TRIGONOMETRY



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Trigonometry

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1: THE SIX TRIGONOMETRIC FUNCTIONS

The trigonometric functions are functions of an angle and relate the angles of a triangle to the lengths of its sides. They are important in the study of triangles and modeling periodic phenomena, among many other applications.

1.1: Angles

1.2: Right Triangle Trigonometry

1.3: Unit Circle

1.4: The Other Trigonometric Functions

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1.1: Angles

Learning Objectives

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Find the area of a sector of a circle

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a *ray*. A ray consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its *endpoint* and any other point on it. The ray in Figure 1.1.1 can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

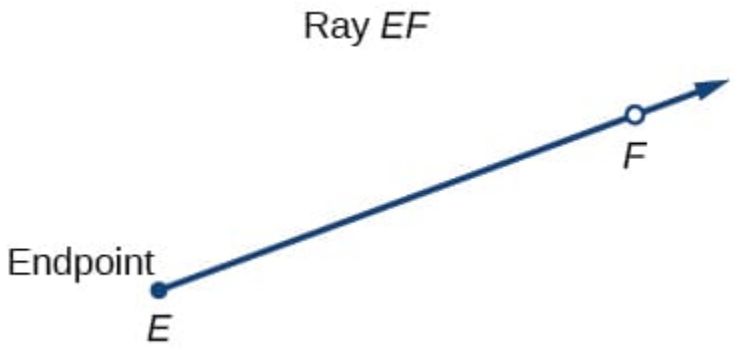


Figure 1.1.1

An *angle* is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in Figure 1.1.2 is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

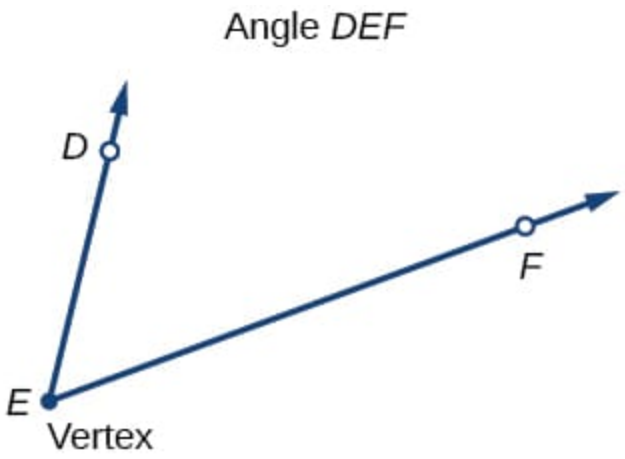


Figure 1.1.2

Greek letters are often used as variables for the measure of an angle. Table 1.1.1 is a list of Greek letters commonly used to represent angles, and a sample angle is shown in Figure 1.1.3.

Table 1.1.1

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

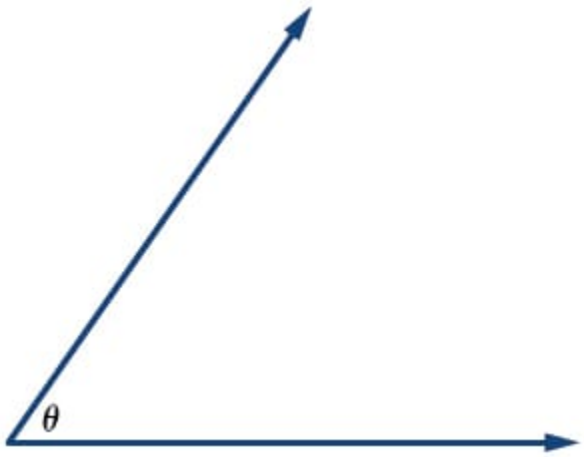


Figure 1.1.3: *Angle theta, shown as* $\angle\theta$

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 1.1.4.

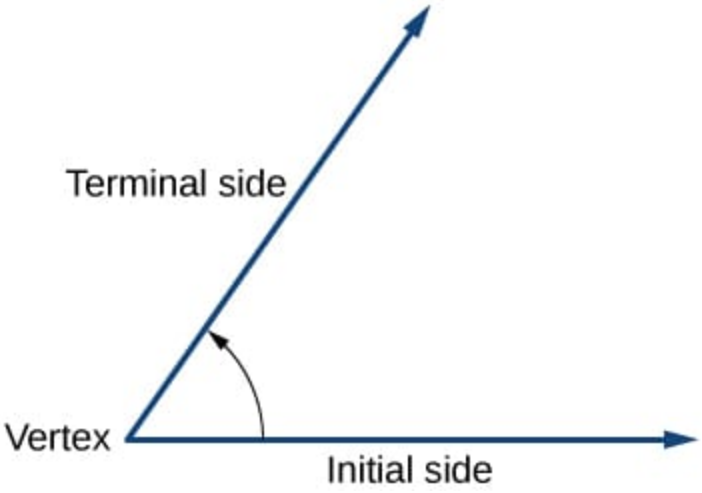


Figure 1.1.4

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol °. For example, 90 degrees = 90°.

To formalize our work, we will begin by drawing angles on an *x*-*y* coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See Figure 1.1.5.

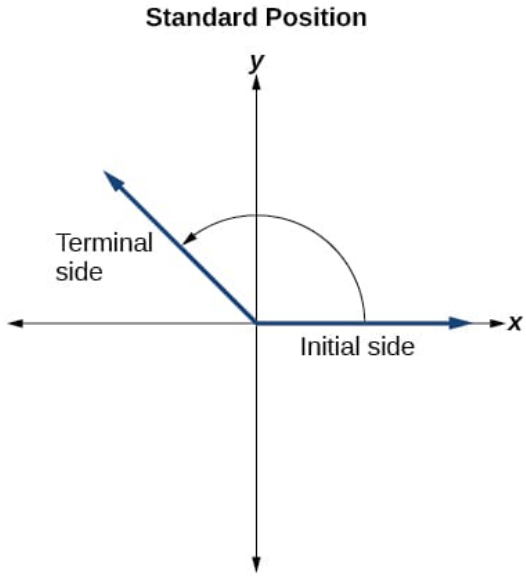


Figure 1.1.5

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90°}{360°} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive *x*-axis. To draw a 360° angle, we calculate that $\frac{360°}{360°} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive *x*-axis. In this case,

the initial side and the terminal side overlap. See Figure 1.1.6 .

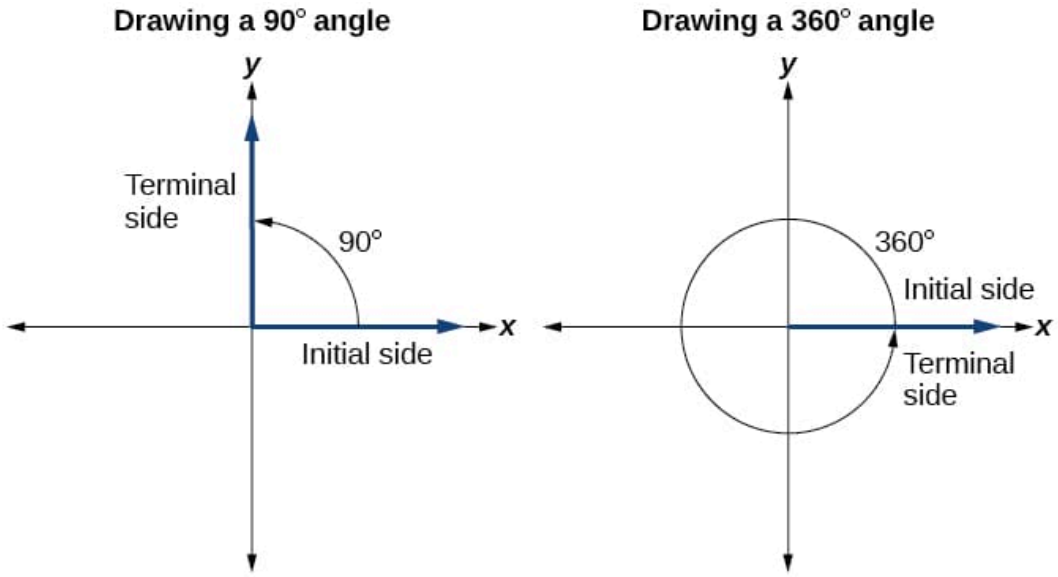


Figure 1.1.6

Since we define an angle in **standard position** by its initial side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See Figure 1.1.7.

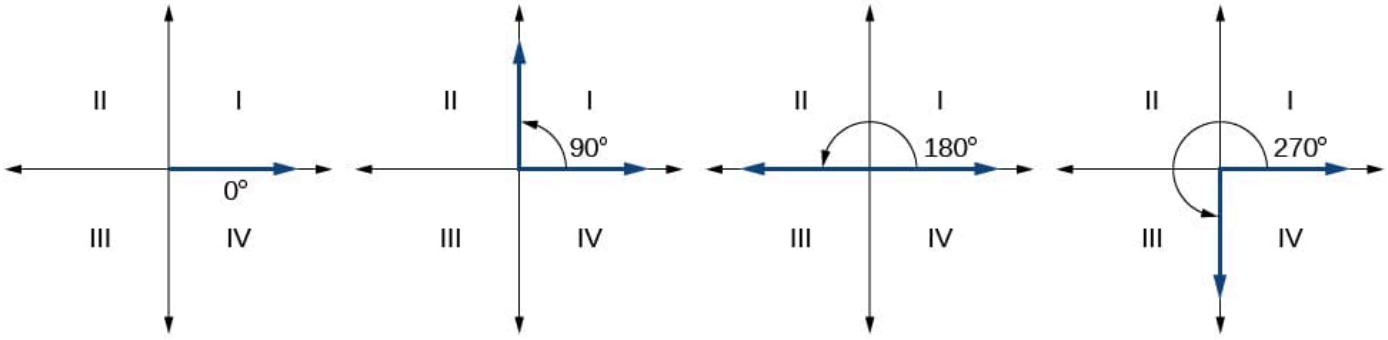


Figure 1.1.7: Quadrantal angles are angles in standard position whose terminal side lies along an axis. Examples are shown.

QUADRANTAL ANGLES

Quadrantal angles are angles in standard position whose terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

Given an angle measure in degrees, draw the angle in standard position

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 1.1.1: Drawing an Angle in Standard Position Measured in Degrees

1. Sketch an angle of 30° in standard position.
2. Sketch an angle of –135° in standard position.

Solution

1. Divide the angle measure by 360°.

$$\frac{30^\circ}{360^\circ} = \frac{1}{12} \tag{1.1.1}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}(\frac{1}{4}) \tag{1.1.2}$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in Figure 1.1.8.

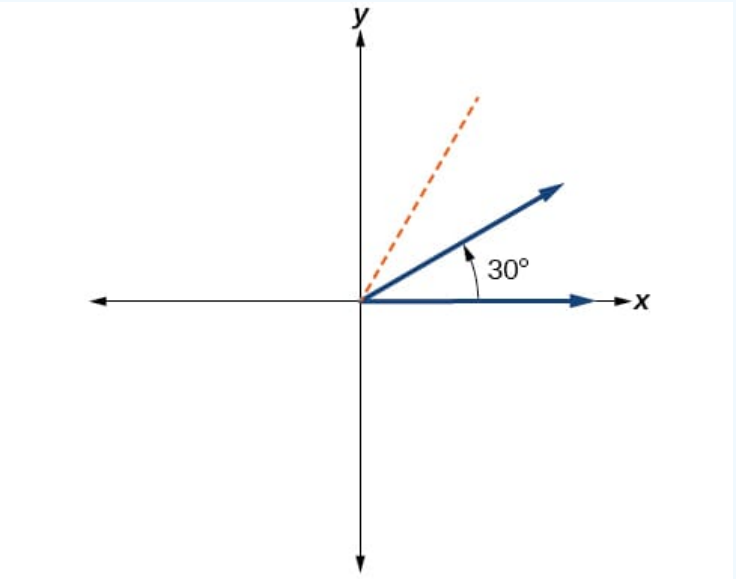


Figure 1.1.8

2. Divide the angle measure by 360°.

$$\frac{-135^\circ}{360^\circ} = -\frac{3}{8} \tag{1.1.3}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}(\frac{1}{4}) \tag{1.1.4}$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in Figure 1.1.9.

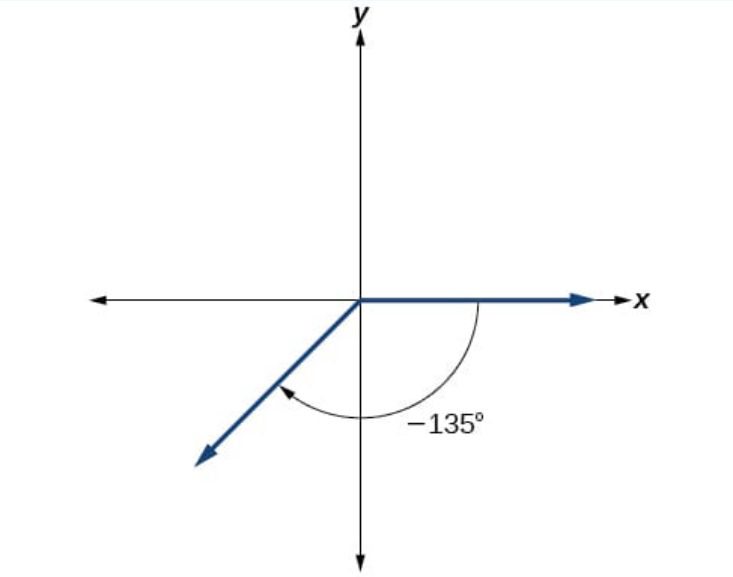


Figure 1.1.9

1.1.1:

Show an angle of 240° on a circle in standard position.

Solution

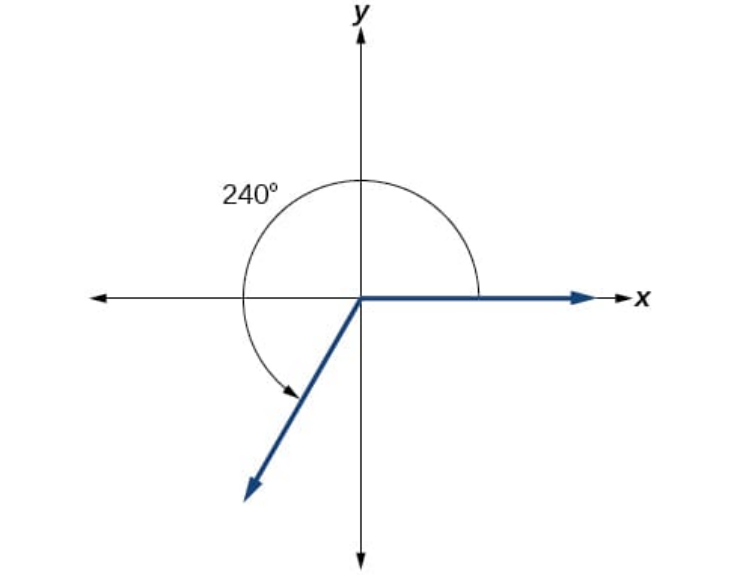


Figure 1.1.10

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the **circumference** of

that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in Figure 1.1.11.

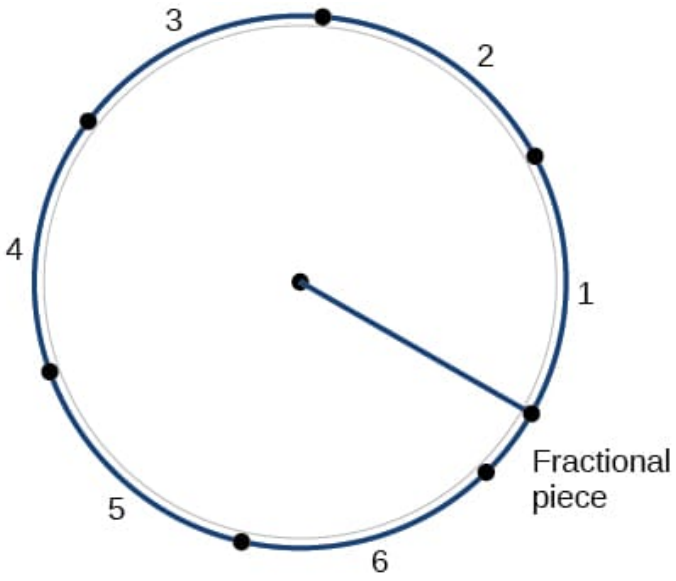


Figure 1.1.11

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians } = 360^\circ \tag{1.1.5}$$

$$\pi \text{ radians } = \frac{360^\circ}{2} = 180^\circ \tag{1.1.6}$$

$$1 \text{ radian } = \frac{180^\circ}{\pi} \approx 57.3^\circ \tag{1.1.7}$$

See Figure 1.1.12. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

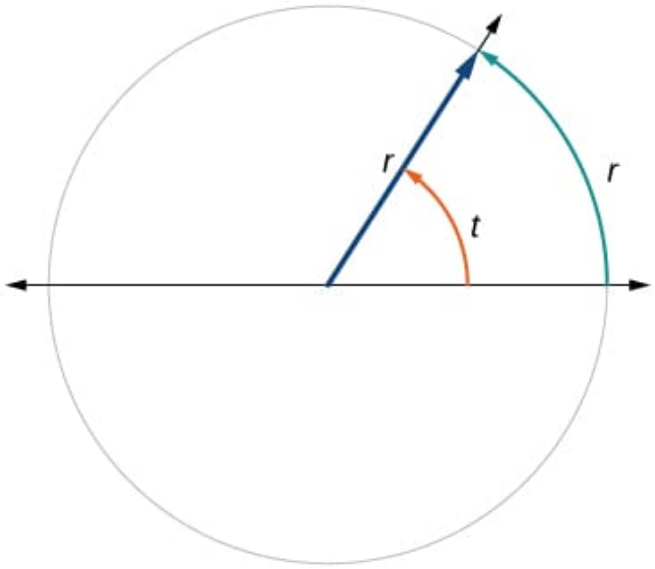


Figure 1.1.12: The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the **radian measure**, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s s to the radius r. See Figure 1.1.13.

$$s = r\theta \tag{1.1.8}$$

$$\theta = \frac{s}{r} \tag{1.1.9}$$

If $s = r$, then $\theta = \frac{r}{r} = 1 \text{ radian}.$

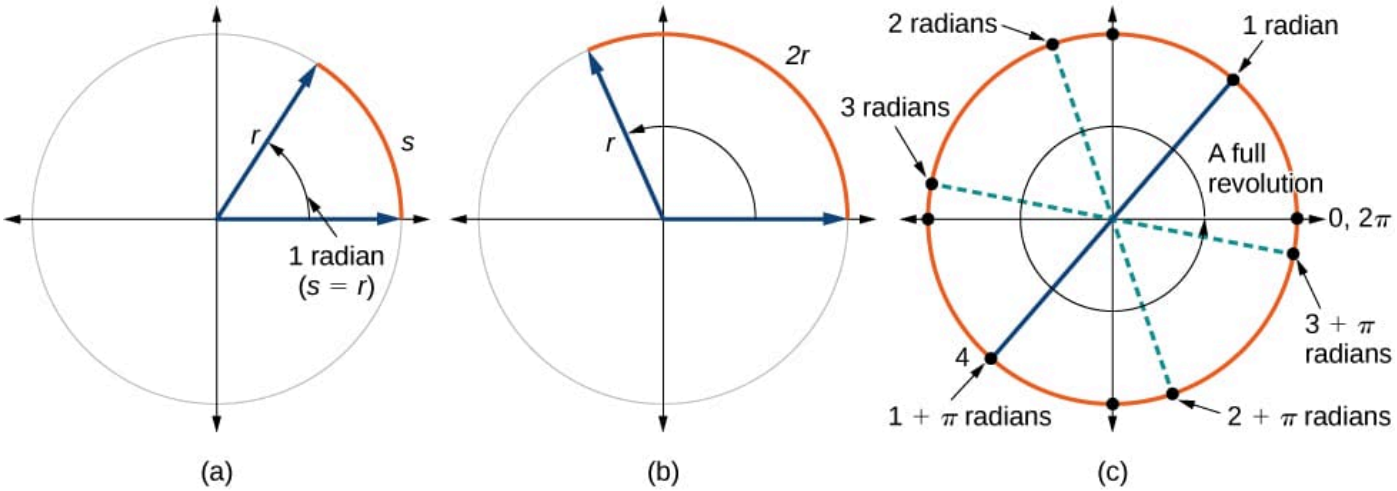


Figure 1.1.13: (a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$,where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in Figure 1.1.14.

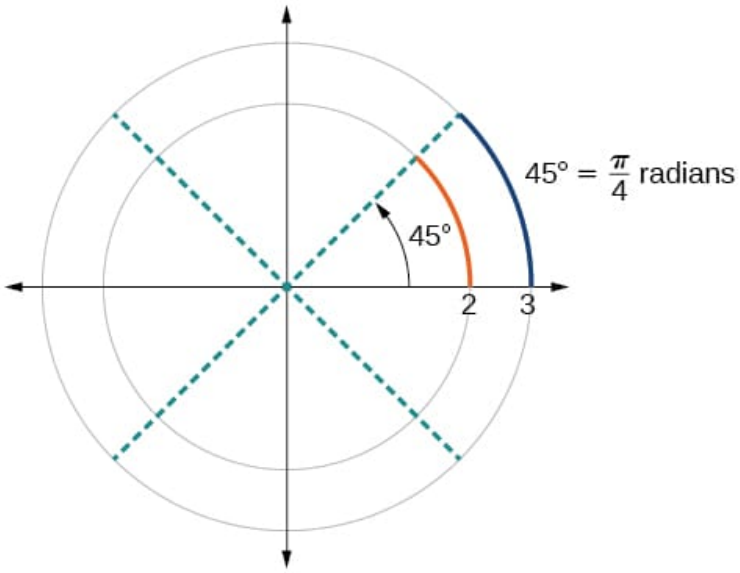


Figure 1.1.14: A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi \tag{1.1.10}$$

$$\text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi \tag{1.1.11}$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

RADIANS

One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π radians equals 360°, 1 radian equals $\frac{360°}{2\pi} \approx 57.3$ °.

Using Radians

Because **radian** measure is the ratio of two lengths, it is a unitless measure. For example, in Figure 1.1.14, suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the “inches” cancel, and we have a result without units. Therefore, it is not necessary to write the label “radians” after a radian measure, and if we see an angle that is not labeled with “degrees” or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the **unit circle** (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{aligned} 1\text{rotation} &= 360° &&= 2\pi &&\text{radians} \\ \frac{1}{2}\text{rotation} &= 180° &&= \pi &&\text{radians} \\ \frac{1}{4}\text{rotation} &= 90° &&= \frac{\pi}{2} &&\text{radians} \end{aligned} \tag{1.1.12}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in Figure 1.1.15. Memorizing these angles will be very useful as we study the properties associated with angles.

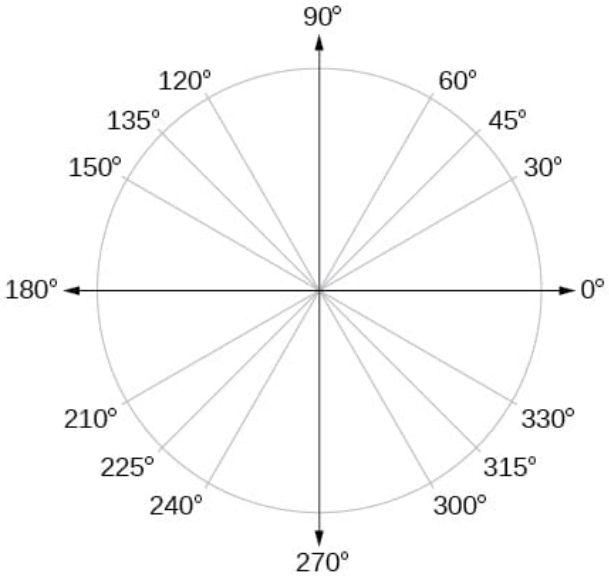


Figure 1.1.15: Commonly encountered angles measured in degrees

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in Figure 1.1.15, which are shown in Figure 1.1.16. Be sure you can verify each of these measures.

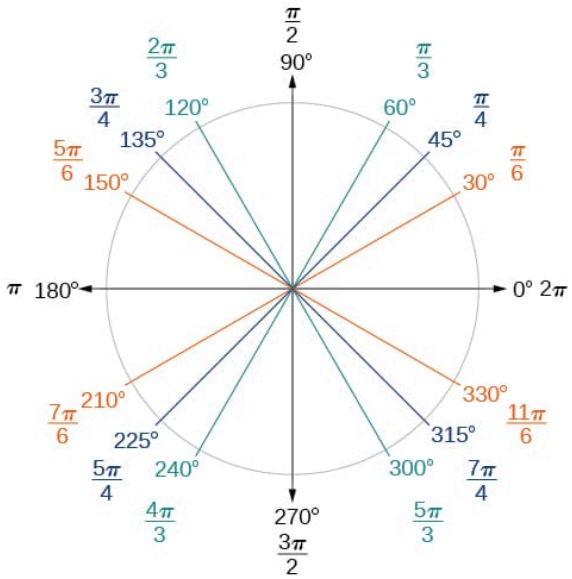


Figure 1.1.16: Commonly encountered angles measured in radians

Example 1.1.2: Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Solution

For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$1 \text{ rotation} = 2\pi r \tag{1.1.13}$$

So,

$$s = \frac{1}{3}(2\pi r) \tag{1.1.14}$$

$$= \frac{2\pi r}{3} \tag{1.1.15}$$

The radian measure would be the arc length divided by the radius.

$$\text{radian measure} = \frac{\frac{2\pi r}{3}}{r} \tag{1.1.16}$$

$$= \frac{2\pi r}{3r} \tag{1.1.17}$$

$$= \frac{2\pi}{3} \tag{1.1.18}$$

1.1.2

Find the radian measure of three-fourths of a full rotation.

Solution

$$\frac{3\pi}{2} \tag{1.1.19}$$

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{1.1.20}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π. We can use fact that $180^\circ = \pi$ radians to convert between degree and radian measure.

CONVERTING BETWEEN RADIANS AND DEGREES

To convert an angle from radian measure to degree measure, multiply the value by the fraction $\frac{180^\circ}{\pi}$ and simplify

To convert an angle from degree measure to radian measure, multiply the value by the fraction $\frac{\pi}{180^\circ}$ and simplify

Example 1.1.3: Converting Radians to Degrees

Convert each radian measure to degrees.

1. $\frac{\pi}{6}$
2. 3

Solution

1. Because we are given radians and we want degrees, we should multiply by $\frac{180^\circ}{\pi}$ and simplify

$$(\frac{\pi}{6})(\frac{180^\circ}{\pi}) = \frac{\pi 180^\circ}{6\pi} = \frac{180^\circ}{6} = 30^\circ \tag{1.1.21}$$

2. Because we are given radians and we want degrees, we should multiply by $\frac{180^\circ}{\pi}$ and simplify

$$3(\frac{180^\circ}{\pi}) = \frac{3 * 180^\circ}{\pi} = \frac{540^\circ}{\pi} \approx 171.89^\circ \tag{1.1.22}$$

1.1.3

Convert $-\frac{3\pi}{4}$ radians to degrees.

Solution

–135°

Example 1.1.4: Converting Degrees to Radians

Convert 15 degrees to radians.

Solution

In this example, we start with degrees and want radians, so we multiply by $\frac{\pi}{180^\circ}$ and simplify

$$15 * (\frac{\pi}{180^\circ}) = \frac{15\pi}{180^\circ} = \frac{\pi}{12} \tag{1.1.23}$$

1.1.4

Convert 126° to radians.

Solution

$\frac{7\pi}{10}$

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at Figure 1.1.17. The angle of 140° is a **positive angle**, measured counterclockwise. The angle of –220° is a **negative angle**, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are **coterminal** angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

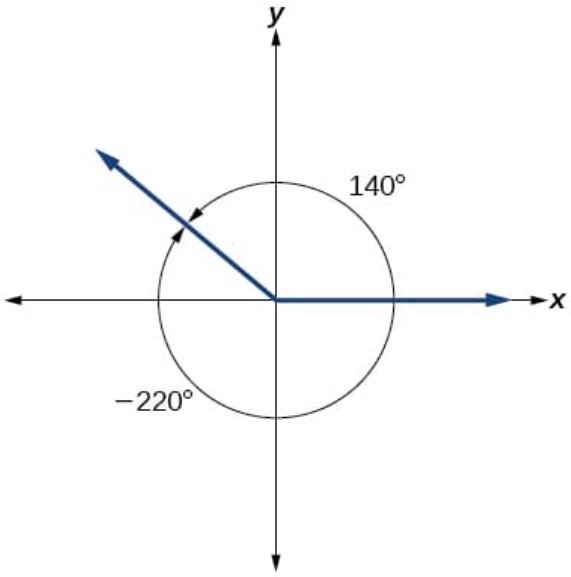


Figure 1.1.17: An angle of 140° and an angle of –220° are coterminal angles.

Any angle has infinitely many **coterminal angles** because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is –260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See Figure 1.1.18 for examples of reference angles for angles in different quadrants. In the figure, t is the given angle and t' is the reference angle.

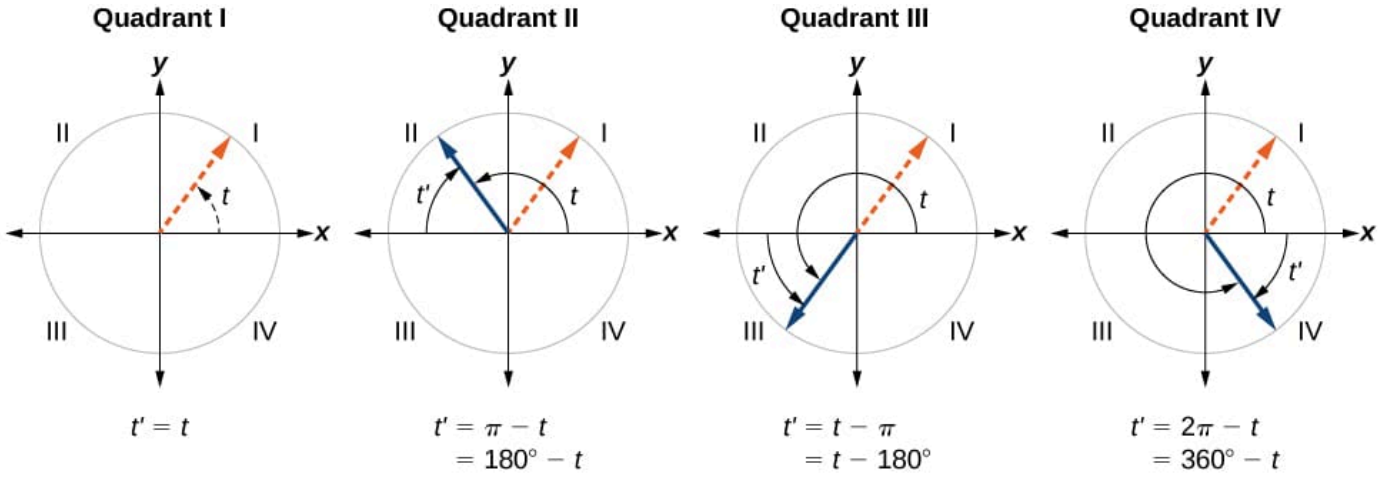


Figure 1.1.18

COTERMINAL AND REFERENCE ANGLES

- *Coterminal angles* are two angles in standard position that have the same terminal side.

- An angle's *reference angle* is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

Given an angle greater than 360°, find a coterminal angle between 0° and 360°

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 1.1.5: Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0^\circ \le \theta < 360^\circ$.

Solution

An angle with measure 800° is coterminal with an angle with measure 800 – 360 = 440°, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: 440 – 360 = 80°.

The angle $\theta = 80^\circ$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in Figure 1.1.19.

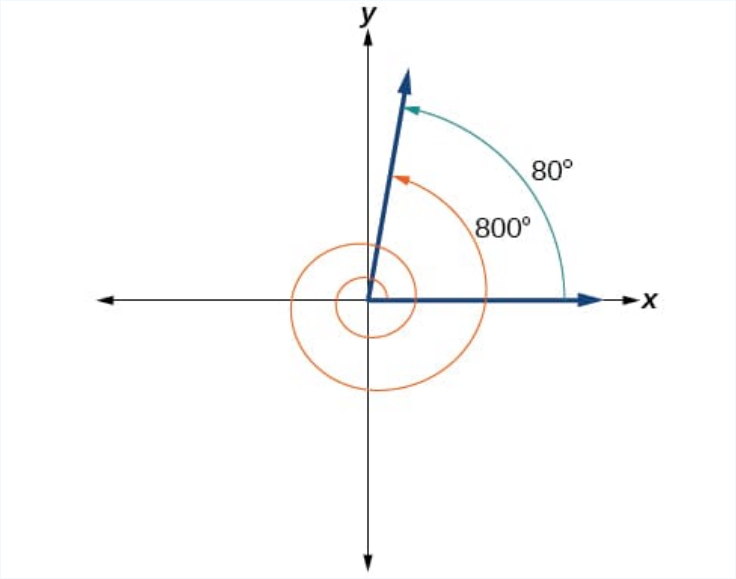


Figure 1.1.19

1.1.5

Find an angle α that is coterminal with an angle measuring 870°, where $0^\circ \le \alpha < 360^\circ$.

Solution

$\alpha = 150^\circ$

Given an angle with measure less than 0°, find a coterminal angle having a measure between 0° and 360°.

1. Add 360° to the given angle.
2. If the result is still less than 0°, add 360° again until the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 1.1.6: Finding an Angle Coterminal with an Angle Measuring Less Than 0°

Show the angle with measure −45° on a circle and find a positive coterminal angle α such that 0° ≤ α < 360°.

Solution

Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$$-45°+360°= 315° \tag{1.1.24}$$

We can then show the angle on a circle, as in Figure 1.1.20.

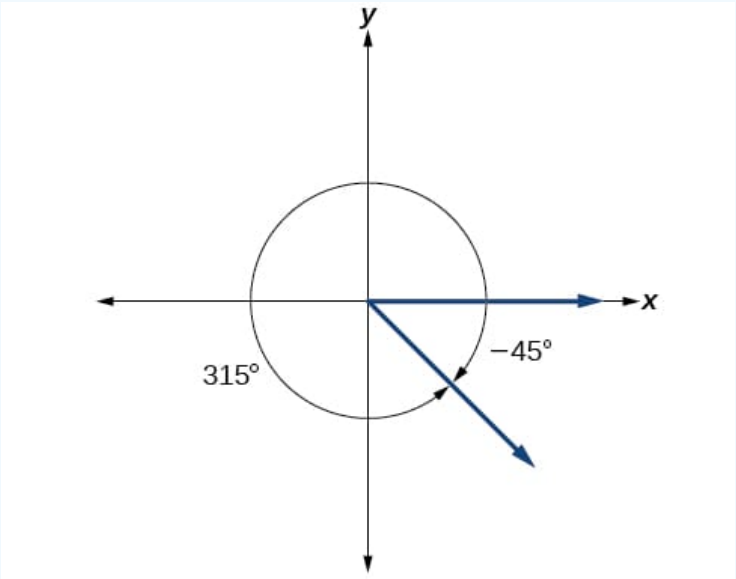


Figure 1.1.20

1.1.6

Find an angle β that is coterminal with an angle measuring −300° such that $0\,^\circ \leq \beta < 360\,^\circ$.

Solution

$\beta = 60\,^\circ$

Finding Coterminal Angles Measured in Radians

We can find **coterminal** angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example 1.1.7: Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \leq \beta < 2\pi$.

Solution

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} \tag{1.1.25}$$

$$= \frac{11\pi}{4} \tag{1.1.26}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} \tag{1.1.27}$$

$$= \frac{3\pi}{4} \tag{1.1.28}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in Figure 1.1.21.

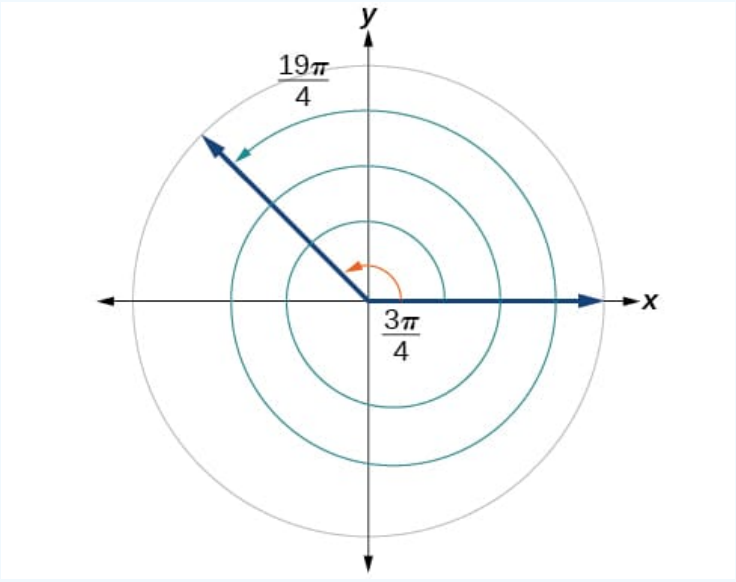


Figure 1.1.21

1.1.7

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Solution

$\frac{7\pi}{6}$

Determining the Length of an Arc

Recall that the **radian measure** θ of an angle was defined as the ratio of the **arc length** s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

ARC LENGTH ON A CIRCLE

In a circle of radius *r*, the length of an arc s subtended by an angle with measure θ in radians, shown in Figure 1.1.22, is

$$s = r\theta \tag{1.1.29}$$

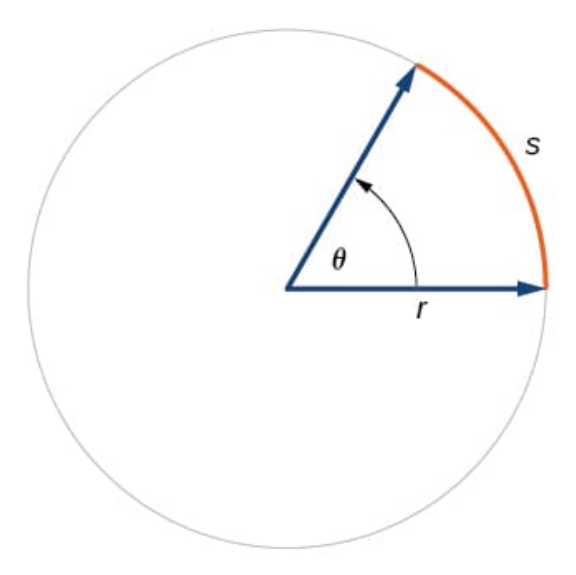


Figure 1.1.22

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ : $s = r\theta$.

Example 1.1.8: Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Solution

1. Let's begin by finding the circumference of Mercury's orbit.

$$\begin{aligned} C &= 2\pi r && (1.1.30)\\ &= 2\pi(36 \text{ million miles}) && (1.1.31)\\ &\approx 226 \text{ million miles} && (1.1.32) \end{aligned}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles} \tag{1.1.33}$$

2. Now, we convert to radians:

$$\text{radian} = \frac{\text{arc length}}{\text{radius}} \tag{1.1.34}$$
$$= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} \tag{1.1.35}$$
$$= 0.0717 \tag{1.1.36}$$

1.1.8

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Solution

$$\frac{215\pi}{18} = 37.525 \text{ units} \tag{1.1.37}$$

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a **sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\text{Area of sector} = (\frac{\theta}{2\pi})\pi r^2 \tag{1.1.38}$$
$$= \frac{\theta \pi r^2}{2\pi} \tag{1.1.39}$$
$$= \frac{1}{2}\theta r^2 \tag{1.1.40}$$

AREA OF A SECTOR

The area of a **sector of a circle** with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2 \tag{1.1.41}$$

See Figure 1.1.23.

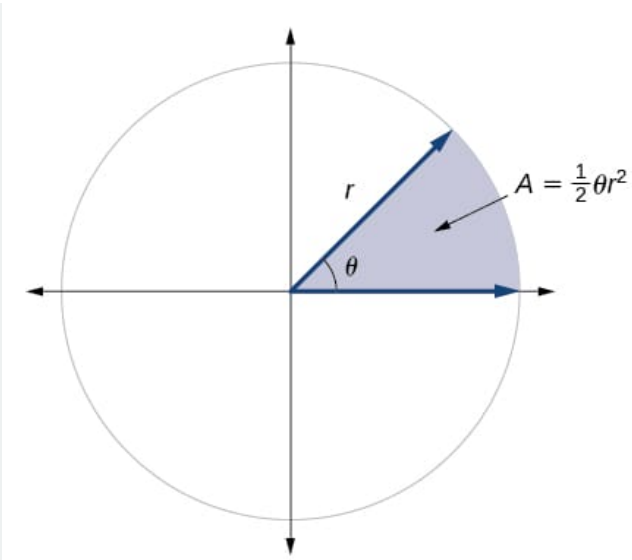


Figure 1.1.23: The area of the sector equals half the square of the radius times the central angle measured in radians.

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r : $A = \frac{1}{2}\theta r^2$.

Example 1.1.9: Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in Figure 1.1.24. What is the area of the sector of grass the sprinkler waters?

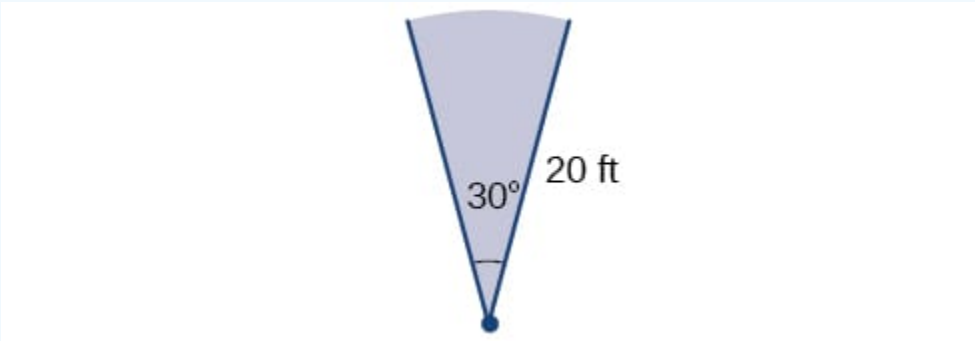


Figure 1.1.24: The sprinkler sprays 20 ft within an arc of 30°.

Solution

First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$30 \text{ degrees} = 30 \cdot \frac{\pi}{180} \tag{1.1.42}$$

$$= \frac{\pi}{6} \text{ radians} \tag{1.1.43}$$

The area of the sector is then

$$\text{Area} = \frac{1}{2}(\frac{\pi}{6})(20)^2 \tag{1.1.44}$$
$$\approx 104.72 \tag{1.1.45}$$

So the area is about 104.72 ft^2.

1.1.9

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Solution

1.88

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See **Example.**
- In addition to degrees, the measure of an angle can be described in radians. See **Example.**
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$. See Example and Example.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π. See Example and Example.
- Coterminal angles can be found using radians just as they are for degrees. See **Example.**
- The length of a circular arc is a fraction of the circumference of the entire circle. See **Example.**
- The area of sector is a fraction of the area of the entire circle. See **Example.**

Glossary

angle
: the union of two rays having a common endpoint

arc length
: the length of the curve formed by an arc

area of a sector
: area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$ multiplied by the area of the entire circle

coterminal angles
: description of positive and negative angles in standard position sharing the same terminal side

degree
: a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

initial side
: the side of an angle from which rotation begins

measure of an angle
: the amount of rotation from the initial side to the terminal side

negative angle
: description of an angle measured clockwise from the positive *x*-axis

positive angle
: description of an angle measured counterclockwise from the positive *x*-axis

quadrantal angle
: an angle whose terminal side lies on an axis

radian measure
: the ratio of the arc length formed by an angle divided by the radius of the circle

radian
: the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray
: one point on a line and all points extending in one direction from that point; one side of an angle

reference angle
: the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

standard position
: the position of an angle having the vertex at the origin and the initial side along the positive *x*-axis

terminal side
: the side of an angle at which rotation ends

vertex
: the common endpoint of two rays that form an angle

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1.2: Right Triangle Trigonometry

Learning Objectives

- Define the six trigonometric functions in terms of x, y, r (rectangular coordinates) and language associated with right triangles.
- Find function values for 30°($\frac{\pi}{6}$),45°($\frac{\pi}{4}$),and 60°($\frac{\pi}{3}$).
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We begin by drawing a circle centered at the origin with radius r , and marking the point on the circle indicated by some angle θ as in Figure $1.2.1$. This point has coordinates (x, y). We can also say that (x, y) is on the terminal side of angle θ.

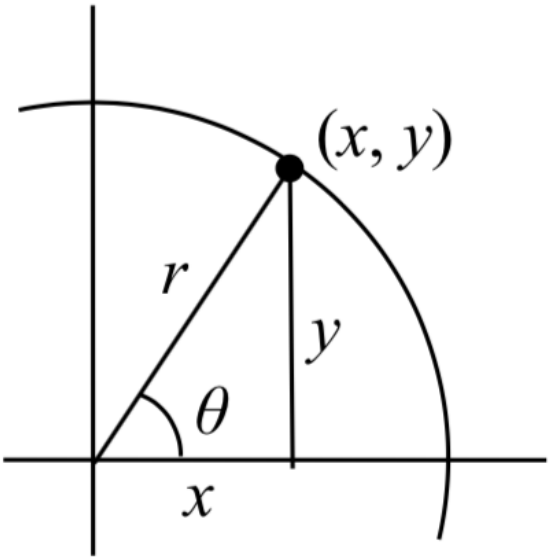


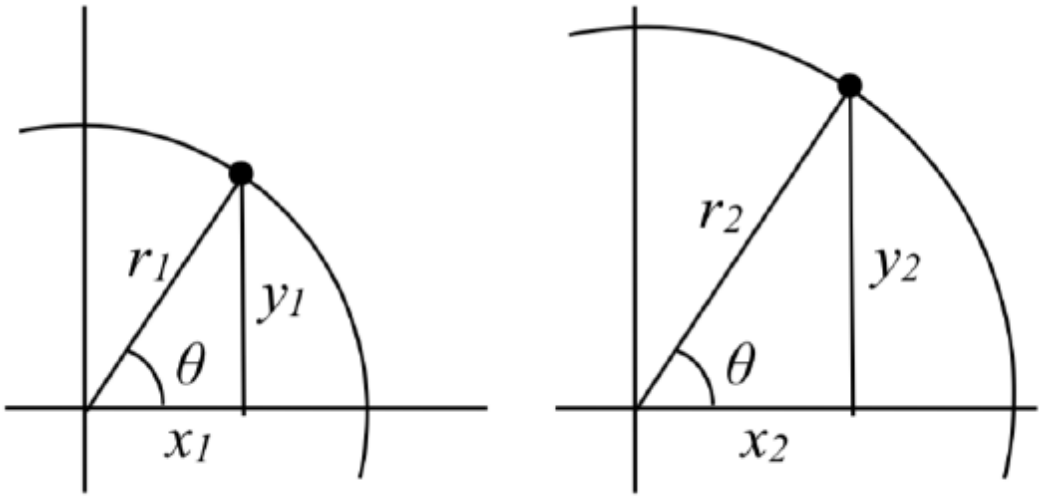
Figure $1.2.1$

If we drop a line segment vertically down from this point to the *x* axis, we would form a right triangle inside of the circle.

No matter which quadrant our angle θ puts us in we can draw a triangle by dropping a perpendicular line segment to the x axis, keeping in mind that the values of x and y may be positive or negative, depending on the quadrant.

Additionally, if the angle θ puts us on an axis, we simply measure the radius as the x or y with the other value being 0, again ensuring we have appropriate signs on the coordinates based on the quadrant.

Triangles obtained from different radii will all be similar triangles, meaning corresponding sides scale proportionally. While the lengths of the sides may change, the *ratios* of the side lengths will always remain constant for any given angle.



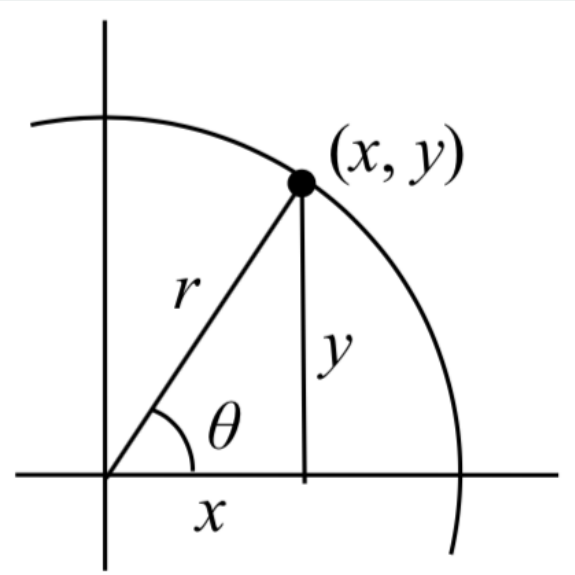
$\frac{y_1}{r_1} = \frac{y_2}{r_2}$

$\frac{x_1}{r_1} = \frac{x_2}{r_2}$

To be able to refer to these ratios more easily, we will give them names. Since the ratios depend on the angle, we will write them as functions of the angle θ.

The Six Trigonometric Functions

For the point (x, y) on a circle of radius r at an angle of θ, we can define the six trigonometric functions as the ratios of the sides of the corresponding triangle:



The **sine** function: $\sin(\theta) = \frac{y}{r}$

The **cosine** function: $\cos(\theta) = \frac{x}{r}$

The **tangent** function: $\tan(\theta) = \frac{y}{x}$

The **cosecant** function: $\csc(\theta) = \dfrac{r}{y}$

The **secant** function: $\sec(\theta) = \dfrac{r}{x}$

The **cotangent** function: $\cot(\theta) = \dfrac{x}{y}$

Example 1.2.1

The point (3, 4) is on the circle of radius 5 at some angle θ. Find the six trigonometric function values of θ

Solution

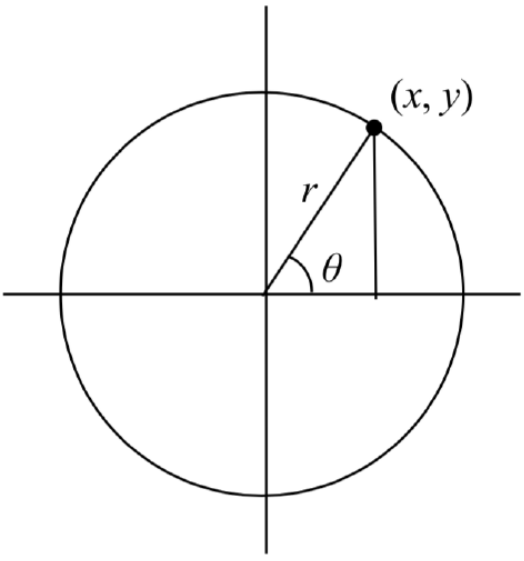
We have $x = 3$, $y = 4$, and $r = 5$. Using the previously listed definitions we have

$$\begin{aligned} \sin(\theta) &= \frac{y}{r} = \frac{4}{5} \\ \cos(\theta) &= \frac{x}{r} = \frac{3}{5} \\ \tan(\theta) &= \frac{y}{x} = \frac{4}{3} \\ \csc(\theta) &= \frac{r}{y} = \frac{5}{4} \\ \sec(\theta) &= \frac{r}{x} = \frac{5}{3} \\ \cot(\theta) &= \frac{x}{y} = \frac{3}{4} \end{aligned} \tag{1.2.1}$$

Consider once again the set-up with a right triangle inscribed in a circle with radius r. The two legs of the right triangle are x and y and the hypotenuse is r. According to the Pythagorean Theorem we have the following relationship:

$$x^2 + y^2 = r^2$$

If we have a given point (x, y) on the terminal side of an angle, we can use the Pythagorean Theorem to find the length of the radius r and can then find the six trigonometric function values of the angle.



Using Right Triangles to Evaluate Trigonometric Functions

We now define the six trigonometric functions in terms of the names of the sides of a right triangle. First, let's recreate our right triangle (shown in Figure $1.2.2$. If we drop a vertical line segment from the point (x, y) to the *x*-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

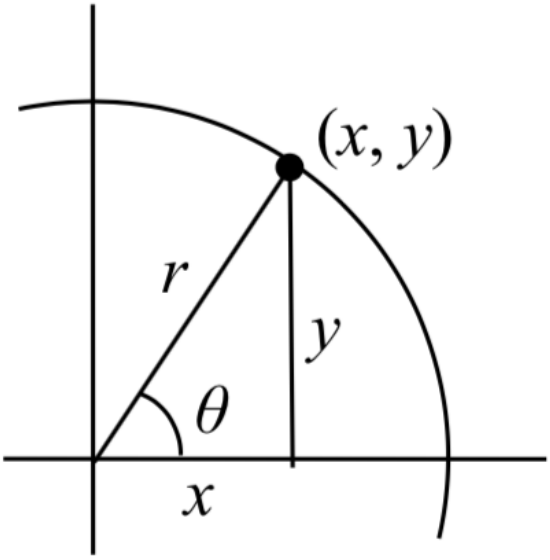


Figure $1.2.2$

We want to have another way to express the ratios without relying on a set of axes. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle θ. (Adjacent means “next to.”) Instead of y,we will call the side most distant from the given angle the **opposite side** from angle θ. And instead of r,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure $1.2.3$.

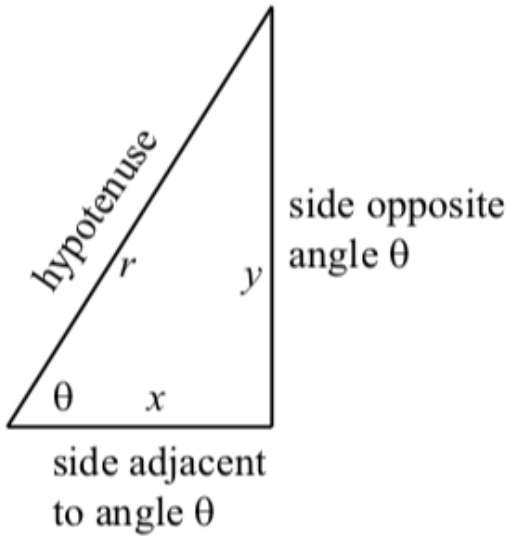


Figure 1.2.3: The sides of a right triangle in relation to angle θ.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of θ,

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{1.2.2}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{1.2.3}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} \tag{1.2.4}$$

$$\csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}} \tag{1.2.5}$$

$$\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} \tag{1.2.6}$$

$$\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}} \tag{1.2.7}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent." To get the other three, we remember that cosecant is the reciprocal of sine, secant is the reciprocal of cosine, and cotangent is the reciprocal of tangent.

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent as the ratio of the opposite side to the adjacent side.
4. Find the cosecant as the ratio of the hypotenuse to the opposite side.
5. Find the secant as the ratio of the hypotenuse to the adjacent side.
6. Find the cotangent as the ratio of the adjacent side to the opposite side.

Also remember that sine and cosecant are reciprocals, cosine and secant and reciprocals, and tangent and cotangent are reciprocals.

Example 1.2.2: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure 1.2.4, find the six trigonometric function values of angle α.

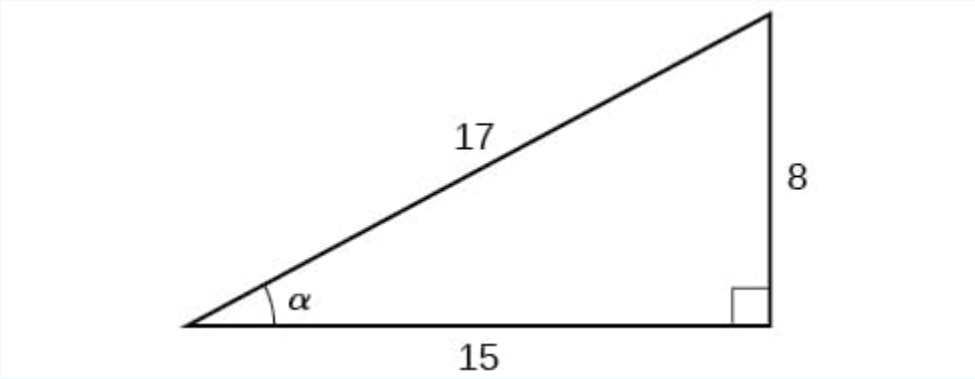


Figure 1.2.4

Solution

From the perspective of angle α, the adjacent side has length 15, the opposite side has length 8, and the hypotenuse has length 17, so we have

$$\begin{aligned}
\sin(\alpha) &= \frac{\text{O}}{\text{H}} = \frac{8}{17} \\
\cos(\alpha) &= \frac{\text{A}}{\text{H}} = \frac{15}{17} \\
\tan(\alpha) &= \frac{\text{O}}{\text{A}} = \frac{8}{15} \\
\csc(\alpha) &= \frac{\text{H}}{\text{O}} = \frac{17}{8} \\
\sec(\alpha) &= \frac{\text{H}}{\text{A}} = \frac{17}{15} \\
\cot(\alpha) &= \frac{\text{A}}{\text{O}} = \frac{15}{8}
\end{aligned}$$

Exercise 1.2.1

Given the triangle shown in Figure 1.2.5, find the value of $\sin t$.

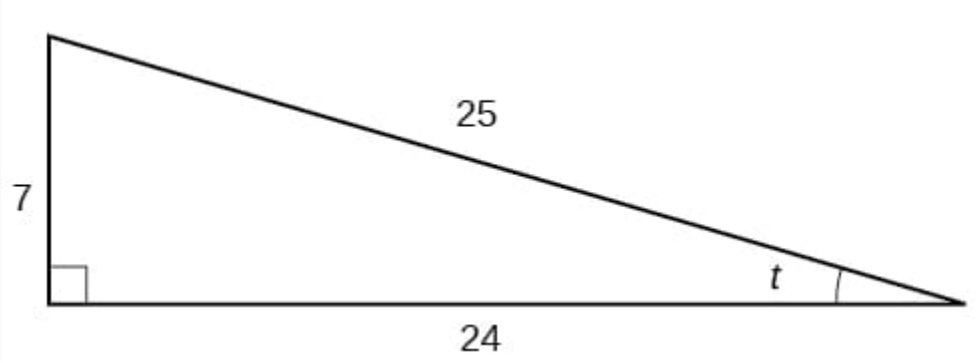


Figure 1.2.5

Answer available at the end of this section.

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure 1.2.6. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

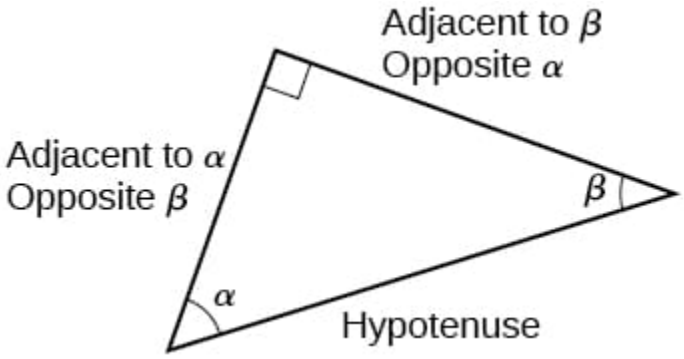


Figure 1.2.6: The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

how to: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 1.2.3: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure 1.2.7, evaluate $\sin\alpha, \cos\alpha, \tan\alpha, \sec\alpha, \csc\alpha,$ and $\cot\alpha$.

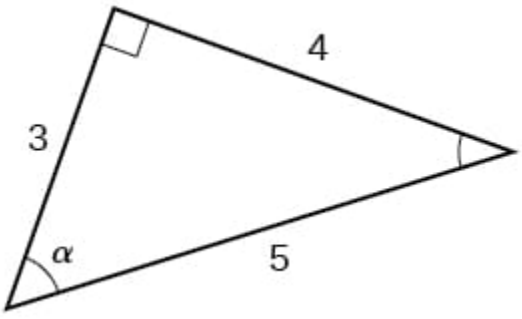


Figure 1.2.7

Solution

$$\sin\alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos\alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan\alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$
$$\sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$
$$\csc\alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$
$$\cot\alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Exercise 1.2.2

Using the triangle shown in Figure 1.2.8, evaluate $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t$.

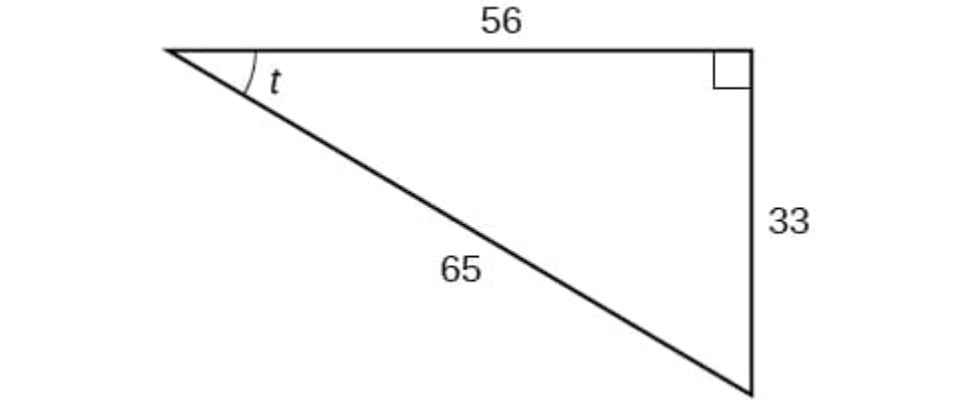


Figure 1.2.8

Answer available at the end of this section.

Finding Trigonometric Functions of Special Angles Using Side Lengths

We now use our right triangle definitions to evaluate trigonometric values of special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. Our special angles are $30\,°, 45\,°,$ and $60\,°$

Recall from geometry the $30\,°, 60\,°, 90\,°$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45\,°, 45\,°, 90\,°$triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure 1.2.9.

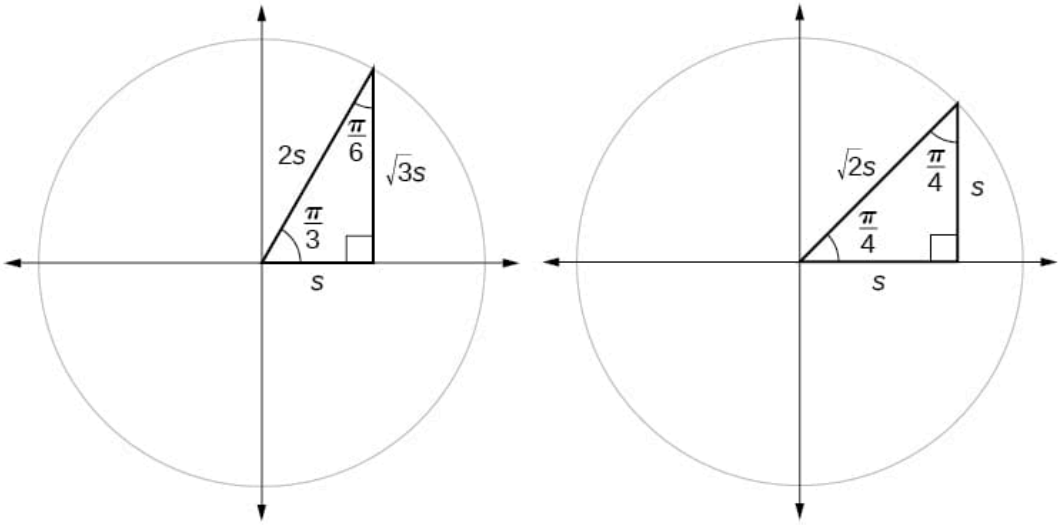


Figure 1.2.9: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 1.2.9 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 1.2.4: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin(\frac{\pi}{3}) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\cos(\frac{\pi}{3}) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$
$$\tan(\frac{\pi}{3}) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$
$$\sec(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot(\frac{\pi}{3}) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 1.2.3

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$ using side lengths.

Answer available at the end of this section.

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure 1.2.10).

$$\sin\frac{\pi}{3}=\cos\frac{\pi}{6}=\frac{\sqrt{3}s}{2s}=\frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6}=\cos\frac{\pi}{3}=\frac{s}{2s}=\frac{1}{2}$$

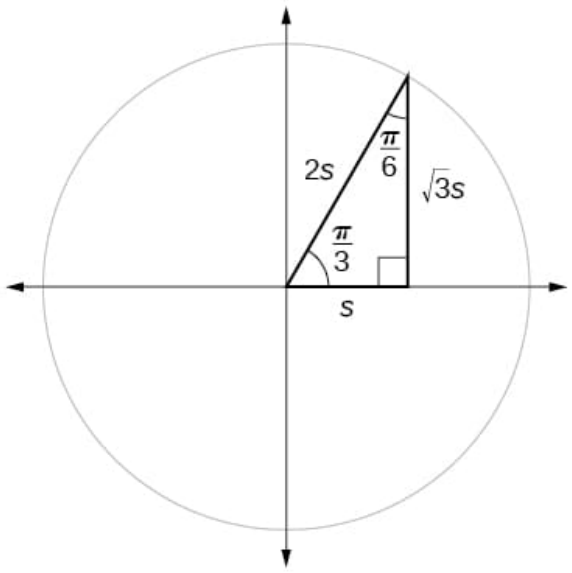


Figure 1.2.10: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure 1.2.10, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, π,and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 1.2.11.

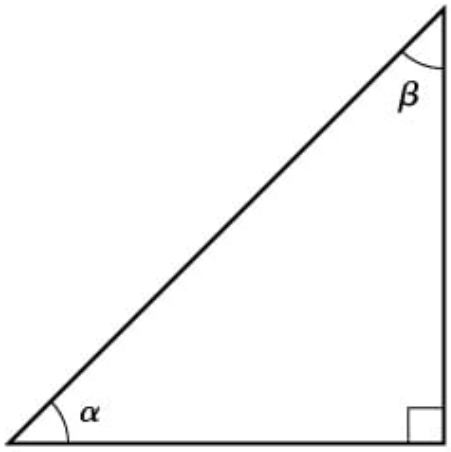


Figure 1.2.11: Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle $t, \cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

COFUNCTION IDENTITIES

The **cofunction identities** in radians are listed in Table 1.2.1.

Table 1.2.1

$\cos t = \sin(\frac{\pi}{2} - t)$	$\sin t = \cos(\frac{\pi}{2} - t)$
$\tan t = \cot(\frac{\pi}{2} - t)$	$\cot t = \tan(\frac{\pi}{2} - t)$
$\sec t = \csc(\frac{\pi}{2} - t)$	$\csc t = \sec(\frac{\pi}{2} - t)$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 1.2.5: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $(\cos \frac{\pi}{2} - t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos(\frac{\pi}{2} - t).$$

So

$$\cos(\frac{\pi}{2} - t) = \frac{5}{12}.$$

Exercise 1.2.4

If $\csc(\frac{\pi}{6}) = 2$, find $\sec(\frac{\pi}{3})$.

Solution

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

how to: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 1.2.6: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure 1.2.12.

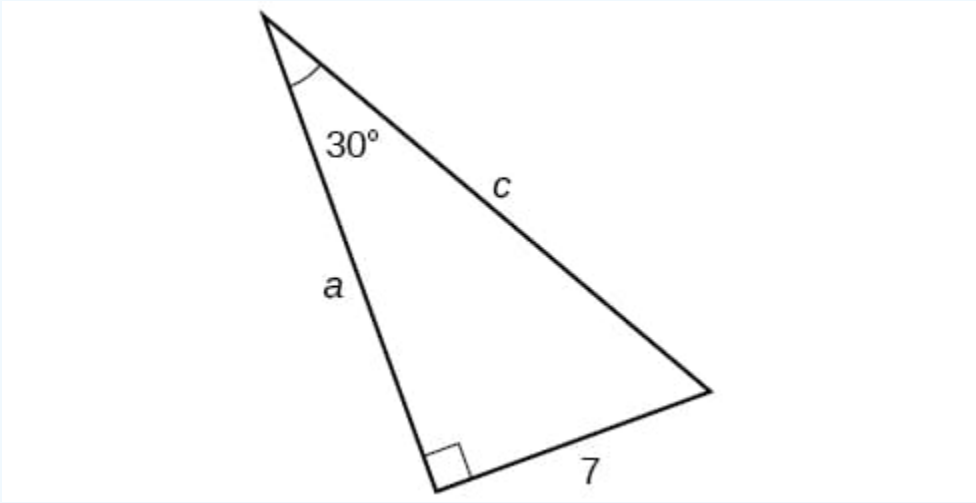


Figure 1.2.12

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30°) = \frac{7}{a}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30°)} \tag{1.2.8}$$
$$= 12.1 \tag{1.2.9}$$

We can use the sine to find the hypotenuse.

$$\sin(30°) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30°)} = 14$$

Exercise 1.2.5:

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Answer available at the end of this section.

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure 1.2.13.

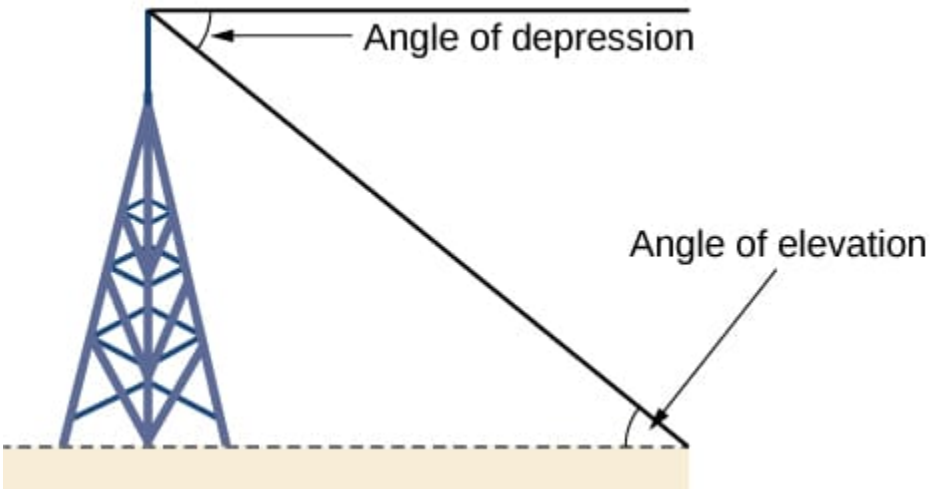


Figure 1.2.13

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 1.2.7: Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° between a line of sight to the top of the tree and the ground, as shown in Figure 1.2.14. Find the height of the tree.

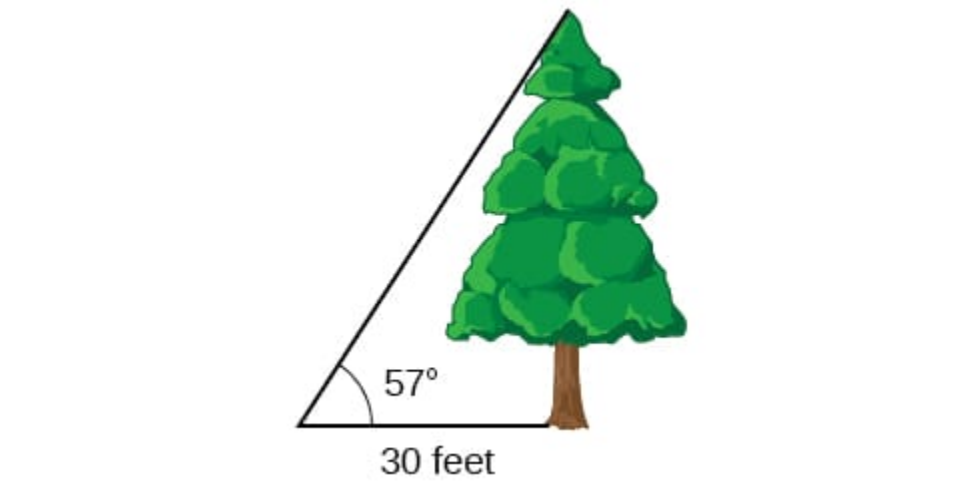


Figure 1.2.14

Solution

We know that the angle of elevation is $57\,^\circ$ and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of $57\,^\circ$, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57°) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30\tan(57°) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.} \end{aligned} \tag{1.2.10}$$

The tree is approximately 46 feet tall.

