MATH 175: PLANE TRIGONOMETRY



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Plane Trigonometry

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Licensing

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CHAPTER OVERVIEW

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Thumbnail: Diagram of two congruent triangles. $\triangle ABC \cong \triangle DEF$ (CC BY-SA; Merovingian via Wikipedia)

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1.1: The Pythagorean Theorem

Learning Objectives

- Use the Pythagorean Theorem to determine if a triangle is a right triangle.
- Use the Pythagorean Theorem to determine the length of one side of a right triangle.
- Use the distance formula to determine the distance between two points on the coordinate plane.

Recall the following definitions from elementary geometry:

a. An angle is **acute** if it is between 0° and 90°.
b. An angle is a **right angle** if it equals 90°.
c. An angle is **obtuse** if it is between 90° and 180°.
d. An angle is a **straight angle** if it equals 180°.

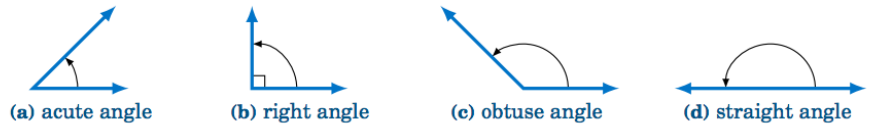


Figure 1.1.1 *Types of angles*

In elementary geometry, angles are always considered to be positive and not larger than 360°. For now we will only consider such angles. The following definitions will be used throughout the text:

a. Two acute angles are **complementary** if their sum equals 90°. In other words, if $0^\circ \leq \angle A, \angle B \leq 90^\circ$ then $\angle A$ and $\angle B$ are complementary if $\angle A + \angle B = 90^\circ$.
b. Two angles between 0° and 180° are **supplementary** if their sum equals 180°. In other words, if $0^\circ \leq \angle A, \angle B \leq 180^\circ$ then $\angle A$ and $\angle B$ are supplementary if $\angle A + \angle B = 180^\circ$.
c. Two angles between 0° and 360° are **conjugate** (or **explementary**) if their sum equals 360°. In other words, if $0^\circ \leq \angle A, \angle B \leq 360^\circ$ then $\angle A$ and $\angle B$ are conjugate if $\angle A + \angle B = 360^\circ$.

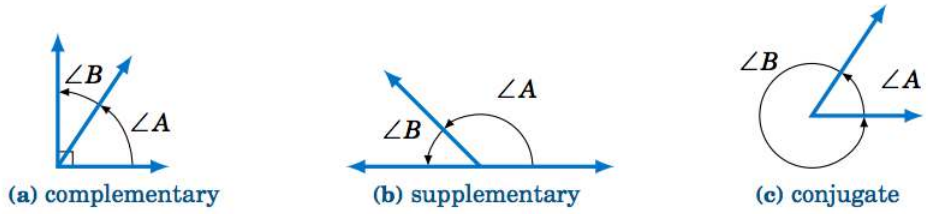


Figure 1.1.2 Types of pairs of angles

Instead of using the angle notation $\angle A$ to denote an angle, we will sometimes use just a capital letter by itself (e.g. A, B, C) or a lowercase variable name (e.g. x, y, t). It is also common to use letters (either uppercase or lowercase) from the Greek alphabet, shown in the table below, to represent angles:

Table 1.1 The Greek alphabet

Letters		Name	Letters		Name	Letters		Name
A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Y	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

In elementary geometry you learned that the sum of the angles in a triangle equals 180°, and that an **isosceles triangle** is a triangle with two sides of equal length. Recall that in a **right triangle** one of the angles is a right angle. Thus, in a right triangle one of the angles is 90° and the other two angles are acute angles whose sum is 90° (i.e. the other two angles are complementary angles).

$$\alpha+3\alpha+\alpha=180^\circ \Rightarrow 5\alpha=180^\circ \Rightarrow \alpha=36^\circ \Rightarrow \boxed{X=36^\circ,\ Y=3\times 36^\circ=108^\circ,\ Z=36^\circ}$$

QED

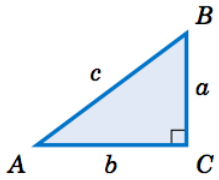


Figure 1.1.3

By knowing the lengths of two sides of a right triangle, the length of the third side can be determined by using the **Pythagorean Theorem**:

$$a^2+b^2=c^2$$

The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs.

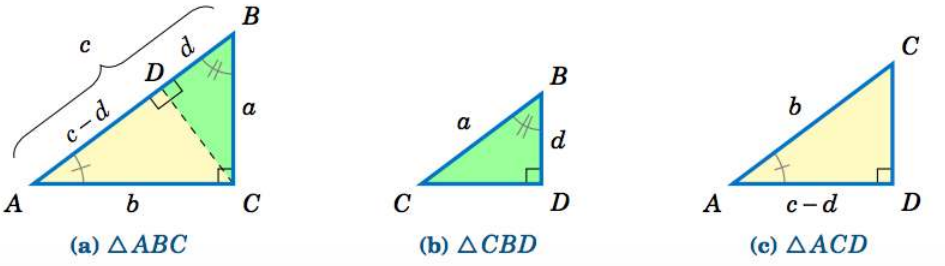


Figure 1.1.4 Similar triangles $\triangle ABC, \triangle CBD, \triangle ACD$

Recall that triangles are **similar** if their corresponding angles are equal, and that similarity implies that corresponding sides are proportional. Thus, since $\triangle ABC$ is similar to $\triangle CBD$, by proportionality of corresponding sides we see that

$$\overline{AB} \text{ is to } \overline{CB} \text{ (hypotenuses) as } \overline{BC} \text{ is to } \overline{BD} \text{ (vertical legs)} \quad \Rightarrow \quad \frac{c}{a}=\frac{a}{d} \quad \Rightarrow \quad cd=a^2\,.$$

Since $\triangle ABC$ is similar to $\triangle ACD$, comparing horizontal legs and hypotenuses gives

$$\frac{b}{c-d}=\frac{c}{b} \quad \Rightarrow \quad b^2=c^2-cd=c^2-a^2 \quad \Rightarrow \quad a^2+b^2=c^2\,.\ \textbf{QED}$$

Note: The symbols $\perp$ and $\sim$ denote perpendicularity and similarity, respectively. For example, in the above proof we had $\overline{CD}\perp\overline{AB}$ and $\triangle ABC\sim\triangle CBD\sim\triangle ACD$.

For triangle $\triangle ABC$, the Pythagorean Theorem says that

$$a^2 + 4^2 = 5^2 \quad \Rightarrow \quad a^2 = 25 - 16 = 9 \quad \Rightarrow \quad \boxed{a = 3}.$$

For triangle $\triangle DEF$, the Pythagorean Theorem says that

$$e^2 + 1^2 = 2^2 \quad \Rightarrow \quad e^2 = 4 - 1 = 3 \quad \Rightarrow \quad \boxed{e = \sqrt{3}}.$$

For triangle $\triangle XYZ$, the Pythagorean Theorem says that

$$1^2 + 1^2 = z^2 \quad \Rightarrow \quad z^2 = 2 \quad \Rightarrow \quad \boxed{z = \sqrt{2}}.$$

Let h be the height at which the ladder touches the wall. We can assume that the ground makes a right angle with the wall, as in the picture on the right. Then we see that the ladder, ground, and wall form a right triangle with a hypotenuse of length 17 ft (the length of the ladder) and legs with lengths 8 ft and h ft. So by the Pythagorean Theorem, we have

$$h^2 + 8^2 = 17^2 \quad \Rightarrow \quad h^2 = 289 - 64 = 225 \quad \Rightarrow \quad \boxed{h = 15 \text{ ft}}.$$

Determining the Distance Using the Pythagorean Theorem

You can use the Pythagorean Theorem is to find the distance between two points.

Consider the points (-1, 6) and (5, -3). If we plot these points on a grid and connect them, they make a diagonal line. Draw a vertical line down from (-1, 6) and a horizontal line to the left of (5, -3) to make a right triangle.

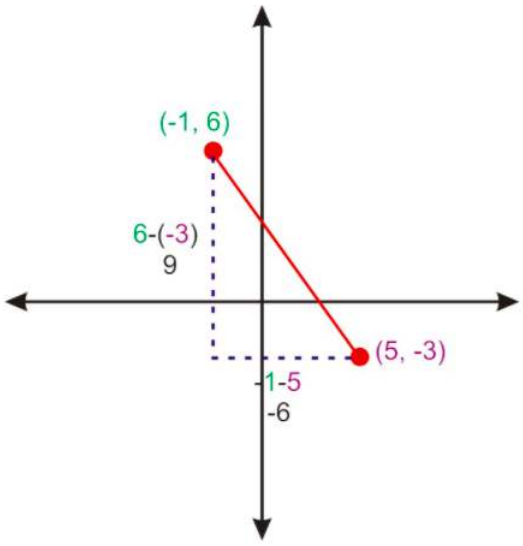


Figure 1.1.5

Now we can find the distance between these two points by using the vertical and horizontal distances that we determined from the graph.

$$\begin{aligned} 9^2 + (-6)^2 &= d^2 \\ 81 + 36 &= d^2 \\ 117 &= d^2 \\ \sqrt{117} &= d \\ 3\sqrt{13} &= d \end{aligned}$$

Notice, that the x−values were subtracted from each other to find the horizontal distance and the y−values were subtracted from each other to find the vertical distance. If this process is generalized for two points $(x_1, y_1)$and (x_2, y_2), the **Distance Formula** is derived.

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2$$

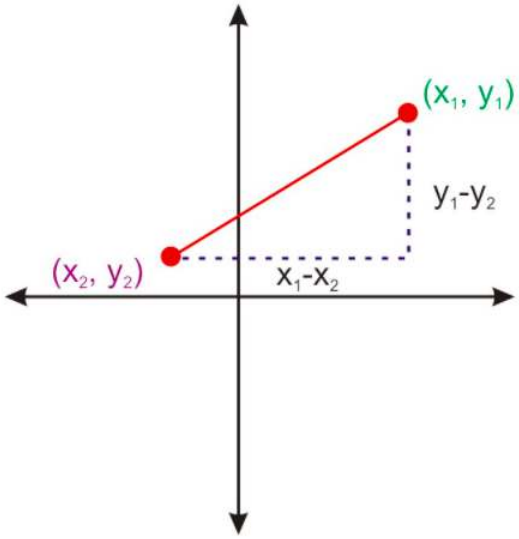


Figure 1.1.6

This is the Pythagorean Theorem with the vertical and horizontal differences between (x_1, y_1) and (x_2, y_2). Taking the square root of both sides will solve the right hand side for d, the distance.

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d$$

This is the Distance Formula.

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1.2: Special Right Triangles

Learning Objectives

- Recognize Special Right Triangles.
- Use the special right triangle rations to solve special right triangles.

30-60-90 Right Triangles

Hypotenuse equals twice the smallest leg, while the larger leg is $\sqrt{3}$ times the smallest.

One of the two special right triangles is called a **30-60-90 triangle**, after its three angles.

30-60-90 Theorem: If a triangle has angle measures 30°, 60° and 90°, then the sides are in the ratio $x : x\sqrt{3} : 2x$.

The shorter leg is always x, the longer leg is always $x\sqrt{3}$, and the **hypotenuse** is always $2x$. If you ever forget these theorems, you can still use the **Pythagorean Theorem**.

What if you were given a 30-60-90 right triangle and the length of one of its side? How could you figure out the lengths of its other sides?



Example 1.2.1

Find the value of x and y.

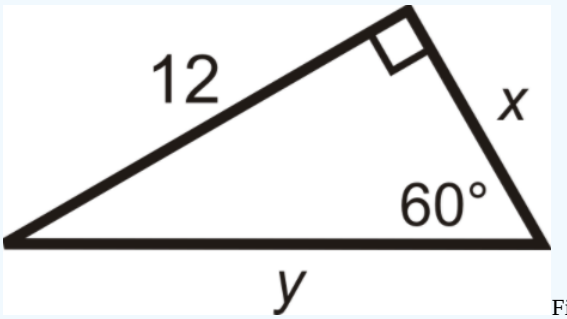


Figure 1.2.1

Solution

We are given the longer leg.

$$x\sqrt{3} = 12$$

$$x = 12\sqrt{3} \cdot \frac{\sqrt{3}}{\sqrt{3}} = 12\frac{\sqrt{3}}{3} = 4\sqrt{3}$$

The hypotenuse is

$$y = 2(4\sqrt{3}) = 8\sqrt{3}$$

Example 1.2.2

Find the value of x and y.

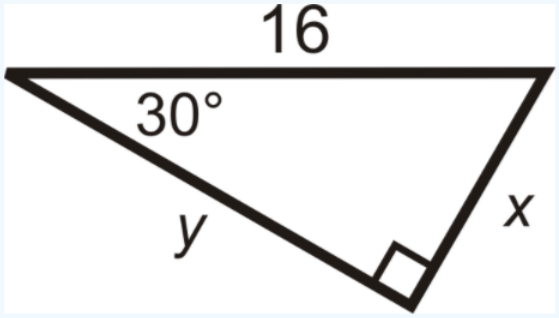


Figure 1.2.2

Solution

We are given the hypotenuse.

$$2x = 16$$
$$x = 8$$

The longer leg is

$$y = 8 \cdot \sqrt{3} \qquad = 8\sqrt{3}$$

Example 1.2.3

Find the length of the missing sides.

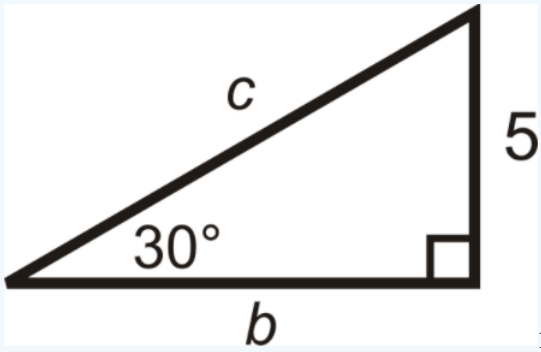


Figure 1.2.3

Solution

We are given the shorter leg. If $x = 5$, then the longer leg, $b = 5\sqrt{3}$, and the hypotenuse, $c = 2(5) = 10$.

Example 1.2.4

Find the length of the missing sides.

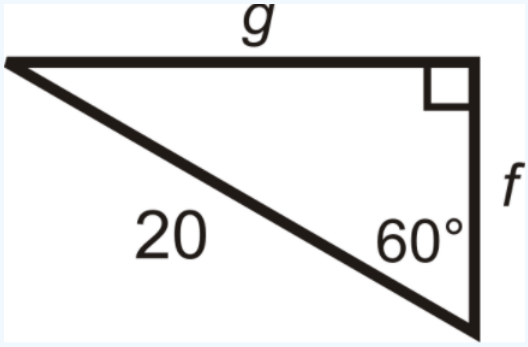


Figure 1.2.4

Solution

We are given the hypotenuse. $2x = 20$, so the shorter leg, $f = \frac{20}{2} = 10$, and the longer leg, $g = 10\sqrt{3}$.

Example 1.2.5

A rectangle has sides 4 and $4\sqrt{3}$. What is the length of the diagonal?

Solution

If you are not given a picture, draw one.

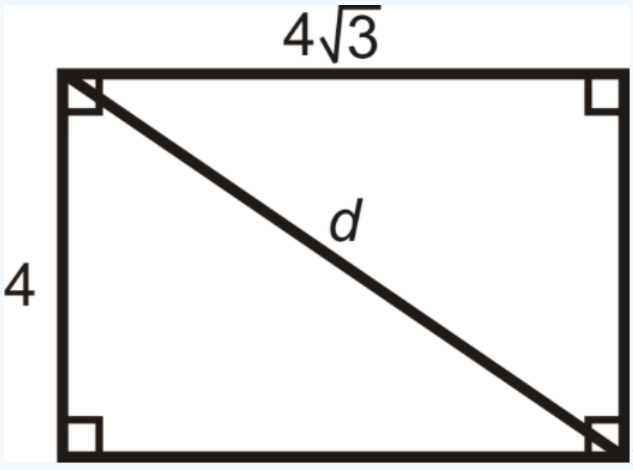


Figure 1.2.5

The two lengths are x, $x\sqrt{3}$, so the diagonal would be $2x$, or $2(4) = 8$.

If you did not recognize this is a 30-60-90 triangle, you can use the Pythagorean Theorem too.

$$\begin{aligned} 4^2 + (4\sqrt{3})^2 &= d^2 \\ 16 + 48 &= d^2 \\ d = \sqrt{64} &= 8 \end{aligned}$$

Review

1. In a 30-60-90 triangle, if the shorter leg is 5, then the longer leg is _____ and the hypotenuse is _____.
2. In a 30-60-90 triangle, if the shorter leg is x, then the longer leg is _____ and the hypotenuse is _____.
3. A rectangle has sides of length 6 and $6\sqrt{3}$. What is the length of the diagonal?
4. Two (opposite) sides of a rectangle are 10 and the diagonal is 20. What is the length of the other two sides

45-45-90 Right Triangles

A right triangle with congruent legs and acute angles is an **Isosceles Right Triangle**. This triangle is also called a **45-45-90 triangle** (named after the angle measures).

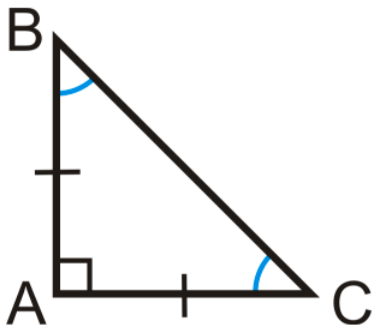


Figure 1.2.6

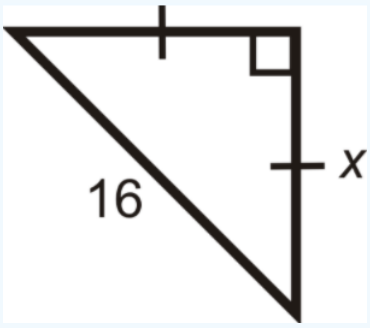
ΔABC is a right triangle with $m\angle A = 90^\circ$, $\overline{AB} \cong \overline{AC}$ and $m\angle B = m\angle C = 45^\circ$.

45-45-90 Theorem: If a right triangle is isosceles, then its sides are in the ratio $x : x : x\sqrt{2}$. For any isosceles right triangle, the legs are x and the **hypotenuse** is always $x\sqrt{2}$.

What if you were given an isosceles right triangle and the length of one of its sides? How could you figure out the lengths of its other sides?

Example 1.2.6

Find the length of x.



Solution

Use the $x : x : x\sqrt{2}$ ratio.

Here, we are given the hypotenuse. Solve for x in the ratio.

$$x\sqrt{2} = 16$$

$$x = 16\sqrt{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$$

Example 1.2.7

Find the length of x, where x is the hypotenuse of a 45-45-90 triangle with leg lengths of $5\sqrt{3}$.

Solution

Use the $x : x : x\sqrt{2}$ ratio.

$x = 5\sqrt{3} \cdot \sqrt{2} = 5\sqrt{6}$

Example 1.2.8

Find the length of the missing side.

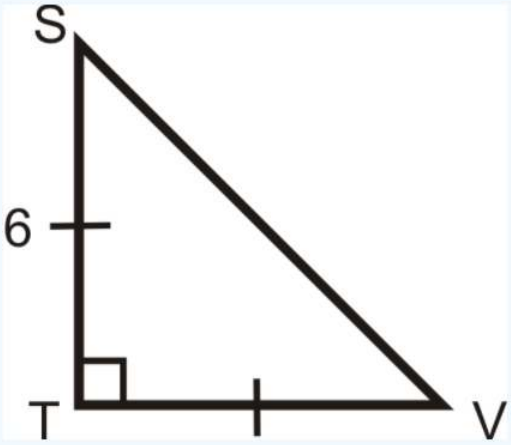


Figure 1.2.8

Solution

Use the $x : x : x\sqrt{2}$ ratio. $TV = 6$ because it is equal to ST. So, $SV = 6 \cdot \sqrt{2} = 6\sqrt{2}$.

Example 1.2.9

Find the length of the missing side.

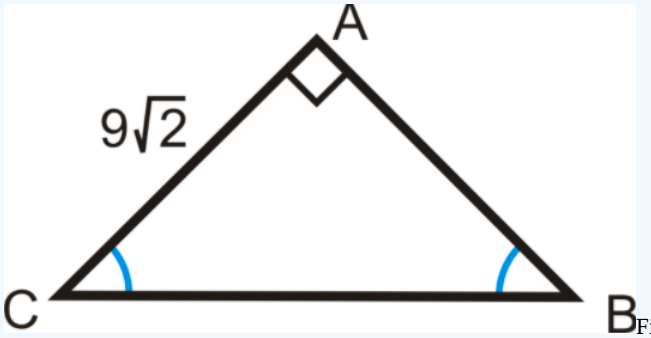


Figure 1.2.9

Solution

Use the $x : x : x\sqrt{2}$ ratio. $AB = 9\sqrt{2}$ because it is equal to AC. So, $BC = 9\sqrt{2} \cdot \sqrt{2} = 9 \cdot 2 = 18$.

Example 1.2.10

A square has a diagonal with length 10, what are the lengths of the sides?

Solution

Draw a picture.

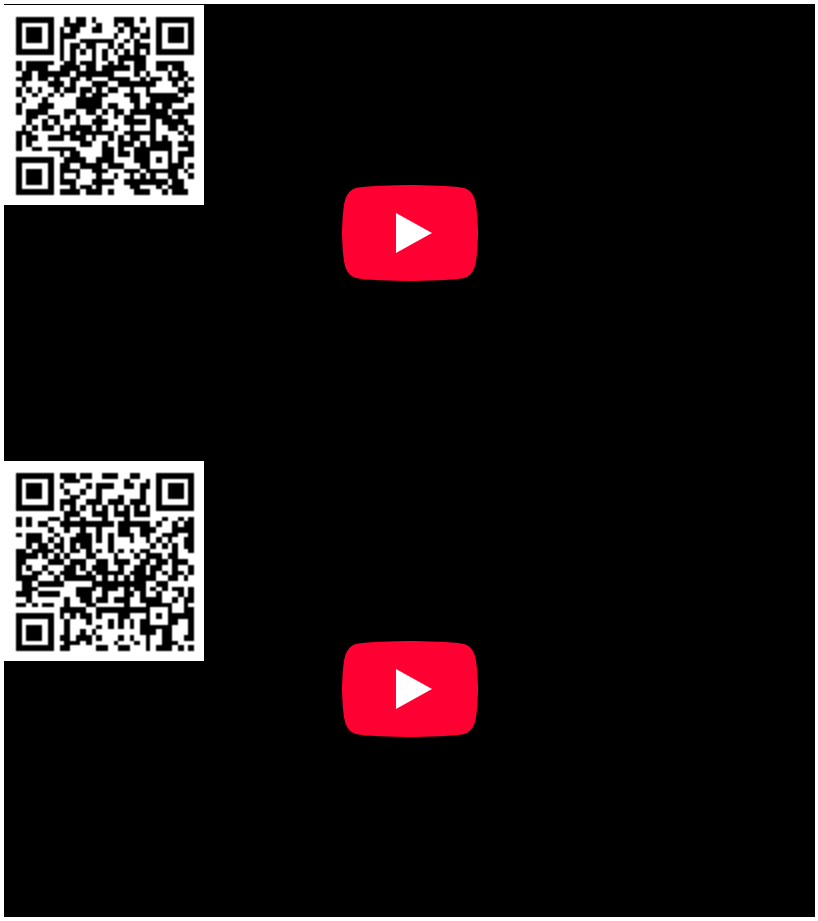
We know half of a square is a 45-45-90 triangle, so $10 = s\sqrt{2}$.

$$s\sqrt{2} = 10$$
$$s = 10\sqrt{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

Review

1. In an isosceles right triangle, if a leg is 4, then the hypotenuse is _____.
2. In an isosceles right triangle, if a leg is x, then the hypotenuse is _____.
3. A square has sides of length 15. What is the length of the diagonal?
4. A square's diagonal is 22. What is the length of each side?

Resources



Vocabulary

Term	Definition
30-60-90 Theorem	If a triangle has angle measures of 30, 60, and 90 degrees, then the sides are in the ratio $x : x\sqrt{3} : 2x$
45-45-90 Theorem	For any isosceles right triangle, if the legs are x units long, the hypotenuse is always $x\sqrt{2}$.
Hypotenuse	The hypotenuse of a right triangle is the longest side of the right triangle. It is across from the right angle.
Legs of a Right Triangle	The legs of a right triangle are the two shorter sides of the right triangle. Legs are adjacent to the right angle.

Term	Definition
Pythagorean Theorem	The Pythagorean Theorem is a mathematical relationship between the sides of a right triangle, given by $a^2 + b^2 = c^2$, where a and b are legs of the triangle and c is the hypotenuse of the triangle.
Radical	The $\sqrt{}$, or square root, sign.

Additional Resources

Video: Solving Special Right Triangles

Activities: 30-60-90 Right Triangles Discussion Questions

Study Aids: Special Right Triangles Study Guide

Practice: 30-60-90 Right Triangles 45-45-90 Right Triangles

Real World: Fighting the War on Drugs Using Geometry and Special Triangles

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1.3: Basic Trigonometric Functions

Learning Objectives

- Find the six trigonometric function values of an angle in a right triangle.

Sine, cosine, tangent, and other ratios of sides of a right triangle.

Sine, Cosine, and Tangent

Trigonometry is the study of the relationships between the sides and angles of right triangles. The legs are called ***adjacent*** or ***opposite*** depending on which **acute angle** is being used.

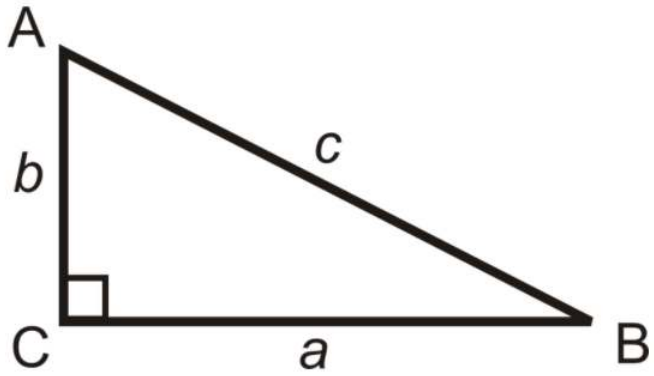


Figure 1.3.1

a is adjacent to $\angle B$ $\quad$ a is opposite $\angle A$
b is adjacent to $\angle A$ $\quad$ b is opposite $\angle B$
c is the hypotenuse

The three basic **trigonometric ratios** are called sine, cosine and tangent. For right triangle $\triangle$ABC, we have:

$$\text{sine Ratio: } \frac{\text{opposite leg}}{\text{hypotenuse}} \qquad \sin A = \frac{a}{c} \text{ or } \sin B = \frac{b}{c}$$

$$\text{cosine Ratio: } \frac{\text{adjacent leg}}{\text{hypotenuse}} \qquad \cos A = \frac{b}{c} \text{ or } \cos B = \frac{a}{c}$$

$$\text{Tangent Ratio: } \frac{\text{opposite leg}}{\text{adjacent leg}} \qquad \tan A = \frac{a}{b} \text{ or } \tan B = \frac{b}{a}$$

An easy way to remember ratios is to use SOH-CAH-TOA.

$$\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}$$

Figure 1.3.2

A few important points:

- Always **reduce ratios** (fractions) when you can.
- Use the **Pythagorean Theorem** to find the missing side (if there is one).
- If there is a **radical** in the denominator, **rationalize the denominator.**

What if you were given a right triangle and told that its sides measure 3, 4, and 5 inches? How could you find the sine, cosine, and tangent of one of the triangle's non-right angles?



Example 1.3.1

Find the sine, cosine and tangent ratios of $\angle A$.

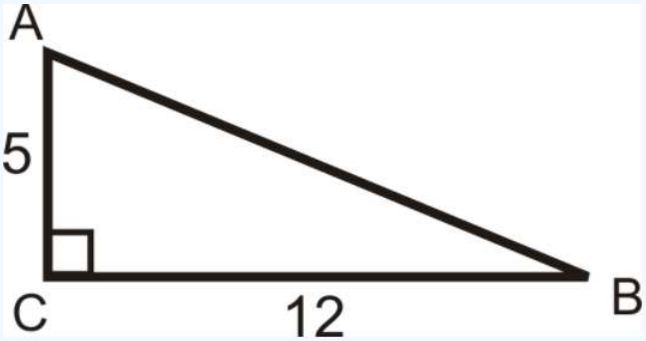


Figure 1.3.3

Solution

First, we need to use the Pythagorean Theorem to find the length of the **hypotenuse**.

$$5^2+12^2 = c^2$$
$$13 = c$$
$$\sin A = \frac{leg\ \text{opposite}\ \angle A}{\text{hypotenuse}} = \frac{12}{13} \qquad \cos A = \frac{\text{leg adjacent to}\ \angle A}{\text{hypotenuse}} = \frac{5}{13},$$
$$\tan A = \frac{\text{leg opposite}\ \angle A}{\text{leg adjacent to}\ \angle A} = \frac{12}{5}$$

Example 1.3.2

Find the sine, cosine, and tangent of $\angle B$.

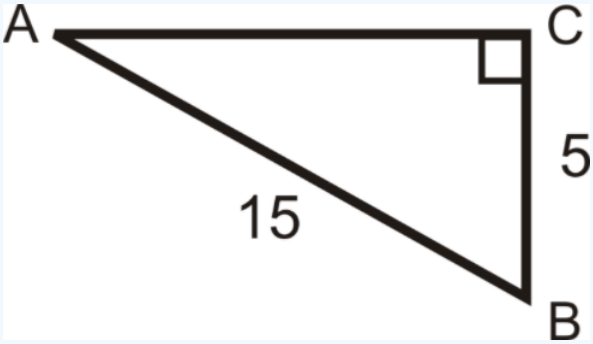


Figure 1.3.4

Find the length of the missing side.

Solution

$$\begin{aligned} AC^2 + 5^2 &= 15^2 \\ AC^2 &= 200 \\ AC &= 10\sqrt{2} \end{aligned}$$

$$\sin B = \frac{10\sqrt{2}}{15} = \frac{2\sqrt{2}}{3} \quad \cos B = \frac{5}{15} = \frac{1}{3} \quad \tan B = \frac{10\sqrt{2}}{5} = 2\sqrt{2}$$

Example 1.3.3

Find the sine, cosine and tangent of 30°.

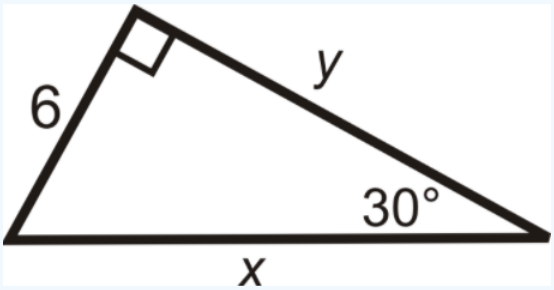


Figure 1.3.5

Solution

This is a 30-60-90 triangle. The short leg is 6, $y = 6\sqrt{3}$ and $x = 12$.

$$\sin 30^\circ = \frac{6}{12} = \frac{1}{2} \qquad \cos 30^\circ = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2} \qquad \tan 30^\circ = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Example 1.3.4

Answer the questions about the following image. Reduce all fractions.

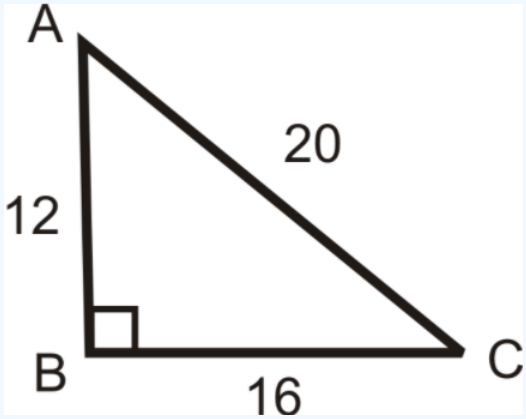


Figure 1.3.6

What is sin A, cos A, and tan A?

Solution

$$\sin A = \tfrac{16}{20} = \tfrac{4}{5}$$
$$\cos A = \tfrac{12}{20} = \tfrac{3}{5}$$
$$\tan A = \tfrac{16}{12} = \tfrac{4}{3}$$

Resources



Vocabulary

Term	Definition
Acute Angle	An acute angle is an angle with a measure of less than 90 degrees.
Adjacent Angles	Two angles are adjacent if they share a side and vertex. The word 'adjacent' means 'beside' or 'next-to'.
Hypotenuse	The hypotenuse of a right triangle is the longest side of the right triangle. It is across from the right angle.
Legs of a Right Triangle	The legs of a right triangle are the two shorter sides of the right triangle. Legs are adjacent to the right angle.
opposite	The opposite of a number x is $-x$. A number and its opposite always sum to zero.
Pythagorean Theorem	The Pythagorean Theorem is a mathematical relationship between the sides of a right triangle, given by $a^2 + b^2 = c^2$, where a and b are legs of the triangle and c is the hypotenuse of the triangle.
Radical	The $\sqrt{}$, or square root, sign.
sine	The sine of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the hypotenuse.
Trigonometric Ratios	Ratios that help us to understand the relationships between sides and angles of right triangles.

Additional Resources

Video: Introduction to Trigonometric Functions Using Triangles

Activities: Sine, Cosine, Tangent Discussion Questions

Study Aids: Trigonometric Ratios Study Guide

Practice: Right Triangle Trigonometry

Real World: Sine Cosine Tangent

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1.4: Solving Right Triangles

Learning Objectives

- Solve right triangles.
- Find the area of any triangle using trigonometry.
- Solve real-world problems using right triangles.
- Find the measure of an angle using inverse trig functions.

Inverse Trigonometric Ratios

In mathematics, the word inverse means "undo." For example, addition and subtraction are inverses of each other because one undoes the other. When we use the inverse trigonometric ratios, we can find acute angle measures as long as we are given two sides.

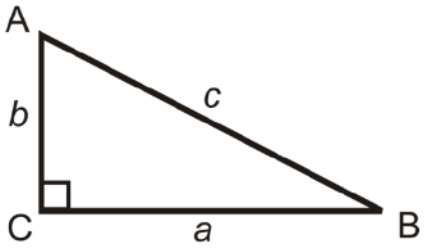


Figure 1.4.1

Inverse Tangent: Labeled $\tan^{-1}$, the "-1" means inverse.

$\tan^{-1}\left(\frac{b}{a}\right) = m\angle B$ and $\tan^{-1}\left(\frac{a}{b}\right) = m\angle A$.

Inverse Sine: Labeled $\sin^{-1}$.

$\sin^{-1}\left(\frac{b}{c}\right) = m\angle B$ and $\sin^{-1}\left(\frac{a}{c}\right) = m\angle A$.

Inverse Cosine: Labeled $\cos^{-1}$.

$\cos^{-1}\left(\frac{a}{c}\right) = m\angle B$ and $\cos^{-1}\left(\frac{b}{c}\right) = m\angle A$.

In most problems, to find the measure of the angles you will need to use your calculator. On most scientific and graphing calculators, the buttons look like $[\sin^{-1}]$, $[\cos^{-1}]$, and $[\backslash(\tan^{-1}]$. You might also have to hit a shift or 2nd button to access these functions.

Now that you know both the trig ratios and the inverse trig ratios you can **solve** a right triangle. To solve a right triangle, you need to find all sides and angles in it. You will usually use sine, cosine, or tangent; inverse sine, inverse cosine, or inverse tangent; or the **Pythagorean Theorem**.

What if you were told the tangent of $\angle Z$ is 0.6494? How could you find the measure of $\angle Z$?



Example 1.4.1

Solve the right triangle.

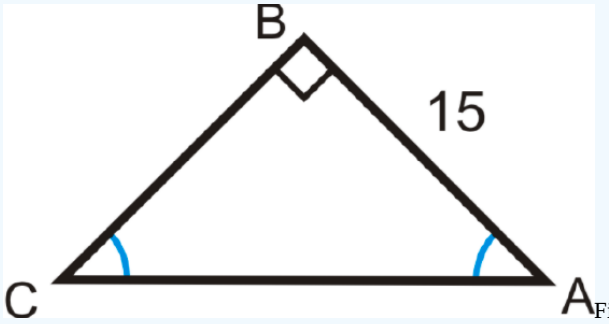


Figure 1.4.2

Solution

The two acute angles are congruent, making them both 45°. This is a 45-45-90 triangle. You can use the trigonometric ratios or the special right triangle ratios.

Trigonometric Ratios

$$\tan 45^\circ = \frac{15}{BC} \qquad \sin 45^\circ = \frac{15}{AC}$$
$$BC = \frac{15}{\tan 45^\circ} = 15 \qquad AC = \frac{15}{\sin 45^\circ} \approx 21.21$$

45-45-90 Triangle Ratios

$$BC = AB = 15,\ AC = 15\sqrt{2} \approx 21.21$$

Example 1.4.2

Use the sides of the triangle and your calculator to find the value of $\angle A$. Round your answer to the nearest tenth of a degree.

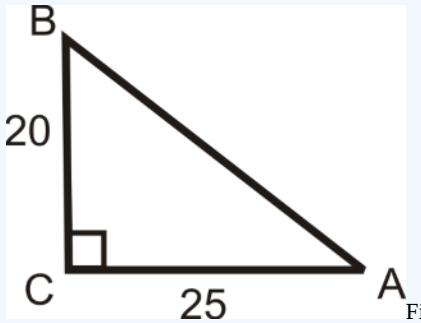


Figure 1.4.3

Solution

In reference to $\angle A$, we are given the *opposite* leg and the *adjacent* leg. This means we should use the *tangent* ratio.

$\tan A = \frac{20}{25} = \frac{4}{5}$. So, $\tan^{-1} \frac{4}{5} = m\angle A$. Now, use your calculator.

If you are using a TI-83 or 84, the keystrokes would be: [2nd][TAN]($\frac{4}{5}$) **[ENTER]** and the screen looks like:

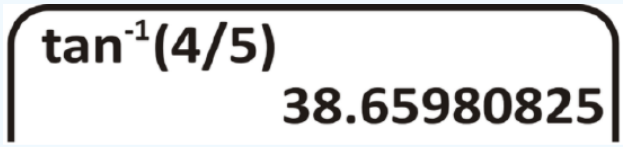


Figure 1.4.4

$$m\angle A \approx 38.7^\circ$$

Example 1.4.3

\angle A is an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree for $\sin A = 0.68$, $\cos A = 0.85$, and $\tan A = 0.34$.

Solution

$$m\angle A = \sin^{-1} 0.68 \approx 42.8^\circ$$
$$m\angle A = \cos^{-1} 0.85 \approx 31.8^\circ$$
$$m\angle A = \tan^{-1} 0.34 \approx 18.8^\circ$$

Example 1.4.4

Solve the right triangle.

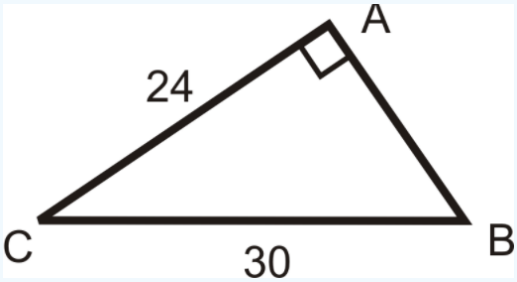


Figure 1.4.5

Solution

To solve this right triangle, we need to find AB, $m\angle C$ and $m\angle B$. Use only the values you are given.

\(\underline{AB}: \text{ Use the Pythagorean Theorem.}

$$\begin{aligned} 24^2 + AB^2 &= 30^2 \\ 576 + AB^2 &= 900 \\ AB^2 &= 324 \\ AB &= \sqrt{324} = 18 \end{aligned}$$

$\underline{m\angle B}:$ Use the inverse sine ratio.

$$\sin B = \frac{24}{30} = \frac{4}{5}$$
$$\sin^{-1}(45) \approx 53.1^\circ = m\angle B$$

$\underline{m\angle C}$: Use the inverse cosine ratio.

$$\cos C = \frac{24}{30} = \frac{4}{5} \rightarrow \cos^{-1}(\frac{4}{5}) \approx 36.9^\circ = m\angle C$$

Example 1.4.5

When would you use sin and when would you use $\sin^{-1}$?

Solution

You would use sin when you are given an angle and you are solving for a missing side. You would use $\sin^{-1}$when you are given sides and you are solving for a missing angle.

Review

Solving the following right triangles. Find all missing sides and angles. Round any decimal answers to the nearest tenth.

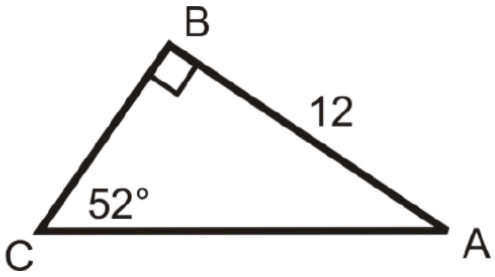


Figure 1.4.6

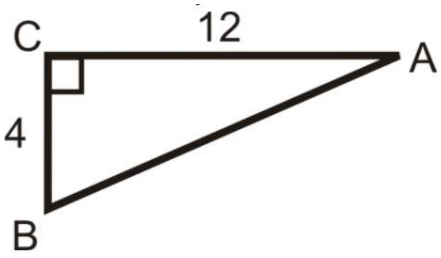


Figure 1.4.7

Additional Resources

Video: Introduction to Inverse Trigonometric Functions

Activities: Inverse Trigonometric Ratios Discussion Questions

Study Aids: Trigonometric Ratios Study Guide

Practice: Solve Right Triangles

Angles of Elevation and Depression

You can use right triangles to find distances, if you know an **angle of elevation** or an **angle of depression**.

The figure below shows each of these kinds of angles.

Figure 1.4.8

The angle of elevation is the angle between the horizontal line of sight and the line of sight up to an object. For example, if you are standing on the ground looking up at the top of a mountain, you could measure the angle of elevation. The angle of depression is

the angle between the horizontal line of sight and the line of sight *down to* an object. For example, if you were standing on top of a hill or a building, looking down at an object, you could measure the angle of depression. You can measure these angles using a clinometer or a theodolite. People tend to use clinometers or theodolites to measure the height of trees and other tall objects. Here we will solve several problems involving these angles and distances.

Finding the angle of elevation

You are standing 20 feet away from a tree, and you measure the angle of elevation to be 38°. How tall is the tree?

Figure 1.4.9

The solution depends on your height, as you measure the angle of elevation from your line of sight. Assume that you are 5 feet tall.

The figure shows us that once we find the value of T, we have to add 5 feet to this value to find the total height of the triangle. To find T, we should use the tangent value:

$$tan38^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{T}{20}$$
$$tan38^\circ = \frac{T}{20}$$
$$T = 20tan38^\circ \approx 15.63$$
$$\text{Height of tree} \approx 20.63 \text{ ft}$$

You are standing on top of a building, looking at a park in the distance. The angle of depression is 53^{\circ} . If the building you are standing on is 100 feet tall, how far away is the park? Does your height matter?

Finding the angle of depression

If we ignore the height of the person, we solve the following triangle:

Figure 1.4.10

Given the angle of depression is 53°, $\angle A$ in the figure above is 37°. We can use the tangent function to find the distance from the building to the park:

$$tan37^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{d}{100}$$
$$tan37^\circ = \frac{d}{100}$$
$$d = 100tan37^\circ \approx 75.36 \text{ ft}$$

If we take into account the height if the person, this will change the value of the adjacent side. For example, if the person is 5 feet tall, we have a different triangle:

Figure 1.4.11

$$tan37^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{d}{105}$$
$$tan37^\circ = \frac{d}{105}$$
$$d = 105tan37^\circ \approx 79.12 \text{ ft}$$

If you are only looking to estimate a distance, then you can ignore the height of the person taking the measurements. However, the height of the person will matter more in situations where the distances or lengths involved are smaller. For example, the height of the person will influence the result more in the tree height problem than in the building problem, as the tree is closer in height to the person than the building is.

Real-World Application: The Horizon

You are on a long trip through the desert. In the distance you can see mountains, and a quick measurement tells you that the angle between the mountaintop and the ground is 13.4°. From your studies, you know that one way to define a mountain is as a pile of land having a height of at least 2,500 meters. If you assume the mountain is the minimum possible height, how far are you away from the center of the mountain?

Figure 1.4.12

$$tan13.4^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{2500}{d}$$
$$tan13.4^\circ = \frac{2500}{d}$$
$$d = \frac{2500}{tan13.4^\circ} \approx 10,494 meters$$

Example 1.4.6

You are six feet tall and measure the angle between the horizontal and a bird in the sky to be 40^{\circ} . You can see that the shadow of the bird is directly beneath the bird, and 200 feet away from you on the ground. How high is the bird in the sky?

Figure 1.4.13

Solution

We can use the tangent function to find out how high the bird is in the sky:

$$tan40^\circ = \frac{height}{200}$$
$$height = 200 tan40^\circ$$
$$height = (200)(.839)$$
$$height = 167.8$$

We then need to add your height to the solution for the triangle. Since you are six feet tall, the total height of the bird in the sky is 173.8 feet.

Example 1.4.7

While out swimming one day you spot a coin at the bottom of the pool. The pool is ten feet deep, and the angle between the top of the water and the coin is 15°. How far away is the coin from you along the bottom of the pool?

Figure 1.4.14

Solution

Since the distance along the bottom of the pool to the coin is the same as the distance along the top of the pool to the coin, we can use the tangent function to solve for the distance to the coin:

$$tan15^\circ = \frac{\text{opposite}}{\text{adjacent}}$$
$$tan15^\circ = \frac{10}{x}$$
$$x = \frac{10}{tan15^\circ}$$
$$x \approx 37.32^\circ$$

Example 1.4.8

You are hiking and come to a cliff at the edge of a ravine. In the distance you can see your campsite at the base of the cliff, on the other side of the ravine. You know that the distance across the ravine is 500 meters, and the angle between your horizontal line of sight and your campsite is 25°. How high is the cliff? (Assume you are five feet tall.)

Figure 1.4.15

Solution

Using the information given, we can construct a solution:

$$tan25^\circ = \frac{\text{opposite}}{\text{adjacent}}$$
$$tan25^\circ = \frac{height}{500}$$
$$height = \frac{500}{tan25^\circ}$$
$$height = (500)(.466)$$
$$height = 233 \text{ meters}$$

This is the total height from the bottom of the ravine to your horizontal line of sight. Therefore, to get the height of the ravine, you should take away five feet for your height, which gives an answer of 228 meters.

Review

1. A 70 foot building casts an 50 foot shadow. What is the angle that the sun hits the building?
2. You are standing 10 feet away from a tree, and you measure the angle of elevation to be 65°. How tall is the tree? Assume you are 5 feet tall up to your eyes.
3. Kaitlyn is swimming in the ocean and notices a coral reef below her. The angle of depression is 35° and the depth of the ocean, at that point is 350 feet. How far away is she from the reef?
4. The angle of depression from the top of a building to the base of a car is 60°. If the building is 78 ft tall, how far away is the car?

Vocabulary

Term	Definition
Angle of Depression	The angle of depression is the angle formed by a horizontal line and the line of sight down to an object when the image of an object is located beneath the horizontal line.
Angle of Elevation	The angle of elevation is the angle formed by a horizontal line and the line of sight up to an object when the image of an object is located above the horizontal line.

Additional Resources

Video: Example: Determine What Trig Function Relates Specific Sides of a Right Triangle

Practice: Angles of Elevation and Depression

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1.5: Measuring Rotation

Learning Objectives

- Identify and draw angles of rotation.
- Identify quadrantal angles.
- Identify co-terminal angles.

Introduction to Angles of Rotation, Coterminal Angles, and Reference Angles

Angles of Rotation

Angles of rotation are formed in the coordinate plane between the positive x-axis **(initial side)** and a ray **(terminal side)**. Positive angle measures represent a counterclockwise rotation while negative angles indicate a clockwise rotation.



Figure 1.5.1

Since the x and y axes are perpendicular, each axis then represents an increment of ninety degrees of rotation. The diagrams below show a variety of angles formed by rotating a ray through the quadrants of the coordinate plane.

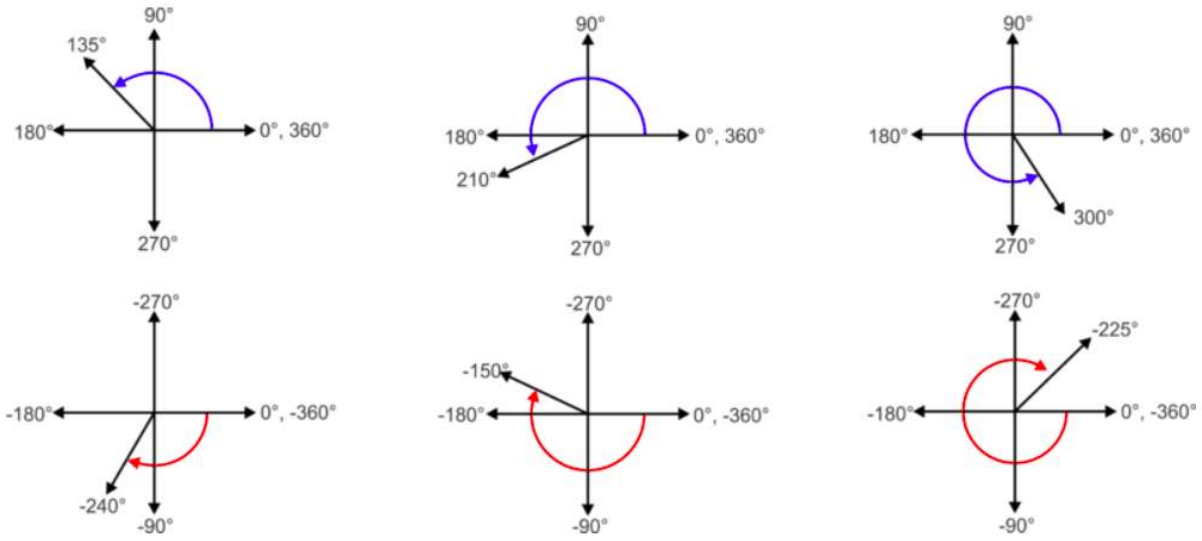


Figure 1.5.2

An angle of rotation can be described infinitely many ways. It can be described by a positive or negative angle of rotation or by making multiple full circle rotations through 360°. The example below illustrates this concept.

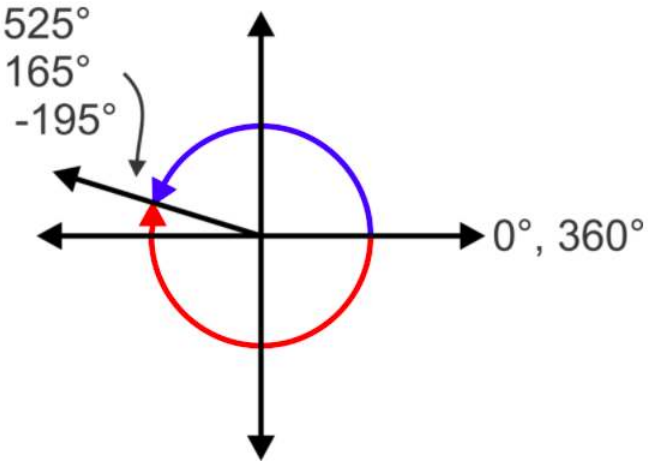


Figure 1.5.3

For the angle 525°, an entire 360° rotation is made and then we keep going another 165° to 525°. Therefore, the resulting angle is equivalent to $525^\circ - 360^\circ$, or 165°. In other words, the terminal side is in the same location as the terminal side for a 165° angle. If we subtract 360° again, we get a negative angle, -195°. Since they all share the same terminal side, they are called **coterminal angles**.

Let's determine two coterminal angles to 837°, one positive and one negative.

To find coterminal angles we simply add or subtract 360° multiple times to get the angles we desire. $837^\circ - 360^\circ = 477^\circ$, so we have a positive coterminal angle. Now we can subtract 360° again to get $477^\circ - 360^\circ = 117^\circ$.



Reference Angle

A **reference angle** is the acute angle between the terminal side of an angle and the x– axis. The diagram below shows the reference angles for terminal sides of angles in each of the four quadrants.

Note: A reference angle is never determined by the angle between the terminal side and the y – axis. This is a common error for students, especially when the terminal side appears to be closer to the y – axis than the x – axis.

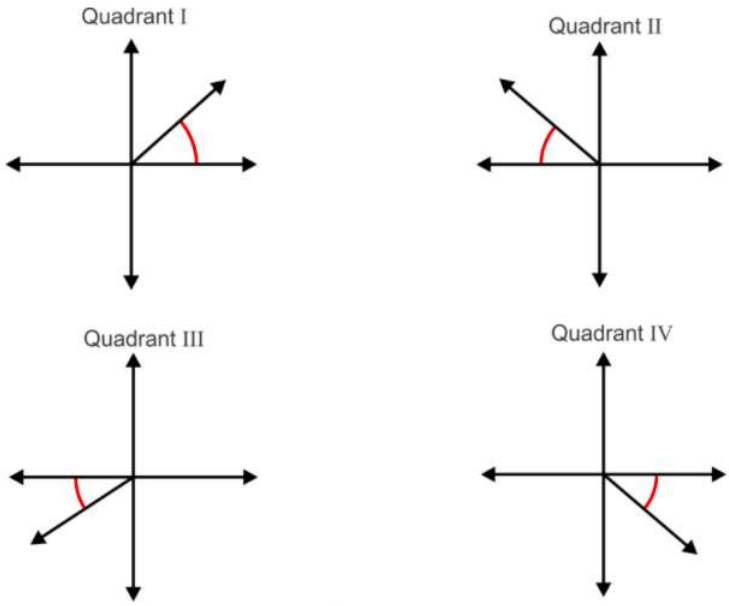


Figure 1.5.4

Now, let's determine the quadrant in which -745° lies and hence determine the reference angle.

Since our angle is more than one rotation, we need to add 360° until we get an angle whose absolute value is less than \)360^\circ$: $-745^\circ + 360^\circ = -385^\circ$, again $-385^\circ + 360^\circ = -25^\circ$.

Now we can plot the angle and determine the reference angle:

Note that the reference angle is positive 25°. All reference angles will be positive as they are acute angles (between 0° and90°).

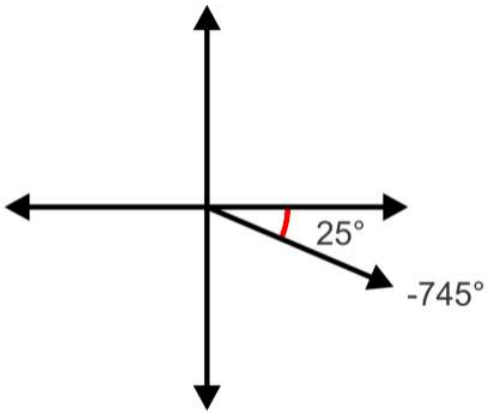


Figure 1.5.5

Finally, let's give two coterminal angles to \(595^{\circ}, one positive and one negative, and find the reference angle.

To find the coterminal angles we can add/subtract 360°. In this case, our angle is greater than 360° so it makes sense to subtract 360° to get a positive coterminal angle: $595^\circ - 360^\circ = 235^\circ$. Now subtract again to get a negative angle: $235^\circ - 360^\circ = -125^\circ$.

By plotting any of these angles we can see that the terminal side lies in the third quadrant as shown.

Since the terminal side lies in the third quadrant, we need to find the angle between 180° and 235°, so $235^\circ - 180^\circ = 55^\circ$.

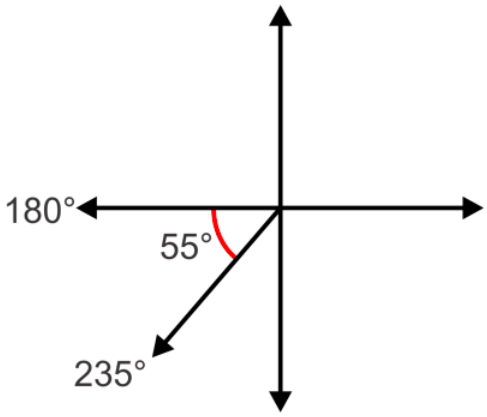


Figure 1.5.6



Example 1.5.1

Earlier, you were asked to find the reference angle of -500° and find the quadrant in which the terminal side lies.

Solution

Since our angle is more than one rotation, we need to add 360° until we get an angle whose absolute value is less than 360°: $-500^\circ + 360^\circ = -200^\circ$.

If we plot this angle we see that it is -200° clockwise from the origin or 160° counterclockwise. 160° lies in the second quadrant.

Now determine the reference angle: $180^\circ - 160^\circ = 20^\circ$.

Example 1.5.2

Find two coterminal angles to 138°, one positive and one negative.

Solution

$138^\circ + 360^\circ = 498^\circ$ and $138^\circ - 360^\circ = -222^\circ$

Example 1.5.3

Find the reference angle for 895°.

Solution

$895^\circ - 360^\circ = 535^\circ$, $535^\circ - 360^\circ = 175^\circ$. The terminal side lies in the second quadrant, so we need to determine the angle between 175° and 180°, which is 5°.

Example 1.5.4

Find the reference angle for 343°.

Solution

343° is in the fourth quadrant so we need to find the angle between 343° and 360° which is 17°.

Review

Find two coterminal angles to each angle measure, one positive and one negative.

1. -1022°
2. 354°
3. -7°

Determine the quadrant in which the terminal side lies and find the reference angle for each of the following angles.

4. 251°
5. -348°

Additional Resources

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1.6: Applying Trigonometric Functions to Angles of Rotation

Learning Objectives

- Find the values of the six trig functions for angles of rotation.
- Recognize angles of a unit circle.

Angles of Rotation and Trigonometric Functions

Just as it is possible to define the six trigonometric functions for angles in right triangles, we can also define the same functions in terms of angles of rotation.



Consider an angle in standard position, whose terminal side intersects a circle of radius r. We can think of the radius as the hypotenuse of a right triangle:

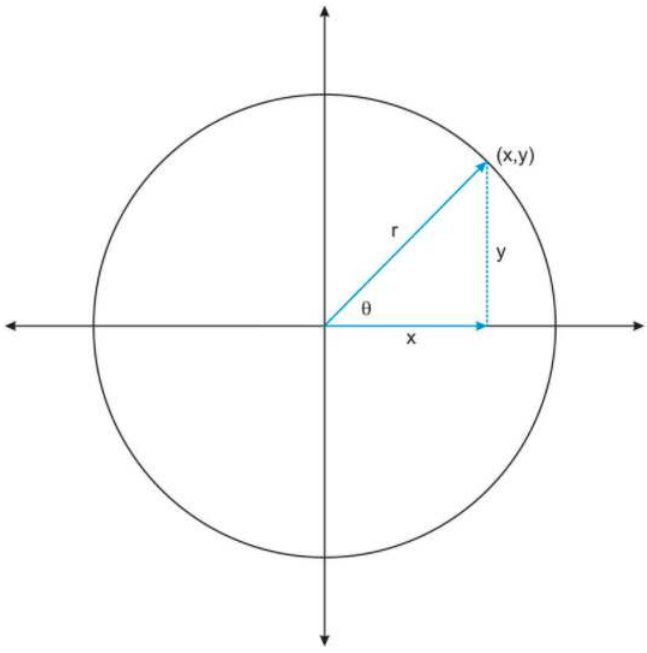


Figure 1.6.1

The point (x, y) where the terminal side of the angle intersects the circle tells us the lengths of the two legs of the triangle. Now, we can define the trigonometric functions in terms of x, y, and r:

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

And, we can extend these functions to include non-acute angles.

Consider an angle in standard position, such that the point (x, y) on the terminal side of the angle is a point on a circle with radius 1.

