MATH 098: INTERMEDIATE ALGEBRA FOR CALCULUS



Highline College

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Math 098: Intermediate Algebra for Calculus

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Glossary

Glossary

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

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1.1: Introduction to Functions and Equations

Difference Between an Equation and a Function

By Intermediate Algebra Students Andrew Medina and Nelly Kihiu

The difference between an equation and function would be that an equation shows “the equality between two expressions”, as indicated by the article “Difference between Equations and Functions” at www.differencebetween.info, and is good for when we must solve a variable when the problem contains another constant or static variable. Functions differ from equations because they show the relationships between inputs and outputs, where one input for the variable, which can change (being the independent variable “x”), will produce a unique output of a dependent variable “y”. So, functions show the mathematical relationships between inputs and outputs.

For example, these variables could be the human population number pertaining to a certain year. The data sets that hold these numbers could be modeled through quadratic regression that fits the curve of data points and would help us predict the population amount at a future year, where the year is the input variable that outputs the population number. A real-world example of this would be the Nihonmachi Fence at Chiyo’s Garden in Seattle that portrays the “growth, decline, and rebirth” of the Seattle’s Japanese American community. In the picture below, the height of the slats refer to the population of the community at a specific year, which depicts a curve of a function with the year being the input and the population being the output. It begins with the year the first steam ship brought over Japanese immigrants in 1896 as indicated by Wing Luke Museum’s website . The artist would have taken the years and the associated population values and through modeling the data the artist would be able to visualize its mathematical representation and turn it into this public and cultural work of art.

References:

Factoring in Algebra. (n.d.). Retrieved April 17, 2022, from https://www.mathsisfun.com/algebra/factoring.html?msclkid=e2d532edbedc11ec91cf522476311c99 Links to an external site.

Flynn, A. (2018, December 16). *How do you use factoring in real life situations?* Greedhead.net. Retrieved April 17, 2022, from https://greedhead.net/how-do-you-use-factoring-in-real-life-situations/#How_do_you_use_factoring_in_real_life_situations

Difference between Equations and Functions. Retrieved April 18, 2022 from http://www.differencebetween.info/difference-between-equations-and-functions

Challenge 6: Is there any Real World reason to factor a quadratic? YouTube Video by MathMachines. https://www.youtube.com/watch?v=_Tf9FeqUVfM&ab_channel=MathMachines

Wing Luke Museum of the Asian Pacific American Experience. *Chiyo’s Garden- Nihonmachi Fence*. http://www.wingluke.org/single-exhibit/?mep_event=759Links to an external site.

Photo of Nihonmachi Fence taken by Alan Alabastro. https://www.alabastrophoto.com/index
Photo source: https://southseattleemerald.com/nihonmachi-fence-2_photo-by-alan-alabastro.

David Arnold. *2.2: Zeros of Polynomials*, LibreTexts (2021, February 15). College of the Redwoods. https://math.libretexts.org/@go/page/58416.

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1.2: Relations and Functions

Learning Objectives

By the end of this section, you will be able to:

- Find the domain and range of a relation
- Determine if a relation is a function
- Find the value of a function

Before you get started, take this readiness quiz.

1. Evaluate $3x-5$ when $x=-2$.
   If you missed this problem, review [link].
2. Evaluate $2x^2-x-3$ when $x=a$.
   If you missed this problem, review [link].
3. Simplify: $7x-1-4x+5$.
   If you missed this problem, review [link].

Find the Domain and Range of a Relation

As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number and our birthday are matched to our name. There is a relationship between our name and each of those items.

When your professor gets her class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a **relation**.

$$(\text{Student name},\ \text{Student ID }\#)$$

The set of all the names of the students in the class is called the **domain** of the relation and the set of all student ID numbers paired with these students is the range of the relation.

There are many similar situations where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation.

Definition: Relation

A **relation** is any set of ordered pairs, (x,y). All the x-values in the ordered pairs together make up the **domain**. All the y-values in the ordered pairs together make up the **range**.

Example 1.2.1

For the relation $(1,1),(2,4),(3,9),(4,16),(5,25)$:

a. Find the domain of the relation.
b. Find the range of the relation.

Answer

$\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$

ⓐ The domain is the set of all x-values of the relation. $\{1,2,3,4,5\}$

ⓑ The range is the set of all y-values of the relation. $\{1,4,9,16,25\}$

Example 1.2.2

For the relation $\{(1,1),(2,8),(3,27),(4,64),(5,125)\}$

a. Find the domain of the relation.
b. Find the range of the relation.

Answer a

$\{1,2,3,4,5\}$

Answer b

$\{1,8,27,64,125\}$

Example 1.2.3

For the relation $\{(1,3),(2,6),(3,9),(4,12),(5,15)\}$

a. Find the domain of the relation.
b. Find the range of the relation.

Answer a

$\{1,2,3,4,5\}$

Answer b

$\{3,6,9,12,15\}$

MAPPING

A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

Example 1.2.4

Use the **mapping** of the relation shown to

a. list the ordered pairs of the relation,
b. find the domain of the relation, and
c. find the range of the relation.



Answer

ⓐ The arrow shows the matching of the person to their birthday. We create ordered pairs with the person's name as the *x*-value and their birthday as the *y*-value.

{(Alison, April 25), (Penelope, May 23), (June, August 2), (Gregory, September 15), (Geoffrey, January 12), (Lauren, May 10), (Stephen, July 24), (Alice, February 3), (Liz, August 2), (Danny, July 24)}

ⓑ The domain is the set of all x-values of the relation.

{Alison, Penelope, June, Gregory, Geoffrey, Lauren, Stephen, Alice, Liz, Danny}

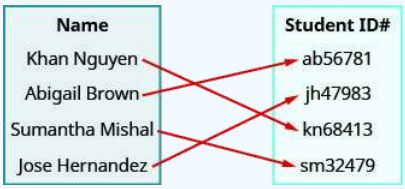
ⓒ The range is the set of all y-values of the relation.

{January 12, February 3, April 25, May 10, May 23, July 24, August 2, September 15}

Example 1.2.5

Use the mapping of the relation shown to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ (Khanh Nguyen, kn68413), (Abigail Brown, ab56781), (Sumantha Mishal, sm32479), (Jose Hern and ez, jh47983)

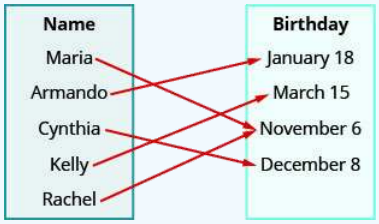
ⓑ {Khanh Nguyen, Abigail Brown, Sumantha Mishal, Jose Hern and ez}

ⓒ {kn68413, ab56781, sm32479, jh47983}

Example 1.2.6

Use the mapping of the relation shown to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ (Maria, November 6), (Arm and o, January 18), (Cynthia, December 8), (Kelly, March 15), (Rachel, November 6)

ⓑ {Maria, Arm and o, Cynthia, Kelly, Rachel}

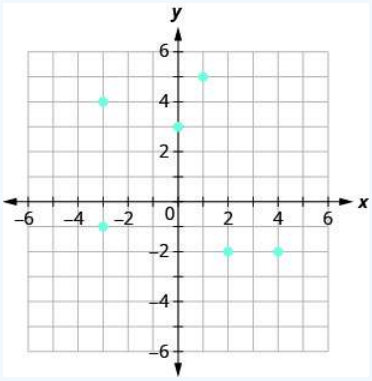
ⓒ{November 6, January 18, December 8, March 15}

A graph is yet another way that a relation can be represented. The set of ordered pairs of all the points plotted is the relation. The set of all x-coordinates is the domain of the relation and the set of all y-coordinates is the range. Generally we write the numbers in ascending order for both the domain and range.

Example 1.2.7

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ The ordered pairs of the relation are:

$$\{(1,5),(-3,-1),(4,-2),(0,3),(2,-2),(-3,4)\}.$$

ⓑ The domain is the set of all x-values of the relation: $\{-3,0,1,2,4\}$

Notice that while -3 repeats, it is only listed once.

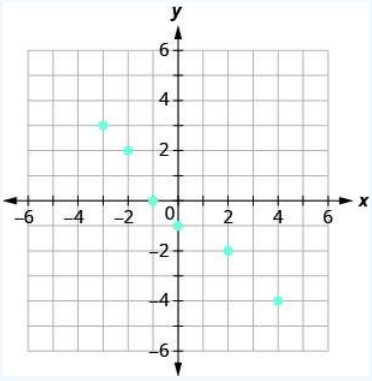
ⓒ The range is the set of all y-values of the relation: $\{-2,-1,3,4,5\}$

Notice that while -2 repeats, it is only listed once.

Example 1.2.8

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

a. $(-3,3),(-2,2),(-1,0),(0,-1),(2,-2),(4,-4)$

b. $\{-3,-2,-1,0,2,4\}$

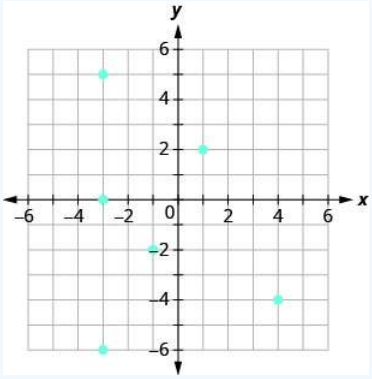
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c. $\{3, 2, 0, -1, -2, -4\}$

Example 1.2.9

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ $(-3, 0), (-3, 5), (-3, -6), (-1, -2), (1, 2), (4, -4)$
ⓑ $\{-3, -1, 1, 4\}$
ⓒ $\{-6, 0, 5, -2, 2, -4\}$

Determine if a Relation is a Function

A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

Definition: Function

A **function** is a relation that assigns to each element in its domain exactly one element in the range.

The birthday example from Example helps us understand this definition. Every person has a birthday but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24th as their birthday and that June and Liz share August 2nd. Since each person has exactly one birthday, the relation in Example is a function.

The relation shown by the graph in Example includes the ordered pairs $(-3, -1)$ and $(-3, 4)$. Is that okay in a function? No, as this is like one person having two different birthdays.

Example 1.2.10

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

a. $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$
b. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

Answer

ⓐ $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

(i) Each x-value is matched with only one y-value. So this relation is a function.

(ii) The domain is the set of all x-values in the relation.
The domain is: $\{-3, -2, -1, 0, 1, 2, 3\}$

(iii) The range is the set of all y-values in the relation. Notice we do not list range values twice.
The range is: $\{27, 8, 1, 0\}$

ⓑ $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

(i) The x-value 9 is matched with two y-values, both 3 and -3. So this relation is not a function.

(ii) The domain is the set of all x-values in the relation. Notice we do not list domain values twice.
The domain is: $\{0, 1, 2, 4, 9\}$

(iii) The range is the set of all y-values in the relation.
The range is: $\{-3, -2, -1, 0, 1, 2, 3\}$

Example 1.2.11

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the function.

a. $\{(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6)\}$
b. $\{(8, -4), (4, -2), (2, -1), (0, 0), (2, 1), (4, 2), (8, 4)\}$

Answer

ⓐ Yes; $\{-3, -2, -1, 0, 1, 2, 3\}$
$\{-6, -4, -2, 0, 2, 4, 6\}$
ⓑ No; $\{0, 2, 4, 8\}$
$\{-4, -2, -1, 0, 1, 2, 4\}$

Example 1.2.12

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

a. $\{(27, -3), (8, -2), (1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$
b. $\{(7, -3), (-5, -4), (8, -0), (0, 0), (-6, 4), (-2, 2), (-1, 3)\}$

Answer

ⓐ No; $\{0, 1, 8, 27\}$
$\{-3, -2, -1, 0, 2, 2, 3\}$
ⓑ Yes; $\{7, -5, 8, 0, -6, -2, -1\}$
$\{-3, -4, 0, 4, 2, 3\}$

Example 1.2.13

Use the mapping to

a. determine whether the relation is a function
b. find the domain of the relation
c. find the range of the relation.

NAME	PHONE

Lydia	321-549-3327 427-658-2314
Eugene	321-964-7324
Janet	684-358-7961
Rick	
Marty	684-369-7231 798-367-8541

Answer

ⓐ Both Lydia and Marty have two phone numbers. So each x-value is not matched with only one y-value. So this relation is not a function.

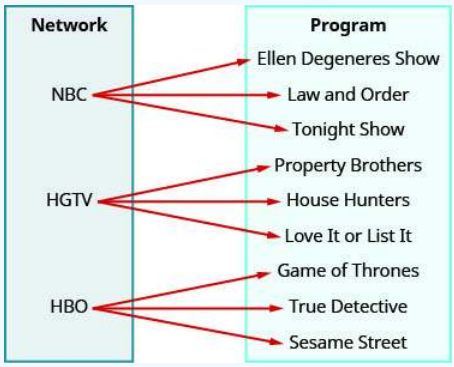
ⓑ The domain is the set of all x-values in the relation. The domain is: {Lydia, Eugene, Janet, Rick, Marty}

ⓒ The range is the set of all y-values in the relation. The range is:

$\{321-549-3327, 427-658-2314, 321-964-7324, 684-358-7961, 684-369-7231, 798-367-8541\}$

Example 1.2.14

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



Answer

ⓐ no ⓑ {NBC, HGTV, HBO} ⓒ {Ellen Degeneres Show, Law and Order, Tonight Show, Property Brothers, House Hunters, Love it or List it, Game of Thrones, True Detective, Sesame Street}

Example 1.2.15

Use the mapping to

a. determine whether the relation is a function
b. find the domain of the relation
c. find the range of the relation.

Neal	123-567-4839 work 231-378-5941 cell
Krystal	743-469-9731 cell
Kelvin	567-534-2970 work
George	684-369-7231 cell
Christa	684-369-7231 cell
Mike	639-847-6971 cell

Answer

ⓐ No ⓑ {Neal, Krystal, Kelvin, George, Christa, Mike} ⓒ {123-567-4839 work, 231-378-5941 cell, 743-469-9731 cell, 567-534-2970 work, 684-369-7231 cell, 798-367-8541 cell, 639-847-6971 cell}

In algebra, more often than not, functions will be represented by an equation. It is easiest to see if the equation is a function when it is solved for *y*. If each value of *x* results in only one value of *y*, then the equation defines a function.

? Example 1.2.16

Determine whether each equation is a function.

a. $2x+y=7$
b. $y=x^2+1$
c. $x+y^2=3$

Answer

ⓐ $2x+y=7$

For each value of *x*, we multiply it by -2 and then add 7 to get the *y*-value

	$y=-2x+7$
For example, if $x=3$:	$y=-2\cdot 3+7$
	$y=1$

We have that when $x=3$, then $y=1$. It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓑ $y=x^2+1$

For each value of *x*, we square it and then add 1 to get the *y*-value.

	$y=x^2+1$
For example, if $x=2$:	$y=2^2+1$
	$y=5$

We have that when $x=2$, then $y=5$. It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓒ

	$x+y^2=3$
Isolate the y term.	$y^2=-x+3$
Let's substitute $x=2$.	$y^2=-2+3$
	$y^2=1$
This give us two values for y.	$y=1\ y=-1$

We have shown that when $x=2$, then $y=1$ and $y=-1$. It would work similarly for any value of x. Since each value of x does not corresponds to only one value of y the equation does not define a function.

Example 1.2.17

Determine whether each equation is a function.

a. $4x+y=-3$
b. $x+y^2=1$
c. $y-x^2=2$

Answer

ⓐ yes ⓑ no ⓒ yes

Example 1.2.18

Determine whether each equation is a function.

a. $x+y^2=4$
b. $y=x^2-7$
c. $y=5x-4$

Answer

ⓐ no ⓑ yes ⓒ yes

Find the Value of a Function

It is very convenient to name a function and most often we name it f, g, h, F, G, or H. In any function, for each x-value from the domain we get a corresponding y-value in the range. For the function f, we write this range value y as $f(x)$. This is called **function notation** and is read f of x or the value of f at x. In this case the parentheses does not indicate multiplication.

Definition: Function Notation

For the function $y=f(x)$

f is the name of the function
x is the domain value
$f(x)$ is the range value y corresponding to the value x

We read $f(x)$ as f of x or the value of f at x.

We call x the independent variable as it can be any value in the domain. We call y the dependent variable as its value depends on x.

INDEPENDENT AND DEPENDENT VARIABLES

For the function $y = f(x)$,

x is the independent variable as it can be any value in the domain
y the dependent variable as its value depends on x

Much as when you first encountered the variable *x*, function notation may be rather unsettling. It seems strange because it is new. You will feel more comfortable with the notation as you use it.

Let's look at the equation $y = 4x - 5$. To find the value of *y* when $x = 2$, we know to substitute $x = 2$ into the equation and then simplify.

	$y = 4x - 5$
Let x=2.	$y = 4 \cdot 2 - 5$
	$y = 3$

The value of the function at $x = 2$ is 3.

We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

	$f(x) = 4x - 5$
Let x=2.	$f(2) = 4 \cdot 2 - 5$
	$f(2) = 3$

The value of the function at $x = 2$ is 3.

This process of finding the value of $f(x)$ for a given value of *x* is called *evaluating the function.*

Example 1.2.19

For the function $f(x) = 2x^2 + 3x - 1$, evaluate the function.

a. $f(3)$
b. $f(-2)$
c. $f(a)$

Answer

ⓐ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(3)$, substitute 3 for *x*.	$f(3) = 2(3)^2 + 3 \cdot 3 - 1$
Simplify.	$f(3) = 2 \cdot 9 + 3 \cdot 3 - 1$
	$f(3) = 18 + 9 - 1$
	$f(3) = 26$

ⓑ

		$f(x) = 2x^2 + 3x - 1$
To evaluate $f(-2)$, substitute -2 for *x*.		$f(-2) = 2(-2)^2 + 3(-2) - 1$

Simplify.		$f(-2) = 2 \cdot 4 + (-6) - 1$
		$f(-2) = 8 + (-6) - 1$
		$f(-2) = 1$

ⓒ

	$f(x) = 2x^2 + 3x - 1$
To evaluate f(a),f(a), substitute a for x.	$f(a) = 2(a)^2 + 3 \cdot a - 1$
Simplify.	$f(a) = 2a^2 + 3a - 1$

Example 1.2.20

For the function $f(x) = 3x^2 - 2x + 1$, evaluate the function.

a. $f(3)$
b. $f(-1)$
c. $f(t)$

Answer

ⓐ $f(3) = 22$ ⓑ $f(-1) = 6$ ⓒ $f(t) = 3t^2 - 2t - 1$

Example 1.2.21

For the function $f(x) = 2x^2 + 4x - 3$, evaluate the function.

a. $f(2)$
b. $f(-3)$
c. $f(h)$

Answer

ⓐ $(2) = 13$ ⓑ $f(-3) = 3$
ⓒ $f(h) = 2h2 + 4h - 3$

In the last example, we found $f(x)$ for a constant value of x. In the next example, we are asked to find $g(x)$ with values of x that are variables. We still follow the same procedure and substitute the variables in for the x.

Example 1.2.22

For the function $g(x) = 3x - 5$, evaluate the function.

a. $g(h^2)$
b. $g(x + 2)$
c. $g(x) + g(2)$

Answer

ⓐ

			$g(x) = 3x - 5$
To evaluate $g(h^2)$, substitute h^2 for x.			$g(h^2) = 3h^2 - 5$

			$g(h^2) = 3h^2 - 5$

ⓑ

	$g(x) = 3x - 5$
To evaluate $g(x+2)$, substitute $x+2$ for x.	$g(x+2) = 3(x+2) - 5$
Simplify.	$g(x+2) = 3x + 6 - 5$
	$g(x+2) = 3x + 1$

ⓒ

		$g(x) = 3x - 5$
To evaluate $g(x) + g(2)$, first find $g(2)$.		$g(2) = 3 \cdot 2 - 5$
		$g(2) = 1$
Now find $g(x) + g(2)$		$g(x) + g(2) = 3x - 5 + 1$ $g(x) \quad g(2)$
Simplify.		$g(x) + g(2) = 3x - 5 + 1$
		$g(x) + g(2) = 3x - 4$

Notice the difference between part ⓑ and ⓒ. We get $g(x+2) = 3x+1$ and $g(x) + g(2) = 3x - 4$. So we see that $g(x+2) \neq g(x) + g(2)$.

Example 1.2.23

For the function $g(x) = 4x - 7$, evaluate the function.

a. $g(m^2)$
b. $g(x-3)$
c. $g(x) - g(3)$

Answer

ⓐ $4m^2 - 7$ ⓑ $4x - 19$
ⓒ $x - 12$

Example 1.2.24

For the function $h(x) = 2x + 1$, evaluate the function.

a. $h(k^2)$
b. $h(x+1)$
c. $h(x) + h(1)$

Answer

ⓐ $2k^2 + 1$ ⓑ $2x + 3$
ⓒ $2x + 4$

Many everyday situations can be modeled using functions.

Example 1.2.25

The number of unread emails in Sylvia's account is 75. This number grows by 10 unread emails a day. The function $N(t) = 75 + 10t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(5)$. Explain what this result means.

Answer

ⓐ The number of unread emails is a function of the number of days. The number of unread emails, *N*, depends on the number of days, *t*. Therefore, the variable *N*, is the dependent variable and the variable tt is the independent variable.

ⓑ Find $N(5)$. Explain what this result means.

	$N(t) = 75 + 10t$
Substitute in t=5.t=5.	$N(5) = 75 + 10 \cdot 5$
Simplify.	$N(5) = 75 + 50$
	$N(5) = 125$

Since 5 is the number of days, $N(5)$, is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

Example 1.2.26

The number of unread emails in Bryan's account is 100. This number grows by 15 unread emails a day. The function $N(t) = 100 + 15t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(7)]$. Explain what this result means.

Answer

ⓐ *t* IND; *N* DEP ⓑ 205; the number of unread emails in Bryan's account on the seventh day.

Example 1.2.27

The number of unread emails in Anthony's account is 110. This number grows by 25 unread emails a day. The function $N(t) = 110 + 25t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(14)$. Explain what this result means.

Answer

ⓐ *t* IND; *N* DEP ⓑ 460; the number of unread emails in Anthony's account on the fourteenth day

Access this online resource for additional instruction and practice with relations and functions.

- Introduction to Functions

Key Concepts

- **Function Notation:** For the function $y = f(x)$
  - f is the name of the function
  - x is the domain value

- $f(x)$ is the range value y corresponding to the value x
  We read $f(x)$ as f of x or the value of f at x.

- **Independent and Dependent Variables:** For the function $y = f(x)$,
  - x is the independent variable as it can be any value in the domain
  - y is the dependent variable as its value depends on x

Glossary

domain of a relation

The domain of a relation is all the x-values in the ordered pairs of the relation.

function

A function is a relation that assigns to each element in its domain exactly one element in the range.

mapping

A mapping is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

range of a relation

The range of a relation is all the y-values in the ordered pairs of the relation.

relation

A relation is any set of ordered pairs,(x,y).(x,y). All the x-values in the ordered pairs together make up the domain. All the y-values in the ordered pairs together make up the range.

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1.2E: Exercises

Practice Makes Perfect

Find the Domain and Range of a Relation

In the following exercises, for each relation a. find the domain of the relation b. find the range of the relation.

1. $\{(1,4),(2,8),(3,12),(4,16),(5,20)\}$

Answer

a. $\{1,2,3,4,5\}$b. $\{4,8,12,16,20\}$

2. $\{(1,-2),(2,-4),(3,-6),(4,-8),(5,-10)\}$

3. $\{(1,7),(5,3),(7,9),(-2,-3),(-2,8)\}$

Answer

a. $\{1,5,7,-2\}$b. $\{7,3,9,-3,8\}$

4. $\{(11,3),(-2,-7),(4,-8),(4,17),(-6,9)\}$

In the following exercises, use the mapping of the relation to a. list the ordered pairs of the relation, b. find the domain of the relation, and c. find the range of the relation.

5.



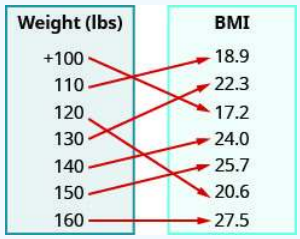
Answer

a. (Rebecca, January 18), (Jennifer, April 1), (John, January 18), (Hector, June 23), (Luis, February 15), (Ebony, April 7), (Raphael, November 6), (Meredith, August 19), (Karen, August 19), (Joseph, July 30)
b. {Rebecca, Jennifer, John, Hector, Luis, Ebony, Raphael, Meredith, Karen, Joseph}
c. {January 18, April 1, June 23, February 15, April 7, November 6, August 19, July 30}

6.



7. For a woman of height $5'4''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5–24.9$is considered healthy.



Answer

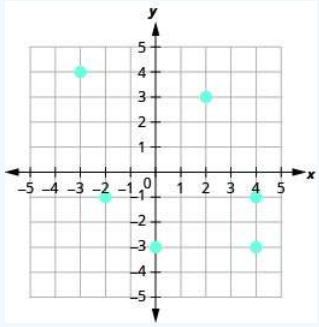
a. $(+100, 17.2), (110, 18.9), (120, 20.6), (130, 22.3), (140, 24.0), (150, 25.7), (160, 27.5)$ b. $\{+100, 110, 120, 130, 140, 150, 160\}$c. $\{17.2, 18.9, 20.6, 22.3, 24.0, 25.7, 27.5\}$

8. For a man of height $5'11''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5–24.9$is considered healthy.



In the following exercises, use the graph of the relation to a. list the ordered pairs of the relation b. find the domain of the relation c. find the range of the relation.

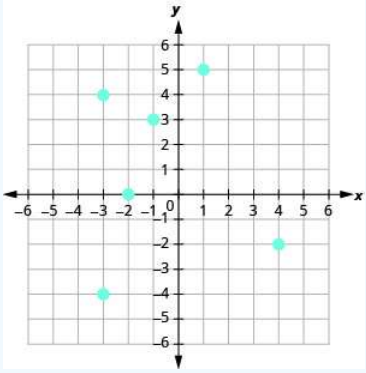
9.



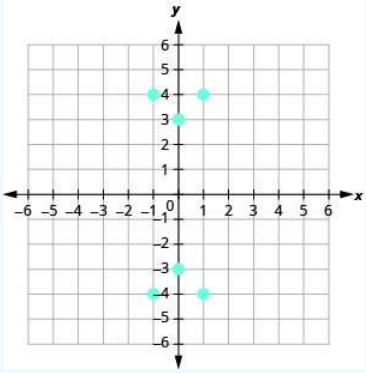
Answer

a. $(2, 3), (4, -3), (-2, -1), (-3, 4), (4, -1), (0, -3)$b. $\{-3, -2, 0, 2, 4\}$
c. $\{-3, -1, 3, 4\}$

10.



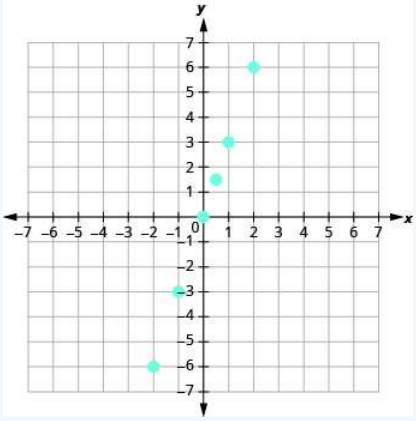
11.



Answer

a. $(1,4),(1,-4),(-1,4),(-1,-4),(0,3),(0,-3)$b. $\{-1,0,1\}$c. $\{-4,-3,3,4\}$

12.



Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to a. determine whether the relation is a function, b. find the domain of the relation, and c. find the range of the relation.

13. $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

Answer

a. yes b. $\{-3,-2,-1,0,1,2,3\}$c. $\{9,4,1,0\}$

14. $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

15. $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

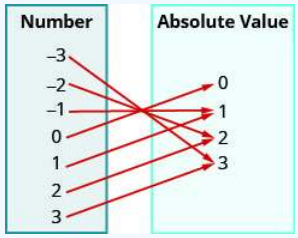
Answer

a. yes b. $\{-3,-2,-1,0,1,2,3\}$c. $\{0,1,8,27\}$

16. $\{(-3,-27),(-2,-8),(-1,-1),(0,0),(1,1),(2,8),(3,27)\}$

In the following exercises, use the mapping to a. determine whether the relation is a function, b. find the domain of the function, and c. find the range of the function.

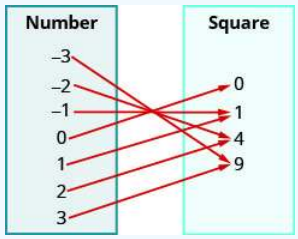
17.



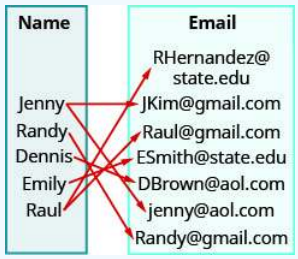
Answer

a. yes b. $\{-3,-2,-1,0,1,2,3\}$c. $\{0,1,2,3\}$

18.



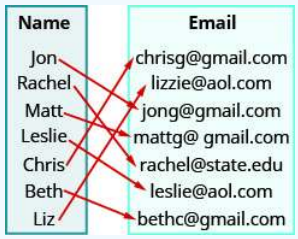
19.



Answer

a. no b. {Jenny, R and y, Dennis, Emily, Raul} c. {RHern and ez@state.edu, JKim@gmail.com, Raul@gmail.com, ESmith@state.edu, DBroen@aol.com, jenny@aol.cvom, R and y@gmail.com}

20.



In the following exercises, determine whether each equation is a function.

21. a. $2x+y=-3$
b. $y=x^2$
c. $x+y^2=-5$

Answer

a. yes b. yes c. no

22. a. $y=3x-5$
b. $y=x^3$
c. $2x+y^2=4$

23. a. $y-3x^3=2$
b. $x+y^2=3$
c. $3x-2y=6$

Answer

a. yes b. no c. yes

24. a. $2x-4y=8$
b. $-4=x^2-y$
c. $y^2=-x+5$

Find the Value of a Function

In the following exercises, evaluate the function: a. $f(2)$ b. $f(-1)$ c. $f(a)$.

25. $f(x)=5x-3$

Answer

a. $f(2)=7$ b. $f(-1)=-8$ c. $f(a)=5a-3$

26. $f(x)=3x+4$

27. $f(x)=-4x+2$

Answer

a. $f(2)=-6$ b. $f(-1)=6$ c. $f(a)=-4a+2$

28. $f(x)=-6x-3$

29. $f(x)=x^2-x+3$

Answer

a. $f(2)=5$ b. $f(-1)=5$
c. $f(a)=a^2-a+3$

30. $f(x)=x^2+x-2$

31. $f(x)=2x^2-x+3$

Answer

a. $f(2)=9$ b. $f(-1)=6$
c. $f(a)=2a^2-a+3$

32. $f(x)=3x^2+x-2$

In the following exercises, evaluate the function: a. $g(h^2)$ b. $g(x+2)$ c. $g(x)+g(2)$.

33. $g(x)=2x+1$

Answer

a. $g(h^2)=2h^2+1$
b. $g(x+2)=4x+5$
c. $g(x)+g(2)=2x+6$

34. $g(x)=5x-8$

35. $g(x)=-3x-2$

Answer

a. $g(h^2)=-3h^2-2$
b. $g(x+2)=-3x-8$
c. $g(x)+g(2)=-3x-10$

36. $g(x)=-8x+2$

37. $g(x)=3-x$

Answer

a. $g(h^2)=3-h^2$
b. $g(x+2)=1-x$
c. $g(x)+g(2)=4-x$

38. $g(x)=7-5x$

In the following exercises, evaluate the function.

39. $f(x)=3x^2-5x;\ f(2)$

Answer

2

40. $g(x) = 4x^2 - 3x$; $g(3)$

41. $F(x) = 2x^2 - 3x + 1$; $F(-1)$

Answer

6

42. $G(x) = 3x^2 - 5x + 2$; $G(-2)$

43. $h(t) = 2|t - 5| + 4$; $f(-4)$

Answer

22

44. $h(y) = 3|y - 1| - 3$; $h(-4)$

45. $f(x) = x + 2x - 1$; $f(2)$

Answer

4

46. $g(x) = x - 2x + 2$; $g(4)$

In the following exercises, solve.

47. The number of unwatched shows in Sylvia's DVR is 85. This number grows by 20 unwatched shows per week. The function $N(t) = 85 + 20t$ represents the relation between the number of unwatched shows, *N*, and the time, *t*, measured in weeks.

a. Determine the independent and dependent variable.

b. Find $N(4)$. Explain what this result means

Answer

a. *t* IND; *N* DEP
b. $N(4) = 165$ the number of unwatched shows in Sylvia's DVR at the fourth week.

48. Every day a new puzzle is downloaded into Ken's account. Right now he has 43 puzzles in his account. The function $N(t) = 43 + t$ represents the relation between the number of puzzles, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.

b. Find $N(30)$. Explain what this result means.

49. The daily cost to the printing company to print a book is modeled by the function $C(x) = 3.25x + 1500$ where *C* is the total daily cost and *x* is the number of books printed.

a. Determine the independent and dependent variable.

b. Find $N(0)$. Explain what this result means.

c. Find $N(1000)$. Explain what this result means.

