12, find the inverse of the function on the given domain.**

5) $f(x)=(x-4)^2, [4,\infty)$

**Answer**

$f^{-1}(x)=\sqrt{x}+4$

6) $f(x)=(x+2)^2, [-2,\infty)$

7) $f(x)=(x+1)^2-3, [-1,\infty)$

**Answer**

$f^{-1}(x)=\sqrt{x+3}-1$

8) $f(x)=2-\sqrt{3+x}$

9) $f(x)=3x^2+5, (-\infty,0]$

**Answer**

$f^{-1}(x)=-\sqrt{\dfrac{x-5}{3}}$

10) $f(x)=12-x^2, [0,\infty)$

11) $f(x)=9-x^2, [0,\infty)$

**Answer**

$f(x)=\sqrt{9-x}$

12) $f(x)=2x^2+4, [0,\infty)$

**For the exercises 13-16, find the inverse of the functions.**

13) $f(x)=x^3+5$

**Answer**

$f^{-1}(x)=\sqrt[3]{x-5}$

14) $f(x) = 3x^3 + 1$

15) $f(x) = 4 - x^3$

**Answer**

$f^{-1}(x) = \sqrt[3]{4 - x}$

16) $f(x) = 4 - 2x^3$

**For the exercises 17-31, find the inverse of the functions.**

17) $f(x) = \sqrt{2x + 1}$

**Answer**

$f^{-1}(x) = \dfrac{x^2 - 1}{2}, [0, \infty)$

18) $f(x) = \sqrt{3 - 4x}$

19) $f(x) = 9 + \sqrt{4x - 4}$

**Answer**

$f^{-1}(x) = \dfrac{(x - 9)^2 + 4}{4}, [9, \infty)$

20) $f(x) = \sqrt{6x - 8} + 5$

21) $f(x) = 9 + 2\sqrt[3]{x}$

**Answer**

$f^{-1}(x) = \left(\dfrac{x - 9}{2}\right)^3$

22) $f(x) = 3 - \sqrt[3]{x}$

23) $f(x) = \dfrac{2}{x + 8}$

**Answer**

$f^{-1}(x) = \dfrac{2 - 8x}{x}$

24) $f(x) = \dfrac{3}{x - 4}$

25) $f(x) = \dfrac{x + 3}{x + 7}$

**Answer**

$f^{-1}(x) = \dfrac{7x - 3}{1 - x}$

26) $f(x) = \dfrac{x - 2}{x + 7}$

27) $f(x) = \dfrac{3x + 4}{5}$

**Answer**

$f^{-1}(x) = \dfrac{5x - 4}{4x + 3}$

28) $f(x)=\dfrac{5x+1}{2-5x}$

29) $f(x)=x^2+2x,[-1,\infty)$

**Answer**

$f^{-1}(x)=\sqrt{x+1}-1$

30) $f(x)=x^2+4x+1,[-2,\infty)$

31) $f(x)=x^2-6x+3,[3,\infty)$

**Answer**

$f^{-1}(x)=\sqrt{x+6}+3$

## Graphical

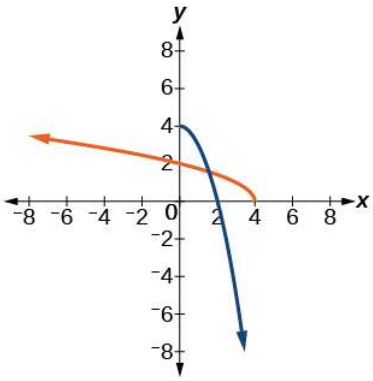
**For the exercises 32-41, find the inverse of the function and graph both the function and its inverse.**

32) $f(x)=x^2+2, x\geq 0$

33) $f(x)=4-x^2, x\geq 0$

**Answer**

$f^{-1}(x)=\sqrt{4-x}$

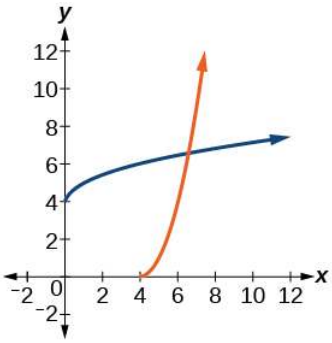



34) $f(x)=(x+3)^2, x\geq -3$

35) $f(x)=(x-4)^2, x\geq 4$

**Answer**

$f^{-1}(x)=\sqrt{x}+4$



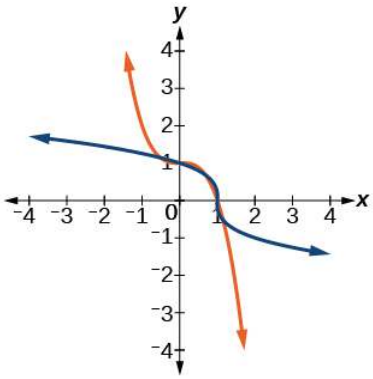

36) $f(x)=x^3+3$

37) $f(x)=1-x^3$


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**Answer**

$f^{-1}(x) = \sqrt[3]{1-x}$

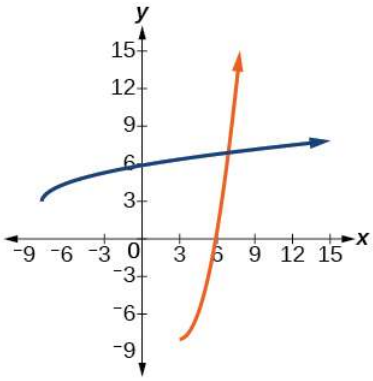



38) $f(x) = x^2 + 4x, x \geq -2$

39) $f(x) = x^2 - 6x + 1, x \geq 3$

**Answer**

$f^{-1}(x) = \sqrt{x+8} + 3$

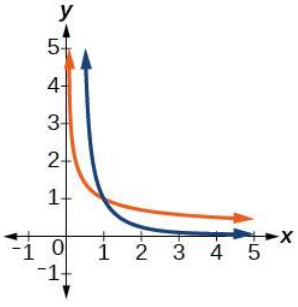



40) $f(x) = \dfrac{2}{x}$

41) $f(x) = \dfrac{1}{x^2}, x \geq 0$

**Answer**

$f^{-1}(x) = \sqrt{\dfrac{1}{x}}$



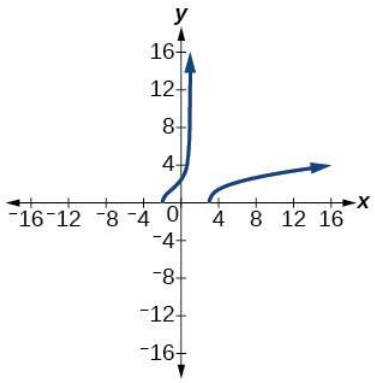

**For the exercises 42-46, use a graph to help determine the domain of the functions.**

42) $f(x)=\sqrt{\dfrac{(x+1)(x-1)}{x}}$

43) $f(x)=\sqrt{\dfrac{(x+2)(x-3)}{x-1}}$

**Answer**

$[-2,1)\cup[3,\infty)$

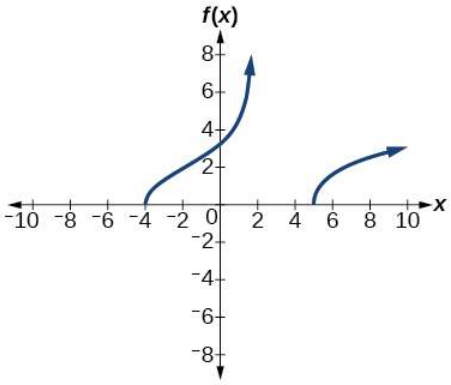



44) $f(x)=\sqrt{\dfrac{x(x+3)}{x-4}}$

45) $f(x)=\sqrt{\dfrac{x^2-x-20}{x-2}}$

**Answer**

$[-4,2)\cup[5,\infty)$




46) $f(x)=\sqrt{\dfrac{9x^2}{x+4}}$

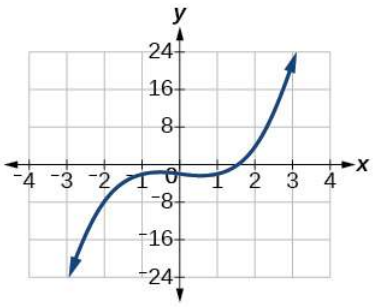
## Technology

**For the exercises 47-51, use a calculator to graph the function. Then, using the graph, give three points on the graph of the inverse with $y$-coordinates given.**

47) $f(x)=x^3-x-2, y=1,2,3$

**Answer**

$(-2,0);(4,2);(22,3)$

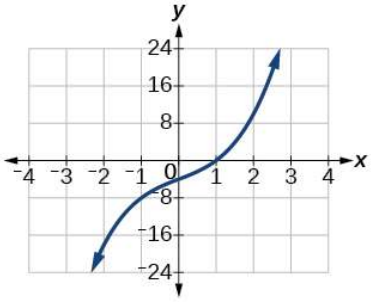



48) $f(x) = x^3 + x - 2, y = 0, 1, 2$

49) $f(x) = x^3 + 3x - 4, y = 0, 1, 2$

**Answer**

$(-4, 0); (0, 1); (10, 2)$



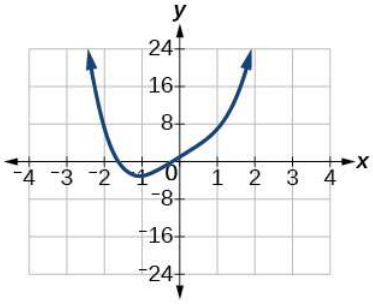

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50) $f(x) = x^3 + 8x - 4, y = -1, 0, 1$

51) $f(x) = x^4 + 5x + 1, y = -1, 0, 1$

**Answer**

$(-3, -1); (1, 0); (7, 1)$




## Extensions

**For the exercises 52-56, find the inverse of the functions with $a, b, c$ positive real numbers.**

52) $f(x) = ax^3 + b$

53) $f(x) = x^2 + bx$

**Answer**

$$f^{-1}(x) = \sqrt{x + \frac{b^2}{4}} - \frac{b}{2}$$

54) $f(x) = \sqrt{ax^2 + b}$

55) $f(x)=\sqrt[3]{ax+b}$

**Answer**

$$f^{-1}(x)=\frac{x^2-b}{a}$$

56) $f(x)=\dfrac{ax+b}{x+c}$

## Real-World Applications

**For the exercises 57-66, determine the function described and then use it to answer the question.**

57) An object dropped from a height of 200 meters has a height, $h(t)$, in meters after $t$ seconds have lapsed, such that $h(t)=200-4.9t^2$. Express $t$ as a function of height, $h$, and find the time to reach a height of 50 meters.

**Answer**

$$t(h)=\sqrt{\frac{200-h}{4.9}},\ 5.53\text{seconds}$$

58) An object dropped from a height of 600 feet has a height, $h(t)$, in feet after $t$ seconds have elapsed, such that $h(t)=600-16t^2$. Express $t$ as a function of height $h$, and find the time to reach a height of 400 feet.

59) The volume, $V$, of a sphere in terms of its radius, $r$, is given by $V(r)=\frac{4}{3}\pi r^3$. Express $r$ as a function of $V$, and find the radius of a sphere with volume of 200 cubic feet.

**Answer**

$$r(V)=\sqrt[3]{\frac{3V}{4\pi}},\ 3.63\text{feet}$$

60) The surface area, $A$, of a sphere in terms of its radius, $r$, is given by $A(r)=4\pi r^2$. Express $r$ as a function of $V$, and find the radius of a sphere with a surface area of 1000 square inches.

61) A container holds 100 ml of a solution that is 25 ml acid. If $n$ ml of a solution that is $60\%$ acid is added, the function $C(n)=\dfrac{25+.6n}{100+n}$ gives the concentration, $C$, as a function of the number of ml added, $n$. Express $n$ as a function of $C$ and determine the number of mL that need to be added to have a solution that is $50\%$ acid.

**Answer**

$$n(C)=\frac{100C-25}{6-C},250\text{mL}$$

62) The period $T$, in seconds, of a simple pendulum as a function of its length $l$, in feet, is given by $T(l)=2\pi\sqrt{\dfrac{l}{322}}$. Express $l$ as a function of $T$ and determine the length of a pendulum with period of 2 seconds.

63) The volume of a cylinder, $V$, in terms of radius, $r$, and height, $h$, is given by $V=\pi r^2 h$. If a cylinder has a height of 6 meters, express the radius as a function of $V$ and find the radius of a cylinder with volume of 300 cubic meters.

**Answer**

$$r(V)=\sqrt{\frac{V}{6\pi}},3.99\text{ meters}$$

64) The surface area, $A$, of a cylinder in terms of its radius, $r$, and height, $h$, is given by $A=2\pi r^2+2\pi rh$. If the height of the cylinder is 4 feet, express the radius as a function of $V$ and find the radius if the surface area is 200 square feet.

65) The volume of a right circular cone, $V$, in terms of its radius, $r$, and its height, $h$, is given by $V=\frac{1}{3}\pi r^2 h$. Express $r$ in terms of $V$ if the height of the cone is 12 feet and find the radius of a cone with volume of 50 cubic inches.

**Answer**

$r(V) = \sqrt{\frac{V}{4\pi}}, 1.99$ inches

66) Consider a cone with height of 30 feet. Express the radius, $r$, in terms of the volume, $V$, and find the radius of a cone with volume of 1000 cubic feet.

## 3.9 Modeling Using Variation

### Verbal

1) What is true of the appearance of graphs that reflect a direct variation between two variables?

**Answer**

The graph will have the appearance of a power function.

2) If two variables vary inversely, what will an equation representing their relationship look like?

3) Is there a limit to the number of variables that can jointly vary? Explain.

**Answer**

No. Multiple variables may jointly vary.

### Algebraic

**For the exercises 4-23, write an equation describing the relationship of the given variables.**

4) $y$ varies directly as $x$ and when $x = 6, y = 12$

5) $y$ varies directly as the square of $x$ and when $x = 4, y = 80$

**Answer**

$y = 5x^2$

6) $y$ varies directly as the square root of $x$ and when $x = 36, y = 24$

7) $y$ varies directly as the cube of $x$ and when $x = 36, y = 24$

**Answer**

$y = 10x^3$

8) $y$ varies directly as the cube root of $x$ and when $x = 27, y = 15$

9) $y$ varies directly as the fourth power of $x$ and when $x = 1, y = 6$

**Answer**

$y = 6x^4$

10) $y$ varies inversely as $x$ and when $x = 4, y = 2$

11) $y$ varies inversely as the square of $x$ and when $x = 3, y = 2$

**Answer**

$y = \frac{18}{x^2}$

12) $y$ varies inversely as the cube of $x$ and when $x = 2, y = 5$

13) $y$ varies inversely as the fourth power of $x$ and when $x = 3, y = 1$

**Answer**

$$y = \frac{81}{x^4}$$

14) $y$ varies inversely as the square root of $x$ and when $x = 25, y = 3$

15) $y$ varies inversely as the cube root of $x$ and when $x = 64, y = 5$

**Answer**

$$y = \frac{20}{\sqrt[3]{x}}$$

16) $y$ varies jointly with $x$ and $z$ and when $x = 2$ and $z = 3, y = 36$

17) $y$ varies jointly as $x$, $z$, and $w$ and when $x = 1, z = 2, w = 5$ , then $y = 100$

**Answer**

$$y = 10xzw$$

18) $y$ varies jointly as the square of $x$ and the square of $z$ and when $x = 3$ and $z = 4$, then $y = 72$

19) $y$ varies jointly as $x$ and the square root of $z$ and when $x = 2$ and $z = 25$, then $y = 100$

**Answer**

$$y = 10x\sqrt{z}$$

20) $y$ varies jointly as the square of $x$ the cube of $z$ and the square root of $w$. When $x = 1, z = 2,$ and $w = 36$, then $y = 48$

21) $y$ varies jointly as $x$ and $z$ and inversely as $w$. When $x = 3, z = 5,$ and $w = 6$, then $y = 10$

**Answer**

$$y = 4\frac{xz}{w}$$

22) $y$ varies jointly as the square of $x$ and the square root of $z$ and inversely as the cube of $w$. When $x = 3, z = 4$, and $w = 3$, then $y = 6$

23) $y$ varies jointly as $x$ and $z$ and inversely as the square root of wand the square of $t$. When $x = 3, z = 1, w = 25$ , and $t = 2$, then $y = 6$

**Answer**

$$y = 40\frac{xz}{\sqrt{w}t^2}$$

## Numeric

**For the exercises 24-40, use the given information to find the unknown value.**

24) $y$ varies directly as $x$. When $x = 3$, then $y = 12$. Find $y$ when $x = 20$.

25) $y$ varies directly as the square of $x$. When $x = 2$, then $y = 16$. Find $y$ when $x = 8$.

**Answer**

$$y = 256$$

26) $y$ varies directly as the cube of $x$. When $x = 3$, then $y = 5$. Find $y$ when $x = 4$.

27) $y$ varies directly as the square root of $x$. When $x = 16$, then $y = 4$. Find $y$ when $x = 36$.

**Answer**

$$y = 6$$

28) $y$ varies directly as the cube root of $x$. When $x = 125$, then $y = 15$. Find $y$ when $x = 1,000$.

29) $y$ varies inversely with $x$. When $x = 3$, then $y = 2$. Find $y$ when $x = 1$.

**Answer**

$y = 6$

30) $y$ varies inversely with the square of $x$. When $x = 4$, then $y = 3$. Find $y$ when $x = 2$.

31) $y$ varies inversely with the cube of $x$. When $x = 3$, then $y = 1$. Find $y$ when $x = 1$.

**Answer**

$y = 27$

32) $y$ varies inversely with the square root of $x$. When $x = 64$, then $y = 12$. Find $y$ when $x = 36$.

33) $y$ varies inversely with the cube root of $x$. When $x = 27$, then $y = 5$. Find $y$ when $x = 125$.

**Answer**

$y = 3$

34) $y$ varies jointly as $x$ and $z$. When $x = 4$ and $z = 2$, then $y = 16$. Find $y$ when $x = 3$ and $z = 3$.

35) $y$ varies jointly as $x$, $z$, and $w$. When $x = 2$, $z = 1$, and $w = 12$, then $y = 72$. Find $y$ when $x = 1$, $z = 2$, and $w = 3$.

**Answer**

$y = 18$

36) $y$ varies jointly as $x$ and the square of $z$. When $x = 2$ and $z = 4$, then $y = 144$. Find $y$ when $x = 4$ and $z = 5$.

37) $y$ varies jointly as the square of $x$ and the square root of $z$. When $x = 9$, then $y = 24$. Find $y$ when $x = 3$ and $z = 25$.

**Answer**

$y = 90$

38) $y$ varies jointly as $x$ and inversely as $w$. When $x = 5$, $z = 2$, and $w = 20$, then $y = 4$. Find $y$ when $x = 3$ and $z = 8$ and $w = 48$.

39) $y$ varies jointly as the square of $x$ and the cube of $z$ and inversely as the square root of $w$. When $x = 2$, $z = 2$, and $w = 64$ then $y = 12$. Find $y$ when $x = 1$, $z = 3$, and $w = 4$.

**Answer**

$y = \frac{81}{2}$

40) $y$ varies jointly as the square of $x$ and of $z$ and inversely as the square root of wand of $t$. When $x = 2, z = 3, w = 16$, and $t = 3$, then $y = 1$. Find $y$ when $x = 3, z = 2, w = 36$, and $t = 5$.

## Technology

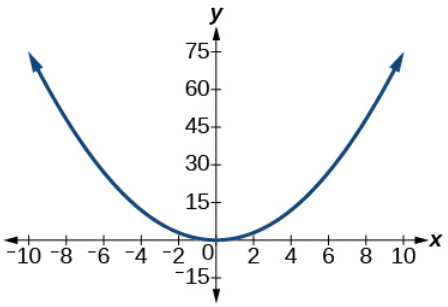
**For the exercises 41-45, use a calculator to graph the equation implied by the given variation.**

41) $y$ varies directly with the square of $x$ and when $x = 2, y = 3$

**Answer**

$y = \frac{3}{4}x^2$


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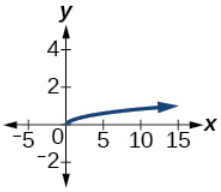



42) $y$ varies directly as the cube of $x$ and when $x = 2, y = 4$.

43) $y$ varies directly as the square root of $x$ and when $x = 36, y = 2$.

**Answer**

$y = \frac{1}{3}\sqrt{x}$

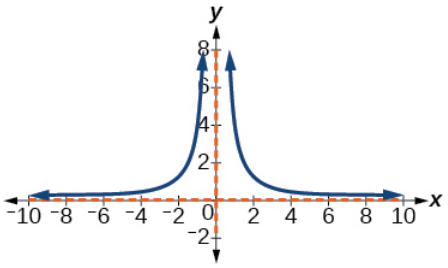



44) $y$ varies inversely with $x$ and when $x = 6, y = 2$.

45) $y$ varies inversely as the square of $x$ and when $x = 1, y = 4$.

**Answer**

$y = \frac{4}{x^2}$




## Extensions

**For the exercises 46-50, use Kepler's Law, which states that the square of the time, $T$, required for a planet to orbit the Sun varies directly with the cube of the mean distance, $a$, that the planet is from the Sun.**

46) Using the Earth's time of 1 year and mean distance of 93 million miles, find the equation relating $T$ and $a$.

47) Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is 142 million miles.

**Answer**

1.89 years

48) Using Earth's distance of 150 million kilometers, find the equation relating $T$ and $a$.

49) Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is 108 million kilometers.


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**Answer**

$0.61$ years

50) Using Earth's distance of 1 astronomical unit (A.U.), determine the time for Saturn to orbit the Sun if its mean distance is $9.54$ A.U.

### Real-World Applications

**For the exercises 51-60, use the given information to answer the questions.**

51) The distance $s$ that an object falls varies directly with the square of the time, $t$, of the fall. If an object falls $16$ feet in one second, how long for it to fall $144$ feet?

**Answer**

$3$ seconds

52) The velocity $v$ of a falling object varies directly to the time, $t$, of the fall. If after $2$ seconds, the velocity of the object is $64$ feet per second, what is the velocity after $5$ seconds?

53) The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is $24$ inches long and vibrates $128$ times per second, what is the length of a string that vibrates $64$ times per second?

**Answer**

$48$ inches

54) The volume of a gas held at constant temperature varies indirectly as the pressure of the gas. If the volume of a gas is $1200$ cubic centimeters when the pressure is $200$ millimeters of mercury, what is the volume when the pressure is $300$ millimeters of mercury?

55) The weight of an object above the surface of the Earth varies inversely with the square of the distance from the center of the Earth. If a body weighs $50$ pounds when it is $3960$ miles from Earth's center, what would it weigh it were $3970$ miles from Earth's center?

**Answer**

$49.75$ pounds

56) The intensity of light measured in foot-candles varies inversely with the square of the distance from the light source. Suppose the intensity of a light bulb is $0.08$ foot-candles at a distance of $3$ meters. Find the intensity level at $8$ meters.

57) The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is $40$ amperes, the resistance is $10$ ohms. Find the current if the resistance is $12$ ohms.

**Answer**

$33.33$ amperes

58) The force exerted by the wind on a plane surface varies jointly with the square of the velocity of the wind and with the area of the plane surface. If the area of the surface is $40$ square feet surface and the wind velocity is $20$ miles per hour, the resulting force is $15$ pounds. Find the force on a surface of $65$ square feet with a velocity of $30$ miles per hour.

59) The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm)) and the cube of the diameter. If the shaft of a certain material $3$ inches in diameter can transmit $45$ hp at $100$ rpm, what must the diameter be in order to transmit $60$ hp at $150$ rpm?

**Answer**

$2.88$ inches

60) The kinetic energy $K$ of a moving object varies jointly with its mass $m$ and the square of its velocity $v$. If an object weighing $40$ kilograms with a velocity of $15$ meters per second has a kinetic energy of $1000$ joules, find the kinetic energy if the velocity is

increased to 20 meters per second.

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# 3.R: Polynomial and Rational Functions(Review)

## 3.1 Complex Numbers

**Perform the indicated operation with complex numbers.**

1) $(4+3i)+(-2-5i)$

**Answer**

$2-2i$

2) $(6-5i)-(10+3i)$

3) $(2-3i)(3+6i)$

**Answer**

$24+3i$

4) $\dfrac{2-i}{2+i}$

**Solve the following equations over the complex number system.**

5) $x^2-4x+5=0$

**Answer**

$\{2+i, 2-i\}$

6) $x^2+2x+10=0$

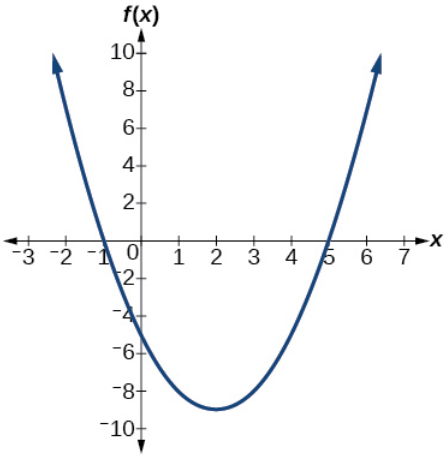
## 3.2 Quadratic Functions

**For the exercises 1-2, write the quadratic function in standard form. Then, give the vertex and axes intercepts. Finally, graph the function.**

1) $f(x)=x^2-4x-5$

**Answer**

$f(x)=(x-2)^2-9$ vertex $(2,-9)$, intercepts $(5,0);(-1,0);(0,-5)$




2) $f(x)=-2x^2-4x$

**For the problems 3-4, find the equation of the quadratic function using the given information.**

3) The vertex is $(-2,3)$ and a point on the graph is $(3,6)$.

**Answer**

$f(x) = \frac{3}{25}(x+2)^2 + 3$

4) The vertex is $(-3, 6.5)$ and a point on the graph is $(2, 6)$.

**Answer the following questions.**

5) A rectangular plot of land is to be enclosed by fencing. One side is along a river and so needs no fence. If the total fencing available is $600$ meters, find the dimensions of the plot to have maximum area.

**Answer**

$300$ meters by $150$ meters, the longer side parallel to river.

6) An object projected from the ground at a $45$ degree angle with initial velocity of $120$ feet per second has height, $h$, in terms of horizontal distance traveled, $x$, given by $h(x) = \frac{-32}{(120)^2}x^2 + x$ . Find the maximum height the object attains.

## 3.3 Power Functions and Polynomial Functions

**For the exercises 1-3, determine if the function is a polynomial function and, if so, give the degree and leading coefficient.**

1) $f(x) = 4x^5 - 3x^3 + 2x - 1$

**Answer**

Yes, degree $= 5$ , leading coefficient $= 4$

2) $f(x) = 5^{x+1} - x^2$

3) $f(x) = x^2\left(3 - 6x + x^2\right)$

**Answer**

Yes, degree $= 4$ , leading coefficient $= 1$

**For the exercises 4-6, determine end behavior of the polynomial function.**

4) $f(x) = 2x^4 + 3x^3 - 5x^2 + 7$

5) $f(x) = 4x^3 - 6x^2 + 2$

**Answer**

As $x \to -\infty, f(x) \to -\infty$ , as $x \to \infty, f(x) \to \infty$

6) $f(x) = 2x^2\left(1 + 3x - x^2\right)$

## 3.4 Graphs of Polynomial Functions

**For the exercises 1-3, find all zeros of the polynomial function, noting multiplicities.**

1) $f(x) = (x+3)^2(2x-1)(x+1)^3$

**Answer**

$-3$ with multiplicity $2$, $-\frac{1}{2}$ with multiplicity $1$, $-1$ with multiplicity $3$

2) $f(x) = x^5 + 4x^4 + 4x^3$

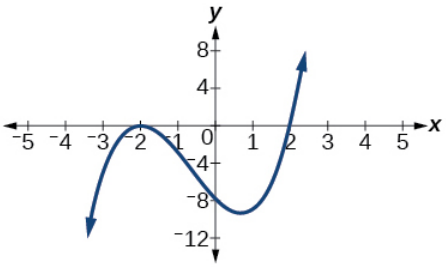
3) $f(x) = x^3 - 4x^2 + x - 4$

**Answer**

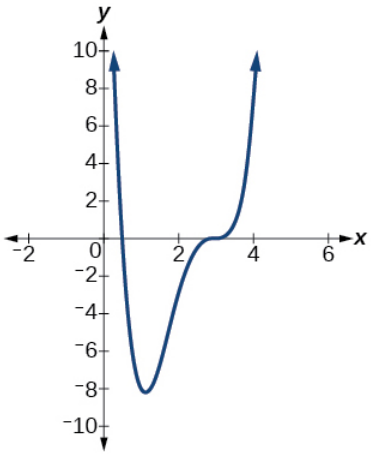
$4$ with multiplicity $1$

**For the exercises 4-5, based on the given graph, determine the zeros of the function and note multiplicity.**

4)




5)




**Answer**

$\frac{1}{2}$ with multiplicity 1, 3 with multiplicity 3

6) Use the Intermediate Value Theorem to show that at least one zero lies between 2 and 3 for the function $f(x) = x^3 - 5x + 1$

## 3.5 Dividing Polynomials

**For the exercises 1-2, use long division to find the quotient and remainder.**

1) $\frac{x^3 - 2x^2 + 4x + 4}{x - 2}$

**Answer**

$x^2 + 4$ with remainder 12

2) $\frac{3x^4 - 4x^2 + 4x + 8}{x + 1}$

**For the exercises 3-6, use synthetic division to find the quotient. If the divisor is a factor, then write the factored form.**

3) $\frac{x^2 - 2x^2 + 5x - 1}{x + 3}$

**Answer**

$x^2 - 5x + 20 - \frac{61}{x + 3}$

4) $\frac{x^2 + 4x + 10}{x - 3}$

5) $\dfrac{2x^3+6x^2-11x-12}{x+4}$

**Answer**

$2x^2-2x-3$ , so factored form is $(x+4)\left(2x^2-2x-3\right)$

6) $\dfrac{3x^4+3x^3+2x+2}{x+1}$

## 3.6 Zeros of Polynomial Functions

**For the exercises 1-4, use the Rational Zero Theorem to help you solve the polynomial equation.**

1) $2x^3-3x^2-18x-8=0$

**Answer**

$\left\{-2, 4, -\dfrac{1}{2}\right\}$

2) $3x^3+11x^2+8x-4=0$

3) $2x^4-17x^3+46x^2-43x+12=0$

**Answer**

$\left\{1, 3, 4, \dfrac{1}{2}\right\}$

4) $4x^4+8x^3+19x^2+32x+12=0$

**For the exercises 5-6, use Descartes' Rule of Signs to find the possible number of positive and negative solutions.**

5) $x^3-3x^2-2x+4=0$

**Answer**

0 or 2 positive, 1 negative

6) $2x^4-x^3+4x^2-5x+1=0$

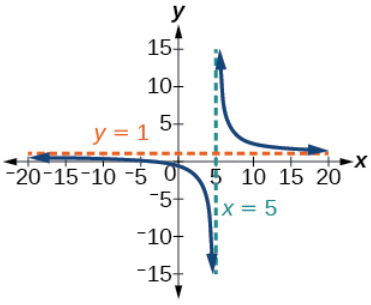
## 3.7 Rational Functions

**For the following rational functions 1-4, find the intercepts and the vertical and horizontal asymptotes, and then use them to sketch a graph.**

1) $f(x)=\dfrac{x+2}{x-5}$

**Answer**

Intercepts $(-2,0)$ and $\left(0,-\dfrac{2}{5}\right)$, Asymptotes $x=5$ and $y=1$

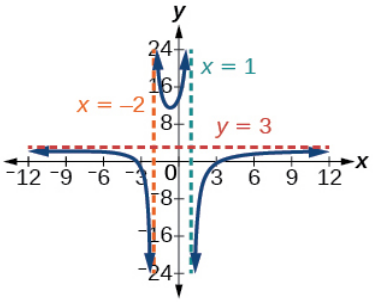


2) $f(x)=\dfrac{x^2+1}{x^2-4}$

3) $f(x)=\dfrac{3x^2-27}{x^2-9}$

**Answer**

Intercepts $(3,0)$, $(-3,0)$, and $\left(0,\dfrac{27}{2}\right)$, Asymptotes $x=1, x=-2, y=3$




4) $f(x)=\dfrac{x+2}{x^2-9}$

**For the exercises 5-6, find the slant asymptote.**

5) $f(x)=\dfrac{x^2-1}{x+2}$

**Answer**

$y=x-2$

6) $f(x)=\dfrac{2x^3-x^2+4}{x^2+1}$

## 3.8 Inverses and Radical Functions

**For the exercises 1-6, find the inverse of the function with the domain given.**

1) $f(x)=(x-2)^2, x\geq 2$

**Answer**

$f^{-1}(x)=\sqrt{x}+2$

2) $f(x)=(x+4)^2-3, x\geq -4$

3) $f(x)=x^2+6x-2, x\geq -3$

**Answer**

$f^{-1}(x)=\sqrt{x+11}-3$

4) $f(x)=2x^3-3$

5) $f(x)=\sqrt{4x+5}-3$

**Answer**

$f^{-1}(x)=\dfrac{(x+3)^2-5}{4}, x\geq -3$

6) $f(x)=\dfrac{x-3}{2x+1}$

## 3.9 Modeling Using Variation

**For the exercises 1-4, find the unknown value.**

1) $y$ varies directly as the square of $x$. If when $x = 3, y = 36$, find $y$ if $x = 4$.

**Answer**

$y = 64$

2) $y$ varies inversely as the square root of $x$. If when $x = 25, y = 2$, find $y$ if $x = 4$.

3) $y$ varies jointly as the cube of $x$ and as $z$. If when $x = 1$ and $z = 2, y = 6$ , find $y$ if $x = 2$ and $z = 3$ .

**Answer**

$y = 72$

4) $y$ varies jointly as $x$ and the square of $z$ and inversely as the cube of $w$. If when $x = 3, z = 4$ , and $w = 2, y = 48$, ind $y$ if $x = 4, z = 5$ , and $w = 3$.

**For the exercises 5-6, solve the application problem.**

5) The weight of an object above the surface of the earth varies inversely with the distance from the center of the earth. If a person weighs $150$ pounds when he is on the surface of the earth ($3,960$ miles from center), find the weight of the person if he is $20$ miles above the surface.

**Answer**

$148.5$pounds

6) The volume $V$ of an ideal gas varies directly with the temperature $T$ and inversely with the pressure$P$. A cylinder contains oxygen at a temperature of $310$ degrees K and a pressure of $18$ atmospheres in a volume of $120$ liters. Find the pressure if the volume is decreased to $100$ liters and the temperature is increased to $320$ degrees K.

## Practice Test

**Perform the indicated operation or solve the equation.**

1) $(3-4i)(4+2i)$

**Answer**

$20-10i$

2) $\dfrac{1-4i}{3+4i}$

3) $x^2-4x+13=0$

**Answer**

$\{2+3i, 2-3i\}$

4) Give the degree and leading coefficient of the following polynomial function.

$$f(x) = x^3\left(3-6x^2-2x^2\right)$$

**Determine the end behavior of the polynomial function.**

5) $f(x) = 8x^3-3x^2+2x-4$

**Answer**

As $x \to -\infty, f(x) \to -\infty$ , as $x \to \infty, f(x) \to \infty$

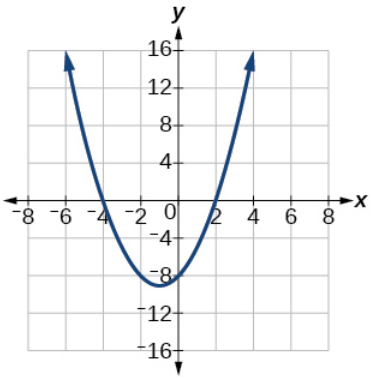
6) $f(x) = -2x^2\left(4-3x-5x^2\right)$

7) Write the quadratic function in standard form. Determine the vertex and axes intercepts and graph the function.

$$f(x) = x^2 + 2x - 8$$

**Answer**

$f(x) = (x+1)^2 - 9$, vertex $(-1, -9)$, intercepts $(2, 0); (-4, 0); (0, -8)$




8) Given information about the graph of a quadratic function, find its equation: Vertex $(2, 0)$ and point on graph $(4, 12)$

**Solve the following application problem.**

9) A rectangular field is to be enclosed by fencing. In addition to the enclosing fence, another fence is to divide the field into two parts, running parallel to two sides. If $1,200$ feet of fencing is available, find the maximum area that can be enclosed.

**Answer**

$60,000$ square feet

**Find all zeros of the following polynomial functions, noting multiplicities.**

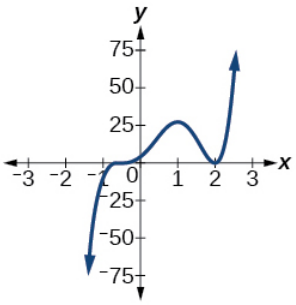
10) $f(x) = (x-3)^3(3x-1)(x-1)^2$

11) $f(x) = 2x^6 - 12x^5 + 18x^4$

**Answer**

0 with multiplicity 4, 3 with multiplicity 2

12) Based on the graph, determine the zeros of the function and multiplicities.




13) Use long division to find the quotient:

$$\frac{2x^2 + 3x - 4}{x + 2}$$

**Answer**

$$2x^2 - 4x + 11 - \frac{26}{x+2}$$

**Use synthetic division to find the quotient. If the divisor is a factor, write the factored form.**

14) $\dfrac{x^4+3x^2-4}{x-2}$

15) $\dfrac{2x^3+5x^2-7x-12}{x+3}$

**Answer**

$2x^2-x-4$ . So factored form is $(x+3)\left(2x^2-x-4\right)$

**Use the Rational Zero Theorem to help you find the zeros of the polynomial functions.**

16) $f(x)=2x^3+5x^2-6x-9$

17) $f(x)=4x^4+8x^3+21x^2+17x+4$

**Answer**

$-\dfrac{1}{2}$ (has multiplicity 2), $\dfrac{-1+i\sqrt{15}}{2}$

18) $f(x)=4x^4+16x^3+13x^2-15x-18$

19) $f(x)=x^5+6x^4+13x^3+14x^2+12x+8$

**Answer**

$-2$ (has multiplicity 3), $\pm i$

**Given the following information about a polynomial function, find the function.**

20) It has a double zero at $x=3$ and zeroes at $x=1$ and $x=-2$. Its $y$ -intercept is $(0,12)$.

21) It has a zero of multiplicity 3 at $x=\dfrac{1}{2}$ and another zero at $x=-3$. It contains the point $(1,8)$.

**Answer**

$f(x)=2(2x-1)^3(x+3)$

22) Use Descartes' Rule of Signs to determine the possible number of positive and negative solutions.

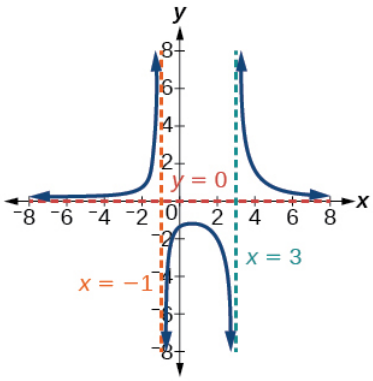
$$8x^3-21x^2+6=0$$

**For the following rational functions, find the intercepts and horizontal and vertical asymptotes, and sketch a graph.**

23) $f(x)=\dfrac{x+4}{x^2-2x-3}$

**Answer**

Intercepts $(-4,0)$, $\left(0,-\dfrac{4}{3}\right)$, Asymptotes $x=3, x=-1, y=0$




24) $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - 4}$

25) Find the slant asymptote of the rational function.

$$f(x) = \frac{x^2 + 3x - 3}{x - 1}$$

**Answer**

$y = x + 4$

**Find the inverse of the function.**

26) $f(x) = \sqrt{x - 2} + 4$

27) $f(x) = 3x^3 - 4$

**Answer**

$f^{-1}(x) = \sqrt[3]{\dfrac{x + 4}{3}}$

28) $f(x) = \dfrac{2x + 3}{3x - 1}$

**Find the unknown value.**

29) $y$ varies inversely as the square of $x$ and when $x = 3, y = 2$. Find $y$ if $x = 1$.

**Answer**

$y = 18$

30) $y$ varies jointly with $x$ and the cube root of $z$. If when $x = 27, y = 12$, find $y$ if $x = 5$ and $z = 8$.

**Solve the following application problem.**

31) The distance a body falls varies directly as the square of the time it falls. If an object falls $64$ feet in $2$ seconds, how long will it take to fall $256$ feet?

**Answer**

4 seconds

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# CHAPTER OVERVIEW

## 4: Exponential and Logarithmic Functions

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.


- 4.0: Prelude to Exponential and Logarithmic Functions
- 4.1: Exponential Functions
- 4.2: Graphs of Exponential Functions
- 4.3: Logarithmic Functions
- 4.4: Graphs of Logarithmic Functions
- 4.5: Logarithmic Properties
- 4.6: Exponential and Logarithmic Equations
- 4.7: Exponential and Logarithmic Models
- 4.8: Fitting Exponential Models to Data
- 4.9: Trigonometry Preview - Circles
- 4.E: Exponential and Logarithmic Functions (Exercises)
- 4.R: Exponential and Logarithmic Functions (Review)


*Thumbnail: The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$. (CC BY-SA; OpenStax).*

# Contributors

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# 4.0: Prelude to Exponential and Logarithmic Functions

Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years. For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. Table 4.0.1 shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Table 4.0.1

| **Hour** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Bacteria** | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

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# 4.1: Exponential Functions

**Learning Objectives**

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base $e$.

India is the second most populous country in the world with a population of about $1.25$ billion people in 2013. The population is growing at a rate of about $.2\%$ each year. If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.

**Identifying Exponential Functions**

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by $3$ each time the input increases by $1$. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

## Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, $2.5\%$ of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up $10\%$ of the U.S. population in 2015, $40\%$ in 2019, and $80\%$ in 2050.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of **exponential growth**, let us contrast exponential growth with **linear growth**. We will construct two functions. The first function is exponential. We will start with an input of $0$, and increase each input by $1$. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of $0$, and increase each input by $1$. We will add $2$ to the corresponding consecutive outputs (Table $4.1.1$).

From Table $4.1.1$ we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Table 4.1.1

| $x$ | $f(x) = 2^x$ | $g(x) = 2x$ |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 2 | 2 |
| 2 | 4 | 4 |

| $x$ | $f(x)=2^x$ | $g(x)=2x$ |
|---|---|---|
| 3 | 8 | 6 |
| 4 | 16 | 8 |
| 5 | 32 | 10 |
| 6 | 64 | 12 |

Apparently, the difference between “the same percentage” and “the same amount” is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

The general form of the **exponential function** is $f(x)=ab^x$, where $a$ is any nonzero number, $b$ is a positive real number not equal to 1.

- If $b>1$,the function grows at a rate proportional to its size.
- If $0<b<1$, the function decays at a rate proportional to its size.

Let’s look at the function $f(x)=2^x$ from our example. We will create a table (Table 4.1.2) to determine the corresponding outputs over an interval from $-3$ to 3.

Table 4.1.2

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)=2^x$ | $2^{-3}=\frac{1}{8}$ | $2^{-2}=\frac{1}{4}$ | $2^{-1}=\frac{1}{2}$ | $2^0=1$ | $2^1=2$ | $2^2=4$ | $2^3=8$ |

Let us examine the graph of $f$ by plotting the ordered pairs from Table 4.1.2 and then make a few observations 4.1.1.

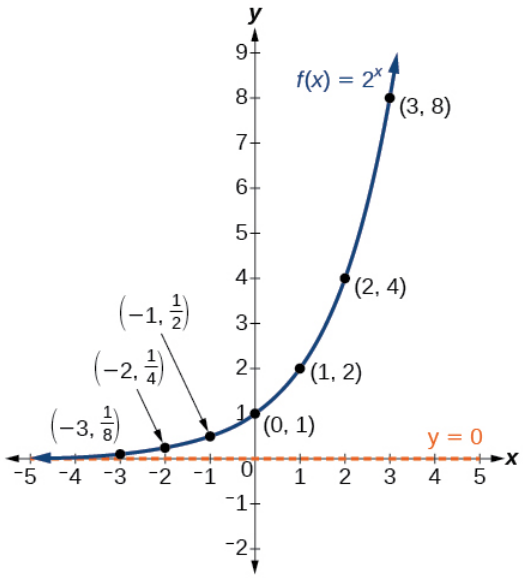



Figure 4.1.1

Let’s define the behavior of the graph of the exponential function $f(x)=2^x$ and highlight some its key characteristics.