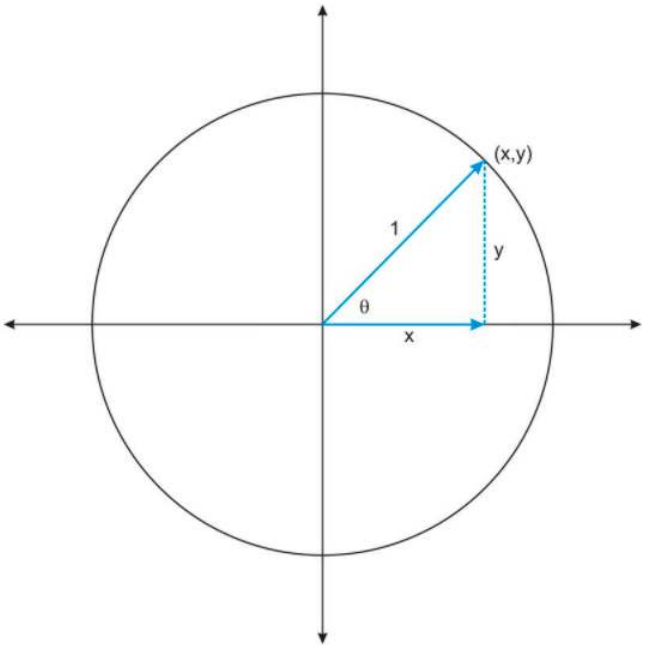


Figure 1.6.2

This circle is called the **unit circle**. With $r = 1$, we can define the trigonometric functions in the unit circle:

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x \qquad \sec\theta = \frac{r}{x} = \frac{1}{x}$$
$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y \qquad \csc\theta = \frac{r}{y} = \frac{1}{y}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Notice that in the unit circle, the sine and cosine of an angle are the y and x coordinates of the point on the terminal side of the angle. Now we can find the values of the trigonometric functions of any angle of rotation, even the quadrantal angles, which are not angles in triangles.

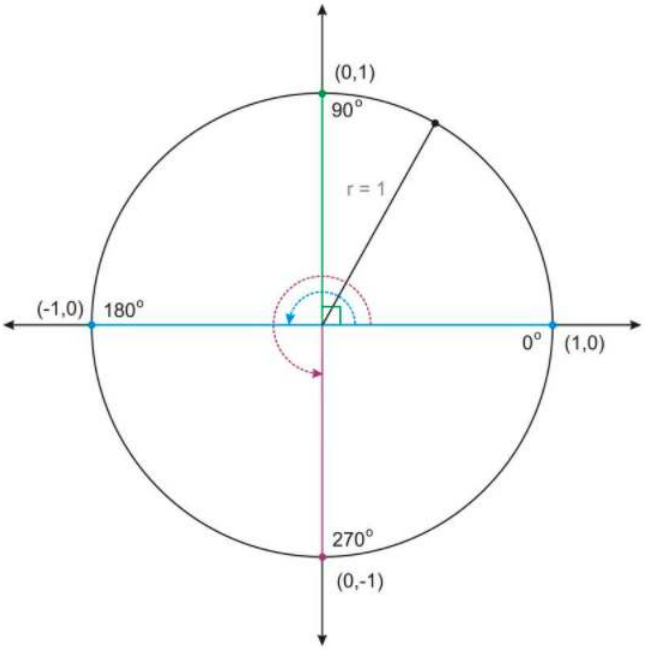


Figure 1.6.3

We can use the figure above to determine values of the trig functions for the quadrantal angles. For example, $sin90^\circ = y = 1$.

Determining the Value of Trigonometric Functions

1. Determine the values of the six trigonometric functions.

The point $(-3, 4)$ is a point on the terminal side of an angle in standard position. Determine the values of the six trigonometric functions of the angle.

Notice that the angle is more than 90 degrees, and that the terminal side of the angle lies in the second quadrant. This will influence the signs of the trigonometric functions.

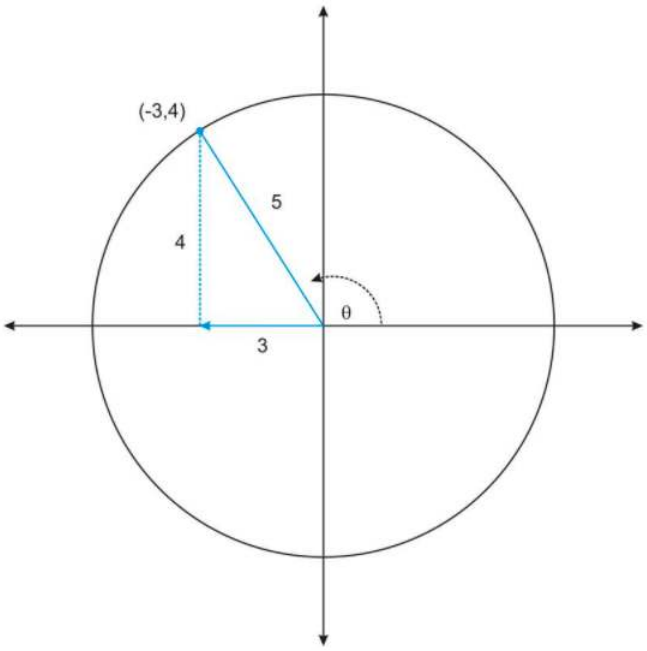


Figure 1.6.4

$$\cos\theta = \frac{-3}{5} \qquad \sec\theta = \frac{5}{-3}$$
$$\sin\theta = \frac{4}{5} \qquad \csc\theta = \frac{5}{4}$$
$$\tan\theta = \frac{4}{-3} \qquad \cot\theta = \frac{-3}{4}$$

Notice that the value of r depends on the coordinates of the given point. You can always find the value of r using the Pythagorean Theorem. However, often we look at angles in a circle with radius 1. As you can see, doing this allows us to simplify the definitions of the trig functions.

2. Use the unit circle above to find the value of $\cos 90^\circ$

$\cos 90^\circ = 0$

The ordered pair for this angle is (0, 1). The cosine value is the x coordinate, .

3. Use the unit circle above to find the value of $\cot 180^\circ$

$\cot 180^\circ$ is undefined

The ordered pair for this angle is (-1, 0). The ratio $\frac{x}{y}$ is $\frac{-1}{0}$, which is undefined.

Example 1.6.1

Earlier, you were asked if you can actually calculate $\sin 150^\circ$.

Solution

Since you now know that it is possible to apply trigonometric functions to angles greater than 90°, you can calculate the $\sin 150^\circ$. The easiest way to do this without difficulty is to consider that an angle of 150° is in the same position as 30°, except it's in the second quadrant. This means that it has the same "x" and "y" values as 30°, except that the "x" value is negative.

Therefore,

$\sin 150^\circ = \frac{1}{2}$

Use this figure:

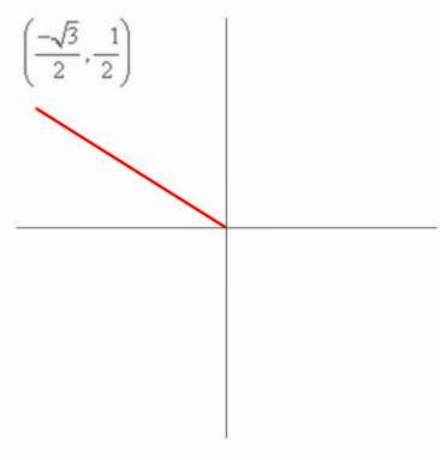


Figure 1.6.5

to answer the following examples.

Example 1.6.2

Find $\cos\theta$ on the circle above.

Solution

We can see from the "x" and "y" axes that the "x" coordinate is $-\frac{\sqrt{3}}{2}$, the "y" coordinate is $\frac{1}{2}$, and the hypotenuse has a length of 1. This means that the cosine function is:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\frac{-\sqrt{3}}{2}}{1} = \frac{-\sqrt{3}}{2}$$

Example 1.6.3

Find cot\theta on the circle above.

Solution

We know that $\cot = \frac{1}{\tan} = \frac{1}{\frac{\text{opposite}}{\text{adjacent}}} = \frac{\text{adjacent}}{\text{opposite}}$. The adjacent side to \theta in the circle is $-\frac{\sqrt{3}}{2}$ and the opposite side is $\frac{1}{2}$. Therefore,

$$\cot\theta = \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

Example 1.6.4

Find csc\theta on the circle above.

Solution

We know that $\csc = \frac{1}{\sin} = \frac{1}{\frac{\text{opposite}}{\text{hypotenuse}}} = \frac{\text{hypotenuse}}{\text{opposite}}$. The opposite side to θ in the circle is $\frac{1}{2}$ and the hypotenuse is 1. Therefore,

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\frac{1}{2}} = 2$$

Review

Find the values of the six trigonometric functions for each angle below.

1. 0°
2. 90°
3. 180°
4. 270°
5. Find the sine of an angle that goes through the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
6. Find the cosine of an angle that goes through the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
7. Find the tangent of an angle that goes through the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
8. Find the secant of an angle that goes through the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
9. Find the cotangent of an angle that goes through the point $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.
10. Find the cosecant of an angle that goes through the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Vocabulary

Term	Definition
Quadrantal Angle	A quadrantal angle is an angle that has its terminal side on one of the four lines of axis: positive x, negative x, positive y or negative y.

Additional Resources

Video: Determine Trigonometric Function Values Using the Unit Circle

This page titled 1.6: Applying Trigonometric Functions to Angles of Rotation is shared under a CK-12 license and was authored, remixed, and/or curated by CK-12 Foundation.

1.7: Trigonometric Functions of Any Angle

Learning Objectives

- Identify reference angles for angles on the unit circle.
- Identify the ordered pair on the unit circle for reference angles.
- Use ordered pairs on the unit circle to determine trig function values.
- Use calculators to find trig function values.

Reference Angles

Reference angles are formed between the terminal side of an angel and the closest part of the x-axis.

Consider the angle 150°. If we graph this angle in standard position, we see that the terminal side of this angle is a reflection of the terminal side of 30°, across the $y-$axis.

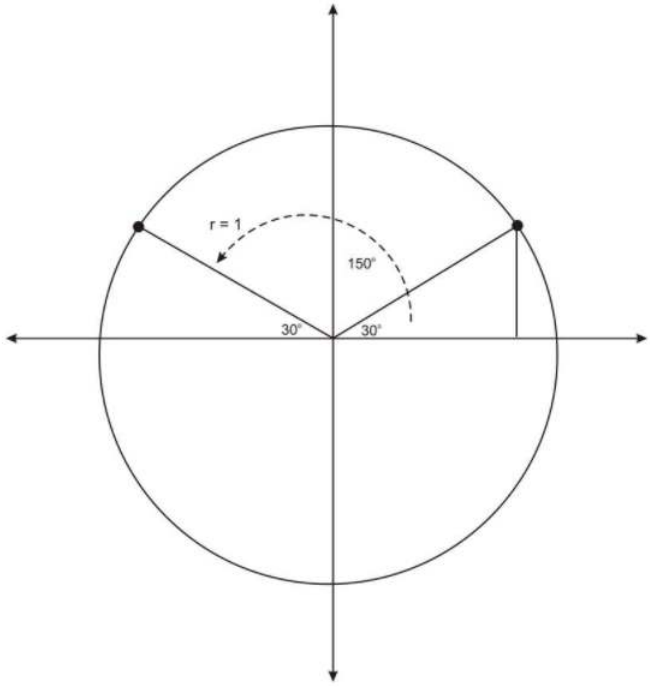


Figure 1.7.1

Notice that 150° makes a 30° angle with the negative x-axis. Therefore we say that 30° is the **reference angle** for 150°. Formally, the **reference angle** of an angle in standard position is the angle formed with the closest portion of the x-axis. Notice that 30° is the reference angle for many angles. For example, it is the reference angle for 210° and for -30°.

In general, identifying the reference angle for an angle will help you determine the values of the trig functions of the angle.



Identifying Reference Angles

Graph each of the following angles and identify their reference angles.

a. 140°

140° makes a 40° angles with the negative x-axis. Therefore the reference angle is 40°.

b. 240°

240° makes a 60° angle with the negative x-axis. Therefore the reference angle is 60°

c. 380°

380° is a full rotation of 360°, plus an additional 20°. So this angle is co-terminal with 20°, and 20° is its reference angle.

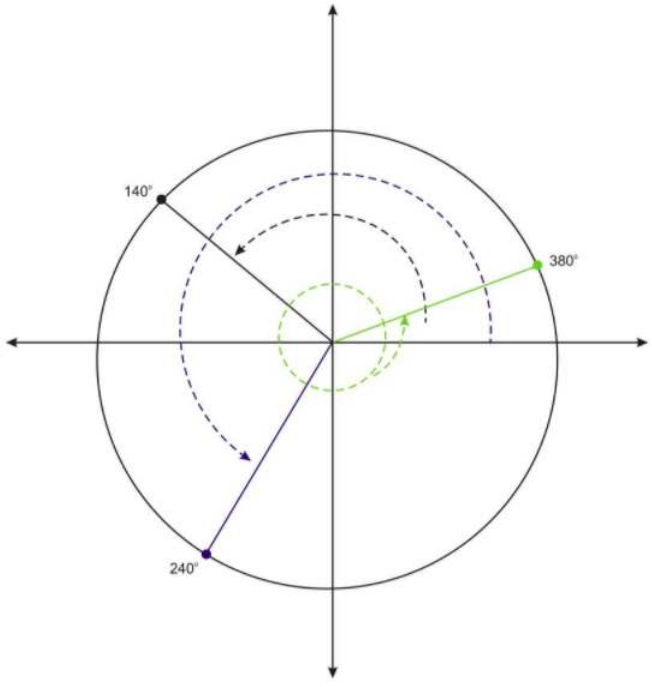


Figure 1.7.2

Determining the Value of Trigonometric Functions

1. Find the ordered pair for 240° and use it to find the value of $\sin 240^\circ$.

$$\sin 240^\circ = -\frac{\sqrt{3}}{2}$$

As we found in part b under the question above, the reference angle for 240° is 60°. The figure below shows 60° and the three other angles in the unit circle that have 60° as a reference angle.

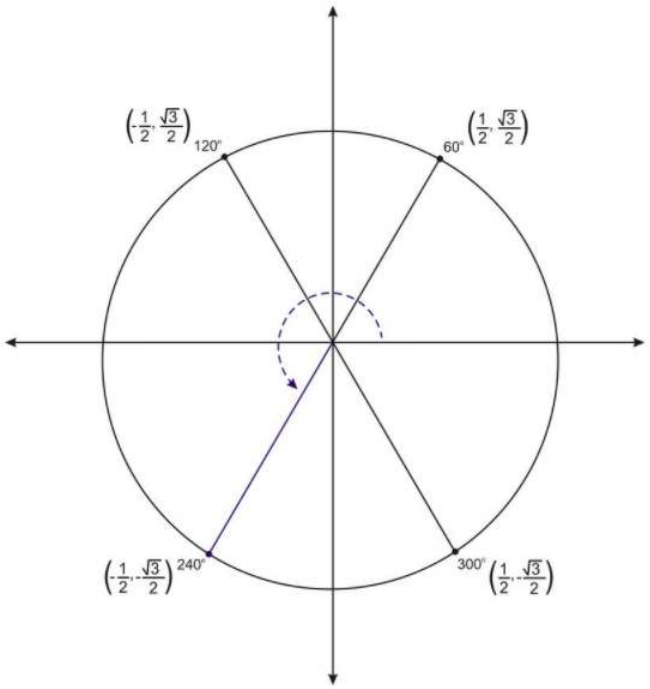


Figure 1.7.3

The terminal side of the angle 240° represents a reflection of the terminal side of 60° over both axes. So the coordinates of the point are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The y−coordinate is the sine value, so $\sin 240^\circ = -\frac{\sqrt{3}}{2}$.

Just as the figure above shows 60° and three related angles, we can make similar graphs for 30° and 45°.

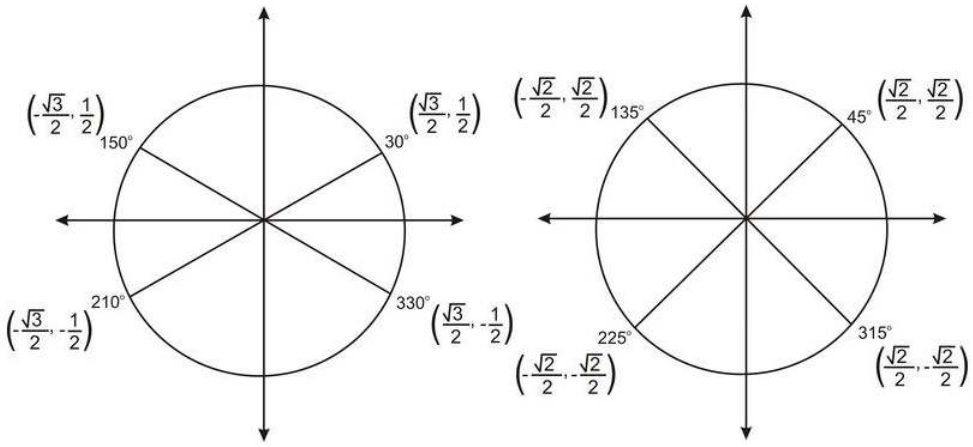


Figure 1.7.4

Knowing these ordered pairs will help you find the value of any of the trig functions for these angles.

2. Find the value of $\cot 300^\circ$

$$\cot 300^\circ = -\frac{1}{\sqrt{3}}$$

Using the graph above, you will find that the ordered pair is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Therefore the cotangent value is

$$\cot 300^\circ = \frac{x}{y} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2} \times -\frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

We can also use the concept of a reference angle and the ordered pairs we have identified to determine the values of the trig functions for other angles.

Example 1.7.1

Graph 210° and identify its reference angle.

Solution

The graph of 210° looks like this:

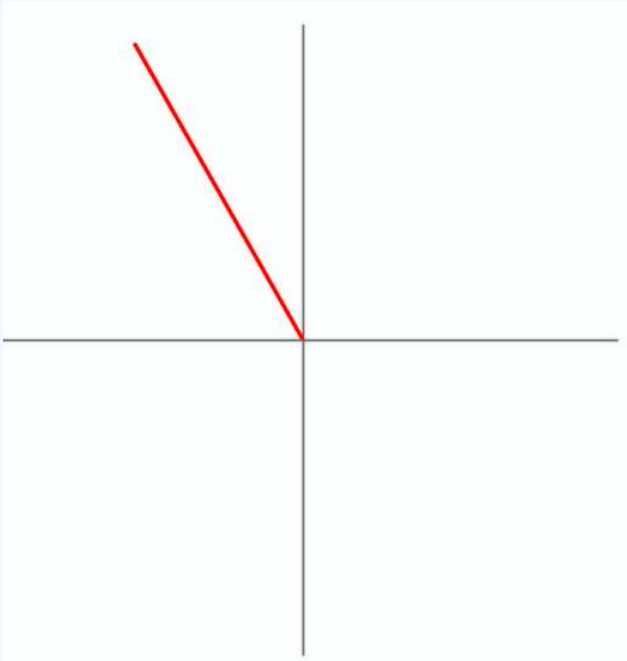


Figure 1.7.5

and since the angle makes a 30° angle with the negative "x" axis, the reference angle is 30°.

Example 1.7.2

Graph 315° and identify its reference angle.

Solution

The graph of 315° looks like this:

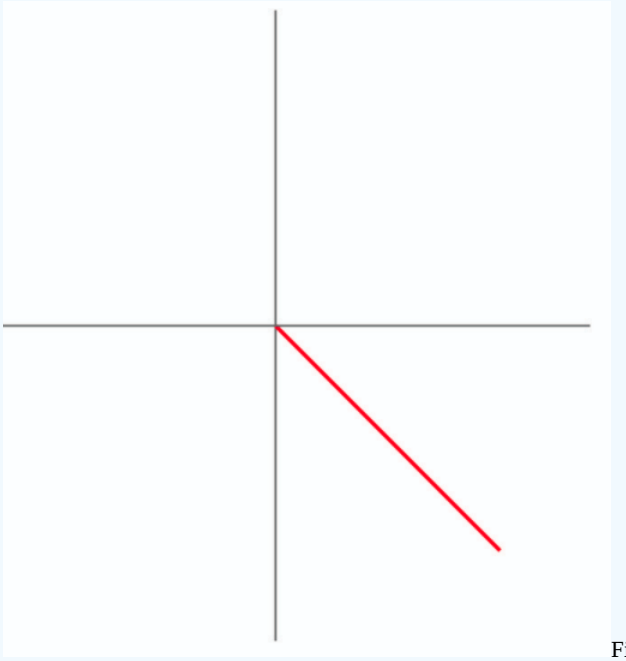


Figure 1.7.6

and since the angle makes a 45° angle with the positive "x" axis, the reference angle is 45°.

Example 1.7.3

Find the ordered pair for 150° and use it to find the value of $\cos 150^\circ$.

Solution

Since the reference angle is 30°, we know that the coordinates for the point on the unit circle are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. This is the same as the value for 30°, except the "x" coordinate is negative instead of positive. Knowing this,

$$\cos 150^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{-\frac{\sqrt{3}}{2}}{1} = -\frac{\sqrt{3}}{2}$$

Review

1. Graph 100° and identify its reference angle.
2. Graph 200° and identify its reference angle.
3. Graph 290° and identify its reference angle.

Calculate each value using the unit circle and special right triangles.

4. $\sin 225^\circ$
5. $\cos 225^\circ$
6. $\sec 225^\circ$

Vocabulary

Term	Definition
Reference Angle	A reference angle is the angle formed between the terminal side of the angle and the closest of either the positive or negative x-axis.

Additional Resources

Video: Reference Angles - Overview

Practice: Reference Angles and Angles in the Unit Circle

Trigonometric Functions of Negative Angles

Recall that graphing a negative angle means rotating clockwise. The graph below shows -30°.

Figure 1.7.7

Notice that this angle is coterminal with 330°. So the ordered pair is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. We can use this ordered pair to find the values of any of the trig functions of -30°. For example, $\cos(-30^\circ) = x = \frac{\sqrt{3}}{2}$.

In general, if a negative angle has a reference angle of 30°, 45°, or 60°, or if it is a quadrantal angle, we can find its ordered pair, and so we can determine the values of any of the trig functions of the angle.



Finding the Value of Trigonometric Expressions

Find the value of the following expressions:

1. $\sin(-45^\circ)$

$\sin(-45^\circ) = -\frac{\sqrt{2}}{2}$

-45° is in the 4^{th} quadrant, and has a reference angle of 45°. That is, this angle is coterminal with 315°. Therefore the ordered pair is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and the sine value is $-\frac{\sqrt{2}}{2}$.

2. $\sec(-300^\circ)$

$\sec(-300^\circ) = 2$

The angle -300° is in the 1^{st} quadrant and has a reference angle of 60°. That is, this angle is coterminal with 60°. Therefore the ordered pair is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the secant value is $\frac{1}{x} = \frac{1}{\frac{1}{2}} = 2$.

3. $\cos(-90^\circ)$

$\cos(-90^\circ) = 0$

The angle -90° is coterminal with 270°. Therefore the ordered pair is (0, -1) and the cosine value is 0.

Figure 1.7.8

We can also use our knowledge of reference angles and ordered pairs to find the values of trig functions of angles with measure greater than 360 degrees.

Example 1.7.4

Earlier, you were asked if it is still possible to find the values of trig functions for the new type of angles.

Solution

What you want to find is the value of the expression: $\cos(-45^\circ)$

$\cos(-45^\circ) = \frac{\sqrt{2}}{2}$

-45° is in the 4^{th} quadrant, and has a reference angle of 45°. That is, this angle is coterminal with 315°. Therefore the ordered pair is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and the cosine value is $\frac{\sqrt{2}}{2}$.

Example 1.7.5

Find the value of the expression: $\cos -180^\circ$

Solution

The angle -180° is coterminal with 180°. Therefore the ordered pair of points is $(-1, 0)$. The cosine is the "x" coordinate, so here it is -1.

Example 1.7.6

Find the value of the expression: $\sin -90^\circ$

Solution

The angle -90° is coterminal with 270°. Therefore the ordered pair of points is $(0, -1)$. The sine is the "y" coordinte, so here it is -1.

Example 1.7.7

Find the value of the expression: $\tan -270^\circ$

Solution

The angle -270° is coterminal with 90°. Therefore the ordered pair of points is $(0, 1)$. The tangent is the "y" coordinate divided by the "x" coordinate. Since the "x" coordinate is 0, the tangent is undefined.

Review

Calculate each value.

1. $\sec -120^\circ$
2. $\csc -135^\circ$
3. $\tan -210^\circ$
4. $\sin -270^\circ$
5. $\cot -90^\circ$

Vocabulary

Term	Definition
Negative Angle	A negative angle is an angle measured by rotating clockwise (instead of counterclockwise) from the positive x axis.

Additional Resources

Video: Evaluating Trigonometric Functions of Any Angle - Overview

Practice: Trigonometric Functions of Negative Angles

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1.8: Relating Trigonometric Functions

Learning Objectives

- State the reciprocal relationships between trig functions, and use these identities to find values of trig functions.
- State quotient relationships between trig functions, and use quotient identities to find values of trig functions.
- State the domain and range of each trig function.
- State the sign of a trig function, given the quadrant in which an angle lies.
- State the Pythagorean identities and use these identities to find values of trig functions.

Reciprocal and Pythagorean Identities

The two most basic types of trigonometric identities are the reciprocal identities and the Pythagorean identities. The reciprocal identities are simply definitions of the reciprocals of the three standard trigonometric ratios:

$$\sec\theta = \frac{1}{\cos\theta} \quad \csc\theta = \frac{1}{\sin\theta} \quad \cot\theta = \frac{1}{\tan\theta} \tag{1.8.1}$$

Also, recall the definitions of the three standard trigonometric ratios (sine, cosine and tangent):

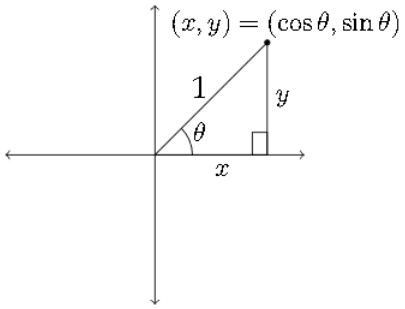
$$\begin{aligned} \sin\theta &= \tfrac{opp}{hyp} \\ \cos\theta &= \tfrac{adj}{hyp} \\ \tan\theta &= \tfrac{opp}{ady} \end{aligned} \tag{1.8.2}$$

If we look more closely at the relationships between the sine, cosine and tangent, we'll notice that $\frac{\sin\theta}{\cos\theta} = \tan\theta$

$$\frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{opp}{hyp}\right)}{\left(\frac{adj}{hyp}\right)} = \frac{opp}{hyp} * \frac{hyp}{adj} = \frac{opp}{adj} = \tan\theta \tag{1.8.3}$$

Pythagorean Identities

The Pythagorean Identities are, of course, based on the Pythagorean Theorem. If we recall a diagram that was introduced in Chapter 2, we can build these identities from the relationships in the diagram:



Using the Pythagorean Theorem in this diagram, we see that $x^2 + y^2 = 1^2$, so $x^2 + y^2 = 1$. But, also remember that, in the unit circle, $x = \cos\theta$ and $y = \sin\theta$

Substituting this equality gives us the first Pythagorean Identity:

$$x^2 + y^2 = 1 \tag{1.8.4}$$

or

$$\cos^2\theta + \sin^2\theta = 1 \tag{1.8.5}$$

This identity is usually stated in the form:

$$\sin^2\theta+\cos^2\theta=1 \tag{1.8.6}$$

If we take this identity and divide through on both sides by $\cos^2\theta$, this will result in the first of two additional Pythagorean Identities:

$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta} \tag{1.8.7}$$

or

$$\tan^2\theta+1=\sec^2\theta \tag{1.8.8}$$

Dividing through by $\sin^2\theta$ gives us the second:

$$\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta} \tag{1.8.9}$$

or

$$1+\cot^2\theta=\csc^2\theta \tag{1.8.10}$$

So, the three Pythagorean Identities we will be using are:

$$\begin{aligned}\sin^2\theta+\cos^2\theta&=1\\ \tan^2\theta+1&=\sec^2\theta\\ 1+\cot^2\theta&=\csc^2\theta\end{aligned} \tag{1.8.11}$$

These Pythagorean Identities are often stated in other terms, such as:

$$\begin{aligned}\sin^2\theta&=1-\cos^2\theta\\ \cos^2\theta&=1-\sin^2\theta\\ \tan^2\theta&=\sec^2\theta-1\\ \cot^2\theta&=\csc^2\theta-1\end{aligned} \tag{1.8.12}$$

Now that we have some basic identities to work with, let's use them to verify the equality of some more complicated statements. The process of verifying trigonometric identities involves changing one side of the given expression into the other side. since these are not really equations, we will not treat them the way we treat equations. That is to say, we won't add or subtract anything to both sides of the statement (or multiply or divide by anything on both sides either).

Another reason for not treating a trigonometric identity as an equation is that, in practice, this process typically involves just one side of the statement. In problem solving, mathematicians typically use trigonometric identities to change the appearance of a problem without changing its value. In this process, a trigonometric expression is changed into another trigonometric expression rather than showing that two trigonometric expressions are the same, which is what we will be doing.

The trigonometric identities we have discussed in this section are summarized below:

Pythagorean Identities	Reciprocal Identities
$\sin^2\theta+\cos^2\theta=1$	$\tan\theta=\frac{\sin\theta}{\cos\theta}$
$\tan^2\theta+1=\sec^2\theta$	$\cot\theta=\frac{\cos\theta}{\sin\theta}$
$1+\cot^2\theta=\csc^2\theta$	$\sec\theta=\frac{1}{\cos\theta}$
	$\csc\theta=\frac{1}{\sin\theta}$

The form $\sin\theta$ or $\cos\theta$ is typically used, however any letter may be used to represent the angle in question so long as it is the SAME letter in all expressions. For example, we can say that:

$$\sin^2\theta+\cos^2\theta=1 \tag{1.8.13}$$

or we can say that

$$\sin^2 x+\cos^2 x=1 \tag{1.8.14}$$

however:

$$\sin^2\theta+\cos^2 x\neq 1 \tag{1.8.15}$$

because θ and x could be different angles!

Quotient Identities

The definitions of the trig functions led us to the reciprocal identities, which can be seen in the Concept about that topic. They also lead us to another set of identities, the quotient identities.



Consider first the sine, cosine, and tangent functions. For angles of rotation (not necessarily in the unit circle) these functions are defined as follows:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$

Given these definitions, we can show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, as long as $\cos\theta \neq 0$:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan\theta \quad .$$

The equation $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is therefore an identity that we can use to find the value of the tangent function, given the value of the sine and cosine.

Let's take a look at some problems involving quotient identities.

1. Find the value of $\tan\theta$?

If $\cos\theta = \frac{5}{13}$ and $\sin\theta = \frac{12}{13}$, what is the value of $\tan\theta$?

$\tan\theta = \frac{12}{5}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{13} \times \frac{13}{5} = \frac{12}{5}$$

2. Show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$$\cos\theta \sin\theta = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y} = \cot\theta$$

3. What is the value of $\cot\theta$?

If $\cos\theta = \frac{7}{25}$ and $\sin\theta = \frac{24}{25}$, what is the value of $\cot\theta$?

$\cot\theta = \frac{7}{24}$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{7}{25}}{\frac{24}{25}} = \frac{7}{25} \times \frac{25}{24} = \frac{7}{24}$$

Example 1.8.3

If $\sin\theta = \frac{63}{65}$ and $\cos\theta = \frac{16}{65}$, what is the value of $\tan\theta$?

Solution

$\tan\theta = \frac{63}{16}$. We can see this from the relationship for the tangent function:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{63}{65}}{\frac{16}{65}} = \frac{63}{65} \times \frac{65}{16} = \frac{63}{16}$$

Example 1.8.4

If $\tan\theta = \frac{40}{9}$ and $\cos\theta = \frac{9}{41}$, what is the value of $\sin\theta$?

Solution

$\sin\theta = \frac{40}{41}$. We can see this from the relationship for the tangent function:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\sin\theta = (\tan\theta)(\cos\theta)$$
$$\sin\theta = \frac{40}{9} \times \frac{9}{41}$$
$$\sin\theta = \frac{40}{41}$$

Review

Fill in each blank with a trigonometric function.

1. If $\cos\theta = \frac{1}{2}$ and $\cot\theta = \frac{\sqrt{3}}{3}$, what is the value of $\sin\theta$?
2. If $\tan\theta = 0$ and $\cos\theta = -1$, what is the value of $\sin\theta$?
3. If $\cot\theta = -1$ and $\sin\theta = -\frac{\sqrt{2}}{2}$, what is the value of $\cos\theta$?

Vocabulary

Term	Definition
Quotient Identity	The quotient identity is an identity relating the tangent of an angle to the sine of the angle divided by the cosine of the angle.

Additional Resources

Video: The Reciprocal, Quotient, and Pythagorean Identities

Cofunction Identities

A cofunction identity is a relationship between one trig function of an angle and another trig function of the complement of that angle.

In a right triangle, you can apply what are called "cofunction identities". These are called cofunction identities because the functions have common values. These identities are summarized below.

$$\sin\theta = \cos(90^\circ - \theta) \qquad \cos\theta = \sin(90^\circ - \theta)$$
$$\tan\theta = \cot(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta)$$

Example 1.8.1

Find the value of $\sin 45^\circ$ using a cofunction identity.

Solution

The sine of 45° is equal to $\cos(90^\circ - 45^\circ) = \cos 45^\circ = \frac{\sqrt{2}}{2}$.

Example 1.8.2

Find the value of $\cos 45^\circ$ using a cofunction identity.

Solution

The cosine of 45° is equal to $\sin(90^\circ - 45^\circ) = \sin 45^\circ = \frac{\sqrt{2}}{2}$.

Example 1.8.3

Find the value of $\cos 60^\circ$ using a cofunction identity.

Solution

The cosine of 60° is equal to $\sin(90^\circ - 60^\circ) = \sin 30^\circ = .5$.

Review

1. Find a value for θ for which $\sin\theta = \cos 15^\circ$ is true.
2. Find a value for θ for which $\cos\theta = \sin 55^\circ$ is true.
3. Find a value for θ for which $\tan\theta = \cot 80^\circ$ is true.
4. Use the sine and cosine cofunction identities to prove that $\tan(90^\circ - \theta) = \cot\theta$.

Vocabulary

Term	Definition
Cofunction Identity	A cofunction identity is a relationship between one trig function of an angle and another trig function of the complement of that angle.

Additional Resources

Video: Cofunctions

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CHAPTER OVERVIEW

Chapter 2: Graphing Trigonometric Functions

2.1: Radian Measure
2.2: Applications of Radian Measure
2.3: Basic Trigonometric Graphs
2.4: Transformations Sine and Cosine Functions
2.5: Graphing Tangent, Cotangent, Secant, and Cosecant

Thumbnail: negative of sine wave (CC BY-NC-SA; Ted Sundstrom & Steven Schlicker via LibreTexts).

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2.1: Radian Measure

Learning Objectives

- Define radian measure.
- Convert angle measure from degrees to radians and radians to degrees.
- Calculate the values of the 6 trigonometric functions for special angles in terms of radians or degrees.

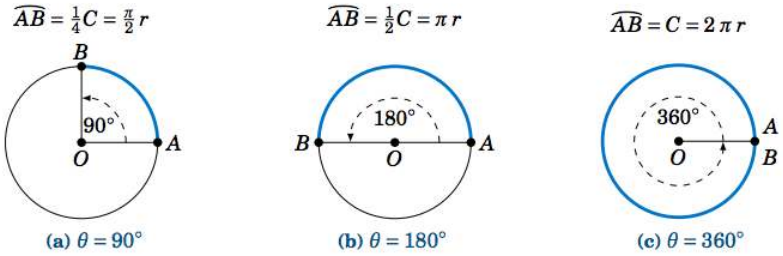


Figure 2.1.1 Angle θ and intercepted arc $\overset{\frown}{AB}$ on circle of circumference $C = 2\pi r$

In Figure 2.1.1 we see that a central angle of 90° cuts off an arc of length $\frac{\pi}{2}\,r$, a central angle of 180° cuts off an arc of length $\pi\,r$, and a central angle of 360° cuts off an arc of length $2\pi\,r$, which is the same as the circumference of the circle. So associating the central angle with its intercepted arc, we could say, for example, that

$$360^\circ \quad \text{"equals''} \quad 2\pi\,r \quad (\text{or } 2\pi \text{ 'radiuses'}).$$

The radius r was arbitrary, but the 2π in front of it stays the same. So instead of using the awkward "radiuses" or "radii", we use the term **radians**:

$$\boxed{360^\circ \;=\; 2\pi \text{ radians}} \tag{2.1.1}$$

The above relation gives us any easy way to convert between degrees and radians:

$$\textbf{Degrees to radians:} \quad x \text{ degrees} \;=\; \left(\frac{\pi}{180}\cdot x\right) \text{ radians} \tag{2.1.2}$$

$$\textbf{Radians to degrees:} \quad x \text{ radians} \;=\; \left(\frac{180}{\pi}\cdot x\right) \text{ degrees} \tag{2.1.3}$$

Equation 2.1.2 follows by dividing both sides of Equation 2.1.1 by 360, so that $1^\circ = \frac{2\pi}{360} = \frac{\pi}{180}$ radians, then multiplying both sides by x. Equation 2.1.3 is similarly derived by dividing both sides of Equation 2.1.1 by 2π then multiplying both sides by x.

The statement $\theta = 2\pi$ radians is usually abbreviated as $\theta = 2\pi$ rad, or just $\theta = 2\pi$ when it is clear that we are using radians. When an angle is given as some multiple of π, you can assume that the units being used are radians.

Example 2.1.1

Convert 18° to radians.

Solution

Using the conversion Equation 2.1.2 for degrees to radians, we get

$$18^\circ \;=\; \frac{\pi}{180}\cdot 18 \;=\; \boxed{\frac{\pi}{10}\text{ rad}}.$$

Example 2.1.2

Convert $\frac{\pi}{9}$ radians to degrees.

Solution

Using the conversion Equation 2.1.3 for radians to degrees, we get

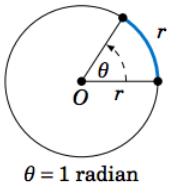
$$\frac{\pi}{9}\ \text{rad} = \frac{180}{\pi} \cdot \frac{\pi}{9} = \boxed{20^\circ}.$$

Table 4.1 Commonly used angles in radians

Degrees	Radians	Degrees	Radians	Degrees	Radians	Degrees	Radians
0°	0	90°	$\frac{\pi}{2}$	180°	π	270°	$\frac{3\pi}{2}$
30°	$\frac{\pi}{6}$	120°	$\frac{2\pi}{3}$	210°	$\frac{7\pi}{6}$	300°	$\frac{5\pi}{3}$
45°	$\frac{\pi}{4}$	135°	$\frac{3\pi}{4}$	225°	$\frac{5\pi}{4}$	315°	$\frac{7\pi}{4}$
60°	$\frac{\pi}{3}$	150°	$\frac{5\pi}{6}$	240°	$\frac{4\pi}{3}$	330°	$\frac{11\pi}{6}$

Table 4.1 shows the conversion between degrees and radians for some common angles. Using the conversion Equation 2.1.3 for radians to degrees, we see that

$$1\ \text{radian} = \frac{180}{\pi}\ \text{degrees} \approx 57.3^\circ.$$



$\theta = 1$ radian

Figure 4.1.2

Formally, a radian is defined as the central angle in a circle of radius r which intercepts an arc of length r, as in Figure 4.1.2. This definition does not depend on the choice of r (imagine resizing Figure 4.1.2).

One reason why radians are used is that the scale is smaller than for degrees. One revolution in radians is $2\pi \approx 6.283185307$ which is much smaller than 360, the number of degrees in one revolution. The smaller scale makes the graphs of trigonometric functions (which we will discuss in Chapter 5) have similar scales for the horizontal and vertical axes. Another reason is that often in physical applications the variables being used are in terms of arc length, which makes radians a natural choice.

The default mode in most scientific calculators is to use degrees for entering angles. On many calculators there is a button labeled $\boxed{DRG}$ for switching between degree mode (D), radian mode (R), and *gradian* mode (G). On some graphing calculators, such as the the TI-83, there is a $\boxed{MODE}$ button for changing between degrees and radians. Make sure that your calculator is in the correct angle mode *before* entering angles, or your answers will likely be way off. For example,

$$\begin{aligned} \sin 4^\circ &= 0.0698, \\ \sin(4\ \text{rad}) &= -0.7568, \end{aligned}$$

so the values are not only off in magnitude, but do not even have the same sign. Using your calculator's $\boxed{\sin^{-1}}$, $\boxed{\cos^{-1}}$, and $\boxed{\tan^{-1}}$ buttons in radian mode will of course give you the angle as a decimal, not an expression in terms of π.

You should also be aware that the math functions in many computer programming languages use radians, so you would have to write your own angle conversions.

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2.2: Applications of Radian Measure

Learning Objectives

- Solve problems involving angles of rotation using radian measure.
- Calculate the length of an arc and the area of a sector.
- Approximate the length of a chord given the central angle and radius.
- Solve problems about angular speed.

In Section 2.1 we saw that one revolution has a radian measure of 2π rad. Note that 2π is the ratio of the circumference (i.e. total arc length) C of a circle to its radius r:

$$\text{Radian measure of 1 revolution} = 2\pi = \frac{2\pi\, r}{r} = \frac{C}{r} = \frac{\text{total arc length}}{\text{radius}}$$

Clearly, that ratio is independent of r. In general, the radian measure of an angle is the ratio of the arc length cut off by the corresponding central angle in a circle to the radius of the circle, independent of the radius.

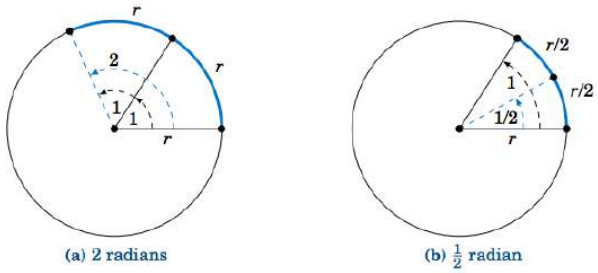


Figure 2.2.1 Radian measure and arc length

Now suppose that we cut the angle with radian measure 1 in half, as in Figure 2.2.1(b). Clearly, this cuts the arc length r in half as well. Thus, we see that

$$\begin{aligned} \text{Angle} &= 1 \text{ radian} &\Rightarrow\quad \text{arc length} &= r\,, \\ \text{Angle} &= 2 \text{ radians} &\Rightarrow\quad \text{arc length} &= 2\,r\,, \\ \text{Angle} &= \tfrac{1}{2} \text{ radian} &\Rightarrow\quad \text{arc length} &= \tfrac{1}{2}\,r\,, \end{aligned}$$

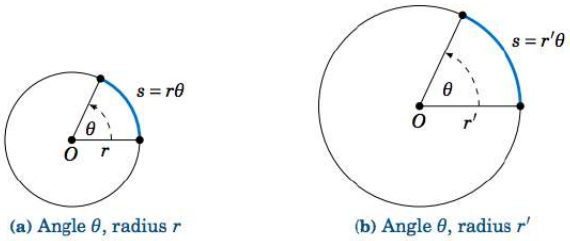
and in general, for any $\theta \geq 0$,

$$\text{Angle} = \theta \text{ radians} \quad\Rightarrow\quad \text{arc length} = \theta\, r\,,$$

so that

$$\theta = \frac{\text{arc length}}{\text{radius}}\,.$$

Intuitively, it is obvious that shrinking or magnifying a circle preserves the measure of a central angle even as the radius changes. The above discussion says more, namely that the ratio of the length s of an intercepted arc to the radius r is preserved, precisely because that ratio is the measure of the central angle in radians (see Figure 2.2.2).



(a) Angle θ, radius r (b) Angle θ, radius r'

Figure 2.2.2 Circles with the same central angle, different radii

We thus get a simple formula for the length of an arc:

In a circle of radius r, let s be the length of an arc intercepted by a central angle with radian measure $\theta \geq 0$. Then the arc length s is:

$$s = r\theta \tag{2.2.1}$$

Example 2.2.1

Find the arc length of a circle with a radius of 10 feet and a central angle of 41°.

Solution

So first convert $\theta = 41^\circ$ to radians, then use $s = r\theta$:

$$\theta = 41^\circ = \frac{\pi}{180} \cdot 41 = 0.716 \text{ rad} \quad \Rightarrow \quad s = r\theta = (10)(0.716) = \boxed{7.16 \text{ ft}}$$

Note that since the arc length s and radius r are usually given in the same units, radian measure is really unitless, since you can think of the units canceling in the ratio $\frac{s}{r}$, which is just θ. This is another reason why radians are so widely used.

For central angles $\theta > 2\pi$ rad, i.e. $\theta > 360^\circ$, it may not be clear what is meant by the intercepted arc, since the angle is larger than one revolution and hence "wraps around" the circle more than once. We will take the approach that such an arc consists of the full circumference plus any additional arc length determined by the angle. In other words, Equation 2.2.1 is still valid for angles $\theta > 2\pi$ rad.

What about negative angles? In this case using $s = r\theta$ would mean that the arc length is negative, which violates the usual concept of length. So we will adopt the convention of only using nonnegative central angles when discussing arc length.

Example 2.2.2

A rope is fastened to a wall in two places 8 ft apart at the same height. A cylindrical container with a radius of 2 ft is pushed away from the wall as far as it can go while being held in by the rope, as in Figure 2.2.3 which shows the top view. If the center of the container is 3 feet away from the point on the wall midway between the ends of the rope, what is the length L of the rope?

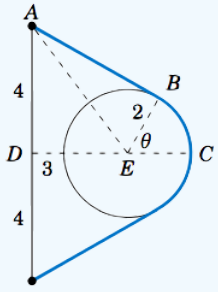


Figure 2.2.3

Solution

We see that, by symmetry, the total length of the rope is $L = 2\,(AB + \overset{\frown}{BC})$. Also, notice that $\triangle ADE$ is a right triangle, so the hypotenuse has length $AE = \sqrt{DE^2 + DA^2} = \sqrt{3^2 + 4^2} = 5$ ft, by the Pythagorean Theorem. Now since $\overline{AB}$ is tangent to the circular container, we know that $\angle ABE$ is a right angle. So by the Pythagorean Theorem we have

$$AB = \sqrt{AE^2 - BE^2} = \sqrt{5^2 - 2^2} = \sqrt{21}\text{ ft}.$$

By Equation 2.2.1 the arc $\overset{\frown}{BC}$ has length $BE \cdot \theta$, where $\theta = \angle BEC$ is the supplement of $\angle AED + \angle AEB$. So since

$$\tan \angle AED = \frac{4}{3} \Rightarrow \angle AED = 53.1^\circ \quad \text{and} \quad \cos \angle AEB = \frac{BE}{AE} = \frac{2}{5} \Rightarrow \angle AEB = 66.4^\circ,$$

we have

$$\theta = \angle BEC = 180^\circ - (\angle AED + \angle AEB) = 180^\circ - (53.1^\circ + 66.4^\circ) = 60.5^\circ.$$

Converting to radians, we get $\theta = \frac{\pi}{180} \cdot 60.5 = 1.06$ rad. Thus,

$$L = 2\,(AB + \cdot\overset{\frown}{BC}) = 2\,(\sqrt{21} + BE \cdot \theta) = 2\,(\sqrt{21} + (2)\,(1.06)) = \boxed{13.4\text{ ft}}.$$

Area of a Sector

In geometry you learned that the area of a circle of radius r is πr^2. We will now learn how to find the area of a **sector** of a circle. A sector is the region bounded by a central angle and its intercepted arc, such as the shaded region in Figure 2.2.1.

Let θ be a central angle in a circle of radius r and let A be the area of its sector. Similar to arc length, the ratio of A to the area of the entire circle is the same as the ratio of θ to one revolution. In other words, again using radian measure,

$$\frac{\text{area of sector}}{\text{area of entire circle}} = \frac{\text{sector angle}}{\text{one revolution}} \quad \Rightarrow \quad \frac{A}{\pi\, r^2} = \frac{\theta}{2\pi}.$$

Solving for A in the above equation, we get the following formula:

In a circle of radius r, the area A of the sector inside a central angle θ is

$$A = \tfrac{1}{2}\, r^2\, \theta, \tag{2.2.2}$$

where θ is measured in radians.

Example 2.2.3

Find the area of a sector whose angle is 117° in a circle of radius 3.5 m.

Solution

As with arc length, we have to make sure that the angle is measured in radians or else the answer will be way off. So converting $\theta = 117^\circ$ to radians and using $r = 3.5$ in Equation 2.2.2 for the area A of the sector, we get

$$\theta = 117^\circ = \frac{\pi}{180} \cdot 117 = 2.042\text{ rad} \quad \Rightarrow \quad A = \tfrac{1}{2}\, r^2\, \theta = \tfrac{1}{2}\,(3.5)^2\,(2.042) = \boxed{12.51\text{ m}^2}.$$

For a sector whose angle is θ in a circle of radius r, the length of the arc cut off by that angle is $s = r\,\theta$. Thus, by Equation 2.2.2 the area A of the sector can be written as:

$$\boxed{A = \tfrac{1}{2}\, rs} \tag{2.2.3}$$

Note: The central angle θ that intercepts an arc is sometimes called the angle **subtended** by the arc.

Example 2.2.4

Find the area of the sector of a circle with a radius of 6 and an arc length of 9.

Solution

Using $s = 6$ and $r = 9$ in Equation 2.2.3 for the area A, we get

$$A = \tfrac{1}{2} rs = \quad = \tfrac{1}{2}(9)(6) = \boxed{27 \text{ cm}^2}.$$

Note that the angle subtended by the arc is $\theta = \frac{s}{r} = \frac{2}{3}$ rad.

Linear and Angular Speed

Radian measure and arc length can be applied to the study of *circular motion*. In physics the *average speed* of an object is defined as:

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}}$$

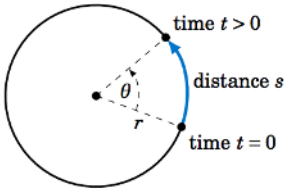


Figure 2.2.4

So suppose that an object moves along a circle of radius r, traveling a distance s over a period of time t, as in Figure 4.4.1. Then it makes sense to define the (average) **linear speed** ν of the object as:

$$\nu = \frac{s}{t} \tag{2.2.4}$$

Let θ be the angle swept out by the object in that period of time. Then we define the (average) **angular speed** ω of the object as:

$$\omega = \frac{\theta}{t} \tag{2.2.5}$$

Angular speed gives the rate at which the central angle swept out by the object changes as the object moves around the circle, and it is thus measured in radians per unit time. Linear speed is measured in distance units per unit time (e.g. feet per second). The word linear is used because straightening out the arc traveled by the object along the circle results in a line of the same length, so that the usual definition of speed as distance over time can be used. We will usually omit the word average when discussing linear and angular speed here.

Since the length s of the arc cut off by a central angle θ in a circle of radius r is $s = r\,\theta$, we see that

$$\nu = \frac{s}{t} = \frac{r\,\theta}{t} = \frac{\theta}{t} \cdot r,$$

so that we get the following relation between linear and angular speed:

$$\nu = \omega r \tag{2.2.6}$$

Example 2.2.5

An object sweeps out a central angle of $\frac{\pi}{3}$ radians in 0.5 seconds as it moves along a circle of radius 3 m. Find its linear and angular speed over that time period.

Solution:

Here we have $t = 0.5$ sec, $r = 3$ m, and $\theta = \frac{\pi}{3}$ rad. So the angular speed ω is

$$\omega = \frac{\theta}{t} = \frac{\frac{\pi}{3}\text{ rad}}{0.5\text{ sec}} \quad \Rightarrow \quad \boxed{\omega = \frac{2\pi}{3}\text{ rad/sec}},$$

and thus the linear speed ν is

$$\nu = \omega r = \left(\frac{2\pi}{3}\text{ rad/sec}\right)(3\text{ m}) \quad \Rightarrow \quad \boxed{\nu = 2\pi\text{ m/sec}}.$$

Note that the units for ω are rad/sec and the units of ν are m/sec. Recall that radians are actually unitless, which is why in the Equation $\nu = \omega r$ the radian units disappear.

Example 2.2.6

An object travels a distance of 35 ft in 2.7 seconds as it moves along a circle of radius 2 ft. Find its linear and angular speed over that time period.

Solution:

Here we have $t = 2.7$ sec, $r = 2$ ft, and $s = 35$ ft. So the linear speed ν is

$$\nu = \frac{s}{t} = \frac{35\text{ feet}}{2.7\text{ sec}} \quad \Rightarrow \quad \boxed{\nu = 12.96\text{ ft/sec}},$$

and thus the angular speed ω is given by

$$\nu = \omega r \quad \Rightarrow \quad 12.96\text{ ft/sec} = \omega(2\text{ ft}) \quad \Rightarrow \quad \boxed{\omega = 6.48\text{ rad/sec}}.$$

Example 2.2.7

An object moves at a constant linear speed of 10 m/sec around a circle of radius 4 m. How large of a central angle does it sweep out in 3.1 seconds?

Solution:

Here we have $t = 3.1$ sec, $\nu = 10$ m/sec, and $r = 4$ m. Thus, the angle θ is given by

$$s = r\theta \quad \Rightarrow \quad \theta = \frac{s}{r} = \frac{\nu t}{r} = \frac{(10\text{ m/sec})(3.1\text{ sec})}{4\text{ m}} = \boxed{7.75\text{ rad}}.$$

In many physical applications angular speed is given in *revolutions per minute*, abbreviated as *rpm*. To convert from rpm to, say, radians per second, notice that since there are 2π radians in one revolution and 60 seconds in one minute, we can convert N rpm to radians per second by "canceling the units" as follows:

$$N\text{ rpm} = N\,\frac{\cancel{\text{rev}}}{\cancel{\text{min}}} \cdot \frac{2\pi\text{ rad}}{1\ \cancel{\text{rev}}} \cdot \frac{1\ \cancel{\text{min}}}{60\text{ sec}} = \frac{N \cdot 2\pi}{60}\text{ rad/sec}$$

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2.3: Basic Trigonometric Graphs

Learning Objectives

- Graph the six trigonometric ratios as functions on the Cartesian plane.
- Identify the domain and range of these six trigonometric functions.
- Identify the radian and degree measure, as well as the coordinates of points on the unit circle and graph for the critical angle.

The first function we will graph is the sine function. We will describe a geometrical way to create the graph, using the *unit circle*. This is the circle of radius 1 in the xy-plane consisting of all points (x, y) which satisfy the equation $x^2 + y^2 = 1$.

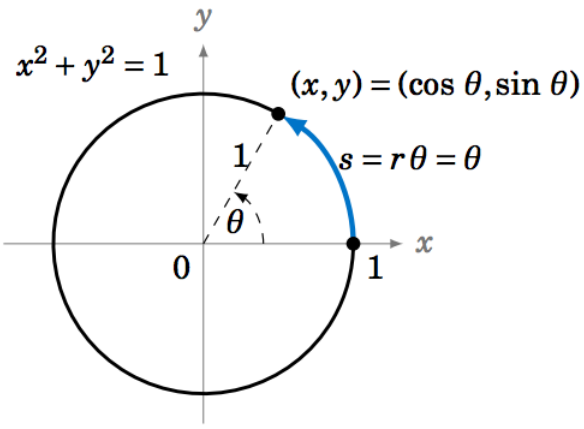


Figure 2.3.1

We see in Figure 2.3.1 that any point on the unit circle has coordinates $(x, y) = (\cos\ \theta, \sin\ \theta)$, where θ is the angle that the line segment from the origin to (x, y) makes with the positive x-axis (by definition of sine and cosine). So as the point (x, y) goes around the circle, its y-coordinate is $\sin\ \theta$.

We thus get a correspondence between the y-coordinates of points on the unit circle and the values $f(\theta) = \sin\ \theta$, as shown by the horizontal lines from the unit circle to the graph of $f(\theta) = \sin\ \theta$ in Figure 2.3.2 for the angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$.