Answer

a. x IND; C DEP
b. $N(0) = 1500$ the daily cost if no books are printed
c. $N(1000) = 4750$the daily cost of printing 1000 books

50. The daily cost to the manufacturing company is modeled by the function $C(x) = 7.25x + 2500$ where $C(x)$ is the total daily cost and x is the number of items manufactured.

a. Determine the independent and dependent variable.

b. Find $C(0)$. Explain what this result means.

c. Find $C(1000)$. Explain what this result means.

Writing Exercises

51. In your own words, explain the difference between a relation and a function.

52. In your own words, explain what is meant by domain and range.

53. Is every relation a function? Is every function a relation?

54. How do you find the value of a function?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the domain and range of a relation.			
determine if a relation is a function.			
find the value of a function.			

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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1.3: The Graph of a Function

Rene Descartes (1596-1650) was a French philosopher and mathematician who is well known for the famous phrase “cogito ergo sum” (I think, therefore I am), which appears in his Discours de la methode pour bien conduire sa raison, et chercher la verite dans les sciences (Discourse on the Method of Rightly Conducting the Reason, and Seeking Truth in the Sciences). In that same treatise, Descartes introduces his coordinate system, a method for representing points in the plane via pairs of real numbers. Indeed, the Cartesian plane of modern day is so named in honor of Rene Descartes, who some call the “Father of Modern Mathematics.”

Descartes’ work, which forever linked geometry and algebra, was continued in an appendix to Discourse on Method, entitled La Geometrie, which some consider the beginning of modern mathematics. Certainly both Newton and Leibniz, in developing the Calculus, built upon the foundation provided in this work by Descartes.

A Cartesian Coordinate System consists of a pair of axes, usually drawn at right angles to one another in the plane, one horizontal (labeled x) and one vertical (labeled y), as shown in the Figure 1.3.1. The quadrants are numbered I, II, III, and IV, in counterclockwise order, and samples of ordered pairs of the form (x, y) are shown in each quadrant of the Cartesian coordinate system in Figure 1.3.1.

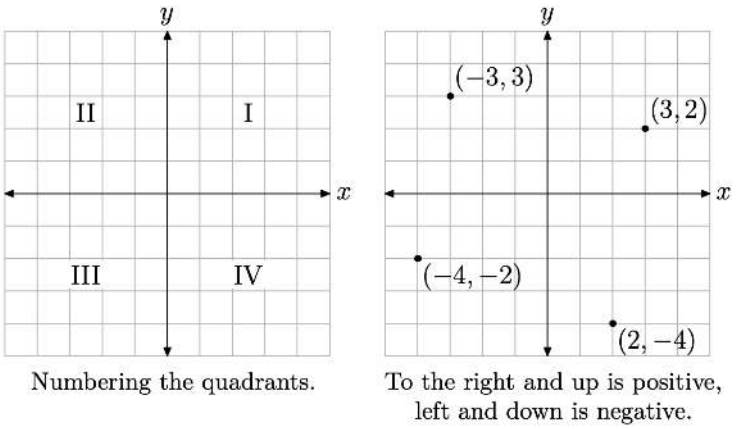


Figure 1.3.1. The Cartesian coordinate system.

Now, suppose that we have a relation

$$R = \{(1, 2), (3, 1), (3, 4), (4, 3)\}$$

Recall that relation is the name given to a collection of ordered pairs. In Figure 1.3.2(b) we’ve plotted each of the ordered pairs in the relation R. This is called the graph of the relation R.

Definition

The graph of a relation is the collection of all ordered pairs of the relation. These are usually represented as points in a Cartesian coordinate system.

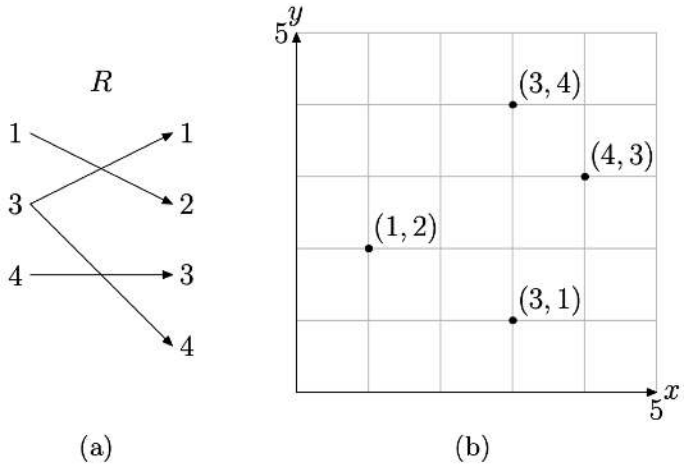


Figure 1.3.2 A mapping diagram and its graph.

In Figure 1.3.2(a), we've created a mapping diagram of the ordered pairs. Note that the domain object 3 is paired with two range elements, namely 1 and 4. Hence the relation R is not a function. It is interesting to note that there are two points in the graph of R in Figure 1.3.2(b) that have the same first coordinate, namely (3, 1) and (3, 4). This is a signal that the graph of the relation R is not a function. In the next section we will discuss the Vertical Line Test, which will use this dual use of the first coordinate to determine when a relation is a not a function.

Creating the Graph of a Function

Some texts will speak of the graph of an equation, such as "Draw the graph of the equation $y = x^2$." This instruction raises a number of difficulties.

- First, the instruction provides no direction to the reader; that is, what does the instruction mean? It's not very helpful.
- Secondly, the instruction is incorrect. You don't draw the graphs of equations. Rather, you draw the graphs of relations and/or functions. A graph is just another way of representing a function, a relation that pairs each element in its domain with exactly one element in its range.

So, what is the proper instruction? First, we will provide the formal definition of the graph of a function, then we will break it down by means of examples.

Definition

The graph of a function is the collection of all ordered pairs of the function. These are usually represented as points in a Cartesian coordinate system.

As an example, consider the function

$$f = \{(1, 2), (2, 4), (3, 1), (4, 3)\}$$

Readers will note that each object in the domain is paired with one and only one object in the range, as seen in the mapping diagram of Figure 1.3.3(a).

Thus, we have two representations of the function f, the collection of ordered pairs (3), and the mapping diagram of in Figure 1.3.3(a). A third representation of the function f is the graph of the ordered pairs of the function, shown in the Cartesian plane in Figure 1.3.3(b).

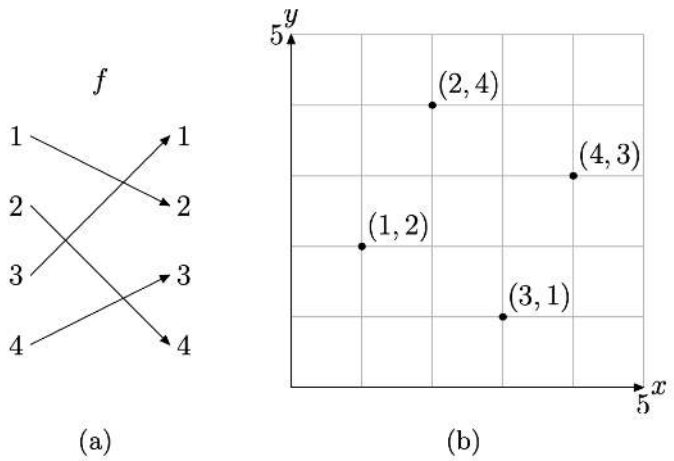


Figure 1.3.3 A mapping diagram and its graph.

When the function is represented by an equation or formula, then we adjust our definition of its graph somewhat.

Definition

The graph of f is the set of all ordered pairs $(x, f(x))$ so that x is in the domain of f. In symbols,

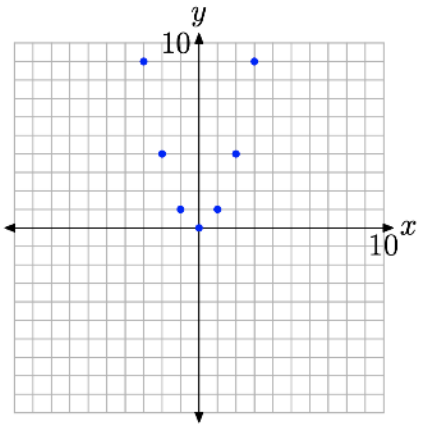
$$\text{Graph of } f = \{(x, f(x)) : x \text{ is in the domain of } f.\}$$

This last definition is most easily explained by example. So, let's define a function f that maps any real number x to the real number x^2; that is, let $f(x) = x^2$. Now, according to Definition, the graph of f is the set of all points $(x, f(x))$, such that x is in the domain of f.

The way is now clear. We begin by creating a table of points $(x, f(x))$, where x is in the domain of the function f defined by $f(x) = x^2$. The choice of x is both subjective and experimental, so we begin by choosing integer values of x between −3 and 3. We then evaluate the function at each of these x-values (e.g., $f(-3) = (-3)^2 = 9$). The results are shown in the table in Figure 1.3.4(a). We then plot the points in our table in the Cartesian plane as shown in Figure 1.3.4(b).

x	$f(x) = x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

(a)



(b)

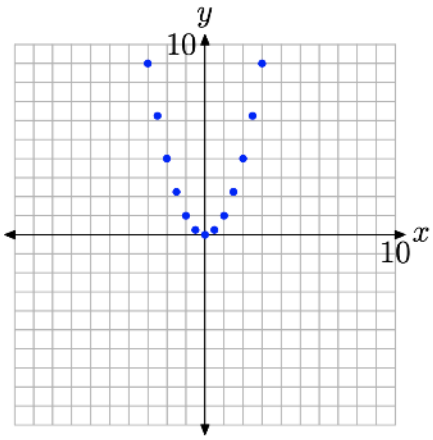
Figure 1.3.4. Plotting pairs satisfying the functional relationship defined by the equation $f(x) = x^2$.

Although this is a good start, the graph in Figure 1.3.4(b) is far from complete. Definition requires that we plot the ordered pairs $(x, f(x))$ for every value of x that is in the domain of f. We've only plotted seven such points, so we're not done. Let's add more points to the graph of f. We'll evaluate the function at each of the x-values shown in the table in Figure 1.3.5(a), then plot the

additional pairs $(x, f(x))$ from the table in the Cartesian plane, as shown in Figure 1.3.5(b).

x	$f(x) = x^2$	$(x, f(x))$
$-5/2$	$25/4$	$(-5/2, 25/4)$
$-3/2$	$9/4$	$(-3/2, 9/4)$
$-1/2$	$1/4$	$(-1/2, 1/4)$
$1/2$	$1/4$	$(1/2, 1/4)$
$3/2$	$9/4$	$(3/2, 9/4)$
$5/2$	$25/4$	$(5/2, 25/4)$

(a)



(b)

Figure 1.3.5. Plotting additional pairs $(x, f(x))$ defined by the equation $f(x) = x^2$.

We're still not finished, because we've only plotted 13 pairs $(x, f(x))$, such that $f(x) = x^2$. Definition 4 requires that we plot the ordered pairs $(x, f(x))$ for every value of x in the domain of f.

However, a pattern is certainly establishing itself, as seen in Figure 1.3.5(b). At some point, we need to "make a leap of faith," and plot all ordered pairs $(x, f(x))$, such that x is in the domain of f. This is done in Figure 1.3.6.

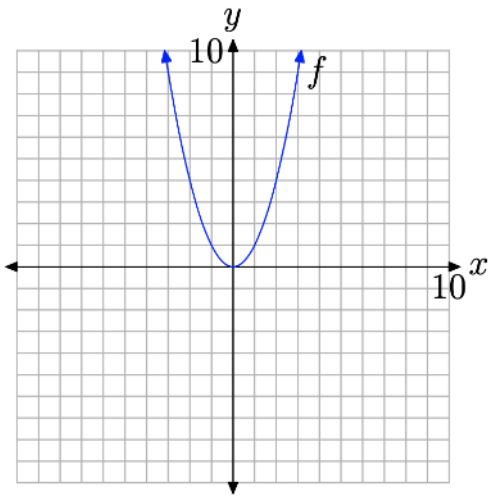


Figure 1.3.6. Plotting all pairs $(x, f(x))$ so that x is in the domain of f.

There are several important points we need to make about the final result in Figure 1.3.6.

- When we draw a smooth curve, such as that shown in Figure 1.3.6, it is important to understand that this is a simply a shortcut for plotting all pairs (x, f(x)), where $f(x) = x^2$ and x is in the domain of f.
- It is important to understand that we are NOT "connecting the dots," neither with a ruler nor with curved segments. Rather, the curve in Figure 1.3.6 is the result of plotting all of the individual pairs $(x, f(x))$.
- The "arrows" at each end of the curve have an important meaning. Much as the ellipsis at the end of the progression $2, 4, 6, \ldots$ mean "et-cetera," the arrows at each end of the curve have a similar meaning. The arrow at the end of the left-half of the curve indicates that the graph continues opening upward and to the left, while the arrow at the end of the right-half of the curve indicates that the graph continues opening upward and to the right.

Creating Graphs by Hand

We're going to look at several basic graphs, which we'll create by employing the strategy used to create the graph of $f(x) = x^2$. First, let's summarize that process.

Summary

If a function is defined by an equation, you can create the graph of the function as follows.

1. Select several values of x in the domain of the function f.
2. Use the selected values of x to create a table of pairs (x, f(x)) that satisfy the equation that defines the function f.
3. Create a Cartesian coordinate system on a sheet of graph paper. Label and scale each axis, then plot the pairs (x, f(x)) from your table on your coordinate system.
4. If the plotted pairs (x, f(x)) provide enough of a pattern for you to intuit the shape of the graph of f, make the "leap of faith" and plot all pairs that satisfy the equation defining f by drawing a smooth curve on your coordinate system. Of course, this curve should contain all previously plotted pairs.
5. If your plotted pairs do not provide enough of a pattern to determine the final shape of the graph of f, then add more pairs to your table and plot them on your Cartesian coordinate system. Continue in this manner until you are confident in the shape of the graph of f.

Let's look at an example.

Example 1.3.1

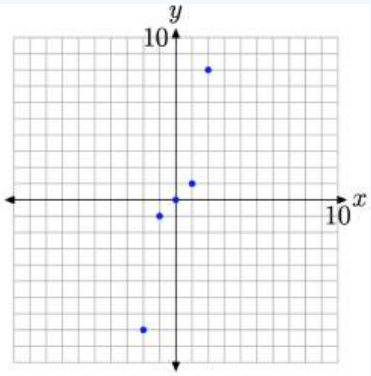
Sketch the graph of the function defined by the equation $f(x) = x^3$.

Solution

We'll start with x-values $-2, -1, 0, 1$, and 2, then use the equation $f(x) = x^3$ to determine pairs (x, f(x)) (e.g., $f(-2) = (-2)^3 = -8$). These are listed in the table in Figure 1.3.7(a). We then plot the points from the table on a Cartesian coordinate system, as shown in Figure 1.3.7(b).

x	$f(x) = x^3$	$(x, f(x))$
-2	-8	$(-2, -8)$
-1	-1	$(-1, -1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	8	$(2, 8)$

(a)



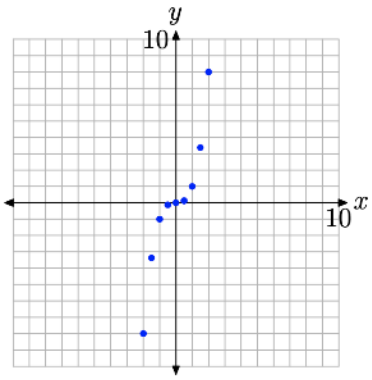
(b)

Figure 1.3.7. Plotting pairs (x, f(x)) defined by the equation $f(x) = x^3$.

We're a bit unsure of the shape of the graph of f, so we'll add a few more pairs to our table and plot them. This is shown in Figures 1.3.8(a) and (b).

x	$f(x) = x^3$	$(x, f(x))$
$-3/2$	$-27/8$	$(-3/2, -27/8)$
$-1/2$	$-1/8$	$(-1/2, -1/8)$
$1/2$	$1/8$	$(1/2, 1/8)$
$3/2$	$27/8$	$(3/2, 27/8)$

(a)



(b)

Figure 1.3.8. Plotting additional pairs (x, f(x)) defined by the equation $f(x) = x^3$.

The additional pairs fill in the shape of f in Figure 1.3.8(b) a bit better than those in Figure 1.3.7(b), enough so that we're confident enough to make a "leap of faith" and draw the final shape of the graph of $f(x) = x^3$ in Figure 1.3.9.

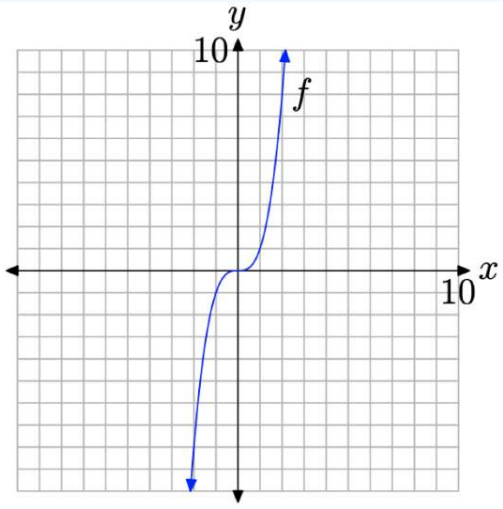


Figure 1.3.9. The final graph of $f(x) = x^3$.

Let's look at another example.

Example 1.3.2

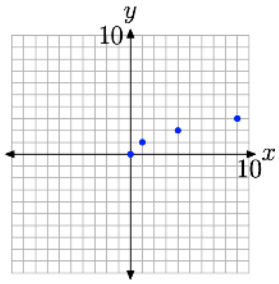
Sketch the graph of $f(x) = \sqrt{x}$

Solution

Again, we'll start by selecting several values of x in the domain of f. In this case, $f(x) = \sqrt{x}$, and it's not possible to take the square root of a negative number. Also, if we're creating a table of pairs by hand, it's good strategy to select known squares. Thus, we'll use x = 0, 1, 4, and 9 for starters.

x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	$(0,0)$
1	1	$(1,1)$
4	2	$(4,2)$
9	3	$(9,3)$

(a)



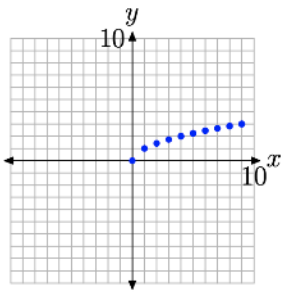
(b)

Figure 1.3.10 Plotting pairs (x, f(x)) defined by the equation $f(x)=\sqrt{x}$.

Some might be ready to make a “leap of faith” based on these initial results. Others might want to use a calculator to compute decimal approximations for additional square roots. The resulting pairs are shown in the table in Figure 1.3.11(a) and the additional pairs are plotted in Figure 1.3.11(b).

x	$f(x)=\sqrt{x}$	$(x, f(x))$
2	1.4	$(2,1.4)$
3	1.7	$(3,1.7)$
5	2.2	$(5,2.2)$
6	2.4	$(6,2.4)$
7	2.6	$(7,2.6)$
8	2.8	$(8,2.8)$

(a)



(b)

Figure 1.3.11 Plotting additional pairs (x, f(x)) defined by the equation $f(x)=\sqrt{x}$

The pattern in Figure 1.3.11(b) is clear enough to make a “leap of faith” and complete the graph as shown in Figure 1.3.12

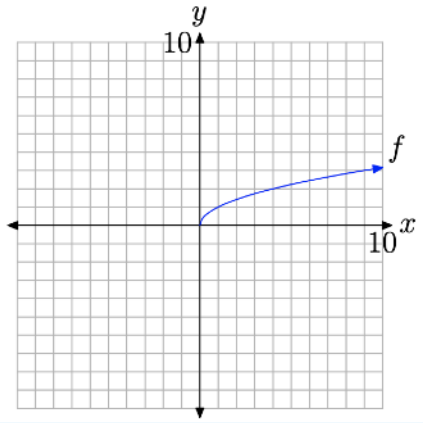


Figure 1.3.12 The graph of f defined by the equation $f(x)=\sqrt{x}$.

Using the Table Feature of the Graphing Calculator

The TABLE feature on your graphing calculator can be of immense help when creating tables of points that satisfy the equation defining the function f. Let’s look at an example

Example 1.3.3

Sketch the graph of $f(x)=|x|$

Solution

Enter the function $f(x) = |x|$ in the Y = menu as follows.

1. Press the Y= button on your calculator. This will open the Y= menu as shown in Figure 1.3.13(a). Use the arrow keys and the CLEAR button on your calculator to delete any existing functions.
2. Press the MATH button to open the menu shown in Figure 1.3.13(b).
3. Press the right-arrow on your calculator to select the NUM submenu as shown in Figure 1.3.13(c).
4. Select 1:abs(, then enter X and close the parentheses, as shown in Figure 1.3.13(d).

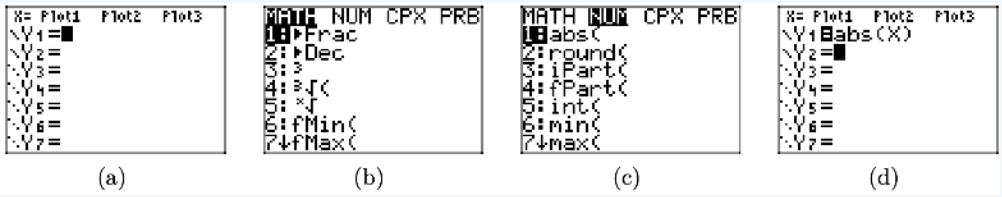


Figure 1.3.13. Entering $f(x) = |x|$ in the Y = menu.

We will now use the TABLE feature of the graphing calculator to help create a table of pairs (x, f(x)) satisfying the equation $f(x) = |x|$. Proceed as follows.

1. Select 2nd TBLSET (i.e., push the 2nd button followed by TBLSET), which is located over the WINDOW button. Enter TblStart=-3, $\Delta Tbl = 1$, and set the independent and dependent variables to Auto (this is done by highlighting Auto and pressing the Enter button), as shown in Figure 1.3.14(a).
2. Press 2nd TABLE, which is located above the GRAPH button, to produce the table of pairs (x, f(x)) shown in Figure 1.3.14(b).

We've plotted the pairs directly from the calculator onto a Cartesian coordinate system on graph paper in Figure 1.3.14(c).

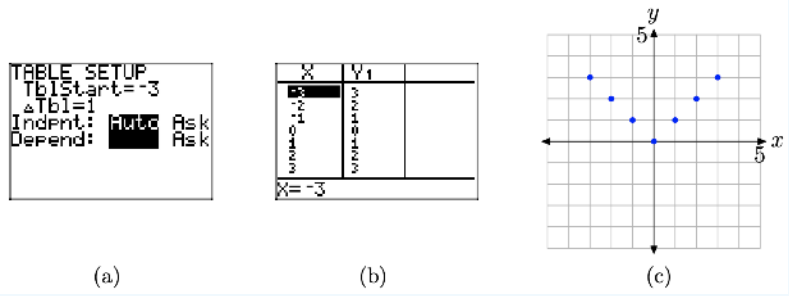


Figure 1.3.14. Creating a table with the TABLE feature of the graphing calculator.

Based on what we see in Figure 1.3.14(c), we're ready to make a "leap of faith" and draw the graph of f shown in Figure 1.3.15.

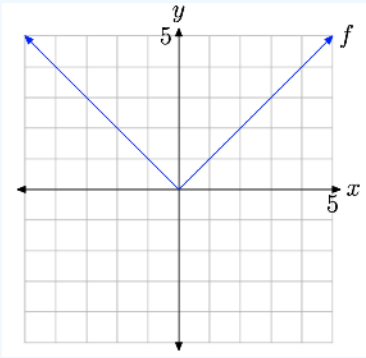


Figure 1.3.15. The graph of f defined by $f(x) = |x|$.

Alternatively, or as a check, we can have the graphing calculator draw the graph for us. Push the ZOOM button, then select 6:ZStandard (shown in Figure 1.3.16(a)) to produce the graph shown in Figure 1.3.16(b).

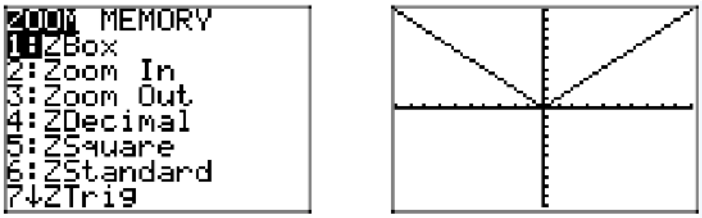


Figure 1.3.16. Creating the graph of $f(x) = |x|$ with the graphing calculator.

Adjusting the Viewing Window

In Example 1.3.3, we used the graphing calculator to draw the graph of the function defined by the equation $f(x) = |x|$. For the functions we've encountered thus far, drawing their graphs using the graphing calculator is pretty trivial. Simply enter the equation in the Y= menu, then press the ZOOM button and select 6:ZStandard. However, if the graph of a function doesn't fit (or even appear) in the "standard" viewing window, it can be quite challenging to find optimal view settings so that the important features of the graph are visible.

Indeed, as one might not even know what "important" features to look for, setting the viewing window is usually highly subjective and experimental by nature. Let's look at some examples.

Example 1.3.4

Use a graphing calculator to sketch the graph of $f(x) = 56 - x - x^2$. Experiment with the WINDOW settings until you feel you have a viewing window that exhibits the important features of the graph.

Solution

First, start by entering the function in the Y= menu, as shown in Figure 1.3.17(a). The caret ^on the keyboard is used for exponents. Press the ZOOM button and select 6:ZStandard to produce the graph shown in Figure 1.3.17(b).



Figure 1.3.17. The graph of $f(x) = 56 - x - x^2$ in the "standard" viewing window.

As the graph draws, observe that the graph rises from the bottom of the screen, leaves the top of the screen, then returns, falling from the top of the screen and leaving again at the bottom of the screen. This would indicate that there must be some sort of "turning point" that is not visible at the top of the screen.

Press the WINDOW button to reveal the "standard viewing window" settings shown in Figure 1.3.18(a). The following legend explains each of the WINDOW parameters in Figure 1.3.18(a).

\(\begin{array} { lll} Xmin & =& x-value of left edge of viewing window \\

Xmax &=& x-value of right edge of viewing window \\

Xscl &=& x-axis tick increment \\

Y min &=& y-value of bottom edge of viewing window \\

Y max &=& y-value of top edge of viewing window \\

Y scl &=& y-axis tick increment \end{array}\)

It is easy to evaluate the function $f(x) = 56 - x - x^2$ at $x = 0$. Indeed, $f(0) = 56 - 0 - 0^2 = 56$. This indicates that the graph of f must pass through the point (0, 56). This gives us a clue at how we should set the upper bound on our viewing

window. Set Ymax = 60, as shown in Figure 1.3.18(b), then press the GRAPH button to produce the graph and viewing window shown in Figure 1.3.18(c).

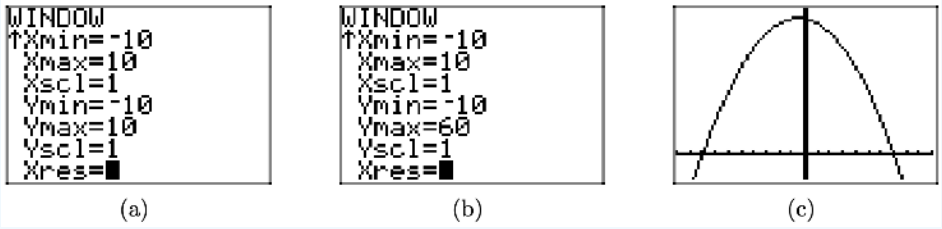


Figure 1.3.18. Changing the viewing window.

Although the viewing window in Figure 1.3.18(c) shows the "turning point" of the graph of f, we will make some additional changes to the window settings, as shown in Figure 1.3.19(a). First, we "widen" the viewing window a bit, setting Xmin = -15 and Xmax = 15, then we set tick marks on the x-axis every 5 units with Xscl = 5. Next, to create a little room at the top of the screen, we set Ymax = 100, then we "balance" this setting with Ymin = -100. Finally, we set tick marks on the y-axis every 10 units with Yscl = 10.

Push the GRAPH button to view the effects of these changes to the WINDOW parameters in Figure 1.3.19(b). Note that these settings are highly subjective, and what one reader might find quite pleasing will not necessarily find favor with other readers.

However, what is important is the fact that we've captured the "important features" of the graph of $f(x) = 56 - x - x^2$. Note that this is a very controversial statement. If one is just beginning to learn about the graphs of functions, how is one to determine what are the "important features" of the graph? Unfortunately, the answer to this question is, "through experience." Undoubtedly, this is a very frustrating phrase for readers to hear, but at least it's truthful. The more graphs that you draw, the more you will learn how to look for "turning points," "end-behavior," "x- and y-intercepts," and the like.

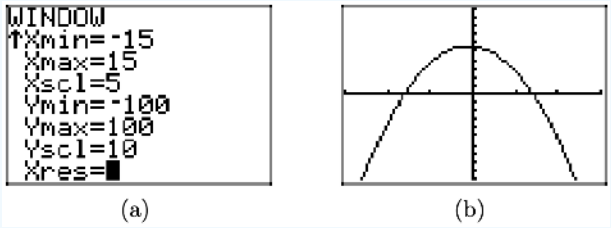


Figure 1.3.19. Improving the WINDOW settings.

For example, how do we know that the WINDOW settings in Figure 1.3.19(a) determine a viewing window (Figure 1.3.19(b)) that reveals all "important features" of the graph? The answer at this point is, "we don't, not without further experiment." For example, the careful reader might want to try the window settings Xmin=-50, Xmax=50, Xscl=10, Ymin=-500, Ymax=500, and Yscl=100 to see if any unexpected behavior crops up.

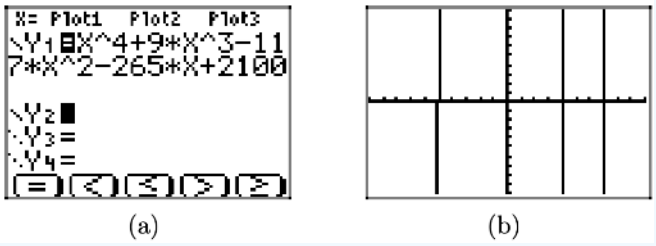
Let's look at one last example.

Example 1.3.5

Sketch the graph of the function f defined by the equation $f(x) = x^4 + 9x^3 - 117x^2 - 265x + 2100$.

Solution

Load the function into the Y= menu (shown in Figure 1.3.20(a)) and select 6:ZStandard to produce the graph shown in Figure 1.3.20(b).

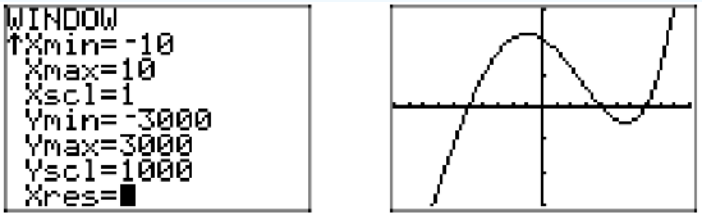


(a) (b)

Figure 1.3.20 Sketching the graph of $f(x) = x^4 + 9x^3 - 117x^2 - 265x + 2100$.

As the graph draws, observe that it rises form the bottom of the viewing window, leaves the top of the viewing window, then returns to fall off the bottom of the viewing window, then returns again and rises off the top of the viewing window.

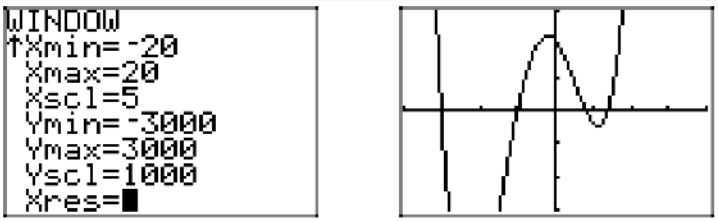
We notice that f(0) = 2100, so we'll need to set the top of the viewing window to that value or higher. With this thought in mind, we'll set Ymax=3000, then set Ymin=-3000 for balance, then to avoid a million little tick marks, we'll set Yscl=1000, all shown in Figure 1.3.21(a). Pressing the GRAPH button then produces the image shown in Figure 1.3.21(b).



(a) (b)

Figure 1.3.21 Adjusting the viewing window.

Does it appear that we have all of the "important features" of this graph displayed in our viewing window? Note that we did not experiment very much. Perhaps we should try expanding the window a bit more to see if we have missed any important behavior. With that thought in mind, we set Xmin=-20, Xmax=20, and to avoid a ton of tick marks, Xscl=5, as shown in Figure 1.3.22(a). Pushing the GRAPH button produces the image in Figure 1.3.22(b).



(a) (b)

Figure 1.3.22 Adjusting the viewing window again reveals behavior not seen.

Note that the viewing window in Figure 1.3.22(b) reveals behavior not seen in the viewing window of Figure 1.3.21(b). If we had not experimented further, if we had not expanded the viewing window, we would not have seen this new behavior. This is an important lesson.