Exercise 1.2.6:

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Answer available at the end of this section.

media:

Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator
- Finding Trig Functions Using a Right Triangle
- Relate Trig Functions to Sides of a Right Triangle
- Determine Six Trig Functions from a Triangle
- Determine Length of Right Triangle Side

Visit this website for additional practice questions from Learningpod.

Key Equations

Cofunction Identities

$$\cos t = \sin\left(\frac{\pi}{2} - t\right)$$
$$\sin t = \cos\left(\frac{\pi}{2} - t\right)$$
$$\tan t = \cot\left(\frac{\pi}{2} - t\right)$$
$$\cot t = \tan\left(\frac{\pi}{2} - t\right)$$
$$\sec t = \csc\left(\frac{\pi}{2} - t\right)$$
$$\csc t = \sec\left(\frac{\pi}{2} - t\right)$$

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See Example.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See Example.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See Example.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See Example.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See Example.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See Example.

Glossary

adjacent side
: in a right triangle, the side between a given angle and the right angle

angle of depression
: the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation
: the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side
in a right triangle, the side most distant from a given angle

hypotenuse
the side of a right triangle opposite the right angle

Answers

Exercise 1.2.1

$\frac{7}{25}$

Exercise 1.2.2

$$\sin t = \frac{33}{65}, \cos t = \frac{56}{65}, \tan t = \frac{33}{56},$$

$$\sec t = \frac{65}{56}, \csc t = \frac{65}{33}, \cot t = \frac{56}{33}$$

Exercise 1.2.3

$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{\pi}{4}) = 1,$
$\sec(\frac{\pi}{4}) = \sqrt{2}, \csc(\frac{\pi}{4}) = \sqrt{2}, \cot(\frac{\pi}{4}) = 1$

Exercise 1.2.5**:**

$\text{adjacent} = 10; \text{opposite} = 10\sqrt{3};$ missing angle is $\frac{\pi}{6}$

Exercise 1.2.6**:**

About 52 ft

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1.3: Unit Circle

Learning Objectives

- Find function values for the sine and cosine of 30° or $(\frac{\pi}{6})$,45° or $(\frac{\pi}{4})$,and 60° or $(\frac{\pi}{3})$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.



Figure 1.3.1: The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in Figure 1.3.2. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a **unit circle**, $s = t$.

Recall that the *x*- and *y*-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

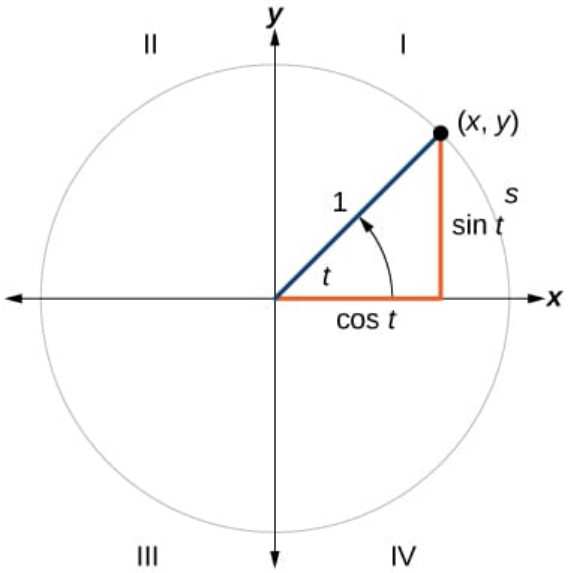


Figure 1.3.2: Unit circle where the central angle is t radians

UNIT CIRCLE

A unit circle has a center at $(0, 0)$ and radius 1. The length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the **arc length** and **angle**. The **sine function** relates a real number t to the y-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t. In Figure 1.3.3, the sine is equal to y. Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y-coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x-value of the endpoint on the unit circle of an arc of length t. In Figure 1.3.1, the cosine is equal to x.

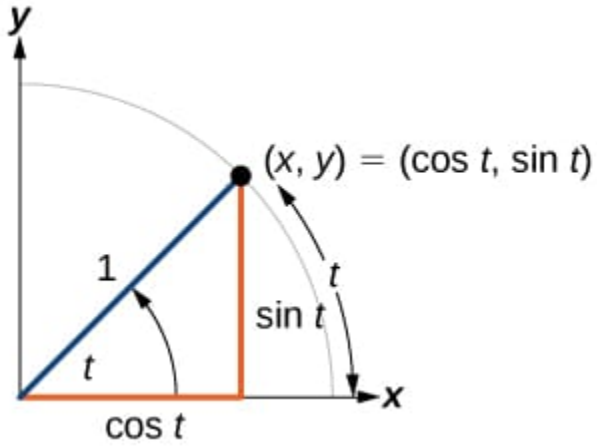


Figure 1.3.3

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

SINE AND COSINE FUNCTIONS

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t, then

$$\cos t = x \tag{1.3.1}$$
$$\sin t = y \tag{1.3.2}$$

How To: Given a point $P(x, y)$ on the unit circle corresponding to an angle of t, find the sine and cosine

1. The sine of t is equal to the y-coordinate of point P: $\sin\ t = y$.
2. The cosine of t is equal to the x-coordinate of point P: $\cos\ t = x$.

Example 1.3.1: Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in Figure 1.3.4. Find $\cos(t)$ and $\sin(t)$.

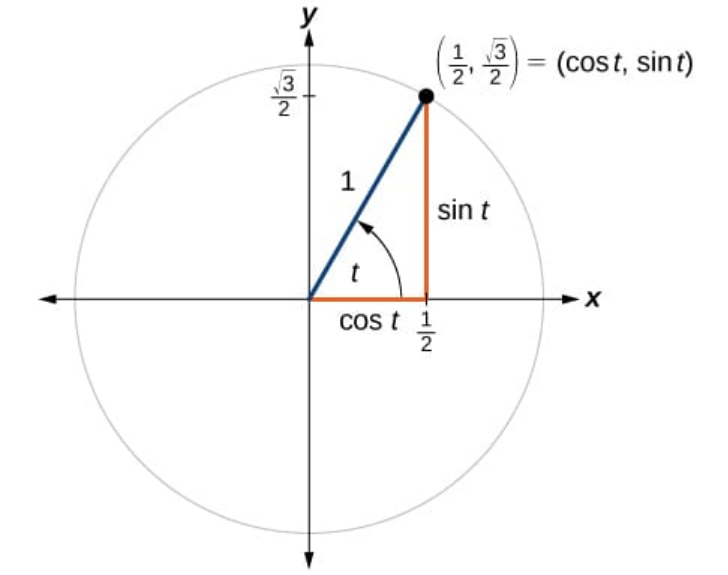


Figure 1.3.4

Solution

We know that $\cos\ t$ is the x-coordinate of the corresponding point on the unit circle and $\sin\ t$ is the y-coordinate of the corresponding point on the unit circle. So:

$$x = \cos\ t = \frac{1}{2}$$
$$y = \sin\ t = \frac{\sqrt{3}}{2}$$

Try It 1.3.1

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure 1.3.5. Find $\cos\ t$ and $\sin\ t$.

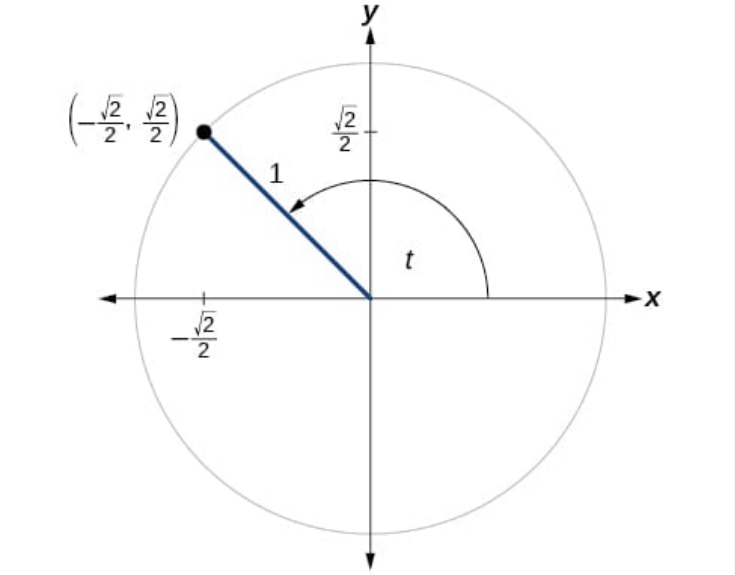


Figure 1.3.5

Answer available at the end of this section.

Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the x- or y-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example 1.3.2: Calculating Sines and Cosines along an Axis

Find $\cos(90\ °)$ and $\sin(90\ °)$.

Solution

Moving $90\ °$ counterclockwise around the unit circle from the positive x-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in Figure 1.3.6.

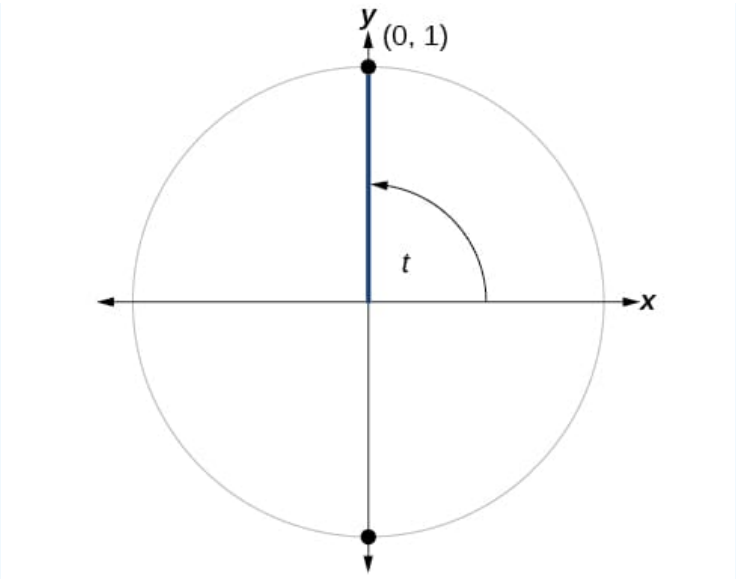


Figure 1.3.6

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90°) = 0$$
$$y = \sin t = \sin(90°) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

Try It 1.3.2

Find cosine and sine of the angle π.

Answer available at the end of this section.

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos\ t$ and $y = \sin\ t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See Figure 1.3.7.

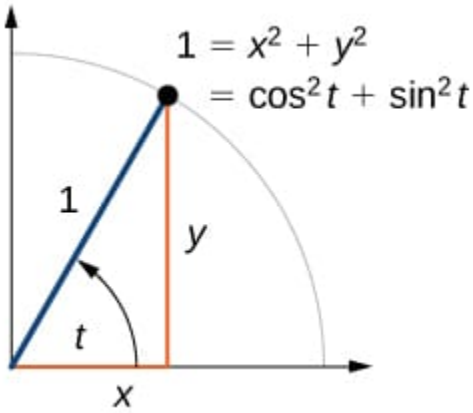


Figure 1.3.7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1 \tag{1.3.3}$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Example 1.3.3: Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution

If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See Figure 1.3.8.

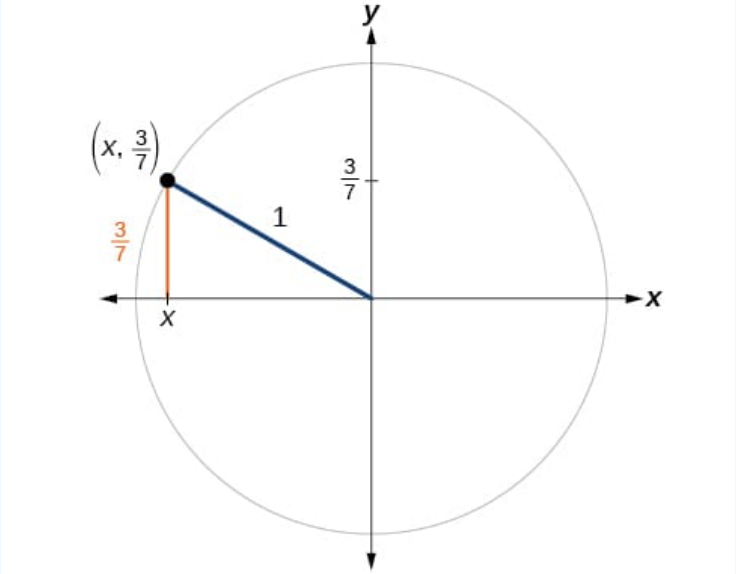


Figure 1.3.8

Substituting the known value for sine into the Pythagorean Identity,

$$\cos^2(t) + \sin^2(t) = 1$$
$$\cos^2(t) + \frac{9}{49} = 1$$
$$\cos^2(t) = \frac{40}{49}$$
$$\cos(t) = \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative. So

$$\cos(t) = -\frac{2\sqrt{10}}{7}$$

Try It 1.3.3

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Answer available at the end of this section.

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the **Pythagorean Identity** and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of $45\,°$ or $\frac{\pi}{4}$, as shown in Figure 1.3.9. A $45\,°$–$45\,°$–$90\,°$ triangle is an isosceles triangle, so the x- and y -coordinates of the corresponding point on the circle are the same. Because the *x*- and y-values are the same, the sine and cosine values will also be equal.

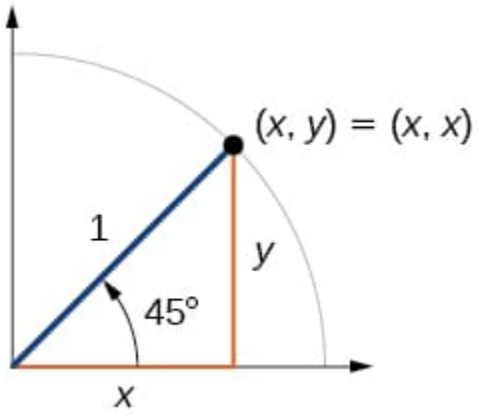


Figure 1.3.9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first **quadrantal angle**. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y (with $y = x$), and a radius = 1. See Figure 1.3.10.

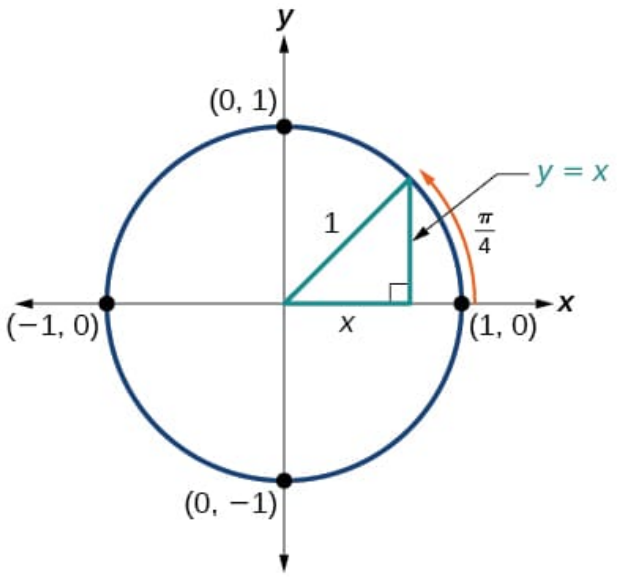


Figure 1.3.10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for x, we get

$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$
$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$$
$$\cos\ t = \frac{1}{\sqrt{2}},\ \sin\ t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\begin{aligned}\cos\, t &= \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \\ \sin\, t &= \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of $45\,^\circ$ are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of $30\,^\circ$, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of $30\,^\circ$, and another at an angle of $-30\,^\circ$, as shown in Figure $1.3.11$. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be $60\,^\circ$, as shown in Figure $1.3.12$.

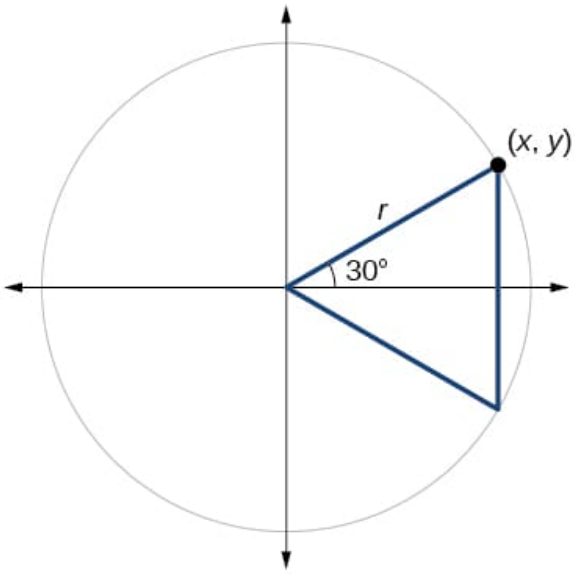


Figure $1.3.11$

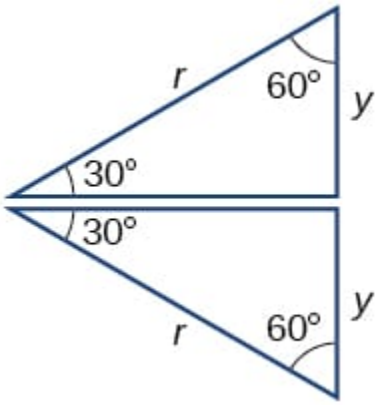


Figure $1.3.12$

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin\, t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

And since $r = 1$ in our **unit circle**,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1)\\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\begin{aligned}\cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right) &= 1\\ \cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 &= 1\\ \cos^2\left(\frac{\pi}{6}\right) &= \frac{3}{4} && \text{Use the square root property.}\\ \cos\left(\frac{\pi}{6}\right) &= \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} && \text{Since } y \text{ is positive, choose the positive root.}\end{aligned}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of $30\,°$ are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in Figure 1.3.13. Angle A has measure 60°. 60°. At point B, we draw an angle ABC with measure of $60\,°$. We know the angles in a triangle sum to $180\,°$, so the measure of angle C is also $60\,°$. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

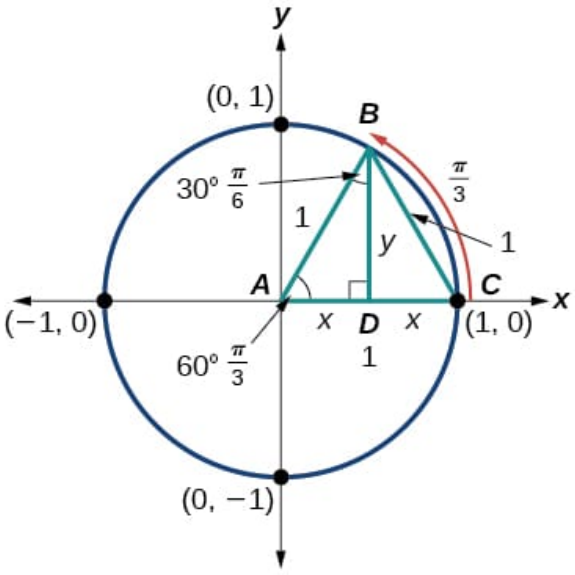


Figure 1.3.13

The measure of angle ABD is 30°. So, if double, angle ABC is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x-coordinate of point B, which is at the intersection of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2 + y^2 = 1$$

Substituting $x = \frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

Solving for y, we get

$$\begin{aligned} \frac{1}{4} + y^2 &= 1 \\ y^2 &= 1 - \frac{1}{4} \\ y^2 &= \frac{3}{4} \\ y &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

Since $t = \frac{\pi}{3}$ has the terminal side in quadrant I where the y-coordinate is positive, we choose $y = \frac{\sqrt{3}}{2}$, the positive value.

At $t = \frac{\pi}{3}$ (60°), the (x, y) coordinates for the point on a circle of radius 1 at an angle of $60\,^\circ$ are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$x = \frac{1}{2}, \quad y = \frac{\sqrt{3}}{2}$$

$$\cos t = \frac{1}{2}, \quad \sin t = \frac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. Table $1.3.1$ summarizes these values.

Table $1.3.1$

Angle	0	$\frac{\pi}{6}$, or 30	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Figure $1.3.14$ shows the common angles in the first quadrant of the unit circle.

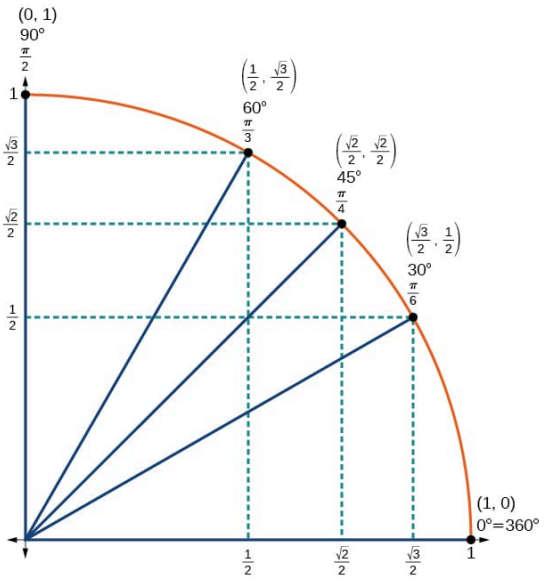


Figure 1.3.14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the **special angles**, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To: Given an angle in radians, use a graphing calculator to find the cosine

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Example 1.3.4: Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution

Enter the following keystrokes:

$$\text{COS}(5 \times \pi \div 3) \text{ ENTER}$$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of $20\ °$, for example, by including the conversion factor to radians as part of the input:

$$\text{SIN}(20 \times \pi \div 180) \text{ ENTER}$$

Try It 1.3.4

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Answer available at the end of this section.

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the **unit circle**, as shown in Figure 1.3.15. The bounds of the x-coordinate are $[-1, 1]$. The bounds of the y-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

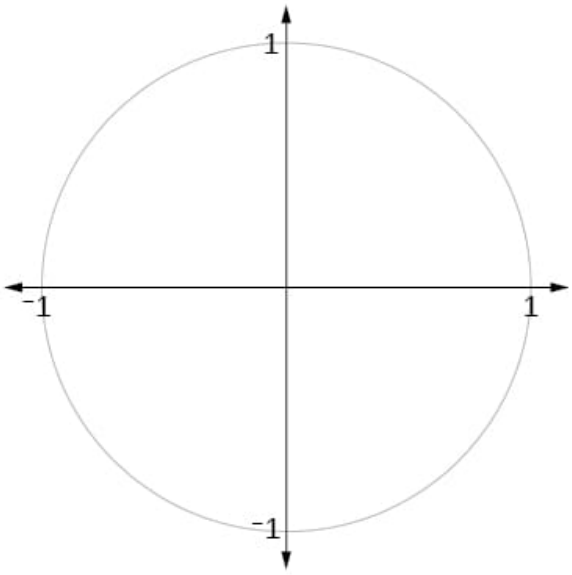


Figure 1.3.15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y-coordinate on the unit circle, the other angle with the same sine will share the same y-value, but have the opposite x-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x-value but will have the opposite y-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in Figure 1.3.16, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\begin{aligned} \sin(t) &= \sin(\alpha) \quad \text{and} \quad \cos(t) = -\cos(\alpha) \\ \sin(t) &= -\sin(\beta) \quad \text{and} \quad \cos(t) = \cos(\beta) \end{aligned}$$

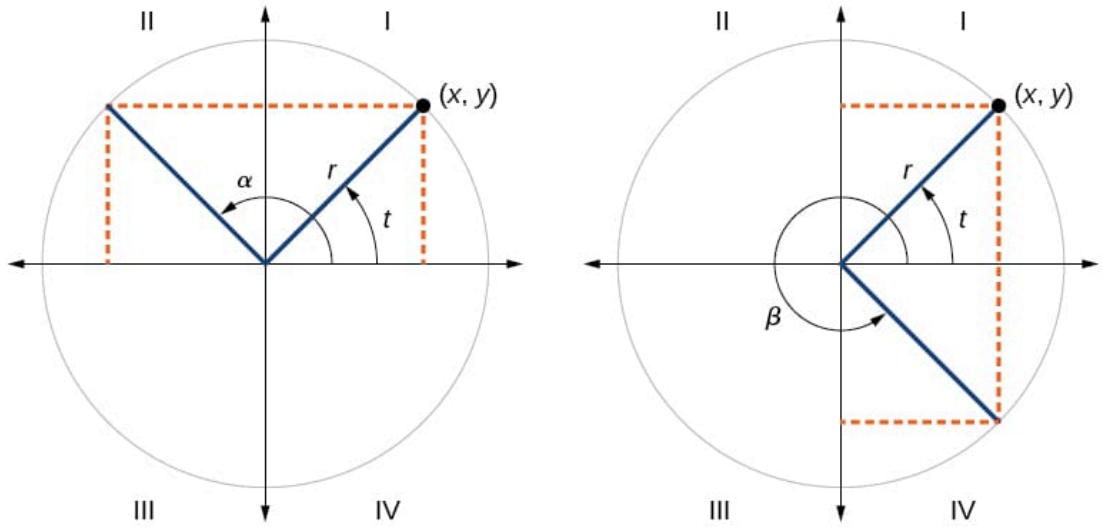


Figure 1.3.16

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and $90\ °$, or 0 and $\frac{\pi}{2}$ radians. As we can see from Figure $1.3.17$, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

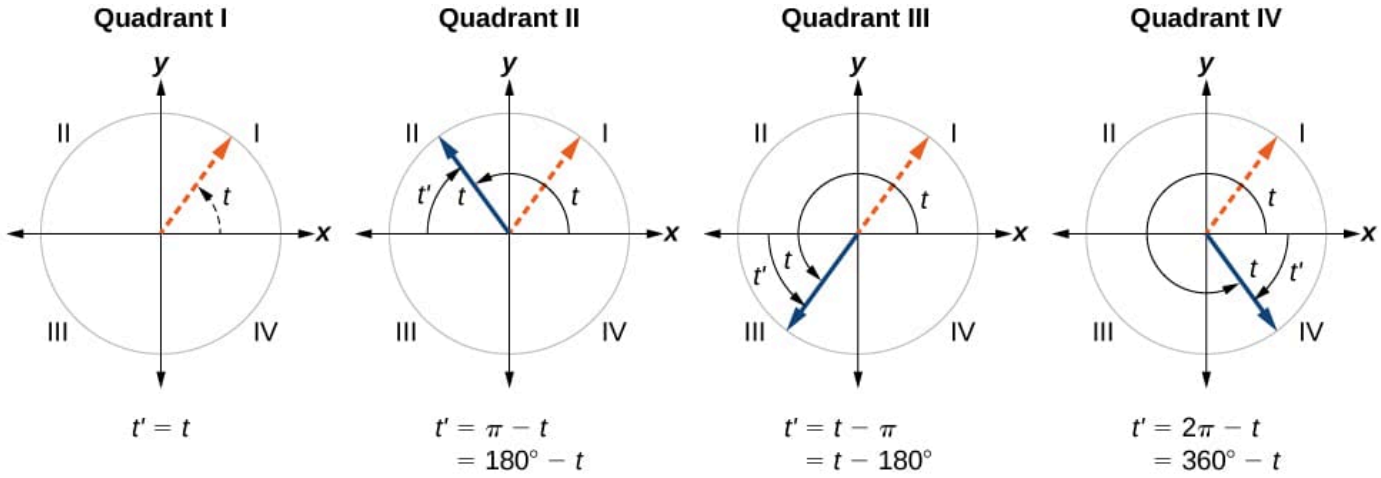


Figure 1.3.17

How To: Given an angle between 0 and 2π, find its reference angle

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180\ °-t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360\ °-t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example $1.3.5$: Finding a Reference Angle

Find the reference angle of $225\ °$ as shown in Figure $1.3.18$.

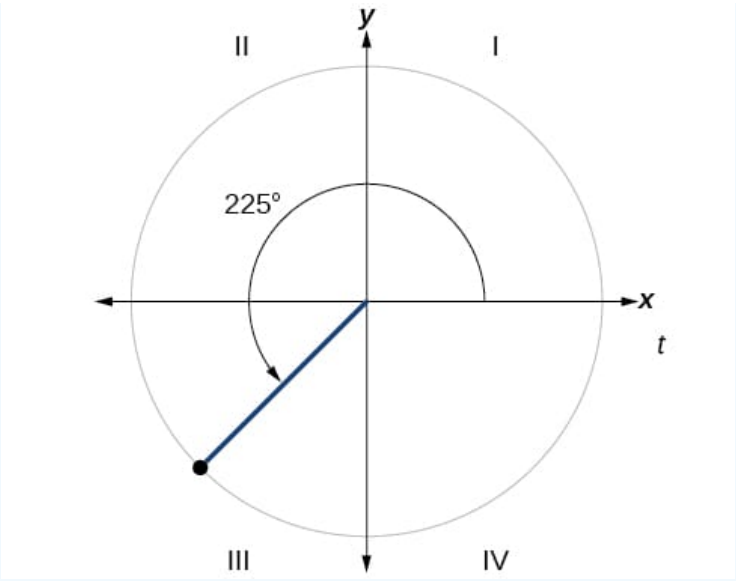


Figure 1.3.18

Solution

Because $225\,^\circ$ is in the third quadrant, the reference angle is

$$|(180°-225°)| = |-45°| = 45°$$

Try It 1.3.5

Find the reference angle of $\frac{5\pi}{3}$.

Answer available at the end of this section.

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a **negative angle**. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the **reference angle** of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

USING REFERENCE ANGLES TO FIND COSINE AND SINE

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To: Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example 1.3.6: Using Reference Angles to Find Sine and Cosine

1. Using a reference angle, find the exact value of $\cos(150\,^\circ)$ and $\sin(150\,^\circ)$.
2. Using the reference angle, find $\cos\,\frac{5\pi}{4}$ and $\sin\,\frac{5\pi}{4}$.

Solution

This tells us that 150° has the same sine and cosine values as 30°, except for the sign. We know that

Since $150\,^\circ$ is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. The y-coordinate is positive, so the sine value is positive.

$\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\left|\frac{5\pi}{4}-\pi\right|=\frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos\,\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}\text{ and }\sin\,\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}$$

Try It 1.3.6

1. Use the reference angle of $315\,^\circ$ to find $\cos(315\,^\circ)$ and $\sin(315\,^\circ)$.
2. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$

Answer available at the end of this section.

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in Figure 1.3.19. Take time to learn the (x,y) coordinates of all of the major angles in the first quadrant.

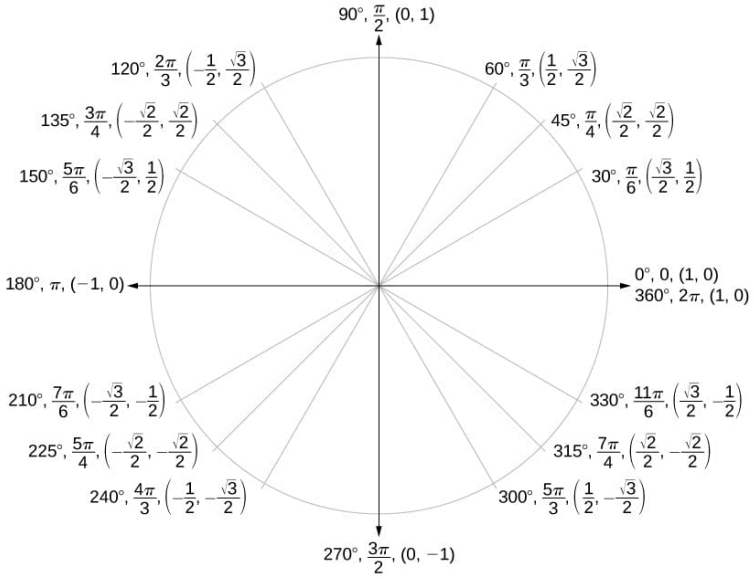


Figure 1.3.19: Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the **identities**

$$x = \cos\ t$$
$$y = \sin\ t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the **reference angle**, and give them the signs corresponding to the y- and x-values of the quadrant.

How To: Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example 1.3.7: Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{3}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$
$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\,\theta$ and $y = \sin\,\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It 1.3.7

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Answer available at the end of this section.

media

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle
- Sine and Cosine from the Unit Circle
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four
- Trigonometric Functions Using Reference Angles

Key Equations

Cosine	$\cos\,t = x$
Sine	$\sin\,t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint. See Example.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See Example.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See Example.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See Example.
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the *x*- and y-values in the quadrant of the original angle.

- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See Example.
- Reference angles can be used to find the sine and cosine of the original angle. See Example.
- Reference angles can also be used to find the coordinates of a point on a circle. See Example.

Glossary

cosine function
: the x-value of the point on a unit circle corresponding to a given angle

Pythagorean Identity
: a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function
: the y-value of the point on a unit circle corresponding to a given angle

unit circle
: a circle with a center at $(0, 0)$ and radius 1.

Answers

Try It 1.3.1

$\cos(t) = -\frac{\sqrt{2}}{2}, \sin(t) = \frac{\sqrt{2}}{2}$

Try It 1.3.2

$\cos(\pi) = -1, \ \sin(\pi) = 0$

Try It 1.3.3

$\sin(t) = -\frac{7}{25}$

Try It 1.3.4

approximately 0.866025403

Try It 1.3.5

(\dfrac{π}{3}\)

Try It 1.3.6

1. $\cos(315\ ^\circ) = \frac{\sqrt{2}}{2}, \sin(315\ ^\circ) = \frac{-\sqrt{2}}{2}$
2. $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

Try It 1.3.7

$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

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1.4: The Other Trigonometric Functions

Learning Objectives

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t,as shown in Figure $1.4.1$. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

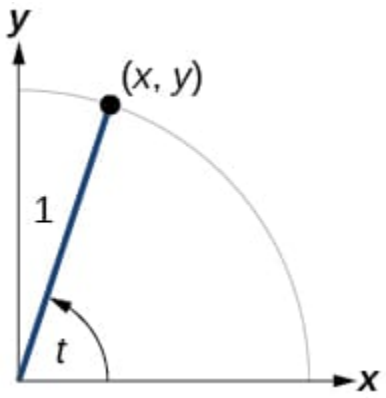


Figure $1.4.1$

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle. In Figure $1.4.1$, the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the *y*-value is equal to the sine of t, and the *x*-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}, \cos t \neq 0$. The tangent function is abbreviated as $\tan$. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In Figure $1.4.1$, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as $\sec$.
- The **cotangent** function is the reciprocal of the tangent function. In Figure $1.4.1$, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as $\cot$.
- The **cosecant** function is the reciprocal of the sine function. In Figure $1.4.1$, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as $\csc$.

TANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0 \tag{1.4.1}$$

$$\sec t = \frac{1}{x}, x \neq 0 \tag{1.4.2}$$

$$\csc t = \frac{1}{y}, y \neq 0 \tag{1.4.3}$$

$$\cot t = \frac{x}{y}, y \neq 0 \tag{1.4.4}$$

Example 1.4.1: Finding Trigonometric Functions from a Point on the Unit Circle

The point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle, as shown in Figure 1.4.2. Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t$.

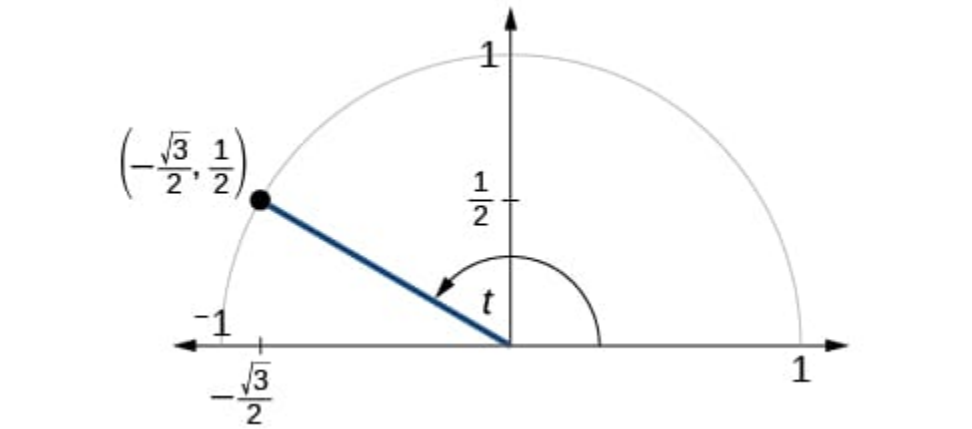


Figure 1.4.2

Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$

$$\cos t = x = -\frac{\sqrt{3}}{2}$$

$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}(-\frac{2}{\sqrt{3}}) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}(\frac{2}{1}) = -\sqrt{3}$$

Exercise 1.4.1:

The point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ is on the unit circle, as shown in Figure 1.4.3. Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t$.

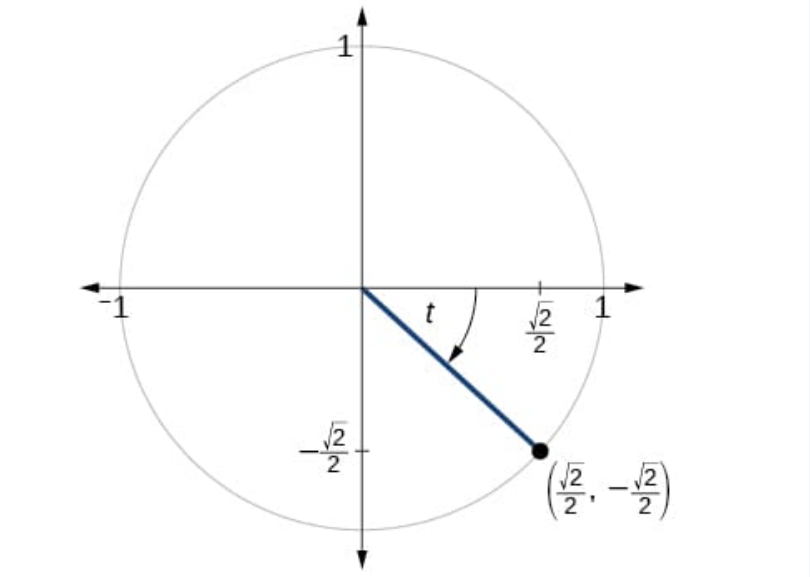


Figure 1.4.3

Solution

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = -1, \sec t = \sqrt{2}, \csc t = -\sqrt{2}, \cot t = -1$

Example 1.4.2: Finding the Trigonometric Functions of an Angle

Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t$ when $t = \frac{\pi}{6}$.

Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\begin{aligned}
\tan \frac{\pi}{6} &= \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}} \\
&= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\
\sec \frac{\pi}{6} &= \frac{1}{\cos \frac{\pi}{6}} \\
&= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\
\csc \frac{\pi}{6} &= \frac{1}{\sin \frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2 \\
\cot \frac{\pi}{6} &= \frac{\cos \frac{\pi}{6}}{\sin \frac{\pi}{6}} \\
&= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}
\end{aligned}$$

Exercise 1.4.2:

Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t$ when $t = \frac{\pi}{3}$.

Solution

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \tag{1.4.5}$$

$$\cos \frac{\pi}{3} = \frac{1}{2} \tag{1.4.6}$$

$$\tan \frac{\pi}{3} = \sqrt{3} \tag{1.4.7}$$

$$\sec \frac{\pi}{3} = 2 \tag{1.4.8}$$

$$\csc \frac{\pi}{3} = \frac{2\sqrt{3}}{3} \tag{1.4.9}$$

$$\cot \frac{\pi}{3} = \frac{\sqrt{3}}{3} \tag{1.4.10}$$

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x x equal to the cosine and y y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in Table 1.4.1.

Table 1.4.1

Angle	0	$\frac{\pi}{6}$, or 30 °	$\frac{\pi}{4}$, or 45 °	$\frac{\pi}{3}$, or 60 °	$\frac{\pi}{2}$, or 90 °
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the **reference angle** formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by *x*- and *y*-values in the original quadrant. Figure 1.4.4 shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.

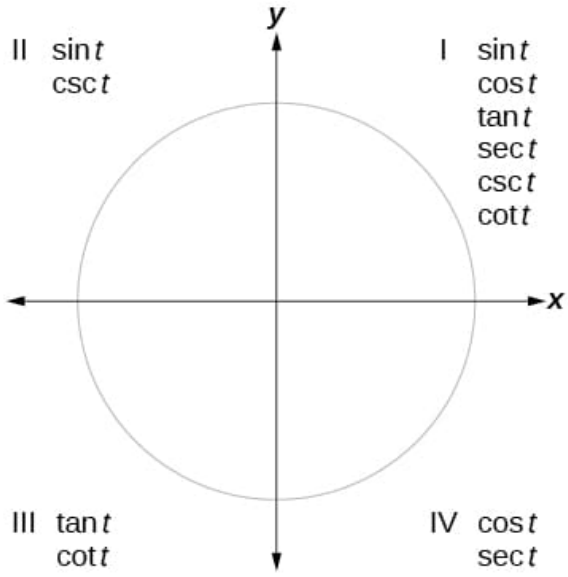


Figure 1.4.4

HOWTO: Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example 1.4.3: Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution

The angle between this angle's terminal side and the *x*-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos(-\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}, \sin(-\frac{5\pi}{6}) = -\frac{1}{2}, \tan(-\frac{5\pi}{6}) = \frac{\sqrt{3}}{3} \tag{1.4.11}$$

$$\sec(-\frac{5\pi}{6}) = -\frac{2\sqrt{3}}{3}, \csc(-\frac{5\pi}{6}) = -2, \cot(-\frac{5\pi}{6}) = \sqrt{3} \tag{1.4.12}$$

Exercise 1.4.3

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Solution

$\sin(-\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{-7\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{-7\pi}{4}) = 1,$

$\sec(\frac{-7\pi}{4}) = \sqrt{2}, \csc(\frac{-7\pi}{4}) = \sqrt{2}, \cot(\frac{-7\pi}{4}) = 1$

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f(x) = x^2$, shown in Figure 1.4.5. The graph of the function is symmetrical about the *y*-axis. All along the curve, any two points with opposite *x*-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2, (-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an **even function**, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

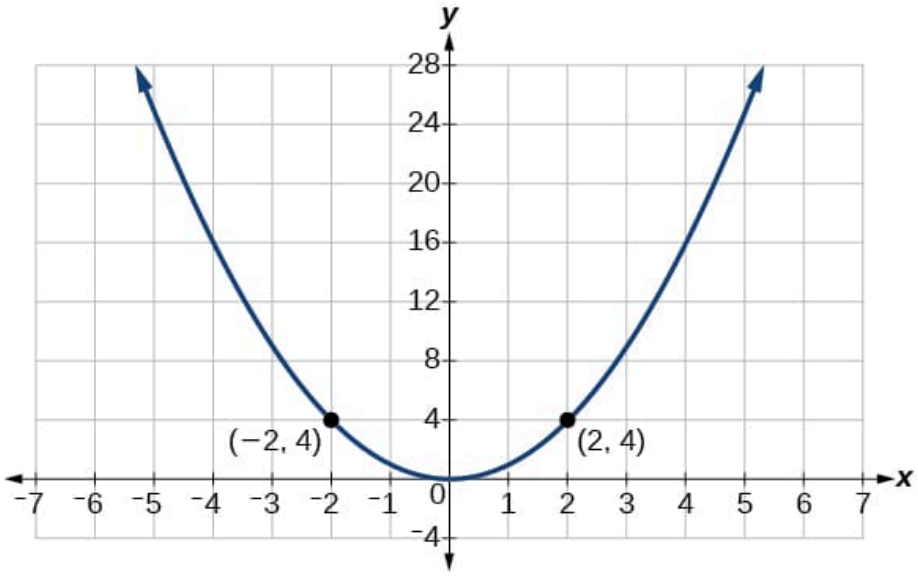


Figure 1.4.5: The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in Figure 1.4.6. The graph is not symmetrical about the *y*-axis. All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f(x) = x^3$ is an **odd function**, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

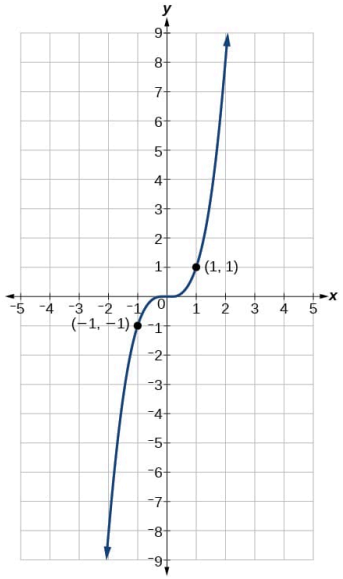


Figure 1.4.6: The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a **unit circle** with a positive and a negative angle, as in Figure 1.4.7. The sine of the positive angle is y. The sine of the negative angle is –*y*. The **sine function**, then, is an odd function.

We can test each of the six trigonometric functions in this fashion. The results are shown in Table 1.4.2.

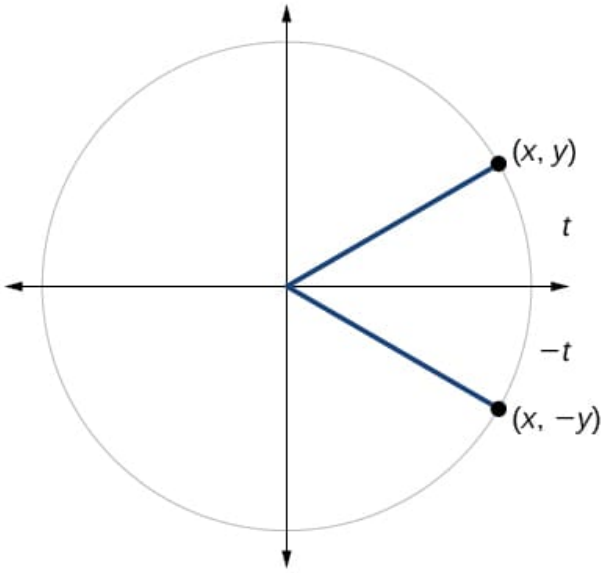


Figure 1.4.7

Table 1.4.2

$\sin t = y$	(1.4.13)
$\sin(-t) = -y$	(1.4.14)
$\sin t \neq sin(-t)$	(1.4.15)

$\sec t = \dfrac{1}{x}$	(1.4.22)
$\sec(-t) = \dfrac{1}{x}$	(1.4.23)
$\sec t = \sec(-t)$	(1.4.24)

EVEN AND ODD TRIGONOMETRIC FUNCTIONS

- An **even function** is one in which $f(-x) = f(x)$.
- An **odd function** is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\cos(-t) = \cos t \tag{1.4.31}$$
$$\sec(-t) = \sec t \tag{1.4.32}$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t \tag{1.4.33}$$
$$\tan(-t) = -\tan t \tag{1.4.34}$$
$$\csc(-t) = -\csc t \tag{1.4.35}$$
$$\cot(-t) = -\cot t \tag{1.4.36}$$

Example 1.4.4: Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Exercise 1.4.4:

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Solution

$-\sqrt{3}$

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

FUNDAMENTAL IDENTITIES

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \tag{1.4.37}$$

$$\sec t = \frac{1}{\cos t} \tag{1.4.38}$$

$$\csc t = \frac{1}{\sin t} \tag{1.4.39}$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \tag{1.4.40}$$

Example 1.4.5: Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45\,^\circ) = \frac{\sqrt{2}}{2}, \cos(45\,^\circ) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45\,^\circ)$.
b. Given $\sin(\frac{5\pi}{6}) = \frac{1}{2}, \cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$, evaluate $\sec(\frac{5\pi}{6})$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.
$$\begin{aligned} \tan(45°) &= \frac{\sin(45°)}{\cos(45°)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1 \end{aligned}$$

b.
$$\begin{aligned}\sec\left(\frac{5\pi}{6}\right) &= \frac{1}{\cos\left(\frac{5\pi}{6}\right)}\\ &= \frac{1}{-\frac{\sqrt{3}}{2}}\\ &= \frac{-2\sqrt{3}}{1}\\ &= \frac{-2}{\sqrt{3}}\\ &= -\frac{2\sqrt{3}}{3}\end{aligned}$$

Exercise 1.4.5

Evaluate $\csc\left(\frac{7\pi}{6}\right)$.

Solution

-2

Example 1.4.6: Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution

We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{aligned}\frac{\sec t}{\tan t} &= \frac{1/\cos t}{\sin t/\cos t} && \text{To divide the functions, we multiply by the reciprocal.}\\ &= \frac{1}{\cos t}\frac{\cos t}{\sin t} && \text{Divide out the cosines.}\\ &= \frac{1}{\sin t} && \text{Simplify and use the identity.}\\ &= \csc t\end{aligned} \tag{1.4.41}$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$,we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Exercise 1.4.6

Simplify $(\tan t)(\cos t)$.

Solution

$\sin t$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the **Pythagorean Identity**, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \tag{1.4.42}$$

$$1 + \tan^2 t = \sec^2 t \tag{1.4.43}$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \tag{1.4.44}$$

$$\cot^2 t + 1 = \csc^2 t \tag{1.4.45}$$

ALTERNATE FORMS OF THE PYTHAGOREAN IDENTITY

$$1 + \tan^2 t = \sec^2 t \tag{1.4.46}$$

$$\cot^2 t + 1 = \csc^2 t \tag{1.4.47}$$

Example 1.4.7: Using Identities to Relate Trigonometric Functions

If cos(t)=1213 cos(t)=1213 and t t is in quadrant IV, as shown in Figure 1.4.8, find the values of the other five trigonometric functions.

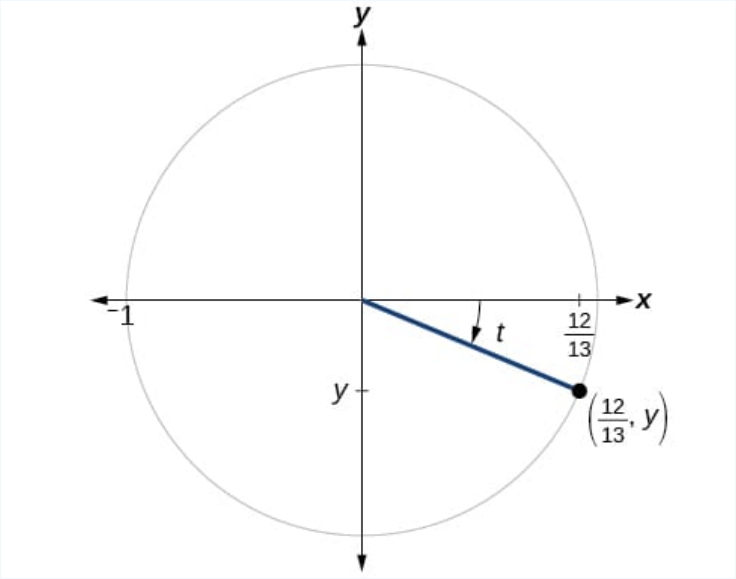


Figure 1.4.8

Solution

We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$(\frac{12}{13})^2 + \sin^2 t = 1 \tag{1.4.48}$$

$$\sin^2 t = 1 - (\frac{12}{13})^2 \tag{1.4.49}$$

$$\sin^2 t = 1 - \frac{144}{169} \tag{1.4.50}$$

$$\sin^2 t = \frac{25}{169} \tag{1.4.51}$$

$$\sin t = \pm\sqrt{\frac{25}{169}} \tag{1.4.52}$$

$$\sin t = \pm\frac{\sqrt{25}}{\sqrt{169}} \tag{1.4.53}$$

$$\sin t = \pm\frac{5}{13} \tag{1.4.54}$$

The sign of the sine depends on the *y*-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the *y*-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12} \tag{1.4.55}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12} \tag{1.4.56}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5} \tag{1.4.57}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5} \tag{1.4.58}$$

Exercise 1.4.7:

If $\sec(t) = -\frac{17}{8}$ and $0 < t < \pi$, find the values of the other five functions.

Solution

$\cos t = -\frac{8}{17}, \sin t = \frac{15}{17}, \tan t = -\frac{15}{8}$

$\csc t = \frac{17}{15}, \cot t = -\frac{8}{15}$

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a **periodic function**. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π,will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4) = f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

PERIOD OF A FUNCTION

The **period** P of a repeating function f f is the number representing the interval such that $f(x+P) = f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

Example 1.4.8: Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on Figure 1.4.9 .

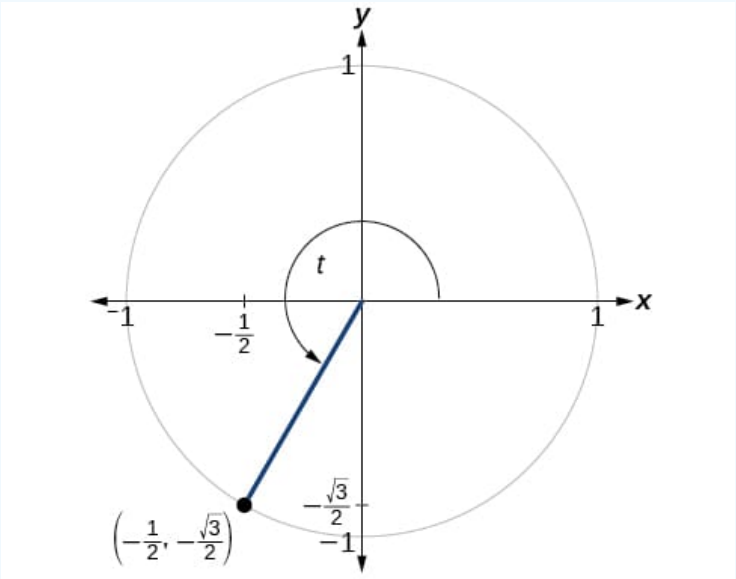


Figure 1.4.9

Solution

$$\sin t = y = -\frac{\sqrt{3}}{2}$$
$$\cos t = x = -\frac{1}{2}$$
$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$
$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2$$
$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$
$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 1.4.8

Find the values of the six trigonometric functions of angle t based on Figure 1.4.10.

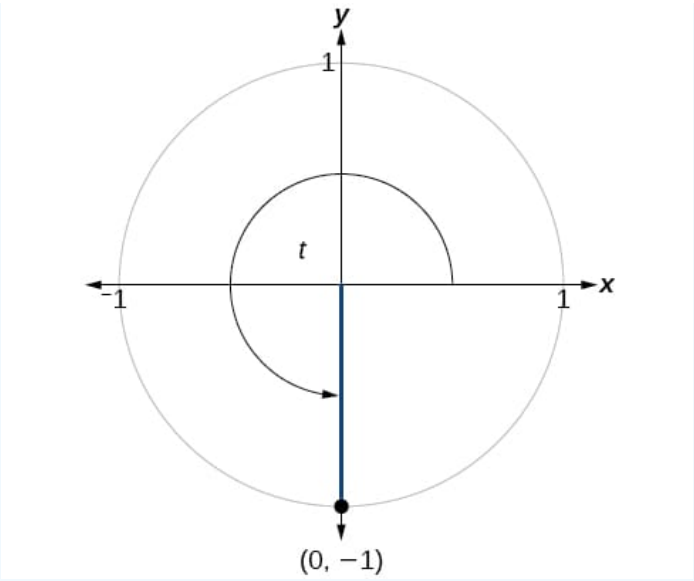


Figure 1.4.10

Solution

$$\sin t = -1, \cos t = 0, \tan t = \text{Undefined} \tag{1.4.59}$$

(1.4.60)

$$sect = \text{Undefined}, \csc t = -1, \cot t = 0 \tag{1.4.61}$$

Example 1.4.9: Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$,find $\sec(t), \csc(t), \tan(t), \cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2 \tag{1.4.62}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} - \frac{2\sqrt{3}}{3} \tag{1.4.63}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \tag{1.4.64}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} \tag{1.4.65}$$

Exercise 1.4.9:

If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t), \csc(t), \tan(t)$, and $\cot(t)$.

Solution

$\sec t = \sqrt{2}, \csc t = \sqrt{2}, \tan t = 1, \cot t = 1$

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as

reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of $30\,°$, we could press

$$(\text{for a scientific calculator}) : \frac{1}{30 \times \frac{\pi}{180}}\text{COS} \tag{1.4.66}$$

or

$$(\text{for a graphing calculator}) : \frac{1}{\cos(\frac{30\pi}{180})} \tag{1.4.67}$$

how to: Given an angle measure in radians, use a scientific calculator to find the cosecant

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: $1\ /$
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

how to: Given an angle measure in radians, use a graphing utility/calculator to find the cosecant

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: $1\ /$
3. Press the SIN key.
4. Enter the value of the angle inside parentheses.
5. Press the ENTER key.

Example 1.4.10: Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1/(5 \times \pi/7)\text{SIN} = \tag{1.4.68}$$

$$\csc(\frac{5\pi}{7}) \approx 1.279 \tag{1.4.69}$$

Exercise 1.4.10:

Evaluate the cotangent of $-\frac{\pi}{8}$.

≈ -2.414

media

Access these online resources for additional instruction and practice with other trigonometric functions.

- Determining Trig Function Values
- More Examples of Determining Trig Functions
- Pythagorean Identities
- Trig Functions on a Calculator

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$
Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See Example.
- Trigonometric functions can also be found from an angle. See Example.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See Example.
- A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See Example.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See Example and Example.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See Example.
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f f is the smallest interval such that $f(x + P) = f(x)$ for any value of x.
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See Example.

Glossary

cosecant
: the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cotangent
: the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

identities
: statements that are true for all values of the input on which they are defined

period
: the smallest interval P of a repeating function f such that $f(x + P) = f(x)$

secant
: the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

tangent
: the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

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2: GRAPHING THE TRIGONOMETRIC FUNCTIONS

In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

2.1: Graphs of the Sine and Cosine Functions

2.2: Graphs of the Secant and Cosecant Functions

2.3: Graphs of the Tangent and Cotangent Functions

2.4: Inverse Trigonometric Functions

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2.1: Graphs of the Sine and Cosine Functions

Learning Objectives

- Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- Use phase shifts of sine and cosine curves.

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.

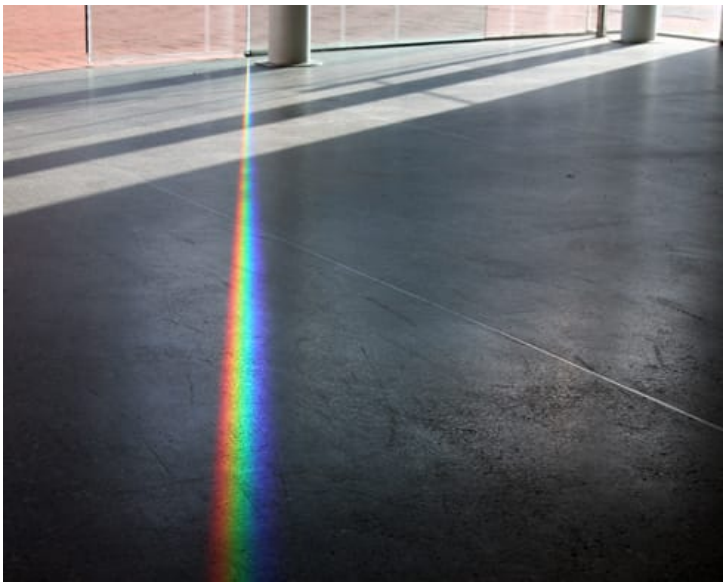


Figure 2.1.1: Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

Light waves can be represented graphically by the sine function. In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. Table 2.1.1 lists some of the values for the sine function on a unit circle.

Table 2.1.1

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the *x*-axis gives the shape of the sine function. See Figure 2.1.2.

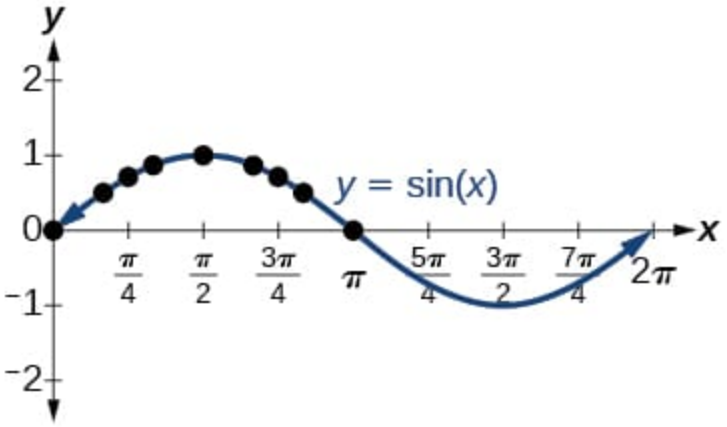


Figure 2.1.2: The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See Figure 2.1.3.

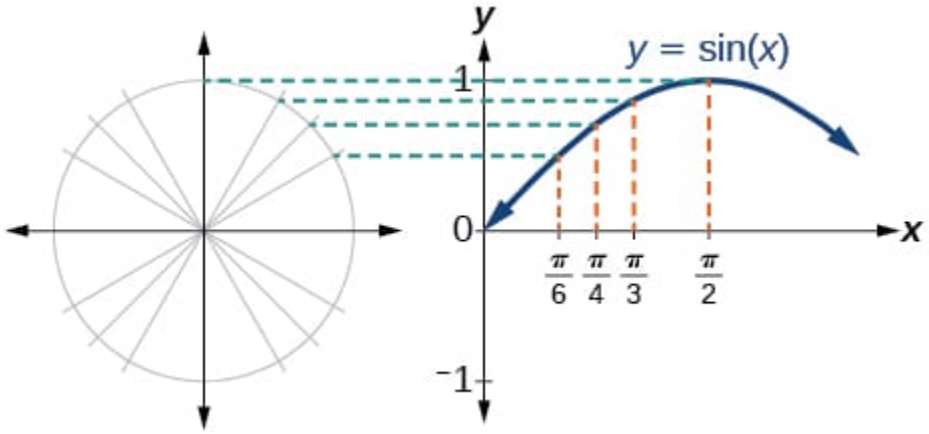


Figure 2.1.3: Plotting values of the sine function

Now let's take a similar look at the **cosine function**. Again, we can create a table of values and use them to sketch a graph. Table 2.1.2 lists some of the values for the cosine function on a unit circle.

Table 2.1.2

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

As with the sine function, we can plots points to create a graph of the cosine function as in Figure 2.1.4.

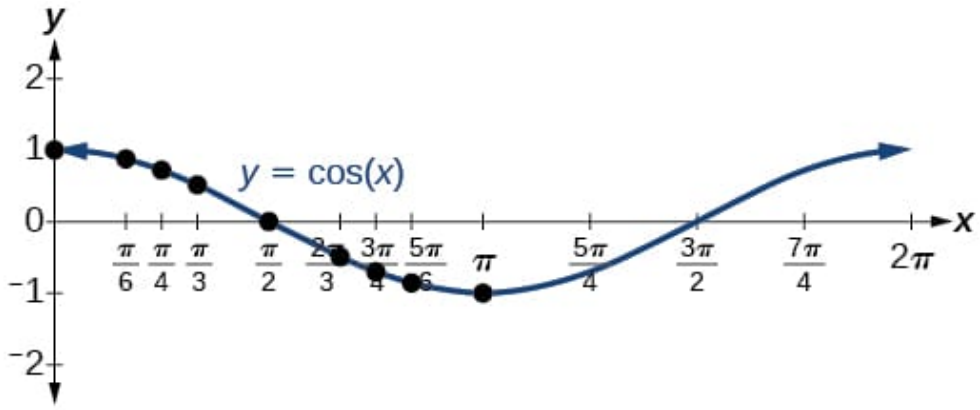


Figure 2.1.4: The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific **horizontal shift**, P, results in a function equal to the original function: $f(x + P) = f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P > 0$ the **period** of the function. Figure 2.1.5 shows several periods of the sine and cosine functions.

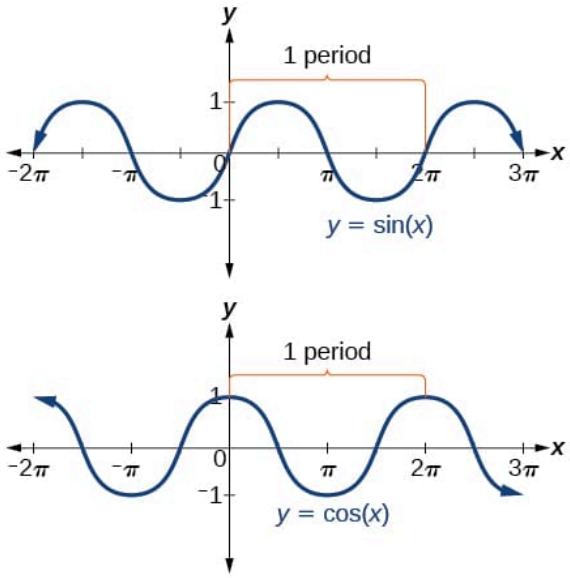


Figure 2.1.5

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in Figure 2.1.6, the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin\ x$. Now we can clearly see this property from the graph.

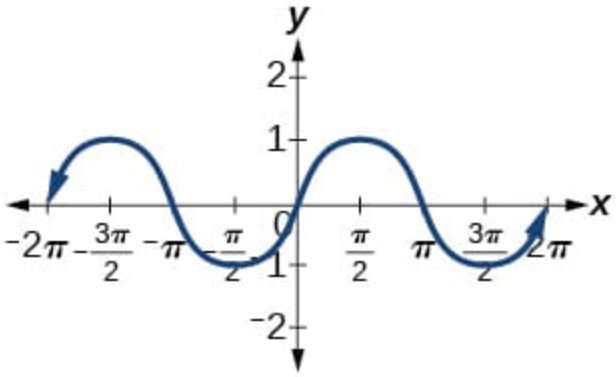


Figure 2.1.6: Odd symmetry of the sine function

Figure 2.1.7 shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos\ x$.

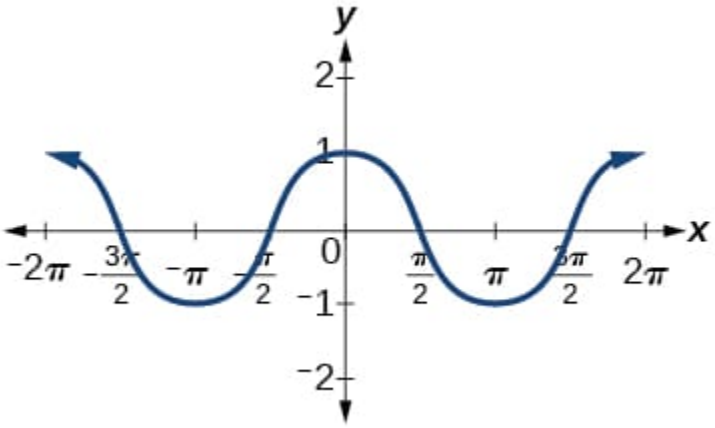


Figure 2.1.7: Even symmetry of the cosine function

CHARACTERISTICS OF SINE AND COSINE FUNCTIONS

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos\ x$ is symmetric about they- y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or **cosine function** is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D \tag{2.1.1}$$

and

$$y = A\cos(Bx - C) + D \tag{2.1.2}$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \frac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π,which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\frac{x}{2}\right)$, then $B = \frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in Figure 2.1.8 how the period is indirectly related to $|B|$.

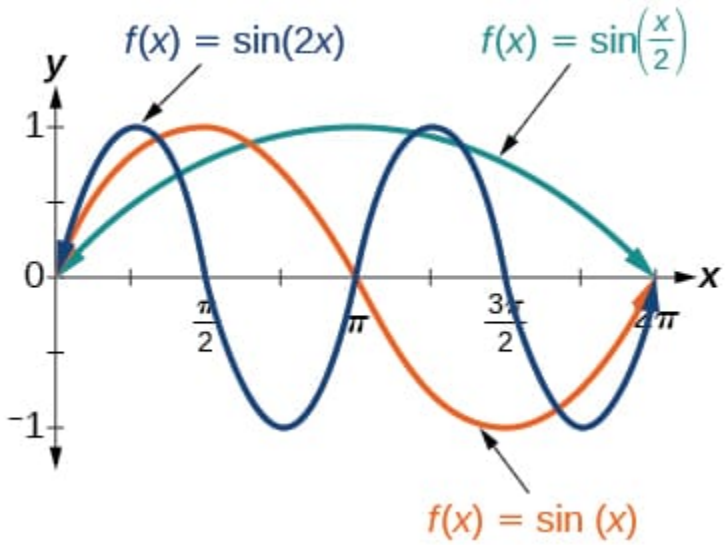


Figure 2.1.8

PERIOD OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

- $y = A\sin(Bx)$
- $y = A\cos(Bx)$

The period is $\frac{2\pi}{|B|}$.

Example 2.1.1: Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\frac{\pi}{6}x\right)$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \frac{\pi}{6}$, so the period will be

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{6}} \\ &= 2\pi \cdot \frac{6}{\pi} \\ &= 12 \end{aligned}$$

Exercise 2.1.1

Determine the period of the function $g(x) = \cos(3x)$.

Answer available at the end of this section.

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the *x*-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4sinx$ is twice the amplitude of

$$f(x) = 2\sin x$$

If $|A| < 1$, the function is compressed. Figure 2.1.9 compares several sine functions with different amplitudes.

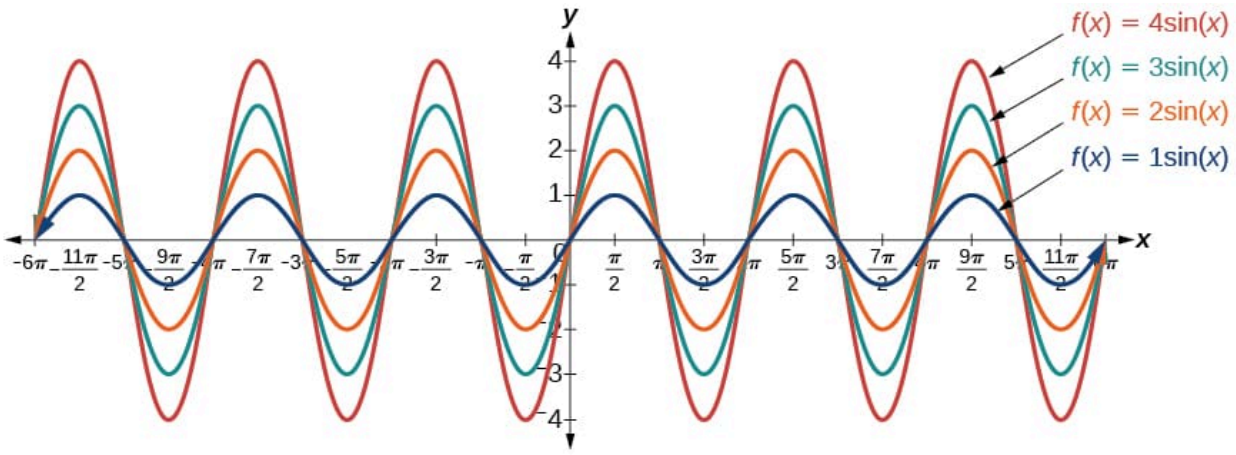


Figure 2.1.9

AMPLITUDE OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx) \text{ and } y = A\cos(Bx) \tag{2.1.3}$$

The amplitude is A, and the vertical height from the midline is $|A|$. Another way to think of the amplitude is that it is half the distance between the maximum and minimum values:

$$|A| = amplitude = \frac{1}{2} \mid maximum - minimum| \tag{2.1.4}$$

Example 2.1.2: Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched. The midline is the x-axis so the maximum values will occur at $y = 4$ and the minimums will occur at $y = -4$.

Analysis

The negative value of A results in a reflection across the x-axis of the sine function, as shown in Figure 2.1.10.