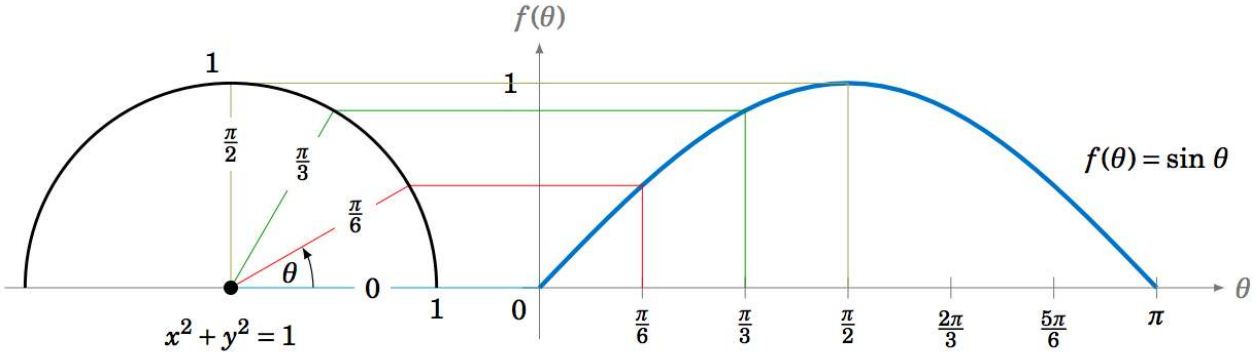


Figure 2.3.2 Graph of sine function based on y-coordinate of points on unit circle

We can extend the above picture to include angles from 0 to 2π radians, as in Figure 2.3.3. This illustrates what is sometimes called the *unit circle definition of the sine function.*

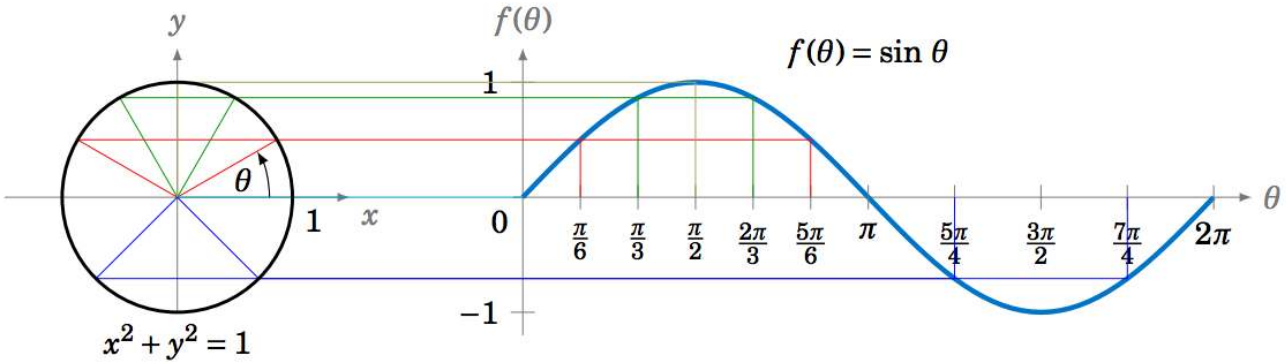


Figure 2.3.3 Unit circle definition of the sine function

Since the trigonometric functions repeat every 2π radians (360°), we get, for example, the following graph of the function $y = \sin\ x$ for x in the interval $[-2\pi, 2\pi]$:

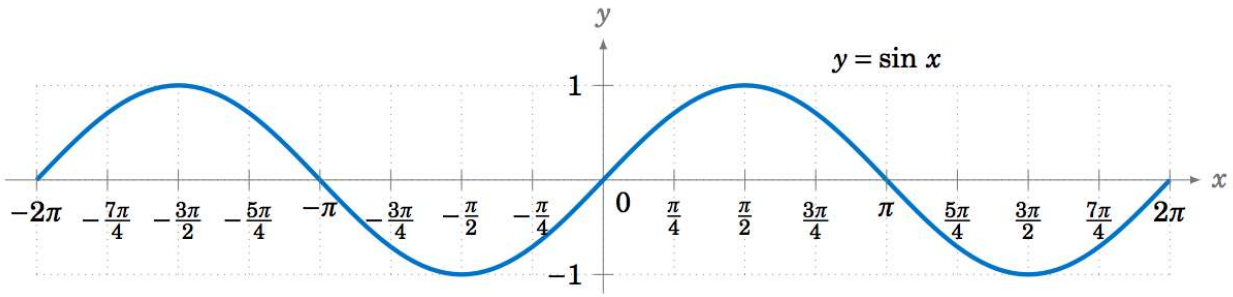


Figure 2.3.4 Graph of $y = \sin x$

To graph the cosine function, we could again use the unit circle idea (using the x-coordinate of a point that moves around the circle), but there is an easier way. Recall from Section 1.5 that $\cos\ x = \sin\ (x + 90^\circ)$ for all x. So $\cos\ 0^\circ$ has the same value as $\sin\ 90^\circ$, $\cos\ 90^\circ$ has the same value as $\sin\ 180^\circ$, $\cos\ 180^\circ$ has the same value as $\sin\ 270^\circ$, and so on. In other words, the graph of the cosine function is just the graph of the sine function shifted to the *left* by $90^\circ = \pi/2$ radians, as in Figure 2.3.5:

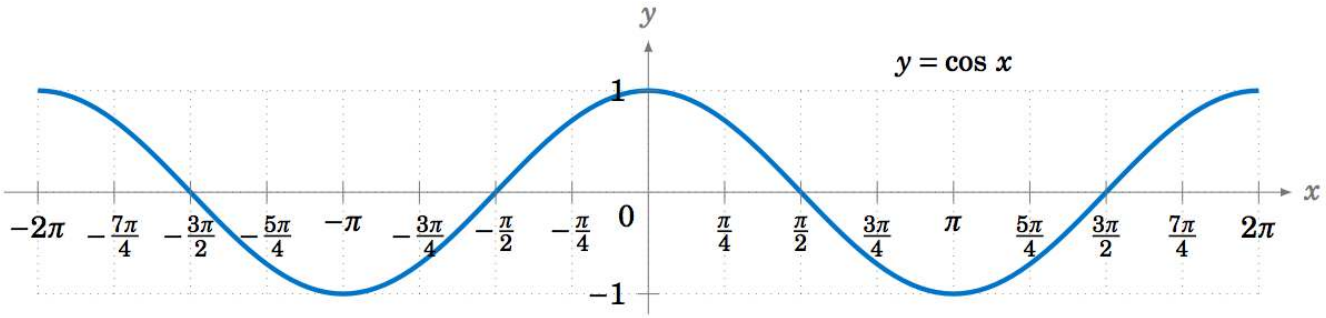


Figure 2.3.5 Graph of $y = \cos x$

To graph the tangent function, use $\tan\ x = \frac{\sin\ x}{\cos\ x}$ to get the following graph:

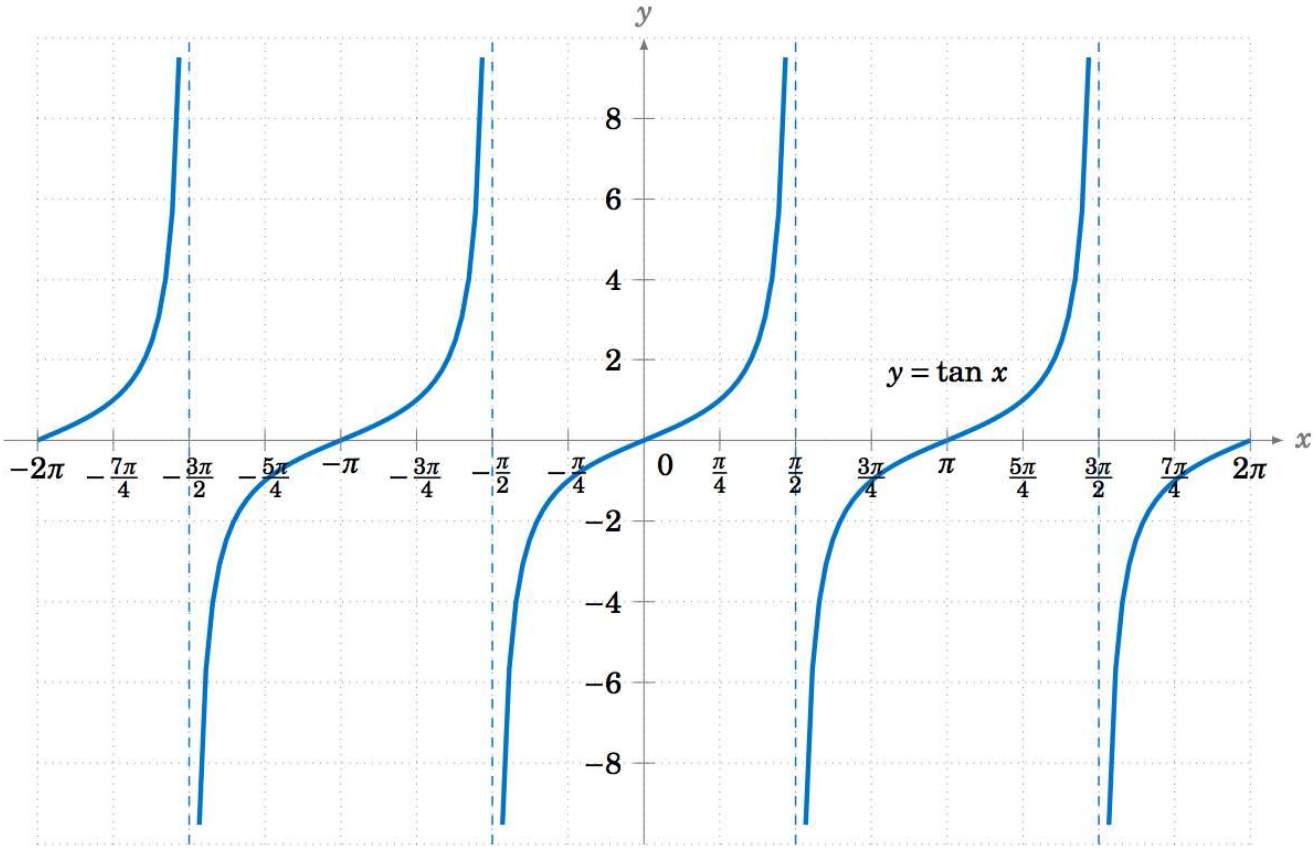


Figure 2.3.6 Graph of $y = \tan x$

Recall that the tangent is positive for angles in QI and QIII, and is negative in QII and QIV, and that is indeed what the graph in Figure 5.1.6 shows. We know that $\tan\ x$ is not defined when $\cos\ x = 0$, i.e. at odd multiples of $\frac{\pi}{2}$: $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, etc. We can figure out what happens *near* those angles by looking at the sine and cosine functions. For example, for x in QI near $\frac{\pi}{2}$, $\sin\ x$ and $\cos\ x$ are both positive, with $\sin\ x$ very close to 1 and $\cos\ x$ very close to 0, so the quotient $\tan\ x = \frac{\sin\ x}{\cos\ x}$ is a positive number that is very large. And the closer x gets to $\frac{\pi}{2}$, the larger $\tan\ x$ gets. Thus, $x = \frac{\pi}{2}$ is a *vertical asymptote* of the graph of $y = \tan\ x$.

Likewise, for x in QII very close to $\frac{\pi}{2}$, $\sin\ x$ is very close to 1 and $\cos\ x$ is negative and very close to 0, so the quotient $\tan\ x = \frac{\sin\ x}{\cos\ x}$ is a negative number that is very large, and it gets larger in the negative direction the closer x gets to $\frac{\pi}{2}$. The graph shows this. Similarly, we get vertical asymptotes at $x = -\frac{\pi}{2}$, $x = \frac{3\pi}{2}$, and $x = -\frac{3\pi}{2}$, as in Figure 5.1.6. Notice that the graph of the tangent function repeats every π radians, i.e. two times faster than the graphs of sine and cosine repeat.

The graphs of the remaining trigonometric functions can be determined by looking at the graphs of their reciprocal functions. For example, using $\csc\ x = \frac{1}{\sin\ x}$ we can just look at the graph of $y = \sin\ x$ and invert the values. We will get vertical asymptotes when $\sin\ x = 0$, namely at multiples of π: $x = 0, \pm\pi, \pm 2\pi$, etc. Figure 2.3.7 shows the graph of $y = \csc\ x$, with the graph of $y = \sin\ x$ (the dashed curve) for reference.

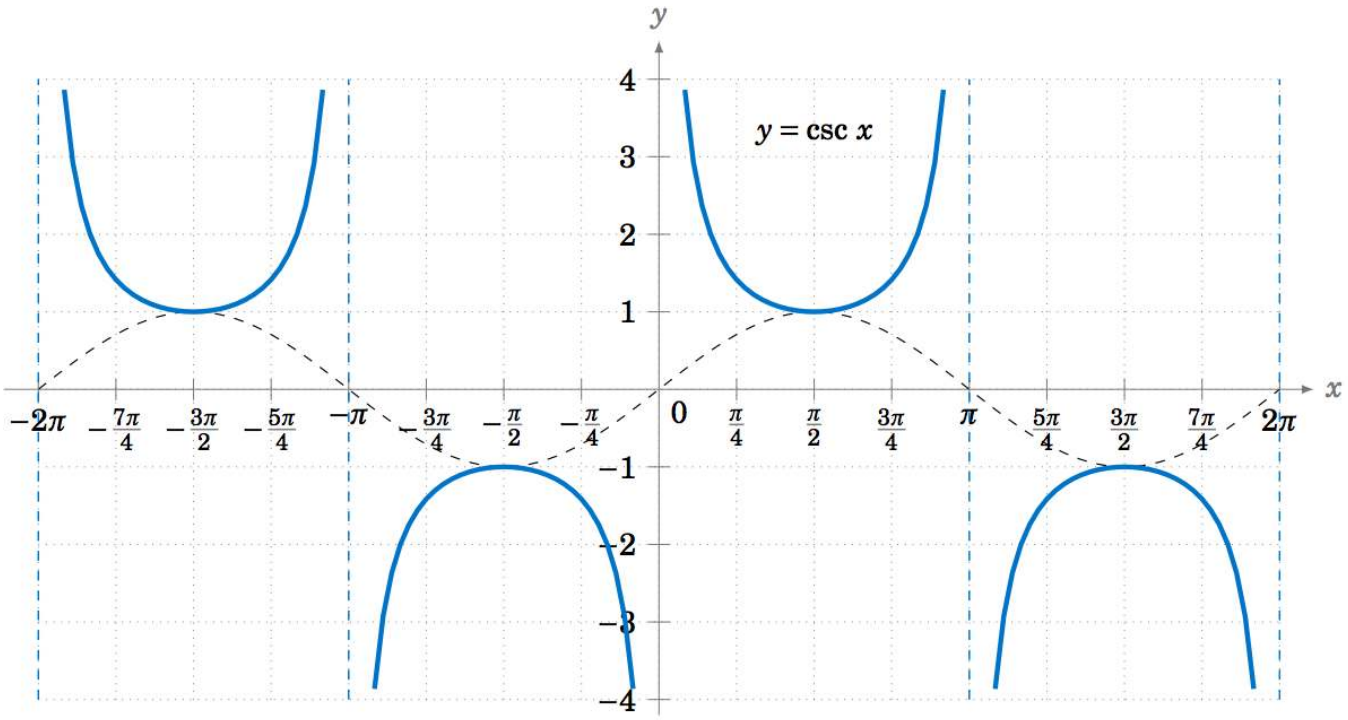


Figure 2.3.7 Graph of $y = \csc x$

Likewise, Figure 2.3.8 shows the graph of $y = \sec\ x$, with the graph of $y = \cos\ x$ (the dashed curve) for reference. Note the vertical asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$. Notice also that the graph is just the graph of the cosecant function shifted to the left by $\frac{\pi}{2}$ radians.

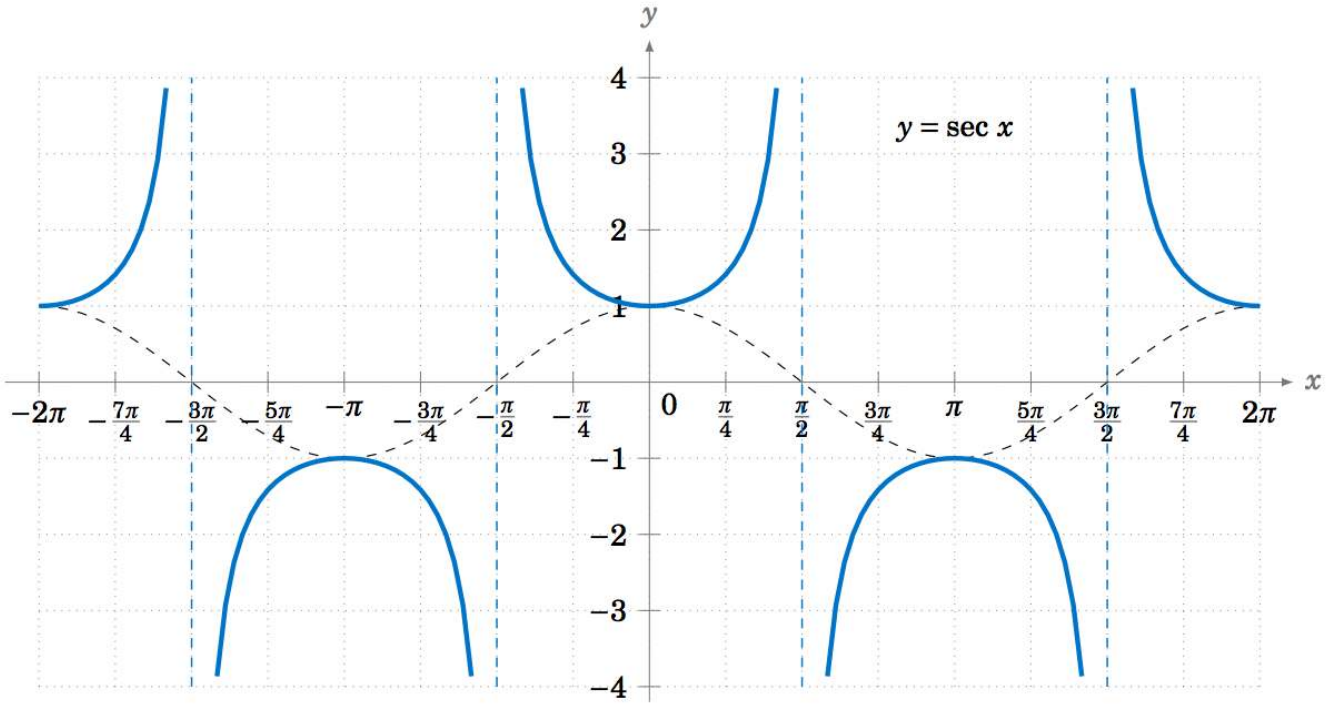


Figure 2.3.8 Graph of $y = \sec x$

The graph of $y = \cot\ x$ can also be determined by using $\cot\ x = \frac{1}{\tan\ x}$. Alternatively, we can use the relation $\cot\ x = -\tan\ (x + 90°)$ from Section 1.5, so that the graph of the cotangent function is just the graph of the tangent function shifted to the left by $\frac{\pi}{2}$ radians and then reflected about the x-axis, as in Figure 2.3.9:

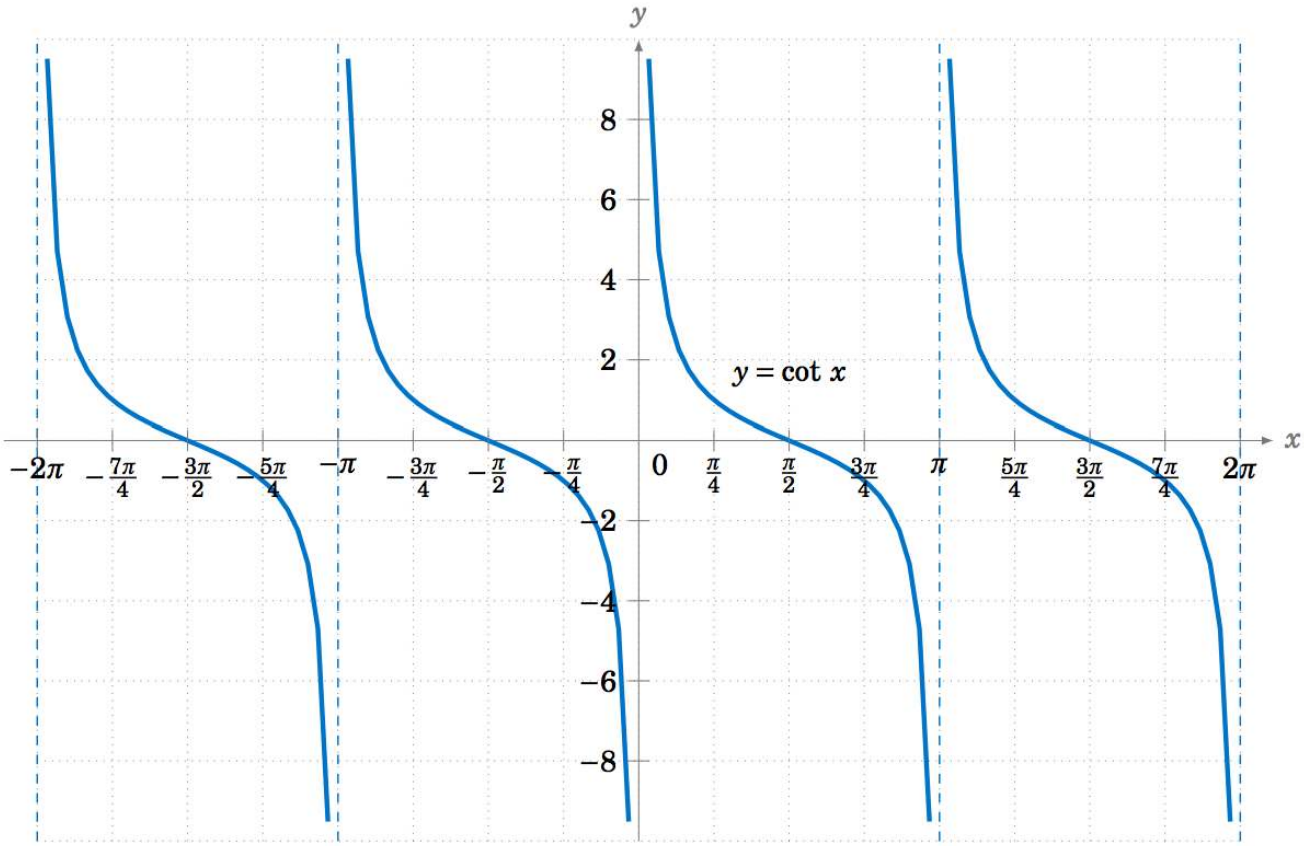


Figure 2.3.9 Graph of $y = \cot x$

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2.4: Transformations Sine and Cosine Functions

Learning Objectives

- Translate sine and cosine functions vertically and horizontally.
- Identify the vertical and horizontal translations of sine and cosine from a graph and an equation.
- Calculate the amplitude and period of a sine or cosine curve.
- Calculate the frequency of a sine or cosine wave.
- Graph transformations of sine and cosine waves involving changes in amplitude and period (frequency).
- Graph any sinusoid given an equation in the form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
- Identify the equation of any sinusoid given a graph and critical values.

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.



Figure 2.4.1: Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

Light waves can be represented graphically by the sine function. In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. Table $2.4.1$ lists some of the values for the sine function on a unit circle.

Table 2.4.1

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the *x*-axis gives the shape of the sine function. See Figure $2.4.2$.

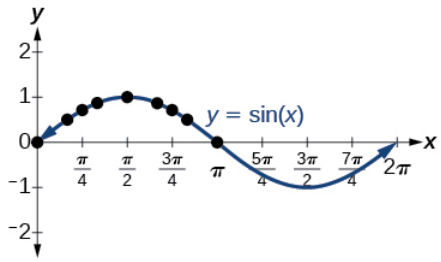


Figure 2.4.2: The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See Figure $2.4.3$.

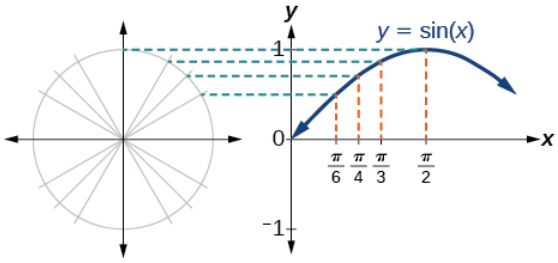


Figure 2.4.3: Plotting values of the sine function

Now let's take a similar look at the **cosine function**. Again, we can create a table of values and use them to sketch a graph. Table $2.4.2$ lists some of the values for the cosine function on a unit circle.

Table 2.4.2

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

As with the sine function, we can plots points to create a graph of the cosine function as in Figure $2.4.4$.

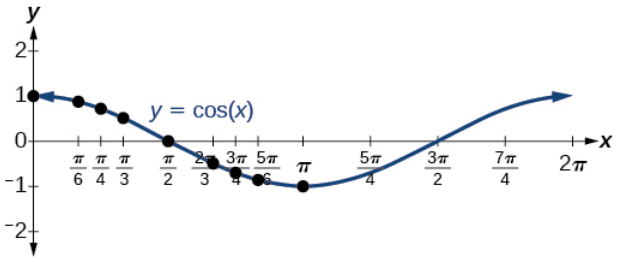


Figure 2.4.4: The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific **horizontal shift**, P, results in a function equal to the original function: $f(x+P) = f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P > 0$ the **period** of the function. Figure $2.4.5$ shows several periods of the sine and cosine functions.

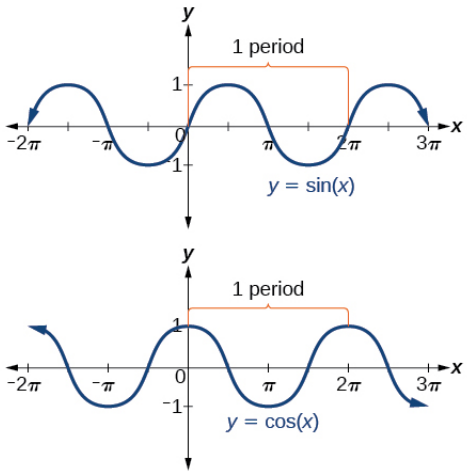


Figure 2.4.5

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in Figure 2.4.6, the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin\ x$. Now we can clearly see this property from the graph.

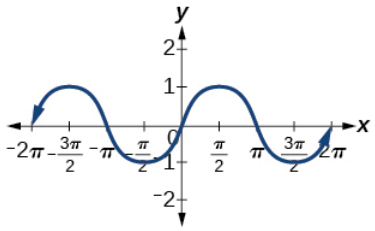


Figure 2.4.6: Odd symmetry of the sine function

Figure 2.4.7 shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos\ x$.

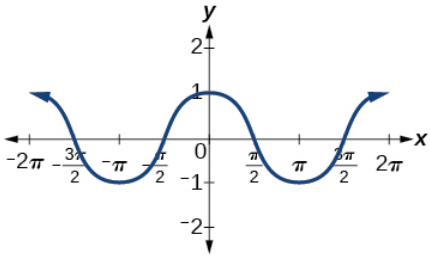


Figure 2.4.7: Even symmetry of the cosine function

CHARACTERISTICS OF SINE AND COSINE FUNCTIONS

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos\ x$ is symmetric about they- y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or **cosine function** is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D \tag{2.4.1}$$

and

$$y = A\cos(Bx - C) + D \tag{2.4.2}$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \frac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π,which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\frac{x}{2}\right)$, then $B = \frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in Figure 2.4.8 how the period is indirectly related to $|B|$.

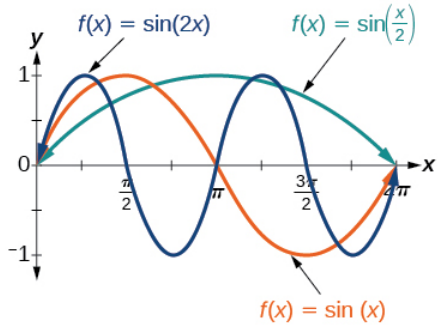


Figure 2.4.8

PERIOD OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

- $y = A\sin(Bx)$
- $y = A\cos(Bx)$

The period is $\frac{2\pi}{|B|}$.

Example 2.4.1: Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\frac{\pi}{6}x\right)$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \frac{\pi}{6}$, so the period will be

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{6}} \\ &= 2\pi \cdot \frac{6}{\pi} \\ &= 12 \end{aligned}$$

Exercise 2.4.1

Determine the period of the function $g(x) = \cos(\frac{x}{3})$.

Answer

6π

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the x-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4sinx$ is twice the amplitude of

$$f(x) = 2\sin x$$

If $|A| < 1$, the function is compressed. Figure 2.4.9 compares several sine functions with different amplitudes.

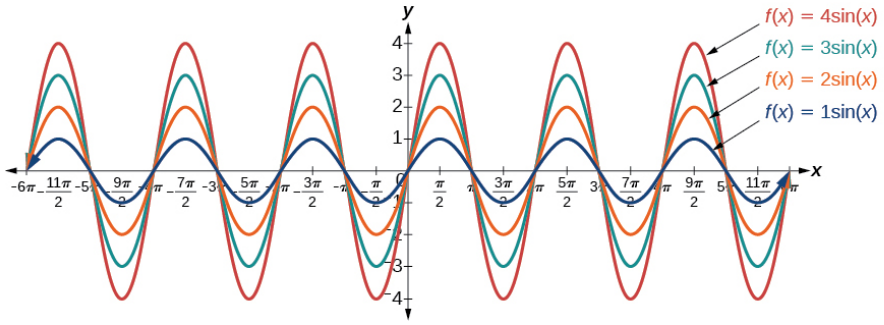


Figure 2.4.9

AMPLITUDE OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx) \text{ and } y = A\cos(Bx) \tag{2.4.3}$$

The amplitude is A, and the vertical height from the midline is $|A|$. In addition, notice in the example that

$$|A| = amplitude = \frac{1}{2}\,|\,maximum - minimum| \tag{2.4.4}$$

Example 2.4.2: Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis

The negative value of A results in a reflection across the x-axis of the sine function, as shown in Figure 2.4.10

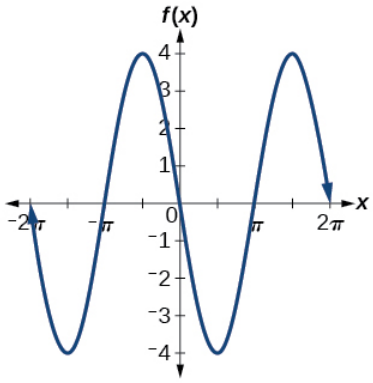


Figure 2.4.10

Exercise 2.4.2

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Answer

$\frac{1}{2}$ compressed

Analyzing Graphs of Variations of $y = \sin\ x$ and $y = \cos\ x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D \tag{2.4.5}$$

or

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \quad \text{and} \quad y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D \tag{2.4.6}$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. Figure 2.4.11 shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

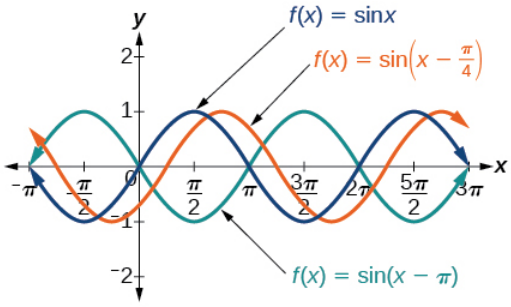


Figure 2.4.11

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See Figure 2.4.12. The function $y = \cos(x) + D$ has its midline at $y = D$.

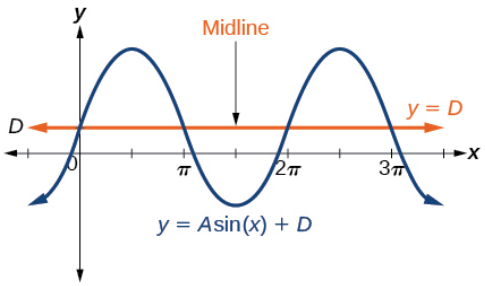


Figure 2.4.12

Any value of D other than zero shifts the graph up or down. Figure 2.4.13 compares $f(x)=\sin x$ with $f(x)=\sin x+2$, which is shifted 2 units up on a graph.

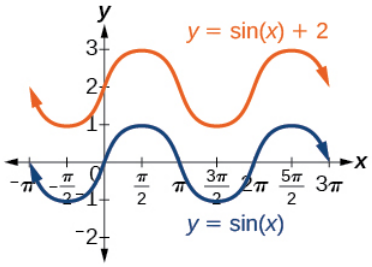


Figure 2.4.13

VARIATIONS OF SINE AND COSINE FUNCTIONS

Given an equation in the form $f(x)=A\sin(Bx-C)+D$ or $f(x)=A\cos(Bx-C)+D$, $\frac{C}{B}$ is the **phase shift** and D is the **vertical shift.**

Example 2.4.3: Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x)=\sin\left(x+\frac{\pi}{6}\right)-2$.

Solution

Let's begin by comparing the equation to the general form $y=A\sin(Bx-C)+D$.

In the given equation, notice that $B=1$ and $C=-\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned}\frac{C}{B}&=-\frac{\frac{\pi}{6}}{1}\\&=-\frac{\pi}{6}\end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x)=\sin(x+\frac{\pi}{6})-2$ can be rewritten as $f(x)=\sin\left(x-\left(-\frac{\pi}{6}\right)\right)-2$. If the value of C is negative, the shift is to the left.

Exercise 2.4.3

Determine the direction and magnitude of the phase shift for $f(x)=3\cos\left(x-\frac{\pi}{2}\right)$.

Answer

$\frac{\pi}{2}$; right

Example 2.4.4: Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

Solution

Let's begin by comparing the equation to the general form $y = A\cos(Bx - C) + D$.

In the given equation, $D = -3$ so the shift is 3 units downward.

Exercise 2.4.4

Determine the direction and magnitude of the vertical shift for $f(x) = 3\sin(x) + 2$.

Answer

2 units up

How to: Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$,identify the midline, amplitude, period, and phase shift

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

Example 2.4.5: Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\dfrac{C}{B} = \dfrac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis

Inspecting the graph, we can determine that the period is π, the midline is $y = 1$, and the amplitude is 3. See Figure 2.4.14

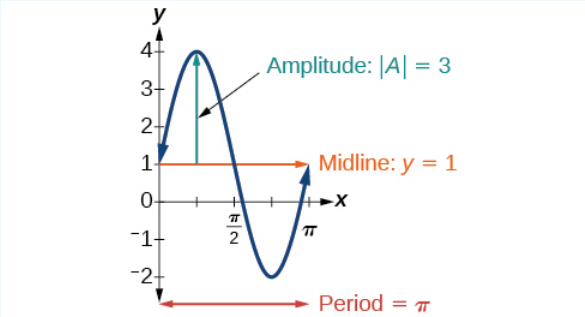


Figure 2.4.14

Exercise 2.4.5

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Answer

midline: $y = 0$; amplitude: $|A| = \frac{1}{2}$; period: $P = \frac{2\pi}{|B|} = 6\pi$; phase shift: $\frac{C}{B} = \pi$

Example 2.4.6: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in Figure 2.4.15.

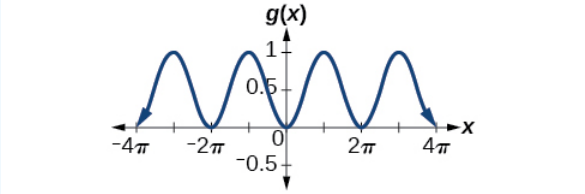


Figure 2.4.15

Solution

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin(Bx - C) + D$$

$$y = A\cos(Bx - C) + D$$

The graph could represent either a sine or a **cosine function** that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the x-axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

Exercise 2.4.6

Determine the formula for the sine function in Figure 2.4.16.

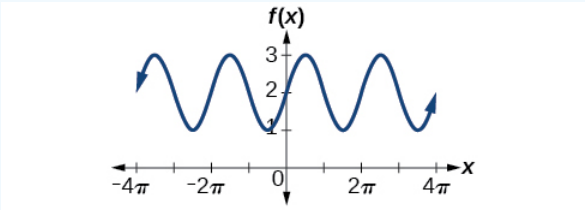


Figure 2.4.16

Answer

$f(x) = \sin(x) + 2$

Example 2.4.7: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in Figure 2.4.17.

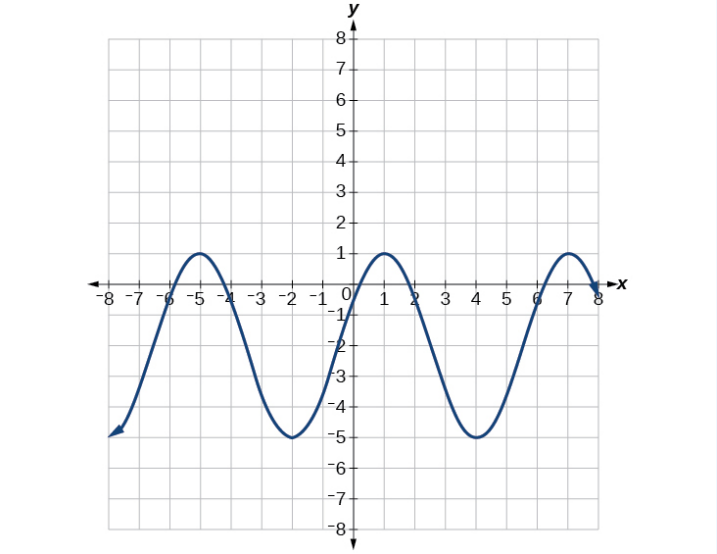


Figure 2.4.17

Solution

With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D=-2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A|=3$.

The period of the graph is 6, which can be measured from the peak at $x=1$ to the next peak at $x=7$,or from the distance between the lowest points. Therefore, $P=\dfrac{2\pi}{|B|}=6$. Using the positive value for B,we find that

$$\begin{aligned} B &= \frac{2\pi}{P} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

So far, our equation is either $y=3\sin\left(\frac{\pi}{3}x-C\right)-2$ or $y=3\cos\left(\frac{\pi}{3}x-C\right)-2$.For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

Again, these functions are equivalent, so both yield the same graph.

Exercise 2.4.7

Write a formula for the function graphed in Figure 2.4.18.

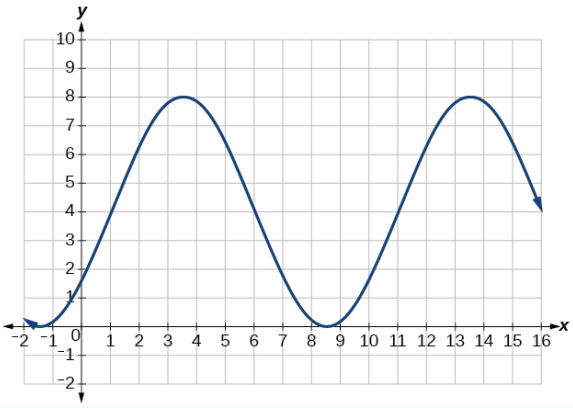


Figure 2.4.18

Answer

two possibilities: $y = 4\sin\left(\frac{\pi}{5}x - \frac{\pi}{5}\right) + 4$ or $y = -4\sin\left(\frac{\pi}{5}x + \frac{4\pi}{5}\right) + 4$

Graphing Variations of $y = \sin\ x$ and $y = \cos\ x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \frac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the *x*-axis at $x = \frac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \frac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the *x*-axis at $x = \frac{\pi}{2|B|}$.

Example 2.4.8: Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$.

Solution

Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

- *Step 1.* We can see from the equation that $A = -2$, so the amplitude is 2.

  $|A| = 2$
- *Step 2.* The equation shows that $B = \frac{\pi}{2}$, so the period is

$$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$

- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4.* The x-intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. Figure 2.4.19 shows the graph of the function.

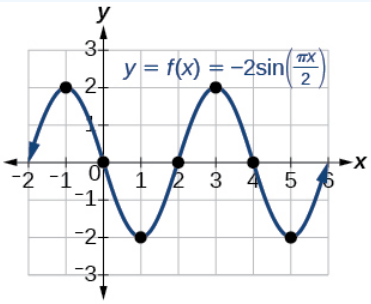


Figure 2.4.19

Exercise 2.4.8

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

Answer

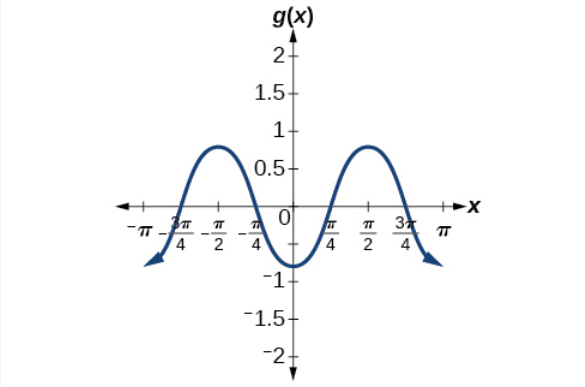


Figure 2.4.20

midline: $y = 0$; amplitude: $|A| = 0.8$; period: $P = \frac{2\pi}{|B|} = \pi$; phase shift: $\frac{C}{B} = 0$ or none

How to: Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \frac{2\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.

5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\frac{C}{B}$ and up or down by D.

Example 2.4.9: Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\frac{\pi}{4x} - \frac{\pi}{4}\right)$.

Solution

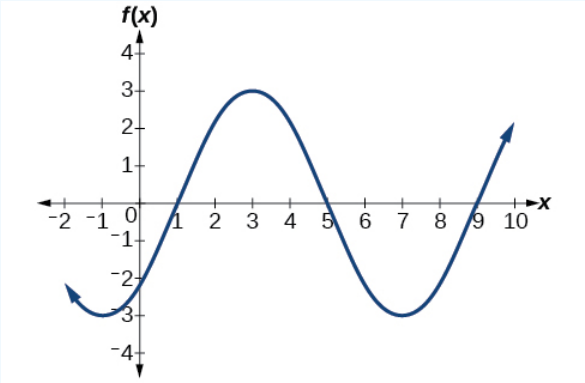


Figure 2.4.21: A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

- *Step 1.* The function is already written in general form: $f(x) = 3\sin\left(\frac{\pi}{4x} - \frac{\pi}{4}\right)$.This graph will have the shape of a sine function, starting at the midline and increasing to the right.
- *Step 2.* $|A| = |3| = 3$. The amplitude is 3.
- *Step 3.* Since $|B| = |\frac{\pi}{4}| = \frac{\pi}{4}$, we determine the period as follows.

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{4}} \\ &= 2\pi \cdot \frac{4}{\pi} \\ &= 8 \end{aligned}$$

The period is 8.

- *Step 4.* Since $C = \frac{\pi}{4}$, the phase shift is

$$\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1 \tag{2.4.7}$$

.

The phase shift is 1 unit.

- *Step 5.* Figure 2.4.21shows the graph of the function.

Exercise 2.4.9

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Answer

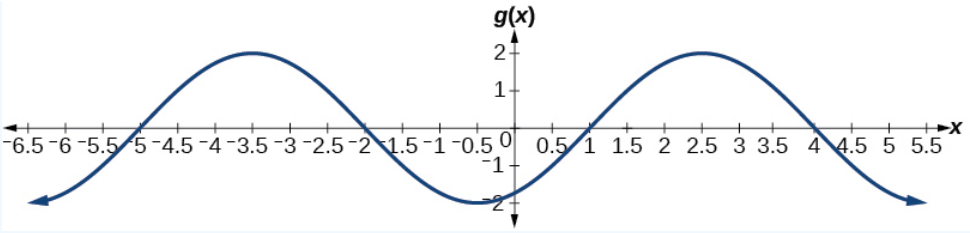


Figure 2.4.22

midline: $y = 0$; amplitude: $|A| = 2$; period: $P = \dfrac{2\pi}{|B|} = 6$; phase shift: $\dfrac{C}{B} = -\dfrac{1}{2}$

Example 2.4.10: Identifying the Properties of a Sinusoidal Function

Given $y = -2cos\left(\dfrac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution

Begin by comparing the equation to the general form.

$$y = A\cos(Bx - C) + D$$

- *Step 1.* The function is already written in general form.
- *Step 2.* Since $A = -2$, the amplitude is $|A| = 2$.
- *Step 3.* $|B| = \dfrac{\pi}{2}$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{\frac{pi}{2}} = 2\pi \cdot \dfrac{2}{\pi} = 4$. The period is 4.
- *Step 4.* $C = -\pi$,so we calculate the phase shift as $\dfrac{C}{B} = -\dfrac{\pi}{\frac{\pi}{2}} = -\pi \cdot \dfrac{2}{\pi} = -2$. The phase shift is -2.
- *Step 5.* $D = 3$,so the midline is $y = 3$, and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the x-axis.

Figure $2.4.23$shows one cycle of the graph of the function.

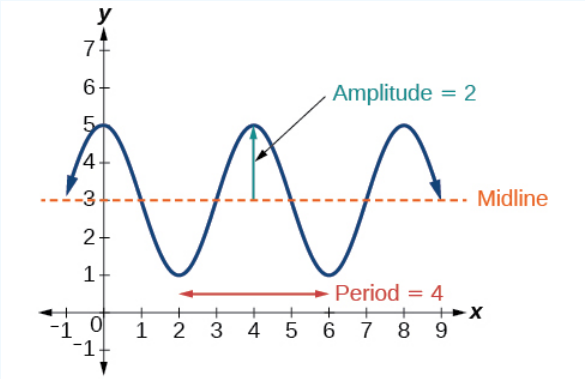


Figure 2.4.23

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter,**circular motion** can be modeled using either the sine or **cosine function**.

Example 2.4.11: Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in Figure $2.4.24$

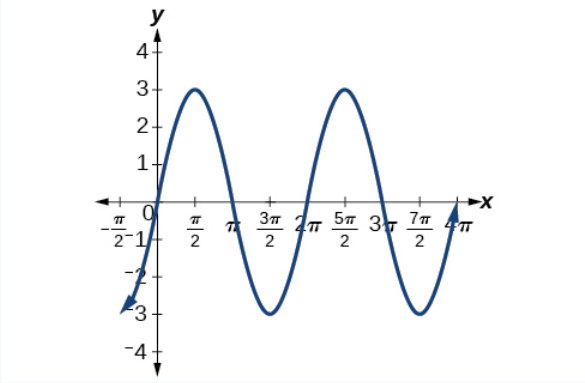


Figure 2.4.24

Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi$,...Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

Exercise 2.4.10

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Answer

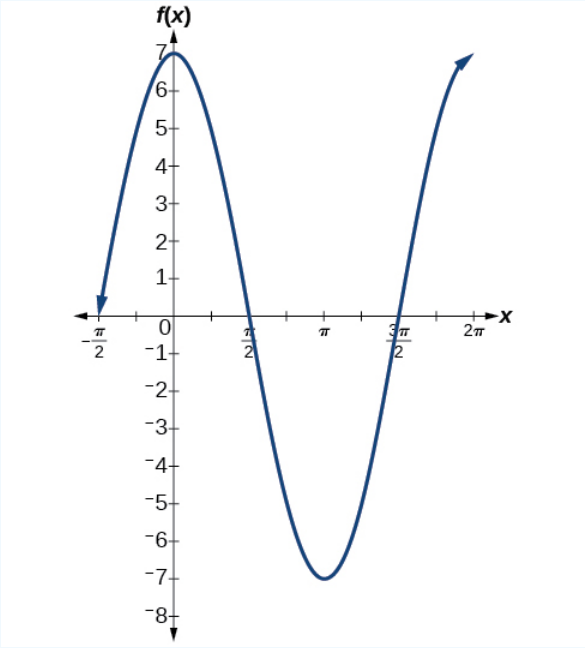


Figure 2.4.25

Example 2.4.12: Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in Figure $2.4.26$. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

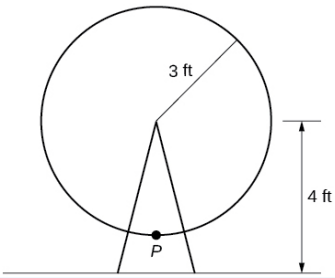


Figure 2.4.26

Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure 2.4.27.

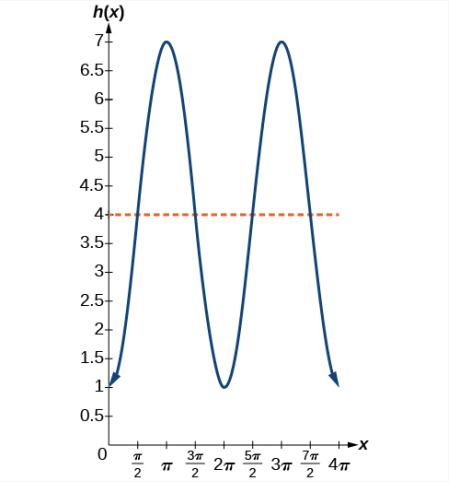


Figure 2.4.27

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a **sine function** starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

Exercise 2.4.11

A weight is attached to a spring that is then hung from a board, as shown in Figure 2.4.28. As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

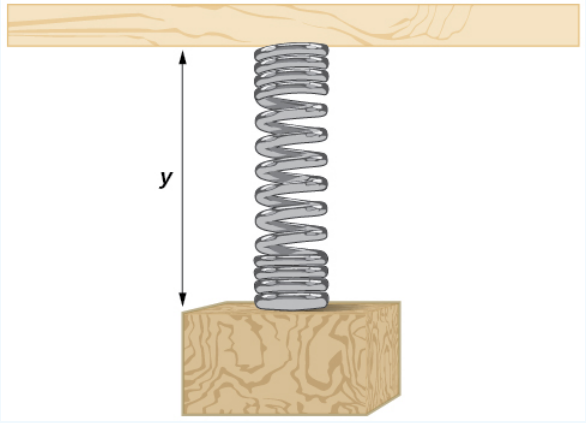


Figure 2.4.28

Answer

$y = 3\cos(x) - 4$

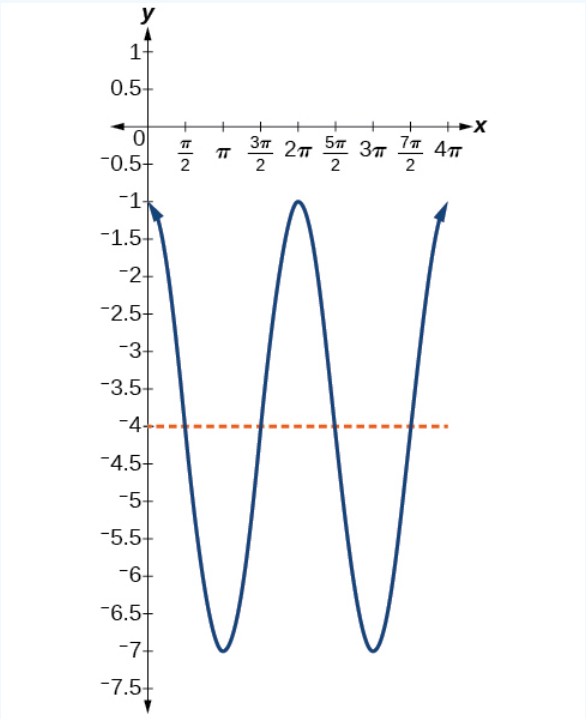


Figure 2.4.29

Example 2.4.13: Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period:30, so $B = \dfrac{2\pi}{30} = \dfrac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$ $f(x) = A\cos(Bx - C) + D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the y-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \dfrac{2\pi}{|B|}$. See Example $2.4.1$.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See Example $2.4.2$.
- The value $\dfrac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See Example $2.4.3$.
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See Example $2.4.4$.
- Combinations of variations of sinusoidal functions can be detected from an equation. See Example $2.4.5$.
- The equation for a sinusoidal function can be determined from a graph. See Example $2.4.6$ and Example $2.4.7$.
- A function can be graphed by identifying its amplitude and period. See Example $2.4.8$ and Example $2.4.9$.
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See Example $2.4.10$.
- Sinusoidal functions can be used to solve real-world problems. See Example $2.4.11$, Example $2.4.12$, and Example $2.4.13$.

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2.5: Graphing Tangent, Cotangent, Secant, and Cosecant

Learning Objectives

- Apply transformations to the remaining four trigonometric functions: tangent, cotangent, secant, and cosecant.
- Identify the equation, given a basic graph.

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan x = \frac{\sin x}{\cos x} \tag{2.5.1}$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned}
\tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent} \\
&= \frac{-\sin x}{\cos x} && \text{Sine is an odd function, cosine is even} \\
&= -\frac{\sin x}{\cos x} && \text{The quotient of an odd and an even function is odd} \\
&= -\tan x && \text{Definition of tangent}
\end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table 2.5.1.

Table 2.5.1

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in Table 2.5.2.

Table 2.5.2

x	1.3	1.5	1.55	1.56
$\tan x$	3.6	14.1	48.1	92.6

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in Table 2.5.3.

Table 2.5.3

x	−1.3	−1.5	−1.55	−1.56
$\tan x$	−3.6	−14.1	−48.1	−92.6

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. Figure 2.5.1 represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

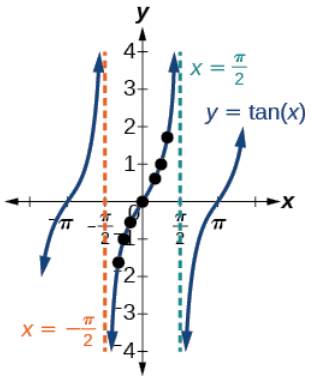


Figure 2.5.1: Graph of the tangent function

Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the *tangent function* can be described by a general equation.

$$y = A\tan(Bx)$$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

FEATURES OF THE GRAPH OF $Y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \dfrac{\pi}{|B|}$.
- The domain is all real numbers x,where $x \neq \dfrac{\pi}{2|B|} + \dfrac{\pi}{|B|}k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{2|B|} + \dfrac{\pi}{|B|}k$ where k is an integer.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the **tangent function** to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single **period** of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is then the interval $\left(-\dfrac{P}{2}, \dfrac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\dfrac{P}{2}$ where $P = \dfrac{\pi}{B}$. On $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x = -\dfrac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at

$x = \frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right) = A\tan\left(B\frac{P}{4}\right) = A\tan\left(B\frac{\pi}{4B}\right) = A$$

because $\tan\left(\frac{\pi}{4}\right) = 1$.

Given the function $f(x) = A\tan(Bx)$, graph one period.

1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x = -\frac{P}{2}$ and $x = \frac{P}{2}$.
4. For $A > 0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A < 0$).
5. Plot reference points at $\left(\frac{P}{4}, A\right)$, $(0, 0)$, and $\left(-\frac{P}{4}, -A\right)$, and draw the graph through these points.

Example 2.5.1: Sketching a Compressed Tangent

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

Solution

First, we identify A and B.

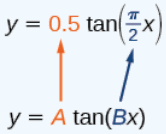


Figure 2.5.2

Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$, so the asymptotes are at $x = \pm 1$. At a quarter period from the origin, we have

$$\begin{aligned} f(0.5) &= 0.5\tan\left(\frac{0.5\pi}{2}\right) \\ &= 0.5\tan\left(\frac{\pi}{4}\right) \\ &= 0.5 \end{aligned}$$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$,and $(-0.5, -0.5)$. The only inflection point is at the origin. Figure 2.5.3 shows the graph of one period of the function.

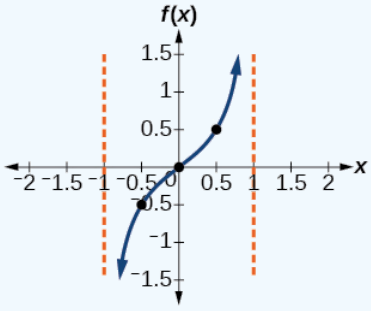


Figure 2.5.3

https://math.libretexts.org/@go/page/61248

Exercise 2.5.1

Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$.

Answer

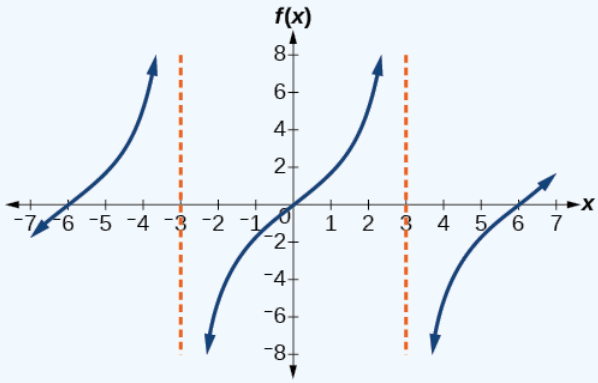


Figure 2.5.4

Graphing One Period of a Shifted Tangent Function

Now that we can graph a **tangent function** that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$$f(x) = A\tan(Bx - C) + D$$

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

FEATURES OF THE GRAPH OF $Y = A\tan(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\tan(Bx)$ is and odd function because it is the qoutient of odd and even functions(sin and cosine perspectively).

Howto: Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the **stretching/compressing** factor, $|A|$.
3. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$,where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Example 2.5.2: Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

Solution

- *Step 1.* The function is already written in the form $y = A\tan(Bx - C) + D$.
- *Step 2.*\(A=−2\), so the stretching factor is $|A| = 2$.
- *Step 3.* \(B=\pi\), so the period is $P = \dfrac{\pi}{|B|} = \dfrac{\pi}{pi} = 1$.
- *Step 4.* \(C=−\pi\), so the phase shift is $CB = \dfrac{-\pi}{\pi} = -1$.
- *Step 5-7.* The asymptotes are at $x = -\dfrac{3}{2}$ and $x = -\dfrac{1}{2}$ and the three recommended reference points are $(-1.25, 1)$, $(-1, -1)$, and $(-0.75, -3)$. The graph is shown in Figure 2.5.5.

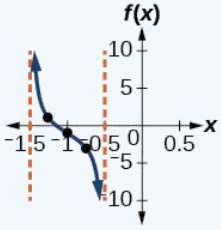


Figure 2.5.5

Analysis

Note that this is a decreasing function because $A < 0$.

Exercise 2.5.2

How would the graph in Example 2.5.2 look different if we made $A = 2$ instead of -2?

Answer

It would be reflected across the line $y = -1$, becoming an increasing function.

Howto: Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or *x*-intercepts.
2. Write $f(x) = A\tan\left(\dfrac{\pi}{P}x\right)$.
3. Determine a convenient point $(x, f(x))$ on the given graph and use it to determine A.

Example 2.5.3: Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in Figure 2.5.6.

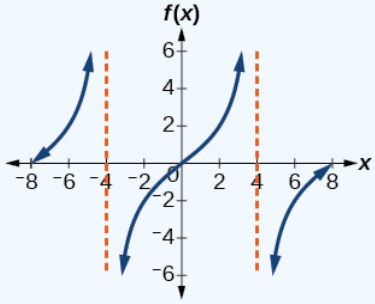


Figure 2.5.6: A stretched tangent function

Solution

The graph has the shape of a tangent function.

- *Step 1.* One cycle extends from -4 to 4, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{\pi}{P} = \frac{\pi}{8}$.
- *Step 2.* The equation must have the form $f(x) = A\tan\left(\frac{\pi}{8}x\right)$.
- *Step 3.* To find the vertical stretch A,we can use the point $(2, 2)$.

$$\begin{aligned} 2 &= A\tan\left(\frac{\pi}{8}\cdot 2\right) \\ &= A\tan\left(\frac{\pi}{4}\right) \end{aligned}$$

Because $\tan\left(\frac{\pi}{4}\right) = 1$, $A = 2$.

This function would have a formula $f(x) = 2\tan\left(\frac{\pi}{8}x\right)$.

Exercise 2.5.3

Find a formula for the function in Figure 2.5.7.

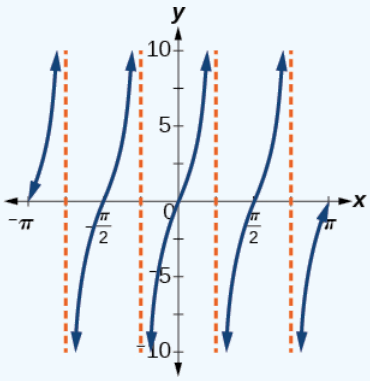


Figure 2.5.7

Answer

$g(x) = 4\tan(2x)$

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

The secant was defined by the reciprocal identity $sec\, x = \frac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\frac{\pi}{2}$, $\frac{3\pi}{2}$ etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure 2.5.8. The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

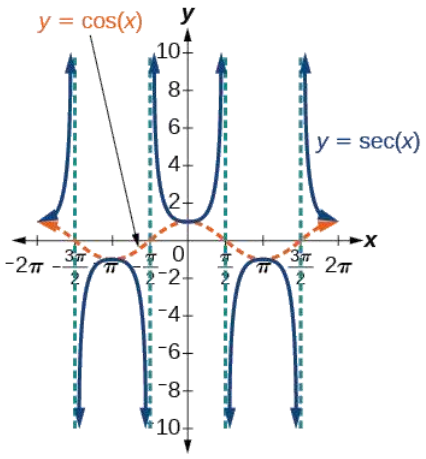


Figure 2.5.8: Graph of the secant function, $f(x) = \sec x = \dfrac{1}{\cos x}$

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

FEATURES OF THE GRAPH OF $Y = A\sec(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the **cosecant** is defined by the reciprocal identity $\csc x = \dfrac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure $2.5.7$. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the **cosecant function** decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in Figure $2.5.9$, is similar to the graph of secant.

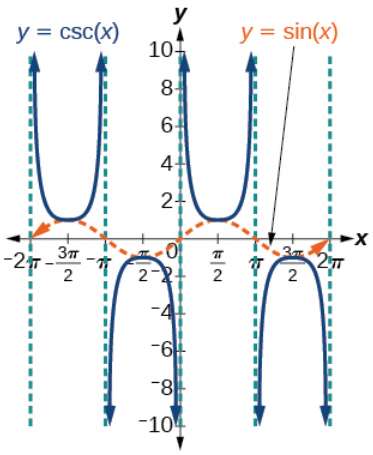


Figure 2.5.9: The graph of the cosecant function, $f(x) = \csc x = \frac{1}{\sin x}$

FEATURES OF THE GRAPH OF $Y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equations become the following.

$$y = A\sec(Bx - C) + D$$

$$y = A\csc(Bx - C) + D$$

FEATURES OF THE GRAPH OF $Y = A\sec(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

FEATURES OF THE GRAPH OF $Y = A\csc(Bx - C) + D$

1. The stretching factor is $|A|$.
2. The period is $\dfrac{2\pi}{|B|}$.
3. The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an integer.
4. The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
5. The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
6. There is no amplitude.
7. $y = A\csc(Bx)$ is an odd function because sine is an odd function.

HOWTO: Given a function of the form $y = A\sec(Bx)$, graph one period

1. Express the function given in the form $y = A\sec(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos(Bx)$.
5. Use the reciprocal relationship between $y = \cos\ x$ and $y = \sec\ x$ to draw the graph of $y = A\sec(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 2.5.4: Graphing a Variation of the Secant Function

Graph one period of $f(x) = 2.5\sec(0.4x)$.

Solution

- *Step 1.* The given function is already written in the general form, $y = A\sec(Bx)$.
- *Step 2.* $A = 2.5$ so the stretching factor is 2.5.
- *Step 3.* $B = 0.4$ so $P = \dfrac{2\pi}{0.4} = 5\pi$. The period is 5π units.
- *Step 4.* Sketch the graph of the function $g(x) = 2.5\cos(0.4x)$.
- *Step 5.* Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.
- *Steps 6–7.* Sketch two asymptotes at $x = 1.25\pi$ and $x = 3.75\pi$. We can use two reference points, the local minimum at $(0, 2.5)$ and the local maximum at $(2.5\pi, -2.5)$ Figure $2.5.10$ shows the graph.

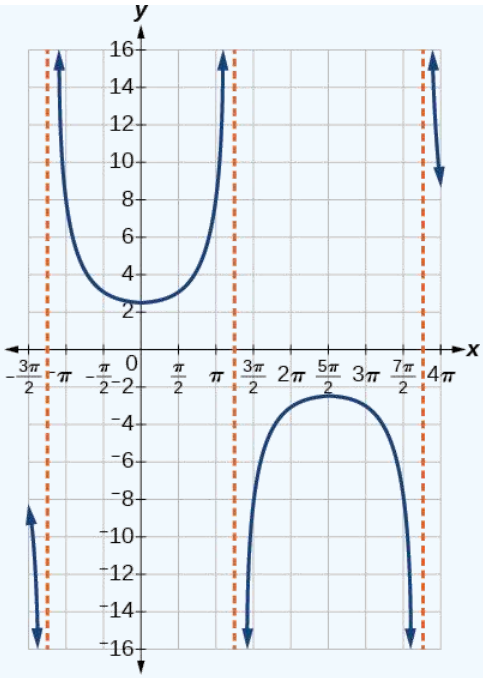


Figure 2.5.10

Exercise 2.5.4

Graph one period of $f(x) = -2.5\sec(0.4x)$.

Answer

This is a vertical reflection of the preceding graph because A is negative.