Note that one of the "turning points" of the graph in Figure 1.3.22(b) lies off the bottom of the viewing window. We'll make one more adjustment to include this important feature. Set Ymin=-10000, Ymax=10000, and Yscl=5000, as shown in Figure 1.3.23(a), then push the GRAPH button to produce the image shown in Figure 1.3.23(b).

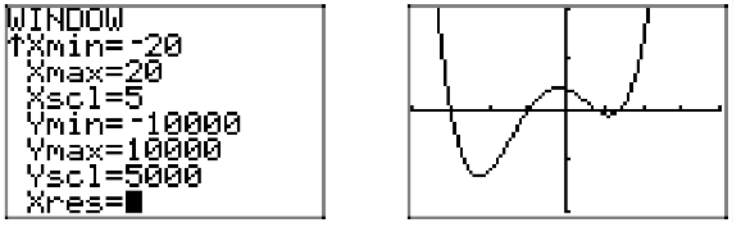


Figure 1.3.23. Adjusting the viewing window again reveals behavior not seen.

The graph in Figure 1.3.23(b) shows all of the "important features" of the graph of f, but the careful reader will continue to experiment, expanding the viewing window to ascertain the truth of this statement.

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1.4: Slope

In the previous section on Linear Models, we saw that if the dependent variable was changing at a constant rate with respect to the independent variable, then the graph was a line. If the rate was positive, then as we swept our eyes from left to right, the line rose upward, the dependent variable increasing with increasing changes in the independent variable. If the rate was negative, then the graph fell downward, the dependent variable decreasing with increasing changes in the independent variable. You may have also learned that higher rates led to steeper lines (lines that rose more quickly) and lower rates led to lines that were less steep.

In this section, we will connect the intuitive concept of rate developed in the previous section with a formal definition of the slope of a line. To start, let's state up front what is meant by the slope of a line.

Definition

Slope is a number that tells us how quickly a line rises or falls.

If slope is a number that is directly connected to the "steepness" of a line, then we should have certain expectations.

Expectations

1. Lines with positive slope should slant uphill (as our eyes sweep from left to right).
2. Lines with negative slope should slant downhill (as our eyes sweep from left to right).
3. Because any horizontal line neither slants uphill nor downhill, we expect that it should have slope equal to zero.
4. Lines with a larger positive slope should rise more quickly than lines with a smaller positive slope.
5. If two lines have negative slope, then the line having the slope with larger absolute value should fall more quickly than the other line.

It remains to define how to compute the slope of a particular line. Whatever definition we choose, it should conform with the expectations outlined above. We also would like the definition of slope to conform with the concept of rate developed in the previous section. Thus, we make the following definition.

Definition: Slope

The **slope** of a line is the rate at which the dependent variable is changing with respect to the independent variable.

Note how the word "change" is used Definition. It is important to understand that the change in some quantity can be positive, negative, or zero. For example, if the temperature outside is $40°\mathrm{F}$ when I leave my home at 6 AM, and at noon the temperature is 65◦ F, then the change in temperature is a positive $25°\mathrm{F}$. On the other hand, if the temperature outside is $65°\mathrm{F}$ at noon, and the temperature is $50°\mathrm{F}$ when I return home in the evening, then the change in temperature is a negative 15 degrees Fahrenheit.

In calculating the change in a quantity, follow this rule.

Definition: Change in Quantity

Change in Quantity = Latter Measurement − Former Measurement.

Thus, if T represents the temperature and ΔT represents the change in the temperature, then in our first case (taking the temperature in the morning then later at noon), the change in temperature is

$$\Delta T = \text{Latter} - \text{Former} = 65°\mathrm{F} - 40°\mathrm{F} = 25°\mathrm{F}$$

This positive result represents an increase in the temperature of $25°\mathrm{F}$.

In the second case (taking the temperature at noon then later in the evening), the change in temperature is

$$\begin{aligned}\Delta T &= \text{Latter} - \text{Former} \\ &= 50^\circ\text{F} - 65^\circ\text{F} \\ &= -15^\circ\text{F}\end{aligned}$$

This negative result represents a decrease in the temperature of 15°F

Tip 3

Readers should note that the direction of subtraction is extremely important. To detect the change in a quantity, always subtract the former (earlier) measurement from the latter (later) measurement.

Example 1.4.1

A ball is perched at rest at the top of a long ramp. It's given a little tap and it begins to roll down the ramp. The speed v of the ball (in meters per second) is plotted versus the time t (in seconds) in Figure **1.4.1**.

Determine the slope of the line.

Solution

We've defined the slope as the rate at which the dependent variable is changing with respect to the independent variable. In this case, the speed v of the ball "depends" upon the amount of time t that has elapsed. Consequently, v is the dependent variable and has been placed on the vertical axis.3 On the other hand, t is the independent variable and has been assigned the horizontal axis.

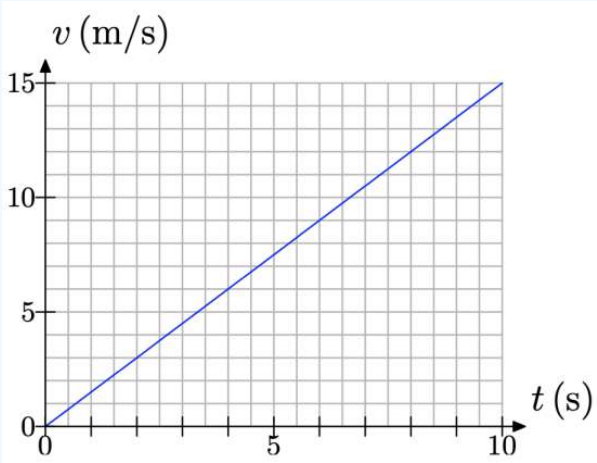


Figure **1.4.1**. Speed versus time.

To determine the rate at which v is changing with respect to t (the slope of the line), we first select two points P(2, 3) and Q(8, 12) on the line, as shown in Figure **1.4.2**. As we sweep our eyes from left to right (a convention we will always follow when dealing with slope), the point P occurs before the point Q. Hence, we consider P the "former" measurement and point Q the "latter" measurement.

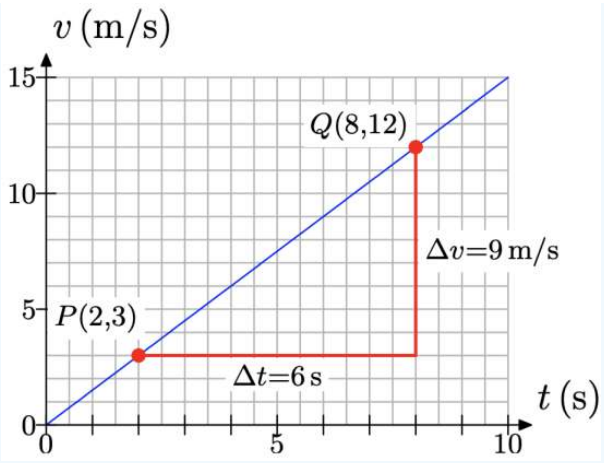


Figure **1.4.2**. Determining the slope of the line.

At point P, the time is t = 2 seconds, then at point Q the time is t = 8 seconds. The change in t is found by subtracting the former measurement from the latter measurement.

$$\Delta t = 8s - 2s = 6s$$

At point P, the speed is v = 3 meters per second, then at point Q the speed is v = 12 meters per second. Hence, the change in v is

$$\Delta v = 12\text{m/s} - 3\text{m/s} = 9\text{m/s}$$

Finally, the slope of the line is defined as the rate at which the dependent variable v is changing with respect to the independent variable t. That is,

$$\text{Slope } = \frac{\Delta v}{\Delta t} = \frac{9\text{m/s}}{6\text{s}} = \frac{3\text{m/s}}{\text{s}}$$

Scientists prefer to write this as 1.5 m/s^2, but this might not be as intuitive as writing 1.5 (m/s)/s, which indicates that the speed is increasing at a rate of 1.5 m/s every second. This makes good sense as a ball rolling down a ramp will pick up speed with the passage of time. The slope provides an exact numerical description of how the speed increases with respect to time.

Note that our definition of the slope of the line satisfies one of our goals: the slope is precisely the same as the notion of rate described in the previous section. Indeed, note the right triangle we've drawn in Figure **1.4.2**. The bottom edge of the triangle is 12 boxes long, but every 2 boxes represents one second, so this displacement in the time t direction is 6 seconds. The vertical side of the right triangle is 9 boxes in height where each box represents 1 meter per second. Consequently, this vertical edge of the right triangle represents a positive displacement of 9 meters per second. Thus, every 6 seconds, there is an increase in speed of 9 meters per second. Hence, the ball is picking up speed at the rate of 9 meters per second every 6 seconds, or equivalently, 1.5 meters per second every second.

Note

In Figure **1.4.2**, the rate at which the speed is increasing with respect to time is equivalent to the slope of the line.

Suppose that we had labeled our points $P\left(t_{\text{initial}}, v_{\text{initial}}\right)$ and $Q\left(t_{\text{final}}, v_{\text{final}}\right)$ as shown in Figure **1.4.3**.

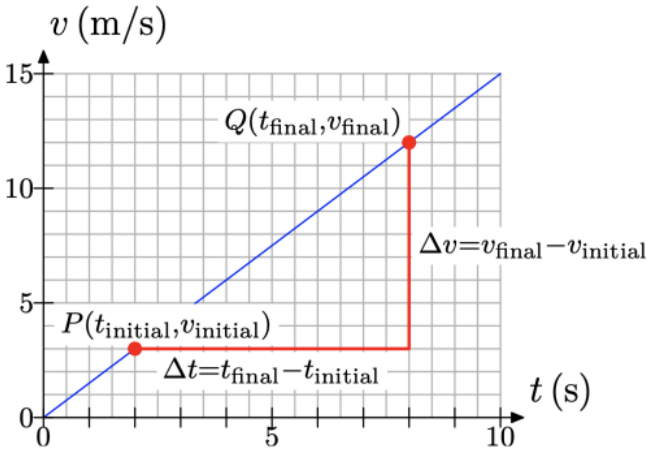


Figure **1.4.3**. Initial and final measurements.

Now the change in speed v would be

$$\Delta v = v_{\text{final}} - v_{\text{initial}}$$

and the change in time t would be

$$\Delta t = t_{\text{final}} - t_{\text{initial}}$$

Therefore, the slope of the line would be computed with the following formula.

$$\text{Slope} = \frac{\Delta v}{\Delta t} = \frac{v_{\text{final}} - v_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}}$$

With $P(t_{\text{initial}}, v_{\text{initial}}) = (2\text{s}, 3\text{m/s})$ and $Q(t_{\text{final}}, v_{\text{final}}) = (8\text{s}, 12\text{m/s})$, this becomes

$$\text{Slope} = \frac{12\text{m/s} - 3\text{m/s}}{8\text{s} - 2\text{s}} = \frac{9\text{m/s}}{6\text{s}} = 1.5\text{m/s}^2$$

The Slope Formula

The last calculation in Example **1.4.1** allows us to discuss the slope of a line as a purely mathematical concept, one that is not rooted in a supporting application as in Example **1.4.1**. Take, for example, the line shown in Figure **1.4.4** that passes through the points $P(-3, -3)$ and $Q(2, 1)$.

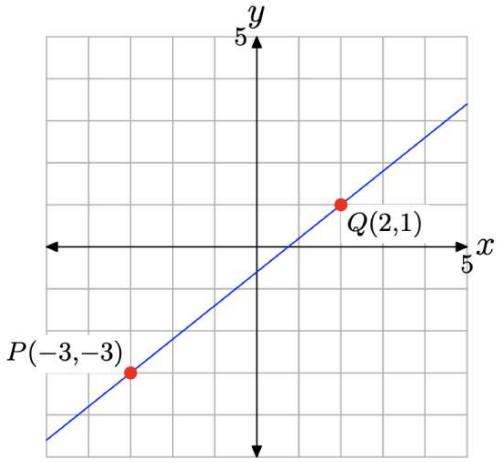


Figure **1.4.4**. Computing the slope of a line in an xy-coordinate system.

In this example, the dependent variable is y and the independent variable is x, so the slope of the line is Δy (the change in y) divided by Δx (the change in x).

$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

Sweeping our eyes from left to right, the point P comes first, followed by the point Q. Keeping “latter minus former” in mind, the change in y is computed by subtracting the y-value of point P from the y-value of point Q. That is,

$$\Delta y = 1 - (-3) = 4$$

Similarly, the change in x is computed by subtracting the x-value of point P from the x-value of point Q. That is,

$$\Delta x = 2 - (-3) = 5$$

Thus, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{4}{5}$$

Alternatively, we can use the points P and Q as vertices of a right triangle with sides parallel to the axes (shown in Figure **1.4.5**(a)). The horizontal edge of the right triangle is 5 boxes (each representing 1 unit), so the displacement in x is 5 units. The vertical edge is 4 boxes (each representing 1 unit), so the displacement in y is 4 units. Hence, each time x is increased by 5 units, y experiences an increase of 4 units. Therefore, the slope of the line is again 4/5.

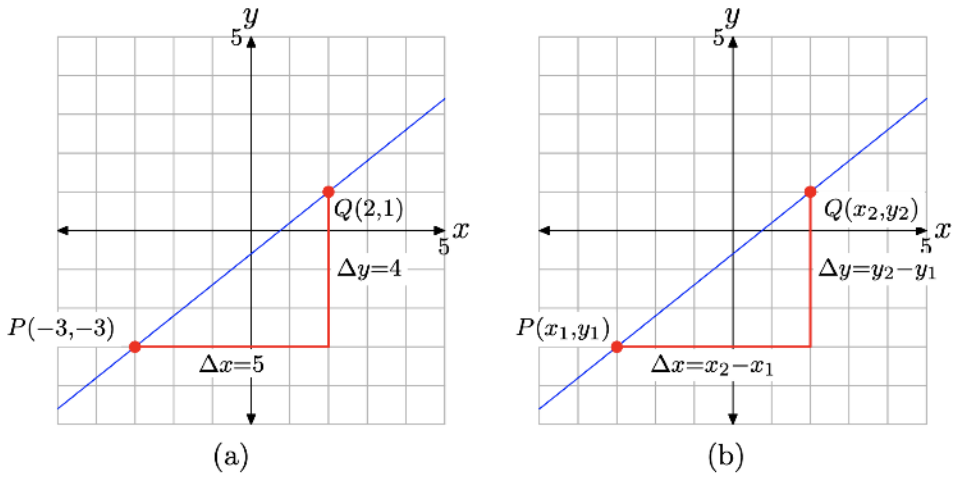


Figure **1.4.5**. Using a right triangle to determine the slope.

Suppose that we had labeled our points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as shown in Figure **1.4.5**(b). Now the change in y would be

$$\Delta y = y_2 - y_1$$

and the change in x would be

$$\Delta x = x_2 - x_1$$

Therefore, the slope of the line would be computed with the following formula.

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

With $P(x_1, y_1) = (-3, -3)$ and $Q(x_2, y_2) = (2, 1)$, this becomes

$$\text{Slope} = \frac{1 - (-3)}{2 - (-3)} = \frac{4}{5}$$

The slope formula is worth summarizing in a definition.

Definition 5

The slope of the line that passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Let's look at some more examples.

Example 1.4.2

Find the slope of the line passing through the points P(−3, −2) and Q(3, 1).

Solution

We can use the slope formula in Definition 5 to determine the slope. With $(x_1, y_1) = P(-3, -2)$ and $(x_2, y_2) = Q(3, 1)$,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-2)}{3 - (-3)} = \frac{3}{6} = \frac{1}{2}$$

Readers will sometimes ask, "Which point should be (x_1, y_1) and which should be (x_2, y_2)?" The short answer is, "It doesn't matter!" Suppose instead, that we let $(x_1, y_1) = Q(3, 1)$ and $(x_2, y_2) = P(-3, -2)$. Then,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 1}{-3 - 3} = \frac{-3}{-6} = \frac{1}{2}$$

Because the change in any quantity is found by subtracting the earlier measurement from the later measurement, we will continue to stress the first order. However, if we reverse the points as we did in our second calculation, both numerator and denominator reverse sign with this interchange, so we get the same answer.

Of course, we can also determine the slope by plotting P(−3, −2) and Q(3, 1) and the line that passes through P and Q, as we've done in Figure **1.4.6**.

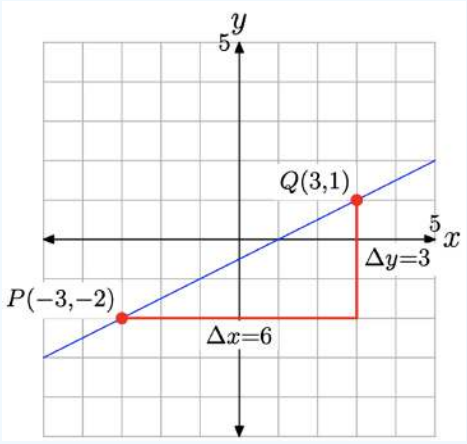


Figure **1.4.6**. Determining the slope from the graph.

Starting at the point P, to get to the point Q, we move 6 boxes to the right, then 3 boxes up, as shown in Figure **1.4.6**. Hence, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{3}{6} = \frac{1}{2}$$

Note that two of our expectations regarding the slope of a line are met with this example.

1. The line through P(−3, −2) and Q(3, 1) in Figure **1.4.6** has slope 1/2. This is a positive number and the line slants uphill (as expected) as we sweep our eyes from left to right.

2. The slope in this example is 1/2, which is less than the slope of the line in Figure **1.4.5**(a), which was 4/5. Note that the line in Figure **1.4.6** is less steep than the line in Figure **1.4.5**(a), which was another of our earlier expectations regarding the slope of a line.

Example 1.4.3

Find the slope of the line passing through the points P(−4, 4) and Q(4, −2).

Solution

We can use the slope formula in Definition 5 to determine the slope. With $(x_1, y_1) = P(-4, 4)$ and $(x_2, y_2) = Q(4, -2)$,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 4}{4 - (-4)} = \frac{-6}{8} = -\frac{3}{4}$$

We can also get the slope of the line from the graph in Figure **1.4.7**. Starting at the point P(−4, 4), move 8 units to the right, then 6 units downward, as shown in Figure **1.4.7**.

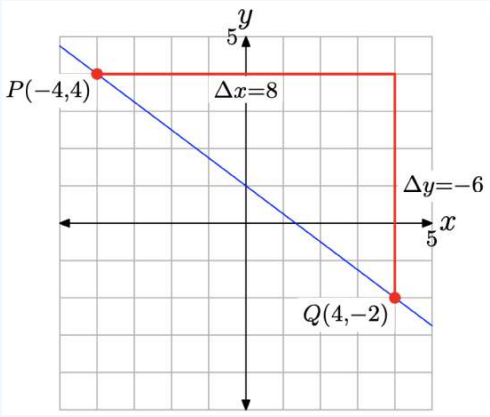


Figure **1.4.7**. Determining the slope from the graph.

Thus, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{-6}{8} = -\frac{3}{4}$$

Again, one of our earlier expectations regarding the slope of a line is met in this example. The slope is −3/4, which is a negative number, and the line in Figure **1.4.7** slants downhill (as we sweep our eyes from left to right).

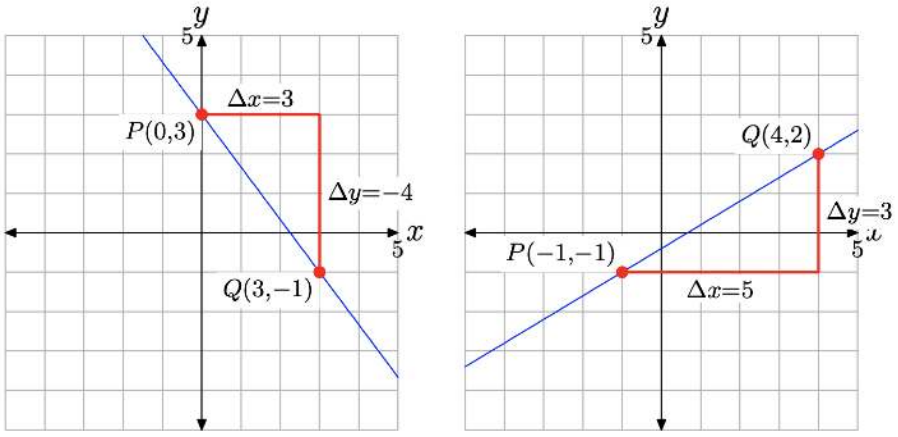
Example 1.4.4

Draw a line that intercepts the y-axis at (0, 3) so that the line has slope −4/3. Draw a second line that passes through the point P(−1, −1) with slope 3/5.

Solution

The slope of the first line is −4/3. This means that our line must slant downhill (as we sweep our eyes from left to right). The slope is the change in y over the change in x. Therefore, every time x increases by 3 units, y must decrease by 4 units. Plot the point P(0, 3), as shown in Figure **1.4.8**(a). Then, starting at P, move 3 units to the right, followed by 4 units downward to the point Q(3, −1), as shown in Figure **1.4.8**(a). Draw the required line, which must pass through the points P and Q.

To draw the second line, first plot the point P(−1, −1), as shown in Figure **1.4.8**(b). Starting at the point P, move 5 units to the right, then upward 3 units to the point Q(4, 2), as shown in Figure **1.4.8**(b). Draw the required line passing through the points P and Q.



(a) A line having y-intercept at $(0, 3)$ with slope $-4/3$.

(b) A line having slope $3/5$ that passes through the point $P(-1, -1)$.

Figure **1.4.8**

Parallel Lines

Because slope controls the “steepness” of a line, it is a simple matter to see that parallel lines must have the same slope.

property

Let L_1 be a line having slope m_1. Let L_2 be a line having slope m_2. If L_1 and L_2 are parallel, then

$$m_1 = m_2$$

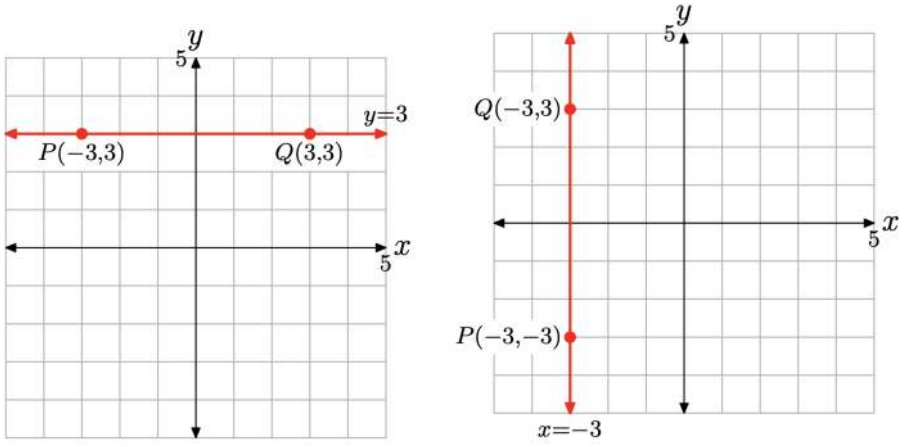
That is, any two parallel lines have the same slope.

Example 1.4.5

What is the slope of any horizontal line? What is the slope of any vertical line?

Solution

One would expect that our definition would verify that the slope of any horizontal line is zero. Select, for example, the horizontal line shown in Figure 9(a). Select the points (−3, 3) and (3, 3) on this line.



(a) Determine the slope of a horizontal line.

(b) Determine the slope of a vertical line.

Figure **1.4.9**

With $(x_1, y_1) = (-3, 3)$ and $(x_2, y_2) = (3, 3)$

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{3-(-3)} = \frac{0}{6} = 0$$

Thus, the horizontal line in Figure **1.4.9**(a) has slope equal to zero, exactly as expected. Further, all horizontal lines are parallel to this horizontal line and have the same slope. Therefore, all horizontal lines have slope zero.

We would surmise that the vertical line in Figure **1.4.9**(b) has undefined slope (we'll explore this more fully in the exercises). In Figure **1.4.9**(b), we've selected the points P(−3, −3) and Q(−3, 3) on the vertical line. With $(x_1, y_1) = P(-3, -3)$ and $(x_2, y_2) = Q(-3, -3)$,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-3)}{-3-(-3)} = \frac{6}{0}, \text{ which is undefined.}$$

The slope of the vertical line in Figure **1.4.9**(b) is undefined because division by zero is meaningless. Further, all vertical lines are parallel to this vertical line and have undefined slope.

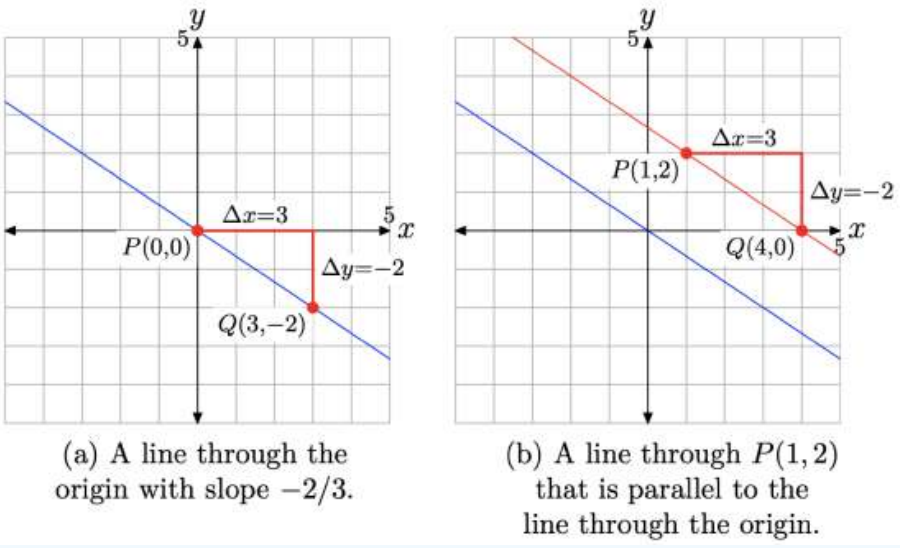
Example 1.4.6

Draw a line through the point P(1, 2) that is parallel to the line passing through the origin with slope −2/3.

Solution

We will first draw a line through the origin with slope −2/3. Plot the point P(0, 0), then move 3 units to the right and 2 units downward to the point Q(3, −2), as shown in Figure **1.4.10**(a). Draw a line through the points P and Q as shown in Figure **1.4.10**(a).

Next, plot the point P(1, 2) as shown in Figure **1.4.10**(b). To draw a line through this point that is parallel to the line through the origin, this second line must have the same slope as the first line. Therefore, start at the point P(1, 2), as shown in Figure **1.4.10**(b), then move 3 units to the right and 2 units downward to the point Q(4, 0). Draw a line through the points P and Q as shown in Figure **1.4.10**(b). Note that this second line is parallel to the first.



(a) A line through the origin with slope $-2/3$.

(b) A line through $P(1,2)$ that is parallel to the line through the origin.

Figure **1.4.10**

Perpendicular Lines

The relationship between the slopes of two perpendicular lines is not as straightforward as the relation between the slopes of two parallel lines. Let's begin by stating the pertinent property.

Property 13

Let L_1 be a line having slope m_1. Let L_2 be a line having slope m_2. If L_1 and L_2 are perpendicular, then

$$m_1 m_2 = -1$$

That is, the product of the slopes of two perpendicular lines is −1.

We can solve equation (14) for m_1 in terms of m_2.

$$m_1 = -\frac{1}{m_2}$$

Equation (15) tells us that the slope of the first line is the negative reciprocal of the slope of the second line.

For example, suppose that L_1 and L_2 are perpendicular lines with slopes m_1 and m_2, respectively

- If $m_2 = 2$, then $m_1 = -\frac{1}{2}$
- If $m_2 = \frac{3}{5}$, then $m_1 = -\frac{5}{3}$
- If $m_2 = -\frac{2}{3}$, then $m_1 = \frac{3}{2}$

Note that in each bulleted item, the product of the slopes is −1.

We won't provide a proof of equation (15), but we will provide some motivating evidence in the form of a graph.

Example 1.4.7

Sketch the graphs of the lines passing through the origin having slopes 2 and −1/2.

Solution

In Figure **1.4.11**(a), we've plotted the point P(0, 0) at the origin, then moved 1 unit to the right and 2 units upward to the point Q(1, 2). The resulting line passes through the origin and has slope $m_1 = 2$ (alternatively, $m_1 = 2/1$).

In Figure **1.4.11**(b), we've again plotted the point P(0, 0) at the origin, then moved 2 units to the right and 1 unit downward to the point Q(2, −1). The resulting line passes through the origin and has slope $m_2 = -1/2$.

There are two important points that need to be made about the lines in Figure **1.4.11**(b).

The two lines in Figure **1.4.11**(b) are perpendicular. They meet and form a right angle of **90°**. If you have a protractor available, you might want to measure the angle between the two lines and note that the measure of the angle is **90°**.

The product of the two slopes is

$$m_1 m_2 = 2 \cdot \left(-\frac{1}{2}\right) = -1$$

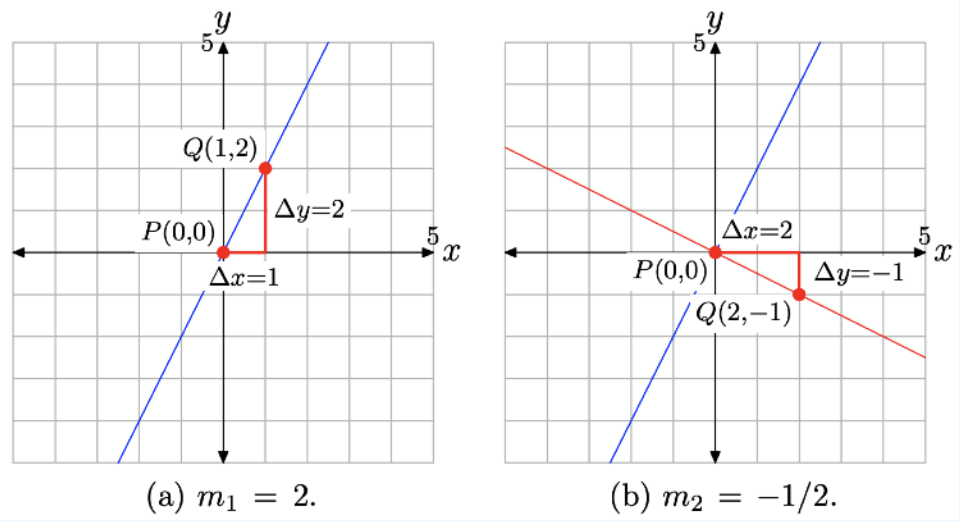


Figure **1.4.11**. Sketching perpendicular lines.

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- **3.2: Slope** by David Arnold is licensed CC BY-NC-SA 2.5. Original source: https://web.archive.org/web/20200814023923/http://msenux2.redwoods.edu/IntAlgText/.

1.5: Equations of Lines

In this section we will develop the slope-intercept form of a line. When you have completed the work in this section, you should be able to look at the graph of a line and determine its equation in slope-intercept form.

The Slope-Intercept Form

In the previous section, we developed the formula for the slope of a line. Let's assume that the dependent variable is y and the independent variable is x and we have a line passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, as shown in Figure 1.5.1.

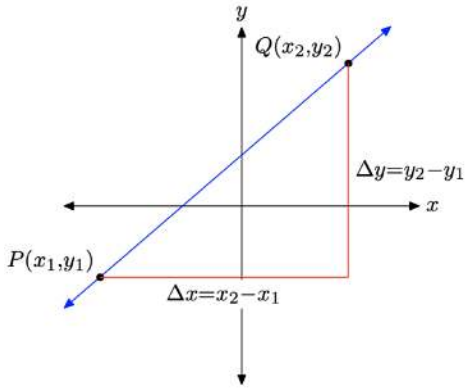


Figure 1.5.1. Determining the slope of a line through two points.

As we sweep our eyes from left to right, note that the change in x is $\Delta x = x_2 - x_1$ and the change in y is $\Delta y = y_2 - y_1$. Thus, the slope of the line is determined by the formula

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Now consider the line in Figure 1.5.2. Suppose that we are given two facts about this line:

1. The point where the line crosses the y-axis (the y-intercept) is (0, b).
2. The "slope" of the line is some number m.

To find the equation of the line pictured in Figure 1.5.2, select an arbitrary point Q(x, y) on the line, then compute the slope of the line using $(x_1, y_1) = P(0, b)$ and $(x_2, y_2) = Q(x, y)$ in the slope formula (1).

$$\text{Slope } = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - b}{x - 0}$$

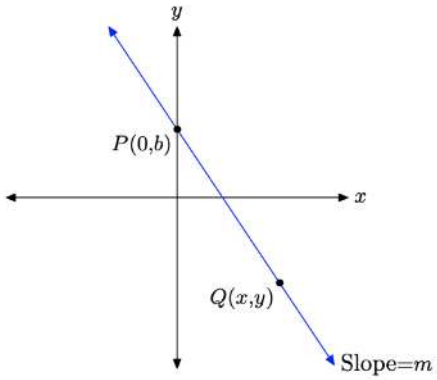


Figure 1.5.2. Find the equation of the line in slope-intercept form.