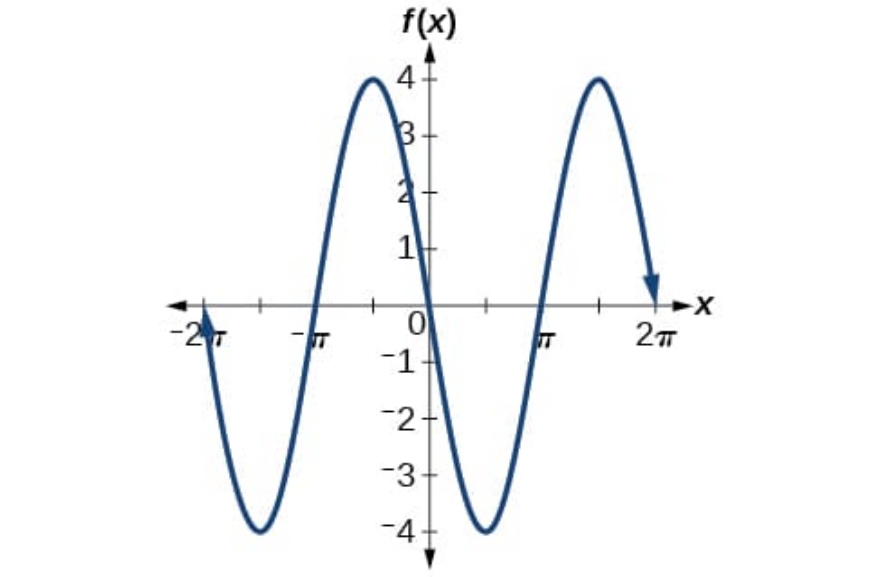


Figure 2.1.10

Exercise 2.1.2

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Answer available at the end of this section.

Analyzing Graphs of Variations of $y = \sin\ x$ and $y = \cos\ x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \qquad \text{and} \qquad y = A\cos(Bx - C) + D \tag{2.1.5}$$

or with the argument factored

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \qquad \text{and} \qquad y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D \tag{2.1.6}$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. Figure 2.1.11 shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the

graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

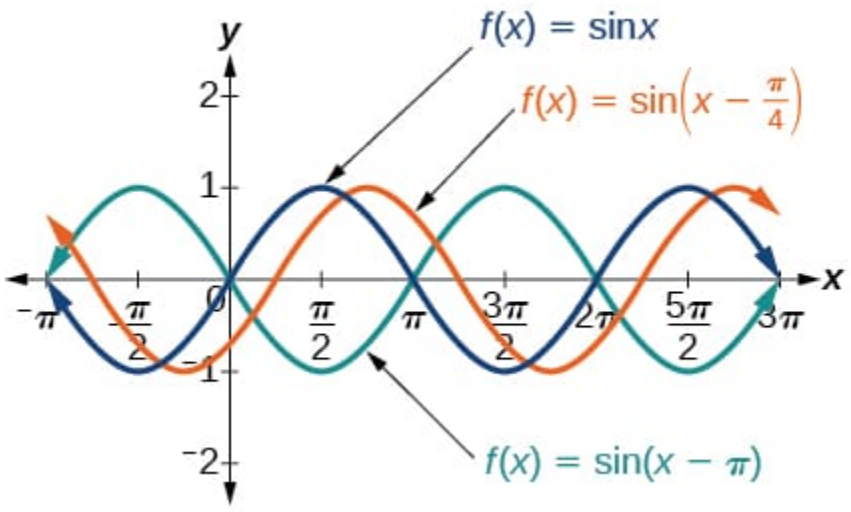


Figure 2.1.11

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See Figure 2.1.12. The function $y = \cos(x) + D$ has its midline at $y = D$.

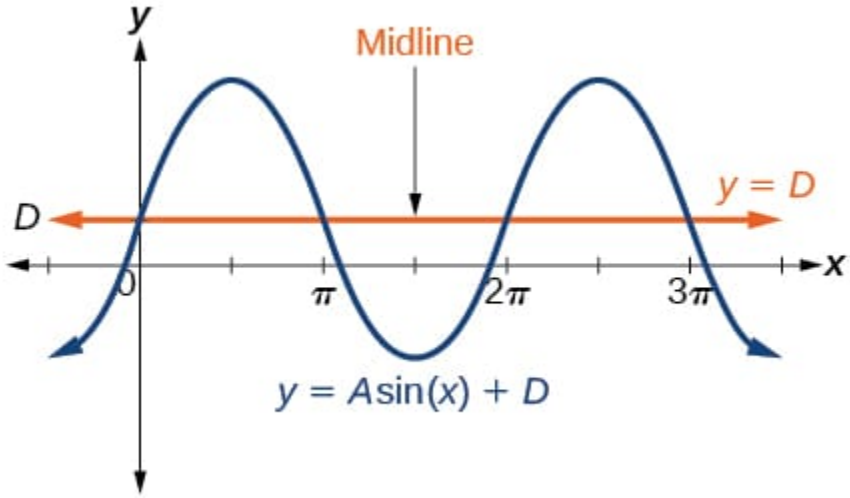


Figure 2.1.12

Any value of D other than zero shifts the graph up or down. Figure 2.1.13 compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

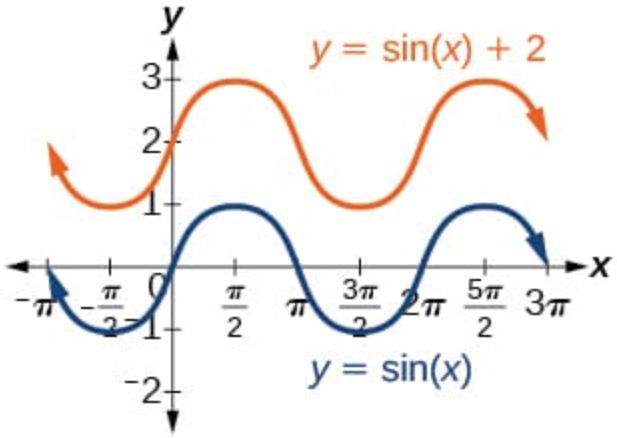


Figure 2.1.13

VARIATIONS OF SINE AND COSINE FUNCTIONS

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and D is the **vertical shift.** Another way to find the **phase shift** is to set the argument of the function equal to zero and solve for the variable:

$$\text{solve the equation } Bx - C = 0 \text{ for } x.$$

If the solution is positive, there is a phase shift to the right. If it is negative, there is a phase shift to the left.

If $C = 0$ there is no phase shift and we can "start" the graph at $x = 0$

Example 2.1.3: Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned}\frac{C}{B} &= -\frac{\frac{\pi}{6}}{1} \\ &= -\frac{\pi}{6}\end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Another way to find the phase shift is to solve the equation $x + \frac{\pi}{6} = 0$, which has the solution $x = -\frac{\pi}{6}$ so there is a phase shift of $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x) = \sin(x + \frac{\pi}{6}) - 2$ can be rewritten as $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2$. If the value of C is negative, the shift is to the left.

Exercise 2.1.3

Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

Answer available at the end of this section.

Example 2.1.4: Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

Solution

Let's begin by comparing the equation to the general form $y = A\cos(Bx - C) + D$.

In the given equation, $D = -3$ so the shift is 3 units downward.

Exercise 2.1.4

Determine the direction and magnitude of the vertical shift for $f(x) = 3\sin(x) + 2$.

Answer available at the end of this section.

How to: Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$,identify the midline, amplitude, period, and phase shift

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$ or solve the equation $Bx - C = 0$ for x
4. Determine the midline as $y = D$.

Example 2.1.5: Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\dfrac{C}{B} = \dfrac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis

Inspecting the graph, we can determine that the period is π, the midline is $y = 1$, and the amplitude is 3. See Figure 2.1.14.

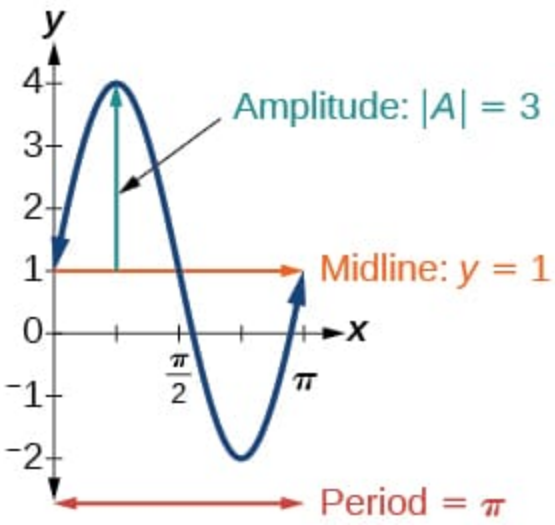


Figure 2.1.14

Exercise 2.1.5

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Answer available at the end of this section.

Example 2.1.6: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the function in Figure 2.1.15.

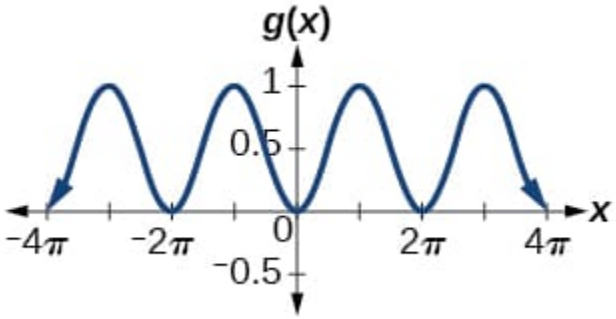


Figure 2.1.15

Solution

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin(Bx - C) + D$$

$$y = A\cos(Bx - C) + D$$

The graph could represent either a **sine** or a **cosine function** that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the x-axis so that $A = -0.5$.

What is the period? We need to find how long it takes the curve to repeat itself. We can find the distance between two maximums, two minimums, or two points on the midline where the function is both increasing or decreasing. There is a minimum at $x = 0$ and the next one to the right is at $2pi$, so the period is $2pi$. The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

Determine the formula for the sine function in Figure $2.1.16$.

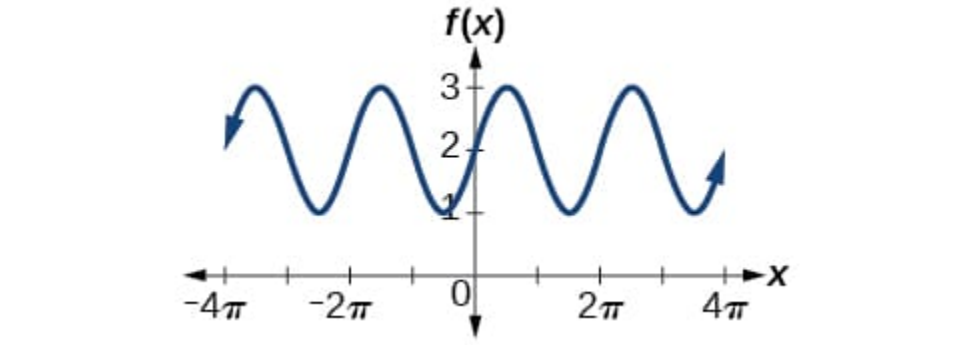


Figure $2.1.16$

Answer available at the end of this section.

Example $2.1.7$: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in Figure $2.1.17$.

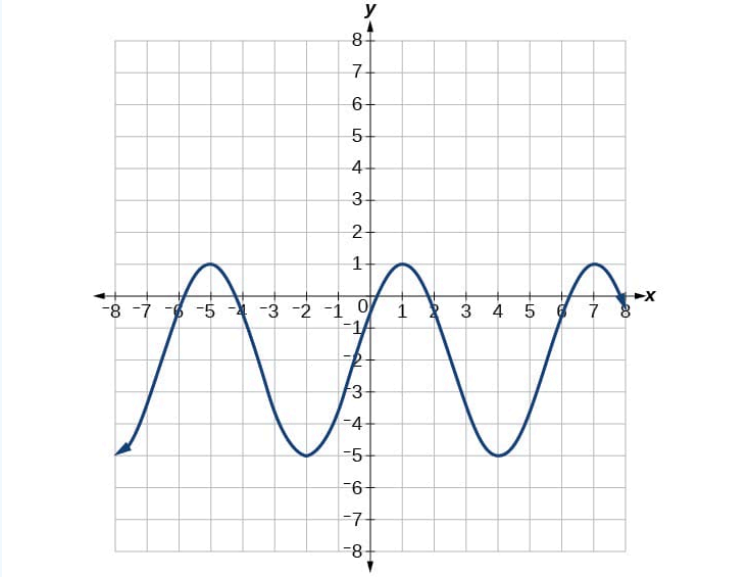


Figure $2.1.17$

Solution

With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D = -2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A| = 3$.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$,or from the distance between the lowest points. Therefore, $P = \frac{2\pi}{|B|} = 6$. Using the positive value for B,we find that

$$\begin{aligned} B &= \frac{2\pi}{P} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

So far, our equation is either $y = 3\sin\left(\frac{\pi}{3}x - C\right) - 2$ or $y = 3\cos\left(\frac{\pi}{3}x - C\right) - 2$.For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. We need to choose which way we want to think of the phase shift. There is an easily identifiable maximum at $(1, 1)$ so let's view it as a phase shift to the right. To find C, we will solve the equation $\frac{\pi}{3}x - C = 0$ where $x = 1$ because there is a phase shift right one unit. So we have

$$\frac{\pi}{3}(1) - C = 0$$

$$\frac{\pi}{3} - C = 0$$

$$\frac{\pi}{3} = C$$

Our equation is $y = 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2$

Write a formula for the function graphed in Figure $2.1.18$.

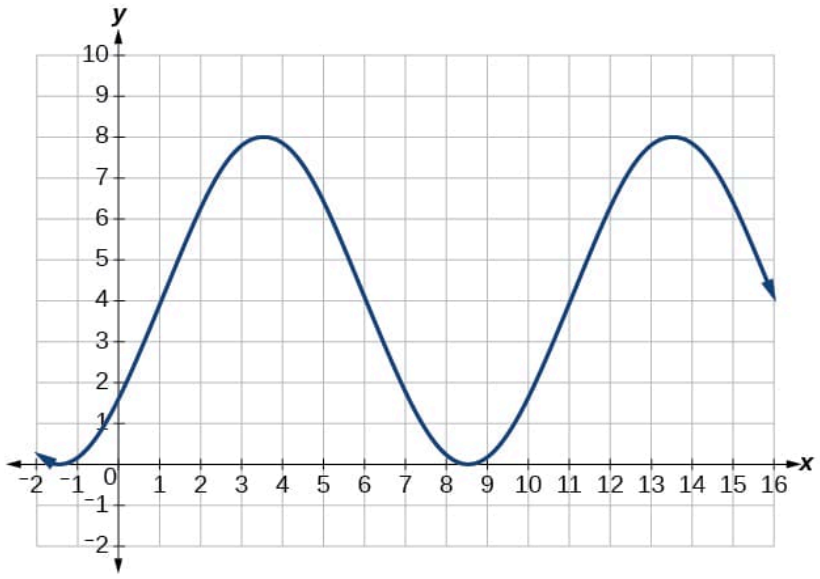


Figure 2.1.18

Answer

two possibilities: $y = 4\sin\left(\frac{\pi}{5}x - \frac{\pi}{5}\right) + 4$ or $y = -4\sin\left(\frac{\pi}{5}x + \frac{4\pi}{5}\right) + 4$

Graphing Variations of $y = \sin\ x$ and $y = \cos\ x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{|B|}$. To find easy points to plot, take the period and divide by 4. Each time this quarter distance is traveled, the curve will have a maximum, be on the midline, or have a minimum.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \frac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the *x*-axis at $x = \frac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \frac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the *x*-axis at $x = \frac{\pi}{2|B|}$.

Example 2.1.8: Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$.

Solution

Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

- *Step 1.* We can see from the equation that $A = -2$, so the amplitude is 2. From the midline (the x-axis, the maximums will occur 2 units above and the minimums will occur 2 units below.

  $|A| = 2$

- *Step 2.* The equation shows that $B = \frac{\pi}{2}$, so the period is

$$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$

Dividing the period by 4 gives 1, so as we move along the x-axis by 1 unit we will hit a maximum, be on the midline, or hit a minimum.

- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4.* The x-intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. Figure $2.1.19$ shows the graph of the function.

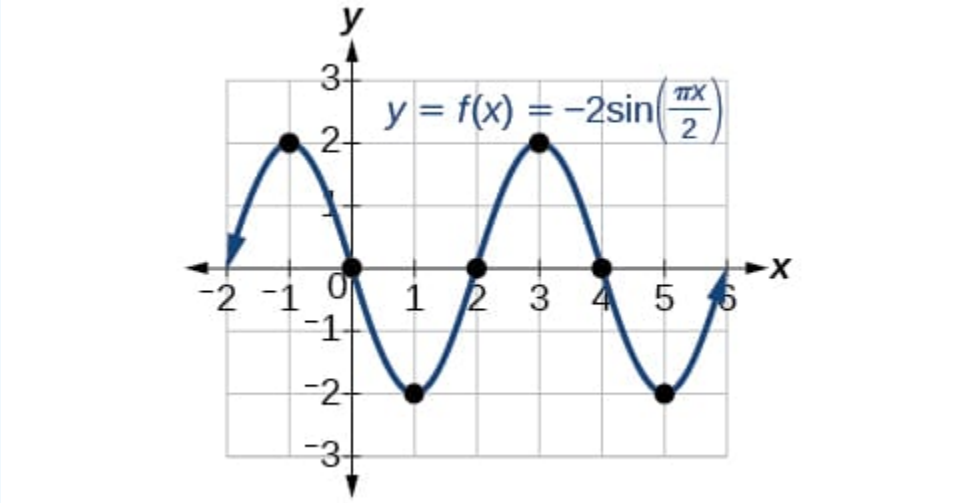


Figure $2.1.19$

Exercise $2.1.8$

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

Answer available at the end of this section.

How to: Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.

2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \frac{2\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$ or solve the equation $Bx - C = 0$ for x.
5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\frac{C}{B}$ and up or down by D.

Example 2.1.9: Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.

Solution

- *Step 1.* The function is already written in general form: $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.This graph will have the shape of a sine function, starting at the midline and increasing to the right.
- *Step 2.* $|A| = |3| = 3$. The amplitude is 3. There is no vertical shift so the midline is the x-axis. The maximums will occur at $y = 3$ and the minimums will occur at $y = -3$.
- *Step 3.* Since $|B| = |\frac{\pi}{4}| = \frac{\pi}{4}$, we determine the period as follows.

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{4}} \\ &= 2\pi \cdot \frac{4}{\pi} \\ &= 8 \end{aligned}$$

The period is 8.

- *Step 4.* To find the phase shift, we solve the equation $\frac{\pi}{4}x - \frac{\pi}{4} = 0$ which has solution $x = 1$, so there is a phase shift 1 unit to the right. We can "start" the curve on the midline (the x-axis) at $(1, 0)$.
- *Step 5.* Figure $2.1.21$ shows the graph of the function.

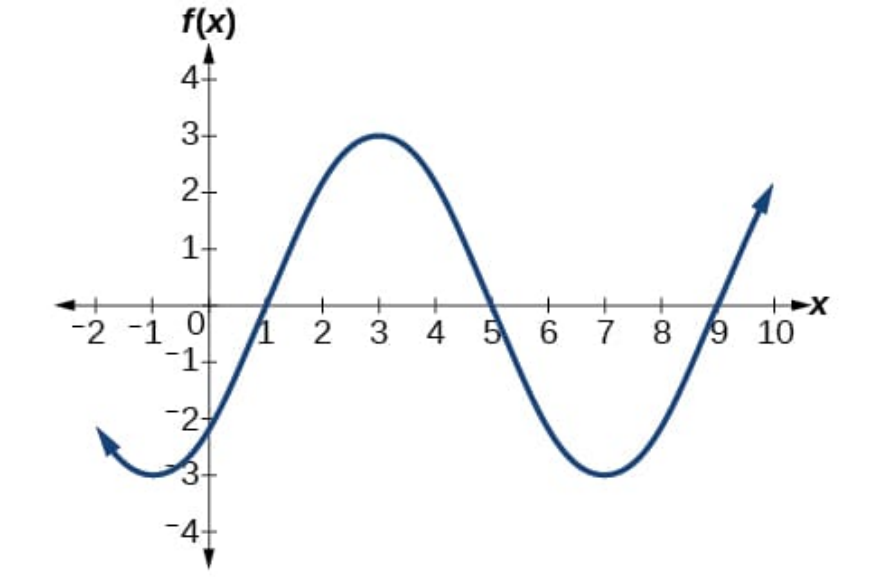


Figure $2.1.21$: A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

Exercise 2.1.9

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Answer available at the end of this section.

Example 2.1.10: Identifying the Properties of a Sinusoidal Function

Given $y = -2\cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution

Begin by comparing the equation to the general form.

$$y = A\cos(Bx - C) + D$$

- *Step 1.* The function is already written in general form.
- *Step 2.* Since $A = -2$, the amplitude is $|A| = 2$.
- *Step 3.* $|B| = \frac{\pi}{2}$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$. The period is 4.
- *Step 4.* $C = -\pi$,so we calculate the phase shift as $\frac{C}{B} = -\frac{\pi}{\frac{\pi}{2}} = -\pi \cdot \frac{2}{\pi} = -2$. Another way to find the phase shift is to solve the equation $\frac{\pi}{2}x + \pi = 0$. The phase shift is 2 units to the left.
- *Step 5.* $D = 3$,so the midline is $y = 3$, and the vertical shift is up 3. Since the amplitude is 2, the maximums will occur at $3 + 2 = 5$ and the minimums will occur at $3 - 2 = 1$.

Since A is negative, the graph of the cosine function has been reflected about the x-axis. According to the phase shift, we can "start" the curve at $x = -2$. Since the curve is reflected, instead of starting at a maximum we will start at a minimum at $(-2, 1)$.

Figure 2.1.23 shows the function over two periods.

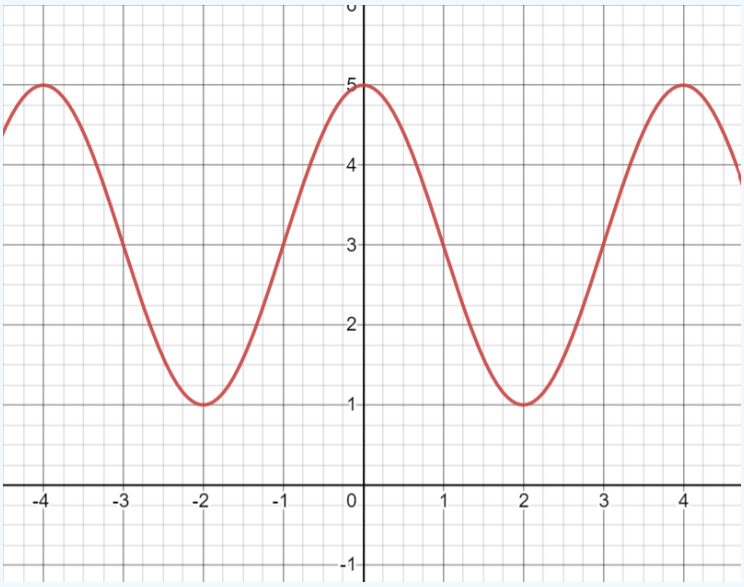


Figure 2.1.23

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter,**circular motion** can be modeled using either the sine or **cosine function**.

Example 2.1.11: Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in Figure 2.1.24.

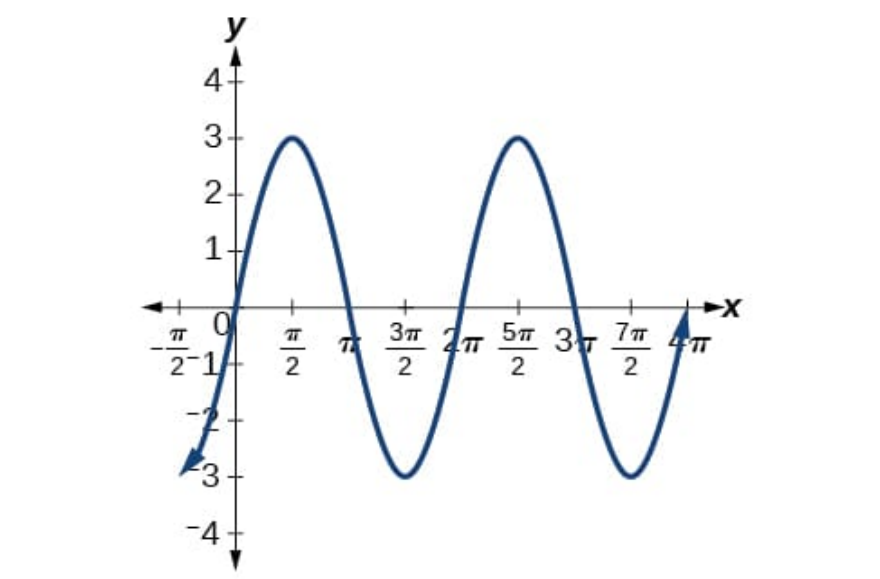


Figure 2.1.24

Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi,$ ....Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

Exercise 2.1.10

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Answer available at the end of this section.

Example 2.1.12: Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in Figure 2.1.26. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

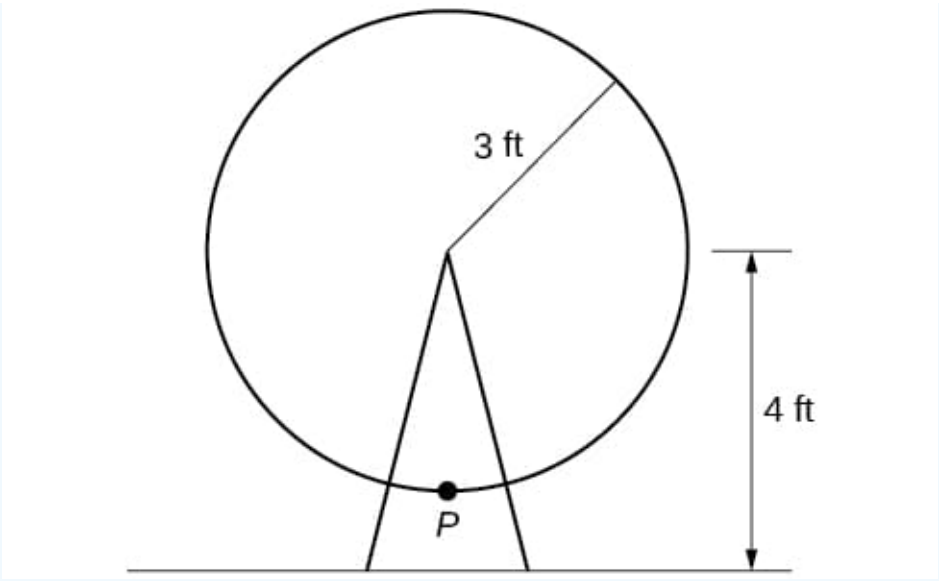


Figure 2.1.26

Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure 2.1.27.

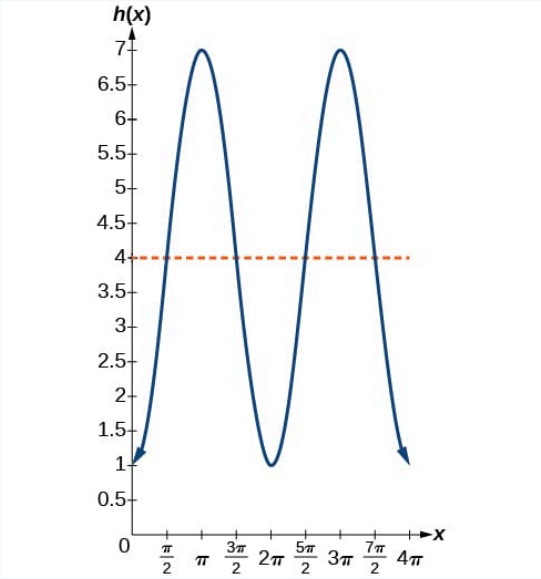


Figure 2.1.27

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a **sine function** starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

Exercise 2.1.11

A weight is attached to a spring that is then hung from a board, as shown in Figure 2.1.28. As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

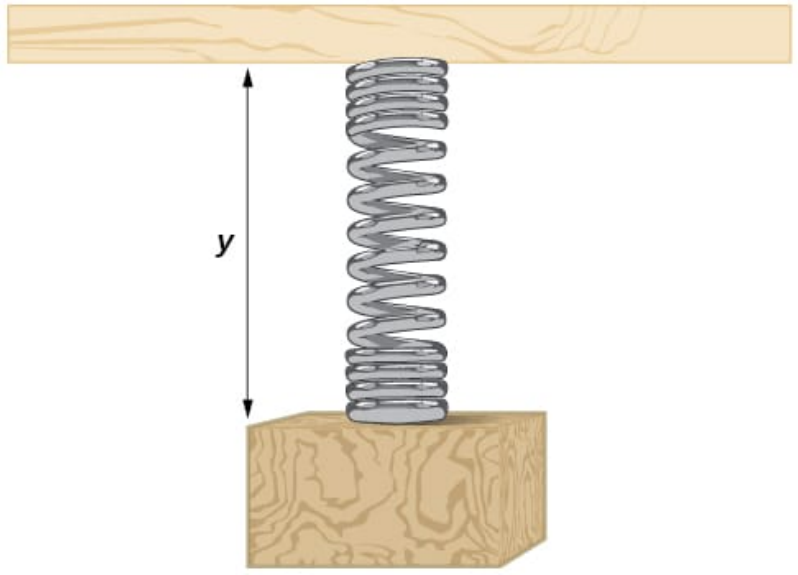


Figure 2.1.28

Answer available at the end of this section.

Example 2.1.13: Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period:30, so $B = \dfrac{2\pi}{30} = \dfrac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5 \cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Media

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine
- Translations of Sine and Cosine
- Graphing Sine and Cosine Transformations
- Graphing the Sine Function

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$ $f(x) = A\cos(Bx - C) + D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the *y*-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \frac{2\pi}{|B|}$. See Example $2.1.1$.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See Example $2.1.2$.
- The value $\frac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See Example $2.1.3$.
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See Example $2.1.4$.
- Combinations of variations of sinusoidal functions can be detected from an equation. See Example $2.1.5$.
- The equation for a sinusoidal function can be determined from a graph. See Example $2.1.6$ and Example $2.1.7$.
- A function can be graphed by identifying its amplitude and period. See Example $2.1.8$ and Example $2.1.9$.
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See Example $2.1.10$.
- Sinusoidal functions can be used to solve real-world problems. See Example $2.1.11$, Example $2.1.12$, and Example $2.1.13$.

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Answers

Exercise $2.1.1$

$\frac{2\pi}{3}$

Exercise 2.1.2

$\frac{1}{2}$ compressed

Exercise 2.1.3

$\frac{\pi}{2}$; right

Exercise 2.1.4

2 units up

Exercise 2.1.5

midline: $y = 0$; amplitude: $|A| = \frac{1}{2}$; period: $P = \frac{2\pi}{|B|} = 6\pi$; phase shift: $\frac{C}{B} = \pi$

$f(x) = \sin(x) + 2$

Exercise 2.1.8

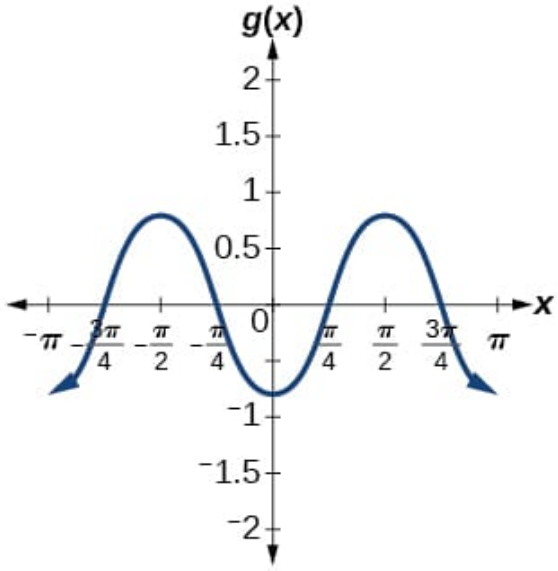


Figure 2.1.20

midline: $y = 0$; amplitude: $|A| = 0.8$; period: $P = \frac{2\pi}{|B|} = \pi$; phase shift: $\frac{C}{B} = 0$ or none

Exercise 2.1.9

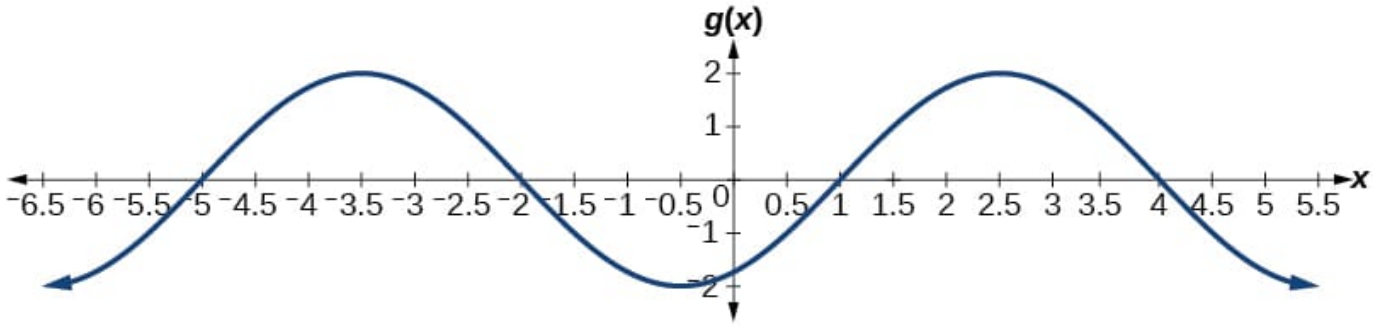


Figure 2.1.22

midline: $y = 0$; amplitude: $|A| = 2$; period: $P = \frac{2\pi}{|B|} = 6$; phase shift: $\frac{C}{B} = -\frac{1}{2}$

Exercise 2.1.10

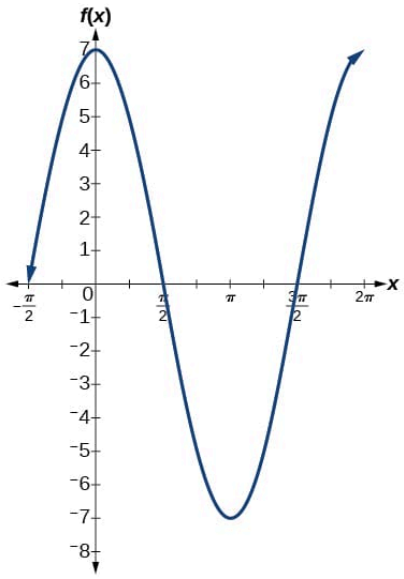


Figure 2.1.25

Exercise 2.1.11

$y = 3\cos(x) - 4$

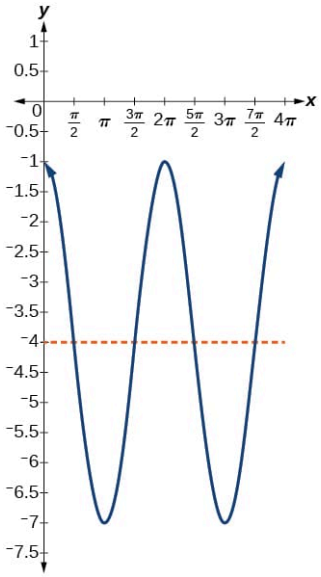


Figure 2.1.29

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2.2: Graphs of the Secant and Cosecant Functions

Learning Objectives

- Analyze the graphs of $y = \sec x$ and $y = \csc x$.
- Graph variations of $y = \sec x$ and $y = \csc x$.

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

The secant function was defined by the reciprocal identity $sec\ x = \dfrac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\dfrac{\pi}{2}$, $\dfrac{3\pi}{2}$ etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure $2.2.1$. The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis - this is because the inverse of 0 is undefined. We show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

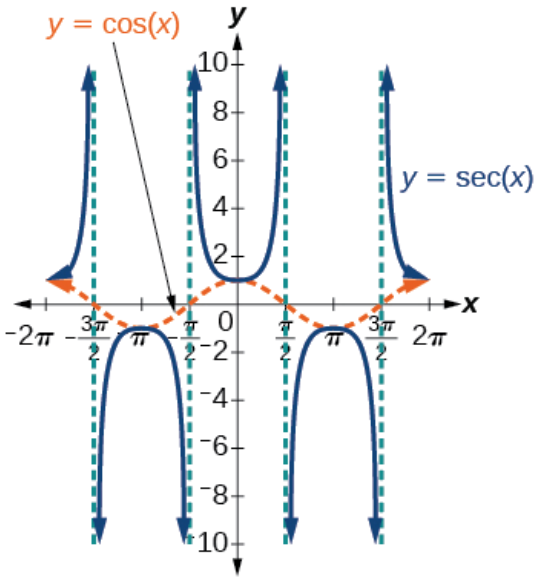


Figure $2.2.1$: Graph of the secant function, $f(x) = \sec x = \dfrac{1}{\cos x}$

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

FEATURES OF THE GRAPH OF $Y = A\sec(Bx)$

- The stretching factor is $|A|$.

- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the **cosecant** is defined by the reciprocal identity $\csc x = \frac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure $2.2.2$. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the **cosecant function** decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in Figure $2.2.2$, is similar to the graph of secant.

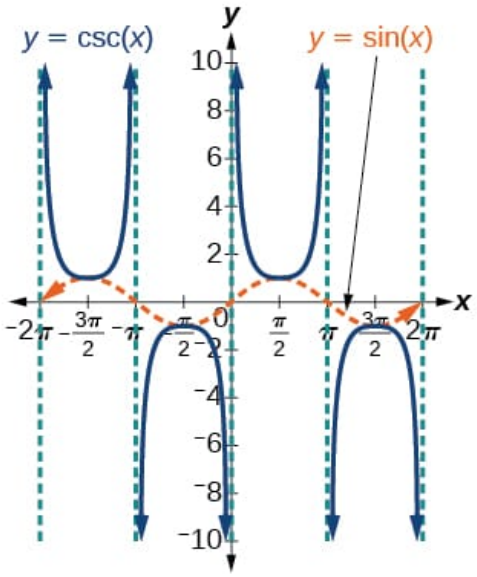


Figure $2.2.2$: The graph of the cosecant function, $f(x) = \csc x = \frac{1}{\sin x}$

FEATURES OF THE GRAPH OF $Y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equations become the following.

$$y = A\sec(Bx - C) + D$$

$$y = A\csc(Bx - C) + D$$

FEATURES OF THE GRAPH OF $Y = A\sec(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{2|B|}k$,where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

FEATURES OF THE GRAPH OF $Y = A\csc(Bx - C) + D$

1. The stretching factor is $|A|$.
2. The period is $\frac{2\pi}{|B|}$.
3. The domain is $x \neq \frac{C}{B} + \frac{\pi}{2|B|}k$,where k is an integer.
4. The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
5. The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
6. There is no amplitude.
7. $y = A\csc(Bx)$ is an odd function because sine is an odd function.

HOWTO: Given a function of the form $y = A\sec(Bx)$, graph one period

1. Sketch the function $y = A\cos(Bx)$.
2. Draw vertical asymptotes where the curve crosses the midline, which is the x-axis
3. Fill in the secant curve in between the asymptotes. Where the cosine curve has a maximum, the secant curve will have an upward U. Where the cosine curve has a minimum, the secant curve will have a downward U.

Example 2.2.1: Graphing a Variation of the Secant Function

Graph one period of $f(x) = 2.5\sec(0.4x)$.

Solution

- *Step 1.* Sketch a graph of the function $f(x) = 2.5\cos(0.4x)$. $A = 2.5$ so the stretching factor is 2.5. $B = 0.4$ so $P = \dfrac{2\pi}{0.4} = 5\pi$. The period is 5π units. Dividing this by 4 gives $\dfrac{5\pi}{4}$. Each time we travel a distance of $\dfrac{5\pi}{4}$ we will be at a maximum, on the midline, or at a minimum.
- *Step 2.* Draw vertical asymptotes where the curve crosses the midline, which is the x-axis
- *Step 3.* Fill in the secant curve in between the asymptotes. Where the cosine curve has a maximum, the secant curve will have an upward U. Where the cosine curve has a minimum, the secant curve will have a downward U. Figure $2.2.3$ shows the graph. The green dashed curve is the cosine function, which acts at the "skeleton" for the secant curve. The blue dashed lines are the vertical asymptotes. The red curves are the acutal secant function.

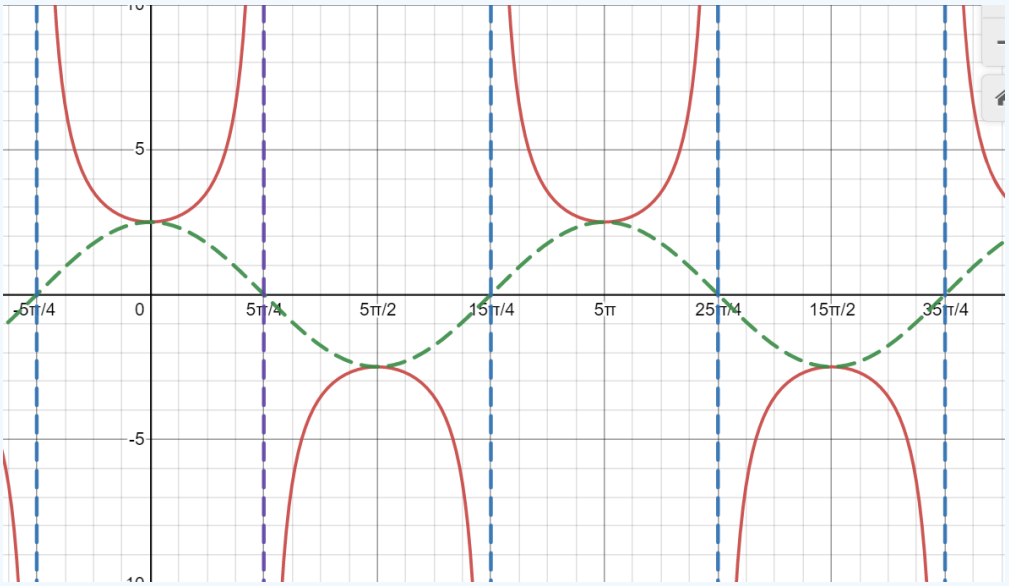
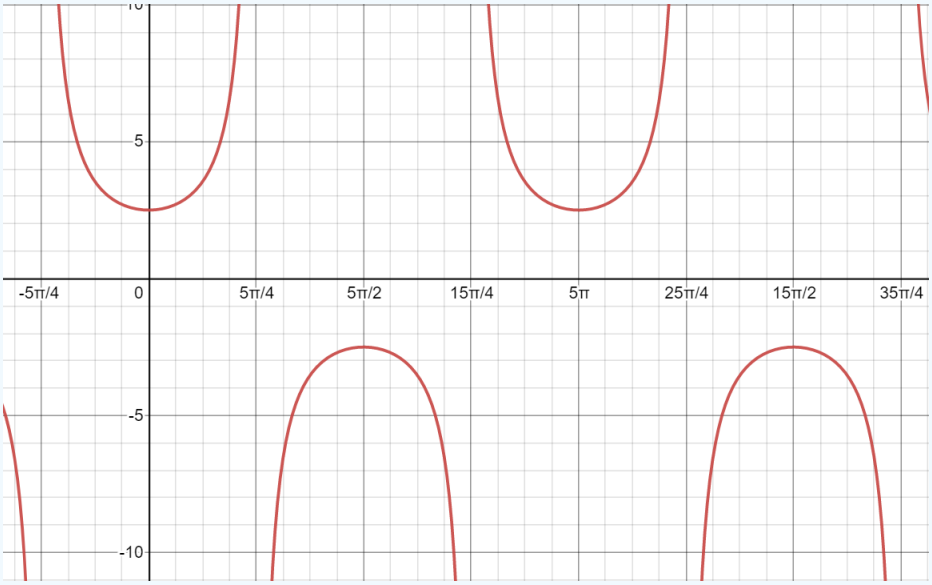


Figure $2.2.3$

Note that in the above graph the green and blue dashed lines are not part of the secant graph, we just use them to obtain the secant graph. Only the red curves are the secant graph, as shown below.



Exercise 2.2.1

Graph one period of $f(x) = -2.5\sec(0.4x)$.

Answer available at the end of this section.

Q&A: Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.

1. Draw the graph of $f(x) = A\cos(Bx - C) + D$
2. Sketch the vertical asymptotes, which occur where the cosine curve passes through its midline at $y = D$
3. Fill in the secant curve in between the asymptotes. Where the cosine curve has a maximum, the secant curve will have an upward U. Where the cosine curve has a minimum, the secant curve will have a downward U.

Example 2.2.2: Graphing a Variation of the Secant Function

Graph one period of $y = \sec\left(2x - \frac{\pi}{2}\right) + 3$.

Solution

- *Step 1.* Sketch the curve $y = \cos\left(2x - \frac{\pi}{2}\right) + 3$.The stretching/compressing factor is $|A| = 1$ so the amplitude is 1. $B = 2$ so the period is $\frac{2\pi}{2} = \pi$. Dividing the period by 4 gives $\frac{\pi}{4}$. Each time we travel this distance we will be at a maximum, on the midline, or a minimum. To find the phase shift, we solve the equation $2x - \frac{\pi}{2} = 0$, which has the solution $x = \frac{\pi}{4}$. There is a phase shift $\frac{\pi}{4}$ to the right. This means we can "start" the cosine curve at a maximum located at $\left(\frac{\pi}{4}, 4\right)$. The midline is the horizontal line $y = 3$.
- *Step 2.* Sketch the vertical asymptotes, which occur where the cosine curve passes through its midline $y = 3$.
- *Step 3.* Fill in the secant curve in between the asymptotes. Where the cosine curve has a maximum, the secant curve will have an upward U. Where the cosine curve has a minimum, the secant curve will have a downward U. Figure 2.2.5 shows the graph. The green curve is the cosine function and the vertical blue lines are the vertical asymptotes. The curve $y = \sec\left(2x - \frac{\pi}{2}\right) + 3$ is in red.

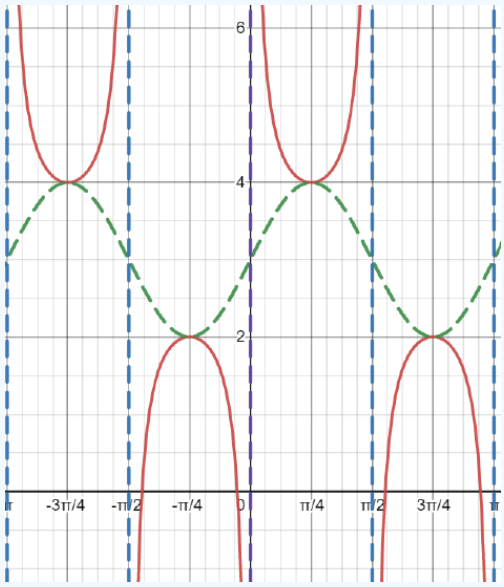


Figure 2.2.5

Exercise 2.2.2

Graph one period of $f(x) = -6\sec(4x + 2) - 8$.

Answer available at the end of this section.

Q&A: The domain of $\csc\ x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \frac{C + k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

Given a function of the form $y = A\csc(Bx)$, graph one period.

1. Sketch the function $y = A\sin(Bx)$.
2. Draw vertical asymptotes where the curve crosses the midline, which is the x-axis
3. Fill in the cosecant curve in between the asymptotes. Where the sine curve has a maximum, the cosecant curve will have an upward U. Where the sine curve has a minimum, the cosecant curve will have a downward U.

Example 2.2.3: Graphing a Variation of the Cosecant Function

Graph one period of $f(x) = -3\csc(4x)$.

Solution

- *Step 1.* Sketch a graph of the function $f(x) = -3\sin(4x)$. $|A| = |-3| = 3$,so the amplitude is 3. $B = 4$,so $P = \frac{2\pi}{4} = \frac{\pi}{2}$. The period is $\frac{\pi}{2}$ units. Dividing the period by 4 gives $\frac{\pi}{8}$. Each time we travel a distance of $\frac{\pi}{8}$ the sine curve will have a maximum, be on the midline, or have a minimum. The value of C is 0 so there is no phase shift, we can "start" the curve on the midline at $x = 0$. The value of D is 0 so there is no vertical shift - the midline is the x-axis.
- *Step 2.* Draw vertical asymptotes where the curve crosses the midline, which is the x-axis.
- *Step 3.* Fill in the cosecant curve in between the asymptotes. Where the sine curve has a maximum, the cosecant curve will have an upward U. Where the sine curve has a minimum, the cosecant curve will have a downward U. Figure 2.2.7 shows the graph over one period.

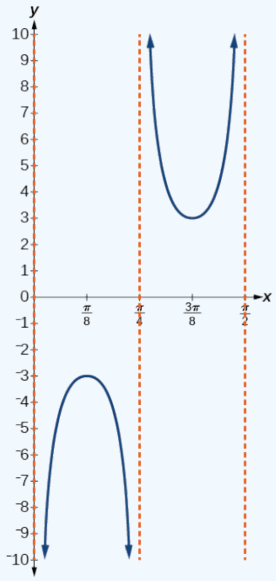


Figure 2.2.7

Exercise 2.2.3

Graph $f(x) = 0.5\csc(2x)$ over at least two periods.

Answer available at the end of this section.

Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period

1. Draw the graph of $f(x) = A\sin(Bx - C) + D$
2. Sketch the vertical asymptotes, which occur where the sine curve passes through its midline at $y = D$
3. Fill in the cosecant curve in between the asymptotes. Where the sine curve has a maximum, the cosecant curve will have an upward U. Where the sine curve has a minimum, the cosecant curve will have a downward U.

Example 2.2.4: Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$.

Solution

- *Step 1.* Sketch a graph of $y = 2\sin\left(\frac{\pi}{2}x\right) + 1$. $|A| = 2$ so the amplitude is 2. The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$. Dividing the period by 4 gives 1. Each time we move a unit of 1 the sine curve will have a maximum, be on the midline, or have a minimum. To find the phase shift we solve the equation $\frac{\pi}{2}x = 0$ which has a solution $x = 0$. We can "start" the sine curve at $x = 0$ on the midline, which is at $y = 1$.
- *Step 2.* Sketch the vertical asymptotes, which occur where the sine curve passes through its midline at $y = 1$.
- *Step 3.* Fill in the cosecant curve in between the asymptotes. Where the sine curve has a maximum, the cosecant curve will have an upward U. Where the sine curve has a minimum, the cosecant curve will have a downward U.

The graph for this function is shown in Figure 2.2.9 in dark blue. The orange dashed line is the sine curve and the dashed vertical blue and green lines are the vertical asymptotes.

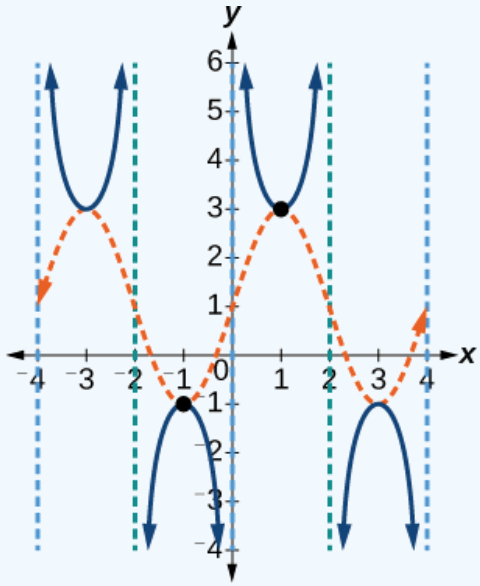


Figure 2.2.9: A transformed cosecant function

Analysis

The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$,shown as the orange dashed wave.

Exercise 2.2.4

Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in Figure 2.2.10, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

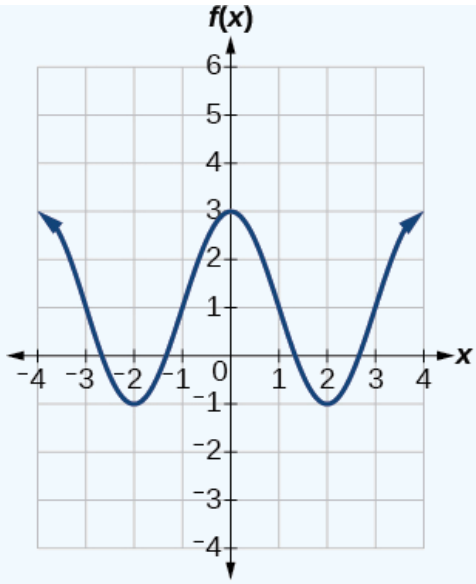


Figure 2.2.10

Answer available at the end of this section.

Key Equations

Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$

Key Concepts

- The secant and cosecant are both periodic functions with a period of 2π.
- $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See Example 2.2.4 and Example 2.2.5.
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See Example 2.2.6 and Example 2.2.7.

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Answers

Exercise 2.2.1

This is a vertical reflection of the preceding graph because A is negative.

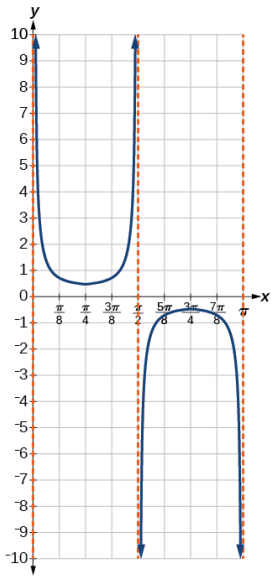


Figure 2.2.4

Exercise 2.2.2

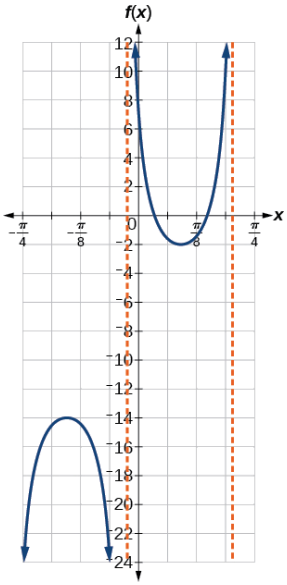


Figure 2.2.6

Exercise 2.2.3

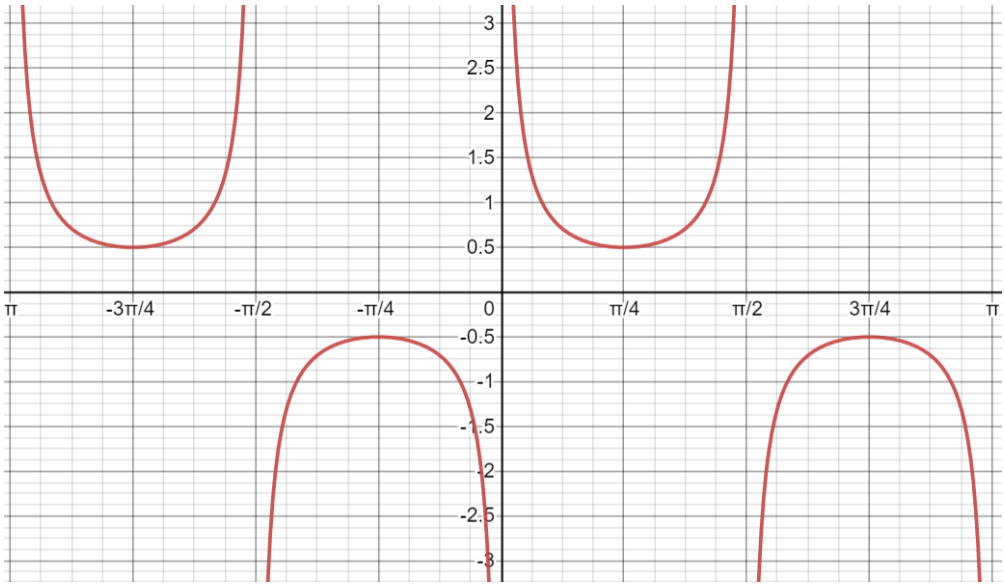


Figure 2.2.8

Exercise 2.2.4

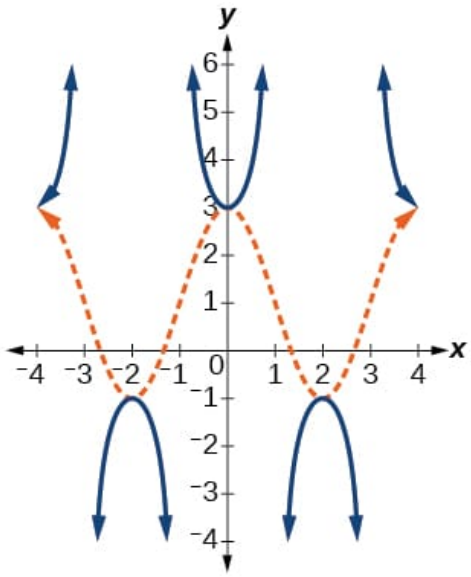


Figure 2.2.18

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2.3: Graphs of the Tangent and Cotangent Functions

Learning Objectives

- Analyze the graph of $y = \tan x$.
- Graph variations of $y = \tan x$.
- Analyze the graph of $y = \cot x$.
- Graph variations of $y = \cot x$.

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and cotangent functions.

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan\, x = \frac{\sin\, x}{\cos\, x} \tag{2.3.1}$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned}
\tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent}\\
&= \frac{-\sin\, x}{\cos\, x} && \text{Sine is an odd function, cosine is even}\\
&= -\frac{\sin\, x}{\cos\, x} && \text{The quotient of an odd and an even function is odd}\\
&= -\tan\, x && \text{Definition of tangent}
\end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table 2.3.1.

Table 2.3.1

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	undefined	$-\sqrt{3}$	–1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in Table 2.3.2.

Table 2.3.2

x	1.3	1.5	1.55	1.56

$\tan x$	3.6	14.1	48.1	92.6

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan\ x$ is an odd function, we see the corresponding table of negative values in Table $2.3.3$.

Table $2.3.3$

x	−1.3	−1.5	−1.55	−1.56
$\tan x$	−3.6	−14.1	−48.1	−92.6

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos\ x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan\ x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. Figure $2.3.1$ represents the graph of $y = \tan\ x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

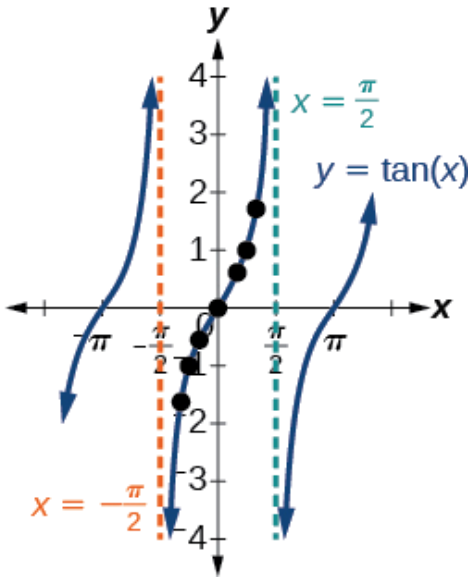


Figure $2.3.1$: Graph of the tangent function

Graphing Variations of $y = \tan\ x$

Let's modify the tangent curve by introducing vertical and horizontal stretching and shrinking. As with the sine and cosine functions, the *tangent function* can be described by a general equation.

$$y = A\tan(Bx)$$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

FEATURES OF THE GRAPH OF $Y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.

- The domain is all real numbers x,where $x \neq \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- To find a pair of asymptotes, solve the equations $Bx = -\frac{\pi}{2}$ and $Bx = \frac{\pi}{2}$. To find other asymptotes from those use the period of the function.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the **tangent function** to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single **period** of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is then the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\frac{P}{2}$ where $P = \frac{\pi}{B}$. On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x = -\frac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x = \frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through.

Given the function $f(x) = A\tan(Bx)$, graph one period.

1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
3. To find a pair of asymptotes, solve the equations $Bx = -\frac{\pi}{2}$ and $Bx = \frac{\pi}{2}$. To find other asymptotes from those use the period of the function.
4. For $A > 0$, the curve increases in between a pair of asymptotes. For $A < 0$ the curve decreases in between a pair of asymptotes.

Example 2.3.1: Sketching a Compressed Tangent

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

Solution

First, we identify A and B.

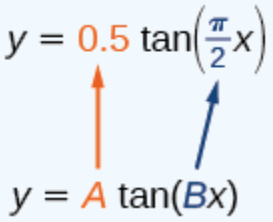


Figure 2.3.2

Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$. To find a pair of asymptotes we solve the equations $\frac{\pi}{2}x = -\frac{\pi}{2}$ and $\frac{\pi}{2}x = \frac{\pi}{2}$, which give solutions of $x = -1$ and $x = 1$, so we know at those two locations we will have vertical asymptotes. At a quarter period from the origin, we have

$$f(0.5) = 0.5\tan\left(\frac{0.5\pi}{2}\right)$$
$$= 0.5\tan\left(\frac{\pi}{4}\right)$$
$$= 0.5$$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$,and $(-0.5, -0.5)$. The only inflection point is at the origin. Figure 2.3.3 shows the graph of one period of the function.

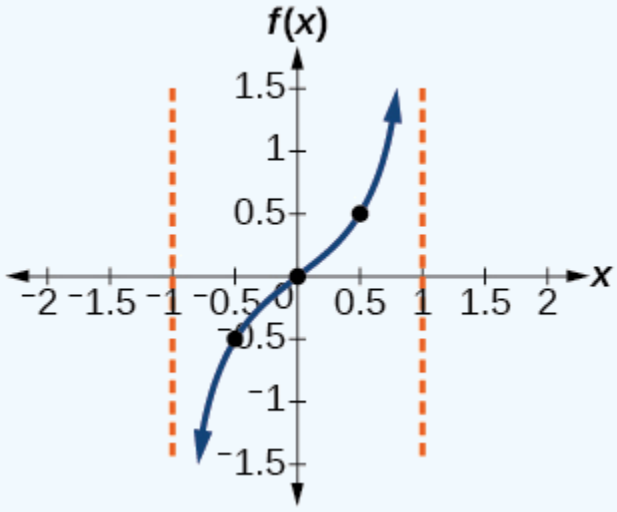


Figure 2.3.3

Exercise 2.3.1

Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$.

Answer available at the end of this section.

Graphing One Period of a Shifted Tangent Function

Now that we can graph a **tangent function** that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$$f(x) = A\tan(Bx - C) + D$$

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

FEATURES OF THE GRAPH OF $Y = A\tan(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- To find a pair of asymptotes, solve the equations $Bx - C = -\frac{\pi}{2}$ and $Bx - C = \frac{\pi}{2}$
- There is no amplitude.
- The "wiggle" point of the curve will happen on the horizontal line $y = D$
- $y = A\tan(Bx)$ is and odd function because it is the quotient of odd and even functions (sin and cosine respectively).

Howto: Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the **stretching/compressing** factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{\pi}{|B|}$.
4. Solve the equations $Bx - C = -\dfrac{\pi}{2}$ and $Bx - C = \dfrac{\pi}{2}$ to find a pair of asymptotes
5. Plot any three reference points and draw the graph through these points.

Example 2.3.2: Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

Solution

- *Step 1.* The function is already written in the form $y = A\tan(Bx - C) + D$.
- *Step 2.* $A = -2$, so the stretching factor is $|A| = 2$. Because of the negative, this curve will be reflected across the x-axis compared to the curve of $y = tanx$, which means the curve will be decreasing in between each pair of asymptotes.
- *Step 3.* $B = \pi$, so the period is $P = \dfrac{\pi}{|B|} = \dfrac{\pi}{\pi} = 1$.
- *Step 4.* Solve the equations $\pi x + \pi = -\dfrac{\pi}{2}$ and $\pi x + \pi = \dfrac{\pi}{2}$. For the first equation:

$$\begin{aligned} \pi x + \pi &= -\frac{\pi}{2} \\ \pi x &= -\frac{\pi}{2} - \pi \\ \pi x &= -\frac{\pi}{2} - \frac{2\pi}{2} \\ \pi x &= -\frac{3\pi}{2} \\ x &= -\frac{3\pi}{2\pi} \\ x &= -\frac{3}{2} \end{aligned}$$

Similarly, for the second equation we have:

$$\begin{aligned} \pi x + \pi &= \frac{\pi}{2} \\ \pi x &= \frac{\pi}{2} - \pi \\ \pi x &= \frac{\pi}{2} - \frac{2\pi}{2} \\ \pi x &= -\frac{\pi}{2} \\ x &= -\frac{\pi}{2\pi} \\ x &= -\frac{1}{2} \end{aligned}$$

- *Step 5.* We can divide the distance of the period by 4 to find three points in between the asymptotes. Taking 1 divided by 4 we have $\dfrac{1}{4}$ or 0.25. Our asymptotes are at -1.5 and -0.5. Starting at the left asymptote -1.5 and increasing by 0.25 we land on the values -1.25, -1, and -0.75. Plugging these values in for x gives us the points $(-1.25, 1)$, $(-1, -1)$, and $(-0.75, -3)$. The graph is shown in Figure 2.3.5.

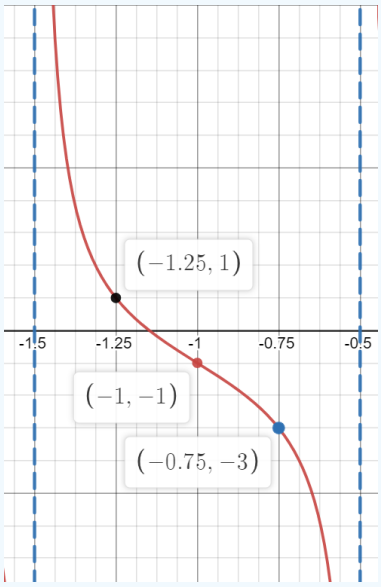


Figure 2.3.5

Analysis

Note that this is a decreasing function because $A < 0$.

Exercise 2.3.2

How would the graph in Example 2.3.2 look different if we made $A = 2$ instead of -2?

Answer available at the end of this section.

Example 2.3.3: Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in Figure 2.3.6.

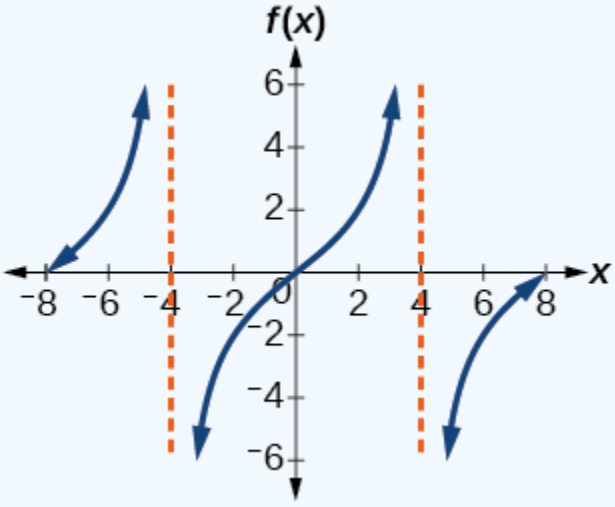


Figure 2.3.6: A stretched tangent function

Solution

The graph has the shape of a tangent function.

- *Step 1.* One cycle extends from -4 to 4, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{\pi}{P} = \frac{\pi}{8}$.
- *Step 2.* The equation must have the form $f(x) = A\tan\left(\frac{\pi}{8}x\right)$.
- *Step 3.* To find the vertical stretch A,we can use the point $(2, 2)$.

$$2 = A\tan\left(\frac{\pi}{8}\cdot 2\right)$$
$$= A\tan\left(\frac{\pi}{4}\right)$$

Because $\tan\left(\frac{\pi}{4}\right) = 1$, $A = 2$.

This function would have a formula $f(x) = 2\tan\left(\frac{\pi}{8}x\right)$.

Exercise 2.3.3

Find a formula for the function in Figure 2.3.7.

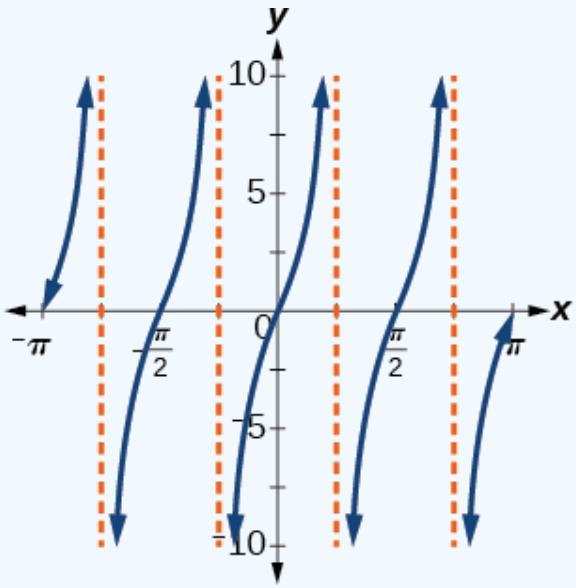


Figure 2.3.7

Answer available at the end of this section.

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $cot\ x = \frac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See Figure 2.3.8. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed

lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

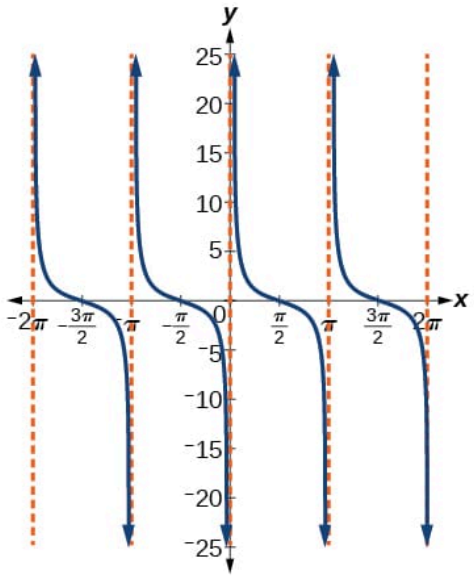


Figure 2.3.8: The cotangent function

FEATURES OF THE GRAPH OF $Y = A\cot(BX)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D \tag{2.3.2}$$

PROPERTIES OF THE GRAPH OF $Y = A\cot(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- A pair of vertical asymptotes occur at the solutions of the equations $Bx - C = 0$ and $Bx - C = \pi$. To find others, use the distance of the period.
- There is no amplitude.
- $y = A\cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

Example 2.3.4: Graphing Variations of the Cotangent Function

Determine the stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph.

Solution

- *Step 1.* The stretching factor is $|A| = 3$.
- *Step 2.* The period is $P = \frac{\pi}{4}$. Dividing this by 4 gives $\frac{\pi}{16}$. Every time we travel a distance of $\frac{\pi}{16}$ we will be at a vertical asymptote, a wiggle point, or a point on either side of the wiggle point.
- *Step 3.* To find a pair of vertical asymptotes, we solve the equations $4x = 0$ and $4x = \pi$, which give solutions $x = 0$ and $x = \frac{\pi}{4}$
- *Step 4.* Plot points in between the vertical asymptotes. Three such points are $\left(\frac{\pi}{16}, 3\right)$, $\left(\frac{\pi}{8}, 0\right)$ $\left(\frac{3\pi}{16}, -3\right)$.