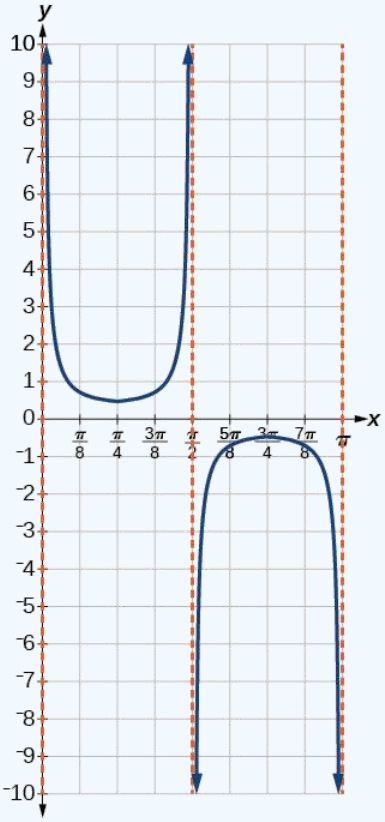


Figure 2.5.11

Q&A: Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\sec(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\dfrac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\sec(Bx)$. but shift it to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.

Example 2.5.5: Graphing a Variation of the Secant Function

Graph one period of $y = 4\sec\left(\dfrac{\pi}{3x} - \dfrac{\pi}{2}\right) + 1$.

Solution

- *Step 1.* Express the function given in the form $y = 4\sec\left(\dfrac{\pi}{3x} - \dfrac{\pi}{2}\right) + 1$.
- *Step 2.* The stretching/compressing factor is $|A| = 4$.
- *Step 3.* The period is

$$\begin{aligned}\frac{2\pi}{|B|} &= \frac{2\pi}{\frac{\pi}{3}} \\ &= 2\pi \cdot \frac{3}{\pi} \\ &= 6\end{aligned}$$

- *Step 4.* The phase shift is

$$\begin{aligned}\frac{C}{B} &= \frac{\frac{\pi}{2}}{\frac{\pi}{3}} \\ &= \frac{\pi}{2} \cdot \frac{3}{\pi} \\ &= 1.5\end{aligned}$$

- *Step 5.* Draw the graph of $y = A\sec(Bx)$, but shift it to the right by $\frac{C}{B} = 1.5$ and up by $D = 6$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 3$, and $x = 6$. There is a local minimum at $(1.5, 5)$ and a local maximum at $(4.5, -3)$. Figure $2.5.12$ shows the graph.

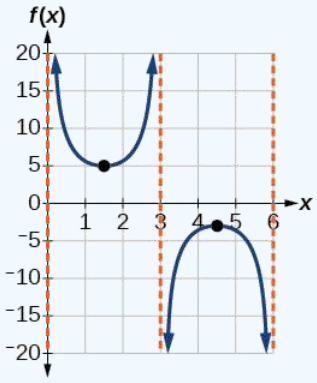


Figure 2.5.12

Exercise 2.5.5

Graph one period of $f(x) = -6\sec(4x + 2) - 8$.

Answer

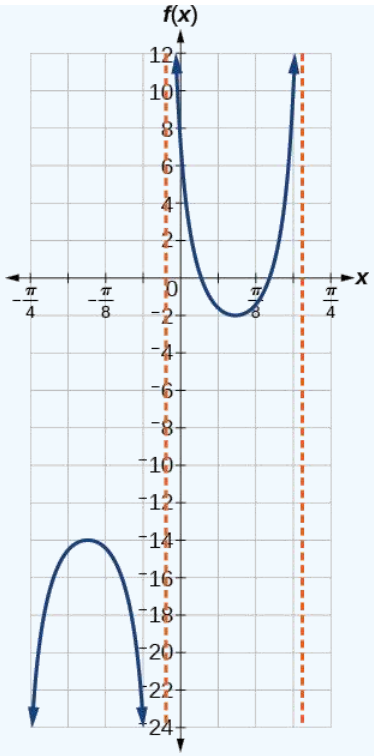


Figure 2.5.13

Q&A: The domain of $\csc\, x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \dfrac{C + k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

Given a function of the form $y = A\csc(Bx)$, graph one period.

1. Express the function given in the form $y = A\csc(Bx)$.
2. $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Draw the graph of $y = A\sin(Bx)$.
5. Use the reciprocal relationship between $y = sin\, x$ and $y = \csc\, x$ to draw the graph of $y = A\csc(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 2.5.6: Graphing a Variation of the Cosecant Function

Graph one period of $f(x) = -3\csc(4x)$.

Solution

- *Step 1.* The given function is already written in the general form, $y = A\csc(Bx)$.
- *Step 2.* $|A| = |-3| = 3$,so the stretching factor is 3.
- *Step 3.* $B = 4$,so $P = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$. The period is $\dfrac{\pi}{2}$ units.
- *Step 4.* Sketch the graph of the function $g(x) = -3\sin(4x)$.
- *Step 5.* Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.

- *Steps 6–7.* Sketch three asymptotes at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We can use two reference points, the local maximum at $\left(\frac{\pi}{8}, -3\right)$ and the local minimum at $\left(\frac{3\pi}{8}, 3\right)$. Figure 2.5.14 shows the graph.

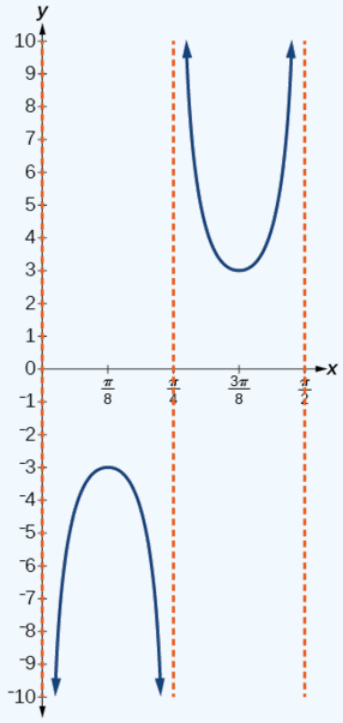


Figure 2.5.14

Exercise 2.5.6

Graph one period of $f(x) = 0.5\csc(2x)$.

Answer

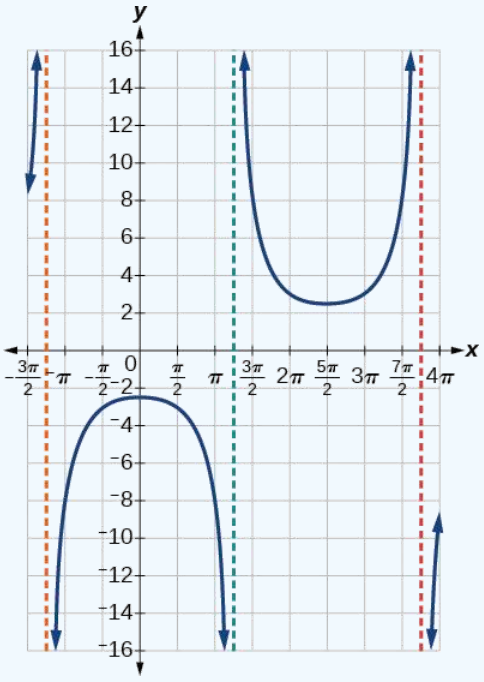


Figure 2.5.15

Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period

1. Express the function given in the form $y = A\csc(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\csc(Bx)$ but shift it to the right by and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.

Example 2.5.7: Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$. What are the domain and range of this function?

Solution

- *Step 1.* Express the function given in the form $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$.
- *Step 2.* Identify the stretching/compressing factor, $|A| = 2$.
- *Step 3.* The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$.
- *Step 4.* The phase shift is $\frac{0}{\frac{\pi}{2}} = 0$.
- *Step 5.* Draw the graph of $y = A\csc(Bx)$ but shift it up $D = 1$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 2$, $x = 4$.

The graph for this function is shown in Figure 2.5.16.

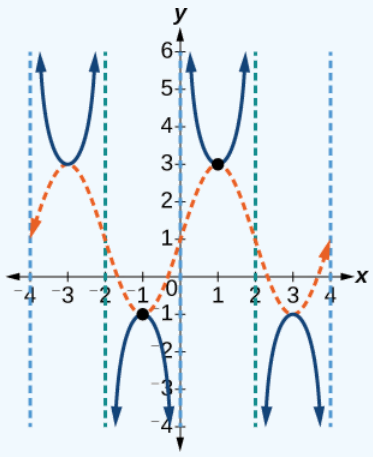


Figure 2.5.16: A transformed cosecant function

Analysis

The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$,shown as the orange dashed wave.

Exercise 2.5.7

Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in Figure 2.5.17, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

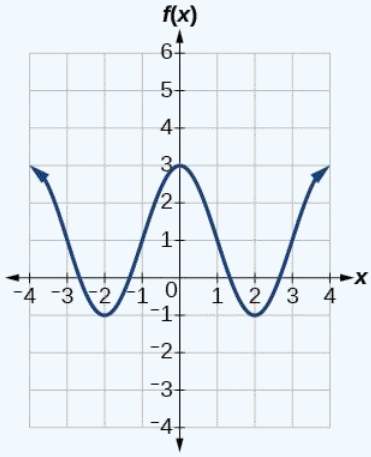


Figure 2.5.17

Answer

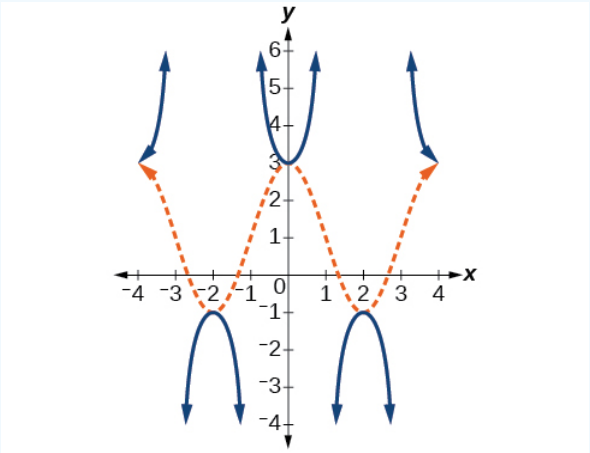


Figure 2.5.18

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $cot\, x = \frac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See Figure 2.5.19. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

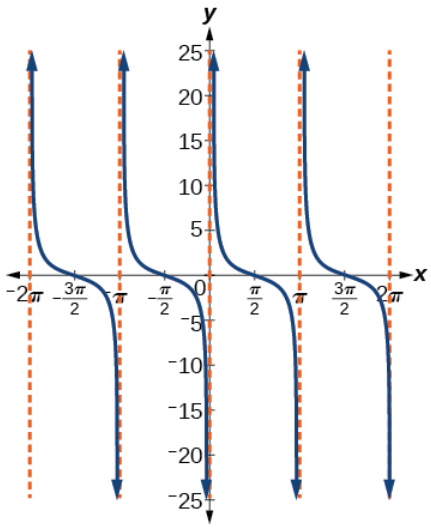


Figure 2.5.19: The cotangent function

FEATURES OF THE GRAPH OF $Y = A\cot(BX)$

- The stretching factor is $|A|$.
- The period is $P = \dfrac{\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D \tag{2.5.2}$$

PROPERTIES OF THE GRAPH OF $Y = A\cot(Bx - c) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{\pi}{|B|}$
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- There is no amplitude.
- $y = A\cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

Given a modified cotangent function of the form $f(x) = A\cot(Bx)$,graph one period.

1. Express the function in the form $f(x) = A\cot(Bx)$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \dfrac{\pi}{|B|}$.
4. Draw the graph of $y = A\tan(Bx)$.
5. Plot any two reference points.

6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y = Acot(Bx)$.
7. Sketch the asymptotes.

Example 2.5.8: Graphing Variations of the Cotangent Function

Determine the stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph.

Solution

- *Step 1.* Expressing the function in the form $f(x) = A\cot(Bx)$ gives $f(x) = 3\cot(4x)$.
- *Step 2.* The stretching factor is $|A| = 3$.
- *Step 3.* The period is $P = \frac{\pi}{4}$.
- *Step 4.* Sketch the graph of $y = 3\tan(4x)$.
- *Step 5.* Plot two reference points. Two such points are $\left(\frac{\pi}{16}, 3\right)$ and $\left(\frac{3\pi}{16}, -3\right)$.
- *Step 6.* Use the reciprocal relationship to draw $y = 3\cot(4x)$.
- *Step 7.* Sketch the asymptotes, $x = 0$, $x = \frac{\pi}{4}$.

The orange graph in Figure 2.5.20 shows $y = 3\tan(4x)$ and the blue graph shows $y = 3\cot(4x)$.

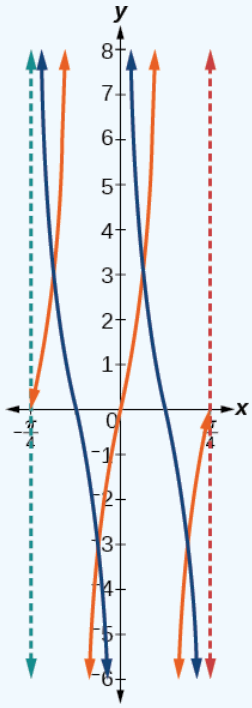


Figure 2.5.20

Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
7. Plot any three reference points and draw the graph through these points.

Example 2.5.9: Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Solution

- *Step 1.* The function is already written in the general form $f(x) = A\cot(Bx - C) + D$.
- *Step 2.* $A = 4$,so the stretching factor is 4.
- *Step 3.* $B = \frac{\pi}{8}$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.
- *Step 4.* $C = \frac{\pi}{2}$,so the phase shift is $CB = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.
- *Step 5.* We draw $f(x) = 4\tan\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 6-7.* Three points we can use to guide the graph are $(6, 2)$, $(8, -2)$, and $(10, -6)$. We use the reciprocal relationship of tangent and cotangent to draw $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 8.* The vertical asymptotes are $x = 4$ and $x = 12$.

The graph is shown in Figure 2.5.21.

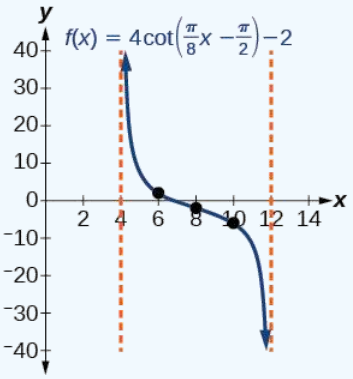


Figure 2.5.21: One period of a modified cotangent function

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the **tangent function**.

Example 2.5.10: Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan(\frac{\pi}{4}t)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.
b. Graph on the interval $[0, 5]$.
c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.

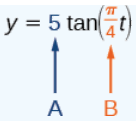


Figure 2.5.22

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See Figure 2.5.23

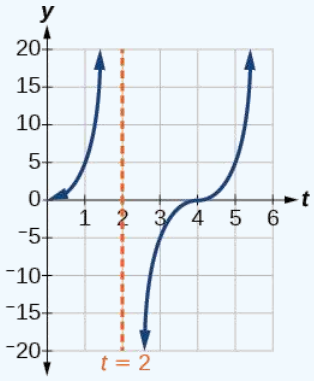


Figure 2.5.23

c. period: $f(1) = 5\tan(\frac{\pi}{4}(1)) = 5(1) = 5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Key Equations

Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See Example 2.5.1, Example 2.5.2, and Example 2.5.3.
- The secant and cosecant are both periodic functions with a period of 2π. $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See Example 2.5.4 and Example 2.5.5.
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See Example 2.5.6 and Example 2.5.7.
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi, ...$
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.
- The cotangent is zero at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See Example 2.5.8 and Example 2.5.9.
- Real-world scenarios can be solved using graphs of trigonometric functions. See Example 2.5.10.

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CHAPTER OVERVIEW

Chapter 3: Trigonometric Identities and Equations

3.1: Fundamental Identities
3.2: Proving Identities
3.3: Solving Trigonometric Equations
3.4: Sum and Difference Identities
3.5: Double Angle Identities
3.6: Half Angle Identities
3.7: Exercises - Double Angle, Half-Angle, and Power Reductions

Thumbnail: Graphs of y=sin(2θ) and y=sin(θ) (CC BY-NC-SA; Ted Sundstrom & Steven Schlicker via LibreTexts).

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3.1: Fundamental Identities

Learning Objectives

- Use the fundamental identities to prove other identities.
- Apply the fundamental identities to values of θ and show that they are true.

Basic Trigonometric Identities

The basic trigonometric identities are ones that can be logically deduced from the definitions and graphs of the six trigonometric functions. Previously, some of these identities have been used in a casual way, but now they will be formalized and added to the toolbox of trigonometric identities.

Trigonometric Identities

An **identity** is a mathematical sentence involving the symbol "=" that is always true for variables within the domains of the expressions on either side.

Reciprocal Identities

The **reciprocal identities** refer to the connections between the trigonometric functions like sine and cosecant. Sine is opposite over hypotenuse and cosecant is hypotenuse over opposite. This logic produces the following six identities.

- $\sin\theta = \dfrac{1}{\csc\theta}$
- $\cos\theta = \dfrac{1}{\sec\theta}$
- $\tan\theta = \dfrac{1}{\cot\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\csc\theta = \dfrac{1}{\sin\theta}$

Quotient Identities

The **quotient identities** follow from the definition of sine, cosine and tangent.

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Odd/Even Identities

An **even function** is a function where the value of the function acting on an argument is the same as the value of the function when acting on the negative of the argument. Or, in short:

$f(x) = f(-x)$

So, for example, if $f(x)$ is some function that is even, then $f(2)$ has the same answer as $f(-2)$. $f(5)$ has the same answer as $f(-5)$, and so on.

In contrast, an **odd function** is a function where the negative of the function's answer is the same as the function acting on the negative argument. In math terms, this is:

$-f(x) = f(-x)$

If a function were negative, then $f(-2) = -f(2)$, $f(-5) = -f(5)$, and so on.

Functions are even or odd depending on how the end behavior of the graphical representation looks. For example, $y = x^2$ is considered an even function because the ends of the parabola both point in the same direction and the parabola is symmetric about the $y-$axis. $y = x^3$ is considered an odd function for the opposite reason. The ends of a cubic function point in opposite directions

and therefore the parabola is not symmetric about the $y-$axis. What about the trig functions? They do not have exponents to give us the even or odd clue (when the degree is even, a function is even, when the degree is odd, a function is odd).

Even Function	Odd Function
$y=(-x)^2=x^2$	$y=(-x)^3=-x^3$

Let's consider sine. Start with $\sin(-x)$. Will it equal $\sin x$ or $-\sin x$? Plug in a couple of values to see.

$$\sin(-30^\circ)=\sin 330^\circ=-\frac{1}{2}=-\sin 30^\circ$$

$$\sin(-135^\circ)=\sin 225^\circ=-\frac{\sqrt{2}}{2}=-\sin 135^\circ$$

From this we see that sine is **odd**. Therefore, $\sin(-x)=-\sin x$, for any value of x. For cosine, we will plug in a couple of values to determine if it's even or odd.

$$\cos(-30^\circ)=\cos 330^\circ=\frac{\sqrt{3}}{2}=\cos 30^\circ$$

$$\cos(-135^\circ)=\cos 225^\circ=-\frac{\sqrt{2}}{2}=\cos 135^\circ$$

This tells us that the cosine is **even**. Therefore, $\cos(-x)=\cos x$, for any value of x. The other four trigonometric functions are as follows:

$$\tan(-x)=-\tan x$$
$$\csc(-x)=-\csc x$$
$$\sec(-x)=\sec x$$
$$\cot(-x)=-\cot x$$

Notice that cosecant is odd like sine and secant is even like cosine.

The **odd-even identities** follow from the fact that only cosine and its reciprocal secant are even and the rest of the trigonometric functions are odd.

- $\sin(-\theta)=-\sin\theta$
- $\cos(-\theta)=\cos\theta$
- $\tan(-\theta)=-\tan\theta$
- $\cot(-\theta)=-\cot\theta$
- $\sec(-\theta)=\sec\theta$
- $\csc(-\theta)=-\csc\theta$



Cofunction Identities

The **cofunction identities** make the connection between trigonometric functions and their "co" counterparts like sine and cosine. Graphically, all of the cofunctions are reflections and horizontal shifts of each other.

- $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$

- $\sin(\frac{\pi}{2}-\theta)=\cos\theta$
- $\tan(\frac{\pi}{2}-\theta)=\cot\theta$
- $\cot(\frac{\pi}{2}-\theta)=\tan\theta$
- $\sec(\frac{\pi}{2}-\theta)=\csc\theta$
- $\csc(\frac{\pi}{2}-\theta)=\sec\theta$



Example 3.1.1

Earlier, you were asked how you could simplify the trigonometric expression:

$$\left(\frac{\sin(\frac{\pi}{2}-\theta)}{\sin(-\theta)}\right)^{-1}$$

Solution

It can be simplified to be equivalent to negative tangent as shown below:

$$\begin{aligned}\left(\frac{\sin(\frac{\pi}{2}-\theta)}{\sin(-\theta)}\right)^{-1} &= \frac{\sin(-\theta)}{\sin(\frac{\pi}{2}-\theta)}\\ &= -\sin\theta\cos\theta\\ &= -\tan\theta\end{aligned}$$

Example 3.1.2

If $\sin\theta=0.87$, find $\cos(\theta-\frac{\pi}{2})$.

Solution

While it is possible to use a calculator to find \theta , using identities works very well too.

First you should factor out the negative from the argument. Next you should note that cosine is even and apply the odd-even identity to discard the negative in the argument. Lastly recognize the cofunction identity.

$$\cos(\theta-\frac{\pi}{2})=\cos(-(\frac{\pi}{2}-\theta))=\cos(\frac{\pi}{2}-\theta)=\sin\theta=0.87$$

Example 3.1.3

If $\cos(\theta - \frac{\pi}{2}) = 0.68$ then determine $\csc(-\theta)$.

Solution

You need to show that $\cos(\theta - \frac{\pi}{2}) = \cos(\frac{\pi}{2} - \theta)$.

$$\begin{aligned} 0.68 &= \cos(\theta - \frac{\pi}{2}) \\ &= \cos(\frac{\pi}{2} - \theta) \\ &= \sin(\theta) \end{aligned}$$

Then, $\csc(-\theta) = -\csc\theta$

$$\begin{aligned} &= -\frac{1}{\sin\theta} \\ &= -(0.68)^{-1} \qquad \approx -1.47 \end{aligned}$$

Example 3.1.4

Use identities to prove the following: $\cot(-\beta)\cot(\frac{\pi}{2} - \beta)\sin(-\beta) = \cos(\beta - \frac{\pi}{2})$.

Solution

When doing trigonometric proofs, it is vital that you start on one side and only work with that side until you derive what is on the other side. Sometimes it may be helpful to work from both sides and find where the two sides meet, but this work is not considered a **proof**. You will have to rewrite your steps so they follow from only one side. In this case, work with the left side and keep rewriting it until you have $\cos(\beta - \frac{\pi}{2})$.

$$\begin{aligned} &\cot(-\beta)\cot(\frac{\pi}{2} - \beta)\sin(-\beta) \\ &= -\cot\beta \cdot \tan\beta \cdot -\sin\beta \\ &= -1 \cdot -sin\beta \\ &= sin\beta \\ &= cos(\frac{\pi}{2} - \beta) \\ &= cos(-(\beta - \frac{\pi}{2})) \\ &= cos(\beta - \frac{\pi}{2}) \end{aligned}$$

Example 3.1.5

Prove the following trigonometric identity by working with only one side.

$$\cos x \sin x \tan x \cot x \sec x \csc x = 1$$

Solution

$$\cos x \sin x \tan x \cot x \sec x \csc x = \cos x \sin x \tan x \cdot \frac{1}{\tan x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\sin x} \qquad = 1$$

Review

1. Prove that $\tan\theta \cdot cot\theta = 1$.
2. Prove that $\sin\theta \cdot \csc\theta = 1$.
3. Prove that $\sin\theta \cdot sec\theta = \tan\theta$.
4. Prove that $\cos\theta \cdot \csc\theta = \cot\theta$.
5. If $\sin\theta = 0.81$, what is $\sin(-\theta)$?
6. If $\cos\theta = 0.5$, what is $\cos(-\theta)$?

7. If $\cos\theta = 0.25$, what is $\sec(-\theta)$?
8. If $\csc\theta = 0.7$, what is $\sin(-\theta)$?

Vocabulary

Term	Definition
cofunction	Cofunctions are functions that are identical except for a reflection and horizontal shift. Examples include: sine and cosine, tangent and cotangent, secant and cosecant.
even	An even function is a function with a graph that is symmetric with respect to the y-axis and has the property that $f(-x) = f(x)$.
identity	An identity is a mathematical sentence involving the symbol "=" that is always true for variables within the domains of the expressions on either side.
Odd Function	An odd function is a function with the property that $f(-x) = -f(x)$. Odd functions have rotational symmetry about the origin.
proof	A proof is a series of true statements leading to the acceptance of truth of a more complex statement.

Additional Resources

Video: Trigonometric Identities - Overview

Practice: Fundamental Trigonometric Identities

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3.2: Proving Identities

Learning Objectives

- Prove identities using several techniques.

Verifying Trigonometric Identities

Now that you are comfortable simplifying expressions, we will extend the idea to verifying entire identities. Here are a few helpful hints to verify an identity:

- Change everything into terms of sine and cosine.
- Use the identities when you can.
- Start with simplifying the left-hand side of the equation, then, once you get stuck, simplify the right-hand side. As long as the two sides end up with the same final expression, the identity is true.

Let's verify the following identities.

1. $\dfrac{\cot^2 x}{\csc x} = \csc x - \sin x$

Rather than have an equal sign between the two sides of the equation, we will draw a vertical line so that it is easier to see what we do to each side of the equation. Start with changing everything into sine and cosine.

$$\begin{array}{l|l} \dfrac{\cot^2 x}{\csc x} & \csc x - \sin x \\ \dfrac{\dfrac{\cos^2 x}{\sin^2 x}}{\dfrac{1}{\sin x}} & \dfrac{1}{\sin x} \\ \dfrac{\cos^2 x}{\sin x} & \end{array}$$

Now, it looks like we are at an impasse with the left-hand side. Let's combine the right-hand side by giving them same denominator.

$$\begin{array}{|l} \dfrac{1}{\sin x} - \dfrac{\sin^2 x}{\sin x} \\ \dfrac{1-\sin^2 x}{\sin x} \\ \dfrac{\cos^2 x}{\sin x} \end{array}$$

The two sides reduce to the same expression, so we can conclude this is a valid identity. In the last step, we used the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$, and isolated the $\cos^2 x = 1 - \sin^2 x$.

There are usually more than one way to verify a trig identity. When proving this identity in the first step, rather than changing the cotangent to $\dfrac{\cos^2 x}{\sin^2 x}$, we could have also substituted the identity $\cot^2 x = \csc^2 x - 1$.

2. $\dfrac{\sin x}{1-\cos x} = \dfrac{1+\cos x}{\sin x}$

Multiply the left-hand side of the equation by $\dfrac{1+\cos x}{1+\cos x}$.

$$\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$$
$$\frac{1+\cos x}{1+\cos x} \cdot \frac{\sin x}{1-\cos x} =$$
$$\frac{\sin(1+\cos x)}{1-\cos^2 x} =$$
$$\frac{\sin(1+\cos x)}{\sin^2 x} =$$
$$\frac{1+\cos x}{\sin x} =$$

The two sides are the same, so we are done.

3. $\sec(-x) = \sec x$

Change secant to cosine.

$$\sec(-x) = \frac{1}{cos(-x)}$$

From the Negative Angle Identities, we know that $\cos(-x) = \cos x$.

$$= \frac{1}{\cos x}$$
$$= \sec x$$

Example 3.2.1

Earlier, you were asked to verify that $\sin^2 x \tan^2 x = 1 - \sin^2 x$.

Solution

Start by simplifying the left-hand side of the equation.

$$\sin^2 x \tan^2 x = \frac{\sin^2 x}{\frac{\sin^2 x}{\cos^2 x}} = \cos^2 x$$

Now simplify the right-hand side of the equation. By manipulating the Trigonometric Identity,

$\sin^2 x + \cos^2 x = 1$, we get $\cos^2 x = 1 - \sin^2 x$.

$\cos^2 x = \cos^2 x$ and the equation is verified.

Verify the following identities.

Example 3.2.2

$\cos x \sec x = 1$

Solution

Change secant to cosine.

$$\cos x \sec x = \cos x \cdot \frac{1}{\cos x}$$
$$= 1$$

Example 3.2.3

$2 - sec^2 x = 1 - \tan^2 x$

Solution

Use the identity $1 + \tan^2 \theta = \sec^2 \theta$.

$$\begin{aligned} 2-\sec^2 x &= 2-(1+\tan^2 x) \\ &= 2-1-\tan^2 x \\ &= 1-\tan^2 x \end{aligned}$$

Example 3.2.4

$$\frac{\cos(-x)}{1+\sin(-x)} = \sec x + \tan x$$

Solution

Here, start with the Negative Angle Identities and multiply the top and bottom by $\frac{1+\sin x}{1+\sin x}$ to make the denominator a monomial.

$$\begin{aligned} \frac{\cos(-x)}{1+\sin(-x)} &= \frac{\cos x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} \\ &= \frac{\cos x(1+\sin x)}{1-\sin^2 x} \\ &= \frac{\cos x(1+\sin x)}{\cos^2 x} \\ &= \frac{1+\sin x}{\cos x} \\ &= \frac{1}{\cos x} + \frac{\sin x}{\cos x} \\ &= \sec x + \tan x \end{aligned}$$

Review

Verify the following identities.

1. $\cot(-x) = -\cot x$
2. $\csc(-x) = -\csc x$
3. $\tan x \csc x \cos x = 1$
4. $\sin x + \cos x \cot x = \csc x$
5. $\csc\left(\frac{\pi}{2} - x\right) = \sec x$
6. $\tan\left(\frac{\pi}{2} - x\right) = \tan x$
7. $\frac{\csc x}{\sin x} - \frac{\cot x}{\tan x} = 1$
8. $\frac{\tan^2 x}{\tan^2 x + 1} = \sin^2 x$
9. $(\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2$
10. $\sin x - \sin x \cos^2 x = \sin^3 x$
11. $\tan^2 x + 1 + \tan x \sec x = \frac{1+\sin x}{\cos^2 x}$
12. $\cos^2 x = \csc x \cos x \tan x + \cot x$
13. $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2\tan x \sec x$
14. $\csc^4 x - \cot^4 x = \csc^2 x + \cot^2 x$
15. $(\sin x - \tan x)(\cos x - \cot x) = (\sin x - 1)(\cos x - 1)$

Additional Resources

Video: Verifying Trigonometric Identities - Example 2

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3.3: Solving Trigonometric Equations

Learning Objectives

- Use the fundamental identities to solve trigonometric equations.
- Express trigonometric expressions in simplest form.
- Solve trigonometric equations by factoring.
- Solve trigonometric equations by using the Quadratic Formula.

Thales of Miletus (circa 625–547 BC) is known as the founder of geometry. The legend is that he calculated the height of the Great Pyramid of Giza in Egypt using the theory of *similar triangles*, which he developed by measuring the shadow of his staff. Based on proportions, this theory has applications in a number of areas, including fractal geometry, engineering, and architecture. Often, the angle of elevation and the angle of depression are found using similar triangles.



Figure 3.3.1: Egyptian pyramids standing near a modern city. (credit: Oisin Mulvihill)

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Solving Linear Trigonometric Equations in Sine and Cosine

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The **period** of both the sine function and the cosine function is 2π. In other words, every 2π units, the *y*-values repeat. If we need to find all possible solutions, then we must add $2\pi k$,where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin\theta = \sin(\theta \pm 2k\pi) \tag{3.3.1}$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence. We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

Example 3.3.1*A*: Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

Solution

From the unit circle, we know that

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \ \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Example 3.3.1B: Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

Solution

Solving for all possible values of t means that solutions include angles beyond the period of 2π. From the section on Sum and Difference Identities, we can see that the solutions are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. But the problem is asking for all possible values that solve the equation. Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where k is an integer.

How to: Given a trigonometric equation, solve using algebra

1. Look for a pattern that suggests an algebraic property, such as the difference of squares or a factoring opportunity.
2. Substitute the trigonometric expression with a single variable, such as x or u.
3. Solve the equation the same way an algebraic equation would be solved.
4. Substitute the trigonometric expression back in for the variable in the resulting expressions.
5. Solve for the angle.

Example 3.3.2: Solve the Linear Trigonometric Equation

Solve the equation exactly: $2\cos\theta - 3 = -5$, $0 \le \theta < 2\pi$.

Solution

Use algebraic techniques to solve the equation.

$$\begin{aligned} 2\cos\theta - 3 &= -5 \\ 2\cos\theta &= -2 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Exercise 3.3.1

Solve exactly the following linear equation on the interval $[0, 2\pi)$: $2\sin x + 1 = 0$.

Answer

$$x = \frac{7\pi}{6}, \ \frac{11\pi}{6}$$

Solving Equations Involving a Single Trigonometric Function

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and the unit circle (see [link]). We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems involving the reciprocals of the primary trigonometric functions need to be viewed from an algebraic perspective. In other words, we will write the reciprocal function, and solve for the angles using the function. Also, an equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π,not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$,unless, of course, a problem places its own restrictions on the domain.

Example 3.3.3A: Solving a Trignometric Equation Involving Sine

Solve the problem exactly: $2\sin^2\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution

As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$\begin{aligned} 2\sin^2\theta - 1 &= 0 \\ 2\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{2} \\ \sqrt{\sin^2\theta} &= \pm\sqrt{\frac{1}{2}} \\ \sin\theta &= \pm\frac{1}{\sqrt{2}} \\ &= \pm\frac{\sqrt{2}}{2} \\ \theta &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

Analysis

As $\sin\theta = -\frac{1}{2}$, notice that all four solutions are in the third and fourth quadrants.

Example 3.3.3B: Solving a Trigonometric Equation Involving Cosecant

Solve the following equation exactly: $\csc\theta = -2$, $0 \le \theta < 4\pi$.

Solution

We want all values of θ for which $\csc\theta = -2$ over the interval $0 \le \theta < 4\pi$.

$$\begin{aligned} \csc\theta &= -2 \\ \frac{1}{\sin\theta} &= -2 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \end{aligned}$$

Example 3.3.3C: Solving an Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta - \frac{\pi}{2}\right) = 1$, $0 \le \theta < 2\pi$.

Solution

Recall that the tangent function has a period of π. On the interval $[0, \pi)$,and at the angle of $\frac{\pi}{4}$,the tangent has a value of 1. However, the angle we want is $\left(\theta - \frac{\pi}{2}\right)$. Thus, if $\tan\left(\frac{\pi}{4}\right) = 1$,then

$$\theta - \frac{\pi}{2} = \frac{\pi}{4}$$
$$\theta = \frac{3\pi}{4} \pm k\pi$$

Over the interval $[0, 2\pi)$,we have two solutions:

$\theta = \frac{3\pi}{4}$ and $\theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$

Exercise 3.3.2

Find all solutions for $\tan x = \sqrt{3}$.

Answer

$\frac{\pi}{3} \pm \pi k$

Example 3.3.4: Identify all Solutions to the Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x + 3) = 5 + \tan x$, $0 \le x < 2\pi$.

Solution

We can solve this equation using only algebra. Isolate the expression $\tan x$ on the left side of the equals sign.

There are two angles on the unit circle that have a tangent value of -1: $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Solve Trigonometric Equations Using a Calculator

Not all functions can be solved exactly using only the unit circle. When we must solve an equation involving an angle other than one of the special angles, we will need to use a calculator. Make sure it is set to the proper mode, either degrees or radians, depending on the criteria of the given problem.

Example 3.3.5A: Using a Calculator to Solve a Trigonometric Equation Involving Sine

Use a calculator to solve the equation $\sin\theta = 0.8$,where θ is in radians.

Solution

Make sure mode is set to radians. To find θ, use the inverse sine function. On most calculators, you will need to push the 2^{ND} button and then the SIN button to bring up the $\sin^{-1}$ function. What is shown on the screen is $\sin^{-1}$.The calculator is ready for the input within the parentheses. For this problem, we enter $\sin^{-1}(0.8)$, and press ENTER. Thus, to four decimals places,

$\sin^{-1}(0.8) \approx 0.9273$

The solution is

$\theta \approx 0.9273 \pm 2\pi k$

The angle measurement in degrees is

$$\theta \approx 53.1^\circ$$
$$\theta \approx 180^\circ - 53.1^\circ$$
$$\approx 126.9^\circ$$

Analysis

Note that a calculator will only return an angle in quadrants I or IV for the sine function since that is the range of the inverse sine. The other angle is obtained by using $\pi - \theta$.

Example 3.3.5B: Using a Calculator to Solve a Trigonometric Equation Involving Secant

Use a calculator to solve the equation $\sec\theta = -4,$ giving your answer in radians.

Solution

We can begin with some algebra.

$$\sec\theta = -4$$
$$\frac{1}{\cos\theta} = -4$$
$$\cos\theta = -\frac{1}{4}$$

Check that the MODE is in radians. Now use the inverse cosine function

$$\cos^{-1}\left(-\frac{1}{4}\right) \approx 1.8235$$
$$\theta \approx 1.8235 + 2\pi k$$

Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$,1.8235 is between these two numbers, thus $\theta \approx 1.8235$ is in quadrant II. Cosine is also negative in quadrant III. Note that a calculator will only return an angle in quadrants I or II for the cosine function, since that is the range of the inverse cosine. See Figure 3.3.2.

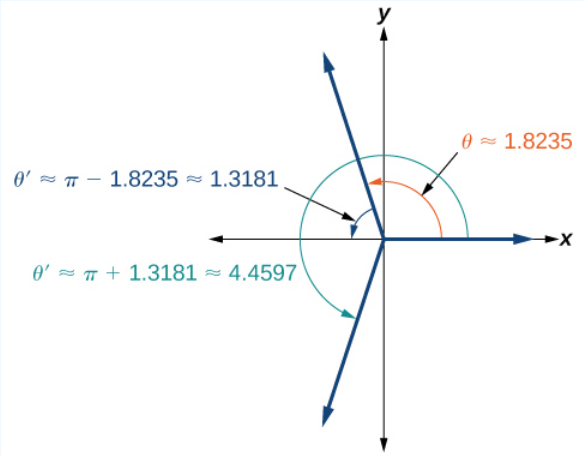


Figure 3.3.2

So, we also need to find the measure of the angle in quadrant III. In quadrant III, the reference angle is $\theta' \approx \pi - 1.8235 \approx 1.3181$. The other solution in quadrant III is $\theta' \approx \pi + 1.3181 \approx 4.4597$.

The solutions are $\theta \approx 1.8235 \pm 2\pi k$ and $\theta \approx 4.4597 \pm 2\pi k$.

Exercise 3.3.3

Solve $\cos\theta = -0.2$.

Answer

$\theta \approx 1.7722 \pm 2\pi k$ and $\theta \approx 4.5110 \pm 2\pi k$

Solving Trigonometric Equations in Quadratic Form

Solving a *quadratic equation* may be more complicated, but once again, we can use algebra as we would for any quadratic equation. Look at the pattern of the equation. Is there more than one trigonometric function in the equation, or is there only one? Which trigonometric function is squared? If there is only one function represented and one of the terms is squared, think about the standard form of a quadratic. Replace the trigonometric function with a variable such as x or u. If substitution makes the equation look like a quadratic equation, then we can use the same methods for solving quadratics to solve the trigonometric equations.

Example 3.3.6A: Solving a Trigonometric Equation in Quadratic Form

Solve the equation exactly: $\cos^2\theta+3\cos\theta-1=0\ ,0\le\theta<2\pi$.

Solution

We begin by using substitution and replacing $\cos\theta$ with x. It is not necessary to use substitution, but it may make the problem easier to solve visually. Let $\cos\theta=x$. We have

$$x^2+3x-1=0$$

The equation cannot be factored, so we will use the **quadratic formula:** $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

$$\begin{aligned} x&=\frac{-3\pm\sqrt{(-3)^2-4(1)(-1)}}{2}\\ &=\frac{-3\pm\sqrt{13}}{2}\end{aligned}$$

Replace x with $\cos\theta$ and solve.

$$\begin{aligned}\cos\theta&=\frac{-3\pm\sqrt{13}}{2}\\ \theta&=\cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)\end{aligned}$$

Note that only the + sign is used. This is because we get an error when we solve $\theta=\cos^{-1}\left(\dfrac{-3-\sqrt{13}}{2}\right)$ on a calculator, since the domain of the inverse cosine function is $[-1,1]$. However, there is a second solution:

$$\begin{aligned}\theta&=\cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)\\ &\approx 1.26\end{aligned}$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\begin{aligned}\theta&=2\pi-\cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)\\ &\approx 5.02\end{aligned}$$

Example 3.3.6B: Solving a Trigonometric Equation in Quadratic Form by Factoring

Solve the equation exactly: $2\sin^2\theta-5\sin\theta+3=0\ ,0\le\theta\le2\pi$.

Solution

Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta=u$,or imagine it, as we factor:

$$\begin{aligned} 2\sin^2\theta - 5\sin\theta + 3 &= 0 \\ (2\sin\theta - 3)(\sin\theta - 1) &= 0 \qquad \text{Now set each factor equal to zero.} \\ 2\sin\theta - 3 &= 0 \\ 2\sin\theta &= 3 \\ \sin\theta &= \frac{3}{2} \\ \sin\theta - 1 &= 0 \\ \sin\theta &= 1 \end{aligned}$$

Next solve for θ: $\sin\theta \neq \frac{3}{2}$, as the range of the sine function is $[-1, 1]$. However, $\sin\theta = 1$, giving the solution $\theta = \frac{\pi}{2}$.

Analysis

Make sure to check all solutions on the given domain as some factors have no solution.

Exercise 3.3.4

Solve $\sin^2\theta = 2\cos\theta + 2$, $0 \leq \theta \leq 2\pi$. [Hint: Make a substitution to express the equation only in terms of cosine.]

Answer

$\cos\theta = -1$, $\theta = \pi$

Example 3.3.7A: Solving a Trigonometric Equation Using Algebra

Solve exactly: $2\sin^2\theta + \sin\theta = 0;\ 0 \leq \theta < 2\pi$

Solution

This problem should appear familiar as it is similar to a quadratic. Let $\sin\theta = x$. The equation becomes $2x^2 + x = 0$. We begin by factoring:

$$\begin{aligned} 2x^2 + x &= 0 \\ x(2x + 1) &= 0 \qquad \text{Set each factor equal to zero.} \\ x &= 0 \\ 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

Then, substitute back into the equation the original expression $\sin\theta$ for x. Thus,

$$\begin{aligned} \sin\theta &= 0 \\ \theta &= 0, \pi \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

The solutions within the domain $0 \leq \theta < 2\pi$ are $\theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

If we prefer not to substitute, we can solve the equation by following the same pattern of factoring and setting each factor equal to zero.

$$\begin{aligned}\sin^2\theta+\sin\theta &= 0\\ \sin\theta(2\sin\theta+1) &= 0\\ \sin\theta &= 0\\ \theta &= 0,\pi\\ 2\sin\theta+1 &= 0\\ 2\sin\theta &= -1\\ \sin\theta &= -\frac{1}{2}\\ \theta &= \frac{7\pi}{6},\frac{11\pi}{6}\end{aligned}$$

Analysis

We can see the solutions on the graph in Figure 3.3.3. On the interval $0 \le \theta < 2\pi$,the graph crosses the x-axis four times, at the solutions noted. Notice that trigonometric equations that are in quadratic form can yield up to four solutions instead of the expected two that are found with quadratic equations. In this example, each solution (angle) corresponding to a positive sine value will yield two angles that would result in that value.

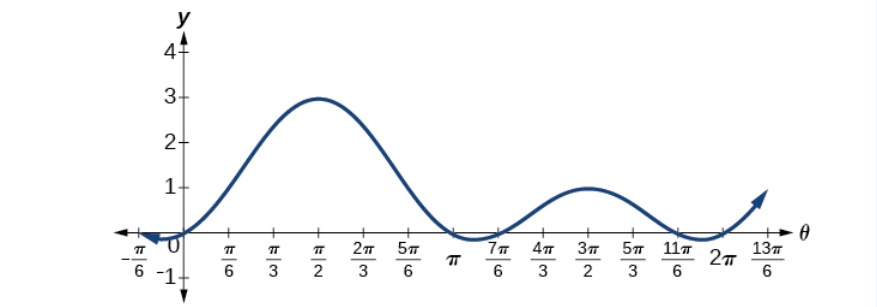


Figure 3.3.3

We can verify the solutions on the unit circle in via the result in the section on Sum and Difference Identities as well.

Example 3.3.7B: Solving a Trigonometric Equation Quadratic in Form

Solve the equation quadratic in form exactly: $2\sin^2\theta-3\sin\theta+1=0$, $0 \le \theta < 2\pi$.

Solution

We can factor using grouping. Solution values of θ can be found on the unit circle.

$$\begin{aligned}(2\sin\theta-1)(\sin\theta-1) &= 0\\ 2\sin\theta-1 &= 0\\ \sin\theta &= \frac{1}{2}\\ \theta &= \frac{\pi}{6},\frac{5\pi}{6}\\ \sin\theta &= 1\\ \theta &= \frac{\pi}{2}\end{aligned}$$

Exercise 3.3.5

Solve the quadratic equation $2\cos^2\theta+\cos\theta=0$.

Answer

$$\frac{\pi}{2},\frac{2\pi}{3},\frac{4\pi}{3},\frac{3\pi}{2}$$

Solving Trigonometric Equations Using Fundamental Identities

While algebra can be used to solve a number of trigonometric equations, we can also use the fundamental identities because they make solving equations simpler. Remember that the techniques we use for solving are not the same as those for verifying identities. The basic rules of algebra apply here, as opposed to rewriting one side of the identity to match the other side. In the next example, we use two identities to simplify the equation.

Example 3.3.8: Solving an Equation Using an Identity

Solve the equation exactly using an identity: $3\cos\theta+3=2\sin^2\theta$, $0\le\theta<2\pi$.

Solution

If we rewrite the right side, we can write the equation in terms of cosine:

$$\begin{aligned} 3\cos\theta+3 &= 2\sin^2\theta \\ 3\cos\theta+3 &= 2(1-\cos^2\theta) \\ 3\cos\theta+3 &= 2-2\cos^2\theta \\ 2\cos^2\theta+3\cos\theta+1 &= 0 \\ (2\cos\theta+1)(\cos\theta+1) &= 0 \\ 2\cos\theta+1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3},\ \frac{4\pi}{3} \\ \cos\theta+1 &= 0 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Our solutions are $\theta=\frac{2\pi}{3},\ \frac{4\pi}{3},\ \pi$.

Solving Trigonometric Equations with Multiple Angles

Sometimes it is not possible to solve a trigonometric equation with identities that have a multiple angle, such as $\sin(2x)$ or $\cos(3x)$. When confronted with these equations, recall that $y=\sin(2x)$ is a horizontal compression by a factor of 2 of the function $y=\sin x$. On an interval of 2π,we can graph two periods of $y=\sin(2x)$,as opposed to one cycle of $y=\sin x$. This compression of the graph leads us to believe there may be twice as many x-intercepts or solutions to $\sin(2x)=0$ compared to $\sin x=0$. This information will help us solve the equation.

Example 3.3.9: Solving a Multiple Angle Trigonometric Equation

Solve exactly: $\cos(2x)=\frac{1}{2}$ on $[0,2\pi)$.

Solution

We can see that this equation is the standard equation with a multiple of an angle. If $\cos(\alpha)=\frac{1}{2}$,we know α is in quadrants I and IV. While $\theta=\cos^{-1}\frac{1}{2}$ will only yield solutions in quadrants I and II, we recognize that the solutions to the equation $\cos\theta=\frac{1}{2}$ will be in quadrants I and IV.

Therefore, the possible angles are $\theta=\frac{\pi}{3}$ and $\theta=\frac{5\pi}{3}$. So, $2x=\frac{\pi}{3}$ or $2x=\frac{5\pi}{3}$, which means that $x=\frac{\pi}{6}$ or $x=\frac{5\pi}{6}$. Does this make sense? Yes, because $\cos\left(2\left(\frac{\pi}{6}\right)\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$.

Are there any other possible answers? Let us return to our first step.

In quadrant I, $2x = \frac{\pi}{3}$, so $x = \frac{\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{\pi}{3} + 2\pi \\ &= \frac{\pi}{3} + \frac{6\pi}{3} \\ &= \frac{7\pi}{3} \\ x &= \frac{7\pi}{6} \end{aligned}$$

One more rotation yields

$$\begin{aligned} 2x &= \frac{\pi}{3} + 4\pi \\ &= \frac{\pi}{3} + \frac{12\pi}{3} \\ &= \frac{13\pi}{3} \end{aligned}$$

$x = \frac{13\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

In quadrant IV, $2x = \frac{5\pi}{3}$, so $x = \frac{5\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 2\pi \\ &= \frac{5\pi}{3} + \frac{6\pi}{3} \\ &= \frac{11\pi}{3} \end{aligned}$$

so $x = \frac{11\pi}{6}$.

One more rotation yields

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 4\pi \\ &= \frac{5\pi}{3} + \frac{12\pi}{3} \\ &= \frac{17\pi}{3} \end{aligned}$$

$x = \frac{17\pi}{6} > 2\pi$,so this value for x is larger than 2π,so it is not a solution on $[0, 2\pi)$.

Our solutions are $x = \frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Note that whenever we solve a problem in the form of $sin(nx) = c$, we must go around the unit circle n times.

Key Concepts

- When solving linear trigonometric equations, we can use algebraic techniques just as we do solving algebraic equations. Look for patterns, like the difference of squares, quadratic form, or an expression that lends itself well to substitution. See Example 3.3.1, Example 3.3.2, and Example 3.3.3.
- Equations involving a single trigonometric function can be solved or verified using the unit circle. See Example 3.3.4, Example 3.3.5, and Example 3.3.6, and Example 3.3.7.
- We can also solve trigonometric equations using a graphing calculator. See Example 3.3.8 and Example 3.3.9.
- Many equations appear quadratic in form. We can use substitution to make the equation appear simpler, and then use the same techniques we use solving an algebraic quadratic: factoring, the quadratic formula, etc. See Example 3.3.10, Example 3.3.11, Example 3.3.12, and Example 3.3.13.

- We can also use the identities to solve trigonometric equation. See Example 3.3.14, Example 3.3.15, and Example 3.3.16.
- We can use substitution to solve a multiple-angle trigonometric equation, which is a compression of a standard trigonometric function. We will need to take the compression into account and verify that we have found all solutions on the given interval. See Example 3.3.17.
- Real-world scenarios can be modeled and solved using the Pythagorean Theorem and trigonometric functions. See Example 3.3.18.

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3.4: Sum and Difference Identities

Learning Objectives

- Use and identify the sum and difference identities.
- Apply the sum and difference identities to solve trigonometric equations.
- Find the exact value of a trigonometric function for certain angles.

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.



Figure 3.4.1: Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown in Figure 3.4.2.

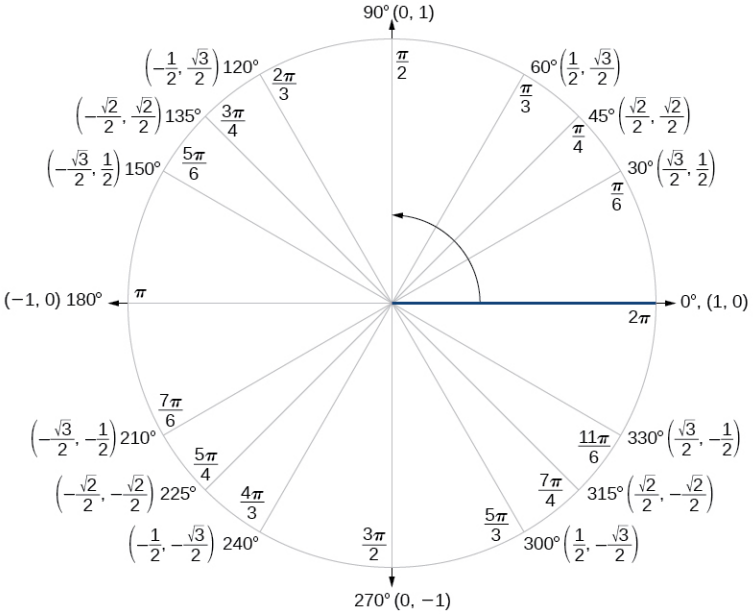


Figure 3.4.2: The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles (Table 3.4.1).

Table 3.4.1

Sum formula for cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle (Figure 3.4.3). Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha,\sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta,\sin\beta)$. Note the measure of angle POQ is $\alpha-\beta$.

Label two more points: A at an angle of $(\alpha-\beta)$ from the positive x-axis with coordinates $(\cos(\alpha-\beta),\sin(\alpha-\beta))$; and point B with coordinates $(1,0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.

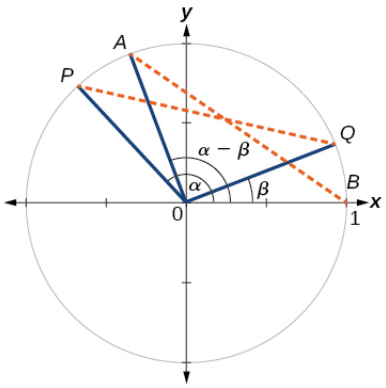


Figure 3.4.3

We can find the distance from P to Q using the distance formula.

$$\begin{aligned} d_{PQ} &= \sqrt{(\cos\alpha-\cos\beta)^2+(sin\alpha-sin\beta)^2} \\ &= \sqrt{\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta} && \text{Apply Pythagorean identity and simplify.} \\ &= \sqrt{(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \\ &= \sqrt{1+1-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \\ &= \sqrt{2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \end{aligned}$$

Similarly, using the distance formula we can find the distance from A to B.

$$\begin{aligned} d_{AB} &= \sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2} \\ &= \sqrt{\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)} && \text{Apply Pythagorean identity and simplify} \\ &= \sqrt{(\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta))-2\cos(\alpha-\beta)+1} \\ &= \sqrt{1-2\cos(\alpha-\beta)+1} \\ &= \sqrt{2-2\cos(\alpha-\beta)} && \text{Subtract 2 from both sides and divide both sides by } -2. \end{aligned}$$

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta = \cos(\alpha-\beta)$$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

SUM AND DIFFERENCE FORMULAS FOR COSINE

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \tag{3.4.1}$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \tag{3.4.2}$$

How to: Given two angles, find the cosine of the difference between the angles

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Example 3.4.1: Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right)$.

Solution

Begin by writing the formula for the cosine of the difference of two angles. Then substitute the given values.

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ \cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= -\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in radian mode.

Exercise 3.4.1:

Find the exact value of $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$.

Answer

$\frac{\sqrt{2}+\sqrt{6}}{4}$

Example 3.4.2: Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of $\cos(75^\circ)$.

Solution

As $75^\circ = 45^\circ + 30^\circ$,we can evaluate $\cos(75^\circ)$ as $\cos(45^\circ + 30^\circ)$.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\ \cos(45^\circ+30^\circ) &= \cos(45^\circ)\cos(30^\circ)-\sin(45^\circ)\sin(30^\circ)\\ &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in degree mode.

Analysis

Note that we could have also solved this problem using the fact that $75^\circ = 135^\circ - 60^\circ$.

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ \cos(135^\circ-60^\circ) &= \cos(135^\circ)\cos(60^\circ)+\sin(135^\circ)\sin(60^\circ)\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= -\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Exercise 3.4.2

Find the exact value of $\cos(105^\circ)$.

Answer

$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

SUM AND DIFFERENCE FORMULAS FOR SINE

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \tag{3.4.3}$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \tag{3.4.4}$$

How to: Given two angles, find the sine of the difference between the angles

1. Write the difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

Example 3.4.3: Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45^\circ-30^\circ)$
b. $\sin(135^\circ-120^\circ)$

Solution

a. Let's begin by writing the formula and substitute the given angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(45^\circ-30^\circ) &= \sin(45^\circ)\cos(30^\circ)-\cos(45^\circ)\sin(30^\circ)\end{aligned}$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45^\circ)=\tfrac{\sqrt{2}}{2},\qquad \cos(30^\circ)=\tfrac{\sqrt{3}}{2},\qquad \cos(45^\circ)=\tfrac{\sqrt{2}}{2},\qquad \sin(30^\circ)=\tfrac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(45^\circ-30^\circ) &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b. Again, we write the formula and substitute the given angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(135^\circ-120^\circ) &= \sin(135^\circ)\cos(120^\circ)-\cos(135^\circ)\sin(120^\circ)\end{aligned}$$

Next, we find the values of the trigonometric expressions.

$$\sin(135^\circ)=\frac{\sqrt{2}}{2},\qquad \cos(120^\circ)=-\frac{1}{2},\qquad \cos(135^\circ)=\frac{\sqrt{2}}{2},\qquad \sin(120^\circ)=\frac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(135^\circ-120^\circ) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example 3.4.4: Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right)$. Then check the answer with a graphing calculator.

Solution

The pattern displayed in this problem is $\sin(\alpha+\beta)$. Let $\alpha=\cos^{-1}\frac{1}{2}$ and $\beta=\sin^{-1}\frac{3}{5}$. Then we can write

$$\begin{aligned}\cos\alpha &= \frac{1}{2}, \quad 0\le\alpha\le\pi\\ \sin\beta &= \frac{3}{5}, \quad -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}\end{aligned}$$

We will use the Pythagorean identities to find $\sin\alpha$ and $\cos\beta$

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha}\\ &= \sqrt{1-\frac{1}{4}}\\ &= \sqrt{\frac{3}{4}}\\ &= \frac{\sqrt{3}}{2}\\ \cos\beta &= \sqrt{1-\sin^2\beta}\\ &= \sqrt{1-\frac{9}{25}}\\ &= \sqrt{\frac{16}{25}}\\ &= \frac{4}{5}\end{aligned}$$

Using the sum formula for sine,

$$\begin{aligned}\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right) &= \sin(\alpha+\beta)\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \frac{\sqrt{3}}{2}\cdot\frac{4}{5}+\frac{1}{2}\cdot\frac{3}{5}\\ &= \frac{4\sqrt{3}+3}{10}\end{aligned}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x=\frac{\sin x}{\cos x}$, when $\cos x\neq 0$.

Let's derive the sum formula for tangent.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\ &= \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\end{aligned}$$

We can derive the difference formula for tangent in a similar way.

SUM AND DIFFERENCE FORMULAS FOR TANGENT

The sum and difference formulas for tangent are:

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{3.4.5}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{3.4.6}$$

How to: Given two angles, find the tangent of the sum of the angles

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

Example 3.4.5: Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$.

Solution

Let's first write the sum formula for tangent and then substitute the given angles into the formula.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\tan\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{4}\right)}{1-\left(\tan\left(\frac{\pi}{6}\right)\right)\left(\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual function values within the formula:

$$\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}},\quad \text{and}\quad \tan\left(\frac{\pi}{4}\right)=1 \tag{3.4.7}$$

So we have,

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\frac{1}{\sqrt{3}}+1}{1-\left(\frac{1}{\sqrt{3}}\right)(1)}$$

$$=\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$$=\frac{1+\sqrt{3}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}-1}$$

$$=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Exercise 3.4.5:

Find the exact value of $\tan\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)$.

Answer

$\frac{1-\sqrt{3}}{1+\sqrt{3}}$

Example 3.4.6: Finding Multiple Sums and Differences of Angles

Given $\sin\alpha=\frac{3}{5}, \quad 0<\alpha<\frac{\pi}{2}$, and $\cos\beta=-\frac{5}{13}, \quad \pi<\beta<\frac{3\pi}{2}$,

find

a. $\sin(\alpha+\beta)$
b. $\cos(\alpha+\beta)$
c. $\tan(\alpha+\beta)$
d. $\tan(\alpha-\beta)$

Solution

We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

a. To find $\sin(\alpha+\beta)$, we begin with $\sin\alpha=\frac{3}{5}$ and $0<\alpha<\frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See Figure 3.4.4. Using the Pythagorean Theorem, we can find the length of side a:

$$a^2+3^2=5^2$$
$$a^2=16$$
$$a=4$$

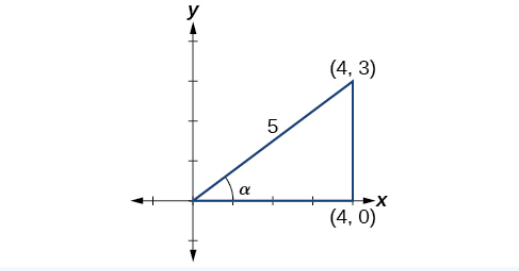


Figure 3.4.4

Since $\cos\beta=-\frac{5}{13}$ and $\pi<\beta<\frac{3\pi}{2}$,the side adjacent to β is -5,the hypotenuse is 13, and β is in the third quadrant. See Figure 3.4.5. Again, using the Pythagorean Theorem, we have

$$\begin{aligned}(-5)^2+a^2 &= 13^2\\ 25+a^2 &= 169\\ a^2 &= 144\\ a &= \pm 12\end{aligned}$$

Since β is in the third quadrant, $a = -12$.