Simplify.

$$\text{Slope } = \frac{y - b}{x}$$

We're given that the slope is the number m, so substitute this number for the word "Slope" in the last result.

$$m = \frac{y-b}{x}$$

Multiply both sides of the last equation by x.

$$mx = y - b$$

Add b to both sides of the last equation to obtain

$$mx + b = y$$

or upon exchanging sides of the equation,

$$y = mx + b$$

The above discussion leads to the following result.

Definition: The Slope-Intercept Form of a Line

If the line L intercepts the y-axis at the point (0, b) and has slope m, then the equation of the line is

$$y = mx + b. \tag{1.5.1}$$

This form of the equation of a line is called the slope-intercept form. The function defined by the equation

$$f(x) = mx + b$$

is called a **linear function**.

It is important to note two key facts about the slope-intercept form y = mx + b.

- The coefficient of x (the m in y = mx + b) is the slope of the line.
- The constant term (the b in y = mx+b) is the y-coordinate of the y-intercept (0, b).

Procedure for Using the Slope-Intercept Form of a Line

When given the slope of a line and the y-intercept of the line, use the slope-intercept form as follows:

1. Substitute the given slope for m in the formula $y = mx + b$.
2. Substitute the y-coordinate of the y-intercept for b in the formula $y = mx + b$.

For example, if the line has slope −2 and the y-intercept (the point where the line crosses the y-axis) is (0, 3), then substitute m = −2 and b = 3 into Equation 1.5.1 to obtain

$$y = -2x + 3$$

.

Let's look at some examples of use of this all-important formula.

Example 1.5.1

What is the equation of the line having slope −2/3 and y-intercept at (0, 3)? Sketch the line on graph paper.

Solution

The equation of the line is

$$y = mx + b$$

We're given that the slope is −2/3. Hence, m = −2/3. Secondly, we're given that the line intercepts the y-axis at the point (0, 3). In the slope-intercept form y = mx + b, recall that b represents the y-coordinate of the y-intercept. Hence, b = 3. Substitute m = −2/3 and b = 3 in equation (4), obtaining

$$y = -\frac{2}{3}x + 3$$

To sketch the graph of the line, first locate the y-intercept at P(0, 3), as shown in Figure 1.5.3. Starting from the y-intercept at P(0, 3), move 3 units to the right and 2 units downward to the point Q(3, 1). The required line passes through the points P and Q.

Note that the line "intercepts" the y-axis at 3 and slants downhill, in accordance with the fact that the slope is negative in this example.

Example 1.5.2

Given the graph of the line in Figure 1.5.4(a), determine the equation of the line.

Solution

First, locate the y-intercept of the line, which we've labeled P(0, −1) in Figure 1.5.4(b). In the formula y = mx + b, recall that b represents the y-coordinate of the y-intercept. Thus, b = −1.

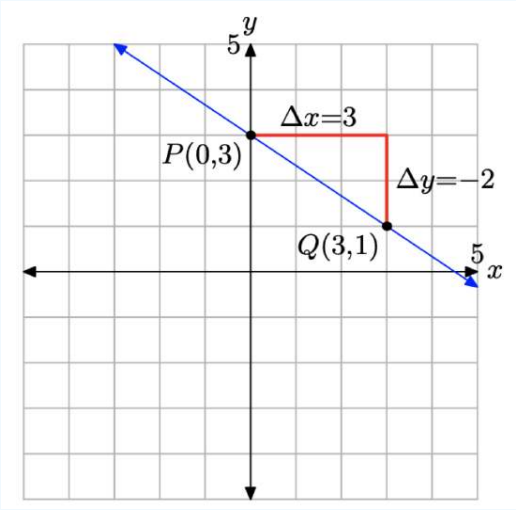


Figure 1.5.3. The line has y-intercept at (0, 3) and slope −2/3.

Secondly, we need to determine the slope of the line. In Figure 1.5.4(b), start at the point P, move 2 units to the right, then 3 units upward to the point Q(2, 2). This makes the slope

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{2}$$

Substitute m = 3/2 and b = −1 into the slope-intercept form y = mx + b to obtain

$$y = \frac{3}{2}x - 1$$

which is the desired equation of the line.

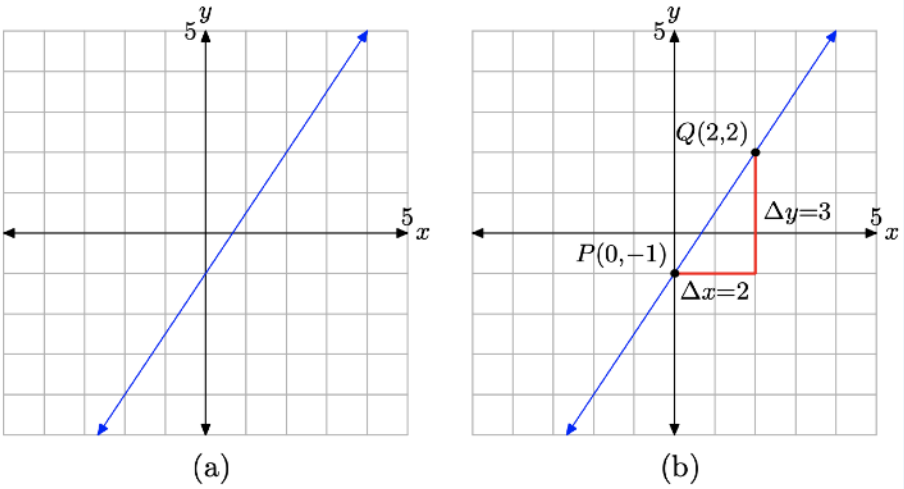


Figure 1.5.4. Determining the equation of a line from its graph.

Making Connections

If the connection between rate and slope is still not clear, let's recall an example we did earlier in the chapter.

Sebastian waves good-bye to his brother, who is talking to a group of his friends approximately 20 feet away. Sebastian then begins to walk away from his brother at a constant rate of 4 feet per second.

The distance between the brothers depends upon the amount of time that has passed, so we set distance d on the vertical axis and time t on the horizontal axis, as shown in Figure 1.5.5. Note that d and t are taking the "usual" place of y and x, respectively. The distance separating the brothers at time t = 0 is d = 20 feet. This is indicated with the "d-intercept" at P(0, 20) in Figure 1.5.5.

Next, the distance between the brothers is increasing at a rate of 4 feet per second. Starting at the point P(0, 20), move 1 second to the right (2 boxes) and 4 units up (1 box) to the point Q(1, 24), as shown in Figure 1.5.5. The line through the points P and Q then models the distance between the brothers as a function of time.

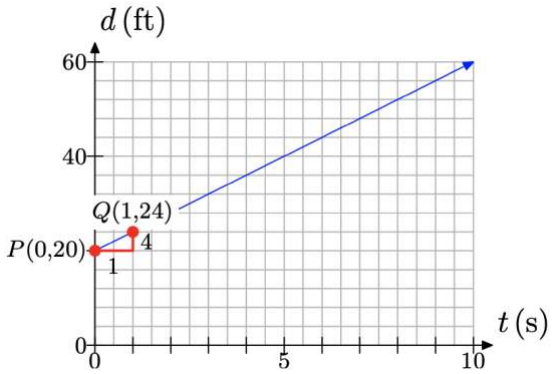


Figure 1.5.5. Distance between brothers as a function of time.

If you recall, we then determined a relation between distance d and time t by examining the distance between the brothers at times t = 0, 1, 2, and 3, and summarizing the results in Table 1.5.1.

Table 1.5.1 Determining a model equation.

t	d
0	20
1	20+4(1)
2	20+4(2)
3	20+4(3)

Intuition then led to the following model, which provides the distance d between the brothers as a function of time t.

$$d = 20 + 4t$$

Again, readers should check that equation (6) produces the results in Table 1.5.1 at t = 0, 1, 2, and 3.

Alternatively, with the theory developed in this section, we would develop the equation of the line by using the slope-intercept form of the line; that is,

$$y = mx + b$$

However, in this case, the dependent variable is d, not y, and the independent variable is t, not x. So, replace y and x in equation (7) with d and t, respectively, obtaining

$$d = mt + b$$

Next, the line intercepts the d-axis at P(0, 20), so b = 20. Furthermore, the slope of the line is 4 feet per second, so m = 4. Substitute m = 4 and b = 20 in equation (8) to obtain

$$d = 4t + 20$$

or using function notation, $d(t) = 4t + 20$. Note that equation (9) is identical to the intuitively generated model of equation (6).

Hopefully, this development should cement for all time the idea that slope of the line is the rate at which the dependent variable is changing with respect to the independent variable.

The Standard Form of a Line

We now know that if our equation has the form $y = mx + b$ (or can be manipulated into this form), the graph will be a line. Let's take a moment to demonstrate that the graph of the equation $Ax + By = C$, where A, B, and C are constants, is a line.

If we can place the form $Ax + By = C$ into slope-intercept form $y = mx + b$, then that will demonstrate that the graph of $Ax + By = C$ is a line. So, start with $Ax + By = C$ and subtract Ax from both sides of the equation.

$$By = -Ax + C$$

Divide both sides of this last equation by B. Note that there is an assumption here that $B \neq 0$. We will handle the case when B = 0 separately, at the end of this section.

$$\frac{By}{B} = \frac{-Ax + C}{B}$$
$$y = -\frac{A}{B}x + \frac{C}{B}$$

When we compare $y = -(A/B)x + (C/B)$ with y = mx + b, we note that the slope is m = −A/B and the y-coordinate of the y-intercept is b = C/B. Because we were successful in placing the equation $Ax + By = C$ into slope-intercept form, we now know that the graph of $Ax + By = C$ is a line (we'll need this result in later work).

The Standard Form of a Line

The graph of the equation Ax + By = C is a line. The form

$$Ax + By = C$$

is called the standard form of a line.

Let's look at an example.

Example 1.5.3

The equation 3x+4y = 12 is in standard form. Place this equation in slope-intercept form, determine the slope and y-intercept, then use these results to draw the graph of the line.

Solution

First, solve the equation 3x + 4y = 12 for y.

$$\begin{aligned} 3x+4y &= 12 \\ 4y &= -3x+12 \\ y &= -\frac{3}{4}x+3 \end{aligned}$$

Note in the last step how the distributive property came into play. When we divided −3x + 12 by 4, we divided each term by 4, getting (−3/4)x + 3.

When we compare y = (−3/4)x+3 with the general slope-intercept form y = mx+b, we determine that the slope is m = −3/4 and the y-coordinate of the y-intercept is b = 3. To sketch the graph of the line, as we've done in Figure 1.5.6, plot the y-intercept at P(0, 3), then move 4 units to the right and 3 units down to the point Q(4, 0). The line passing through the points P and Q is the required line.

Note again that the slope is m = −3/4 and the line slants "downhill." Also, b = 3 and the line "intercepts" the y-axis at P(0, 3).

Let's look at another example.

Example 1.5.4

In Example 1.5.1, we determined that the given line had the equation

$$y = \frac{3}{2}x - 1$$

Place this equation in standard from $Ax + By = C$, where A, B, and C are integers and $A > 0$.

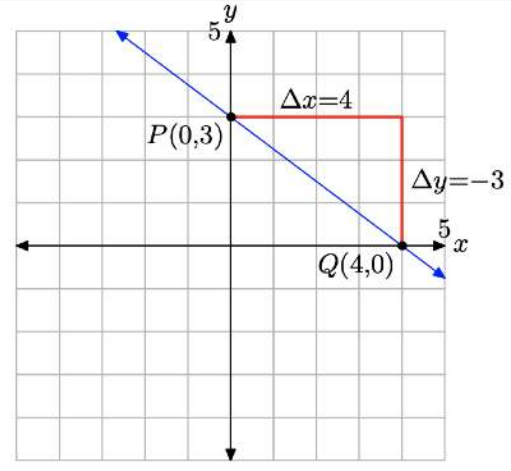


Figure 1.5.6. The graph of $3x + 4y = 12$.

Solution

We're requested to place the equation $y = (3/2)x - 1$ in the form $Ax + By = C$, where A, B, and C are integers, so let's begin by clearing fractions from the equation. Multiply both sides of the equation by the common denominator 2.

$$\begin{aligned} y &= \frac{3}{2}x - 1 \\ 2y &= 2\left(\frac{3}{2}x - 1\right) \\ 2y &= 3x - 2 \end{aligned}$$

Now, subtract 2y from both sides of the equation, then add 2 to both sides of the equation to obtain

$$2 = 3x - 2y$$

or equivalently,

$$3x - 2y = 2$$

Note that this last result is in standard form $Ax + By = C$, where A, B, and C are integers and A > 0.

Intercepts

We now know that the graph of the equation $Ax + By = C$, where A, B, and C are constants, is a line. Because the graph of $Ax + By = C$ is a line, to draw the graph of the line, we need only find two points that satisfy the equation, plot them, then draw a line through them. Our two favorite points to work with are the x- and y-intercepts, because each involves the number zero, an easy number to work with.

Consider the graph in Figure 1.5.7(a). Note that the graph passes through the x-axis three times. The points where the graph intercepts the x-axis are called x-intercepts. Note that each of these points has a defining value in common: the y-value of each of these x-intercepts is equal to zero.

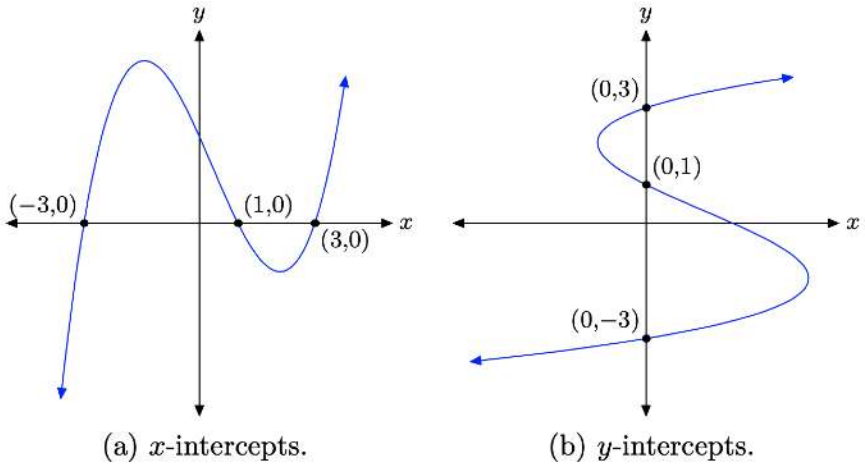


Figure 1.5.7.

How to Find an x-intercept

To find an x intercept, let y = 0 in the equation and solve for x.

On the other hand, consider the graph in Figure 1.5.7(b). Note that this is not a function (fails the vertical line test) but the graph intercepts the y-axis three separate times. The points where the graph intercepts the y-axis are called y-intercepts. Note that each of the y-intercepts in Figure 1.5.7(b) has a defining value in common: the x-value of each of the y-intercepts is equal to zero.

How to Find a y-intercept

To find a y-intercept, let x = 0 in the equation and solve for y.

Let's put what we've learned into practice.

Example 1.5.5

Sketch the graph of 3x + 4y = 12.

Solution

We drew the graph of the equation 3x + 4y = 12 in Figure 1.5.6. There we solved the equation for y to determine the slope and the y-intercept. These, in turn, were used to draw the graph of 3x + 4y = 12 in Figure 1.5.6.

Here, our approach will be different. We will first determine the x- and y-intercepts, plot them, then draw a line through these intercepts. Hopefully, we will get a result that matches that in Figure 1.5.6.

To find the x-intercept, let $y = 0$ in $3x + 4y = 12$ and solve for x.

$$\begin{aligned} 3x+4y &= 12 \\ 3x+4(0) &= 12 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

Hence, the x-intercept is the point Q(4, 0). To find the y-intercept, let x = 0 in $3x+4y=12$ and solve for y.

$$\begin{aligned} 3x+4y &= 12 \\ 3(0)+4y &= 12 \\ 4y &= 12 \\ y &= 3 \end{aligned}$$

Hence, the y-intercept is the point P(0, 3). In Figure 1.5.8, we've plotted both x- and y-intercepts and drawn a line through them. Note that the resulting line in Figure 1.5.8 matches the same line drawn in Figure 1.5.6 (where we used a different method).

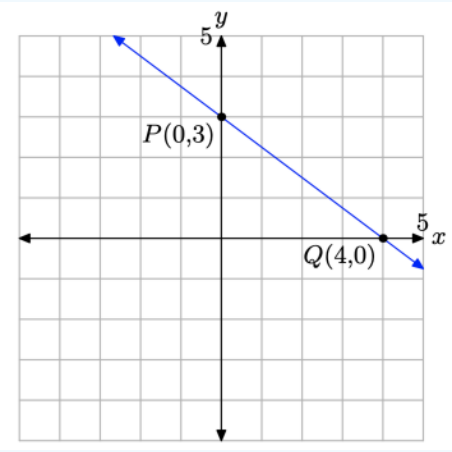


Figure 1.5.8. Plotting the x- and y-intercepts.

We recommend that whenever the line is given in Standard Form $Ax+By=C$, find the x- and y-intercepts, plot them, then draw a line through them. This technique is quite efficient because working with the number zero greatly simplifies the calculations.

Horizontal and Vertical Lines

We've introduced the standard form of the line $Ax+By=C$. The case where A and B are simultaneously equal to zero is not very interesting.4 However, the following two cases are of interest.

1. If we let A = 0 and B 6= 0 in the standard form $Ax+By=C$, then By = C, or equivalently y = C/B. Note that this has the form y = b, where b is some constant.
2. Similarly, if we let B = 0 and A 6= 0 in the standard form $Ax+By=C$, then Ax = C, or equivalently, x = C/A. Note that this has the form x = a, where a is some constant.

The lines having the form x = a and y = b are two of the easiest lines to plot. Let's look at an example of each.

Example 1.5.6

Sketch the graph of the equation x = 3.

Solution

The direction "sketch the graph of the equation x = 3" can be quite vexing unless one remembers that a graph of an equation is the set of all points that satisfy the equation. Thus, the direction is better posed if we say "sketch the set of all points (x, y) that satisfy x = 3," or equivalently, "sketch the set of all points (x, y) that have an x-value of 3." Then it is an easy matter to sketch the vertical line shown in Figure 1.5.9.

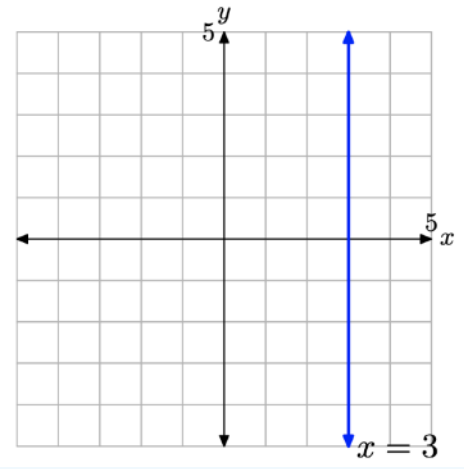


Figure 1.5.9. The graph of the equation x = 3.

Note that each point on the line has an x-value equal to 3. Also, note that the slope of this vertical line is undefined.

Example 1.5.7

Sketch the graph of the equation y = 3.

Solution

This direction is better posed if we say "sketch the set of all points (x, y) that satisfy y = 3," or equivalently, "sketch the set of all points (x, y) that have a y-value of 3." Then it is an easy matter to sketch the horizontal line shown in Figure 1.5.10

Note that each point on the line has a y-value equal to 3. Also, note that this horizontal line has slope zero.

Two final comments are in order. Because the line in Figure 10 has slope zero and y-intercept (0, 3), we can insert m = 0 and b = 3 into the slope intercept form y = mx + b and obtain

$$y = 0x + 3$$

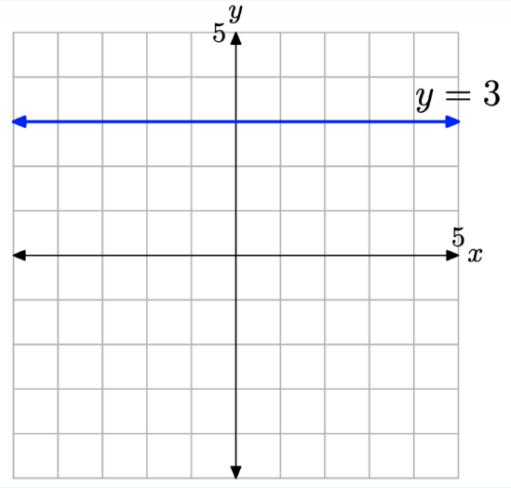


Figure 1.5.10. The graph of the equation y = 3.

which of course, is equivalent to y = 3. However, the vertical line shown in Figure 1.5.9 has "undefined" slope, so this approach is unavailable. We must simply recognize that the vertical line in Figure 1.5.9 consists of all points having an x-value equal to 3, and then intuit that its equation is x = 3.

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1.6: The Point-Slope Form of a Line

In the last section, we developed the slope-intercept form of a line (y = mx + b). The slope-intercept form of a line is applicable when you're given the slope and y-intercept of the line. However, there will be times when the y-intercept is unknown.

Suppose for example, that you are asked to find the equation of a line that passes through a particular point $P\,(x_0, y_0)$ with slope = m. This situation is pictured in Figure 1.6.1.

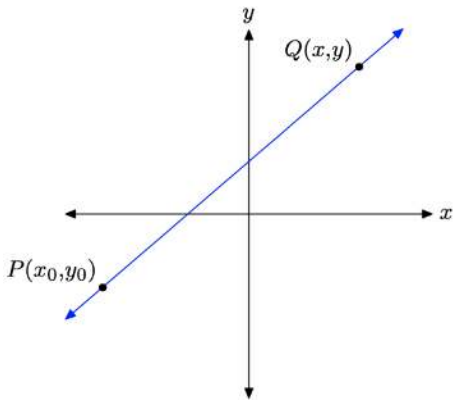


Figure 1.6.1. A line through (x_0, y_0) with slope m.

Let the point Q(x, y) be an arbitrary point on the line. We can determine the equation of the line by using the slope formula with points P and Q. Hence,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0}$$

Because the slope equals m, we can set Slope = m in this last result to obtain

$$m = \frac{y - y_0}{x - x_0}$$

If we multiply both sides of this last equation by $x - x_0$, we get

$$m\,(x - x_0) = y - y_0$$

or exchanging sides of this last equation,

$$y - y_0 = m\,(x - x_0)$$

This last result is the equation of the line.

The Point-Slope Form of a Line

If line L passes through the point (x_0, y_0) and has slope m, then the equation of the line is

$$y - y_0 = m\,(x - x_0)$$

This form of the equation of a line is called the point-slope form.

To use the point-slope form of a line, follow these steps.

Procedure for Using the Point-Slope Form of a Line

When given the slope of a line and a point on the line, use the point-slope form as follows:

1. Substitute the given slope for m in the formula $y - y_0 = m\,(x - x_0)$.
2. Substitute the coordinates of the given point for x0 and y0 in the formula $y - y_0 = m\,(x - x_0)$.

For example, if the line has slope −2 and passes through the point (3, 4), then substitute $m = -2, x_0 = 3,$ and $y_0 = 4$ in the formula $y - y_0 = m\,(x - x_0)$ to obtain

$$y-4=-2(x-3)$$

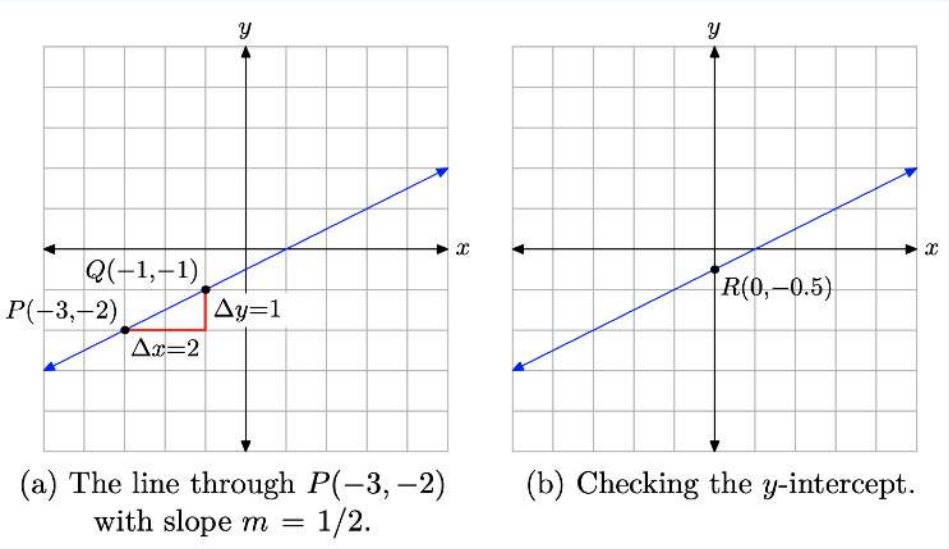
.

Example 1.6.1

Draw the line that passes through the point P(−3, −2) and has slope m = 1/2. Use the point-slope form to determine the equation of the line.

Solution

First, plot the point P(−3, −2), as shown in Figure 1.6.2(a). Starting from the point P(−3, −2), move 2 units to the right and 1 unit up to the point Q(−1, −1). The line through the points P and Q in Figure 1.6.2(a) now has slope m = 1/2.



(a) The line through $P(-3,-2)$ with slope $m = 1/2$.

(b) Checking the y-intercept.

Figure 1.6.2

To determine the equation of the line in Figure 1.6.2(a), we will use the point-slope form of the line

$$y-y_0=m\,(x-x_0)$$

The slope of the line is m = 1/2 and the given point is P(−3, −2), so $(x_0, y_0) = (-3, -2)$. In equation (3), set $m = 1/2, x_0 = -3,$ and $y_0 = -2$, obtaining

$$y-(-2)=\frac{1}{2}(x-(-3))$$

or equivalently,

$$y+2=\frac{1}{2}(x+3)$$

This is the equation of the line in Figure 1.6.2(a).

As a check, we've estimated the y-intercept of the line in Figure 1.6.2(b) as R(0, −0.5). Let's place equation (4) in slope-intercept form to determine the exact value of the y-intercept. First, distribute 1/2 to get

$$y+2=\frac{1}{2}x+\frac{3}{2}$$

Subtract 2 from both sides of this last equation.

$$y=\frac{1}{2}x+\frac{3}{2}-2$$

Make equivalent fractions with a common denominator and simplify.

$$y = \frac{1}{2}x + \frac{3}{2} - \frac{4}{2}$$
$$y = \frac{1}{2}x - \frac{1}{2}$$

Comparing equation (5) with y = mx+b gives us b = −1/2. This is the exact y-value of the y-intercept. Note that this result compares exactly with the y-value of point R in Figure 1.6.2(b). This is a bit lucky. Don't expect to get an exact comparison every time. However, if the comparison is not close, look for an error in your work, either in your computations or in your graph.

Let's look at another example.

Example 1.6.2

Find the equation of the line passing through the points P(−3, 2) and Q(2, −1). Place your final answer in standard form.

Solution

Again, to help keep our focus, we draw the line passing through the points P(−3, 2) and Q(2, −1) in Figure 1.6.3.

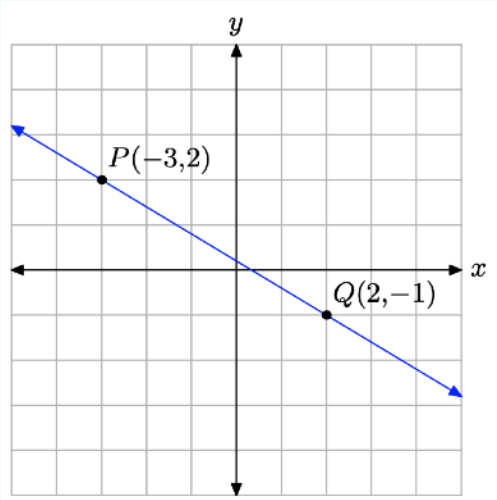


Figure 1.6.3. The line through points P(−3, 2) and Q(2, −1).

Use the slope formula to determine the slope of the line through the points P(−3, 2) and Q(2, −1).

$$m = \frac{\Delta y}{\Delta x} = \frac{-1-2}{2-(-3)} = -\frac{3}{5}$$

We'll use the point-slope form of the line

$$y - y_0 = m\,(x - x_0)$$

Let's use point P(−3, 2) as the given point (x_0, y_0). That is, $(x_0, y_0) = (-3, 2)$. Substitute $m = -3/5, x_0 = -3,$ and $y_0 = 2$ in equation (7), obtaining

$$y - 2 = -\frac{3}{5}(x - (-3))$$

This is the equation of the line passing through the points P and Q.

Alternatively, we could also use the point Q(2, −1) as the given point (x_0, y_0). That is, $(x_0, y_0) = (2, -1)$. Substitute $m = -3/5, x_0 = 2,$ and $y_0 = -1$ in the point-slope form (7), obtaining

$$y - (-1) = -\frac{3}{5}(x - 2)$$

This too, is the equation of the line passing through the points P and Q.

How can the equations (8) and (9) both be the equation of the line through P and Q, yet look so distinctly different? Let's place each equation in standard form $Ax + By = C$ and compare the results.

If we start with equation (8) and distribute the slope,

$$y-2=-\tfrac{3}{5}(x-(-3))$$
$$y-2=-\tfrac{3}{5}x-\tfrac{9}{5}$$

Multiply both sides by the common denominator 5 to clear the fractions.

$$\begin{aligned}5(y-2)&=5\left(-\frac{3}{5}x-\frac{9}{5}\right)\\5y-10&=-3x-9\end{aligned}$$

Add 3x to both sides of the equation, then add 10 to both sides of the equation to obtain

$$3x+5y=1$$

Place equation (9) in standard form in a similar manner. First, start with equation (9) and distribute the slope,

$$\begin{aligned}y-(-1)&=-\frac{3}{5}(x-2)\\y+1&=-\frac{3}{5}x+\frac{6}{5}\end{aligned}$$

Next, multiply both sides of this last result by 5 to clear the fractions from the equation.

$$\begin{aligned}5(y+1)&=5\left(-\frac{3}{5}x+\frac{6}{5}\right)\\5y+5&=-3x+6\end{aligned}$$

Finally, add 3x to both sides of the equation, then subtract 5 from both sides of the equation to obtain

$$3x+5y=1$$

Note that equation (11) is identical to equation (10). Thus, it doesn't matter which point you use in the point-slope form. Both lead to the same result.

Parallel Lines

Recall that slope controls the "steepness" of a line. Consequently, if two lines are parallel, they must have the same "steepness" or slope. Let's look at an example of parallel lines.

Example 1.6.3

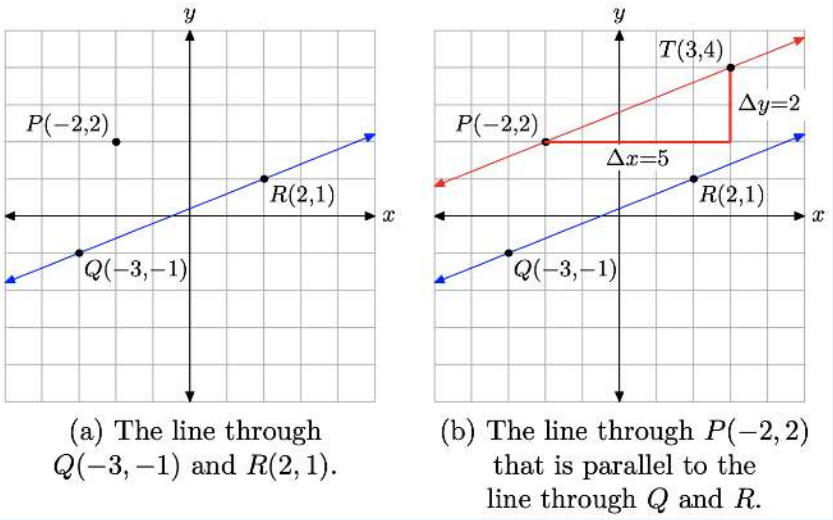
Find the equation of the line that passes through the point P(−2, 2) that is parallel to the line passing through the points Q(−3, −1) and R(2, 1).

Solution

First, to help us stay focused, we draw the line through the points Q(−3, −1) and R(2, 1), then plot the point P(−2, 2), as shown in Figure 1.6.4(a).

We can use the slope formula to calculate the slope of the line passing through the points Q(−3, −1) and R(2, 1).

$$m=\frac{\Delta y}{\Delta x}=\frac{1-(-1)}{2-(-3)}=\frac{2}{5}$$



(a) The line through $Q(-3,-1)$ and $R(2,1)$.

(b) The line through $P(-2,2)$ that is parallel to the line through Q and R.

Figure 1.6.4.

We now draw a line through the point P(−2, 2) that is parallel to the line through the points Q and R. Parallel lines must have the same slope, so we start at the point P(−2, 2), "run" 5 units to the right, then "rise" 2 units up to the point T(3, 4), as shown in Figure 1.6.4(b).