- the domain is $(-\infty,\infty)$,
- the range is $(0,\infty)$,
- as $x\to\infty$, $f(x)\to\infty$,
- as $x\to-\infty$, $f(x)\to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.

- $y = 0$ is the horizontal asymptote.
- the $y$-intercept is $1$.

**Definition: Exponential Functions**

For any real number $x$,an exponential function is a function with the form

$$f(x) = ab^x \tag{4.1.1}$$

where

- $a$ is a non-zero real number called the initial value and
- $b$ is any positive real number such that $b \neq 1$.
- The domain of $f$ is all real numbers.
- The range of $f$ is all positive real numbers if $a > 0$.
- The range of $f$ is all negative real numbers if $a < 0$.
- The $y$-intercept is $(0, a)$,and the horizontal asymptote is $y = 0$.

**Example 4.1.1: Identifying Exponential Functions**

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

**Solution**

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base $b$ of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, $-2$, is less than $0$.

**Exercise 4.1.1**

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

**Answer**

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions.

## Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1.Why do we limit the base bb to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{\frac{1}{2}} = \sqrt{-9}$ ,which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1:

- Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of $x$.

To evaluate an exponential function with the form $f(x) = b^x$,we simply substitute $x$ with the given value, and calculate the resulting power. For example:

Let $f(x) = 2^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 2^x \\ f(3) &= 2^3 \qquad \text{Substitute } x = 3 \\ &= 8 \qquad \text{Evaluate the power} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x) = 30(2)^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 30(2)^x \\ f(3) &= 30(2)^3 \qquad \text{Substitute } x = 3 \\ &= 30(8) \qquad \text{Simplify the power first} \\ &= 240 \qquad \text{Multiply} \end{aligned}$$

Note that if the order of operations were not followed, the result would be incorrect:

$$f(3) = 30(2)^3 \neq 60^3 = 216,000$$

### Example 4.1.2: Evaluating Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

**Solution**

Follow the order of operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} f(x) &= 5(3)^{x+1} \\ f(2) &= 5(3)^{2+1} \qquad \text{Substitute } x = 2 \\ &= 5(3)^3 \qquad \text{Add the exponents} \\ &= 5(27) \qquad \text{Simplify the power} \\ &= 135 \qquad \text{Multiply} \end{aligned}$$

### Exercise 4.1.2

Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

**Answer**

5.5556

## Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term “exponential growth” is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

### Definition: Exponential Growth

A function that models exponential growth grows by a rate proportional to the amount present. For any real number $x$ and any positive real numbers $a$ and $b$ such that $b \neq 1$,an exponential growth function has the form

$$f(x) = ab^x \tag{4.1.2}$$

where

- $a$ is the initial or starting value of the function.
- $b$ is the growth factor or growth multiplier per unit $x$.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A(x) = 100 + 50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in Table 4.1.3.

Table 4.1.3

| Year, $x$ | Stores, Company A | Stores, Company B |
|---|---|---|
| 0 | $100 + 50(0) = 100$ | $100(1 + 0.5)^0 = 100$ |
| 1 | $100 + 50(1) = 150$ | $100(1 + 0.5)^1 = 150$ |
| 2 | $100 + 50(2) = 200$ | $100(1 + 0.5)^2 = 225$ |
| 3 | $100 + 50(3) = 250$ | $100(1 + 0.5)^3 = 337.5$ |
| $x$ | $A(x) = 100 + 50x$ | $B(x) = 100(1 + 0.5)^x$ |

The graphs comparing the number of stores for each company over a five-year period are shown in Figure 4.1.2. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

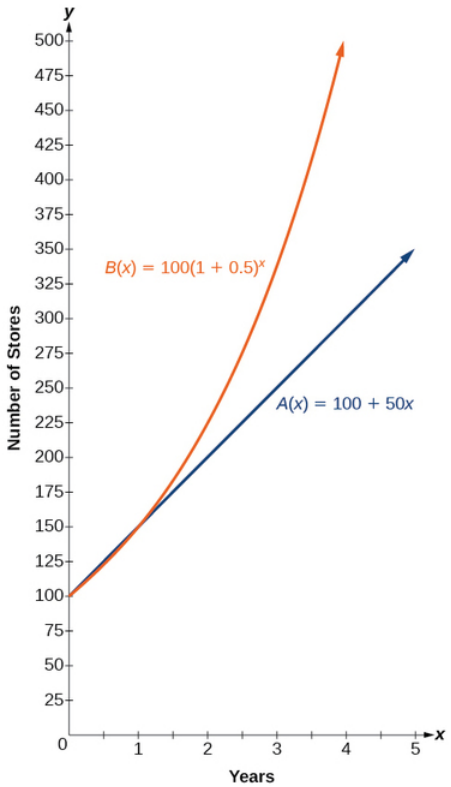



Figure 4.1.2: The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is $[0, \infty)$,and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company $B$, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and $1 + 0.5 = 1.5$ represents the

growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$,where $100$ is the initial value, $1.5$ is called the *base*, and $x$ is called the *exponent*.

### Example 4.1.3: Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about $1.25$ billion in the year 2013, with an annual growth rate of about $1.2\%$. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where $t$ is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

**Solution**

To estimate the population in 2031, we evaluate the models for $t = 18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

There will be about $1.549$ billion people in India in the year 2031.

### Exercise 4.1.3

The population of China was about $1.39$ billion in the year 2013, with an annual growth rate of about $0.6\%$. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where $t$ is the number of years since 2013.To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example 4.1.3?

**Answer**

About $1.548$ billion people; by the year 2031, India's population will exceed China's by about $0.001$ billion, or 1 million people.

## Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants $a$, $a$ and $b$, $b$,and evaluate the function.

### How to: Given two data points, write an exponential model

1. If one of the data points has the form $(0, a)$, then $a$ is the initial value. Using $a$, substitute the second point into the equation $f(x) = a(b)^x$, and solve for $b$.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find $a$ and $b$.
3. Using the $a$ and $b$ found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

### Example 4.1.4: Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population $(N)$ of deer over time $t$.

**Solution**

We let our independent variable $t$ be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find $b$:

$$
\begin{aligned}
N(t) &= 80b^t \\
180 &= 80b^6 && \text{Substitute using point } (6, 180) \\
\frac{9}{4} &= b^6 && \text{Divide and write in lowest terms} \\
b &= \left(\frac{9}{4}\right)^{\frac{1}{6}} && \text{Isolate b using properties of exponents} \\
b &\approx 1.1447 && \text{Round to 4 decimal places}
\end{aligned}
$$

> *Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.*

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure 4.1.3 passes through the initial points given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$,and the range for the function is $[80, \infty)$.

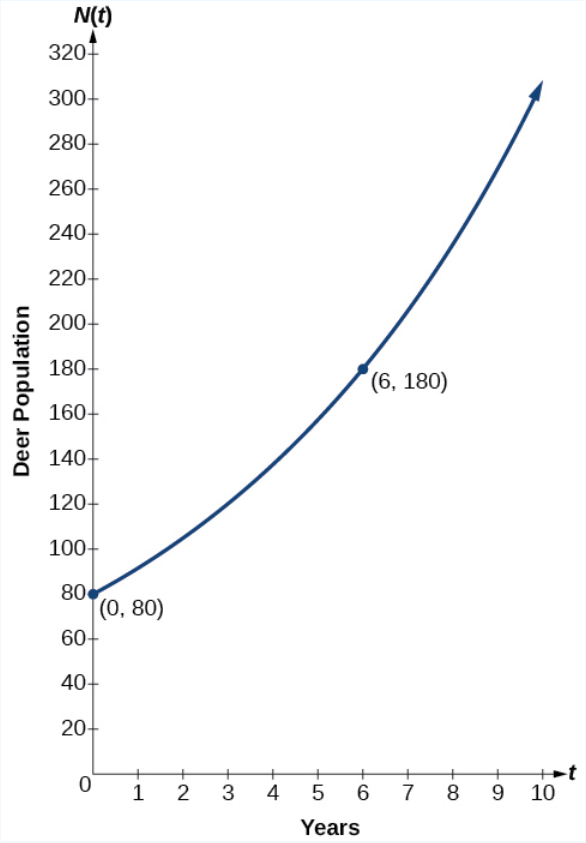



Figure 4.1.3: Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, $t$ years after 2006

## Exercise 4.1.4

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population $N$ of wolves over time $t$.

**Answer**

$(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

### Example 4.1.5: Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

**Solution**

Because we don't have the initial value, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for $a$ and $b$.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for $a$ in terms of $b$:

$$\begin{aligned} 6 &= ab^{-2} \\ \frac{6}{b^{-2}} &= a \qquad \text{Divide} \\ a &= 6b^2 \qquad \text{Use properties of exponents to rewrite the denominator} \end{aligned}$$

Substitute a in the second equation, and solve for $b$:

$$\begin{aligned} 1 &= ab^2 \\ 1 &= 6b^2b^2 \\ &= 6b^4 \qquad \text{Substitute a} \\ b &= \left(\frac{1}{6}\right)^{\frac{1}{4}} \qquad \text{Round 4 decimal places rewrite the denominator} \\ b &\approx 0.6389 \end{aligned}$$

Use the value of $b$ in the first equation to solve for the value of $a$:

$$\begin{aligned} a &= 6b^2 \\ &\approx 6(0.6389)^2 \\ &\approx 2.4492 \end{aligned}$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in Figure 4.1.4 passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

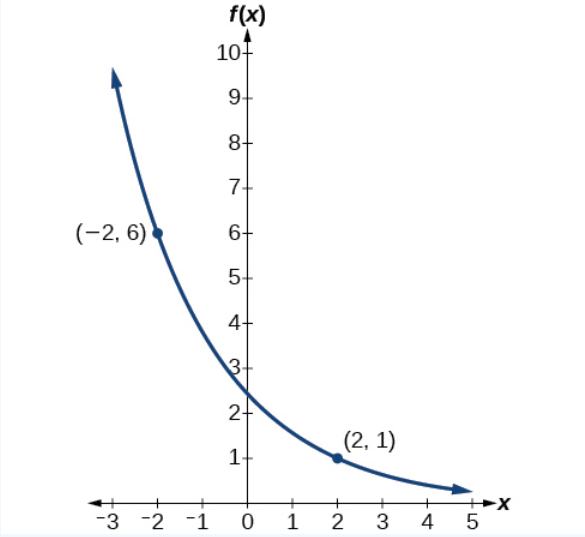



Figure 4.1.4: The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.


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## Exercise 4.1.5

Given the two points $(1, 3)$ and $(2, 4.5)$, find the equation of the exponential function that passes through these two points.

**Answer**

$f(x) = 2(1.5)^x$

## Q&A: Do two points always determine a unique exponential function?

*Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in $x$, which in many real world cases involves time.*

## How to: Given the graph of an exponential function, write its equation

1. First, identify two points on the graph. Choose the $y$-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the $y$-intercept $(0, a)$, then $a$ is the initial value. Using $a$, substitute the second point into the equation $f(x) = a(b)^x$, and solve for $b$
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find $a$ and $b$.
4. Write the exponential function, $f(x) = a(b)^x$.

## Example 4.1.6: Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in Figure 4.1.5.

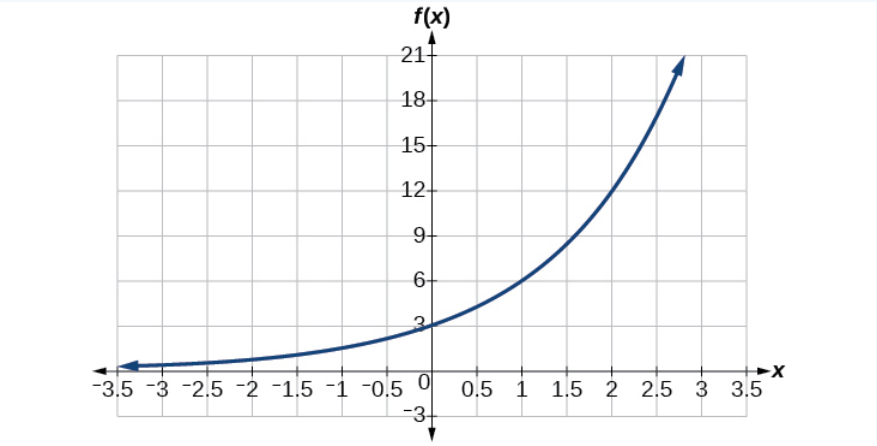



Figure 4.1.5

**Solution**

We can choose the $y$-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$$
\begin{aligned}
y &= ab^x && \text{Write the general form of an exponential equation} \\
y &= 3b^x && \text{Substitute the initial value 3 for } a \\
12 &= 3b^2 && \text{Substitute in 12 for } y \text{ and 2 for } x \\
4 &= b^2 && \text{Divide by 3} \\
b &= \pm 2 && \text{Take the square root}
\end{aligned}
$$


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Because we restrict ourselves to positive values of $b$, we will use $b = 2$. Substitute $a$ and $b$ into the standard form to yield the equation $f(x) = 3(2)^x$.

### Exercise 4.1.6

Find an equation for the exponential function graphed in Figure 4.1.6.

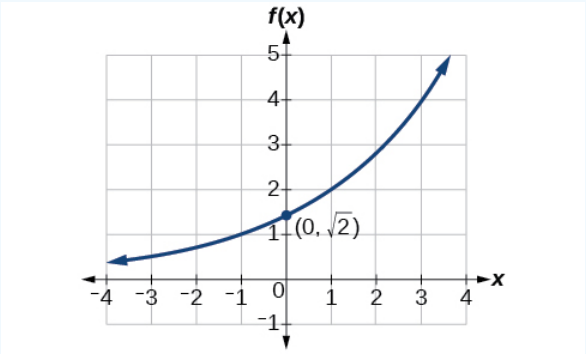



Figure 4.1.6

**Answer**

$f(x) = \sqrt{2}(\sqrt{2})^x$ . Answers may vary due to round-off error. The answer should be very close to $1.4142(1.4142)^x$.

### How to: Given two points on the curve of an exponential function, use a graphing calculator to find the equation

1. Press **[STAT].**
2. Clear any existing entries in columns **L1** or **L2.**
3. In **L1**, enter the *x*-coordinates given.
4. In **L2**, enter the corresponding *y*-coordinates.
5. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential Regression)**, and press **[ENTER].**
6. The screen displays the values of *a* and *b* in the exponential equation $y = a \cdot b^x$ .

### Example 4.1.7: Using a Graphing Calculator to Find an Exponential Function

Use a graphing calculator to find the exponential equation that includes the points $(2, 24.8)$ and $(5, 198.4)$.

**Solution**

Follow the guidelines above. First press **[STAT]**, **[EDIT]**, **[1: Edit…],** and clear the lists **L1** and **L2**. Next, in the **L1** column, enter the $x$-coordinates, 2 and 5. Do the same in the **L2** column for the $y$-coordinates, 24.8 and 198.4.

Now press **[STAT]**, **[CALC]**, **[0: ExpReg]** and press **[ENTER]**. The values $a = 6.2$ and $b = 2$ will be displayed. The exponential equation is $y = 6.2 \cdot 2^x$ .

### Exercise 4.1.7

Use a graphing calculator to find the exponential equation that includes the points $(3, 75.98)$ and $(6, 481.07)$.

**Answer**

$y \approx 12 \cdot 1.85^x$

## Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the

account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time $t$, principal $P$, $APR$ $r$, and number of compounding periods in a year $n$:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe Table 4.1.4, which shows the result of investing $\$1,000$ at $10\%$ for one year. Notice how the value of the account increases as the compounding frequency increases.

Table 4.1.4

| Frequency | Value after 1 year |
|---|---|
| Annually | $1100 |
| Semiannually | $1102.50 |
| Quarterly | $1103.81 |
| Monthly | $1104.71 |
| Daily | $1105.16 |

**Definition: Compound Interest**

*Compound interest* can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \tag{4.1.3}$$

where

- $A(t)$ is the account value,
- $t$ is measured in years,
- $P$ is the starting amount of the account, often called the principal, or more generally present value,
- $r$ is the annual percentage rate (APR) expressed as a decimal, and
- $n$ is the number of compounding periods in one year.

**Example 4.1.8: Calculating Compound Interest**

If we invest $\$3,000$ in an investment account paying $3\%$ interest compounded quarterly, how much will the account be worth in 10 years?

**Solution**

Because we are starting with $\$3,000$, $P = 3000$. Our interest rate is $3\%$, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$,the value when $t = 10$.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula}$$

$$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{(4)\cdot(10)} \quad \text{Substitute using given values}$$

$$\approx \$4045.05 \quad \text{Round to two decimal places}$$

The account will be worth about $\$4,045.05$ in 10 years.

**Exercise 4.1.8**

An initial investment of $\$100,000$ at $12\%$ interest is compounded weekly (use $52$ weeks in a year). What will the investment be worth in $30$ years?

**Answer**

about $\$3,644,675.88$

**Example 4.1.9: Using the Compound Interest Formula to Solve for the Principal**

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $\$40,000$ over $18$ years. She believes the account will earn $6\%$ compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

**Solution**

The nominal interest rate is $6\%$, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, $P$, needed so that the value of the account will be worth $\$40,000$ in $18$ years. Substitute the given values into the compound interest formula, and solve for $P$.

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula}$$

$$40,000 = P\left(1+\frac{0.06}{2}\right)^{2(18)} \quad \text{Substitute using given values } A, r, n, t$$

$$40,000 = P(1.03)^{36} \quad \text{Simplify}$$

$$\frac{40,000}{(1.03)^{36}} = P \quad \text{Isolate } P$$

$$P \approx \$13,801 \quad \text{Divide and round to the nearest dollar}$$

Lily will need to invest $\$13,801$ to have $\$40,000$ in $18$ years.

**Exercise 4.1.9**

Refer to Example 4.1.9. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

**Answer**

$\$13,693$

## Evaluating Functions with Base $e$

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table 4.1.5 shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $\$1$ invested at $100\%$ interest for $1$ year, compounded at various frequencies, listed in Table 4.1.5.

Table 4.1.5

| Frequency | $A(t) = \left(1+\frac{1}{n}\right)^n$ | Value |
|---|---|---|
| Annually | $\left(1+\frac{1}{1}\right)^1$ | $\$2$ |
| Semiannually | $\left(1+\frac{1}{2}\right)^2$ | $\$2.25$ |

| Frequency | $A(t)=\left(1+\frac{1}{n}\right)^n$ | Value |
|---|---|---|
| Quarterly | $\left(1+\frac{1}{4}\right)^4$ | $2.441406 |
| Monthly | $\left(1+\frac{1}{12}\right)^{12}$ | $2.613035 |
| Daily | $\left(1+\frac{1}{365}\right)^{365}$ | $2.714567 |
| Hourly | $\left(1+\frac{1}{8760}\right)^{8760}$ | $2.718127 |
| Once per minute | $\left(1+\frac{1}{525600}\right)^{525600}$ | $2.718279 |
| Once per second | $\left(1+\frac{1}{31536000}\right)^{31536000}$ | $2.718282 |

These values appear to be approaching a limit as $n$ increases without bound. In fact, as $n$ gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter $e$. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

**Definition: The Number e**

The letter $e$ represents the irrational number

$$\left(1+\frac{1}{n}\right)^n \tag{4.1.4}$$

as $n$ increases without bound

The letter $e$ is used as a base for many real-world exponential models. To work with base $e$, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

**Example 4.1.10: Using a Calculator to Find Powers of $e$**

Calculate $e^{3.14}$. Round to five decimal places.

**Solution**

On a calculator, press the button labeled $[e^x]$. The window shows $[e^(]$. Type $3.14$ and then close parenthesis, $[)]$. Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387$. Caution: Many scientific calculators have an "Exp" button, which is used to enter numbers in scientific notation. It is not used to find powers of $e$.

**Exercise 4.1.10**

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

**Answer**

$e^{-0.5} \approx 0.60653$

## Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, $e$ is used as the base for exponential functions. Exponential models that use $e$ as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

### Definition: The Continuous Growth/decay Formula

For all real numbers $t$,and all positive numbers $a$ and $r$, continuous growth or decay is represented by the formula

$$A(t) = ae^{rt} \tag{4.1.5}$$

where

- $a$ is the initial value,
- $r$ is the continuous growth rate per unit time,
- $t$ is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt} \tag{4.1.6}$$

where

- $P$ is the principal or the initial invested,
- $r$ is the growth or interest rate per unit time,
- $t$ is the period or term of the investment.

### How to: Given the initial value, rate of growth or decay, and time $t$, solve a continuous growth or decay function

1. Use the information in the problem to determine $a$, the initial value of the function.
2. Use the information in the problem to determine the growth rate $r$.
   - If the problem refers to continuous growth, then $r > 0$.
   - If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time $t$.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

### Example 4.1.11: Calculating Continuous Growth

A person invested $\$1,000$ in an account earning a nominal $10\%$ per year compounded continuously. How much was in the account at the end of one year?

**Solution**

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $\$1,000$, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P, r, t \\ &\approx 1105.17 && \text{Use a calculator to approximate} \end{aligned}$$

The account is worth $\$1,105.17$after one year.

### Exercise 4.1.11

A person invests $\$100,000$ at a nominal $12\%$ interest per year compounded continuously. What will be the value of the investment in $30$ years?

**Answer**

$\$3,659,823.44$

### Example 4.1.12: Calculating Continuous Decay

$Radon-222$ decays at a continuous rate of $17.3\%$ per day. How much will $100mg$ of $Radon-222$ decay to in 3 days?

**Solution**

Since the substance is decaying, the rate, $17.3\%$, is negative. So, $r=-0.173$. The initial amount of $Radon-222$ was 100 mg, so $a=100$. We use the continuous decay formula to find the value after $t=3$ days:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula} \\ &= 100e6-0.173(3) && \text{Substitute known values for } a, r, t \\ &\approx 59.5115 && \text{Use a calculator to approximate} \end{aligned}$$

So $59.5115$mg of $Radon-222$ will remain.

### Exercise 4.1.12

Using the data in Example 4.1.12, how much $Radon-222$ will remain after one year?

**Answer**

$3.77E-26$ (This is calculator notation for the number written as $3.77\times10^{-26}$ in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

### Media

Access these online resources for additional instruction and practice with exponential functions.

- Exponential Growth Function
- Compound Interest

## Key Equations

| | |
|---|---|
| definition of the exponential function | $f(x)=b^x$, where $b>0, b\neq1$ |
| definition of exponential growth | $f(x)=ab^x$, where $a>0, b>0, b\neq1$ |
| compound interest formula | $A(t)=P(1+\frac{r}{n})^{nt}$,<br>where $A(t)$ is the account value at time $t$<br>$t$ is the number of years<br>$P$ is the initial investment, often called the principal<br>$r$ is the annual percentage rate (APR), or nominal rate<br>$n$ is the number of compounding periods in one year |
| continuous growth formula | $A(t)=ae^{rt}$, where $t$ is the number of unit time periods of growth $a$ is the starting amount (in the continuous compounding formula a is replaced with $P$, the principal) $e$ is the mathematical constant, $e\approx2.718282$ |

## Key Concepts

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent. See Example.
- A function is evaluated by solving at a specific value. See Example and Example.

- An exponential model can be found when the growth rate and initial value are known. See Example.
- An exponential model can be found when the two data points from the model are known. See Example.
- An exponential model can be found using two data points from the graph of the model. See Example.
- An exponential model can be found using two data points from the graph and a calculator. See Example.
- The value of an account at any time $t$ can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See Example.
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See Example.
- The number $e$ is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282$.
- Scientific and graphing calculators have the key $[ex]$ or $[exp(x)]$ for calculating powers of $e$. See Example.
- Continuous growth or decay models are exponential models that use $e$ as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See Example and Example.

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# 4.2: Graphs of Exponential Functions

**Learning Objectives**

- Graph exponential functions.
- Graph exponential functions using transformations.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

## Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table 4.2.1 change as the input increases by 1.

Table 4.2.1

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 2^x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ | $8$ |

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x) = ab^x$, $b$ is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of $a$.

Notice from the table that

- the output values are positive for all values of $x$;
- as $x$ increases, the output values increase without bound; and
- as $x$ decreases, the output values grow smaller, approaching zero.

Figure 4.2.1 shows the exponential growth function $f(x) = 2^x$.

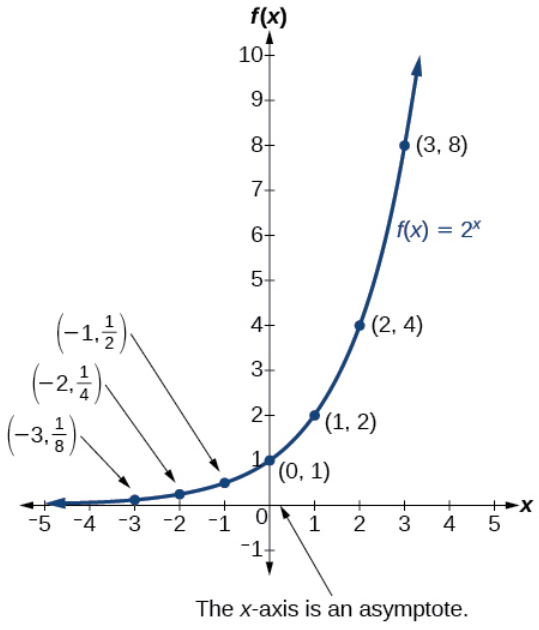



Figure 4.2.1: Notice that the graph gets close to the x-axis, but never touches it.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table 4.2.2 change as the input increases by 1.

Table 4.2.2

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $g(x) = \left(\frac{1}{2}\right)^x$ | $8$ | $4$ | $2$ | $1$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of $x$;
- as $x$ increases, the output values grow smaller, approaching zero; and
- as $x$ decreases, the output values grow without bound.

Figure 4.2.2 shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

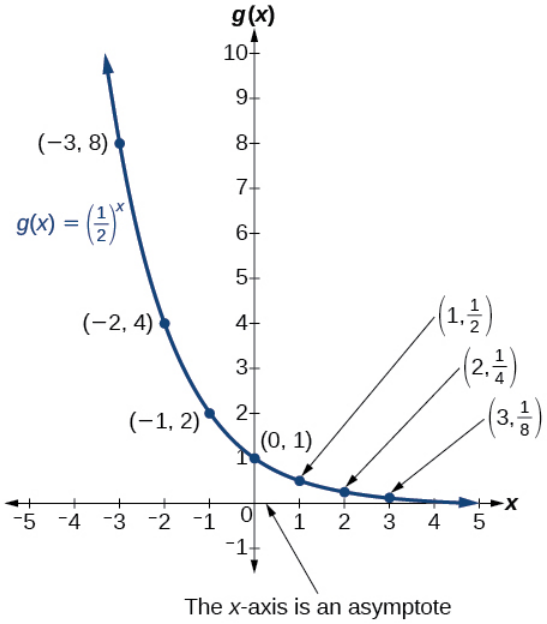



Figure 4.2.2

The domain of $g(x) = (\frac{1}{2})^x$ is all real numbers, the range is $(0, \infty)$,and the horizontal asymptote is $y = 0$.

### CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION $F(X) = B^X$

An exponential function with the form $f(x) = b^x$, $b > 0$, $b \neq 1$,has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- $x$-intercept: none

- $y$-intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

Figure $4.2.3$ compares the graphs of exponential growth and decay functions.



1 and the second graph is of the same function when b is 0

**Figure 4.2.3**

**How to: Given an exponential function of the form $f(x) = b^x$ ,graph the function**

1. Create a table of points.
2. Plot at least 3 point from the table, including the $y$-intercept $(0, 1)$.
3. Draw a smooth curve through the points.
4. State the domain, $(-\infty, \infty)$,the range, $(0, \infty)$,and the horizontal asymptote, $y = 0$.

**Example 4.2.1: Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$**

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

**Solution**

Before graphing, identify the behavior and create a table of points for the graph.

Table 4.2.3

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 0.25^x$ | 64 | 16 | 4 | 1 | 0.25 | \(0.0625\0 | 0.015625 |

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in Table $4.2.3$.
- Plot the $y$-intercept, $(0, 1)$,along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in Figure $4.2.4$.

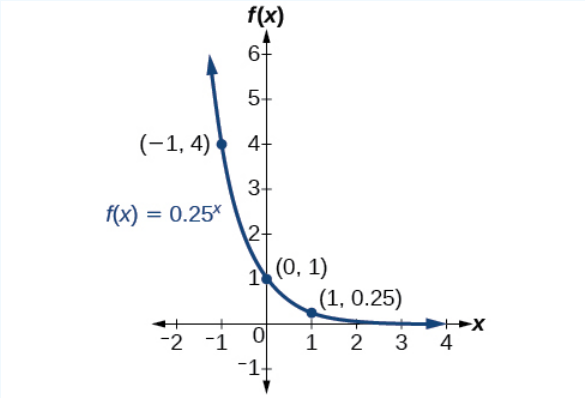



Figure 4.2.4

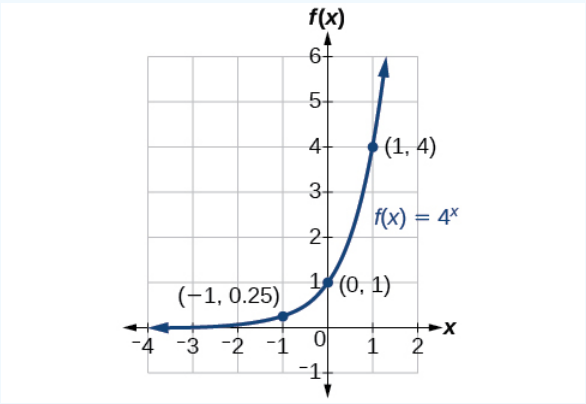
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

**Exercise 4.2.1**

Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

**Answer**

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.




## Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

### Graphing a Vertical Shift

The first transformation occurs when we add a constant $d$ to the parent function $f(x) = b^x$, giving us a vertical shiftd d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure 4.2.5.

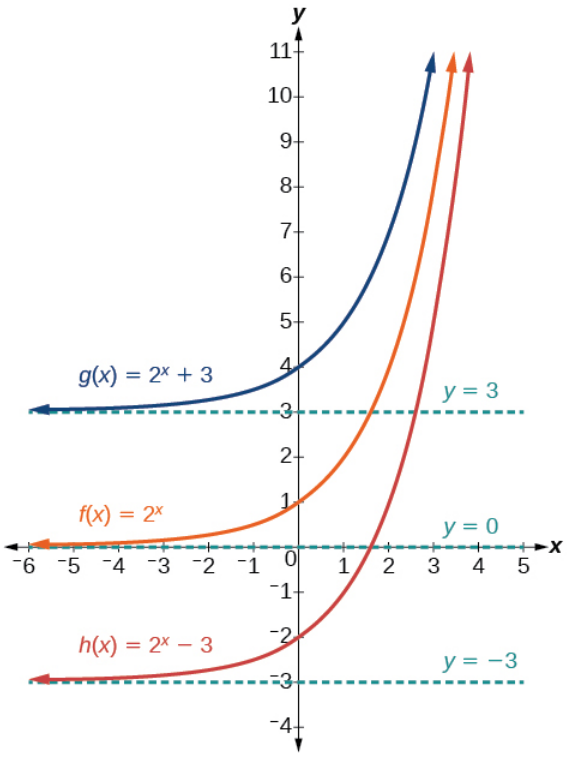



Figure 4.2.5

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.

- When the function is shifted up 3 units to $g(x) = 2^x + 3$ :
  - The *y*-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$ :
  - The *y*-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

### Graphing a Horizontal Shift

The next transformation occurs when we add a constant $c$ to the input of the parent function $f(x) = b^x$, giving us a horizontal shift $c$ units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$ , and the shift right, $h(x) = 2^{x-3}$ . $h(x) = 2^{x-3}$ . Both horizontal shifts are shown in Figure $4.2.6$.

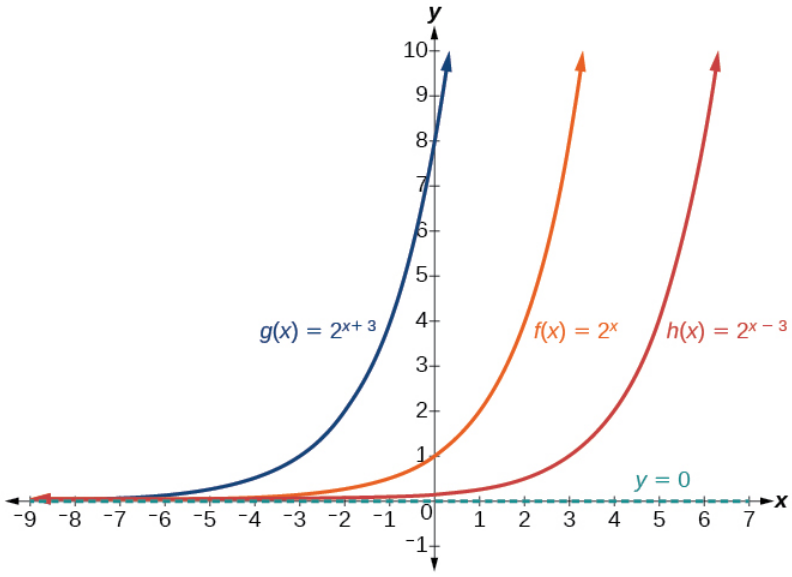



Figure 4.2.6

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$,remains unchanged.
- The asymptote, $y = 0$,remains unchanged.
- The *y*-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$ ,the *y*-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$ ,so the initial value of the function is $8$.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$ ,the *y*-intercept becomes $(0, \frac{1}{8})$. Again, see that $2^{x-3} = (\frac{1}{8})2^x$ ,so the initial value of the function is $\frac{1}{8}$.

**SHIFTS OF THE PARENT FUNCTION $F(X) = b^x$**

For any constants $c$ and $d$,the function $f(x) = b^{x+c} + d$ shifts the parent function $f(x) = b^x$

- vertically $d$ units, in the *same* direction of the sign of $d$.
- horizontally $c$ units, in the *opposite* direction of the sign of $c$.
- The *y*-intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes $(d, \infty)$.

- The domain, $(-\infty, \infty)$,remains unchanged.

**How to: Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation**

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left $c$ units if $c$ is positive, and right $c$ units if $c$ is negative.
3. Shift the graph of $f(x) = b^x$ up $d$ units if $d$ is positive, and down $d$ units if $d$ is negative.
4. State the domain, $(-\infty, \infty)$,the range, $(d, \infty)$,and the horizontal asymptote $y = d$.

**Example 4.2.2: Graphing a Shift of an Exponential Function**

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

**Solution**

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

Draw the horizontal asymptote $y = d$, so draw $y = -3$.

Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.

Shift the graph of $f(x) = b^x$ left 1 units and down 3 units.

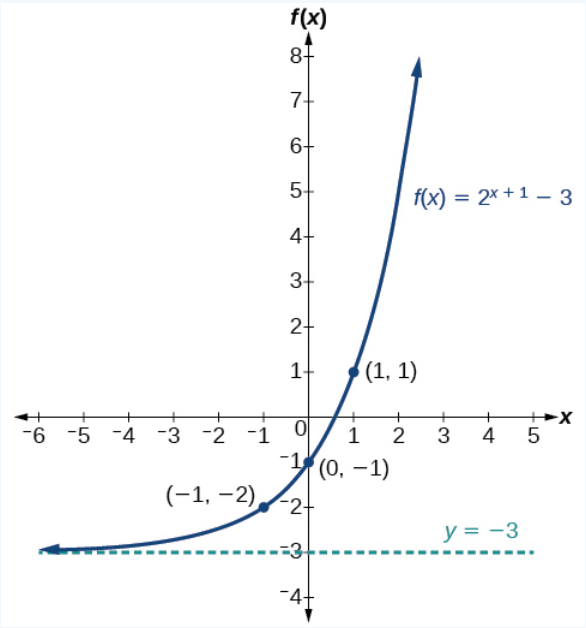



Figure 4.2.7

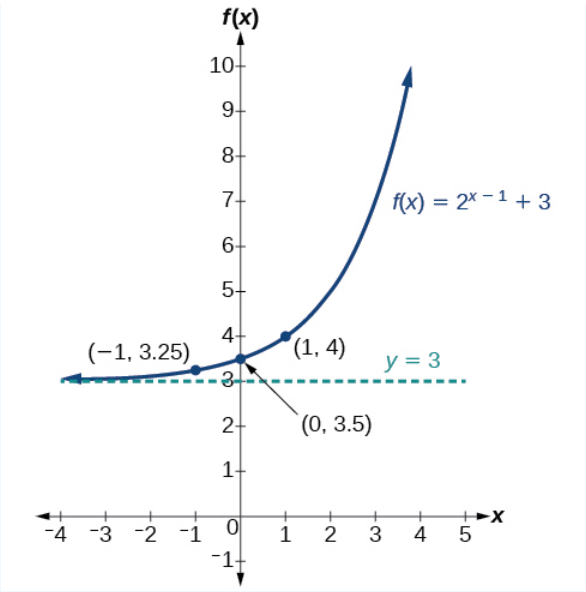
The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

**Exercise 4.2.2**

Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

**Answer**

The domain is $(-\infty, \infty)$ ; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.




**How to: Given an equation of the form $f(x) = b^{x+c} + d$ for $x$, use a graphing calculator to approximate the solution**

- Press **[Y=]**. Enter the given exponential equation in the line headed "$\mathbf{Y_1}$=".
- Enter the given value forf(x) f(x) in the line headed "$\mathbf{Y_2}$=".
- Press **[WINDOW]**. Adjust the *y*-axis so that it includes the value entered for "$\mathbf{Y_2}$=".
- Press **[GRAPH]** to observe the graph of the exponential function along with the line for the specified value off(x). f(x).
- To find the value ofx, x,we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of *x* for the indicated value of the function.

**Example 4.2.3: Approximating the Solution of an Exponential Equation**

Solve $42 = 1.2(5)^x + 2.8$ graphically. Round to the nearest thousandth.

**Solution**

Press **[Y=]** and enter $1.2(5)^x + 2.8$ next to $\mathbf{Y_1}$=. Then enter $42$ next to **Y2=**. For a window, use the values $-3$ to $3$ for $x$ and $-5$ to $55$ for $y$. Press **[GRAPH]**. The graphs should intersect somewhere near $x = 2$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as $2.1661943$ (Your answer may be different if you use a different window or use a different value for **Guess?**) To the nearest thousandth, $x \approx 2.166$.

**Exercise 4.2.3**

Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

**Answer**

$x \approx -1.608$

## Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a **stretch** or **compression** occurs when we multiply the parent function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the parent function $f(x) = 2^x$,we can then graph the stretch, using $a = 3$,to get $g(x) = 3(2)^x$ as shown on the left in Figure 4.2.8, and the compression, using $a = \frac{1}{3}$,to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure 4.2.8.

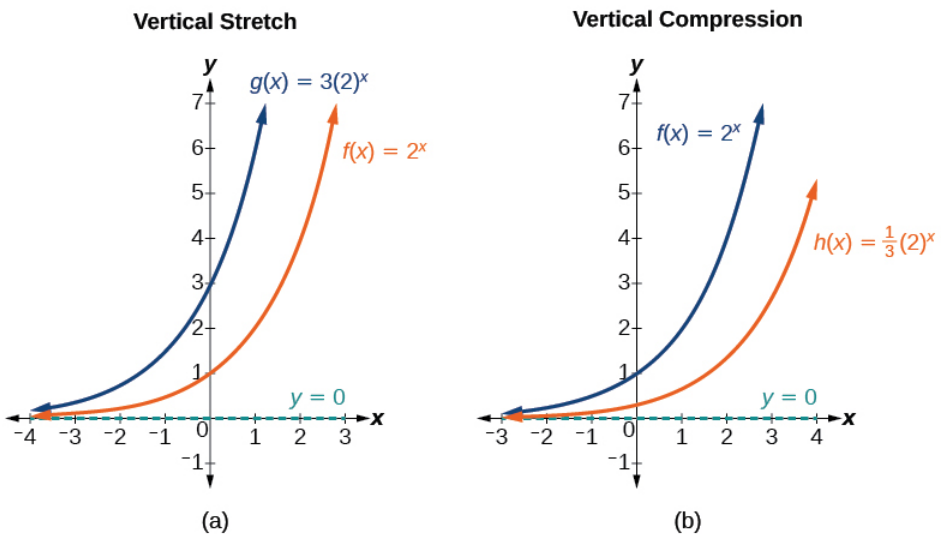



Figure 4.2.8: (a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

### STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $F(X) = B^X$

For any factor $a > 0$,the function $f(x) = a(b)^x$

- is stretched vertically by a factor of $a$ if $|a| > 1$.
- is compressed vertically by a factor of $a$ if $|a| < 1$.
- has a $y$-intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$,which are unchanged from the parent function.

### Example 4.2.4: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4(\frac{1}{2})^x$. State the domain, range, and asymptote.

**Solution**

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as $x$ decreases, and the right tail will approach the $x$-axis as $x$ increases.
- Since $a = 4$,the graph of $f(x) = (\frac{1}{2})^x$ will be stretched by a factor of 4.
- Create a table of points as shown in Table 4.2.4.

Table 4.2.4

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 4(\frac{1}{2})^x$ | 32 | 16 | 8 | 4 | 2 | 1 | 0.5 |

- Plot the $y$-intercept, $(0, 4)$,along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in Figure 4.2.9.

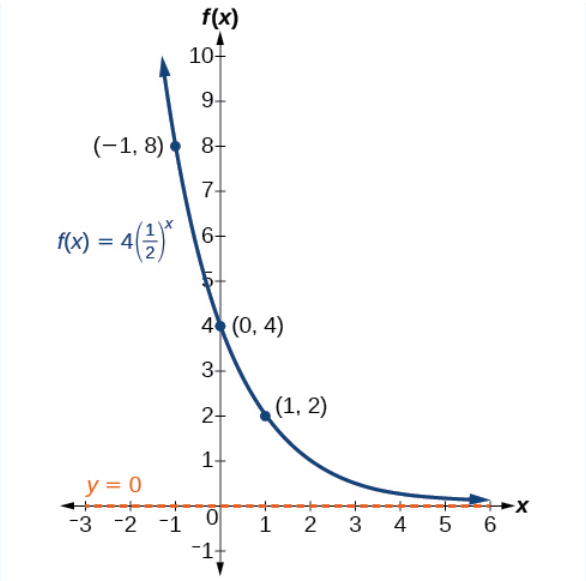



Figure 4.2.9

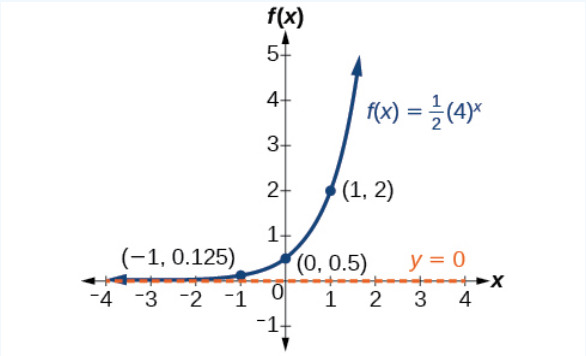
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

**? Exercise 4.2.4**

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

**Answer**

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.




## Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the *x*-axis or the *y*-axis. When we multiply the parent function $f(x) = b^x$ by $-1$,we get a reflection about the *x*-axis. When we multiply the input by $-1$,we get a reflection about the *y*-axis. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the *x*-axis, $g(x) = -2^x$ ,is shown on the left side of Figure 4.2.10, and the reflection about the y-axis $h(x) = 2^{-x}$ , is shown on the right side of Figure 4.2.10.