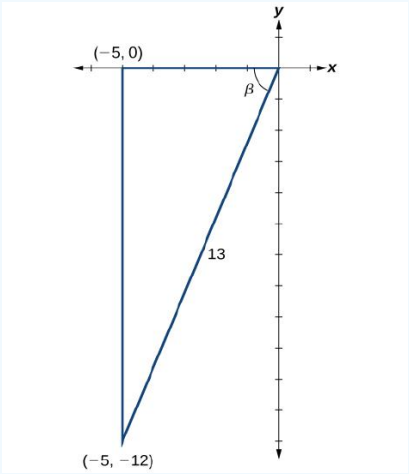


Figure 3.4.5

The next step is finding the cosine of α and the sine of β. The cosine ofα α is the adjacent side over the hypotenuse. We can find it from the triangle in Figure 3.4.5: $\cos\alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in Figure 3.4.5, as opposite side over the hypotenuse: $\sin\beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha+\beta)$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)+\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)\\ &= -\frac{15}{65}-\frac{48}{65}\\ &= -\frac{63}{65}\end{aligned}$$

b. We can find $\cos(\alpha+\beta)$ in a similar manner. We substitute the values according to the formula.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right)-\left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)\\ &= -\frac{20}{65}+\frac{36}{65}\\ &= \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha+\beta)$,if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, then

$$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

If $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$, then

$$\tan\beta = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$$

Then,

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{\frac{63}{20}}{-\frac{16}{20}}\\ &= -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha-\beta)$, we have the values we need. We can substitute them in and evaluate.

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4}-\frac{12}{5}}{1+\frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{-\frac{33}{20}}{\frac{56}{20}}\\ &= -\frac{33}{56}\end{aligned}$$

Analysis

A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that

$$\tan(\alpha+\beta) = \sin(\alpha+\beta)\cos(\alpha+\beta)$$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from Right Triangle Trigonometry that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In Figure 3.4.6, notice that if one of the acute angles is labeled as θ, then the other acute angle must be labeled $\frac{\pi}{2}-\theta$.

Notice also that $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$,which is opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

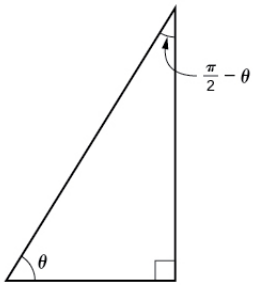


Figure 3.4.6

From these relationships, the cofunction identities are formed. Recall that you first encountered these identities in The Unit Circle: Sine and Cosine Functions.

COFUNCTION IDENTITIES

The cofunction identities are summarized in Table 3.4.2.

Table 3.4.2

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Notice that the formulas in the table may also justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

We can write

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta \\ &= \sin\theta\end{aligned}$$

Example 3.4.7: Finding a Cofunction with the Same Value as the Given Expression

Write $\tan\frac{\pi}{9}$ in terms of its cofunction.

Solution

The cofunction of $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$. Thus,

$$\begin{aligned}\tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{9}\right) \\ &= \cot\left(\frac{9\pi}{18} - \frac{2\pi}{18}\right) \\ &= \cot\left(\frac{7\pi}{18}\right)\end{aligned}$$

Exercise 3.4.7

Write $\sin\frac{\pi}{7}$ in terms of its cofunction.

Answer

$\cos\left(\frac{5\pi}{14}\right)$

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules presented earlier may help simplify the process of verifying an identity.

How to: Given an identity, verify using sum and difference formulas

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Example 3.4.8A: Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

Solution

We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ &= 2\sin\alpha\cos\beta\end{aligned}$$

We see that the identity is verified.

Example 3.4.8B: Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\tan\beta$$

Solution

We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned}\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} &= \frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\\ &= \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} && \text{Rewrite using a common denominator}\\ &= \frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta} && \text{Cancel}\\ &= \tan\alpha-\tan\beta && \text{Rewrite in terms of tangent}\end{aligned}$$

Exercise 3.4.8:

Verify the identity: $\tan(\pi-\theta)=-\tan\theta$.

Answer

$$\begin{aligned}\tan(\pi-\theta) &= \frac{\tan(\pi)-\tan\theta}{1+\tan(\pi)\tan\theta}\\ &= \frac{0-\tan\theta}{1+0\cdot\tan\theta}\\ &= -\tan\theta\end{aligned}$$

Example 3.4.9A: Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2. See Figure 3.4.7. Show that

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan\theta_1=m_1$ and $\tan\theta_2=m_2$.)

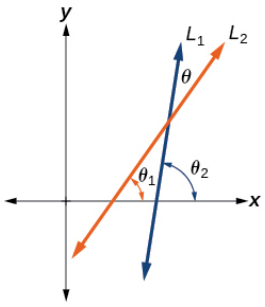


Figure 3.4.7

Solution

Using the difference formula for tangent, this problem does not seem as daunting as it might.

$$\begin{aligned}\tan\theta &= \tan(\theta_2 - \theta_1)\\ &= \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_1\tan\theta_2}\\ &= \frac{m_2 - m_1}{1+m_1 m_2}\end{aligned}$$

Extra Practice

For the following exercises, find the exact value.

1. $\tan\left(\frac{7\pi}{12}\right)$
2. $\cos\left(\frac{25\pi}{12}\right)$
3. $\sin(70^\circ)\cos(25^\circ) - \cos(70^\circ)\sin(25^\circ)$
4. $\cos(83^\circ)\cos(23^\circ) + \sin(83^\circ)\sin(23^\circ)$

For the following exercises, prove the identity.

1. $\cos(4x) - \cos(3x)\cos x = \sin^2 x - 4\cos^2 x \sin^2 x$
2. $\cos(3x) - \cos^3 x = -\cos x \sin^2 x - \sin x \sin(2x)$

For the following exercise, simplify the expression.

1. $\frac{\tan\left(\frac{1}{2}x\right)+\tan\left(\frac{1}{8}x\right)}{1-\tan\left(\frac{1}{8}x\right)\tan\left(\frac{1}{2}x\right)}$

For the following exercises, find the exact value.

1. $\cos\left(\sin^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right)\right)$
2. $\tan\left(\sin^{-1}(0) + \sin^{-1}\left(\frac{1}{2}\right)\right)$

Key Equations

Sum Formula for Cosine	$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
Difference Formula for Cosine	$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
Sum Formula for Sine	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
Difference Formula for Sine	$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
Sum Formula for Tangent	$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
Difference Formula for Tangent	$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Cofunction identities	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$ $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Key Concepts

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle. See Example $3.4.1$ and Example $3.4.2$.
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle. See Example $3.4.3$.
- The sum and difference formulas for sine and cosine can also be used for inverse trigonometric functions. See Example $3.4.4$.
- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles. The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles. See Example $3.4.5$.
- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles. See Example $3.4.6$.
- The cofunction identities apply to complementary angles and pairs of reciprocal functions. See Example $3.4.7$.
- Sum and difference formulas are useful in verifying identities. See Example $3.4.8$ and Example $3.4.9$.
- Application problems are often easier to solve by using sum and difference formulas. See Example $3.4.10$ and Example $3.4.11$.

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3.5: Double Angle Identities

Learning Objectives

- Use the double angle identities to solve other identities.
- Use the double angle identities to solve equations.

Two special cases of the sum of angles identities arise often enough that we choose to state these identities separately.

IDENTITIES

The double angle identities

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{3.5.1}$$

$$\begin{aligned} \cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) \\ &= 1 - 2\sin^2(\alpha) \\ &= 2\cos^2(\alpha) - 1 \end{aligned} \tag{3.5.2}$$

These identities follow from the sum of angles identities.

Proof of the sine double angle identity

$$\sin(2\alpha)$$

$$= \sin(\alpha + \alpha)$$

Apply the sum of angles identity

$$= \sin(\alpha)\cos(\alpha) + \cos(\alpha)\sin(\alpha)$$

Simplify

$$= 2\sin(\alpha)\cos(\alpha)$$

Establishing the identity

Exercise 3.5.1

Show $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ by using the sum of angles identity for cosine.

Answer

$$\cos(2\alpha) = \cos(\alpha + \alpha)$$
$$\cos(\alpha)\cos(\alpha) - \sin(\alpha)\sin(\alpha)$$
$$\cos^2(\alpha) - \sin^2(\alpha)$$

For the cosine double angle identity, there are three forms of the identity stated because the basic form, $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$, can be rewritten using the Pythagorean Identity. Rearranging the Pythagorean Identity results in the equality $\cos^2(\alpha) = 1 - \sin^2(\alpha)$, and by substituting this into the basic double angle identity, we obtain the second form of the double angle identity.

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

Substituting using the Pythagorean identity

$$\cos(2\alpha) = 1 - \sin^2(\alpha) - \sin^2(\alpha)$$

Simplifying

$$\cos(2\alpha) = 1 - 2\sin^2(\alpha)$$

Example 3.5.1

If $\sin(\theta) = \frac{3}{5}$ and θ is in the second quadrant, find exact values for $\sin(2\theta)$ and $\cos(2\theta)$.

Solution

To evaluate $\cos(2\theta)$, since we know the value for $\sin(\theta)$ we can use the version of the double angle that only involves sine.

$$\cos(2\theta) = 1 - 2\sin^2(\theta) = 1 - 2\left(\frac{3}{5}\right)^2 = 1 - \frac{18}{25} = \frac{7}{25}$$

Since the double angle for sine involves both sine and cosine, we'll need to first find $\cos(\theta)$, which we can do using the Pythagorean Identity.

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\left(\frac{3}{5}\right)^2 + \cos^2(\theta) = 1$$

$$\cos^2(\theta) = 1 - \frac{9}{25}$$

$$\cos(\theta) = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$$

Since θ is in the second quadrant, we know that $\cos(\theta) < 0$, so

$$\cos(\theta) = -\frac{4}{5}$$

Now we can evaluate the sine double angle

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$$

Example 3.5.2

Simplify the expressions

a) $2\cos^2(12^\circ) - 1$

b) $8\sin(3x)\cos(3x)$

Solution

a) Notice that the expression is in the same form as one version of the double angle identity for cosine: $\cos(2\theta) = 2\cos^2(\theta) - 1$. Using this,

$$2\cos^2(12^\circ) - 1 = \cos(2 \cdot 12^\circ) = \cos(24^\circ)$$

b) This expression looks similar to the result of the double angle identity for sine.

$$8\sin(3x)\cos(3x)$$

Factoring a 4 out of the original expression

$$4 \cdot 2\sin(3x)\cos(3x)$$

Applying the double angle identity

$$4\sin(6x)$$

We can use the double angle identities to simplify expressions and prove identities.

Example 3.5.3

Simplify $\dfrac{\cos(2t)}{\cos(t)-\sin(t)}$.

Solution

With three choices for how to rewrite the double angle, we need to consider which will be the most useful. To simplify this expression, it would be great if the denominator would cancel with something in the numerator, which would require a factor of $\cos(t)-\sin(t)$ in the numerator, which is most likely to occur if we rewrite the numerator with a mix of sine and cosine.

$$\frac{\cos(2t)}{\cos(t)-\sin(t)}$$

Apply the double angle identity

$$=\frac{\cos^2(t)-\sin^2(t)}{\cos(t)-\sin(t)}$$

Factor the numerator

$$=\frac{(\cos(t)-\sin(t))(\cos(t)+\sin(t))}{\cos(t)-\sin(t)}$$

Cancelling the common factor

$$=\cos(t)+\sin(t)$$

Resulting in the most simplified form

Example 3.5.4

Prove $\sec(2\alpha)=\dfrac{\sec^2(\alpha)}{2-\sec^2(\alpha)}$.

Solution

Since the right side seems a bit more complicated than the left side, we begin there.

$$\frac{\sec^2(\alpha)}{2-\sec^2(\alpha)}$$

Rewrite the secants in terms of cosine

$$=\frac{\dfrac{1}{\cos^2(\alpha)}}{2-\dfrac{1}{\cos^2(\alpha)}}$$

At this point, we could rewrite the bottom with common denominators, subtract the terms, invert and multiply, then simplify. Alternatively, we can multiple both the top and bottom by $\cos^2(\alpha)$, the common denominator:

$$=\frac{\dfrac{1}{\cos^2(\alpha)}\cdot\cos^2(\alpha)}{\left(2-\dfrac{1}{\cos^2(\alpha)}\right)\cdot\cos^2(\alpha)}$$

Distribute on the bottom

$$= \frac{\frac{\cos^2(\alpha)}{\cos^2(\alpha)}}{2\cos^2(\alpha) - \frac{\cos^2(\alpha)}{\cos^2(\alpha)}}.$$

Simplify

$$= \frac{1}{2\cos^2(\alpha) - 1}$$

Rewrite the denominator as a double angle

$$= \frac{1}{\cos(2\alpha)}$$

Rewrite as a secant

$$= \sec(2\alpha)$$

Establishing the identity

Exercise 3.5.2

Use an identity to find the exact value of $\cos^2(75^\circ) - \sin^2(75^\circ)$.

Answer

$$\cos^2(75^\circ) - \sin^2(75^\circ) = \cos(2 \cdot 75^\circ)$$

$$= \cos(150^\circ) = \frac{-\sqrt{3}}{2}$$

As with other identities, we can also use the double angle identities for solving equations.

Example 3.5.5

Solve $\cos(2t) = \cos(t)$ for all solutions with $0 \leq t < 2\pi$.

Solution

In general when solving trig equations, it makes things more complicated when we have a mix of sines and cosines and when we have a mix of functions with different periods. In this case, we can use a double angle identity to rewrite the cos(2*t*). When choosing which form of the double angle identity to use, we notice that we have a cosine on the right side of the equation. We try to limit our equation to one trig function, which we can do by choosing the version of the double angle formula for cosine that only involves cosine.

$$\cos(2t) = \cos(t)$$

Apply the double angle identity

$$2\cos^2(t) - 1 = \cos(t)$$

This is quadratic in cosine, so make one side 0

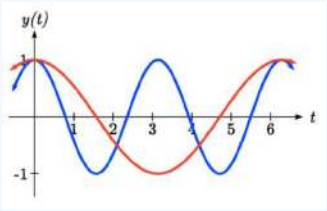
$$2\cos^2(t) - \cos(t) - 1 = 0$$

Factor

$$(2\cos(t) + 1)(\cos(t) - 1) = 0$$

Break this apart to solve each part separately

$$2\cos(t)+1=0 \text{ or } \cos(t)-1=0$$



$$\cos(t)=-\frac{1}{2} \text{ or } \cos(t)=1$$

$$t=\frac{2\pi}{3} \text{ or } t=\frac{4\pi}{3} \text{ or } t=0$$

Looking at a graph of cos(2t) and cos(t) shown together, we can verify that these three solutions on [0, 2 π) seem reasonable.

Important Topics of This Section

- Double angle identity
- Using identities
- Simplify equations
- Prove identities
- Solve equations

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3.6: Half Angle Identities

Learning Objectives

- Apply the half-angle identities to expressions, equations and other identities.
- Use the half-angle identities to find the exact value of trigonometric functions for certain angles.

Power Reduction and Half Angle Identities

Another use of the cosine double angle identities is to use them in reverse to rewrite a squared sine or cosine in terms of the double angle. Starting with one form of the cosine double angle identity:

$$\cos(2\alpha) = 2\cos^2(\alpha) - 1$$

Isolate the cosine squared term

$$\cos(2\alpha) + 1 = 2\cos^2(\alpha)$$

Add 1

$$\cos^2(\alpha) = \frac{\cos(2\alpha) + 1}{2}$$

Divide by 2

$$\cos^2(\alpha) = \frac{\cos(2\alpha) + 1}{2}$$

This is called a **power reduction identity**

Exercise 3.6.1

Use another form of the cosine double angle identity to prove the identity $\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2}$.

Answer

$$\frac{1 - \cos(2\alpha)}{2}$$
$$\frac{1 - \left(\cos^2(\alpha) - \sin^2(\alpha)\right)}{2}$$
$$\frac{1 - \cos^2(\alpha) + \sin^2(\alpha)}{2}$$
$$\frac{\sin^2(\alpha) + \sin^2(\alpha)}{2}$$
$$\frac{2\sin^2(\alpha)}{2} = \sin^2(\alpha)$$

The cosine double angle identities can also be used in reverse for evaluating angles that are half of a common angle. Building from our formula $\cos^2(\alpha) = \frac{\cos(2\alpha) + 1}{2}$, if we let $\theta = 2\alpha$, then $\alpha = \frac{\theta}{2}$ this identity becomes $\cos^2\left(\frac{\theta}{2}\right) = \frac{\cos(\theta) + 1}{2}$. Taking the square root, we obtain

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta) + 1}{2}}$$

where the sign is determined by the quadrant.

This is called a **half-angle identity**.

Exercise 3.6.2

Use your results from the last Try it Now to prove the identity $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$.

Answer

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$$
$$\sin(\alpha) = \pm\sqrt{\frac{1-\cos(2\alpha)}{2}}$$
$$\alpha = \frac{\theta}{2}$$
$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}}$$
$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

IDENTITIES

Half-Angle Identities

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta)+1}{2}} \tag{3.6.1}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}} \tag{3.6.2}$$

Power Reduction Identities

$$\cos^2(\alpha) = \frac{\cos(2\alpha)+1}{2} \tag{3.6.3}$$

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{3.6.4}$$

Since these identities are easy to derive from the double-angle identities, the power reduction and half-angle identities are not ones you should need to memorize separately.

Example 3.6.1

Rewrite $\cos^4(x)$ without any powers.

Solution

$$\cos^4(x) = \left(\cos^2(x)\right)^2$$

Using the power reduction formula

$$= \left(\frac{\cos(2x)+1}{2}\right)^2$$

Square the numerator and denominator

$$= \frac{(\cos(2x)+1)^2}{4}$$

Expand the numerator

$$= \frac{\cos^2(2x) + 2\cos(2x) + 1}{4}$$

Split apart the fraction

$$= \frac{\cos^2(2x)}{4} + \frac{2\cos(2x)}{4} + \frac{1}{4}$$

Apply the formula above to $\cos^2(2x)$

$$\cos^2(2x) = \frac{\cos(2 \cdot 2x) + 1}{2}$$

$$= \frac{\left(\frac{\cos(4x) + 1}{2}\right)}{4} + \frac{2\cos(2x)}{4} + \frac{1}{4}$$

Simplify

$$= \frac{\cos(4x)}{8} + \frac{1}{8} + \frac{1}{2}\cos(2x) + \frac{1}{4}$$

Combine the constants

$$= \frac{\cos(4x)}{8} + \frac{1}{2}\cos(2x) + \frac{3}{8}$$

Example 3.6.2

Find an exact value for $\cos(15^\circ)$.

Solution

Since 15 degrees is half of 30 degrees, we can use our result from above:

$$\cos(15^\circ) = \cos\left(\frac{30^\circ}{2}\right) = \pm\sqrt{\frac{\cos(30^\circ) + 1}{2}}$$

We can evaluate the cosine. Since 15 degrees is in the first quadrant, we need the positive result.

$$\sqrt{\frac{\cos(30^\circ) + 1}{2}} = \sqrt{\frac{\frac{\sqrt{3}}{2} + 1}{2}}$$

$$= \sqrt{\frac{\sqrt{3}}{4} + \frac{1}{2}}$$

Exercise 3.6.3

If $\csc(x) = 7$ and $90^\circ < x < 180^\circ$, then find exact values for (without solving for x):

a. $\sin\left(\frac{x}{2}\right)$
b. $\cos\left(\frac{x}{2}\right)$
c. $\tan\left(\frac{x}{2}\right)$

Answer

a. $\sqrt{\frac{1}{2}+\frac{2+\sqrt{7}}{7}}$

b. $\sqrt{\frac{1}{2}-\frac{2+\sqrt{7}}{7}}$

c. $\frac{1}{7-4\sqrt{3}}$

Important Topics of This Section

- Power reduction identity
- Half angle identity
- Using identities
- Simplify equations
- Prove identities
- Solve equations

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3.7: Exercises - Double Angle, Half-Angle, and Power Reductions

Chapter 3 Practice:

1. If $\sin(x) = \frac{1}{8}$ and x is in quadrant I, then find exact values for (without solving for x):
   a. $\sin(2x)$
   b. $\cos(2x)$
   c. $\tan(2x)$
2. If $\cos(x) = \frac{2}{3}$ and x is in quadrant I, then find exact values for (without solving for x):
   a. $\sin(2x)$
   b. $\cos(2x)$
   c. $\tan(2x)$

Simplify each expression.

3. $\cos^2(28^\circ) - \sin^2(28^\circ)$
4. $2\cos^2(37^\circ) - 1$
5. $1 - 2\sin^2(17^\circ)$
6. $\cos^2(37^\circ) - \sin^2(37^\circ)$
7. $\cos^2(9x) - \sin^2(9x)$
8. $\cos^2(6x) - \sin^2(6x)$
9. $4\sin(8x)\cos(8x)$
10. $6\sin(5x)\cos(5x)$

Solve for all solutions on the interval $[0, 2\pi)$.

11. $6\sin(2t) + 9\sin(t) = 0$
12. $2\sin(2t) + 3\cos(t) = 0$
13. $9\cos(2\theta) = 9\cos^2(\theta) - 4$
14. $8\cos(2\alpha) = 8\cos^2(\alpha) - 1$
15. $\sin(2t) = \cos(t)$
16. $\cos(2t) = \sin(t)$
17. $\cos(6x) - \cos(3x) = 0$
18. $\sin(4x) - \sin(2x) = 0$

Use a double angle, half angle, or power reduction formula to rewrite without exponents.

19. $\cos^2(5x)$
20. $\cos^2(6x)$
21. $\sin^4(8x)$
22. $\sin^4(3x)$
23. $\cos^2 x \sin^4 x$
24. $\cos^4 x \sin^2 x$
25. If $\csc(x) = 7$ and $90^\circ < x < 180^\circ$, then find exact values for (without solving for x):

a. $\sin\left(\frac{x}{2}\right)$

b. $\cos\left(\frac{x}{2}\right)$

c. $\tan\left(\frac{x}{2}\right)$

26. If $\sec(x)=4$ and $270^\circ < x < 360^\circ$, then find exact values for (without solving for x):

a. $\sin\left(\frac{x}{2}\right)$

b. $\cos\left(\frac{x}{2}\right)$

c. $\tan\left(\frac{x}{2}\right)$

Prove the identity.

27. $(\sin t - \cos t)^2 = 1 - \sin(2t)$

28. $\left(\sin^2 x - 1\right)^2 = \cos(2x) + \sin^4 x$

29. $\sin(2x) = \frac{2\tan(x)}{1+\tan^2(x)}$

30. $\tan(2x) = \frac{2\sin(x)\cos(x)}{2\cos^2(x)-1}$

31. $\cot(x) - \tan(x) = 2\cot(2x)$

32. $\frac{\sin(2\theta)}{1+\cos(2\theta)} = \tan(\theta)$

33. $\cos(2\alpha) = \frac{1-\tan^2(\alpha)}{1+\tan^2(\alpha)}$

34. $\frac{1+\cos(2t)}{\sin(2t)-\cos(t)} = \frac{2\cos(t)}{2\sin(t)-1}$

35. $\sin(3x) = 3\sin(x)\cos^2(x) - \sin^3(x)$

36. $\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x)$

Answer

1. a. $\frac{3\sqrt{7}}{32}$

b. $\frac{31}{32}$

c. $\frac{3\sqrt{7}}{31}$

3. $\cos(56^\circ)$

5. $\cos(34^\circ)$

7. $\cos(18x)$

9. $2\sin(16x)$

11. 0, π, 2.4189,3.8643

13. 0.7297, 2.4119, 3.8713, 5.5535

15. $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

17. a. $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$, 0, $\frac{2\pi}{3}, \frac{4\pi}{3}$

19. $\frac{1+\cos(10x)}{2}$

21. $\frac{3}{8}-\frac{1}{2}\cos(16x)+\frac{1}{8}\cos(32x)$

23. $\frac{1}{16}-\frac{1}{16}\cos(2x)+\frac{1}{16}\cos(4x)-\frac{1}{16}\cos(2x)\cos(4x)$

25. a. $\sqrt{\frac{1}{2}+\frac{2+\sqrt{7}}{7}}$

b. $\sqrt{\frac{1}{2}-\frac{2+\sqrt{7}}{7}}$

c. $\frac{1}{7-4\sqrt{3}}$

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CHAPTER OVERVIEW

Chapter 4: Inverse Trigonometric Functions

4.1: Basic Inverse Trigonometric Functions
4.2: Graphing Inverse Trigonometric Functions
4.3: Inverse Trigonometric Properties
4.4: Applications

Thumbnail: inverse of sine graph (GNU GPL; Michael Corral via LibreTexts).

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4.1: Basic Inverse Trigonometric Functions

Learning Objectives

- Understand and evaluate inverse trigonometric functions.
- Extend the inverse trigonometric functions to include the $\csc^{-1}$, $\sec^{-1}$, and $\cot^{-1}$ functions.
- Apply inverse trigonometric functions to the critical values on the unit circle.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in Figure 4.1.1.

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

Figure 4.1.1

For example, if $f(x) = \sin\ x$, then we would write $f^{-1}(x) = \sin^{-1} x$. Be aware that $\sin^{-1} x$ does not mean $\dfrac{1}{\sin\ x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$, then $\dfrac{\pi}{6} = \sin^{-1}\left(\dfrac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\dfrac{\pi}{4}\right) = 1$, then $\dfrac{\pi}{4} = \tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the **domain** of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. Figure 4.1.2 shows the graph of the sine function limited to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.

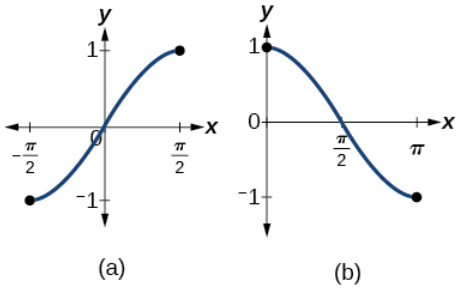


Figure 4.1.2: (a) Sine function on a restricted domain of $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$; (b) Cosine function on a restricted domain of $[0, \pi]$

Figure 4.1.3 shows the graph of the tangent function limited to $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

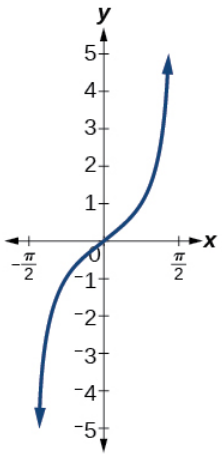


Figure 4.1.3: Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one **vertical asymptote** to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin\ y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin\ x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos\ y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos\ x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan\ y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan\ x$.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

The graphs of the inverse functions are shown in Figures 4.1.4 - 4.1.6. Notice that the output of each of these inverse functions is a *number*, an angle in radian measure. We see that $\sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

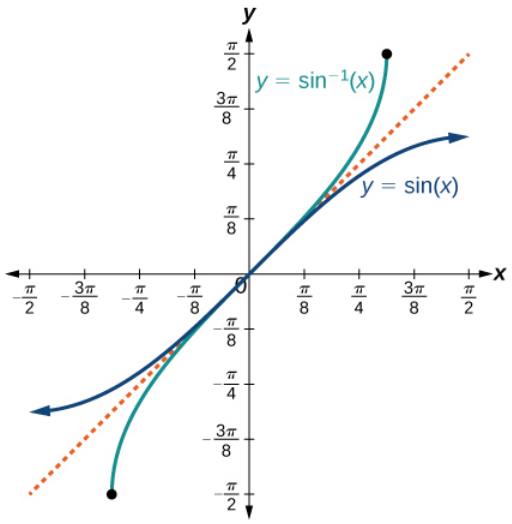


Figure 4.1.4: The sine function and inverse sine (or arcsine) function

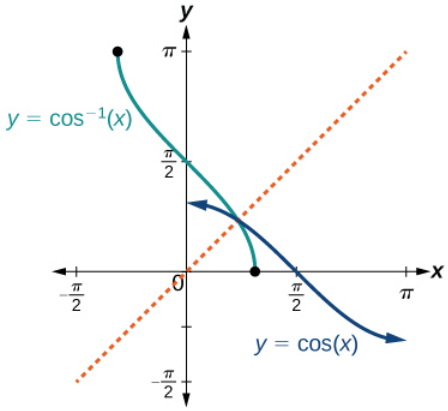


Figure 4.1.5: The cosine function and inverse cosine (or arccosine) function

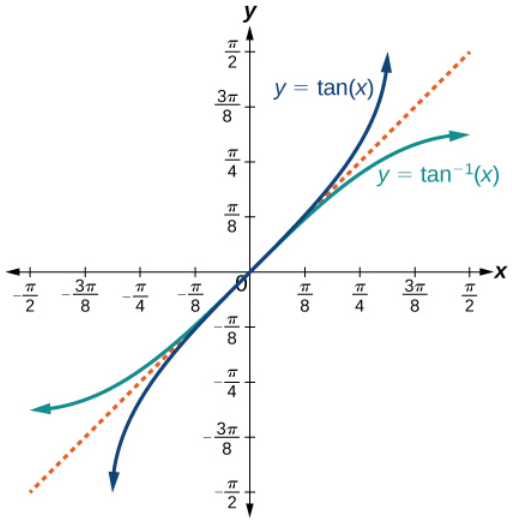


Figure 4.1.6: The tangent function and inverse tangent (or arctangent) function

RELATIONS FOR INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

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For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$,then $\tan^{-1} x = y$.

Example 4.1.1: Writing a Relation for an Inverse Function

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Solution

Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$

Exercise 4.1.1

Given $\cos(0.5) \approx 0.8776$,write a relation involving the inverse cosine.

Answer

$\arccos(0.8776) \approx 0.5$

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}(30°)$, $\frac{\pi}{4}(45°)$, and $\frac{\pi}{3}(60°)$, and their reflections into other quadrants.

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x,depending on which corresponds to the given inverse function.

Example 4.1.2: Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$

b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

c. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

d. $\tan^{-1}(1)$

Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x) = \frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a

function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0, \pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1) = \frac{\pi}{4}$.

Exercise 4.1.2

Evaluate each of the following.

a. $\sin^{-1}(-1)$
b. $\tan^{-1}(-1)$
c. $\cos^{-1}(-1)$
d. $\cos^{-1}\left(\frac{1}{2}\right)$

Answer a

$-\frac{\pi}{2}$

Answer b

$-\frac{\pi}{4}$

Answer c

π

Answer d

$\frac{\pi}{3}$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate **inverse trigonometric functions** that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN-1, ARCSIN, or ASIN.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

Example 4.1.3: Evaluating the Inverse Sine on a Calculator

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Solution

Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93^{\circ}$. Note that in calculus and beyond we will use radians in almost all cases.

Exercise 4.1.3

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer

$1.9823 \text{or } 113.578^{\circ}$

Given two sides of a right triangle like the one shown in Figure 8.4.7, find an angle.

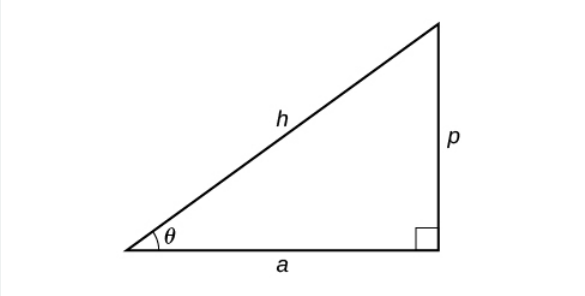


Figure 4.1.7

1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

Example 4.1.4: Applying the Inverse Cosine to a Right Triangle

Solve the triangle in Figure 4.1.8 for the angle θ.

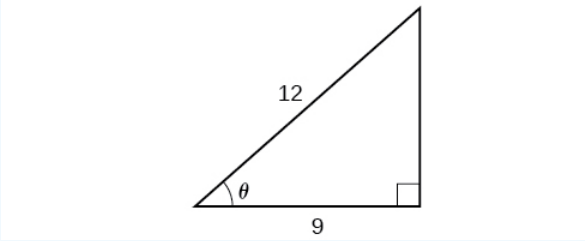


Figure 4.1.8

Solution

Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos\theta = \frac{9}{12}$$

$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \qquad \text{Apply definition of the inverse}$$

$$\theta \approx 0.7227 \qquad \text{or about } 41.4096^{\circ} \text{ Evaluate}$$

Exercise 4.1.4

Solve the triangle in Figure 4.1.9 for the angle θ.

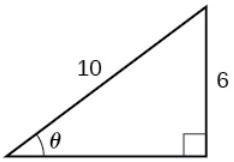


Figure 4.1.9

Answer

$\sin^{-1}(0.6) = 36.87^\circ = 0.6435$radians

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set{ $\sin(x)$,$\cos(x)$,$\tan(x)$ } and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function,$f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

COMPOSITIONS OF A TRIGONOMETRIC FUNCTION AND ITS INVERSE

$$\begin{aligned}
\sin(\sin^{-1}x) &= x \qquad \text{for } -1 \le x \le 1 \\
\cos(\cos^{-1}x) &= x \qquad \text{for } -1 \le x \le 1 \\
\tan(\tan^{-1}x) &= x \qquad \text{for } -\infty < x < \infty \\
\sin^{-1}(\sin x) &= x \qquad \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos^{-1}(\cos x) &= x \qquad \text{only for } 0 \le x \le \pi \\
\tan^{-1}(\tan x) &= x \qquad \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Q&A

Is it correct that $\sin^{-1}(\sin x) = x$?

No. This equation is correct ifx x belongs to the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but sine is defined for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

Given an expression of the form $f^{-1}(f(\theta))$ where $f(\theta) = \sin\theta$, $\cos\theta$, or $\tan\theta$, evaluate.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain off f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Example 4.1.5: Using Inverse Trigonometric Functions

Evaluate the following:

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
c. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$
d. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution

a. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.
b. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $sin\left(\frac{2\pi}{3}\right) = sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$.
c. $\frac{2\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$.
d. $-\frac{\pi}{3}$ is not in $[0, \pi]$, but $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ because cosine is an even function. $\frac{\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

Exercise 4.1.5

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Answer

$\frac{\pi}{8}; \frac{2\pi}{9}$

Key Concepts

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f(x)$, if $x = f^{-1}(y)$, then $f(x) = y$. However, $f(x) = y$ only implies $x = f^{-1}(y)$ if x is in the restricted domain of f. See Example 4.1.1.
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4} = \tan^{-1}(1)$ and $\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right)$ .See Example 4.1.2.
- A calculator will return an angle within the restricted domain of the original trigonometric function. See Example 4.1.3.
- Inverse functions allow us to find an angle when given two sides of a right triangle. See Example 4.1.4.

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4.2: Graphing Inverse Trigonometric Functions

Learning Objectives

- Understand the meaning of restricted domain as it applies to the inverses of the six trigonometric functions.
- Apply the domain, range, and quadrants of the six inverse trigonometric functions to evaluate expressions.

we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin\ y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin\ x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos\ y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos\ x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan\ y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan\ x$.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

The graphs of the inverse functions are shown in Figures 4.2.1 - 4.2.3. Notice that the output of each of these inverse functions is a *number,* an angle in radian measure. We see that $\sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

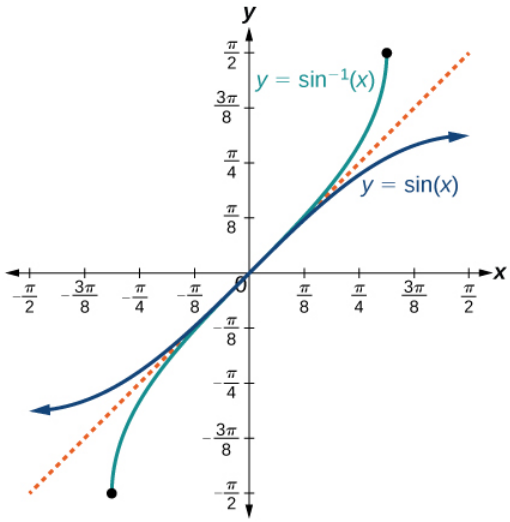


Figure 4.2.1: The sine function and inverse sine (or arcsine) function

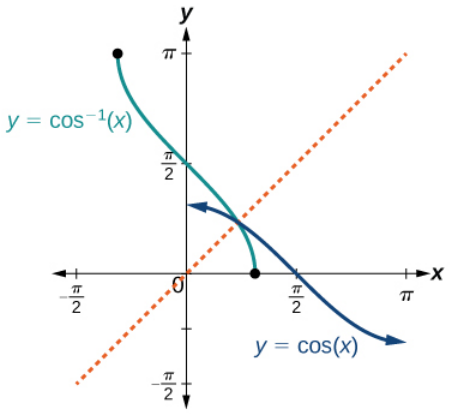


Figure 4.2.2: The cosine function and inverse cosine (or arccosine) function

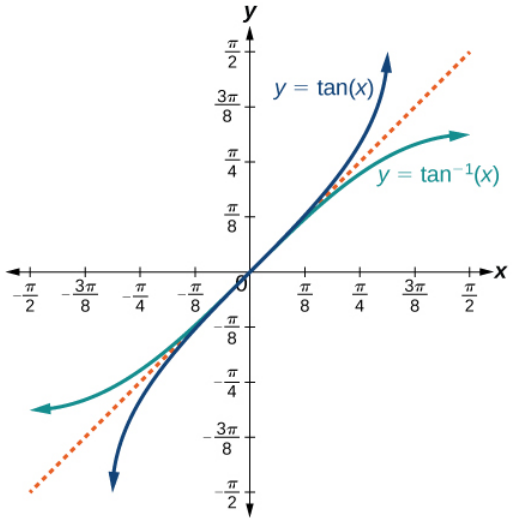


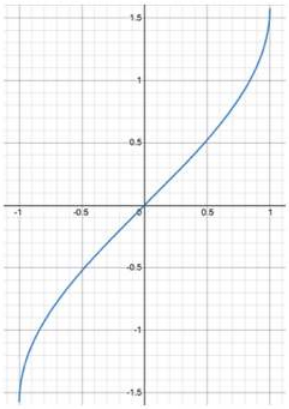
Figure 4.2.3: The tangent function and inverse tangent (or arctangent) function

Inverse Trigonometric Functions

$f(x) = sin^{-1}(x) = \arcsin(x)$

Domain: $[-1,1]$

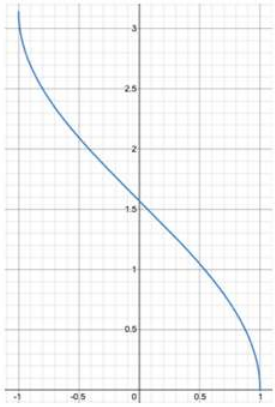
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$f(x) = cos^{-1}(x) = \arccos(x)$

Domain: $[-1,1]$

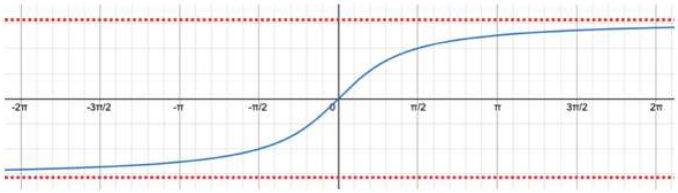
Range: $[0, \pi]$



$f(x) = tan^{-1}(x) = \arctan(x)$

Domain: $(-\infty, \infty)$

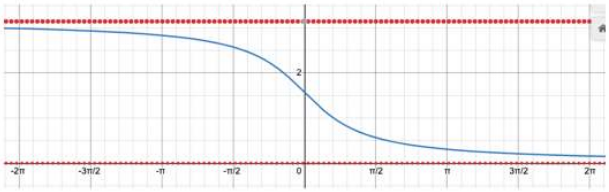
Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$f(x) = cot^{-1}(x) = \operatorname{arccot}(x)$

Domain: $(-\infty, \infty)$

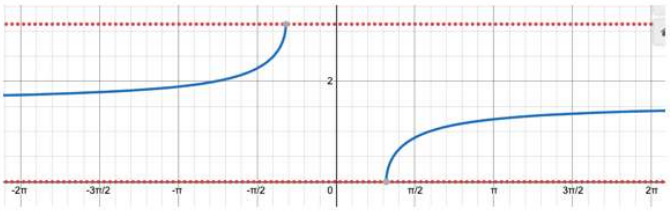
Range: $(0, \pi)$



$f(x) = sec^{-1}(x) = \operatorname{arcsec}(x)$

Domain: $(-\infty, -1] \cup [1, \infty)$

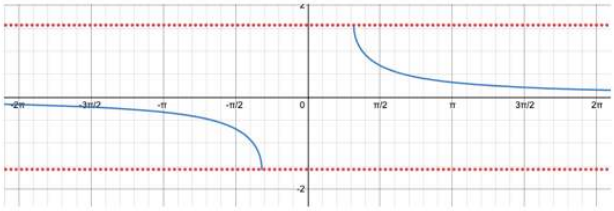
Range: $[0, \pi], y \neq \frac{\pi}{2}$



$f(x) = cos^{-1}(x) = \arccos(x)$

Domain: $(-\infty, -1] \cup [1, \infty)$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq 0$



RELATIONS FOR INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$,then $\tan^{-1} x = y$.

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4.3: Inverse Trigonometric Properties

Learning Objectives

- Relate the concept of inverse functions to trigonometric functions.
- Reduce the composite function to an algebraic expression involving no trigonometric functions.
- Use the inverse reciprocal properties.
- Compose each of the six basic trigonometric functions with $\tan^{-1}(x)$.

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}(g(x))$. For special values of x,we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2}-\theta$.Consider the sine and cosine of each angle of the right triangle in Figure 4.3.1.

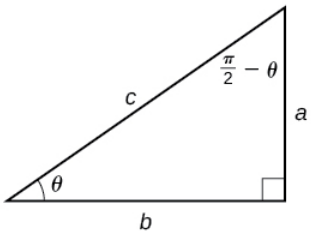


Figure 4.3.1: Right triangle illustrating the cofunction relationships

Because $\cos\theta=\frac{b}{c}=sin\left(\frac{\pi}{2}-\theta\right)$, we have $\sin^{-1}(\cos\theta)=\frac{\pi}{2}-\theta$ if $0\le\theta\le\pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$.Similarly, $\sin\theta=\frac{a}{c}=\cos\left(\frac{\pi}{2}-\theta\right)$, so $\cos^{-1}(\sin\theta)=\frac{\pi}{2}-\theta$ if $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0,\pi]$, then $\sin^{-1}(\cos x)=\frac{\pi}{2}-x$.
2. If x is not in $[0,\pi]$, then find another angle y in $[0,\pi]$ such that $\cos y=\cos x$.

$$\sin^{-1}(\cos x)=\frac{\pi}{2}-y \tag{4.3.1}$$

3. If x is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x)=\frac{\pi}{2}-x$.
4. If x is not in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then find another angle y in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ such that $\sin y=\sin x$.

$$\cos^{-1}(\sin x)=\frac{\pi}{2}-y \tag{4.3.2}$$

Example 4.3.1: Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

a. by direct evaluation.
b. by the method described previously.

Solution

a. Here, we can directly evaluate the inside of the composition.

$$\cos\left(\frac{13\pi}{6}\right) = \cos\left(\frac{\pi}{6}+2\pi\right)$$
$$= \cos\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{2}$$

Now, we can evaluate the inverse function as we did earlier.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \tag{4.3.3}$$

b. We have $x = \frac{13\pi}{6}$, $y = \frac{\pi}{6}$, and

$$\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) = \frac{\pi}{2} - \frac{\pi}{6}$$
$$= \frac{\pi}{3}$$

Exercise 4.3.1

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Answer

$\frac{3\pi}{4}$

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin(\cos^{-1}x) = \sqrt{1-x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Example 4.3.2: Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution

Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{Use our known value for cosine}$$
$$\sin^2\theta + \left(\frac{4}{5}\right)^2 = 1 \qquad \text{Solve for sine}$$
$$\sin^2\theta = 1 - \frac{16}{25}$$
$$\sin\theta = \pm\frac{9}{25}$$
$$= \pm\frac{3}{5}$$

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in quadrant I, $\sin\theta$ must be positive, so the solution is 35. See Figure 4.3.11.

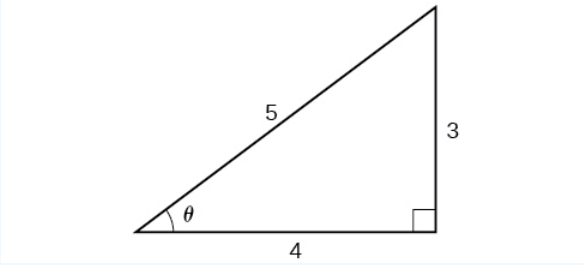


Figure 4.3.11: Right triangle illustrating that if $\cos\theta = \frac{4}{5}$, then $\sin\theta = \frac{3}{5}$

We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive; therefore $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \frac{3}{5}$.

Exercise 4.3.2

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Answer

$\frac{12}{13}$

Example 4.3.3: Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Solution

While we could use a similar technique as in Example 4.3.6, we will demonstrate a different technique here. From the inside, we know there is an angle such that $\tan\theta = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in Figure 4.3.12.

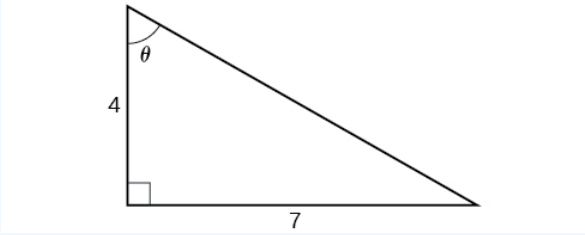


Figure 4.3.12: A right triangle with two sides known

Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

$$4^2+7^2 = hypotenuse^2$$
$$hypotenuse = \sqrt{65}$$

Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{7}{\sqrt{65}}$$

This gives us our desired composition.

$$\begin{aligned}\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) &= \sin\theta \\ &= \frac{7}{\sqrt{65}} \\ &= \frac{7\sqrt{65}}{65}\end{aligned}$$

Exercise 4.3.3

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Answer

$\frac{4\sqrt{2}}{9}$

Example 4.3.4: Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

Solution

We know there is an angle θ such that $\sin\theta = \frac{x}{3}$.

$$\begin{aligned}\sin^2\theta+\cos^2\theta &= 1 && \text{Use the Pythagorean Theorem} \\ \left(\frac{x}{3}\right)^2+\cos^2\theta &= 1 && \text{Solve for cosine} \\ \cos^2\theta &= 1-\frac{x^2}{9} \\ \cos\theta &= \pm\sqrt{\frac{9-x^2}{9}} \\ &= \pm\sqrt{\frac{9-x^2}{3}}\end{aligned}$$

Because we know that the inverse sine must give an angle on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we can deduce that the cosine of that angle must be positive.

$$cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \sqrt{\frac{9-x^2}{3}}$$

Exercise 4.3.4

Find a simplified expression for $\sin(\tan^{-1}(4x))$ for $-\frac{1}{4} \le x \le \frac{1}{4}$.

Answer

$\frac{4x}{\sqrt{16x^2+1}}$

Key Concepts

- If the inside function is a trigonometric function, then the only possible combinations are $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ if $0 \le x \le \pi$ and $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. See Example $4.3.1$ and Example $4.3.2$.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See Example $4.3.3$.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See Example $4.3.4$.

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4.4: Applications

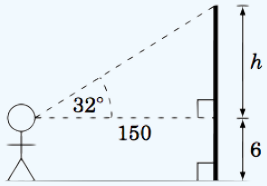
Learning Objectives

- Apply inverse trigonometric functions to real-life situations.

Throughout its early development, trigonometry was often used as a means of indirect measurement, e.g. determining large distances or lengths by using measurements of angles and small, known distances. Today, trigonometry is widely used in physics, astronomy, engineering, navigation, surveying, and various fields of mathematics and other disciplines. In this section we will see some of the ways in which trigonometry can be applied. Your calculator should be in degree mode for these examples.

Example 4.4.1

A person stands 150 ft away from a flagpole and measures an *angle of elevation* of 32° from his horizontal line of sight to the top of the flagpole. Assume that the person's eyes are a vertical distance of 6 ft from the ground. What is the height of the flagpole?



Solution:

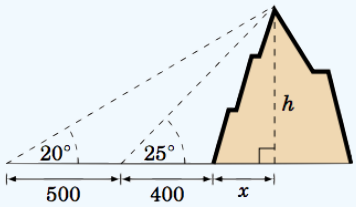
The picture on the right describes the situation. We see that the height of the flagpole is $h+6$ ft, where

$$\frac{h}{150} = \tan 32^\circ \quad \Rightarrow \quad h = 150 \tan 32^\circ = 150\,(0.6249) = 94\,. \tag{4.4.1}$$

How did we know that $\tan 32^\circ = 0.6249$? By using a calculator. And since none of the numbers we were given had decimal places, we rounded off the answer for h to the nearest integer. Thus, the height of the flagpole is $h+6 = 94+6 = \boxed{100 \text{ ft}}$.

Example 4.4.2

A person standing 400 ft from the base of a mountain measures the angle of elevation from the ground to the top of the mountain to be 25°. The person then walks 500 ft straight back and measures the angle of elevation to now be 20°. How tall is the mountain?



Solution:

We will assume that the ground is flat and not inclined relative to the base of the mountain. Let h be the height of the mountain, and let x be the distance from the base of the mountain to the point directly beneath the top of the mountain, as in the picture on the right. Then we see that

$$\frac{h}{x+400} = \tan 25^\circ \quad \Rightarrow \quad h = (x+400) \tan 25^\circ\,, \text{ and} \tag{4.4.2}$$

$$\frac{h}{x+400+500} = \tan 20^\circ \quad \Rightarrow \quad h = (x+900) \tan 20^\circ\,, \text{ so} \tag{4.4.3}$$

$(x+400) \tan 25^\circ = (x+900) \tan 20^\circ$, since they both equal h. Use that equation to solve for x:

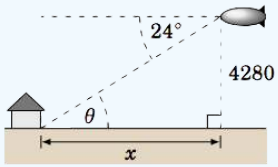
$$x \tan 25^\circ - x \tan 20^\circ = 900 \tan 20^\circ - 400 \tan 25^\circ \quad \Rightarrow \quad x = \frac{900 \tan 20^\circ - 400 \tan 25^\circ}{\tan 25^\circ - \tan 20^\circ} = 1378 \text{ ft} \tag{4.4.4}$$

Finally, substitute x into the first formula for h to get the height of the mountain:

$$h = (1378+400)\tan 25^\circ = 1778\,(0.4663) = \boxed{829 \text{ ft}} \tag{4.4.5}$$

Example 4.4.3

A blimp 4280 ft above the ground measures an *angle of depression* of 24° from its horizontal line of sight to the base of a house on the ground. Assuming the ground is flat, how far away along the ground is the house from the blimp?



Solution:

Let x be the distance along the ground from the blimp to the house, as in the picture to the right. Since the ground and the blimp's horizontal line of sight are parallel, we know from elementary geometry that the angle of elevation θ from the base of the house to the blimp is equal to the angle of depression from the blimp to the base of the house, i.e. $\theta = 24^\circ$. Hence,

$$\frac{4280}{x} = \tan 24^\circ \quad \Rightarrow \quad x = \frac{4280}{\tan 24^\circ} = \boxed{9613 \text{ ft}}. \tag{4.4.6}$$

Example 4.4.4

An observer at the top of a mountain 3 miles above sea level measures an angle of depression of 2.23° to the ocean horizon. Use this to estimate the radius of the earth.

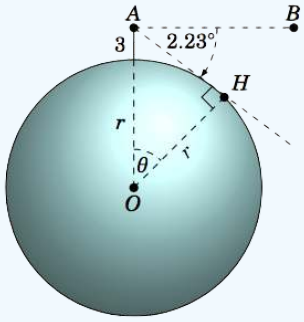


Figure 4.4.1

Solution:

We will assume that the earth is a sphere. Let r be the radius of the earth. Let the point A represent the top of the mountain, and let H be the ocean horizon in the line of sight from A, as in Figure 1.3.1. Let O be the center of the earth, and let B be a point on the horizontal line of sight from A (i.e. on the line perpendicular to $\overline{OA}$). Let θ be the angle $\angle AOH$.

Since A is 3 miles above sea level, we have $OA = r+3$. Also, $OH = r$. Now since $\overline{AB} \perp \overline{OA}$, we have $\angle OAB = 90^\circ$, so we see that $\angle OAH = 90^\circ - 2.23^\circ = 87.77^\circ$. We see that the line through A and H is a tangent line to the surface of the earth (considering the surface as the circle of radius r through H as in the picture). So by Exercise 14 in Section 1.1, $\overline{AH} \perp \overline{OH}$ and hence $\angle OHA = 90^\circ$. Since the angles in the triangle $\triangle OAH$ add up to 180°, we have $\theta = 180^\circ - 90^\circ - 87.77^\circ = 2.23^\circ$. Thus,

$$\cos\theta = \frac{OH}{OA} = \frac{r}{r+3} \quad \Rightarrow \quad \frac{r}{r+3} = \cos 2.23^\circ, \tag{4.4.7}$$

so solving for r we get

$$r = (r+3)\cos 2.23^\circ \quad \Rightarrow \quad r - r\cos 2.23^\circ = 3\cos 2.23^\circ \tag{4.4.8}$$

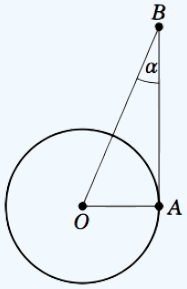
$$\Rightarrow \quad r = \frac{3\cos 2.23^\circ}{1-\cos 2.23^\circ} \tag{4.4.9}$$

$$\Rightarrow \quad \boxed{r = 3958.3 \text{ miles}}. \tag{4.4.10}$$

Note: This answer is very close to the earth's actual (mean) radius of 3956.6miles.

Example 4.4.5

As another application of trigonometry to astronomy, we will find the distance from the earth to the sun. Let O be the center of the earth, let A be a point on the equator, and let B represent an object (e.g. a star) in space, as in the picture on the right. If the earth is positioned in such a way that the angle $\angle OAB = 90^\circ$, then we say that the angle $\alpha = \angle OBA$ is the *equatorial parallax* of the object. The equatorial parallax of the sun has been observed to be approximately $\alpha = 0.00244^\circ$. Use this to estimate the distance from the center of the earth to the sun.



Solution:

Let B be the position of the sun. We want to find the length of $\overline{OB}$. We will use the actual radius of the earth, mentioned at the end of Example 1.14, to get $OA = 3956.6$ miles. Since $\angle OAB = 90^\circ$, we have

$$\frac{OA}{OB} = \sin\,\alpha \quad \Rightarrow \quad OB = \frac{OA}{\sin\,\alpha} = \frac{3956.6}{\sin\,0.00244^\circ} = 92908394\,, \tag{4.4.11}$$

so the distance from the center of the earth to the sun is approximately $\boxed{93 \text{ million miles}}$.

Note: The earth's orbit around the sun is an ellipse, so the actual distance to the sun varies.

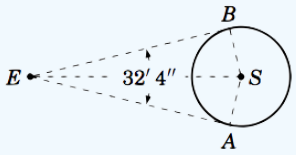
In the above example we used a very small angle (0.00244°). A degree can be divided into smaller units: a **minute** is one-sixtieth of a degree, and a **second** is one-sixtieth of a minute. The symbol for a minute is $'$ and the symbol for a second is $''$. For example, $4.5^\circ = 4^\circ\ 30'$. And $4.505^\circ = 4^\circ\ 30'\ 18''$:

$$4^\circ\ 30'\ 18'' = 4 + \frac{30}{60} + \frac{18}{3600}\ \text{degrees} = 4.505^\circ \tag{4.4.12}$$

In Example 1.15 we used $\alpha = 0.00244^\circ \approx 8.8''$, which we mention only because some angle measurement devices do use minutes and seconds.

Example 4.4.6

An observer on earth measures an angle of $32'\ 4''$ from one visible edge of the sun to the other (opposite) edge, as in the picture on the right. Use this to estimate the radius of the sun.



Solution:

Let the point E be the earth and let S be the center of the sun. The observer's lines of sight to the visible edges of the sun are tangent lines to the sun's surface at the points A and B. Thus, $\angle EAS = \angle EBS = 90^\circ$. The radius of the sun equals AS. Clearly $AS = BS$. So since $EB = EA$ (why?), the triangles $\triangle EAS$ and $\triangle EBS$ are similar. Thus, $\angle AES = \angle BES = \frac{1}{2}\ \angle AEB = \frac{1}{2}\ (32'\ 4'') = 16'\ 2'' = (16/60) + (2/3600) = 0.26722^\circ$.

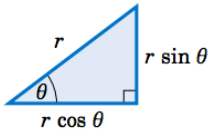
Now, ES is the distance from the *surface* of the earth (where the observer stands) to the center of the sun. In Example 1.15 we found the distance from the *center* of the earth to the sun to be $92,908,394$miles. Since we treated the sun in that example as a point, then we are justified in treating that distance as the distance between the centers of the earth and sun. So $ES = 92908394 -$ radius of earth $= 92908394 - 3956.6 = 92904437.4$miles. Hence,

$$\sin(\angle AES) = \frac{AS}{ES} \quad \Rightarrow \quad AS = ES \sin 0.26722^\circ = (92904437.4) \sin 0.26722^\circ = \boxed{433,293 \text{ miles}}. \tag{4.4.13}$$

Note: This answer is close to the sun's actual (mean) radius of $432,200$ miles.

You may have noticed that the solutions to the examples we have shown required at least one right triangle. In applied problems it is not always obvious which right triangle to use, which is why these sorts of problems can be difficult. Often no right triangle will be immediately evident, so you will have to create one. There is no general strategy for this, but remember that a right triangle requires a right angle, so look for places where you can form perpendicular line segments. When the problem contains a circle, you can create right angles by using the perpendicularity of the tangent line to the circle at a point with the line that joins that point to the center of the circle.

For some problems, it may help to remember that when a right triangle has a hypotenuse of length r and an acute angle θ, as in the picture below, the adjacent side will have length $r \cos\theta$ and the opposite side will have length $r \sin\theta$. You can think of those lengths as the horizontal and vertical ``components'' of the hypotenuse.



Notice in the above right triangle that we were given two pieces of information: one of the acute angles and the length of the hypotenuse. From this we determined the lengths of the other two sides, and the other acute angle is just the complement of the known acute angle. In general, a triangle has six parts: three sides and three angles. **Solving a triangle** means finding the unknown parts based on the known parts. In the case of a right triangle, one part is always known: one of the angles is 90°.

Example 4.4.7

Solve the right triangle in Figure 4.4.2 using the given information:

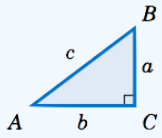


Figure 4.4.2

(a) $c = 10,\ A = 22^\circ$
Solution: The unknown parts are a, b, and B. Solving yields:

$$a = c \sin A = 10 \sin 22^\circ = 3.75 \tag{4.4.14}$$
$$b = c \cos A = 10 \cos 22^\circ = 9.27 \tag{4.4.15}$$
$$B = 90^\circ - A = 90^\circ - 22^\circ = 68^\circ \tag{4.4.16}$$

(b) $b = 8,\ A = 40^\circ$
Solution: The unknown parts are a, c, and B. Solving yields:

$$\frac{a}{b} = \tan A \quad \Rightarrow \quad a = b \tan A = 8 \tan 40^\circ = 6.71 \tag{4.4.17}$$

$$\frac{b}{c} = \cos A \quad \Rightarrow \quad c = \frac{b}{\cos A} = \frac{8}{\cos 40^\circ} = 10.44 \tag{4.4.18}$$

(c) $a = 3,\ b = 4$
Solution: The unknown parts are c, A, and B. By the Pythagorean Theorem,

$$c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5\,. \tag{4.4.19}$$

Now, $\tan A = \frac{a}{b} = \frac{3}{4} = 0.75$. So how do we find A? There should be a key labeled $\boxed{\tan^{-1}}$ on your calculator, which works like this: give it a number x and it will tell you the angle θ such that $\tan\theta = x$. In our case, we want the angle A such that $\tan A = 0.75$:

$$\text{Enter: } 0.75 \quad \text{Press: } \boxed{\tan^{-1}} \quad \text{Answer: } 36.86989765 \tag{4.4.20}$$

This tells us that $A = 36.87^\circ$, approximately. Thus $B = 90^\circ - A = 90^\circ - 36.87^\circ = 53.13^\circ$.

Note: The $\boxed{\sin^{-1}}$ and $\boxed{\cos^{-1}}$ keys work similarly for sine and cosine, respectively. These keys use the *inverse trigonometric functions*, which we will discuss in Chapter 5.