We seek the equation of the line through the points P and T. We'll use the pointslope form of the line

$$y - y_0 = m\,(x - x_0)$$

We'll use the point P(−2, 2) as the given point (x_0, y_0). That is, $(x_0, y_0) = (-2, 2)$. The line through P has slope 2/5. Substitute $m = 2/5, x_0 = -2,$ and $y_0 = 2$ in equation (13) to obtain

$$y - 2 = \frac{2}{5}(x - (-2))$$

Let's place the equation (14) in standard form. Distribute the slope, then clear fractions by multiplying both sides of the resulting equation by 5.

$$\begin{aligned} y - 2 &= \frac{2}{5}x + \frac{4}{5} \\ 5(y-2) &= 5\left(\frac{2}{5}x + \frac{4}{5}\right) \\ 5y - 10 &= 2x + 4 \end{aligned}$$

Finally, subtract 5y from both sides of the last equation, then subtract 4 from both sides of the equation, obtaining

$$-14 = 2x - 5y$$

or equivalently,

$$2x - 5y = -14$$

This is the standard form of the equation of the line passing through the point P and parallel to the line passing through the points Q and R.

Perpendicular Lines

Suppose that two lines L_1 and L_2 have slopes m_1 and m_2, respectively. Recall (see the section on Slope) that if L_1 and L_2 are perpendicular, then the product of their slopes is $m_1 m_2 = -1$. Alternatively, the slope of the first line is the negative reciprocal of the second line, and vice-versa; i.e., $m_1 = -1/m_2$ and $m_2 = -1/m_1$. Let's look at an example of perpendicular lines.

Example 1.6.4

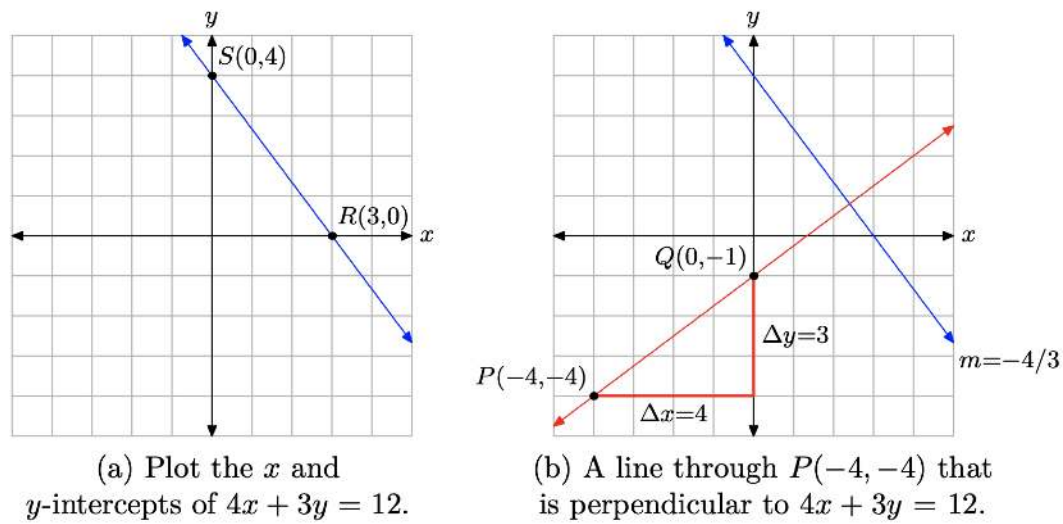
Find the equation of the line passing through the point P(−4, −4) that is perpendicular to the line 4x + 3y = 12.

Solution

It will help our focus if we draw the given line 4x + 3y = 12. The easiest way to plot a line in standard form Ax + By = C is to find the x- and y-intercepts.

$$\begin{aligned} 4x+3y &= 12 \\ 4x+3(0) &= 12 \\ 4x &= 12 \\ x &= 3 \end{aligned} \qquad \begin{aligned} 4x+3y &= 12 \\ 4(0)+3y &= 12 \\ 3y &= 12 \\ y &= 4 \end{aligned}$$

Plot the x- and y-intercepts R(3, 0) and S(0, 4) as shown in Figure 1.6.5(a). The line through points R and S is the graph of the equation 4x + 3y = 12.



(a) Plot the x and y-intercepts of $4x + 3y = 12$.

(b) A line through $P(-4, -4)$ that is perpendicular to $4x + 3y = 12$.

Figure 1.6.5

Next, determine the slope of the line 4x + 3y = 12 by placing this equation in slope-intercept form (i.e., solve the equation 4x + 3y = 12 for y)

$$\begin{aligned} 4x+3y &= 12 \\ 3y &= -4x+12 \\ y &= -\frac{4}{3}x+4 \end{aligned}$$

If two lines are perpendicular, then their slopes are negative reciprocals of one another. Therefore, the slope of the line that is perpendicular to the line 4x + 3y = 12 (which has slope −4/3) is m = 3/4. Our second line must pass through the point P(−4, −4). To draw this second line, first plot the point P(−4, −4), then move 4 units to the right and 3 units upward to the point Q(0, −1), as shown in Figure 1.6.5(b). The line through the points P and Q is perpendicular to the line 4x + 3y = 12.

To determine the equation of the line through the points P and Q, we will use the point-slope form of the line, namely

$$y - y_0 = m\,(x - x_0)$$

The slope of the line through points P and Q is m = 3/4. If we use the point P(−4, −4), then $(x_0, y_0) = (-4, -4)$. Set $m = 3/4$, $x_0 = -4$, and $y_0 = -4$ in equation (16), obtaining

$$y - (-4) = \frac{3}{4}(x - (-4))$$

or equivalently,

$$y+4=\frac{3}{4}(x+4)$$

Alternatively, we could use the slope-intercept form of the line. We know that the line through points P and Q in Figure 1.6.5(b) crosses the y-axis at Q(0, −1). So, with slope m = 3/4 and y-coordinate of the y-intercept b = −1, the slope-intercept form y = mx + b becomes

$$y=\frac{3}{4}x-1$$

On the other hand, if we solve equation (17) for y,

$$\begin{aligned} y+4 &= \frac{3}{4}(x+4) \\ y+4 &= \frac{3}{4}x+3 \\ y &= \frac{3}{4}x-1 \end{aligned}$$

Note that this is identical to the result found using the slope-intercept form above.

It is comforting to note that the two forms (point-slope and slope-intercept) give the same result, but how do we determine the most efficient form to use for a particular problem? Here's a good hint.

Determining the Form of the Line to Use

Here is some sound advice when you are trying to determine whether to use the slope-intercept form or the pointslope form of a line.

- If you are given the slope and the y-intercept, use the slope-intercept form y = mx + b.
- If you are given a point (other than the y-intercept) and the slope, use the point-slope form $y-y_0=m\,(x-x_0)$.

Applications of Linear Functions

In this section we will look at some applications of linear functions. We begin by developing a function relating Fahrenheit and Celsius temperature.

Example 1.6.5

Water freezes at $32^\circ F$ and 0°C. Water boils at $212^\circ F$ and 100°C. F and C are abbreviations for Fahrenheit and Celsius temperature scales, respectively. Assuming a linear relationship, develop a model relating Fahrenheit and Celsius temperature.

Solution

First, to help keep our focus, we set up a coordinate system on a sheet of graph paper. In Figure 1.6.6, we've decided to make the Celsius temperature the dependent variable and have assigned the Celsius temperature to the vertical axis. Similarly, we've declared the Fahrenheit temperature the independent variable and assigned it to the horizontal axis.

Interpret the given data:

- Water freezes at $32^\circ F$ and $0^\circ C$. This gives us the point (F, C) = (32, 0), which we plot in Figure 1.6.6.
- Water boils at $212^\circ F$ and $100^\circ C$. This gives us the point (F, C) = (212, 100), which we plot in Figure 1.6.6.

Now we are on familiar ground. We want to find the equation of the line through these two points, which is the same type of problem we tackled in Example 6. First, use the points (32, 0) and (212, 100) to determine the slope of the line.

$$m=\frac{\Delta C}{\Delta F}=\frac{100-0}{212-32}=\frac{100}{180}=\frac{5}{9}$$

We will now use the point-slope form of the line, $y-y_0=m\,(x-x_0)$ with m = 5/9 and $(x_0,y_0)=(32,0)$. Substitute $m=5/9, x_0=32$, and $y_0=0$ in $y-y_0=m\,(x-x_0)$ to obtain

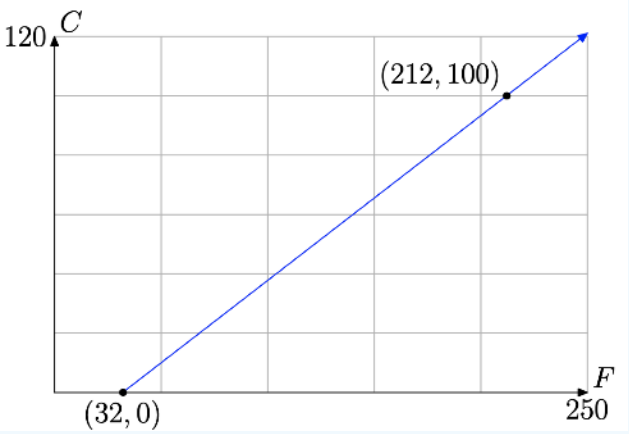


Figure 1.6.6. Plotting Celsius temperature versus Fahrenheit temperature

$$y - 0 = \frac{5}{9}(x - 32)$$

However, our dependent axis is labeled C, not y, and our independent axis is labeled F, not x. So, we must replace y and x in equation (20) with C and F, respectively, obtaining

$$C = \frac{5}{9}(F - 32)$$

This result in equation (21) expresses the Celsius temperature as a function of the Fahrenheit temperature. Alternatively, we could also use function notation and write

$$C(F) = \frac{5}{9}(F - 32)$$

Suppose that we know that the Fahrenheit temperature outside is $80^\circ\mathrm{F}$ and we wish to express this using the Celsius scale. To do so, we simply evaluate C(80), as in

$$C(80) = \frac{5}{9}(80 - 32) \approx 26.6$$

Hence, the Celsius temperature is approximately $26.6^\circ\mathrm{C}$.

On the other hand, suppose that we know the Celsius temperature on a metal roof is $80^\circ\mathrm{C}$ and we wish to find the Fahrenheit temperature. To do so, we need to solve

$$C(F) = 80$$

for F, or equivalently,

$$\frac{5}{9}(F - 32) = 80$$

Multiply both sides by 9 to obtain

$$5(F - 32) = 720$$

then divide both sides of the result by 5 to obtain

$$F - 32 = 144$$

Adding 32 to both sides of this last result produces the Fahrenheit temperature $F = 176^\circ\mathrm{F}$. Wow, that's hot!

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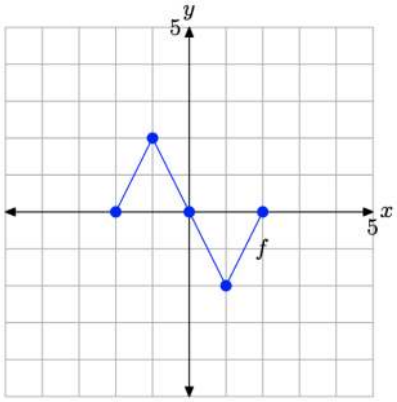
1.7: Vertical Transformations

In this section we study the art of transformations: scalings, reflections, and translations. We will restrict our attention to transformations in the vertical or y-direction. Our goal is to apply certain transformations to the equation of a function, then ask what effect it has on the graph of the function.

We begin our task with an example that requires that we read the graph of a function to capture several key points that lie on the graph of the function.

Example 1.7.1

Consider the graph of f presented in Figure 1.7.1(a). Use the graph of f to complete the table in Figure 1.7.1(b).



(a) The graph of f.

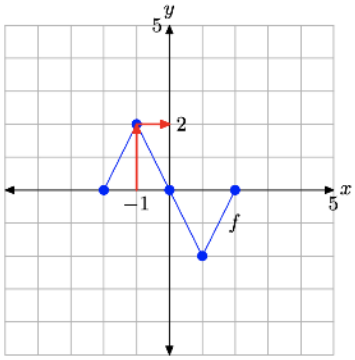
x	$f(x)$	$(x, f(x))$
-2		
-1		
0		
1		
2		

(b) The table.

Figure 1.7.1. Reading key values from the graph of f.

Solution

To compute f(−1), we would locate −1 on the x-axis, draw a vertical arrow to the graph of f, then a horizontal arrow to the y-axis, as shown in Figure 1.7.2(a). The y-value of this final destination is the value of f(−1). That is, f(−1) = 2. This allows us to complete one entry in the table, as shown in Figure 1.7.2(b). Continue in this manner to complete all of the entries in the table. The result is shown in Figure 1.7.2(c).



(a) The graph of f.

x	$f(x)$	$(x, f(x))$
-2		
-1	2	$(-1, 2)$
0		
1		
2		

(b) Recording $f(-1) = 2$.

x	$f(x)$	$(x, f(x))$
-2	0	$(-2, 0)$
-1	2	$(-1, 2)$
0	0	$(0, 0)$
1	-2	$(1, -2)$
2	0	$(2, 0)$

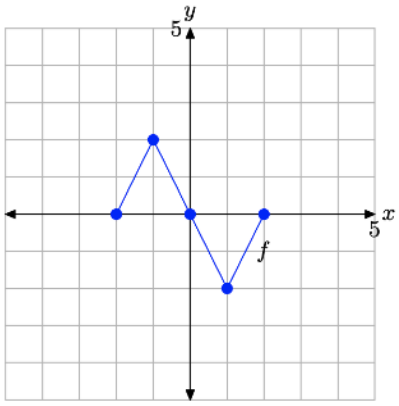
(c) Completed table.

Figure 1.7.2. Recording coordinates of points on the graph of f in the tables.

Vertical Scaling

In the narrative that follows, we will have repeated need of the graph in Figure 1.7.2(a) and the table in Figure 1.7.2(c). They characterize the basic function that will be the starting point for the concepts of scaling, reflection, and translation that we develop

in this section. Consequently, let's place them side-by-side for emphasis in Figure 1.7.3.



(a)

x	$f(x)$	$(x, f(x))$
-2	0	$(-2, 0)$
-1	2	$(-1, 2)$
0	0	$(0, 0)$
1	-2	$(1, -2)$
2	0	$(2, 0)$

(b)

Figure 1.7.3. The original graph of f and a table of key points on the graph of f.

We are now going to scale the graph of f in the vertical direction.

Example 1.7.2

If y = f(x) has the graph shown in Figure 1.7.3(a), sketch the graph of y = 2f(x).

Solution

What do we do when we meet a graph whose shape we are unsure of? The answer to this question is we plot some points that satisfy the equation in order to get an idea of the shape of the graph. With that thought in mind, let's evaluate the function y = 2f(x) at x = −2.

The letter f refers to the original function shown in Figure 1.7.3(a) and the table in Figure 1.7.3(b) contains the values of that function at the given values of x. Thus, in computing y = 2f(−2), the first step is to look up the value of f(−2) in the table in Figure 1.7.3(b). There we find that f(−2) = 0. Thus, we can write

$$y = 2f(-2) = 2(0) = 0$$

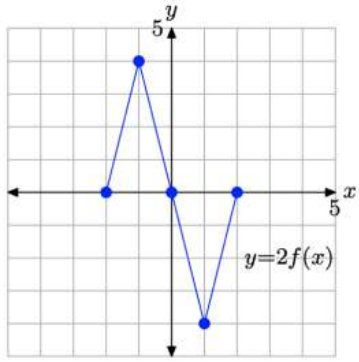
In similar fashion, let's evaluate the function y = 2f(x) at x = −1. First, look up the value of f(−1) in the table in Figure 1.7.3(b). There we find that f(−1) = 2. Thus, we can write

$$y = 2f(-1) = 2(2) = 4$$

We finish by evaluating the function y = 2f(x) at x = 0, 1, and 2. Each time you need to evaluate the function f at a number, take the result from the table or graph in Figure 3. What follows are the evaluations of y = 2f(x) at x = −2, −1, 0, 1, and 2.

$$y = 2f(-2) = 2(0) = 0$$
$$y = 2f(-1) = 2(2) = 4$$
$$y = 2f(0) = 2(0) = 0$$
$$y = 2f(1) = 2(-2) = -4$$
$$y = 2f(2) = 2(0) = 0$$

We can arrange these results in a table shown in Figure 1.7.4(b), then plot them in the figure shown in Figure 1.7.4(a).



(a)

x	$y=2f(x)$	$(x,2f(x))$
-2	0	$(-2,0)$
-1	4	$(-1,4)$
0	0	$(0,0)$
1	-4	$(1,-4)$
2	0	$(2,0)$

(b)

Figure 1.7.4. The points in the table are points on the graph of y = 2f(x).

At this point, there are a number of comparisons you can make.

1. Compare the data in the tables in Figure 1.7.3(b) and Figure 1.7.4(b). Note that the x-values are identical. In both tables, x = −2, −1, 0, 1, and 2. However, note that each y-value in the table in Figure 1.7.4(b) is precisely double the corresponding y-value in the table in Figure 1.7.3(b).
2. Compare the graphs in Figure 1.7.3(a) and Figure 1.7.4(a). Note that the y-value of each point in the graph of y = 2f(x) in Figure 1.7.4(a) is precisely double the y-value of the corresponding point in Figure 1.7.3(a).

Note the result. The graph of y = 2f(x) has been stretched vertically (away from the x-axis) , both positively and negatively, by a factor of 2.

Let's look at another example.

✔ Example 1.7.3

If y = f(x) has the graph shown in Figure 1.7.3(a), sketch the graph of y = (1/2)f(x).

Solution

Let's begin by evaluating the function y = (1/2)f(x) at x = −2. First, look up the value of f(−2) in the table in Figure 1.7.3(b). There we find that f(−2) = 0. Thus, we can write

$$y=(1/2)f(-2)=(1/2)(0)=0$$

In similar fashion, let's evaluate the function y = (1/2)f(x) at x = −1. First, look up the value of f(−1) in the table in Figure 1.7.3(b). There we find that f(−1) = 2. Thus, we can write

$$y=(1/2)f(-1)=(1/2)(2)=1$$

Continuing in this manner, we can evaluate the function y = (1/2)f(x) at x = 0, 1, and 2.

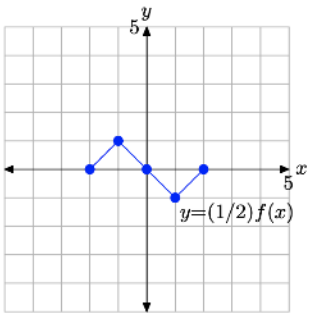
$$y=(1/2)f(0)=(1/2)(0)=0$$
$$y=(1/2)f(1)=(1/2)(-2)=-1$$
$$y=(1/2)f(2)=(1/2)(0)=0$$

The results are recorded in the table in Figure 1.7.5(b). Rather than double each value of y as did the function y = 2f(x) in Example 1.7.2, this function y = (1/2)f(x) halves each value of y. The graph of y = (1/2)f(x) and a table of key points on the graph are presented in Figures 1.7.5(a) and (b), respectively.

Again, there are a number of comparisons.

1. Compare the data in the tables in Figure 1.7.5(b) and Figure 1.7.3(b). Note that the x-values are identical. In both tables x = −2, −1, 0, 1, and 2. However, note that each y-value in the table in Figure 1.7.5(b) is precisely half the corresponding y-value in the table in Figure 1.7.3(b).

2. When you compare the graph of y = (1/2)f(x) in Figure 1.7.5(a) with the original graph of y = f(x) in Figure 1.7.3(a), note that each point on the graph of y = (1/2)f(x) has a y-value that is precisely half of the corresponding y-value on the original graph of y = f(x) in Figure 1.7.3(a).



(a)

x	$y=(1/2)f(x)$	$(x,(1/2)f(x))$
-2	0	$(-2,0)$
-1	1	$(-1,1)$
0	0	$(0,0)$
1	-1	$(1,-1)$
2	0	$(2,0)$

(b)

Figure 1.7.5. The points in the table are points on the graph of y = (1/2)f(x).

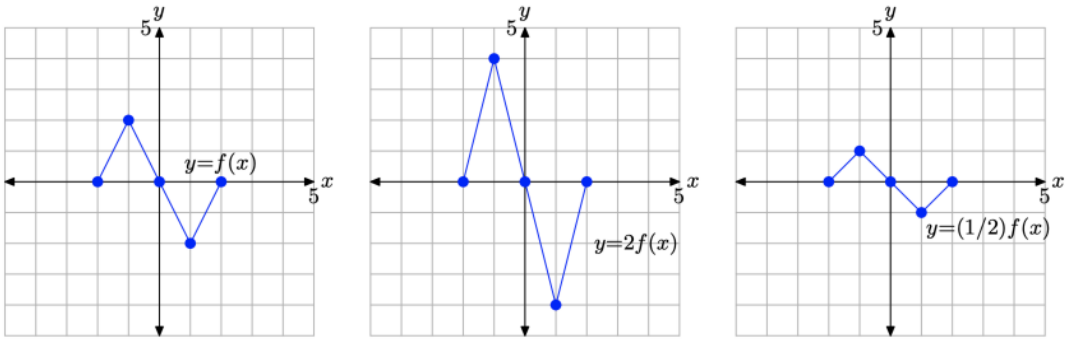
Note the result. The graph of f has been compressed vertically (toward the x-axis), both positively and negatively, by a factor of 2.

Let's summarize our findings.

A Visual Summary — Vertical Scaling.

Consider the images in Figure 1.7.6.

- In Figure 1.7.6(a), we see pictured the graph of the original function y = f(x).
- In Figure 1.7.6(b), note that each key point on the graph of y = 2f(x) has a y-value that is precisely double the y-value of the corresponding point on the graph of y = f(x) in Figure 1.7.6(a).
- In Figure 1.7.6(c), note that each key point on the graph of y = (1/2)f(x) has a y-value that is precisely half the y-value of the corresponding point on the graph of y = f(x) in Figure 1.7.6(a).
- Note that the x-value of each transformed point remains the same.



(a) $y = f(x)$ (b) $y = 2f(x)$ (c) $y = (1/2)f(x)$

Figure 1.7.6. The graph of y = 2f(x) stretches vertically (away from the x-axis) by a factor of 2. The graph of y = (1/2)f(x) compresses vertically (toward the x-axis) by a factor of 2.

The visual summary in Figure 1.7.6 makes sketching the graphs of y = 2f(x) and y = (1/2)f(x) an easy task.

- Given the graph of y = f(x), to sketch the graph of y = 2f(x), simply take each point on the graph of y = f(x) and double its y-value, keeping the same x-value.
- Given the graph of y = f(x), to sketch the graph of y = (1/2)f(x), simply take each point on the graph of y = f(x) and halve its y-value, keeping the same x-value.

Follow the same procedures for other scaling factors. For example, in the case of y = 3f(x), take each point on the graph of y = f(x) and multiply its y-value by 3, keeping the same x-value. On the other hand, to draw the graph of y = (1/3)f(x), take each point on the graph of f and multiply its y-value by 1/3, keeping the same y-value.

In general, we can state the following.

summary

Suppose we are given the graph of y = f(x).

- If a > 1, then the graph of y = af(x) is stretched vertically (away from the x-axis), both positively and negatively, by a factor of a.
- If 0 < a < 1, then the graph of y = af(x) is compressed vertically (toward the x-axis), both positively and negatively, by a factor of 1/a.

The second item in Summary warrants a word of explanation. Compare the general form y = af(x) with the function of Example 1.7.3, y = (1/2)f(x). In this case, a = 1/2, so

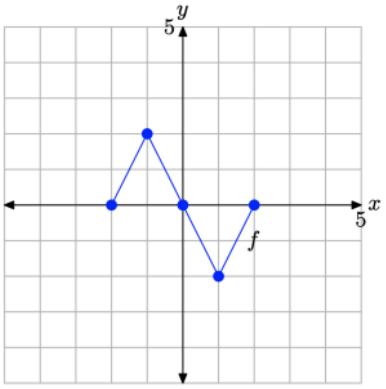
$$\frac{1}{a} = \frac{1}{1/2} = 1 \times 2 = 2$$

The second item says that when 0 < a < 1, the graph of y = af(x) is compressed vertically by a factor of 1/a. Indeed, this is exactly what happens in the case of y = (1/2)f(x), which is compressed by a factor of 1/(1/2), or 2.

Vertical Reflections

For convenience, we begin by repeating the original graph of y = f(x) and its accompanying data.

We are now going to reflect the graph in the vertical direction (across the x-axis).



(a)

x	$f(x)$	$(x, f(x))$
-2	0	$(-2, 0)$
-1	2	$(-1, 2)$
0	0	$(0, 0)$
1	-2	$(1, -2)$
2	0	$(2, 0)$

(b)

Figure 1.7.7. The original graph of f and a table of key points on the graph of f.

Example 1.7.4

If y = f(x) has the graph shown in Figure 1.7.7(a), sketch the graph of y = −f(x).

Solution

To set up a table of points in preparation for the plot of y = −f(x), we'll use exactly the same values of x that you see in the table in Figure 1.7.7(b), namely x = −2, −1, 0, 1, and 2. To evaluate y = −f(x) at the first value of x, namely x = −2, we make the following calculation,

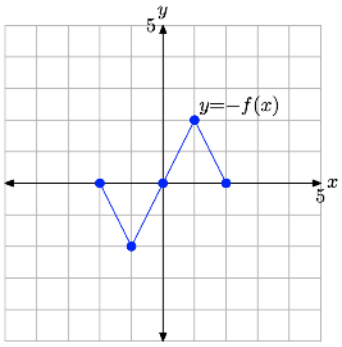
$$y = -f(-2) = -(0) = 0$$

where we've used the fact that f(−2) = 0 from the table in Figure 1.7.7(b). In similar fashion, we evaluate y = −f(x) at each of the remaining values of x, namely x = −1, 0, 1, and 2.

$$y = -f(-1) = -(2) = -2$$
$$y = -f(0) = -(0) = 0$$
$$y = -f(1) = -(-2) = 2$$
$$y = -f(2) = -(0) = 0$$

We assemble these points in the table in Figure 1.7.8(b) and plot them in Figure 1.7.8(a).

Note that the graph of y = −f(x) in Figure 1.7.8(a) is a reflection of the graph of y = f(x) in Figure 1.7.7(a) across the x-axis.



(a)

x	$y = -f(x)$	$(x, -f(x))$
-2	0	$(-2, 0)$
-1	-2	$(-1, -2)$
0	0	$(0, 0)$
1	2	$(1, 2)$
2	0	$(2, 0)$

(b)

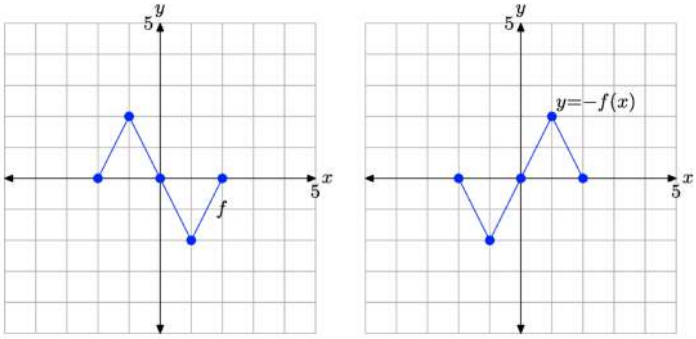
Figure 1.7.8. The graph of y = −f(x) and a table of key points on the graph.

Let's summarize what we've learned about vertical reflections.

A Visual Summary — Vertical Reflections

Consider the images in Figure 1.7.9.

- In Figure 1.7.9(a), we see pictured the original graph of y = f(x).
- In Figure 1.7.9(b), the graph of y = −f(x) is a reflection of the graph of y = f(x) across the x-axis.



(a) $y = f(x)$ (b) $y = -f(x)$

Figure 1.7.9. The graph of y = −f(x) is a reflection of the graph of y = f(x) across the x-axis.

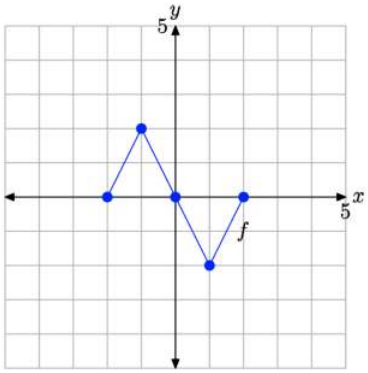
Thus, given the graph of y = f(x), it is a simple task to draw the graph of y = −f(x).

- To draw the graph of y = −f(x), take each point on the graph of y = f(x) and reflect it across the x-axis, keeping the x-value the same, but negating the y-value.

Vertical Translations

Translations are perhaps the easiest transformation of all. A translation is a "shift" or a "slide." Pretend, for a moment, that you've placed a transparent sheet of thin plastic over a sheet of graph paper. You've drawn a Cartesian coordinate system on your graph paper, but you've plotted your graph on the transparent sheet of plastic. Now, "shift" or "slide" the transparency over your graph paper in a constant direction without rotating the transparency. This is what we mean by a "translation." In this section, we will focus strictly on vertical translations.

For convenience, we begin by repeating the original graph of y = f(x) and its accompanying data in Figure 1.7.10(a) and(b), respectively. We will now translate this graph in the vertical direction.



(a)

x	$f(x)$	$(x, f(x))$
-2	0	$(-2,0)$
-1	2	$(-1,2)$
0	0	$(0,0)$
1	-2	$(1,-2)$
2	0	$(2,0)$

(b)

Figure 1.7.10. The original graph of f and a table of key points on the graph of f.

Example 1.7.5

If y = f(x) has the graph shown in Figure 1.7.10(a), sketch the graph of y = f(x) + 1.

Solution

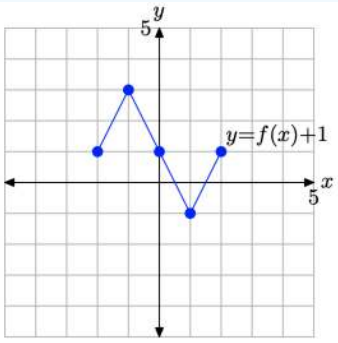
We will evaluate y = f(x)+1 at the same values shown in the table in Figure 1.7.10(b), namely x = −2, −1, 0, 1, and 2. To evaluate y = f(x)+1 at the first value of x, namely x = −2, we make the following calculation

$$y = f(-2) + 1 = 0 + 1 = 1$$

where we've used that fact that f(−2) = 0 from the table in Figure 1.7.10(b). In similar fashion, we can evaluate y = f(x) + 1 at each of the remaining values of x, namely x = −1, 0, 1, and 2.

$$y = f(-1) + 1 = 2 + 1 = 3$$
$$y = f(0) + 1 = 0 + 1 = 1$$
$$y = f(1) + 1 = -2 + 1 = -1$$
$$y = f(2) + 1 = 0 + 1 = 1$$

We assemble these points in the table in Figure 1.7.11(b) and plot them in Figure 1.7.11(a).



(a)

x	$y = f(x) + 1$	$(x, f(x) + 1)$
-2	1	$(-2,1)$
-1	3	$(-1,3)$
0	1	$(0,1)$
1	-1	$(1,-1)$
2	1	$(2,1)$

(b)

Figure 1.7.11. The graph of y = f(x) + 1 and a table of key points on the graph.

When you compare the entries in the table in Figure 1.7.11(b) with the original values in the table in Figure 1.7.10(b), you'll note that the x-values in each table are identical, but the y-values in the table in Figure 1.7.11(b) are all increased by 1. This makes sense, because these are the y-values of the points associated with the function y = f(x) + 1. Of course, all the y-values should be 1 larger than the y-values associated with the original equation y = f(x).

Note the result. The graph of y = f(x) + 1 in Figure 1.7.11(a), when compared with the graph of y = f(x) in Figure 1.7.10(a), is shifted 1 unit upwards.

Let's look at another example.

Example 1.7.6

If y = f(x) has the graph shown in Figure 1.7.10(a), sketch the graph of y = f(x) − 2.

Solution

Evaluate the function y = f(x) − 2 at each value of x in the table in Figure 1.7.10(b). At x = −2,

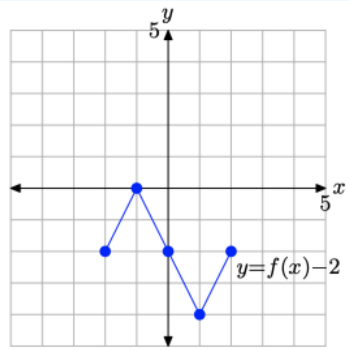
$$y = f(-2) - 2 = 0 - 2 = -2$$

In similar fashion, evaluate y = f(x) − 2 at each remaining x-value in the table in Figure 1.7.10(b).

$$\begin{aligned} y &= f(-1) - 2 = 2 - 2 = 0 \\ y &= f(0) - 2 = 0 - 2 = -2 \\ y &= f(1) - 2 = -2 - 2 = -4 \\ y &= f(2) - 2 = 0 - 2 = -2 \end{aligned}$$

We assemble these points in the table in Figure 1.7.12(b) and plot them in Figure 1.7.12(a).