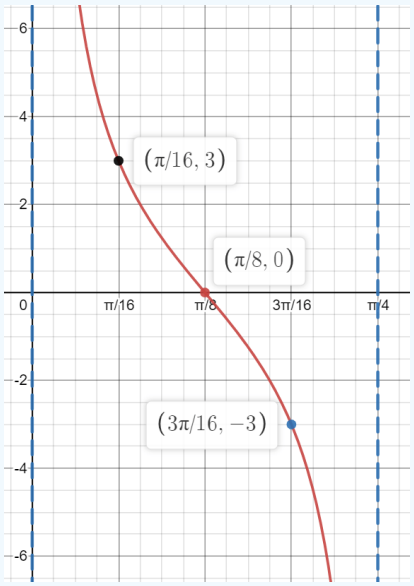


Figure 2.3.9

Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Identify the stretching factor, $|A|$. If $A > 0$ the function is decreasing in between a pair of asymptotes. If $A < 0$ then it is increasing.
2. Identify the period, $P = \frac{\pi}{|B|}$.
3. Find a pair of asymptotes by solving the equations $Bx - C = 0$ and $Bx - C = \pi$. To find others, use the distance of the period.
4. Plot any three reference points and draw the graph through these points. Points that will be nice to work with are those a distance of a quarter of the period away from the asymptotes and in the center of the asymptotes. In the center of the asymptotes is where you should have your wiggle point. The wiggle point will be on the line $y = D$

Example 2.3.5: Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Solution

- *Step 1.* $A = 4$,so the stretching factor is 4.
- *Step 2.* $B = \frac{\pi}{8}$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.
- *Step 3.* To find a pair of asymptotes we solve the equations $\frac{\pi}{8}x - \frac{\pi}{2} = 0$ and $\frac{\pi}{8}x - \frac{\pi}{2} = \pi$. For the first equation:

$$\begin{aligned} \frac{\pi}{8}x - \frac{\pi}{2} &= 0 \\ \frac{\pi}{8}x &= \frac{\pi}{2} \\ x &= \frac{\frac{\pi}{2}}{\frac{\pi}{8}} \\ x &= \frac{\pi}{2} \times \frac{8}{\pi} \\ x &= 4 \end{aligned}$$

For the second equation:

$$\begin{aligned} \frac{\pi}{8}x - \frac{\pi}{2} &= \pi \\ \frac{\pi}{8}x &= \pi + \frac{\pi}{2} \\ \frac{\pi}{8}x &= \frac{2\pi}{2} + \frac{\pi}{2} \\ \frac{\pi}{8}x &= \frac{3\pi}{2} \\ x &= \frac{\frac{3\pi}{2}}{\frac{\pi}{8}} \\ x &= \frac{3\pi}{2} \times \frac{8}{\pi} \\ x &= 12 \end{aligned}$$

The vertical asymptotes are located at $x = 4$ and $x = 12$

- *Step 4.* Dividing the period 8 by 4 gives 2. Every 2 units we will hit an asymptote, wiggle point, or a point on either side of the wiggle point. The wiggle point will happen half way between the asymptotes on the line $y = -2$. Starting at the left asymptote and increasing by 2 gives the values 6, 8, and 10. Three points we can use to guide the graph are $(6, 2)$, $(8, -2)$, and $(10, -6)$.

The graph is shown in Figure $2.3.10$.

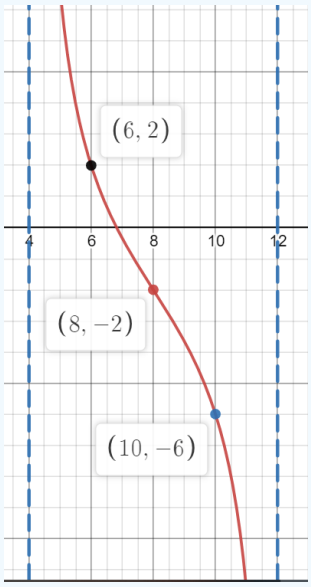


Figure 2.3.10: One period of a modified cotangent function

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the **tangent function**.

Example 2.3.6: Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan(\frac{\pi}{4}t)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.
b. Graph on the interval $[0, 5]$.
c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.

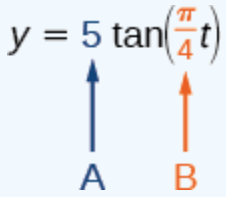


Figure 2.3.11

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See Figure $2.3.12$

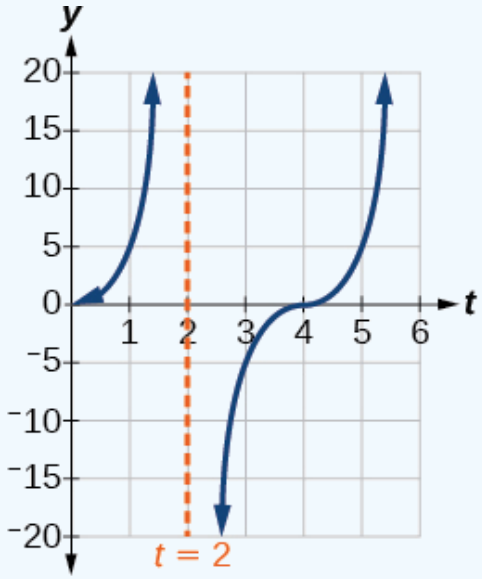


Figure $2.3.12$

c. period: $f(1) = 5\tan(\frac{\pi}{4}(1)) = 5(1) = 5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Key Equations

Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$

Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift.
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi$,....
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.
- The cotangent is zero at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$,....
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift.
- Real-world scenarios can be solved using graphs of trigonometric functions.

Answers

Exercise $2.3.1$

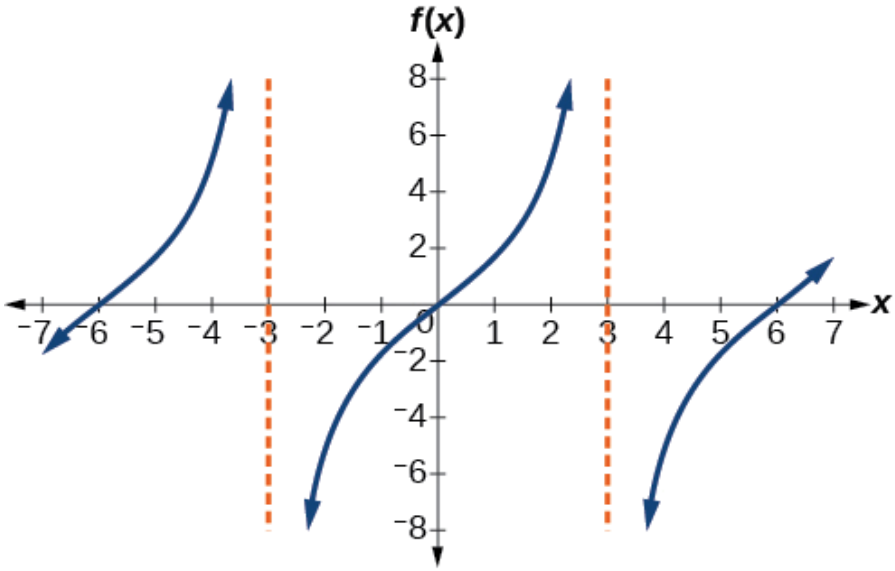


Figure 2.3.4

Exercise 2.3.2

It would be reflected across the line $y = -1$, becoming an increasing function.

Exercise 2.3.3

$g(x) = 4\tan(2x)$

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2.4: Inverse Trigonometric Functions

Learning Objectives

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in Figure 2.4.1.



Figure 2.4.1

For example, if $f(x) = \sin\ x$, then we would write $f^{-1}(x) = \sin^{-1}x$. Be aware that $\sin^{-1}x$ does not mean $\frac{1}{\sin\ x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, then $\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\frac{\pi}{4}\right) = 1$, then $\frac{\pi}{4} = \tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the **domain** of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. Figure 2.4.2 shows the graph of the sine function limited to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.

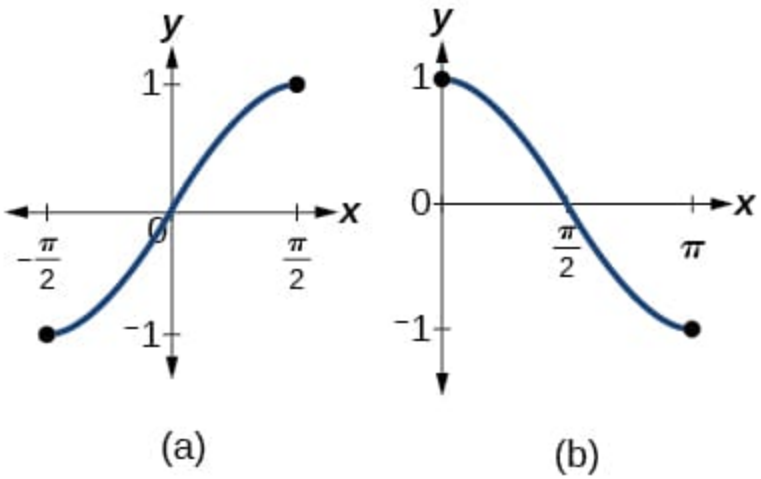


Figure 2.4.2: (a) Sine function on a restricted domain of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; (b) Cosine function on a restricted domain of $[0, \pi]$

Figure 2.4.3 shows the graph of the tangent function limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

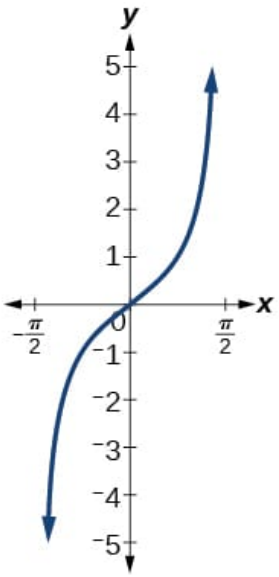


Figure 2.4.3: Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one **vertical asymptote** to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin\ y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin\ x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos\ y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos\ x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The inverse tangent function $y = \tan^{-1}x$ means $x = \tan\ y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan\ x$.

$$y = \tan^{-1}x \text{ has domain } (-\infty, \infty) \text{ and range } (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

The graphs of the inverse functions are shown in Figures 2.4.4 - 2.4.6. Notice that the output of each of these inverse functions is a *number,* an angle in radian measure. We see that $\sin^{-1}x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1}x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1}x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

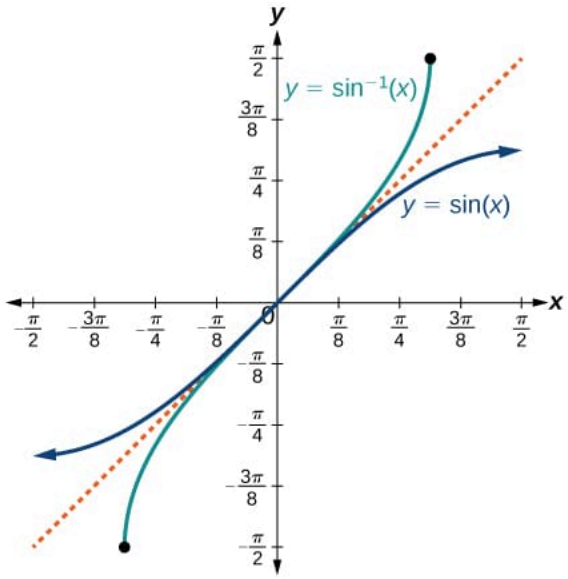


Figure 2.4.4: The sine function and inverse sine (or arcsine) function

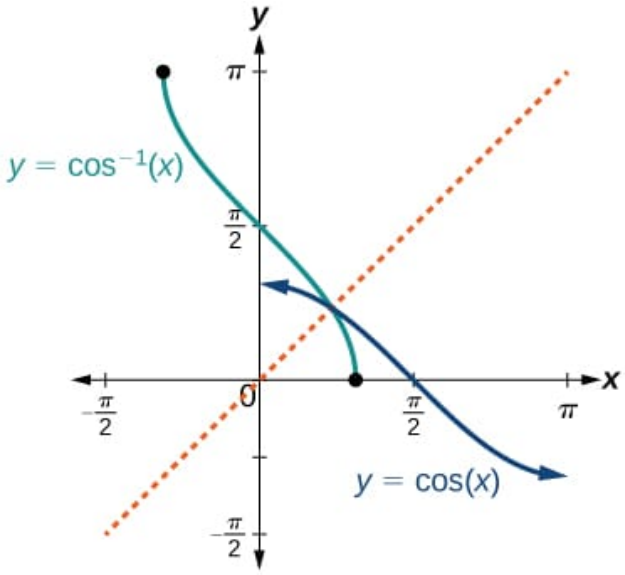


Figure 2.4.5: The cosine function and inverse cosine (or arccosine) function

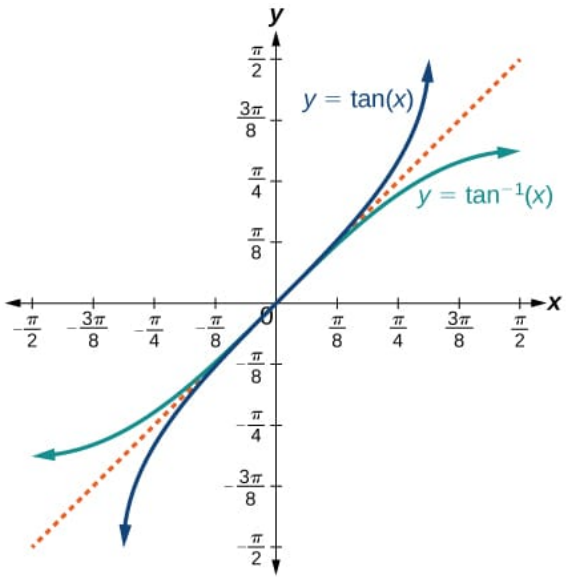


Figure 2.4.6: The tangent function and inverse tangent (or arctangent) function

RELATIONS FOR INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$,then $\tan^{-1} x = y$.

Example 2.4.1: Writing a Relation for an Inverse Function

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Solution

Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$

Exercise 2.4.1

Given $\cos(0.5) \approx 0.8776$,write a relation involving the inverse cosine.

Answer available at the end of this section.

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the

inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}$(30°), $\frac{\pi}{4}$(45°), and $\frac{\pi}{3}$(60°), and their reflections into other quadrants.

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x,depending on which corresponds to the given inverse function.

Example 2.4.2: Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$
b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
c. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
d. $\tan^{-1}(1)$

Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x)=\frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$. Remember that the inverse is a function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$.

c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0,\pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$.

d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1)=\frac{\pi}{4}$.

Exercise 2.4.2

Evaluate each of the following.

a. $\sin^{-1}(-1)$
b. $\tan^{-1}(-1)$
c. $\cos^{-1}(-1)$
d. $\cos^{-1}\left(\frac{1}{2}\right)$

Answer a available at the end of this section.

Answer b available at the end of this section.

Answer c available at the end of this section.

Answer d available at the end of this section.

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate **inverse trigonometric functions** that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN-1, ARCSIN, or ASIN.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

Example 2.4.3: Evaluating the Inverse Sine on a Calculator

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Solution

Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93\ °$. Note that in calculus and beyond we will use radians in almost all cases.

Exercise 2.4.3

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer available at the end of this section.

Given two sides of a right triangle like the one shown in Figure 8.4.7, find an angle.

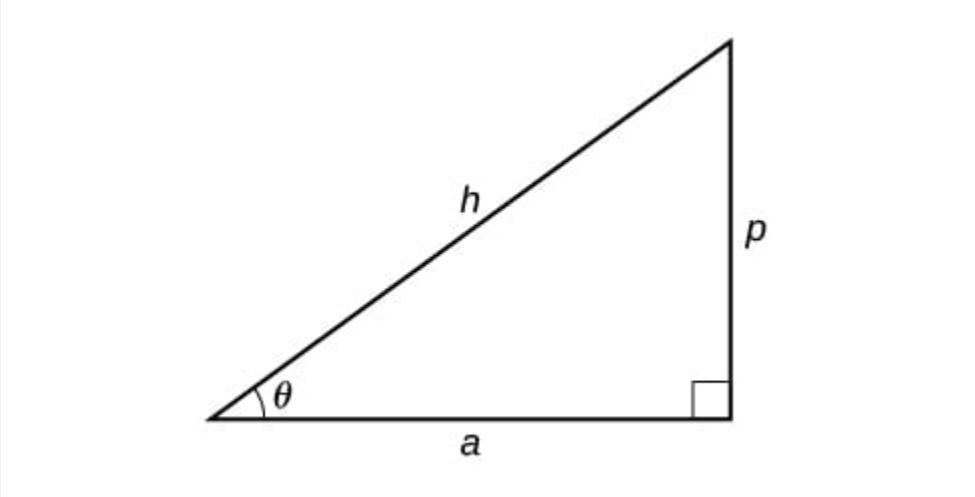


Figure 2.4.7

1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

Example 2.4.4: Applying the Inverse Cosine to a Right Triangle

Solve the triangle in Figure 2.4.8 for the angle θ.

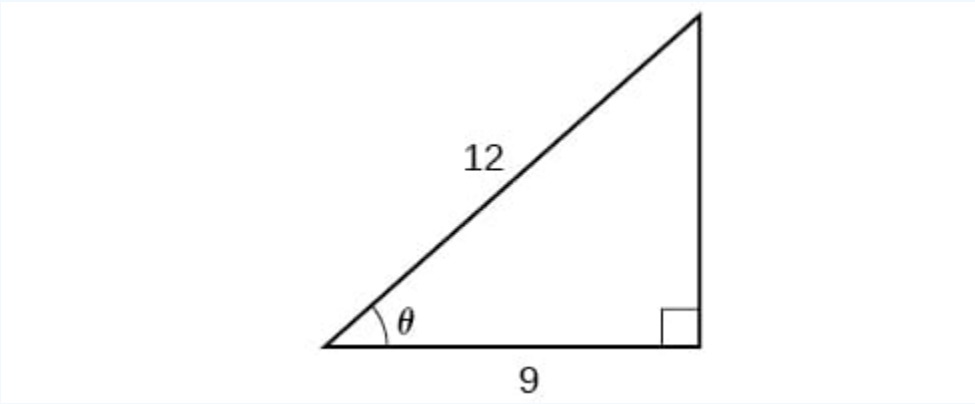


Figure 2.4.8

Solution

Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos\theta = \frac{9}{12}$$
$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \quad \text{Apply definition of the inverse}$$
$$\theta \approx 0.7227 \quad \text{or about } 41.4096^\circ \text{ Evaluate}$$

Exercise 2.4.4

Solve the triangle in Figure 2.4.9 for the angle θ.

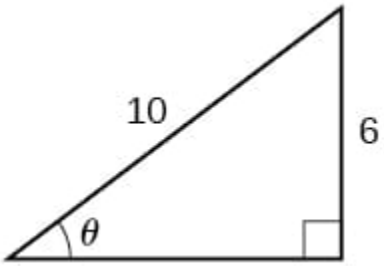


Figure 2.4.9

Answer available at the end of this section.

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set{ $\sin(x)$,$\cos(x)$,$\tan(x)$ } and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function,$f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

COMPOSITIONS OF A TRIGONOMETRIC FUNCTION AND ITS INVERSE

$$\begin{aligned}
\sin(\sin^{-1}x) &= x \quad \text{for } -1 \le x \le 1\\
\cos(\cos^{-1}x) &= x \quad \text{for } -1 \le x \le 1\\
\tan(\tan^{-1}x) &= x \quad \text{for } -\infty < x < \infty\\
\sin^{-1}(\sin x) &= x \quad \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}\\
\cos^{-1}(\cos x) &= x \quad \text{only for } 0 \le x \le \pi\\
\tan^{-1}(\tan x) &= x \quad \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Q&A

Is it correct that $\sin^{-1}(\sin x) = x$?

No. This equation is correct ifx x belongs to the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but sine is defined for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain off f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Example 2.4.5: Using Inverse Trigonometric Functions

Evaluate the following:

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
c. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$
d. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution

a. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.
b. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $sin\left(\frac{2\pi}{3}\right) = sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$.
c. $\frac{2\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$.
d. $-\frac{\pi}{3}$ is not in $[0, \pi]$, but $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ because cosine is an even function. $\frac{\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

Exercise 2.4.5

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Answer available at the end of this section.

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}(g(x))$. For special values of x,we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2} - \theta$.Consider the sine and cosine of each angle of the right triangle in Figure 2.4.10.

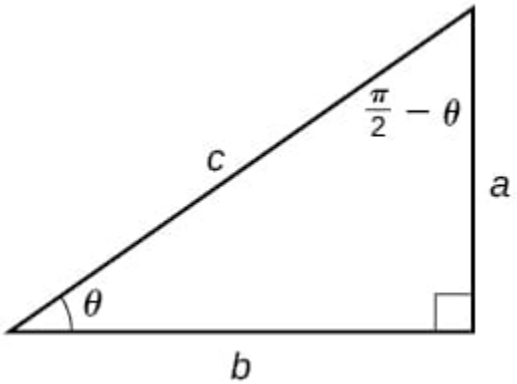


Figure 2.4.10: Right triangle illustrating the cofunction relationships

Because $\cos\theta = \frac{b}{c} = sin\left(\frac{\pi}{2} - \theta\right)$, we have $\sin^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$ if $0 \le \theta \le \pi$. If θ is not in this domain, then we need to find another

angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$.Similarly, $\sin\theta = \frac{a}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$, so $\cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$ if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle y in $[0, \pi]$ such that $\cos y = \cos x$.

$$\sin^{-1}(\cos x) = \frac{\pi}{2} - y \tag{2.4.1}$$

3. If x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then find another angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \sin x$.

$$\cos^{-1}(\sin x) = \frac{\pi}{2} - y \tag{2.4.2}$$

Example 2.4.6: Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

a. by direct evaluation.
b. by the method described previously.

Solution

a. Here, we can directly evaluate the inside of the composition.

$$\begin{aligned}\cos\left(\frac{13\pi}{6}\right) &= \cos\left(\frac{\pi}{6} + 2\pi\right) \\ &= \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Now, we can evaluate the inverse function as we did earlier.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \tag{2.4.3}$$

b. We have $x = \frac{13\pi}{6}$, $y = \frac{\pi}{6}$, and

$$\begin{aligned}\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) &= \frac{\pi}{2} - \frac{\pi}{6} \\ &= \frac{\pi}{3}\end{aligned}$$

Exercise 2.4.6

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Answer available at the end of this section.

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin(\cos^{-1}x) = \sqrt{1-x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Example 2.4.7: Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution

Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 && \text{Use our known value for cosine} \\ \sin^2\theta + \left(\frac{4}{5}\right)^2 &= 1 && \text{Solve for sine} \\ \sin^2\theta &= 1 - \frac{16}{25} \\ \sin\theta &= \pm\frac{9}{25} \\ &= \pm\frac{3}{5} \end{aligned}$$

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in quadrant I, $\sin\theta$ must be positive, so the solution is 35. See Figure 2.4.11.

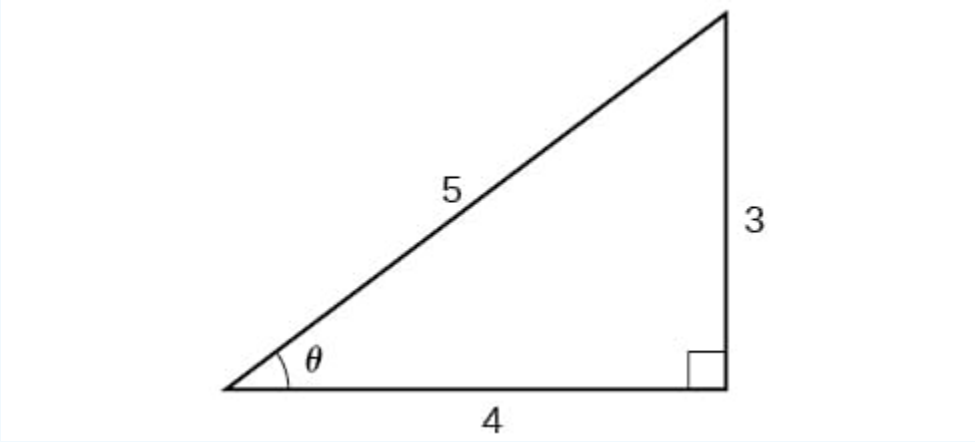


Figure 2.4.11: Right triangle illustrating that if $\cos\theta = \frac{4}{5}$, then $\sin\theta = \frac{3}{5}$

We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive; therefore $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \frac{3}{5}$.

Exercise 2.4.7

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Answer available at the end of this section.

Example 2.4.8: Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Solution

While we could use a similar technique as in Example 2.4.6, we will demonstrate a different technique here. From the inside, we know there is an angle such that $\tan\theta = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in Figure 2.4.12.

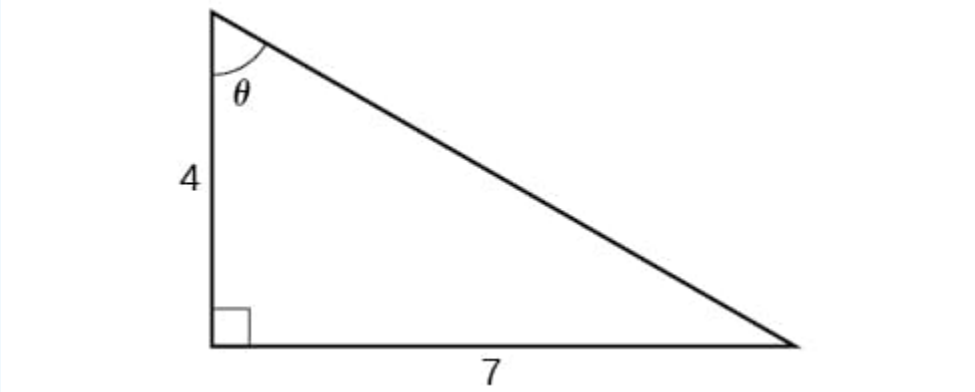


Figure 2.4.12: A right triangle with two sides known

Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

$$4^2 + 7^2 = hypotenuse^2$$
$$hypotenuse = \sqrt{65}$$

Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{7}{\sqrt{65}}$$

This gives us our desired composition.

$$\begin{aligned}\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) &= \sin\theta \\ &= \frac{7}{\sqrt{65}} \\ &= \frac{7\sqrt{65}}{65}\end{aligned}$$

Exercise 2.4.8

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Answer available at the end of this section.

Example 2.4.9: Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

Solution

We know there is an angle θ such that $\sin\theta = \frac{x}{3}$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \qquad \text{Use the Pythagorean Theorem} \\ \left(\frac{x}{3}\right)^2 + \cos^2\theta &= 1 \qquad \text{Solve for cosine} \\ \cos^2\theta &= 1 - \frac{x^2}{9} \\ \cos\theta &= \pm\sqrt{\frac{9-x^2}{9}} \\ &= \pm\sqrt{\frac{9-x^2}{3}} \end{aligned}$$

Because we know that the inverse sine must give an angle on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we can deduce that the cosine of that angle must be positive.

$cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \sqrt{\frac{9-x^2}{3}}$

Exercise 2.4.9

Find a simplified expression for $\sin(\tan^{-1}(4x))$ for $-\frac{1}{4} \le x \le \frac{1}{4}$.

Answer available at the end of this section.

Media

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- Evaluate Expressions Involving Inverse Trigonometric Functions

Visit this website for additional practice questions from Learningpod.

Key Concepts

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f(x)$, if $x = f^{-1}(y)$, then $f(x) = y$. However, $f(x) = y$ only implies $x = f^{-1}(y)$ if x is in the restricted domain of f. See Example 2.4.1.
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4} = \tan^{-1}(1)$ and $\frac{\pi}{6} = \sin^{-1}(\frac{1}{2})$.See Example 2.4.2.
- A calculator will return an angle within the restricted domain of the original trigonometric function. See Example 2.4.3.
- Inverse functions allow us to find an angle when given two sides of a right triangle. See Example 2.4.4.
- In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example,$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$. See Example 2.4.5.
- If the inside function is a trigonometric function, then the only possible combinations are $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ if $0 \le x \le \pi$ and $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. See Example 2.4.6 and Example 2.4.7.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See Example 2.4.8.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig

identities to assist in determining the ratio of sides. See Example $2.4.9$.

Answers

Exercise $2.4.1$

$\arccos(0.8776) \approx 0.5$

Exercise $2.4.2$

$-\frac{\pi}{2}$

Exercise $2.4.2$

$-\frac{\pi}{4}$

Exercise $2.4.2$

π

Exercise $2.4.2$

$\frac{\pi}{3}$

Exercise $2.4.3$

1.9823 or 113.578°

Exercise $2.4.4$

$\sin^{-1}(0.6) = 36.87^{\circ} = 0.6435$ radians

Exercise $2.4.5$

$\frac{\pi}{8}; \frac{2\pi}{9}$

Exercise $2.4.6$

$\frac{3\pi}{4}$

Exercise $2.4.7$

$\frac{12}{13}$

Exercise $2.4.8$

$\frac{4\sqrt{2}}{9}$

Exercise $2.4.9$

$\frac{4x}{\sqrt{16x^2+1}}$

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3: TRIGONOMETRIC IDENTITIES AND EQUATIONS

In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

3.1: Solving Trigonometric Equations with Identities

3.2: Sum and Difference Identities

3.3: Double-Angle, Half-Angle, and Reduction Formulas

3.4: Sum-to-Product and Product-to-Sum Formulas

3.5: Solving Trigonometric Equations

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3.1: Solving Trigonometric Equations with Identities

Learning Objectives

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports—there are many ways to represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.



Figure 3.1.1: International passports and travel documents

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the **Pythagorean identities**, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean identities** (Table 3.1.1), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identifies, but now we will also use additional identities.

Table 3.1.1*: Pythagorean Identities*

$\sin^2\theta + \cos^2\theta = 1$	$1 + \cot^2\theta = \csc^2\theta$	$1 + \tan^2\theta = \sec^2\theta$

The second and third identities can be obtained by manipulating the first. The identity $1 + \cot^2\theta = \csc^2\theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1 + \cot^2\theta = \csc^2\theta$

$$\begin{aligned} 1 + \cot^2\theta &= (1 + \frac{\cos^2}{\sin^2}) && \text{Rewrite the left side} \\ &= \left(\frac{\sin^2}{\sin^2}\right) + \left(\frac{\cos^2}{\sin^2}\right) && \text{Write both terms with the common denominator} \\ &= \frac{\sin^2 + \cos^2}{\sin^2} \\ &= \frac{1}{\sin^2} \\ &= \csc^2 \end{aligned}$$

Similarly,$1 + \tan^2\theta = \sec^2\theta$can be obtained by rewriting the left side of this identity in terms of sine and cosine. This gives

$$\begin{aligned} 1 + \tan^2\theta &= 1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Rewrite left side} \\ &= \left(\frac{\cos\theta}{\cos\theta}\right)^2 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Write both terms with the common denominator} \\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} \\ &= \sec^2\theta \end{aligned}$$

Recall that we determined which trigonometric functions are odd and which are even. The next set of fundamental identities is the set of **even-odd identities.** The **even-odd identities** relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle (Table 3.1.2).

Table 3.1.2*: Even-Odd Identities*

$\tan(-\theta) = -\tan\theta$	$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$
$\cot(-\theta) = -\cot\theta$	$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$

Recall that an **odd function** is one in which $f(-x) = -f(x)$ for all x in the domain off. f. The sine function is an odd function because $\sin(-\theta) = -\sin\theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is opposite the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$\begin{aligned} \sin\left(\frac{\pi}{2}\right) &= 1 \\ \sin\left(-\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

This is shown in Figure 3.1.2.

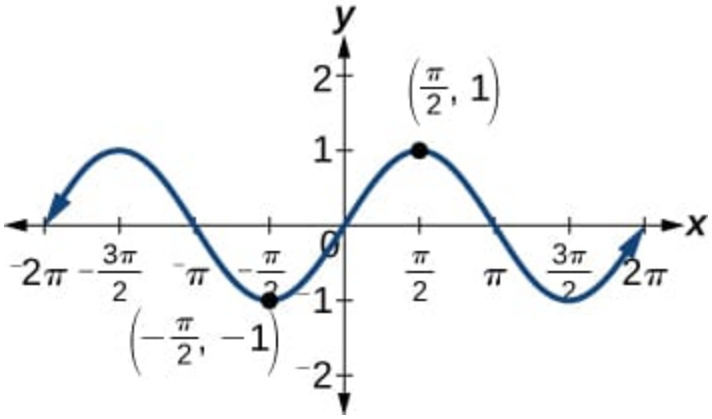


Figure 3.1.2: Graph of $y = \sin\theta$

Recall that an **even function** is one in which

$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$

The graph of an even function is symmetric about the *y*-axis. The cosine function is an even function because $\cos(-\theta) = \cos\theta$. For example, consider corresponding inputs $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. The output of $\cos\left(\frac{\pi}{4}\right)$ is the same as the output of $\cos\left(-\frac{\pi}{4}\right)$. Thus,

$$\begin{aligned}\cos\left(-\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right)\\ &\approx 0.707\end{aligned}$$

See Figure 3.1.3.

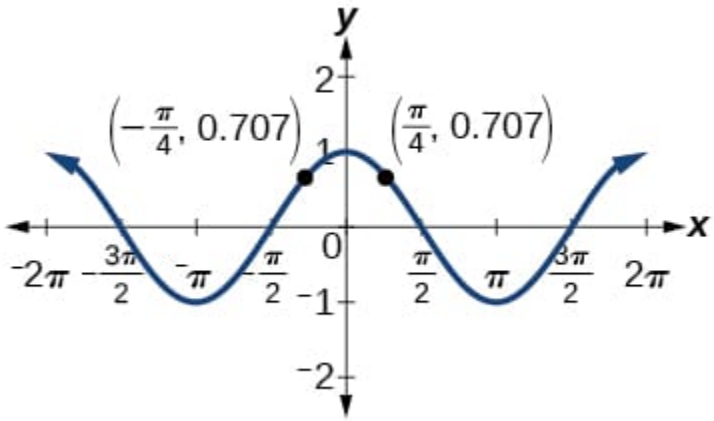


Figure 3.1.3: Graph of $y = \cos\theta$

For all θ in the domain of the sine and cosine functions, respectively, we can state the following:

- Since $\sin(-\theta) = -\sin\theta$,sine is an odd function.
- Since $\cos(-\theta) = \cos\theta$,cosine is an even function.

The other even-odd identities follow from the even and odd nature of the sine and cosine functions. For example, consider the tangent identity,$\tan(-\theta) = -\tan\theta$. We can interpret the tangent of a negative angle as

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$$

Tangent is therefore an odd function, which means that $\tan(-\theta) = -\tan(\theta)$ for all θ in the domain of the **tangent function.**

The cotangent identity, $\cot(-\theta) = -\cot\theta$,also follows from the sine and cosine identities. We can interpret the cotangent of a

negative angle as

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$$

Cotangent is therefore an odd function, which means that $\cot(-\theta) = -\cot(\theta)$ for all θ in the domain of the **cotangent function.**

The **cosecant function** is the reciprocal of the sine function, which means that the cosecant of a negative angle will be interpreted as

$$\csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta.$$

The cosecant function is therefore odd.

Finally, the secant function is the reciprocal of the cosine function, and the secant of a negative angle is interpreted as

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta.$$

The secant function is therefore even.

To sum up, only two of the trigonometric functions, cosine and secant, are even. The other four functions are odd, verifying the even-odd identities.

The next set of fundamental identities is the set of reciprocal identities, which, as their name implies, relate trigonometric functions that are reciprocals of each other. (Table 3.1.3). Recall that we first encountered these identities when defining trigonometric functions from right angles in Right Angle Trigonometry.

Table 3.1.3: Reciprocal Identities

$\sin\theta = \frac{1}{\csc\theta}$	$\csc\theta = \frac{1}{\sin\theta}$
$\cos\theta = \frac{1}{\sec\theta}$	$\sec\theta = \frac{1}{\cos\theta}$
$\tan\theta = \frac{1}{\cot\theta}$	$\cot\theta = \frac{1}{\tan\theta}$

The final set of identities is the set of **quotient identities,** which define relationships among certain trigonometric functions and can be very helpful in verifying other identities (Table 3.1.4).

Table 3.1.4: Quotient Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$

The reciprocal and quotient identities are derived from the definitions of the basic trigonometric functions.

SUMMARIZING TRIGONOMETRIC IDENTITIES

The Pythagorean identities are based on the properties of a right triangle.

$$\cos^2\theta + \sin^2\theta = 1 \tag{3.1.1}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{3.1.2}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{3.1.3}$$

The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle.

$$\tan(-\theta) = -\tan\theta \tag{3.1.4}$$

$$\cot(-\theta) = -\cot\theta \tag{3.1.5}$$

$$\sin(-\theta) = -\sin\theta \tag{3.1.6}$$

$$\csc(-\theta) = -\csc\theta \tag{3.1.7}$$

$$\cos(-\theta) = \cos\theta \tag{3.1.8}$$

$$\sec(-\theta) = \sec\theta \tag{3.1.9}$$

The reciprocal identities define reciprocals of the trigonometric functions.

$$\sin\theta = \frac{1}{\csc\theta} \tag{3.1.10}$$

$$\cos\theta = \frac{1}{\sec\theta} \tag{3.1.11}$$

$$\tan\theta = \frac{1}{\cot\theta} \tag{3.1.12}$$

$$\csc\theta = \frac{1}{\sin\theta} \tag{3.1.13}$$

$$\sec\theta = \frac{1}{\cos\theta} \tag{3.1.14}$$

$$\cot\theta = \frac{1}{\tan\theta} \tag{3.1.15}$$

The **quotient identities** define the relationship among the trigonometric functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{3.1.16}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{3.1.17}$$

Example 3.1.1: Graphing the Equations of an Identity

Graph both sides of the identity $\cot\theta = \frac{1}{\tan\theta}$. In other words, on the graphing calculator, graph $y = \cot\theta$ and $y = \frac{1}{\tan\theta}$.

Solution

See Figure 3.1.4.

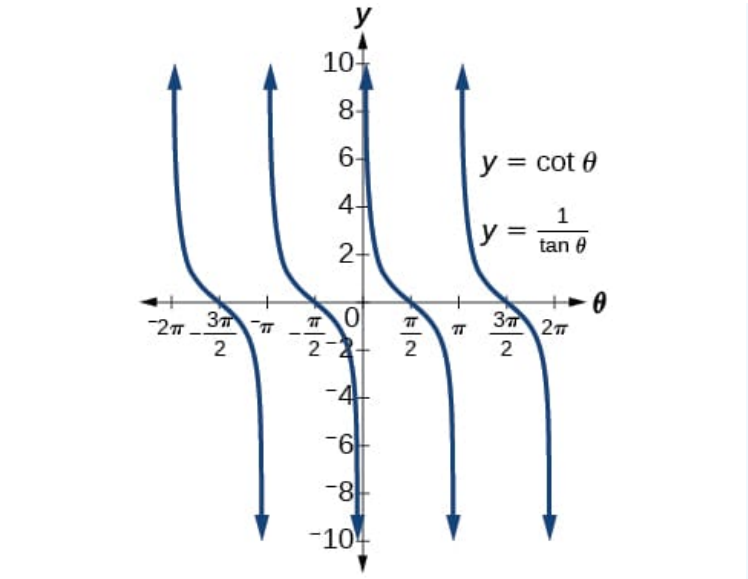


Figure 3.1.4

Analysis

We see only one graph because both expressions generate the same image. One is on top of the other. This is a good way to prove any identity. If both expressions give the same graph, then they must be identities.

How to: Given a trigonometric identity, verify that it is true.

1. Work on one side of the equation. It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. Noting which functions are in the final expression, look for opportunities to use the identities and make the proper substitutions.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

Example 3.1.2: Verifying a Trigonometric Identity

Verify $\tan\theta\cos\theta = \sin\theta$.

Solution

We will start on the left side, as it is the more complicated side:

$$\begin{aligned}\tan\theta\cos\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta \\ &= \sin\theta\end{aligned}$$

Analysis

This identity was fairly simple to verify, as it only required writing $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Exercise 3.1.1

Verify the identity $\csc\theta\cos\theta\tan\theta = 1$.

Answer available at the end of this section.

Example 3.1.3A: Verifying the Equivalency Using the Even-Odd Identities

Verify the following equivalency using the even-odd identities:

$$(1+\sin x)[1+\sin(-x)] = \cos^2 x$$

Solution

Working on the left side of the equation, we have

$$(1+\sin x)[1+\sin(-x)] = (1+\sin x)(1-\sin x)$$

Since

$$\begin{aligned} \sin(-x) &= -\sin x \\ [5pt] &= 1-\sin^2 x \qquad \text{Difference of squares} \\ [5pt] &= \cos^2 x \\ \cos^2 x &= 1-\sin^2 x \end{aligned}$$

Example 3.1.3B: Verifying a Trigonometric Identity Involving $\sec^2\theta$

Verify the identity $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$

Solution

As the left side is more complicated, let's begin there.

$$\begin{aligned} \frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{(\tan^2\theta + 1) - 1}{\sec^2\theta} \\ \sec^2\theta &= \tan^2\theta + 1 \\ &= \frac{\tan^2\theta}{\sec^2\theta} \\ &= \tan^2\theta\left(\frac{1}{\sec^2\theta}\right) \\ &= \tan^2\theta\left(\cos^2\theta\right) \\ \cos^2\theta &= \frac{1}{\sec^2\theta} \\ &= \left(\frac{\sin^2\theta}{\cos^2\theta}\right) \\ \tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta} \\ &= \sin^2\theta \end{aligned}$$

There is more than one way to verify an identity. Here is another possibility. Again, we can start with the left side.

$$\begin{aligned} \frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta} \\ &= 1 - \cos^2\theta \\ &= \sin^2\theta \end{aligned}$$

Analysis

In the first method, we used the identity $\sec^2\theta = \tan^2\theta + 1$ and continued to simplify. In the second method, we split the fraction, putting both terms in the numerator over the common denominator. This problem illustrates that there are multiple ways we can verify an identity. Employing some creativity can sometimes simplify a procedure. As long as the substitutions are correct, the answer will be the same.

Exercise 3.1.2

Show that $\dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

Answer available at the end of this section.

Example 3.1.4: Creating and Verifying an Identity

Create an identity for the expression $2\tan\theta\sec\theta$ by rewriting strictly in terms of sine.

Solution

There are a number of ways to begin, but here we will use the quotient and reciprocal identities to rewrite the expression:

$$\begin{aligned} 2\tan\theta\sec\theta &= 2\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right) \\ &= \frac{2\sin\theta}{\cos^2\theta} \\ &= \frac{2\sin\theta}{1-\sin^2\theta} \qquad \text{Substitute } 1-\sin^2\theta \text{ for } \cos^2\theta \end{aligned}$$

Thus,

$$2\tan\theta\sec\theta = \frac{2\sin\theta}{1-\sin^2\theta}$$

Example 3.1.5: Verifying an Identity Using Algebra and Even/Odd Identities

Verify the identity:

$$\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} = \cos\theta - \sin\theta$$

Solution

Let's start with the left side and simplify:

$$\begin{aligned} \frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} &= \frac{[\sin(-\theta)]^2-[\cos(-\theta)]^2}{\sin(-\theta)-\cos(-\theta)} \\ &= \frac{(-\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} \qquad ,\sin(-x) = -\sin\ x \text{ and } \cos(-x) = \cos\ x \\ &= \frac{(\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} \qquad \text{Difference of squares} \\ &= \frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{-(\sin\theta+\cos\theta)} \\ &= \cos\theta-\sin\theta \end{aligned}$$

Exercise 3.1.3

Verify the identity $\dfrac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta} = \dfrac{\sin\theta + 1}{\tan\theta}$.

Answer available at the end of this section.

Example 3.1.6: Verifying an Identity Involving Cosines and Cotangents

Verify the identity: $(1 - \cos^2 x)(1 + \cot^2 x) = 1$.

Solution

$$\begin{aligned}(1-\cos^2 x)(1+\cot^2 x) &= (1-\cos^2 x)\left(1+\frac{\cos^2 x}{\sin^2 x}\right)\\ &= (1-\cos^2 x)\left(\frac{\sin^2 x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}\right) \qquad \text{Find the common denominator}\\ &= (1-\cos^2 x)\left(\frac{\sin^2 x+\cos^2 x}{\sin^2 x}\right)\\ &= (\sin^2 x)\left(\frac{1}{\sin^2 x}\right)\\ &= 1\end{aligned}$$

Using Algebra to Simplify Trigonometric Expressions

We have seen that algebra is very important in verifying trigonometric identities, but it is just as critical in simplifying trigonometric expressions before solving. Being familiar with the basic properties and formulas of algebra, such as the difference of squares formula, the perfect square formula, or substitution, will simplify the work involved with trigonometric expressions and equations.

For example, the equation $(\sin x + 1)(\sin x - 1) = 0$ resembles the equation $(x + 1)(x - 1) = 0$, which uses the factored form of the difference of squares. Using algebra makes finding a solution straightforward and familiar. We can set each factor equal to zero and solve. This is one example of recognizing algebraic patterns in trigonometric expressions or equations.

Another example is the difference of squares formula, $a^2 - b^2 = (a - b)(a + b)$, which is widely used in many areas other than mathematics, such as engineering, architecture, and physics. We can also create our own identities by continually expanding an expression and making the appropriate substitutions. Using algebraic properties and formulas makes many trigonometric equations easier to understand and solve.

Example 3.1.7A: Writing the Trigonometric Expression as an Algebraic Expression

Write the following trigonometric expression as an algebraic expression: $2\cos^2\theta + \cos\theta - 1$.

Solution

Notice that the pattern displayed has the same form as a standard quadratic expression,$ax^2 + bx + c$. Letting $\cos\theta = x$,we can rewrite the expression as follows:

$$2x^2 + x - 1$$

This expression can be factored as $(2x + 1)(x - 1)$. If it were set equal to zero and we wanted to solve the equation, we would use the zero factor property and solve each factor for x. At this point, we would replace x with $\cos\theta$ and solve for θ.

Example 3.1.7B: Rewriting a Trigonometric Expression Using the Difference of Squares

Rewrite the trigonometric expression using the difference of squares: $4cos^2\theta - 1$.

Solution

Notice that both the coefficient and the trigonometric expression in the first term are squared, and the square of the number 1 is 1. This is the difference of squares.

$$\begin{aligned} 4\cos^2\theta - 1 &= (2\cos\theta)^2 - 1 \\ &= (2\cos\theta - 1)(2\cos\theta + 1) \end{aligned}$$

Analysis

If this expression were written in the form of an equation set equal to zero, we could solve each factor using the zero factor property. We could also use substitution like we did in the previous problem and let $\cos\theta = x$, rewrite the expression as $4x^2 - 1$, and factor $(2x - 1)(2x + 1)$. Then replace x with $\cos\theta$ and solve for the angle.

Exercise 3.1.4

Rewrite the trigonometric expression using the difference of squares: $25 - 9\sin^2\theta$.

Answer available at the end of this section.

Example 3.1.8: Simplify by Rewriting and Using Substitution

Simplify the expression by rewriting and using identities:

$$\csc^2\theta - \cot^2\theta$$

Solution

We can start with the Pythagorean identity.

$$1 + \cot^2\theta = \csc^2\theta$$

$$\text{Now we can simplify by substituting } 1 + \cot^2\theta \text{ for } \csc^2\theta$$

$$\begin{aligned} \csc^2\theta - \cot^2\theta &= 1 + \cot^2\theta - \cot^2\theta \\ &= 1 \end{aligned}$$

Exercise 3.1.5

Use algebraic techniques to verify the identity: $\dfrac{\cos\theta}{1 + \sin\theta} = \dfrac{1 - \sin\theta}{\cos\theta}$.

(Hint: Multiply the numerator and denominator on the left side by $1 - \sin\theta$.)

Answer available at the end of this section.

Media

Access these online resources for additional instruction and practice with the fundamental trigonometric identities.

- Fundamental Trigonometric Identities

- Verifying Trigonometric Identities

Key Equations

Pythagorean identities	$\cos^2\theta + \sin^2\theta = 1$ $1 + \cot^2\theta = \csc^2\theta$ $1 + \tan^2\theta = \sec^2\theta$
Even-odd identities	$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$ $\sin(-\theta) = -\sin\theta$ $\csc(-\theta) = -\csc\theta$ $\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$
Reciprocal identities	$\sin\theta = \frac{1}{\csc\theta}$ $\cos\theta = \frac{1}{\sec\theta}$ $\tan\theta = \frac{1}{\cot\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$
Quotient identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Key Concepts

- There are multiple ways to represent a trigonometric expression. Verifying the identities illustrates how expressions can be rewritten to simplify a problem.
- Graphing both sides of an identity will verify it. See Example $3.1.1$.
- Simplifying one side of the equation to equal the other side is another method for verifying an identity. See Example $3.1.2$ and Example $3.1.3$.
- The approach to verifying an identity depends on the nature of the identity. It is often useful to begin on the more complex

side of the equation. See Example 3.1.4.
- We can create an identity and then verify it. See Example 3.1.5.
- Verifying an identity may involve algebra with the fundamental identities. See Example 3.1.6 and Example 3.1.7.
- Algebraic techniques can be used to simplify trigonometric expressions. We use algebraic techniques throughout this text, as they consist of the fundamental rules of mathematics. See Example 3.1.8, Example 3.1.9, and Example 3.1.10.

Answers

Exercise 3.1.1

$$\begin{aligned}\csc\theta\cos\theta\tan\theta &= \left(\frac{1}{\sin\theta}\right)\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\cos\theta}{\sin\theta}\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\\ &= 1\end{aligned}$$

Exercise 3.1.2

$$\begin{aligned}\frac{\cot\theta}{\csc\theta} &= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}}\\ &= \frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1}\\ &= \cos\theta\end{aligned}$$

Exercise 3.1.3

$$\begin{aligned}\frac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta} &= \frac{(\sin\theta+1)(\sin\theta-1)}{\tan\theta(\sin\theta-1)}\\ &= \frac{\sin\theta+1}{\tan\theta}\end{aligned}$$

Exercise 3.1.4

This is a difference of squares formula: $25-9\sin^2\theta=(5-3\sin\theta)(5+3\sin\theta)$.

Exercise 3.1.5

$$\begin{aligned}\frac{\cos\theta}{1+\sin\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right) &= \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}\\ &= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}\\ &= \frac{1-\sin\theta}{\cos\theta}\end{aligned}$$

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3.2: Sum and Difference Identities

Learning Objectives

- Use sum and difference formulas for cosine.
- Use sum and difference formulas for sine.
- Use sum and difference formulas for tangent.
- Use sum and difference formulas for cofunctions.
- Use sum and difference formulas to verify identities.

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.



Figure 3.2.1: Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown in Figure 3.2.2.

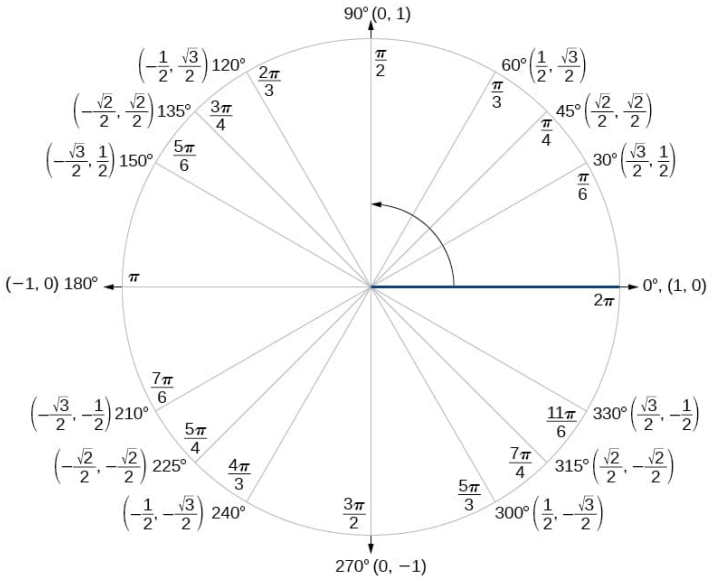


Figure 3.2.2: The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles (Table 3.2.1).

Table 3.2.1

Sum formula for cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle (Figure 3.2.3). Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha,\sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta,\sin\beta)$. Note the measure of angle POQ is $\alpha-\beta$.

Label two more points: A at an angle of $(\alpha-\beta)$ from the positive x-axis with coordinates $(\cos(\alpha-\beta),\sin(\alpha-\beta))$; and point B with coordinates $(1,0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.

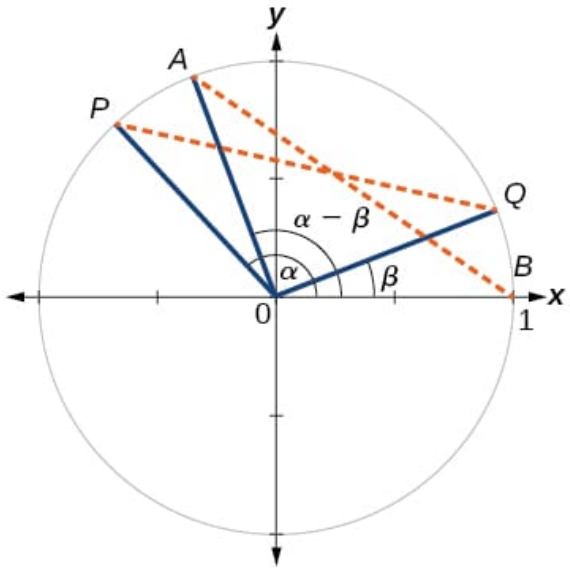


Figure 3.2.3

We can find the distance from P to Q using the distance formula.

$$\begin{aligned} d_{PQ} &= \sqrt{(\cos\alpha - \cos\beta)^2 + (sin\alpha - sin\beta)^2} \\ &= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} && \text{Apply Pythagorean identity and simplify.} \\ &= \sqrt{(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\ &= \sqrt{1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\ &= \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \end{aligned}$$

Similarly, using the distance formula we can find the distance from A to B.

$$\begin{aligned} d_{AB} &= \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2} \\ &= \sqrt{\cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)} && \text{Apply Pythagorean identity and simplify} \\ &= \sqrt{(\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta)) - 2\cos(\alpha-\beta) + 1} \\ &= \sqrt{1 - 2\cos(\alpha-\beta) + 1} \\ &= \sqrt{2 - 2\cos(\alpha-\beta)} && \text{Subtract 2 from both sides and divide both sides by } -2. \end{aligned}$$

$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

SUM AND DIFFERENCE FORMULAS FOR COSINE

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{3.2.1}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{3.2.2}$$

How to: Given two angles, find the cosine of the difference between the angles

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Example 3.2.1: Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$.

Solution

Begin by writing the formula for the cosine of the difference of two angles. Then substitute the given values.

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\begin{aligned}\cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right)&=\cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\&=\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\&=-\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\&=\frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in radian mode.

Exercise 3.2.1:

Find the exact value of $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$.

Answer available at the end of this section.

Example 3.2.2: Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of $\cos(75\text{ °})$.

Solution

As $75\text{ °}=45\text{ °}+30\text{ °}$,we can evaluate $\cos(75\text{ °})$ as $\cos(45\text{ °}+30\text{ °})$.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\begin{aligned}\cos(45^\circ+30^\circ)&=\cos(45^\circ)\cos(30^\circ)-\sin(45^\circ)\sin(30^\circ)\\&=\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\&=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\&=\frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in degree mode.

Analysis

Note that we could have also solved this problem using the fact that $75\text{ °}=135\text{ °}-60\text{ °}$.

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\begin{aligned}\cos(135^\circ-60^\circ)&=\cos(135^\circ)\cos(60^\circ)+\sin(135^\circ)\sin(60^\circ)\\&=\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\&=-\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}\\&=\frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Exercise 3.2.2

Find the exact value of $\cos(105\ °)$.

Answer available at the end of this section.

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

SUM AND DIFFERENCE FORMULAS FOR SINE

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \tag{3.2.3}$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \tag{3.2.4}$$

How to: Given two angles, find the sine of the difference between the angles

1. Write the difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

Example 3.2.3: Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45\ °-30\ °)$
b. $\sin(135\ °-120\ °)$

Solution

a. Let's begin by writing the formula and substitute the given angles.

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\sin(45^\circ-30^\circ)=\sin(45^\circ)\cos(30^\circ)-\cos(45^\circ)\sin(30^\circ)$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45\ °)=\tfrac{\sqrt{2}}{2},\qquad \cos(30\ °)=\tfrac{\sqrt{3}}{2},\qquad \cos(45\ °)=\tfrac{\sqrt{2}}{2},\qquad \sin(30\ °)=\tfrac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\sin(45°-30°)=\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$
$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$

b. Again, we write the formula and substitute the given angles.

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\sin(135^\circ-120^\circ)=\sin(135^\circ)\cos(120^\circ)-\cos(135^\circ)\sin(120^\circ)$$

Next, we find the values of the trigonometric expressions.

$$\sin(135\,^\circ)=\tfrac{\sqrt{2}}{2},\qquad \cos(120\,^\circ)=-\tfrac{1}{2},\qquad \cos(135\,^\circ)=\tfrac{\sqrt{2}}{2},\qquad \sin(120\,^\circ)=\tfrac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(135^\circ-120^\circ)&=\frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\&=\frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example 3.2.4: Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right)$. Then check the answer with a graphing calculator.

Solution

The pattern displayed in this problem is $\sin(\alpha+\beta)$. Let $\alpha=\cos^{-1}\frac{1}{2}$ and $\beta=\sin^{-1}\frac{3}{5}$. Then we can write

$$\cos\alpha=\frac{1}{2},\quad 0\le\alpha\le\pi$$
$$\sin\beta=\frac{3}{5},\quad -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}$$

We will use the Pythagorean identities to find $\sin\alpha$ and $\cos\beta$

$$\begin{aligned}\sin\alpha&=\sqrt{1-\cos^2\alpha}\\&=\sqrt{1-\frac{1}{4}}\\&=\sqrt{\frac{3}{4}}\\&=\frac{\sqrt{3}}{2}\\\cos\beta&=\sqrt{1-\sin^2\beta}\\&=\sqrt{1-\frac{9}{25}}\\&=\sqrt{\frac{16}{25}}\\&=\frac{4}{5}\end{aligned}$$

Using the sum formula for sine,

$$\begin{aligned}\sin\left(\cos^{-1}\tfrac{1}{2}+\sin^{-1}\tfrac{3}{5}\right)&=\sin(\alpha+\beta)\\&=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\&=\frac{\sqrt{3}}{2}\cdot\frac{4}{5}+\frac{1}{2}\cdot\frac{3}{5}\\&=\frac{4\sqrt{3}+3}{10}\end{aligned}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x = \dfrac{\sin x}{\cos x}$, when $\cos x \neq 0$.

Let's derive the sum formula for tangent.

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\
&= \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\
&= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\
&= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\
&= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}
\end{aligned}$$

We can derive the difference formula for tangent in a similar way.

SUM AND DIFFERENCE FORMULAS FOR TANGENT

The sum and difference formulas for tangent are:

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{3.2.5}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{3.2.6}$$

How to: Given two angles, find the tangent of the sum of the angles

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

Example 3.2.5: Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\dfrac{\pi}{6}+\dfrac{\pi}{4}\right)$.

Solution

Let's first write the sum formula for tangent and then substitute the given angles into the formula.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\tan\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{4}\right)}{1-\left(\tan\left(\frac{\pi}{6}\right)\right)\left(\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual function values within the formula:

$$\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}},\quad \text{and}\quad \tan\left(\frac{\pi}{4}\right)=1 \tag{3.2.7}$$

So we have,

$$\begin{aligned}\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)&=\frac{\frac{1}{\sqrt{3}}+1}{1-\left(\frac{1}{\sqrt{3}}\right)(1)}\\&=\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}\\&=\frac{1+\sqrt{3}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}-1}\\&=\frac{\sqrt{3}+1}{\sqrt{3}-1}\end{aligned}$$

Exercise 3.2.5:

Find the exact value of $\tan\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)$.

Answer available at the end of this section.

Example 3.2.6: Finding Multiple Sums and Differences of Angles

Given $\sin\alpha=\frac{3}{5},\quad 0<\alpha<\frac{\pi}{2}$, and $\cos\beta=-\frac{5}{13},\quad \pi<\beta<\frac{3\pi}{2}$,

find

a. $\sin(\alpha+\beta)$
b. $\cos(\alpha+\beta)$
c. $\tan(\alpha+\beta)$
d. $\tan(\alpha-\beta)$

Solution

We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

a. To find $\sin(\alpha+\beta)$, we begin with $\sin\alpha=\frac{3}{5}$ and $0<\alpha<\frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See Figure 3.2.4. Using the Pythagorean Theorem, we can find the length of side a:

$$\begin{aligned} a^2+3^2&=5^2\\ a^2&=16\\ a&=4 \end{aligned}$$

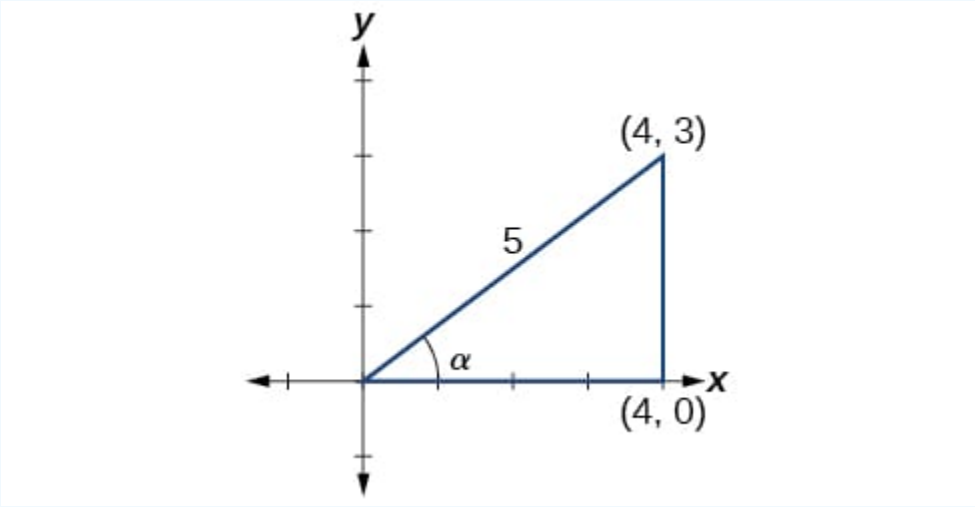


Figure 3.2.4

Since $\cos\beta=-\frac{5}{13}$ and $\pi<\beta<\frac{3\pi}{2}$,the side adjacent to β is -5,the hypotenuse is 13, and β is in the third quadrant. See Figure 3.2.5. Again, using the Pythagorean Theorem, we have

$$\begin{aligned} (-5)^2+a^2&=13^2\\ 25+a^2&=169\\ a^2&=144\\ a&=\pm12 \end{aligned}$$

Since β is in the third quadrant, $a=-12$.

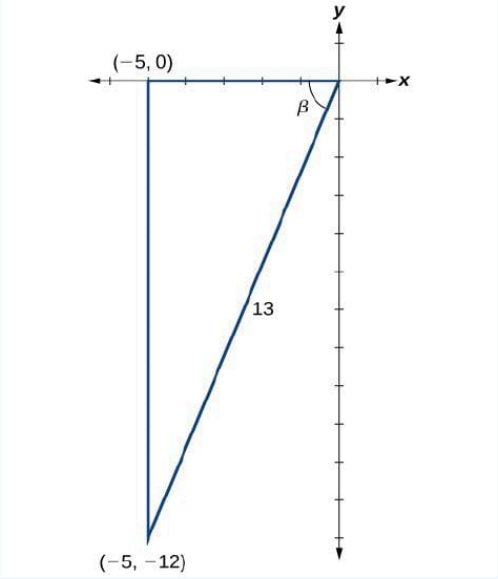


Figure 3.2.5

The next step is finding the cosine of α and the sine of β. The cosine ofα α is the adjacent side over the hypotenuse. We can find it from the triangle in Figure $3.2.5$: $\cos\alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in Figure $3.2.5$, as opposite side over the hypotenuse: $\sin\beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha+\beta)$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{15}{65} - \frac{48}{65} \\ &= -\frac{63}{65}\end{aligned}$$

b. We can find $\cos(\alpha+\beta)$ in a similar manner. We substitute the values according to the formula.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{20}{65} + \frac{36}{65} \\ &= \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha+\beta)$,if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, then

$$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

If $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$, then

$$\tan\beta = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$$

Then,

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\ &= \frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}\left(\frac{12}{5}\right)} \\ &= \frac{\frac{63}{20}}{-\frac{16}{20}} \\ &= -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha-\beta)$, we have the values we need. We can substitute them in and evaluate.

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4}-\frac{12}{5}}{1+\frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{-\frac{33}{20}}{\frac{56}{20}}\\ &= -\frac{33}{56}\end{aligned}$$

Analysis

A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that

$$\tan(\alpha+\beta) = \sin(\alpha+\beta)\cos(\alpha+\beta)$$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from Right Triangle Trigonometry that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In Figure $3.2.6$, notice that if one of the acute angles is labeled as θ, then the other acute angle must be labeled $\frac{\pi}{2}-\theta$.

Notice also that $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$,which is opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

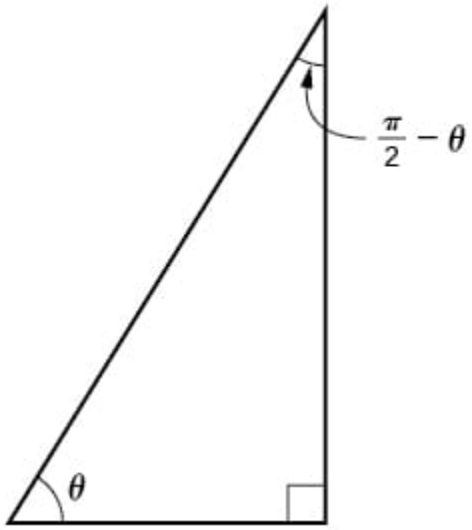


Figure $3.2.6$

From these relationships, the cofunction identities are formed. Recall that you first encountered these identities in The Unit Circle: Sine and Cosine Functions.

COFUNCTION IDENTITIES

The cofunction identities are summarized in Table 3.2.2.

Table 3.2.2

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Notice that the formulas in the table may also justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

We can write

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta \\ &= \sin\theta\end{aligned}$$

Example 3.2.7: Finding a Cofunction with the Same Value as the Given Expression

Write $\tan\frac{\pi}{9}$ in terms of its cofunction.

Solution

The cofunction of $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$. Thus,

$$\begin{aligned}\tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{9}\right) \\ &= \cot\left(\frac{9\pi}{18} - \frac{2\pi}{18}\right) \\ &= \cot\left(\frac{7\pi}{18}\right)\end{aligned}$$

Exercise 3.2.7

Write $\sin\frac{\pi}{7}$ in terms of its cofunction.

Answer available at the end of this section.

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules presented earlier may help simplify the process of verifying an identity.

How to: Given an identity, verify using sum and difference formulas

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Example $3.2.8A$: Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

Solution

We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta\\&=2\sin\alpha\cos\beta\end{aligned}$$

We see that the identity is verified.

Example $3.2.8B$: Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\tan\beta$$

Solution

We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned}\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}&=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\\&=\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}&&\text{Rewrite using a common denominator}\\&=\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}&&\text{Cancel}\\&=\tan\alpha-\tan\beta&&\text{Rewrite in terms of tangent}\end{aligned}$$

Exercise $3.2.8$:

Verify the identity: $\tan(\pi-\theta)=-\tan\theta$.

Answer available at the end of this section.

Example $3.2.9A$: Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2. See Figure $3.2.7$. Show that

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.)

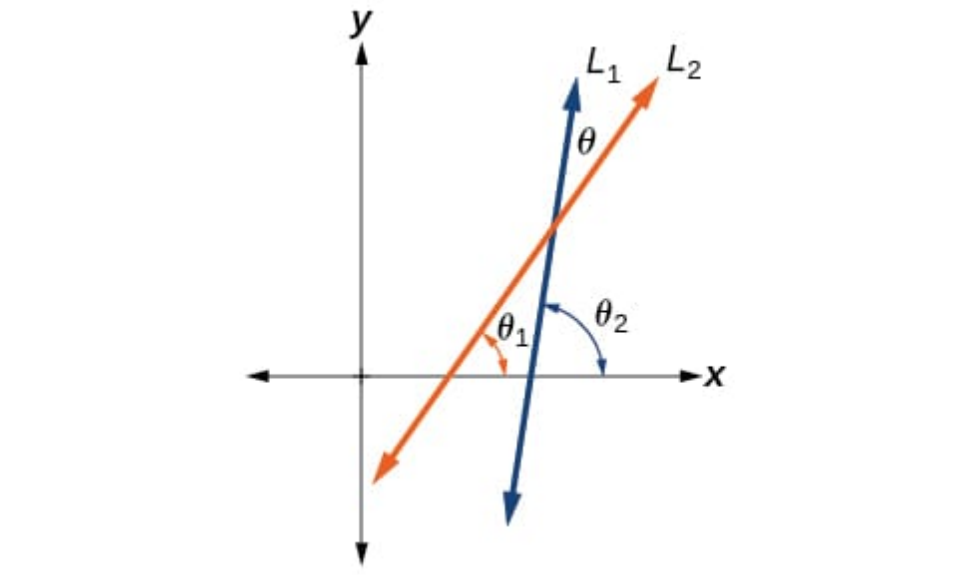


Figure $3.2.7$

Solution

Using the difference formula for tangent, this problem does not seem as daunting as it might.

$$\begin{aligned}\tan\theta &= \tan(\theta_2 - \theta_1)\\ &= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2}\\ &= \frac{m_2 - m_1}{1 + m_1 m_2}\end{aligned}$$

Example $3.2.9B$: Investigating a Guy-wire Problem

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See Figure $3.2.8$.

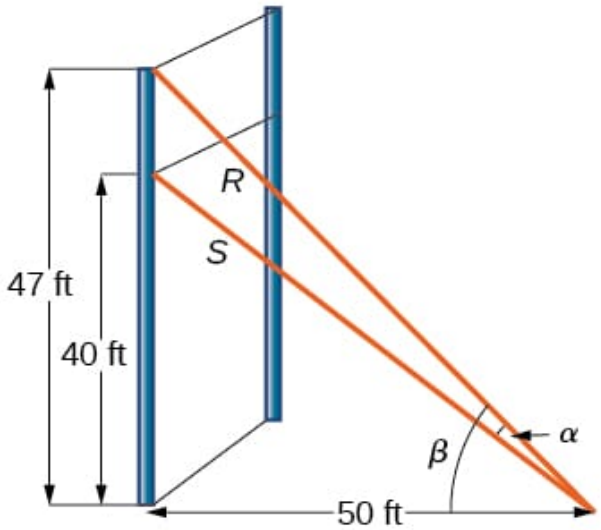


Figure 3.2.8

Solution

Let's first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan\beta = \frac{47}{50}$, and $\tan(\beta - \alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha} \tag{3.2.8}$$

Now, substituting the values we know into the formula, we have,

$$\frac{4}{5} = \frac{\frac{47}{50} - \tan\alpha}{1 + \frac{47}{50}\tan\alpha}$$

$$4\left(1 + \tfrac{47}{50}\tan\alpha\right) = 5\left(\tfrac{47}{50} - \tan\alpha\right)$$

Use the distributive property, and then simplify the functions.

$$\begin{aligned} 4(1) + 4\left(\tfrac{47}{50}\right)\tan\alpha &= 5\left(\tfrac{47}{50}\right) - 5\tan\alpha \\ 4 + 3.76\tan\alpha &= 4.7 - 5\tan\alpha \\ 5\tan\alpha + 3.76\tan\alpha &= 0.7 \\ 8.76\tan\alpha &= 0.7 \\ \tan\alpha &\approx 0.07991 \\ \tan^{-1}(0.07991) &\approx .079741 \end{aligned}$$

Now we can calculate the angle in degrees.

$$\begin{aligned} \alpha &\approx 0.079741\left(\frac{180}{\pi}\right) \\ &\approx 4.57^\circ \end{aligned}$$

Analysis

Occasionally, when an application appears that includes a right triangle, we may think that solving is a matter of applying the Pythagorean Theorem. That may be partially true, but it depends on what the problem is asking and what information is given.

Media

Access these online resources for additional instruction and practice with sum and difference identities.