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CHAPTER OVERVIEW

Chapter 5: Triangles and Vectors

5.1: Non-right Triangles - Law of Cosines
5.2: Area of a Triangle
5.3: Non-right Triangles - Law of Sines
5.4: Vectors

Thumbnail: right triangle (CC BY-NC-SA; Ted Sundstrom & Steven Schlicker via LibreTexts).

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5.1: Non-right Triangles - Law of Cosines

Learning Objectives

- Understand how the Law of Cosines is derived.
- Apply the Law of Cosines when you know two sides and the inclusion of an oblique (non-right) triangle (SAS).
- Apply the Law of Cosines when you know all three sides of an oblique triangle.
- Identify accurate drawings of oblique triangles.
- Use the Law of Cosines in real-world and applied problems.

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in Figure 5.1.1 How far from port is the boat?

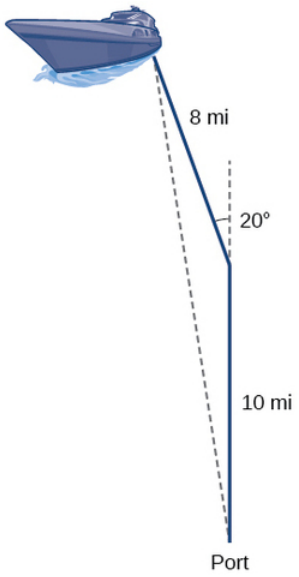


Figure 5.1.1

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the *Law of Cosines*, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the x-axis, and vertex C located at some point (x, y) in the plane, as illustrated in Figure 5.1.2. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

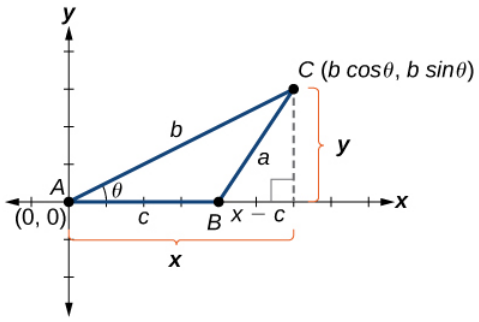


Figure 5.1.2

We can drop a perpendicular from C to the x-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x(adjacent)}{b(hypotenuse)} \text{ and } \sin\theta = \frac{y(opposite)}{b(hypotenuse)}$$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta, b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned} a^2 &= (x-c)^2 + y^2 \\ &= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y \\ &= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta && \text{Expand the perfect square.} \\ &= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1 \\ &= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{Factor out } b^2 \end{aligned}$$

$$a^2 = b^2 + c^2 - 2bc\cos\theta$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

The LAW OF COSINES

The Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle.

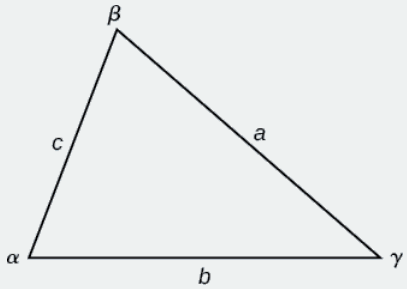


Figure 5.1.3

For triangles labeled as in Figure 5.1.3, with angles α, β and γ, and opposite corresponding sides a, b, and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \tag{5.1.1}$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \tag{5.1.2}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{5.1.3}$$

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \tag{5.1.4}$$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \tag{5.1.5}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \tag{5.1.6}$$

How to: Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Example 5.1.1: Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in Figure 5.1.4.

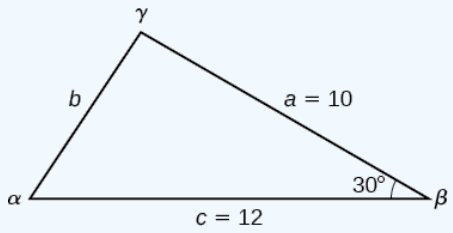


Figure 5.1.4

Solution

First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$b^2 = 10^2 + 12^2 - 2(10)(12)\cos(30^\circ) \quad \text{Substitute the measurements for the known quantities.}$$

$$b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right) \quad \text{Evaluate the cosine and begin to simplify.}$$

$$b^2 = 244 - 120\sqrt{3}$$

$$b = \sqrt{244 - 120\sqrt{3}} \quad \text{Use the square root property.}$$

$$b \approx 6.013$$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle α, we have

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{10}=\frac{\sin(30^\circ)}{6.013}$$

$$\sin\alpha=\frac{10\sin(30^\circ)}{6.013}\qquad \text{Multiply both sides of the equation by } 10.$$

$$\alpha=\sin^{-1}\left(\frac{10\sin(30^\circ)}{6.013}\right)\qquad \text{Find the inverse sine of } \frac{10\sin(30^\circ)}{6.013}.$$

$$\alpha\approx 56.3^\circ$$

The other possibility for α would be $\alpha=180^\circ-56.3^\circ\approx 123.7^\circ$. In the original diagram,α is adjacent to the longest side, so α is an acute angle and, therefore, 123.7° does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between 0° and 180°. Proceeding with $\alpha\approx 56.3^\circ$, we can then find the third angle of the triangle.

$$\begin{aligned}\gamma &=180^\circ-30^\circ-56.3^\circ\\ &\approx 93.7^\circ\end{aligned}$$

The complete set of angles and sides is

$$\alpha\approx 56.3^\circ \; a=10$$

$$\beta=30^\circ \; b\approx 6.013$$

$$\gamma\approx 93.7^\circ \; c=12$$

Exercise 5.1.1

Find the missing side and angles of the given triangle: $\alpha=30^\circ$, $b=12$, $c=24$.

Answer

$a\approx 14.9$, $\beta\approx 23.8^\circ$, $\gamma\approx 126.2^\circ$.

Example 5.1.2: Solving for an Angle of a SSS Triangle

Find the angle α for the given triangle if side $a=20$, side $b=25$, and side $c=18$.

Solution

For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$a^2=b^2+c^2-2bc\cos\alpha$$

$$20^2=25^2+18^2-2(25)(18)\cos\alpha\qquad \text{Substitute the appropriate measurements.}$$

$$400=625+324-900\cos\alpha\qquad \text{Simplify in each step.}$$

$$400=949-900\cos\alpha$$

$$-549=-900\cos\alpha\qquad \text{Isolate } \cos\alpha.$$

$$-549-900=\cos\alpha$$

$$0.61\approx\cos\alpha$$

$$0.61\approx\cos\alpha\qquad \text{Find the inverse cosine.}$$

$$\alpha\approx 52.4^\circ$$

See Figure 5.1.5.

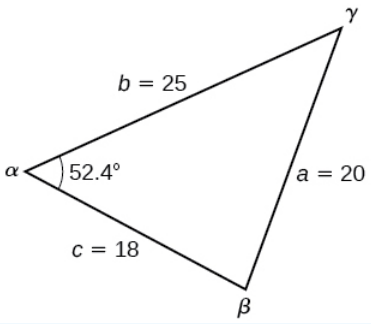


Figure 5.1.5

Analysis

Because the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases using this method.

Exercise 5.1.2

Given $a = 5$, $b = 7$, and $c = 10$, find the missing angles.

Answer

$\alpha \approx 27.7^\circ, \beta \approx 40.5^\circ, \gamma \approx 111.8^\circ$

Solving Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

Example 5.1.3A: Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Solution

For simplicity, we start by drawing a diagram similar to Figure $5.1.6$ and labeling our given information.

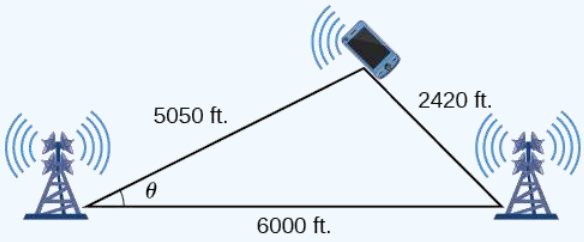


Figure 5.1.6

Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a = 2420$, $b = 5050$, and $c = 6000$. Thus, θ corresponds to the opposite side $a = 2420$.

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos\theta \\
(2420)^2 &= (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta \\
\cos\theta &\approx 0.9183 \\
\cos\theta &\approx 0.9183 \\
\theta &\approx \cos^{-1}(0.9183) \\
\theta &\approx 23.3^\circ
\end{aligned}
$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in Figure 5.1.7. This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.

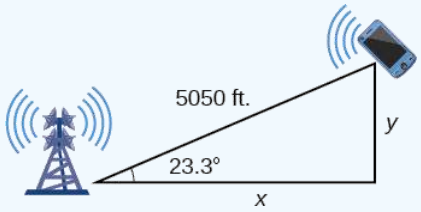


Figure 5.1.7

Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions. Thus

$$
\begin{aligned}
\cos(23.3^\circ) &= \frac{x}{5050} \\
x &= 5050\cos(23.3^\circ) \\
x &\approx 4638.15\ feet \\
\sin(23.3^\circ) &= \frac{y}{5050} \\
y &= 5050\sin(23.3^\circ) \\
y &\approx 1997.5\ feet
\end{aligned}
$$

The cell phone is approximately 4638 feet east and 1998 feet north of the first tower, and 1998 feet from the highway.

Example 5.1.3B: Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in Figure 5.1.8.

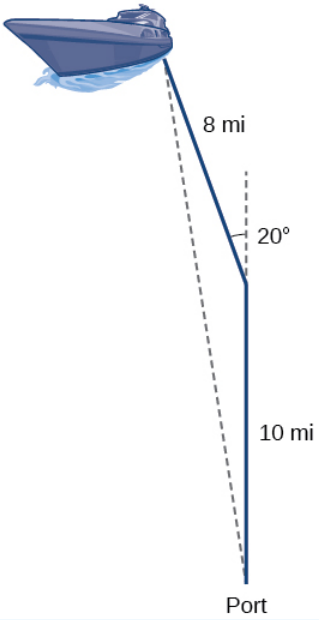


Figure 5.1.8

Solution

The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180^\circ - 20^\circ = 160^\circ$. With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle—the distance of the boat to the port.

$$\begin{aligned} x^2 &= 8^2 + 10^2 - 2(8)(10)\cos(160^\circ) \\ x^2 &= 314.35 \\ x &= \sqrt{314.35} \\ x &\approx 17.7\, miles \end{aligned}$$

The boat is about 17.7 miles from port.

Key Equations

Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$ $b^2 = a^2 + c^2 - 2ac\cos\beta$ $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Key Concepts

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See Example $5.1.1$ and Example $5.1.2$.
- The Law of Cosines is useful for many types of applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law of Cosines to find a solution. See Example $5.1.3$ and Example $5.1.4$.

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5.2: Area of a Triangle

Learning Objectives

- Apply the area formula to triangles where you know two sides and the included angle.
- Apply the area formula to triangles where you know all three sides, Heron's Formula.
- Use the area formulas in real-world and applied problems.

Area of a Triangle

We will now develop a few different ways to calculate the area of a triangle. Perhaps the most familiar formula for the area is the following:

The triangles in Figure $5.2.1$ illustrate the use of the variables in this formula.

The area A of a triangle is

$$A = \frac{1}{2}bh. \tag{5.2.1}$$

where b is the length of the base of a triangle and h is the length of the altitude that is perpendicular to that base.

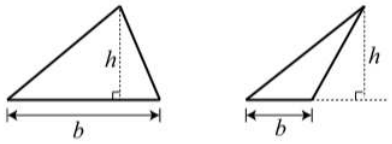
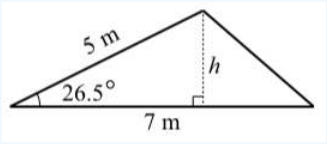


Figure $5.2.1$: Diagrams for the Formula for the Area of a Triangle

Exercise $5.2.1$



Suppose that the length of two sides of a triangle are 5 meters and 7 meters and that the angle formed by these two sides is 26.5°. See the diagram on the right.

For this problem, we are using the side of length 7 meters as the base. The altitude of length h that is perpendicular to this side is shown.

1. Use right triangle trigonometry to determine the value of h.
2. Determine the area of this triangle.

Answer

Using the right triangle, we see that $\sin(26.5^\circ) = \frac{h}{5}$. So $h = 5\sin(26.5^\circ)$, and the area of the triangle is

$$A = \frac{1}{2}(7)[5\sin(26.5^\circ)] = \frac{35}{2}\sin(26.5^\circ) \approx 7.8085 \tag{5.2.2}$$

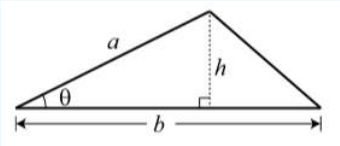
The area of the triangle is approximately 7.8085square meters.

If we know the length of two sides of a triangle and the angle formed by these two sides, then we can determine the area of that triangle.

The Area of a Triangle

The area of a triangle equals one-half the product of two of its sides times the sine of the angle formed by these two sides.

Exercise 5.2.2



In the diagram on the right, b is the length of the base of a triangle, a is the length of another side, and θ is the angle formed by these two sides. We let A be the area of the triangle.

Prove that

$$A = \frac{1}{2}ab\sin(\theta) \tag{5.2.3}$$

Explain why this proves the formula for the area of a triangle.

Answer

Using the right triangle, we see that $\sin(\theta) = \frac{h}{a}$. So $h = a\sin(\theta)$, and the area of the triangle is

$$A = \frac{1}{2}b(a\sin(\theta)) = \frac{1}{2}ab\sin(\theta) \tag{5.2.4}$$

Example 5.2.1: Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90$, $b = 52$, and angle $\gamma = 102^\circ$. Round the area to the nearest integer.

Solution

Using the formula, we have

$$Area = \frac{1}{2}ab\sin\gamma$$
$$Area = \frac{1}{2}(90)(52)\sin(102^\circ)$$
$$Area \approx 2289 \text{ square units}$$

Exercise 5.2.3

Find the area of the triangle given $\beta = 42^\circ$, $a = 7.2\ ft$, $c = 3.4\ ft$. Round the area to the nearest tenth.

Answer

about 8.2 square feet

There is another common formula for the area of a triangle known as Heron's Formula named after Heron of Alexandria (circa 75 CE). This formula shows that the area of a triangle can be computed if the lengths of the three sides of the triangle are known.

Heron's Formula

The area A of a triangle with sides of length a, b, and c is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{5.2.5}$$

where $s = \frac{1}{2}(a+b+c)$.

For example, suppose that the lengths of the three sides of a triangle are $a = 3ft$, $b = 5ft$, and $c = 6ft$. Using Heron's Formula (Equation 5.2.5), we get

$$s = \frac{1}{2}(a+b+c) \tag{5.2.6}$$

$$s = 7 \tag{5.2.7}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{5.2.8}$$

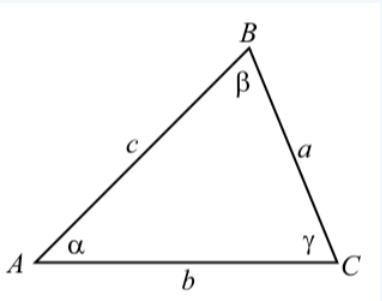
$$A = \sqrt{7(7-3)(7-5)(7-6)} \tag{5.2.9}$$

$$A = \sqrt{42} \tag{5.2.10}$$

This fairly complex formula is actually derived from the previous formula for the area of a triangle and the Law of Cosines. We begin our exploration of the proof of this formula in Progress Check

Exercise 5.2.4

Suppose we have a triangle as shown in the diagram below.



1. Use the Law of Cosines that involves the angle γ and solve this formula for $\cos(\gamma)$. This gives a formula for $\cos(\gamma)$ in terms of a, b, and c.
2. Use the Pythagorean Identity $\cos^2(\gamma) + \sin^2(\gamma) = 1$ 1 to write $\sin(\gamma)$ in terms of $\cos^2(\gamma)$. Substitute for $\cos^2(\gamma)$ using the formula in (1). This gives a formula for $\sin(\gamma)$ in terms of a, b, and c. (Do not do any algebraic simplification.)
3. We also know that a formula for the area of this triangle is $A = \frac{1}{2}ab\sin(\gamma)$

Substitute for $\sin(\gamma)$ using the formula in (2). (Do not do any algebraic simplification.) This gives a formula for the area A in terms of a, b, and c.

The formula obtained in Progress Check 3.23 was

$$A = \frac{1}{2}ab\sqrt{1-(\frac{a^2+b^2-c^2}{2ab})^2} \tag{5.2.11}$$

This is a formula for the area of a triangle in terms of the lengths of the three sides of the triangle. It does not look like Heron's Formula, but we can use some substantial algebra to rewrite this formula to obtain Heron's Formula. This algebraic work is completed in the appendix for this section.

Answer

1. Using the Law of Cosines, we see that

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma) \tag{5.2.12}$$

$$2ab\cos(\gamma) = a^2 + b^2 - c^2 \tag{5.2.13}$$

$$\cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab} \tag{5.2.14}$$

2. We see that

$$\sin^2(\gamma) = 1 - \cos^2(\gamma). \tag{5.2.15}$$

Since γ is between 0° and 180°, we know that $\sin(\gamma) > 0$ and so

$$\sin(\gamma) = \sqrt{1 - (\frac{a^2 + b^2 - c^2}{2ab})^2} \tag{5.2.16}$$

3. Substituting the equation in part (2) into the formula $A = \frac{1}{2}ab\sin(\gamma)$, we obtain

$$A = \frac{1}{2}ab\sin(\gamma) = \frac{1}{2}ab\sqrt{1 - (\frac{a^2 + b^2 - c^2}{2ab})^2} \tag{5.2.17}$$

Appendix – Proof of Heron's Formula

The formula for the area of a triangle obtained in Progress Check 3.23 was

$$A = \frac{1}{2}ab\sqrt{1 - (\frac{a^2 + b^2 - c^2}{2ab})^2} \tag{5.2.18}$$

We now complete the algebra to show that this is equivalent to Heron's formula. The first step is to rewrite the part under the square root sign as a single fraction.

$$A = \frac{1}{2}ab\sqrt{1 - (\frac{a^2 + b^2 - c^2}{2ab})^2} \tag{5.2.19}$$

$$= \frac{1}{2}ab\sqrt{\frac{(2ab)^2 - (a^2 + b^2 - c^2)^2}{(2ab)^2}} \tag{5.2.20}$$

$$= \frac{1}{2}ab\frac{\sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2}}{2ab} \tag{5.2.21}$$

$$= \frac{\sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2}}{4} \tag{5.2.22}$$

Squaring both sides of the last equation, we obtain

$$A^2 = \frac{(2ab)^2 - (a^2 + b^2 - c^2)^2}{16} \tag{5.2.23}$$

The numerator on the right side of the last equation is a difference of squares. We will now use the difference of squares formula, $x^2 - y^2 = (x - y)(x + y)$ to factor the numerator.

$$A^2 = \frac{(2ab)^2 - (a^2 + b^2 - c^2)^2}{16} \tag{5.2.24}$$

$$= \frac{(2ab - (a^2 + b^2 - c^2))(2ab + (a^2 + b^2 - c^2))}{16} \tag{5.2.25}$$

$$= \frac{(-a^2 + 2ab - b^2 + c^2)(a^2 + 2ab + b^2 - c^2)}{16} \tag{5.2.26}$$

We now notice that $-a^2 + 2ab - b^2 = -(a - b)^2$ and $a^2 + 2ab + b^2 = (a + b)^2$. So using these in the last equation, we have

$$A^2 = \frac{(-(a - b)^2 + c^2)((a + b)^2 - c^2)}{16} \tag{5.2.27}$$

$$= \frac{(-[(a-b)^2 - c^2])((a+b)^2 - c^2)}{16} \tag{5.2.28}$$

We can once again use the difference of squares formula as follows:

$$(a-b)^2 - c^2 = (a-b-c)(a-b+c) \tag{5.2.29}$$

$$(a+b)^2 - c^2 = (a+b-c)(a+b+c) \tag{5.2.30}$$

Substituting this information into the last equation for A^2, we obtain

$$A^2 = \frac{-(a-b-c)(a-b+c)(a+b-c)(a+b+c)}{16} \tag{5.2.31}$$

Since $s = \frac{1}{2}(a+b+c)$, $2s = a+b+c$ Now notice that

$$-(a-b-c) = -a+b+c = a+b+c-2a = 2s-2a \tag{5.2.32}$$

$$a-b+c = a+b+c-2b = 2s-2b \tag{5.2.33}$$

$$a+b+c = a+b+c-2c = 2s-2c \tag{5.2.34}$$

$$a+b+c = 2s \tag{5.2.35}$$

so

$$A^2 = \frac{-(a-b-c)(a-b+c)(a+b-c)(a+b+c)}{16} \tag{5.2.36}$$

$$= \frac{(2s-2a)(2s-2b)(2s-2c)(2s)}{16} \tag{5.2.37}$$

$$= \frac{16s(s-a)(s-b)(s-c)}{16} \tag{5.2.38}$$

$$= s(s-a)(s-b)(s-c) \tag{5.2.39}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{5.2.40}$$

This completes the proof of Heron's formula.

Summary

In this section, we studied the following important concepts and ideas:

Three ways to determine the area A of a triangle.

$A = \frac{1}{2}bh$, where b is the length of the base and h is the length of the altitude.

$A = \frac{1}{2}ab\sin(\theta)$, where a and b are the lengths of two sides of the triangle and θ is the angle formed by the sides of length a and b.

Heron's Formula. If a, b, and c are the lengths of the sides of a triangle and $s = \frac{1}{2}(a+b+c)$, then

$$A = \sqrt{s(s-a)(s-b)(s-c)} \ . \tag{5.2.41}$$

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5.3: Non-right Triangles - Law of Sines

Learning Objectives

- Understand how both forms of the Law of Sines are obtained.
- Apply the Law of Sines when you know two angles and a non-included side and if you know two angles and the included side.
- Use the Law of Sines in real-world and applied problems.

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the **angle of elevation** measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in Figure 5.3.1 that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving **non-right triangles**.

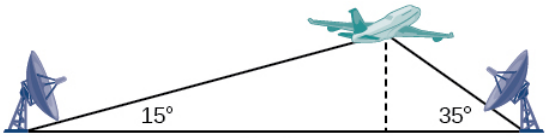


Figure 5.3.1

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

ASA (angle-side-angle) We know the measurements of two angles and the included side. See Figure 5.3.2.

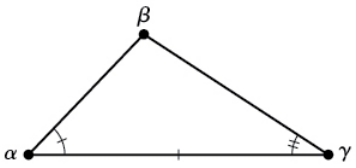


Figure 5.3.2

AAS (angle-angle-side) We know the measurements of two angles and a side that is not between the known angles. See Figure 5.3.3.

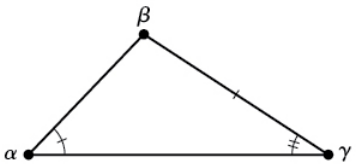


Figure 5.3.3

SSA (side-side-angle) We know the measurements of two sides and an angle that is not between the known sides. See Figure 5.3.4.

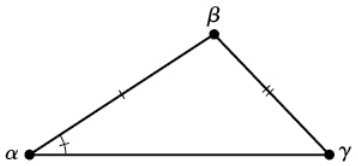


Figure 5.3.4

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in Figure 5.3.5.

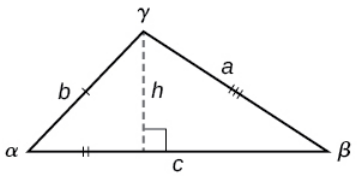


Figure 5.3.5

Using the right triangle relationships, we know that $\sin\alpha = \frac{h}{b}$ and $\sin\beta = \frac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin\alpha \text{ and } h = a\sin\beta$$

We then set the expressions equal to each other.

$$\begin{aligned} b\sin\alpha &= a\sin\beta \\ \left(\frac{1}{ab}\right)(b\sin\alpha) &= (a\sin\beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab} \\ \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \end{aligned}$$

Similarly, we can compare the other ratios.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \text{ and } \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Collectively, these relationships are called the **Law of Sines**.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See Figure 5.3.6.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

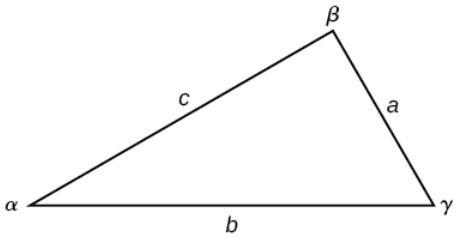


Figure 5.3.6

LAW OF SINES

Given a triangle with angles and opposite sides labeled as in Figure 5.3.6, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c} \tag{5.3.1}$$

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma} \tag{5.3.2}$$

To solve an oblique triangle, use any pair of applicable ratios.

Example 5.3.1: Solving for Two Unknown Sides and Angle of an AAS Triangle

Solve the triangle shown in Figure 5.3.7 to the nearest tenth.

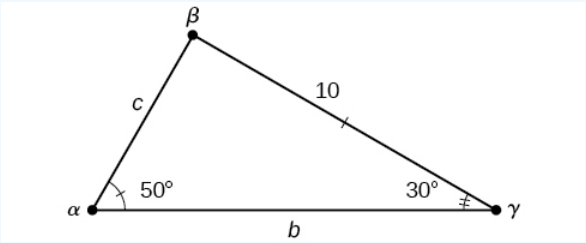


Figure 5.3.7

Solution

The three angles must add up to 180 degrees. From this, we can determine that

$$\begin{aligned}\beta &= 180^\circ - 50^\circ - 30^\circ\\ &= 100^\circ\end{aligned}$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50^\circ$ and its corresponding side $a = 10$. We can use the following proportion from the Law of Sines to find the length of c.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(30^\circ)}{c}\\ c\frac{\sin(50^\circ)}{10} &= \sin(30^\circ) && \text{Multiply both sides by } c\\ c &= \sin(30^\circ)\frac{10}{\sin(50^\circ)} && \text{Multiply by the reciprocal to isolate } c\\ c &\approx 6.5\end{aligned}$$

Similarly, to solve for b, we set up another proportion.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(100^\circ)}{b}\\ b\sin(50^\circ) &= 10\sin(100^\circ) && \text{Multiply both sides by } b\\ b &= \frac{10\sin(100^\circ)}{\sin(50^\circ)} && \text{Multiply by the reciprocal to isolate } b\\ b &\approx 12.9\end{aligned}$$

Therefore, the complete set of angles and sides is

$$\begin{aligned}\alpha &= 50^\circ & a &= 10\\ \beta &= 100^\circ & b &\approx 12.9\\ \gamma &= 30^\circ & c &\approx 6.5\end{aligned}$$

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Exercise 5.3.1

Solve the triangle shown in Figure 5.3.8 to the nearest tenth.

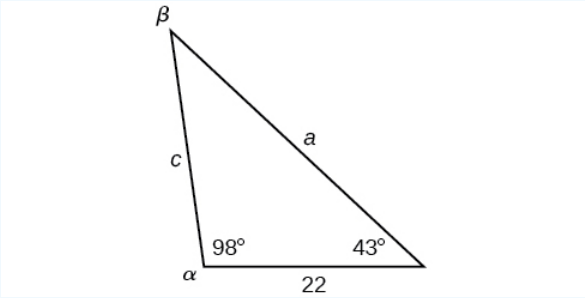


Figure 5.3.8

Answer

$\alpha = 98^\circ \quad a = 34.6$
$\beta = 39^\circ \quad b = 22$
$\gamma = 43^\circ \quad c = 23.8$

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an ambiguous case. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

POSSIBLE OUTCOMES FOR SSA TRIANGLES

Oblique triangles in the category SSA may have four different outcomes. Figure 5.3.9 illustrates the solutions with the known sides a and b and known angle α.

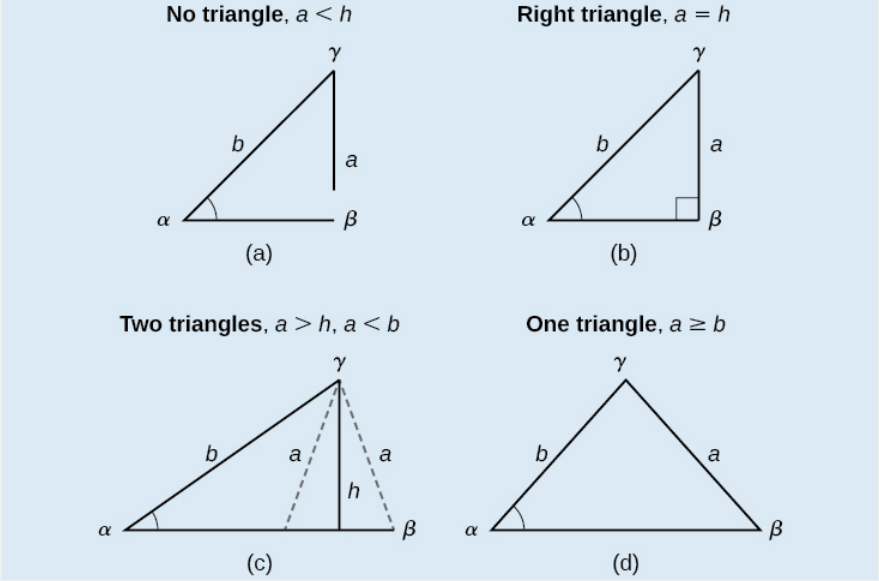


Figure 5.3.9

Solving an Oblique SSA Triangle

Solve the triangle in Figure 5.3.10 for the missing side and find the missing angle measures to the nearest tenth.

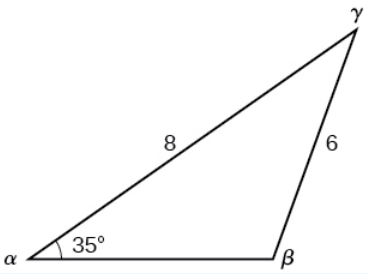


Figure 5.3.10

Solution

Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\begin{aligned} \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \\ \frac{\sin(35^\circ)}{6} &= \frac{\sin\beta}{8} \\ \frac{8\sin(35^\circ)}{6} &= \sin\beta \\ 0.7648 &\approx \sin\beta \\ \sin^{-1}(0.7648) &\approx 49.9^\circ \\ \beta &\approx 49.9^\circ \end{aligned}$$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than 90°. How did we get an acute angle, and how do we find the measurement of β? Let's investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have Figure 5.3.11. It appears that there may be a second triangle that will fit the given criteria.

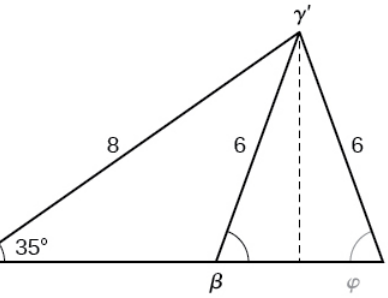


Figure 5.3.11

The angle supplementary to β is approximately equal to 49.9°, which means that $\beta = 180^\circ - 49.9^\circ = 130.1^\circ$. (Remember that the sine function is positive in both the first and second quadrants.) Solving for γ, we have

$$\begin{aligned} \gamma &= 180^\circ - 35^\circ - 130.1^\circ \\ &\approx 14.9^\circ \end{aligned}$$

We can then use these measurements to solve the other triangle. Since γ' is supplementary to γ, we have

$$\begin{aligned} \gamma' &= 180^\circ - 35^\circ - 49.5^\circ \\ &\approx 95.1^\circ \end{aligned}$$

Now we need to find c and c'.

We have

$$\frac{c}{\sin(14.9^\circ)} = \frac{6}{\sin(35^\circ)}$$
$$c = \frac{6\sin(14.9^\circ)}{\sin(35^\circ)}$$
$$\approx 2.7$$

Finally,

$$\frac{c'}{\sin(95.1^\circ)} = \frac{6}{\sin(35^\circ)}$$
$$c' = \frac{6\sin(95.1^\circ)}{\sin(35^\circ)}$$
$$\approx 10.4$$

To summarize, there are two triangles with an angle of 35°, an adjacent side of 8, and an opposite side of 6, as shown in Figure 5.3.12

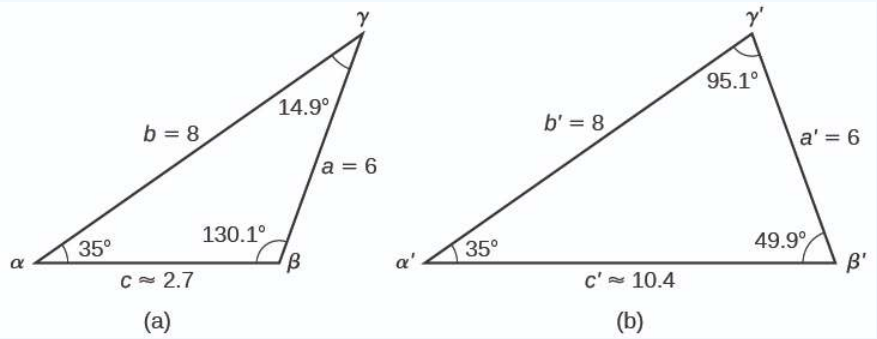


Figure 5.3.12

However, we were looking for the values for the triangle with an obtuse angle β. We can see them in the first triangle (a) in Figure 5.3.12

Exercise 5.3.2

Given $\alpha = 80^\circ$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.

Answer

Solution 1

$\alpha = 80^\circ$	$a = 120$
$\beta \approx 83.2^\circ$	$b = 121$
$\gamma \approx 16.8^\circ$	$c \approx 35.2$

Solution 2

$\alpha' = 80^\circ$	$a' = 120$
$\beta' \approx 96.8^\circ$	$b' = 121$
$\gamma' \approx 3.2^\circ$	$c' \approx 6.8$

Example 5.3.2: Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in Figure 5.3.13, solve for the unknown side and angles. Round your answers to the nearest tenth.

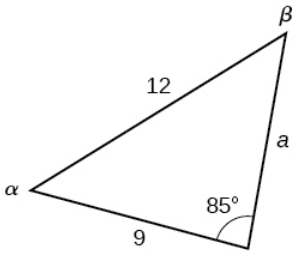


Figure 5.3.13

Solution

In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle, $\gamma = 85^\circ$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\frac{\sin(85^\circ)}{12} = \frac{\sin\beta}{9} \qquad \text{Isolate the unknown.}$$
$$\frac{9\sin(85^\circ)}{12} = \sin\beta$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85^\circ)}{12}\right)$$
$$\beta \approx \sin^{-1}(0.7471)$$
$$\beta \approx 48.3^\circ$$

In this case, if we subtract β from 180°, we find that there may be a second possible solution. Thus, $\beta = 180^\circ - 48.3^\circ \approx 131.7^\circ$. To check the solution, subtract both angles, 131.7° and 85°, from 180°. This gives

$$\alpha = 180^\circ - 85^\circ - 131.7^\circ$$
$$\approx -36.7^\circ$$

which is impossible, and so $\beta \approx 48.3^\circ$.

To find the remaining missing values, we calculate $\alpha = 180^\circ - 85^\circ - 48.3^\circ \approx 46.7^\circ$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85^\circ)}{12} = \frac{\sin(46.7^\circ)}{a}$$
$$a\frac{\sin(85^\circ)}{12} = \sin(46.7^\circ)$$
$$a = \frac{12\sin(46.7^\circ)}{\sin(85^\circ)}$$
$$\approx 8.8$$

The complete set of solutions for the given triangle is

$$\alpha \approx 46.7^\circ \quad a \approx 8.8$$
$$\beta \approx 48.3^\circ \quad b = 9$$
$$\gamma = 85^\circ \quad c = 12$$

Exercise 5.3.3

Given $\alpha = 80^\circ$, $a = 100$, $b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Answer

$\beta \approx 5.7^\circ, \gamma \approx 94.3^\circ, c \approx 101.3$

Example 5.3.3: Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Solution

Using the given information, we can solve for the angle opposite the side of length 10. See Figure 5.3.14

$$\frac{\sin\alpha}{10} = \frac{\sin(50^\circ)}{4}$$
$$\sin\alpha = \frac{10\sin(50^\circ)}{4}$$
$$\sin\alpha \approx 1.915$$

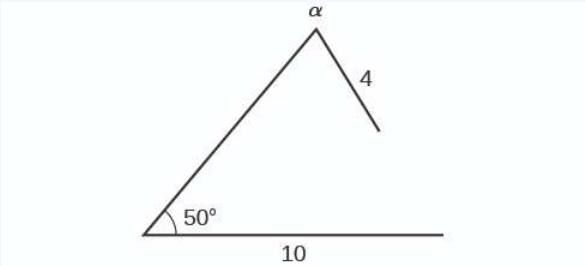


Figure 5.3.14

We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$in a graphing calculator generates an ERROR DOMAIN. Therefore, no triangles can be drawn with the provided dimensions.

Exercise 5.3.4

Determine the number of triangles possible given $a = 31, b = 26, \beta = 48^\circ$.

Answer

two

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Example 5.3.4: Finding an Altitude

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in Figure 5.3.15. Round the altitude to the nearest tenth of a mile.

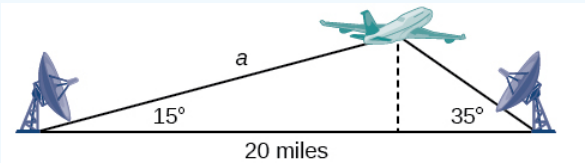


Figure 5.3.15

Solution

To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side a, and then use right triangle relationships to find the height of the aircraft, h.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180^\circ - 15^\circ - 35^\circ = 130^\circ$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\begin{aligned} \frac{\sin(130^\circ)}{20} &= \frac{\sin(35^\circ)}{a} \\ a\sin(130^\circ) &= 20\sin(35^\circ) \\ a &= \frac{20\sin(35^\circ)}{\sin(130^\circ)} \\ a &\approx 14.98 \end{aligned}$$

The distance from one station to the aircraft is about 14.98 miles.

Now that we know a, we can use right triangle relationships to solve for h.

$$\begin{aligned} \sin(15^\circ) &= \frac{opposite}{hypotenuse} \\ \sin(15^\circ) &= \frac{h}{a} \\ \sin(15^\circ) &= \frac{h}{14.98} \\ h &= 14.98\sin(15^\circ) \\ h &\approx 3.88 \end{aligned}$$

The aircraft is at an altitude of approximately 3.9 miles.

Exercise 5.3.5

The diagram shown in Figure 5.3.16 represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point A, is 70°, the angle of elevation from the northern end zone, point B, is 62°, and the distance between the viewing points of the two end zones is 145 yards.

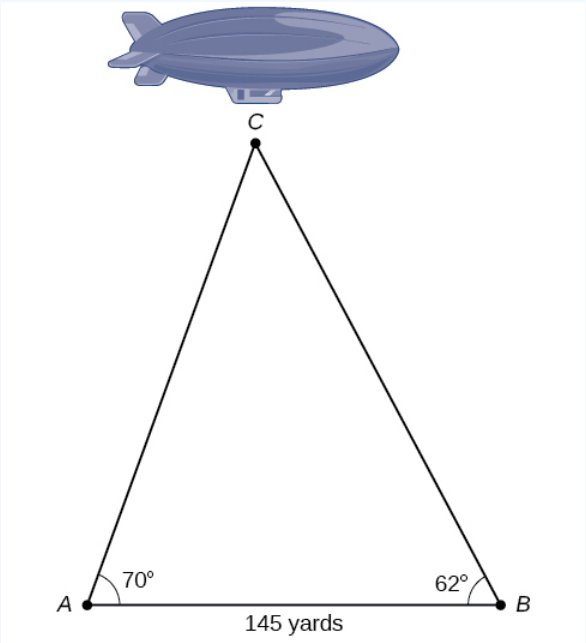


Figure 5.3.16

Answer

161.9yd.

Key Equations

Law of Sines	$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$ $\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$

Key Concepts

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.
- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See Example $5.3.1$.
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See Example $5.3.2$ and Example $5.3.3$.
- The Law of Sines can be used to solve triangles with given criteria. See Example $5.3.4$.
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See Example $5.3.6$.

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5.4: Vectors

Learning Objectives

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of v.
- Perform operations with vectors in terms of i and j.
- Find the dot product of two vectors.
- Find the angle between two vectors.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in Figure $5.4.1$. What are the ground speed and actual bearing of the plane?

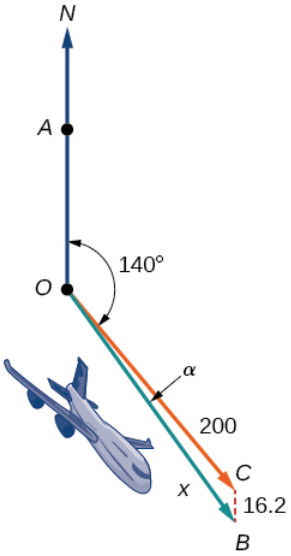


Figure 5.4.1

Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. In an earlier section, we used triangles to solve a similar problem involving the movement of boats. Later in this section, we will find the airplane's ground speed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

A Geometric View of Vectors

A vector is a specific quantity drawn as a line segment with an arrowhead at one end. It has an **initial point**, where it begins, and a **terminal point**, where it ends. A vector is defined by its magnitude, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- Lower case, boldfaced type, with or without an arrow on top such as $\boldsymbol{u}$, $\boldsymbol{w}$, $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{w}}$.
- Given initial point P and terminal point Q, a vector can be represented as $\overrightarrow{PQ}$. The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of $(0, 0)$ and terminal point (a, b), a vector may be represented as $\langle a, b \rangle$.

This last symbol $\langle a, b \rangle$ has special significance. It is called the **standard position**. The position vector has an initial point $(0, 0)$ and a terminal point $\langle a, b \rangle$. To change any vector into the position vector, we think about the change in the x-coordinates and the change in the y-coordinates. Thus, if the initial point of a vector $\overrightarrow{CD}$ is $C(x_1, y_1)$ and the terminal point is $D(x_2, y_2)$, then the position vector is found by calculating

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$$
$$= \langle a, b \rangle$$

In Figure 5.4.2, we see the original vector $\overrightarrow{CD}$ and the position vector $\overrightarrow{AB}$.

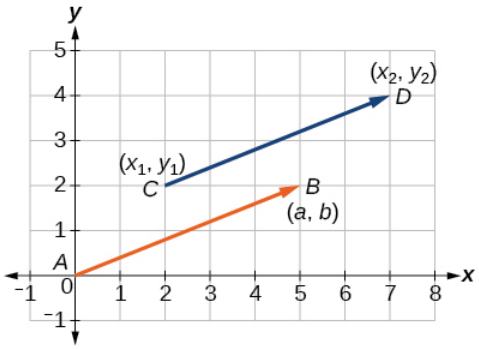


Figure 5.4.2

PROPERTIES OF VECTORS

A vector is a directed line segment with an initial point and a terminal point. Vectors are identified by magnitude, or the length of the line, and direction, represented by the arrowhead pointing toward the terminal point. The position vector has an initial point at $(0, 0)$ and is identified by its terminal point $\langle a, b \rangle$.

Example 5.4.1A: Find the Position Vector

Consider the vector whose initial point is $P(2, 3)$ and terminal point is $Q(6, 4)$. Find the position vector.

Solution

The position vector is found by subtracting one x-coordinate from the other x-coordinate, and one y-coordinate from the other y-coordinate. Thus

$$v = \langle 6 - 2, 4 - 3 \rangle$$
$$= \langle 4, 1 \rangle$$

The position vector begins at $(0, 0)$ and terminates at $(4, 1)$. The graphs of both vectors are shown in Figure 5.4.3.

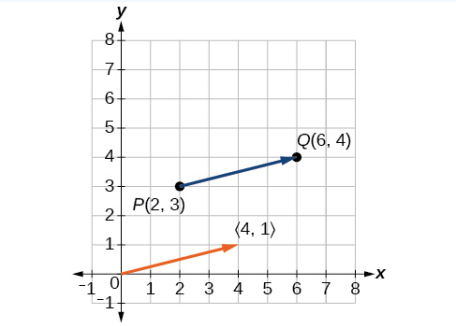


Figure 5.4.3

We see that the position vector is $\langle 4, 1 \rangle$.

Example 5.4.1B: Drawing a Vector with the Given Criteria and Its Equivalent Position Vector

Find the position vector given that vector v has an initial point at $(-3, 2)$ and a terminal point at $(4, 5)$, then graph both vectors in the same plane.

Solution

The position vector is found using the following calculation:

$$v = \langle 4-(-3), 5-2\rangle$$
$$= \langle 7, 3\rangle$$

Thus, the position vector begins at $(0, 0)$ and terminates at $(7, 3)$. See Figure 5.4.4.

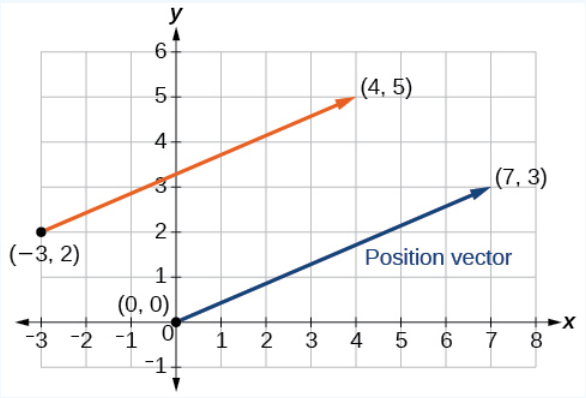


Figure 5.4.4

Exercise 5.4.1

Draw a vector $\vec{v}$ that connects from the origin to the point $(3, 5)$.

Answer

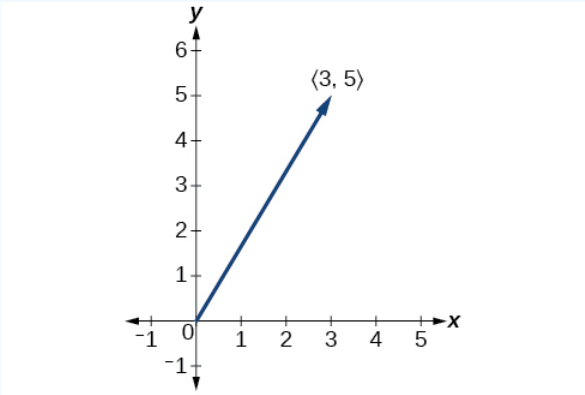


Figure 5.4.5

Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

MAGNITUDE AND DIRECTION OF A VECTOR

Given a position vector $\vec{v} = \langle a, b\rangle$,the magnitude is found by $|v| = \sqrt{a^2 + b^2}$ .The direction is equal to the angle formed with the x-axis, or with the y-axis, depending on the application. For a position vector, the direction is found by $\tan\theta = \left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in Figure 5.4.6.

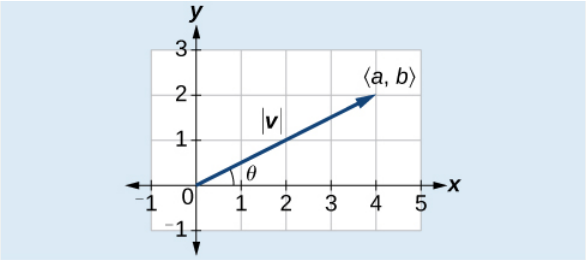


Figure 5.4.6

Two vectors $\vec{v}$ and $\vec{u}$ are considered equal if they have the same magnitude and the same direction. Additionally, if both vectors have the same position vector, they are equal.

Example $5.4.2A$: Finding the Magnitude and Direction of a Vector

Find the magnitude and direction of the vector with initial point $P(-8, 1)$ and terminal point $Q(-2, -5)$.Draw the vector.

Solution

First, find the position vector.

$$\begin{aligned} u &= \langle -2, -(-8), -5-1 \rangle \\ &= \langle 6, -6 \rangle \end{aligned}$$

We use the Pythagorean Theorem to find the magnitude.

$$\begin{aligned} |u| &= \sqrt{(6)^2 + (-6)^2} \\ &= \sqrt{72} \\ &= \sqrt{62} \end{aligned}$$

The direction is given as

$$\begin{aligned} \tan\theta &= \frac{-6}{6} = -1 \rightarrow \theta = \tan^{-1}(-1) \\ &= -45^\circ \end{aligned}$$

However, the angle terminates in the fourth quadrant, so we add 360° to obtain a positive angle. Thus, $-45^\circ + 360^\circ = 315^\circ$. See Figure $5.4.7$.

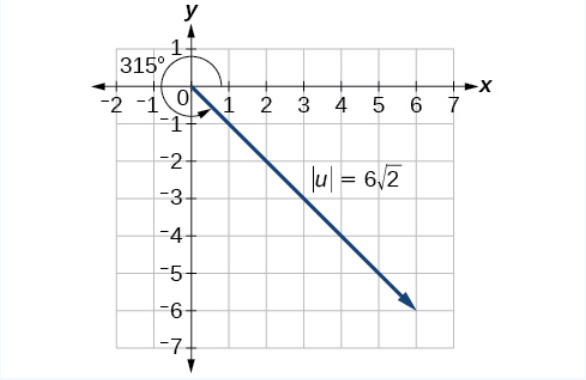


Figure 5.4.7

Example $5.4.2B$: Showing That Two Vectors Are Equal

Show that vector $\vec{v}$ with initial point at $(5, -3)$ and terminal point at $(-1, 2)$ is equal to vector $\vec{u}$ with initial point at $(-1, -3)$ and terminal point at $(-7, 2)$. Draw the position vector on the same grid as $\vec{v}$ and $\vec{u}$. Next, find the magnitude and direction of each vector.

Solution

As shown in Figure 5.4.8, draw the vector $\vec{v}$ starting at initial $(5,-3)$ and terminal point $(-1,2)$. Draw the vector $\vec{u}$ with initial point $(-1,-3)$ and terminal point $(-7,2)$. Find the standard position for each.

Next, find and sketch the position vector for $\vec{v}$ and $\vec{u}$. We have

$$\begin{aligned} v &= \langle -1-5, 2-(-3)\rangle \\ &= \langle -6,5\rangle u \\ &= \langle -7-(-1), 2-(-3)\rangle \\ &= \langle -6,5\rangle \end{aligned}$$

Since the position vectors are the same, $\vec{v}$ and $\vec{u}$ are the same.

An alternative way to check for vector equality is to show that the magnitude and direction are the same for both vectors. To show that the magnitudes are equal, use the Pythagorean Theorem.

$$\begin{aligned} |v| &= \sqrt{(-1-5)^2+(2-(-3))^2} \\ &= \sqrt{(-6)^2+(5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \\ |u| &= \sqrt{(-7-(-1))^2+(2-(-3))^2} \\ &= \sqrt{(-6)^2+(5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \end{aligned}$$

As the magnitudes are equal, we now need to verify the direction. Using the tangent function with the position vector gives

$$\begin{aligned} \tan\theta &= -\frac{5}{6} \Rightarrow \theta = \tan^{-1}\left(-\frac{5}{6}\right) \\ &= -39.8^\circ \end{aligned}$$

However, we can see that the position vector terminates in the second quadrant, so we add 180°. Thus, the direction is $-39.8^\circ + 180^\circ = 140.2^\circ$.

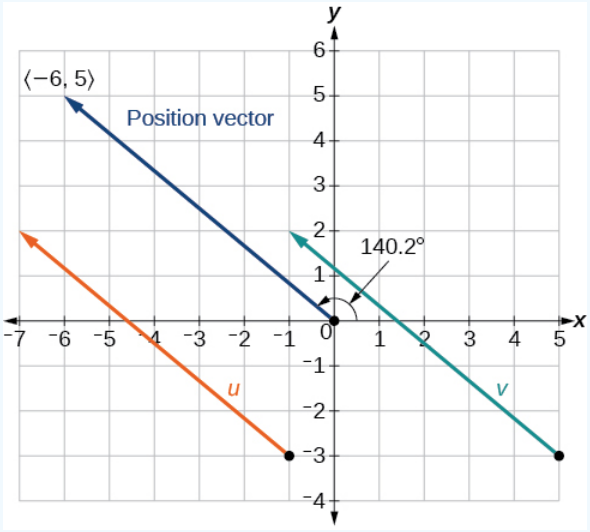


Figure 5.4.8

Performing Vector Addition and Scalar Multiplication

Now that we understand the properties of vectors, we can perform operations involving them. While it is convenient to think of the vector $u = \langle x, y \rangle$ as an arrow or directed line segment from the origin to the point (x, y), vectors can be situated anywhere in the plane. The sum of two vectors $\vec{u}$ and $\vec{v}$, or vector addition, produces a third vector $\overrightarrow{u+v}$, the resultant vector.

To find $\overrightarrow{u+v}$, we first draw the vector $\vec{u}$, and from the terminal end of $\vec{u}$, we drawn the vector $\vec{v}$. In other words, we have the initial point of $\vec{v}$ meet the terminal end of $\vec{u}$. This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector. The sum $\overrightarrow{u+v}$ is the resultant vector because it results from addition or subtraction of two vectors. The resultant vector travels directly from the beginning of $\vec{u}$ to the end of $\vec{v}$ in a straight path, as shown in Figure 5.4.9.

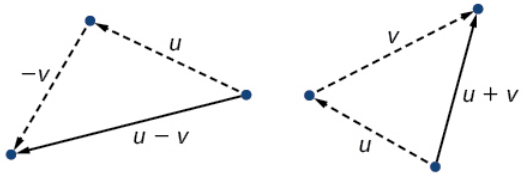


Figure 5.4.9

Vector subtraction is similar to vector addition. To find $\overrightarrow{u-v}$, view it as $\overrightarrow{u+(-v)}$. Adding $\overrightarrow{-v}$ is reversing direction of $\vec{v}$ and adding it to the end of $\vec{u}$. The new vector begins at the start of $\vec{u}$ and stops at the end point of $\overrightarrow{-v}$. See Figure 5.4.10 for a visual that compares vector addition and vector subtraction using parallelograms.

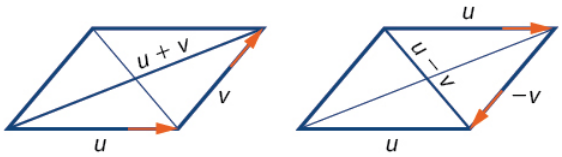


Figure 5.4.10

Example 5.4.3: Adding and Subtracting Vectors

Given $u = \langle 3, -2 \rangle$ and $v = \langle -1, 4 \rangle$, find two new vectors $\overrightarrow{u+v}$, and $\overrightarrow{u-v}$.

Solution

To find the sum of two vectors, we add the components. Thus,

$$\begin{aligned} u+v &= \langle 3, -2 \rangle + \langle -1, 4 \rangle \\ &= \langle 3+(-1), -2+4 \rangle \\ &= \langle 2, 2 \rangle \end{aligned}$$

See Figure 5.4.11a.

To find the difference of two vectors, add the negative components of $\vec{v}$ to $\vec{u}$. Thus,

$$\begin{aligned} u+(-v) &= \langle 3, -2 \rangle + \langle 1, -4 \rangle \\ &= \langle 3+1, -2+(-4) \rangle \\ &= \langle 4, -6 \rangle \end{aligned}$$

See Figure 5.4.11b

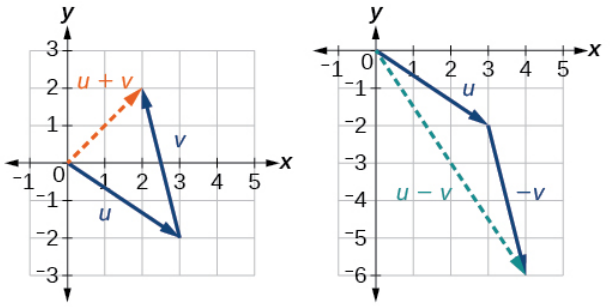


Figure 5.4.11: (a) Sum of two vectors (b) Difference of two vectors

Multiplying By a Scalar

While adding and subtracting vectors gives us a new vector with a different magnitude and direction, the process of multiplying a vector by a **scalar**, a constant, changes only the magnitude of the vector or the length of the line. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

SCALAR MULTIPLICATION

Scalar multiplication involves the product of a vector and a scalar. Each component of the vector is multiplied by the scalar. Thus, to multiply $v = \langle a, b \rangle$ by k, we have

$kv = \langle ka, kb \rangle$

Only the magnitude changes, unless k is negative, and then the vector reverses direction.

Example 5.4.4: Performing Scalar Multiplication

Given vector $\vec{v} = \langle 3, 1 \rangle$, find $3\vec{v}$, $\frac{1}{2}$, and $\overrightarrow{-v}$.

Solution

See Figure 5.4.12 for a geometric interpretation. If $\vec{v} = \langle 3, 1 \rangle$, then

$$\begin{aligned} 3v &= \langle 3 \cdot 3, 3 \cdot 1 \rangle \\ &= \langle 9, 3 \rangle \\ \frac{1}{2}v &= \langle \frac{1}{2} \cdot 3, \frac{1}{2} \cdot 1 \rangle \\ &= \langle \frac{3}{2}, \frac{1}{2} \rangle \\ -v &= \langle -3, -1 \rangle \end{aligned}$$

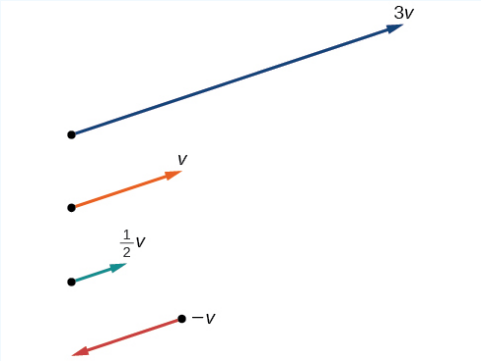


Figure 5.4.12

Analysis

Notice that the vector $3\vec{v}$ is three times the length of $\vec{v}$, $\frac{1}{2}\vec{v}$ is half the length of $\vec{v}$, and $\overrightarrow{-v}$ is the same length of $\vec{v}$, but in the opposite direction.

Exercise 5.4.2

Find the scalar multiple $3\vec{u}$ given $\vec{u} = \langle 5, 4 \rangle$.

Answer

$3u = \langle 15, 12 \rangle$

Example 5.4.5

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

Solution

First, we must multiply each vector by the scalar.

$$\begin{aligned} 3u &= 3\langle 3, -2 \rangle \\ &= \langle 9, -6 \rangle \\ 2v &= 2\langle -1, 4 \rangle \\ &= \langle -2, 8 \rangle \end{aligned}$$

Then, add the two together.

$$\begin{aligned} w &= 3u + 2v \\ &= \langle 9, -6 \rangle + \langle -2, 8 \rangle \\ &= \langle 9 - 2, -6 + 8 \rangle \\ &= \langle 7, 2 \rangle \end{aligned}$$

So, $w = \langle 7, 2 \rangle$.

Finding Component Form

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Vectors are comprised of two components: the horizontal component is the x direction, and the vertical component is the y direction. For example, we can see in the graph in Figure $5.4.13$ that the position vector $\langle 2, 3 \rangle$ comes from adding the vectors v_1 and v_2. We have v_2 with initial point $(0, 0)$ and terminal point $(2, 0)$.

$$\begin{aligned} v_1 &= \langle 2 - 0, 0 - 0 \rangle \\ &= \langle 2, 0 \rangle \end{aligned}$$

We also have v_2 with initial point $(0, 0)$ and terminal point $(0, 3)$.

$$\begin{aligned} v_2 &= \langle 0 - 0, 3 - 0 \rangle \\ &= \langle 0, 3 \rangle \end{aligned}$$

Therefore, the position vector is

$$\begin{aligned} v &= \langle 2 + 0, 3 + 0 \rangle \\ &= \langle 2, 3 \rangle \end{aligned}$$

Using the Pythagorean Theorem, the magnitude of v_1 is 2, and the magnitude of v_2 is 3. To find the magnitude of v, use the formula with the position vector.

$$\begin{aligned} |v| &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

The magnitude of v is $\sqrt{13}$. To find the direction, we use the tangent function $\tan\theta = \frac{y}{x}$.

$$\tan\theta = \frac{v_2}{v_1}$$

$$\tan\theta = \frac{3}{2}$$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.3^\circ$$

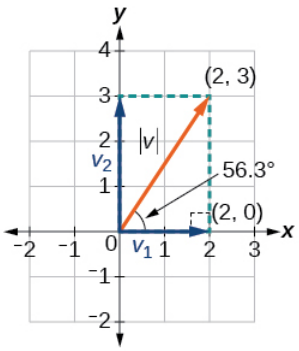


Figure 5.4.13

Thus, the magnitude of $\vec{v}$ is $\sqrt{13}$ and the direction is 56.3° off the horizontal.

Example 5.4.6: Finding the Components of the Vector

Find the components of the vector $\vec{v}$ with initial point $(3, 2)$ and terminal point $(7, 4)$.