When you compare the entries in the table in Figure 1.7.12(b) with the original values in the table in Figure 1.7.10(b), you'll note that the x-values in each table are identical, but the y-values in the table in Figure 1.7.12(b) are all decremented by 2. This makes



(a)

x	$y = f(x) - 2$	$(x, f(x) - 2)$
−2	−2	(−2, −2)
−1	0	(−1, 0)
0	−2	(0, −2)
1	−4	(1, −4)
2	−2	(2, −2)

(b)

Figure 1.7.12. The graph of y = f(x) − 2 and a table of key points on the graph.

sense, because these are the y-values of the points associated with the function y = f(x) − 2. Of course, all the y-values should be 2 less than the y-values associated with the original equation y = f(x).

Note the result. The graph of y = f(x) − 2 in Figure 1.7.12(a), when compared with the graph of y = f(x) in Figure 1.7.10(a), is shifted downward 2 units.

Let's summarize what we've learned about vertical translations.

A Visual Summary — Vertical Translations (Shifts)

Consider the images in Figure 1.7.13.

- In Figure 1.7.13(a), we see pictured the graph of the original function y = f(x).
- In Figure 1.7.13(b), note that each key point on the graph of y = f(x) + 1 has a y-value that is precisely 1 unit larger than the y-value of the corresponding point on the graph of y = f(x) in in Figure 1.7.13(a).
- In Figure 1.7.13(c), note that each key point on the graph of y = f(x) − 2 has a y-value that is precisely 2 units smaller than the y-value of the corresponding point on the graph of y = f(x) in in Figure 1.7.13(a).

- Note that the x-value of each transformed point remains the same.

The visual summary in Figure 1.7.13 makes sketching the graphs of y = f(x) + 1 and y = f(x) − 2 an easy task.

- Given the graph of y = f(x), to sketch the graph of y = f(x) + 1, simply take each point on the graph of y = f(x) and move it upwards 1 unit, keeping the same x-value.
- Given the graph of y = f(x), to sketch the graph of y = f(x) − 2, simply take each point on the graph of y = f(x) and move it downwards 2 units, keeping the same x-value.

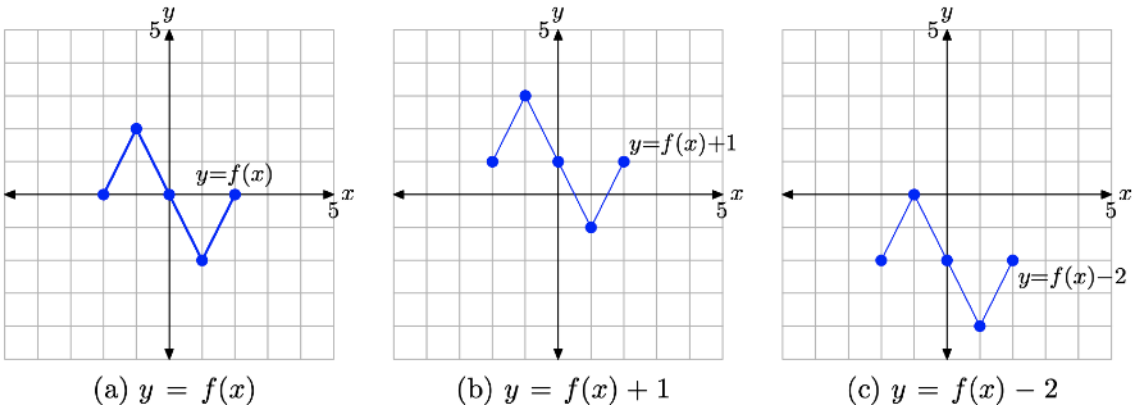


Figure 1.7.13 The graph of y = f(x) + 1 is formed by shifting (vertically) the graph of y = f(x) upward 1 unit. The graph of y = f(x) − 2 is formed by shifting (vertically) the graph of y = f(x) downward 2 units.

In general, we can state the following.

summary

Suppose that we are given the graph of y = f(x) and suppose that c is any positive real number.

- The graph of y = f(x)+c is shifted c units upward from the graph of y = f(x).
- The graph of y = f(x) − c is shifted c units downward from the graph of y = f(x).

Composing Transformations

Sometimes we will want to perform one transformation, then take the result of the first transformation and apply a second transformation. Let's look at an example.

Example 1.7.7

Consider the graph of y = f(x) presented in Figure 1.7.14

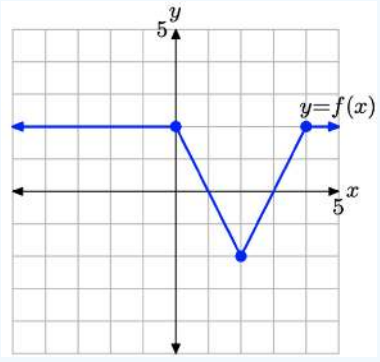


Figure 1.7.14 The graph of y = f(x) that will be transformed in Example 1.7.7.

Use the concepts discussed in the Visual Summaries to sketch the graph of y = −2f(x) without creating and referring to a table of points.

Solution

Note that the equation y = −2f(x) can be formed by a sequence of two transformations.

1. First, scale the original function y = f(x) to obtain the equation y = 2f(x).
2. Second, negate the resulting function y = 2f(x) to obtain the equation y = −2f(x).

Thus, the graph of y = −2f(x) can be formed as follows:

1. Start with the graph of y = f(x) and double the y-value of each point on the graph of y = f(x), keeping the same x-value. The result is the graph of y = 2f(x) shown in Figure 1.7.15(b).
2. Next, negate the y-value of each point on the graph of y = 2f(x), keeping the same x-value. The result is the graph of y = −2f(x) in Figure 1.7.15(c).

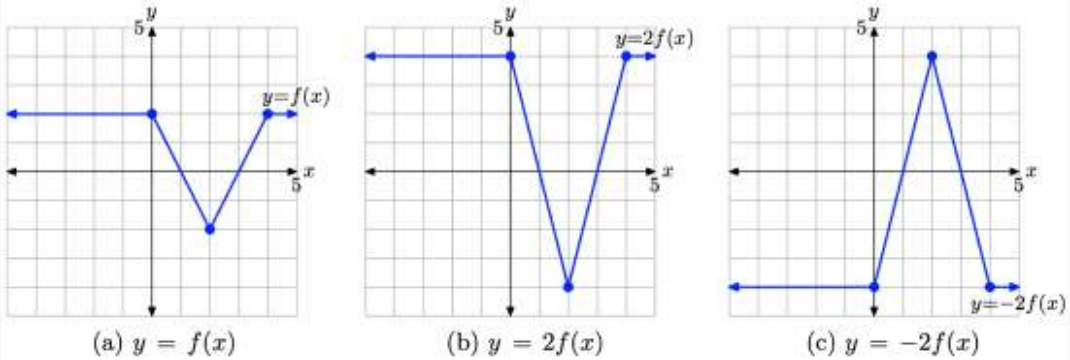


Figure 1.7.15. Transforming the graph of y = f(x) with a sequence of two transformations.

It is interesting to note that you will get the same result if you negate first, then scale the result. We will leave it to our readers to check that this is true.

Let's look at one final example.

Example 1.7.8

Consider the graph of y = f(x) presented in Figure 1.7.16.

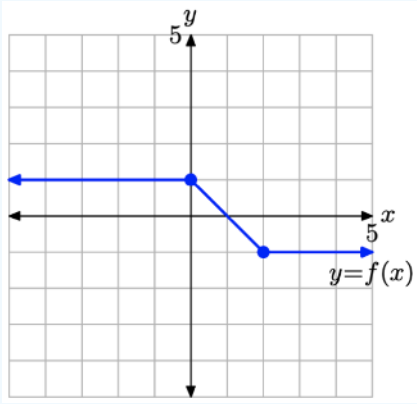
Use the concepts discussed in the Visual Summaries to sketch the graph of y = −f(x)+2 without creating and referring to a table of points.

Solution

Note that the equation y = −f(x) + 2 can be formed by a sequence of two transformations.

1. First, negate the original function y = f(x) to obtain the equation y = −f(x).
2. Second, add 2 to the resulting function y = −f(x) to obtain the equation y = −f(x) + 2.

Thus, the graph of y = −f(x) + 2 can be formed as follows.



https://math.libretexts.org/@go/page/58412

Figure 1.7.16. The graph of y = f(x) that will be transformed in Example 1.7.8.

1. First, start with the graph of y = f(x) in Figure 1.7.17(a) and negate the y-value of each point to produce the graph of y = −f(x) Figure 1.7.17(b).

2. Next, add 2 to the y-value of each point on the graph of y = −f(x) in Figure 1.7.17(b) to produce the graph of y = −f(x) + 2 in Figure 1.7.17(c).

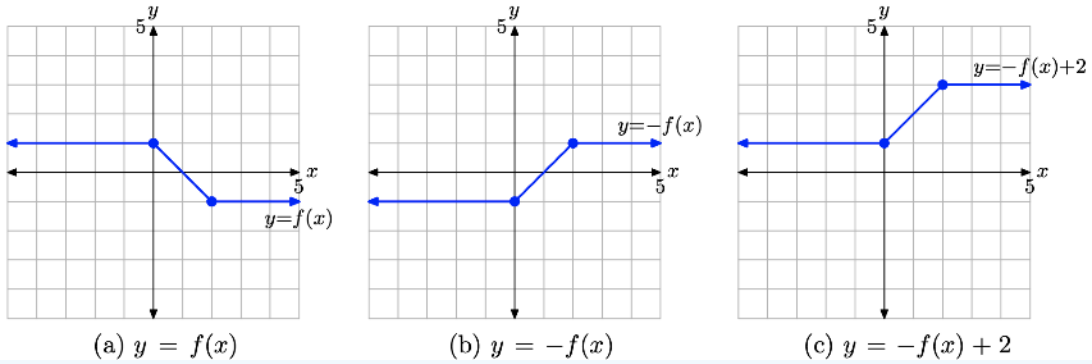


Figure 1.7.17. Transforming the graph of y = f(x), first reflecting across the x-axis, then shifting 2 units upward to obtain the graph of y = −f(x) + 2.

In Example 1.7.7, where we started with the graph of y = f(x) and then graphed y = 2f(x), the order of the transformations did not matter. Scale by 2, then negate, or negate and scale by 2, you get the same result (readers should verify this claim). However, in this example, the order in which the transformations are applied does matter. To see this, let's do the following:

1. Add 2 to shift the graph of y = f(x) in Figure 1.7.18(a) two units upward to obtain the graph of y = f(x) + 2 in Figure 1.7.18(b).
2. Negate the y-value of each point on the graph of y = f(x) + 2 in Figure 1.7.18(b) to obtain the graph of y = −(f(x) + 2) in Figure 1.7.18(c). Note that we must negate the entire y-value. Hence the parentheses.

Unfortunately, the graph of y = −(f(x)+ 2) in Figure 1.7.18(c) is not the same as the graph of y = −f(x) + 2 in Figure 1.7.17(c). But of course, this makes complete sense, as the equations (in the case of Figure 1.7.18(c))

$$y = -(f(x) + 2) = -f(x) - 2$$

and (in the case of Figure 1.7.17(c))

$$y = -f(x) + 2$$

are also not the same.

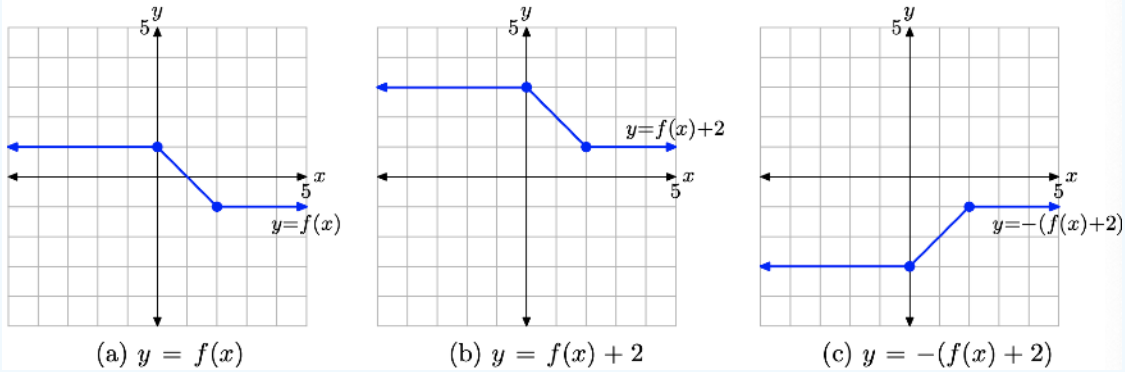


Figure 1.7.18. Transforming the graph of y = f(x), shifting 2 units upward to obtain the graph of y = f(x) + 2, then reflecting across the x-axis to obtain the graph of y = −(f(x) + 2).

Therefore, care must be taken when applying more than one transformation. Here is a good rule of thumb to live by.

Do Vertical Scalings and Reflections First, then Vertical Translations. When performing a sequence of vertical transformations, it is usually easier (less confusing) to apply vertical scalings and reflections before vertical translations.

However, as long as you perform the transformations correctly, you should obtain the correct result. In Example 1.7.8, if you want to sketch the graph of y = −f(x)+2 by doing the translation first, the correct way to proceed is as follows (though somewhat counterintuitive):

1. First, shift the graph of y = f(x) downward 2 units to obtain the graph of y = f(x) − 2.
2. Second, reflect the graph of y = f(x) − 2 across the x-axis to obtain the graph of y = −(f(x) − 2). Again, note the use of parentheses as we negate the entire y-value.

Finally, note that

$$y = -(f(x) - 2) = -f(x) + 2$$

We will leave it to our readers to show that this sequence produces the correct result, a graph identical to the correct answer shown in Figure 1.7.17(c).

Summary

In this section we've seen how a handful of transformations greatly enhance our graphing capability. We end this section by listing the transformations presented in this section and their effects on the graph of a function.

Vertical Transformations.

Vertical Transformations. Suppose we are given the graph of y = f(x).

- If $a > 1$, then the graph of y = af(x) is stretched vertically (away from the x-axis), both positively and negatively, by a factor of a.
- If $0 < a < 1$, then the graph of y = af(x) is compressed vertically (toward the x-axis), both positively and negatively, by a factor of 1/a.
- The graph of y = −f(x) is a reflection of the graph of y = f(x) across the x-axis.
- If $c > 0$, then the graph of y = f(x) + c is shifted c units upward from the graph of y = f(x).
- If $c > 0$, then the graph of y = f(x) − c is shifted c units downward from the graph of y = f(x).

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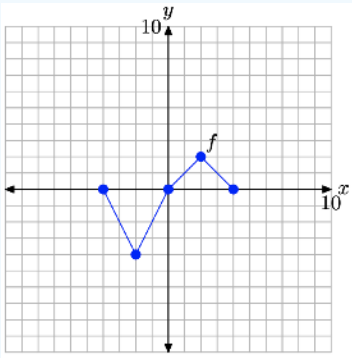
1.8: Horizontal Transformations

In the previous section, we introduced the concept of transformations. We made a change to the basic equation y = f(x), such as y = af(x), y = −f(x), y = f(x) − c, or y = f(x) + c, then studied how these changes affected the shape of the graph of y = f(x). In that section, we concentrated strictly on transformations that applied in th vertical direction. In this section, we will study transformations that will affect the shape of the graph in the horizontal direction.

We begin our task with an example that requires that we read the graph of a function to capture several key points that lie on the graph of the function.

Example 1.8.1

Consider the graph of f presented in Figure 1.8.1(a). Use the graph of f to complete the table in Figure 1.8.1(b).



(a) The graph of f.

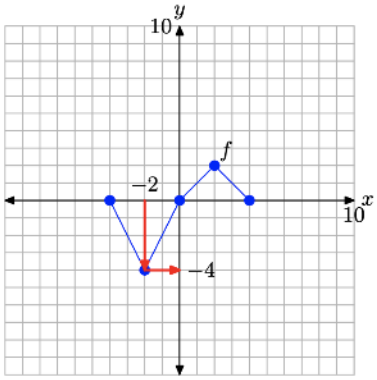
x	$f(x)$	$(x, f(x))$
-4		
-2		
0		
2		
4		

(b) The table.

Figure 1.8.1. Reading key values from the graph of f.

Solution

To compute f(−2), for example, we would first locate −2 on the x-axis, draw a vertical arrow to the graph of f, then a horizontal arrow to the y-axis, as shown in Figure 1.8.2(a). The y-value of this final destination is the value of f(−2). That is, f(−2) = −4. This allows us to complete one entry in the table, as shown in Figure 1.8.2(b). Continue in this manner to complete all of the entries in the table. The result is shown in Figure 1.8.2(c).



(a) The graph of f.

x	$f(x)$	$(x, f(x))$
-4		
-2	-4	$(-2, -4)$
0		
2		
4		

(b) Recording $f(-2) = -4$.

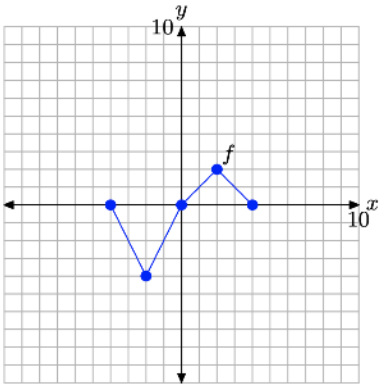
x	$f(x)$	$(x, f(x))$
-4	0	$(-4, 0)$
-2	-4	$(-2, -4)$
0	0	$(0, 0)$
2	2	$(2, 2)$
4	0	$(4, 0)$

(c) Completed table.

Figure 1.8.2. Recording coordinates of points on the graph of f in the tables

Horizontal Scaling

In the narrative that follows, we will have repeated need of the graph in Figure 1.8.2(a) and the table in Figure 1.8.2(c). They characterize the basic function that will be the starting point for the concepts of scaling, reflection, and translation that we develop in this section. Consequently, let's place them side-by-side for emphasis in Figure 1.8.3.



(a)

x	$f(x)$	$(x, f(x))$
-4	0	$(-4, 0)$
-2	-4	$(-2, -4)$
0	0	$(0, 0)$
2	2	$(2, 2)$
4	0	$(4, 0)$

(b)

Figure 1.8.3. The original graph of f and a table of key points on the graph of f

We are now going to scale the graph of f in the horizontal direction.

Example 1.8.2

If y = f(x) has the graph shown in Figure 1.8.3(a), sketch the graph of y = f(2x).

Solution

In the previous section, we investigated the graph of y = 2f(x). The number 2 was outside the function notation and as a result we stretched the graph of y = f(x) vertically by a factor of 2. However, note that the 2 is now inside the function notation y = f(2x). Intuition would demand that this might have something to do with scaling in the x-direction (horizontal direction), but how?

Again, when we're unsure of the shape of the graph, we rely on plotting a table of points. We begin by picking these x-values: x = −2, −1, 0, 1, and 2. Note that these are precisely half of each of the x-values presented in the table in Figure 1.8.3(b). We will now evaluate the function y = f(2x) at each of these x-values. For example, to compute y = f(2x) at x = −2, we first insert x = −2 for x to obtain

$$y = f(2(-2)) = f(-4)$$

To complete the computation, we must now evaluate f(−4). However, this result is recorded in the table in Figure 1.8.3(b). There we find that f(−4) = 0, and we can complete the computation started above.

$$y = f(2(-2)) = f(-4) = 0$$

In similar fashion, to evaluate the function y = f(2x) at x = −1, first substitute x = −1 in y = f(2x) to obtain

$$y = f(2(-1)) = f(-2)$$

Now, note that f(−2) is the next recorded value in the table in Figure 1.8.3(b). There we find that f(−2) = −4, so we can complete the computation started above.

$$y = f(2(-1)) = f(-2) = -4$$

At this point, you might see why we chose x-values: −2, −1, 0, 1, and 2. These are precisely half of the x-values in the table of original values for the function y = f(x) in Figure 1.8.3(b). When the values −2, −1, 0, 1, and 2 are substituted into the function y = f(2x), they are first doubled before we go to look up the function value in the table in Figure 1.8.3(b).

Continuing in this manner, we evaluate the function y = f(2x) at the remaining values of x, namely, 0, 1, and 2.

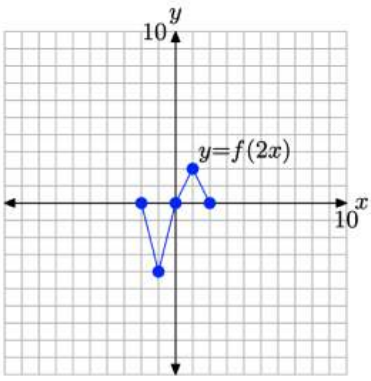
$$\begin{aligned} y &= f(2(0)) = f(0) = 0 \\ y &= f(2(1)) = f(2) = 2 \\ y &= f(2(2)) = f(4) = 0 \end{aligned}$$

We enter these values into the table in Figure 4(b) and plot them to determine the graph of y = f(2x) in Figure 1.8.4(a).

At this point, there are a number of comparisons you can make.

1. Compare the data in the table in Figure 1.8.4(b) with the original function data in the table in Figure 1.8.3(b). Note that the y-values in each table are identical. However, note that each x-value in the table of Figure 1.8.4(b) is precisely half of the corresponding x-value in the table of Figure 1.8.3(b).

2. Compare the graph of y = f(2x) in Figure 1.8.4(a) with the original graph of y = f(x) in Figure 1.8.3(a). Note that each x-value at each point on the graph of y = f(2x) in



(a)

x	$y = f(2x)$	$(x, f(2x))$
-2	0	$(-2, 0)$
-1	-4	$(-1, -4)$
0	0	$(0, 0)$
1	2	$(1, 2)$
2	0	$(2, 0)$

(b)

Figure 1.8.4. The points in the table are points on the graph of y = f(2x).

Figure 1.8.4(a) is precisely half the x-value of the corresponding point on the graph of y = f(x) in Figure 1.8.3(a).

Note the result. The graph of y = f(2x) is compressed horizontally (toward the y-axis), both positively and negatively, by a factor of 2. Note that this is exactly the opposite of what you might expect by intuition, but a careful examination of the data in the tables in Figures 1.8.3(b) and 1.8.4(b) will explain why.

Let's look at another example.

Example 1.8.3

If y = f(x) has the graph shown in Figure 1.8.3(a), sketch the graph of y = f((1/2)x).

Solution

Rather than doubling each value of x at the start, this function first halves each value of x. Thus, we will want to evaluate the function y = f((1/2)x) at x = −8, −4, 0, 4, and 8. For example, to evaluate the function y = f((1/2)x) at x = −8, first substitute x = −8 to obtain

$$y = f((1/2)(-8)) = f(-4)$$

Now, look up this value in the table in Figure 1.8.3(b) and note that f(−4) = 0. Thus, we can complete the computation as follows.

$$y = f((1/2)(-8)) = f(-4) = 0$$

Similarly, to evaluate the function y = f((1/2)x) at x = −4, first substitute x = −4 to obtain

$$y = f((1/2)(-4)) = f(-2)$$

Now, look up this value in the table in Figure 1.8.3(b) and note that f(−2) = −4. Thus, we can complete the computation as follows.

$$y = f((1/2)(-4)) = f(-2) = -4$$

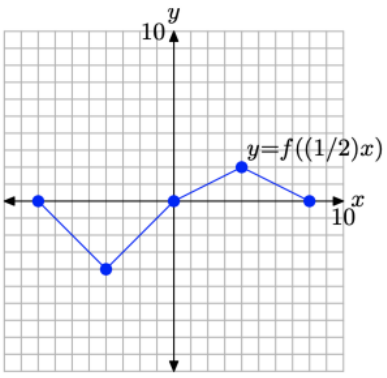
At this point, you will see why we chose x-values: −8, −4, 0, 4, and 8. These values are precisely double the x-values in the table of original values for the function y = f(x) in Figure 1.8.3(b). When the values −8, −4, 0, 4, and 8 are substituted into the

function y = f((1/2)x), they are first halved before we go to look up the function value in the table in Figure 1.8.3(b). This halving leads to the values −4, −2, 0, 2, and 4, which are precisely the values available in the table in Figure 1.8.3(b).

We make similar computations at the remaining values of x, namely x = 0, 4, and 8.

$$y = f((1/2)(0)) = f(0) = 0$$
$$y = f((1/2)(4)) = f(2) = 2$$
$$y = f((1/2)(8)) = f(4) = 0$$

Hopefully, these computations explain our choice of x-values above. Each of these results is recorded in the table in Figure 1.8.5(b) and plotted on the graph shown in Figure 1.8.5(a).



(a)

x	$y = f((1/2)x)$	$(x, f((1/2)x))$
−8	0	(−8, 0)
−4	−4	(−4, −4)
0	0	(0, 0)
4	2	(4, 2)
8	0	(8, 0)

(b)

Figure 1.8.5. The points in the table are points on the graph of y = f((1/2)x).

Again, note that the y-values in the table in Figure 1.8.5(b) are identical to the y-values in the table in Figure 1.8.3(b). However, each x-value in the table in Figure 1.8.5(b) is precisely double the corresponding x-value in the table in Figure 1.8.3(b).

This doubling of the x-values is apparent in the graph of y = f((1/2)x) shown in Figure 1.8.5(a), where the graph is stretched by a factor of 2 horizontally (away from the y-axis), both positively and negatively. Note that this is exactly the opposite of what you might expect by intuition, but a careful examination of the data in the tables in Figures 1.8.3(b) and 1.8.5(b) will explain why.

Let's summarize our findings.

A Visual Summary — Horizontal Scaling

Consider the images in Figure 1.8.6.

- In Figure 1.8.6(a), we see pictured the graph of the original function y = f(x).
- In Figure 1.8.6(b), note that each key point on the graph of y = f(2x) has an x-value that is precisely half the x-value of the corresponding point on the graph of y = f(x) in Figure 1.8.6(a).
- In Figure 1.8.6(c), note that each key point on the graph of y = f((1/2)x) has an x-value that is twice the x-value of the corresponding point on the graph of y = f(x) in Figure 1.8.6(a).
- Note that the y-value of each transformed point remains the same.

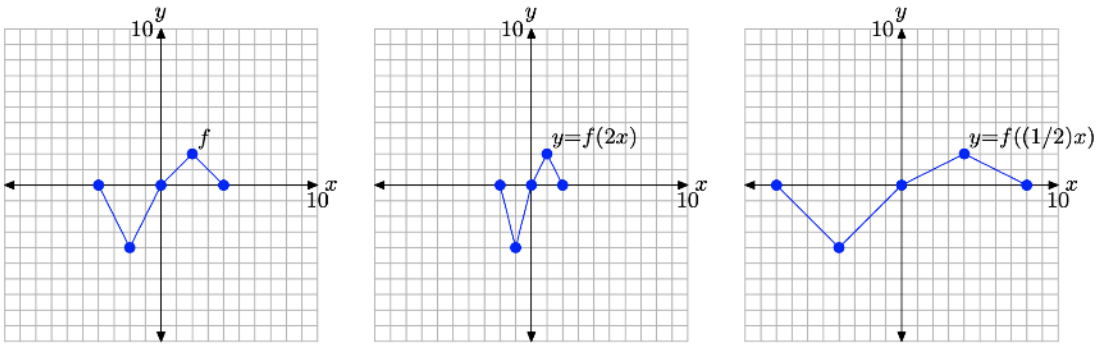


Figure 1.8.6. The graph of y = f(2x) compresses horizontally (toward the y-axis) by a factor of 2. The graph of y = f((1/2)x) stretches horizontally (away from the y-axis) by a factor of 2.

The visual summary in Figure 1.8.6 makes sketching the graphs of y = f(2x) and y = f(1/2)x) an easy task.

- Given the graph of y = f(x), to sketch the graph of y = f(2x), simply take each point on the graph of y = f(x) and cut its x-value in half, keeping the same y-value.
- Given the graph of y = f(x), to sketch the graph of y = f((1/2)x), simply take each point on the graph of y = f(x) and double its x-value, keeping the same y-value.

Follow the same procedures for other scaling factors. For example, in the case of y = f(3x), take each point on the graph of y = f(x) and divide its x-value by 3, keeping the same y-value. On the other hand, to draw the graph of y = f((1/3)x), take each point on the graph of f and multiply its x-value by 3, keeping the same y-value.

In general, we can state the following.

summary

Suppose we are given the graph of y = f(x).

- If a > 1, the graph of y = f(ax) compresses horizontally (toward the y-axis) , both positively and negatively, by a factor of a.
- If 0 < a < 1, the graph of y = f(ax) stretches horizontally (away from the y-axis), both positively and negatively, by a factor of 1/a.

In the case of the first item in Summary, when we compare the general form y = f(ax) with y = f(2x), we see that a = 2. In this case, note that a > 1 and the graph of y = f(2x) compresses horizontally by a factor of 2 when compared with the graph of y = f(x) (see Figure 1.8.6(b)).

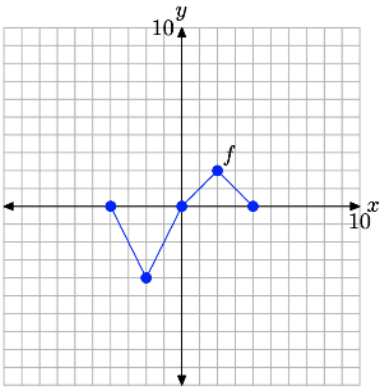
In the case of the second item in Summary, when we compare the general form y = f(ax) with the equation y = f((1/2)x), we see that a = 1/2, so

$$\frac{1}{a} = \frac{1}{1/2} = 2$$

The second item in Summary 4 says that when 0 < a < 1, the graph of y = f(ax) stretches horizontally by a factor of 1/a. Indeed, this is exactly what happened in the case of y = f((1/2)x), which stretched in the horizontal direction by a factor of 1/(1/2), or 2 (see Figure 1.8.6(c)).

Horizontal Reflections

For convenience, we begin by repeating the original graph of y = f(x) and its accompanying data in Figure 1.8.7. We are now going to reflect the graph of y = f(x) in the horizontal direction (across the y-axis).



(a)

x	$f(x)$	$(x, f(x))$
-4	0	$(-4, 0)$
-2	-4	$(-2, -4)$
0	0	$(0, 0)$
2	2	$(2, 2)$
4	0	$(4, 0)$

(b)

Figure 1.8.7. The original graph of f and a table of key points on the graph of f

Example 1.8.4

If y = f(x) has the graph shown in Figure 1.8.7(a), draw the graph of y = f(−x).

Solution

In the previous section, we were asked to draw the graph of y = −f(x). Note how the minus sign appears on the outside of the function. Clearly, the y-values of y = −f(x) must be opposite in sign to the y-values of y = f(x). That is why the graph of y = −f(x) was a reflection of the graph of y = f(x) across the x-axis.

However, in this example, the minus sign is inside the function, leaving one to intuit that it is the x-values, not the y-values, that are being negated. We will choose the following x-values: x = 4, 2, 0, −2, and −4. This is a bit deceptive, as it looks like we are choosing the same x-values, only in reverse order. This is not the case. We are choosing the negative of each x-value in the table in Figure 1.8.7(b).

To evaluate y = f(−x) at our first x-value, namely x = 4, we perform the following calculation. First substitute x = 4 to obtain

$$y = f(-(4)) = f(-4)$$

Now, look up this value in the table in Figure 1.8.7(b) and note that f(−4) = 0. Thus, we can complete the computation as follows.

$$y = f(-(4)) = f(-4) = 0$$

Similarly, to evaluate the function y = f(−x) at x = 2, first substitute x = 2 to obtain

$$y = f(-(2)) = f(-2)$$

Now, look up this value in the table in Figure 1.8.7(b) and note that f(−2) = −4. Thus, we can complete the computation as follows.

$$y = f(-(2)) = f(-2) = -4$$

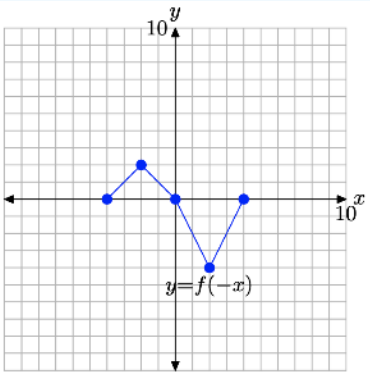
At this point, you will see why we chose x-values: 4, 2, 0, −2, and −4. These values are the negatives of the x-values in the table of original values for the function y = f(x) in Figure 1.8.7(b). When the values 4, 2, 0, −2, and −4 are substituted into the function y = f(−x), they are first negated before we go to look up the function value in the table in Figure 1.8.7(b). This negating leads to the values −4, −2, 0, 2, and 4, which are precisely the values available in the table in Figure 1.8.7(b).

We make similar computations at the remaining values of x, namely x = 0, −2, and −4.

$$y = f(-(0)) = f(0) = 0$$
$$y = f(-(-2)) = f(2) = 2$$
$$y = f(-(-4)) = f(4) = 0$$

We organize these points in the table in Figure 1.8.8(b), then plot them in Figure 1.8.8(a).

When you compare the entries in the table in Figure 1.8.8(b) with those in the table in Figure 1.8.7(b), note that the y-values appear in the same order, but the x-values of the table in Figure 1.8.7(b) have been negated in the table in Figure 1.8.8(b). This means that a former point such as (−2, −4) is transformed to the point (2, −4), which is a reflection of the point (−2, −4) across the y-axis.