- Sum and Difference Identities for Cosine
- Sum and Difference Identities for Sine
- Sum and Difference Identities for Tangent

Key Equations

Sum Formula for Cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference Formula for Cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
Sum Formula for Sine	$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
Difference Formula for Sine	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
Sum Formula for Tangent	$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
Difference Formula for Tangent	$\tan(\alpha-\beta)=\dfrac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$
Cofunction identities	$\sin\theta=\cos\left(\dfrac{\pi}{2}-\theta\right)$ $\cos\theta=\sin\left(\dfrac{\pi}{2}-\theta\right)$ $\tan\theta=\cot\left(\dfrac{\pi}{2}-\theta\right)$ $\cot\theta=\tan\left(\dfrac{\pi}{2}-\theta\right)$ $\sec\theta=\csc\left(\dfrac{\pi}{2}-\theta\right)$ $\csc\theta=\sec\left(\dfrac{\pi}{2}-\theta\right)$

Key Concepts

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle. See Example $3.2.1$ and Example $3.2.2$.
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle. See Example $3.2.3$.
- The sum and difference formulas for sine and cosine can also be used for inverse trigonometric functions. See Example $3.2.4$.
- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles. The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles. See Example $3.2.5$.

- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles. See Example $3.2.6$.
- The cofunction identities apply to complementary angles and pairs of reciprocal functions. See Example $3.2.7$.
- Sum and difference formulas are useful in verifying identities. See Example $3.2.8$ and Example $3.2.9$.
- Application problems are often easier to solve by using sum and difference formulas. See Example $3.2.10$ and Example $3.2.11$.

Answers

Exercise $3.2.1$**:**
$\frac{\sqrt{2}+\sqrt{6}}{4}$

Exercise $3.2.2$
$\frac{\sqrt{2}-\sqrt{6}}{4}$

Exercise $3.2.5$**:**
$\frac{1-\sqrt{3}}{1+\sqrt{3}}$

Exercise $3.2.7$
$\cos\left(\frac{5\pi}{14}\right)$

Exercise $3.2.8$**:**

$$\begin{aligned}\tan(\pi-\theta) &= \frac{\tan(\pi)-\tan\theta}{1+\tan(\pi)\tan\theta}\\ &= \frac{0-\tan\theta}{1+0\cdot\tan\theta}\\ &= -\tan\theta\end{aligned}$$

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3.3: Double-Angle, Half-Angle, and Reduction Formulas

Learning Objectives

- Use double-angle formulas to find exact values
- Use double-angle formulas to verify identities
- Use reduction formulas to simplify an expression
- Use half-angle formulas to find exact values

Bicycle ramps made for competition (see Figure 3.3.1) must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be θ such that $\tan\ \theta = \frac{5}{3}$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

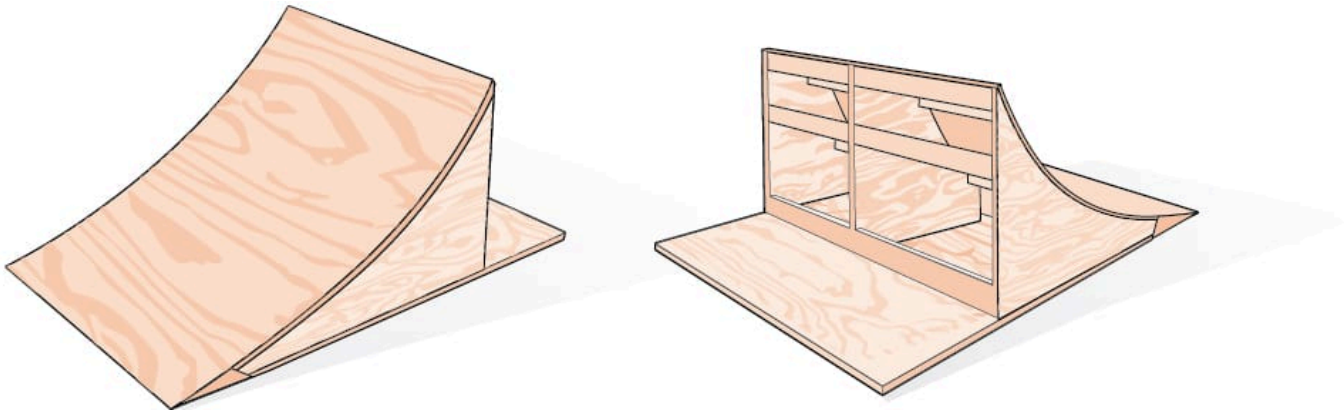


Figure 3.3.1: Bicycle ramps for advanced riders have a steeper incline than those designed for novices.

Using Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The **double-angle formulas** are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{3.3.1}$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned}\sin(\theta + \theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ \sin(2\theta) &= 2\sin\theta\cos\theta\end{aligned}$$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha + \beta) = \cos\ \alpha\ \cos\ \beta - \sin\ \alpha\ \sin\ \beta$,and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned}\cos(\theta + \theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more variations. The first variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= (1 - \sin^2\theta) - \sin^2\theta\end{aligned}$$

The second variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1\end{aligned}$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

DOUBLE-ANGLE FORMULAS

The *double-angle formulas* are summarized as follows:

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{3.3.2}$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{3.3.3}$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} \tag{3.3.4}$$

How to: Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 3.3.1: Using a Double-Angle Formula to Find the Exact Value Involving Tangent

Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$
b. $\cos(2\theta)$
c. $\tan(2\theta)$

Solution

If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4}$,such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the *x*-axis and is negative. Use the **Pythagorean Theorem** to find the length of the hypotenuse:

$$\begin{aligned}(-4)^2 + (3)^2 &= c^2 \\ 16 + 9 &= c^2 \\ 25 &= c^2 \\ c &= 5\end{aligned}$$

Now we can draw a triangle similar to the one shown in Figure 3.3.2.

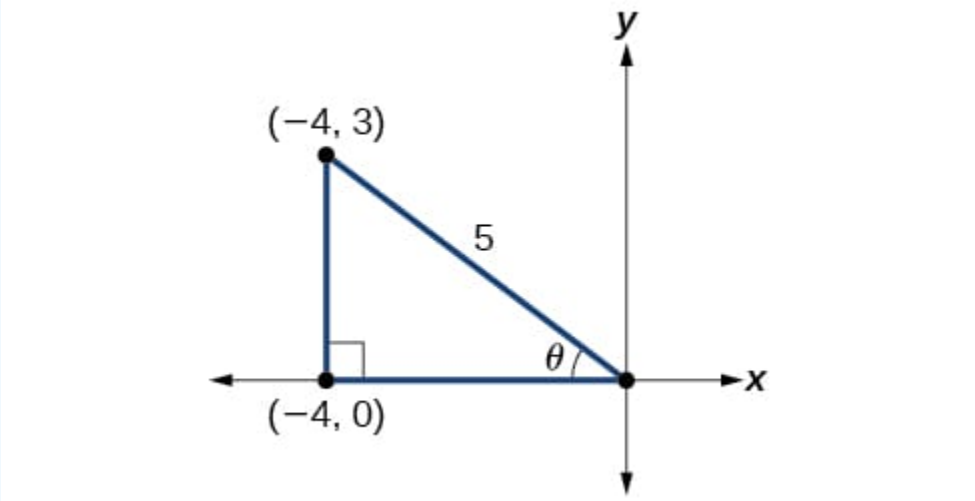


Figure 3.3.2

a. Let's begin by writing the double-angle formula for sine.

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

We see that we to need to find $\sin \theta$ and $\cos \theta$. Based on Figure 3.3.2, we see that the hypotenuse equals 5, so $\sin\theta = 35$, $\sin \theta = 35$, and $\cos\theta = -45$. Substitute these values into the equation, and simplify.

Thus,

$$\begin{aligned}\sin(2\theta) &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

b. Write the double-angle formula for cosine.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Again, substitute the values of the sine and cosine into the equation, and simplify.

$$\begin{aligned}\cos(2\theta) &= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2\\ &= \frac{16}{25} - \frac{9}{25}\\ &= \frac{7}{25}\end{aligned}$$

c. Write the double-angle formula for tangent.

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2\theta}$$

In this formula, we need the tangent, which we were given as $\tan \theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2} \\ &= \frac{-\frac{3}{2}}{1-\frac{9}{16}} \\ &= -\frac{3}{2}\left(\frac{16}{7}\right) \\ &= -\frac{24}{7}\end{aligned}$$

Exercise 3.3.1

Given $\sin\ \alpha = \frac{5}{8}$,with θ in quadrant I, find $\cos(2\alpha)$.

Answer available at the end of this section.

Example 3.3.2: Using the Double-Angle Formula for Cosine without Exact Values

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $cos(3x)$.

Solution

$$\begin{aligned}\cos(6x) &= \cos(3x+3x) \\ &= \cos 3x \cos 3x - \sin 3x \sin 3x \\ &= \cos^2 3x - \sin^2 3x\end{aligned}$$

Analysis

This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.

Using Double-Angle Formulas to Verify Identities

Establishing identities using the double-angle formulas is performed using the same steps we used to derive the sum and difference formulas. Choose the more complicated side of the equation and rewrite it until it matches the other side.

Example 3.3.3: Using the Double-Angle Formulas to Verify an Identity

Verify the following identity using double-angle formulas:

$$1+\sin(2\theta) = (\sin\theta + \cos\theta)^2$$

Solution

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= (\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta\\ &= 1+2\sin\theta\cos\theta\\ &= 1+\sin(2\theta)\end{aligned}$$

Analysis

This process is not complicated, as long as we recall the perfect square formula from algebra:

$$(a\pm b)^2 = a^2\pm 2ab+b^2$$

where $a=\sin\theta$ and $b=\cos\theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

Exercise 3.3.2

Verify the identity: $\cos^4\theta-\sin^4\theta=\cos(2\theta)$.

Answer available at the end of this section.

Example 3.3.4: Verifying a Double-Angle Identity for Tangent

Verify the identity: $\tan(2\theta)=2\cot\theta-\tan\theta$

Solution

In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}\tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} && \text{Double-angle formula}\\ &= \frac{2\tan\theta\left(\frac{1}{\tan\theta}\right)}{(1-\tan^2\theta)\left(\frac{1}{\tan\theta}\right)} && \text{Multiply by a term that results in desired numerator}\\ &= \frac{2}{\frac{1}{\tan\theta}-\frac{\tan^2\theta}{\tan\theta}}\\ &= \frac{2}{\cot\theta-\tan\theta} && \text{Use reciprocal identity for } \frac{1}{\tan\theta}\end{aligned}$$

Analysis

Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{2}{\cot\theta - \tan\theta}$$

Lets work on the right side

$$\begin{aligned}\frac{2}{\cot\theta - \tan\theta} &= \frac{2}{\frac{1}{\tan\theta} - \tan\theta}\left(\frac{\tan\theta}{\tan\theta}\right)\\ &= \frac{2\tan\theta}{\frac{1}{\tan\theta}(\tan\theta) - \tan\theta(\tan\theta)}\\ &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

Exercise 3.3.3

Verify the identity: $\cos(2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$.

Answer available at the end of this section.

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the **reduction formulas,** which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\sin^2\theta$. Solve for $\sin^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 1 - 2\sin^2\theta\\ 2\sin^2\theta &= 1 - \cos(2\theta)\\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2}\end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2\cos^2\theta - 1$. Solve for $\cos^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 2\cos^2\theta - 1\\ 1 + \cos(2\theta) &= 2\cos^2\theta\\ \frac{1+\cos(2\theta)}{2} &= \cos^2\theta\end{aligned}$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned}\tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta} \\ &= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} \qquad \text{(Substitute the reduction formulas)} \\ &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{2}{1+\cos(2\theta)}\right) \\ &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)}\end{aligned}$$

REDUCTION FORMULAS

The reduction formulas are summarized as follows:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \tag{3.3.5}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \tag{3.3.6}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \tag{3.3.7}$$

Example 3.3.5: Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution

We will apply the reduction formula for cosine twice.

$$\begin{aligned}\cos^4 x &= \left(\cos^2 x\right)^2 \\ &= \left(\frac{1+\cos(2x)}{2}\right)^2 \qquad \text{Substitute reduction formula} \\ &= \frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{8}+\frac{1}{8}\cos(4x) \\ &= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\end{aligned}$$

Analysis

The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.

Example 3.3.6: Using the Power-Reducing Formulas to Prove an Identity

Use the power-reducing formulas to prove $\sin^3(2x) = \left[\frac{1}{2}\sin(2x)\right][1-\cos(4x)]$

Solution

We will work on simplifying the left side of the equation:

$$\begin{aligned}\sin^3(2x) &= [\sin(2x)][\sin^2(2x)] \\ &= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right] \qquad \text{Substitute the power-reduction formula.} \\ &= \sin(2x)\left(\frac{1}{2}\right)[1-\cos(4x)] \\ &= \frac{1}{2}[\sin(2x)][1-\cos(4x)]\end{aligned}$$

Analysis

Note that in this example, we substituted $\frac{1-\cos(4x)}{2}$ for $\sin^2(2x)$. The formula states $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

We let $\theta = 2x$, so $2\theta = 4x$.

Exercise 3.3.4

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

Answer available at the end of this section.

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$,the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$\begin{aligned}\sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\ \sin^2\left(\frac{\alpha}{2}\right) &= \frac{1-\left(\cos 2\cdot\frac{\alpha}{2}\right)}{2} \\ &= \frac{1-\cos\alpha}{2} \\ \sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{2}}\end{aligned}$$

To derive the half-angle formula for cosine, we have

$$\begin{aligned}\cos^2\theta &= \frac{1+\cos(2\theta)}{2} \\ \cos^2\left(\frac{\alpha}{2}\right) &= \frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2} \\ &= \frac{1+\cos\alpha}{2} \\ \cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}}\end{aligned}$$

For the tangent identity, we have

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

HALF-ANGLE FORMULAS

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}} \tag{3.3.8}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \tag{3.3.9}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} \tag{3.3.10}$$

Example 3.3.7

Using a Half-Angle Formula to Find the Exact Value of a Sine Function. Find $\sin(15\,^\circ)$ using a half-angle formula.

Solution

Since $15\,^\circ = \frac{30^\circ}{2}$,we use the half-angle formula for sine (Equation 3.3.8):

$$\begin{aligned}\sin\frac{30^\circ}{2} &= \sqrt{\frac{1-\cos 30^\circ}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Remember that we can check the answer with a graphing calculator.

Analysis

Notice that we used only the positive root because $\sin(15\,^\circ)$ is positive.

Howto: Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 3.3.8: Finding Exact Values Using Half-Angle Identities

Given that $\tan\ \alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$
b. $\cos\left(\frac{\alpha}{2}\right)$
c. $\tan\left(\frac{\alpha}{2}\right)$

Solution

Using the given information, we can draw the triangle shown in Figure 3.3.3. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin\ \alpha = -\frac{8}{17}$ and $\cos\ \alpha = -\frac{15}{17}$.

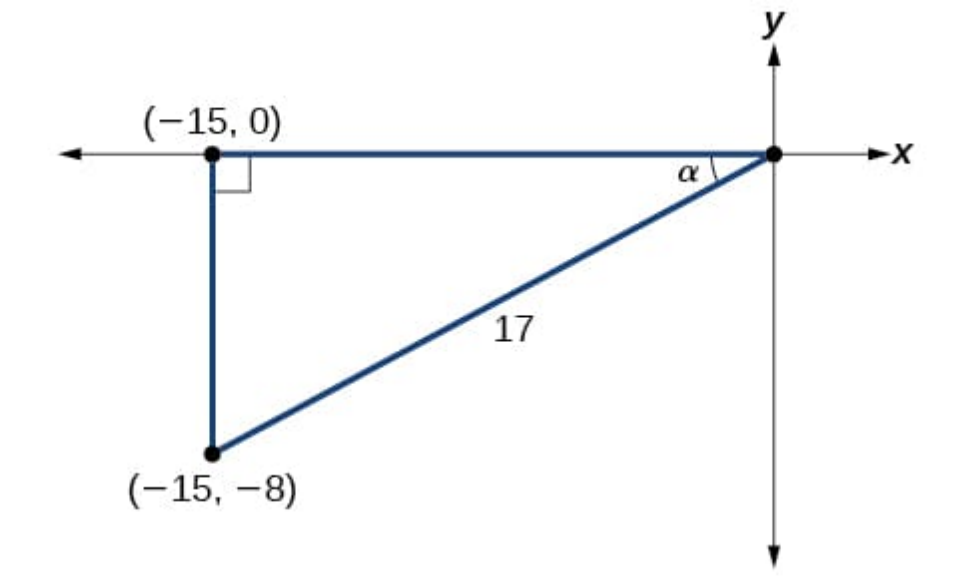


Figure 3.3.3

a. Before we start, we must remember that if α is in quadrant III, then $180\ ° < \alpha < 270\ °$,so $\frac{180°}{2} < \frac{\alpha}{2} < \frac{270°}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90\ ° < \frac{\alpha}{2} < 135\ °$. To find $\sin\ \frac{\alpha}{2}$,we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in Figure 3.3.3 and simplify.

$$\begin{aligned}\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}}\\ &= \pm\sqrt{\frac{1-(-\frac{15}{17})}{2}}\\ &= \pm\sqrt{\frac{\frac{32}{17}}{2}}\\ &= \pm\sqrt{\frac{32}{17}\cdot\frac{1}{2}}\\ &= \pm\sqrt{\frac{16}{17}}\\ &= \pm\frac{4}{\sqrt{17}}\\ &= \frac{4\sqrt{17}}{17}\end{aligned}$$

We choose the positive value of $\sin\frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos\frac{\alpha}{2}$,we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in Figure $3.3.3$, and simplify.

$$\begin{aligned}\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}}\\ &= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}}\\ &= \pm\sqrt{\frac{\frac{2}{17}}{2}}\\ &= \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}}\\ &= \pm\sqrt{\frac{1}{17}}\\ &= -\frac{\sqrt{17}}{17}\end{aligned}$$

We choose the negative value of $\cos\frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan\frac{\alpha}{2}$,we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in Figure $3.3.3$ and simplify.

$$\begin{aligned}\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\\ &= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}}\\ &= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}}\\ &= \pm\sqrt{\frac{32}{2}}\\ &= -\sqrt{16}\\ &= -4\end{aligned}$$

We choose the negative value of $\tan\ \frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

Exercise 3.3.5

Given that $\sin\ \alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\ \left(\frac{\alpha}{2}\right)$.

Answer available at the end of this section.

Example 3.3.9: Finding the Measurement of a Half Angle

Now, we will return to the problem posed at the beginning of the section. A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $tan\,\theta = 53$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution

Since the angle for novice competition measures half the steepness of the angle for the high level competition, and $\tan\ \theta = \frac{5}{3}$ for high competition, we can find $\cos\ \theta$ from the right triangle and the Pythagorean theorem so that we can use the half-angle identities. See Figure 3.3.4.

$$\begin{aligned}3^2 + 5^2 &= 34\\ c &= \sqrt{34}\end{aligned}$$

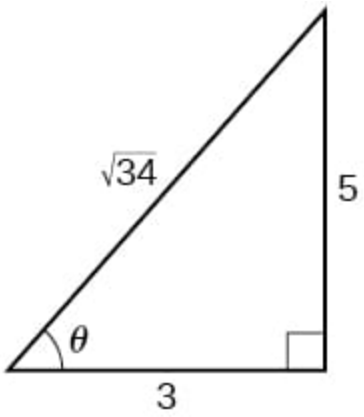


Figure 3.3.4

We see that $\cos\,\theta = \dfrac{3}{\sqrt{34}} = \dfrac{3\sqrt{34}}{34}$. We can use the half-angle formula for tangent: $\tan\,\dfrac{\theta}{2} = \sqrt{\dfrac{1-\cos\,\theta}{1+\cos\,\theta}}$. Since $\tan\,\theta$ is in the first quadrant, so is $\tan\,\dfrac{\theta}{2}$.

$$\begin{aligned}\tan\frac{\theta}{2} &= \sqrt{\frac{1-\dfrac{3\sqrt{34}}{34}}{1+\dfrac{3\sqrt{34}}{34}}} \\ &= \sqrt{\frac{\dfrac{34-3\sqrt{34}}{34}}{\dfrac{34+3\sqrt{34}}{34}}} \\ &= \sqrt{\frac{34-3\sqrt{34}}{34+3\sqrt{34}}} \\ &\approx 0.57\end{aligned}$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7\,°$. So the angle of the ramp for novice competition is $\approx 29.7\,°$.

Media

Access these online resources for additional instruction and practice with double-angle, half-angle, and reduction formulas.

- Double-Angle Identities
- Half-Angle Identities

Key Equations

Double-angle formulas	$\sin(2\theta) = 2\sin\ \theta\ \cos\ \theta$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $= 1 - 2\sin^2\theta$ $= 2\cos^2\theta - 1$ $\tan(2\theta) = \dfrac{2\tan\ \theta}{1 - \tan^2\theta}$
Reduction formulas	$\sin^2\theta = \dfrac{1 - \cos(2\theta)}{2}$ $\cos^2\theta = \dfrac{1 + \cos(2\theta)}{2}$ $\tan^2\theta = \dfrac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$
Half-angle formulas	$\sin\ \dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1 - \cos\ \alpha}{2}}$ $\cos\ \dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1 + \cos\ \alpha}{2}}$ $\tan\ \dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1 - \cos\ \alpha}{1 + \cos\ \alpha}}$ $= \dfrac{\sin\ \alpha}{1 + \cos\ \alpha}$ $= \dfrac{1 - \cos\ \alpha}{\sin\ \alpha}$

Key Concepts

- Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. See Example 3.3.1, Example 3.3.2, Example 3.3.3, and Example 3.3.4.
- Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. See Example 3.3.5 and Example 3.3.6.
- Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not. See Example 3.3.7, Example 3.3.8, and Example 3.3.9.

Answers

Exercise 3.3.1

$\cos(2\alpha) = \dfrac{7}{32}$

Exercise 3.3.2

$\cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = \cos(2\theta)$

Exercise 3.3.3

$\cos(2\theta)\cos\ \theta = (\cos^2\theta - \sin^2\theta)\cos\ \theta = \cos^3\theta - \cos\ \theta\sin^2\theta$

Exercise 3.3.4

$$
\begin{aligned}
10\cos^4 x &= 10(\cos^2 x)^2 \\
&= 10\left[\frac{1+\cos(2x)}{2}\right]^2 \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}[1+2\cos(2x)+\cos^2(2x)] \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{8}+\frac{10}{8}\cos(4x) \\
&= \frac{30}{8}+5\cos(2x)+\frac{10}{8}\cos(4x) \\
&= \frac{15}{4}+5\cos(2x)+\frac{5}{4}\cos(4x)
\end{aligned}
$$

Exercise 3.3.5

$$-\frac{2}{\sqrt{5}}$$

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3.4: Sum-to-Product and Product-to-Sum Formulas

Skills to Develop

- Express products as sums.
- Express sums as products.

A band marches down the field creating an amazing sound that bolsters the crowd. That sound travels as a wave that can be interpreted using trigonometric functions.



Figure 3.4.1***:*** *The UCLA marching band (credit: Eric Chan, Flickr).*

For example, Figure 3.4.2 represents a sound wave for the musical note A. In this section, we will investigate trigonometric identities that are the foundation of everyday phenomena such as sound waves.

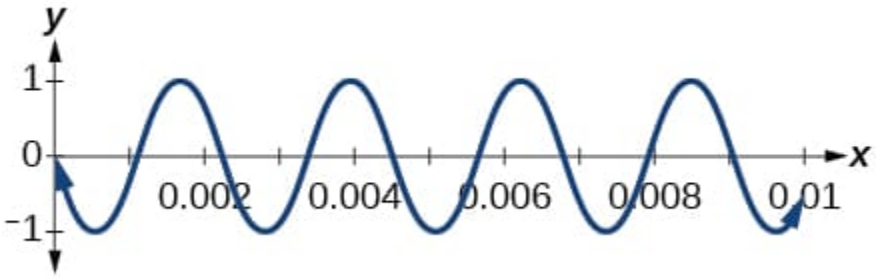


Figure 3.4.2

Expressing Products as Sums

We have already learned a number of formulas useful for expanding or simplifying trigonometric expressions, but sometimes we may need to express the product of cosine and sine as a sum. We can use the **product-to-sum formulas,** which express products of trigonometric functions as sums. Let's investigate the cosine identity first and then the sine identity.

Expressing Products as Sums for Cosine

We can derive the product-to-sum formula from the sum and difference identities for **cosine**. If we add the two equations, we get:

$$\begin{aligned} \cos\alpha\cos\beta + \sin\alpha\sin\beta &= \cos(\alpha-\beta) \\ \underline{+\cos\alpha\cos\beta - \sin\alpha\sin\beta} &= \underline{\cos(\alpha+\beta)} \\ 2\cos\alpha\cos\beta &= \cos(\alpha-\beta)+\cos(\alpha+\beta) \end{aligned}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{3.4.1}$$

How to: Given a product of cosines, express as a sum

1. Write the formula for the product of cosines.
2. Substitute the given angles into the formula.
3. Simplify.

Example 3.4.1: Writing the Product as a Sum Using the Product-to-Sum Formula for Cosine

Write the following product of cosines as a sum: $2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)$.

Solution

We begin by writing the formula for the product of cosines (Equation 3.4.1):

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

We can then substitute the given angles into the formula and simplify.

$$\begin{aligned} 2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right) &= 2\left(\frac{1}{2}\right)\left[\cos\left(\frac{7x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{7x}{2}+\frac{3x}{2}\right)\right] \\ &= \cos\left(\frac{4x}{2}\right)+\cos\left(\frac{10x}{2}\right) \\ &= \cos 2x+\cos 5x \end{aligned}$$

Exercise 3.4.1

Use the product-to-sum formula (Equation 3.4.1) to write the product as a sum or difference: $\cos(2\theta)\cos(4\theta)$.

Answer

$$\frac{1}{2}(\cos 6\theta+\cos 2\theta)$$

Expressing the Product of Sine and Cosine as a Sum

Next, we will derive the product-to-sum formula for sine and cosine from the sum and difference formulas for **sine**. If we add the sum and difference identities, we get:

$$\begin{aligned}\cos\alpha\cos\beta+\sin\alpha\sin\beta&=\cos(\alpha-\beta)\\ +\cos\alpha\cos\beta-\sin\alpha\sin\beta&=\cos(\alpha+\beta)\\ 2\cos\alpha\cos\beta&=\cos(\alpha-\beta)+\cos(\alpha+\beta)\end{aligned}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

Example 3.4.2: Writing the Product as a Sum Containing only Sine or Cosine

Express the following product as a sum containing only sine or cosine and no products: $\sin(4\theta)\cos(2\theta)$.

Solution

Write the formula for the product of sine and cosine. Then substitute the given values into the formula and simplify.

$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\begin{aligned}\sin(4\theta)\cos(2\theta)&=\frac{1}{2}[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)]\\ &=\frac{1}{2}[\sin(6\theta)+\sin(2\theta)]\end{aligned}$$

Exercise 3.4.2

Use the product-to-sum formula to write the product as a sum: $\sin(x+y)\cos(x-y)$.

Answer

$$\frac{1}{2}(\sin 2x+\sin 2y)$$

Expressing Products of Sines in Terms of Cosine

Expressing the product of sines in terms of **cosine** is also derived from the sum and difference identities for cosine. In this case, we will first subtract the two cosine formulas:

$$\begin{aligned}\cos(\alpha-\beta)&=\cos\alpha\cos\beta+\sin\alpha\sin\beta\\ -\cos(\alpha+\beta)&=-(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\\ \cos(\alpha-\beta)-\cos(\alpha+\beta)&=2\sin\alpha\sin\beta\end{aligned}$$

Then, we divide by 2 to isolate the product of sines:

$$\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

Similarly we could express the product of cosines in terms of sine or derive other product-to-sum formulas.

THE PRODUCT-TO-SUM FORMULAS

The **product-to-sum** formulas are as follows:

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{3.4.2}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \tag{3.4.3}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \tag{3.4.4}$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)] \tag{3.4.5}$$

Example 3.4.3: Express the Product as a Sum or Difference

Write $\cos(3\theta)\cos(5\theta)$ as a sum or difference.

Solution

We have the product of cosines, so we begin by writing the related formula. Then we substitute the given angles and simplify.

$$\begin{aligned}\cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]\\ \cos(3\theta)\cos(5\theta) &= \frac{1}{2}[\cos(3\theta-5\theta)+\cos(3\theta+5\theta)]\\ &= \frac{1}{2}[\cos(2\theta)+\cos(8\theta)] \qquad \text{Use even-odd identity}\end{aligned}$$

Exercise 3.4.3

Use the product-to-sum formula to evaluate $\cos\frac{11\pi}{12}\cos\frac{\pi}{12}$.

Answer

$$\frac{-2-\sqrt{3}}{4}$$

Expressing Sums as Products

Some problems require the reverse of the process we just used. The **sum-to-product formulas** allow us to express sums of sine or cosine as products. These formulas can be derived from the product-to-sum identities. For example, with a few substitutions, we can derive the sum-to-product identity for **sine**. Let $\frac{u+v}{2}=\alpha$ and $\frac{u-v}{2}=\beta$.

Then,

$$\begin{aligned}\alpha+\beta &= \frac{u+v}{2}+\frac{u-v}{2}\\ &= \frac{2u}{2}\\ &= u\end{aligned}$$

$$\begin{aligned}\alpha-\beta &= \frac{u+v}{2}-\frac{u-v}{2}\\ &= \frac{2v}{2}\\ &= v\end{aligned}$$

Thus, replacing α and β in the product-to-sum formula with the substitute expressions, we have

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \frac{1}{2}[\sin u+\sin v] \qquad \text{Substitute for } (\alpha+\beta) \text{ and } (\alpha\beta)$$

$$2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \sin u+\sin v$$

The other sum-to-product identities are derived similarly.

SUM-TO-PRODUCT FORMULAS

The **sum-to-product formulas** are as follows:

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{3.4.6}$$

$$\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{3.4.7}$$

$$\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{3.4.8}$$

$$\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{3.4.9}$$

Example 3.4.4: Writing the Difference of Sines as a Product

Write the following difference of sines expression as a product: $\sin(4\theta)-\sin(2\theta)$.

Solution

We begin by writing the formula for the difference of sines.

$$\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

Substitute the values into the formula, and simplify.

$$\begin{aligned}\sin(4\theta)-\sin(2\theta) &= 2\sin\left(\frac{4\theta-2\theta}{2}\right)\cos\left(\frac{4\theta+2\theta}{2}\right)\\ &= 2\sin\left(\frac{2\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right)\\ &= 2\sin\theta\cos(3\theta)\end{aligned}$$

Exercise 3.4.4

Use the sum-to-product formula to write the sum as a product: $\sin(3\theta)+\sin(\theta)$.

Answer

$2\sin(2\theta)\cos(\theta)$

Example 3.4.5: Evaluating Using the Sum-to-Product Formula

Evaluate $\cos(15\,^\circ)-\cos(75\,^\circ)$. Check the answer with a graphing calculator.

Solution

We begin by writing the formula for the difference of cosines.

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Then we substitute the given angles and simplify.

$$\begin{aligned}\cos(15^\circ) - \cos(75^\circ) &= -2\sin\left(\frac{15^\circ+75^\circ}{2}\right)\sin\left(\frac{15^\circ-75^\circ}{2}\right)\\ &= -2\sin(45^\circ)\sin(-30^\circ)\\ &= -2\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Example 3.4.6: Proving an Identity

Prove the identity:

$$\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} = -\tan t \tag{3.4.10}$$

Solution

We will start with the left side, the more complicated side of the equation, and rewrite the expression until it matches the right side.

$$\begin{aligned}\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} &= \frac{-2\sin\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)}{2\sin\left(\frac{4t+2t}{2}\right)\cos\left(\frac{4t-2t}{2}\right)}\\ &= \frac{-2\sin(3t)\sin t}{2\sin(3t)\cos t}\\ &= -\frac{\sin t}{\cos t}\\ &= -\tan t\end{aligned}$$

Analysis

Recall that verifying trigonometric identities has its own set of rules. The procedures for solving an equation are not the same as the procedures for verifying an identity. When we prove an identity, we pick one side to work on and make substitutions until that side is transformed into the other side.

Example 3.4.7: Verifying the Identity Using Double-Angle Formulas and Reciprocal Identities

Verify the identity $\csc^2\theta - 2 = \cos(2\theta)\sin 2\theta$.

Solution

For verifying this equation, we are bringing together several of the identities. We will use the double-angle formula and the reciprocal identities. We will work with the right side of the equation and rewrite it until it matches the left side.

$$\begin{aligned}\cos(2\theta)\sin 2\theta &= \frac{1-2\sin^2\theta}{\sin^2\theta}\\ &= \frac{1}{\sin^2\theta}-\frac{2\sin^2\theta}{\sin^2\theta}\\ &= \csc^2\theta-2\end{aligned}$$

Exercise 3.4.5

Verify the identity $\tan\theta\cot\theta-\cos^2\theta=\sin^2\theta$.

Answer

$$\begin{aligned}\tan\theta\cot\theta-\cos^2\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)-\cos^2\theta\\ &= 1-\cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

Media

Access these online resources for additional instruction and practice with the product-to-sum and sum-to-product identities.

- Sum to Product Identities
- Sum to Product and Product to Sum Identities

Key Equations

Product-to-sum Formulas

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

Sum-to-product Formulas

$$\sin\alpha+\sin\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha-\sin\beta=2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Key Concepts

- From the sum and difference identities, we can derive the product-to-sum formulas and the sum-to-product formulas for sine and cosine.
- We can use the product-to-sum formulas to rewrite products of sines, products of cosines, and products of sine and cosine as sums or differences of sines and cosines. See Example $3.4.1$, Example $3.4.2$, and Example $3.4.3$.
- We can also derive the sum-to-product identities from the product-to-sum identities using substitution.
- We can use the sum-to-product formulas to rewrite sum or difference of sines, cosines, or products sine and cosine as products of sines and cosines. See Example $3.4.4$.
- Trigonometric expressions are often simpler to evaluate using the formulas. See Example $3.4.5$.
- The identities can be verified using other formulas or by converting the expressions to sines and cosines. To verify an identity, we choose the more complicated side of the equals sign and rewrite it until it is transformed into the other side. See Example $3.4.6$ and Example $3.4.7$.

Contributors

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3.5: Solving Trigonometric Equations

Learning Objectives

- Solve linear trigonometric equations in sine and cosine.
- Solve equations involving a single trigonometric function.
- Solve trigonometric equations using a calculator.
- Solve trigonometric equations that are quadratic in form.
- Solve trigonometric equations using fundamental identities.
- Solve trigonometric equations with multiple angles.
- Solve right triangle problems.

Thales of Miletus (circa 625–547 BC) is known as the founder of geometry. The legend is that he calculated the height of the Great Pyramid of Giza in Egypt using the theory of *similar triangles*, which he developed by measuring the shadow of his staff. Based on proportions, this theory has applications in a number of areas, including fractal geometry, engineering, and architecture. Often, the angle of elevation and the angle of depression are found using similar triangles.



Figure 3.5.1: Egyptian pyramids standing near a modern city. (credit: Oisin Mulvihill)

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Solving Linear Trigonometric Equations in Sine and Cosine

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The **period** of both the sine function and the cosine function is 2π. In other words, every 2π units, the *y*-values repeat. If we need to find all possible solutions, then we must add $2\pi k$,where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin\theta = \sin(\theta \pm 2k\pi) \tag{3.5.1}$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence.

We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

Example $3.5.1A$: Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

Solution

From the unit circle, we know that

$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3},\ \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Example $3.5.1B$: Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

Solution

Solving for all possible values of t means that solutions include angles beyond the period of 2π. From the section on Sum and Difference Identities, we can see that the solutions are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. But the problem is asking for all possible values that solve the equation. Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where k is an integer.

Howto: Given a trigonometric equation, solve using algebra

1. Look for a pattern that suggests an algebraic property, such as the difference of squares or a factoring opportunity.
2. Substitute the trigonometric expression with a single variable, such as x or u.
3. Solve the equation the same way an algebraic equation would be solved.
4. Substitute the trigonometric expression back in for the variable in the resulting expressions.
5. Solve for the angle.

Example $3.5.2$: Solve the Linear Trigonometric Equation

Solve the equation exactly: $2\cos\theta - 3 = -5,\ 0 \le \theta < 2\pi$.

Solution

Use algebraic techniques to solve the equation.

$$\begin{aligned} 2\cos\theta - 3 &= -5 \\ 2\cos\theta &= -2 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Exercise 3.5.2

Solve exactly the following linear equation on the interval $[0, 2\pi)$: $2\sin x + 1 = 0$.

Answer available at the end of this section.

Solving Equations Involving a Single Trigonometric Function

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and the unit circle (see [link]). We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems involving the reciprocals of the primary trigonometric functions need to be viewed from an algebraic perspective. In other words, we will write the reciprocal function, and solve for the angles using the function. Also, an equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π,not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$,unless, of course, a problem places its own restrictions on the domain.

Solving a Problem Involving a Single Trigonometric Function

Solve the problem exactly: $2\sin^2\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution

As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$\begin{aligned} 2\sin^2\theta - 1 &= 0 \\ 2\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{2} \\ \sqrt{\sin^2\theta} &= \pm\sqrt{\frac{1}{2}} \\ \sin\theta &= \pm\frac{1}{\sqrt{2}} \\ &= \pm\frac{\sqrt{2}}{2} \\ \theta &= \frac{\pi}{4}, \ \frac{3\pi}{4}, \ \frac{5\pi}{4}, \ \frac{7\pi}{4} \end{aligned}$$

Example 3.5.3B: Solving a Trigonometric Equation Involving Cosecant

Solve the following equation exactly: $\csc\theta = -2$, $0 \le \theta < 4\pi$.

Solution

We want all values of θ for which $\csc\theta = -2$ over the interval $0 \le \theta < 4\pi$.

$$\csc\theta = -2$$
$$\frac{1}{\sin\theta} = -2$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6},\ \frac{11\pi}{6},\ \frac{19\pi}{6},\ \frac{23\pi}{6}$$

Analysis

As $\sin\theta = -\frac{1}{2}$, notice that all four solutions are in the third and fourth quadrants.

Example $3.5.3C$: Solving an Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta - \frac{\pi}{2}\right) = 1,\ 0 \le \theta < 2\pi$.

Solution

Recall that the tangent function has a period of π. On the interval $[0, \pi)$,and at the angle of $\frac{\pi}{4}$,the tangent has a value of 1. However, the angle we want is $\left(\theta - \frac{\pi}{2}\right)$. Thus, if $\tan\left(\frac{\pi}{4}\right) = 1$,then

$$\theta - \frac{\pi}{2} = \frac{\pi}{4}$$
$$\theta = \frac{3\pi}{4} \pm k\pi$$

Over the interval $[0, 2\pi)$,we have two solutions:

$\theta = \frac{3\pi}{4}$ and $\theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$

Exercise $3.5.3$

Find all solutions for $\tan x = \sqrt{3}$.

Answer available at the end of this section.

Example $3.5.4$: Identify all Solutions to the Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x + 3) = 5 + \tan x,\ 0 \le x < 2\pi$.

Solution

We can solve this equation using only algebra. Isolate the expression $\tan x$ on the left side of the equals sign.

$$2(\tan x) + 2(3) = 5 + \tan x$$
$$2\tan x + 6 = 5 + \tan x$$
$$2\tan x - \tan x = 5 - 6$$
$$\tan x = -1$$

There are two angles on the unit circle that have a tangent value of -1: $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Solve Trigonometric Equations Using a Calculator

Not all functions can be solved exactly using only the unit circle. When we must solve an equation involving an angle other than one of the special angles, we will need to use a calculator. Make sure it is set to the proper mode, either degrees or radians, depending on the criteria of the given problem.

Example $3.5.5A$: Using a Calculator to Solve a Trigonometric Equation Involving Sine

Use a calculator to solve the equation $\sin\theta = 0.8$,where θ is in radians.

Solution

Make sure mode is set to radians. To find θ, use the inverse sine function. On most calculators, you will need to push the 2ND button and then the SIN button to bring up the $\sin^{-1}$ function. What is shown on the screen is $\sin^{-1}$.The calculator is ready for the input within the parentheses. For this problem, we enter $\sin^{-1}(0.8)$, and press ENTER. Thus, to four decimals places,

$\sin^{-1}(0.8) \approx 0.9273$

The solution is

$\theta \approx 0.9273 \pm 2\pi k$

The angle measurement in degrees is

$$\begin{aligned}\theta &\approx 53.1^\circ \\ \theta &\approx 180^\circ - 53.1^\circ \\ &\approx 126.9^\circ\end{aligned}$$

Analysis

Note that a calculator will only return an angle in quadrants I or IV for the sine function, since that is the range of the inverse sine. The other angle is obtained by using $\pi - \theta$.

Example $3.5.5B$: Using a Calculator to Solve a Trigonometric Equation Involving Secant

Use a calculator to solve the equation $\sec\theta = -4$, giving your answer in radians.

Solution

We can begin with some algebra.

$$\begin{aligned}\sec\theta &= -4 \\ \frac{1}{\cos\theta} &= -4 \\ \cos\theta &= -\frac{1}{4}\end{aligned}$$

Check that the MODE is in radians. Now use the inverse cosine function

$$\begin{aligned}\cos^{-1}\left(-\frac{1}{4}\right) &\approx 1.8235 \\ \theta &\approx 1.8235 + 2\pi k\end{aligned}$$

Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$,1.8235 is between these two numbers, thus $\theta \approx 1.8235$ is in quadrant II. Cosine is also negative in quadrant III. Note that a calculator will only return an angle in quadrants I or II for the cosine function, since that is the range of the inverse cosine. See Figure $3.5.2$.

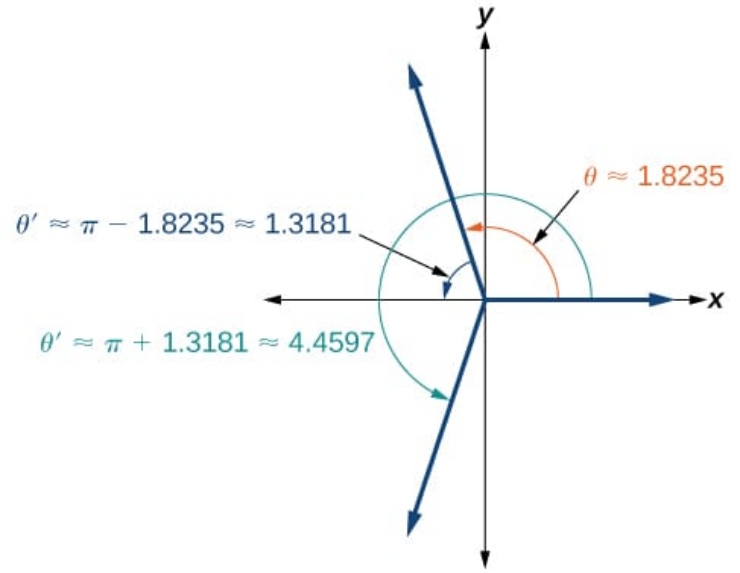


Figure 3.5.2

So, we also need to find the measure of the angle in quadrant III. In quadrant III, the reference angle is $\theta' \approx \pi - 1.8235 \approx 1.3181$. The other solution in quadrant III is $\theta' \approx \pi + 1.3181 \approx 4.4597$.

The solutions are $\theta \approx 1.8235 \pm 2\pi k$ and $\theta \approx 4.4597 \pm 2\pi k$.

Exercise 3.5.5

Solve $\cos\theta = -0.2$.

Answer available at the end of this section.

Solving Trigonometric Equations in Quadratic Form

Solving a *quadratic equation* may be more complicated, but once again, we can use algebra as we would for any quadratic equation. Look at the pattern of the equation. Is there more than one trigonometric function in the equation, or is there only one? Which trigonometric function is squared? If there is only one function represented and one of the terms is squared, think about the standard form of a quadratic. Replace the trigonometric function with a variable such as x or u. If substitution makes the equation look like a quadratic equation, then we can use the same methods for solving quadratics to solve the trigonometric equations.

Example 3.5.6A: Solving a Trigonometric Equation in Quadratic Form

Solve the equation exactly: $\cos^2\theta + 3\cos\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution

We begin by using substitution and replacing $\cos\theta$ with x. It is not necessary to use substitution, but it may make the problem easier to solve visually. Let $\cos\theta = x$. We have

$$x^2 + 3x - 1 = 0$$

The equation cannot be factored, so we will use the **quadratic formula:** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2}$$
$$= \frac{-3 \pm \sqrt{13}}{2}$$

Replace x with $\cos\theta$ and solve.

$$\cos\theta = \frac{-3 \pm \sqrt{13}}{2}$$
$$\theta = \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right)$$

Note that only the + sign is used. This is because we get an error when we solve $\theta = \cos^{-1}\left(\frac{-3 - \sqrt{13}}{2}\right)$ on a calculator, since the domain of the inverse cosine function is $[-1, 1]$. However, there is a second solution:

$$\theta = \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right)$$
$$\approx 1.26$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\theta = 2\pi - \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right)$$
$$\approx 5.02$$

Example $3.5.6B$: Solving a Trigonometric Equation in Quadratic Form by Factoring

Solve the equation exactly: $2\sin^2\theta - 5\sin\theta + 3 = 0$, $0 \le \theta \le 2\pi$.

Solution

Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta = u$,or imagine it, as we factor:

$$\begin{aligned} 2\sin^2\theta - 5\sin\theta + 3 &= 0 \\ (2\sin\theta - 3)(\sin\theta - 1) &= 0 \qquad \text{Now set each factor equal to zero.} \\ 2\sin\theta - 3 &= 0 \\ 2\sin\theta &= 3 \\ \sin\theta &= \frac{3}{2} \\ \sin\theta - 1 &= 0 \\ \sin\theta &= 1 \end{aligned}$$

Next solve for θ: $\sin\theta \neq \frac{3}{2}$, as the range of the sine function is $[-1, 1]$. However, $\sin\theta = 1$, giving the solution $\theta = \frac{\pi}{2}$.

Analysis

Make sure to check all solutions on the given domain as some factors have no solution.

Exercise $3.5.6$

Solve $\sin^2\theta = 2\cos\theta + 2$, $0 \le \theta \le 2\pi$. [Hint: Make a substitution to express the equation only in terms of cosine.]

Answer available at the end of this section.

Example $3.5.7A$: Solving a Trigonometric Equation Using Algebra

Solve exactly: $2\sin^2\theta + \sin\theta = 0;\ 0 \le \theta < 2\pi$

Solution

This problem should appear familiar as it is similar to a quadratic. Let $\sin\theta = x$. The equation becomes $2x^2 + x = 0$. We begin by factoring:

$$\begin{aligned} 2x^2 + x &= 0 \\ x(2x+1) &= 0 \qquad \text{Set each factor equal to zero.} \\ x &= 0 \\ 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

Then, substitute back into the equation the original expression $\sin\theta$ for x. Thus,

$$\begin{aligned} \sin\theta &= 0 \\ \theta &= 0, \pi \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

The solutions within the domain $0 \le \theta < 2\pi$ are $\theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

If we prefer not to substitute, we can solve the equation by following the same pattern of factoring and setting each factor equal to zero.

$$\begin{aligned} \sin^2\theta + \sin\theta &= 0 \\ \sin\theta(2\sin\theta + 1) &= 0 \\ \sin\theta &= 0 \\ \theta &= 0, \pi \\ 2\sin\theta + 1 &= 0 \\ 2\sin\theta &= -1 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

Analysis

We can see the solutions on the graph in Figure $3.5.3$. On the interval $0 \le \theta < 2\pi$,the graph crosses the x-axis four times, at the solutions noted. Notice that trigonometric equations that are in quadratic form can yield up to four solutions instead of the expected two that are found with quadratic equations. In this example, each solution (angle) corresponding to a positive sine value will yield two angles that would result in that value.

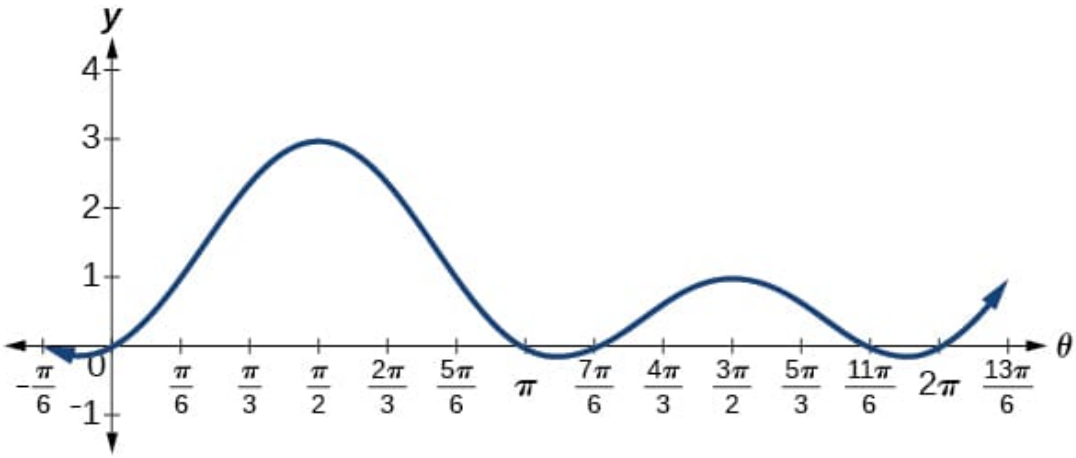


Figure 3.5.3

We can verify the solutions on the unit circle in via the result in the section on Sum and Difference Identities as well.

Example $3.5.7B$: Solving a Trigonometric Equation Quadratic in Form

Solve the equation quadratic in form exactly: $2\sin^2\theta - 3\sin\theta + 1 = 0$, $0 \le \theta < 2\pi$.

Solution

We can factor using grouping. Solution values of θ can be found on the unit circle.

$$\begin{aligned}(2\sin\theta - 1)(\sin\theta - 1) &= 0\\ 2\sin\theta - 1 &= 0\\ \sin\theta &= \frac{1}{2}\\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}\\ \sin\theta &= 1\\ \theta &= \frac{\pi}{2}\end{aligned}$$

Exercise 3.5.7

Solve the quadratic equation $2\cos^2\theta + \cos\theta = 0$.

Answer available at the end of this section.

Solving Trigonometric Equations Using Fundamental Identities

While algebra can be used to solve a number of trigonometric equations, we can also use the fundamental identities because they make solving equations simpler. Remember that the techniques we use for solving are not the same as those for verifying identities. The basic rules of algebra apply here, as opposed to rewriting one side of the identity to match the other side. In the next example, we use two identities to simplify the equation.

Example $3.5.8A$: Use Identities to Solve an Equation

Use identities to solve exactly the trigonometric equation over the interval $0 \le x < 2\pi$.

$$\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$$

Solution

Notice that the left side of the equation is the difference formula for cosine.

$$\begin{aligned} \cos x \cos(2x) + \sin x \sin(2x) &= \frac{\sqrt{3}}{2} \\ \cos(x - 2x) &= \frac{\sqrt{3}}{2} && \text{Difference formula for cosine} \\ \cos(-x) &= \frac{\sqrt{3}}{2} && \text{Use the negative angle identity.} \\ \cos x &= \frac{\sqrt{3}}{2} \end{aligned}$$

From the unit circle in the section on Sum and Difference Identities, we see that $\cos x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}, \frac{11\pi}{6}$.

Example $3.5.8B$: Solving the Equation Using a Double-Angle Formula

Solve the equation exactly using a double-angle formula: $\cos(2\theta) = \cos\theta$.

Solution

We have three choices of expressions to substitute for the double-angle of cosine. As it is simpler to solve for one trigonometric function at a time, we will choose the double-angle identity involving only cosine:

$$\begin{aligned} \cos(2\theta) &= \cos\theta \\ 2\cos^2\theta - 1 &= \cos\theta \\ 2\cos^2\theta - \cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta - 1) &= 0 \\ 2\cos\theta + 1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \cos\theta - 1 &= 0 \\ \cos\theta &= 1 \end{aligned}$$

So, if $\cos\theta = -\frac{1}{2}$,then $\theta = \frac{2\pi}{3} \pm 2\pi k$ and $\theta = \frac{4\pi}{3} \pm 2\pi k$; if $\cos\theta = 1$,then $\theta = 0 \pm 2\pi k$.

Example $3.5.8C$: Solving an Equation Using an Identity

Solve the equation exactly using an identity: $3\cos\theta + 3 = 2\sin^2\theta$, $0 \le \theta < 2\pi$.

Solution

If we rewrite the right side, we can write the equation in terms of cosine:

$$
\begin{aligned}
3\cos\theta + 3 &= 2\sin^2\theta \\
3\cos\theta + 3 &= 2(1-\cos^2\theta) \\
3\cos\theta + 3 &= 2 - 2\cos^2\theta \\
2\cos^2\theta + 3\cos\theta + 1 &= 0 \\
(2\cos\theta + 1)(\cos\theta + 1) &= 0 \\
2\cos\theta + 1 &= 0 \\
\cos\theta &= -\frac{1}{2} \\
\theta &= \frac{2\pi}{3},\ \frac{4\pi}{3} \\
\cos\theta + 1 &= 0 \\
\cos\theta &= -1 \\
\theta &= \pi
\end{aligned}
$$

Our solutions are $\theta = \frac{2\pi}{3},\ \frac{4\pi}{3},\ \pi$.

Solving Trigonometric Equations with Multiple Angles

Sometimes it is not possible to solve a trigonometric equation with identities that have a multiple angle, such as $\sin(2x)$ or $\cos(3x)$. When confronted with these equations, recall that $y = \sin(2x)$ is a horizontal compression by a factor of 2 of the function $y = \sin x$. On an interval of 2π,we can graph two periods of $y = \sin(2x)$,as opposed to one cycle of $y = \sin x$. This compression of the graph leads us to believe there may be twice as many *x*-intercepts or solutions to $\sin(2x) = 0$ compared to $\sin x = 0$. This information will help us solve the equation.

Example 3.5.9: Solving a Multiple Angle Trigonometric Equation

Solve exactly: $\cos(2x) = \frac{1}{2}$ on $[0, 2\pi)$.

Solution

We can see that this equation is the standard equation with a multiple of an angle. If $\cos(\alpha) = \frac{1}{2}$,we know α is in quadrants I and IV. While $\theta = \cos^{-1}\frac{1}{2}$ will only yield solutions in quadrants I and II, we recognize that the solutions to the equation $\cos\theta = \frac{1}{2}$ will be in quadrants I and IV.

Therefore, the possible angles are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. So, $2x = \frac{\pi}{3}$ or $2x = \frac{5\pi}{3}$, which means that $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Does this make sense? Yes, because $\cos\left(2\left(\frac{\pi}{6}\right)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Are there any other possible answers? Let us return to our first step.

In quadrant I, $2x = \frac{\pi}{3}$, so $x = \frac{\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{\pi}{3} + 2\pi \\ &= \frac{\pi}{3} + \frac{6\pi}{3} \\ &= \frac{7\pi}{3} \\ x &= \frac{7\pi}{6} \end{aligned}$$

One more rotation yields

$$\begin{aligned} 2x &= \frac{\pi}{3} + 4\pi \\ &= \frac{\pi}{3} + \frac{12\pi}{3} \\ &= \frac{13\pi}{3} \end{aligned}$$

$x = \frac{13\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

In quadrant IV, $2x = \frac{5\pi}{3}$, so $x = \frac{5\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 2\pi \\ &= \frac{5\pi}{3} + \frac{6\pi}{3} \\ &= \frac{11\pi}{3} \end{aligned}$$

so $x = \frac{11\pi}{6}$.

One more rotation yields

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 4\pi \\ &= \frac{5\pi}{3} + \frac{12\pi}{3} \\ &= \frac{17\pi}{3} \end{aligned}$$

$x = \frac{17\pi}{6} > 2\pi$,so this value for x is larger than 2π,so it is not a solution on $[0, 2\pi)$.

Our solutions are $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Note that whenever we solve a problem in the form of $sin(nx) = c$, we must go around the unit circle n times.

Solving Right Triangle Problems

We can now use all of the methods we have learned to solve problems that involve applying the properties of right triangles and the Pythagorean Theorem. We begin with the familiar Pythagorean Theorem,

$$a^2 + b^2 = c^2 \tag{3.5.2}$$

and model an equation to fit a situation.

Example 3.5.10A: Using the Pythagorean Theorem to Model an Equation

One of the cables that anchors the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris wheel is 69.5 meters above the ground, and the second anchor on the ground is 23 meters from the base of the Ferris wheel. Approximately how long is the cable, and what is the angle of elevation (from ground up to the center of the Ferris wheel)? See Figure $3.5.4$.

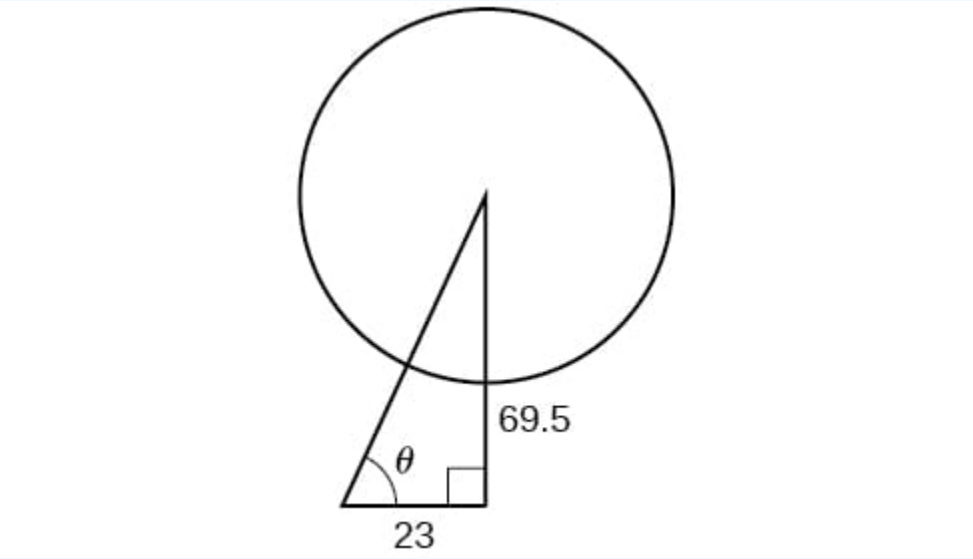


Figure $3.5.4$

Solution

Use the Pythagorean Theorem (Equation $3.5.2$) and the properties of right triangles to model an equation that fits the problem. Using the information given, we can draw a right triangle. We can find the length of the cable with the Pythagorean Theorem.

$$\begin{aligned} a^2+b^2 &= c^2 \\ (23)^2+(69.5)^2 &\approx 5359 \\ \sqrt{5359} &\approx 73.2\ m \end{aligned}$$

The angle of elevation is θ,formed by the second anchor on the ground and the cable reaching to the center of the wheel. We can use the tangent function to find its measure. Round to two decimal places.

$$\begin{aligned} \tan\theta &= 69.523 \\ \tan^{-1}(69.523) &\approx 1.2522 \\ &\approx 71.69^\circ \end{aligned}$$

The angle of elevation is approximately 71.7 °, and the length of the cable is 73.2 meters.

Example 3.5.10B: Using the Pythagorean Theorem to Model an Abstract Problem

OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length. Find the angle that a ladder of any length forms with the ground and the height at which the ladder touches the wall.

Solution

For any length of ladder, the base needs to be a distance from the wall equal to one fourth of the ladder's length. Equivalently, if the base of the ladder is "*a*" feet from the wall, the length of the ladder will be $4a$ feet. See Figure $3.5.5$.

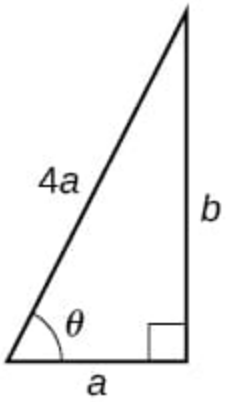


Figure 3.5.5

The side adjacent to θ is a and the hypotenuse is $4a$. Thus,

$$\begin{aligned}\cos\theta &= \frac{a}{4a}\\ &= \frac{1}{4}\\ \cos^{-1}\left(\frac{1}{4}\right) &\approx 75.5^\circ\end{aligned}$$

The elevation of the ladder forms an angle of 75.5 ° with the ground. The height at which the ladder touches the wall can be found using the Pythagorean Theorem:

$$\begin{aligned}a^2+b^2 &= (4a)^2\\ b^2 &= (4a)^2-a^2\\ b^2 &= 16a^2-a^2\\ b^2 &= 15a^2\\ b &= a\sqrt{15}\end{aligned}$$

Thus, the ladder touches the wall at $a\sqrt{15}$ feet from the ground.

Media

Access these online resources for additional instruction and practice with solving trigonometric equations.

- Solving Trigonometric Equations I
- Solving Trigonometric Equations II
- Solving Trigonometric Equations III
- Solving Trigonometric Equations IV
- Solving Trigonometric Equations V
- Solving Trigonometric Equations VI

Key Concepts

- When solving linear trigonometric equations, we can use algebraic techniques just as we do solving algebraic equations. Look for patterns, like the difference of squares, quadratic form, or an expression that lends itself well to substitution. See Example $3.5.1$, Example $3.5.2$, and Example $3.5.3$.
- Equations involving a single trigonometric function can be solved or verified using the unit circle. See Example $3.5.4$, Example $3.5.5$, and Example $3.5.6$, and Example $3.5.7$.
- We can also solve trigonometric equations using a graphing calculator. See Example $3.5.8$ and Example $3.5.9$.
- Many equations appear quadratic in form. We can use substitution to make the equation appear simpler, and then use the

same techniques we use solving an algebraic quadratic: factoring, the quadratic formula, etc. See Example $3.5.10$, Example $3.5.11$, Example $3.5.12$, and Example $3.5.13$.
- We can also use the identities to solve trigonometric equation. See Example $3.5.14$, Example $3.5.15$, and Example $3.5.16$.
- We can use substitution to solve a multiple-angle trigonometric equation, which is a compression of a standard trigonometric function. We will need to take the compression into account and verify that we have found all solutions on the given interval. See Example $3.5.17$.
- Real-world scenarios can be modeled and solved using the Pythagorean Theorem and trigonometric functions. See Example $3.5.18$.

Answers

Exercise $3.5.2$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

Exercise $3.5.3$

$\frac{\pi}{3} \pm \pi k$

Exercise $3.5.5$

$\theta \approx 1.7722 \pm 2\pi k$ and $\theta \approx 4.5110 \pm 2\pi k$

Exercise $3.5.6$

$\cos\theta = -1, \theta = \pi$

Exercise $3.5.7$

$\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

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4: FURTHER APPLICATIONS OF TRIGONOMETRY

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in Trigonometric Functions and investigate applications more deeply and meaningfully.

4.1: Non-right Triangles - Law of Sines

4.2: Non-right Triangles - Law of Cosines

4.3: Polar Coordinates

4.4: Polar Coordinates - Graphs

4.5: Polar Form of Complex Numbers

4.6: Vectors

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4.1: Non-right Triangles - Law of Sines

Learning Objectives

- Use the Law of Sines to solve oblique triangles.
- Find the area of an oblique triangle using the sine function.
- Solve applied problems using the Law of Sines.

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the **angle of elevation** measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in Figure $4.1.1$ that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving **non-right triangles**.

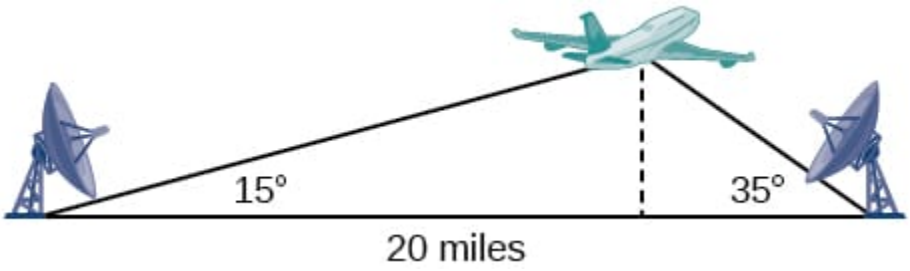


Figure $4.1.1$

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

ASA (angle-side-angle) We know the measurements of two angles and the included side. See Figure $4.1.2$.

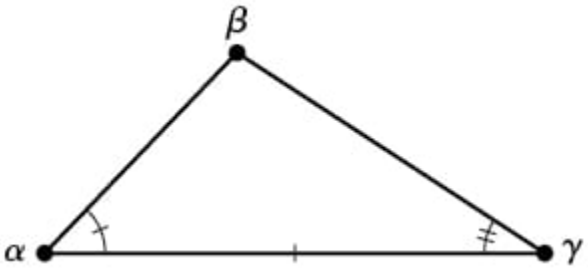


Figure $4.1.2$

AAS (angle-angle-side) We know the measurements of two angles and a side that is not between the known angles. See Figure $4.1.3$.

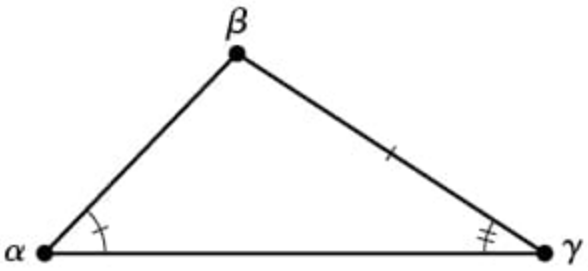


Figure 4.1.3

SSA (side-side-angle) We know the measurements of two sides and an angle that is not between the known sides. See Figure 4.1.4.

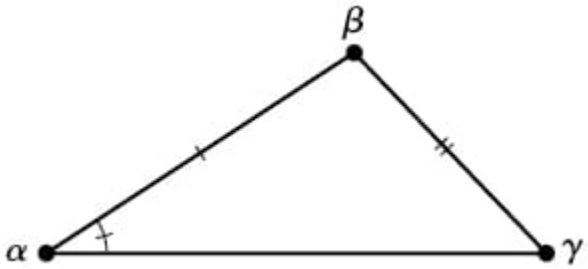


Figure 4.1.4

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in Figure 4.1.5.

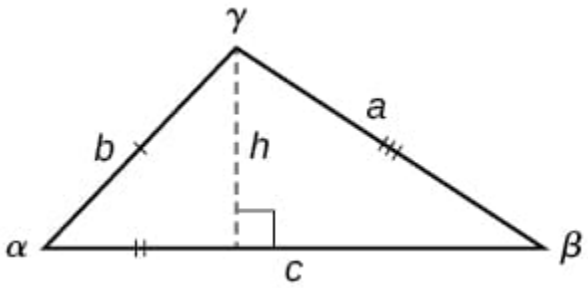


Figure 4.1.5

Using the right triangle relationships, we know that $\sin\ \alpha = \frac{h}{b}$ and $\sin\ \beta = \frac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin\ \alpha \text{ and } h = a\sin\ \beta$$

We then set the expressions equal to each other.

$$\begin{aligned} b\sin\alpha &= a\sin\beta \\ \left(\frac{1}{ab}\right)(b\sin\alpha) &= (a\sin\beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab} \\ \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \end{aligned}$$

Similarly, we can compare the other ratios.

$$\frac{\sin\ \alpha}{a} = \frac{\sin\ \gamma}{c} \text{ and } \frac{\sin\ \beta}{b} = \frac{\sin\ \gamma}{c}$$

Collectively, these relationships are called the **Law of Sines**.

$$\frac{\sin\,\alpha}{a} = \frac{\sin\,\beta}{b} = \frac{\sin\,\gamma}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See Figure 4.1.6.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

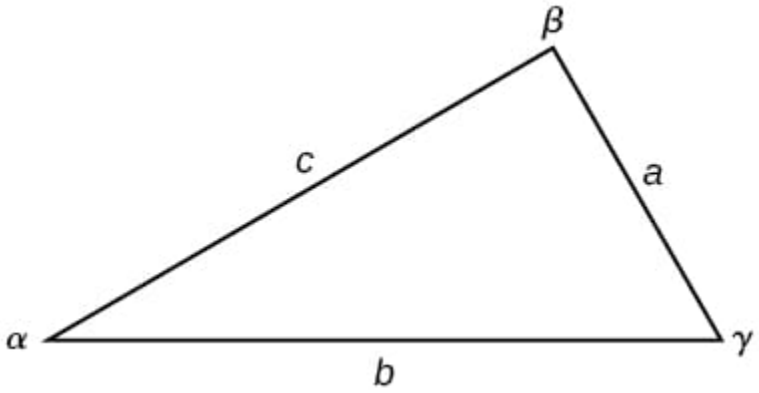


Figure 4.1.6

Note: LAW OF SINES

Given a triangle with angles and opposite sides labeled as in Figure 4.1.6, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin\,\alpha}{a} = \frac{\sin\,\beta}{b} = \frac{\sin\,\gamma}{c} \tag{4.1.1}$$

$$\frac{a}{\sin\,\alpha} = \frac{b}{\sin\,\beta} = \frac{c}{\sin\,\gamma} \tag{4.1.2}$$

To solve an oblique triangle, use any pair of applicable ratios.

Example 4.1.1: Solving for Two Unknown Sides and Angle of an AAS Triangle

Solve the triangle shown in Figure 4.1.7 to the nearest tenth.

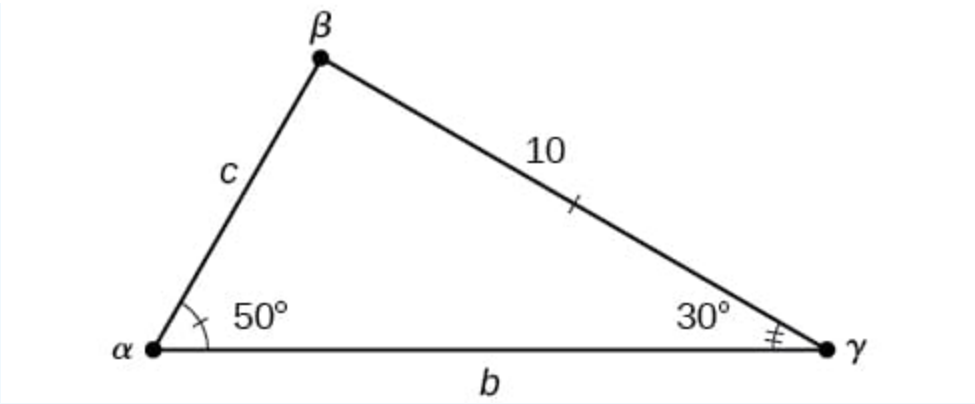


Figure 4.1.7

Solution

The three angles must add up to 180 degrees. From this, we can determine that

$$\begin{aligned}\beta &= 180^\circ - 50^\circ - 30^\circ \\ &= 100^\circ\end{aligned}$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50$ °and its corresponding side $a = 10$. We can use the following proportion from the Law of Sines to find the length of c.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(30^\circ)}{c} \\ c\frac{\sin(50^\circ)}{10} &= \sin(30^\circ) \qquad \text{Multiply both sides by } c \\ c &= \sin(30^\circ)\frac{10}{\sin(50^\circ)} \qquad \text{Multiply by the reciprocal to isolate } c \\ c &\approx 6.5\end{aligned}$$

Similarly, to solve for b, we set up another proportion.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(100^\circ)}{b} \\ b\sin(50^\circ) &= 10\sin(100^\circ) \qquad \text{Multiply both sides by } b \\ b &= \frac{10\sin(100^\circ)}{\sin(50^\circ)} \qquad \text{Multiply by the reciprocal to isolate } b \\ b &\approx 12.9\end{aligned}$$

Therefore, the complete set of angles and sides is

$$\begin{aligned}\alpha &= 50^\circ \quad a = 10 \\ \beta &= 100^\circ \quad b \approx 12.9 \\ \gamma &= 30^\circ \quad c \approx 6.5\end{aligned}$$

Exercise 4.1.1

Solve the triangle shown in Figure 4.1.8 to the nearest tenth.