Solution

First find the standard position.

$$\begin{aligned} v &= \langle 7-3, 4-2 \rangle \\ &= \langle 4, 2 \rangle \end{aligned}$$

See the illustration in Figure 5.4.14

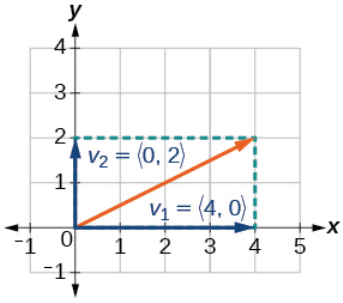


Figure 5.4.14

The horizontal component is $\overrightarrow{v_1} = \langle 4, 0 \rangle$ and the vertical component is $\overrightarrow{v_2} = \langle 0, 2 \rangle$.

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Finding the Unit Vector in the Direction of v

In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a unit vector. We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components. The horizontal unit vector is written as $\vec{i} = \langle 1, 0 \rangle$ and is directed along the positive horizontal axis. The vertical unit vector is written as $\vec{j} = \langle 0, 1 \rangle$ and is directed along the positive vertical axis. See Figure 5.4.15.

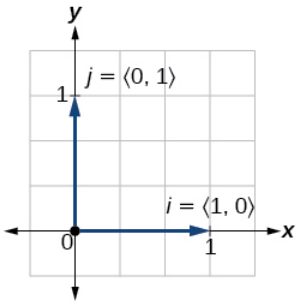


Figure 5.4.15

THE UNIT VECTORS

If $\vec{v}$ is a nonzero vector, then $\dfrac{v}{|v|}$ is a unit vector in the direction of $\vec{v}$. Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

Example 5.4.7: Finding the Unit Vector in the Direction of v

Find a unit vector in the same direction as $v = \langle -5, 12 \rangle$.

Solution

First, we will find the magnitude.

$$\begin{aligned} |v| &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Then we divide each component by $|v|$, which gives a unit vector in the same direction as $\vec{v}$:

$$\frac{v}{|v|} = -\frac{5}{13}i + \frac{12}{13}j$$

or, in component form

$$\frac{v}{|v|} = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$$

See Figure 5.4.16.

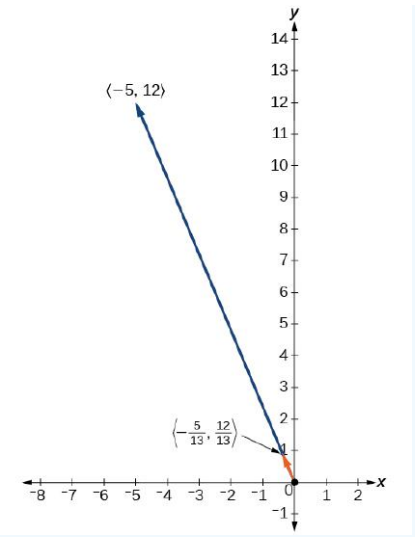


Figure 5.4.16

Verify that the magnitude of the unit vector equals 1. The magnitude of $-\frac{5}{13}i+\frac{12}{13}j$ is given as

$$\begin{aligned}\sqrt{\left(-\frac{5}{13}\right)^2+\left(\frac{12}{13}\right)^2} &= \sqrt{\frac{25}{169}+\frac{144}{169}} \\ &= \sqrt{\frac{169}{169}} \\ &= 1\end{aligned}$$

The vector $u=\frac{5}{13}i+\frac{12}{13}j$ is the unit vector in the same direction as $v=\langle -5,12\rangle$.

Performing Operations with Vectors in Terms of i and j

So far, we have investigated the basics of vectors: magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically. Now that we are familiar with the general strategies used in working with vectors, we will represent vectors in rectangular coordinates in terms of i and j.

VECTORS IN THE RECTANGULAR PLANE

Given a vector $\vec{v}$ with initial point $P=(x_1,y_1)$ and terminal point $Q=(x_2,y_2)$, $\vec{v}$ is written as

$$v=(x_2-x_1)i+(y_1-y_2)j \tag{5.4.1}$$

The position vector from $(0,0)$ to (a,b), where $(x_2-x_1)=a$ and $(y_2-y_1)=b$, is written as $\vec{v}=\overrightarrow{ai}+\overrightarrow{bj}$. This vector sum is called a linear combination of the vectors $\vec{i}$ and $\vec{j}$.

The magnitude of $\vec{v}=\overrightarrow{ai}+\overrightarrow{bj}$ is given as $|v|=\sqrt{a^2+b^2}$. See Figure 5.4.17.

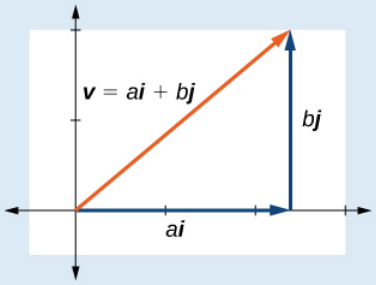


Figure 5.4.17

Example 5.4.8A: Writing a Vector in Terms of i and j

Given a vector $\vec{v}$ with initial point $P=(2,-6)$ and terminal point $Q=(-6,6)$, write the vector in terms of $\vec{i}$ and $\vec{j}$.

Solution

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2-x_1)i+(y_2-y_1)j \\ &= (-6-2)i+(6-(-6))j \\ &= -8i+12j \end{aligned}$$

Example 5.4.8B: Writing a Vector in Terms of i and j Using Initial and Terminal Points

Given initial point $P_1=(-1,3)$ and terminal point $P_2=(2,7)$, write the vector $\vec{v}$ in terms of $\vec{i}$ and $\vec{j}$.

Solution

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2-x_1)i+(y_2-y_1)j \\ v &= (2-(-1))i+(7-3)j \\ &= 3i+4j \end{aligned}$$

Exercise 5.4.3

Write the vector $\vec{u}$ with initial point $P=(-1,6)$ and terminal point $Q=(7,-5)$ in terms of $\vec{i}$ and $\vec{j}$.

Answer

$u=8i-11j$

Performing Operations on Vectors in Terms of i and j

When vectors are written in terms of i and j, we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

ADDING AND SUBTRACTING VECTORS IN RECTANGULAR COORDINATES

Given $v=ai+bj$ and $u=ci+dj$, then

$$\begin{aligned} v+u &= (a+c)i+(b+d)j \\ v-u &= (a-c)i+(b-d)j \end{aligned}$$

Example 5.4.9: Finding the Sum of the Vectors

Find the sum of $v_1=2i-3j$ and $v_2=4i+5j$.

Solution

$$\begin{aligned} v_1+v_2 &= (2+4)i+(-3+5)j \\ &= 6i+2j \end{aligned}$$

Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using i and j. For any of these vectors, we can calculate the magnitude. Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates. We find the direction of the vector by finding the angle to the horizontal. We do this by using the basic trigonometric identities, but with $|v|$ replacing r.

VECTOR COMPONENTS IN TERMS OF MAGNITUDE AND DIRECTION

Given a position vector $\vec{u} = \langle x, y \rangle$ and a direction angle θ,

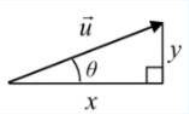


Figure 5.4.18

$$\cos\theta = \frac{x}{|\vec{u}|} \text{ and } \sin\theta = \frac{y}{|\vec{u}|}$$

$$x = |\vec{u}|\cos\theta$$

$$y = |\vec{u}|\sin\theta$$

Thus, $\vec{u} = xi + yj = |\vec{u}|\cos\theta i + |\vec{u}|\sin\theta j$, and magnitude is expressed as $|\vec{u}| = \sqrt{x^2 + y^2}$.

Example 5.4.10: Writing a Vector in Terms of Magnitude and Direction

Write a vector with length 7 at an angle of 135° to the positive *x*-axis in terms of magnitude and direction.

Solution

Using the conversion formulas $x = |\vec{u}|\cos\theta i$ and $y = |\vec{u}|\sin\theta j$, we find that

$$x = 7\cos(135^\circ)i$$

$$= -\frac{7\sqrt{2}}{2}$$

$$y = 7\sin(135^\circ)j$$

$$= \frac{7\sqrt{2}}{2}$$

This vector can be written as $\vec{u} = 7\cos(135^\circ)i + 7\sin(135^\circ)j$ or simplified as

$$\vec{u} = -\frac{7\sqrt{2}}{2}i + \frac{7\sqrt{2}}{2}j$$

Exercise 5.4.4

A vector travels from the origin to the point $(3, 5)$. Write the vector in terms of magnitude and direction.

Answer

$v = \sqrt{34}\cos(59^\circ)i + \sqrt{34}\sin(59^\circ)j$

Magnitude = 34

$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.04^\circ$

Finding the Dot Product of Two Vectors

As we discussed earlier in the section, scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication. If we multiply a vector by a vector, there are two possibilities: the ***dot product*** and the ***cross product***. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.

Definitions of the Dot Product

Given two vectors $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 \quad \text{Component definition} \tag{5.4.2}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta) \quad \text{Geometric definition} \tag{5.4.3}$$

The first definition, $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$, gives us a simple way to calculate the dot product from components. The second definition, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$, gives us a geometric interpretation of the dot product, and gives us a way to find the angle between two vectors, as we desired.

Example 5.4.11

Find the dot product $\langle 3, -2 \rangle \cdot \langle 5, 1 \rangle$.

Solution

Using the first definition, we can calculate the dot product by multiplying the x components and adding that to the product of the y components.

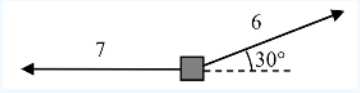
$$\langle 3, -2 \rangle \cdot \langle 5, 1 \rangle = (3)(5) + (-2)(1) = 15 - 2 = 13$$

Example 5.4.12

find the dot product of the two vectors shown.

Solution

We can immediately see that the magnitudes of the two vectors are 7 and 6, We quickly calculate that the angle between the vectors is 150°. Using the geometric definition of the dot product,



$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta) = (6)(7)\cos(150^\circ) = 42 \cdot \frac{\sqrt{3}}{2} = 21\sqrt{3}$$

Exercise 5.4.5

Calculate the dot product $\langle -7, 3 \rangle \cdot \langle -2, -6 \rangle$

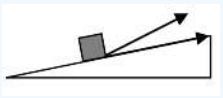
Answer

$$\langle -7, 3 \rangle \cdot \langle -2, -6 \rangle = (-7)(-2) + (3)(-6) = 14 - 18 = -4$$

Now we can return to our goal of finding the angle between vectors.

Example 5.4.13

An object is being pulled up a ramp in the direction $\langle 5, 1 \rangle$ by a rope pulling in the direction $\langle 4, 2 \rangle$. What is the angle between the rope and the ramp?



Solution

Using the component form, we can easily calculate the dot product.

$$\vec{a} \cdot \vec{b} = \langle 5, 1 \rangle \cdot \langle 4, 2 \rangle = (5)(4) + (1)(2) = 20 + 2 = 22$$

We can also calculate the magnitude of each vector.

$$|\vec{a}| = \sqrt{5^2 + 1^2} = \sqrt{26} \quad |\vec{b}| = \sqrt{4^2 + 2^2} = \sqrt{20}$$

Substituting these values into the geometric definition, we can solve for the angle between the vectors.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$$

$$22 = \sqrt{26}20 \cos(\theta)$$

$$\theta = \cos^{-1}(\frac{22}{\sqrt{26}\sqrt{20}}) \approx 15.255^\circ$$

Example 5.4.14

Calculate the angle between the vectors $\langle 6, 4 \rangle$ and $\langle -2, 3 \rangle$.

Solution

Calculating the dot product, $\langle 6, 4 \rangle \cdot \langle -2, 3 \rangle = (6)(-2) + (4)(3) = -12 + 12 = 0$

We don't even need to calculate the magnitudes in this case since the dot product is 0.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$$

$$0 = |\vec{a}||\vec{b}| \cos(\theta)$$

$$\theta = \cos^{-1}(\frac{0}{|\vec{a}||\vec{b}|}) = \cos^{-1}(0) = 90^\circ$$

With the dot product equaling zero, as in the last example, the angle between the vectors will always be 90°, indicating that the vectors are **orthogonal**, a more general way of saying perpendicular. This gives us a quick way to check if vectors are orthogonal. Also, if the dot product is positive, then the inside of the inverse cosine will be positive, giving an angle less than 90°. A negative dot product will then lead to an angle larger than 90°.

SIGN OF THE DOT PRODUCT

If the dot product is:

Zero The vectors are **orthogonal** (perpendicular).

Positive The angle between the vectors is less than 90°

Negative The angle between the vectors is greater than 90°

Exercise 5.4.6

Are the vectors $\langle -7, 3 \rangle$ and $\langle -2, -6 \rangle$ orthogonal? If not, find the angle between them.

Answer

In the previous Exercise, we found the dot product was -4, so the vectors are not orthogonal. The magnitudes of the vectors are $\sqrt{(-7)^2 + 3^2} = \sqrt{58}$ and $\sqrt{(-2)^2 + 6^2} = \sqrt{40}$. The angle between the vectors will be $\theta = \cos^{-1}(\frac{-4}{\sqrt{58}\sqrt{40}}) \approx 94.764^\circ$

Example 5.4.15: Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane? See Figure 5.4.20.

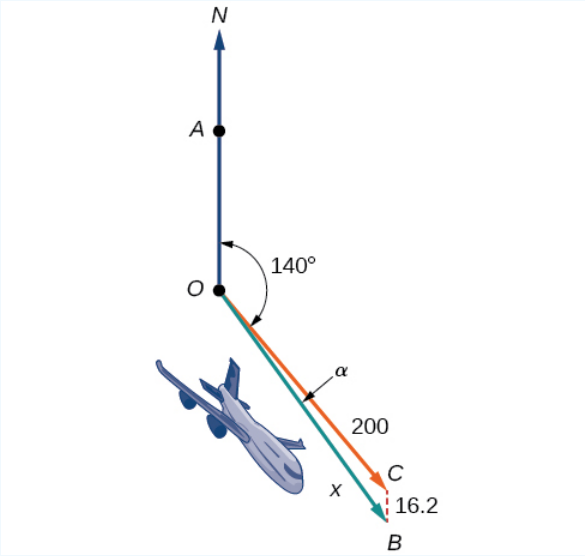


Figure 5.4.20

Solution

The ground speed is represented by x in the diagram, and we need to find the angle α in order to calculate the adjusted bearing, which will be $140^\circ + \alpha$.

Notice in Figure 5.4.20, that angle $\angle BCO$ must be equal to angle $\angle AOC$ by the rule of alternating interior angles, so angle $\angle BCO$ is 140°. We can find x by the Law of Cosines:

$$\begin{aligned} x^2 &= (16.2)^2 + (200)^2 - 2(16.2)(200)\cos(140^\circ) \\ x^2 &= 45,226.41 \\ x &= \sqrt{45,226.41} \\ x &= 212.7 \end{aligned}$$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

$$\begin{aligned} \frac{\sin\alpha}{16.2} &= \frac{\sin(140^\circ)}{212.7} \\ \sin\alpha &= \frac{16.2\sin(140^\circ)}{212.7} \\ &= 0.04896 \\ \sin^{-1}(0.04896) &= 2.8^\circ \end{aligned}$$

Therefore, the plane has a SE bearing of $140^\circ + 2.8^\circ = 142.8^\circ$. The ground speed is 212.7 miles per hour.

Key Concepts

- The position vector has its initial point at the origin. See Example 5.4.1.
- If the position vector is the same for two vectors, they are equal. See Example 5.4.2.
- Vectors are defined by their magnitude and direction. See Example 5.4.3.
- If two vectors have the same magnitude and direction, they are equal. See Example 5.4.4.

- Vector addition and subtraction result in a new vector found by adding or subtracting corresponding elements. See Example 5.4.5.
- Scalar multiplication is multiplying a vector by a constant. Only the magnitude changes; the direction stays the same. See Example 5.4.6 and Example 5.4.7.
- Vectors are comprised of two components: the horizontal component along the positive x-axis, and the vertical component along the positive y-axis. See Example 5.4.8.
- The unit vector in the same direction of any nonzero vector is found by dividing the vector by its magnitude.
- The magnitude of a vector in the rectangular coordinate system is $|v| = \sqrt{a^2 + b^2}$. See Example 5.4.9.
- In the rectangular coordinate system, unit vectors may be represented in terms of ii and jj where i represents the horizontal component and j represents the vertical component. Then, $v = ai + bj$ is a scalar multiple of v by real numbers a and b. See Example 5.4.10 and Example 5.4.11.
- Adding and subtracting vectors in terms of i and j consists of adding or subtracting corresponding coefficients of i and corresponding coefficients of j. See Example 5.4.12.
- A vector $v = ai + bj$ is written in terms of magnitude and direction as $v = |v| \cos\theta i + |v| \sin\theta j$. See Example 5.4.13.
- The dot product of two vectors is the product of the i terms plus the product of the j terms. See Example 5.4.11.
- We can use the dot product to find the angle between two vectors. Example 5.4.13 and Example 5.4.14.
- Dot products are useful for many types of physics applications. See Example 5.4.15.

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CHAPTER OVERVIEW

Chapter 6: The Polar System

6.1: Polar Coordinates
6.2: Graphing Basic Polar Equations
6.3: Converting Between Systems
6.4: The Polar Form of Complex Numbers
6.5: De Moivre's and the nth Root Theorem

Thumbnail: two complex numbers (CC BY-NC-SA; Ted Sundstrom & Steven Schlicker via LibreTexts).

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6.1: Polar Coordinates

Learning Objectives

- Distinguish between and understand the difference between a rectangular coordinate system and a polar coordinate system.
- Plot points with polar coordinates on a polar plane.

Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see Figure 6.1.1). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.

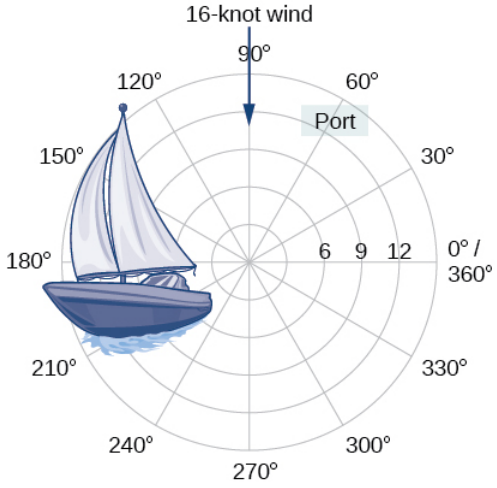
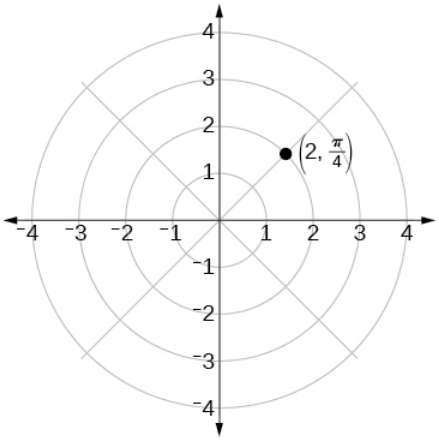


Figure 6.1.1

Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of *rectangular coordinates* (x, y) in the *Cartesian coordinate plane*. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to polar coordinates, which are points labeled (r, θ) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the pole, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive x-axis now viewed as the polar axis and the origin as the pole. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ, measured in radians, indicates the direction of r. We move counterclockwise from the polar axis by an angle of θ,and measure a directed line segment the length of r in the direction of θ. Even though we measure θ first and then r, the polar point is written with the r-coordinate first. For example, to plot the point $\left(2, \frac{\pi}{4}\right)$, we would move $\frac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in Figure 6.1.2.



Polar Grid

Figure 6.1.2

Example 6.1.1: Plotting a Point on the Polar Grid

Plot the point $\left(3, \frac{\pi}{2}\right)$ on the polar grid.

Solution

The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction 90° from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$ direction, as shown in Figure $6.1.3$.

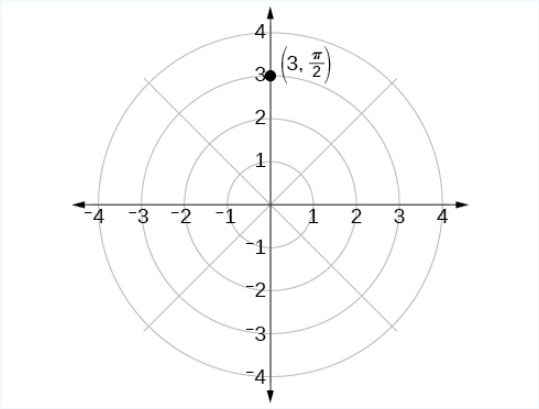


Figure 6.1.3

Exercise 6.1.1

Plot the point $\left(2, \frac{\pi}{3}\right)$ in the polar grid.

Answer

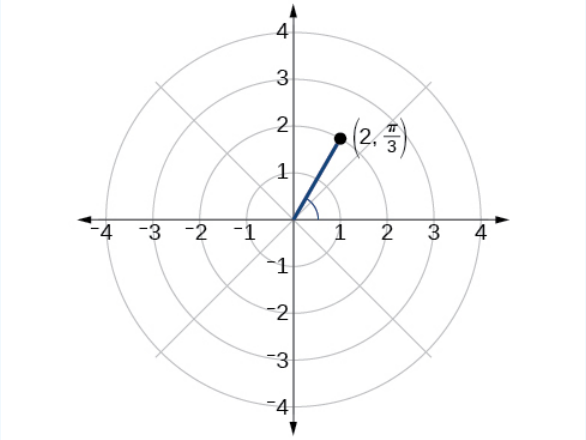


Figure 6.1.4

Example 6.1.2: Plotting a Point in the Polar Coordinate System with a Negative Component

Plot the point $\left(-2, \frac{\pi}{6}\right)$ on the polar grid.

Solution

We know that $\frac{\pi}{6}$ is located in the first quadrant. However, $r = -2$. We can approach plotting a point with a negative r in two ways:

1. Plot the point $\left(2, \frac{\pi}{6}\right)$ by moving $\frac{\pi}{6}$ in the counterclockwise direction and extending a directed line segment 2 units into the first quadrant. Then retrace the directed line segment back through the pole, and continue 2 units into the third quadrant;
2. Move $\frac{\pi}{6}$ in the counterclockwise direction, and draw the directed line segment from the pole 2 units in the negative direction, into the third quadrant.

See Figure $6.1.5a$. Compare this to the graph of the polar coordinate $\left(2, \frac{\pi}{6}\right)$ shown in Figure $6.1.5b$

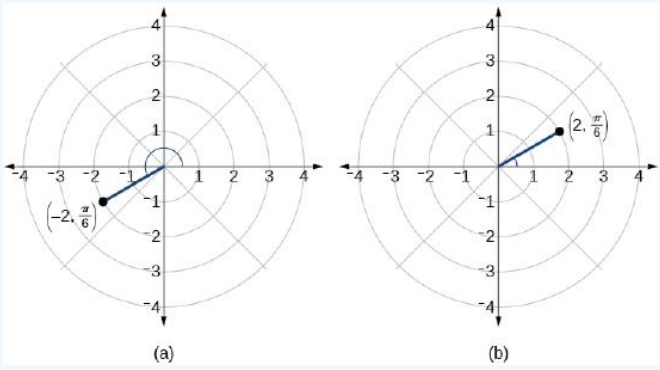


Figure 6.1.5

Exercise 6.1.2

Plot the points $\left(3, -\frac{\pi}{6}\right)$ and $\left(2, \frac{9\pi}{4}\right)$ on the same polar grid.

Answer

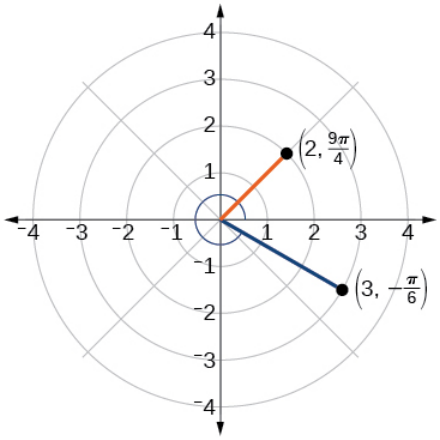


Figure 6.1.6

Converting from Polar Coordinates to Rectangular Coordinates

When given a set of **polar coordinates**, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables x, y, r, and θ.

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$$
$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$$

Dropping a perpendicular from the point in the plane to the *x*-axis forms a right triangle, as illustrated in Figure 6.1.7. An easy way to remember the equations above is to think of $\cos\theta$ as the adjacent side over the hypotenuse and $\sin\theta$ as the opposite side over the hypotenuse.

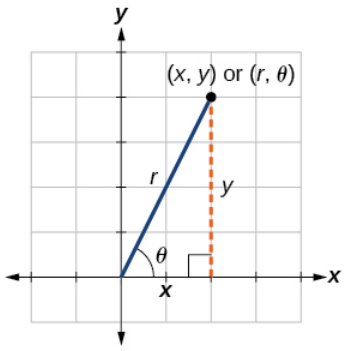


Figure 6.1.7

CONVERTING FROM POLAR COORDINATES TO RECTANGULAR COORDINATES

To convert polar coordinates (r, θ) to rectangular coordinates (x, y), let

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta \tag{6.1.1}$$

$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta \tag{6.1.2}$$

How to: Given polar coordinates, convert to rectangular coordinates.

1. Given the polar coordinate (r, θ), write $x = r\cos\theta$ and $y = r\sin\theta$.
2. Evaluate $\cos\theta$ and $\sin\theta$.
3. Multiply $\cos\theta$ by r to find the x-coordinate of the rectangular form.
4. Multiply $\sin\theta$ by r to find the y-coordinate of the rectangular form.

Example 6.1.3A: Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $\left(3, \frac{\pi}{2}\right)$ as rectangular coordinates.

Solution

Use the equivalent relationships.

$$\begin{aligned} x &= r\cos\theta \\ x &= 3\cos\frac{\pi}{2} \\ &= 0 \\ y &= r\sin\theta \\ y &= 3\sin\frac{\pi}{2} \\ &= 3 \end{aligned}$$

The rectangular coordinates are $(0, 3)$. See Figure 6.1.8.

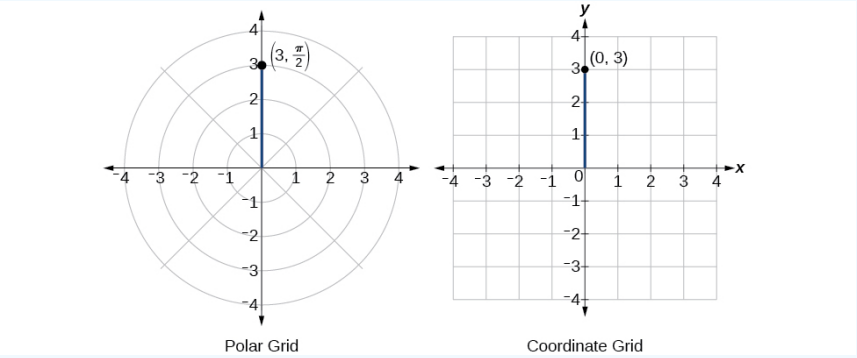


Figure 6.1.8

Example 6.1.3B: Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

Solution

See Figure 6.1.9. Writing the polar coordinates as rectangular, we have

$$\begin{aligned} x &= r\cos\theta \\ x &= -2\cos(0) \\ &= -2 \\ y &= r\sin\theta \\ y &= -2\sin(0) \\ &= 0 \end{aligned}$$

The rectangular coordinates are also $(-2, 0)$.

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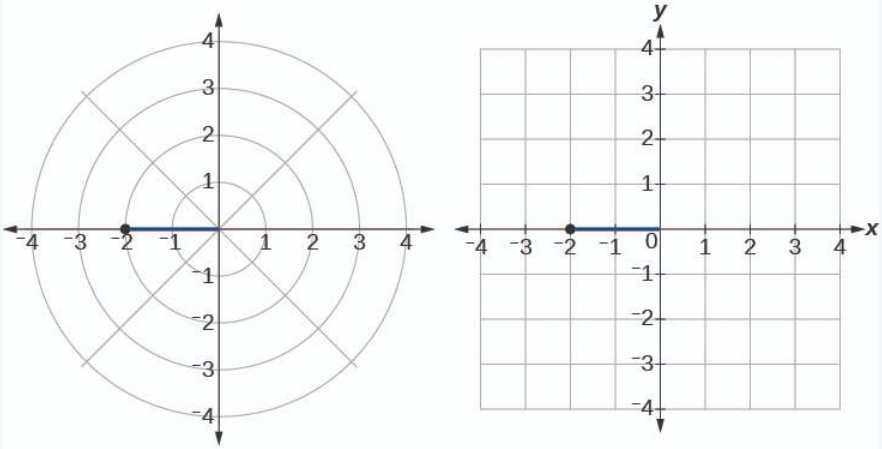


Figure 6.1.9

Exercise 6.1.3

Write the polar coordinates $\left(-1, \frac{2\pi}{3}\right)$ as rectangular coordinates.

Answer

$$(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

CONVERTING FROM RECTANGULAR COORDINATES TO POLAR COORDINATES

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in Figure 6.1.10.

$$\cos\theta = \frac{x}{r} \text{ or } x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \text{ or } y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

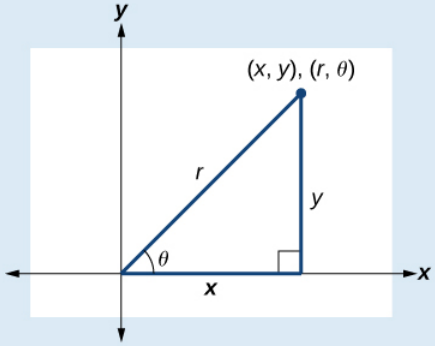


Figure 6.1.10

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Example 6.1.4: Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates $(3, 3)$ to polar coordinates.

Solution

We see that the original point $(3, 3)$ is in the first quadrant. To find θ, use the formula $\tan\theta = \frac{y}{x}$. This gives

$$\begin{aligned} \tan\theta &= \frac{3}{3} \\ \tan\theta &= 1 \\ \tan^{-1}(1) &= \frac{\pi}{4} \end{aligned}$$

To find r, we substitute the values for x and y into the formula $r = \sqrt{x^2 + y^2}$. We know that r must be positive, as $\frac{\pi}{4}$ is in the first quadrant. Thus

$$\begin{aligned} r &= \sqrt{3^2 + 3^2} \\ r &= \sqrt{9 + 9} \\ r &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

So, $r = 3\sqrt{2}$ and $\theta = \frac{\pi}{4}$, giving us the polar point $(3\sqrt{2}, \frac{\pi}{4})$. See Figure 6.1.11.

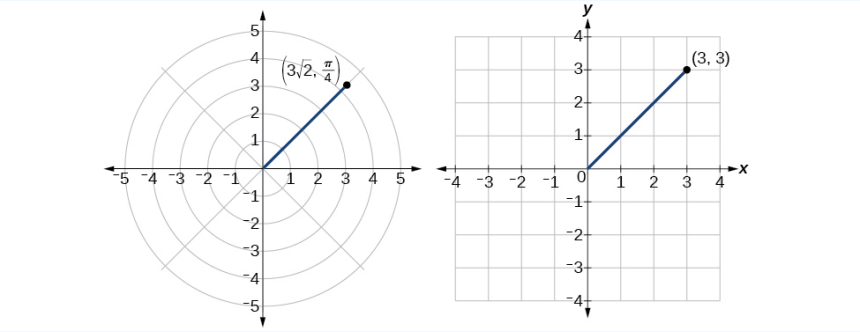


Figure 6.1.11

Analysis

There are other sets of polar coordinates that will be the same as our first solution. For example, the points $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ and $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ will coincide with the original solution of $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. The point $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ indicates a move further counterclockwise by π, which is directly opposite $\frac{\pi}{4}$. The radius is expressed as $-3\sqrt{2}$. However, the angle $\frac{5\pi}{4}$ is located in the third quadrant and, as r is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. The point $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ is a move further clockwise by $-\frac{7\pi}{4}$, from $\frac{\pi}{4}$. The radius, $3\sqrt{2}$, is the same.

Extra Practice

1. Plot the point with polar coordinates $\left(3, \frac{\pi}{6}\right)$.
2. Plot the point with polar coordinates $\left(5, -\frac{2\pi}{3}\right)$
3. Convert $\left(6, -\frac{3\pi}{4}\right)$ to rectangular coordinates.
4. Convert $\left(-2, \frac{3\pi}{2}\right)$ to rectangular coordinates.
5. Convert (7,-2) to polar coordinates.
6. Convert (-9,-4) to polar coordinates.

Key Equations

Conversion formulas	$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$ $\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$ $r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$

Key Concepts

- The polar grid is represented as a series of concentric circles radiating out from the pole, or origin.
- To plot a point in the form (r, θ), $\theta > 0$, move in a counterclockwise direction from the polar axis by an angle of θ, and then extend a directed line segment from the pole the length of r in the direction of θ. If θ is negative, move in a clockwise direction, and extend a directed line segment the length of r in the direction of θ. See Example $6.1.1$.
- If r is negative, extend the directed line segment in the opposite direction of θ. See Example $6.1.2$.
- To convert from polar coordinates to rectangular coordinates, use the formulas $x = r\cos\theta$ and $y = r\sin\theta$. See Example $6.1.3$ and Example $6.1.4$.
- To convert from rectangular coordinates to polar coordinates, use one or more of the formulas: $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, $\tan\theta = \frac{y}{x}$, and $r = \sqrt{x^2 + y^2}$. See Example $6.1.5$.

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6.2: Graphing Basic Polar Equations

Learning Objectives

- Graph polar equations.
- Graph and recognize limaçons and cardioids.
- Determine the shape of a limaçon from the polar equation.

Keplar's First Law of Planetary Motion argues that the planets move through space in elliptical, periodic orbits about the sun, as shown in Figure 6.2.1. They are in constant motion, so fixing an exact position of any planet is valid only for a moment. In other words, we can fix only a planet's instantaneous position. This is one application of polar coordinates, represented as (r, θ). We interpret r as the distance from the sun and θ as the planet's angular bearing, or its direction from a fixed point on the sun. In this section, we will focus on the polar system and the graphs that are generated directly from polar coordinates.

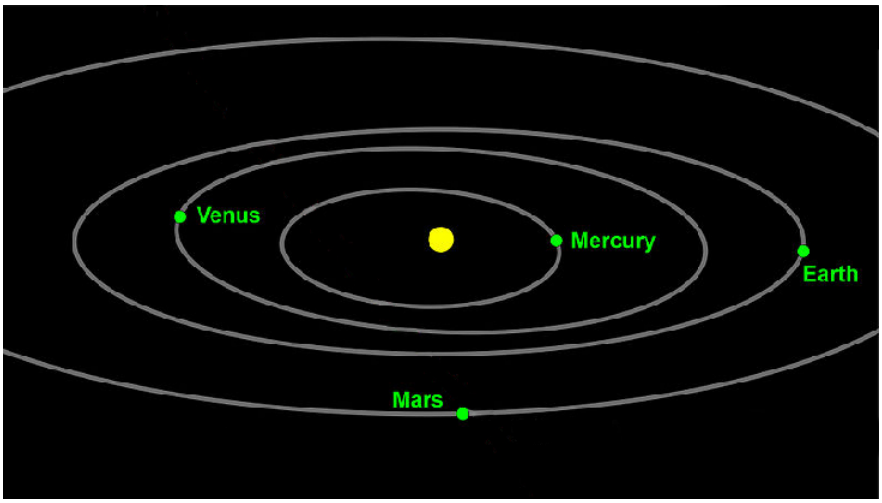


Figure 6.2.1: Planets follow elliptical paths as they orbit around the Sun. (credit: modification of work by NASA/JPL-Caltech)

Testing Polar Equations for Symmetry

Just as a rectangular equation such as $y = x^2$ describes the relationship between x and y on a Cartesian grid, a **polar equation** describes a relationship between r and θ on a polar grid. Recall that the coordinate pair (r, θ) indicates that we move counterclockwise from the polar axis (positive x-axis) by an angle of θ, and extend a ray from the pole (origin) r units in the direction of θ. All points that satisfy the polar equation are on the graph.

Symmetry is a property that helps us recognize and plot the graph of any equation. If an equation has a graph that is symmetric with respect to an axis, it means that if we folded the graph in half over that axis, the portion of the graph on one side would coincide with the portion on the other side. By performing three tests, we will see how to apply the properties of symmetry to polar equations. Further, we will use symmetry (in addition to plotting key points, zeros, and maximums of r) to determine the graph of a polar equation.

In the first test, we consider symmetry with respect to the line $\theta = \dfrac{\pi}{2}$ (y-axis). We replace (r, θ) with $(-r, -\theta)$ to determine if the new equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin\theta$;

$$\begin{aligned} r &= 2\sin\theta \\ -r &= 2\sin -\theta \qquad \text{Replace } (r, \theta) \text{ with } (-r, -\theta). \\ -r &= -2\sin\theta \qquad \text{Identity: } \sin(-\theta) = -\sin\theta. \\ r &= 2\sin\theta \qquad \text{Multiply both sides by } -1 \end{aligned}$$

This equation exhibits symmetry with respect to the line $\theta = \dfrac{\pi}{2}$.

In the second test, we consider symmetry with respect to the polar axis (x-axis). We replace (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ to determine equivalency between the tested equation and the original. For example, suppose we are given the equation $r = 1 - 2\cos\theta$.

$$
\begin{aligned}
r &= 1 - 2\cos\theta \\
r &= 1 - 2\cos(-\theta) \qquad \text{Replace } (r,\theta) \text{ with } (r,-\theta). \\
r &= 1 - 2\cos\theta \qquad \text{Even/Odd identity}
\end{aligned}
$$

The graph of this equation exhibits symmetry with respect to the polar axis.

In the third test, we consider symmetry with respect to the pole (origin). We replace (r,θ) with $(-r,\theta)$ to determine if the tested equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin(3\theta)$.

$$r = 2\sin(3\theta)$$

$$-r = 2\sin(3\theta)$$

The equation has failed the symmetry test, but that does not mean that it is not symmetric with respect to the pole. Passing one or more of the symmetry tests verifies that symmetry will be exhibited in a graph. However, failing the symmetry tests does not necessarily indicate that a graph will not be symmetric about the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole. In these instances, we can confirm that symmetry exists by plotting reflecting points across the apparent axis of symmetry or the pole. Testing for symmetry is a technique that simplifies the graphing of polar equations, but its application is not perfect.

Note: SYMMETRY TESTS

A polar equation describes a curve on the polar grid. The graph of a polar equation can be evaluated for three types of symmetry, as shown in Figure 6.2.2.

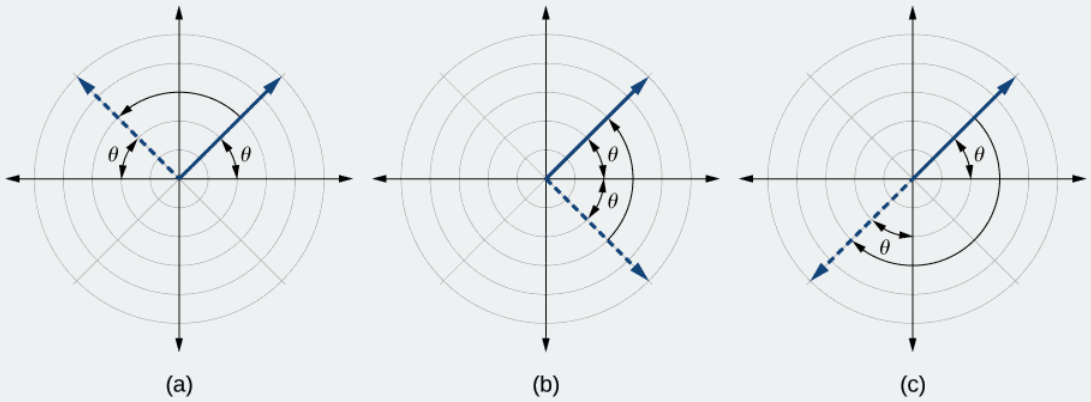


Figure 6.2.2: (a) A graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$ (y-axis) if replacing (r,θ) with $(-r,-\theta)$ yields an equivalent equation. (b) A graph is symmetric with respect to the polar axis (x-axis) if replacing (r,θ) with $(r,-\theta)$ or $(-r,\pi-\theta)$ yields an equivalent equation. (c) A graph is symmetric with respect to the pole (origin) if replacing (r,θ) with $(-r,\theta)$ yields an equivalent equation.

How to: Given a polar equation, test for symmetry.

1. Substitute the appropriate combination of components for (r,θ): $(-r,-\theta)$ for $\theta = \frac{\pi}{2}$ symmetry; $(r,-\theta)$ for polar axis symmetry; and $(-r,\theta)$ for symmetry with respect to the pole.
2. If the resulting equations are equivalent in one or more of the tests, the graph produces the expected symmetry.

Example 6.2.1: Testing a Polar Equation for Symmetry

Test the equation $r = 2\sin\theta$ for symmetry.

Solution

Test for each of the three types of symmetry.

1) Replacing (r, θ) with $(-r, -\theta)$ yields the same result. Thus, the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.	$-r = 2\sin(-\theta)$ $-r = -2\sin\theta$ Even-odd identity $r = 2\sin\theta$ Multiply by -1 Passed
2) Replacing θ with $-\theta$ does not yield the same equation. Therefore, the graph fails the test and may or may not be symmetric with respect to the polar axis.	$r = 2\sin(-\theta)$ $r = -2\sin\theta$ Even-odd identity $r = -2\sin\theta \neq 2\sin\theta$ Failed
3) Replacing r with $-r$ changes the equation and fails the test. The graph may or may not be symmetric with respect to the pole.	$-r = 2\sin\theta$ $r = -2\sin\theta \neq 2\sin\theta$ Failed

Analysis

Using a graphing calculator, we can see that the equation $r = 2\sin\theta$ is a circle centered at $(0, 1)$ with radius $r = 1$ and is indeed symmetric to the line $\theta = \frac{\pi}{2}$. We can also see that the graph is not symmetric with the polar axis or the pole. See Figure 6.2.3.

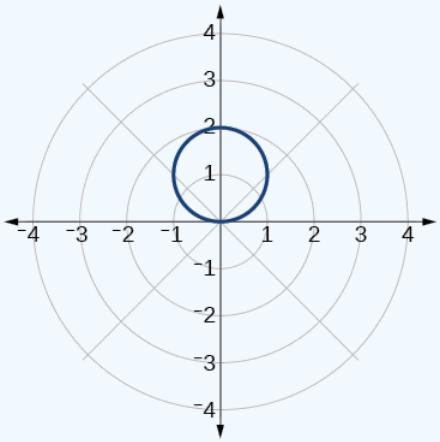


Figure 6.2.3

Exercise 6.2.1

Test the equation for symmetry: $r = -2\cos\theta$.

Answer

The equation fails the symmetry test with respect to the line $\theta = \frac{\pi}{2}$ and with respect to the pole. It passes the polar axis symmetry test.

Graphing Polar Equations by Plotting Points

To graph in the rectangular coordinate system we construct a table of x and y values. To graph in the polar coordinate system we construct a table of θ and r values. We enter values of θ into a polar equation and calculate r. However, using the properties of symmetry and finding key values of θ and r means fewer calculations will be needed.

Finding Zeros and Maxima

To find the zeros of a polar equation, we solve for the values of θ that result in $r = 0$. Recall that, to find the zeros of polynomial functions, we set the equation equal to zero and then solve for x. We use the same process for polar equations. Set $r = 0$, and solve for θ.

For many of the forms we will encounter, the maximum value of a polar equation is found by substituting those values of θ into the equation that result in the maximum value of the trigonometric functions. Consider $r = 5\cos\theta$; the maximum distance between the curve and the pole is 5 units. The maximum value of the cosine function is 1 when $\theta = 0$, so our polar equation is $5\cos\theta$, and the value $\theta = 0$ will yield the maximum $|r|$.

Similarly, the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, and if our polar equation is $r = 5\sin\theta$, the value $\theta = \frac{\pi}{2}$ will yield the maximum $|r|$. We may find additional information by calculating values of r when $\theta = 0$. These points would be polar axis intercepts, which may be helpful in drawing the graph and identifying the curve of a polar equation.

Example 6.2.2: Finding Zeros and Maximum Values for a Polar Equation

Using the equation in Example 6.2.1, find the zeros and maximum $|r|$ and, if necessary, the polar axis intercepts of $r = 2\sin\theta$.

Solution

To find the zeros, set r equal to zero and solve for θ.

$$\begin{aligned} 2\sin\theta &= 0 \\ \sin\theta &= 0 \\ \theta &= \sin^{-1}0 \\ \theta &= n\pi \qquad \text{where n is an integer} \end{aligned}$$

Substitute any one of the θ values into the equation. We will use 0.

$$\begin{aligned} r &= 2\sin(0) \\ r &= 0 \end{aligned}$$

The points $(0, 0)$ and $(0, \pm n\pi)$ are the zeros of the equation. They all coincide, so only one point is visible on the graph. This point is also the only polar axis intercept.

To find the maximum value of the equation, look at the maximum value of the trigonometric function $\sin\theta$, which occurs when $\theta = \frac{\pi}{2} \pm 2k\pi$ resulting in $\sin\left(\frac{\pi}{2}\right) = 1$. Substitute $\frac{\pi}{2}$ for θ.

$$\begin{aligned} r &= 2\sin\left(\frac{\pi}{2}\right) \\ r &= 2(1) \\ r &= 2 \end{aligned}$$

Analysis

The point $\left(2, \frac{\pi}{2}\right)$ will be the maximum value on the graph. Let's plot a few more points to verify the graph of a circle. See Table 6.2.1 and Figure 6.2.4.

Table 6.2.1

θ	$r = 2\sin\theta$	r
0	$r = 2\sin(0) = 0$	0
$\frac{\pi}{6}$	$r = 2\sin\left(\frac{\pi}{6}\right) = 1$	1
$\frac{\pi}{3}$	$r = 2\sin\left(\frac{\pi}{3}\right) \approx 1.73$	1.73
$\frac{pi}{2}$	$r = 2\sin\left(\frac{\pi}{2}\right) = 2$	2
$\frac{2\pi}{3}$	$r = 2\sin\left(\frac{2\pi}{3}\right) \approx 1.73$	1.73
$\frac{5\pi}{6}$	$r = 2\sin\left(\frac{5\pi}{6}\right) = 1$	1

θ	$r = 2\sin\theta$	r
π	$r = 2\sin(\pi) = 0$	0

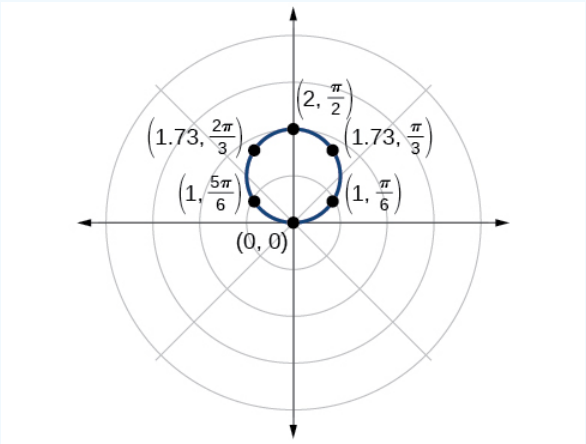


Figure 6.2.4

Exercise 6.2.2

Without converting to Cartesian coordinates, test the given equation for symmetry and find the zeros and maximum values of $|r|$: $r = 3\cos\theta$.

Answer

Tests will reveal symmetry about the polar axis. The zero is $\left(0, \frac{\pi}{2}\right)$, and the maximum value is $(3, 0)$.

Investigating Circles

Now we have seen the equation of a circle in the polar coordinate system. In the last two examples, the same equation was used to illustrate the properties of symmetry and demonstrate how to find the zeros, maximum values, and plotted points that produced the graphs. However, the circle is only one of many shapes in the set of polar curves.

There are five classic polar curves**: cardioids**, **limaçons, lemniscates, rose curves**, and **Archimedes' spirals**. We will briefly touch on the polar formulas for the circle before moving on to the classic curves and their variations.

FORMULAS FOR THE EQUATION OF A CIRCLE

Some of the formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$, wherea a is the diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is $\frac{|a|}{2}$, or one-half the diameter. For $r = a\cos\theta$, the center is $\left(\frac{a}{2}, 0\right)$. For $r = a\sin\theta$, the center is $\left(\frac{a}{2}, \pi\right)$. Figure 6.2.5 shows the graphs of these four circles.

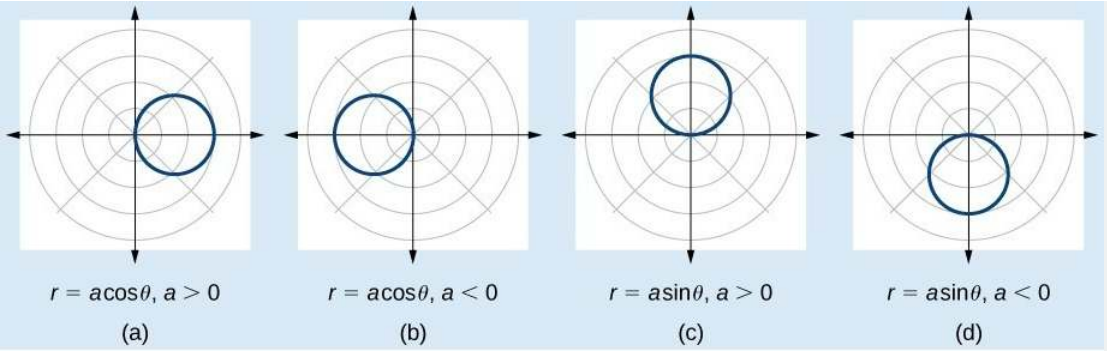


Figure 6.2.5

Example 6.2.3: Sketching the Graph of a Polar Equation for a Circle

Sketch the graph of $r = 4\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph is symmetric about the polar axis. Next, we find the zeros and maximum $|r|$ for $r = 4\cos\theta$. First, set $r = 0$, and solve for θ. Thus, a zero occurs at $\theta = \frac{\pi}{2} \pm k\pi$. A key point to plot is $\left(0, \frac{\pi}{2}\right)$.

To find the maximum value of r, note that the maximum value of the cosine function is 1 when $\theta = 0 \pm 2k\pi$. Substitute $\theta = 0$ into the equation:

$$\begin{aligned} r &= 4\cos\theta \\ r &= 4\cos(0) \\ r &= 4(1) \\ &= 4 \end{aligned}$$

The maximum value of the equation is 4. A key point to plot is $(4, 0)$.

As $r = 4\cos\theta$ is symmetric with respect to the polar axis, we only need to calculate r-values for θ over the interval $[0, \pi]$. Points in the upper quadrant can then be reflected to the lower quadrant. Make a table of values similar to Table 6.2.2. The graph is shown in Figure 6.2.6.

Table 6.2.2

$\boldsymbol{\theta}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\boldsymbol{\pi}$
$\boldsymbol{r}$	4	3.46	2.83	2	0	-2	-2.83	-3.46	4

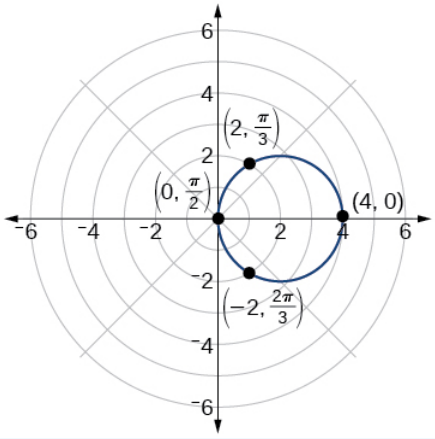


Figure 6.2.6

Investigating Cardioids

While translating from polar coordinates to Cartesian coordinates may seem simpler in some instances, graphing the classic curves is actually less complicated in the polar system. The next curve is called a cardioid, as it resembles a heart. This shape is often included with the family of curves called limaçons, but here we will discuss the cardioid on its own.

FORMULAS FOR A CARDIOID

The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. The cardioid graph passes through the pole, as we can see in Figure 6.2.7.

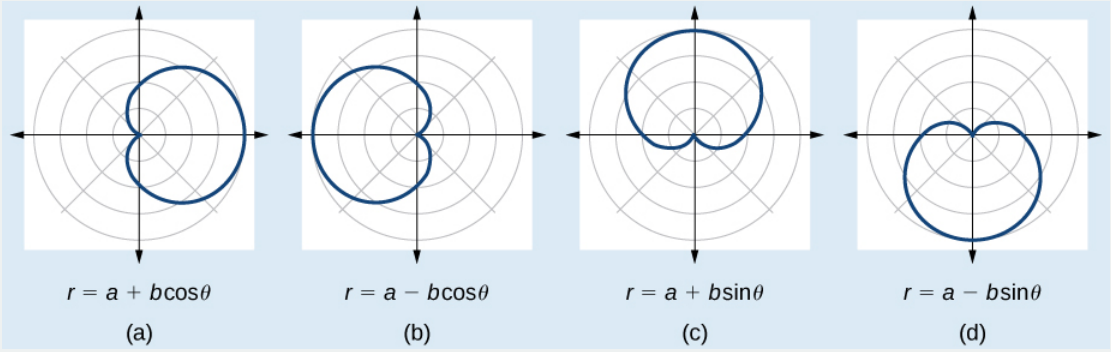


Figure 6.2.7

How to: Given the polar equation of a cardioid, sketch its graph

1. Check equation for the three types of symmetry.
2. Find the zeros. Set $r = 0$.
3. Find the maximum value of the equation according to the maximum value of the trigonometric expression.
4. Make a table of values for r and θ.
5. Plot the points and sketch the graph.

Example 6.2.4: Sketching the Graph of a Cardioid

Sketch the graph of $r = 2 + 2\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph of this equation will be symmetric about the polar axis. Next, we find the zeros and maximums. Setting $r = 0$, we have $\theta = \pi + 2k\pi$. The zero of the equation is located at $(0, \pi)$. The graph passes through this point.

The maximum value of $r = 2 + 2\cos\theta$ occurs when $\cos\theta$ is a maximum, which is when $\cos\theta = 1$ or when $\theta = 0$. Substitute $\theta = 0$ into the equation, and solve for r.

$$\begin{aligned} r &= 2 + 2\cos(0) \\ r &= 2 + 2(1) \\ &= 4 \end{aligned}$$

The point $(4, 0)$ is the maximum value on the graph.

We found that the polar equation is symmetric with respect to the polar axis, but as it extends to all four quadrants, we need to plot values over the interval $[0, \pi]$. The upper portion of the graph is then reflected over the polar axis. Next, we make a table of values, as in Table 6.2.3, and then we plot the points and draw the graph. See Figure 6.2.8.

Table 6.2.3

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	3.41	2	1	0

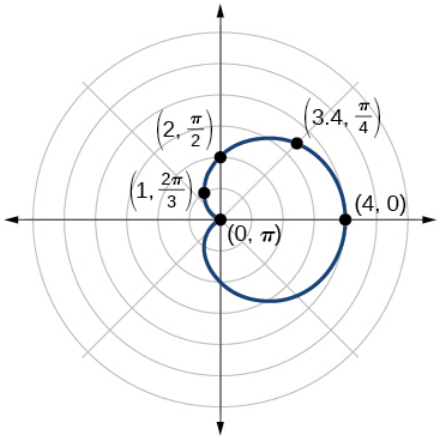


Figure 6.2.8

Investigating Limaçons

The word *limaçon* is Old French for "snail," a name that describes the shape of the graph. As mentioned earlier, the cardioid is a member of the limaçon family, and we can see the similarities in the graphs. The other images in this category include the one-loop limaçon and the two-loop (or inner-loop) limaçon. **One-loop limaçons** are sometimes referred to as dimpled limaçons when $1 < \frac{a}{b} < 2$ and convex limaçons when $\frac{a}{b} \geq 2$.

FORMULAS FOR ONE-LOOP LIMAÇONS

The formulas that produce the graph of a dimpled one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $1 < ab < 2$. All four graphs are shown in Figure 6.2.9.

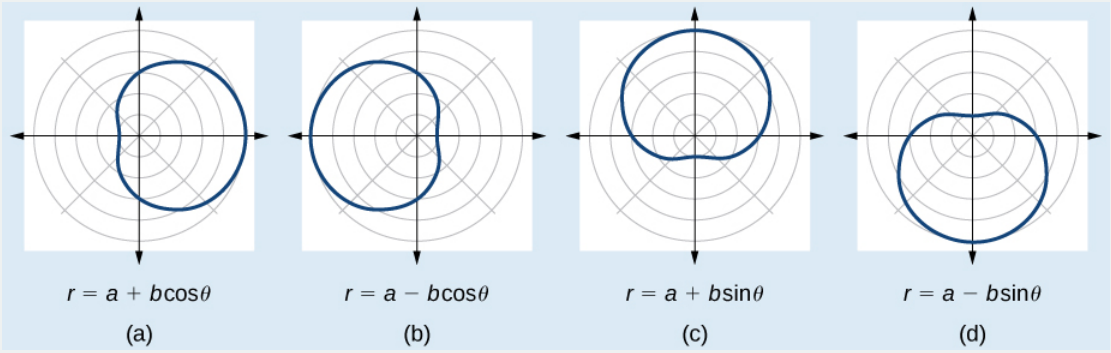


Figure 6.2.9

How to: Given a polar equation for a one-loop limaçon, sketch the graph

1. Test the equation for symmetry. Remember that failing a symmetry test does not mean that the shape will not exhibit symmetry. Often the symmetry may reveal itself when the points are plotted.
2. Find the zeros.
3. Find the maximum values according to the trigonometric expression.
4. Make a table.
5. Plot the points and sketch the graph.

Example 6.2.5: Sketching the Graph of a One-Loop Limaçon

Graph the equation $r = 4 - 3\sin\theta$.

Solution

First, testing the equation for symmetry, we find that it fails all three symmetry tests, meaning that the graph may or may not exhibit symmetry, so we cannot use the symmetry to help us graph it. However, this equation has a graph that clearly displays

symmetry with respect to the line $\theta = \frac{\pi}{2}$, yet it fails all the three symmetry tests. A graphing calculator will immediately illustrate the graph's reflective quality.

Next, we find the zeros and maximum, and plot the reflecting points to verify any symmetry. Setting $r = 0$ results in θ being undefined. What does this mean? How could θ be undefined? The angle θ is undefined for any value of $\sin\theta > 1$. Therefore, θ is undefined because there is no value of θ for which $\sin\theta > 1$. Consequently, the graph does not pass through the pole. Perhaps the graph does cross the polar axis, but not at the pole. We can investigate other intercepts by calculating r when $\theta = 0$.

$$\begin{aligned} r(0) &= 4 - 3\sin(0) \\ r &= 4 - 3 \cdot 0 \\ &= 4 \end{aligned}$$

So, there is at least one polar axis intercept at $(4, 0)$.

Next, as the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, we will substitute $\theta = \frac{\pi}{2}$ into the equation and solve for r. Thus, $r = 1$.

Make a table of the coordinates similar to Table 6.2.4.

Table 6.2.4

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	4	2.5	1.4	1	1.4	2.5	4	5.5	6.6	7	6.6	5.5	4

The graph is shown in Figure 6.2.10.

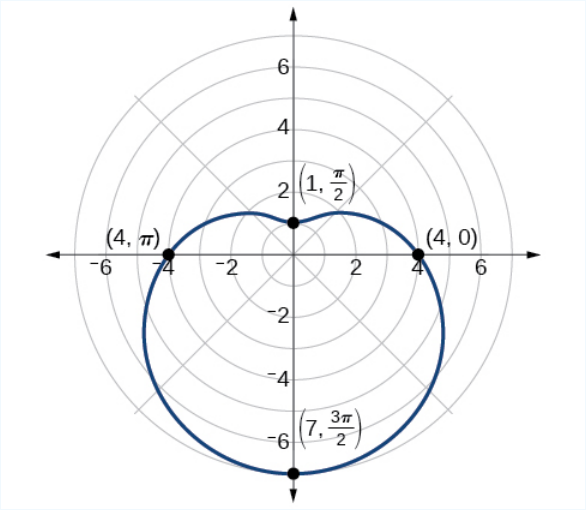


Figure 6.2.10: One-loop limaçon

Analysis

This is an example of a curve for which making a table of values is critical to producing an accurate graph. The symmetry tests fail; the zero is undefined. While it may be apparent that an equation involving $\sin\theta$ is likely symmetric with respect to the line $\theta = \frac{\pi}{2}$, evaluating more points helps to verify that the graph is correct.

Exercise 6.2.3

Sketch the graph of $r = 3 - 2\cos\theta$.