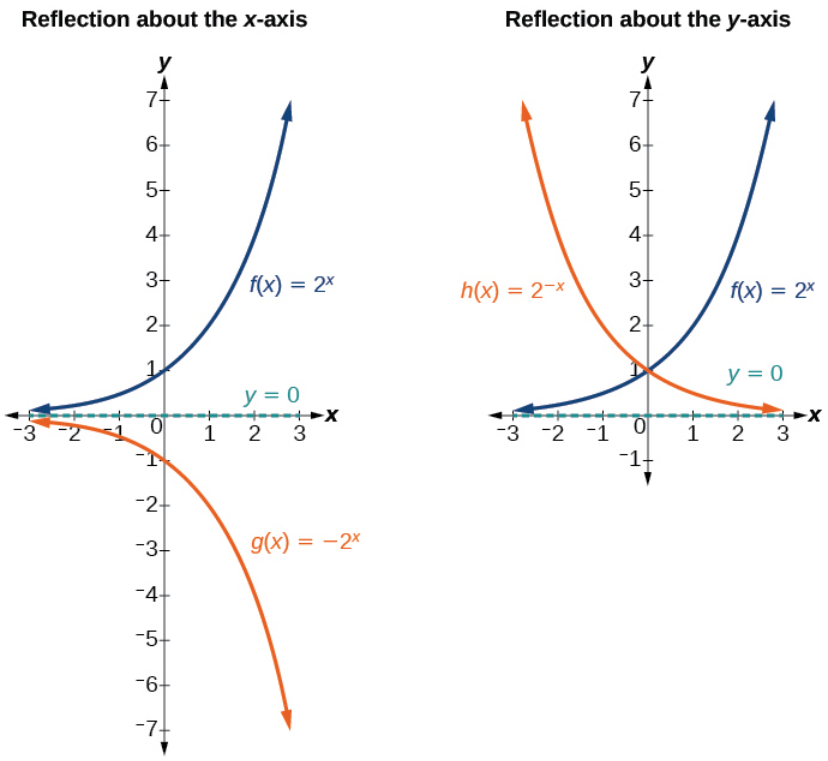



Figure 4.2.10: (a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.

### REFLECTIONS OF THE PARENT FUNCTION $F(X) = B^x$

The function $f(x) = -b^x$

- reflects the parent function $f(x) = b^x$ about the *x*-axis.
- has a *y*-intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$,which are unchanged from the parent function.

The function $f(x) = b^{-x}$

- reflects the parent function $f(x) = b^x$ about the *y*-axis.
- has a *y*-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

### Example 4.2.5: Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$,that reflects $f(x) = (\frac{1}{4})^x$ about the *x*-axis. State its domain, range, and asymptote.

**Solution**

Since we want to reflect the parent function $f(x) = (\frac{1}{4})^x$ about the *x*-axis, we multiply $f(x)$ by $-1$ to get, $g(x) = -(\frac{1}{4})^x$. Next we create a table of points as in Table 4.2.5.

Table 4.2.5

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $g(x) = -(\frac{1}{4})^x$ | $-64$ | $-16$ | $-4$ | $-1$ | $-0.25$ | $\mathbf{-0.0625}$ | $\mathbf{-0.0156}$ |

Plot the $y$-intercept, $(0, -1)$,along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

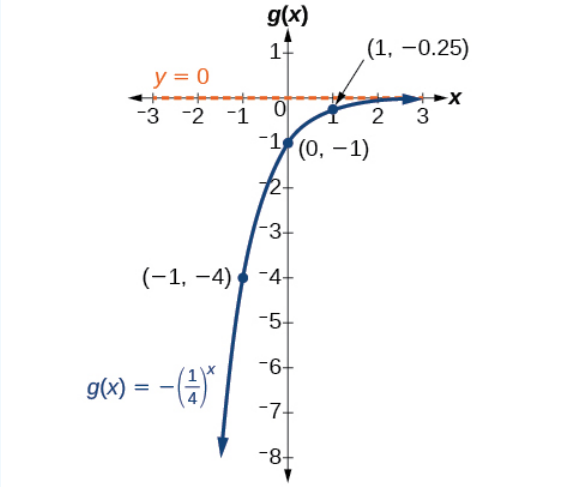



Figure 4.2.11

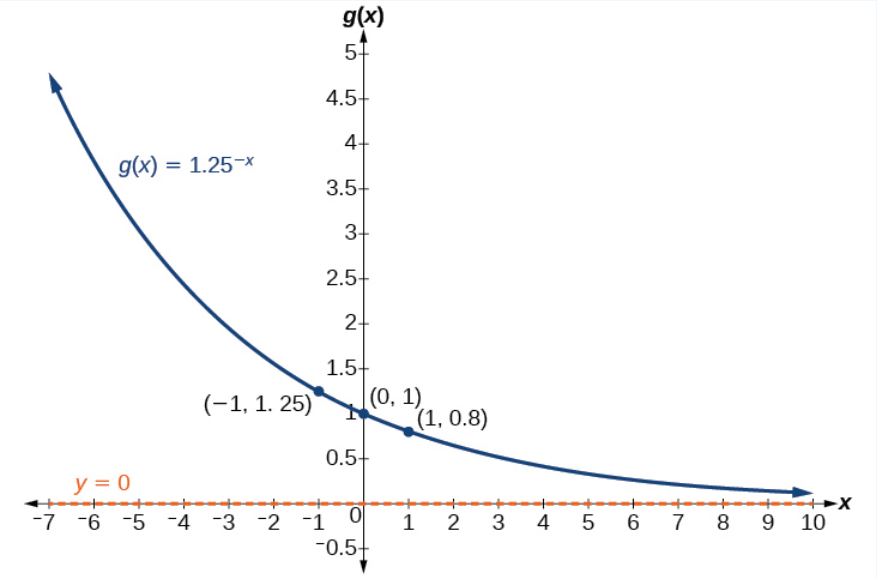
The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

## Exercise 4.2.5

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the $y$-axis. State its domain, range, and asymptote.

**Answer**

The domain is $(-\infty, \infty)$ ; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.




### Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in **Table 4.2.6** to arrive at the general equation for translating exponential functions.

1, and notes the following changes: the reflected function is decreasing as x moves from 0 to infinity, the asymptote remains x=0, the x-intercept remains (1, 0), the key point changes to (b^(-1), 1), the domain remains (0, infinity), and the range remains (-infinity, infinity). The second column shows the left shift of the equation g(x)=log_b(x) when b>1, and notes the following

changes: the reflected function is decreasing as x moves from 0 to infinity, the asymptote remains x=0, the x-intercept changes to (-1, 0), the key point changes to (-b, 1), the domain changes to (-infinity, 0), and the range remains (-infinity, infinity)."> 

Table 4.2.6: Translations of the Parent Function $f(x) = b^x$

| Translation | Form |
|---|---|
| Shift<br>• Horizontally $c$ units to the left<br>• Vertically $d$ units up | $f(x) = b^{x+c} + d$ |
| Stretch and Compress<br>• Stretch if $\|a\| > 1$<br>• Compression if $0 < \|a\| < 1$ | $f(x) = ab^x$ |
| Reflect about the x-axis | $f(x) = -b^x$ |
| Reflect about the y-axis | $f(x) = b^{-x} = (\frac{1}{b})^x$ |
| General equation for all translations | $f(x) = ab^{x+c} + d$ |

## TRANSLATIONS OF EXPONENTIAL FUNCTIONS

A translation of an exponential function has the form

$$f(x) = ab^{x+c} + d$$

Where the parent function, $y = b^x$, $b > 1$,is

- shifted horizontally $c$ units to the left.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- shifted vertically $d$ units.
- reflected about the $x$-axis when $a < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

## Example 4.2.6: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is vertically stretched by a factor of $2$, reflected across the $y$-axis, and then shifted up $4$ units.

### Solution

We want to find an equation of the general form f(x)=abx+c+d. f(x)=abx+c+d. We use the description provided to finda, a, b,b, c,c, and d. d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of $2$, so $a = 2$.
- The function is reflected about the $y$-axis. We replace $x$ with $-x$ to get: $e^{-x}$.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$\begin{aligned} f(x) &= ab^{x+c} + d \\ &= 2e^{-x+0} + 4 \\ &= 2e^{-x} + 4 \end{aligned}$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

**Exercise 4.2.6**

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the *x*-axis and then shifted down 2 units.

**Answer**

$f(x) = -\frac{1}{3}e^x - 2$ ; the domain is $(-\infty, \infty)$ ; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

**Media**

Access this online resource for additional instruction and practice with graphing exponential functions.

- Graph Exponential Functions

## Key Equations

| | |
|---|---|
| General Form for the Translation of the Parent Function $f(x) = b^x$ | $f(x) = ab^{x+c} + d$ |

## Key Concepts

- The graph of the function $f(x) = b^x$ has a *y*-intercept at $(0, 1)$,domain $(-\infty, \infty)$,range $(0, \infty)$, and horizontal asymptote $y = 0$. See Example.
- If $b > 1$ ,the function is increasing. The left tail of the graph will approach the asymptote $y = 0$, and the right tail will increase without bound.
- If $0 < b < 1$ , the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- The equation $f(x) = b^x + d$ represents a vertical shift of the parent function $f(x) = b^x$.
- The equation $f(x) = b^{x+c}$ represents a horizontal shift of the parent function $f(x) = b^x$. See Example.
- Approximate solutions of the equation $f(x) = b^{x+c} + d$ can be found using a graphing calculator. See Example.
- The equation $f(x) = ab^x$ , where $a > 0$ , represents a vertical stretch if $|a| > 1$ or compression if $0 < |a| < 1$ of the parent function $f(x) = b^x$. See Example.
- When the parent function $f(x) = b^x$ is multiplied by $-1$,the result, $f(x) = -b^x$ , is a reflection about the *x*-axis. When the input is multiplied by $-1$,the result, $f(x) = b^{-x}$ , is a reflection about the *y*-axis. See Example.
- All translations of the exponential function can be summarized by the general equation $f(x) = ab^{x+c} + d$ . See Table.
- Using the general equation $f(x) = ab^{x+c} + d$ , we can write the equation of a function given its description. See Example.

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# 4.3: Logarithmic Functions

**Learning Objectives**

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure 4.3.1. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.



Figure 4.3.1: Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce).

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is

$$10^{8-4} = 10^4 = 10,000$$

times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

## Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where $x$ represents the difference in magnitudes on the Richter Scale. How would we solve for $x$?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that $x$ must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure 4.3.1, to better estimate the solution.

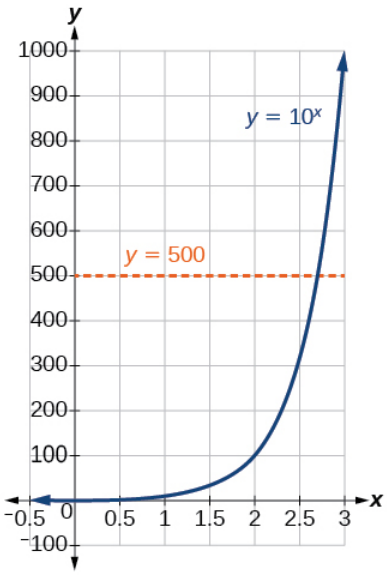



Figure 4.3.2

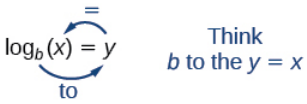
Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure $4.3.2$ passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange $x$ and $y$ and solve for $y$ to find the inverse function. To represent $y$ as a function of $x$, we use a logarithmic function of the form $y = \log_b(x)$. The base $b$ **logarithm** of a number is the exponent by which we must raise $b$ to get that number.

We read a logarithmic expression as, "The logarithm with base $b$ of $x$ is equal to $y$," or, simplified, "log base $b$ of $x$ is $y$." We can also say, "$b$ raised to the power of $y$ is $x$," because logs are exponents. For example, the base $2$ logarithm of $32$ is $5$, because $5$ is the exponent we must apply to $2$ to get $32$. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base $2$ of $32$ is $5$."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

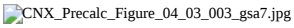
$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1 \tag{4.3.1}$$

Note that the base $b$ is always positive.




Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the $x$.

We can illustrate the notation of logarithms as follows:




Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log^b(x)$ and $y = b^x$ are inverse functions.

### DEFINITION OF THE LOGARITHMIC FUNCTION

A logarithm base $b$ of a positive number $x$ satisfies the following definition.

For $x > 0, b > 0, b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x \tag{4.3.2}$$

where,

- we read $\log_b(x)$ as, "the logarithm with base $b$ of $x$" or the "log base $b$ of $x$."

- the logarithm $y$ is the exponent to which $b$ must be raised to get $x$.

Also, since the logarithmic and exponential functions switch the $x$ and $y$ values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base $b$ is $(0, \infty)$.
- the range of the logarithm function with base $b$ is $(-\infty, \infty)$.

### Q&A: Can we take the logarithm of a negative number?

*No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.*

### How to: Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form

1. Examine the equation $y = \log_b x$ and identify $b$, $y$,and $x$.
2. Rewrite $\log_b x = y$ as $b^y = x$.

### Example 4.3.1: Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$
b. $\log_3(9) = 2$

**Solution**

First, identify the values of $b$, $y$,and $x$. Then, write the equation in the form $b^y = x$.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$

Here, $b = 6$, $y = \frac{1}{2}$,and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to

$$6^{\frac{1}{2}} = \sqrt{6}$$

b. $\log_3(9) = 2$

Here, $b = 3$, $y = 2$,and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to

$$3^2 = 9$$

### Exercise 4.3.1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

**Answer a**

$\log_{10}(1,000,000) = 6$is equivalent to $10^6 = 1,000,000$

**Answer b**

$\log_5(25) = 2$ is equivalent to $5^2 = 25$

## Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base $b$,exponent $x$,and output $y$. Then we write $x = \log_b(y)$.

### Example 4.3.2: Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \dfrac{1}{10,000}$

**Solution**

First, identify the values of $b$, $y$,and $x$. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$

Here, $b = 2$, $x = 3$,and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$

Here, $b = 5$, $x = 2$,and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \dfrac{1}{10,000}$

Here, $b = 10$, $x = -4$,and $y = \dfrac{1}{10,000}$. Therefore, the equation $10^{-4} = \dfrac{1}{10,000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10,000}\right) = -4$.

### Exercise 4.3.2

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \dfrac{1}{2}$

**Answer a**

$3^2 = 9$ is equivalent to $\log_3(9) = 2$

**Answer b**

$5^3 = 125$ is equivalent to $\log_5(125) = 3$

**Answer c**

$2^{-1} = \dfrac{1}{2}$ is equivalent to $\log_2\left(\dfrac{1}{2}\right) = -1$

## Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must $2$ be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must $7$ be raised in order to get $49$?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, "To what exponent must $3$ be raised in order to get $27$?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$?" We know $2^2 = 4$ and $3^2 = 9$, so

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Therefore,

$$\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2.$$

**How to: Given a logarithm of the form $y = \log_b(x)$, evaluate it mentally**

1. Rewrite the argument $x$ as a power of $b$: $b^y = x$.
2. Use previous knowledge of powers of $b$ identify $y$ by asking, "To what exponent should $b$ be raised in order to get $x$?"

**Example 4.3.3: Solving Logarithms Mentally**

Solve $y = \log_4(64)$ without using a calculator.

**Solution**

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know

$$4^3 = 64$$

Therefore,

$$\log_4(64) = 3$$

**Exercise 4.3.3**

Solve $y = \log_{121}(11)$ without using a calculator.

**Answer**

$\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{\frac{1}{2}} = 11$)

**Example 4.3.4: Evaluating the Logarithm of a Reciprocal**

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

**Solution**

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$, but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$3^{-3} = \frac{1}{3^3}$$
$$= \frac{1}{27}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

**? Exercise 4.3.4**

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

**Answer**

$$\log_2\left(\frac{1}{32}\right) = -5$$

## Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base $-10$ logarithm a *common logarithm*. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

**DEFINITION OF THE COMMON LOGARITHM**

A common logarithm is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number $x$ satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x \tag{4.3.3}$$

We read $\log(x)$ as, "the logarithm with base 10 of $x$ " or "log base 10 of $x$."

The logarithm $y$ is the exponent to which 10 must be raised to get $x$.

**How to: Given a common logarithm of the form $y = \log(x)$, evaluate it mentally**

1. Rewrite the argument $x$ as a power of 10: $10^y = x$.
2. Use previous knowledge of powers of 10 to identify $y$ by asking, "To what exponent must 10 be raised in order to get $x$?"

**✓ Example 4.3.5: Finding the Value of a Common Logarithm Mentally**

Evaluate $y = \log(1000)$ without using a calculator.

**Solution**

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

## Exercise 4.3.5

Evaluate $y = \log(1,000,000)$

**Answer**

$\log(1,000,000) = 6$

### How to: Given a common logarithm with the form $y = \log(x)$,evaluate it using a calculator

1. Press **[LOG]**.
2. Enter the value given for $x$,followed by **[ ) ]**.
3. Press **[ENTER]**.

## Example 4.3.6: Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$to four decimal places using a calculator.

**Solution**

- Press **[LOG]**.
- Enter 321, followed by **[ ) ]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$

**Analysis**

Note that $10^2 = 100$ and that $10^3 = 1000$. Since $321$ is between $100$ and $1000$, we know that $\log(321)$must be between $\log(100)$and $\log(1000)$. This gives us the following:

$$100 < 321 < 1000$$

$$2 < 2.5065 < 3$$

## Exercise 4.3.6

Evaluate $y = \log(123)$to four decimal places using a calculator.

**Answer**

$\log(123) \approx 2.0899$

## Example 4.3.7: Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was $500$ times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where $x$ is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

**Solution**

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$

$$\log(500) = x \text{ Use the definition of the common log.}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter $500$,followed by **[ ) ]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$

The difference in magnitudes was about $2.699$.

**Exercise 4.3.7**

The amount of energy released from one earthquake was $8,500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where $x$ is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

**Answer**

The difference in magnitudes was about $3.929$

## Using Natural Logarithms

The most frequently used base for logarithms is $e$. Base $e$ logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base $e$ logarithm, $\log_e(x)$, has its own notation, $\ln(x)$. Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the ln key that can be found on most scientific calculators. We can also find the natural logarithm of any power of $e$ using the inverse property of logarithms.

**DEFINITION OF THE NATURAL LOGARITHM**

A natural logarithm is a logarithm with base $e$. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number $x$ satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base $e$ of $x$" or "the natural logarithm of $x$."

The logarithm $y$ is the exponent to which $e$ must be raised to get $x$.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all $x$ and $e^{\ln(x)} = x$ for $x > 0$.

**How to: Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator**

1. Press **[LN]**.
2. Enter the value given for $x$, followed by **[ ) ]**.
3. Press **[ENTER]**.

**Example 4.3.8: Evaluating a Natural Logarithm Using a Calculator**

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

**Solution**

- Press **[LN]**.
- Enter $500$, followed by **[ ) ]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

**Exercise 4.3.8**

Evaluate $\ln(-500)$.

**Answer**

It is not possible to take the logarithm of a negative number in the set of real numbers.

Media

Access this online resource for additional instruction and practice with logarithms.

- Introduction to Logarithms

## Key Equations

| | |
|---|---|
| Definition of the logarithmic function | For $x > 0$, $b > 0$, $b \neq 1$, $y = \log_b(x)$ if and only if $b^y = x$. |
| Definition of the common logarithm | For $x > 0$, $y = \log(x)$ if and only if $10^y = x$. |
| Definition of the natural logarithm | For $x > 0$, $y = \ln(x)$ if and only if $e^y = x$. |

## Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See Example 4.3.1.
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See Example 4.3.2.
- Logarithmic functions with base $b$ can be evaluated mentally using previous knowledge of powers of $b$. See Example 4.3.3 and Example 4.3.4.
- Common logarithms can be evaluated mentally using previous knowledge of powers of 10. See Example 4.3.5.
- When common logarithms cannot be evaluated mentally, a calculator can be used. See Example 4.3.6.
- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator. See Example 4.3.7.
- Natural logarithms can be evaluated using a calculator Example 4.3.8.

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# 4.4: Graphs of Logarithmic Functions

**Learning Objectives**

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

In the Section on Graphs of Exponential Functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year $t$ can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure $4.4.1$ shows this point on the logarithmic graph.

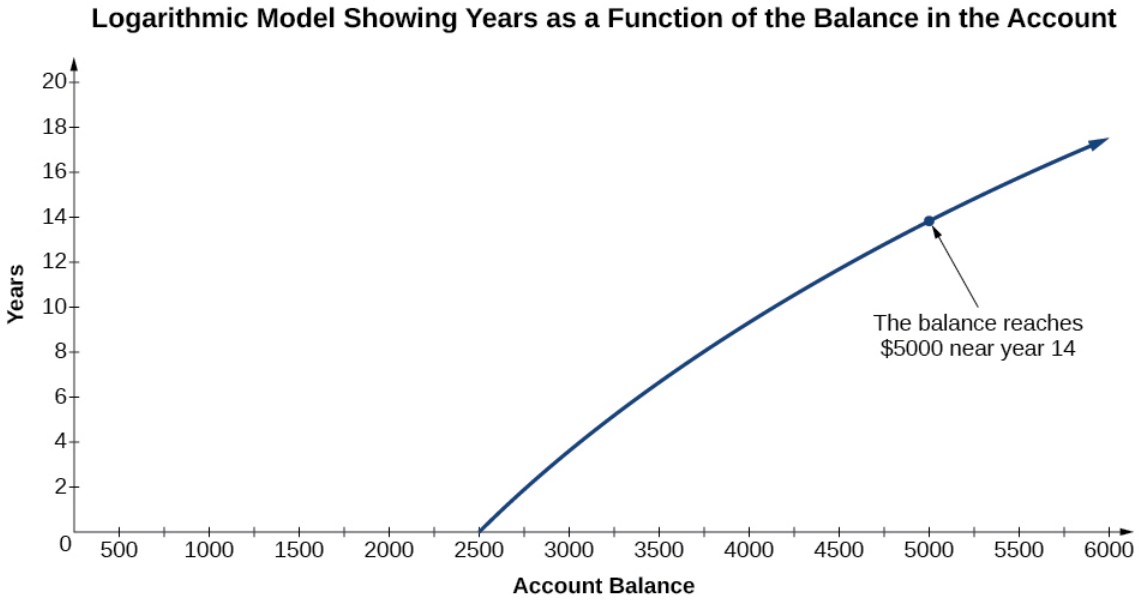



Figure 4.4.1

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

## Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number $x$ and constant $b > 0$, $b \neq 1$, where

- The domain of $y$ is $(-\infty, \infty)$.
- The range of $y$ is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of $x$ such that the argument, in this case $2x - 3$,is greater than zero. To find the domain, we set up an inequality and solve for $x$:

$$\begin{aligned} 2x - 3 &> 0 && \text{Show the argument greater than zero} \\ 2x &> 3 && \text{Add 3} \\ x &> 1.5 && \text{Divide by 2} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

**Given a logarithmic function, identify the domain**

1. Set up an inequality showing the argument greater than zero.
2. Solve for $x$.
3. Write the domain in interval notation.

**Example 4.4.1: Identifying the Domain of a Logarithmic Shift**

What is the domain of $f(x) = \log_2(x + 3)$?

**Solution**

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$\begin{aligned} x + 3 &> 0 && \text{The input must be positive} \\ x &> -3 && \text{Subtract 3} \end{aligned}$$

The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

**Exercise 4.4.1**

What is the domain of $f(x) = \log_5(x - 2) + 1$ ?

**Answer**

$(2, \infty)$

**Example 4.4.2: Identifying the Domain of a Logarithmic Shift and Reflection**

What is the domain of $f(x) = \log(5 - 2x)$?

**Solution**

The logarithmic function is defined only when the input is positive, so this function is defined when $5 - 2x > 0$. Solving this inequality,

$$\begin{aligned} 5 - 2x &> 0 && \text{The input must be positive} \\ -2x &> -5 && \text{Subtract 5} \\ x &< \frac{5}{2} && \text{Divide by -2 and switch the inequality} \end{aligned}$$

The domain of $f(x) = \log(5 - 2x)$ is $\left(-\infty, \frac{5}{2}\right)$.

**? Exercise 4.4.2**

What is the domain of $f(x) = \log(x - 5) + 2$ ?

**Answer**

$(5, \infty)$

## Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table 4.4.1.

Table 4.4.1

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $2^x = y$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ | $8$ |
| $\log_2(y) = x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |

Using the inputs and outputs from Table 4.4.1, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table 4.4.2.

Table 4.4.2

| $f(x) = 2^x$ | $\left(-3, \frac{1}{8}\right)$ | $\left(-2, \frac{1}{4}\right)$ | $\left(-1, \frac{1}{2}\right)$ | $(0, 1)$ | $(1, 2)$ | $(2, 4)$ | $(3, 8)$ |
|---|---|---|---|---|---|---|---|
| $g(x) = \log_2(x)$ | $\left(\frac{1}{8}, -3\right)$ | $\left(\frac{1}{4}, -2\right)$ | $\left(\frac{1}{2}, -1\right)$ | $(1, 0)$ | $(2, 1)$ | $(4, 2)$ | $(8, 3)$ |

As we'd expect, the $x$- and $y$-coordinates are reversed for the inverse functions. Figure 4.4.2 shows the graph of $f$ and $g$.

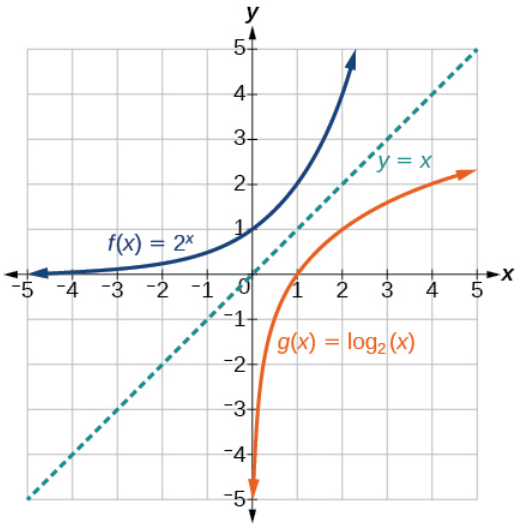



Figure 4.4.2: Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

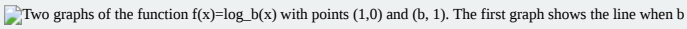
- $f(x) = 2^x$ has a $y$-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an $x$- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

## CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION, $F(X) = LOG_B(X)$

For any real number $x$ and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- $x$-intercept: $(1, 0)$ and key point $(b, 1)$
- $y$-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

See Figure 4.4.3.




1, and the second graph shows the line when 0

**<1.">**

**Figure 4.4.3**

Figure 4.4.4 shows how changing the base $b$ in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (Note: recall that the function $\ln(x)$ has base $e \approx 2.718$.)

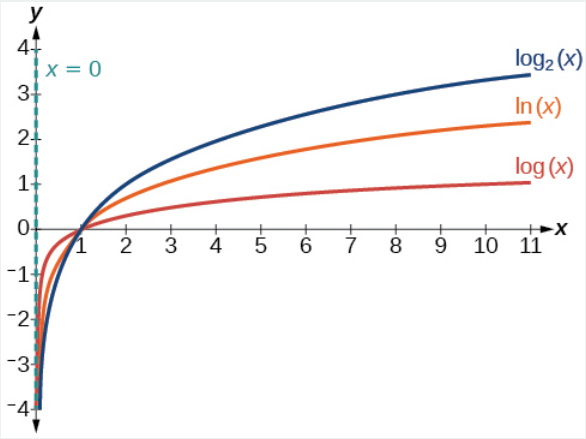



Figure 4.4.4: The graphs of three logarithmic functions with different bases, all greater than 1.

## Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the $x$-intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$,the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

## Example 4.4.3: Graphing a Logarithmic Function with the Form $f(x) = log_b(x)$

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

### Solution

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The $x$-intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure 4.4.5).

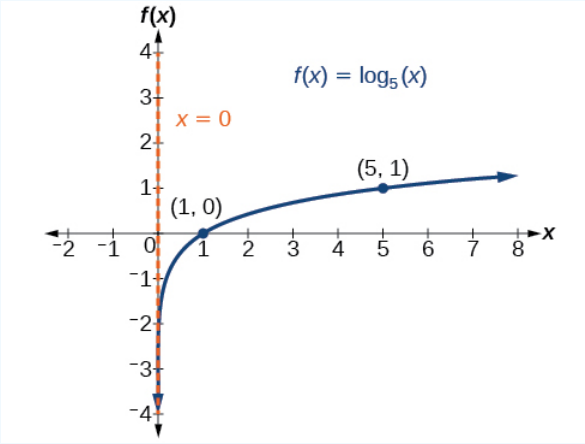



Figure 4.4.5

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

**? Exercise 4.4.3**

Graph $f(x) = \log_{\frac{1}{5}}(x)$. State the domain, range, and asymptote.

**Answer**

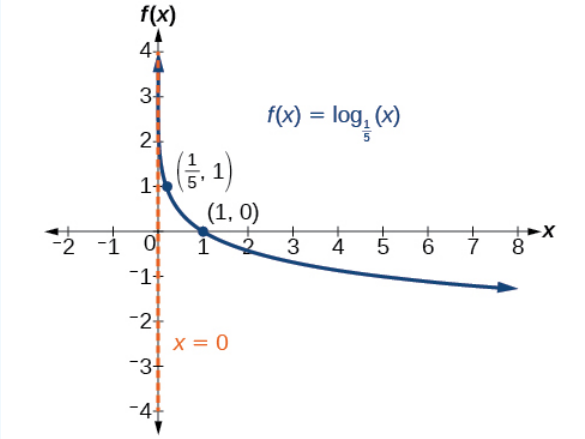



Figure 4.4.6

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

## Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

### Graphing a Horizontal Shift of $f(x) = \log_b(x)$

When a constant $c$ is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift $c$ units in the *opposite* direction of the sign on $c$. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 4.4.7.

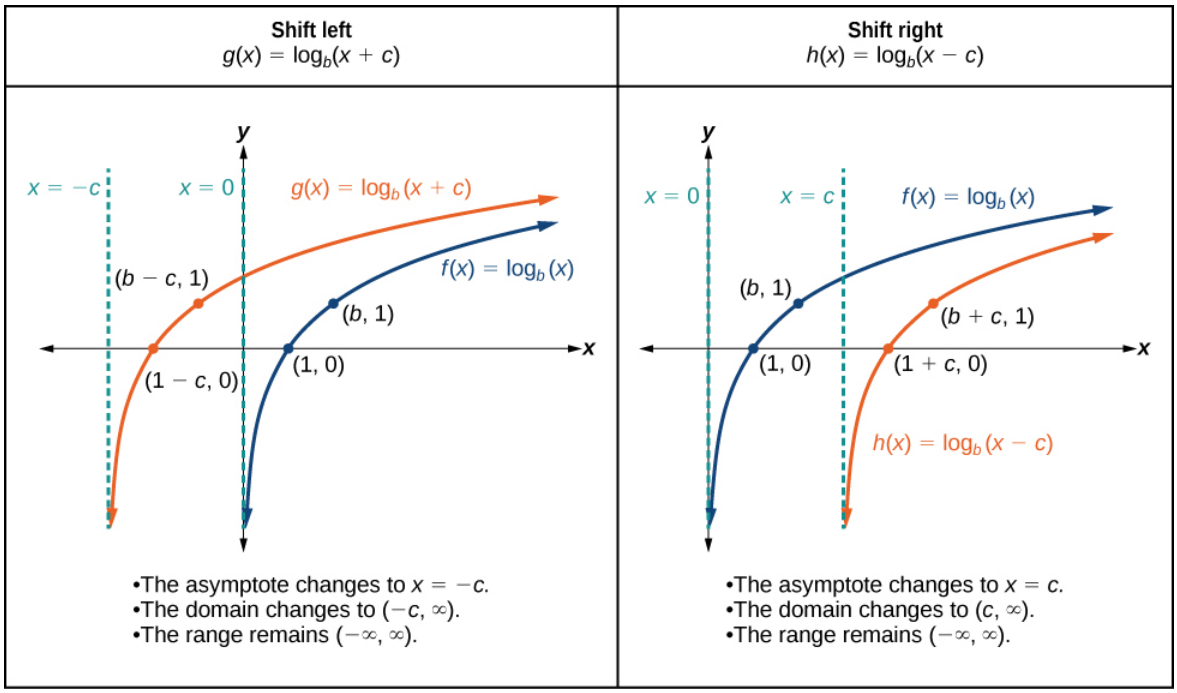



Figure 4.4.7

### HORIZONTAL SHIFTS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant $c$,the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left $c$ units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right $c$ units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

### Given a logarithmic function with the form $f(x) = \log_b(x + c)$, graph the translation.

1. Identify the horizontal shift:
   - If $c > 0$,shift the graph of $f(x) = \log_b(x)$ left $c$ units.
   - If $c < 0$,shift the graph of $f(x) = \log_b(x)$ right $c$ units.
2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting $c$ from the $x$ coordinate.
4. Label the three points.
5. The Domain is $(-c, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

### Example 4.4.4: Graphing a Horizontal Shift of the Parent Function $y = log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x - 2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

**Solution**

Since the function is $f(x) = \log_3(x - 2)$, we notice $x + (-2) = x - 2$.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.


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Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$,and $(3, 1)$.

The new coordinates are found by adding $2$ to the $x$ coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$, $(3, 0)$,and $(5, 1)$.

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

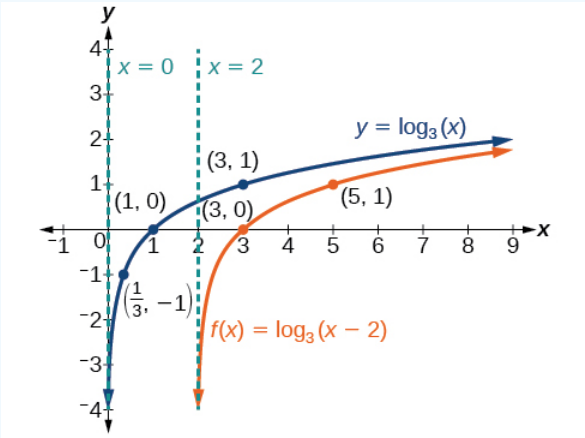



Figure 4.4.8

## Exercise 4.4.4

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

**Answer**

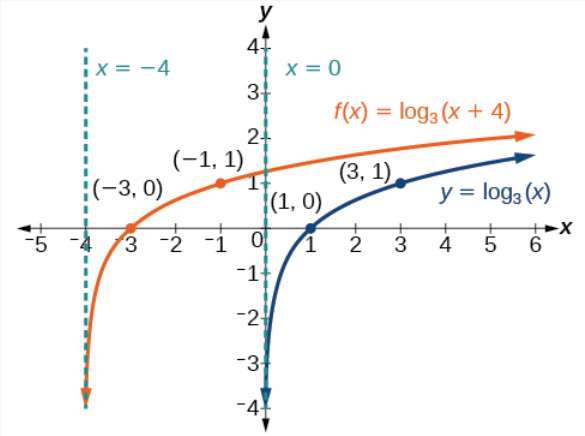



Figure 4.4.9

The domain is $(-4, \infty)$,the range $(-\infty, \infty)$ ,and the asymptote $x = -4$.

### Graphing a Vertical Shift of $y = log_b(x)$

When a constant $d$ is added to the parent function $f(x) = \log_b(x)$,the result is a vertical shift $d$ units in the direction of the sign on $d$. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$ .See Figure 4.4.10

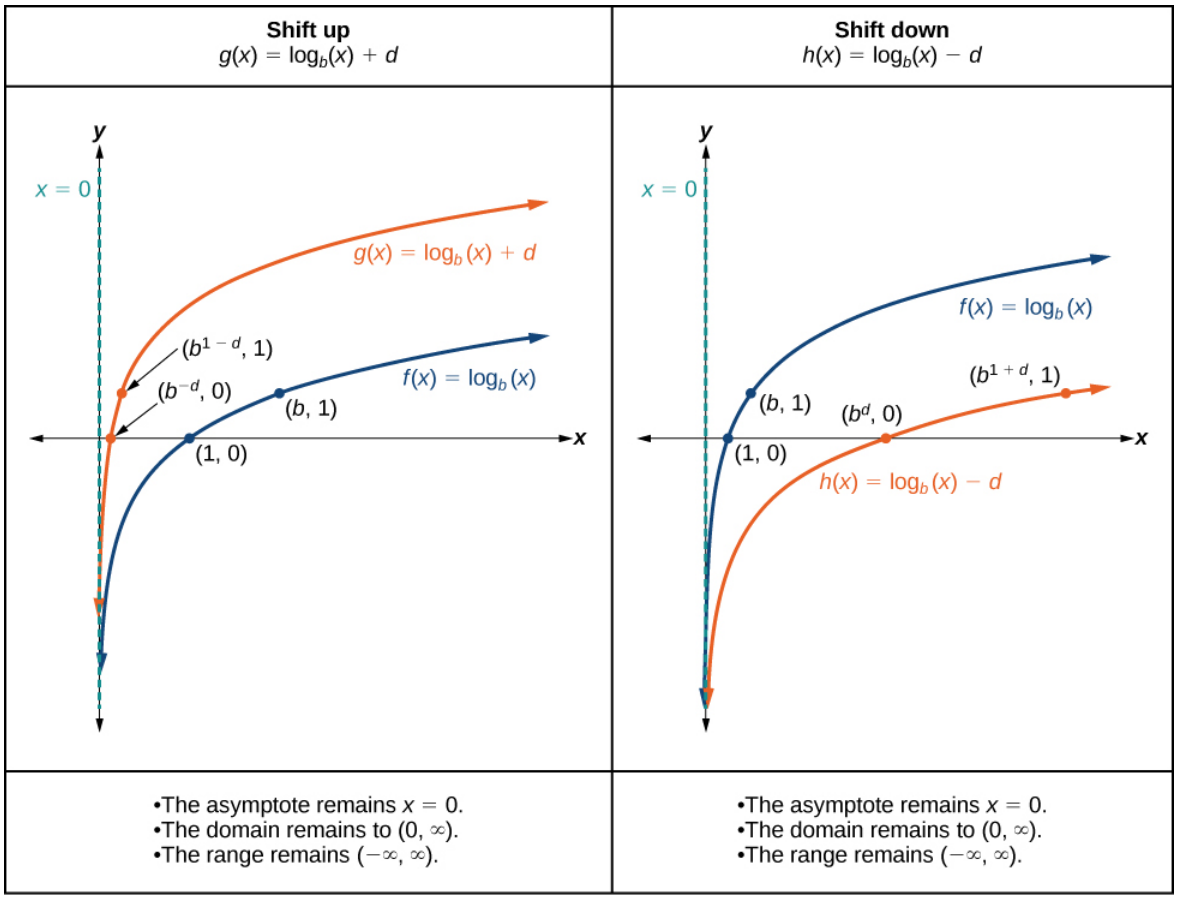



Figure 4.4.10

### VERTICAL SHIFTS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant $d$,the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up $d$ units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down $d$ units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

### Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up $d$ units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$ down $d$ units.
2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding $d$ to the $y$ coordinate.
4. Label the three points.
5. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

### Example 4.4.5: Graphing a Vertical Shift of the Parent Function $y = log_b(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Solution**

Since the function is $f(x) = \log_3(x) - 2$ ,we will notice $d = -2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$,and $(3, 1)$.

The new coordinates are found by subtracting 2 from the *y* coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

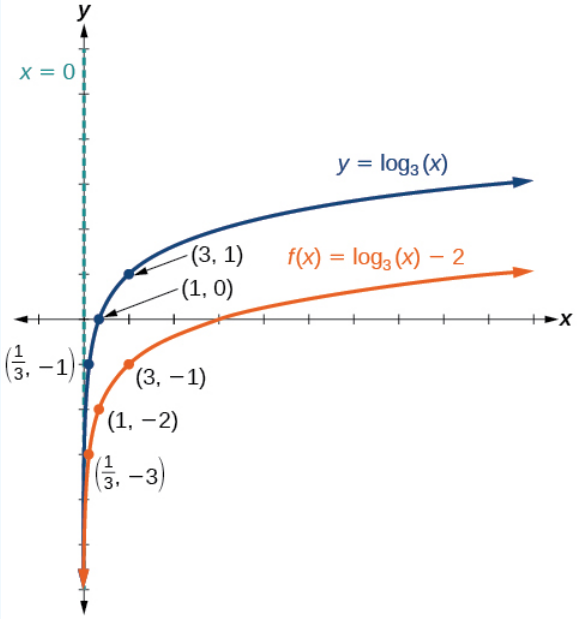



Figure 4.4.11

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

**? Exercise 4.4.5**

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Answer**

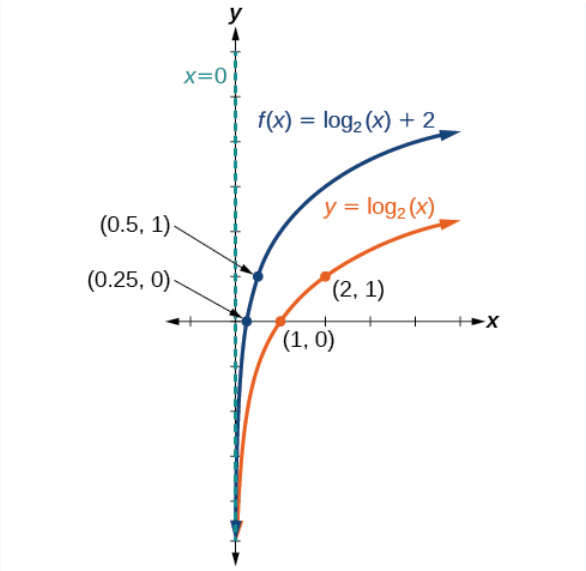



Figure 4.4.12

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$ , and the vertical asymptote is $x = 0$.

## Graphing Stretches and Compressions of $y = log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$.See Figure 4.4.13

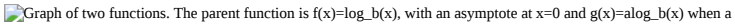

Graph of two functions. The parent function is f(x)=log_b(x), with an asymptote at x=0 and g(x)=alog_b(x) when a 1 is the translation function with an asymptote at x=0. The graph note the intersection of the two lines at (1, 0). This shows the translation of a vertical stretch." src="/@api/deki/files/12175/fig_6.5.13.jpg">

Figure 4.4.13

### VERTICAL STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant $a > 1$ ,the function $f(x) = a\log_b(x)$

- stretches the parent function $y = \log_b(x)$ vertically by a factor of $a$ if $a > 1$.
- compresses the parent function $y = \log_b(x)$ vertically by a factor of $a$ if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the $x$-intercept $(1, 0)$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

### Given a logarithmic function with the form $f(x) = a\log_b(x)$ , $a > 0$ ,graph the translation.

1. Identify the vertical stretch or compressions:
   - If $|a| > 1$, the graph of $f(x) = \log_b(x)$ is stretched by a factor of $a$ units.
   - If $|a| < 1$, the graph of $f(x) = \log_b(x)$ is compressed by a factor of $a$ units.
2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying they y coordinates by $a$.
4. Label the three points.
5. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

## Example 4.4.6: Graphing a Stretch or Compression of the Parent Function $y = log_b(x)$

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

### Solution

Since the function is $f(x) = 2\log_4(x)$,we will notice $a = 2$.

This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{4}, -1\right)$, $(1, 0)$, and $(4, 1)$.

The new coordinates are found by multiplying the $y$ coordinates by 2.

Label the points $\left(\frac{1}{4}, -2\right)$, $(1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$. See Figure 4.4.14

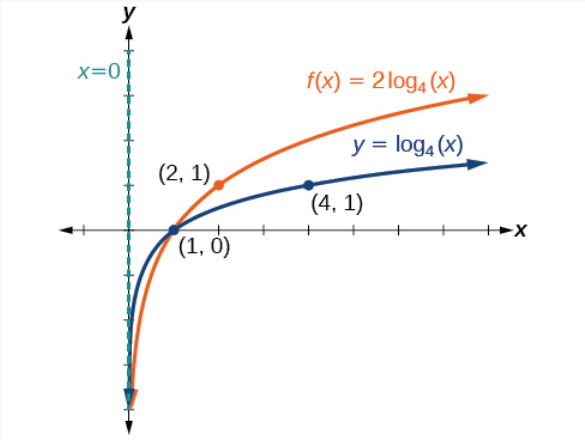



Figure 4.4.14

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

## Exercise 4.4.6

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Answer**


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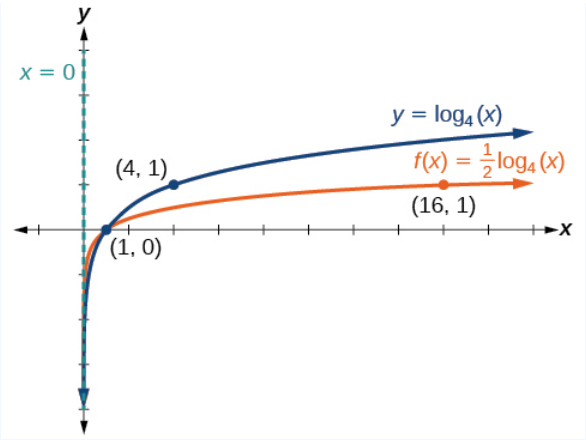



Figure 4.4.15

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$ ,and the vertical asymptote is $x = 0$.