(a)

x	$y=f(-x)$	$(x, f(-x))$
4	0	$(4,0)$
2	−4	$(2,-4)$
0	0	$(0,0)$
−2	2	$(-2,2)$
−4	0	$(-4,0)$

(b)

Figure 1.8.8. The graph of y = f(−x) and a table of key points on the graph.

Thus, to produce the graph of y = f(−x), simply reflect the graph of y = f(x) across the y-axis.

Let's summarize what we've learned about horizontal reflections.

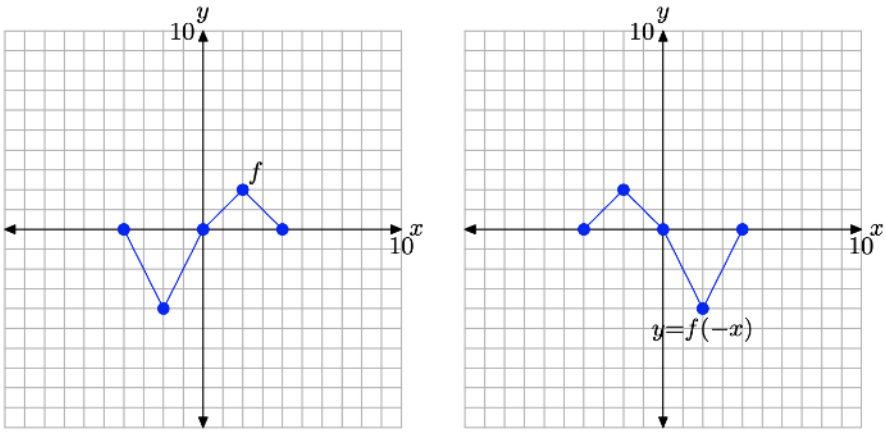


Figure 1.8.9. The graph of y = f(−x) is a reflection of the graph of y = f(x) across the y-axis.

A Visual Summary — Horizontal Reflections

Consider the images in Figure 1.8.9.

- In Figure 1.8.9(a), we see pictured the original graph of y = f(x).
- In Figure 1.8.9(b), the graph of y = f(−x) is a reflection of the graph of y = f(x) across the y-axis.

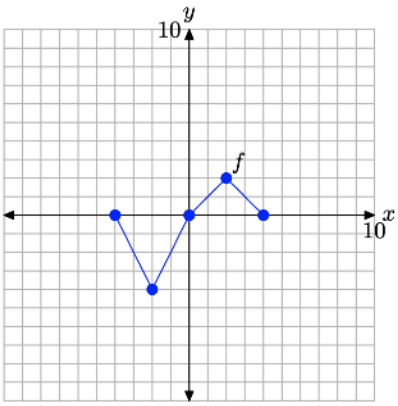
Thus, given the graph of y = f(x), it is a simple task to draw the graph of y = f(−x).

- To draw the graph of y = f(−x), take each point on the graph of y = f(x) and reflect it across the y-axis, keeping the y-value the same, but negating the x-value.

Horizontal Translations

In the previous section, we saw that the graphs of y = f(x) + c and y = f(x) − c were vertical translations of the graph of y = f(x). If c is a positive number, then the graph of y = f(x) + c shifts c units upward while the graph of y = f(x) − c shifts c units downward.

In this section, we will study horizontal translations. For convenience, we begin by repeating the original graph of y = f(x) and its accompanying data in Figure 10.



(a)

x	$f(x)$	$(x, f(x))$
-4	0	$(-4, 0)$
-2	-4	$(-2, -4)$
0	0	$(0, 0)$
2	2	$(2, 2)$
4	0	$(4, 0)$

(b)

Figure 1.8.10. The original graph of f and a table of key points on the graph of f

Example 1.8.5

If y = f(x) has the graph shown in Figure 1.8.10(a), sketch the graph of y = f(x + 1).

Solution

In the previous section, we drew the graph of y = f(x)+1. Note that in y = f(x)+1, the number 1 is outside the function. The result was a graph that was shifted 1 unit upwards in the y-direction.

In this case, y = f(x + 1) and the 1 is inside the function notation, leading one to intuit that the translation might be in the horizontal direction (x-direction). But how?

Again, we will set up a table of points that satisfy the equation y = f(x + 1), then plot them. Because this function requires that we first add 1 to each x-value before inserting it into the function, we will choose x-values appropriately, namely x = −5, −3, −1, 1, and 3. In a moment, it will be clear why we have chosen these particular values of x. Perhaps you already see why?

We need to evaluate the function y = f(x + 1) at each of these chosen values of x. To evaluate y = f(x+ 1) at the first value, namely x = −5, we insert x = −5 and make the calculation

$$y = f(-5+1) = f(-4)$$

To complete the calculation, we must now evaluate f(−4). However, this result is recorded in the table in Figure 1.8.10(b). There we find that f(−4) = 0, and we can complete the calculation started above.

$$y = f(-5+1) = f(-4) = 0$$

In similar fashion, we can evaluate the function y = f(x+1) at x = −3. First, substitute x = −3 in y = f(x + 1) to obtain

$$y = f(-3+1) = f(-2)$$

To complete the calculation, we must now evaluate f(−2). However, this result is recorded in the table in Figure 1.8.10(b). There we find that f(−2) = −4, and we can complete the calculation started above.

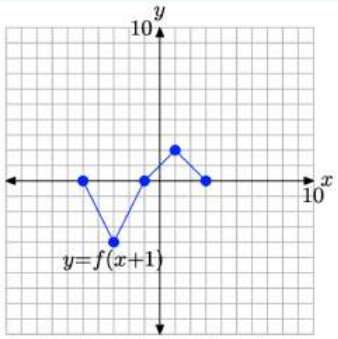
$$y = f(-3+1) = f(-2) = -4$$

At this point, you might see why we chose x-values: −5, −3, −1, 1, and 3. These are precisely one less than the x-values in the table of original values for the function y = f(x) in Figure 1.8.10(b). When the values −5, −3, −1, 1, and 3 are substituted into the function y = f(x + 1), we first add 1 to each value before we go to look up the function value in the table in Figure 1.8.10(b). This adding of 1 leads to the values −4, −2, 0, 2, and 4, which are precisely the values available in the table in Figure 1.8.10(b).

Continuing in this manner, we evaluate the function y = f(x + 1) at the remaining values of x, namely, −1, 1, and 3.

$$y = f(-1+1) = f(0) = 0$$
$$y = f(1+1) = f(2) = 2$$
$$y = f(3+1) = f(4) = 0$$

We assemble these results in the table in Figure 1.8.11(b) and plot them in Figure 1.8.11(a).



(a)

x	$y = f(x+1)$	$(x, f(x+1))$
-5	0	$(-5, 0)$
-3	-4	$(-3, -4)$
-1	0	$(-1, 0)$
1	2	$(1, 2)$
3	0	$(3, 0)$

(b)

Figure 1.8.11. The graph of y = f(x + 1) and a table of key points on the graph

When you compare the points on the graph of y = f(x+1) in the table in Figure 1.8.11(b) with the original points on the graph of y = f(x) in the table in Figure 1.8.10(b), note that the y-values are identical, but the x-values in the table in Figure 1.8.11(b) are all 1 unit less than the corresponding x-values in the table in Figure 1.8.10(b). It is no wonder that the graph of y = f(x + 1) in Figure 1.8.11(a) is shifted 1 unit to the left of the original graph of y = f(x) in Figure 1.8.10(a).

Note that this is somewhat counterintuitive, because we're seemingly adding 1 to each x-value in y = f(x + 1). Why doesn't the graph move one unit to the right? Well, a careful comparison of the x-values in the tables in Figures 1.8.10(b) and 1.8.11(b) reveals the answer. In order to use the data in the table in Figure 1.8.10(b), we must first subtract 1 from each x-value to produce the x-values in the table in Figure 1.8.11(b). This is why the graph of y = f(x + 1) moves 1 unit to the left instead of 1 unit to the right.

You might also recall that the function y = f(2x) compressed by a factor of 2, which is also the opposite of what intuition might dictate. Similarly, the function y = f((1/2)x) stretches by a factor of 2, which also goes counter to intuition. With these thoughts in mind, it is not surprising that y = f(x+ 1) shifts one unit to the left. Still, a comparison of the x-values in the tables in Figures 1.8.10(b) and 1.8.11(b) provide irrefutable evidence that the shift is 1 unit to the left.

Let's look at another example.

Example 1.8.6

If y = f(x) has the graph shown in Figure 1.8.10(a), sketch the graph of y = f(x − 2).

Solution

Again, we will set up a table of points that satisfy the equation y = f(x − 2), then plot them. Because this function requires that we first subtract 2 from each x-value before inserting it into the function, we will choose x-values: −2, 0, 2, 4, and 6. We need to evaluate the function y = f(x − 2) at each of these values of x.

To evaluate y = f(x − 2) at the first value, namely x = −2, insert x = −2 into the function y = f(x − 2) to obtain

$$y = f(-2-2) = f(-4)$$

In the table in Figure 1.8.10(b), we find that f(−4) = 0, which allows us to complete the calculation above.

$$y = f(-2-2) = f(-4) = 0$$

In similar fashion, we evaluate y = f(x − 2) at x = 0 to obtain

$$y = f(0-2) = f(-2)$$

In the table in Figure 1.8.10(b), we find that f(−2) = −4, which allows us to complete the calculation above.

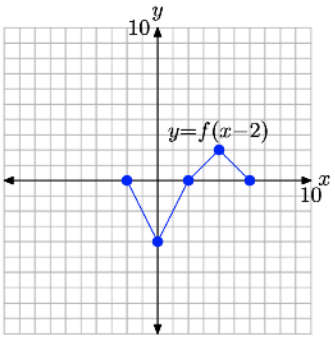
$$y = f(0-2) = f(-2) = -4$$

Hopefully, you see why we chose the x-values: −2, 0, 2, 4, and 6. These values are 2 larger than the x-values in the table of original values for the function y = f(x) in Figure 1.8.10(b). When the values −2, 0, 2, 4, and 6 are substituted into the function y = f(x − 2), we first subtract 2 from each value before we go to look up the function value in the table in Figure 1.8.10(b). This subtracting of 2 leads to −4, −2, 0, 2, and 4, precisely the values that are available in the table in Figure 1.8.10(b).

Continuing in this manner, we evaluate y = f(x − 2) at the remaining values of x, namely, x = 2, 4, and 6.

$$y = f(2-2) = f(0) = 0$$
$$y = f(4-2) = f(2) = 2$$
$$y = f(6-2) = f(4) = 0$$

We assemble these results in the table in Figure 1.8.12(b) and plot them in Figure 1.8.12(a).



(a)

x	$y = f(x-2)$	$(x, f(x-2))$
−2	0	(−2, 0)
0	−4	(0, −4)
2	0	(2, 0)
4	2	(4, 2)
6	0	(6, 0)

(b)

Figure 1.8.12. The graph of y = f(x − 2) and a table of key points on the graph.

When you compare the points on the graph of y = f(x−2) in the table in Figure 1.8.12(b) with the original points on the graph of y = f(x) in the table in Figure 1.8.10(b), note that the y-values are identical, but the x-values in the table in Figure 1.8.12(b) are all 2 larger than the corresponding x-values in the table in Figure 1.8.10(b). It is no wonder that the graph of y = f(x − 2) in Figure 1.8.12(a) is shifted 2 units to the right of the original graph of y = f(x) in Figure 1.8.10(a).

Again, this runs counterintuitive (why doesn't the graph of y = f(x − 2) shift 2 units to the left?), but a comparison of the x-values in the tables in Figures 1.8.10(b) and 1.8.12(b) clearly indicates a shift to the right.

Let's summarize what we've learned about horizontal translations.

Visual Summary — Horizontal Translations (Shifts)

Consider the images in Figure 1.8.13.

- In Figure 1.8.13(a), we see pictured the graph of the original function y = f(x).
- In Figure 1.8.13(b), note that each point on the graph of y = f(x + 1) has an x-value that is 1 unit less than the x-value of the corresponding point on the graph of y = f(x) in Figure 1.8.13(a).
- In Figure 1.8.13(c), note that each point on the graph of y = f(x − 2) has an x-value that is 2 units greater than the x-value of the corresponding point on the graph of y = f(x) in Figure 1.8.13(a).
- Note that the y-value of each transformed point remains the same.

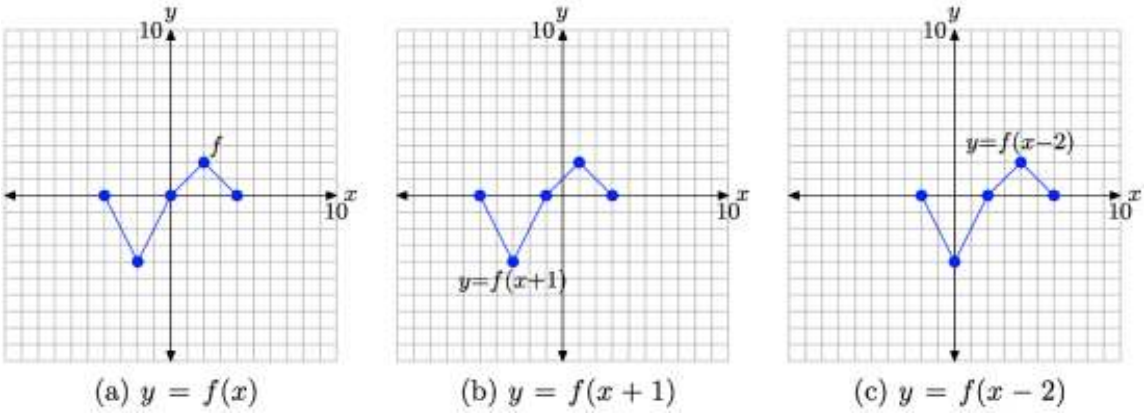


Figure 1.8.13 The graph of y = f(x + 1) is formed by shifting (horizontally) the graph of y = f(x) one unit to the left. The graph of y = f(x − 2) is formed by shifting (horizontally) the graph of y = f(x) two units to the right.

The visual summary in Figure 1.8.13 makes sketching the graphs of y = f(x + 1) and y = f(x − 2) an easy task.

- Given the graph of y = f(x), to sketch the graph of y = f(x + 1), simply take each point on the graph of y = f(x) and shift it 1 unit to the left, keeping the same y-value.
- Given the graph of y = f(x), to sketch the graph of y = f(x − 2), simply take each point on the graph of y = f(x) and shift it 2 units to the right, keeping the same y-value.

In general, we can state the following.

Summary

Suppose that we are given the graph of y = f(x) and suppose that c is any positive real number.

- The graph of y = f(x+c) is shifted c units to the left of the graph of y = f(x).
- The graph of y = f(x−c) is shifted c units to the right of the graph of y = f(x).

When we looked at vertical translations in the previous section, a translation was described by first imagining a graph on a sheet of transparent plastic, then sliding the transparency (without rotating it) over a coordinate system on a sheet of graph paper. Horizontal translations can be thought of in the same way, as sliding the graph on the transparency c units to the left, or c units to the right.

Extra Practice

In this section, let's take the concepts from the Visual Summaries and put them to work on some final examples.

Example 1.8.7

Consider the graph of f in Figure 1.8.14

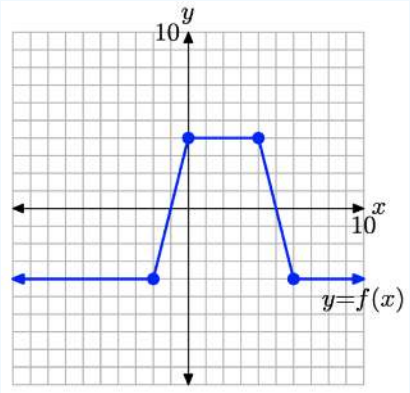


Figure 1.8.14 The graph of y = f(x) for Example 1.8.7.

Use the concepts from the Visual Summaries (scaling, reflection, and translation) to sketch the graphs of y = f(2x), y = f(−x), and y = f(x + 2) without creating and referring to tables.

Solution

To sketch the graph of y = f(2x), simply take each point on the graph of y = f(x) in Figure 1.8.15(a) and divide its x-value by 2, keeping the same y-value. The result is shown in Figure 1.8.15(b).

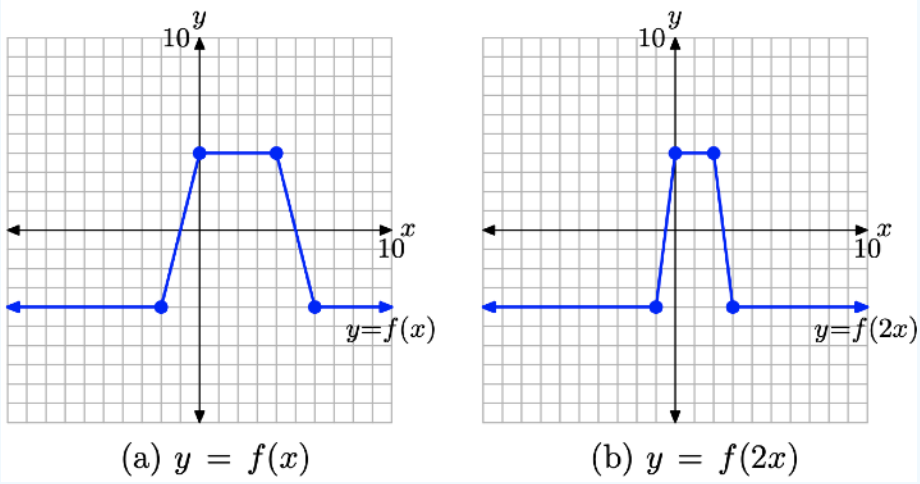


Figure 1.8.15: Compress the graph of y = f(x) by a factor of 2 to produce the graph of y = f(2x).

To sketch the graph of y = f(−x), simply take each point on the graph of y = f(x) in Figure 1.8.16(a) and negate its x-value, keeping the same y-value. The result is shown in Figure 1.8.16(b).

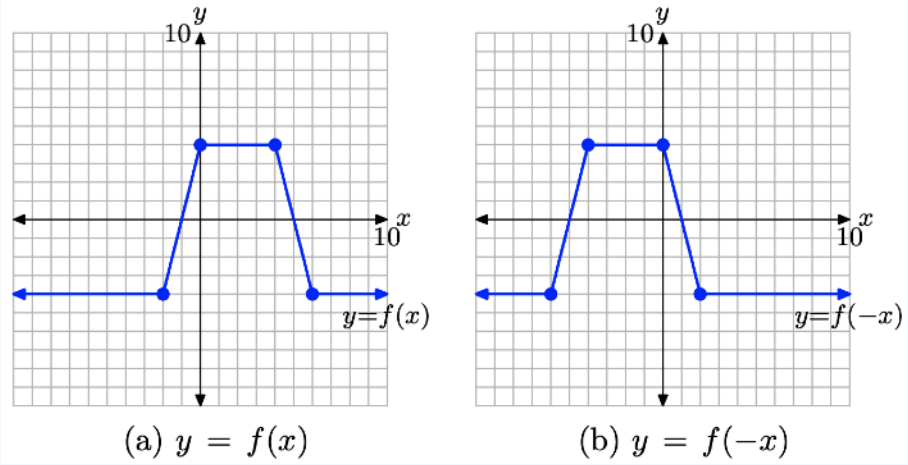


Figure 1.8.16: Reflect the graph of y = f(x) across the y-axis to produce the graph of y = f(−x).

To sketch the graph of y = f(x + 2), simply take each point on the graph of y = f(x) in Figure 1.8.17(a) and subtract 2 from its x-value, keeping the same y-value. The result is shown in Figure 1.8.17(b).

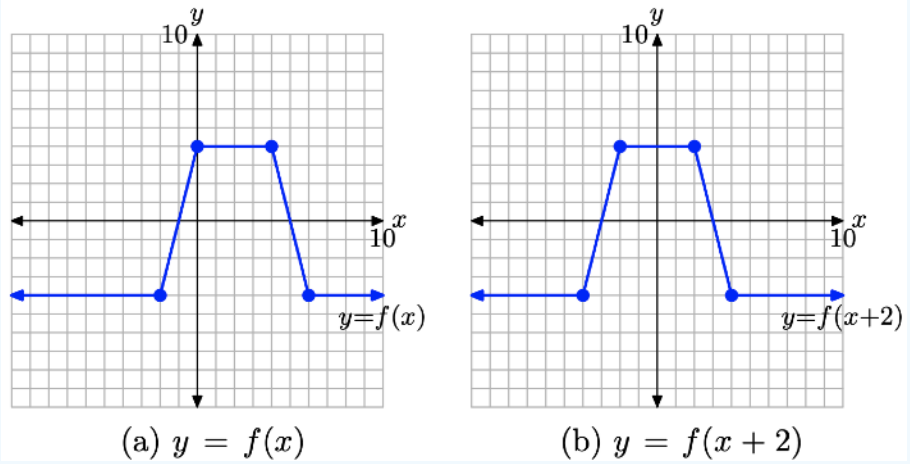


Figure 1.8.17. Shift the graph of y = f(x) to the left 2 units to produce the graph of y = f(x + 2).

Summary

In this section we've seen how a handful of transformations greatly enhance our graphing capability. We end this section by listing the transformations presented in this section and their effects on the graph of a function.

Vertical Transformations

Suppose we are given the graph of y = f(x).

- If a > 1, the graph of y = f(ax) compresses horizontally (toward the y-axis) , both positively and negatively, by a factor of a.
- If 0 < a < 1, the graph of y = f(ax) stretches horizontally (away from the y-axis), both positively and negatively, by a factor of 1/a.
- The graph of y = f(−x) is a reflection of the graph of y = f(x) across the y-axis.
- If c > 0, then the graph of y = f(x + c) is shifted c units to the left of the graph of y = f(x).
- If c > 0, then the graph of y = f(x − c) is shifted c units to the right of the graph of y = f(x).

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CHAPTER OVERVIEW

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2.1: Polynomial Functions

- We've seen in previous sections that a **monomial** is the product of a number and one or more variable factors, each raised to a positive integral power, as in $-3x^2$ or $4x^3y^4$.
- We've also seen that a **binomial** is the sum or difference of two monomial terms, as in $3x+5, x^2+4,$ or $3xy^2=2x^2y$.
- We've also seen that a **trinomial** is the sum or difference of three monomial terms, as in x^2-2x-3 or $x^2-4xy+5y^2$.

The root word "poly" means "many," as in polygon (many sides) or polyglot (speaking many languages—multilingual).

- In algebra, the word **polynomial** means "many terms," where the phrase "many terms" can be construed to mean anywhere from one to an arbitrary, but finite, number of terms. Consequently, a monomial could be considered a polynomial, as could binomials and trinomials. In our work, we will concentrate for the most part on polynomials of a single variable. What follows is a more formal definition of a polynomial in a single variable x.

Definition: Polynomials

The function p, defined by

$$p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n \tag{2.1.1}$$

is called a **polynomial** in x.

There are several important points to be made about this definition.

Note

1. The polynomial in our definition in Equation 2.1.1 is arranged in ascending powers of x. We could just as easily arrange our polynomial in descending powers of x, as in
$$p(x)=a_nx^n+\cdots+a_2x^2+a_1x+a_0$$
2. The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the coefficients of the polynomial p. − If all of the coefficients are integers, then we say that "p is a polynomial with integer coefficients." − If all of the coefficients are rational numbers, then we say that "p is a polynomial with rational coefficients." − If all of the coefficients are real numbers, then we say that "p is a polynomial with real coefficients."
3. The degree of the polynomial p is n, the highest power of x.
4. The leading term of the polynomial p is the term with the highest power of x. In the case of Equation ???, the leading term is an x^n.

Let's look at an example.

Example 2.1.1

Consider the polynomial

$$p(x)=3-4x^2+5x^3-6x$$

Find the degree, the leading term, and make a statement about the coefficients of p by referencing the Note above to determine integer or rational coefficients.

Solution

First, put the polynomial terms in order. Whether you use ascending or descending powers of x makes no difference. Choose one or the other. In descending powers of x,

$$p(x)=5x^3-4x^2-6x+3 \tag{2.1.2}$$

but in ascending powers of x,

$$p(x)=3-6x-4x^2+5x^3 \tag{2.1.3}$$

In either case, Equation 2.1.2 or Equation 2.1.3, the degree of the polynomial is 3. Also, in either case, the leading term of the polynomial is $5x^3$. Because all coefficients of this polynomial are integers, we say that "p is a polynomial with integer coefficients." However, all the coefficients are also rational numbers, so we could say that p is a polynomial with rational coefficients. For that matter, all of the coefficients of p are real numbers, so we could also say that p is a polynomial with real coefficients.

Let's look at another example.

Example 2.1.2

Consider the polynomial

$$p(x) = 3 - \frac{4}{3}x + \frac{2}{5}x^2 - 9x^3 + 12x^4.$$

Find the degree, the leading term, and make a statement about the coefficients of p.

Solution

Fortunately, the polynomial p is already arranged in ascending powers of x. The degree of p is 4 and the leading term is $12x^4$. Not all of the coefficients are integers, so we cannot say that "p is a polynomial with integer coefficients." However, all of the coefficients are rational numbers, so we can say that "p is a polynomial with rational coefficients." Because all of the coefficients of p are real numbers, we could also say that "p is a polynomial with real coefficients."

Example 2.1.3

Consider the polynomial

$$p(x) = 3 - \frac{4}{3}x + \sqrt{2}x^2 - 9x^3 + \pi x^5$$

. Find the degree, the leading term, and make a statement about the coefficients of p.

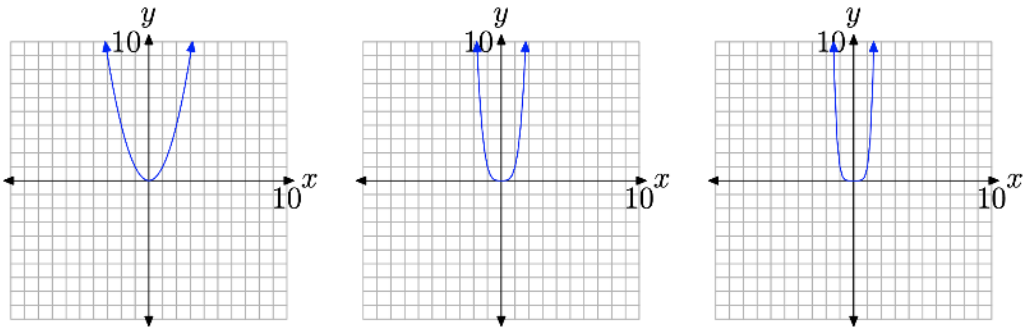
Solution

Fortunately, the polynomial p is already arranged in ascending powers of x. The degree of p is 5 and the leading term is πx^5. Not all of the coefficients are integers, so we cannot say that "p is a polynomial with integer coefficients." Not all of the coefficients are rational numbers, so we cannot say that "p is a polynomial with rational coefficients." However, because all of the coefficients of p are real numbers, we can say that "p is a polynomial with real coefficients."

The Graph of $y = x^n$

The primary goal in this section is to discuss the end-behavior of arbitrary polynomials. By "end-behavior," we mean the behavior of the polynomial for very small values of x (like −1 000, −10 000, −100 000, etc.) or very large values of x (like 1 000, 10 000, 100 000, etc.). Before we can explore the end-behavior of arbitrary polynomials, we must first examine the end-behavior of some very basic monomials. Specifically, we need to investigate the end-behavior of the graphs of $y = x^n$, where n = 1, 2, 3, ...

Let's first examine the graph of $y = x^n$, when n is even. The graphs are simple enough to draw, either by creating a table of points or by using your graphing calculator. In Figure 2.1.1(a), (b), and (c), we've drawn the graphs of $y = x^2, y = x^4$, and $y = x^6$, respectively.



(a) The graph of $y = x^2$. (b) The graph of $y = x^4$. (c) The graph of $y = x^6$.

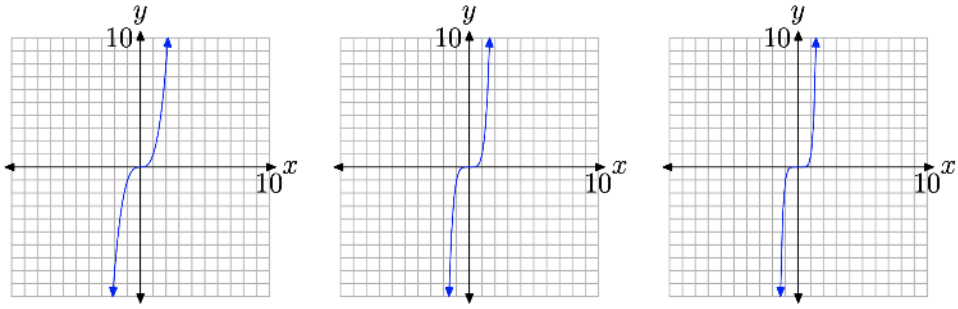
Figure 2.1.1. Examples of the graph of $y = x^n$, when n is an even integer.

The graphs in Figure 2.1.1 share an important trait. As you sweep your eyes from left to right, each graph falls from positive infinity, wiggles through the origin, then rises back to positive infinity.

Next, let's examine the graph of $y = x^n$, when n is odd. Again, a table of points or a graphing calculator will help produce the graphs of $y = x^3, y = x^5,$ and $y = x^7$, as shown in Figure 2.1.2(a), (b), and (c), respectively.

The graphs in Figure 2.1.2 share an important trait. As you sweep your eyes from left to right, each graph rises from negative infinity, wiggles through the origin, then rises up to positive infinity.

The behavior shown in Figure 2.1.1 and Figure 2.1.2 is typical.



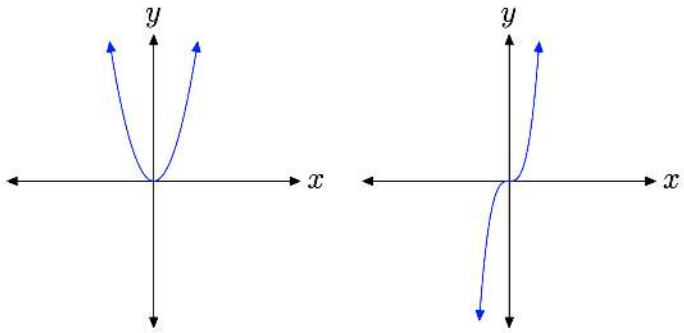
(a) The graph of $y = x^3$. (b) The graph of $y = x^5$. (c) The graph of $y = x^7$.

Figure 2.1.2. Examples of the graph of $y = x^n$, when n is an odd integer.

Property 9

When n is an even natural number, the graph of $y = x^n$ will look like that shown in Figure 2.1.3(a). If n is an odd natural number, then the graph of $y = x^n$ will be similar to that shown in Figure 2.1.3(b).

1. When n is even, as you sweep your eyes from left to right, the graph of $y = x^n$ falls from positive infinity, wiggles through the origin, then rises back to positive infinity.
2. If n is odd, as you sweep your eyes from left to right, the graph of $y = x^n$ rises from negative infinity, wiggles through the origin, then rises to positive infinity.



(a) The graph of $y = x^n$, n even. (b) The graph of $y = x^n$, n odd.

Figure 2.1.3

The Graph of $y = ax^n$

Now that we know the general shape of the graph of $y = x^n$, let's scale this function by multiplying by a constant, as in $y = ax^n$.

In our study of the parabola, we learned that if we multiply by a factor of a, where a > 1, then we will stretch the graph in the vertical direction by a factor of a. Conversely, if we multiply the graph by a factor of a, where 0 < a < 1, then we will compress the graph in the vertical direction by a factor of 1/a. If a < 0, then not only will we scale the graph, but multiplying by this factor will also reflect the graph across the horizontal axis.

Let's look at a few examples.

Example 2.1.4

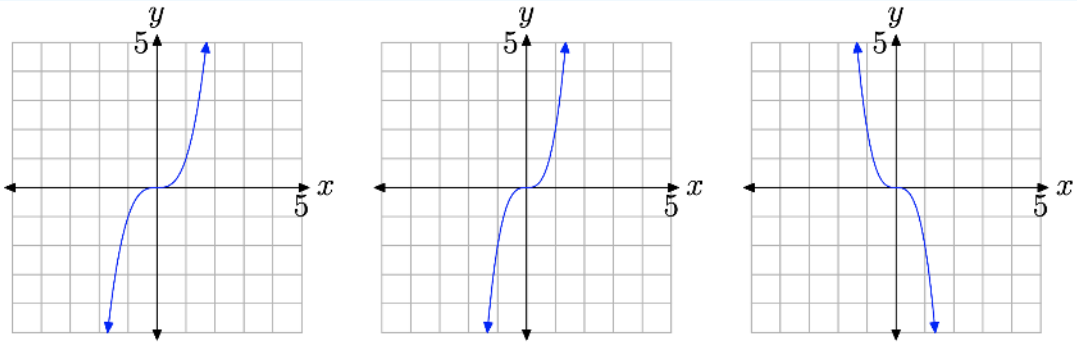
Sketch the graph of $y = -2x^3$.

Solution

We know what the graph of $y = x^3$ looks like. As we sweep our eyes from left to right, the graph rises from negative infinity, wiggles through the origin, then rises to positive infinity. This behavior is shown in Figure 2.1.4(a).