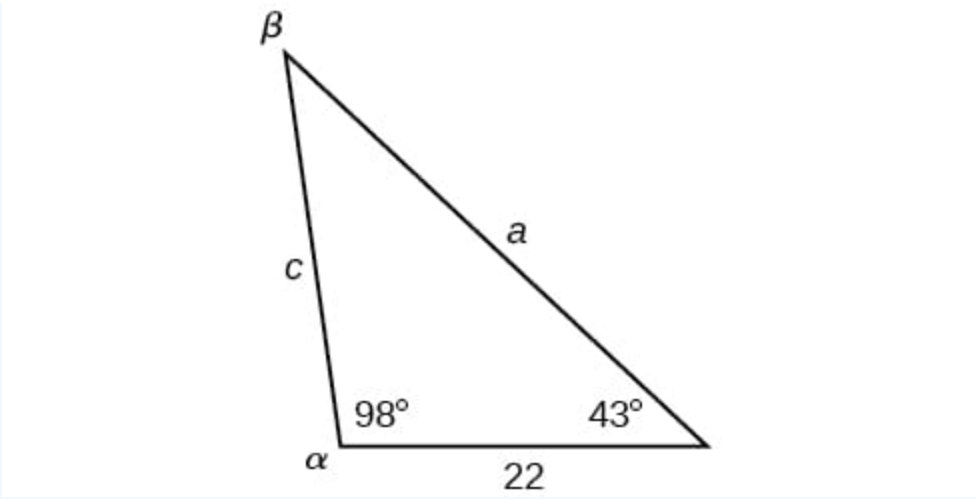


Figure 4.1.8

Answer available at the end of this section.

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an ambiguous case. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

Note: POSSIBLE OUTCOMES FOR SSA TRIANGLES

Oblique triangles in the category SSA may have four different outcomes. Figure 4.1.9 illustrates the solutions with the known sides a and b and known angle α.

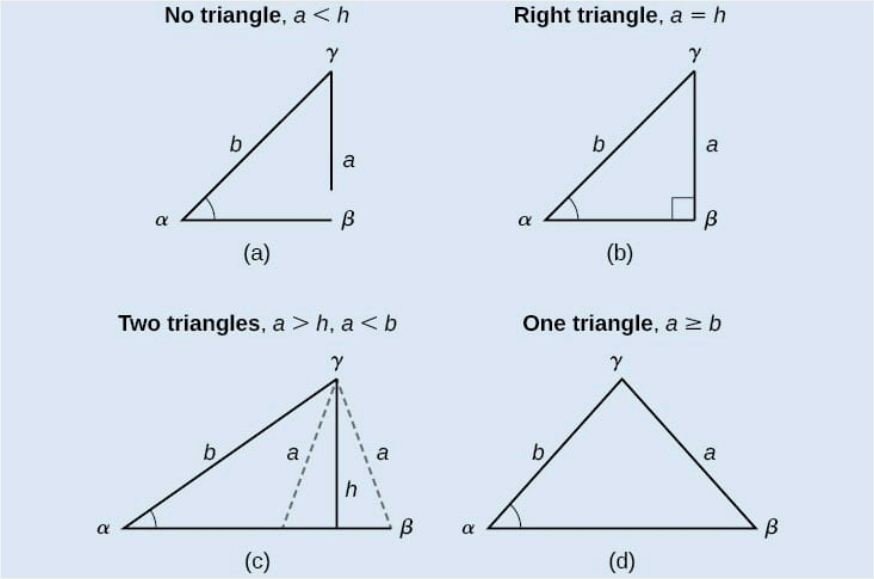


Figure 4.1.9

Example 4.1.2: Solving an Oblique SSA Triangle

Solve the triangle in Figure 4.1.10 for the missing side and find the missing angle measures to the nearest tenth.

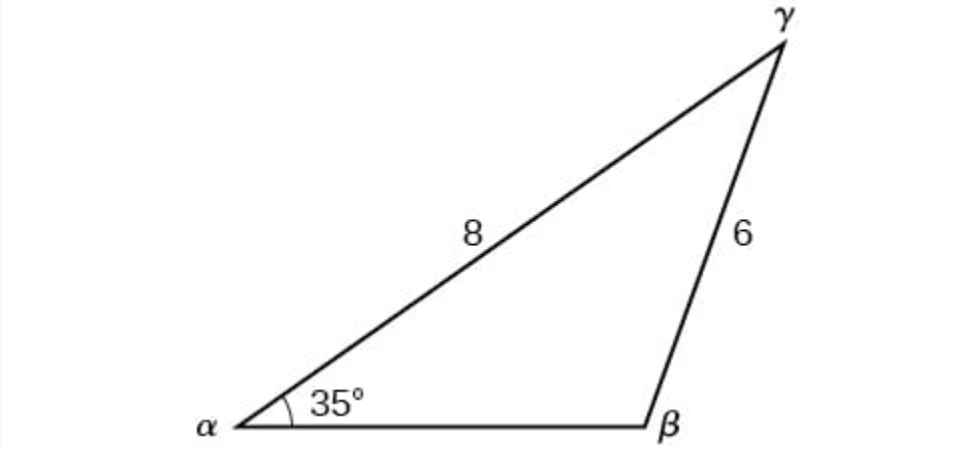


Figure 4.1.10

Solution

Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\begin{aligned} \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \\ \frac{\sin(35^\circ)}{6} &= \frac{\sin\beta}{8} \\ \frac{8\sin(35^\circ)}{6} &= \sin\beta \\ 0.7648 &\approx \sin\beta \\ \sin^{-1}(0.7648) &\approx 49.9^\circ \\ \beta &\approx 49.9^\circ \end{aligned}$$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than $90\,^\circ$. How did we get an acute angle, and how do we find the measurement of β? Let's investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have Figure 4.1.11. It appears that there may be a second triangle that will fit the given criteria.

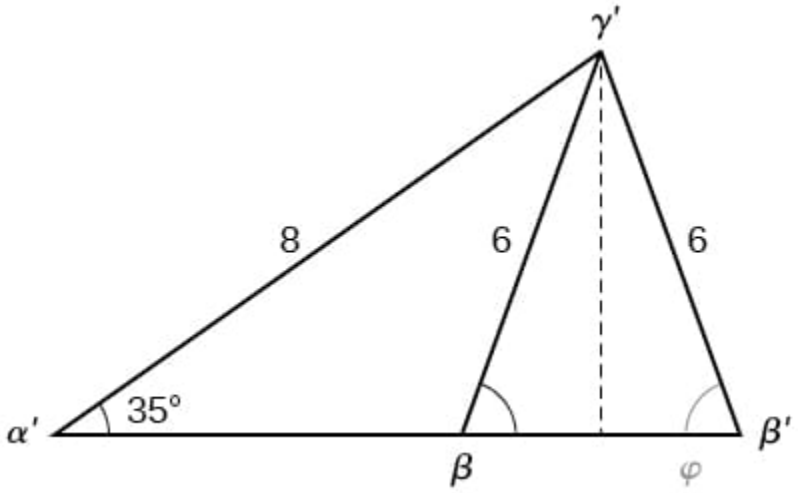


Figure 4.1.11

The angle supplementary to β is approximately equal to $49.9\,^\circ$, which means that $\beta = 180\,^\circ - 49.9\,^\circ = 130.1\,^\circ$. (Remember that the

sine function is positive in both the first and second quadrants.) Solving for γ, we have

$$\begin{aligned}\gamma &= 180^\circ - 35^\circ - 130.1^\circ \\ &\approx 14.9^\circ\end{aligned}$$

We can then use these measurements to solve the other triangle. Since β' is supplementary to β, we have

$$\begin{aligned}\gamma' &= 180^\circ - 35^\circ - 49.5^\circ \\ &\approx 95.1^\circ\end{aligned}$$

Now we need to find c and c'.

We have

$$\begin{aligned}\frac{c}{\sin(14.9^\circ)} &= \frac{6}{\sin(35^\circ)} \\ c &= \frac{6\sin(14.9^\circ)}{\sin(35^\circ)} \\ &\approx 2.7\end{aligned}$$

Finally,

$$\begin{aligned}\frac{c'}{\sin(95.1^\circ)} &= \frac{6}{\sin(35^\circ)} \\ c' &= \frac{6\sin(95.1^\circ)}{\sin(35^\circ)} \\ &\approx 10.4\end{aligned}$$

To summarize, there are two triangles with an angle of 35°, an adjacent side of 8, and an opposite side of 6, as shown in Figure 4.1.12.

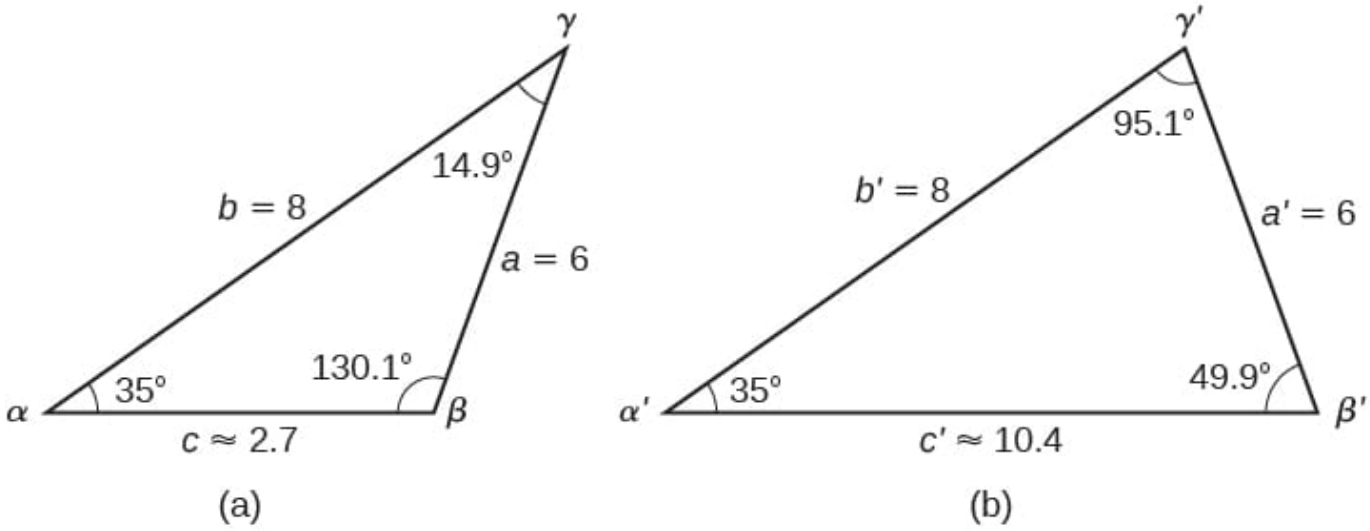


Figure 4.1.12

However, we were looking for the values for the triangle with an obtuse angle β. We can see them in the first triangle (a) in Figure 4.1.12.

Exercise 4.1.2

Given $\alpha = 80^\circ$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.

Answer available at the end of this section.

Example 4.1.3: Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in Figure 4.1.13, solve for the unknown side and angles. Round your answers to the nearest tenth.

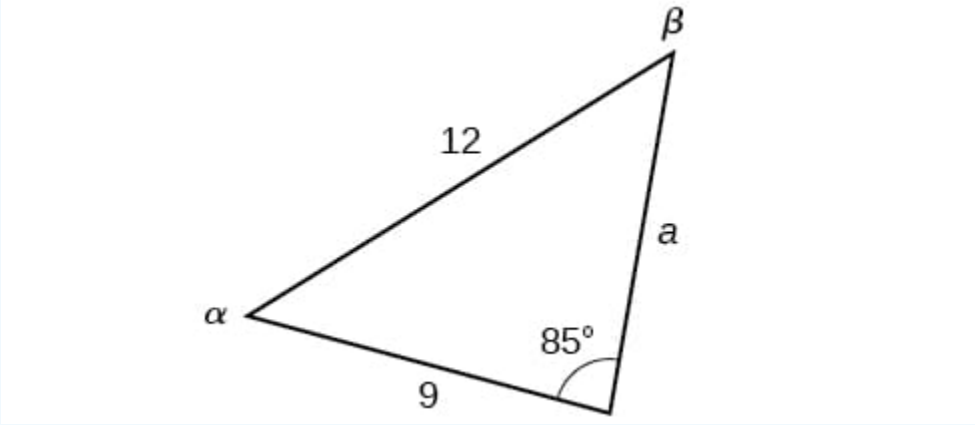


Figure 4.1.13

Solution

In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle, $\gamma = 85\,°$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\begin{aligned} \frac{\sin(85^\circ)}{12} &= \frac{\sin\beta}{9} \qquad \text{Isolate the unknown.} \\ \frac{9\sin(85^\circ)}{12} &= \sin\beta \end{aligned}$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\begin{aligned} \beta &= \sin^{-1}\left(\frac{9\sin(85^\circ)}{12}\right) \\ \beta &\approx \sin^{-1}(0.7471) \\ \beta &\approx 48.3^\circ \end{aligned}$$

In this case, if we subtract β from $180\,°$, we find that there may be a second possible solution. Thus, $\beta = 180\,° - 48.3\,° \approx 131.7\,°$. To check the solution, subtract both angles, $131.7\,°$ and $85\,°$, from $180\,°$. This gives

$$\begin{aligned} \alpha &= 180^\circ - 85^\circ - 131.7^\circ \\ &\approx -36.7^\circ \end{aligned}$$

which is impossible, and so $\beta \approx 48.3\,°$.

To find the remaining missing values, we calculate $\alpha = 180\,° - 85\,° - 48.3\,° \approx 46.7\,°$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\begin{aligned} \frac{\sin(85^\circ)}{12} &= \frac{\sin(46.7^\circ)}{a} \\ a\frac{\sin(85^\circ)}{12} &= \sin(46.7^\circ) \\ a &= \frac{12\sin(46.7^\circ)}{\sin(85^\circ)} \\ &\approx 8.8 \end{aligned}$$

The complete set of solutions for the given triangle is

$$\begin{array}{ll} \alpha \approx 46.7^\circ & a \approx 8.8 \\ \beta \approx 48.3^\circ & b = 9 \\ \gamma = 85^\circ & c = 12 \end{array}$$

Exercise 4.1.3

Given $\alpha = 80\,°$, $a = 100$, $b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Answer available at the end of this section.

Example 4.1.4: Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of $50\,°$, and a second side has length 10.

Solution

Using the given information, we can solve for the angle opposite the side of length 10. See Figure 4.1.14.

$$\begin{aligned} \frac{\sin\alpha}{10} &= \frac{\sin(50^\circ)}{4} \\ \sin\alpha &= \frac{10\sin(50^\circ)}{4} \\ \sin\alpha &\approx 1.915 \end{aligned}$$

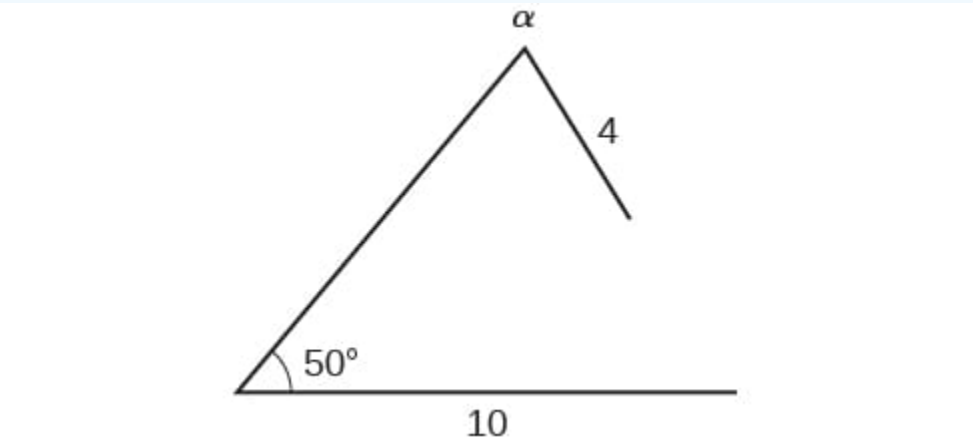


Figure 4.1.14

We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$ in a graphing calculator generates an ERROR DOMAIN. Therefore, no triangles can be drawn with the provided dimensions.

Exercise 4.1.4

Determine the number of triangles possible given $a = 31$, $b = 26$, $\beta = 48\,°$.

Answer available at the end of this section.

Finding the Area of an Oblique Triangle Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as $Area = \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in Figure $4.1.15$, one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin\alpha = \frac{opposite}{hypotenuse}$ to write an equation for area in oblique triangles. In the acute triangle, we have $\sin\ \alpha = \frac{h}{c}$ or $c\sin\ \alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is $\alpha\,'$, or $180 - \alpha$.

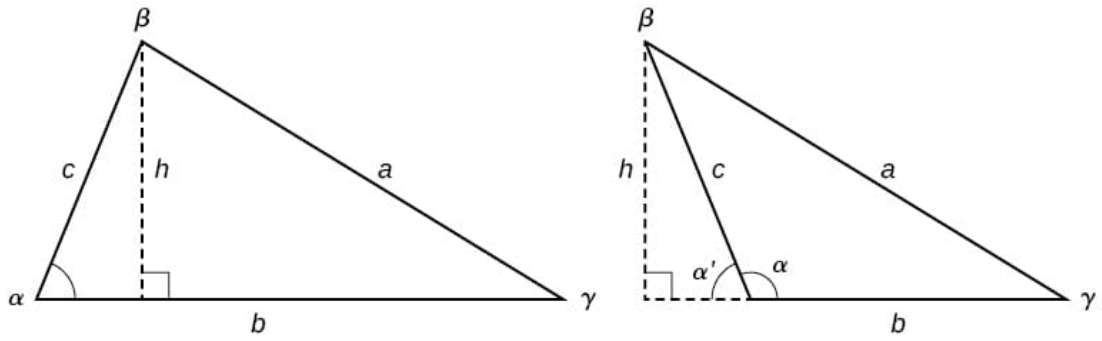


Figure $4.1.15$

Thus,

$$Area = \frac{1}{2}(base)(height) = \frac{1}{2}b(c\sin\ \alpha)$$

Similarly,

$$Area = \frac{1}{2}a(b\sin\ \gamma) = \frac{1}{2}a(c\sin\ \beta)$$

Note: AREA OF AN OBLIQUE TRIANGLE

The formula for the area of an oblique triangle is given by

$$Area = \frac{1}{2}bc\sin\ \alpha \tag{4.1.3}$$

$$Area = \frac{1}{2}ac\sin\ \beta \tag{4.1.4}$$

$$Area = \frac{1}{2}ab\sin\ \gamma \tag{4.1.5}$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Example $4.1.5$: Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90$, $b = 52$, and angle $\gamma = 102\ °$. Round the area to the nearest integer.

Solution

Using the formula, we have

$$Area = \frac{1}{2}ab\sin\gamma$$
$$Area = \frac{1}{2}(90)(52)\sin(102^\circ)$$
$$Area \approx 2289 \text{ square units}$$

Exercise 4.1.5

Find the area of the triangle given $\beta = 42\,^\circ, a = 7.2\ ft, c = 3.4\ ft$. Round the area to the nearest tenth.

Answer available at the end of this section.

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Example 4.1.6: Finding an Altitude

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in Figure 4.1.16. Round the altitude to the nearest tenth of a mile.

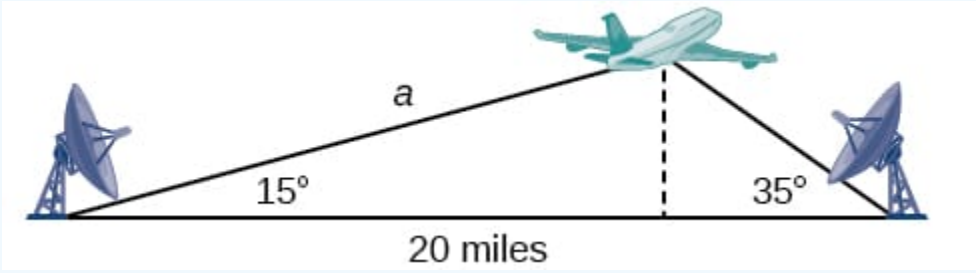


Figure 4.1.16

Solution

To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side a, and then use right triangle relationships to find the height of the aircraft, h.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180\,^\circ - 15\,^\circ - 35\,^\circ = 130\,^\circ$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\frac{\sin(130^\circ)}{20} = \frac{\sin(35^\circ)}{a}$$
$$a\sin(130^\circ) = 20\sin(35^\circ)$$
$$a = \frac{20\sin(35^\circ)}{\sin(130^\circ)}$$
$$a \approx 14.98$$

The distance from one station to the aircraft is about 14.98 miles.

Now that we know a, we can use right triangle relationships to solve for h.

$$\sin(15^\circ) = \frac{opposite}{hypotenuse}$$
$$\sin(15^\circ) = \frac{h}{a}$$
$$\sin(15^\circ) = \frac{h}{14.98}$$
$$h = 14.98\sin(15^\circ)$$
$$h \approx 3.88$$

The aircraft is at an altitude of approximately 3.9 miles.

Exercise 4.1.6

The diagram shown in Figure $4.1.17$ represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point A, is $70\,^\circ$, the angle of elevation from the northern end zone, point B, is $62\,^\circ$, and the distance between the viewing points of the two end zones is 145 yards.

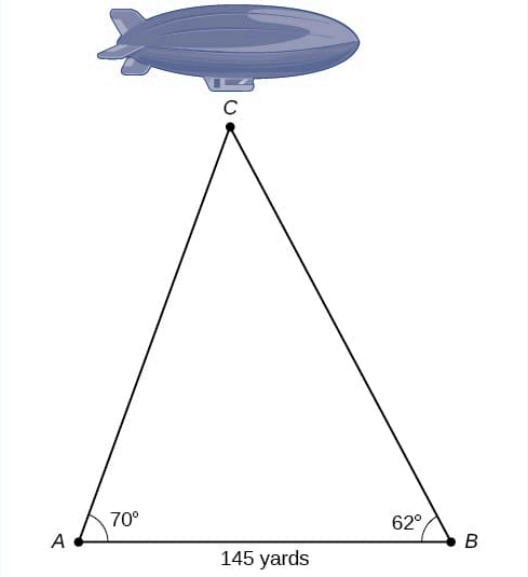


Figure $4.1.17$

Answer available at the end of this section.

Media

Access these online resources for additional instruction and practice with trigonometric applications.

- Law of Sines: The Basics
- Law of Sines: The Ambiguous Case

Key Equations

Law of Sines	$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$ $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

Area for oblique triangles	$$Area = \frac{1}{2}bc\sin\alpha$$ $$= \frac{1}{2}ac\sin\beta$$ $$= \frac{1}{2}ab\sin\gamma$$

Key Concepts

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.
- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See Example $4.1.1$.
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See Example $4.1.2$ and Example $4.1.3$.
- The Law of Sines can be used to solve triangles with given criteria. See Example $4.1.4$.
- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value. See Example $4.1.5$.
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See Example $4.1.6$.

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Answers

Exercise $4.1.1$

$\alpha = 98^\circ \quad a = 34.6$
$\beta = 39^\circ \quad b = 22$
$\gamma = 43^\circ \quad c = 23.8$

Exercise $4.1.2$

Solution 1

$\alpha = 80^\circ \quad a = 120$
$\beta \approx 83.2^\circ \quad b = 121$
$\gamma \approx 16.8^\circ \quad c \approx 35.2$

Solution 2

$\alpha' = 80^\circ \quad a' = 120$
$\beta' \approx 96.8^\circ \quad b' = 121$
$\gamma' \approx 3.2^\circ \quad c' \approx 6.8$

Exercise $4.1.3$

$\beta \approx 5.7\,^\circ, \gamma \approx 94.3\,^\circ, c \approx 101.3$

Exercise $4.1.4$

two

Exercise $4.1.5$

about 8.2 square feet

Exercise $4.1.6$

161.9 yd.

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4.2: Non-right Triangles - Law of Cosines

Learning Objectives

In this section, you will:

- Use the Law of Cosines to solve oblique triangles.
- Solve applied problems using the Law of Cosines.
- Use Heron's formula to find the area of a triangle.

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in Figure $4.2.1$ How far from port is the boat?

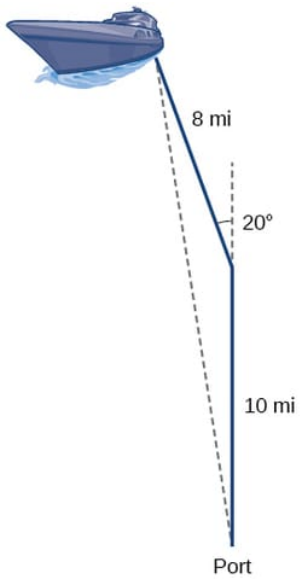


Figure $4.2.1$

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the *Law of Cosines*, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the *x*-axis, and vertex C located at some point (x, y) in the plane, as illustrated in Figure $4.2.2$. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

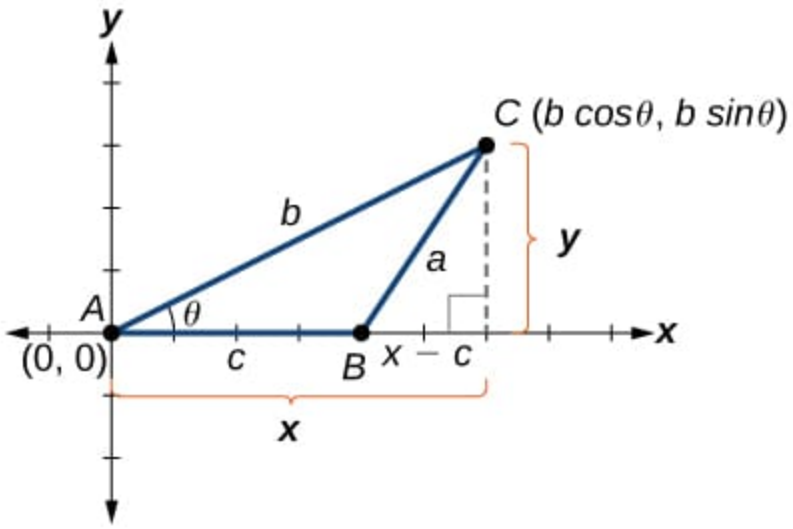


Figure 4.2.2

We can drop a perpendicular from C to the *x*-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x(adjacent)}{b(hypotenuse)} \text{ and } \sin\theta = \frac{y(opposite)}{b(hypotenuse)}$$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta, b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned}
a^2 &= (x-c)^2 + y^2 \\
&= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y \\
&= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta && \text{Expand the perfect square.} \\
&= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1 \\
&= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{Factor out } b^2 \\
a^2 &= b^2 + c^2 - 2bc\cos\theta
\end{aligned}$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

The LAW OF COSINES

The Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle.

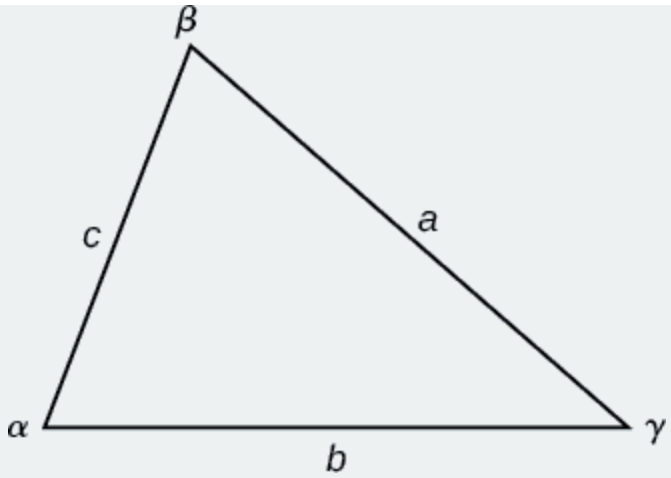


Figure 4.2.3

For triangles labeled as in Figure 4.2.3, with angles α, β and γ, and opposite corresponding sides a, b, and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \tag{4.2.1}$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \tag{4.2.2}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{4.2.3}$$

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \tag{4.2.4}$$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \tag{4.2.5}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \tag{4.2.6}$$

How to: Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Example 4.2.1: Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in Figure 4.2.4.

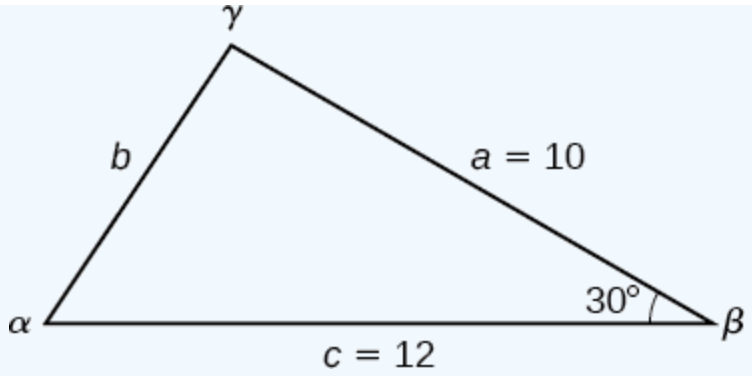


Figure 4.2.4

Solution

First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos\beta \\ b^2 &= 10^2 + 12^2 - 2(10)(12)\cos(30°) && \text{Substitute the measurements for the known quantities.} \\ b^2 &= 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right) && \text{Evaluate the cosine and begin to simplify.} \\ b^2 &= 244 - 120\sqrt{3} \\ b &= \sqrt{244 - 120\sqrt{3}} && \text{Use the square root property.} \\ b &\approx 6.013 \end{aligned}$$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle α, we have

$$\begin{aligned} \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \\ \frac{\sin\alpha}{10} &= \frac{\sin(30°)}{6.013} \\ \sin\alpha &= \frac{10\sin(30°)}{6.013} && \text{Multiply both sides of the equation by 10.} \\ \alpha &= \sin^{-1}\left(\frac{10\sin(30°)}{6.013}\right) && \text{Find the inverse sine of } \frac{10\sin(30°)}{6.013}. \\ \alpha &\approx 56.3° \end{aligned}$$

The other possibility for α would be $\alpha = 180\,° - 56.3\,° \approx 123.7\,°$. In the original diagram,α is adjacent to the longest side, so α is an acute angle and, therefore, $123.7\,°$ does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between $0\,°$ and $180\,°$. Proceeding with $\alpha \approx 56.3\,°$, we can then find the third angle of the triangle.

$$\begin{aligned} \gamma &= 180° - 30° - 56.3° \\ &\approx 93.7° \end{aligned}$$

The complete set of angles and sides is

$$\alpha \approx 56.3\,°\ a = 10$$

$$\beta = 30\,^\circ\ b \approx 6.013$$

$$\gamma \approx 93.7\,^\circ\ c = 12$$

Exercise 4.2.1

Find the missing side and angles of the given triangle: $\alpha = 30\,^\circ$, $b = 12$, $c = 24$.

Answer available at the end of this section.

Example 4.2.2: Solving for an Angle of a SSS Triangle

Find the angle α for the given triangle if side $a = 20$, side $b = 25$, and side $c = 18$.

Solution

For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$\begin{aligned}
&a^2 = b^2 + c^2 - 2bc\cos\alpha \\
&20^2 = 25^2 + 18^2 - 2(25)(18)\cos\alpha && \text{Substitute the appropriate measurements.} \\
&400 = 625 + 324 - 900\cos\alpha && \text{Simplify in each step.} \\
&400 = 949 - 900\cos\alpha \\
&-549 = -900\cos\alpha && \text{Isolate } \cos\alpha. \\
&0.61 \approx \cos\alpha && \text{Find the inverse cosine.} \\
&\alpha \approx 52.4^\circ
\end{aligned}$$

See Figure 4.2.5.

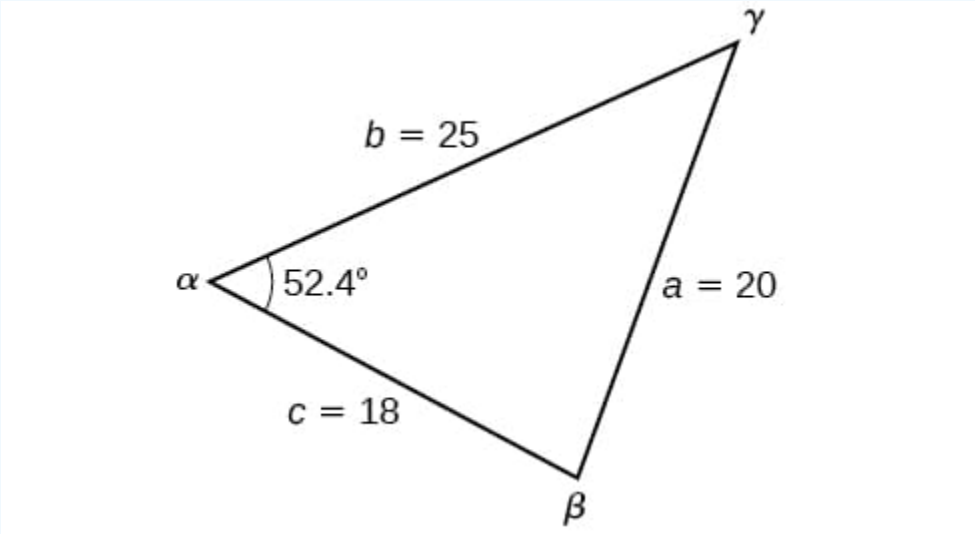


Figure 4.2.5

Analysis

Because the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases using this method.

Exercise 4.2.2

Given $a = 5$, $b = 7$, and $c = 10$, find the missing angles.

Answer available at the end of this section.

Solving Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

Example 4.2.3A: Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Solution

For simplicity, we start by drawing a diagram similar to Figure 4.2.6 and labeling our given information.

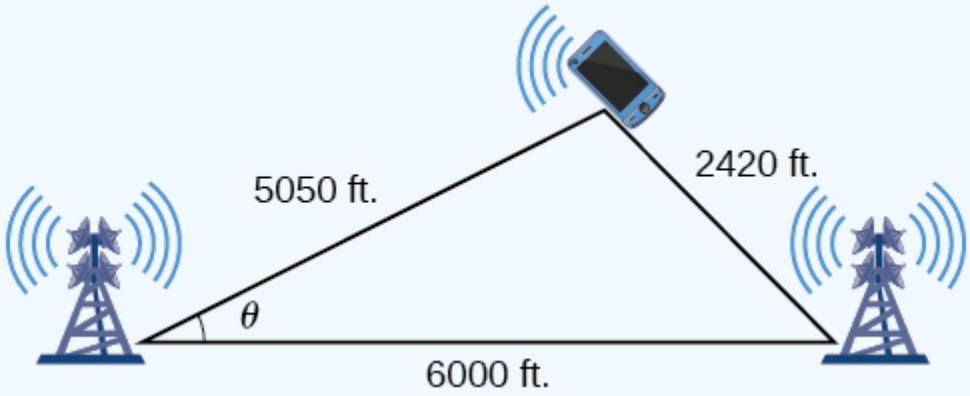


Figure 4.2.6

Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a = 2420$, $b = 5050$, and $c = 6000$. Thus, θ corresponds to the opposite side $a = 2420$.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\theta \\ (2420)^2 &= (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta \\ \cos\theta &\approx 0.9183 \\ \theta &\approx \cos^{-1}(0.9183) \\ \theta &\approx 23.3° \end{aligned}$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in Figure 4.2.7. This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.

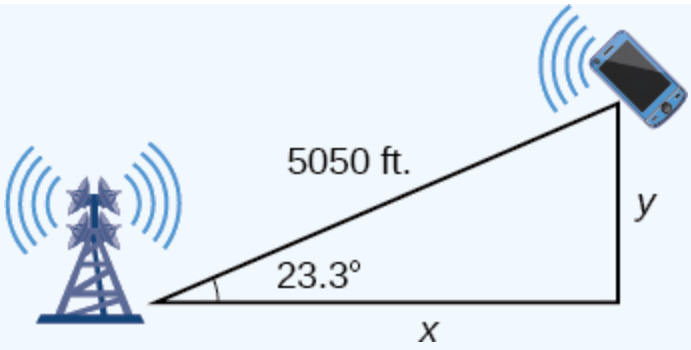


Figure 4.2.7

Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions. Thus

$$\begin{aligned} \cos(23.3°) &= \frac{x}{5050} \\ x &= 5050\cos(23.3°) \\ x &\approx 4638.15\ feet \\ \sin(23.3°) &= \frac{y}{5050} \\ y &= 5050\sin(23.3°) \\ y &\approx 1997.5\ feet \end{aligned}$$

The cell phone is approximately 4638 feet east and 1998 feet north of the first tower, and 1998 feet from the highway.

Example $4.2.3B$: Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in Figure $4.2.8$.

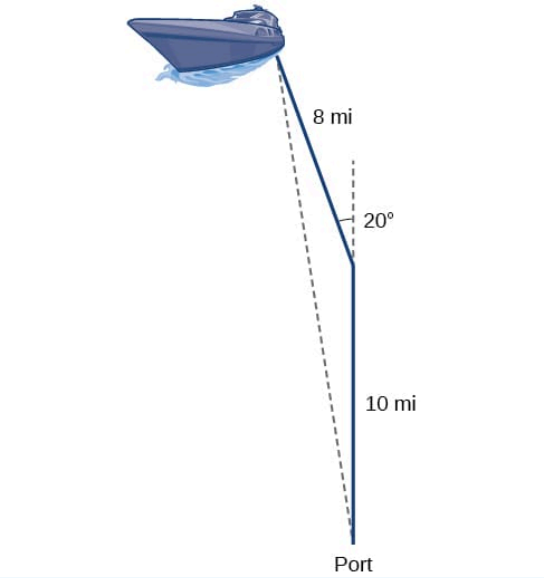


Figure 4.2.8

Solution

The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180\,°-20\,° = 160\,°$. With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle—the distance of the boat to the port.

$$x^2 = 8^2 + 10^2 - 2(8)(10)\cos(160°)$$
$$x^2 = 314.35$$
$$x = \sqrt{314.35}$$
$$x \approx 17.7\, miles$$

The boat is about 17.7 miles from port.

Using Heron's Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use **Heron's formula** instead of finding the height. **Heron of Alexandria** was a geometer who lived during the first century A.D. He discovered a formula for finding the area of oblique triangles when three sides are known.

HERON'S FORMULA

Heron's formula finds the area of oblique triangles in which sides a, b,and c are known.

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \tag{4.2.7}$$

where $s = \dfrac{(a+b+c)}{2}$ is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

Example 4.2.4: Using Heron's Formula to Find the Area of a Given Triangle

Find the area of the triangle in Figure $4.2.9$ using Heron's formula.

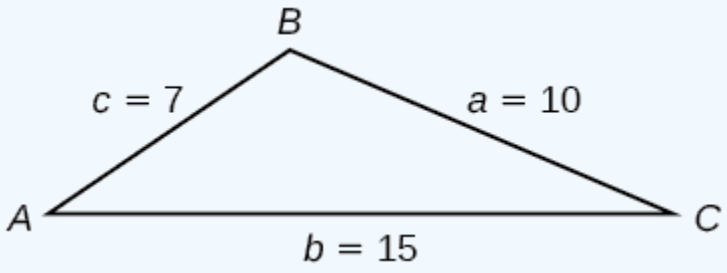


Figure $4.2.9$

Solution

First, we calculate s.

$$s = \frac{(a+b+c)}{2}$$
$$s = \frac{(10+15+7)}{2}$$
$$= 16$$

Then we apply the formula.

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$
$$Area = \sqrt{16(16-10)(16-15)(16-7)}$$
$$Area \approx 29.4$$

The area is approximately 29.4 square units.

Exercise 4.2.3

Use Heron's formula to find the area of a triangle with sides of lengths $a = 29.7$ ft, $b = 42.3$ ft, and $c = 38.4$ ft.

Answer available at the end of this section.

Example 4.2.5: Applying Heron's Formula to a Real-World Problem

A Chicago city developer wants to construct a building consisting of artist's lofts on a triangular lot bordered by Rush Street, Wabash Avenue, and Pearson Street. The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters. How many square meters are available to the developer? See Figure $4.2.10$ for a view of the city property.



Figure $4.2.10$

Solution

Find the measurement for s, which is one-half of the perimeter.

$$s = \frac{(62.4 + 43.5 + 34.1)}{2}$$
$$s = 70\ m$$

Apply Heron's formula.

$$Area = \sqrt{70(70 - 62.4)(70 - 43.5)(70 - 34.1)}$$
$$Area = \sqrt{506,118.2}$$
$$Area \approx 711.4$$

The developer has about 711.4 square meters.

Exercise 4.2.4

Find the area of a triangle given $a = 4.38$ ft , $b = 3.79$ ft, and $c = 5.22$ ft.

Answer available at the end of this section.

Media

Access these online resources for additional instruction and practice with the Law of Cosines.

- Law of Cosines
- Law of Cosines: Applications
- Law of Cosines: Applications 2

Key Equations

Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$ $b^2 = a^2 + c^2 - 2ac\cos\beta$ $c^2 = a^2 + b^2 - 2ab\cos\gamma$
Heron's formula	$Area = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{(a+b+c)}{2}$

Key Concepts

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See Example $4.2.1$ and Example $4.2.2$.
- The Law of Cosines is useful for many types of applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law of Cosines to find a solution. See Example $4.2.3$ and Example $4.2.4$.
- Heron's formula allows the calculation of area in oblique triangles. All three sides must be known to apply Heron's formula. See Example $4.2.5$ and See Example $4.2.6$.

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Answers

Exercise $4.2.1$

$a \approx 14.9$, $\beta \approx 23.8\,°$, $\gamma \approx 126.2\,°$.

Exercise $4.2.2$

$\alpha \approx 27.7\,°$, $\beta \approx 40.5\,°$, $\gamma \approx 111.8\,°$

Exercise $4.2.3$

Area = 552 square feet

Exercise $4.2.4$

about 8.15 square feet

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4.3: Polar Coordinates

Skills to Develop

- Plot points using polar coordinates.
- Convert from polar coordinates to rectangular coordinates.
- Convert from rectangular coordinates to polar coordinates.
- Transform equations between polar and rectangular forms.
- Identify and graph polar equations by converting to rectangular equations.

Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see Figure $4.3.1$). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.

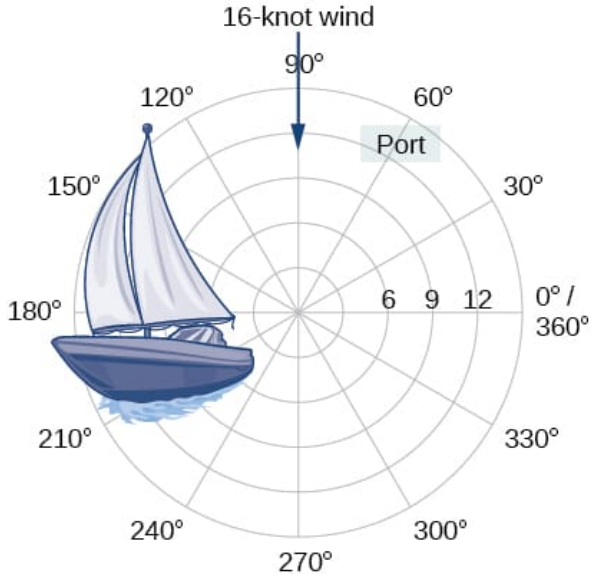
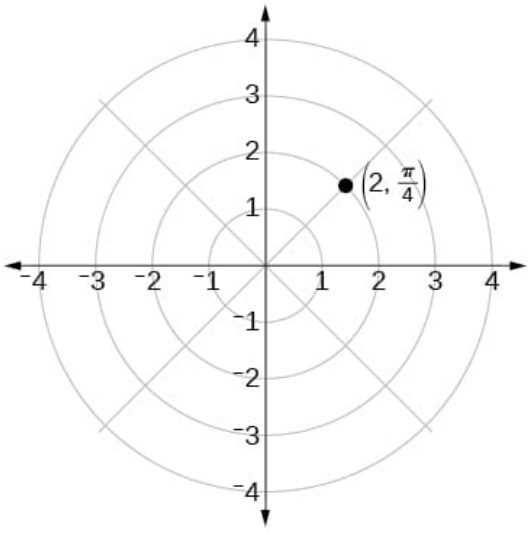


Figure $4.3.1$

Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of *rectangular coordinates* (x, y) in the *Cartesian coordinate plane*. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to polar coordinates, which are points labeled (r, θ) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the pole, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive x-axis now viewed as the polar axis and the origin as the pole. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ, typically measured in radians (but we can also use degree measure if desired), indicates the direction of r. We move counterclockwise from the polar axis by an angle of θ,and measure a directed line segment the length of r in the direction of θ. Even though we measure θ first and then r, the polar point is written with the r-coordinate first. For example, to plot the point $\left(2, \frac{\pi}{4}\right)$,we would move $\frac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in Figure $4.3.2$.



Polar Grid

Figure 4.3.2

Example 4.3.1: Plotting a Point on the Polar Grid

Plot the point $\left(3, \frac{\pi}{2}\right)$ on the polar grid.

Solution

The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction $90\,^\circ$ from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$ direction, as shown in Figure 4.3.3.

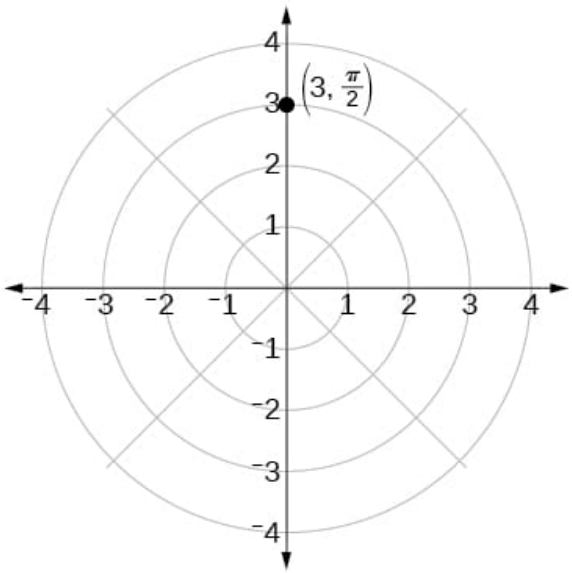


Figure 4.3.3

Exercise $4.3.1$

Plot the point $\left(2, \frac{\pi}{3}\right)$ in the polar grid.

Answer

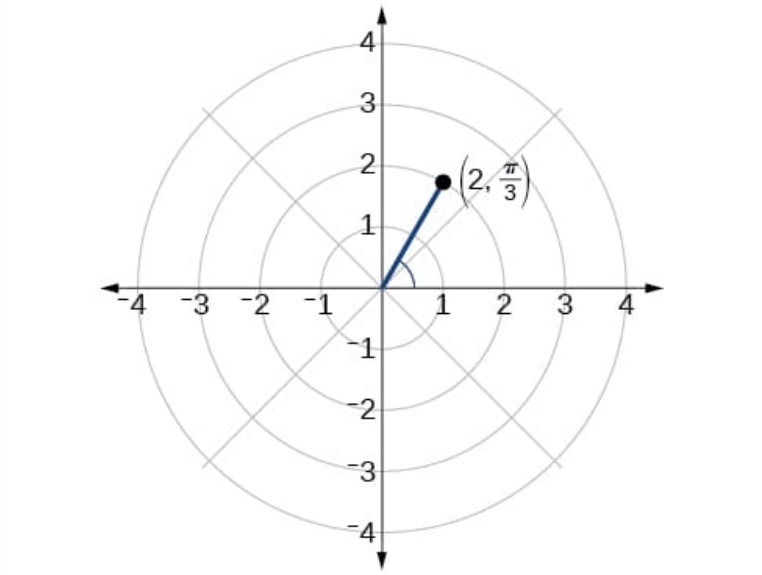


Figure $4.3.4$

Example $4.3.2$: Plotting a Point in the Polar Coordinate System with a Negative Component

Plot the point $\left(-2, \frac{\pi}{6}\right)$ on the polar grid.

Solution

We know that $\frac{\pi}{6}$ is located in the first quadrant. However, $r = -2$. We can approach plotting a point with a negative r in two ways:

1. Plot the point $\left(2, \frac{\pi}{6}\right)$ by moving $\frac{\pi}{6}$ in the counterclockwise direction and extending a directed line segment 2 units into the first quadrant. Then retrace the directed line segment back through the pole, and continue 2 units into the third quadrant;
2. Move $\frac{\pi}{6}$ in the counterclockwise direction, and draw the directed line segment from the pole 2 units in the negative direction, into the third quadrant.

See Figure $4.3.5a$. Compare this to the graph of the polar coordinate $(2, \pi 6)$ shown in Figure $4.3.5b$.

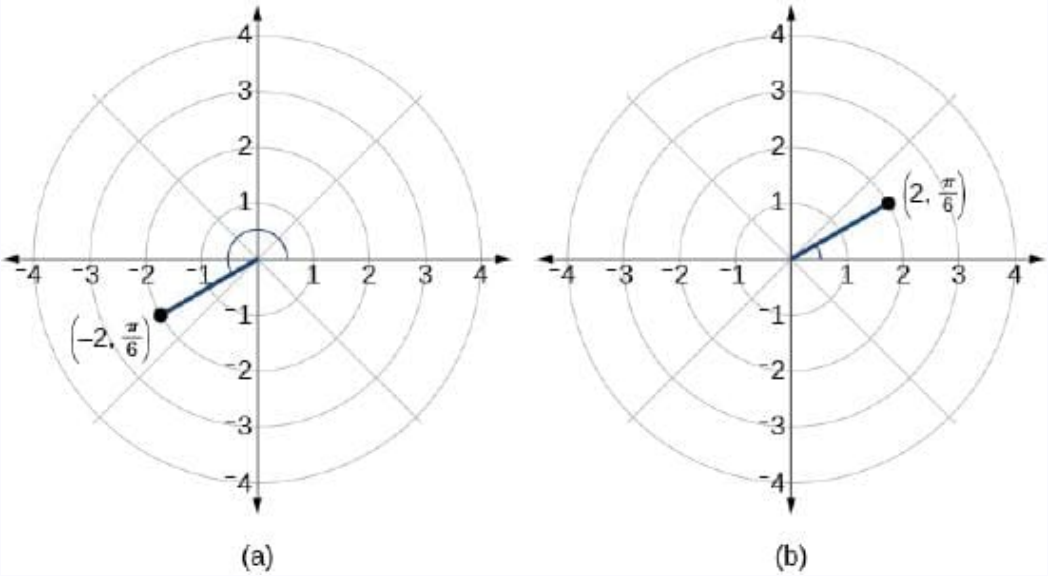


Figure 4.3.5

Exercise 4.3.2

Plot the points $\left(3,-\frac{\pi}{6}\right)$ and $\left(2,\frac{9\pi}{4}\right)$ on the same polar grid.

Answer

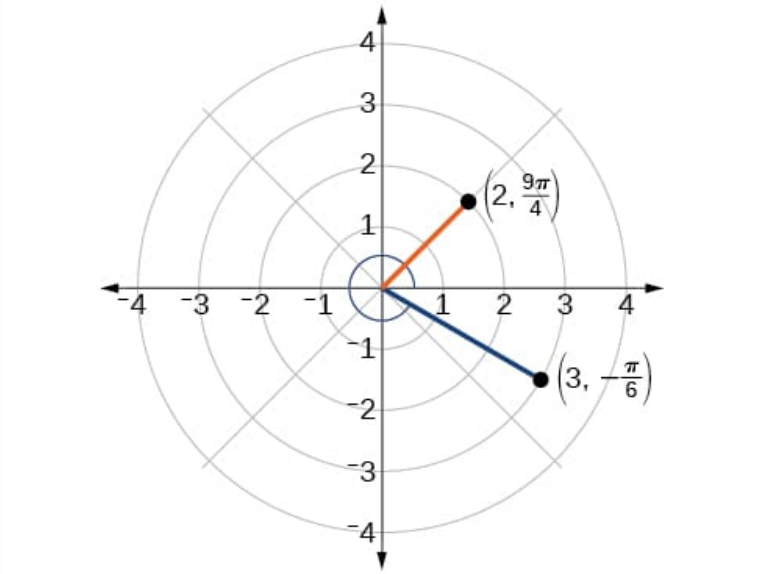


Figure 4.3.6

Converting from Polar Coordinates to Rectangular Coordinates

When given a set of **polar coordinates**, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables x, y, r, and θ.

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$$

Dropping a perpendicular from the point in the plane to the *x*-axis forms a right triangle, as illustrated in Figure $4.3.7$. An easy way to remember the equations above is to think of $\cos\theta$ as the adjacent side over the hypotenuse and $\sin\theta$ as the opposite side over the hypotenuse.

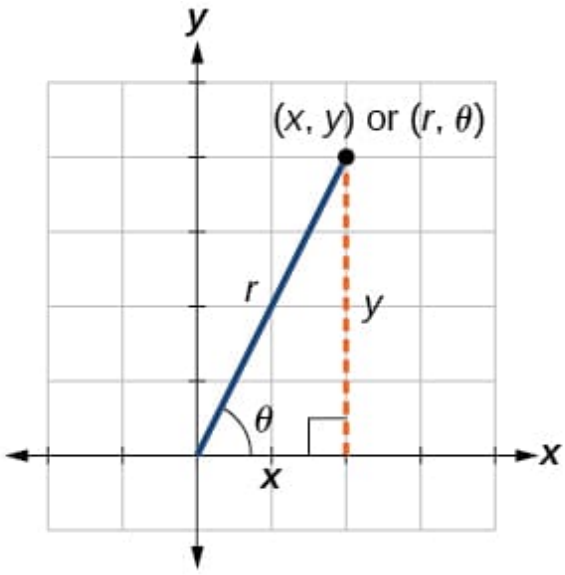


Figure $4.3.7$

CONVERTING FROM POLAR COORDINATES TO RECTANGULAR COORDINATES

To convert polar coordinates (r, θ) to rectangular coordinates (x, y), let

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta \tag{4.3.1}$$

$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta \tag{4.3.2}$$

How to: Given polar coordinates, convert to rectangular coordinates.

1. Given the polar coordinate (r, θ), write $x = r\cos\theta$ and $y = r\sin\theta$.
2. Evaluate $\cos\theta$ and $\sin\theta$.
3. Multiply $\cos\theta$ by r to find the x-coordinate of the rectangular form.
4. Multiply $\sin\theta$ by r to find the y-coordinate of the rectangular form.

Example $4.3.3A$: Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $\left(3, \frac{\pi}{2}\right)$ as rectangular coordinates.

Solution

Use the equivalent relationships.

$$
\begin{aligned}
x &= r\cos\theta \\
x &= 3\cos\frac{\pi}{2} \\
&= 0 \\
y &= r\sin\theta \\
y &= 3\sin\frac{\pi}{2} \\
&= 3
\end{aligned}
$$

The rectangular coordinates are $(0, 3)$. See Figure $4.3.8$.

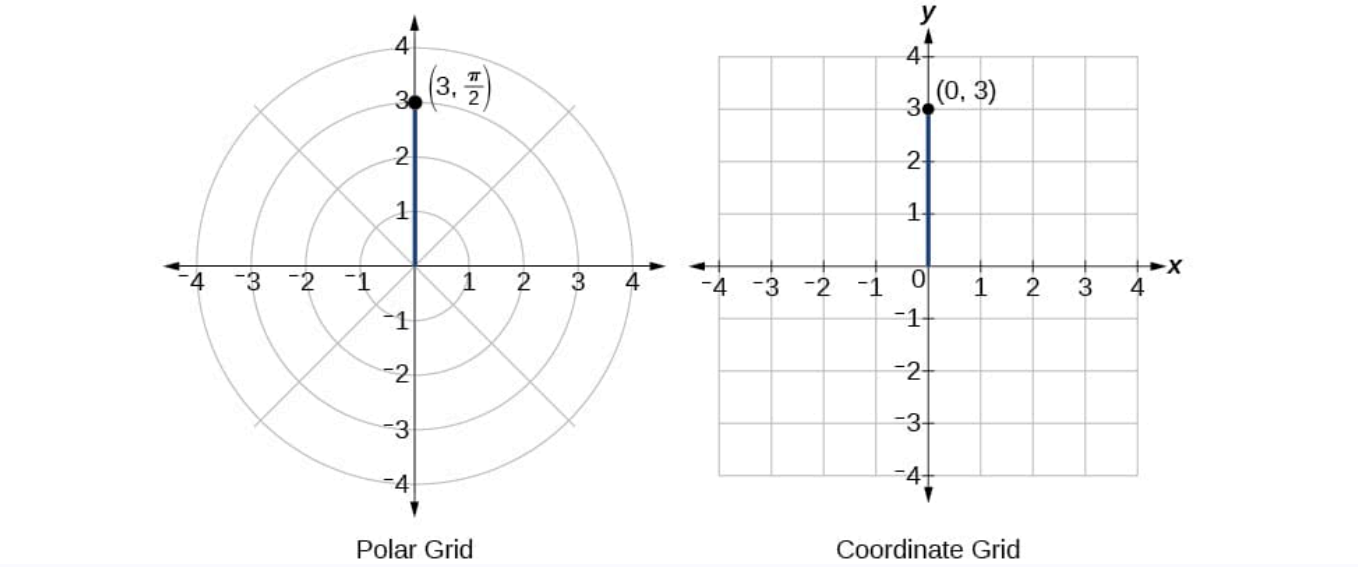


Figure $4.3.8$

Example $4.3.3B$: Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

Solution

See Figure $4.3.9$. Writing the polar coordinates as rectangular, we have

$$
\begin{aligned}
x &= r\cos\theta \\
x &= -2\cos(0) \\
&= -2 \\
y &= r\sin\theta \\
y &= -2\sin(0) \\
&= 0
\end{aligned}
$$

The rectangular coordinates are also $(-2, 0)$.

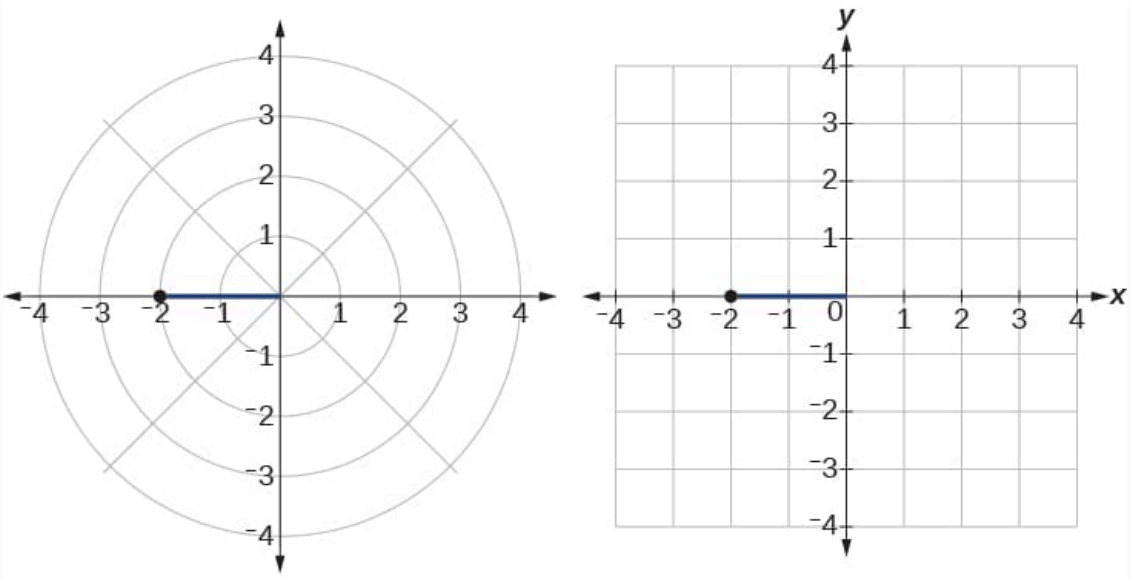


Figure 4.3.9

Exercise 4.3.3

Write the polar coordinates $\left(-1, \frac{2\pi}{3}\right)$ as rectangular coordinates.

Answer

$$(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

CONVERTING FROM RECTANGULAR COORDINATES TO POLAR COORDINATES

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in Figure 4.3.10.

$$\cos\theta = \frac{x}{r} \text{ or } x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \text{ or } y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

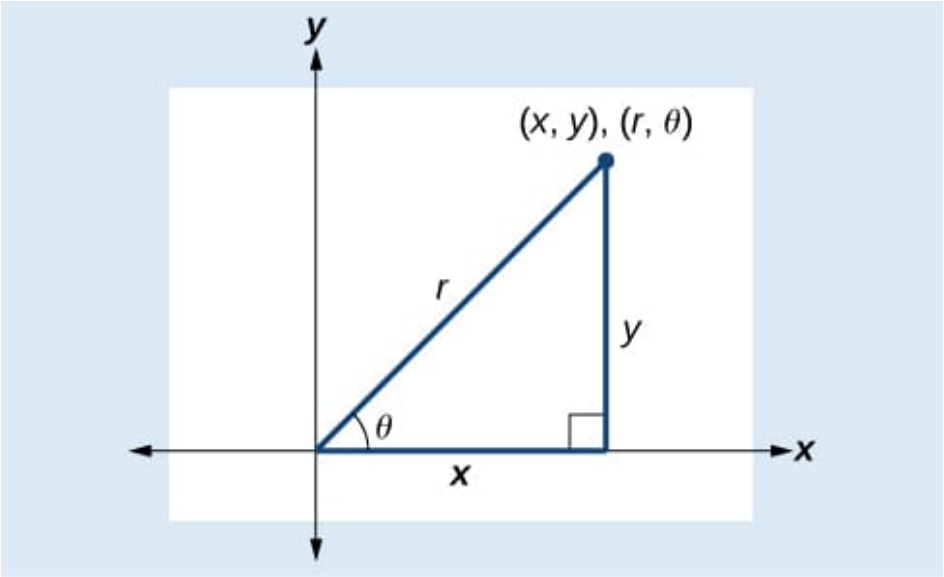


Figure 4.3.10

Example 4.3.4: Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates $(3, 3)$ to polar coordinates.

Solution

We see that the original point $(3, 3)$ is in the first quadrant. To find θ, use the formula $\tan\theta = \frac{y}{x}$. This gives

$$\begin{aligned} \tan\theta &= \frac{3}{3} \\ \tan\theta &= 1 \\ \tan^{-1}(1) &= \frac{\pi}{4} \end{aligned}$$

To find r, we substitute the values for x and y into the formula $r = \sqrt{x^2 + y^2}$. We know that r must be positive, as $\frac{\pi}{4}$ is in the first quadrant. Thus

$$\begin{aligned} r &= \sqrt{3^2 + 3^2} \\ r &= \sqrt{9 + 9} \\ r &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

So, $r = 3\sqrt{2}$ and $\theta = \frac{\pi}{4}$, giving us the polar point $(3\sqrt{2}, \frac{\pi}{4})$. See Figure 4.3.11.

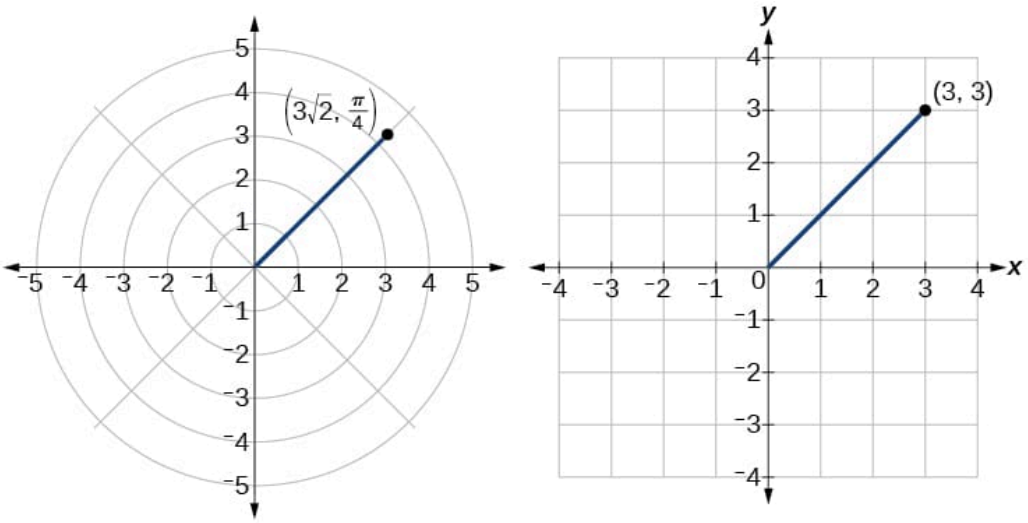


Figure 4.3.11

Analysis

There are other sets of polar coordinates that will be the same as our first solution. For example, the points $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ and $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ will coincide with the original solution of $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. The point $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ indicates a move further counterclockwise by π, which is directly opposite $\frac{\pi}{4}$. The radius is expressed as $-3\sqrt{2}$. However, the angle $\frac{5\pi}{4}$ is located in the third quadrant and, as r is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. The point $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ is a move further clockwise by $-\frac{7\pi}{4}$, from $\frac{\pi}{4}$. The radius, $3\sqrt{2}$, is the same.

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

How to: Given an equation in polar form, graph it using a graphing calculator

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: r_1, r_2,..., r_6.
3. Enter the polar equation, set equal to r.
4. Press **GRAPH**.

Example $4.3.5A$: Writing a Cartesian Equation in Polar Form

Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.

Solution

The goal is to eliminate x and y from the equation and introduce r and θ. Ideally, we would write the equation r as a function of θ. To obtain the polar form, we will use the relationships between (x, y) and (r, θ). Since $x = r\cos\theta$ and $y = r\sin\theta$, we can

substitute and solve for r.

$$\begin{aligned}(r\cos\theta)^2+(r\sin\theta)^2&=9\\ r^2\cos^2\theta+r^2\sin^2\theta&=9\\ r^2(\cos^2\theta+\sin^2\theta)&=9\\ r^2(1)&=9 \quad \text{Substitute } \cos^2\theta+\sin^2\theta=1\\ r&=\pm 3 \quad \text{Use the square root property.}\end{aligned}$$

Thus, $x^2+y^2=9$,$r=3$,and $r=-3$ should generate the same graph. See Figure $4.3.12$.

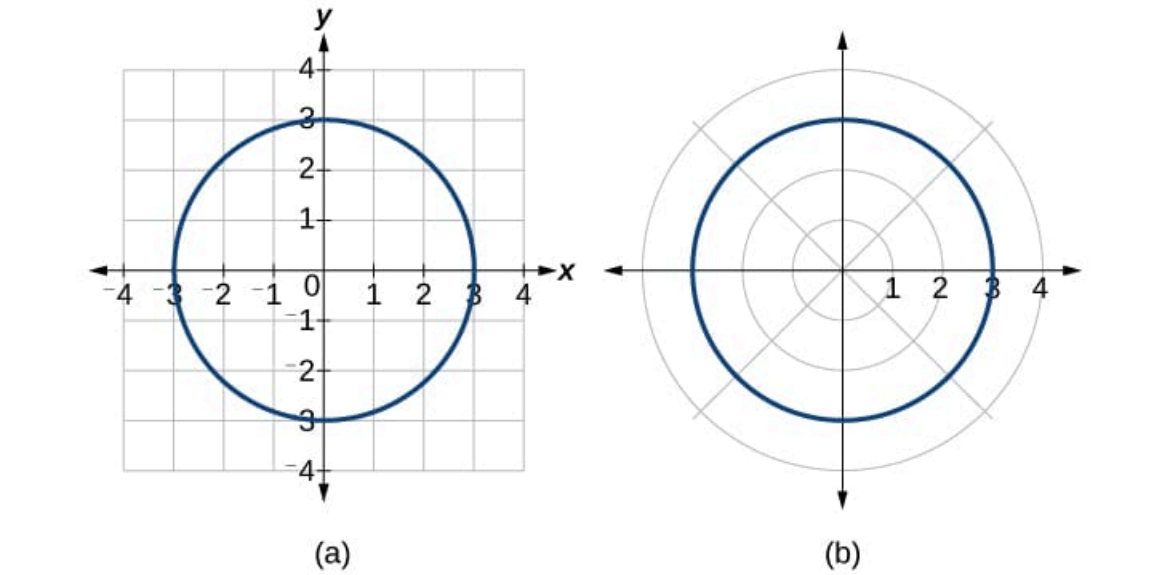


Figure $4.3.12$***:*** *(a) Cartesian form* $x^2+y^2=9$ *(b) Polar form* $r=3$

To graph a circle in rectangular form, we must first solve for y.

$$\begin{aligned}x^2+y^2&=9\\ y^2&=9-x^2\\ y&=\pm\sqrt{9-x^2}\end{aligned}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1=\sqrt{9-x^2}$ and $Y_2=-\sqrt{9-x^2}$. Press **GRAPH.**

Example $4.3.5B$: Rewriting a Cartesian Equation as a Polar Equation

Rewrite the **Cartesian equation** $x^2+y^2=6y$ as a polar equation.

Solution

This equation appears similar to the previous example, but it requires different steps to convert the equation.

We can still follow the same procedures we have already learned and make the following substitutions:

$r^2 = 6y$	Use $x^2 + y^2 = r^2$.
$r^2 = 6r\sin\theta$	Substitute $y = r\sin\theta$.
$r^2 - 6r\sin\theta = 0$	Set equal to 0.
$r(r - 6\sin\theta) = 0$	Factor and solve.
$r = 0$	We reject $r = 0$, as it only represents one point, $(0, 0)$.
or $r = 6\sin\theta$	

Therefore, the equations $x^2 + y^2 = 6y$ and $r = 6\sin\theta$ should give us the same graph. See Figure 4.3.13.

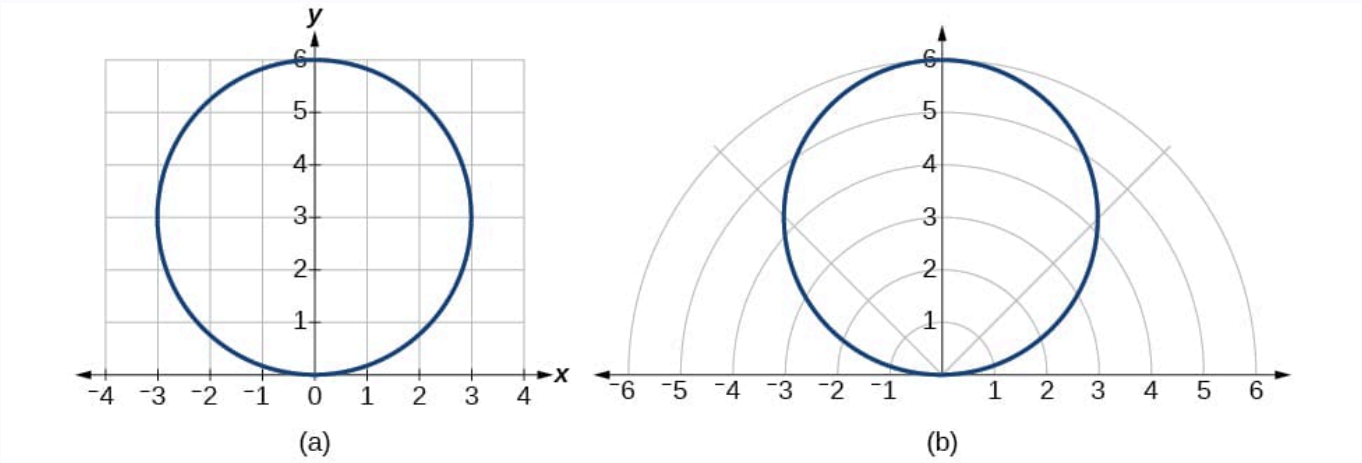


Figure 4.3.13***:*** *(a) Cartesian form $x^2 + y^2 = 6y$ (b) polar form $r = 6\sin\theta$*

The Cartesian or rectangular equation is plotted on the rectangular grid, and the **polar equation** is plotted on the polar grid. Clearly, the graphs are identical.

Exercise 4.3.4A:

Rewriting a Cartesian Equation in Polar Form

Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

Answer

We will use the relationships $x = r\cos\theta$ and $y = r\sin\theta$.

$y = 3x + 2$	
$r\sin\theta = 3r\cos\theta + 2$	
$r\sin\theta - 3r\cos\theta = 2$	
$r(\sin\theta - 3\cos\theta) = 2$	Isolate r.
$r = 2\sin\theta - 3\cos\theta$	Solve for r.