### Example 4.4.7: Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x + 2)$. State the domain, range, and asymptote.

**Solution**

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in Figure 4.4.16. The vertical asymptote will be shifted to $x = -2$. The $x$-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the $x$-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

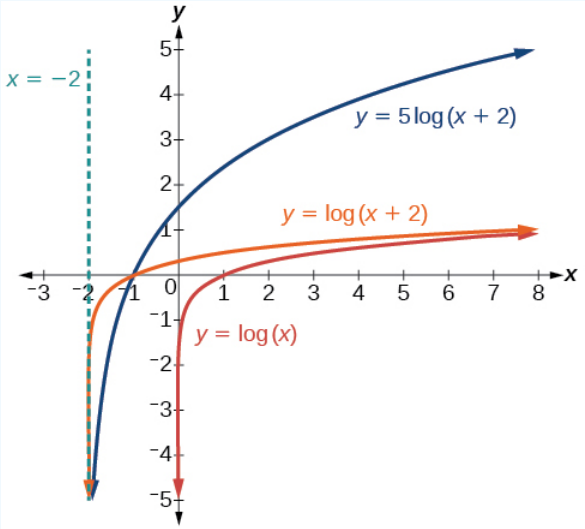



Figure 4.4.16

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = -2$.

### Exercise 4.4.7

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

**Answer**


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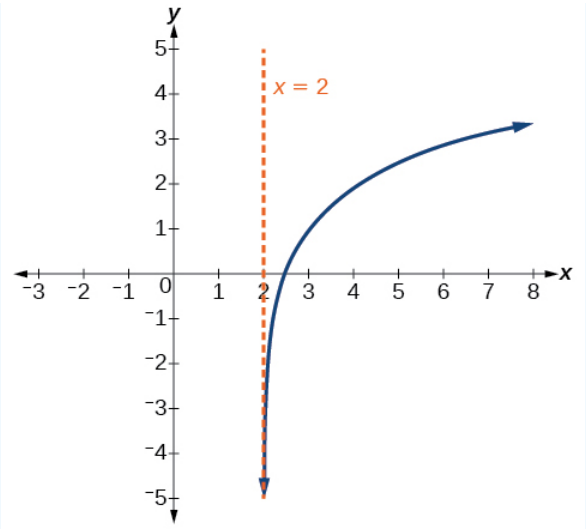



Figure 4.4.17

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$ , and the vertical asymptote is $x = 2$.

## Graphing Reflections of $f(x) = log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by $-1$,the result is a reflection about the $x$-axis. When the *input* is multiplied by $-1$,the result is a reflection about the $y$-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the $x$-axis, $g(x) = -\log_b(x)$ and the reflection about the $y$-axis, $h(x) = \log_b(-x)$.

Graph of two functions. The parent function is f(x)=log_b(x), with an asymptote at x=0 and g(x)=-log_b(x) when b
1 is the translation function with an asymptote at x=0. The graph note the intersection of the two lines at (1, 0). This shows the translation of a reflection about the x-axis." src="/@api/deki/files/12180/fig_6.5.18.jpg">

Figure 4.4.18

### REFLECTIONS OF THE PARENT FUNCTION $Y = LOG_B(X)$

The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the $x$-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the $y$-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

### Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

Table 4.4.3

| If $f(x) = -\log_b(x)$ | If $f(x) = \log_b(-x)$ |
|---|---|
| 1. Draw the vertical asymptote, $x = 0$. | 1. Draw the vertical asymptote, $x = 0$. |
| 2. Plot the *x*-intercept, $(1, 0)$. | 2. Plot the *x*-intercept, $(1, 0)$. |
| 3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *x*-axis. | 3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *y*-axis. |
| 4. Draw a smooth curve through the points. | 4. Draw a smooth curve through the points. |

| If $f(x) = -\log_b(x)$ | If $f(x) = \log_b(-x)$ |
|---|---|
| 5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$. | 5. State the domain, $(-\infty, 0)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$. |

### Example 4.4.8: Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Solution**

Before graphing $f(x) = log(-x)$, $f(x) = log(-x)$,identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by $-1$, $f(x)$ is a reflection of the parent graph about the $y$-axis. Thus, $f(x) = \log(-x)$ will be decreasing as $x$ moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The $x$-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

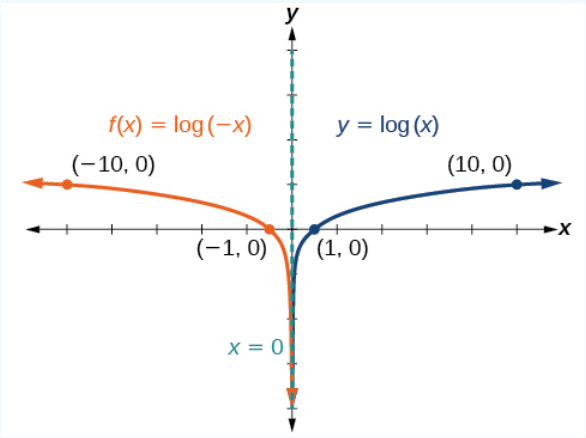



Figure 4.4.19

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

### Exercise 4.4.8

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

**Answer**

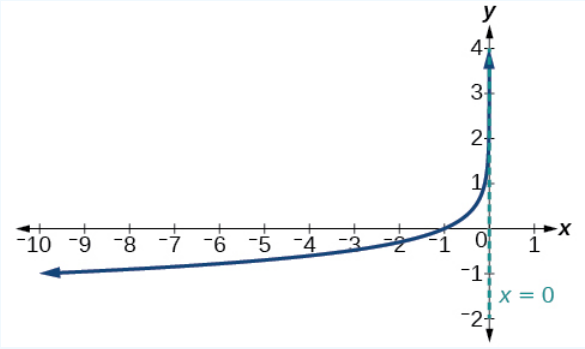



Figure 4.4.20

The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$ ,and the vertical asymptote is $x = 0$.

**Given a logarithmic equation, use a graphing calculator to approximate solutions.**

1. Press **[Y=]**. Enter the given logarithm equation or equations as $\mathbf{Y_1}$= and, if needed, $\mathbf{Y_2}$=.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of $x$, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of $x$,for the point(s) of intersection.

**Example 4.4.9: Approximating the Solution of a Logarithmic Equation**

Solve $4\ln(x)+1=-2\ln(x-1)$ graphically. Round to the nearest thousandth.

**Solution**

Press **[Y=]** and enter $4\ln(x)+1$ next to $\mathbf{Y_1}$=. Then enter $-2\ln(x-1)$ next to $\mathbf{Y_2}$=. For a window, use the values $0$ to $5$ for $x\backslash 0\ and\backslash(-10$ to $10$ for $y$. Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x=1$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as $1.3385297$ (Your answer may be different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x\approx 1.339$.

**Exercise 4.4.9**

Solve $5\log(x+2)=4-\log(x)$ graphically. Round to the nearest thousandth.

**Answer**

$x\approx 3.049$

## Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in Table 4.4.4 to arrive at the general equation for translating exponential functions.

1 and a compression is 0<|a|<1. Note its form is f(x)=alog_b(x). The third translation is a reflection about the x-axis with the form f(x) = -log_b(x). The fourth translation is a reflection about the y-axis with the form f(x)=log_b(-x). The general equation for all translations is f(x)=alog_b(x+c)+d.">

***Table* 4.4.4**

| Translations of the Parent Function $y=\log_b(x)$ | |
|---|---|
| Translation | Form |
| Shift<br>Horizontally $c$ units to the left<br>Vertically $d$ units up | $y=\log_b(x+c)+d$ |
| Stretch and Compress<br>Stretch if $\lvert a\rvert>1$<br>Compression if $\lvert a\rvert<1$ | $y=a\log_b(x)$ |
| Reflect about the x-axis | $y=-\log_b(x)$ |
| Reflect about the y-axis | $y=\log_b(-x)$ |
| General equation for all translations | $y=a\log_b(x+c)+d$ |

## TRANSLATIONS OF LOGARITHMIC FUNCTIONS

All translations of the parent logarithmic function, $y = \log_b(x)$, have the form

$$f(x) = a\log_b(x + c) + d$$

where the parent function, $y = \log_b(x)$, $b > 1$,is

- shifted vertically up $d$ units.
- shifted horizontally to the left $c$ units.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- reflected about the $x$-axis when $a < 0$.

For $f(x) = \log(-x)$, the graph of the parent function is reflected about the $y$-axis.

### Example 4.4.10: Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x + 4) + 5$ ?

**Solution**

The vertical asymptote is at $x = -4$.

**Analysis**

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to$x = -4$.

### Exercise 4.4.10

What is the vertical asymptote of $f(x) = 3 + \ln(x - 1)$ ?

**Answer**

$x = 1$

### Example 4.4.11: Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in Figure 4.4.21.

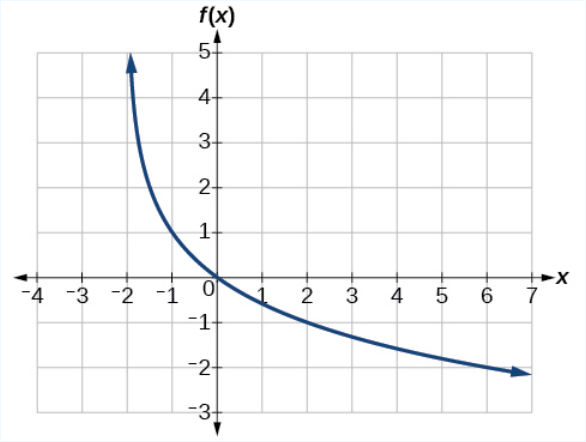



Figure 4.4.21

**Solution**

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$$f(x) = -a\log(x+2)+k$$

It appears the graph passes through the points $(-1,1)$ and $(2,-1)$. Substituting $(-1,1)$,

$$\begin{aligned} 1 &= -a\log(-1+2)+k \qquad \text{Substitute}(-1,1) \\ 1 &= -a\log(1)+k \qquad \text{Arithmetic} \\ 1 &= k\log(1) \\ &= 0 \end{aligned}$$

Next, substituting in $(2,-1)$,

$$\begin{aligned} -1 &= -a\log(2+2)+1 \qquad \text{Substitute}(2,-1) \\ -2 &= -a\log(4) \qquad \text{Arithmetic} \\ a &= \frac{2}{\log(4)} \qquad \text{Solve for a} \end{aligned}$$

This gives us the equation $f(x) = -\frac{2}{\log(4)}\log(x+2)+1$.

**Analysis**

We can verify this answer by comparing the function values in Table 4.4.5 with the points on the graph in Figure 4.4.21.

Table 4.4.5

| $x$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 0 | −0.58496 | −1 | −1.3219 |
| $x$ | 4 | 5 | 6 | 7 | 8 |
| $f(x)$ | −1.5850 | −1.8074 | −2 | −2.1699 | −2.3219 |

## Exercise 4.4.11

Give the equation of the natural logarithm graphed in Figure 4.4.22

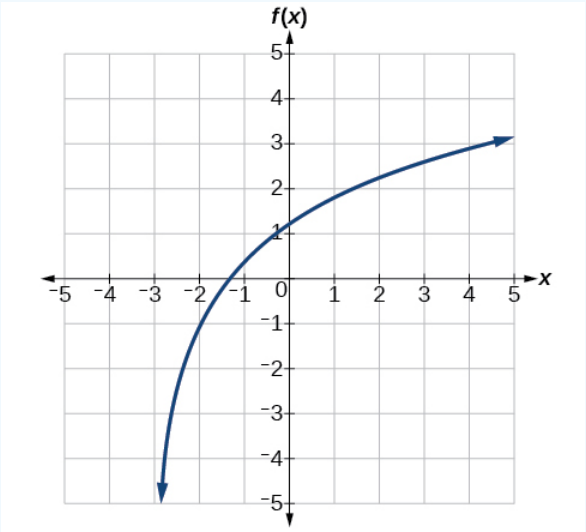



Figure 4.4.22

**Answer**

$f(x) = 2\ln(x+3)-1$

**Media: Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?**

Yes, if we know the function is a general logarithmic function. For example, look at the graph in Figure 4.4.22 The graph approaches $x = -3$ (or thereabouts) more and more closely, so $x = -3$ is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right, $x|x > -3$. The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

**Media**

Access these online resources for additional instruction and practice with graphing logarithms.

- Graph an Exponential Function and Logarithmic Function
- Match Graphs with Exponential and Logarithmic Functions
- Find the Domain of Logarithmic Functions

## Key Equations

| | |
|---|---|
| General Form for the Translation of the Parent Logarithmic Function $f(x) = \log_b(x)$ | $f(x) = a\log_b(x + c) + d$ |

## Key Concepts

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for $x$. See Example 4.4.1 and Example 4.4.2
- The graph of the parent function $f(x) = \log_b(x)$ has an *x*-intercept at $(1, 0)$,domain $(0, \infty)$,range $(-\infty, \infty)$,vertical asymptote $x = 0$,and
- - if $b > 1$,the function is increasing.
  - if $0 < b < 1$, the function is decreasing.

  See Example 4.4.3.
- The equation $f(x) = \log_b(x + c)$ shifts the parent function $y = \log_b(x)$ horizontally
  - left $c$ units if $c > 0$.
  - right $c$ units if $c < 0$.

  See Example 4.4.4.
- The equation $f(x) = \log_b(x) + d$ shifts the parent function $y = \log_b(x)$ vertically
  - up $d$ units if $d > 0$.
  - down $d$ units if $d < 0$.

  See Example 4.4.5.
- For any constant $a > 0$, the equation $f(x) = a\log_b(x)$
  - stretches the parent function $y = \log_b(x)$ vertically by a factor of $a$ if $|a| > 1$.
  - compresses the parent function $y = \log_b(x)$ vertically by a factor of $a$ if $|a| < 1$.

  See Example 4.4.6 and Example 4.4.7.
- When the parent function $y = \log_b(x)$ is multiplied by $-1$, the result is a reflection about the *x*-axis. When the input is multiplied by $-1$, the result is a reflection about the *y*-axis.
  - The equation $f(x) = -\log_b(x)$ represents a reflection of the parent function about the *x*-axis.
  - The equation $f(x) = \log_b(-x)$ represents a reflection of the parent function about the *y*-axis.

  See Example 4.4.8.
  - A graphing calculator may be used to approximate solutions to some logarithmic equations See Example 4.4.9.
- All translations of the logarithmic function can be summarized by the general equation $f(x) = a\log_b(x + c) + d$ . See Table 4.4.4.

- Given an equation with the general form $f(x) = a\log_b(x + c) + d$ ,we can identify the vertical asymptote $x = -c$ for the transformation. See Example 4.4.10.
- Using the general equation $f(x) = a\log_b(x + c) + d$ , we can write the equation of a logarithmic function given its graph. See Example 4.4.11.

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# 4.5: Logarithmic Properties

**Learning Objectives**

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

In chemistry, the pH scale is used as a measure of the acidity or alkalinity of a substance. Substances with a pH less than $7$ are considered acidic, and substances with a pH greater than $7$ are said to be alkaline. Our bodies, for instance, must maintain a pH close to $7.35$ in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: $0.8$
- Stomach acid: $2.7$
- Orange juice: $3.3$
- Pure water: $7$ at $25^{\circ}C$
- Human blood: $7.35$
- Fresh coconut: $7.8$
- Sodium hydroxide (lye): $14$

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where $[\mathrm{H}^+]$ is the concentration of hydrogen ions in the solution

$$pH = -\log([\mathrm{H}^+]) \tag{4.5.1}$$

$$= \log\left(\frac{1}{[\mathrm{H}^+]}\right) \tag{4.5.2}$$

The equivalence of Equations 4.5.1 and 4.5.2 is one of the logarithm properties we will examine in this section.

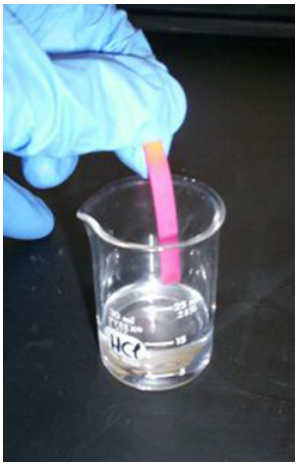


Figure 4.5.1: The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan).

## Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0$$

$$\log_b b = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b(b^x) = x$$
$$b^{\log_b x} = x,\ x > 0$$

For example, to evaluate $\log(100)$, we can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

$$\log_b M = \log_b N \text{ if and only if } M = N \tag{4.5.3}$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x+5)$ for $x$. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for $x$:

$3x = 2x + 5$ Set the arguments equal.

$x = 5$ Subtract $2x$.

But what about the equation $\log_3(3x) + \log_3(2x+5) = 2$ ? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number $x$ and positive real numbers $M$, $N$, and $b$, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N)\ .$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for M and N} \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents} \\ &= m+n && \text{Apply the inverse property of logs} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for m and n} \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

### The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0 \tag{4.5.4}$$

### How to: Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

### Example 4.5.1: Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4))$.

**Solution**

We begin by factoring the argument completely, expressing 30 as a product of primes.

$$\log_3(30x(3x+4)) = \log_3(2\cdot 3\cdot 5\cdot x\cdot(3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30x(3x+4)) = \log_3(2)+\log_3(3)+\log_3(5)+\log_3(x)+\log_3(3x+4)$$

### Exercise 4.5.1

Expand $\log_b(8k)$.

**Answer**

$$\log_b 2+\log_b 2+\log_b 2+\log_b k = 3\log_b 2+\log_b k$$

## Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $x^{\frac{a}{b}} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms.

### The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N \tag{4.5.5}$$

Just as with the product rule, we can use the inverse property to derive the quotient rule.

### Proof

Given any real number $x$ and positive real numbers $M$, $N$, and b, b, where $b\neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M)-\log_b(N)\ .$$

Let $m=\log_b M$ and $n=\log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}
\log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for M and N}\\
&= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents}\\
&= m-n && \text{Apply the inverse property of logs}\\
&= \log_b(M)-\log_b(N) && \text{Substitute for m and n}
\end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\log\left(\frac{2x^2+6x}{3x+9}\right)=\log\left(\frac{2x(x+3)}{3(x+3)}\right) \qquad \text{Factor the numerator and denominator}$$

$$=\log\left(\frac{2x}{3}\right) \qquad \text{Cancel the common factors}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\log\left(\frac{2x}{3}\right)=\log(2x)-\log(3)$$

$$=\log(2)+\log(x)-\log(3)$$

**How to: Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms**

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

**Example 4.5.2: Using the Quotient Rule for Logarithms**

Expand $log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

**Solution**

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2(\frac{15x(x-1)}{(3x+4)(2-x)})=\log_2(15x(x-1))-\log_2((3x+4)(2-x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\log_2(15x(x-1))-\log_2((3x+4)(2-x)) = [\log_2(3)+\log_2(5)+\log_2(x)+\log_2(x-1)]-[\log_2(3x+4)+\log_2(2-x)]$$

$$=\log_2(3)+\log_2(5)+\log_2(x)+\log_2(x-1)-\log_2(3x+4)-\log_2(2-x)$$

**Analysis**

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x=-\frac{4}{3}$ and $x=2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x>0$, $x>1$, $x>-\frac{4}{3}$, and $x<2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

**Exercise 4.5.2**

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

**Answer**

$\log_3(x+3)-\log_3(x-1)-\log_3(x-2)$

## Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as $x^2$? One method is as follows:

$$\begin{aligned}\log_b(x^2) &= \log_b(x \cdot x)\\ &= \log_b x + \log_b x\\ &= 2\log_b x\end{aligned}$$

Notice that we used the **product rule for logarithms** to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{\frac{1}{2}} \quad \frac{1}{e} = e^{-1}$$

### The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b M \tag{4.5.6}$$

### How to: Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

### Example 4.5.3: Expanding a Logarithm with Powers

Expand $\log_2 x^5$.

**Solution**

The argument is already written as a power, so we identify the exponent, 5, and the base, $x$, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

### Exercise 4.5.3

Expand $\ln x^2$.

**Answer**

$2\ln x$

### Example 4.5.4: Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3(25)$ using the power rule for logs.

**Solution**

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(52) = 2\log_3(5)$$

**Exercise 4.5.4**

Expand $\ln\left(\frac{1}{x^2}\right)$.

**Answer**

$-2\ln(x)$

**Example 4.5.5: Using the Power Rule in Reverse**

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

**Solution**

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, $x$, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

**Exercise 4.5.5**

Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

**Answer**

$\log_3 16$

## Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called "laws of logs." Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A) + \log_b(C^{-1}) \\ &= \log_b A + (-1)\log_b C \\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

**Example 4.5.6: Expanding Logarithms Using Product, Quotient, and Power Rules**

Rewrite $ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs.

**Solution**

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

### Exercise 4.5.6

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

**Answer**

$2\log x + 3\log y - 4\log z$

### Example 4.5.7: Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log(\sqrt{x})$.

**Solution**

$$\begin{aligned}\log(\sqrt{x}) &= \log x^{\left(\frac{1}{2}\right)} \\ &= \frac{1}{2}\log x\end{aligned}$$

### Exercise 4.5.7

Expand $\ln(\sqrt[3]{x^2})$.

**Answer**

$\frac{2}{3}\ln x$

### Q&A: Can we expand $\ln(x^2 + y^2)$?

*No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.*

### Example 4.5.8: Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

**Solution**

We can expand by applying the Product and Quotient Rules.

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) && \text{Apply the Quotient Rule} \\ &= \log_6 2^6 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) && \text{Simplify by writing } 64 \text{ as } 2^6 \\ &= 6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1) && \text{Apply the Power Rule}\end{aligned}$$

**Exercise 4.5.8**

Expand $\ln\left(\dfrac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

**Answer**

$\dfrac{1}{2}\ln(x-1)+\ln(2x+1)-\ln(x+3)-\ln(x-3)$

## Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

**How to: Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm**

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

**Example 4.5.9: Using the Product and Quotient Rules to Combine Logarithms**

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

**Solution**

Using the product and quotient rules

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40)$$

This reduces our original expression to

$$\log_3(40)-\log_3(2)$$

Then, using the quotient rule

$$\log_3(40)-\log_3(2)=\log_3\left(\frac{40}{2}\right)=\log_3(20)$$

**Exercise 4.5.9**

Condense $\log_3-\log_4+\log_5-\log_6$ .

**Answer**

$\log\left(\dfrac{3\cdot 5}{4\cdot 6}\right)$; can also be written $\log\left(\dfrac{5}{8}\right)$ by reducing the fraction to lowest terms.

**Example 4.5.10: Condensing Complex Logarithmic Expressions**

Condense $\log_2(x^2)+\dfrac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)$ .

**Solution**

We apply the power rule first:

$$\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)=\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2((x+3)^6)$$

Next we apply the product rule to the sum:

$$\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2((x+3)^6)=\log_2(x^2\sqrt{x-1})-\log_2((x+3)^6)$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2\sqrt{x-1})-\log_2((x+3)^6)=\log_2\frac{x^2\sqrt{x-1}}{(x+3)^6}$$

### Example 4.5.11: Rewriting as a Single Logarithm

Rewrite $2\log x-4\log(x+5)+\frac{1}{x}\log(3x+5)$ as a single logarithm.

**Solution**

We apply the power rule first:

$$2\log x-4\log(x+5)+\frac{1}{x}\log(3x+5)=\log(x^2)-\log(x+5)^4+\log((3x+5)^{x^{-1}})$$

Next we rearrange and apply the product rule to the sum:

$$\begin{aligned}\log(x^2)-\log(x+5)^4+\log((3x+5)^{x^{-1}}) &=\log(x^2)+\log((3x+5)^{x^{-1}}-\log(x+5)^4\\ &=\log(x^2(3x+5)^{x^{-1}})-\log(x+5)^4\\ &=\log\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4} \qquad \text{Apply the quotient rule to the difference}\end{aligned}$$

### Exercise 4.5.10

Rewrite $\log(5)+0.5\log(x)-\log(7x-1)+3\log(x-1)$ as a single logarithm.

**Answer**

$$\log\frac{5(x-1)^3\sqrt{x}}{(7x-1)}$$

### Exercise 4.5.11

Condense $4(3\log(x)+\log(x+5)-\log(2x+3))$.

**Answer**

$\log\frac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log\left(\frac{x^3(x+5)}{(2x+3)}\right)^4$

### Example 4.5.12: Applying of the Laws of Logs

Recall that, in chemistry, $pH=-\log[H+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

**Solution**

Suppose $C$ is the original concentration of hydrogen ions, and $P$ is the original pH of the liquid. Then $P=-\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH=-\log(2C)$$

Using the product rule of logs

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$,the new pH is

$$pH = P - \log(2) \approx P - 0.301$$

**? Exercise 4.5.12**

When the concentration of hydrogen ions is doubled, the pH decreases by about $0.301$.

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

**Answer**

The pH increases by about $0.301$.

## Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 ore, e,we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the **one-to-one** property and **power rule for logarithms**.

Given any positive real numbers $M$, $b$, and $n$, where $n \neq 1$ and $b \neq 1$,we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$. By taking the log base $n$ of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{aligned} \log_n(b^y) &= \log_n M && \text{Apply the one-to-one property} \\ y\log_n b &= \log_n M && \text{Apply the power rule for logarithms} \\ y &= \frac{\log_n M}{\log_n b} && \text{Isolate y} \\ \log_b M &= \frac{\log_n M}{\log_n b} && \text{Substitute for y} \end{aligned}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{aligned} \log_5 36 &= \frac{\log(36)}{\log(5)} && \text{Apply the change of base formula using base 10} \\ &\approx 2.2266 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

**THE CHANGE-OF-BASE FORMULA**

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers $M$, $b$, and $n$, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b} \tag{4.5.7}$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b} \tag{4.5.8}$$

and

$$\log_b M = \frac{\log M}{\log b} \tag{4.5.9}$$

**How to: Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base $n$, where $n \neq 1$**

1. Determine the new base $n$, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$,has base $e$.
2. Rewrite the log as a quotient using the change-of-base formula
   - The numerator of the quotient will be a logarithm with base $n$ and argument $M$.
   - The denominator of the quotient will be a logarithm with base $n$ and argument $b$.

**Example 4.5.13: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs**

Change $\log_5 3$ to a quotient of natural logarithms.

**Solution**

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$

$$\log_5 3 = \frac{\ln 3}{\ln 5}$$

**Exercise 4.5.13**

Change $\log 0.58$ to a quotient of natural logarithms.

**Answer**

$$\frac{\ln 8}{\ln 0.5}$$

**Q&A: Can we change common logarithms to natural logarithms?**

*Yes. Remember that* $\log 9$ *means*$\log_{10} 9$. *So,* $\log 9 = \frac{\ln 9}{\ln 10}$.

**Example 4.5.14: Using the Change-of-Base Formula with a Calculator**

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

**Solution**

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base $e$.

$$\log_2 10 = \frac{\ln 10}{\ln 2} \qquad \text{Apply the change of base formula using base } e$$

$$\approx 3.3219 \qquad \text{Use a calculator to evaluate to 4 decimal places}$$

**Exercise 4.5.14**

Evaluate $\log_5(100)$ using the change-of-base formula.

**Answer**

$$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$$

**Media**

Access these online resources for additional instruction and practice with laws of logarithms.

- The Properties of Logarithms
- Expand Logarithmic Expressions
- Evaluate a Natural Logarithmic Expression

## Key Equations

| | |
|---|---|
| The Product Rule for Logarithms | $\log_b(MN) = \log_b(M) + \log_b(N)$ |
| The Quotient Rule for Logarithms | $\log_b(\frac{M}{N}) = \log_b M - \log_b N$ |
| The Power Rule for Logarithms | $\log_b(M^n) = n\log_b M$ |
| The Change-of-Base Formula | $\log_b M = \frac{\log_n M}{\log_n b} \quad n > 0, n \neq 1, b \neq 1$ |

## Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See Example 4.5.1.
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See Example 4.5.2.
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See Example 4.5.3, Example 4.5.4, and Example 4.5.5.
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See Example 4.5.6, Example 4.5.7, and Example 4.5.8.
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See Example 4.5.9, Example 4.5.10, Example 4.5.11, and Example 4.5.12
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See Example 4.5.13
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and $e$ as the quotient of natural or common logs. That way a calculator can be used to evaluate. See Example 4.5.14

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# 4.6: Exponential and Logarithmic Equations

**Learning Objectives**

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.



Figure 4.6.1: Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

## Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers $b$, $S$, and $T$, where $b > 0$, $b \neq 1$, $b^S = b^T$ if and only if $S = T$.

In other words, when an **exponential equation** has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \dfrac{3^{2x}}{3}$. To solve for $x$, we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for $x$:

$$
\begin{aligned}
3^{4x-7} &= \frac{3^{2x}}{3} && \\
3^{4x-7} &= \frac{3^{2x}}{3^1} && \text{Rewrite 3 as } 3^1 \\
3^{4x-7} &= 3^{2x-1} && \text{Use the division property of exponents} \\
4x-7 &= 2x-1 && \text{Apply the one-to-one property of exponents} \\
2x &= 6 && \text{Subtract 2x and add 7 to both sides} \\
x &= 3 && \text{Divide by 3}
\end{aligned}
$$

### USING THE ONE-TO-ONE PROPERTY OF EXPONENTIAL FUNCTIONS TO SOLVE EXPONENTIAL EQUATIONS

For any algebraic expressions $S$ and $T$, and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{4.6.1}$$

### How to: Given an exponential equation with the form $b^S = b^T$, where $S$ and $T$ are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

### Example 4.6.1: Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

**Solution**

| | |
|---|---|
| $2^{x-1} = 2^{2x-4}$ | The common base is 2 |
| $x - 1 = 2x - 4$ | By the one-to-one property the exponents must be equal |
| $x = 3$ | Solve for x |

### Exercise 4.6.1

Solve $5^{2x} = 5^{3x+2}$.

**Answer**

$x = -2$

## Rewrite Equations So All Powers Have the Same Base

Sometimes the *common base* for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for $x$:

| | |
|---|---|
| $256 = 4^{x-5}$ | |
| $2^8 = (2^2)^{x-5}$ | Rewrite each side as a power with base 2 |
| $2^8 = 2^{2x-10}$ | Use the one-to-one property of exponents |
| $8 = 2x - 10$ | Apply the one-to-one property of exponents |
| $18 = 2x$ | Add 10 to both sides |
| $x = 9$ | Divide by 2 |

### How to: Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
3. Use the one-to-one property to set the exponents equal.
4. Solve the resulting equation, $S = T$, for the unknown.

### Example 4.6.2: Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

**Solution**

$$
\begin{aligned}
8^{x+2} &= 16^{x+1} && \\
(2^3)^{x+2} &= (2^4)^{x+1} && \text{Write 8 and 16 as powers of 2} \\
2^{3x+6} &= 2^{4x+4} && \text{To take a power of a power, multiply exponents} \\
3x+6 &= 4x+4 && \text{Use the one-to-one property to set the exponents equal} \\
x &= 2 && \text{Solve for x}
\end{aligned}
$$

### Exercise 4.6.2

Solve $5^{2x} = 25^{3x+2}$.

**Answer**

$x = -1$

### Example 4.6.3: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

**Solution**

$$
\begin{aligned}
2^{5x} &= 2^{\frac{1}{2}} && \text{Write the square root of 2 as a power of 2} \\
5x &= \frac{1}{2} && \text{Use the one-to-one property} \\
x &= \frac{1}{10} && \text{Solve for x}
\end{aligned}
$$

### Exercise 4.6.3

Solve $5^x = \sqrt{5}$.

**Answer**

$x = \frac{1}{2}$

### Q&A: Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?

*No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.*

### Example 4.6.4: Solving an Equation with Positive and Negative Powers

Solve $3^{x+1} = -2$.

**Solution**

This equation has no solution. There is no real value of $x$ that will make the equation a true statement because any power of a positive number is positive.

Analysis

Figure 4.6.2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

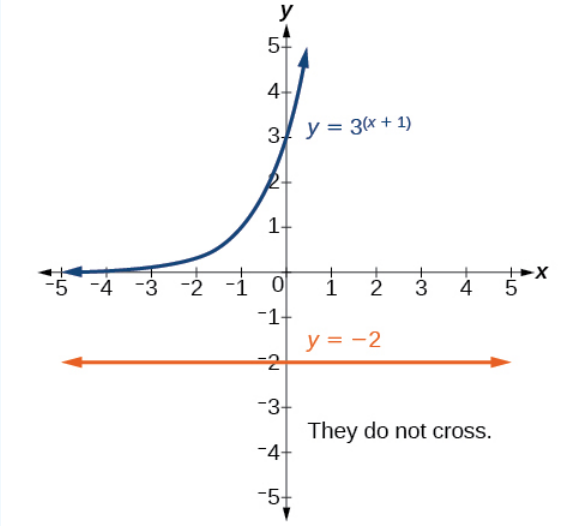



Figure 4.6.2

**Exercise 4.6.4**

Solve $2^x = -100$.

**Answer**

The equation has no solution.

## Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

**How to: Given an exponential equation in which a common base cannot be found, solve for the unknown**

1. Apply the logarithm of both sides of the equation.
   - If one of the terms in the equation has base 10, use the common logarithm.
   - If none of the terms in the equation has base 10, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

**Example 4.6.5: Solving an Equation Containing Powers of Different Bases**

Solve $5^{x+2} = 4^x$.

**Solution**

| | | |
|---|---|---|
| $5^{x+2}$ | $=4^{x}$ | There is no easy way to get the powers to have the same base |
| $\ln 5^{x+2}$ | $=\ln 4^{x}$ | Take ln of both sides |
| $(x+2)\ln 5$ | $=x\ln 4$ | Use laws of logs |
| $x\ln 5+2\ln 5$ | $=x\ln 4$ | Use the distributive law |
| $x\ln 5-x\ln 4$ | $=-2\ln 5$ | Get terms containing x on one side, terms without x on the other |
| $x(\ln 5-\ln 4)$ | $=-2\ln 5$ | On the left hand side, factor out an x |
| $x\ln\left(\frac{5}{4}\right)$ | $=\ln\left(\frac{1}{25}\right)$ | Use the laws of logs |
| $x$ | $=\frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$ | Divide by the coefficient of x |

**? Exercise 4.6.5**

Solve $2^{x}=3^{x+1}$.

**Answer**

$$x=\frac{\ln 3}{\ln\left(\frac{2}{3}\right)}$$

**Q&A: Is there any way to solve $2^{x}=3^{x}$?**

*Yes. The solution is* $0$.

## Equations Containing $e$

One common type of exponential equations are those with base $e$. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base $e$ on either side, we can use the natural logarithm to solve it.

**How to: Given an equation of the form $y=Ae^{kt}$, solve for $t$.**

1. Divide both sides of the equation by $A$.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by $k$.

**Example 4.6.6: Solve an Equation of the Form $y=Ae^{kt}$**

Solve $100=20e^{2t}$.

**Solution**

| | | |
|---|---|---|
| $100$ | $=20e^{2t}$ | |
| $5$ | $=e^{2t}$ | Divide by the coefficient of the power |
| $\ln 5$ | $=2t$ | Take ln of both sides. Use the fact that $ln(x)$ and $e^{x}$ are inverse functions |
| $t$ | $=\frac{\ln 5}{2}$ | Divide by the coefficient of t |

**Analysis**

Using laws of logs, we can also write this answer in the form $t=\ln\sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

**Exercise 4.6.6**

Solve $3e^{0.5t} = 11$.

**Answer**

$$t = 2\ln\left(\frac{11}{3}\right) \text{ or } \ln\left(\frac{11}{3}\right)^2$$

**Q&A: Does every equation of the form $y = Ae^{kt}$ have a solution?**

No. There is a solution when $k \neq 0$ ,and when $y$ and $A$ are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2 = -3e^t$.

**Example 4.6.7: Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$**

Solve $4e^{2x} + 5 = 12$.

**Solution**

| | |
|---|---|
| $4e^{2x} + 5 = 12$ | |
| $4e^{2x} = 7$ | Combine like terms |
| $e^{2x} = \frac{7}{4}$ | Divide by the coefficient of the power |
| $2x = \ln\left(\frac{7}{4}\right)$ | Take ln of both sides |
| $x = \frac{1}{2}\ln\left(\frac{7}{4}\right)$ | Solve for x |

**Exercise 4.6.7**

Solve $3 + e^{2t} = 7e^{2t}$.

**Answer**

$$t = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2)$$

## Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

**Example 4.6.8: Solving Exponential Functions in Quadratic Form**

Solve $e^{2x} - e^x = 56$.

**Solution**

$$
\begin{aligned}
e^{2x}-e^{x} &= 56 & \\
e^{2x}-e^{x}-56 &= 0 & \text{Get one side of the equation equal to zero} \\
(e^{x}+7)(e^{x}-8) &= 0 & \text{Factor by the FOIL method} \\
e^{x}+7 &= 0 & \text{or} \\
e^{x}-8 &= 0 & \text{If a product is zero, then one factor must be zero} \\
e^{x} &= -7 & \text{or} \\
e^{x} &= 8 & \text{Isolate the exponentials} \\
e^{x} &= 8 & \text{Reject the equation in which the power equals a negative number} \\
x &= \ln 8 & \text{Solve the equation in which the power equals a positive number}
\end{aligned}
$$

**Analysis**

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^{x}=-7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

**? Exercise 4.6.8**

Solve $e^{2x}=e^{x}+2$ .

**Answer**

$x=\ln 2$

**Q&A: Does every logarithmic equation have a solution?**

*No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.*

## Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x)=y$ is equivalent to the exponential equation $b^{y}=x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2)+\log_2(3x-5)=3$ . To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for $x$:

$$
\begin{aligned}
\log_2(2)+\log_2(3x-5) &= 3 & \\
\log_2(2(3x-5)) &= 3 & \text{Apply the product rule of logarithms} \\
\log_2(6x-10) &= 3 & \text{Distribute} \\
2^{3} &= 6x-10 & \text{Apply the definition of a logarithm} \\
8 &= 6x-10 & \text{Calculate } 2^{3} \\
18 &= 6x & \text{Add 10 to both sides} \\
x &= 3 & \text{Divide by 6}
\end{aligned}
$$

**USING THE DEFINITION OF A LOGARITHM TO SOLVE LOGARITHMIC EQUATIONS**

For any algebraic expression $S$ and real numbers $b$ and $c$,where $b>0$, $b\neq 1$,

$$\log_b(S)=c \text{ if and only if } b^{c}=S \tag{4.6.2}$$

**✓ Example 4.6.9: Using Algebra to Solve a Logarithmic Equation**

Solve $2\ln x+3=7$.

**Solution**

$$
\begin{aligned}
2\ln x + 3 &= 7 \\
2\ln x &= 4 && \text{Subtract 3} \\
\ln x &= 2 && \text{Divide by 2} \\
x &= e^2 && \text{Rewrite in exponential form}
\end{aligned}
$$

**? Exercise 4.6.9**

Solve $6 + \ln x = 10$.

**Answer**

$x = e^4$

**✓ Example 4.6.10: Using Algebra Before and After Using the Definition of the Natural Logarithm**

Solve $2\ln(6x) = 7$.

**Solution**

$$
\begin{aligned}
2\ln(6x) &= 7 \\
\ln(6x) &= \frac{7}{2} && \text{Divide by 2} \\
6x &= e^{\left(\frac{7}{2}\right)} && \text{Use the definition of ln} \\
x &= \frac{1}{6}e^{\left(\frac{7}{2}\right)} && \text{Divide by 6}
\end{aligned}
$$

**? Exercise 4.6.10**

Solve $2\ln(x+1) = 10$.

**Answer**

$x = e^5 - 1$

**✓ Example 4.6.11: Using a Graph to Understand the Solution to a Logarithmic Equation**

Solve $\ln x = 3$.

**Solution**

$$
\begin{aligned}
\ln x &= 3 \\
x &= e^3 && \text{Use the definition of the natural logarithm}
\end{aligned}
$$

Figure 4.6.3 represents the graph of the equation. On the graph, the *x*-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

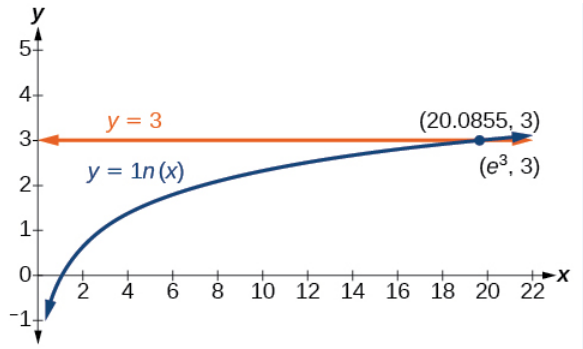



Figure 4.6.3: The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately $(20.0855, 3)$.

**? Exercise 4.6.11**

Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

**Answer**

$x \approx 9.97$

## Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number $b$, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for $x$,and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$ . In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for $x$:

$$\begin{aligned}
\log(3x-2) - \log(2) &= \log(x+4) \\
\log\left(\frac{3x-2}{2}\right) &= \log(x+4) && \text{Apply the quotient rule of logarithms} \\
\frac{3x-2}{2} &= x+4 && \text{Apply the one to one property of a logarithm} \\
3x-2 &= 2x+8 && \text{Multiply both sides of the equation by 2} \\
x &= 10 && \text{Subtract 2x and add 2}
\end{aligned}$$

To check the result, substitute $x = 10$ into $\log(3x-2) - \log(2) = \log(x+4)$.

$$\begin{aligned}
\log(3(10)-2) - \log(2) &= \log((10)+4) \\
\log(28) - \log(2) &= \log(14) \\
\log\left(\frac{28}{2}\right) &= \log(14) && \text{The solution checks}
\end{aligned}$$

### USING THE ONE-TO-ONE PROPERTY OF LOGARITHMS TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expressions $S$ and $T$ and any positive real number $b$, where $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{4.6.3}$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

### How to: Given an equation containing logarithms, solve it using the one-to-one property

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S = \log_b T$.
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S = T$, for the unknown.

### Example 4.6.12: Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x+3)$.

**Solution**

$$\begin{aligned} \ln(x^2) &= \ln(2x+3) \\ x^2 &= 2x+3 \quad \text{Use the one-to-one property of the logarithm} \\ x^2-2x-3 &= 0 \quad \text{Get zero on one side before factoring} \\ (x-3)(x+1) &= 0 \quad \text{Factor using FOIL} \\ x-3 &= 0 \quad \text{or } x+1=0 \text{ If a product is zero, one of the factors must be zero} \\ x &= 3 \quad \text{or} \\ x &= -11 \quad \text{Solve for x} \end{aligned}$$

**Analysis**

There are two solutions: 3 or $-1$. The solution $-1$ is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

### Exercise 4.6.12

Solve $\ln(x^2) = \ln 1$.

**Answer**

$x = 1$ or $x = -1$

## Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its **half-life**. Table 4.6.1 lists the half-life for several of the more common radioactive substances.

Table 4.6.1

| *Substance* | *Use* | *Half-life* |
|---|---|---|
| *gallium-67* | *nuclear medicine* | *80 hours* |
| *cobalt-60* | *manufacturing* | *5.3 years* |

| Substance | Use | Half-life |
|---|---|---|
| technetium-99m | nuclear medicine | 6 hours |
| americium-241 | construction | 432 years |
| carbon-14 | archeological dating | 5,715 years |
| uranium-235 | atomic power | 703,800,000 years |

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$A(t) = A_0 e^{\frac{\ln(0.5)}{T}t} \tag{4.6.4}$$

$$A(t) = A_0 e^{\frac{\ln(0.5)t}{T}} \tag{4.6.5}$$

$$A(t) = A_0 (e^{\ln(0.5)})^{\frac{t}{T}} \tag{4.6.6}$$

$$A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}} \tag{4.6.7}$$

where

- $A_0$ is the amount initially present
- $T$ is the half-life of the substance
- $t$ is the time period over which the substance is studied
- $y$ is the amount of the substance present after time $t$

**Example 4.6.13: Using the Formula for Radioactive Decay to Find the Quantity of a Substance**

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

**Solution**

$$\begin{aligned} y &= 1000e^{\frac{\ln(0.5)}{703,800,000}t} \\ 900 &= 1000e^{\frac{\ln(0.5)}{703,800,000}t} && \text{After } 10\% \text{ decays, 900 grams are left} \\ 0.9 &= e^{\frac{\ln(0.5)}{703,800,000}t} && \text{Divide by 1000} \\ \ln(0.9) &= \ln\left(e^{\frac{\ln(0.5)}{703,800,000}t}\right) && \text{Take ln of both sides} \\ \ln(0.9) &= \frac{\ln(0.5)}{703,800,000}t && \ln(e^M) = M \\ t &= 703,800,000 \times \frac{\ln(0.9)}{\ln(0.5)} && \text{years Solve for t} \\ t &\approx 106,979,777 && \text{years} \end{aligned}$$

**Analysis**

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

**Exercise 4.6.13**

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

**Answer**

$t = 703,800,000 \times \frac{\ln(0.8)}{\ln(0.5)}$ years ≈ 226,572,993 years.