Answer

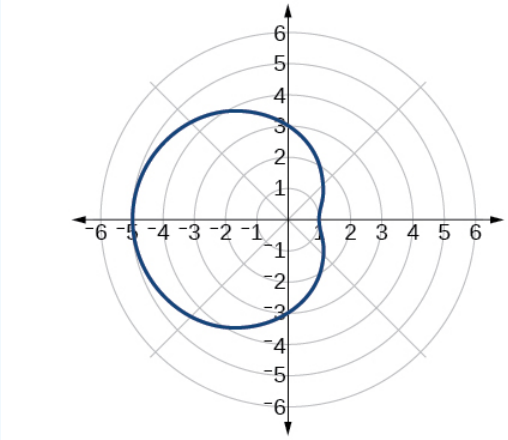


Figure 6.2.11

Another type of limaçon, the **inner-loop limaçon**, is named for the loop formed inside the general limaçon shape. It was discovered by the German artist Albrecht Dürer(1471-1528), who revealed a method for drawing the inner-loop limaçon in his 1525 book *Underweysung der Messing*. A century later, the father of mathematician Blaise Pascal, Étienne Pascal(1588-1651), rediscovered it.

FORMULAS FOR INNER-LOOP LIMAÇONS

The formulas that generate the inner-loop limaçons are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $a < b$. The graph of the inner-loop limaçon passes through the pole twice: once for the outer loop, and once for the inner loop. See Figure 10.5.12 for the graphs.

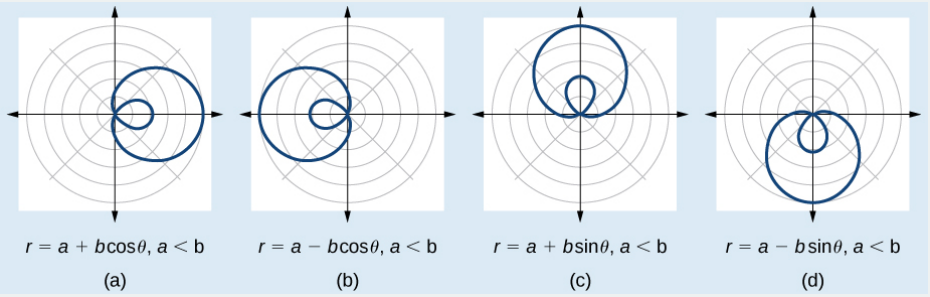


Figure 6.2.12

Example 6.2.6: Sketching the Graph of an Inner-Loop Limaçon

Sketch the graph of $r = 2 + 5\cos\theta$.

Solution

Testing for symmetry, we find that the graph of the equation is symmetric about the polar axis. Next, finding the zeros reveals that when $r = 0$, $\theta = 1.98$. The maximum $|r|$ is found when $\cos\theta = 1$ or when $\theta = 0$. Thus, the maximum is found at the point $(7, 0)$.

Even though we have found symmetry, the zero, and the maximum, plotting more points will help to define the shape, and then a pattern will emerge. See Table 6.2.5.

Table 6.2.5

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	7	6.3	4.5	2	$\mathbf{-0.5}$	$\mathbf{-2.3}$	-3	$\mathbf{-2.3}$	$\mathbf{-0.5}$	2	4.5	6.3	7

As expected, the values begin to repeat after $\theta = \pi$. The graph is shown in Figure 6.2.13.

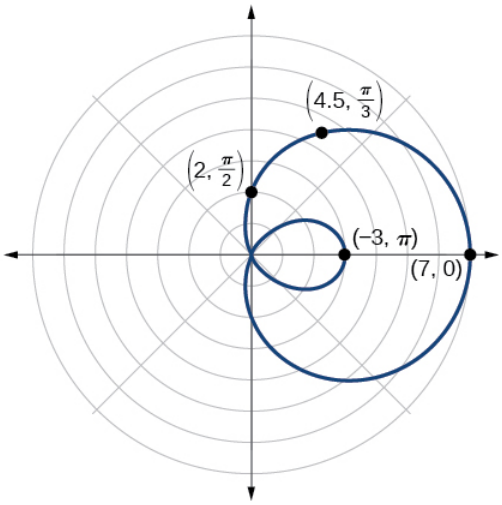


Figure 6.2.13: Inner-loop limaçon

Investigating Lemniscates

The lemniscate is a polar curve resembling the infinity symbol ∞ or a figure 8. Centered at the pole, a lemniscate is symmetrical by definition.

FORMULAS FOR LEMNISCATES

The formulas that generate the graph of a lemniscate are given by $r^2 = a^2 \cos 2\theta$ and $r^2 = a^2 \sin 2\theta$ where $a \neq 0$. The formula $r^2 = a^2 \sin 2\theta$ is symmetric with respect to the pole. The formula $r^2 = a^2 \cos 2\theta$ is symmetric with respect to the pole, the line $\theta = \dfrac{\pi}{2}$, and the polar axis. See Figure 6.2.14 for the graphs.

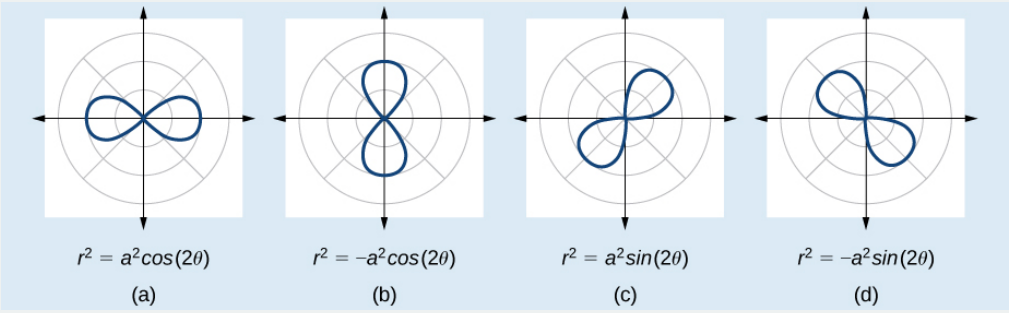


Figure 6.2.14

Example 6.2.7: Sketching the Graph of a Lemniscate

Sketch the graph of $r^2 = 4 \cos 2\theta$.

Solution

The equation exhibits symmetry with respect to the line $\theta = \dfrac{\pi}{2}$, the polar axis, and the pole.

Let's find the zeros. It should be routine by now, but we will approach this equation a little differently by making the substitution $u = 2\theta$.

$$
\begin{aligned}
0 &= 4\cos 2\theta \\
0 &= 4\cos u \\
0 &= \cos u \\
\cos^{-1} 0 &= \frac{\pi}{2} \\
u &= \frac{\pi}{2} \qquad \text{Substitute } 2\theta \text{ back in for } u. \\
2\theta &= \frac{\pi}{2} \\
\theta &= \frac{\pi}{4}
\end{aligned}
$$

So, the point $\left(0, \frac{\pi}{4}\right)$ is a zero of the equation.

Now let's find the maximum value. Since the maximum of $\cos u = 1$ when $u = 0$, the maximum $\cos 2\theta = 1$ when $2\theta = 0$. Thus,

$$
\begin{aligned}
r^2 &= 4\cos(0) \\
r^2 &= 4(1) \\
r^2 &= 4 \\
r &= \pm 4 \\
&= 2
\end{aligned}
$$

We have a maximum at $(2, 0)$. Since this graph is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis, we only need to plot points in the first quadrant.

Make a table similar to Table 6.2.6.

Table 6.2.6

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$\sqrt{2}$	0

Plot the points on the graph, such as the one shown in Figure 6.2.15.

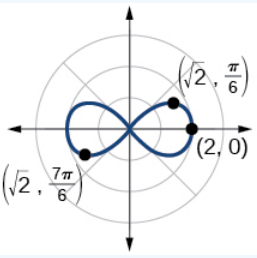


Figure 6.2.15: Lemniscate

Analysis

Making a substitution such as $u = 2\theta$ is a common practice in mathematics because it can make calculations simpler. However, we must not forget to replace the substitution term with the original term at the end, and then solve for the unknown.

Some of the points on this graph may not show up using the Trace function on the TI-84 graphing calculator, and the calculator table may show an error for these same points of r. This is because there are no real square roots for these values of θ. In other words, the corresponding r-values of $\sqrt{4\cos(2\theta)}$ are complex numbers because there is a negative number under the radical.

Investigating Rose Curves

The next type of polar equation produces a petal-like shape called a rose curve. Although the graphs look complex, a simple polar equation generates the pattern.

ROSE CURVES

The formulas that generate the graph of a **rose curve** are given by $r = a\cos n\theta$ and $r = a\sin n\theta$ where $a \neq 0$. If n is even, the curve has $2n$ petals. If n is odd, the curve has n petals. See Figure 6.2.16.

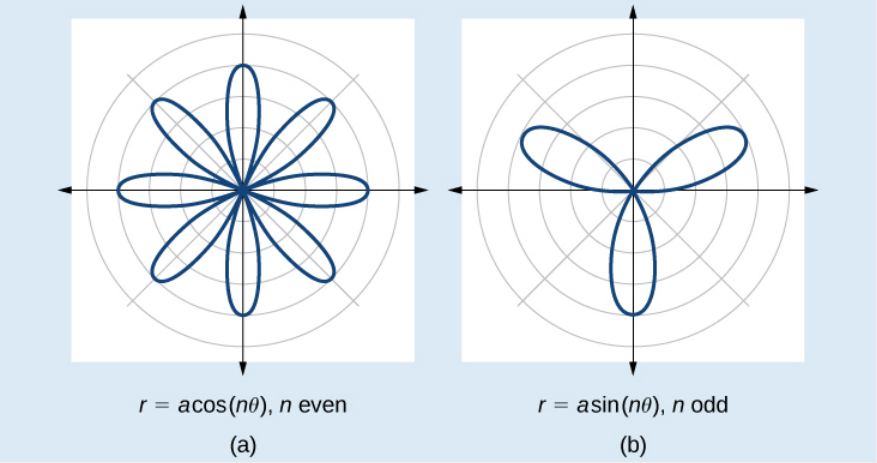


Figure 6.2.16

Example 6.2.8: Sketching the Graph of a Rose Curve (n Even)

Sketch the graph of $r = 2\cos 4\theta$.

Solution

Testing for symmetry, we find again that the symmetry tests do not tell the whole story. The graph is not only symmetric with respect to the polar axis, but also with respect to the line $\theta = \frac{\pi}{2}$ and the pole.

Now we will find the zeros. First make the substitution $u = 4\theta$.

$$\begin{aligned} 0 &= 2\cos 4\theta \\ 0 &= \cos 4\theta \\ 0 &= \cos u \\ \cos^{-1} 0 &= u \\ u &= \frac{\pi}{2} \\ 4\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{8} \end{aligned}$$

The zero is $\theta = \frac{\pi}{8}$. The point $\left(0, \frac{\pi}{8}\right)$ is on the curve.

Next, we find the maximum $|r|$. We know that the maximum value of $\cos u = 1$ when $\theta = 0$. Thus,

$$\begin{aligned} r &= 2\cos(4 \cdot 0) \\ r &= 2\cos(0) \\ r &= 2(1) \\ &= 2 \end{aligned}$$

The point $(2, 0)$ is on the curve.

The graph of the rose curve has unique properties, which are revealed in Table 6.2.7.

Table 6.2.7

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$5\pi 8$	$3\pi 4$

r	2	0	-2	0	2	0	-2

As $r = 0$ when $\theta = \frac{\pi}{8}$, it makes sense to divide values in the table by $\frac{\pi}{8}$ units. A definite pattern emerges. Look at the range of r-values: $2, 0, -2, 0$ and so on. This represents the development of the curve one petal at a time. Starting at $r = 0$, each petal extends out a distance of $r = 2$, and then turns back to zero $2n$ times for a total of eight petals. See the graph in Figure 6.2.17.

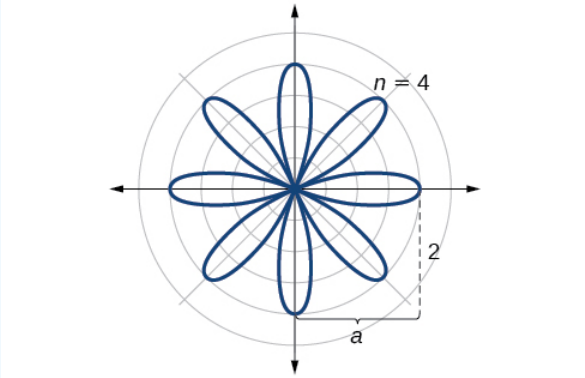


Figure 6.2.17: Rose curve, n even

Analysis

When these curves are drawn, it is best to plot the points in order, as in the Table 6.2.7. This allows us to see how the graph hits a maximum (the tip of a petal), loops back crossing the pole, hits the opposite maximum, and loops back to the pole. The action is continuous until all the petals are drawn.

Exercise 6.2.4

Sketch the graph of $r = 4\sin(2\theta)$.

Answer

The graph is a rose curve, n even

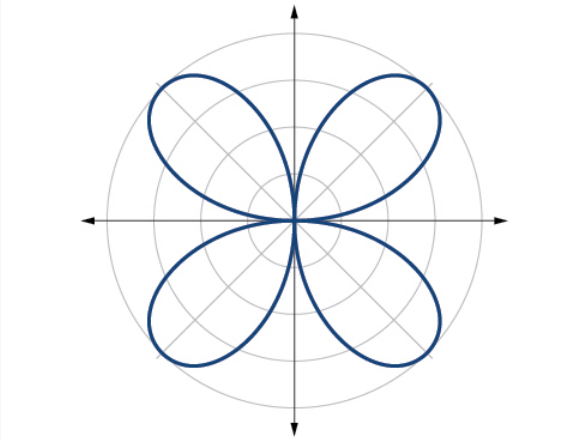


Figure 6.2.18

Example 6.2.9: Sketching the Graph of a Rose Curve (n Odd)

Sketch the graph of $r = 2\sin(5\theta)$.

Solution

The graph of the equation shows symmetry with respect to the line $\theta = \frac{\pi}{2}$. Next, find the zeros and maximum. We will want to make the substitution $u = 5\theta$.

$$\begin{aligned} 0 &= 2\sin(5\theta) \\ 0 &= \sin u \\ \sin^{-1} 0 &= 0 \\ u &= 0 \\ 5\theta &= 0 \\ \theta &= 0 \end{aligned}$$

The maximum value is calculated at the angle where $\sin\theta$ is a maximum. Therefore,

$$\begin{aligned} r &= 2\sin\left(5 \cdot \frac{\pi}{2}\right) \\ r &= 2(1) \\ &= 2 \end{aligned}$$

Thus, the maximum value of the polar equation is 2. This is the length of each petal. As the curve for n odd yields the same number of petals as n, there will be five petals on the graph. See Figure 6.2.19.

Create a table of values similar to Table 6.2.8.

Table 6.2.8

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	\(0\0	1	$\mathbf{-1.73}$	2	$\mathbf{-1.73}$	1	0

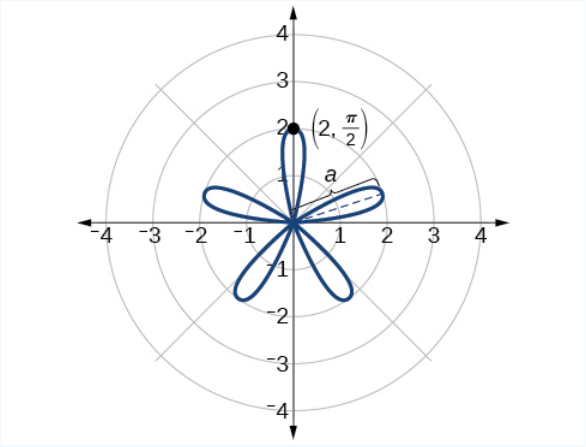


Figure 6.2.19: Rose curve, n odd

Exercise 6.2.5

Sketch the graph of $r = 3\cos(3\theta)$.

Answer

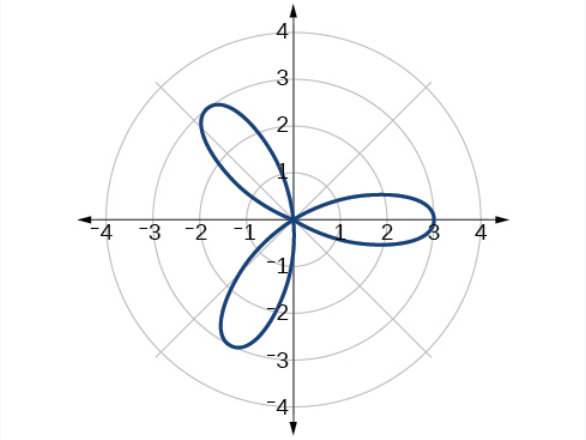


Figure 6.2.20

Rose curve, n odd

Investigating the Archimedes' Spiral

The final polar equation we will discuss is the Archimedes' spiral, named for its discoverer, the Greek mathematician Archimedes (c. 287 BCE - c. 212 BCE), who is credited with numerous discoveries in the fields of geometry and mechanics.

ARCHIMEDES' SPIRAL

The formula that generates the graph of the Archimedes' spiral is given by $r = \theta$ for $\theta \geq 0$. As θ increases, r increases at a constant rate in an ever-widening, never-ending, spiraling path. See Figure 6.2.21.

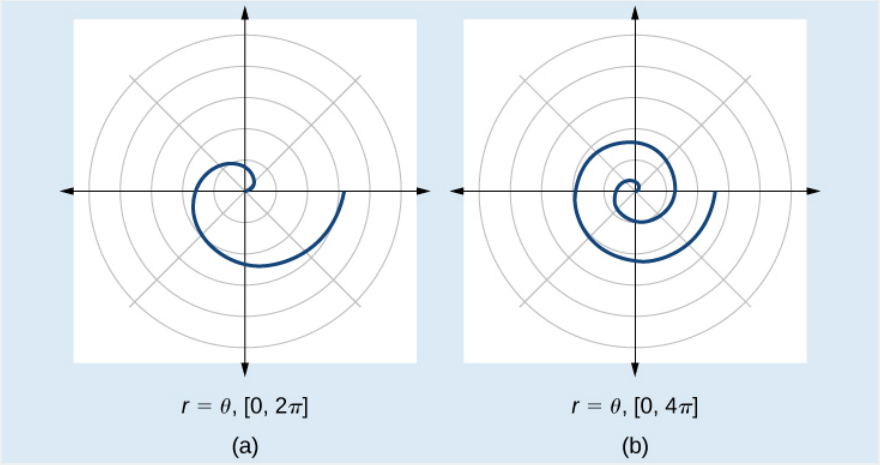


Figure 6.2.21

How to: Given an Archimedes' spiral over $[0, 2\pi]$,sketch the graph

1. Make a table of values for r and θ over the given domain.
2. Plot the points and sketch the graph.

Example 6.2.10: Sketching the Graph of an Archimedes' Spiral

Sketch the graph of $r = \theta$ over $[0, 2\pi]$.

Solution

As r is equal to θ, the plot of the Archimedes' spiral begins at the pole at the point $(0, 0)$. While the graph hints of symmetry, there is no formal symmetry with regard to passing the symmetry tests. Further, there is no maximum value, unless the domain is restricted.

Create a table such as Table 6.2.9.

Table 6.2.9

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0.785	1.57	3.14	4.71	5.50	6.28

Notice that the r-values are just the decimal form of the angle measured in radians. We can see them on a graph in Figure 6.2.22

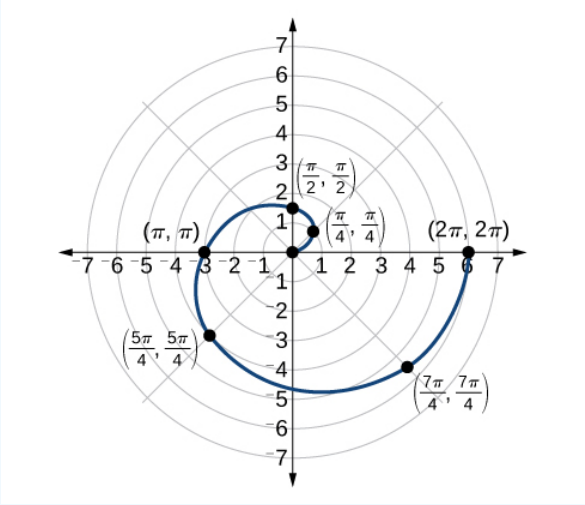


Figure 6.2.22: Archimedes' spiral

Analysis

The domain of this polar curve is $[0, 2\pi]$. In general, however, the domain of this function is $(-\infty, \infty)$. Graphing the equation of the Archimedes' spiral is rather simple, although the image makes it seem like it would be complex.

Exercise 6.2.6

Sketch the graph of $r = -\theta$ over the interval $[0, 4\pi]$.

Answer

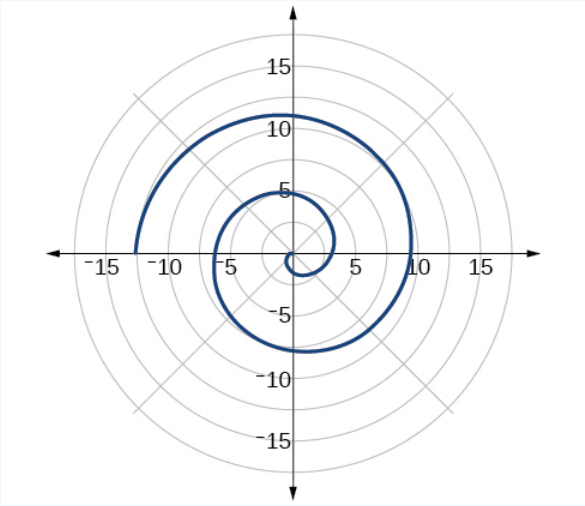


Figure 6.2.23

Summary of Curves

We have explored a number of seemingly complex polar curves in this section. Figure 6.2.24 and Figure 6.2.25 summarize the graphs and equations for each of these curves.

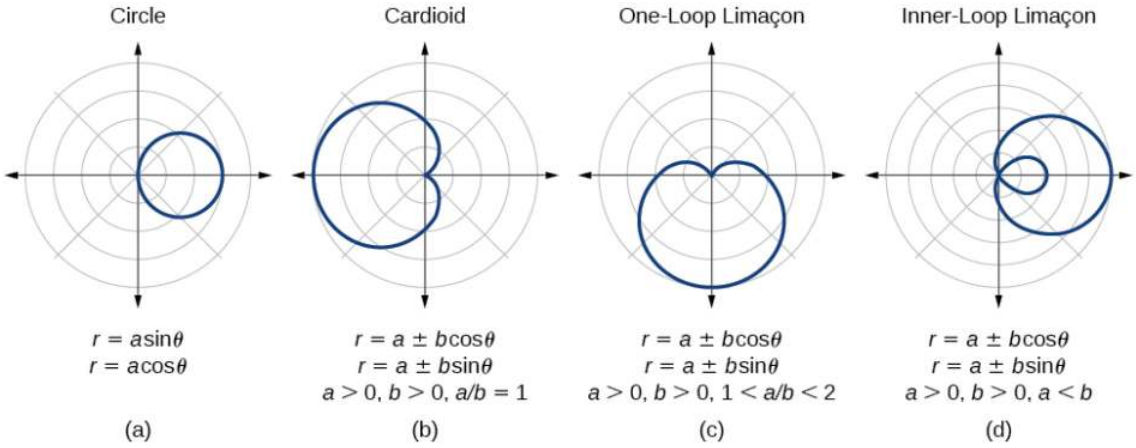


Figure 6.2.24

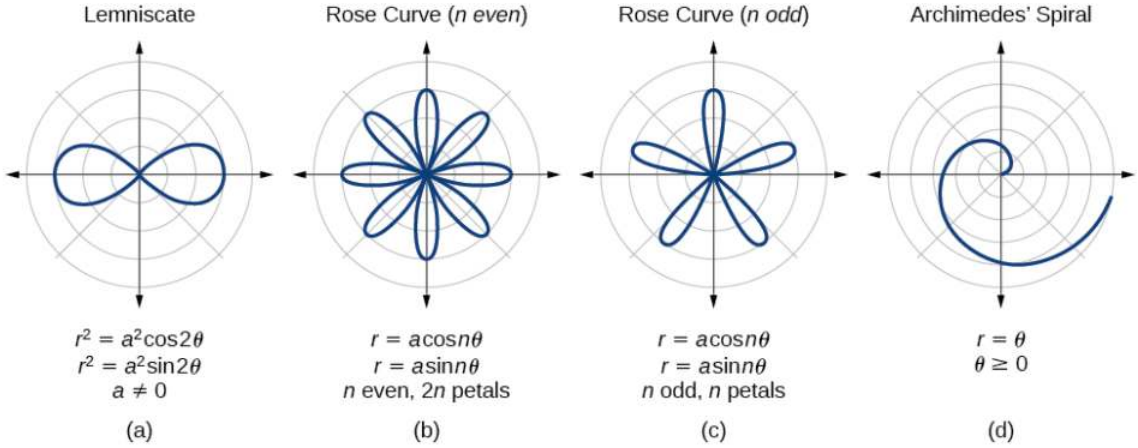


Figure 6.2.25

Extra Practice

1. For the following exercises, test each equation for symmetry.
2. $r = 4 + 4\sin\theta$
3. $r = 7$
4. Sketch a graph of the polar equation $r = 1 - 5\sin\theta$. Label the axis intercepts.
5. Sketch a graph of the polar equation $r = 5\sin(7\theta)$.
6. Sketch a graph of the polar equation $r = 3 - 3\cos\theta$

Key Concepts

- It is easier to graph polar equations if we can test the equations for symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole.
- There are three symmetry tests that indicate whether the graph of a polar equation will exhibit symmetry. If an equation fails a symmetry test, the graph may or may not exhibit symmetry. See Example 6.2.1.
- Polar equations may be graphed by making a table of values for θ and r.
- The maximum value of a polar equation is found by substituting the value θ that leads to the maximum value of the trigonometric expression.
- The zeros of a polar equation are found by setting $r = 0$ and solving for θ. See Example 6.2.2.

- Some formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$. See Example 6.2.3.
- The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, for $a > 0$, $b > 0$, and $ab = 1$. See Example 6.2.4.
- The formulas that produce the graphs of a one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $1 < ab < 2$. See Example 6.2.5.
- The formulas that produce the graphs of an inner-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $a > 0$, $b > 0$, and $a < b$. See Example 6.2.6.
- The formulas that produce the graphs of a lemniscates are given by $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$, where $a \neq 0$.See Example 6.2.7.
- The formulas that produce the graphs of rose curves are given by $r = a\cos n\theta$ and $r = a\sin n\theta$, where $a \neq 0$; if n is even, there are $2n$ petals, and if n is odd, there aren n petals. See Example 6.2.8 and Example 6.2.9.
- The formula that produces the graph of an Archimedes' spiral is given by $r = \theta$, $\theta \geq 0$. See Example 6.2.10.

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6.3: Converting Between Systems

Learning Objectives

- Convert rectangular coordinates to polar coordinates.
- Convert equations given in rectangular form to equations in polar form and vise versa.

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

How to: Given an equation in polar form, graph it using a graphing calculator

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: r_1, r_2,..., r_6.
3. Enter the polar equation, set equal to r.
4. Press **GRAPH**.

Example 6.3.1A: Writing a Cartesian Equation in Polar Form

Write the Cartesian equation $x^2+y^2=9$ in polar form.

Solution

The goal is to eliminate x and y from the equation and introduce r and θ. Ideally, we would write the equation r as a function of θ. To obtain the polar form, we will use the relationships between (x,y) and (r,θ). Since $x=r\cos\theta$ and $y=r\sin\theta$, we can substitute and solve for r.

$$\begin{aligned}(r\cos\theta)^2+(r\sin\theta)^2&=9\\ r^2\cos^2\theta+r^2\sin^2\theta&=9\\ r^2(\cos^2\theta+\sin^2\theta)&=9\\ r^2(1)&=9 \qquad \text{Substitute } \cos^2\theta+\sin^2\theta=1\\ r&=\pm 3 \qquad \text{Use the square root property.}\end{aligned}$$

Thus, $x^2+y^2=9$, $r=3$, and $r=-3$ should generate the same graph. See Figure 6.3.1.

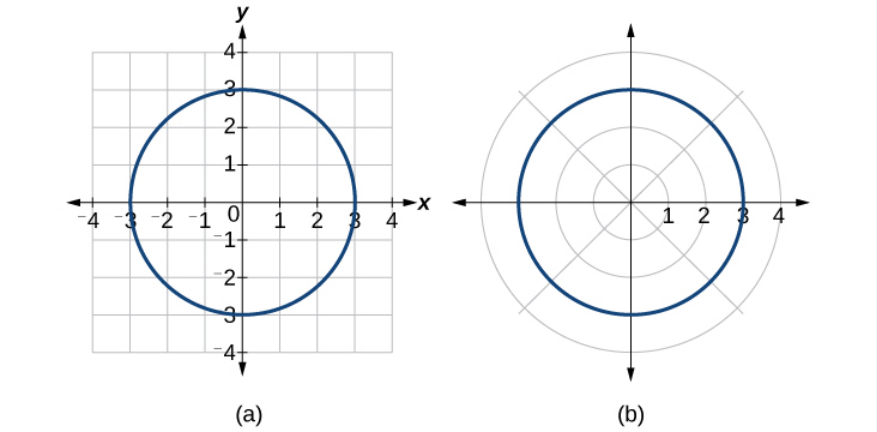


Figure 6.3.1: (a) Cartesian form $x^2+y^2=9$ (b) Polar form $r=3$

To graph a circle in rectangular form, we must first solve for y.

$$\begin{aligned} x^2+y^2 &= 9 \\ y^2 &= 9-x^2 \\ y &= \pm\sqrt{9-x^2} \end{aligned}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1 = \sqrt{9-x^2}$ and $Y_2 = -\sqrt{9-x^2}$. Press **GRAPH.**

Example 6.3.1B: Rewriting a Cartesian Equation as a Polar Equation

Rewrite the **Cartesian equation** $x^2+y^2=6y$ as a polar equation.

Solution

This equation appears similar to the previous example, but it requires different steps to convert the equation.

We can still follow the same procedures we have already learned and make the following substitutions:

$r^2 = 6y$	Use $x^2+y^2=r^2$.
$r^2 = 6r\sin\theta$	Substitute $y = r\sin\theta$.
$r^2 - 6r\sin\theta = 0$	Set equal to 0.
$r(r-6\sin\theta)=0$	Factor and solve.
$r=0$	We reject $r=0$, as it only represents one point, $(0,0)$.
or $r = 6\sin\theta$	

Therefore, the equations $x^2+y^2=6y$ and $r=6\sin\theta$ should give us the same graph. See Figure 6.3.2.

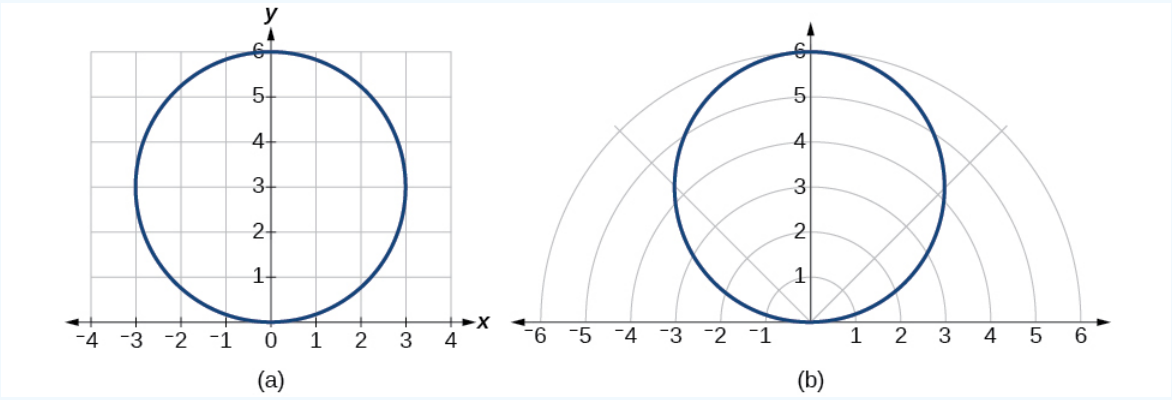


Figure 6.3.2: (a) Cartesian form $x^2+y^2=6y$ (b) polar form $r=6\sin\theta$

The Cartesian or rectangular equation is plotted on the rectangular grid, and the **polar equation** is plotted on the polar grid. Clearly, the graphs are identical.

Exercise 6.3.1A: Rewriting a Cartesian Equation in Polar Form

Rewrite the Cartesian equation $y=3x+2$ as a polar equation.

Answer

We will use the relationships $x = r\cos\theta$ and $y = r\sin\theta$.

$$
\begin{aligned}
y &= 3x + 2 \\
r\sin\theta &= 3r\cos\theta + 2 \\
r\sin\theta - 3r\cos\theta &= 2 \\
r(\sin\theta - 3\cos\theta) &= 2 && \text{Isolate } r. \\
r &= \frac{2}{\sin\theta - 3\cos\theta} && \text{Solve for } r.
\end{aligned}
$$

Exercise 6.3.1B:

Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form.

Answer

$r = \sqrt{3}$

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example 6.3.2A: Graphing a Polar Equation by Converting to a Rectangular Equation

Covert the polar equation $r = 2\sec\theta$ to a rectangular equation, and draw its corresponding graph.

Solution

The conversion is

$$
\begin{aligned}
r &= 2\sec\theta \\
r &= \frac{2}{\cos\theta} \\
r\cos\theta &= 2 \\
x &= 2
\end{aligned}
$$

Notice that the equation $r = 2\sec\theta$ drawn on the polar grid is clearly the same as the vertical line $x = 2$ drawn on the rectangular grid (see Figure 6.3.14). Just as $x = c$ is the standard form for a vertical line in rectangular form, $r = c\sec\theta$ is the standard form for a vertical line in polar form.

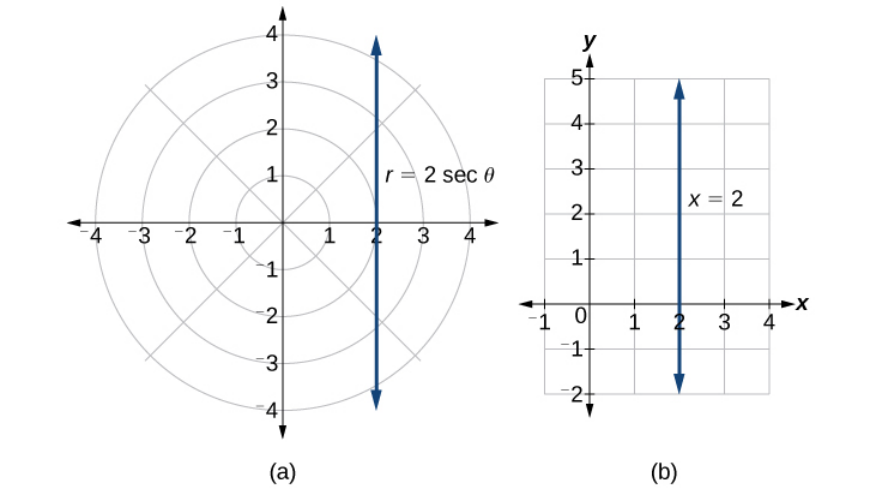


Figure 6.3.3: (a) Polar grid (b) Rectangular coordinate system

A similar discussion would demonstrate that the graph of the function $r = 2\csc\theta$ will be the horizontal line $y = 2$. In fact, $r = c\csc\theta$ is the standard form for a horizontal line in polar form, corresponding to the rectangular form $y = c$.

Example $6.3.2B$: Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \dfrac{3}{1-2\cos\theta}$ as a Cartesian equation.

Solution

The goal is to eliminate θ and r, and introduce x and y. We clear the fraction, and then use substitution. To replace r with x and y, we must use the expression $x^2 + y^2 = r^2$.

$$\begin{aligned} r &= \frac{3}{1-2\cos\theta} \\ r(1-2\cos\theta) &= 3 \\ r\left(1-2\left(\frac{x}{r}\right)\right) &= 3 && \text{Use } \cos\theta = \frac{x}{r} \text{ to eliminate } \theta. \\ r-2x &= 3 \\ r &= 3+2x && \text{Isolate } r. \\ r^2 &= (3+2x)^2 && \text{Square both sides.} \\ x^2+y^2 &= (3+2x)^2 && \text{Use } x^2+y^2=r^2. \end{aligned}$$

The Cartesian equation is $x^2 + y^2 = (3+2x)^2$. However, to graph it, especially using a graphing calculator or computer program, we want to isolate y.

$$\begin{aligned} x^2+y^2 &= (3+2x)^2 \\ y^2 &= (3+2x)^2 - x^2 \\ y &= \pm\sqrt{(3+2x)^2 - x^2} \end{aligned}$$

When our entire equation has been changed from r and θ to x and y, we can stop, unless asked to solve for y or simplify. See Figure 6.3.4.

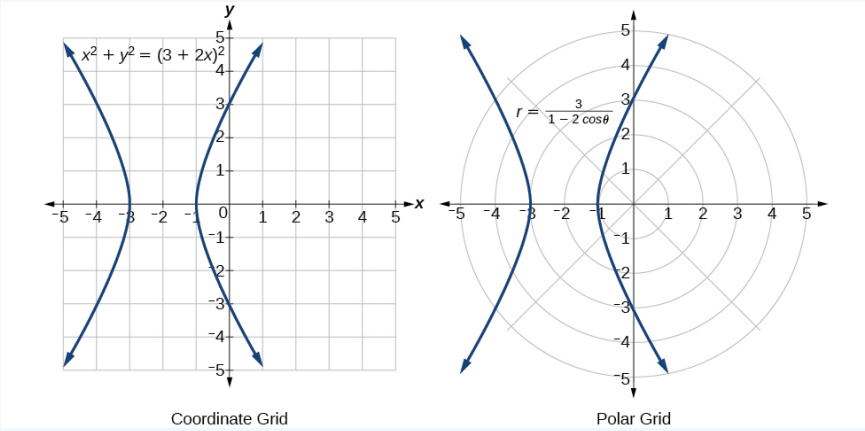


Figure 6.3.4

The "hour-glass" shape of the graph is called a *hyperbola*. Hyperbolas have many interesting geometric features and applications, which we will investigate further in Analytic Geometry.

Analysis

In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However, the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola's standard form. To do this, we can start with the initial equation.

$$
\begin{aligned}
x^2+y^2 &= (3+2x)^2 \\
x^2+y^2-(3+2x)^2 &= 0 \\
x^2+y^2-(9+12x+4x^2) &= 0 \\
x^2+y^2-9-12x-4x^2 &= 0 \\
-3x^2-12x+y^2 &= 9 && \text{Multiply through by } -1. \\
3x^2+12x-y^2 &= -9 \\
3(x^2+4x)-y^2 &= -9 && \text{Organize terms to complete the square for } x. \\
3(x^2+4x+4)-y^2 &= -9+12 \\
3(x+2)^2-y^2 &= 3 \\
(x+2)^2-\frac{y^2}{3} &= 1
\end{aligned}
$$

Exercise 6.3.2

Rewrite the polar equation $r = 2\sin\theta$ in Cartesian form.

Answer

$x^2+y^2=2y$ or, in the standard form for a circle, $x^2+(y-1)^2=1$

Example 6.3.3: Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r=\sin(2\theta)$ in Cartesian form.

Solution

$$
\begin{aligned}
r &= \sin(2\theta) && \text{Use the double angle identity for sine.} \\
r &= 2\sin\theta\cos\theta && \text{Use } \cos\theta=\frac{x}{r} \text{ and } \sin\theta=\frac{y}{r}. \\
r &= 2\left(\frac{x}{r}\right)\left(\frac{y}{r}\right) && \text{Simplify.} \\
r &= \frac{2xy}{r^2} && \text{Multiply both sides by } r^2. \\
r^3 &= 2xy \\
(x^2+y^2)^3 &= 2xy && \text{As } x^2+y^2=r^2, r=\sqrt{x^2+y^2}.
\end{aligned}
$$

This equation can also be written as

$$(x^2+y^2)^{\frac{3}{2}} = 2xy \text{ or } x^2+y^2=(2xy)^{\frac{2}{3}}$$

Extra Practice

For the following exercises, convert the given Cartesian equation to a polar equation.

1. $x=-2$
2. $x^2+y^2=64$
3. $x^2+y^2=-2y$

For the following exercises, convert the given polar equation to a Cartesian equation.

1. $r=7\cos\theta$
2. $r=\frac{-2}{4\cos\theta+\sin\theta}$

For the following exercises, convert to rectangular form and graph.

1. $\theta = \frac{3\pi}{4}$
2. $r = 5 \sec \theta$

Key Equations

Conversion formulas	$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$ $\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$ $r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$

Key Concepts

- Transforming equations between polar and rectangular forms means making the appropriate substitutions based on the available formulas, together with algebraic manipulations. See Example $6.3.1$.
- Using the appropriate substitutions makes it possible to rewrite a polar equation as a rectangular equation, and then graph it in the rectangular plane. See Example $6.3.2$ and Example $6.3.3$.

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6.4: The Polar Form of Complex Numbers

Learning Objectives

- Plot complex numbers in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.

"God made the integers; all else is the work of man." This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

We first encountered complex numbers in the section on Complex Numbers. In this section, we will focus on the mechanics of working with complex numbers: translation of complex numbers from polar form to rectangular form and vice versa, interpretation of complex numbers in the scheme of applications, and application of De Moivre's Theorem.

Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a+bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, a, and the vertical axis represents the imaginary part of the number, bi.

How to: Given a complex number $a+bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary axis.*
2. Plot the point in the complex plane by moving a units in the horizontal direction and b units in the vertical direction.

Example 6.4.1: Plotting a Complex Number in the Complex Plane

Plot the complex number $2-3i$ in the complex plane.

Solution

From the origin, move two units in the positive horizontal direction and three units in the negative vertical direction. See Figure 6.4.1.

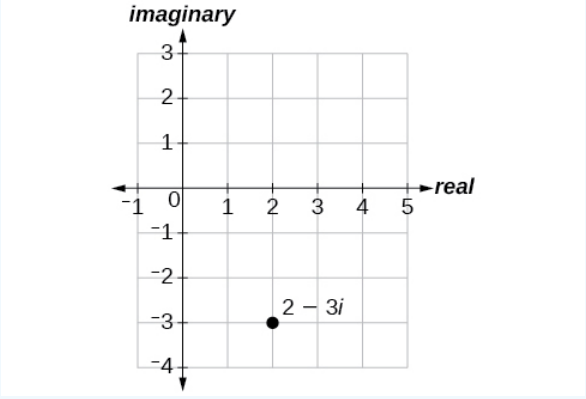


Figure 6.4.1

Exercise 6.4.1

Plot the point $1+5i$ in the complex plane.

Answer

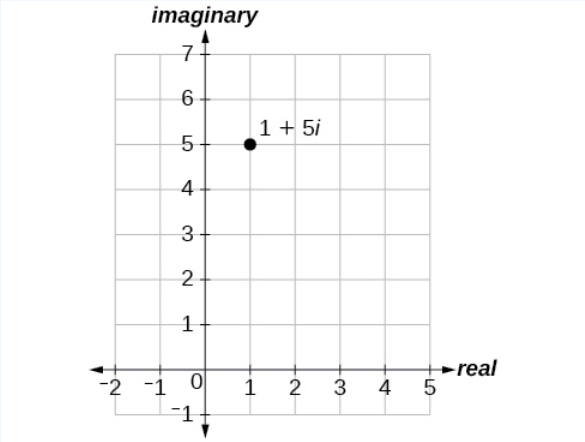


Figure 6.4.2

Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z=2+4i$, in Figure 6.4.3, shows $|z|$.

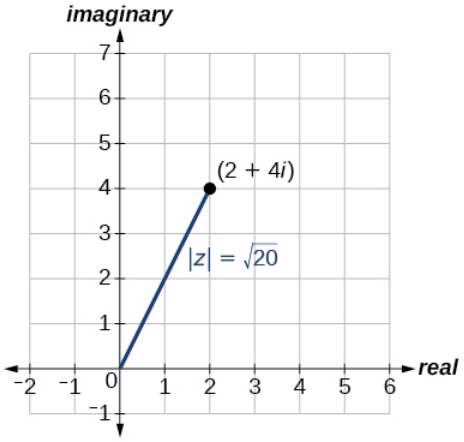


Figure 6.4.3

ABSOLUTE VALUE OF A COMPLEX NUMBER

Given $z=x+yi$, a complex number, the absolute value of z is defined as

$$|z|=\sqrt{x^2+y^2} \tag{6.4.1}$$

It is the distance from the origin to the point (x,y).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, $(0,0)$.

Example 6.4.2: Finding the Absolute Value of a Complex Number with a Radical

Find the absolute value of $z = \sqrt{5} - i$.

Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(\sqrt{5})^2 + (-1)^2}$$
$$|z| = \sqrt{5 + 1}$$
$$|z| = \sqrt{6}$$

See Figure 6.4.4.

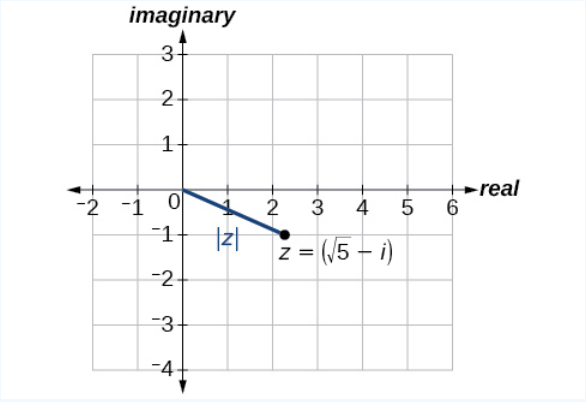


Figure 6.4.4

Exercise 6.4.2

Find the absolute value of the complex number $z = 12 - 5i$.

Answer

13

Example 6.4.3: Finding the Absolute Value of a Complex Number

Given $z = 3 - 4i$, find $|z|$.

Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(3)^2 + (-4)^2}$$
$$|z| = \sqrt{9 + 16}$$
$$|z| = \sqrt{25}$$
$$|z| = 5$$

The absolute value z is 5. See Figure 6.4.5.

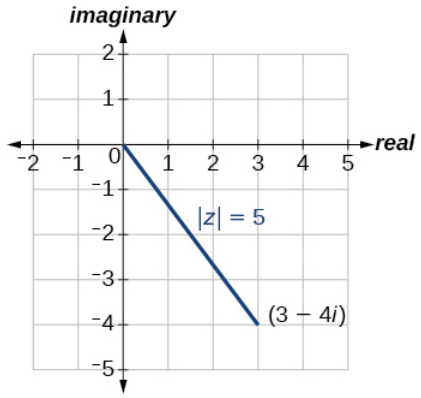


Figure 6.4.5

Exercise 6.4.3

Given $z = 1 - 7i$, find $|z|$.

Answer

$|z| = \sqrt{50} = 5\sqrt{2}$

Writing Complex Numbers in Polar Form

The polar form of a complex number expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

We review these relationships in Figure 6.4.6.

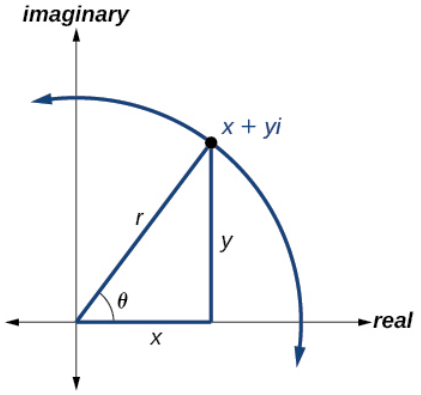


Figure 6.4.6

We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point (x, y). The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$\begin{aligned} z &= x + yi \\ z &= r\cos\theta + (r\sin\theta)i \\ z &= r(\cos\theta + i\sin\theta) \end{aligned}$$

https://math.libretexts.org/@go/page/61271

POLAR FORM OF A COMPLEX NUMBER

Writing a complex number in polar form involves the following conversion formulas:

$$x = r\cos\theta \tag{6.4.2}$$
$$y = r\sin\theta \tag{6.4.3}$$
$$r = \sqrt{x^2+y^2} \tag{6.4.4}$$

Making a direct substitution, we have

$$z = x + yi \tag{6.4.5}$$
$$z = (r\cos\theta) + i(r\sin\theta) \tag{6.4.6}$$
$$z = r(\cos\theta + i\sin\theta) \tag{6.4.7}$$

where r is the modulus and θ is the argument. We often use the abbreviation $r\ cis\theta$ to represent $r(\cos\theta + i\sin\theta)$.

Example 6.4.4: Expressing a Complex Number Using Polar Coordinates

Express the complex number $4i$ using polar coordinates.

Solution

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{0^2+4^2}$$
$$r = \sqrt{16}$$
$$r = 4$$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \frac{\pi}{2}$. In polar coordinates, the complex number $z = 0 + 4i$ can be written as $z = 4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ or $4\ cis\left(\frac{\pi}{2}\right)$. See Figure 6.4.7.

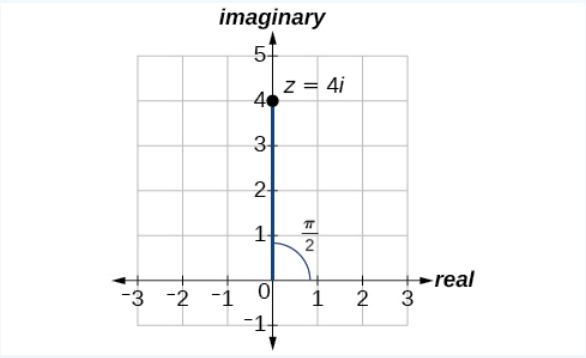


Figure 6.4.7

Exercise 6.4.4

Express $z = 3i$ as $r\ cis\theta$ in polar form.

Answer

$$z = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

Example 6.4.5: Finding the Polar Form of a Complex Number

Find the polar form of $-4+4i$.

Solution

First, find the value of r.

$$r=\sqrt{x^2+y^2}$$
$$r=\sqrt{(-4)^2+(4^2)}$$
$$r=\sqrt{32}$$
$$r=4\sqrt{2}$$

Find the angle θ using the formula:

$$\cos\theta=\frac{x}{r}$$
$$\cos\theta=\frac{-4}{4\sqrt{2}}$$
$$\cos\theta=-\frac{1}{\sqrt{2}}$$
$$\theta=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$
$$=\frac{3\pi}{4}$$

Thus, the solution is $4\sqrt{2}\,cis\left(\frac{3\pi}{4}\right)$.

Exercise 6.4.5

Write $z=\sqrt{3}+i$ in polar form.

Answer

$$z=2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given $z=r(\cos\theta+i\sin\theta)$, first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$. Then, multiply through by r.

Example 6.4.6A: Converting from Polar to Rectangular Form

Convert the polar form of the given complex number to rectangular form:

$$z=12\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

Solution

We begin by evaluating the trigonometric expressions.

$$\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\text{ and }\sin(\frac{\pi}{6})=\frac{1}{2}$$

After substitution, the complex number is

$$z=12\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

We apply the distributive property:

$$\begin{aligned} z &= 12\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) \\ &= (12)\frac{\sqrt{3}}{2}+(12)\frac{1}{2}i \\ &= 6\sqrt{3}+6i \end{aligned}$$

The rectangular form of the given point in complex form is $6\sqrt{3}+6i$.

Example 6.4.6B: Finding the Rectangular Form of a Complex Number

Find the rectangular form of the complex number given $r=13$ and $\tan\theta=\frac{5}{12}$.

Solution

If $\tan\theta=\frac{5}{12}$, and $\tan\theta=\frac{y}{x}$, we first determine $r=\sqrt{x^2+y^2}=\sqrt{122+52}=13$. We then find $\cos\theta=\frac{x}{r}$ and $\sin\theta=\frac{y}{r}$.

$$\begin{aligned} z &= 13\left(\cos\theta+i\sin\theta\right) \\ &= 13\left(\frac{12}{13}+\frac{5}{13}i\right) \\ &= 12+5i \end{aligned}$$

The rectangular form of the given number in complex form is $12+5i$.

Exercise 6.4.6

Convert the complex number to rectangular form:

$$z=4\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$$

Answer

$z=2\sqrt{3}-2i$

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham **de Moivre** (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

PRODUCTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)] \tag{6.4.8}$$

$$z_1z_2 = r_1r_2\,cis(\theta_1+\theta_2) \tag{6.4.9}$$

Notice that the product calls for multiplying the moduli and adding the angles.

Example 6.4.7: Finding the Product of Two Complex Numbers in Polar Form

Find the product of z_1z_2, given $z_1=4(\cos(80^\circ)+i\sin(80^\circ))$ and $z_2=2(\cos(145^\circ)+i\sin(145^\circ))$.

Solution

Follow the formula

$$\begin{aligned} z_1 z_2 &= 4 \cdot 2[\cos(80^\circ + 145^\circ) + i \sin(80^\circ + 145^\circ)] \\ z_1 z_2 &= 8[\cos(225^\circ) + i \sin(225^\circ)] \\ z_1 z_2 &= 8\left[\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)\right] \\ z_1 z_2 &= 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ z_1 z_2 &= -4\sqrt{2} - 4i\sqrt{2} \end{aligned}$$

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

QUOTIENTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i \sin\theta_2)$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)],\ z_2 \neq 0 \tag{6.4.10}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\, cis(\theta_1 - \theta_2),\ z_2 \neq 0 \tag{6.4.11}$$

Notice that the moduli are divided, and the angles are subtracted.

How to: Given two complex numbers in polar form, find the quotient

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r(\cos\theta + i \sin\theta)$. Replace r with $\frac{r_1}{r_2}$, and replace θ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by r.

Example 6.4.8: Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2(\cos(213^\circ) + i \sin(213^\circ))$ and $z_2 = 4(\cos(33^\circ) + i \sin(33^\circ))$.

Solution

Using the formula, we have

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2}{4}[\cos(213^\circ - 33^\circ) + i \sin(213^\circ - 33^\circ)] \\ \frac{z_1}{z_2} &= \frac{1}{2}[\cos(180^\circ) + i \sin(180^\circ)] \\ \frac{z_1}{z_2} &= \frac{1}{2}[-1 + 0i] \\ \frac{z_1}{z_2} &= -\frac{1}{2} + 0i \\ \frac{z_1}{z_2} &= -\frac{1}{2} \end{aligned}$$

Exercise 6.4.8

Find the product and the quotient of $z_1 = 2\sqrt{3}(\cos(150^\circ) + i \sin(150^\circ))$ and $z_2 = 2(\cos(30^\circ) + i \sin(30^\circ))$.

Answer

$z_1 z_2 = -4\sqrt{3}\,;\ \dfrac{z_1}{z_2} = -\dfrac{\sqrt{3}}{2} + \dfrac{3}{2}i$

Extra Practice

For the following exercises, plot the complex number in the complex plane.

1. $6 - 2\mathbf{i}$
2. $-1 + 3\mathbf{i}$

For the following exercises, find the absolute value of each complex number.

1. $-2 + 6\mathbf{i}$
2. $4 - 3\mathbf{i}$

Write the complex number in polar form.

1. $5 + 9\mathbf{i}$
2. $\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}$

For the following exercises, convert the complex number from polar to rectangular form.

1. $z = 5\,\text{cis}\left(\frac{5\pi}{6}\right)$
2. $z = 3\,\text{cis}(40^\circ)$

For the following exercises, find the product $z_1 z_2$ in polar form.

1.

$z_1 = 2\,\text{cis}(89^\circ)$

$z_2 = 5\,\text{cis}(23^\circ)$

2.

$z_1 = 10\ \text{cis}\ \left(\frac{\pi}{6}\right)$

$z_2 = 6\,\text{cis}\left(\frac{\pi}{3}\right)$

For the following exercises, find the quotient $\frac{z_1}{z_2}$ in polar form.

1.

$z_1 = 12\,\text{cis}(55^\circ)$

$z_2 = 3\,\text{cis}(18^\circ)$

2.

$z_1 = 27\,\text{cis}\left(\frac{5\pi}{3}\right)$

$z_2 = 9\,\text{cis}\left(\frac{\pi}{3}\right)$

Key Concepts

- Complex numbers in the form $a + bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the x-axis as the real axis and the y-axis as the *imaginary* axis. See Example 6.4.1.
- The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2 + b^2}$. See Example 6.4.2 and Example 6.4.3.
- To write complex numbers in polar form, we use the formulas $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$. Then, $z = r(\cos\theta + i\sin\theta)$. See Example 6.4.4 and Example 6.4.5.
- To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by r. See Example 6.4.6.
- To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See Example 6.4.7.

- To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See Example 6.4.8.

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6.5: De Moivre's and the nth Root Theorem

Learning Objectives

- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using **De Moivre's Theorem**. It states that, for a positive integer n, z^n is found by raising the modulus to the n^{th} power and multiplying the argument by n. It is the standard method used in modern mathematics.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] \tag{6.5.1}$$
$$z^n = r^n\, cis(n\theta) \tag{6.5.2}$$

where n is a positive integer.

Example 6.5.1: Evaluating an Expression Using De Moivre's Theorem

Evaluate the expression $(1+i)^5$ using De Moivre's Theorem.

Solution

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1+i)$ in polar form. Let us find r.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(1)^2 + (1)^2}$$
$$r = \sqrt{2}$$

Then we find θ. Using the formula $\tan\theta = \dfrac{y}{x}$ gives

$$\tan\theta = \frac{1}{1}$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

Use De Moivre's Theorem to evaluate the expression.

$$(a+bi)^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$
$$(1+i)^5 = (\sqrt{2})^5\left[\cos\left(5\cdot\frac{\pi}{4}\right) + i\sin\left(5\cdot\frac{\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$(1+i)^5 = -4 - 4i$$

Finding Roots of Complex Numbers in Polar Form

To find the n^{th} root of a complex number in polar form, we use the n^{th} Root Theorem or De Moivre's Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding n^{th} roots of

complex numbers in polar form.

THE N^{TH} ROOT THEOREM

To find the n^{th} root of a complex number in polar form, use the formula given as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right] \tag{6.5.3}$$

where $k = 0, 1, 2, 3, \ldots, n-1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

Example 6.5.2: the Root of a Complex Number

Evaluate the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)$.

Solution

We have

$$\begin{aligned} z^{\frac{1}{3}} &= 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)\right] \\ z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)\right] \end{aligned}$$

There will be three roots: $k = 0, 1, 2$. When $k = 0$, we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)$$

When $k = 1$, we have

$$\begin{aligned} z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)\right] \qquad \text{Add } \frac{2(1)\pi}{3} \text{ to each angle.} \\ z^{\frac{1}{3}} &= 2\left(\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)\right) \end{aligned}$$

When $k = 2$, we have

$$\begin{aligned} z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)\right] \qquad \text{Add } \frac{2(2)\pi}{3} \text{ to each angle.} \\ z^{\frac{1}{3}} &= 2\left(\cos\left(\frac{14\pi}{9}\right)+i\sin\left(\frac{14\pi}{9}\right)\right) \end{aligned}$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k = 1$, the angle simplification is

$$\begin{aligned} \frac{\frac{2\pi}{3}}{3}+\frac{2(1)\pi}{3} &= \frac{2\pi}{3}\left(\frac{1}{3}\right)+\frac{2(1)\pi}{3}\left(\frac{3}{3}\right) \\ &= \frac{2\pi}{9}+\frac{6\pi}{9} \\ &= \frac{8\pi}{9} \end{aligned}$$

Exercise 6.5.1

Find the four fourth roots of $16(\cos(120^\circ)+i\sin(120^\circ))$.

Answer

$$z_0 = 2(\cos(30^\circ) + i\sin(30^\circ))$$
$$z_1 = 2(\cos(120^\circ) + i\sin(120^\circ))$$
$$z_2 = 2(\cos(210^\circ) + i\sin(210^\circ))$$
$$z_3 = 2(\cos(300^\circ) + i\sin(300^\circ))$$

Extra Practice

For the following exercises, find the powers of each complex number in polar form.

1. Find z^4 when $z = 2 \text{ cis } (70^\circ)$
2. Find z^2 when $z = 5 \text{ cis } \left(\frac{3\pi}{4}\right)$

For the following exercises, evaluate each root.

1. Evaluate the cube root of z when $z = 64 \text{ cis } (210^\circ)$.
2. Evaluate the square root of z when $z = 25 \text{ cis } \left(\frac{3\pi}{2}\right)$.

Key Concepts

- To find the power of a complex number z^n, raise r to the power n, and multiply θ by n. See Example $6.5.1$.
- Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See Example $6.5.2$.

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