Exercise 4.3.4B:

Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form.

Answer

$r = \sqrt{3}$

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example $4.3.6A$: Graphing a Polar Equation by Converting to a Rectangular Equation

Covert the polar equation $r = 2\sec\theta$ to a rectangular equation, and draw its corresponding graph.

Solution

The conversion is

$$\begin{aligned} r &= 2\sec\theta \\ r &= \frac{2}{\cos\theta} \\ r\cos\theta &= 2 \\ x &= 2 \end{aligned}$$

Notice that the equation $r = 2\sec\theta$ drawn on the polar grid is clearly the same as the vertical line $x = 2$ drawn on the rectangular grid (see Figure $4.3.14$). Just as $x = c$ is the standard form for a vertical line in rectangular form, $r = c\sec\theta$ is the standard form for a vertical line in polar form.

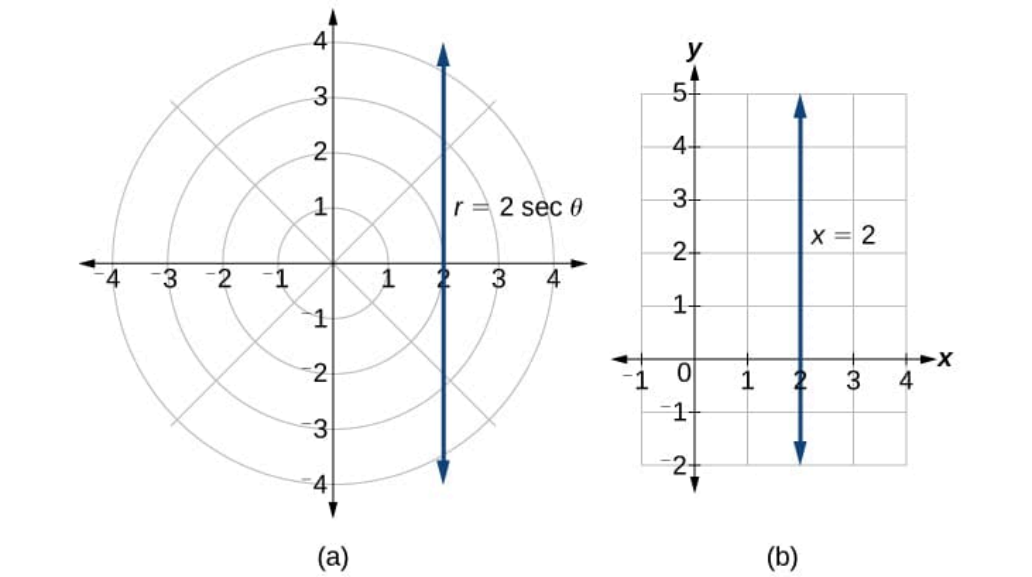


Figure $4.3.14$***:*** *(a) Polar grid (b) Rectangular coordinate system*

A similar discussion would demonstrate that the graph of the function $r = 2\csc\theta$ will be the horizontal line $y = 2$. In fact, $r = c\csc\theta$ is the standard form for a horizontal line in polar form, corresponding to the rectangular form $y = c$.

Example $4.3.6B$: Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \dfrac{3}{1 - 2\cos\theta}$ as a Cartesian equation.

Solution

The goal is to eliminate θ and r,and introduce x and y. We clear the fraction, and then use substitution. In order to replace r with x and y, we must use the expression $x^2 + y^2 = r^2$.

$$\begin{aligned} &= \frac{3}{1 - 2\cos\theta} \\ r(1 - 2\cos\theta) &= 3 \\ r\left(1 - 2\left(\frac{x}{r}\right)\right) &= 3 \quad \text{Use } \cos\theta = \frac{x}{r} \text{ to eliminate } \theta. \\ r - 2x &= 3 \\ r &= 3 + 2x \qquad \text{Isolate } r. \\ r^2 &= (3 + 2x)^2 \qquad \text{Square both sides.} \\ x^2 + y^2 &= (3 + 2x)^2 \qquad \text{Use } x^2 + y^2 = r^2. \end{aligned}$$

The Cartesian equation is $x^2 + y^2 = (3 + 2x)^2$. However, to graph it, especially using a graphing calculator or computer program, we want to isolate y.

$$\begin{aligned} x^2 + y^2 &= (3 + 2x)^2 \\ y^2 &= (3 + 2x)^2 - x^2 \\ y &= \pm(3 + 2x)^2 - x^2 \end{aligned}$$

When our entire equation has been changed from r and θ to x and y, we can stop, unless asked to solve for y or simplify. See Figure 4.3.15.

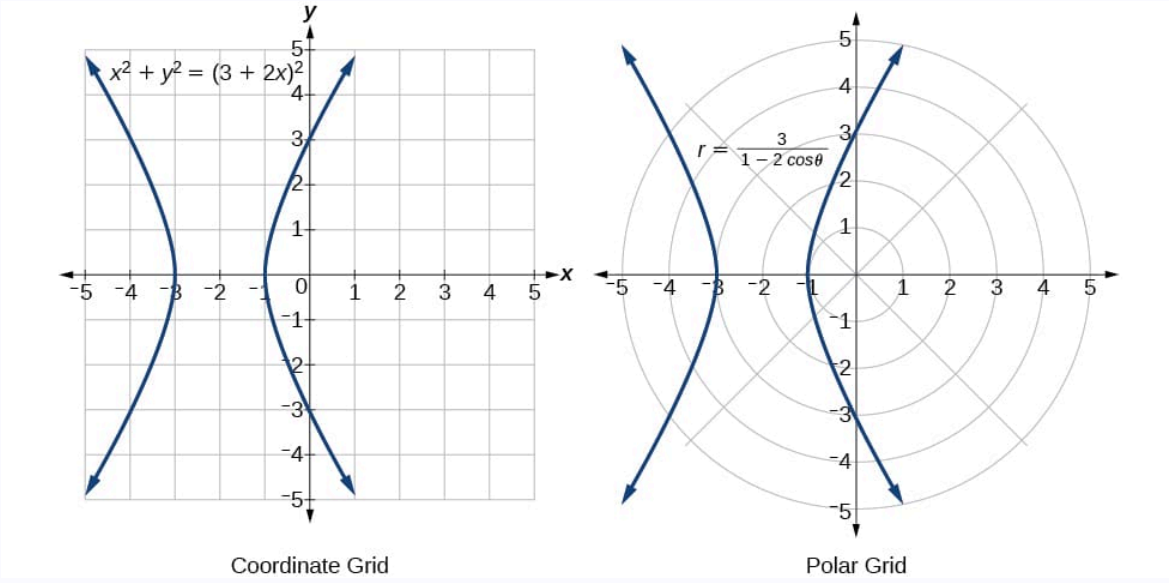


Figure 4.3.15

The "hour-glass" shape of the graph is called a *hyperbola*. Hyperbolas have many interesting geometric features and applications, which we will investigate further in Analytic Geometry.

Analysis

In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However, the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola's standard form. To do this, we can start with the initial equation.

$$
\begin{aligned}
x^2+y^2 &= (3+2x)^2 \\
x^2+y^2-(3+2x)^2 &= 0 \\
x^2+y^2-(9+12x+4x^2) &= 0 \\
x^2+y^2-9-12x-4x^2 &= 0 \\
-3x^2-12x+y^2 &= 9 && \text{Multiply through by } -1. \\
3x^2+12x-y^2 &= -9 \\
3(x^2+4x)-y2 &= -9 && \text{Organize terms to complete the square for } x. \\
3(x^2+4x+4)-y^2 &= -9+12 \\
3(x+2)^2-y^2 &= 3 \\
(x+2)^2-\frac{y^2}{3} &= 1
\end{aligned}
$$

Exercise 4.3.5

Rewrite the polar equation $r = 2\sin\theta$ in Cartesian form.

Answer

$x^2+y^2=2y$ or, in the standard form for a circle, $x^2+(y-1)^2=1$

Example 4.3.7: Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

Solution

$$
\begin{aligned}
r &= \sin(2\theta) && \text{Use the double angle identity for sine.} \\
r &= 2\sin\theta\cos\theta && \text{Use } \cos\theta=\frac{x}{r} \text{ and } \sin\theta=\frac{y}{r}. \\
r &= 2\frac{x}{r})(\frac{y}{r}) && \text{Simplify.} \\
r &= \frac{2xy}{r^2} && \text{Multiply both sides by } r^2. \\
r^3 &= 2xy \\
(x^2+y^2)^3 &= 2xy && \text{As } x^2+y^2=r^2, r=\sqrt{x^2+y^2}.
\end{aligned}
$$

This equation can also be written as

$$(x^2+y^2)^{\frac{3}{2}} = 2xy \text{ or } x^2+y^2=(2xy)^{\frac{2}{3}}$$

Media

Access these online resources for additional instruction and practice with polar coordinates.

- Introduction to Polar Coordinates
- Comparing Polar and Rectangular Coordinates

Key Equations

Conversion formulas	$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$ $\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$ $r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$

Key Concepts

- The polar grid is represented as a series of concentric circles radiating out from the pole, or origin.
- To plot a point in the form (r, θ), $\theta > 0$, move in a counterclockwise direction from the polar axis by an angle of θ, and then extend a directed line segment from the pole the length of r in the direction of θ. If θ is negative, move in a clockwise direction, and extend a directed line segment the length of r in the direction of θ. See Example $4.3.1$.
- If r is negative, extend the directed line segment in the opposite direction of θ. See Example $4.3.2$.
- To convert from polar coordinates to rectangular coordinates, use the formulas $x = r\cos\theta$ and $y = r\sin\theta$. See Example $4.3.3$ and Example $4.3.4$.
- To convert from rectangular coordinates to polar coordinates, use one or more of the formulas: $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, $\tan\theta = \frac{y}{x}$, and $r = \sqrt{x^2 + y^2}$. See Example $4.3.5$.
- Transforming equations between polar and rectangular forms means making the appropriate substitutions based on the available formulas, together with algebraic manipulations. See Example $4.3.6$, Example $4.3.7$, and Example $4.3.8$.
- Using the appropriate substitutions makes it possible to rewrite a polar equation as a rectangular equation, and then graph it in the rectangular plane. See Example $4.3.9$, Example $4.3.10$, and Example $4.3.11$.

Contributors

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4.4: Polar Coordinates - Graphs

Learning Objectives

- Test polar equations for symmetry.
- Graph polar equations by plotting points.

Keplar's First Law of Planetary Motion argues that the planets move through space in elliptical, periodic orbits about the sun, as shown in Figure 4.4.1. They are in constant motion, so fixing an exact position of any planet is valid only for a moment. In other words, we can fix only a planet's instantaneous position. This is one application of polar coordinates, represented as (r, θ). We interpret r as the distance from the sun and θ as the planet's angular bearing, or its direction from a fixed point on the sun. In this section, we will focus on the polar system and the graphs that are generated directly from polar coordinates.

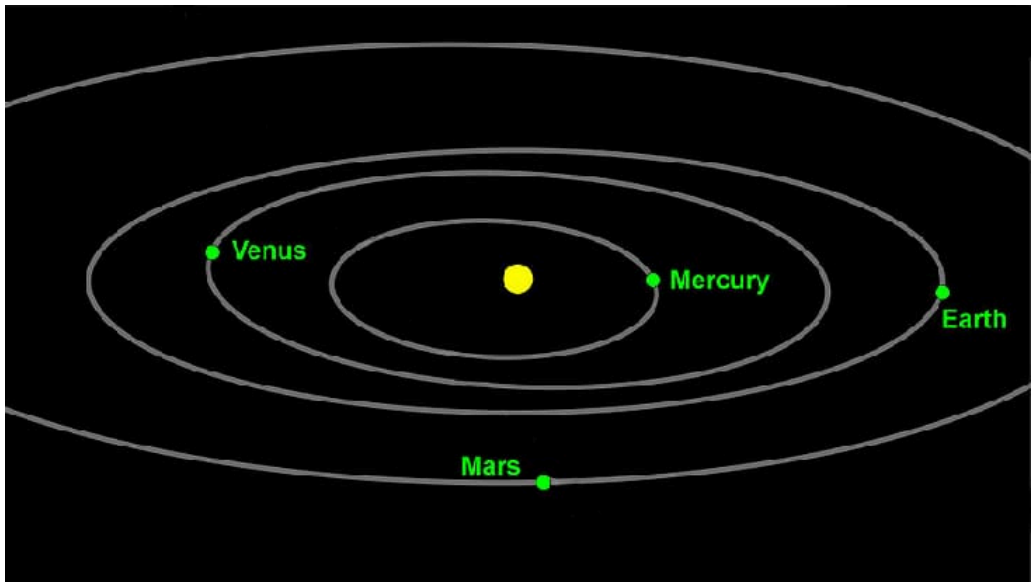


Figure 4.4.1: Planets follow elliptical paths as they orbit around the Sun. (credit: modification of work by NASA/JPL-Caltech)

Testing Polar Equations for Symmetry

Just as a rectangular equation such as $y = x^2$ describes the relationship between x and y on a Cartesian grid, a **polar equation** describes a relationship between r and θ on a polar grid. Recall that the coordinate pair (r, θ) indicates that we move counterclockwise from the polar axis (positive x-axis) by an angle of θ, and extend a ray from the pole (origin) r units in the direction of θ. All points that satisfy the polar equation are on the graph.

Symmetry is a property that helps us recognize and plot the graph of any equation. If an equation has a graph that is symmetric with respect to an axis, it means that if we folded the graph in half over that axis, the portion of the graph on one side would coincide with the portion on the other side. By performing three tests, we will see how to apply the properties of symmetry to polar equations. Further, we will use symmetry (in addition to plotting key points, zeros, and maximums of r) to determine the graph of a polar equation.

In the first test, we consider symmetry with respect to the line $\theta = \frac{\pi}{2}$ (y-axis). We replace (r, θ) with $(-r, -\theta)$ to determine if the new equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin\theta$;

$$\begin{aligned} r &= 2\sin\theta \\ -r &= 2\sin -\theta \qquad \text{Replace } (r,\theta) \text{ with } (-r,-\theta). \\ -r &= -2\sin\theta \qquad \text{Identity: } \sin(-\theta) = -\sin\theta. \\ r &= 2\sin\theta \qquad \text{Multiply both sides by } -1 \end{aligned}$$

This equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$.

In the second test, we consider symmetry with respect to the polar axis (x-axis). We replace (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ to determine equivalency between the tested equation and the original. For example, suppose we are given the equation $r = 1 - 2\cos\theta$.

$$\begin{aligned} r &= 1 - 2\cos\theta \\ r &= 1 - 2\cos(-\theta) \qquad \text{Replace } (r, \theta) \text{ with } (r, -\theta). \\ r &= 1 - 2\cos\theta \qquad \text{Even/Odd identity} \end{aligned}$$

The graph of this equation exhibits symmetry with respect to the polar axis.

In the third test, we consider symmetry with respect to the pole (origin). We replace (r, θ) with $(-r, \theta)$ to determine if the tested equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin(3\theta)$.

$$r = 2\sin(3\theta)$$

$$-r = 2\sin(3\theta)$$

The equation has failed the symmetry test, but that does not mean that it is not symmetric with respect to the pole. Passing one or more of the symmetry tests verifies that symmetry will be exhibited in a graph. However, failing the symmetry tests does not necessarily indicate that a graph will not be symmetric about the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole. In these instances, we can confirm that symmetry exists by plotting reflecting points across the apparent axis of symmetry or the pole. Testing for symmetry is a technique that simplifies the graphing of polar equations, but its application is not perfect.

Note: SYMMETRY TESTS

A polar equation describes a curve on the polar grid. The graph of a polar equation can be evaluated for three types of symmetry, as shown in Figure 4.4.2.

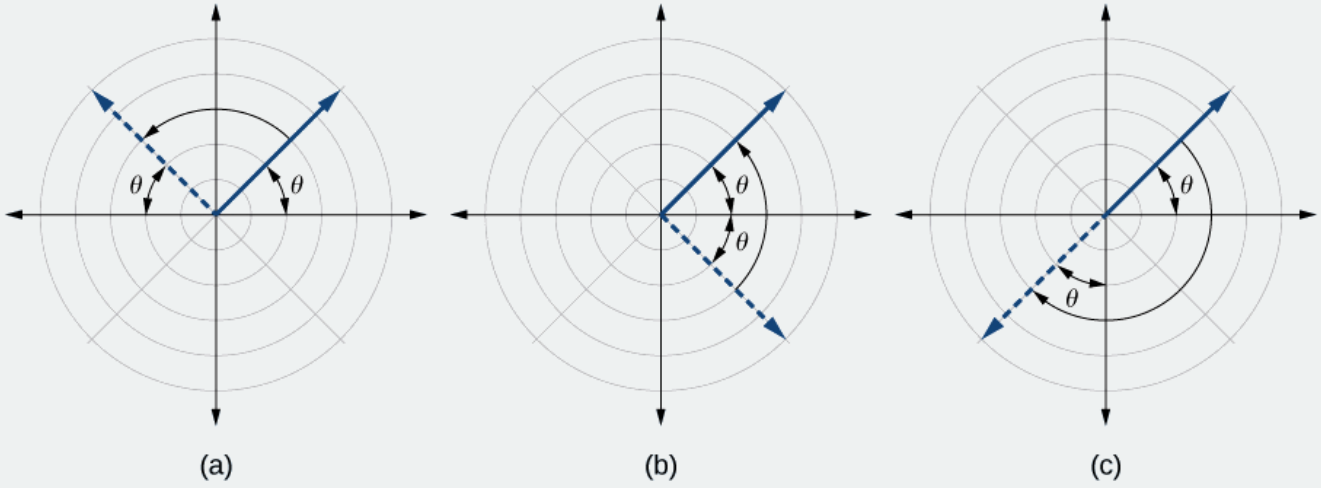


Figure 4.4.2: (a) A graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$ (y-axis) if replacing (r, θ) with $(-r, -\theta)$ yields an equivalent equation. (b) A graph is symmetric with respect to the polar axis (x-axis) if replacing (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ yields an equivalent equation. (c) A graph is symmetric with respect to the pole (origin) if replacing (r, θ) with $(-r, \theta)$ yields an equivalent equation.

How to: Given a polar equation, test for symmetry.

1. Substitute the appropriate combination of components for (r, θ): $(-r, -\theta)$ for $\theta = \frac{\pi}{2}$ symmetry; $(r, -\theta)$ for polar axis symmetry; and $(-r, \theta)$ for symmetry with respect to the pole.
2. If the resulting equations are equivalent in one or more of the tests, the graph produces the expected symmetry.

Example 4.4.1: Testing a Polar Equation for Symmetry

Test the equation $r = 2\sin\theta$ for symmetry.

Solution

Test for each of the three types of symmetry.

1) Replacing (r, θ) with $(-r, -\theta)$ yields the same result. Thus, the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.	$-r = 2\sin(-\theta)$ $-r = -2\sin\theta$ Even-odd identity $r = 2\sin\theta$ Multiply by -1 Passed
2) Replacing θ with $-\theta$ does not yield the same equation. Therefore, the graph fails the test and may or may not be symmetric with respect to the polar axis.	$r = 2\sin(-\theta)$ $r = -2\sin\theta$ Even-odd identity $r = -2\sin\theta \neq 2\sin\theta$ Failed
3) Replacing r with $-r$ changes the equation and fails the test. The graph may or may not be symmetric with respect to the pole.	$-r = 2\sin\theta$ $r = -2\sin\theta \neq 2\sin\theta$ Failed

Analysis

Using a graphing calculator, we can see that the equation $r = 2\sin\theta$ is a circle centered at $(0, 1)$ with radius $r = 1$ and is indeed symmetric to the line $\theta = \frac{\pi}{2}$. We can also see that the graph is not symmetric with the polar axis or the pole. See Figure 4.4.3.

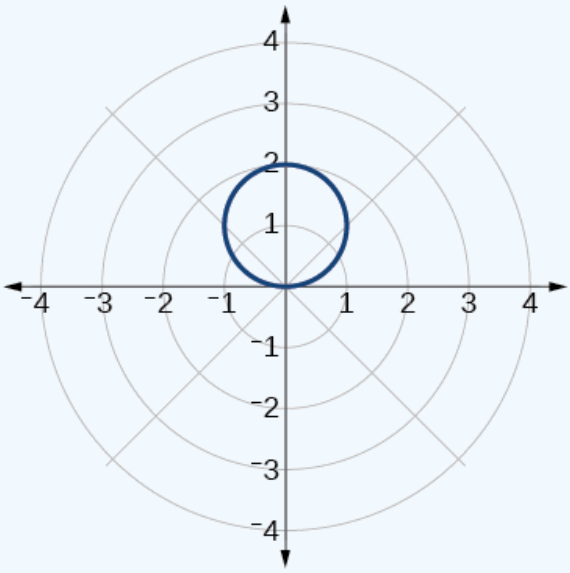


Figure 4.4.3

Exercise 4.4.1

Test the equation for symmetry: $r = -2\cos\theta$.

Answer available at the end of this section.

Graphing Polar Equations by Plotting Points

To graph in the rectangular coordinate system we construct a table of x and y values. To graph in the polar coordinate system we construct a table of θ and r values. We enter values of θ into a polar equation and calculate r. However, using the properties of symmetry and finding key values of θ and r means fewer calculations will be needed.

Finding Zeros and Maxima

To find the zeros of a polar equation, we solve for the values of θ that result in $r = 0$. Recall that, to find the zeros of polynomial functions, we set the equation equal to zero and then solve for x. We use the same process for polar equations. Set $r = 0$, and solve for θ.

For many of the forms we will encounter, the maximum value of a polar equation is found by substituting those values of θ into the equation that result in the maximum value of the trigonometric functions. Consider $r = 5\cos\theta$; the maximum distance between the curve and the pole is 5 units. The maximum value of the cosine function is 1 when $\theta = 0$, so our polar equation is $5\cos\theta$, and the value $\theta = 0$ will yield the maximum $|r|$.

Similarly, the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, and if our polar equation is $r = 5\sin\theta$, the value $\theta = \frac{\pi}{2}$ will yield the maximum $|r|$. We may find additional information by calculating values of r when $\theta = 0$. These points would be polar axis intercepts, which may be helpful in drawing the graph and identifying the curve of a polar equation.

Example 4.4.2: Finding Zeros and Maximum Values for a Polar Equation

Using the equation in Example 4.4.1, find the zeros and maximum $|r|$ and, if necessary, the polar axis intercepts of $r = 2\sin\theta$.

Solution

To find the zeros, set r equal to zero and solve for θ.

$$\begin{aligned} 2\sin\theta &= 0 \\ \sin\theta &= 0 \\ \theta &= \sin^{-1}0 \\ \theta &= n\pi \qquad \text{where n is an integer} \end{aligned}$$

Substitute any one of the θ values into the equation. We will use 0.

$$\begin{aligned} r &= 2\sin(0) \\ r &= 0 \end{aligned}$$

The points $(0,0)$ and $(0,\pm n\pi)$ are the zeros of the equation. They all coincide, so only one point is visible on the graph. This point is also the only polar axis intercept.

To find the maximum value of the equation, look at the maximum value of the trigonometric function $\sin\theta$, which occurs when $\theta = \frac{\pi}{2} \pm 2k\pi$ resulting in $\sin\left(\frac{\pi}{2}\right) = 1$. Substitute $\frac{\pi}{2}$ for θ.

$$\begin{aligned} r &= 2\sin\left(\frac{\pi}{2}\right) \\ r &= 2(1) \\ r &= 2 \end{aligned}$$

Analysis

The point $\left(2, \frac{\pi}{2}\right)$ will be the maximum value on the graph. Let's plot a few more points to verify the graph of a circle. See Table $4.4.1$ and Figure $4.4.4$.

Table $4.4.1$

θ	$r = 2\sin\theta$	r
0	$r = 2\sin(0) = 0$	0
$\frac{\pi}{6}$	$r = 2\sin\left(\frac{\pi}{6}\right) = 1$	1
$\frac{\pi}{3}$	$r = 2\sin\left(\frac{\pi}{3}\right) \approx 1.73$	1.73
$\frac{pi}{2}$	$r = 2\sin\left(\frac{\pi}{2}\right) = 2$	2
$\frac{2\pi}{3}$	$r = 2\sin\left(\frac{2\pi}{3}\right) \approx 1.73$	1.73
$\frac{5\pi}{6}$	$r = 2\sin\left(\frac{5\pi}{6}\right) = 1$	1
π	$r = 2\sin(\pi) = 0$	0

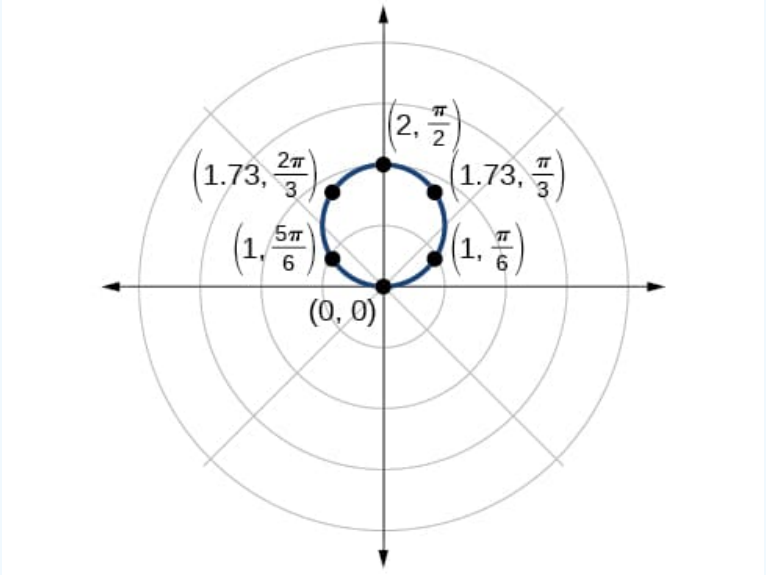


Figure 4.4.4

Exercise 4.4.2

Without converting to Cartesian coordinates, test the given equation for symmetry and find the zeros and maximum values of $|r|$: $r = 3\cos\theta$.

Answer available at the end of this section.

Investigating Circles

Now we have seen the equation of a circle in the polar coordinate system. In the last two examples, the same equation was used to illustrate the properties of symmetry and demonstrate how to find the zeros, maximum values, and plotted points that produced the graphs. However, the circle is only one of many shapes in the set of polar curves.

There are five classic polar curves**: cardioids**, **limaçons, lemniscates, rose curves**, and **Archimedes' spirals**. We will briefly touch on the polar formulas for the circle before moving on to the classic curves and their variations.

FORMULAS FOR THE EQUATION OF A CIRCLE

Some of the formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$, wherea a is the diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is $\frac{|a|}{2}$, or one-half the diameter. For $r = a\cos\theta$, the center is $\left(\frac{a}{2}, 0\right)$. For $r = a\sin\theta$, the center is $\left(\frac{a}{2}, \pi\right)$. Figure 4.4.5 shows the graphs of these four circles.

Four graphs side by side. All have radius absolute value of a / 2. First is r=acos(theta), a
0. The center is at (a/2,0). Second is r=acos(theta), a<0. The center is at (a/2,0). Third is r=asin(theta), a>0. The center is at (a/2, pi). Fourth is r=asin(theta), a<0. The center is at (a/2, pi)." src="/@api/deki/files/7429/CNX_Precalc_Figure_08_04_005new.jpg">

Figure 4.4.5

Example 4.4.3: Sketching the Graph of a Polar Equation for a Circle

Sketch the graph of $r = 4\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph is symmetric about the polar axis. Next, we find the zeros and maximum $|r|$ for $r = 4\cos\theta$. First, set $r = 0$, and solve for θ. Thus, a zero occurs at $\theta = \frac{\pi}{2} \pm k\pi$. A key point to plot is $\left(0, \frac{\pi}{2}\right)$.

To find the maximum value of r, note that the maximum value of the cosine function is 1 when $\theta = 0 \pm 2k\pi$. Substitute $\theta = 0$ into the equation:

$$\begin{aligned} r &= 4\cos\theta \\ r &= 4\cos(0) \\ r &= 4(1) \\ &= 4 \end{aligned}$$

The maximum value of the equation is 4. A key point to plot is $(4, 0)$.

As $r = 4\cos\theta$ is symmetric with respect to the polar axis, we only need to calculate r-values for θ over the interval $[0, \pi]$. Points in the upper quadrant can then be reflected to the lower quadrant. Make a table of values similar to Table $4.4.2$. The graph is shown in Figure $4.4.6$.

Table $4.4.2$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	3.46	2.83	2	0	-2	-2.83	-3.46	4

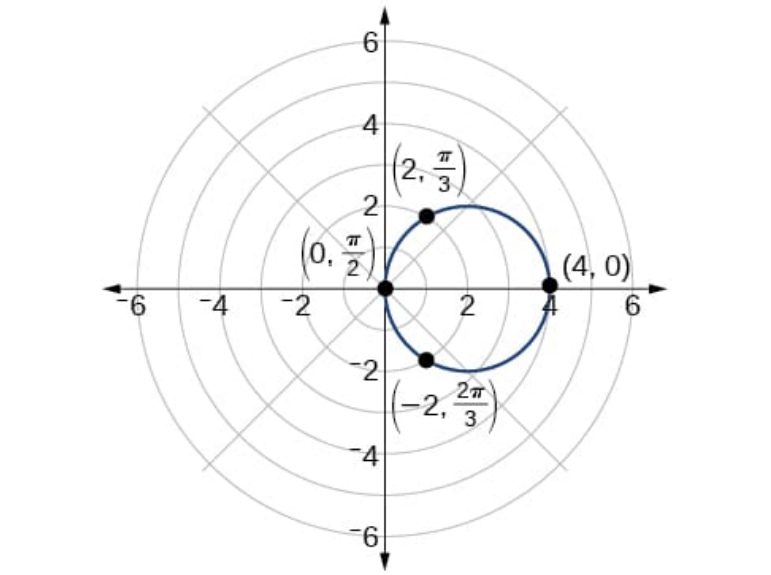


Figure $4.4.6$

Investigating Cardioids

While translating from polar coordinates to Cartesian coordinates may seem simpler in some instances, graphing the classic curves is actually less complicated in the polar system. The next curve is called a cardioid, as it resembles a heart. This shape is often included with the family of curves called limaçons, but here we will discuss the cardioid on its own.

FORMULAS FOR A CARDIOID

The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. The cardioid graph passes through the pole, as we can see in Figure $4.4.7$.

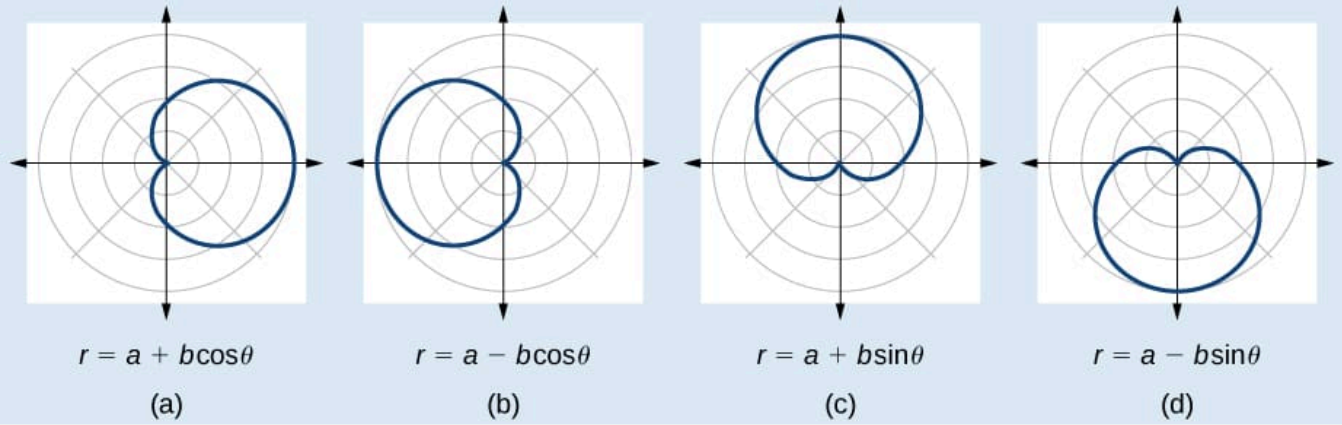


Figure 4.4.7

How to: Given the polar equation of a cardioid, sketch its graph

1. Check equation for the three types of symmetry.
2. Find the zeros. Set $r = 0$.
3. Find the maximum value of the equation according to the maximum value of the trigonometric expression.
4. Make a table of values for r and θ.
5. Plot the points and sketch the graph.

Example 4.4.4: Sketching the Graph of a Cardioid

Sketch the graph of $r = 2 + 2\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph of this equation will be symmetric about the polar axis. Next, we find the zeros and maximums. Setting $r = 0$, we have $\theta = \pi + 2k\pi$. The zero of the equation is located at $(0, \pi)$. The graph passes through this point.

The maximum value of $r = 2 + 2\cos\theta$ occurs when $\cos\theta$ is a maximum, which is when $\cos\theta = 1$ or when $\theta = 0$. Substitute $\theta = 0$ into the equation, and solve for r.

$$\begin{aligned} r &= 2 + 2\cos(0) \\ r &= 2 + 2(1) \\ &= 4 \end{aligned}$$

The point $(4, 0)$ is the maximum value on the graph.

We found that the polar equation is symmetric with respect to the polar axis, but as it extends to all four quadrants, we need to plot values over the interval $[0, \pi]$. The upper portion of the graph is then reflected over the polar axis. Next, we make a table of values, as in Table 4.4.3, and then we plot the points and draw the graph. See Figure 4.4.8.

Table 4.4.3

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	3.41	2	1	0

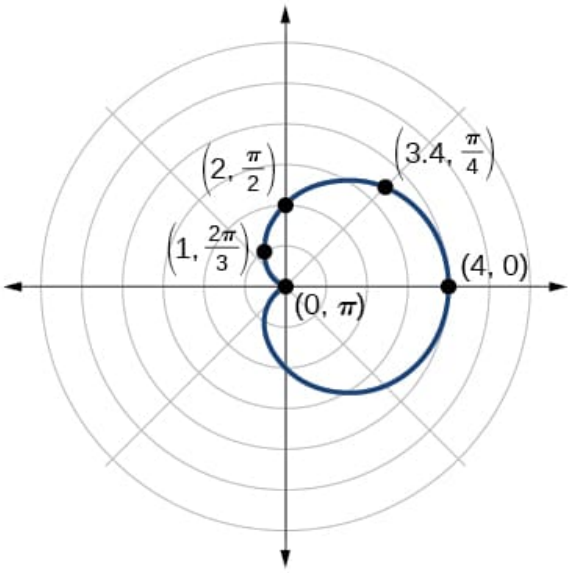


Figure 4.4.8

Investigating Limaçons

The word *limaçon* is Old French for "snail," a name that describes the shape of the graph. As mentioned earlier, the cardioid is a member of the limaçon family, and we can see the similarities in the graphs. The other images in this category include the one-loop limaçon and the two-loop (or inner-loop) limaçon. **One-loop limaçons** are sometimes referred to as dimpled limaçons when $1 < \frac{a}{b} < 2$ and convex limaçons when $\frac{a}{b} \geq 2$.

FORMULAS FOR ONE-LOOP LIMAÇONS

The formulas that produce the graph of a dimpled one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $1 < ab < 2$. All four graphs are shown in Figure 4.4.9.

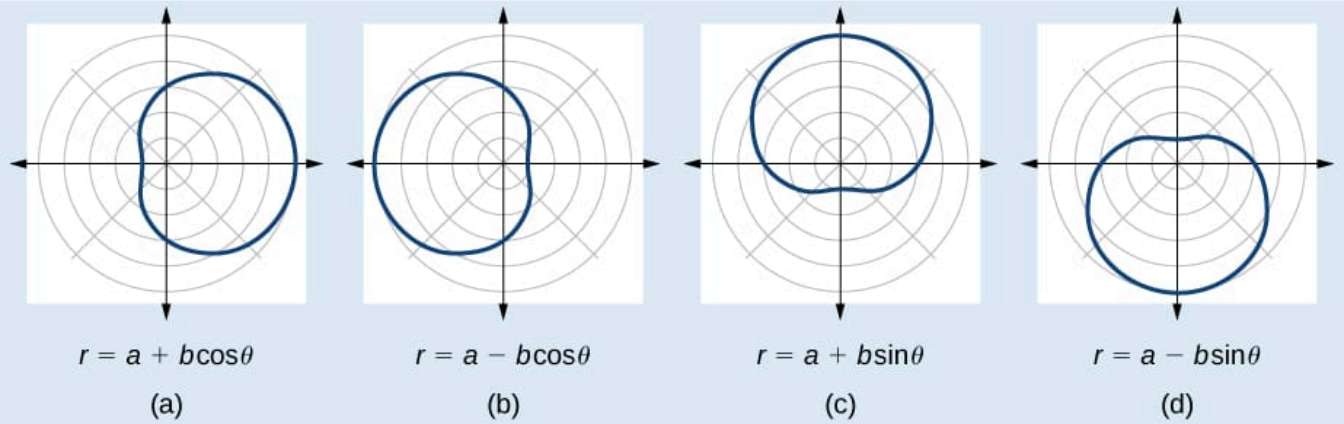


Figure 4.4.9

How to: Given a polar equation for a one-loop limaçon, sketch the graph

1. Test the equation for symmetry. Remember that failing a symmetry test does not mean that the shape will not exhibit symmetry. Often the symmetry may reveal itself when the points are plotted.
2. Find the zeros.
3. Find the maximum values according to the trigonometric expression.

4. Make a table.
5. Plot the points and sketch the graph.

Example 4.4.5: Sketching the Graph of a One-Loop Limaçon

Graph the equation $r = 4 - 3\sin\theta$.

Solution

First, testing the equation for symmetry, we find that it fails all three symmetry tests, meaning that the graph may or may not exhibit symmetry, so we cannot use the symmetry to help us graph it. However, this equation has a graph that clearly displays symmetry with respect to the line $\theta = \frac{\pi}{2}$, yet it fails all the three symmetry tests. A graphing calculator will immediately illustrate the graph's reflective quality.

Next, we find the zeros and maximum, and plot the reflecting points to verify any symmetry. Setting $r = 0$ results in θ being undefined. What does this mean? How could θ be undefined? The angle θ is undefined for any value of $\sin\theta > 1$. Therefore, θ is undefined because there is no value of θ for which $\sin\theta > 1$. Consequently, the graph does not pass through the pole. Perhaps the graph does cross the polar axis, but not at the pole. We can investigate other intercepts by calculating r when $\theta = 0$.

$$\begin{aligned} r(0) &= 4 - 3\sin(0) \\ r &= 4 - 3 \cdot 0 \\ &= 4 \end{aligned}$$

So, there is at least one polar axis intercept at $(4, 0)$.

Next, as the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, we will substitute $\theta = \frac{\pi}{2}$ into the equation and solve for r. Thus, $r = 1$.

Make a table of the coordinates similar to Table 4.4.4.

Table 4.4.4

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	4	2.5	1.4	1	1.4	2.5	4	5.5	6.6	7	6.6	5.5	4

The graph is shown in Figure 4.4.10.

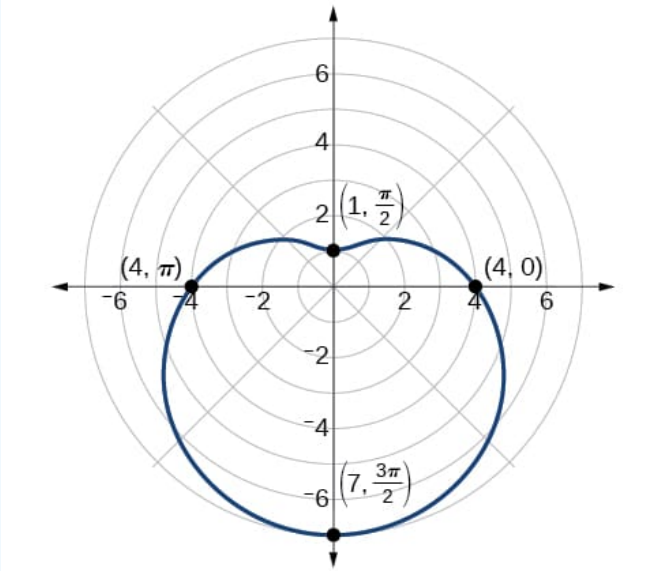


Figure 4.4.10: One-loop limaçon

Analysis

This is an example of a curve for which making a table of values is critical to producing an accurate graph. The symmetry tests fail; the zero is undefined. While it may be apparent that an equation involving $\sin\theta$ is likely symmetric with respect to the line $\theta = \frac{\pi}{2}$, evaluating more points helps to verify that the graph is correct.

Exercise 4.4.3

Sketch the graph of $r = 3 - 2\cos\theta$.

Answer available at the end of this section.

Another type of limaçon, the **inner-loop limaçon**, is named for the loop formed inside the general limaçon shape. It was discovered by the German artist Albrecht Dürer(1471-1528), who revealed a method for drawing the inner-loop limaçon in his 1525 book *Underweysung der Messing*. A century later, the father of mathematician Blaise Pascal, Étienne Pascal(1588-1651), rediscovered it.

FORMULAS FOR INNER-LOOP LIMAÇONS

The formulas that generate the inner-loop limaçons are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $a < b$. The graph of the inner-loop limaçon passes through the pole twice: once for the outer loop, and once for the inner loop. See Figure 10.5.12 for the graphs.

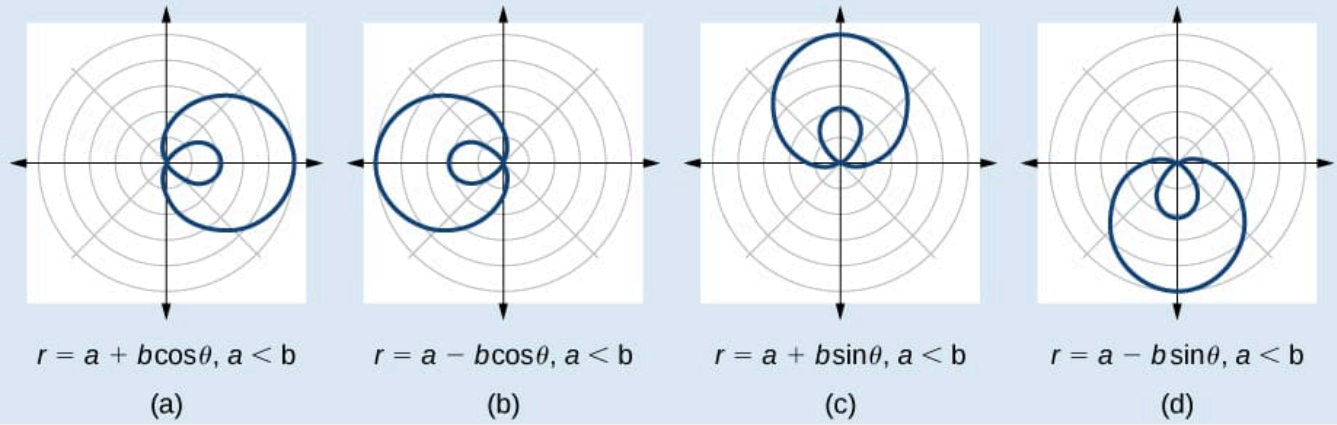


Figure 4.4.12

Example 4.4.6: Sketching the Graph of an Inner-Loop Limaçon

Sketch the graph of $r = 2 + 5\cos\theta$.

Solution

Testing for symmetry, we find that the graph of the equation is symmetric about the polar axis. Next, finding the zeros reveals that when $r = 0$, $\theta = 1.98$. The maximum $|r|$ is found when $\cos\theta = 1$ or when $\theta = 0$. Thus, the maximum is found at the point $(7, 0)$.

Even though we have found symmetry, the zero, and the maximum, plotting more points will help to define the shape, and then a pattern will emerge. See Table 4.4.5.

Table 4.4.5

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	7	6.3	4.5	2	-0.5	-2.3	-3	-2.3	-0.5	2	4.5	6.3	7

As expected, the values begin to repeat after $\theta = \pi$. The graph is shown in Figure 4.4.13.

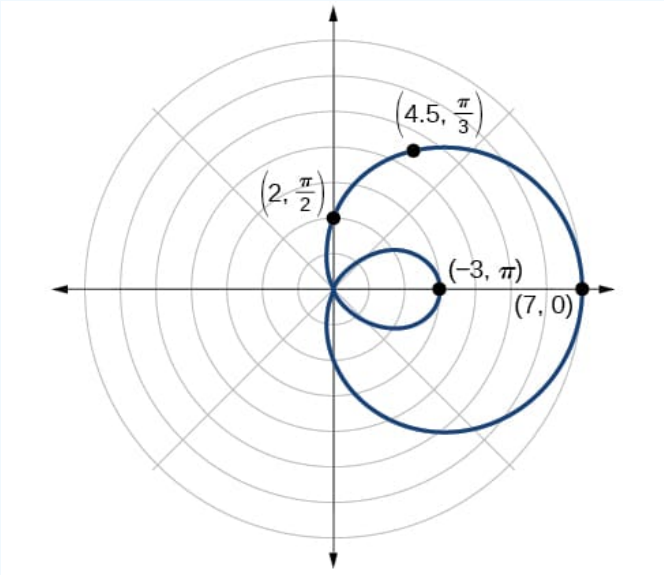


Figure 4.4.13: Inner-loop limaçon

Investigating Lemniscates

The lemniscate is a polar curve resembling the infinity symbol ∞ or a figure 8. Centered at the pole, a lemniscate is symmetrical by definition.

FORMULAS FOR LEMNISCATES

The formulas that generate the graph of a lemniscate are given by $r^2 = a^2 \cos 2\theta$ and $r^2 = a^2 \sin 2\theta$ where $a \neq 0$. The formula $r^2 = a^2 \sin 2\theta$ is symmetric with respect to the pole. The formula $r^2 = a^2 \cos 2\theta$ is symmetric with respect to the pole, the line $\theta = \dfrac{\pi}{2}$, and the polar axis. See Figure $4.4.14$ for the graphs.

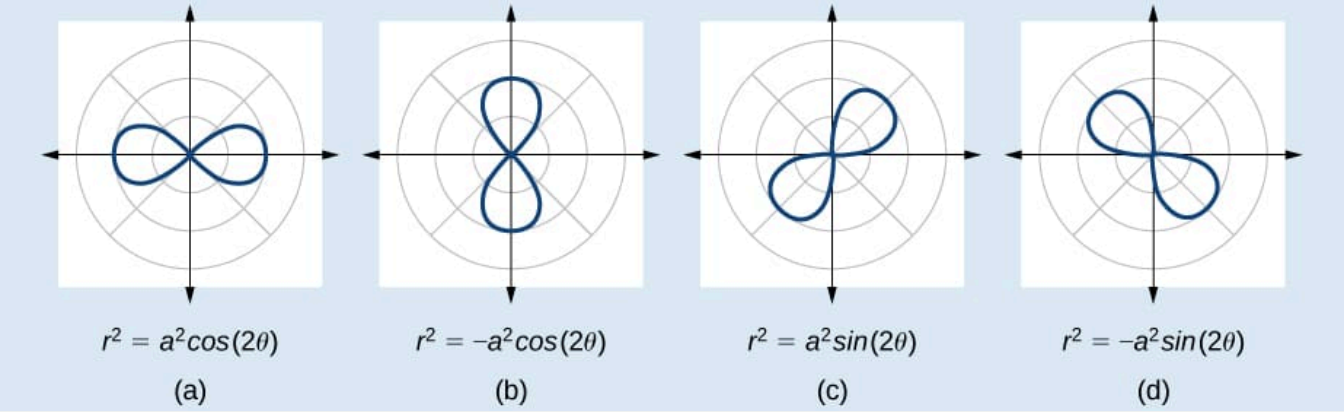


Figure $4.4.14$

Example $4.4.7$: Sketching the Graph of a Lemniscate

Sketch the graph of $r^2 = 4 \cos 2\theta$.

Solution

The equation exhibits symmetry with respect to the line $\theta = \dfrac{\pi}{2}$, the polar axis, and the pole.

Let's find the zeros. It should be routine by now, but we will approach this equation a little differently by making the substitution $u = 2\theta$.

$$\begin{aligned} 0 &= 4 \cos 2\theta \\ 0 &= 4 \cos u \\ 0 &= \cos u \\ \cos^{-1} 0 &= \frac{\pi}{2} \\ u &= \frac{\pi}{2} \qquad \text{Substitute } 2\theta \text{ back in for } u. \\ 2\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{4} \end{aligned}$$

So, the point $\left(0, \dfrac{\pi}{4}\right)$ is a zero of the equation.

Now let's find the maximum value. Since the maximum of $\cos u = 1$ when $u = 0$, the maximum $\cos 2\theta = 1$ when $2\theta = 0$. Thus,

$$\begin{aligned} r^2 &= 4\cos(0) \\ r^2 &= 4(1) \\ r^2 &= 4 \\ r &= \pm 4 \\ &= 2 \end{aligned}$$

We have a maximum at $(2, 0)$. Since this graph is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis, we only need to plot points in the first quadrant.

Make a table similar to Table $4.4.6$.

Table $4.4.6$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$\sqrt{2}$	0

Plot the points on the graph, such as the one shown in Figure $4.4.15$.

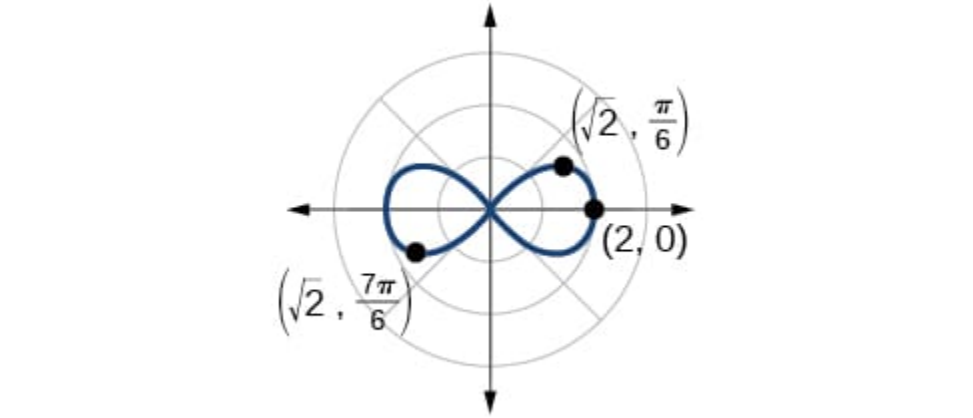


Figure $4.4.15$: Lemniscate

Analysis

Making a substitution such as $u = 2\theta$ is a common practice in mathematics because it can make calculations simpler. However, we must not forget to replace the substitution term with the original term at the end, and then solve for the unknown.

Some of the points on this graph may not show up using the Trace function on the TI-84 graphing calculator, and the calculator table may show an error for these same points of r. This is because there are no real square roots for these values of θ. In other words, the corresponding r-values of $\sqrt{4\cos(2\theta)}$ are complex numbers because there is a negative number under the radical.

Investigating Rose Curves

The next type of polar equation produces a petal-like shape called a rose curve. Although the graphs look complex, a simple polar equation generates the pattern.

ROSE CURVES

The formulas that generate the graph of a **rose curve** are given by $r = a\cos n\theta$ and $r = a\sin n\theta$ where $a \neq 0$. If n is even, the curve has $2n$ petals. If n is odd, the curve has n petals. See Figure $4.4.16$.

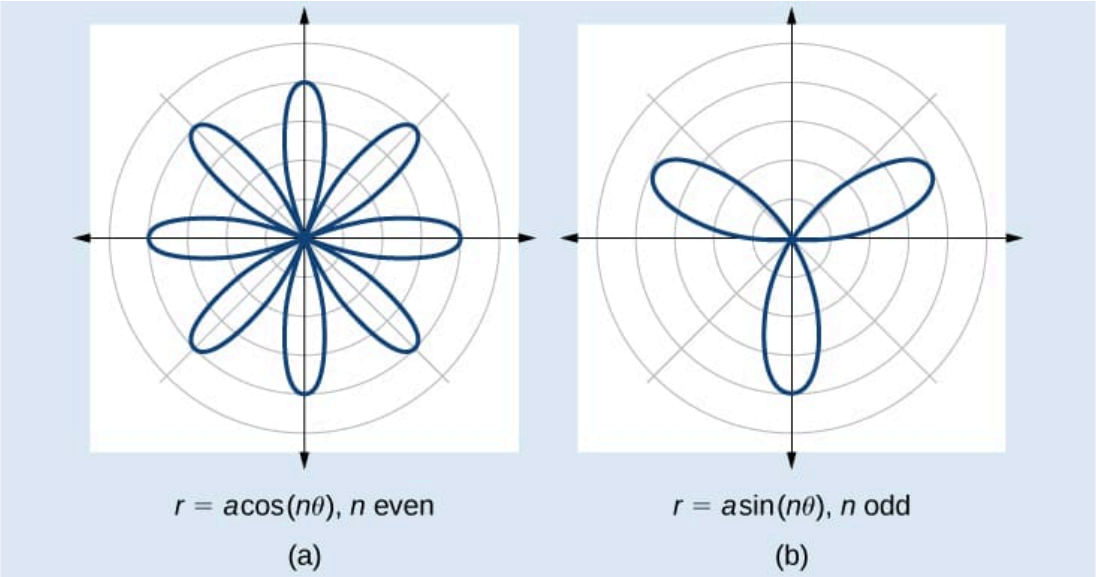


Figure 4.4.16

Example 4.4.8: Sketching the Graph of a Rose Curve (n Even)

Sketch the graph of $r = 2\cos 4\theta$.

Solution

Testing for symmetry, we find again that the symmetry tests do not tell the whole story. The graph is not only symmetric with respect to the polar axis, but also with respect to the line $\theta = \frac{\pi}{2}$ and the pole.

Now we will find the zeros. First make the substitution $u = 4\theta$.

$$\begin{aligned} 0 &= 2\cos 4\theta \\ 0 &= \cos 4\theta \\ 0 &= \cos u \\ \cos^{-1} 0 &= u \\ u &= \frac{\pi}{2} \\ 4\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{8} \end{aligned}$$

The zero is $\theta = \frac{\pi}{8}$. The point $\left(0, \frac{\pi}{8}\right)$ is on the curve.

Next, we find the maximum $|r|$. We know that the maximum value of $\cos u = 1$ when $\theta = 0$. Thus,

$$\begin{aligned} r &= 2\cos(4 \cdot 0) \\ r &= 2\cos(0) \\ r &= 2(1) \\ &= 2 \end{aligned}$$

The point $(2, 0)$ is on the curve.

The graph of the rose curve has unique properties, which are revealed in Table 4.4.7.

Table 4.4.7

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$5\pi 8$	$3\pi 4$
r	2	0	-2	0	2	0	-2

As $r = 0$ when $\theta = \frac{\pi}{8}$, it makes sense to divide values in the table by $\frac{\pi}{8}$ units. A definite pattern emerges. Look at the range of r-values: $2, 0, -2, 0$ and so on. This represents the development of the curve one petal at a time. Starting at $r = 0$, each petal extends out a distance of $r = 2$, and then turns back to zero $2n$ times for a total of eight petals. See the graph in Figure 4.4.17 .

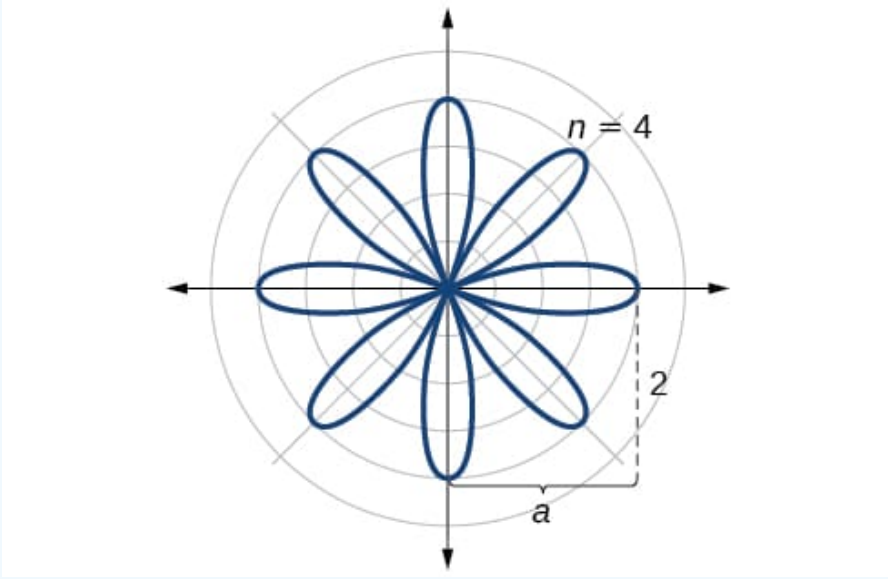


Figure 4.4.17: Rose curve, n even

Analysis

When these curves are drawn, it is best to plot the points in order, as in the Table 4.4.7. This allows us to see how the graph hits a maximum (the tip of a petal), loops back crossing the pole, hits the opposite maximum, and loops back to the pole. The action is continuous until all the petals are drawn.

Exercise 4.4.4

Sketch the graph of $r = 4\sin(2\theta)$.

Answer available at the end of this section.

Example 4.4.9: Sketching the Graph of a Rose Curve (n Odd)

Sketch the graph of $r = 2\sin(5\theta)$.

Solution

The graph of the equation shows symmetry with respect to the line $\theta = \frac{\pi}{2}$. Next, find the zeros and maximum. We will want to make the substitution $u = 5\theta$.

$$\begin{aligned} 0 &= 2\sin(5\theta) \\ 0 &= \sin u \\ \sin^{-1}0 &= 0 \\ u &= 0 \\ 5\theta &= 0 \\ \theta &= 0 \end{aligned}$$

The maximum value is calculated at the angle where $\sin\theta$ is a maximum. Therefore,

$$\begin{aligned} r &= 2\sin\left(5\cdot\frac{\pi}{2}\right) \\ r &= 2(1) \\ &= 2 \end{aligned}$$

Thus, the maximum value of the polar equation is 2. This is the length of each petal. As the curve for n odd yields the same number of petals as n, there will be five petals on the graph. See Figure $4.4.19$.

Create a table of values similar to Table $4.4.8$.

Table $4.4.8$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	\(0\0	1	-1.73	2	-1.73	1	0

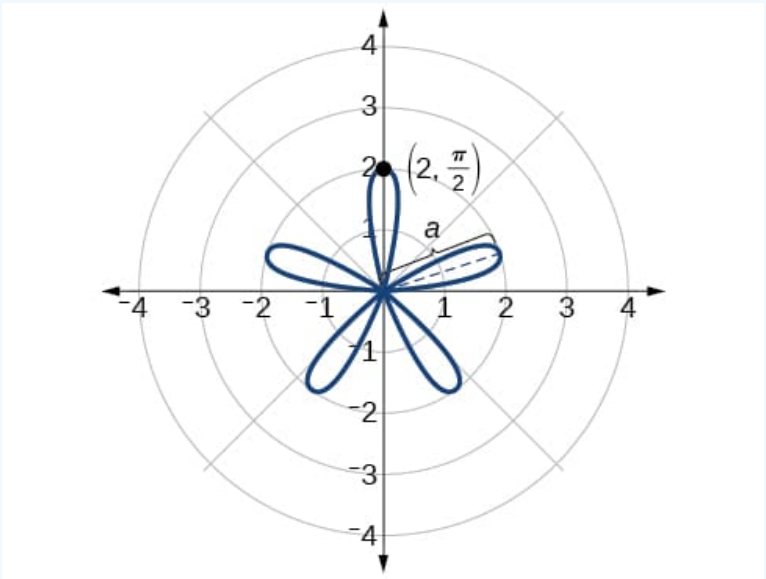


Figure $4.4.19$: Rose curve, n odd

Exercise $4.4.5$

Sketch the graph of $r = 3\cos(3\theta)$.

Answer available at the end of this section.

Investigating the Archimedes' Spiral

The final polar equation we will discuss is the Archimedes' spiral, named for its discoverer, the Greek mathematician Archimedes (c. 287 BCE - c. 212 BCE), who is credited with numerous discoveries in the fields of geometry and mechanics.

ARCHIMEDES' SPIRAL

The formula that generates the graph of the Archimedes' spiral is given by $r = \theta$ for $\theta \geq 0$. As θ increases, r increases at a constant rate in an ever-widening, never-ending, spiraling path. See Figure 4.4.21.

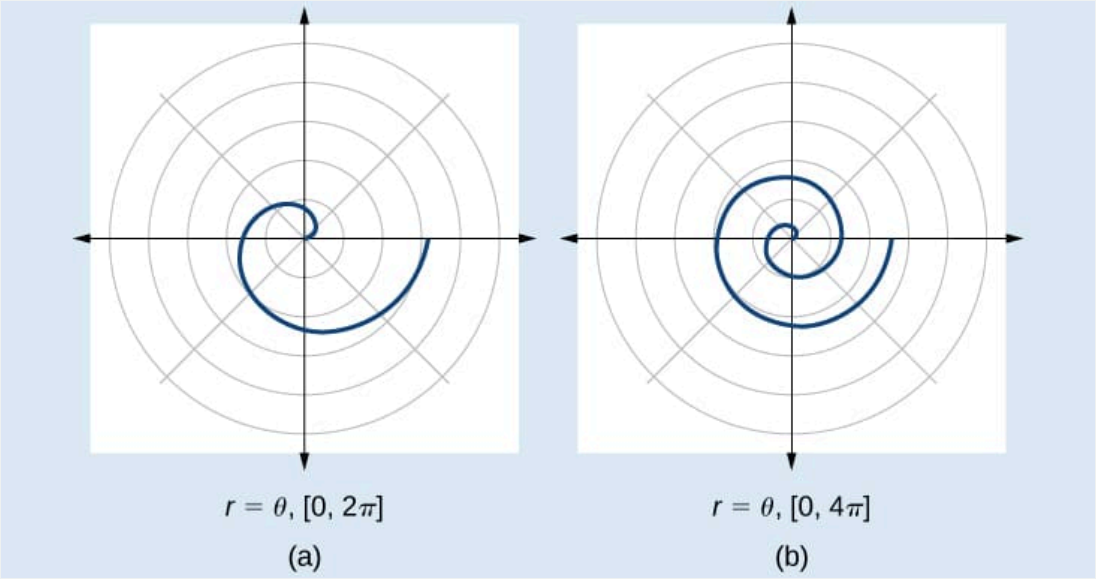


Figure 4.4.21

How to: Given an Archimedes' spiral over $[0, 2\pi]$,sketch the graph

1. Make a table of values for r and θ over the given domain.
2. Plot the points and sketch the graph.

Example 4.4.10: Sketching the Graph of an Archimedes' Spiral

Sketch the graph of $r = \theta$ over $[0, 2\pi]$.

Solution

As r is equal to θ, the plot of the Archimedes' spiral begins at the pole at the point $(0, 0)$. While the graph hints of symmetry, there is no formal symmetry with regard to passing the symmetry tests. Further, there is no maximum value, unless the domain is restricted.

Create a table such as Table 4.4.9.

Table 4.4.9

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0.785	1.57	3.14	4.71	5.50	6.28

Notice that the r-values are just the decimal form of the angle measured in radians. We can see them on a graph in Figure 4.4.22.

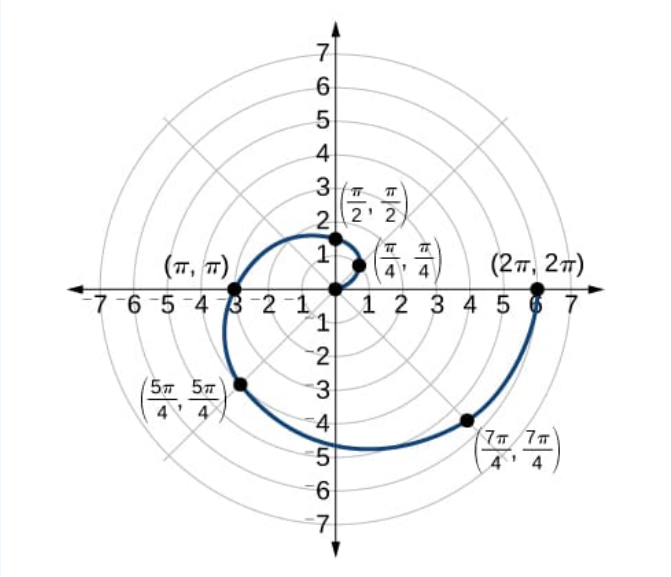


Figure 4.4.22: Archimedes' spiral

Analysis

The domain of this polar curve is $[0, 2\pi]$. In general, however, the domain of this function is $(-\infty, \infty)$. Graphing the equation of the Archimedes' spiral is rather simple, although the image makes it seem like it would be complex.

Exercise 4.4.6

Sketch the graph of $r = -\theta$ over the interval $[0, 4\pi]$.

Answer available at the end of this section.

Summary of Curves

We have explored a number of seemingly complex polar curves in this section. Figure 4.4.24 and Figure 4.4.25 summarize the graphs and equations for each of these curves.

Four graphs side by side - a summary. (A) is a circle: r=asin(theta) or r=acos(theta). (B) is a cardioid: r= a + or - bcos(theta), or r = a + or - b sin(theta). a
0, b>0, a/b=1. (C) is one-loop limaçons. r= a + or - bcos(theta), or r= a + or - bsin(theta). a>0, b>0, 10, b>0, a<b.">

Figure 4.4.24

Four graphs side by side - a summary. (A) is lemniscates. R^2 = a^2cos(2theta), or r^2=a^2sin(2theta). a is not equal to 0. (B) is a rsose curve (n even). R = acos(ntheta), or r=asin(ntheta). N is even, and there are 2n petals. (C) is a rose curve (n odd). R = acos(ntheta), or r=asin(theta). N is odd, and there are n petals. (D) is an Archimedes's spiral. R=theta, and theta
=0." src="/@api/deki/files/7449/CNX_Precalc_Figure_08_04_024n.jpg">

Figure 4.4.25

Media

Access these online resources for additional instruction and practice with graphs of polar coordinates.

- Graphing Polar Equations Part 1
- Graphing Polar Equations Part 2
- Animation: The Graphs of Polar Equations

- Graphing Polar Equations on the TI-84

Key Concepts

- It is easier to graph polar equations if we can test the equations for symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole.
- There are three symmetry tests that indicate whether the graph of a polar equation will exhibit symmetry. If an equation fails a symmetry test, the graph may or may not exhibit symmetry. See Example $4.4.1$.
- Polar equations may be graphed by making a table of values for θ and r.
- The maximum value of a polar equation is found by substituting the value θ that leads to the maximum value of the trigonometric expression.
- The zeros of a polar equation are found by setting $r = 0$ and solving for θ. See Example $4.4.2$.
- Some formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$. See Example $4.4.3$.
- The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, for $a > 0$, $b > 0$, and $ab = 1$. See Example $4.4.4$.
- The formulas that produce the graphs of a one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $1 < ab < 2$. See Example $4.4.5$.
- The formulas that produce the graphs of an inner-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $a > 0$, $b > 0$, and $a < b$. See Example $4.4.6$.
- The formulas that produce the graphs of a lemniscates are given by $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$, where $a \neq 0$.See Example $4.4.7$.
- The formulas that produce the graphs of rose curves are given by $r = a\cos n\theta$ and $r = a\sin n\theta$, where $a \neq 0$; if n is even, there are $2n$ petals, and if n is odd, there aren n petals. See Example $4.4.8$ and Example $4.4.9$.
- The formula that produces the graph of an Archimedes' spiral is given by $r = \theta$, $\theta \geq 0$. See Example $4.4.10$.

Answers

Exercise $4.4.1$

The equation fails the symmetry test with respect to the line $\theta = \frac{\pi}{2}$ and with respect to the pole. It passes the polar axis symmetry test.

Exercise $4.4.2$

Tests will reveal symmetry about the polar axis. The zero is $\left(0, \frac{\pi}{2}\right)$, and the maximum value is $(3, 0)$.

Exercise $4.4.3$

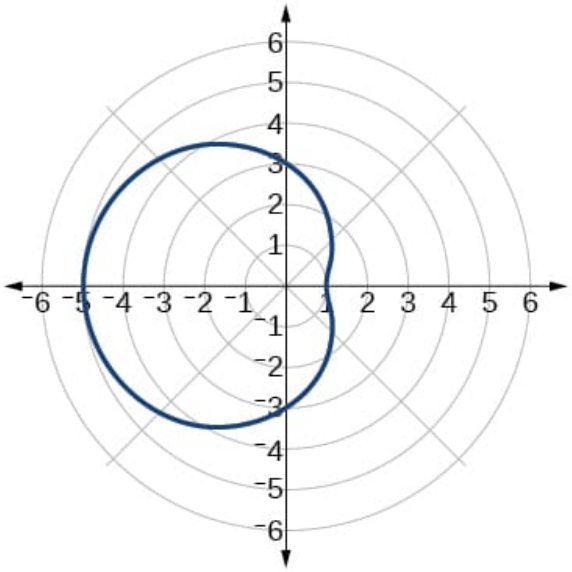


Figure $4.4.11$

Exercise $4.4.4$

The graph is a rose curve, n even

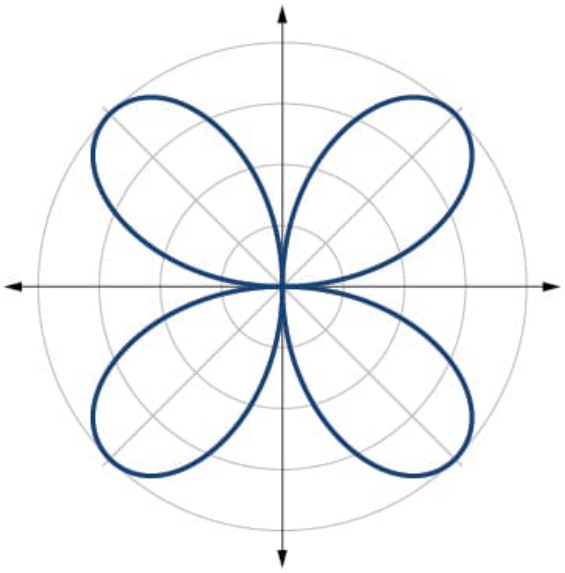


Figure 4.4.18

Exercise 4.4.5

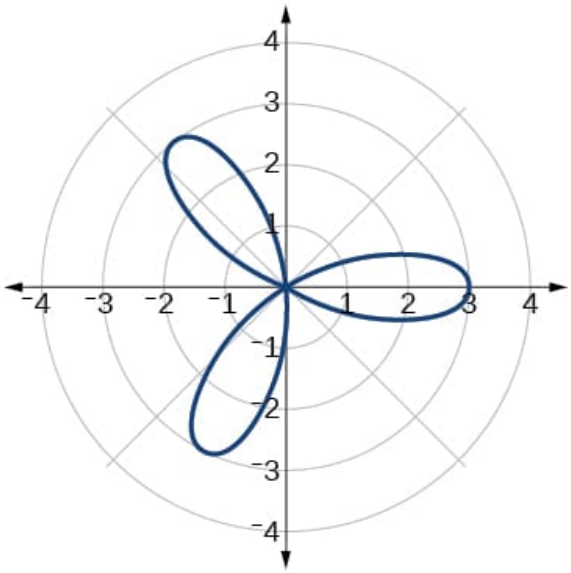


Figure 4.4.20

Rose curve, n odd

Exercise 4.4.6

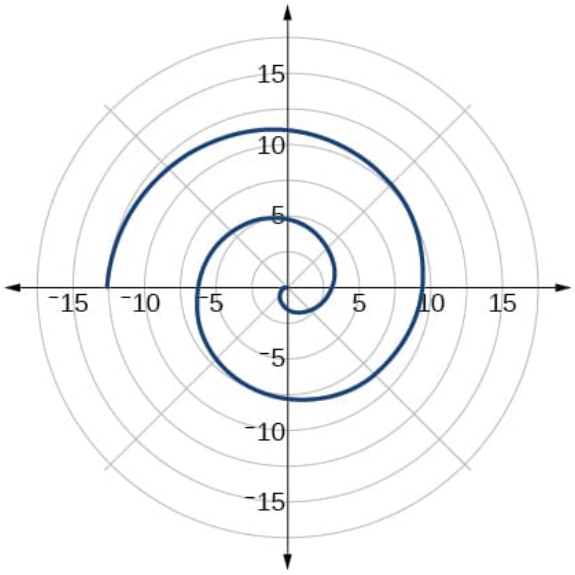


Figure 4.4.23

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4.5: Polar Form of Complex Numbers

Learning Objectives

- Plot complex numbers in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

"God made the integers; all else is the work of man." This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

We first encountered complex numbers in the section on Complex Numbers. In this section, we will focus on the mechanics of working with complex numbers: translation of complex numbers from polar form to rectangular form and vice versa, interpretation of complex numbers in the scheme of applications, and application of De Moivre's Theorem.

Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a+bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, a, and the vertical axis represents the imaginary part of the number, bi.

How to: Given a complex number $a+bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary axis*.
2. Plot the point in the complex plane by moving a units in the horizontal direction and b units in the vertical direction.

Example $4.5.1$: Plotting a Complex Number in the Complex Plane

Plot the complex number $2-3i$ in the complex plane.

Solution

From the origin, move two units in the positive horizontal direction and three units in the negative vertical direction. See Figure $4.5.1$.

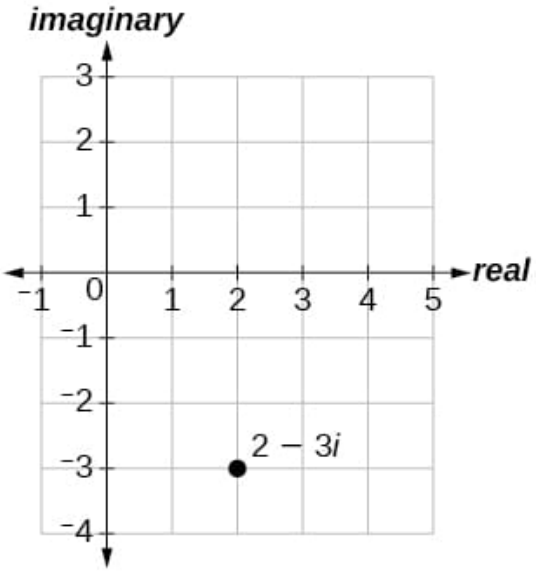


Figure 4.5.1

Exercise 4.5.1

Plot the point $1 + 5i$ in the complex plane.

Answer available at the end of this section.

Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z = 2 + 4i$, in Figure 4.5.3, shows $|z|$.

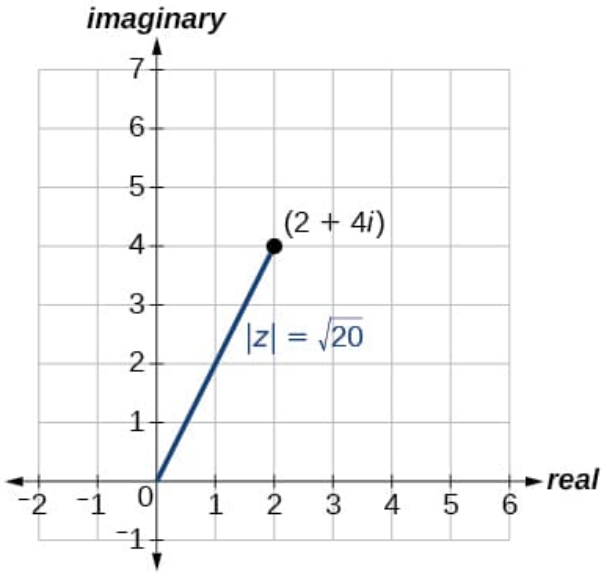


Figure 4.5.3

ABSOLUTE VALUE OF A COMPLEX NUMBER

Given $z = x + yi$, a complex number, the absolute value of z is defined as

$$|z| = \sqrt{x^2 + y^2} \tag{4.5.1}$$

It is the distance from the origin to the point (x, y).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, $(0, 0)$.

Example 4.5.2: Finding the Absolute Value of a Complex Number with a Radical

Find the absolute value of $z = \sqrt{5} - i$.

Solution

Using the formula, we have

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ |z| &= \sqrt{(\sqrt{5})^2 + (-1)^2} \\ |z| &= \sqrt{5 + 1} \\ |z| &= \sqrt{6} \end{aligned}$$

See Figure 4.5.4.

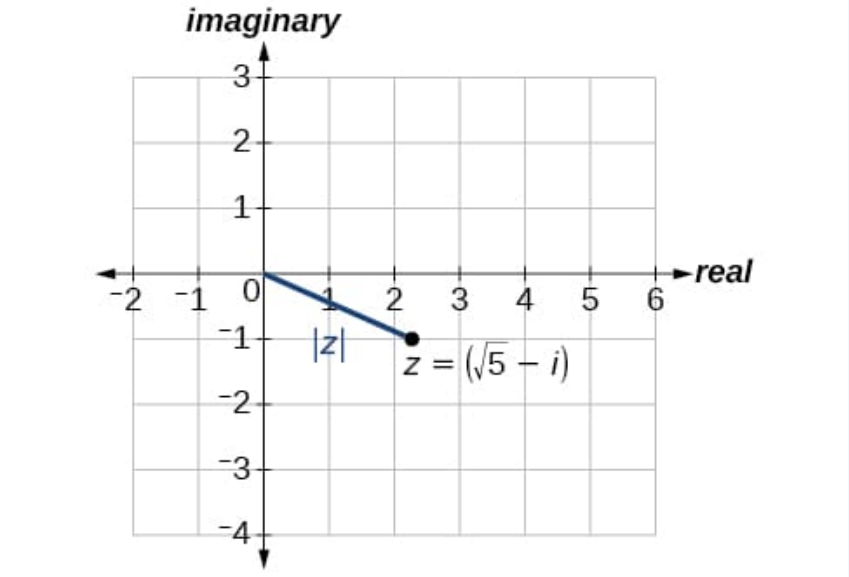


Figure 4.5.4

Exercise 4.5.2

Find the absolute value of the complex number $z = 12 - 5i$.

Answer available at the end of this section.

Example 4.5.3: Finding the Absolute Value of a Complex Number

Given $z = 3 - 4i$, find $|z|$.

Solution

Using the formula, we have

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ |z| &= \sqrt{(3)^2 + (-4)^2} \\ |z| &= \sqrt{9 + 16} \\ |z| &= \sqrt{25} \\ |z| &= 5 \end{aligned}$$

The absolute value z is 5. See Figure $4.5.5$.

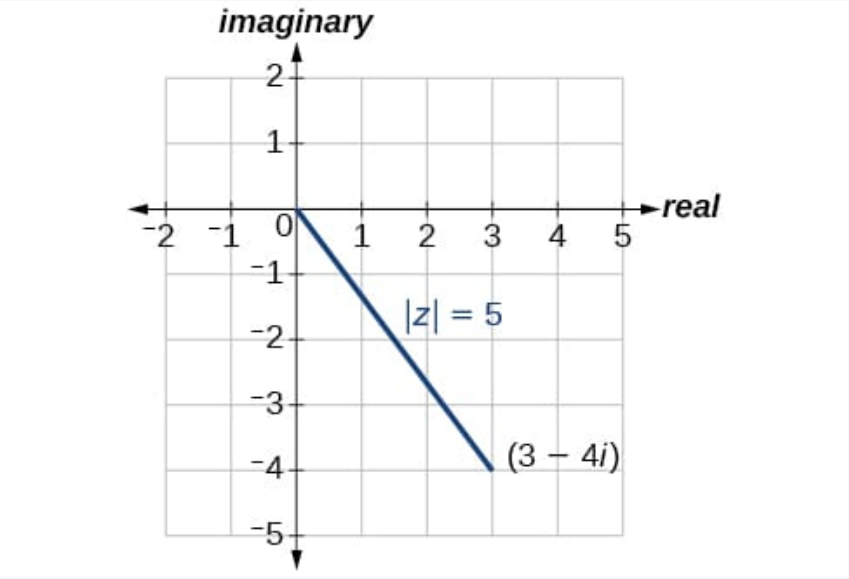


Figure $4.5.5$

Exercise 4.5.3

Given $z = 1 - 7i$, find $|z|$.

Answer available at the end of this section.

Writing Complex Numbers in Polar Form

The polar form of a complex number expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

We review these relationships in Figure $4.5.6$.

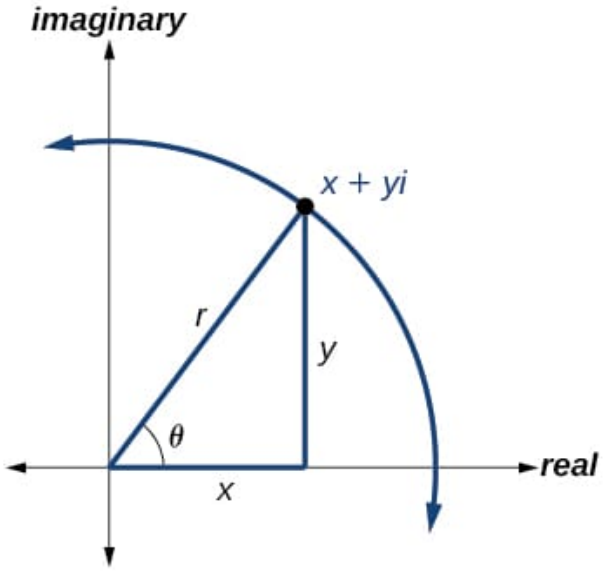


Figure 4.5.6

We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point (x, y). The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$\begin{aligned} z &= x + yi \\ z &= r\cos\theta + (r\sin\theta)i \\ z &= r(\cos\theta + i\sin\theta) \end{aligned}$$

POLAR FORM OF A COMPLEX NUMBER

Writing a complex number in polar form involves the following conversion formulas:

$$x = r\cos\theta \tag{4.5.2}$$
$$y = r\sin\theta \tag{4.5.3}$$
$$r = \sqrt{x^2 + y^2} \tag{4.5.4}$$

Making a direct substitution, we have

$$z = x + yi \tag{4.5.5}$$
$$z = (r\cos\theta) + i(r\sin\theta) \tag{4.5.6}$$
$$z = r(\cos\theta + i\sin\theta) \tag{4.5.7}$$

where r is the modulus and θ is the argument. We often use the abbreviation $r\ cis\theta$ to represent $r(\cos\theta + i\sin\theta)$.

Example 4.5.4: Expressing a Complex Number Using Polar Coordinates

Express the complex number $4i$ using polar coordinates.

Solution

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{0^2 + 4^2}$$
$$r = \sqrt{16}$$
$$r = 4$$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \frac{\pi}{2}$. In polar coordinates, the complex number $z = 0 + 4i$ can be written as $z = 4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ or $4\ cis\left(\frac{\pi}{2}\right)$. See Figure 4.5.7.

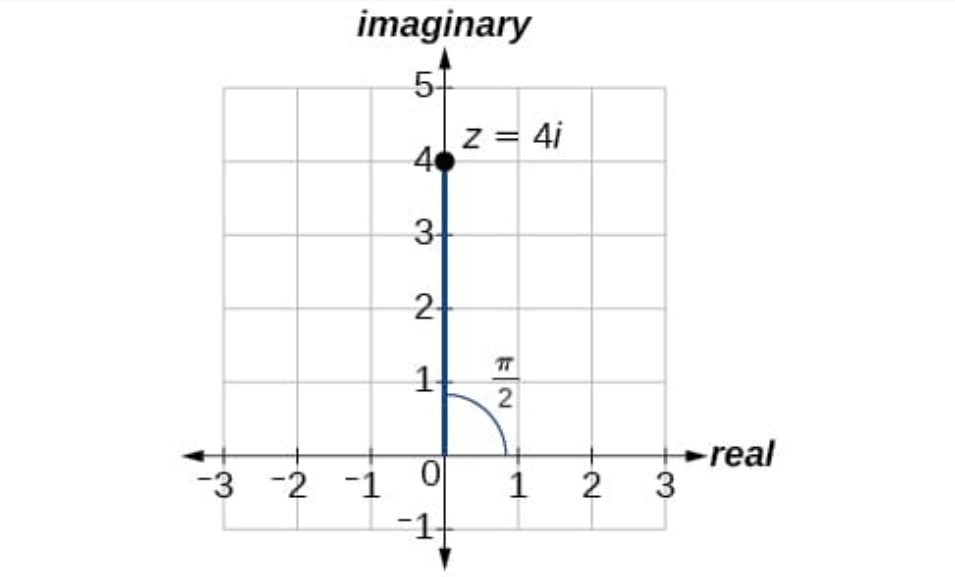


Figure 4.5.7

Exercise 4.5.4

Express $z = 3i$ as $r\ cis\theta$ in polar form.

Answer available at the end of this section.

Example 4.5.5: Finding the Polar Form of a Complex Number

Find the polar form of $-4 + 4i$.

Solution

First, find the value of r.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(-4)^2 + (4^2)}$$
$$r = \sqrt{32}$$
$$r = 4\sqrt{2}$$

Find the angle θ using the formula:

$$\cos\theta = \frac{x}{r}$$
$$\cos\theta = \frac{-4}{4\sqrt{2}}$$
$$\cos\theta = -\frac{1}{\sqrt{2}}$$
$$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$
$$= \frac{3\pi}{4}$$

Thus, the solution is $4\sqrt{2}\ cis\left(\frac{3\pi}{4}\right)$.

Exercise 4.5.5

Write $z = \sqrt{3} + i$ in polar form.

Answer available at the end of this section.

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given $z = r(\cos\theta + i\sin\theta)$, first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$. Then, multiply through by r.

Example 4.5.6A: Converting from Polar to Rectangular Form

Convert the polar form of the given complex number to rectangular form:

$$z = 12\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

Solution

We begin by evaluating the trigonometric expressions.

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ and } \sin(\frac{\pi}{6}) = \frac{1}{2}$$

After substitution, the complex number is

$$z = 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

We apply the distributive property:

$$z = 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= (12)\frac{\sqrt{3}}{2} + (12)\frac{1}{2}i$$
$$= 6\sqrt{3} + 6i$$

The rectangular form of the given point in complex form is $6\sqrt{3} + 6i$.

Example $4.5.6B$: Finding the Rectangular Form of a Complex Number

Find the rectangular form of the complex number given $r = 13$ and $\tan\theta = \frac{5}{12}$.

Solution

If $\tan\theta = \frac{5}{12}$, and $\tan\theta = \frac{y}{x}$, we first determine $r = \sqrt{x^2 + y^2} = \sqrt{122 + 52} = 13$. We then find $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.

$$\begin{aligned} z &= 13\,(\cos\theta + i\sin\theta) \\ &= 13\left(\frac{12}{13} + \frac{5}{13}i\right) \\ &= 12 + 5i \end{aligned}$$

The rectangular form of the given number in complex form is $12 + 5i$.

Exercise $4.5.6$

Convert the complex number to rectangular form:

$$z = 4\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

Answer available at the end of this section.

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham **de Moivre** (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

PRODUCTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \tag{4.5.8}$$

$$z_1 z_2 = r_1 r_2\ cis(\theta_1 + \theta_2) \tag{4.5.9}$$

Notice that the product calls for multiplying the moduli and adding the angles.

Example $4.5.7$: Finding the Product of Two Complex Numbers in Polar Form

Find the product of $z_1 z_2$, given $z_1 = 4(\cos(80\,^\circ) + i\sin(80\,^\circ))$ and $z_2 = 2(\cos(145\,^\circ) + i\sin(145\,^\circ))$.

Solution

Follow the formula

$$z_1 z_2 = 4 \cdot 2[\cos(80°+145°) + i\sin(80°+145°)]$$
$$z_1 z_2 = 8[\cos(225°) + i\sin(225°)]$$
$$z_1 z_2 = 8\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$z_1 z_2 = 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$z_1 z_2 = -4\sqrt{2} - 4i\sqrt{2}$$

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

QUOTIENTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],\ z_2 \neq 0 \tag{4.5.10}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,cis(\theta_1 - \theta_2),\ z_2 \neq 0 \tag{4.5.11}$$

Notice that the moduli are divided, and the angles are subtracted.

How to: Given two complex numbers in polar form, find the quotient

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r(\cos\theta + i\sin\theta)$. Replace r with $\frac{r_1}{r_2}$, and replace θ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by r.

Example 4.5.8: Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2(\cos(213\,°) + i\sin(213\,°))$ and $z_2 = 4(\cos(33\,°) + i\sin(33\,°))$.

Solution

Using the formula, we have

$$\frac{z_1}{z_2} = \frac{2}{4}[\cos(213°-33°) + i\sin(213°-33°)]$$
$$\frac{z_1}{z_2} = \frac{1}{2}[\cos(180°) + i\sin(180°)]$$
$$\frac{z_1}{z_2} = \frac{1}{2}[-1 + 0i]$$
$$\frac{z_1}{z_2} = -\frac{1}{2} + 0i$$
$$\frac{z_1}{z_2} = -\frac{1}{2}$$

Exercise 4.5.8

Find the product and the quotient of $z_1 = 2\sqrt{3}(\cos(150\,^\circ) + i\sin(150\,^\circ))$ and $z_2 = 2(\cos(30\,^\circ) + i\sin(30\,^\circ))$.

Answer available at the end of this section.

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using **De Moivre's Theorem**. It states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the argument by n. It is the standard method used in modern mathematics.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] \tag{4.5.12}$$
$$z^n = r^n\ cis(n\theta) \tag{4.5.13}$$

where n is a positive integer.

Example 4.5.9: Evaluating an Expression Using De Moivre's Theorem

Evaluate the expression $(1+i)^5$ using De Moivre's Theorem.

Solution

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1+i)$ in polar form. Let us find r.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(1)^2 + (1)^2}$$
$$r = \sqrt{2}$$

Then we find θ. Using the formula $\tan\theta = \dfrac{y}{x}$ gives

$$\tan\theta = \frac{1}{1}$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

Use De Moivre's Theorem to evaluate the expression.

$$(a+bi)^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$
$$(1+i)^5 = (\sqrt{2})^5\left[\cos\left(5\cdot\frac{\pi}{4}\right) + i\sin\left(5\cdot\frac{\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$(1+i)^5 = -4 - 4i$$

Finding Roots of Complex Numbers in Polar Form

To find the nth root of a complex number in polar form, we use the nth Root Theorem or De Moivre's Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding n^{th} roots of complex numbers in polar form.

THE N^{TH} ROOT THEOREM

To find the nth root of a complex number in polar form, use the formula given as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right] \tag{4.5.14}$$

where $k = 0, 1, 2, 3, ..., n-1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

th Root of a Complex Number

Evaluate the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)$.

Solution

We have

$$z^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)\right]$$
$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)\right]$$

There will be three roots: $k = 0, 1, 2$. When $k = 0$, we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)$$

When $k = 1$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)\right] \qquad \text{Add } \frac{2(1)\pi}{3} \text{ to each angle.}$$
$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)\right)$$

When $k = 2$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)\right] \qquad \text{Add } \frac{2(2)\pi}{3} \text{ to each angle.}$$
$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{14\pi}{9}\right)+i\sin\left(\frac{14\pi}{9}\right)\right)$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k = 1$, the angle simplification is

$$\begin{aligned}\frac{\frac{2\pi}{3}}{3}+\frac{2(1)\pi}{3} &= \frac{2\pi}{3}\left(\frac{1}{3}\right)+\frac{2(1)\pi}{3}\left(\frac{3}{3}\right)\\ &= \frac{2\pi}{9}+\frac{6\pi}{9}\\ &= \frac{8\pi}{9}\end{aligned}$$

Exercise 4.5.10

Find the four fourth roots of $16(\cos(120\,°) + i\sin(120\,°))$.

Answer available at the end of this section.

Media

Access these online resources for additional instruction and practice with polar forms of complex numbers.

- The Product and Quotient of Complex Numbers in Trigonometric Form
- De Moivre's Theorem

Key Concepts

- Complex numbers in the form $a + bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the x-axis as the real axis and the y-axis as the *imaginary* axis. See Example 4.5.1.
- The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2 + b^2}$. See Example 4.5.2 and Example 4.5.3.
- To write complex numbers in polar form, we use the formulas $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$. Then, $z = r(\cos\theta + i\sin\theta)$. See Example 4.5.4 and Example 4.5.5.
- To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by r. See Example 4.5.6 and Example 4.5.7.
- To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See Example 4.5.8.
- To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See Example 4.5.9.
- To find the power of a complex number z^n, raise r to the power n, and multiply θ by n. See Example 4.5.10.
- Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See Example 4.5.11.

Answers

Exercise 4.5.1

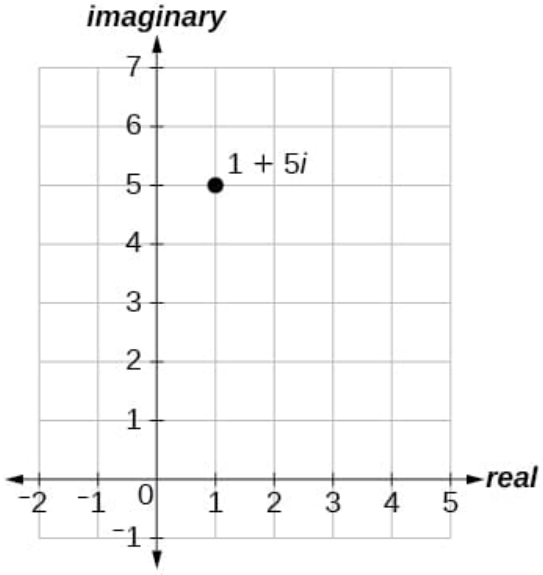


Figure 4.5.2

Exercise 4.5.2

13

Exercise 4.5.3

$|z| = \sqrt{50} = 5\sqrt{2}$

Exercise 4.5.4

$z = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

Exercise 4.5.5

$z = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

Exercise 4.5.6

$z = 2\sqrt{3} - 2i$

Exercise 4.5.8

$z_1 z_2 = -4\sqrt{3};\ \frac{z_1}{z_2} = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

Exercise 4.5.10

$z_0 = 2(\cos(30\ °) + i\sin(30\ °))$

$z_1 = 2(\cos(120\ °) + i\sin(120\ °))$

$z_2 = 2(\cos(210\ °) + i\sin(210\ °))$

$z_3 = 2(\cos(300\ °) + i\sin(300\ °))$

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4.6: Vectors

Learning Objectives

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of v.
- Perform operations with vectors in terms of i and j.
- Find the dot product of two vectors.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of $140\,^\circ$. A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in Figure $4.6.1$. What are the ground speed and actual bearing of the plane?

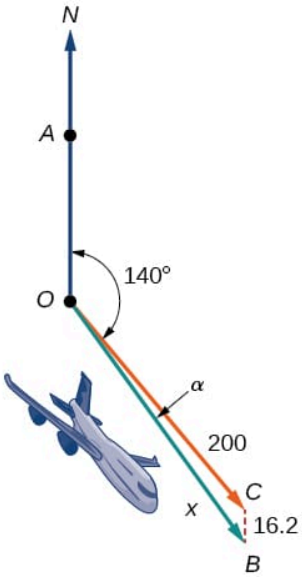


Figure $4.6.1$

Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. In an earlier section, we used triangles to solve a similar problem involving the movement of boats. Later in this section, we will find the airplane's ground speed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

A Geometric View of Vectors

A vector is a specific quantity drawn as a line segment with an arrowhead at one end. It has an **initial point**, where it begins, and a **terminal point**, where it ends. A vector is defined by its magnitude, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- Lower case, boldfaced type, with or without an arrow on top such as u, w, $\vec{v}$, $\vec{u}$, $\vec{w}$.
- Given initial point P and terminal point Q, a vector can be represented as $\overrightarrow{PQ}$. The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of $(0,0)$ and terminal point (a,b), a vector may be represented as $\langle a,b\rangle$.

This last symbol $\langle a,b\rangle$ has special significance. It is called the standard position. The position vector has an initial point $(0,0)$

and a terminal point $\langle a, b\rangle$. To change any vector into the position vector, we think about the change in the *x*-coordinates and the change in the *y*-coordinates. Thus, if the initial point of a vector $\overrightarrow{CD}$ is $C(x_1, y_1)$ and the terminal point is $D(x_2, y_2)$, then the position vector is found by calculating

$$\begin{aligned}\overrightarrow{AB} &= \langle x_2 - x_1, y_2 - y_1\rangle \\ &= \langle a, b\rangle\end{aligned}$$

In Figure $4.6.2$, we see the original vector $\overrightarrow{CD}$ and the position vector $\overrightarrow{AB}$.

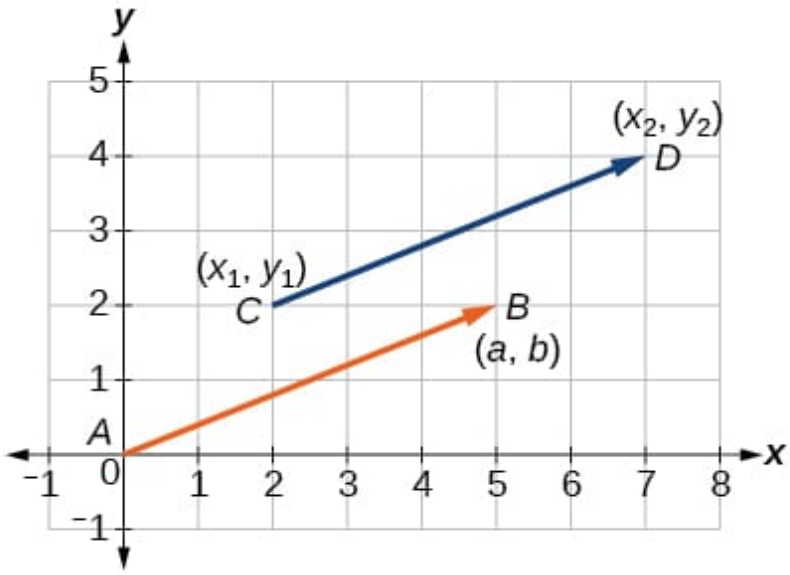


Figure $4.6.2$

PROPERTIES OF VECTORS

A vector is a directed line segment with an initial point and a terminal point. Vectors are identified by magnitude, or the length of the line, and direction, represented by the arrowhead pointing toward the terminal point. The position vector has an initial point at $(0, 0)$ and is identified by its terminal point $\langle a, b\rangle$.

Example $4.6.1A$: Find the Position Vector

Consider the vector whose initial point is $P(2, 3)$ and terminal point is $Q(6, 4)$. Find the position vector.

Solution

The position vector is found by subtracting one x-coordinate from the other x-coordinate, and one y-coordinate from the other y-coordinate. Thus

$$\begin{aligned}v &= \langle 6 - 2, 4 - 3\rangle \\ &= \langle 4, 1\rangle\end{aligned}$$

The position vector begins at $(0, 0)$ and terminates at $(4, 1)$. The graphs of both vectors are shown in Figure $4.6.3$.

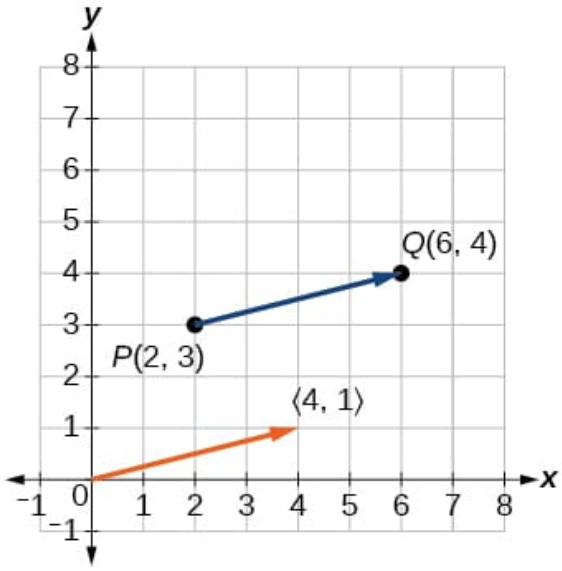


Figure 4.6.3

We see that the position vector is $\langle 4, 1\rangle$.

Example $4.6.1B$: Drawing a Vector with the Given Criteria and Its Equivalent Position Vector

Find the position vector given that vector v has an initial point at $(-3, 2)$ and a terminal point at $(4, 5)$, then graph both vectors in the same plane.

Solution

The position vector is found using the following calculation:

$$\begin{aligned} v &= \langle 4-(-3), 5-2\rangle \\ &= \langle 7, 3\rangle \end{aligned}$$

Thus, the position vector begins at $(0, 0)$ and terminates at $(7, 3)$. See Figure 4.6.4.

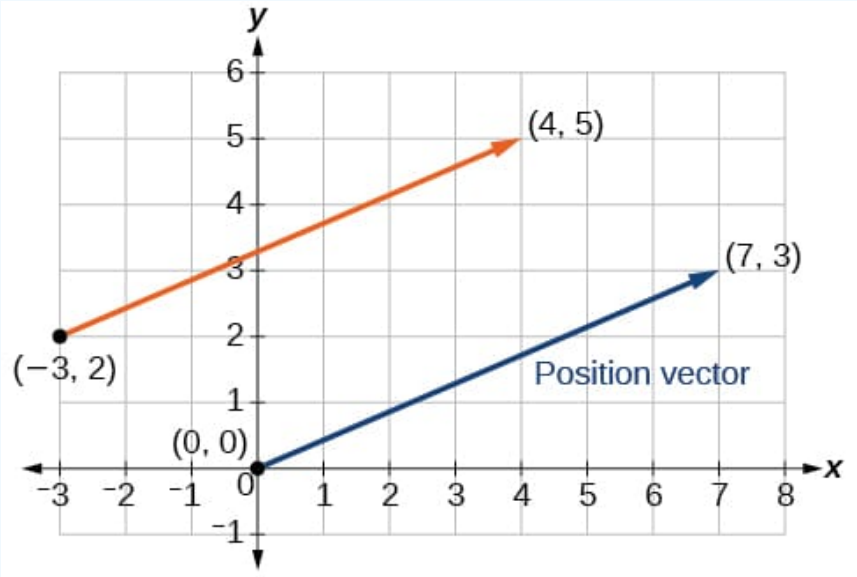


Figure 4.6.4

Exercise 4.6.1

Draw a vector $\vec{v}$ that connects from the origin to the point $(3, 5)$.

Answer available at the end of this section.

Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

MAGNITUDE AND DIRECTION OF A VECTOR

Given a position vector $\vec{v} = \langle a, b \rangle$,the magnitude is found by $|v| = \sqrt{a^2 + b^2}$.The direction is equal to the angle formed with the x-axis, or with the y-axis, depending on the application. For a position vector, the direction is found by $\tan\theta = \left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in Figure 4.6.6.

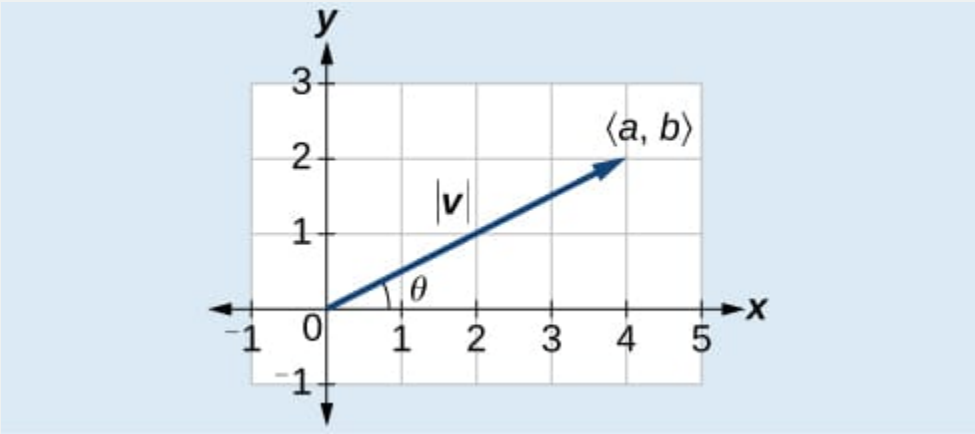


Figure 4.6.6

Two vectors $\vec{v}$ and $\vec{u}$ are considered equal if they have the same magnitude and the same direction. Additionally, if both vectors have the same position vector, they are equal.

Example 4.6.2A: Finding the Magnitude and Direction of a Vector

Find the magnitude and direction of the vector with initial point $P(-8, 1)$ and terminal point $Q(-2, -5)$.Draw the vector.

Solution

First, find the position vector.

$$\begin{aligned} u &= \langle -2, -(-8), -5 - 1 \rangle \\ &= \langle 6, -6 \rangle \end{aligned}$$

We use the Pythagorean Theorem to find the magnitude.

$$|u| = \sqrt{(6)^2 + (-6)^2}$$
$$= \sqrt{72}$$
$$= \sqrt{62}$$

The direction is given as

$$\tan\theta = \frac{-6}{6} = -1 \rightarrow \theta = \tan^{-1}(-1)$$
$$= -45°$$

However, the angle terminates in the fourth quadrant, so we add $360\,°$ to obtain a positive angle. Thus, $-45\,°+360\,° = 315\,°$. See Figure $4.6.7$.

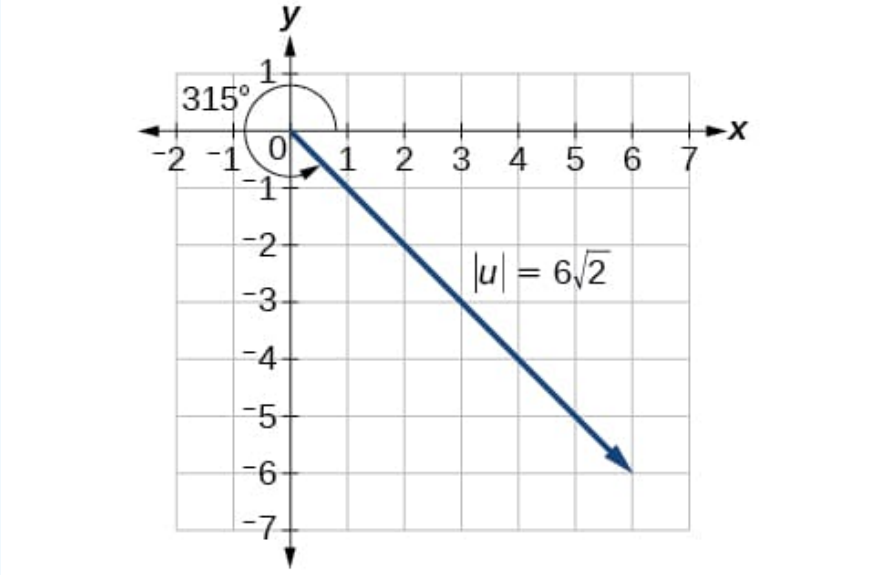


Figure $4.6.7$

Example $4.6.2B$: Showing That Two Vectors Are Equal

Show that vector $\vec{v}$ with initial point at $(5,-3)$ and terminal point at $(-1,2)$ is equal to vector $\vec{u}$ with initial point at $(-1,-3)$ and terminal point at $(-7,2)$. Draw the position vector on the same grid as $\vec{v}$ and $\vec{u}$. Next, find the magnitude and direction of each vector.

Solution

As shown in Figure $4.6.8$, draw the vector $\vec{v}$ starting at initial $(5,-3)$ and terminal point $(-1,2)$. Draw the vector $\vec{u}$ with initial point $(-1,-3)$ and terminal point $(-7,2)$. Find the standard position for each.

Next, find and sketch the position vector for $\vec{v}$ and $\vec{u}$. We have

$$v = \langle -1-5, 2-(-3)\rangle$$
$$= \langle -6,5\rangle u$$
$$= \langle -7-(-1), 2-(-3)\rangle$$
$$= \langle -6,5\rangle$$

Since the position vectors are the same, $\vec{v}$ and $\vec{u}$ are the same.

An alternative way to check for vector equality is to show that the magnitude and direction are the same for both vectors. To show that the magnitudes are equal, use the Pythagorean Theorem.

$$\begin{aligned} |v| &= \sqrt{(-1-5)^2+(2-(-3))^2} \\ &= \sqrt{(-6)^2+(5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \\ |u| &= \sqrt{(-7-(-1))^2+(2-(-3))^2} \\ &= \sqrt{(-6)^2+(5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \end{aligned}$$

As the magnitudes are equal, we now need to verify the direction. Using the tangent function with the position vector gives

$$\begin{aligned} \tan\theta = -\frac{5}{6} \Rightarrow \theta &= \tan^{-1}\left(-\frac{5}{6}\right) \\ &= -39.8° \end{aligned}$$

However, we can see that the position vector terminates in the second quadrant, so we add $180\,°$. Thus, the direction is $-39.8\,°+180\,° = 140.2\,°$.

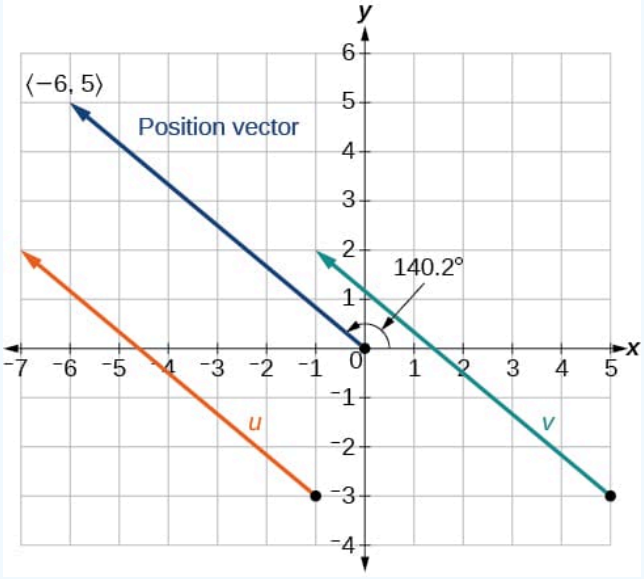


Figure 4.6.8

Performing Vector Addition and Scalar Multiplication

Now that we understand the properties of vectors, we can perform operations involving them. While it is convenient to think of the vector $u = \langle x, y\rangle$ as an arrow or directed line segment from the origin to the point (x, y), vectors can be situated anywhere in the plane. The sum of two vectors $\vec{u}$ and $\vec{v}$, or vector addition, produces a third vector $\overrightarrow{u+v}$, the resultant vector.

To find $\overrightarrow{u+v}$, we first draw the vector $\vec{u}$, and from the terminal end of $\vec{u}$, we drawn the vector $\vec{v}$. In other words, we have the initial point of $\vec{v}$ meet the terminal end of $\vec{u}$. This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector. The sum $\overrightarrow{u+v}$ is the resultant vector because it results from addition or subtraction of two vectors. The resultant vector travels directly from the beginning of $\vec{u}$ to the end of $\vec{v}$ in a straight path, as shown

in Figure 4.6.9.

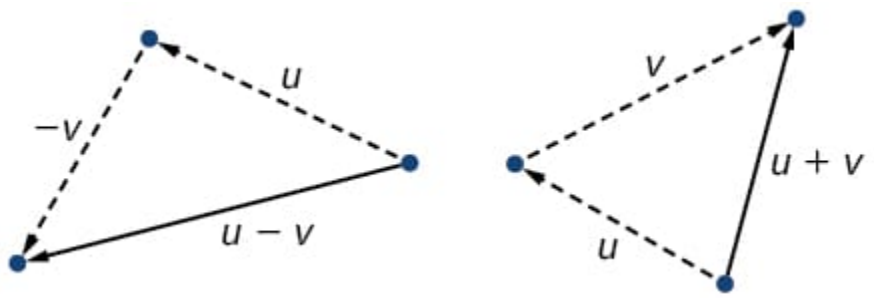


Figure 4.6.9

Vector subtraction is similar to vector addition. To find $\overrightarrow{u-v}$, view it as $\overrightarrow{u+(-v)}$. Adding $\overrightarrow{-v}$ is reversing direction of $\vec{v}$ and adding it to the end of $\vec{u}$. The new vector begins at the start of $\vec{u}$ and stops at the end point of $\overrightarrow{-v}$. See Figure 4.6.10 for a visual that compares vector addition and vector subtraction using parallelograms.

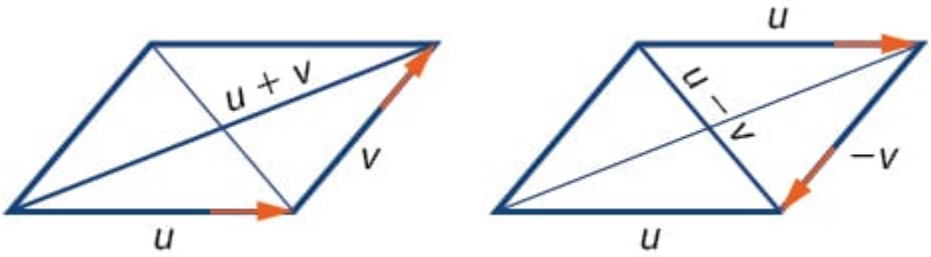


Figure 4.6.10

Example 4.6.3: Adding and Subtracting Vectors

Given $u=\langle 3,-2\rangle$ and $v=\langle -1,4\rangle$, find two new vectors $\overrightarrow{u+v}$, and $\overrightarrow{u-v}$.

Solution

To find the sum of two vectors, we add the components. Thus,

$$\begin{aligned} u+v &= \langle 3,-2\rangle + \langle -1,4\rangle \\ &= \langle 3+(-1),-2+4\rangle \\ &= \langle 2,2\rangle \end{aligned}$$

See Figure 4.6.11a.

To find the difference of two vectors, add the negative components of $\vec{v}$ to $\vec{u}$. Thus,

$$\begin{aligned} u+(-v) &= \langle 3,-2\rangle + \langle 1,-4\rangle \\ &= \langle 3+1,-2+(-4)\rangle \\ &= \langle 4,-6\rangle \end{aligned}$$

See Figure 4.6.11b.

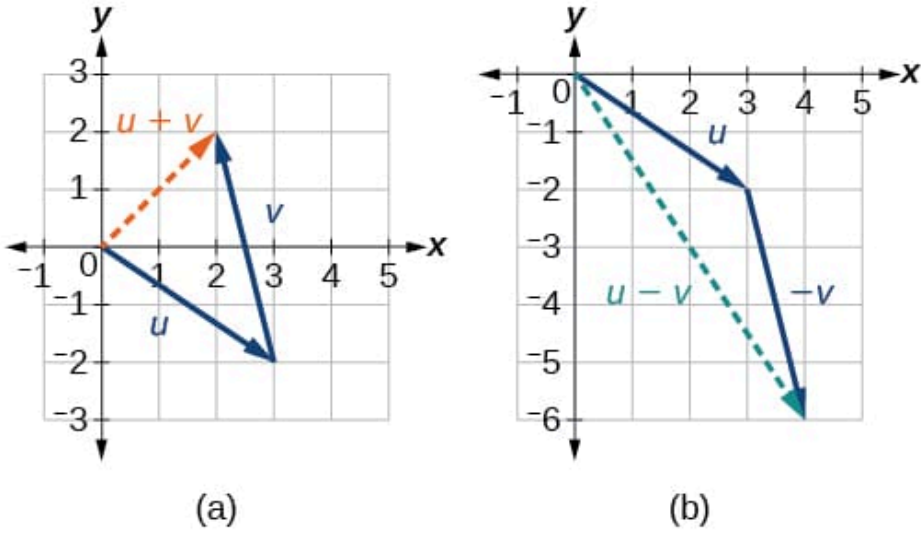


Figure 4.6.11: (a) Sum of two vectors (b) Difference of two vectors

Multiplying By a Scalar

While adding and subtracting vectors gives us a new vector with a different magnitude and direction, the process of multiplying a vector by a **scalar**, a constant, changes only the magnitude of the vector or the length of the line. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

SCALAR MULTIPLICATION

Scalar multiplication involves the product of a vector and a scalar. Each component of the vector is multiplied by the scalar. Thus, to multiply $v = \langle a, b \rangle$ by k, we have

$$kv = \langle ka, kb \rangle$$

Only the magnitude changes, unless k is negative, and then the vector reverses direction.

Example 4.6.4: Performing Scalar Multiplication

Given vector $\vec{v} = \langle 3, 1 \rangle$, find $3\vec{v}$, $\frac{1}{2}$, and $-\vec{v}$.

Solution

See Figure 4.6.12 for a geometric interpretation. If $\vec{v} = \langle 3, 1 \rangle$, then

$$\begin{aligned} 3v &= \langle 3 \cdot 3, 3 \cdot 1 \rangle \\ &= \langle 9, 3 \rangle \\ \frac{1}{2}v &= \langle \frac{1}{2} \cdot 3, \frac{1}{2} \cdot 1 \rangle \\ &= \langle \frac{3}{2}, \frac{1}{2} \rangle \\ -v &= \langle -3, -1 \rangle \end{aligned}$$

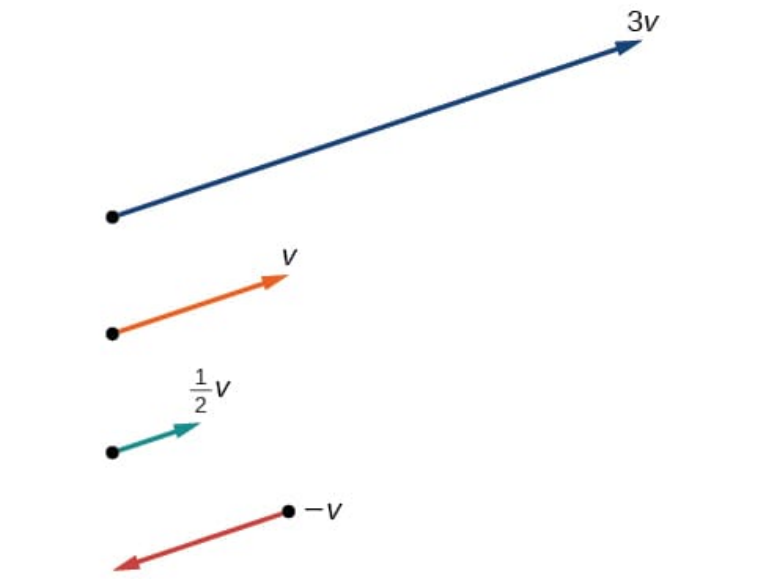


Figure 4.6.12

Analysis

Notice that the vector $3\vec{v}$ is three times the length of $\vec{v}$, $\frac{1}{2}\vec{v}$ is half the length of $\vec{v}$, and $\overrightarrow{-v}$ is the same length of $\vec{v}$, but in the opposite direction.

Exercise 4.6.2

Find the scalar multiple $3u$ given $\vec{u} = \langle 5, 4\rangle$.

Answer available at the end of this section.

Example 4.6.5

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

Solution

First, we must multiply each vector by the scalar.

$$\begin{aligned} 3u &= 3\langle 3, -2\rangle \\ &= \langle 9, -6\rangle \\ 2v &= 2\langle -1, 4\rangle \\ &= \langle -2, 8\rangle \end{aligned}$$

Then, add the two together.

$$\begin{aligned} w &= 3u + 2v \\ &= \langle 9, -6\rangle + \langle -2, 8\rangle \\ &= \langle 9 - 2, -6 + 8\rangle \\ &= \langle 7, 2\rangle \end{aligned}$$

So, $w = \langle 7, 2\rangle$.

Finding Component Form

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Vectors are comprised of two components: the horizontal component is the x direction, and the vertical component is the y direction. For example, we can see in the graph in Figure $4.6.13$ that the position vector $\langle 2,3\rangle$ comes from adding the vectors v_1 and v_2. We have v_2 with initial point $(0,0)$ and terminal point $(2,0)$.

$$\begin{aligned} v_1 &= \langle 2-0, 0-0\rangle \\ &= \langle 2,0\rangle \end{aligned}$$

We also have v_2 with initial point $(0,0)$ and terminal point $(0,3)$.

$$\begin{aligned} v_2 &= \langle 0-0, 3-0\rangle \\ &= \langle 0,3\rangle \end{aligned}$$

Therefore, the position vector is

$$\begin{aligned} v &= \langle 2+0, 3+0\rangle \\ &= \langle 2,3\rangle \end{aligned}$$

Using the Pythagorean Theorem, the magnitude of v_1 is 2, and the magnitude of v_2 is 3. To find the magnitude of v, use the formula with the position vector.

$$\begin{aligned} |v| &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{2^2+3^2} \\ &= \sqrt{13} \end{aligned}$$

The magnitude of v is $\sqrt{13}$. To find the direction, we use the tangent function $\tan\theta = \dfrac{y}{x}$.

$$\begin{aligned} \tan\theta &= \frac{v_2}{v_1} \\ \tan\theta &= \frac{3}{2} \\ \theta &= \tan^{-1}\left(\frac{3}{2}\right) = 56.3^\circ \end{aligned}$$

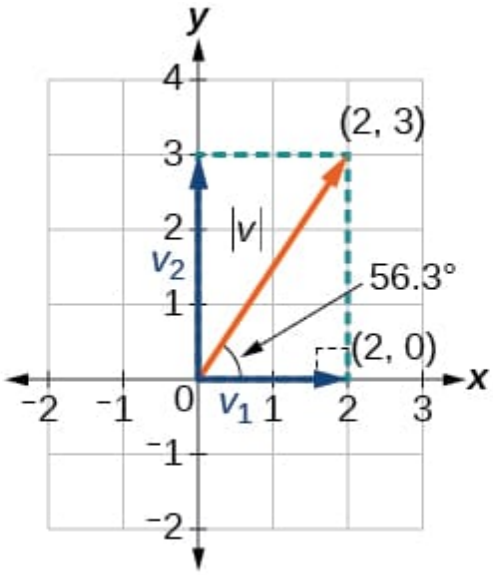


Figure $4.6.13$

Thus, the magnitude of $\vec{v}$ is $\sqrt{13}$ and the direction is 56.3° off the horizontal.

Example 4.6.6: Finding the Components of the Vector

Find the components of the vector $\vec{v}$ with initial point $(3, 2)$ and terminal point $(7, 4)$.

Solution

First find the standard position.

$$\begin{aligned} v &= \langle 7-3, 4-2 \rangle \\ &= \langle 4, 2 \rangle \end{aligned}$$

See the illustration in Figure 4.6.14.

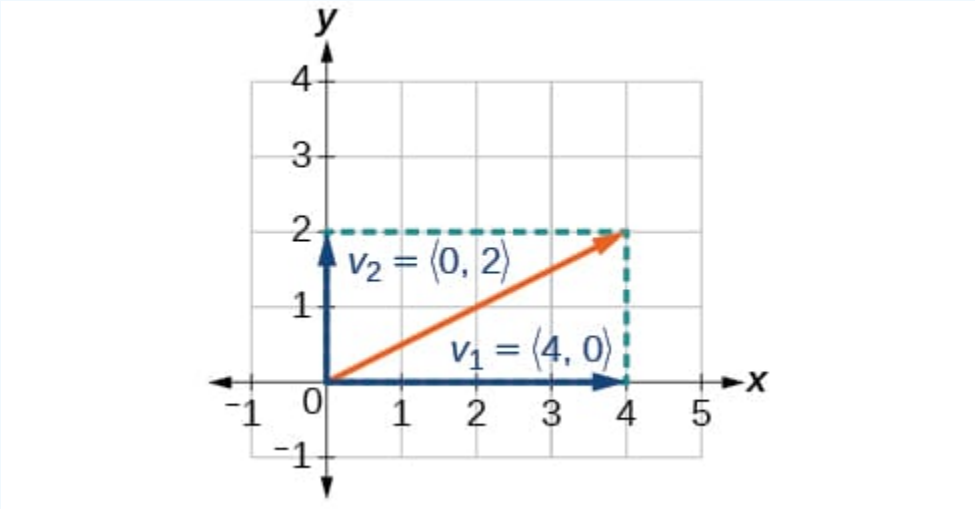


Figure 4.6.14

The horizontal component is $\vec{v_1} = \langle 4, 0 \rangle$ and the vertical component is $\vec{v_2} = \langle 0, 2 \rangle$.

Finding the Unit Vector in the Direction of v

In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a unit vector. We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components. The horizontal unit vector is written as $\vec{i} = \langle 1, 0 \rangle$ and is directed along the positive horizontal axis. The vertical unit vector is written as $\vec{j} = \langle 0, 1 \rangle$ and is directed along the positive vertical axis. See Figure 4.6.15.

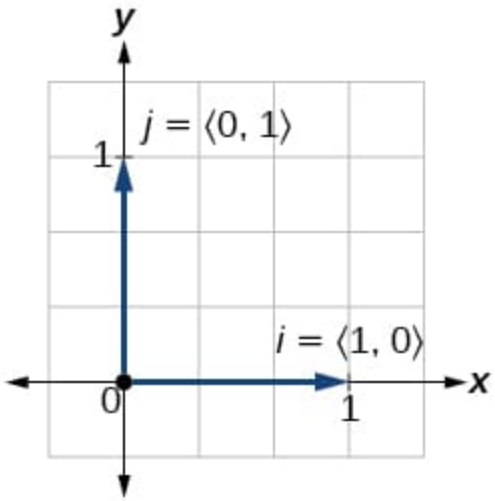


Figure 4.6.15

THE UNIT VECTORS

If $\vec{v}$ is a nonzero vector, then $\frac{v}{|v|}$ is a unit vector in the direction of v. Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

Example 4.6.7: Finding the Unit Vector in the Direction of v

Find a unit vector in the same direction as $v = \langle -5, 12 \rangle$.

Solution

First, we will find the magnitude.

$$\begin{aligned} |v| &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Then we divide each component by $|v|$, which gives a unit vector in the same direction as $\vec{v}$:

$$\frac{v}{|v|} = -\frac{5}{13}i + \frac{12}{13}j$$

or, in component form

$$\frac{v}{|v|} = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$$

See Figure 4.6.16.

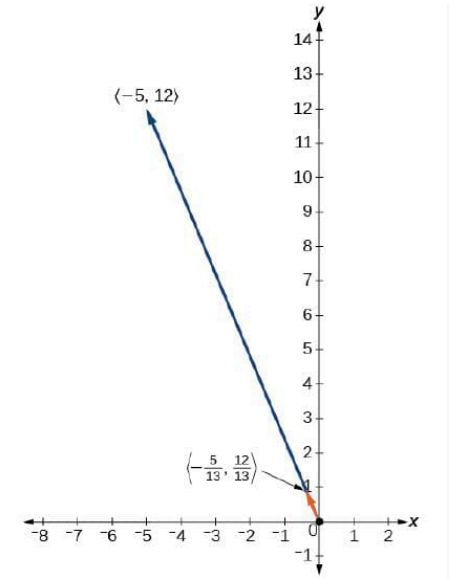


Figure 4.6.16

Verify that the magnitude of the unit vector equals 1. The magnitude of $-\frac{5}{13}i + \frac{12}{13}j$ is given as

$$\begin{aligned}\sqrt{\left(-\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2} &= \sqrt{\frac{25}{169} + \frac{144}{169}} \\ &= \sqrt{\frac{169}{169}} \\ &= 1\end{aligned}$$

The vector $u = \frac{5}{13}i + \frac{12}{13}j$ is the unit vector in the same direction as $v = \langle -5, 12\rangle$.

Performing Operations with Vectors in Terms of i and j

So far, we have investigated the basics of vectors: magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically. Now that we are familiar with the general strategies used in working with vectors, we will represent vectors in rectangular coordinates in terms of i and j.

VECTORS IN THE RECTANGULAR PLANE

Given a vector $\vec{v}$ with initial point $P = (x_1, y_1)$ and terminal point $Q = (x_2, y_2)$, $\vec{v}$ is written as

$$v = (x_2 - x_1)i + (y_1 - y_2)j \tag{4.6.1}$$

The position vector from $(0, 0)$ to (a, b), where $(x_2 - x_1) = a$ and $(y_2 - y_1) = b$, is written as $\vec{v} = \vec{a}i + \vec{b}j$. This vector sum is called a linear combination of the vectors $\vec{i}$ and $\vec{j}$.

The magnitude of $\vec{v} = \overrightarrow{a}i + \overrightarrow{b}j$ is given as $|v| = \sqrt{a^2 + b^2}$. See Figure 4.6.17.

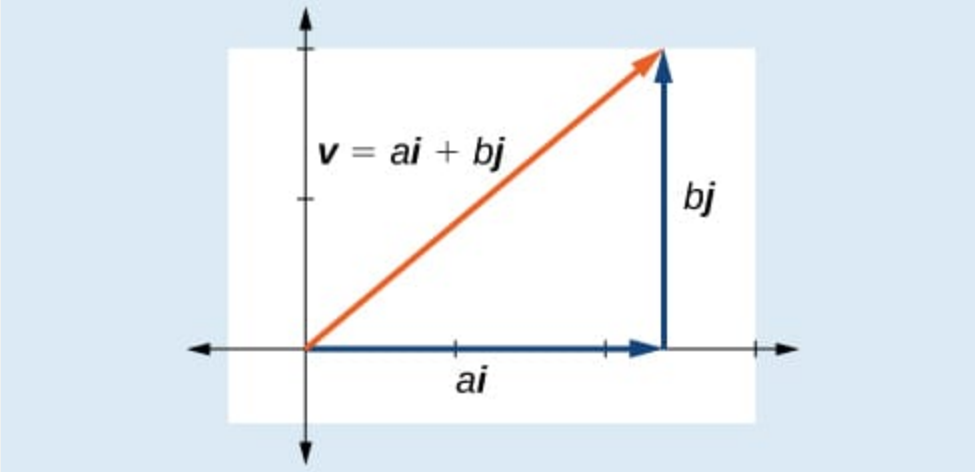


Figure 4.6.17

Example 4.6.8A: Writing a Vector in Terms of i and j

Given a vector $\vec{v}$ with initial point $P = (2, -6)$ and terminal point $Q = (-6, 6)$, write the vector in terms of $\vec{i}$ and $\vec{j}$.

Solution

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2 - x_1)i + (y_2 - y_1)j \\ &= (-6 - 2)i + (6 - (-6))j \\ &= -8i + 12j \end{aligned}$$

Example 4.6.8B: Writing a Vector in Terms of i and j Using Initial and Terminal Points

Given initial point $P_1 = (-1, 3)$ and terminal point $P_2 = (2, 7)$, write the vector $\vec{v}$ in terms of $\vec{i}$ and $\vec{j}$.

Solution

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2 - x_1)i + (y_2 - y_1)j \\ v &= (2 - (-1))i + (7 - 3)j \\ &= 3i + 4j \end{aligned}$$

Exercise 4.6.3

Write the vector $\vec{u}$ with initial point $P = (-1, 6)$ and terminal point $Q = (7, -5)$ in terms of $\vec{i}$ and $\vec{j}$.

Answer available at the end of this section.

Performing Operations on Vectors in Terms of i and j

When vectors are written in terms of i and j, we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

ADDING AND SUBTRACTING VECTORS IN RECTANGULAR COORDINATES

Given $v = ai + bj$ and $u = ci + dj$, then

$$v + u = (a + c)i + (b + d)j$$
$$v - u = (a - c)i + (b - d)j$$

Example 4.6.9: Finding the Sum of the Vectors

Find the sum of $v_1 = 2i - 3j$ and $v_2 = 4i + 5j$.

Solution

$$\begin{aligned} v_1 + v_2 &= (2 + 4)i + (-3 + 5)j \\ &= 6i + 2j \end{aligned}$$

Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using i and j. For any of these vectors, we can calculate the magnitude. Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates. We find the direction of the vector by finding the angle to the horizontal. We do this by using the basic trigonometric identities, but with $|v|$ replacing r.

VECTOR COMPONENTS IN TERMS OF MAGNITUDE AND DIRECTION

Given a position vector $v = \langle x, y \rangle$ and a direction angle θ,

$$\cos\theta = \frac{x}{|v|} \text{ and } \sin\theta = y|v|$$
$$x = |v|\cos\theta$$
$$y = |v|\sin\theta$$

Thus, $v = xi + yj = |v|\cos\theta i + |v|\sin\theta j$, and magnitude is expressed as $|v| = \sqrt{x^2 + y^2}$.

Example 4.6.10: Writing a Vector in Terms of Magnitude and Direction

Write a vector with length 7 at an angle of $135\,°$ to the positive *x*-axis in terms of magnitude and direction.

Solution

Using the conversion formulas $x = |v|\cos\theta i$ and $y = |v|\sin\theta j$, we find that

$$\begin{aligned} x &= 7\cos(135°)i \\ &= -\frac{7\sqrt{2}}{2} \\ y &= 7\sin(135°)j \\ &= \frac{7\sqrt{2}}{2} \end{aligned}$$

This vector can be written as $v = 7\cos(135\ °)i + 7\sin(135\ °)j$ or simplified as

$$v = -\frac{7\sqrt{2}}{2}i + \frac{7\sqrt{2}}{2}j$$

Exercise 4.6.4

A vector travels from the origin to the point $(3, 5)$. Write the vector in terms of magnitude and direction.

Answer available at the end of this section.

Finding the Dot Product of Two Vectors

As we discussed earlier in the section, scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication. If we multiply a vector by a vector, there are two possibilities: the *dot product* and the *cross product*. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.

DOT PRODUCT

The dot product of two vectors $v = \langle a, b\rangle$ and $u = \langle c, d\rangle$ is the sum of the product of the horizontal components and the product of the vertical components.

$$v \cdot u = ac + bd \tag{4.6.2}$$

To find the angle between the two vectors, use the formula below.

$$\cos\theta = \frac{v}{|v|} \cdot \frac{u}{|u|} \tag{4.6.3}$$

Example 4.6.11A: Finding the Dot Product of Two Vectors

Find the dot product of $v = \langle 5, 12\rangle$ and $u = \langle -3, 4\rangle$.

Solution

Using the formula, we have

$$\begin{aligned} v \cdot u &= \langle 5, 12\rangle \cdot \langle -3, 4\rangle \\ &= 5 \cdot (-3) + 12 \cdot 4 \\ &= -15 + 48 \\ &= 33 \end{aligned}$$

Example 4.6.11B: Finding the Dot Product of Two Vectors and the Angle between Them

Find the dot product of $v_1 = 5i + 2j$ and $v_2 = 3i + 7j$. Then, find the angle between the two vectors.

Solution

Finding the dot product, we multiply corresponding components.

$$\begin{aligned} v_1 \cdot v_2 &= \langle 5, 2\rangle \cdot \langle 3, 7\rangle \\ &= 5 \cdot 3 + 2 \cdot 7 \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

To find the angle between them, we use the formula $\cos\theta = \dfrac{v}{|v|} \cdot \dfrac{u}{|u|}$.

$$\begin{aligned} \frac{v}{|v|} \cdot \frac{u}{|u|} &= \left\langle \frac{5}{\sqrt{29}} + \frac{2}{\sqrt{29}} \right\rangle \cdot \left\langle \frac{3}{\sqrt{58}} + \frac{7}{\sqrt{58}} \right\rangle \\ &= \frac{5}{\sqrt{29}} \cdot \frac{3}{\sqrt{58}} + \frac{2}{\sqrt{29}} \cdot \frac{7}{\sqrt{58}} \\ &= \frac{15}{\sqrt{1682}} + \frac{14}{\sqrt{1682}} \\ &= \frac{29}{\sqrt{1682}} \\ &= 0.707107 \end{aligned}$$

$$\cos^{-1}(0.707107) = 45°$$

See Figure 4.6.18.

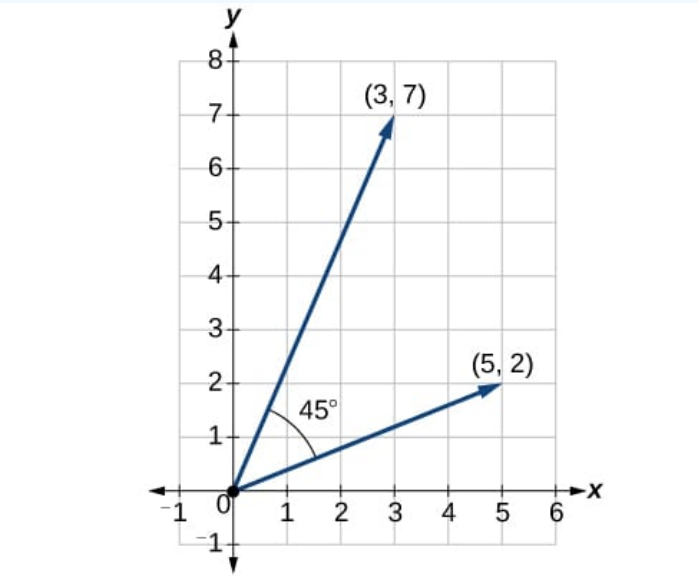


Figure 4.6.18

Example 4.6.11C: Finding the Angle between Two Vectors

Find the angle between $u = \langle -3, 4\rangle$ and $v = \langle 5, 12\rangle$.

Solution

Using the formula, we have

$$\theta = \cos^{-1}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right)$$

$$\begin{aligned}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right) &= \frac{-3i+4j}{5} \cdot \frac{5i+12j}{13} \\ &= \left(-\frac{3}{5} \cdot \frac{5}{13}\right) + \left(\frac{4}{5} \cdot \frac{12}{13}\right) \\ &= -\frac{15}{65} + \frac{48}{65} \\ &= \frac{33}{65}\end{aligned}$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{33}{65}\right) \\ &= 59.5^\circ\end{aligned}$$

See Figure 4.6.19.

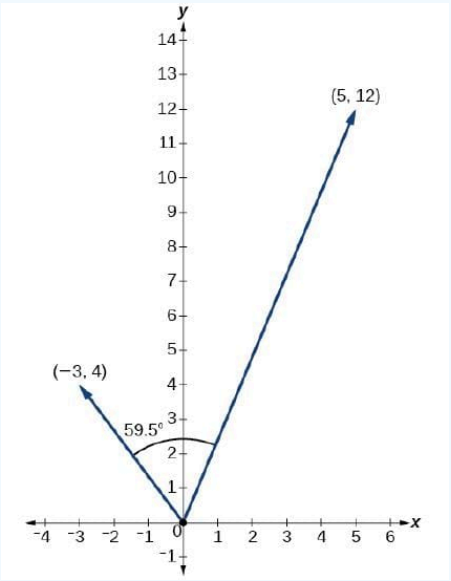


Figure 4.6.19

Example 4.6.11D: Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140 °. A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane? See Figure 4.6.20.

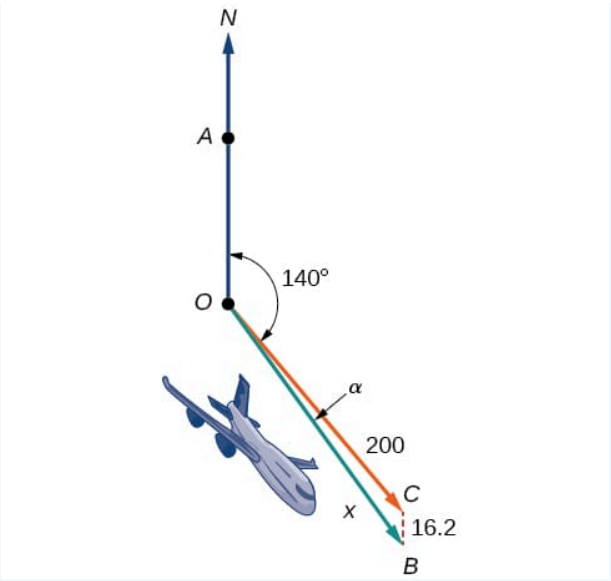


Figure 4.6.20

Solution

The ground speed is represented by x in the diagram, and we need to find the angle α in order to calculate the adjusted bearing, which will be $140\,°+\alpha$.

Notice in Figure 4.6.20, that angle $\angle BCO$ must be equal to angle $\angle AOC$ by the rule of alternating interior angles, so angle $\angle BCO$ is 140°. We can find x by the Law of Cosines:

$$\begin{aligned} x^2 &= (16.2)^2 + (200)^2 - 2(16.2)(200)\cos(140°) \\ x^2 &= 45,226.41 \\ x &= \sqrt{45,226.41} \\ x &= 212.7 \end{aligned}$$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

$$\begin{aligned} \frac{\sin\alpha}{16.2} &= \frac{\sin(140°)}{212.7} \\ \sin\alpha &= \frac{16.2\sin(140°)}{212.7} \\ &= 0.04896 \\ \sin^{-1}(0.04896) &= 2.8° \end{aligned}$$

Therefore, the plane has a SE bearing of $140\,°+2.8\,° = 142.8\,°$. The ground speed is 212.7 miles per hour.

Media: Access these online resources for additional instruction and practice with vectors.

- Introduction to Vectors
- Vector Operations
- The Unit Vector

Key Concepts

- The position vector has its initial point at the origin. See Example 4.6.1.
- If the position vector is the same for two vectors, they are equal. See Example 4.6.2.

- Vectors are defined by their magnitude and direction. See Example 4.6.3.
- If two vectors have the same magnitude and direction, they are equal. See Example 4.6.4.
- Vector addition and subtraction result in a new vector found by adding or subtracting corresponding elements. See Example 4.6.5.
- Scalar multiplication is multiplying a vector by a constant. Only the magnitude changes; the direction stays the same. See Example 4.6.6 and Example 4.6.7.
- Vectors are comprised of two components: the horizontal component along the positive x-axis, and the vertical component along the positive y-axis. See Example 4.6.8.
- The unit vector in the same direction of any nonzero vector is found by dividing the vector by its magnitude.
- The magnitude of a vector in the rectangular coordinate system is $|v| = \sqrt{a^2 + b^2}$. See Example 4.6.9.
- In the rectangular coordinate system, unit vectors may be represented in terms of ii and jj where i represents the horizontal component and j represents the vertical component. Then, $v = ai + bj$ is a scalar multiple of v by real numbers a and b. See Example 4.6.10 and Example 4.6.11.
- Adding and subtracting vectors in terms of i and j consists of adding or subtracting corresponding coefficients of i and corresponding coefficients of j. See Example 4.6.12.
- A vector $v = ai + bj$ is written in terms of magnitude and direction as $v = |v| \cos\theta i + |v| \sin\theta j$. See Example 4.6.13.
- The dot product of two vectors is the product of the i terms plus the product of the j terms. See Example 4.6.14.
- We can use the dot product to find the angle between two vectors. Example 4.6.15 and Example 4.6.16.
- Dot products are useful for many types of physics applications. See Example 4.6.17.

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Answers

Exercise 4.6.1

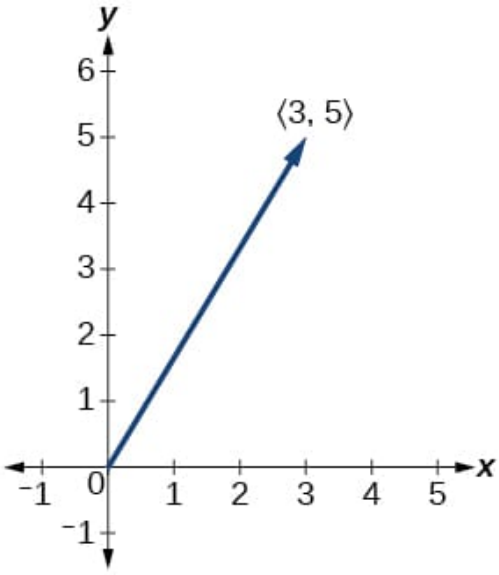


Figure 4.6.5

Exercise 4.6.2

$3u = \langle 15, 12 \rangle$

Exercise 4.6.3

$u = 8i - 11j$

Exercise 4.6.4

$v = \sqrt{34}\cos(59\,°)i + \sqrt{34}\sin(59\,°)j$

Magnitude = 34

$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.04\,°$

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