**Media**

Access these online resources for additional instruction and practice with exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Exponential Equations with Logarithms

## Key Equations

| | |
|---|---|
| One-to-one property for exponential functions | For any algebraic expressions $S$ and $T$ and any positive real number $b$, where $b^S = b^T$ if and only if $S = T$. |
| Definition of a logarithm | For any algebraic expression S and positive real numbers $b$ and $c$, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$. |
| One-to-one property for logarithmic functions | For any algebraic expressions $S$ and $T$ and any positive real number $b$, where $b \neq 1$, $\log_b S = \log_b T$ if and only if $S = T$. |

## Key Concepts

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown. See Example 4.6.1.
- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See Example 4.6.2, Example 4.6.3, and Example 4.6.4.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See Example 4.6.5.
- We can solve exponential equations with base $e$,by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See Example 4.6.6 and Example 4.6.7.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions. See Example 4.6.8.
- When given an equation of the form $\log_b(S) = c$, where $S$ is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See Example 4.6.9 and Example 4.6.10.
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the *x*-value of the intersecting point. See Example 4.6.11.
- When given an equation of the form $\log_b S = \log_b T$, where $S$ and $T$ are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See Example 4.6.12.
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See Example 4.6.13.

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# 4.7: Exponential and Logarithmic Models

**Learning Objectives**

- Model exponential growth and decay.
- Use Newton's Law of Cooling.
- Use logistic-growth models.
- Choose an appropriate model for data.
- Express an exponential model in base $e$.

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton's Law of Cooling.

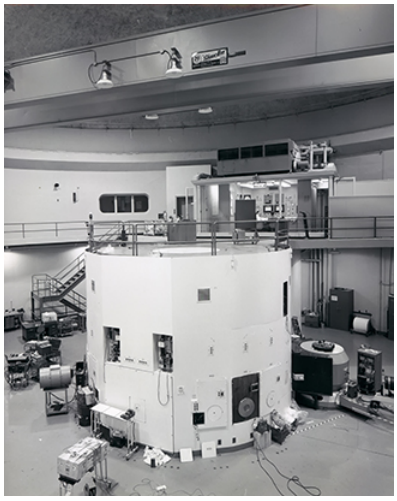


Figure 4.7.1: A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus (credit: Georgia Tech Research Institute)

## Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function:

$$y = A_0 e^{kt} \tag{4.7.1}$$

where $A_0$ is equal to the value at time zero, $e$ is Euler's constant, and $k$ is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double. Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth based on a doubling time. In some applications, however, as we will see when we discuss the logistic equation, the logistic model sometimes fits the data better than the exponential model.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the **exponential decay** model. Again, we have the form $y = A_0 e^{kt}$ where $A_0$ is the starting value, and $e$ is Euler's constant. Now $k$ is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating half-life, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in Figure 4.7.2 and Figure 4.7.3. It is important to remember that, although parts of each of the two graphs seem to lie on the $x$-axis, they are really a tiny distance above the $x$-axis.

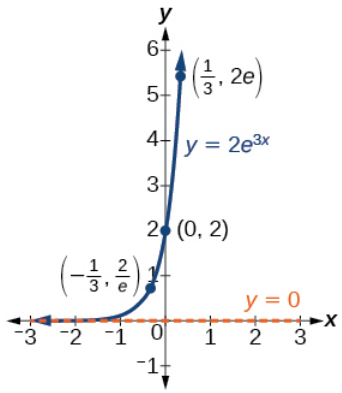



Figure 4.7.2: A graph showing exponential growth. The equation is $y = 2e^{3x}$.

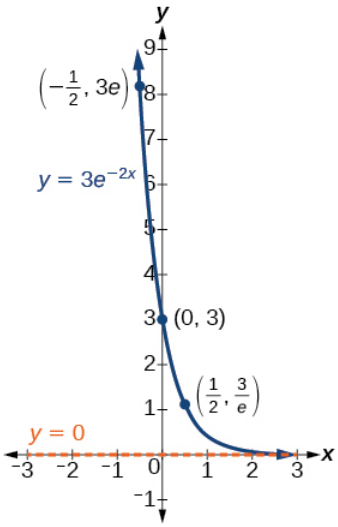



Figure 4.7.3: A graph showing exponential decay. The equation is $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude. The **order of magnitude** is the power of ten, when the number is expressed in scientific notation, with one digit to the left of the decimal. For example, the distance to the nearest star, **Proxima Centauri**, measured in kilometers, is $40,113,497,200,000$kilometers. Expressed in scientific notation, this is $4.01134972 \times 1013$ So, we could describe this number as having order of magnitude $1013$.

### $y = a_0 e^{kt}$

An exponential function with the form $y = A_0 e^{kt}$ has the following characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- $x$ intercept: none
- $y$-intercept: $(0, A_0)$
- increasing if $k > 0$ (see Figure 4.7.4)
- decreasing if $k < 0$ (see Figure 4.7.4)

Two graphs of y=(A_0)(e^(kt)) with the asymptote at y=0. The first graph is of when k
0 and with the labeled points (1/k, (A_0)e), (0, A_0), and (-1/k, (A_0)/e). The second graph is of when k<0 and with the labeled points (-1/k, (A_0)e), (0, A_0), and (1/k, (A_0)/e)." src="/@api/deki/files/12238/fig_6.8.4.jpg">

Figure 4.7.4: An exponential function models exponential growth when $k > 0$ and exponential decay when $k < 0$.

### Example 4.7.1: Graphing Exponential Growth

A population of bacteria doubles every hour. If the culture started with 10 bacteria, graph the population as a function of time.

**Solution**

When an amount grows at a fixed percent per unit time, the growth is exponential. To find $A_0$ we use the fact that $A_0$ is the amount at time zero, so $A_0 = 10$. To find $k$, use the fact that after one hour $(t = 1)$ the population doubles from 10 to 20.The formula is derived as follows

$$\begin{aligned} 20 &= 10e^{k\cdot 1} \\ 2 &= e^{k} \qquad \text{Divide by 10} \\ \ln 2 &= k \qquad \text{Take the natural logarithm} \end{aligned}$$

so $k = \ln(2)$. Thus the equation we want to graph is $y = 10e^{(\ln 2)t} = 10(e^{\ln 2})^{t} = 10 \cdot 2^{t}$ . The graph is shown in Figure 4.7.5.

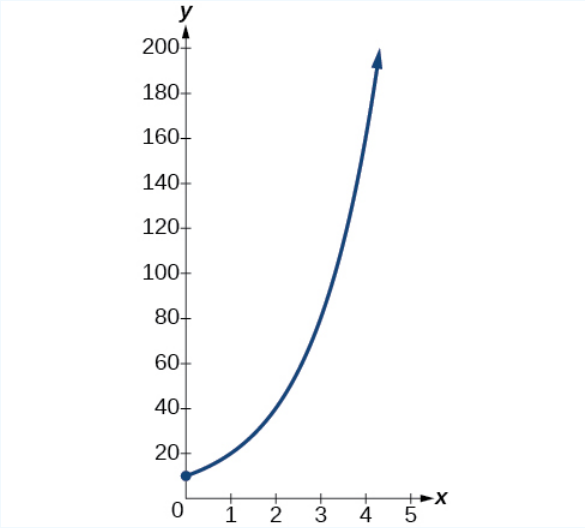



Figure 4.7.5: The graph of $y = 10e^{(\ln 2)t}$

**Analysis**

The population of bacteria after ten hours is $10,240$. We could describe this amount is being of the order of magnitude $10^4$. The population of bacteria after twenty hours is $10,485,760$which is of the order of magnitude $10^7$, so we could say that the population has increased by three orders of magnitude in ten hours.

## Half-Life

We now turn to **exponential decay**. One of the common terms associated with exponential decay, as stated above, is **half-life**, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay.

To find the half-life of a function describing exponential decay, solve the following equation:

$$\frac{1}{2}A_0 = A_0 e^{kt}$$

We find that the half-life depends only on the constant $k$ and not on the starting quantity $A_0$.

The formula is derived as follows

$$\begin{aligned}
\frac{1}{2}A_0 &= A_0 e^{kt} \\
\frac{1}{2} &= e^{kt} && \text{Divide by } A_0 \\
\ln\left(\frac{1}{2}\right) &= ktv && \text{Take the natural log} \\
-\ln(2) &= kt && \text{Apply laws of logarithms} \\
-\ln(2)k &= t && \text{Divide by k}
\end{aligned}$$

Since $t$, the time, is positive, $k$ must, as expected, be negative. This gives us the half-life formula

$$t = -\frac{\ln(2)}{k} \tag{4.7.2}$$

**Given the half-life, find the decay rate.**

1. Write $A = A_0 e^{kt}$.
2. Replace $A$ by $\frac{1}{2}A_0$ and replace $t$ by the given half-life.
3. Solve to find $k$. Express $k$ as an exact value (do not round).

Note: It is also possible to find the decay rate using $k = -\ln(2)t$.

**Example 4.7.2: Finding the Function that Describes Radioactive Decay**

The half-life of carbon-14 is $5,730$ years. Express the amount of carbon-14 remaining as a function of time, $t$.

**Solution**

This formula is derived as follows.

$$\begin{aligned}
A &= A_0 e^{kt} && \text{The continuous growth formula} \\
0.5A_0 &= A_0 e^{k\cdot 5730} && \text{Substitute the half-life for t and } 0.5A_0 \text{ for } f(t) \\
0.5 &= e^{5730k} && \text{Divide by } A_0 \\
\ln(0.5) &= 5730k && \text{Take the natural log of both sides} \\
k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of k} \\
A &= A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} && \text{Substitute for r in the continuous growth formula}
\end{aligned}$$

The function that describes this continuous decay is $f(t) = A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$. We observe that the coefficient of $t$, $\frac{\ln(0.5)}{5730} \approx -1.2097 \times 10^{-4}$ is negative, as expected in the case of exponential decay.

**Exercise 4.7.1**

The half-life of plutonium-244 is $80,000,000$ years. Find function gives the amount of carbon-14 remaining as a function of time, measured in years.

**Answer**

$f(t) = A_0 e^{-0.0000000087t}$

## Radiocarbon Dating

The formula for radioactive decay is important in **radiocarbon dating**, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares

the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about $1\%$ error for plants or animals that died within the last $60,000$ years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of $5,730$ years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of $12$ and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last $60,000$ years, using tree rings and other organic samples of known dates—although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is $5,730$ years, the formula for the amount of carbon-14 remaining after $t$ years is

$$A \approx A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$$

where

- $A$ is the amount of carbon-14 remaining
- $A_0$ is the amount of carbon-14 when the plant or animal began decaying.

This formula is derived as follows:

$$\begin{aligned} A &= A_0 e^{kt} && \text{The continuous growth formula} \\ 0.5A_0 &= A_0 e^{k \cdot 5730} && \text{Substitute the half-life for t and } 0.5A_0 \text{ for f(t)} \\ 0.5 &= e^{5730k} && \text{Divide by } A_0 \\ \ln(0.5) &= 5730k && \text{Take the natural log of both sides} \\ k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of k} \\ A &= A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} && \text{Substitute for r in the continuous growth formula} \end{aligned}$$

To find the age of an object, we solve this equation for $t$:

$$t = \frac{\ln\left(\frac{A}{A_0}\right)}{-0.000121}$$

Out of necessity, we neglect here the many details that a scientist takes into consideration when doing carbon-14 dating, and we only look at the basic formula. The ratio of carbon-14 to carbon-12 in the atmosphere is approximately $0.0000000001$ Let $r$ be the ratio of carbon-14 to carbon-12 in the organic artifact or fossil to be dated, determined by a method called liquid scintillation. From the equation $A \approx A_0 e^{-0.000121t}$ we know the ratio of the percentage of carbon-14 in the object we are dating to the percentage of carbon-14 in the atmosphere is $r = \frac{A}{A_0} \approx e^{-0.000121t}$. We solve this equation for $t$, to get

$$t = \frac{\ln(r)}{-0.000121}$$

1. Express the given percentage of carbon-14 as an equivalent decimal, $k$.
2. Substitute for $k$ in the equation $t = \frac{\ln(r)}{-0.000121}$ and solve for the age, $t$.

### Example 4.7.3: Finding the Age of a Bone

A bone fragment is found that contains $20\%$ of its original carbon-14. To the nearest year, how old is the bone?

**Solution**

We substitute $20\% = 0.20$ for $k$ in the equation and solve for $t$:

$$\begin{aligned} t &= \frac{\ln(r)}{-0.000121} && \text{Use the general form of the equation} \\ &= \frac{\ln(0.20)}{-0.000121} && \text{Substitute for r} \\ &\approx 13301 && \text{Round to the nearest year} \end{aligned}$$

The bone fragment is about $13,301$ years old.

**Analysis**

The instruments that measure the percentage of carbon-14 are extremely sensitive and, as we mention above, a scientist will need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about $1\%$, so this age should be given as $13,301$ years $\pm 1\%$ or $13,301$ years $\pm 133$ years.

**? Exercise 4.7.2**

Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

**Answer**

less than $230$ years, $229.3157$ to be exact

### Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the **doubling time**.

Given the basic **exponential growth** equation $A = A_0 e^{kt}$, doubling time can be found by solving for when the original quantity has doubled, that is, by solving $2A_0 = A_0 e^{kt}$.

The formula is derived as follows:

$$\begin{aligned} 2A_0 &= A_0 e^{kt} \\ 2 &= e^{kt} && \text{Divide by } A_0 \\ \ln 2 &= kt && \text{Take the natural logarithm} \\ t &= \frac{\ln 2}{k} && \text{Divide by the coefficient of t} \end{aligned}$$

Thus the doubling time is

$$t = \frac{\ln 2}{k} \tag{4.7.3}$$

**✓ Example 4.7.4: Finding a Function That Describes Exponential Growth**

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

**Solution**

The formula is derived as follows:

$$\begin{aligned} t &= \frac{\ln 2}{k} && \text{The doubling time formula} \\ 2 &= \frac{\ln 2}{k} && \text{Use a doubling time of two years} \\ k &= \frac{\ln 2}{2} && \text{Multiply by k and divide by 2} \\ A &= A_0 e^{\frac{\ln 2}{2} t} && \text{Substitute k into the continuous growth formula} \end{aligned}$$

The function is $A = A_0 e^{\frac{\ln 2}{2} t}$.

**Exercise 4.7.3**

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

**Answer**

$f(t) = A_0 e^{\frac{\ln 2}{3} t}$

## Using Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a **vertical shift** of the generic **exponential decay** function. This translation leads to **Newton's Law of Cooling**, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature

$$T(t) = ae^{kt} + T_s$$

This formula is derived as follows:

$$\begin{aligned} T(t) &= Ab^{ct} + T_s \\ T(t) &= Ae^{\ln(b^{ct})} + T_s && \text{Laws of logarithms} \\ T(t) &= Ae^{ct \ln b} + T_s && \text{Laws of logarithms} \\ T(t) &= Ae^{kt} + T_s && \text{Rename the constant c } lnb\text{, calling it } k \end{aligned}$$

**NEWTON'S LAW OF COOLING**

The temperature of an object, $T$, in surrounding air with temperature $T_s$ will behave according to the formula

$$T(t) = Ae^{kt} + T_s \tag{4.7.4}$$

where

- $t$ is time
- $A$ is the difference between the initial temperature of the object and the surroundings
- $k$ is a constant, the continuous rate of cooling of the object

**Given a set of conditions, apply Newton's Law of Cooling.**

1. Set $T_s$ equal to the $y$-coordinate of the horizontal asymptote (usually the ambient temperature).
2. Substitute the given values into the continuous growth formula $T(t) = Ae^{kt} + T_s$ to find the parameters $A$ and $k$.
3. Substitute in the desired time to find the temperature or the desired temperature to find the time.

**Example 4.7.5: Using Newton's Law of Cooling**

A cheesecake is taken out of the oven with an ideal internal temperature of $165^\circ F$, and is placed into a $35^\circ F$ refrigerator. After 10 minutes, the cheesecake has cooled to $150^\circ F$. If we must wait until the cheesecake has cooled to $70^\circ F$ before we eat it, how long will we have to wait?

**Solution**

Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35, following the equation

$$T(t) = Ae^{kt} + 35$$

We know the initial temperature was 165, so $T(0) = 165$.

$$\begin{aligned} 165 &= Ae^{k0} + 35 && \text{Substitute } (0, 165) \\ A &= 130 && \text{Solve for A} \end{aligned}$$

We were given another data point, $T(10) = 150$, which we can use to solve for $k$.

$$\begin{aligned} 150 &= 130e^{k10} + 35 && \text{Substitute } (10, 150) \\ 115 &= 130e^{k10} && \text{Subtract 35} \\ \frac{115}{130} &= e^{10k} && \text{Divide by 130} \\ \ln\left(\frac{115}{130}\right) &= 10k && \text{Take the natural log of both sides} \\ k &= \frac{\ln\left(\frac{115}{130}\right)}{10} \\ &= -0.0123 && \text{Divide by the coefficient of k} \end{aligned}$$

This gives us the equation for the cooling of the cheesecake: $T(t) = 130e^{-0.0123t} + 35$.

Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$\begin{aligned} 70 &= 130e^{-0.0123t} + 35 && \text{Substitute in 70 for } T(t) \\ 35 &= 130e^{-0.0123t} && \text{Subtract 35} \\ \frac{35}{130} &= e^{-0.0123t} && \text{Divide by 130} \\ \ln\left(\frac{35}{130}\right) &= -0.0123t && \text{Take the natural log of both sides} \\ t &= \frac{\ln\left(\frac{35}{130}\right)}{-0.0123} \\ &\approx 106.68 && \text{Divide by the coefficient of t} \end{aligned}$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to $70^\circ F$.

**? Exercise 4.7.4**

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

**Answer**

6.026hours

## Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the **exponential growth** model is still useful over a short term, before approaching the limiting value.

The **logistic growth model** is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, called the **carrying capacity**. For constants $a$, $b$, and $c$, the logistic growth of a population over time $x$ is

represented by the model

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

The graph in Figure $4.7.6$ shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.

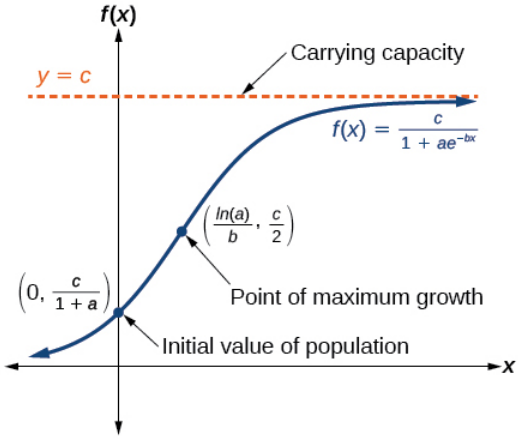



Figure 4.7.6

## LOGISTIC GROWTH

The logistic growth model is

$$f(x) = \frac{c}{1 + ae^{-bx}} \tag{4.7.5}$$

where

- $\frac{c}{1+a}$ is the initial value
- $c$ is the *carrying capacity*, or *limiting value*
- $b$ is a constant determined by the rate of growth.

## Example 4.7.6: Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of $1,000$ people who has the flu. So, in that community, at most $1,000$ people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

### Solution

We substitute the given data into the logistic growth model

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

Because at most $1,000$ people, the entire population of the community, can get the flu, we know the limiting value is $c = 1000$. To find $a$,we use the formula that the number of cases at time $t = 0$ is $\frac{c}{1+a} = 1$, from which it follows that $a = 999$.This model predicts that, after ten days, the number of people who have had the flu is

$f(x) = \dfrac{1000}{1 + 999e^{-0.6030x}} \approx 293.8$. Because the actual number must be a whole number (a person has either had the flu or not) we round to $294$. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000$.

**Analysis**

Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.

The graph in Figure $4.7.7$ gives a good picture of how this model fits the data.

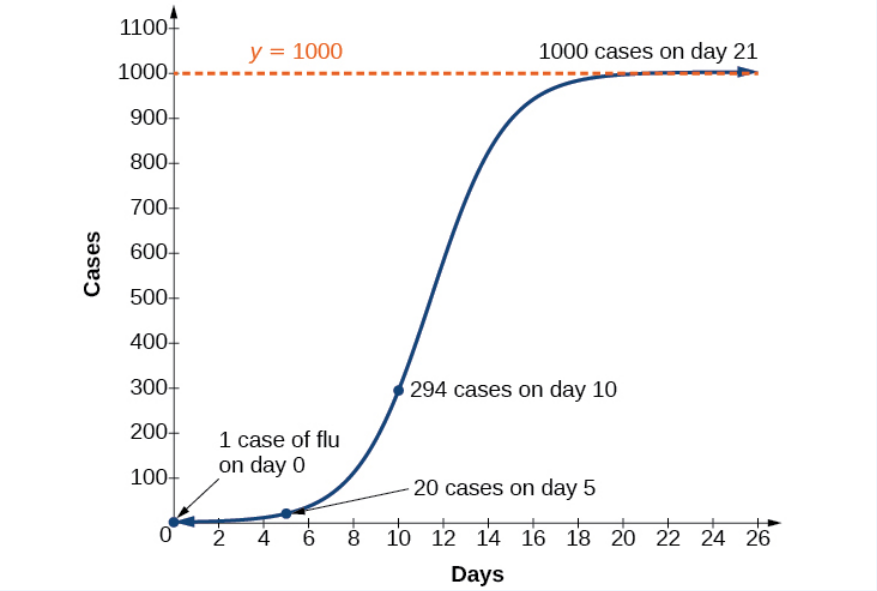



Figure 4.7.7: The graph of $f(x) = \dfrac{1000}{1 + 999e^{-0.6030x}}$

**? Exercise 4.7.5**

Using the model in the Example above, estimate the number of cases of flu on day $15$.

**Answer**

$895$ cases on day $15$

## Choosing an Appropriate Model for Data

Now that we have discussed various mathematical models, we need to learn how to choose the appropriate model for the raw data we have. Many factors influence the choice of a mathematical model, among which are experience, scientific laws, and patterns in the data itself. Not all data can be described by elementary functions. Sometimes, a function is chosen that approximates the data over a given interval. For instance, suppose data were gathered on the number of homes bought in the United States from the years 1960 to 2013. After plotting these data in a scatter plot, we notice that the shape of the data from the years 2000 to 2013 follow a logarithmic curve. We could restrict the interval from 2000 to 2010, apply regression analysis using a logarithmic model, and use it to predict the number of home buyers for the year 2015.

Three kinds of functions that are often useful in mathematical models are linear functions, exponential functions, and logarithmic functions. If the data lies on a straight line, or seems to lie approximately along a straight line, a linear model may be best. If the data is non-linear, we often consider an exponential or logarithmic model, though other models, such as quadratic models, may also be considered.

In choosing between an exponential model and a logarithmic model, we look at the way the data curves. This is called the concavity. If we draw a line between two data points, and all (or most) of the data between those two points lies above that line, we say the curve is concave down. We can think of it as a bowl that bends downward and therefore cannot hold water. If all (or most)

of the data between those two points lies below the line, we say the curve is concave up. In this case, we can think of a bowl that bends upward and can therefore hold water. An exponential curve, whether rising or falling, whether representing growth or decay, is always concave up away from its horizontal asymptote. A logarithmic curve is always concave away from its vertical asymptote. In the case of positive data, which is the most common case, an exponential curve is always concave up, and a logarithmic curve always concave down.

A logistic curve changes concavity. It starts out concave up and then changes to concave down beyond a certain point, called a point of inflection.

After using the graph to help us choose a type of function to use as a model, we substitute points, and solve to find the parameters. We reduce round-off error by choosing points as far apart as possible.

### Example 4.7.7: Choosing a Mathematical Model

Does a linear, exponential, logarithmic, or logistic model best fit the values listed in Table 4.7.1? Find the model, and use a graph to check your choice.

Table 4.7.1

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0 | 1.386 | 2.197 | 2.773 | 3.219 | 3.584 | 3.892 | 4.159 | 4.394 |

**Solution**

First, plot the data on a graph as in Figure 4.7.8. For the purpose of graphing, round the data to two significant digits.

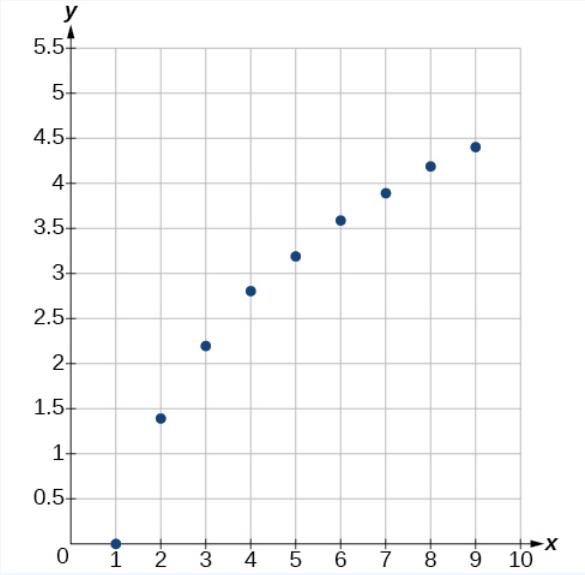



Figure 4.7.8

Clearly, the points do not lie on a straight line, so we reject a linear model. If we draw a line between any two of the points, most or all of the points between those two points lie above the line, so the graph is concave down, suggesting a logarithmic model. We can try $y = a\ln(bx)$. Plugging in the first point, $(1, 0)$, gives $0 = a\ln b$. We reject the case that $a = 0$ (if it were, all outputs would be 0), so we know $\ln(b) = 0$. Thus $b = 1$ and $y = a\ln(x)$. Next we can use the point $(9, 4.394)$ to solve for $a$:

$$\begin{aligned} y &= a\ln(x) \\ 4.394 &= a\ln(9) \\ a &= \frac{4.394}{\ln(9)} \end{aligned}$$

Because $a = \dfrac{4.394}{\ln(9)} \approx 2$ ,an appropriate model for the data is $y = 2\ln(x)$.

To check the accuracy of the model, we graph the function together with the given points as in Figure 4.7.9.

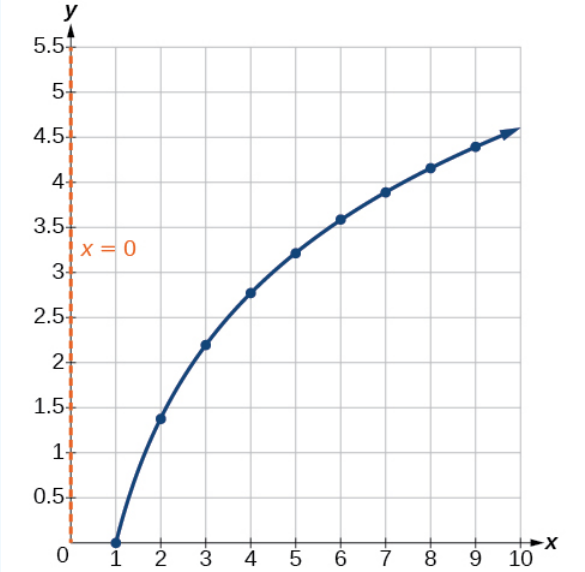



Figure 4.7.9: The graph of $y = 2\ln x$.

We can conclude that the model is a good fit to the data.

Compare Figure 4.7.9 to the graph of $y = \ln(x^2)$ shown in Figure 4.7.10.

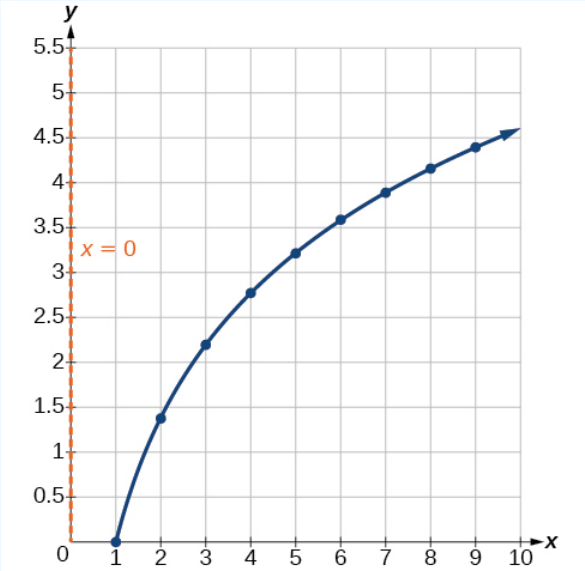



Figure 4.7.10: The graph of $y = \ln(x^2)$

The graphs appear to be identical when $x > 0$. A quick check confirms this conclusion: $y = \ln(x^2) = 2\ln(x)$ for $x > 0$.

However, if $x < 0$, the graph of $y = \ln(x^2)$ includes a "extra" branch, as shown in Figure 4.7.11. This occurs because, while $y = 2\ln(x)$ cannot have negative values in the domain (as such values would force the argument to be negative), the function $y = \ln(x^2)$ can have negative domain values.

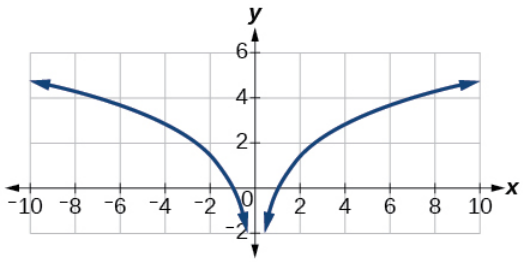



Figure 4.7.11

**Exercise 4.7.6**

Does a linear, exponential, or logarithmic model best fit the data in Table 4.7.2? Find the model.

Table 4.7.2

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 3.297 | 5.437 | 8.963 | 14.778 | 24.365 | 40.172 | 66.231 | 109.196 | 180.034 |

**Answer**

Exponential $y = 2e^{0.5x}$.

## Expressing an Exponential Model in Base $e$

While powers and logarithms of any base can be used in modeling, the two most common bases are 10 and $e$. In science and mathematics, the base $e$ is often preferred. We can use laws of exponents and laws of logarithms to change any base to base $e$.

**Given a model with the form $y = ab^x$, change it to the form $y = A_0e^{kx}$.**

1. Rewrite $y = ab^x$ as $y = ae^{\ln(b^x)}$.
2. Use the power rule of logarithms to rewrite $y$ as $y = ae^{x\ln(b)} = ae^{\ln(b)^x}$.
3. Note that $a = A_0$ and $k = \ln(b)$ in the equation $y = A_0e^{kx}$.

**Example 4.7.8: Changing to base $e$**

Change the function $y = 2.5(3.1)^x$ so that this same function is written in the form $y = A_0e^{kx}$.

**Solution**

The formula is derived as follows

$$\begin{aligned} y &= 2.5(3.1)^x \\ &= 2.5e^{\ln(3.1^x)} && \text{Insert exponential and its inverse} \\ &= 2.5e^{x\ln 3.1} && \text{Laws of logs} \\ &= 2.5e^{(\ln 3.1)x} && \text{Commutative law of multiplication} \end{aligned}$$

**Exercise 4.7.7**

Change the function $y = 3(0.5)^x$ to one having $e$ as the base.

**Answer**

$y = 3e^{(\ln 0.5)x}$


https://math.libretexts.org/@go/page/69103

## Media

Access these online resources for additional instruction and practice with exponential and logarithmic models.

- Logarithm Application – pH
- Exponential Model – Age Using Half-Life
- Newton's Law of Cooling
- Exponential Growth Given Doubling Time
- Exponential Growth – Find Initial Amount Given Doubling Time

## Key Equations

| | |
|---|---|
| Half-life formula | If $A = A_0 e^{kt}$, $k < 0$, the half-life is $t = -\frac{\ln(2)}{k}$. |
| Carbon-14 dating | $t = \frac{\ln(\frac{A}{A_0})}{-0.000121}$.<br>$A_0$ $A$ is the amount of carbon-14 when the plant or animal died<br>$t$ is the amount of carbon-14 remaining today<br>is the age of the fossil in years |
| Doubling time formula | If $A = A_0 e^{kt}$, $k > 0$, the doubling time is $t = \frac{\ln 2}{k}$ |
| Newton's Law of Cooling | $T(t) = Ae^{kt} + T_s$ , where $T_s$ is the ambient temperature, $A = T(0) - T_s$ , and $k$ is the continuous rate of cooling. |

## Key Concepts

- The basic exponential function is $f(x) = ab^x$. If $b > 1$,we have exponential growth; if $0 < b < 1$, we have exponential decay.
- We can also write this formula in terms of continuous growth as $A = A_0 e^{kx}$, where $A_0$ is the starting value. If $A_0$ is positive, then we have exponential growth when $k > 0$ and exponential decay when $k < 0$. See Example 4.7.1.
- In general, we solve problems involving exponential growth or decay in two steps. First, we set up a model and use the model to find the parameters. Then we use the formula with these parameters to predict growth and decay. See Example 4.7.2.
- We can find the age, $t$, of an organic artifact by measuring the amount, $k$, of carbon-14 remaining in the artifact and using the formula $t = \frac{\ln(k)}{-0.000121}$ to solve for $t$. See Example 4.7.3.
- Given a substance's doubling time or half-time, we can find a function that represents its exponential growth or decay. See Example 4.7.4.
- We can use Newton's Law of Cooling to find how long it will take for a cooling object to reach a desired temperature, or to find what temperature an object will be after a given time. See Example 4.7.5.
- We can use logistic growth functions to model real-world situations where the rate of growth changes over time, such as population growth, spread of disease, and spread of rumors. See Example 4.7.6.
- We can use real-world data gathered over time to observe trends. Knowledge of linear, exponential, logarithmic, and logistic graphs help us to develop models that best fit our data. See Example 4.7.7.
- Any exponential function with the form $y = abx$ can be rewritten as an equivalent exponential function with the form $y = A_0 e^{kx}$ where $k = \ln b$. See Example 4.7.8.

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# 4.8: Fitting Exponential Models to Data

**Learning Objectives**

- Build an exponential model from data.
- Build a logarithmic model from data.
- Build a logistic model from data.

In previous sections of this chapter, we were either given a function explicitly to graph or evaluate, or we were given a set of points that were guaranteed to lie on the curve. Then we used algebra to find the equation that fit the points exactly. In this section, we use a modeling technique called *regression analysis* to find a curve that models data collected from real-world observations. With regression analysis, we don't expect all the points to lie perfectly on the curve. The idea is to find a model that best fits the data. Then we use the model to make predictions about future events.

Do not be confused by the word *model*. In mathematics, we often use the terms *function*, *equation*, and *model* interchangeably, even though they each have their own formal definition. The term *model* is typically used to indicate that the equation or function approximates a real-world situation.

We will concentrate on three types of regression models in this section: exponential, logarithmic, and logistic. Having already worked with each of these functions gives us an advantage. Knowing their formal definitions, the behavior of their graphs, and some of their real-world applications gives us the opportunity to deepen our understanding. As each regression model is presented, key features and definitions of its associated function are included for review. Take a moment to rethink each of these functions, reflect on the work we've done so far, and then explore the ways regression is used to model real-world phenomena.

## Building an Exponential Model from Data

As we've learned, there are a multitude of situations that can be modeled by exponential functions, such as investment growth, radioactive decay, atmospheric pressure changes, and temperatures of a cooling object. What do these phenomena have in common? For one thing, all the models either increase or decrease as time moves forward. But that's not the whole story. It's the *way* data increase or decrease that helps us determine whether it is best modeled by an exponential equation. Knowing the behavior of exponential functions in general allows us to recognize when to use exponential regression, so let's review exponential growth and decay.

Recall that exponential functions have the form $y = ab^x$ or $y = A_0e^{kx}$. When performing regression analysis, we use the form most commonly used on graphing utilities, $y = ab^x$. Take a moment to reflect on the characteristics we've already learned about the exponential function $y = ab^x$ (assume $a > 0$):

- $b$ must be greater than zero and not equal to one.
- The initial value of the model is $y = a$.
  - If $b > 1$, the function models exponential growth. As $x$ increases, the outputs of the model increase slowly at first, but then increase more and more rapidly, without bound.
  - If $0 < b < 1$, the function models **exponential decay**. As $x$ increases, the outputs for the model decrease rapidly at first and then level off to become asymptotic to the *x*-axis. In other words, the outputs never become equal to or less than zero.

As part of the results, your calculator will display a number known as the *correlation coefficient*, labeled by the variable $r$, or $r^2$. (You may have to change the calculator's settings for these to be shown.) The values are an indication of the "goodness of fit" of the regression equation to the data. We more commonly use the value of $r^2$ instead of $r$, but the closer either value is to 1, the better the regression equation approximates the data.

**EXPONENTIAL REGRESSION**

*Exponential regression* is used to model situations in which growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero. We use the command "ExpReg" on a graphing utility to fit an exponential function to a set of data points. This returns an equation of the form,

$$y = ab^x \tag{4.8.1}$$

Note that:

- $b$ must be non-negative.
- when $b > 1$, we have an exponential growth model.
- when $0 < b < 1$, we have an exponential decay model.

**How to: Given a set of data, perform exponential regression using a graphing utility**

1. Use the STAT then EDIT menu to enter given data.
   i. Clear any existing data from the lists.
   ii. List the input values in the L1 column.
   iii. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   i. Use ZOOM [9] to adjust axes to fit the data.
   ii. Verify the data follow an exponential pattern.
3. Find the equation that models the data.
   i. Select "ExpReg" from the STAT then CALC menu.
   ii. Use the values returned for $a$ and $b$ to record the model, $y = ab^x$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

In 2007, a university study was published investigating the crash risk of alcohol impaired driving. Data from $2,871$ crashes were used to measure the association of a person's blood alcohol level (BAC) with the risk of being in an accident. Table $4.8.1$ shows results from the study. The *relative risk* is a measure of how many times more likely a person is to crash. So, for example, a person with a BAC of $0.09$ is $3.54$ times as likely to crash as a person who has not been drinking alcohol.

Table 4.8.1

| BAC | 0 | 0.01 | 0.03 | 0.05 | 0.07 | 0.09 |
|---|---|---|---|---|---|---|
| Relative Risk of Crashing | 1 | 1.03 | 1.06 | 1.38 | 2.09 | 3.54 |
| BAC | 0.11 | 0.13 | 0.15 | 0.17 | 0.19 | 0.21 |
| Relative Risk of Crashing | 6.41 | 12.6 | 22.1 | 39.05 | 65.32 | 99.78 |

a. Let $x$ represent the BAC level, and let $y$ represent the corresponding relative risk. Use exponential regression to fit a model to these data.
b. After $6$ drinks, a person weighing $160$ pounds will have a BAC of about $0.16$. How many times more likely is a person with this weight to crash if they drive after having a 6-pack of beer? Round to the nearest hundredth.

**Solution**

a. Using the STAT then EDIT menu on a graphing utility, list the BAC values in L1 and the relative risk values in L2. Then use the STATPLOT feature to verify that the scatterplot follows the exponential pattern shown in Figure $4.8.1$:

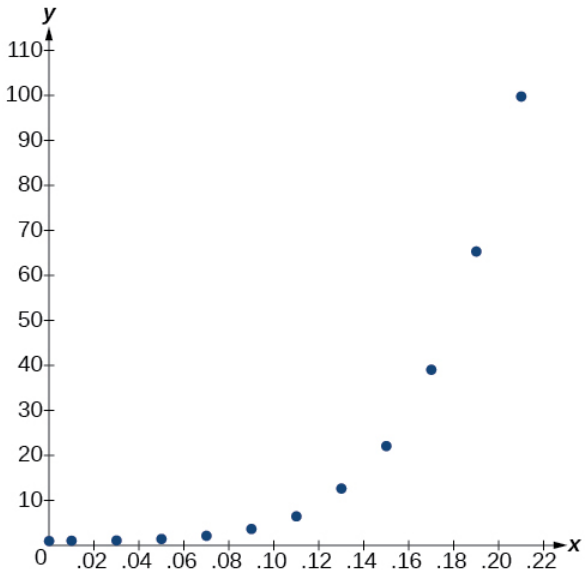


Figure 4.8.1

Use the "ExpReg" command from the STAT then CALC menu to obtain the exponential model,

$$y = 0.58304829(2.20720213E10)^{x}$$

Converting from scientific notation, we have:

$$y = 0.58304829(22,072,021,300)^{x}$$

Notice that $r^2 \approx 0.97$ which indicates the model is a good fit to the data. To see this, graph the model in the same window as the scatterplot to verify it is a good fit as shown in Figure 4.8.2:

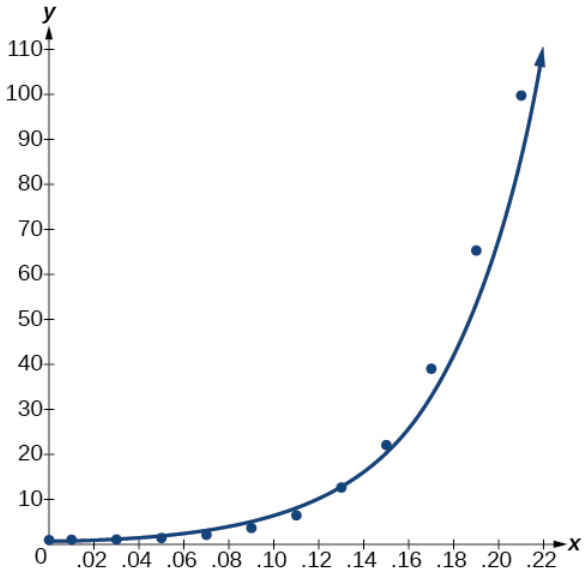


Figure 4.8.2

b. Use the model to estimate the risk associated with a BAC of $0.16$. Substitute $0.16$ for $x$ in the model and solve for $y$.

$$\begin{aligned} y &= 0.58304829(22,072,021,300)^{x} && \text{Use the regression model found in part } (a) \\ &= 0.58304829(22,072,021,300)^{0.16} && \text{Substitute } 0.16 \text{ for x} \\ &\approx 26.35 && \text{Round to the nearest hundredth} \end{aligned}$$

If a $160$-pound person drives after having $6$ drinks, he or she is about $26.35$ times more likely to crash than if driving while sober.


https://math.libretexts.org/@go/page/69104

**Exercise 4.8.1**

Table 4.8.2 shows a recent graduate's credit card balance each month after graduation.

Table 4.8.2

| Month | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Debt ($) | 620.00 | 761.88 | 899.80 | 1039.93 | 1270.63 | 1589.04 | 1851.31 | 2154.92 |

a. Use exponential regression to fit a model to these data.
b. If spending continues at this rate, what will the graduate's credit card debt be one year after graduating?

**Answer a**

The exponential regression model that fits these data is $y = 522.88585984(1.19645256)^x$

**Answer b**

If spending continues at this rate, the graduate's credit card debt will be $\$4,499.38$ after one year.

**Q&A: Is it reasonable to assume that an exponential regression model will represent a situation indefinitely?**

*No. Remember that models are formed by real-world data gathered for regression. It is usually reasonable to make estimates within the interval of original observation (interpolation). However, when a model is used to make predictions, it is important to use reasoning skills to determine whether the model makes sense for inputs far beyond the original observation interval (extrapolation).*

## Building a Logarithmic Model from Data

Just as with exponential functions, there are many real-world applications for logarithmic functions: intensity of sound, pH levels of solutions, yields of chemical reactions, production of goods, and growth of infants. As with exponential models, data modeled by logarithmic functions are either always increasing or always decreasing as time moves forward. Again, it is the *way* they increase or decrease that helps us determine whether a **logarithmic model** is best.

Recall that logarithmic functions increase or decrease rapidly at first, but then steadily slow as time moves on. By reflecting on the characteristics we've already learned about this function, we can better analyze real world situations that reflect this type of growth or decay. When performing logarithmic **regression analysis**, we use the form of the logarithmic function most commonly used on graphing utilities, $y = a + b\ln(x)$ . For this function

- All input values, $x$,must be greater than zero.
- The point $(1, a)$ is on the graph of the model.
- If $b > 0$,the model is increasing. Growth increases rapidly at first and then steadily slows over time.
- If $b < 0$,the model is decreasing. Decay occurs rapidly at first and then steadily slows over time.

**LOGARITHMIC REGRESSION**

*Logarithmic regression* is used to model situations where growth or decay accelerates rapidly at first and then slows over time. We use the command "LnReg" on a graphing utility to fit a logarithmic function to a set of data points. This returns an equation of the form,

$$y = a + b\ln(x) \tag{4.8.2}$$

Note that

- all input values, $x$,must be non-negative.
- when $b > 0$, the model is increasing.
- when $b < 0$, the model is decreasing.

**How to: Given a set of data, perform logarithmic regression using a graphing utility**

1. Use the STAT then EDIT menu to enter given data.
   i. Clear any existing data from the lists.
   ii. List the input values in the L1 column.
   iii. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   i. Use ZOOM [9] to adjust axes to fit the data.
   ii. Verify the data follow a logarithmic pattern.
3. Find the equation that models the data.
   i. Select "LnReg" from the STAT then CALC menu.
   ii. Use the values returned for $a$ and $b$ to record the model, $y = a + b\ln(x)$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

**Example 4.8.2: Using Logarithmic Regression to Fit a Model to Data**

Due to advances in medicine and higher standards of living, life expectancy has been increasing in most developed countries since the beginning of the 20th century. Table 4.8.3 shows the average life expectancies, in years, of Americans from 1900–2010.

**Table 4.8.3**

| Year | 1900 | 1910 | 1920 | 1930 | 1940 | 1950 |
|---|---|---|---|---|---|---|
| Life Expectancy(Years) | 47.3 | 50.0 | 54.1 | 59.7 | 62.9 | 68.2 |
| Year | 1960 | 1970 | 1980 | 1990 | 2000 | 2010 |
| Life Expectancy(Years) | 69.7 | 70.8 | 73.7 | 75.4 | 76.8 | 78.7 |

a. Let $x$ represent time in decades starting with $x = 1$ for the year 1900, $x = 2$ for the year 1910, and so on. Let $y$ represent the corresponding life expectancy. Use logarithmic regression to fit a model to these data.
b. Use the model to predict the average American life expectancy for the year 2030.

**Solution**

a. Using the STAT then EDIT menu on a graphing utility, list the years using values $1-12$ in L1 and the corresponding life expectancy in L2. Then use the STATPLOT feature to verify that the scatterplot follows a logarithmic pattern as shown in Figure 4.8.3:

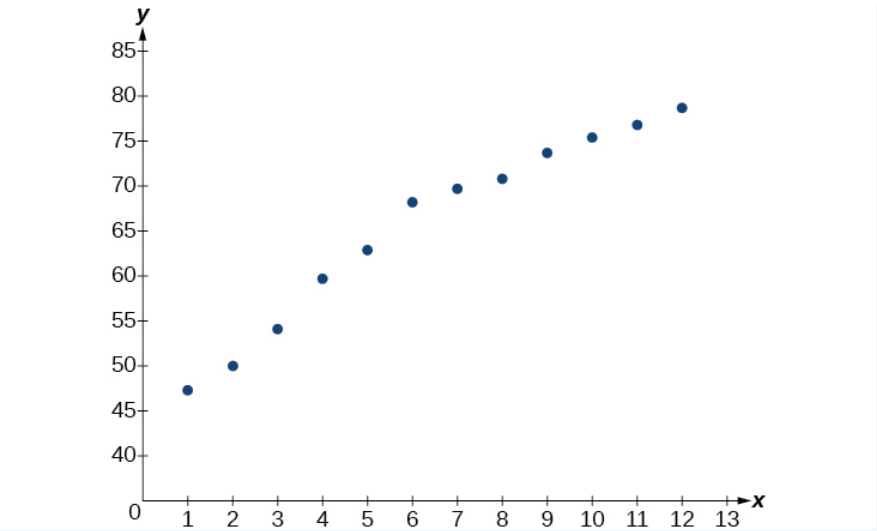



Figure 4.8.3

Use the "LnReg" command from the STAT then CALC menu to obtain the logarithmic model,

$$y = 42.52722583 + 13.85752327 \ln(x)$$

Next, graph the model in the same window as the scatterplot to verify it is a good fit as shown in Figure 4.8.4:

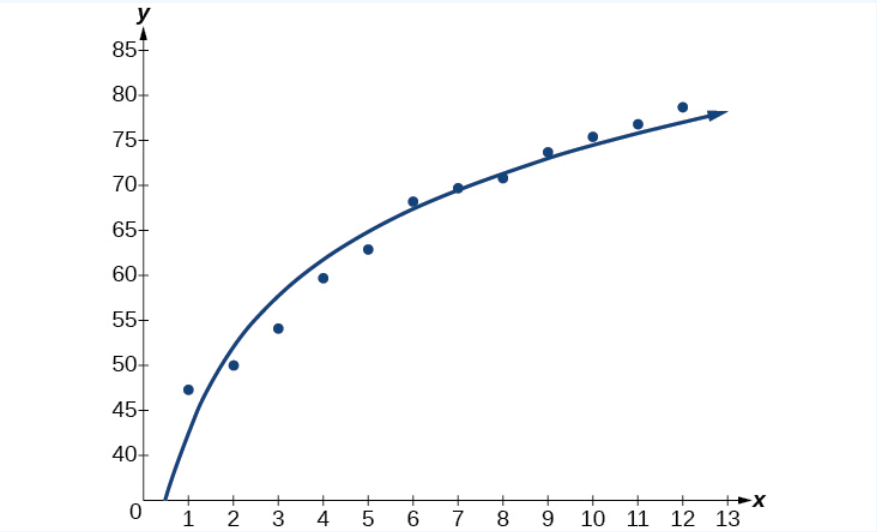



Figure 4.8.4

b. To predict the life expectancy of an American in the year 2030, substitute $x = 14$ for the in the model and solve for $y$:

$$\begin{aligned} y &= 42.52722583 + 13.85752327 \ln(x) && \text{Use the regression model found in part } (a) \\ &= 42.52722583 + 13.85752327 \ln(14) && \text{Substitute 14 for x} \\ &\approx 79.1 && \text{Round to the nearest tenth} \end{aligned}$$

If life expectancy continues to increase at this pace, the average life expectancy of an American will be 79.1 by the year 2030.

## ? Exercise 4.8.2

Sales of a video game released in the year 2000 took off at first, but then steadily slowed as time moved on. Table 4.8.4 shows the number of games sold, in thousands, from the years 2000–2010.

Table 4.8.4

| Year | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
|---|---|---|---|---|---|---|

| | | | | | | |
|---|---|---|---|---|---|---|
| Number Sold (thousands) | 142 | 149 | 154 | 155 | 159 | 161 |
| Year | 2006 | 2007 | 2008 | 2009 | 2010 | - |
| Number Sold (thousands) | 163 | 164 | 164 | 166 | 167 | - |

Let $x$ represent time in years starting with $x = 1$ for the year 2000. Let $y$ represent the number of games sold in thousands.

a. Use logarithmic regression to fit a model to these data.
b. If games continue to sell at this rate, how many games will sell in 2015? Round to the nearest thousand.

**Answer a**

The logarithmic regression model that fits these data is $y = 141.91242949 + 10.45366573 \ln(x)$

**Answer b**

If sales continue at this rate, about $171,000$games will be sold in the year $2015$

## Building a Logistic Model from Data

Like exponential and logarithmic growth, logistic growth increases over time. One of the most notable differences with logistic growth models is that, at a certain point, growth steadily slows and the function approaches an upper bound, or *limiting value*. Because of this, logistic regression is best for modeling phenomena where there are limits in expansion, such as availability of living space or nutrients.

It is worth pointing out that logistic functions actually model resource-limited exponential growth. There are many examples of this type of growth in real-world situations, including population growth and spread of disease, rumors, and even stains in fabric. When performing logistic regression analysis, we use the form most commonly used on graphing utilities:

$$y = \frac{c}{1 + ae^{-bx}}$$

Recall that:

- $\frac{c}{1+a}$ is the initial value of the model.
- when $b > 0$, the model increases rapidly at first until it reaches its point of maximum growth rate, $(\frac{\ln(a)}{b}, \frac{c}{2})$. At that point, growth steadily slows and the function becomes asymptotic to the upper bound $y = c$.
- $c$ is the limiting value, sometimes called the *carrying capacity*, of the model.

### LOGISTIC REGRESSION

*Logistic regression* is used to model situations where growth accelerates rapidly at first and then steadily slows to an upper limit. We use the command “Logistic” on a graphing utility to fit a logistic function to a set of data points. This returns an equation of the form

$$y = \frac{c}{1 + ae^{-bx}} \tag{4.8.3}$$

Note that

- The initial value of the model is $\frac{c}{1+a}$.
- Output values for the model grow closer and closer to $y = c$ as time increases.

**How to: Given a set of data, perform logistic regression using a graphing utility**

1. Use the STAT then EDIT menu to enter given data.
   i. Clear any existing data from the lists.
   ii. List the input values in the L1 column.
   iii. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   i. Use ZOOM [9] to adjust axes to fit the data.
   ii. Verify the data follow a logistic pattern.
3. Find the equation that models the data.
   i. Select "Logistic" from the STAT then CALC menu.
   ii. Use the values returned for $a$, $b$, and $c$ to record the model, $y = \dfrac{c}{1 + ae^{-bx}}$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

**Example 4.8.3: Using Logistic Regression to Fit a Model to Data**

Mobile telephone service has increased rapidly in America since the mid 1990s. Today, almost all residents have cellular service. Table 4.8.5 shows the percentage of Americans with cellular service between the years 1995 and 2012.

Table 4.8.5

| Year | Americans with Cellular Service (%) | Year | Americans with Cellular Service (%) |
|---|---|---|---|
| 1995 | 12.69 | 2004 | 62.852 |
| 1996 | 16.35 | 2005 | 68.63 |
| 1997 | 20.29 | 2006 | 76.64 |
| 1998 | 25.08 | 2007 | 82.47 |
| 1999 | 30.81 | 2008 | 85.68 |
| 2000 | 38.75 | 2009 | 89.14 |
| 2001 | 45.00 | 2010 | 91.86 |
| 2002 | 49.16 | 2011 | 95.28 |
| 2003 | 55.15 | 2012 | 98.17 |

a. Let $x$ represent time in years starting with $x = 0$ for the year 1995. Let $y$ represent the corresponding percentage of residents with cellular service. Use logistic regression to fit a model to these data.
b. Use the model to calculate the percentage of Americans with cell service in the year 2013. Round to the nearest tenth of a percent.
c. Discuss the value returned for the upper limit, $c$. What does this tell you about the model? What would the limiting value be if the model were exact?

**Solution**

a. Using the STAT then EDIT menu on a graphing utility, list the years using values $0–15$ in L1 and the corresponding percentage in L2. Then use the STATPLOT feature to verify that the scatterplot follows a logistic pattern as shown in Figure 4.8.5:

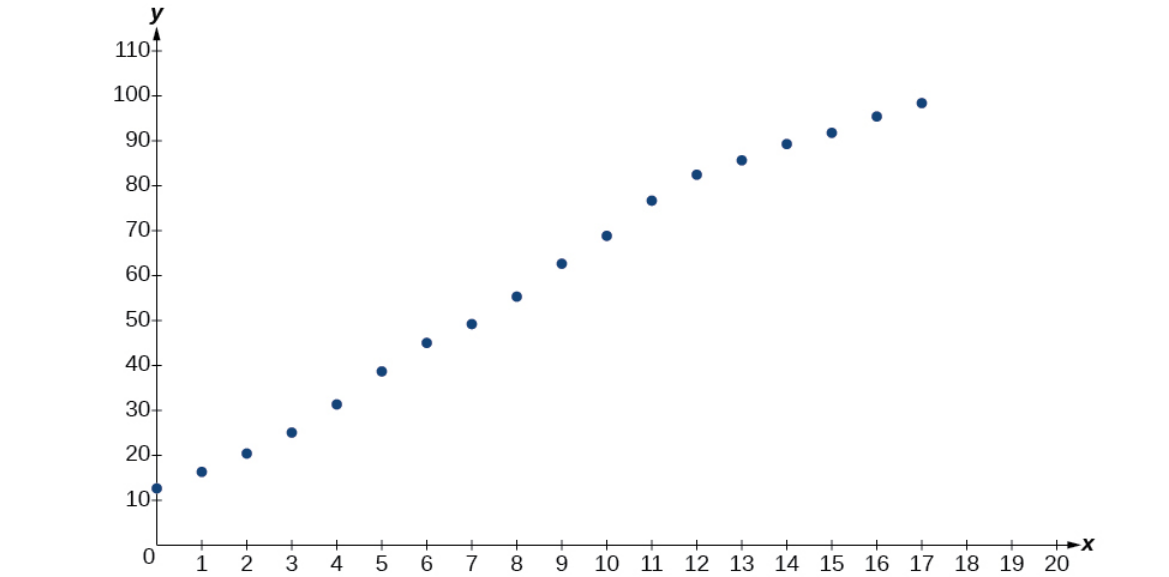



Figure 4.8.5

Use the "Logistic" command from the STAT then CALC menu to obtain the logistic model,

$$y = 105.73795261 + 6.88328979e^{-0.2595440013x} \tag{4.8.4}$$

Next, graph the model in the same window as shown in Figure $4.8.6$ the scatterplot to verify it is a good fit:

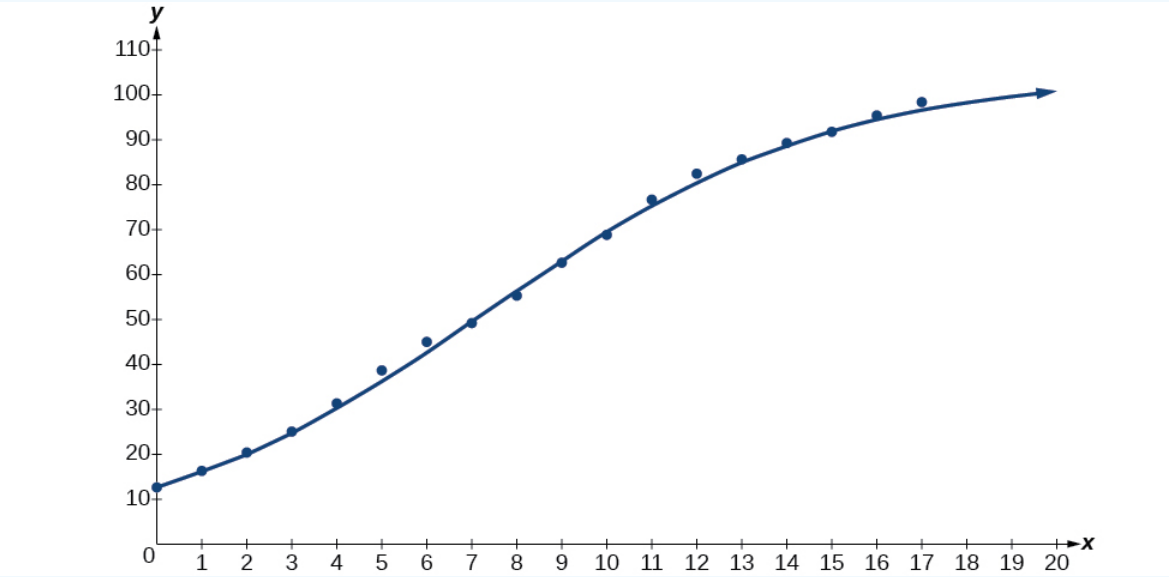



Figure 4.8.6

b. To approximate the percentage of Americans with cellular service in the year 2013, substitute $x = 18$ for the in the model and solve for $y$:

$$\begin{aligned} y &= \frac{105.7379526}{1 + 6.88328979e^{-0.2595440013x}} && \text{Use the regression model found in part } (a) \\ &= \frac{105.7379526}{1 + 6.88328979e^{-0.2595440013(18)}} && \text{Substitute 18 for x} \\ &\approx 99.3 && \text{Round to the nearest tenth} \end{aligned}$$

According to the model, about 98.8% of Americans had cellular service in 2013.

c. The model gives a limiting value of about 105. This means that the maximum possible percentage of Americans with cellular service would be 105, which is impossible. (How could over 100 of a population have cellular service?) If the model were exact, the limiting value would be $c = 100$ and the model's outputs would get very close to, but never actually reach 100. After all, there will always be someone out there without cellular service!

## Exercise 4.8.3

Table 4.8.6 shows the population, in thousands, of harbor seals in the Wadden Sea over the years 1997 to 2012.

Table 4.8.6

| Year | Seal Population (Thousands) | Year | Seal Population (Thousands) |
|---|---|---|---|
| 1997 | 3.493 | 2005 | 19.590 |
| 1998 | 5.282 | 2006 | 21.955 |
| 1999 | 6.357 | 2007 | 22.862 |
| 2000 | 9.201 | 2008 | 23.869 |
| 2001 | 11.224 | 2009 | 24.243 |
| 2002 | 12.964 | 2010 | 24.344 |
| 2003 | 16.226 | 2011 | 24.919 |
| 2004 | 18.137 | 2012 | 25.108 |

Let $x$ represent time in years starting with $x = 0$ for the year 1997. Let $y$ represent the number of seals in thousands.

a. Use logistic regression to fit a model to these data.
b. Use the model to predict the seal population for the year 2020.
c. To the nearest whole number, what is the limiting value of this model?

**Answer a**

The logistic regression model that fits these data is $y = \dfrac{25.65665979}{1 + 6.113686306e^{-0.3852149008x}}$.

**Answer b**

If the population continues to grow at this rate, there will be about $25,634$seals in 2020.

**Answer c**

To the nearest whole number, the carrying capacity is $25,657$.

## Media

Access this online resource for additional instruction and practice with exponential function models.

- Exponential Regression on a Calculator

Visit this website for additional practice questions from Learningpod.

## Key Concepts

- Exponential regression is used to model situations where growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero.
- We use the command "ExpReg" on a graphing utility to fit function of the form $y = ab^x$ to a set of data points. See Example 4.8.1.
- Logarithmic regression is used to model situations where growth or decay accelerates rapidly at first and then slows over time.

- We use the command “LnReg” on a graphing utility to fit a function of the form $y = a + b\ln(x)$ to a set of data points. See Example 4.8.2.
- Logistic regression is used to model situations where growth accelerates rapidly at first and then steadily slows as the function approaches an upper limit.
- We use the command “Logistic” on a graphing utility to fit a function of the form $y = \dfrac{c}{1 + ae^{-bx}}$ to a set of data points. See Example 4.8.3.

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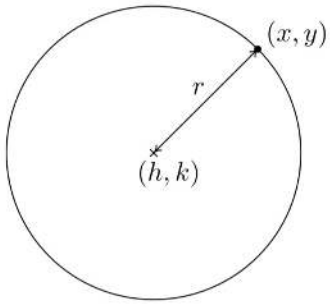
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# 4.9: Trigonometry Preview - Circles

Recall from Geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

**Definition 4.9.1: Circles**

A **circle** with center $(h,k)$ and radius $r>0$ is the set of all points $(x,y)$ in the plane whose distance to $(h,k)$ is $r$.




From the picture, we see that a point $(x,y)$ is on the circle if and only if its distance to $(h,k)$ is $r$. We express this relationship algebraically using the Distance Formula (Equation 4.9.1), as

$$r=\sqrt{(x-h)^2+(y-k)^2} \tag{4.9.1}$$

By squaring both sides of this equation, we get an equivalent equation (since $r>0$) which gives us the standard equation of a circle.

**Note 4.9.1: The Standard Equation of a Circle**

The equation of a circle with center $(h,k)$ and radius $r>0$ is

$$(x-h)^2+(y-k)^2=r^2. \tag{4.9.2}$$

**Example 4.9.1: The Standard Equation**

Write the standard equation of the circle with center (−2,3) and radius 5.

**Solution**.

Here, $(h,k)=(-2,3)$ and $r=5$, so we get

$$\begin{aligned}(x-(-2))^2+(y-3)^2 &= (5)^2\\ (x+2)^2+(y-3)^2 &= 25\end{aligned}$$

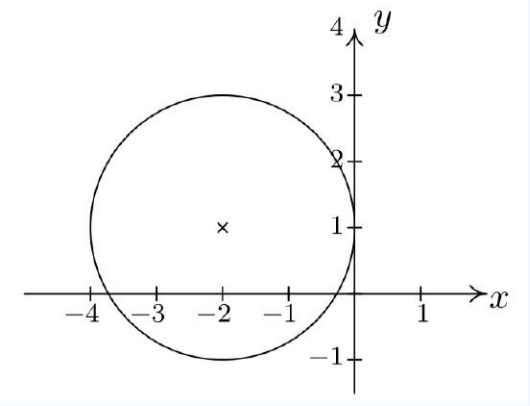
□

**Example 4.9.2:**

Graph $(x+2)^2+(y-1)^2=4$ . Find the center and radius.

**Solution**

From the standard form of a circle, Equation 4.9.2, we have that $x+2$ is $x-h$, so $h=-2$ and $y-1$ is $y-k$ so $k=1$. This tells us that our center is $(-2,1)$. Furthermore, $r^2=4$, so $r=2$. Thus we have a circle centered at $(-2,1)$ with a radius of 2. Graphing gives us




□

If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2+(y-1)^2=4$ , we'd be contending with $x^2+4x+y^2-2y+1=0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in Equation 4.9.2 by following the steps given below.

**To Write the Equation of a Circle in Standard Form**

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

**Example 4.9.3:**

Complete the square to find the center and radius of $3x^2-6x+3y^2+4y-4=0$ .

**Solution**

$$
\begin{aligned}
3x^2-6x+3y^2+4y-4 &= 0 && \\
3x^2-6x+3y^2+4y &= 4 && \text{add 4 to both sides} \\
3\left(x^2-2x\right)+3\left(y^2+\frac{4}{3}y\right) &= 4 && \text{factor out leading coefficients} \\
3\left(x^2-2x+\underline{1}\right)+3\left(y^2+\frac{4}{3}y+\underline{\frac{4}{9}}\right) &= 4+3\underline{(1)}+3\underline{\left(\frac{4}{9}\right)} && \text{complete the square in } x, y \qquad (4.9.3) \\
3(x-1)^2+3\left(y+\frac{2}{3}\right)^2 &= \frac{25}{3} && \text{factor} \\
(x-1)^2+\left(y+\frac{2}{3}\right)^2 &= \frac{25}{9} && \text{divide both sides by 3}
\end{aligned}
$$

From Equation 4.9.2, we identify $x-1$ as $x-h$, so $h=1$, and $y+\frac{2}{3}$ as $y-k$, so $k=-\frac{2}{3}$. Hence, the center is $(h,k)=\left(1,-\frac{2}{3}\right)$. Furthermore, we see that $r^2=\frac{25}{9}$ so the radius is $r=\frac{5}{3}$.

□

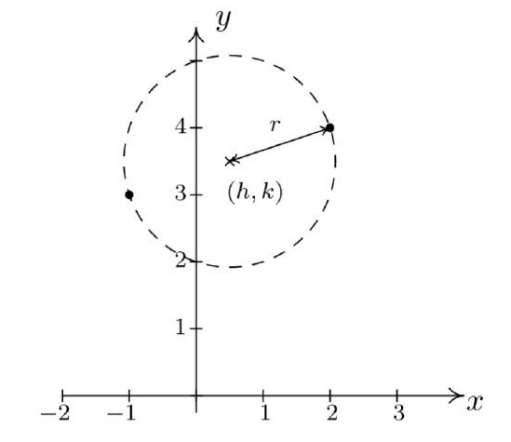
It is possible to obtain equations like $(x-3)^2+(y+1)^2=0$ or $(x-3)^2+(y+1)^2=-1$ , neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations. The next example uses the Midpoint Formula, Equation ???, in conjunction with the ideas presented so far in this section.

**Example 4.9.4:**

Write the standard equation of the circle which has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

**Solution**

We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields




Since the given points are endpoints of a diameter, we know their midpoint $(h,k)$ is the center of the circle. Equation ??? gives us

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-1+2}{2},\frac{3+4}{2}\right)\\ &= \left(\frac{1}{2},\frac{7}{2}\right)\end{aligned} \tag{4.9.4}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned} \tag{4.9.5}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2=\frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\frac{10}{4}$

□

We close this section with the most important\footnote{While this may seem like an opinion, it is indeed a fact. See Chapters ??? and ??? for details.} circle in all of mathematics: the **Unit Circle**.

**Definition 4.9.2: Unit Circle**

The **Unit Circle** is the circle centered at $(0,0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2+y^2=1$.

**Example 4.9.5:**

Find the points on the unit circle with $y$-coordinate $\frac{\sqrt{3}}{2}$.

**Solution**

We replace $y$ with $\frac{\sqrt{3}}{2}$ in the equation $x^2+y^2=1$ to get

$$\begin{aligned} x^2+y^2 &= 1 \\ x^2+\left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ \frac{3}{4}+x^2 &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} \\ x &= \pm\frac{1}{2} \end{aligned} \tag{4.9.6}$$

Our final answers are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

□

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# 4.E: Exponential and Logarithmic Functions (Exercises)

## 4.1: Exponential Functions

When populations grow rapidly, we often say that the growth is “exponential,” meaning that something is growing very rapidly. To a mathematician, however, the term exponential growth has a very specific meaning. In this section, we will take a look at exponential functions, which model this kind of rapid growth.

### Verbal

1) Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

**Answer:**

Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

2) Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

3)The Oxford Dictionary defines the word nominal as a value that is “stated or expressed but not necessarily corresponding exactly to the real value.” Develop a reasonable argument for why the term nominal rate is used to describe the annual percentage rate of an investment account that compounds interest.

**Answer:**

When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of nominal.

### Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

4) The average annual population increase of a pack of wolves is $25$.

5) A population of bacteria decreases by a factor of $\frac{1}{8}$ every $24$ hours.

**Answer:**

exponential; the population decreases by a proportional rate.

6) The value of a coin collection has increased by $3.25\%$annually over the last $20$ years.

7) For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.

**Answer:**

not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function.

8) The height of a projectile at time $t$ is represented by the function $h(t) = -4.9t^2 + 18t + 40$

**For the following exercises, consider this scenario: For each year $t$, the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)**

9) Which forest’s population is growing at a faster rate?

**Answer:**

The forest represented by the function $B(t) = 82(1.029)^t$.

10) Which forest had a greater number of trees initially? By how many?

11) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after $20$ years? By how many?

**Answer:**

After $t = 20$ years, forest A will have $43$ more trees than forest B.

12) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after $100$ years? By how many?

13) Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

**Answer:**

Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

**For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.**

14) $y = 300(1 - t)^5$

15) $y = 220(1.06)^x$

**Answer:**

exponential growth; The growth factor, $1.06$ is greater than $1$.

16) $y = 16.5(1.025)^{\frac{1}{x}}$

17) $y = 11,701(0.97)^t$

**Answer:**

exponential decay; The decay factor, $0.97$, is between $0$ and $1$.

For the following exercises, find the formula for an exponential function that passes through the two points given.

18) $(0, 6)$ and $(3, 750)$

19) $(0, 2000)$ and $(2, 20)$

**Answer:**

$f(x) = 2000(0.1)^x$

20) $\left(-1, \frac{3}{2}\right)$ and $(3, 24)$

21) $(-2, 6)$ and $(3, 1)$

**Answer:**

$f(x) = \left(\frac{1}{6}\right)^{-\frac{3}{5}} \left(\frac{1}{6}\right)^{\frac{x}{5}} \approx 2.93(0.699)^x$

22) $(3, 1)$ and $(5, 4)$

**For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.**

23)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 70 | 40 | 10 | -20 |

**Answer:**

Linear

24)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $h(x)$ | 70 | 49 | 34.3 | 24.01 |

25)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $m(x)$ | 80 | 61 | 42.9 | 25.61 |

**Answer:**

Neither

26)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 10 | 20 | 40 | 80 |

27)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $g(x)$ | -3.25 | 2 | 7.25 | 12.5 |

**Answer:**

Linear

For the following exercises, use the compound interest formula, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

28) After a certain number of years, the value of an investment account is represented by the equation $10,250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the value of the account?

29) What was the initial deposit made to the account in the previous exercise?

**Answer:**

$\$10,250$

30) How many years had the account from the previous exercise been accumulating interest?

31) An account is opened with an initial deposit of $\$6,500$ and earns $3.6\%$ interest compounded semi-annually. What will the account be worth in $20$ years?

**Answer:**

$\$13,268.58$

32) How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

33) Solve the compound interest formula for the principal, $P$.

**Answer:**

$P = A(t) \cdot \left(1 + \frac{r}{n}\right)^{-nt}$

34) Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning $5.5\%$ interest compounded monthly for 5 years. (Round to the nearest dollar.)

35) How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?

**Answer:**

$\$4,572.56$

36) Use properties of rational exponents to solve the compound interest formula for the interest rate, $r$.

37) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $\$9,000$ and was worth $\$13,373.53$ after 10 years.

**Answer:**

$4\%$

38) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $\$5,500$ and was worth $\$38,455$ after 30 years.

**For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.**

39) $y = 3742(e)^{0.75t}$

**Answer:**

continuous growth; the growth rate is greater than 0.

40) $y = 150(e)^{\frac{3.25}{t}}$

41) $y = 2.25(e)^{-2t}$

**Answer:**

continuous decay; the growth rate is less than 0.

42) Suppose an investment account is opened with an initial deposit of $\$12,000$ earning $7.2\%$ interest compounded continuously. How much will the account be worth after 30 years?

43) How much less would the account from Exercise 42 be worth after 30 years if it were compounded monthly instead?

**Answer:**

$\$669.42$

## Numeric

**For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.**

44) $f(x) = 2(5)^x$ for $f(-3)$

45) $f(x) = -4^{2x+3}$ for $f(-1)$

**Answer:**

$f(-1) = -4$

46) $f(x) = e^x$, for $f(3)$

47) $f(x) = -2e^{x-1}$, for $f(-1)$

**Answer:**

$f(-1) \approx -0.2707$

48) $f(x) = 2.7(4)^{-x+1} + 1.5$, for $f(-2)$

49) $f(x) = 1.2e^{2x} - 0.3$, for $f(3)$

**Answer:**

$f(3) \approx 483.8146$

50) $f(x) = -\frac{3}{2}(3)^{-x} + \frac{3}{2}$ , for $f(2)$

## Technology

**For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.**

51) $(0, 3)$ and $(3, 375)$

**Answer:**

$y = 3 \cdot 5^x$

52) $(3, 222.62)$and $(10, 77.456)$

53) $(20, 29.495)$and $(150, 730.89)$

**Answer:**

$y \approx 18 \cdot 1.025^x$

54) $(5, 2.909)$and $(13, 0.005)$

55) ((11,310.035)\) and $(25, 356.3652)$

**Answer:**

$y \approx 0.2 \cdot 1.95^x$

## Extensions

56) The annual percentage yield (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $APY = \left(1 + \frac{r}{12}\right)^{12} - 1$

57) Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds $n$ times per year.

**Answer:**

$$\begin{aligned} APY &= \frac{A(t) - a}{a} \\ &= \frac{a\left(1 + \frac{r}{365}\right)^{365(1)} - a}{a} \\ &= \frac{a\left[\left(1 + \frac{r}{365}\right)^{365} - 1\right]}{a} \\ &= \left(1 + \frac{r}{365}\right)^{365} - 1 \end{aligned} \quad ; I(n) = \left(1 + \frac{r}{n}\right)^{n} - 1$$

58) Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that $a$ and $b$ are positive numbers and $b \neq 1$. Any positive number $b$ can be written as $b = e^n$ for some value of $n$. Use this fact to rewrite the formula for an exponential function that uses the number $e$ as a base.

59) In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$, the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f(x) = a \cdot (e)^{-nx}$ for some positive number $n$.

**Answer:**

Let $f$ be the exponential decay function $f(x) = a \cdot \left(\frac{1}{b}\right)^x$ such that $b > 1$. Then for some number $n > 0$,

$$\begin{aligned} f(x) &= a \cdot \left(\frac{1}{b}\right)^x \\ &= a\left(b^{-1}\right)^x \\ &= a\left(\left(e^n\right)^{-1}\right)^x \\ &= a\left(e^{-n}\right)^x \\ &= a(e)^{-nx} \end{aligned}$$

60) The formula for the amount $A$ in an investment account with a nominal interest rate $r$ at any time $t$ is given by $A(t) = a(e)^{rt}$, where $a$ is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time $t$ can be calculated with the formula $I(t) = e^{rt} - 1$

## Real-World Applications

61) The fox population in a certain region has an annual growth rate of $9\%$ per year. In the year 2012, there were $23,900$ fox counted in the area. What is the fox population predicted to be in the year 2020?

**Answer:**

$47,622$fox

62) A scientist begins with $100$ milligrams of a radioactive substance that decays exponentially. After $35$ hours, $50$mg of the substance remains. How many milligrams will remain after $54$ hours?

63) In the year 1985, a house was valued at $\$110,000$ By the year 2005, the value had appreciated to $\$145,000$ What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

**Answer:**

$1.39\%$, $\$155,368.09$

64) A car was valued at $\$38,000$ in the year 2007. By 2013, the value had depreciated to $\$11,000$ If the car's value continues to drop by the same percentage, what will it be worth by 2017?

65) Jamal wants to save $\$54,000$ for a down payment on a home. How much will he need to invest in an account with $8.2\%$ APR, compounding daily, in order to reach his goal in $5$ years?

**Answer:**

$\$35,838.76$

66) Kyoko has $\$10,000$ that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $\$15,000$ by the time she finishes graduate school in $6$ years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (Hint: solve the compound interest formula for the interest rate.)

67) Alyssa opened a retirement account with $7.25\%$ APR in the year 2000. Her initial deposit was $\$13,500$ How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

**Answer:**

$\$82,247.78$ $\$449.75$

68) An investment account with an annual interest rate of $7\%$ was opened with an initial deposit of $\$4,000$ Compare the values of the account after $9$ years when the interest is compounded annually, quarterly, monthly, and continuously.

## 4.2: Graphs of Exponential Functions

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool.

### Verbal

1) What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?

**Answer:**

An asymptote is a line that the graph of a function approaches, as $x$ either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

2) What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

### Algebraic

3) The graph of $f(x)=3^x$ is reflected about the $y$-axis and stretched vertically by a factor of $4$. What is the equation of the new function, $g(x)$. State its $y$-intercept, domain, and range.

**Answer:**

$g(x)=4(3)^{-x}$ ; y-intercept: $(0,4)$; Domain: all real numbers; Range: all real numbers greater than $0$.

4) The graph of $f(x)=\left(\frac{1}{2}\right)^{-x}$ is reflected about the $y$-axis and compressed vertically by a factor of $\left(\frac{1}{5}\right)$. What is the equation of the new function, $g(x)$? State its $y$-intercept, domain, and range.

5) The graph of $f(x)=10^x$ is reflected about the $x$-axis and shifted upward $7$ units. What is the equation of the new function, $g(x)$? State its $y$-intercept, domain, and range.

**Answer:**

$g(x)=-10^x+7$ ; $y$-intercept: $(0,6)$; Domain: all real numbers; Range: all real numbers less than $7$.

6) The graph of $f(x)=(1.68)^x$ is shifted right $3$ units, stretched vertically by a factor of $2$,reflected about the $x$-axis, and then shifted downward $3$ units. What is the equation of the new function, $g(x)$? State its $y$-intercept (to the nearest thousandth), domain, and range.

7) The graph of $f(x)=2\left(\frac{1}{4}\right)^{x-20}$ is shifted left $2$ units, stretched vertically by a factor of $4$,reflected about the $x$-axis, and then shifted downward $4$ units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

**Answer:**

$g(x)=2\left(\frac{1}{4}\right)^{x}$ ; $y$-intercept: $(0,2)$; Domain: all real numbers; Range: all real numbers greater than $0$.

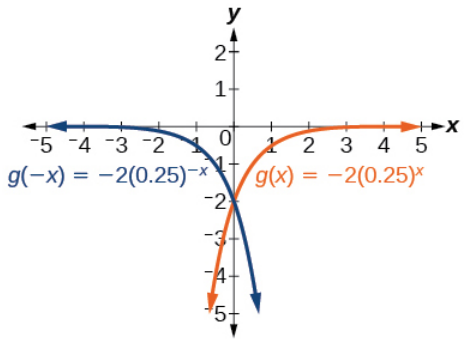
### Graphical

**For the following exercises, graph the function and its reflection about the $y$-axis on the same axes, and give the $y$-intercept.**

8) $f(x)=3\left(\frac{1}{2}\right)^{x}$

9) $g(x)=-2(0.25)^x$

**Answer:**



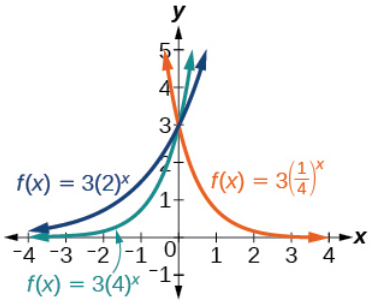

$y$-intercept: $(0, -2)$

10) $h(x) = 6(1.75)^{-x}$

**For the following exercises, graph each set of functions on the same axes.**

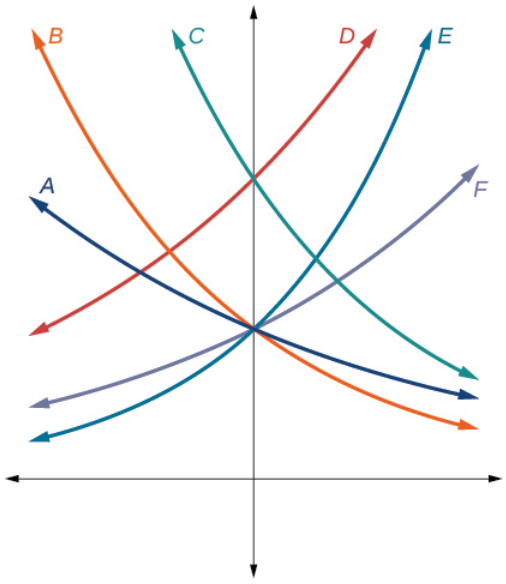
11) $f(x) = 3\left(\frac{1}{4}\right)^x, g(x) = 3(2)^x, h(x) = 3(4)^x$

**Answer:**




12) $f(x) = \frac{1}{4}(3)^x, g(x) = 2(3)^x, h(x) = 4(3)^x$

**For the following exercises, match each function with one of the graphs in Figure below.**




13) $f(x) = 2(0.69)^x$

**Answer:**

B

14) $f(x) = 2(1.28)^x$

15) $f(x) = 2(0.81)^x$

**Answer:**

A

16) $f(x) = 4(1.28)^x$

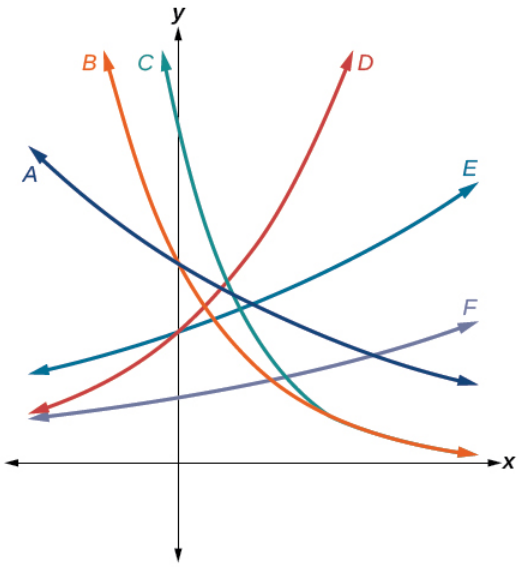
17) $f(x) = 2(1.59)^x$

**Answer:**

E

18) $f(x) = 4(0.69)^x$

**For the following exercises, use the graphs shown in Figure below. All have the form $f(x) = ab^x$.**




19) Which graph has the largest value for $b$?

**Answer:**

D

20) Which graph has the smallest value for $b$?

21) Which graph has the largest value for $a$?

**Answer:**

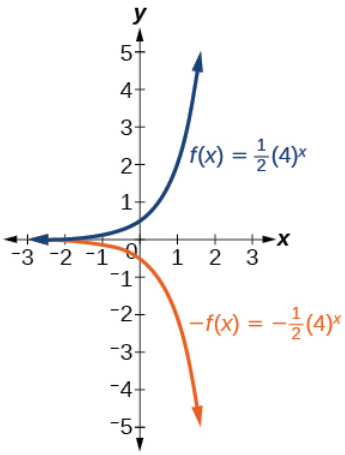
C

22) Which graph has the smallest value for $a$?

**For the following exercises, graph the function and its reflection about the $x$-axis on the same axes.**

23) $f(x) = \frac{1}{2}(4)^x$

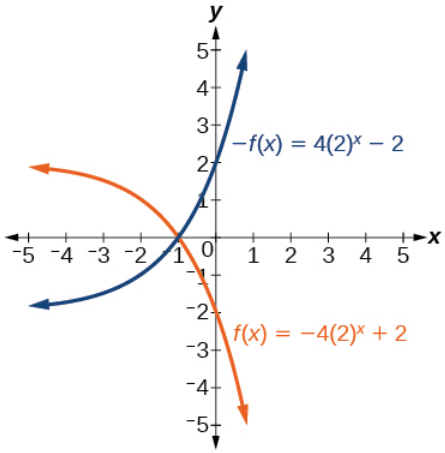
**Answer:**




24) $f(x) = 3(0.75)^x - 1$

25) $f(x) = -4(2)^x + 2$

**Answer:**

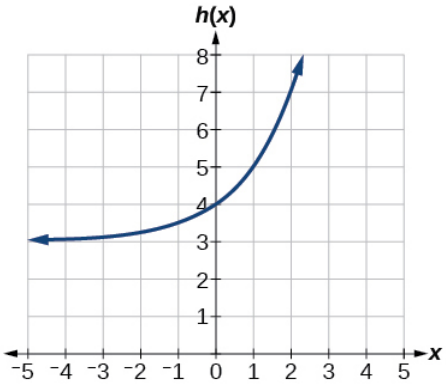



**For the following exercises, graph the transformation of $f(x) = 2^x$. Give the horizontal asymptote, the domain, and the range.**

26) $f(x) = 2^{-x}$

27) $h(x) = 2^x + 3$

**Answer:**



Horizontal asymptote: $h(x) = 3$ ; Domain: all real numbers; Range: all real numbers strictly greater than 3.

28) $f(x) = 2^{x-2}$

**For the following exercises, describe the end behavior of the graphs of the functions.**

29) $f(x) = -5(4)^x - 1$

**Answer:**

As $x \to \infty$, $f(x) \to -\infty$

As $x \to -\infty$, $f(x) \to -1$

30) $f(x) = 3\left(\frac{1}{2}\right)^x - 2$

31) $f(x) = 3(4)^{-x} + 2$

**Answer:**

As $x \to \infty$, $f(x) \to 2$

As $x \to -\infty$, $f(x) \to \infty$

**For the following exercises, start with the graph of $f(x) = 4^x$. Then write a function that results from the given transformation.**

32) Shift $f(x)$ 4 units upward

33) Shift $f(x)$ 3 units downward

**Answer:**

$f(x) = 4^x - 3$

34) Shift $f(x)$ 2 units left

35) Shift $f(x)$ 5 units right

**Answer:**

$f(x) = 4^{x-5}$

36) Reflect $f(x)$ about the $x$-axis

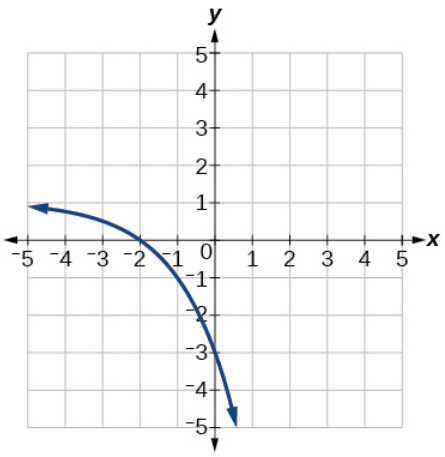
37) Reflect $f(x)$ about the $y$-axis

**Answer:**

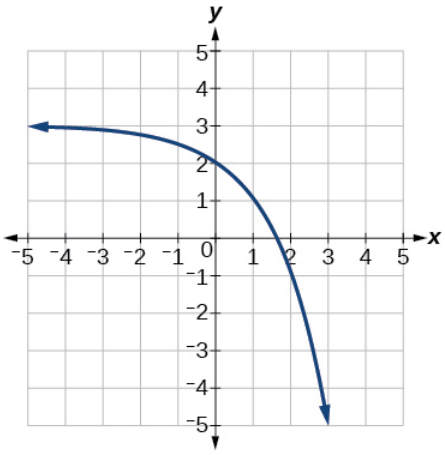
$f(x) = 4^{-x}$

**For the following exercises, each graph is a transformation of $f(x) = 2^x$. Write an equation describing the transformation.**

38)



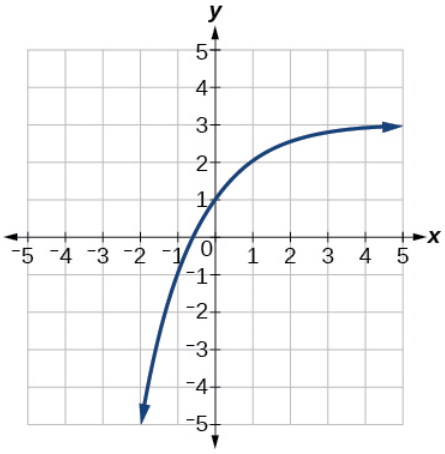

39)




**Answer:**

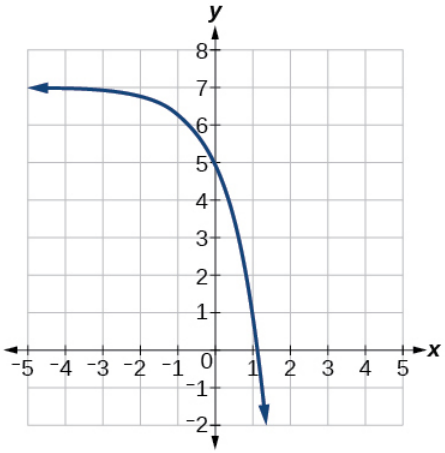
$y = -2^x + 3$

40)




**For the following exercises, find an exponential equation for the graph.**

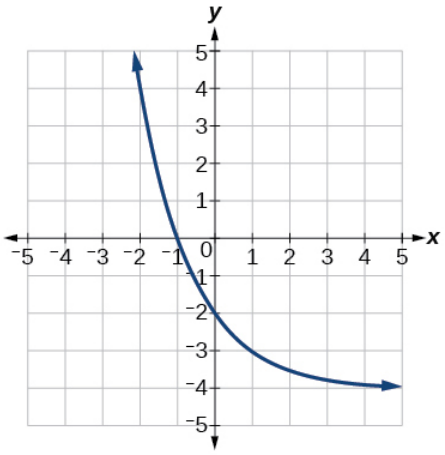
41)




**Answer:**

$y = -2(3)^x + 7$

42)




## Numeric

**For the following exercises, evaluate the exponential functions for the indicated value of $x$.**

43) $g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.

**Answer:**

$g(6) = 800 + \frac{1}{3} \approx 800.3333$

44) $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.

45) $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^x + 6$ for $h(-7)$.

**Answer:**

$h(-7) = -58$

## Technology

**For the following exercises, use a graphing calculator to approximate the solutions of the equation. Round to the nearest thousandth.**

46) $-50 = \left(\frac{1}{2}\right)^{-x}$

47) $116 = \left(\frac{1}{4}\right)\left(\frac{1}{8}\right)^x$

**Answer:**

$x \approx -2.953$

48) $12 = 2(3)^x + 1$

49) $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$

**Answer:**

$x \approx -0.222$

50) $-30 = -4(2)^{x+2} + 2$

### Extensions

51) Explore and discuss the graphs of $F(x) = (b)^x$ and $G(x) = \left(\frac{1}{b}\right)^x$. Then make a conjecture about the relationship between the graphs of the functions $b^x$ and $\left(\frac{1}{b}\right)^x$ for any real number $b > 0$.

**Answer:**

The graph of $G(x) = \left(\frac{1}{b}\right)^x$ is the refelction about the $y$-axis of the graph of $F(x) = (b)^x$; For any real number $b > 0$ and function $f(x) = (b)^x$, the graph of $\left(\frac{1}{b}\right)^x$ is the the reflection about the $y$-axis, $F(-x)$.

52) Prove the conjecture made in the previous exercise.

53) Explore and discuss the graphs of $f(x) = 4^x$, $g(x) = 4^{x-2}$, and $h(x) = \left(\frac{1}{16}\right)4^x$. Then make a conjecture about the relationship between the graphs of the functions $b^x$ and $\left(\frac{1}{b^n}\right)b^x$ for any real number $n$ and real number $b > 0$.

**Answer:**

The graphs of $g(x)$ and $h(x)$ are the same and are a horizontal shift to the right of the graph of $f(x)$; For any real number $n$, real number $b > 0$, and function $f(x) = b^x$, the graph of $\left(\frac{1}{b^n}\right)b^x$ is the horizontal shift $f(x - n)$.

54) Prove the conjecture made in the previous exercise.

## 4.3: Logarithmic Functions

The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.

### Verbal

1) What is a base $b$ logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1$.

**Answer**

A logarithm is an exponent. Specifically, it is the exponent to which a base $b$ is raised to produce a given value. In the expressions given, the base $b$ has the same value. The exponent, $y$, in the expression $b^y$ can also be written as the logarithm, $\log_b x = y$ ,and the value of $x$ is the result of raising $b$ to the power of $y$.

2) How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?

3) How can the logarithmic equation $\log_b x = y$ be solved for $x$ using the properties of exponents?

**Answer**

Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$ , and then properties of exponents can be applied to solve for $x$.

4) Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base $b$, and how does the notation differ?

5) Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base $b$, and how does the notation differ?

**Answer**

The natural logarithm is a special case of the logarithm with base $b$ in that the natural log always has base $e$. Rather than notating the natural logarithm as $\log_e(x)$,the notation used is $\ln(x)$.

### Algebraic

**For the following exercises, rewrite each equation in exponential form.**

6) $\log_4(q) = m$

7) $\log_a(b) = c$

**Answer**

$a^c = b$

8) $\log_{16}(y) = x$

9) $\log_x(64) = y$

**Answer**

$x^y = 64$

10) $\log_y(x) = -11$

11) $\log_{15}(a) = b$

**Answer**

$15^b = a$

12) $\log_y(137) = x$

13) $\log_{13}(142) = a$

**Answer**

$13^a = 142$

14) $\log(v) = t$

15) $\ln(w) = n$

**Answer**

$e^n = w$

**For the following exercises, rewrite each equation in logarithmic form.**

16) $4^x = y$

17) $c^d = k$

**Answer**

$\log_c(k) = d$

18) $m^{-7} = n$

19) $19^{x} = y$

**Answer**

$\log_{19}(y) = x$

20) $x^{-\frac{10}{13}} = y$

21) $n^{4} = 103$

**Answer**

$\log_{n}(103) = 4$

22) $\left(\frac{7}{5}\right)^{m} = n$

23) $y^{x} = \frac{39}{100}$

**Answer**

$\log_{y}\left(\frac{39}{100}\right) = x$

24) $10^{a} = b$

25) $e^{k} = h$

**Answer**

$\ln(w) = n$

**For the following exercises, solve for $x$ by converting the logarithmic equation to exponential form.**

26) $\log_{3}(x) = 2$

27) $\log_{2}(x) = -3$

**Answer**

$x = 2^{-3} = \frac{1}{8}$

28) $\log_{5}(x) = 2$

29) $\log_{3}(x) = 3$

**Answer**

$x = 3^{3} = 27$

30) $\log_{2}(x) = 6$

31) $\log_{9}(x) = \frac{1}{2}$

**Answer**

$x = 9^{\frac{1}{2}} = 3$

32) $\log_{18}(x) = 2$

33) $\log_{6}(x) = -3$

**Answer**

$$x = 6^{-3} = \frac{1}{216}$$

34) $\log(x) = 3$

35) $\ln(x) = 2$

**Answer**

$$x = e^2$$

**For the following exercises, use the definition of common and natural logarithms to simplify.**

36) $\log(100^8)$

37) $10^{\log(32)}$

**Answer**

$$32$$

38) $2\log(.0001)$

39) $e^{\ln(1.06)}$

**Answer**

$$1.06$$

40) $\ln(e^{-5.03})$

41) $e^{\ln(10.125)} + 4$

**Answer**

$$14.125$$

### Numeric

**For the following exercises, evaluate the base $b$ logarithmic expression without using a calculator.**

42) $\log_3\left(\frac{1}{27}\right)$

43) $\log_6(\sqrt{6})$

**Answer**

$$\frac{1}{2}$$

44) $\log_2\left(\frac{1}{8}\right) + 4$

45) $6\log_8(4)$

**Answer**

$$4$$

**For the following exercises, evaluate the common logarithmic expression without using a calculator.**

46) $\log(10,000)$

47) $\log(0.001)$

**Answer**

$$-3$$

48) $\log(1)+7$

49) $2\log(100^{-3})$

**Answer**

$-12$

**For the following exercises, evaluate the natural logarithmic expression without using a calculator.**

50) $\ln\left(e^{\frac{1}{3}}\right)$

51) $\ln(1)$

**Answer**

$0$

52) $\ln\left(e^{-0.225}\right)-3$

53) $25\ln\left(e^{\frac{2}{5}}\right)$

**Answer**

$10$

## Technology

**For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.**

54) $\log(0.04)$

55) $\ln(15)$

**Answer**

$2.708$

56) $\ln\left(\frac{4}{5}\right)$

57) $\log(\sqrt{2})$

**Answer**

$0.151$

58) $\ln(\sqrt{2})$

## Extensions

59) Is $x=0$ in the domain of the function $f(x)=\log x$? If so, what is the value of the function when $x=0$? Verify the result.

**Answer**

No, the function has no defined value for $x=0$. To verify, suppose $x=0$ is in the domain of the function $f(x)=\log(x)$. Then there is some number $n$ such that $n=\log(0)$. Rewriting as an exponential equation gives: $10^n=0$, which is impossible since no such real number $n$ exists. Therefore, $x=0$ is not the domain of the function $f(x)=\log(x)$.

60) Is $f(x)=0$ in the range of the function $f(x)=\log(x)$? If so, for what value of $x$? Verify the result.

61) Is there a number $x$ such that $\ln x=2$? If so, what is that number? Verify the result.

**Answer**

Yes. Suppose there exists a real number $x$ such that $\ln x=2$. Rewriting as an exponential equation gives $x=e^2$, which is a real number. To verify, let $x=e^2$. Then, by definition, $\ln(x)=\ln\left(e^2\right)=2$.

62) Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$ log3(27)log4(164)=−1? log3(27)log4(164)=−1? Verify the result.

63) Is the following true: $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$ Verify the result.

**Answer**

No; $\ln(1) = 0$, so $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$ is undefined.

### Real-World Applications

64) The exposure index $EI$ for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where $f$ is the "f-stop" setting on the camera, and $t$ is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?

65) Refer to the previous exercise. Suppose the light meter on a camera indicates an $EI$ of $-2$, and the desired exposure time is 16 seconds. What should the f-stop setting be?

**Answer**

2

66) The intensity levels $I$ of two earthquakes measured on a seismograph can be compared by the formula $\log\left(\frac{I_1}{I_2}\right) = M_1 - M_2$ where $M$ is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0. How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

## 4.4: Graphs of Logarithmic Functions

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

### Verbal

1) The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

**Answer**

Since the functions are inverses, their graphs are mirror images about the line $y - x$ . So for every point $(a, b)$ on the graph of a logarithmic function, there is a corresponding point $(b, a)$ on the graph of its inverse exponential function.

2) What type(s) of translation(s), if any, affect the range of a logarithmic function?

3) What type(s) of translation(s), if any, affect the domain of a logarithmic function?

**Answer**

Shifting the function right or left and reflecting the function about the $y$-axis will affect its domain.

4) Consider the general logarithmic function $f(x) = \log_b(x)$. Why can't $x$ be zero?

5) Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

**Answer**

No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

### Algebraic

**For the following exercises, state the domain and range of the function.**

6) $f(x) = \log_3(x+4)$

7) $h(x) = \ln\left(\frac{1}{2} - x\right)$

**Answer**

Domain: $\left(-\infty, \frac{1}{2}\right)$; Range: $(-\infty, \infty)$

8) $g(x) = \log_5(2x+9) - 2$

9) $h(x) = \ln(4x+17) - 5$

**Answer**

Domain: $\left(-\frac{17}{4}, \infty\right)$; Range: $(-\infty, \infty)$

10) $f(x) = \log_2(12 - 3x) - 3$

**For the following exercises, state the domain and the vertical asymptote of the function.**

11) $f(x) = \log_b(x-5)$

**Answer**

Domain: $(5, \infty)$; Vertical asymptote: $x = 5$

12) $g(x) = \ln(3-x)$

13) $f(x) = \log(3x+1)$

**Answer**

Domain: $\left(-\frac{1}{3}, \infty\right)$; Vertical asymptote: $x = -\frac{1}{3}$

14) $f(x) = 3\log(-x) + 2$

15) $g(x) = -\ln(3x+9) - 7$

**Answer**

Domain: $(-3, \infty)$; Vertical asymptote: $x = -3$

**For the following exercises, state the domain, vertical asymptote, and end behavior of the function.**

16) $f(x) = \ln(2-x)$

17) $f(x) = \log\left(x - \frac{3}{7}\right)$

**Answer**

Domain: $\left(\frac{3}{7}, \infty\right)$

Vertical asymptote: $x = \frac{3}{7}$

End behavior: as $x \to \left(\frac{3}{7}\right)^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

18) $h(x) = -\log(3x-4) + 3$

19) $g(x) = \ln(2x+6) - 5$

**Answer**

Domain: $(-3, \infty)$

Vertical asymptote: $x = -3$

End behavior: as $x \to -3^+$ , $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

20) $f(x) = \log_3(15 - 5x) + 6$

**For the following exercises, state the domain, range, and x- and y-intercepts, if they exist. If they do not exist, write DNE.**

21) $h(x) = \log_4(x - 1) + 1$

**Answer**

Domain: $(1, \infty)$

Range: $-\infty, \infty$

Vertical asymptote: $x = 1$

$x$-intercept: $\left(\frac{5}{4}, 0\right)$

$y$-intercept: DNE

22) $f(x) = \log(5x + 10) + 3$

23) $g(x) = \ln(-x) - 2$

**Answer**

Domain: $(-\infty, 0)$

Range: $-\infty, \infty$

Vertical asymptote: $x = 0$

$x$-intercept: $\left(-e^2, 0\right)$

$y$-intercept: DNE

24) $f(x) = \log_2(x + 2) - 5$

25) $h(x) = 3\ln(x) - 9$

**Answer**

Domain: $(0, \infty)$

Range: $-\infty, \infty$

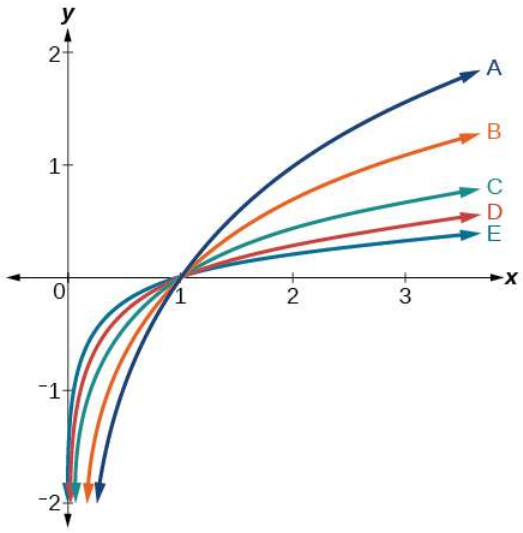
Vertical asymptote: $x = 0$

$x$-intercept: $\left(e^3, 0\right)$

$y$-intercept: DNE

## Graphical

**For the following exercises, match each function in Figure below with the letter corresponding to its graph.**




26) $d(x) = \log(x)$

27) $f(x) = \ln(x)$

**Answer**

$B$

28) $g(x) = \log_2(x)$

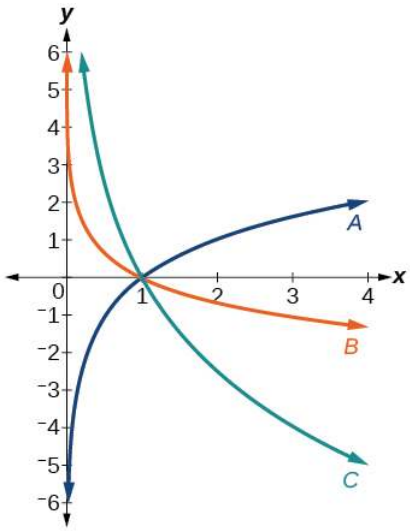
29) $h(x) = \log_5(x)$

**Answer**

$C$

30) $j(x) = \log_{25}(x)$

**For the following exercises, match each function in Figure with the letter corresponding to its graph.**




31) $f(x) = \log_{\frac{1}{3}}(x)$

**Answer**

$B$

32) $g(x) = \log_2(x)$

33) $h(x) = \log_{\frac{3}{4}}(x)$

**Answer**

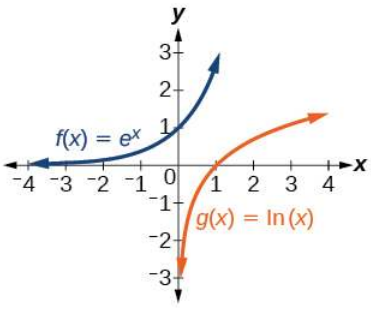
$C$

**For the following exercises, sketch the graphs of each pair of functions on the same axis.**

34) $f(x) = \log(x)$ and $g(x) = 10^x$

35) $f(x) = e^x$ and $g(x) = \ln(x)$

**Answer**




**For the following exercises, match each function in Figure with the letter corresponding to its graph.**

36) $f(x) = \log_4(-x+2)$

37) $g(x) = -\log_4(x+2)$

**Answer**

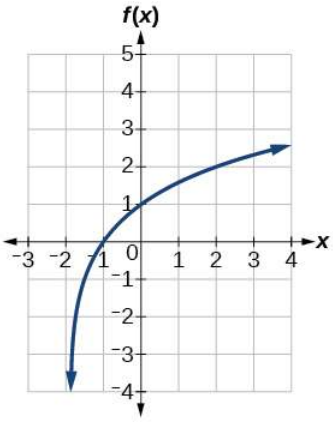
$C$

38) $h(x) = \log_4(x+2)$

**For the following exercises, sketch the graph of the indicated function.**

39) $f(x) = \log_2(x+2)$

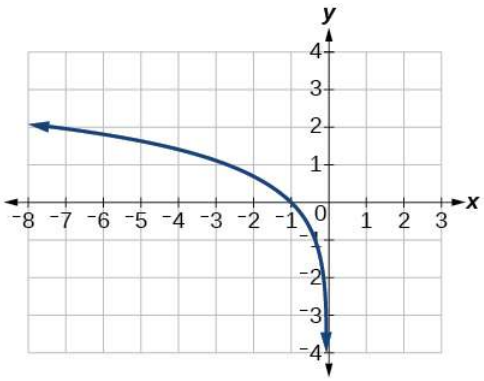
**Answer**




40) $f(x) = 2\log(x)$

41) $f(x) = \ln(-x)$

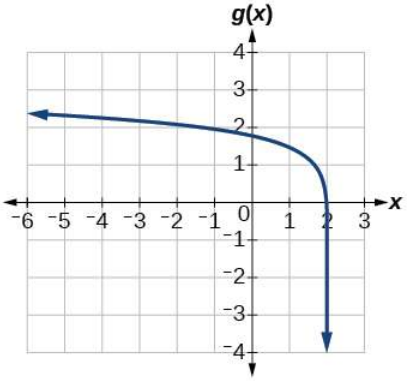
**Answer**




42) $g(x) = \log(4x + 16) + 4$

43) $g(x) = \log(6 - 3x) + 1$

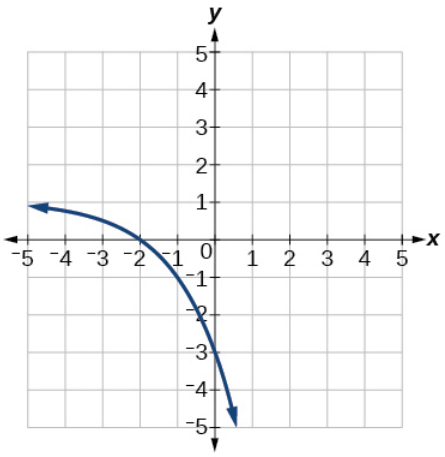
**Answer**




44) $h(x) = -\frac{1}{2}\log(x + 1) - 3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

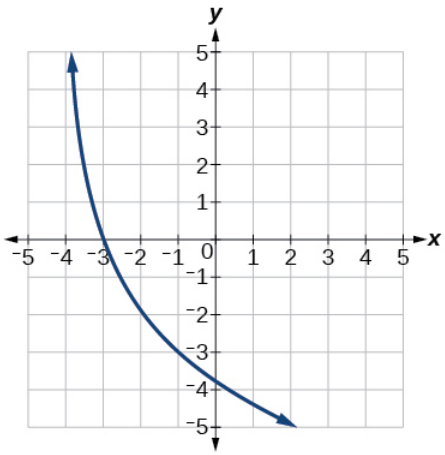
45) Use $y = \log_2(x)$ as the parent function.



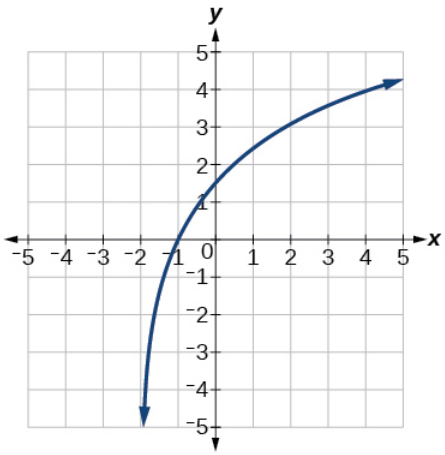

**Answer**

$f(x) = \log_2(-(x - 1))$

46) Use $f(x) = \log_3(x)$ as the parent function.



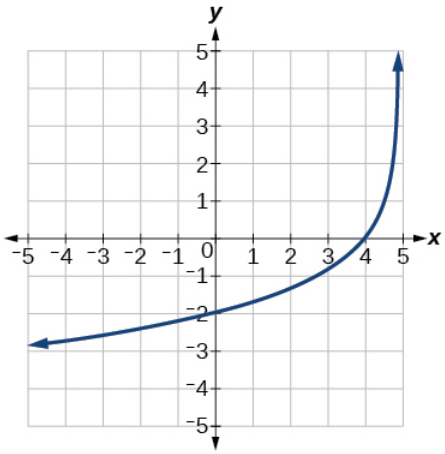

47) Use $f(x) = \log_4(x)$ as the parent function.




**Answer**

$f(x) = 3\log_4(x+2)$

48) Use $f(x) = \log_5(x)$ as the parent function.



## Technology

**For the following exercises, use a graphing calculator to find approximate solutions to each equation.**

49) $\log(x-1)+2=\ln(x-1)+2$

**Answer**

$x=2$

50) $\log(2x-3)+2=-\log(2x-3)+5$

51) $\ln(x-2)+2=-\ln(x+1)$

**Answer**

$x\approx 2.303$

52) $2\ln(5x+1)=\dfrac{1}{2}\ln(-5x)+1$

53) $\dfrac{1}{3}\log(1-x)=\log(x+1)+\dfrac{1}{3}$

**Answer**

$x\approx -0.472$

## Extensions

54) Let $b$ be any positive real number such that $b\neq 1$. What must $\log_b 1$be equal to? Verify the result.

55) Explore and discuss the graphs of $f(x)=\log_{\frac{1}{2}}(x)$ and $g(x)=-\log_2(x)$. Make a conjecture based on the result.

**Answer**

The graphs of $f(x)=\log_{\frac{1}{2}}(x)$ and $g(x)=-\log_2(x)$ appear to be the same;

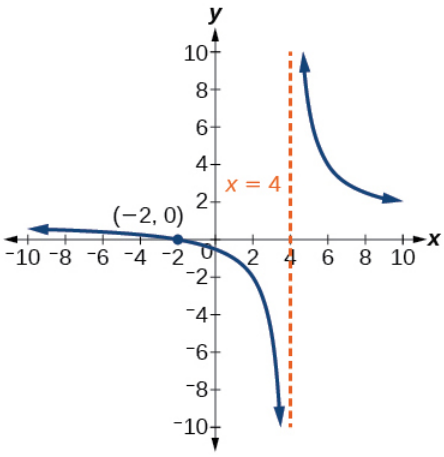
Conjecture: for any positive base $b\neq 1$, $\log_b(x)=\log_{\frac{1}{b}}(x)$

56) Prove the conjecture made in the previous exercise.

57) What is the domain of the function $f(x)=\ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.

**Answer**

Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4}>0$. From the graph of the function $f(x)=\frac{x+2}{x-4}$, note that the graph lies above the $x$-axis on the interval $(-\infty,-2)$ and again to the right of the vertical asymptote, that is $(4,\infty)$. Therefore, the domain is $(-\infty,-2)\cup(4,\infty)$.




58) Use properties of exponents to find the $x$-intercepts of the function $f(x) = \log\left(x^2 + 4x + 4\right)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.

## 4.5: Logarithmic Properties

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here.

### Verbal

1) How does the power rule for logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?

**Answer**

Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_b(x)$.

2) What does the change-of-base formula do? Why is it useful when using a calculator?

### Algebraic

**For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.**

3) $\log_b(7x \cdot 2y)$

**Answer**

$\log_b(2) + \log_b(7) + \log_b(x) + \log_b(y)$

4) $\ln(3ab \cdot 5c)$

5) $\log_b\left(\frac{13}{17}\right)$

**Answer**

$\log_b(13) - \log_b(17)$

6) $\log_4\left(\frac{\frac{x}{z}}{w}\right)$

7) $\ln\left(\frac{1}{4^k}\right)$

**Answer**

$-k\ln(4)$

8) $\log_2(y^x)$

**For the following exercises, condense to a single logarithm if possible.**

9) $\ln(7)+\ln(x)+\ln(y)$

**Answer**

$\ln(7xy)$

10) $\log_3(2)+\log_3(a)+\log_3(11)+\log_3(b)$

11) $\log_b(28)-\log_b(7)$

**Answer**

$\log_b(4)$

12) $\ln(a)-\ln(d)-\ln(c)$

13) $-\log_b\left(\frac{1}{7}\right)$

**Answer**

$\log_b(7)$

14) $\frac{1}{3}\ln(8)$

**For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.**

15) $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

**Answer**

$15\log(x)+13\log(y)-19\log(z)$

16) $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$

17) $\log\left(\sqrt{x^3y^{-4}}\right)$

**Answer**

$\frac{3}{2}\log(x)-2\log(y)$

18) $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

19) $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$

**Answer**

$\frac{8}{3}\log(x)+\frac{14}{3}\log(y)$

**For the following exercises, condense each expression to a single logarithm using the properties of logarithms.**

20) $\log\left(2x^4\right)+\log\left(3x^5\right)$

21) $\ln\left(6x^9\right)-\ln\left(3x^2\right)$

**Answer**

$\ln(2x^7)$

22) $2\log(x)+3\log(x+1)$

23) $\log(x)-\frac{1}{2}\log(y)+3\log(z)$

**Answer**

$\log\left(\frac{xz^3}{\sqrt{y}}\right)$

24) $4\log_7(c)+\frac{\log_7(a)}{3}+\frac{\log_7(b)}{3}$

**For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.**

25) $\log_7(15)$ to base $e$

**Answer**

$\log_7(15)=\frac{\ln(15)}{\ln(7)}$

**For the following exercises, suppose $\log_5(6)=a$ and $\log_5(11)=b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of $a$ and $b$. Show the steps for solving.**

27) $\log_{11}(5)$

**Answer**

$\log_{11}(5)=\frac{\log_5(5)}{\log_5(11)}=\frac{1}{b}$

28) $\log_6(55)$

29) $\log_{11}\left(\frac{6}{11}\right)$

**Answer**

$\log_{11}\left(\frac{6}{11}\right)=\frac{\log_{11}\left(\frac{6}{11}\right)}{\log_5(11)}=\frac{\log_5(6)-\log_5(11)}{\log_5(11)}=\frac{a-b}{b}=\frac{a}{b}-1$

## Numeric

**For the following exercises, use properties of logarithms to evaluate without using a calculator.**

30) $\log_3\left(\frac{1}{9}\right)-3\log_3(3)$

31) $6\log_8(2)+\frac{\log_8(64)}{3\log_8(4)}$

**Answer**

$3$

32) $2\log_9(3)-4\log_9(3)+\log_9\left(\frac{1}{729}\right)$

**For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.**

33) $\log_3(22)$

**Answer**

$2.81359$

34) $\log_8(65)$

35) $\log_6(5.38)$

**Answer**

$0.93913$

36) $\log_4\left(\frac{15}{2}\right)$

37) $\log_{\frac{1}{2}}(4.7)$

**Answer**

$-2.23266$

### Extensions

38) Use the product rule for logarithms to find all $x$ values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2$ . Show the steps for solving.

39) Use the quotient rule for logarithms to find all $x$ values such that $\log_6(x+2)-\log_6(x-3)=1$ . Show the steps for solving.

**Answer**

Rewriting as an exponential equation and solving for $x$:

$$6^1=\frac{x+2}{x-3}$$
$$0=\frac{x+2}{x-3}-6$$
$$0=\frac{x+2}{x-3}-\frac{6(x-3)}{(x-3)}$$
$$0=\frac{x+2-6x+18}{x-3}$$
$$0=\frac{x-4}{x-3}$$
$$x=4$$

Checking, we find that $\log_6(4+2)-\log_6(4-3)=\log_6(6)-\log_6(1)$ is defined, so $x=4$

40) Can the power property of logarithms be derived from the power property of exponents using the equation $b^x=m$? If not, explain why. If so, show the derivation.

41) Prove that $\log_b(n)=\frac{1}{\log_b(n)}$ for any positive integers $b>1$ and $n>1$.

**Answer**

Let $b$ and $n$ be positive integers greater than 1. Then, by the change-of-base formula, $\log_b(n)=\frac{\log_n(n)}{\log_n(b)}=\frac{1}{\log_n(b)}$

42) Does $\log_{81}(2401)=\log_3(7)$? Verify the claim algebraically.

## 4.6: Exponential and Logarithmic Equations

Uncontrolled population growth can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

## Verbal

1) How can an exponential equation be solved?

**Answer**

Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

2) When does an extraneous solution occur? How can an extraneous solution be recognized?

3) When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

**Answer**

The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

## Algebraic

**For the following exercises, use like bases to solve the exponential equation.**

4) $4^{-3v-2} = 4^{-v}$

5) $64 \cdot 4^{3x} = 16$

**Answer**

$x = -\frac{1}{3}$

6) $3^{2x+1} \cdot 3^{x} = 243$

7) $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$

**Answer**

$n = -1$

8) $625 \cdot 5^{3x+3} = 125$

9) $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$

**Answer**

$b = \frac{6}{5}$

10) $\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^{6}$

**For the following exercises, use logarithms to solve.**

11) $9^{x-10} = 1$

**Answer**

$x = 10$

12) $2e^{6x} = 13$

13) $e^{r+10} - 10 = -42$

**Answer**

No solution

14) $2 \cdot 10^{9a} = 29$

15) $-8 \cdot 10^{p+7} - 7 = -24$

**Answer**

$p = \log\left(\frac{17}{8}\right) - 7$

16) $7e^{3n-5} + 5 = -89$

17) $e^{-3k} + 6 = 44$

**Answer**

$k = -\frac{\ln(38)}{3}$

18) $-5e^{9x-8} - 8 = -62$

19) $-6e^{9x+8} + 2 = -74$

**Answer**

$x = \frac{\frac{\ln(38)}{3} - 8}{9}$

20) $2^{x+1} = 5^{2x-1}$

21) $e^{2x} - e^{x} - 132 = 0$

**Answer**

$x = \ln 12$

22) $7e^{8x+8} - 5 = -95$

23) $10e^{8x+3} + 2 = 8$

**Answer**

$x = \frac{\frac{\ln(3)}{5} - 3}{8}$

24) $4e^{3x+3} - 7 = 53$

25) $8e^{-5x-2} - 4 = -90$

**Answer**

No solution

26) $3^{2x+1} = 7^{x-2}$

27) $e^{2x} - e^{x} - 6 = 0$

**Answer**

$x = \ln 3$

28) $3e^{3-3x} + 6 = -31$

**For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.**

29) $\log\left(\frac{1}{100}\right) = -2$

**Answer**

$$10^{-2} = \frac{1}{100}$$

30) $\log_{324}(18) = \frac{1}{2}$

**For the following exercises, use the definition of a logarithm to solve the equation.**

31) $5 \log_7 n = 10$

**Answer**

$n = 49$

32) $-8 \log_9 x = 16$

33) $4 + \log_2(9k) = 2$

**Answer**

$k = \frac{1}{36}$

34) $2 \log(8n + 4) + 6 = 10$

35) $10 - 4 \ln(9 - 8x) = 6$

**Answer**

$x = \frac{9 - e}{8}$

**For the following exercises, use the one-to-one property of logarithms to solve.**

36) $\ln(10 - 3x) = \ln(-4x)$

37) $\log_{13}(5n - 2) = \log_{13}(8 - 5n)$

**Answer**

$n = 1$

38) $\log(x + 3) - \log(x) = \log(74)$

39) $\ln(-3x) = \ln(x^2 - 6x)$

**Answer**

No solution

40) $\log_4(6 - m) = \log_4(3m)$

41) $\ln(x - 2) - \ln(x) = \ln(54)$

**Answer**

No solution

42) $\log_9(2n^2 - 14n) = \log_9(-45 + n^2)$

43) $\ln(x^2 - 10) + \ln(9) = \ln(10)$

**Answer**

$x = \pm\frac{10}{3}$

For the following exercises, solve each equation for $x$.

44) $\log(x + 12) = \log(x) + \log(12)$

45) $\ln(x)+\ln(x-3)=\ln(7x)$

**Answer**

$x=10$

46) $\log_2(7x+6)=3$

47) $\ln(7)+\ln(2-4x^2)=\ln(14)$

**Answer**

$x=0$

48) $\log_8(x+6)-\log_8(x)=\log_8(58)$

49) $\ln(3)-\ln(3-3x)=\ln(4)$

**Answer**

$x=\frac{3}{4}$

50) $\log_3(3x)-\log_3(6)=\log_3(77)$

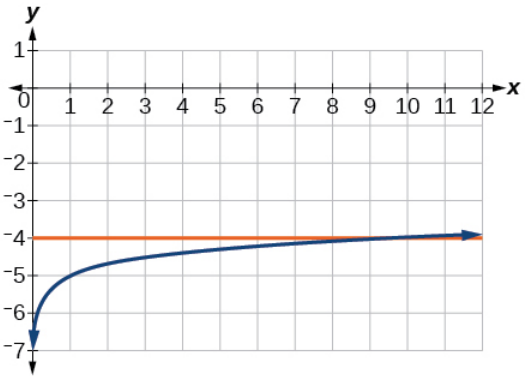
### Graphical

**For the following exercises, solve the equation for $x$, if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.**

51) $\log_9(x)-5=-4$

**Answer**

$x=9$

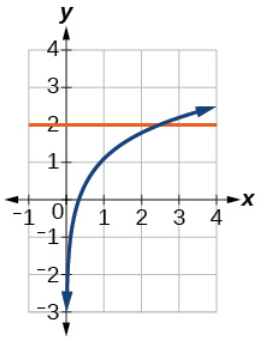



52) $\log_3(x)+3=2$

53) $\ln(3x)=2$

**Answer**

$x=\frac{e^2}{3}\approx 2.5$

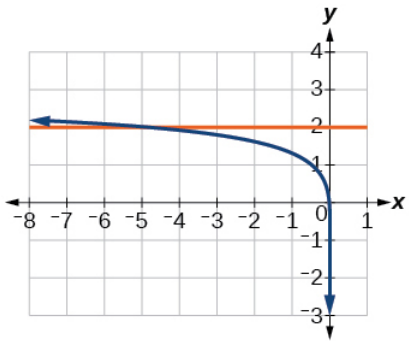



54) $\ln(x-5)=1$

55) $\log(4)+\log(-5x)=2$

**Answer**

$x=-5$

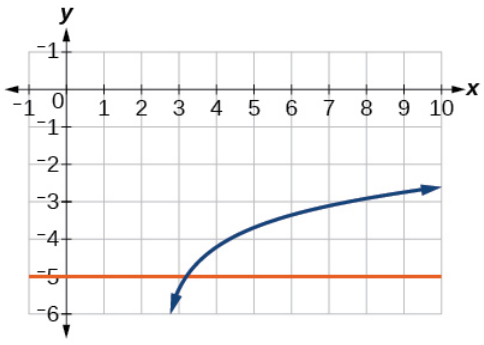



56) $-7+\log_3(4-x)=-6$

57) $\ln(4x-10)-6=-5$

**Answer**

$x=\dfrac{e+10}{4}\approx 3.2$

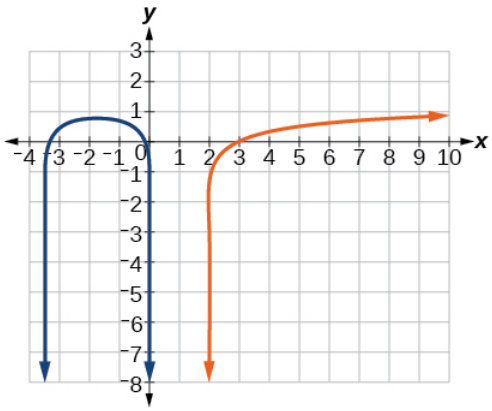



58) $\log(4-2x)=\log(-4x)$

59) $\log_{11}(-2x^2-7x)=\log_{11}(x-2)$

**Answer**

No solution

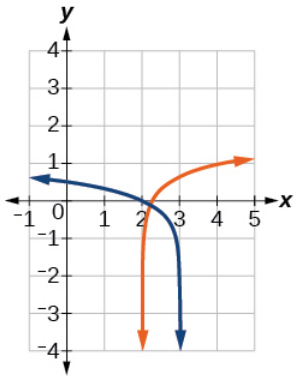


60) $\ln(2x+9) = \ln(-5x)$

61) $\log_9(3-x) = \log_9(4x-8)$

**Answer**

$x = \frac{11}{5} \approx 2.2$

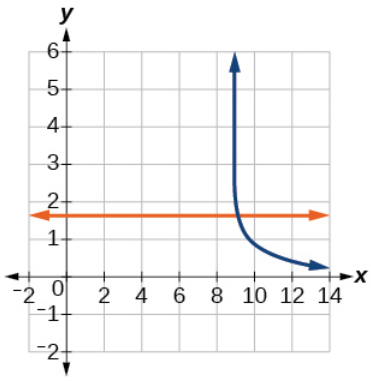


62) $\log(x^2+13) = \log(7x+3)$

63) $\frac{3}{\log_2(10)} - \log(x-9) = \log(44)$

**Answer**

$x = \frac{101}{11} \approx 9.2$



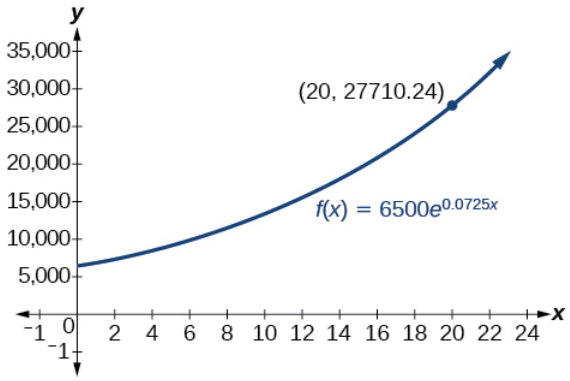
64) $\ln(x) - \ln(x+3) = \ln(6)$

**For the following exercises, solve for the indicated value, and graph the situation showing the solution point.**

65) An account with an initial deposit of $\$6,500$ earns $7.25\%$ annual interest, compounded continuously. How much will the account be worth after $20$ years?

**Answer**

about $\$27,710.24$

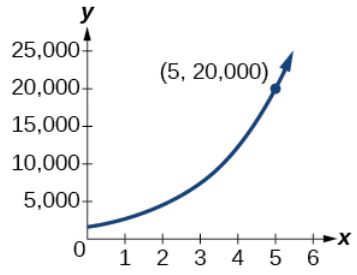



66) The formula for measuring sound intensity in decibels $D$ is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where $I$ is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8 \cdot 3 \cdot 10^2$ watts per square meter?

67) The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where $t$ is measured in years. In approximately how many years will the town's population reach $20,000$?

**Answer**

about $5$ years




## Technology

**For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to 3 decimal places.**

68) $1000(1.03)^t = 5000$using the common log.

69) $e^{5x} = 17$ using the natural log.

**Answer**

$$\frac{\ln(17)}{5} \approx 0.567$$

70) $3(1.04)^{3t} = 8$ using the common log

71) $3^{4x-5} = 38$ using the common log

**Answer**

$$x = \frac{\log(38+5\log(3))}{4\log(3)} \approx 2.078$$

72) $50e^{-0.12t} = 10$ using the natural log

**For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.**

73) $7e^{3x-5} + 7.9 = 47$

**Answer**

$x \approx 2.2401$

74) $\ln(3) + \ln(4.4x + 6.8) = 2$

75) $\log(-0.7x - 9) = 1 + 5\log(5)$

**Answer**

$x \approx -44655.7143$

76) Atmospheric pressure $P$ in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where $x$ is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of $8.369$ pounds per square inch? (Hint: there are $5280$ feet in a mile)

77) The magnitude $M$ of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where $E$ is the amount of energy released by the earthquake in joules $E_0 = 10^{4.4}$ and is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

**Answer**

about $5.83$

## Extensions

78) Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b x} = x$.

79) Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time $t$ such that $t$ is equal to a single logarithm.

**Answer**

$$t = \ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$$

80) Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time $t$.

81) Newton's Law of Cooling states that the temperature $T$ of an object at any time $t$ can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$, where $T_s$ is the temperature of the surrounding environment, $T_0$ is the initial temperature of the object, and $k$ is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time $t$ such that $t$ is equal to a single logarithm.

**Answer**

$$t = \ln\left(\left(\frac{T-T_s}{T_0-T_s}\right)^{-\frac{1}{k}}\right)$$

## 4.7: Exponential and Logarithmic Models

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton's Law of Cooling.

### Verbal

1) With what kind of exponential model would *half-life* be associated? What role does half-life play in these models?

**Answer**

Half-life is a measure of decay and is thus associated with exponential decay models. The half-life of a substance or quantity is the amount of time it takes for half of the initial amount of that substance or quantity to decay.

2) What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.

3) With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?

**Answer**

Doubling time is a measure of growth and is thus associated with exponential growth models. The doubling time of a substance or quantity is the amount of time it takes for the initial amount of that substance or quantity to double in size.

4) Define Newton's Law of Cooling. Then name at least three real-world situations where Newton's Law of Cooling would be applied.

5) What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

**Answer**

An order of magnitude is the nearest power of ten by which a quantity exponentially grows. It is also an approximate position on a logarithmic scale; Sample response: Orders of magnitude are useful when making comparisons between numbers that differ by a great amount. For example, the mass of Saturn is $95$ times greater than the mass of Earth. This is the same as saying that the mass of Saturn is about $10^2$ times, or $2$ orders of magnitude greater, than the mass of Earth.

### Numeric

6) The temperature of an object in degrees Fahrenheit after $t$ minutes is represented by the equation $T(t) = 68e^{-0.0174t} + 72$. To the nearest degree, what is the temperature of the object after one and a half hours?

**For the following exercises, use the logistic growth model** $f(x) = \dfrac{150}{1+8e^{-2x}}$

7) Find and interpret $f(0)$. Round to the nearest tenth.

**Answer**

$f(0) \approx 16.7$; The amount initially present is about $16.7$units.

8) Find and interpret $f(4)$. Round to the nearest tenth.

9) Find the carrying capacity.

**Answer**

$150$

10) Graph the model.

11) Determine whether the data from the table could best be represented as a function that is linear, exponential, or logarithmic. Then write a formula for a model that represents the data.

| $x$ | $f(x)$ |
|---|---|
| –2 | 0.694 |

| $x$ | $f(x)$ |
|---|---|
| –1 | 0.833 |
| 0 | 1 |
| 1 | 1.2 |
| 2 | 1.44 |
| 3 | 1.728 |
| 4 | 2.074 |
| 5 | 2.488 |

**Answer**

exponential; $f(x) = 1.2^x$

12) Rewrite $f(x) = 1.68(0.65)^x$ as an exponential equation with base $e$ to five significant digits.

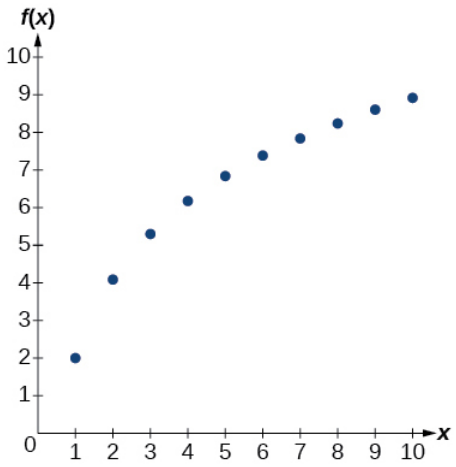
### Technology

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table could represent a function that is linear, exponential, or logarithmic.

13)

| $x$ | $f(x)$ |
|---|---|
| 1 | 2 |
| 2 | 4.079 |
| 3 | 5.296 |
| 4 | 6.159 |
| 5 | 6.828 |
| 6 | 7.375 |
| 7 | 7.838 |
| 8 | 8.238 |
| 9 | 8.592 |
| 10 | 8.908 |

**Answer**



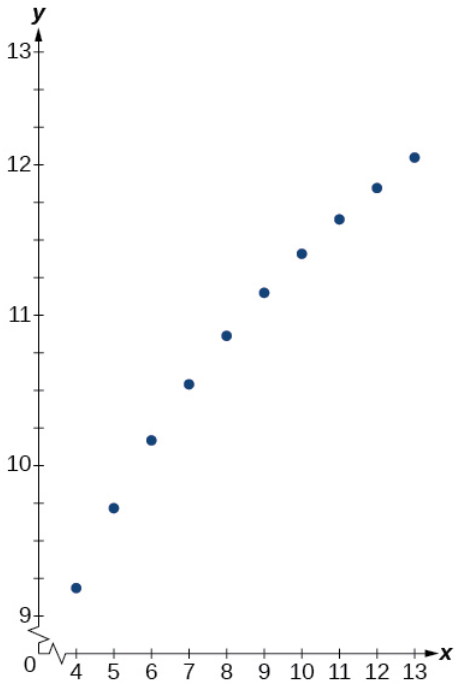

14)

| $x$ | $f(x)$ |
|---|---|
| 1 | 2.4 |
| 2 | 2.88 |
| 3 | 3.456 |
| 4 | 4.147 |
| 5 | 4.977 |
| 6 | 5.972 |
| 7 | 7.166 |
| 8 | 8.6 |
| 9 | 10.32 |
| 10 | 12.383 |

15)

| $x$ | $f(x)$ |
|---|---|
| 4 | 9.429 |
| 5 | 9.972 |
| 6 | 10.415 |
| 7 | 10.79 |
| 8 | 11.115 |
| 9 | 11.401 |
| 10 | 11.657 |
| 11 | 11.889 |
| 12 | 12.101 |
| 13 | 12.295 |

**Answer**



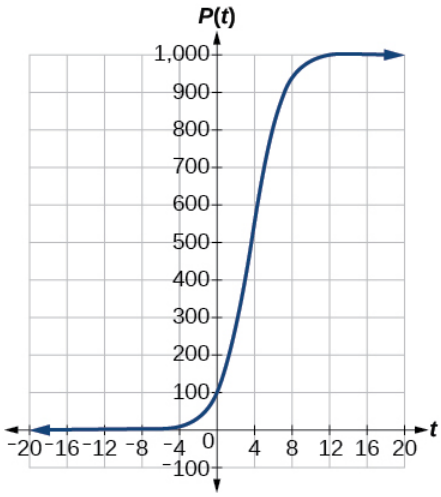

16)

| $x$ | $f(x)$ |
|---|---|
| 1.25 | 5.75 |
| 2.25 | 8.75 |
| 3.56 | 12.68 |
| 4.2 | 14.6 |
| 5.65 | 18.95 |
| 6.75 | 22.25 |
| 7.25 | 23.75 |
| 8.6 | 27.8 |
| 9.25 | 29.75 |
| 10.5 | 33.5 |

**For the following exercises, use a graphing calculator and this scenario: the population of a fish farm in $t$ years is modeled by the equation $P(t) = \dfrac{1000}{1+9e^{-0.6t}}$**

17) Graph the function.

**Answer**




18) What is the initial population of fish?

19) To the nearest tenth, what is the doubling time for the fish population?

**Answer**

about $1.4$ years

20) To the nearest whole number, what will the fish population be after $2$ years?

21) To the nearest tenth, how long will it take for the population to reach $900$?

**Answer**

about $7.3$ years

22) What is the carrying capacity for the fish population? Justify your answer using the graph of $P$.

## Extensions

23) A substance has a half-life of $2.045$ minutes. If the initial amount of the substance was $132.8$ grams, how many half-lives will have passed before the substance decays to $8.3$ grams? What is the total time of decay?

**Answer**

4 half-lives; $8.18$ minutes

24) The formula for an increasing population is given by $P(t) = P_0 e^{rt}$ where $P_0$ is the initial population and $r > 0$. Derive a general formula for the time $t$ it takes for the population to increase by a factor of $M$.

25) Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. Show each step for solving this equation algebraically for the seismic moment $S$.

**Answer**

$$M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$
$$\log\left(\frac{S}{S_0}\right) = \frac{3}{2}M$$
$$\frac{S}{S_0} = 10^{\frac{3M}{2}}$$
$$S = S_0 10^{\frac{3M}{2}}$$

26) What is the $y$-intercept of the logistic growth model $y = \dfrac{c}{1+ae^{-rx}}$ ? Show the steps for calculation. What does this point tell us about the population?

27) Prove that $b^x = e^{x\ln(b)}$ for positive $b \neq 1$.

**Answer**

Let $y = b^x$ for some non-negative real number $b$ such that $b \neq 1$. Then,

$$\ln(y) = \ln(b^x)$$
$$\ln(y) = x\ln(b)$$
$$e^{\ln(y)} = e^{x\ln(b)}$$
$$y = e^{x\ln(b)}$$

## Real-World Applications

**For the following exercises, use this scenario: A doctor prescribes $125$ milligrams of a therapeutic drug that decays by about $30\%$ each hour.**

28) To the nearest hour, what is the half-life of the drug?

29) Write an exponential model representing the amount of the drug remaining in the patient's system after $t$ hours. Then use the formula to find the amount of the drug that would remain in the patient's system after $3$ hours. Round to the nearest milligram.

**Answer**

$A = 125e^{(-0.3567t)}$; $A \approx 43$ mg

30) Using the model found in the previous exercise, find $f(10$ and interpret the result. Round to the nearest hundredth.

**For the following exercises, use this scenario: A tumor is injected with $0.5$ grams of Iodine-125, which has a decay rate of $1.15\%$ per day.**

31) To the nearest day, how long will it take for half of the Iodine-125 to decay?

**Answer**

about $60$ days

32) Write an exponential model representing the amount of Iodine-125 remaining in the tumor after $t$ days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after $60$ days. Round to the nearest tenth of a gram.

33) A scientist begins with $250$ grams of a radioactive substance. After $250$ minutes, the sample has decayed to $32$ grams. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?

**Answer**

$f(t) = 250e^{(-0.00914t)}$; half-life: about $76$ minutes

34) The half-life of Radium-226 is $1590$ years. What is the annual decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

35) The half-life of Erbium-165 is $10.4$ hours. What is the hourly decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

**Answer**

$r \approx -0.0667$, So the hourly decay rate is about $6.67\%$

36) A wooden artifact from an archeological dig contains $60$ percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is $5730$ years.)

37) A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was 1350 bacteria. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest whole number, what is the population size after 3 hours?

**Answer**

$f(t) = 1350e^{(0.03466t)}$; after 3 hours: $P(180) \approx 691,200$

**For the following exercises, use this scenario: A biologist recorded a count of 360 bacteria present in a culture after 5 minutes and 1000 bacteria present after 20 minutes.**

38) To the nearest whole number, what was the initial population in the culture?

39) Rounding to six significant digits, write an exponential equation representing this situation. To the nearest minute, how long did it take the population to double?

**Answer**

$f(t) = 256e^{(0.068110t)}$; doubling time: about 10 minutes

**For the following exercises, use this scenario: A pot of boiling soup with an internal temperature of 100° Fahrenheit was taken off the stove to cool in a 69° F room. After fifteen minutes, the internal temperature of the soup was 95° F.**

40) Use Newton's Law of Cooling to write a formula that models this situation.

41) To the nearest minute, how long will it take the soup to cool to 80° F?

**Answer**

about 88 minutes

42) To the nearest degree, what will the temperature be after 2 and a half hours?

**For the following exercises, use this scenario: A turkey is taken out of the oven with an internal temperature of 165° F and is allowed to cool in a 75° F room. After half an hour, the internal temperature of the turkey is 145° F.**

43) Write a formula that models this situation.

**Answer**

$T(t) = 90e^{(-0.008377t)} + 75$, where $t$ is in minutes.

44) To the nearest degree, what will the temperature be after 50 minutes?

45) To the nearest minute, how long will it take the turkey to cool to 110° F?

**Answer**

about 113 minutes

**For the following exercises, find the value of the number shown on each logarithmic scale. Round all answers to the nearest thousandth.**



Figure_04_07_207.jpg" src="/@api/deki/files/13777/CNX_PreCalc_Figure_04_07_207.jpg" />

**Answer**

$\log(x) = 1.5; x \approx 31.623$

48) Plot each set of approximate values of intensity of sounds on a logarithmic scale: Whisper: $10^{-10}\frac{W}{m^2}$, Vacuum: $10^{-4}\frac{W}{m^2}$, Jet: $10^{2}\frac{W}{m^2}$

49) Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. One earthquake has magnitude $3.9$ on the MMS scale. If a second earthquake has $750$ times as much energy as the first, find the magnitude of the second quake. Round to the nearest hundredth.

**Answer**

MMS magnitude: $5.82$

**For the following exercises, use this scenario: The equation $N(t) = \frac{500}{1+49e^{-0.7t}}$ models the number of people in a town who have heard a rumor after $t$ days.**

50) How many people started the rumor?

51) To the nearest whole number, how many people will have heard the rumor after $3$ days?

**Answer**

$N(3) \approx 71$

52) As t t t increases without bound, what value does $N(t)$ approach? Interpret your answer.

**For the following exercise, choose the correct answer choice.**

53) A doctor and injects a patient with $13$ milligrams of radioactive dye that decays exponentially. After $12$ minutes, there are $4.75$ milligrams of dye remaining in the patient's system. Which is an appropriate model for this situation?

a. $f(t) = 13(0.0805)^t$
b. $f(t) = 13e^{0.9195t}$
c. $f(t) = 13e^{(-0.0839t)}$
d. $f(t) = \frac{4.75}{1+13e^{-0.83925t}}$

**Answer**

c

## 4.8: Fitting Exponential Models to Data

We will concentrate on three types of regression models in this section: exponential, logarithmic, and logistic. Having already worked with each of these functions gives us an advantage. Knowing their formal definitions, the behavior of their graphs, and some of their real-world applications gives us the opportunity to deepen our understanding. As each regression model is presented, key features and definitions of its associated function are included for review.

### Verbal

1) What situations are best modeled by a logistic equation? Give an example, and state a case for why the example is a good fit.

**Answer**

Logistic models are best used for situations that have limited values. For example, populations cannot grow indefinitely since resources such as food, water, and space are limited, so a logistic model best describes populations.

2) What is a carrying capacity? What kind of model has a carrying capacity built into its formula? Why does this make sense?

3) What is regression analysis? Describe the process of performing regression analysis on a graphing utility.

**Answer**

Regression analysis is the process of finding an equation that best fits a given set of data points. To perform a regression analysis on a graphing utility, first list the given points using the STAT then EDIT menu. Next graph the scatter plot using the STAT PLOT feature. The shape of the data points on the scatter graph can help determine which regression feature to use. Once this is determined, select the appropriate regression analysis command from the STAT then CALC menu.

4) What might a scatterplot of data points look like if it were best described by a logarithmic model?

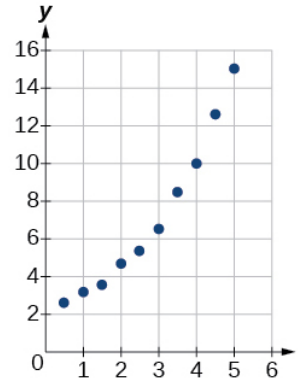
5) What does the y-intercept on the graph of a logistic equation correspond to for a population modeled by that equation?

**Answer**

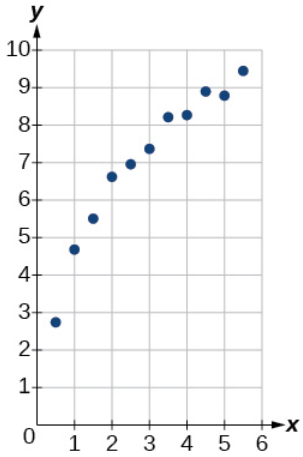
The $y$-intercept on the graph of a logistic equation corresponds to the initial population for the population model.

## Graphical

**For the following exercises, match the given function of best fit with the appropriate scatterplot in Figure (a) through Figure (e). Answer using the letter beneath the matching graph.**

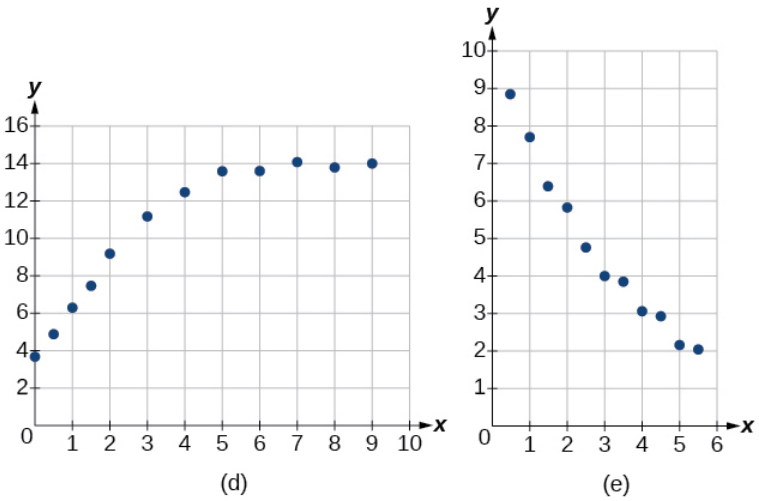



(a)




(b)

Figure_04_08_203.jpg" src="/@api/deki/files/13803/CNX_PreCalc_Figure_04_08_203.jpg" />




6) $y = 10.209e^{-0.294x}$

7) $y = 5.598 - 1.912\ln(x)$

**Answer**

c

8) $y = 2.104(1.479)^x$

9) $y = 4.607 + 2.733\ln(x)$

**Answer**

b

10) $y = \dfrac{14.005}{1 + 2.79e^{-0.812x}}$

## Numeric

11) To the nearest whole number, what is the initial value of a population modeled by the logistic equation $P(t) = \dfrac{175}{1 + 6.995e^{-0.68t}}$? What is the carrying capacity?

**Answer**

$P(0) = 22;\ 175$

12) Rewrite the exponential model $A(t) = 1550(1.085)^x$ as an equivalent model with base $e$. Express the exponent to four significant digits.

13) A logarithmic model is given by the equation $h(p) = 67.682 - 5.792\ln(p)$. To the nearest hundredth, for what value of $p$ does $h(p) = 62$?

**Answer**

$p \approx 2.67$

14) A logistic model is given by the equation $P(t) = \dfrac{90}{1 + 5e^{-0.42t}}$. To the nearest hundredth, for what value of $t$ does $P(t) = 45$?

15) What is the $y$-intercept on the graph of the logistic model given in the previous exercise?

**Answer**

$y$-intercept: $(0, 15)$

Technology

**For the following exercises, use this scenario: The population $P$ of a koi pond over $x$ months is modeled by the function $P(x) = \frac{68}{1+16e^{-0.28x}}$.**

16) Graph the population model to show the population over a span of 3 years.

17) What was the initial population of koi?

**Answer**

\(4)\ koi

18) How many koi will the pond have after one and a half years?

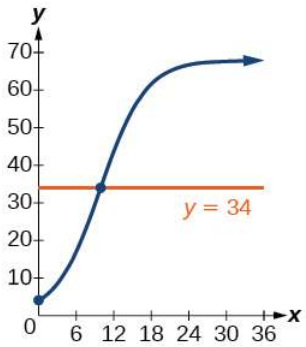
19) How many months will it take before there are 20 koi in the pond?

**Answer**

about 6.8 months

20) Use the intersect feature to approximate the number of months it will take before the population of the pond reaches half its carrying capacity.

**Answer**




**For the following exercises, use this scenario: The population $P$ of an endangered species habitat for wolves is modeled by the function $P(x) = \frac{558}{1+54.8e^{-0.462x}}$, where $x$ is given in years.**

21) Graph the population model to show the population over a span of 10 years.

22) What was the initial population of wolves transported to the habitat?

**Answer**

\(10)\ wolves

23) How many wolves will the habitat have after 3 years?

24) How many years will it take before there are 100 wolves in the habitat?

**Answer**

about 5.4 years

25) Use the intersect feature to approximate the number of years it will take before the population of the habitat reaches half its carrying capacity.

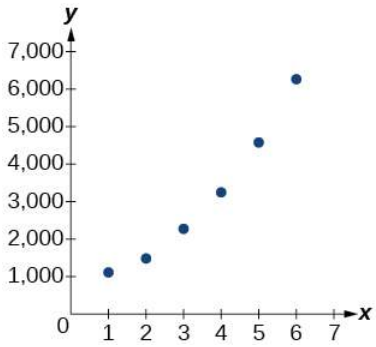
**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|

| $x$ | $f(x)$ |
|---|---|
| 1 | 1125 |
| 2 | 1495 |
| 3 | 2310 |
| 4 | 3294 |
| 5 | 4650 |
| 6 | 6361 |

26) Use a graphing calculator to create a scatter diagram of the data.

**Answer**




27) Use the regression feature to find an exponential function that best fits the data in the table.

28) Write the exponential function as an exponential equation with base $e$.

**Answer**

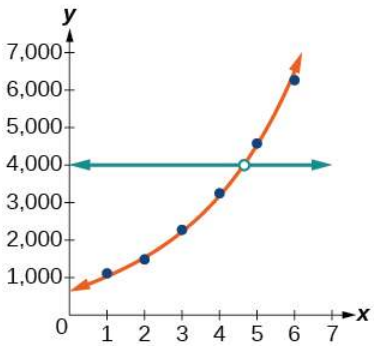
$f(x) = 776.682e^{0.3549x}$

29) Graph the exponential equation on the scatter diagram.

30) Use the intersect feature to find the value of $x$ for which $f(x) = 4000$.

**Answer**

When $f(x) = 4000$, $x \approx 4.6$




**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 1 | 555 |

| $x$ | $f(x)$ |
|---|---|
| 2 | 383 |
| 3 | 307 |
| 4 | 210 |
| 5 | 158 |
| 6 | 122 |

31) Use a graphing calculator to create a scatter diagram of the data.

32) Use the regression feature to find an exponential function that best fits the data in the table.

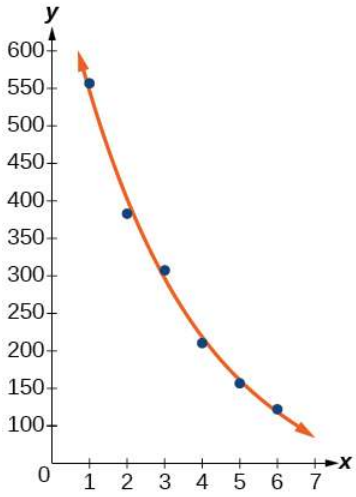
**Answer**

$f(x) = 731.92(0.738)^x$

33) Write the exponential function as an exponential equation with base $e$.

34) Graph the exponential equation on the scatter diagram.

**Answer**



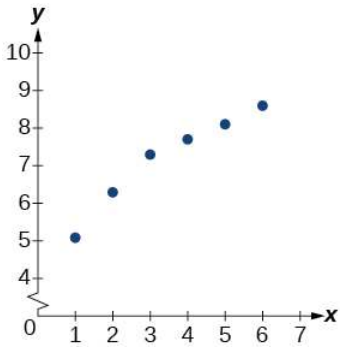

35) Use the intersect feature to find the value of $x$ for which $f(x) = 250$.

**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 1 | 5.1 |
| 2 | 6.3 |
| 3 | 7.3 |
| 4 | 7.7 |
| 5 | 8.1 |
| 6 | 8.6 |

36) Use a graphing calculator to create a scatter diagram of the data.

**Answer**




37) Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

38) Use the logarithmic function to find the value of the function when $x = 10$.

**Answer**

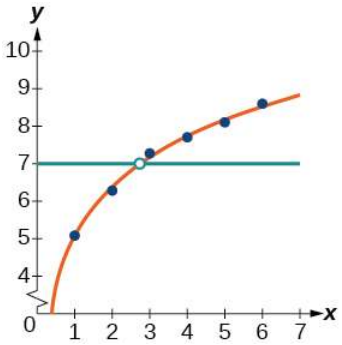
$f(10) \approx 9.5$

39) Graph the logarithmic equation on the scatter diagram.

40) Use the intersect feature to find the value of $x$ for which $f(x) = 7$.

**Answer**

When $f(x) = 7$, $x \approx 2.7$




**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 1 | 7.5 |
| 2 | 6 |
| 3 | 5.2 |
| 4 | 4.3 |
| 5 | 3.9 |
| 6 | 3.4 |
| 7 | 3.1 |
| 8 | 2.9 |

41) Use a graphing calculator to create a scatter diagram of the data.

42) Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

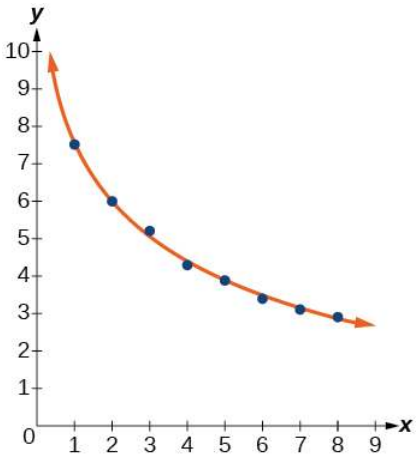
**Answer**

$f(x) = 7.544 - 2.268\ln(x)$

43) Use the logarithmic function to find the value of the function when $x = 10$.

44) Graph the logarithmic equation on the scatter diagram.

**Answer**



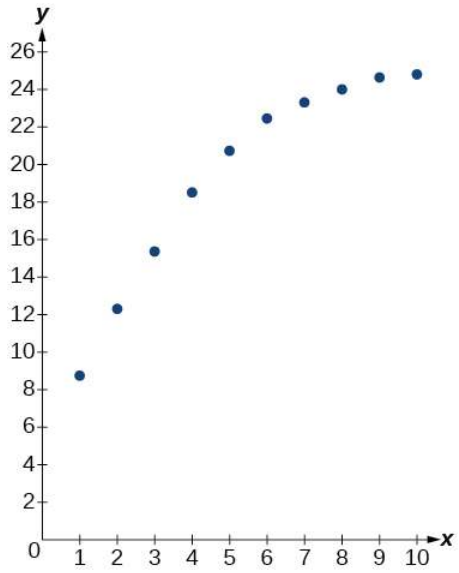

45) Use the intersect feature to find the value of $x$ for which $f(x) = 8$.

**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 1 | 8.7 |
| 2 | 12.3 |
| 3 | 15.4 |
| 4 | 18.5 |
| 5 | 20.7 |
| 6 | 22.5 |
| 7 | 23.3 |
| 8 | 24 |
| 9 | 24.6 |
| 10 | 24.8 |

46) Use a graphing calculator to create a scatter diagram of the data.

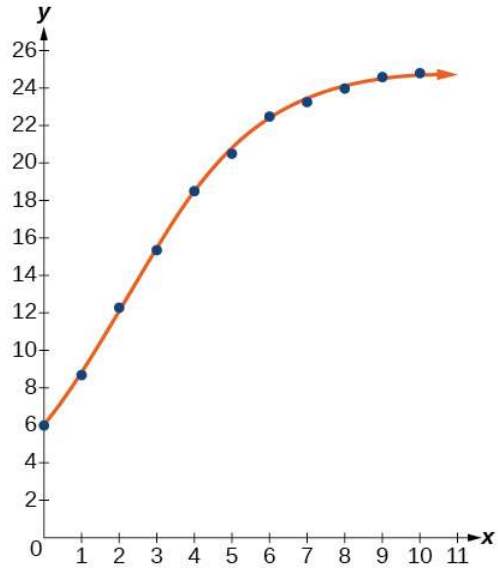
**Answer**




47) Use the LOGISTIC regression option to find a logistic growth model of the form $y = \dfrac{c}{1 + ae^{-bx}}$ that best fits the data in the table.

48) Graph the logistic equation on the scatter diagram.

**Answer**

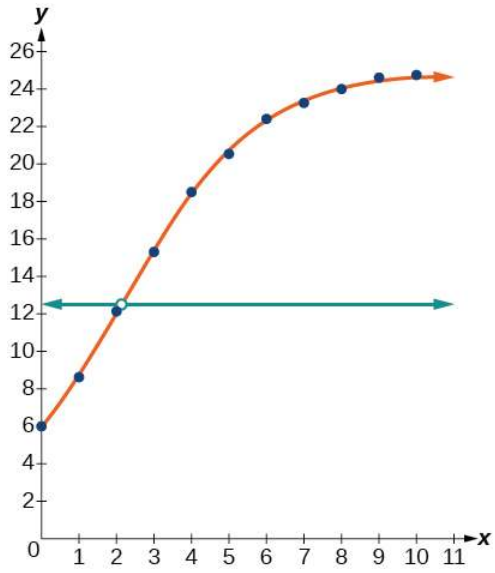



49) To the nearest whole number, what is the predicted carrying capacity of the model?

50) Use the intersect feature to find the value of $x$ for which the model reaches half its carrying capacity.

**Answer**

When $f(x) = 12.5$, $x \approx 2.1$




**For the following exercises, refer to Table below.**

| $x$ | $f(x)$ |
|---|---|
| 0 | 12 |
| 2 | 28.6 |
| 4 | 52.8 |
| 5 | 70.3 |
| 7 | 99.9 |
| 8 | 112.5 |
| 10 | 125.8 |
| 11 | 127.9 |
| 15 | 135.1 |
| 17 | 135.9 |

51) Use a graphing calculator to create a scatter diagram of the data.

52) Use the LOGISTIC regression option to find a logistic growth model of the form $y = \dfrac{c}{1 + ae^{-bx}}$ that best fits the data in the table.

**Answer**

$$y = \frac{136.068}{1 + 10.324e^{-0.480x}}$$

53) Graph the logistic equation on the scatter diagram.

54) To the nearest whole number, what is the predicted carrying capacity of the model?

**Answer**

about $136$

55) Use the intersect feature to find the value of $x$ for which the model reaches half its carrying capacity.

## Extensions

56) Recall that the general form of a logistic equation for a population is given by $P(t)=\dfrac{c}{1+ae^{-bt}}$, such that the initial population at time $t=0$ is $P(0)=P_0$. Show algebraically that $\dfrac{c-P(t)}{P(t)}=\dfrac{c-P_0}{P_0}e^{-bt}$.

**Answer**

Working with the left side of the equation, we see that it can be rewritten as $ae^{-bt}$:

$$\begin{aligned}\frac{c-P(t)}{P(t)} &= \frac{c-\frac{c}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}}\\ &= \frac{\frac{c(1+ae^{-bt})-c}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}}\\ &= \frac{\frac{c(1+ae^{-bt}-1)}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}}\\ &= 1+ae^{-bt}-1\\ &= ae^{-bt}\end{aligned}$$

$$\begin{aligned}P_0 &= \frac{c}{1+ae^{-b(0)}}\\ &= \frac{c}{1+a}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{c-P_0}{P_0}e^{-bt} &= \frac{c-\frac{c}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= \frac{\frac{c(1+a)-c}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= \frac{\frac{c(1+a-1)}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= (1+a-1)e^{-bt}\\ &= ae^{-bt}\end{aligned}$$

Thus,

$$\frac{c-P(t)}{P(t)}=\frac{c-P_0}{P_0}e^{-bt}$$

57) Use a graphing utility to find an exponential regression formula $f(x)$ and a logarithmic regression formula $g(x)$ for the points $(1.5,1.5)$ and $(8.5,8.5$ Round all numbers to 6 decimal places. Graph the points and both formulas along with the line $y=x$ on the same axis. Make a conjecture about the relationship of the regression formulas.

58) Verify the conjecture made in the previous exercise. Round all numbers to six decimal places when necessary.

**Answer**

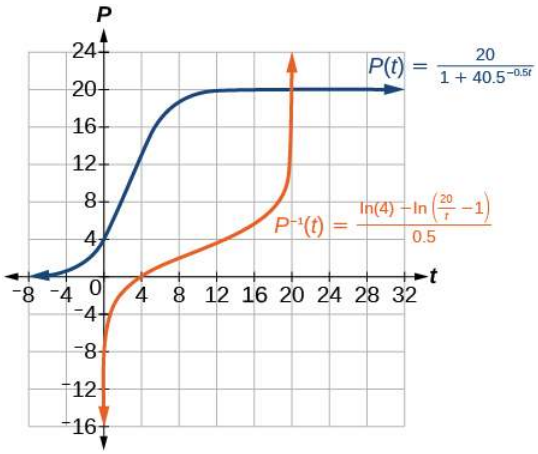
First rewrite the exponential with base e: $f(x)=1.034341e^{0.247800x}$. Then test to verify that $f(g(x))=x$ ,taking rounding error into consideration:

$$
\begin{aligned}
g(f(x)) &= 4.035510\ln\left(1.034341e^{0.247800x}\right) - 0.136259 \\
&= 4.03551\left(\ln(1.034341) + \ln\left(e^{0.2478x}\right)\right) - 0.136259 \\
&= 4.03551(\ln(1.034341) + 0.2478x) - 0.136259 \\
&= 0.136257 + 0.999999x - 0.136259 \\
&= -0.000002 + 0.999999x \\
&\approx 0 + x \\
&= x
\end{aligned}
$$

59) Find the inverse function $f^{-1}(x)$ for the logistic function $f(x) = \dfrac{c}{1+ae^{-bx}}$. Show all steps.

60) Use the result from the previous exercise to graph the logistic model $P(t) = \dfrac{20}{1+4e^{-0.5t}}$ along with its inverse on the same axis. What are the intercepts and asymptotes of each function?

**Answer**




The graph of $P(t)$ has a $y$-intercept at $(0, 4)$ and horizontal asymptotes at $y = 0$ and $y = 20$. The graph of $P^{-1}(t)$ has an $x$-intercept at $(4, 0)$ and vertical asymptotes at $x = 0$ and $x = 20$.

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# 4.R: Exponential and Logarithmic Functions (Review)

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# Index


## A

absolute value equation
2.7: Absolute Value Functions
Absolute Value Function
2.2: Functions and Function Notation
absolute value inequality
2.7: Absolute Value Functions
action plan
1.8: Goal Setting and Motivation
annotating notes
1.5: Taking Notes
Arrow Notation
3.7: Rational Functions
average rate of change
2.4: Rates of Change and Behavior of Graphs
axis of symmetry
3.2: Quadratic Functions Review

## B

breaking tasks into steps
1.7: Enhanced Strategies for Time and Task Management

## C

category of learning
1.0: The Power to Learn
circles
4.9: Trigonometry Preview - Circles
coefficient
3.3: Power Functions and Polynomial Functions Behavior
common logarithms
4.3: Logarithmic Functions
completing the square
2.1.0: Extracting Square Roots and Completing the Square (Review)
complex conjugate
3.1: Complex Numbers Review
3.E: Polynomial and Rational Functions(Exercises)
3.R: Polynomial and Rational Functions(Review)
complex number
3.1: Complex Numbers Review
3.E: Polynomial and Rational Functions(Exercises)
3.R: Polynomial and Rational Functions(Review)
complex plane
3.1: Complex Numbers Review
3.E: Polynomial and Rational Functions(Exercises)
3.R: Polynomial and Rational Functions(Review)
composite function
2.6: Composition of Functions
compound interest
4.1: Exponential Functions
concept mapping
1.5: Taking Notes
constant function
2.2: Functions and Function Notation
continuous function
3.3: Power Functions and Polynomial Functions Behavior
continuous growth formula
4.1: Exponential Functions
Cornell Method
1.5: Taking Notes
Cube root function
2.2: Functions and Function Notation
cubic function
2.2: Functions and Function Notation

## D

daily top three tasks
1.7: Enhanced Strategies for Time and Task Management
decipherable shorthand note taking
1.5: Taking Notes
definition of a logarithm
4.6: Exponential and Logarithmic Equations
Definition of exponential functions
4.1: Exponential Functions
Definition of exponential growth
4.1: Exponential Functions
degree
3.3: Power Functions and Polynomial Functions Behavior
dependent variable
2.2: Functions and Function Notation
Descartes' Rule of Signs
3.6: Zeros of Polynomial Functions
direct variation
3.9: Modeling Using Variation (optional)
division algorithm
3.5: Dividing Polynomials
domain
2.2: Functions and Function Notation
2.3: Domain and Range
3.7: Rational Functions
3.8: Inverses and Radical Functions
doubling time
4.7: Exponential and Logarithmic Models

## E

effective scheduling
1.7: Enhanced Strategies for Time and Task Management
end behavior
3.3: Power Functions and Polynomial Functions Behavior
3.4: Graphs of Polynomial Functions
enhancing notes
1.5: Taking Notes
even function
2.5: Transformation of Functions
exponential regression
4.8: Fitting Exponential Models to Data
extracting the root
2.1.0: Extracting Square Roots and Completing the Square (Review)
extrema
2.4: Rates of Change and Behavior of Graphs

## F

Factor Theorem
3.6: Zeros of Polynomial Functions
fixed vs. growth mindset
1.1: It's All in the Mindset
function
2.2: Functions and Function Notation
Fundamental Theorem of Algebra
3.6: Zeros of Polynomial Functions

## G

General form for the translation of the parent function
4.2: Graphs of Exponential Functions
General form for the translation of the parent logarithmic function
4.4: Graphs of Logarithmic Functions
general form of a polynomial function
3.3: Power Functions and Polynomial Functions Behavior
general form of a quadratic function
3.2: Quadratic Functions Review
global maximum
3.4: Graphs of Polynomial Functions
global minimum
3.4: Graphs of Polynomial Functions

## H

horizontal asymptote
3.7: Rational Functions
horizontal line test
2.2: Functions and Function Notation
horizontal reflection
2.5: Transformation of Functions
Horizontal shifts
2.5: Transformation of Functions
4.2: Graphs of Exponential Functions
horizontal stretch
2.5: Transformation of Functions
4.2: Graphs of Exponential Functions

## I

identity function
2.2: Functions and Function Notation
imaginary number
3.1: Complex Numbers Review
3.E: Polynomial and Rational Functions(Exercises)
3.R: Polynomial and Rational Functions(Review)
imaginary unit
3.1: Complex Numbers Review
3.E: Polynomial and Rational Functions(Exercises)
3.R: Polynomial and Rational Functions(Review)
independent variable
2.2: Functions and Function Notation
input
2.2: Functions and Function Notation
Intermediate Value Theorem
3.4: Graphs of Polynomial Functions
Interval Notation
2.3: Domain and Range
inverse function
2.8: Inverse Functions
3.8: Inverses and Radical Functions
Inverse of a polynomial function
3.8: Inverses and Radical Functions
inverse variation
3.9: Modeling Using Variation (optional)

## J

joint variation
3.9: Modeling Using Variation (optional)

## L

leading coefficient
3.3: Power Functions and Polynomial Functions Behavior
leading term
3.3: Power Functions and Polynomial Functions Behavior
local maximum
2.4: Rates of Change and Behavior of Graphs
local minimum
2.4: Rates of Change and Behavior of Graphs
Logarithmic Function
4.3: Logarithmic Functions
logarithmic regression
4.8: Fitting Exponential Models to Data
logistic growth
4.7: Exponential and Logarithmic Models
4.8: Fitting Exponential Models to Data
long division
3.5: Dividing Polynomials

## M

maxima
2.4: Rates of Change and Behavior of Graphs
minima
2.4: Rates of Change and Behavior of Graphs
motivational goals
1.8: Goal Setting and Motivation
multiplicity
3.4: Graphs of Polynomial Functions

## N

natural logarithm
4.3: Logarithmic Functions
Newton's Law of Cooling
4.7: Exponential and Logarithmic Models
nota taking systems
1.5: Taking Notes
note organizing systems
1.5: Taking Notes

## O

odd function
2.5: Transformation of Functions
outlining notes
1.5: Taking Notes
output
2.2: Functions and Function Notation

## P

parabola
3.2: Quadratic Functions Review
polynomial division
3.5: Dividing Polynomials
polynomial function
3: Polynomial and Rational Functions
3.3: Power Functions and Polynomial Functions Behavior
polynomial inequality
2.1.4: Solving Polynomial and Rational Inequalities
Pomodoro Technique
1.7: Enhanced Strategies for Time and Task Management
population growth
4.1: Exponential Functions
power functions
3.3: Power Functions and Polynomial Functions Behavior
power rule
4.5: Logarithmic Properties
prioritizing tasks
1.6: How to Manage Time
product rule for logarithms
4.5: Logarithmic Properties

## Q

quadratic formula
3.2: Quadratic Functions Review
quadratic function
2.2: Functions and Function Notation
quadratic inequality
2.1.3: Solving Quadratic Inequalities
quotient rule
4.5: Logarithmic Properties

## R

range
2.2: Functions and Function Notation
2.3: Domain and Range
rational function
3: Polynomial and Rational Functions
3.7: Rational Functions
Rational Zero Theorem
3.6: Zeros of Polynomial Functions
Reciprocal function
2.2: Functions and Function Notation
Reciprocal squared function
2.2: Functions and Function Notation
reflection
4.2: Graphs of Exponential Functions
regression
4.8: Fitting Exponential Models to Data
relation
2.2: Functions and Function Notation
Remainder Theorem
3.6: Zeros of Polynomial Functions
reviewing notes
1.5: Taking Notes

## S

shorter work periods
1.7: Enhanced Strategies for Time and Task Management
skill acquired
1.0: The Power to Learn
SMART goals
1.8: Goal Setting and Motivation
smooth curve
3.3: Power Functions and Polynomial Functions Behavior
square root function
2.2: Functions and Function Notation
standard form of a quadratic function
3.2: Quadratic Functions Review
surface area
3.8: Inverses and Radical Functions
synthetic division
3.5: Dividing Polynomials

## T

tabular function
2.8: Inverse Functions
term of a polynomial function
3.3: Power Functions and Polynomial Functions Behavior
time management
1.6: How to Manage Time
turning point
3.3: Power Functions and Polynomial Functions Behavior

## U

unit circle
4.9: Trigonometry Preview - Circles

## V

vertex
3.2: Quadratic Functions Review
vertex form of a quadratic function
3.2: Quadratic Functions Review
vertical asymptote
3.7: Rational Functions
vertical compression
2.5: Transformation of Functions
vertical line test
2.2: Functions and Function Notation
vertical reflection
2.5: Transformation of Functions
Vertical shifts
2.5: Transformation of Functions
4.2: Graphs of Exponential Functions
vertical stretch
2.5: Transformation of Functions
4.2: Graphs of Exponential Functions
visual note taking
1.5: Taking Notes

## Z

zeros of a function
3.2: Quadratic Functions Review

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**Sample Word 1** | Sample Definition 1

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## By Page

- Math 141: Precalculus I (old edition) - *Undeclared*
  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Academic Success Skills - *Undeclared*
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    - 1.1: It's All in the Mindset - *CC BY 4.0*
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    - 1.6: How to Manage Time - *CC BY 4.0*
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    - 1.10: Pre- Mid- and Post-Test-Taking Strategies - *CC BY 4.0*
    - 1.11: Test-Taking Strategy Specifics - *CC BY 4.0*
  - 2: Inequalities and Functions - *CC BY 4.0*
    - 2.0: Prelude to Inequalities and Functions - *CC BY 4.0*
    - 2.1: Equations and Inequalities - *CC BY-NC-SA 4.0*
      - 2.1.0: Extracting Square Roots and Completing the Square (Review) - *CC BY-NC-SA 4.0*
      - 2.1.1: Solving Equations Quadratic in Form (Review) - *CC BY-NC-SA 4.0*
      - 2.1.2: Quadratic Functions and Their Graphs (Review) - *CC BY-NC-SA 4.0*
      - 2.1.3: Solving Quadratic Inequalities - *CC BY-NC-SA 4.0*
      - 2.1.4: Solving Polynomial and Rational Inequalities - *CC BY-NC-SA 4.0*
      - 2.1.E: Solving Equations and Inequalities (Exercises) - *CC BY-NC-SA 4.0*
    - 2.2: Functions and Function Notation - *CC BY 4.0*
    - 2.3: Domain and Range - *CC BY 4.0*
    - 2.4: Rates of Change and Behavior of Graphs - *CC BY 4.0*
    - 2.5: Transformation of Functions - *CC BY 4.0*
    - 2.6: Composition of Functions - *CC BY 4.0*
    - 2.7: Absolute Value Functions - *CC BY 4.0*
    - 2.8: Inverse Functions - *CC BY 4.0*
    - 2.E: Functions (Exercises) - *CC BY 4.0*
    - 2.R: Functions (Review) - *CC BY 4.0*
  - 3: Polynomial and Rational Functions - *CC BY 4.0*
    - 3.0: Prelude to Polynomial and Rational Functions - *CC BY 4.0*
    - 3.1: Complex Numbers Review - *CC BY 4.0*
    - 3.2: Quadratic Functions Review - *CC BY 4.0*
    - 3.3: Power Functions and Polynomial Functions Behavior - *CC BY 4.0*
    - 3.4: Graphs of Polynomial Functions - *CC BY 4.0*
    - 3.5: Dividing Polynomials - *CC BY 4.0*
    - 3.6: Zeros of Polynomial Functions - *CC BY 4.0*
    - 3.7: Rational Functions - *CC BY 4.0*
    - 3.8: Inverses and Radical Functions - *CC BY 4.0*
    - 3.9: Modeling Using Variation (optional) - *CC BY 4.0*
    - 3.E: Polynomial and Rational Functions(Exercises) - *CC BY 4.0*
    - 3.R: Polynomial and Rational Functions(Review) - *CC BY 4.0*
  - 4: Exponential and Logarithmic Functions - *CC BY 4.0*
    - 4.0: Prelude to Exponential and Logarithmic Functions - *CC BY 4.0*
    - 4.1: Exponential Functions - *CC BY 4.0*
    - 4.2: Graphs of Exponential Functions - *CC BY 4.0*
    - 4.3: Logarithmic Functions - *CC BY 4.0*

- 4.4: Graphs of Logarithmic Functions - *CC BY 4.0*
- 4.5: Logarithmic Properties - *CC BY 4.0*
- 4.6: Exponential and Logarithmic Equations - *CC BY 4.0*
- 4.7: Exponential and Logarithmic Models - *CC BY 4.0*
- 4.8: Fitting Exponential Models to Data - *CC BY 4.0*
- 4.9: Trigonometry Preview - Circles - *CC BY-NC-SA 3.0*
- 4.E: Exponential and Logarithmic Functions (Exercises) - *Undeclared*
- 4.R: Exponential and Logarithmic Functions (Review) - *Undeclared*

Back Matter - *Undeclared*

- Index - *Undeclared*
- Glossary - *Undeclared*
- Detailed Licensing - *Undeclared*