If we multiply by a factor of 2, then we stretch the original graph by a factor of 2 in the vertical direction. The graph of $y = 2x^3$ is shown in Figure 2.1.4(b). Note the stretching in the vertical direction.

Finally, if we negate by multiplying by −2, this will stretch the graph by a factor of 2, as in Figure 2.1.4(b), but it will also reflect the graph across the x-axis. The graph of $y = -2x^3$ is shown in Figure 2.1.4(c).



(a) The graph of $y = x^3$. (b) The graph of $y = 2x^3$. (c) The graph of $y = -2x^3$.

Figure 2.1.4. Scaling by −2 stretches vertically by a factor of 2, then reflects the graph across the x-axis.

Let's look at another example.

Example 2.1.5

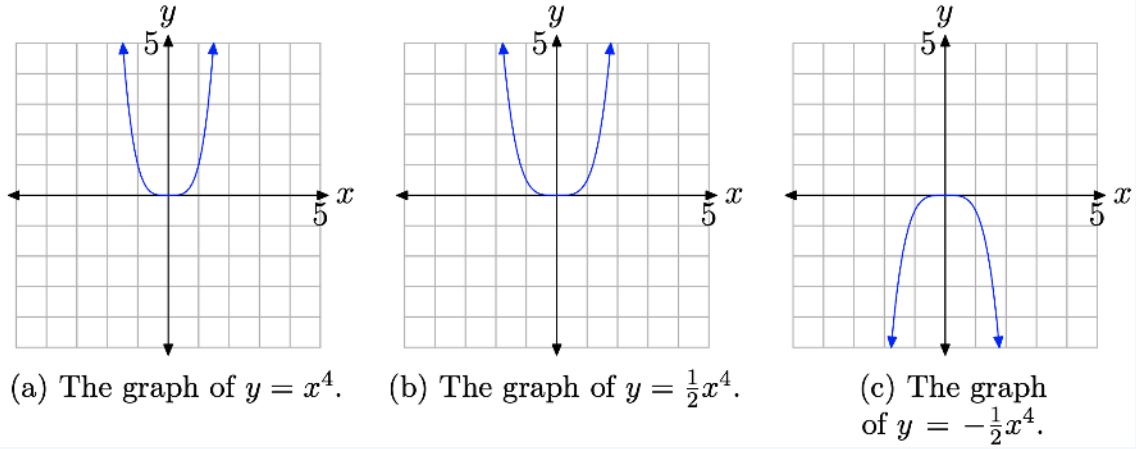
Sketch the graph of $y = -\frac{1}{2}x^4$

Solution

We know what the graph of $y = x^4$ looks like. As we sweep our eyes from left to right, the graph falls from positive infinity, wiggles through the origin, then rises back to positive infinity. This behavior is shown in Figure 2.1.5(a).

If we multiply by 1/2, then we will compress the graph by a factor of 2. Note that the graph of $y = \frac{1}{2}x^4$ in Figure 2.1.5(b) is compressed by a factor of 2 in the vertical direction.

Finally, if we multiply by −1/2, not only will we compress the graph by a factor of 2, we will also reflect the graph across the x-axis. The graph of $y = -\frac{1}{2}x^4$ is shown in Figure 2.1.5(c).



(a) The graph of $y = x^4$. (b) The graph of $y = \frac{1}{2}x^4$. (c) The graph of $y = -\frac{1}{2}x^4$.

Figure 2.1.5. Scaling by −1/2 compresses vertically by a factor of 2, then reflects the graph across the x-axis.

Hopefully, at this point you can now sketch the graph of $y = ax^n$ for any real number a and any natural number n, either even or odd, without the use of a calculator. Let's put this new-found knowledge to use in investigating the end-behavior of polynomials.

End Behavior

Consider the polynomial

$$p(x) = x^3 - 7x^2 + 7x + 15$$

Here's a key fact that we will use to determine the end-behavior of any polynomial.

Property 13

A polynomial's end-behavior is completely determined by its leading term. That is, the end-behavior of the graph of the polynomial will match the end-behavior of the graph of its leading term.

In a moment, we will show why this property is true. In the meantime, let's accept the veracity of this statement and apply it to the polynomial defined by equation (12). The leading term of the polynomial $p(x) = x^3 - 7x^2 + 7x + 15$ is x^3. We know the end behavior of graph of $y = x^3$. As we sweep our eyes from left to right, the graph of $y = x^3$ will rise from negative infinity, wiggle through the origin, then continue to rise to positive infinity. We pictured this behavior earlier in Figure 2.1.4(a).

Property 13 tells us that the graph of the polynomial $p(x) = x^3 - 7x^2 + 7x + 15$ will exhibit the same end-behavior as the graph of its leading term, $y = x^3$. We can predict that, as we sweep our eyes from left to right, the graph of the polynomial $p(x) = x^3 - 7x^2 + 7x + 15$ will rise from negative infinity, wiggle a bit, then rise to positive infinity. We don't know what happens in-between, but we do know what happens at far left- and right-hand ends.

Our conjecture is verified by drawing the graph (use a graphing calculator). The graph of the polynomial $p(x) = x^3 - 7x^2 + 7x + 15$ is shown in Figure 2.1.6. Sure enough, as we sweep our eyes from left to right, the graph in Figure 2.1.6 rises from negative infinity as predicted, wiggles a bit, then continues its rise to positive infinity.

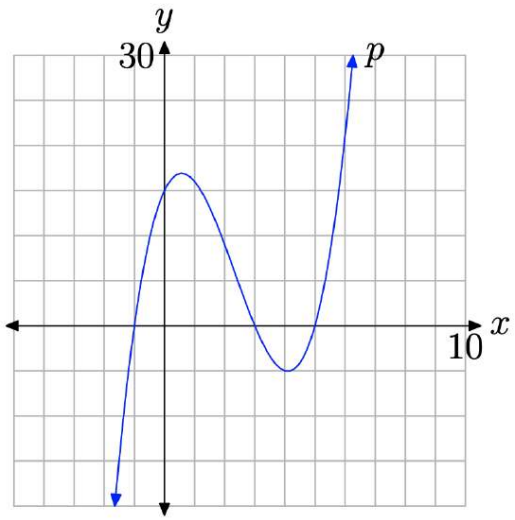


Figure 2.1.6. Note that this graph of $p(x) = x^3 - 7x^2 + 7x + 15$ has the same end-behavior as the graph of $y = x^3$.

Why Does it Work? Why does Property 13 predict so accurately the end-behavior of this polynomial?

$$p(x) = x^3 - 7x^2 + 7x + 15$$

We can demonstrate why by first factoring out the leading term.

$$p(x) = x^3 \left(1 - \frac{7}{x} + \frac{7}{x^2} + \frac{15}{x^3}\right)$$

Now, ask the following question. What happens to the polynomial as we move to the right end? That is, what happens to the polynomial as we use large values of x, such as 1 000, 10 000, or even 100 000?

Consider the fraction 7/x. Because the numerator is fixed at 7, and the denominator is getting bigger and bigger (growing without bound), the fraction is getting closer and closer to zero. Calculus students would use the notation

$$\lim_{x \to \infty} \frac{7}{x} = 0$$

Don't be put off by the notation. We're using sophisticated mathematical notation for a very simple idea that says "As x approaches infinity, the fraction 7/x approaches zero."

Using similar reasoning, each of the fractions in equation (14) go to zero as x goes to infinity (increases without bound). Thus, as x gets larger and larger (as we move further and further to the right),

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} x^3 \left(1 - \frac{7}{x} + \frac{7}{x^2} + \frac{15}{x^3}\right) \approx x^3(1 - 0 + 0 + 0 + 0) \approx x^3$$

That is, as x increases without bound, the graph of $p(x) = x^3 - 7x^2 + 7x + 15$ should approximate the graph of $y = x^3$.

Using similar reasoning, each of the fractions in equation (14) go to zero as x goes to minus infinity. That is, if you are putting in numbers for x such as −1 000, −10 000, −100 000, and the like, the fractions in equation (14) will go to zero. Hence, the polynomial p(x) must still approach its leading term x 3 for very small values of x (as x approaches $-\infty$).

If you superimpose the graph of $y = x^3$ on the graph of $p(x) = x^3 - 7x^2 + 7x + 15$, as in Figure 2.1.7, it's clear that the polynomial p has the same end-behavior as the graph of its leading term $y = x^3$.

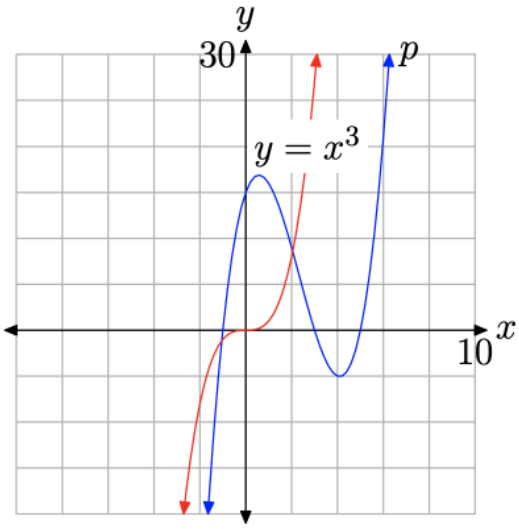


Figure 2.1.7. As you move to the extreme left or right, the graph of $p(x) = x^3 - 7x^2 + 7x + 15$ approaches the graph of its leading term $y = x^3$.

You can provide a more striking demonstration of the validity of the claim in equation (15) by plotting both the polynomial p and its leading term $y = x^3$ on your calculator, then zooming out by adjusting the window parameters as shown in Figure 2.1.8(b). Note how the graph of $p(x) = x^3 - 7x^2 + 7x + 15$ more closely resembles the graph of its leading term $y = x^3$, at least at the right and left edges of the viewing window. When we zoom further out, by adjusting the window parameters as shown in Figure 2.1.8(d), note how that graph of p approaches the graph of its leading term $y = x^3$ even more closely at each edge of the viewing window.

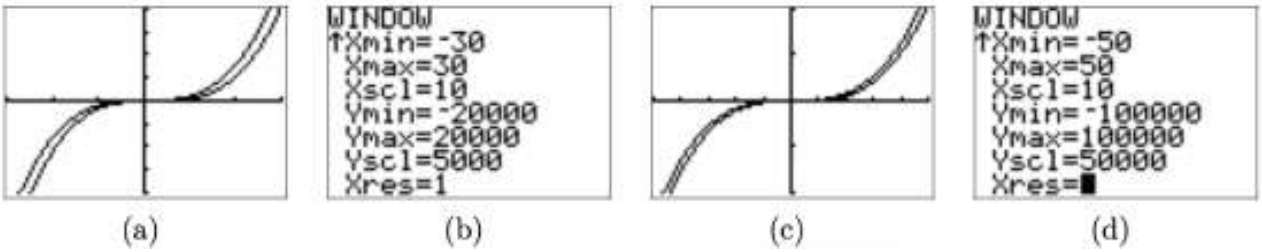


Figure 2.1.8. Zooming out clearly demonstrates that the end-behavior of $p(x) = x^3 - 7x^2 + 7x + 15$ matches that of its leading term $y = x^3$.

Let's look at another example.

Example 2.1.6

Consider the polynomial

$$p(x) = -x^4 + 37x^2 + 24x - 180$$

. Comment on the end-behavior of p and use your graphing calculator to sketch its graph.

Solution

The leading term of $p(x) = -x^4 + 37x^2 + 24x - 180$ is $y = -x^4$. We know the end-behavior of the graph of the leading term. As we sweep our eyes from left to right, the graph of $y = -x^4$ rises from negative infinity, wiggles through the origin, then falls back to minus infinity. The graph of p should exhibit the same end-behavior. Indeed, in Figure 2.1.9, note that the graph of $y = -x^4$ and $y = -x^4 + 37x^2 + 24x - 180$ both share the same end-behavior.

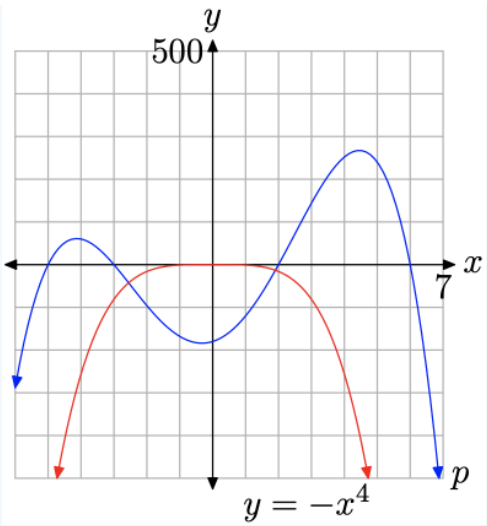


Figure 2.1.9. The polynomial $p(x) = -x^4 + 37x^2 + 24x - 180$ has the same end-behavior as the graph of its leading term $y = -x^4$.

Exercise

In **Exercises 1-8**, arrange each polynomial in descending powers of x, state the degree of the polynomial, identify the leading term, then make a statement about the coefficients of the given polynomial by referencing the Note above Example 2.1.1 to determine integer or rational coefficients.

Exercise 2.1.1

$p(x) = 3x - x^2 + 4 - x^3$

Answer

$p(x) = -x^3 - x^2 + 3x + 4$, degree = 3, leading term = $-x^3$, "p is a polynomial with integer coefficients, polynomial with rational coefficients," or "p is

a polynomial with real coefficients."

Exercise 2.1.2

$p(x) = 4 + 3x^2 - 5x + x^3$

Exercise 2.1.3

$p(x) = 3x^2 + x^4 - x - 4$

Answer

$p(x) = x^4 + 3x^2 - x - 4$, degree = 4, leading term = x^4, "p is a polynomial with integer coefficients," "p is a polynomial with rational coefficients," or "p is

a polynomial with real coefficients."

Exercise 2.1.4

$p(x) = -3 + x^2 - x^3 + 5x^4$

Exercise 2.1.5

$p(x) = 5x - \frac{3}{2}x^3 + 4 - \frac{2}{3}x^5$

Answer

$p(x) = -\frac{2}{3}x^5 - \frac{3}{2}x^3 + 5x + 4$, degree = 5, leading term = $-\frac{2}{3}x^5$, "p is a polynomial with rational coefficients," or p is a polynomial with real coefficients."

Exercise 2.1.6

$p(x) = -\frac{3}{2}x + 5 - \frac{7}{3}x^5 + \frac{4}{3}x^3$

Exercise 2.1.7

$p(x) = -x + \frac{2}{3}x^3 - \sqrt{2}x^2 + \pi x^6$

Answer

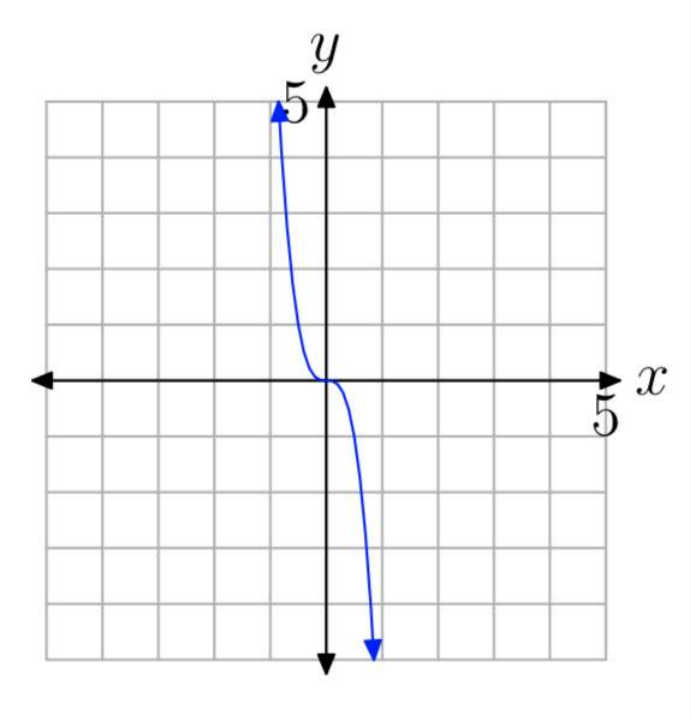
$p(x) = \pi x^6 + \frac{2}{3}x^3 - \sqrt{2}x^2 - x$, degree = 6, leading term = πx^6, "p is a polynomial with real coefficients."

Exercise 2.1.8

$p(x) = 3 + \sqrt{2}x^4 + \sqrt{3}x - 2x^2 + \sqrt{5}x^6$

In **Exercises 9-14**, you are presented with the graph of $y = ax^n$. In each case, state whether the degree is even or odd, then state whether a is a positive or negative number.

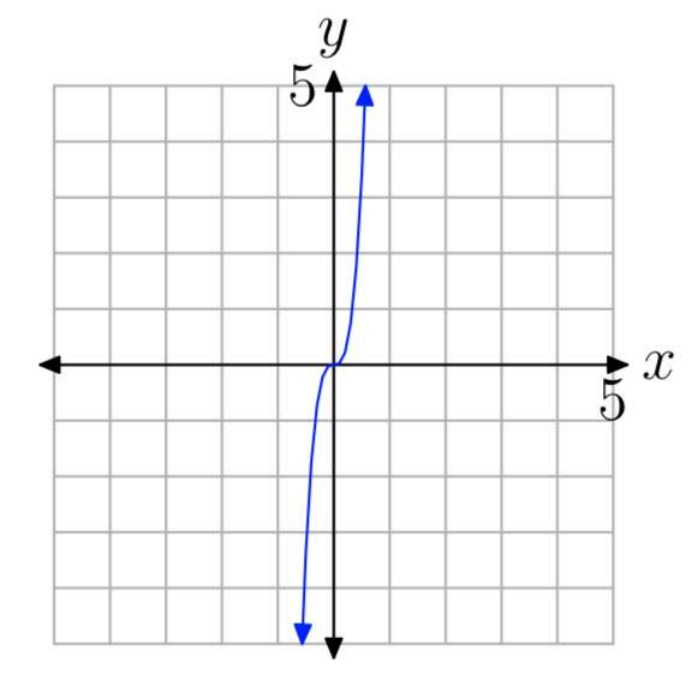
Exercise 2.1.9



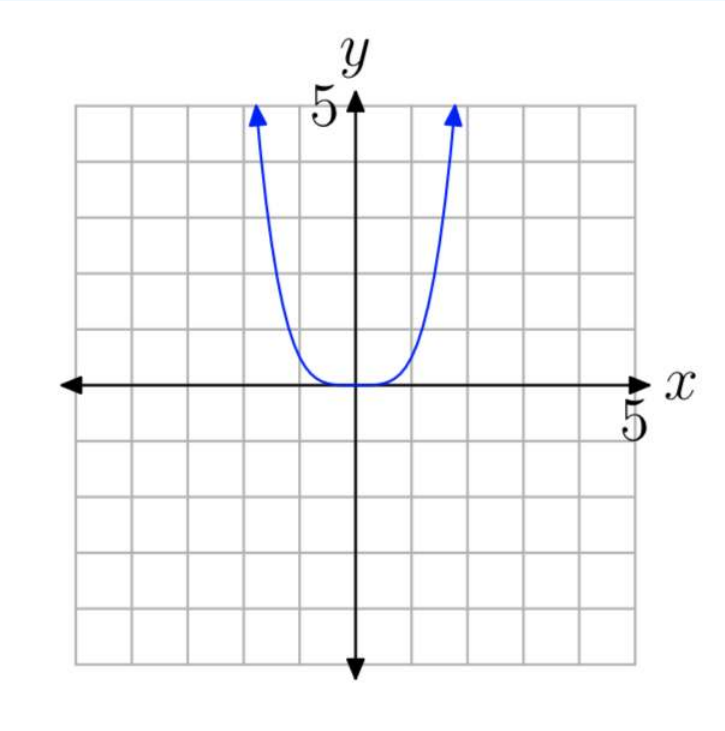
Answer

$y = ax^n$, n odd, a < 0.

Exercise 2.1.10



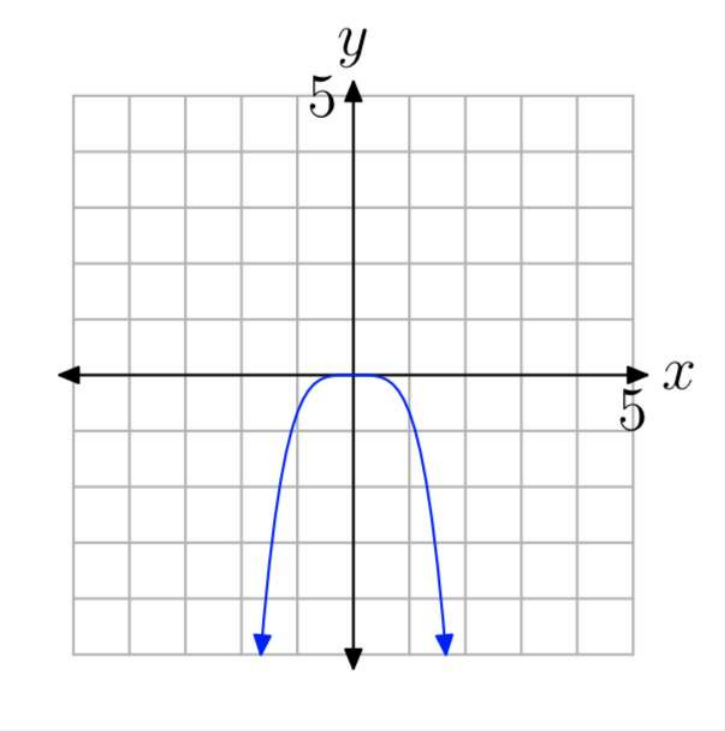
Exercise 2.1.11



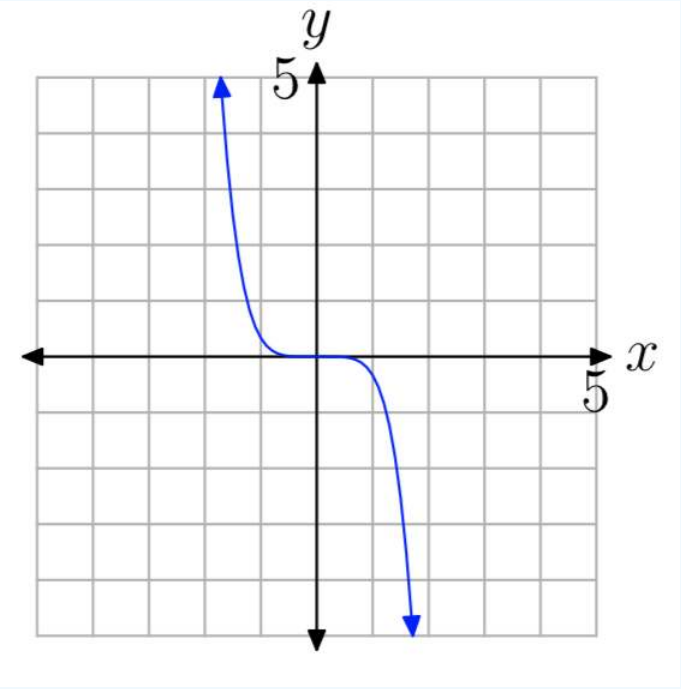
Answer

$y = ax^n$, n even, a > 0.

Exercise 2.1.12



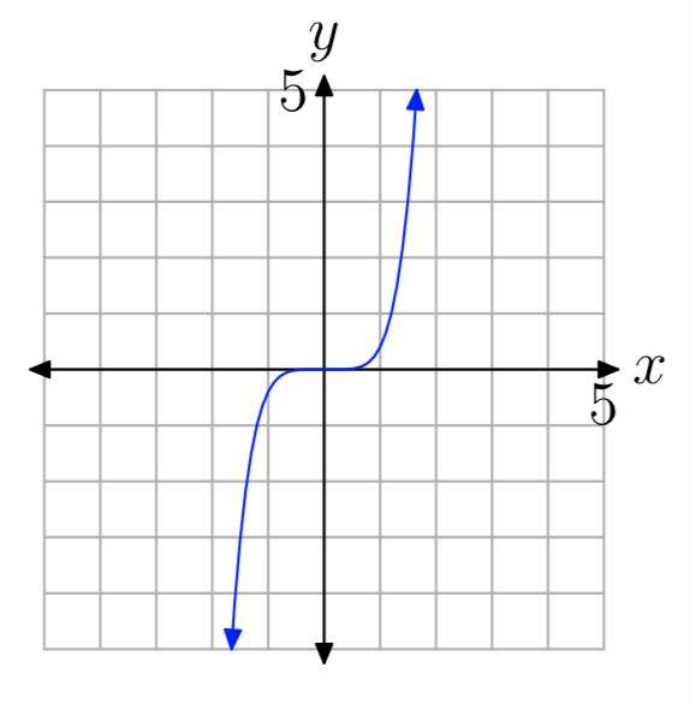
Exercise 2.1.13



Answer

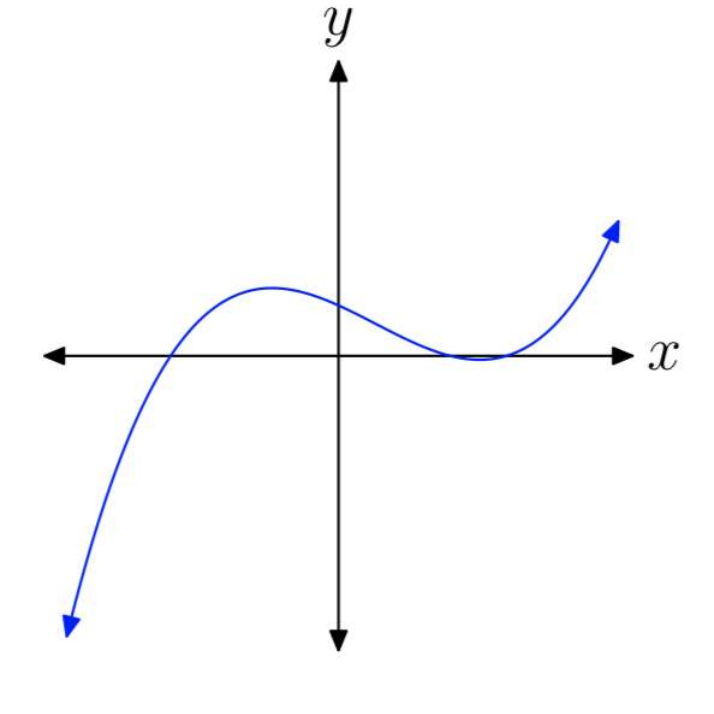
$y = ax^n$, n odd, a < 0.

Exercise 2.1.14



In **Exercises 15-20**, you are presented with the graph of the polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$. In each case, state whether the degree of the polynomial is even or odd, then state whether the leading coefficient an is positive or negative.

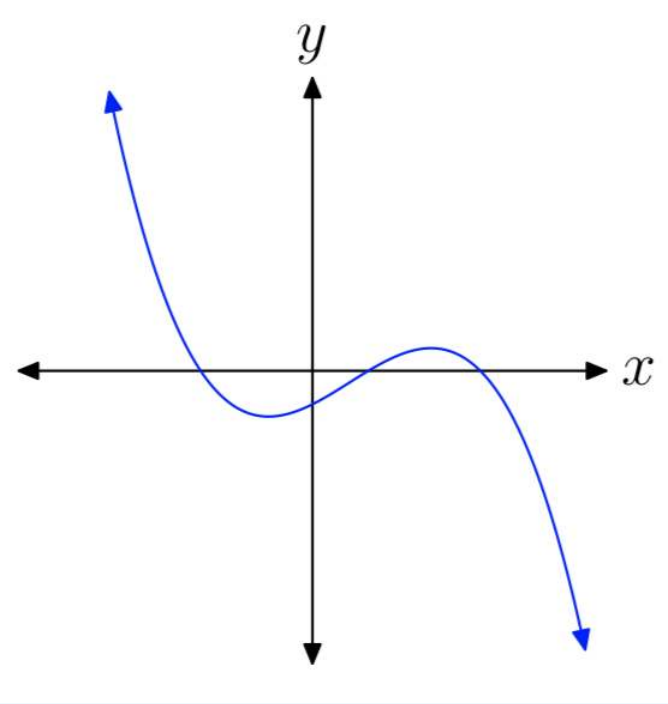
Exercise 2.1.15



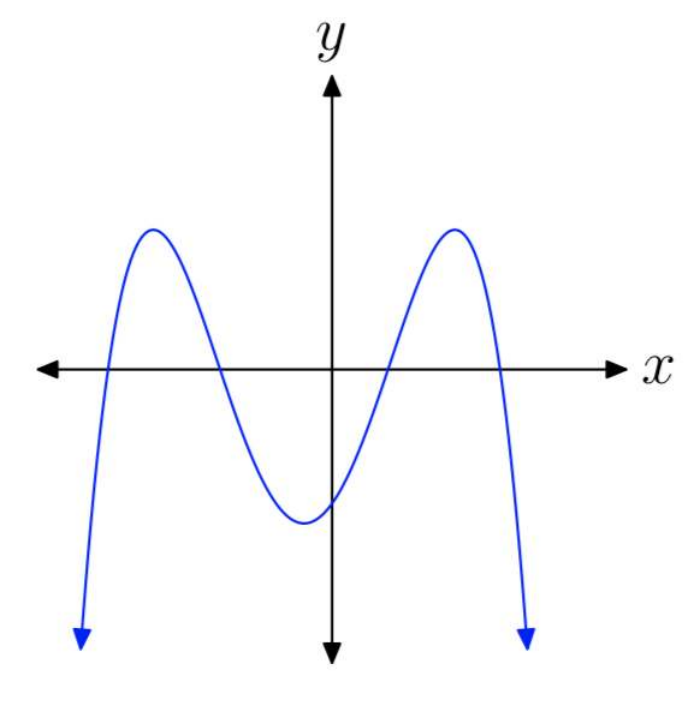
Answer

odd, positive

Exercise 2.1.16



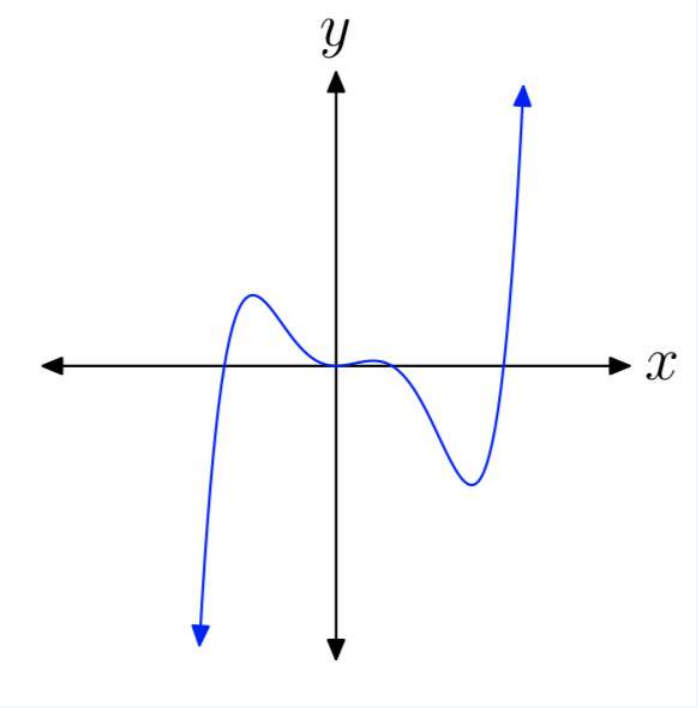
Exercise 2.1.17



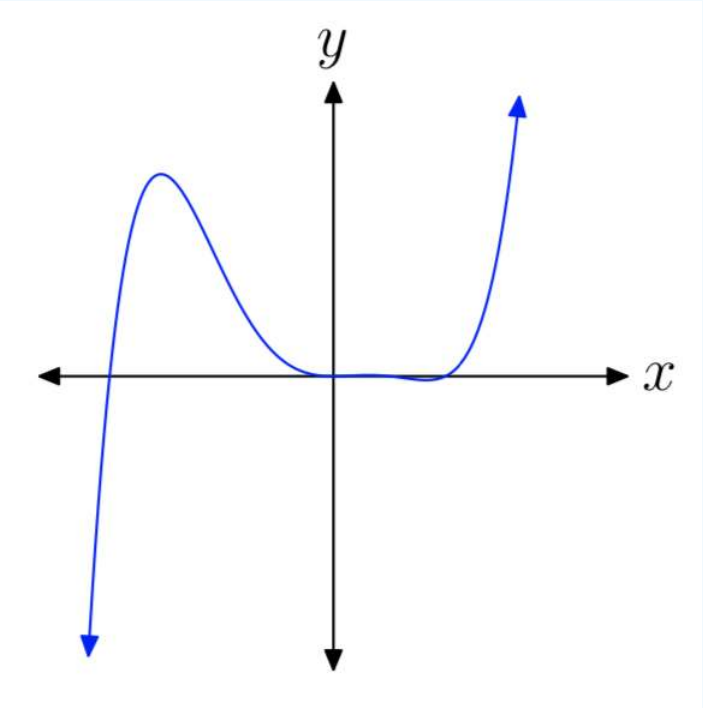
Answer

even, negative

Exercise 2.1.18



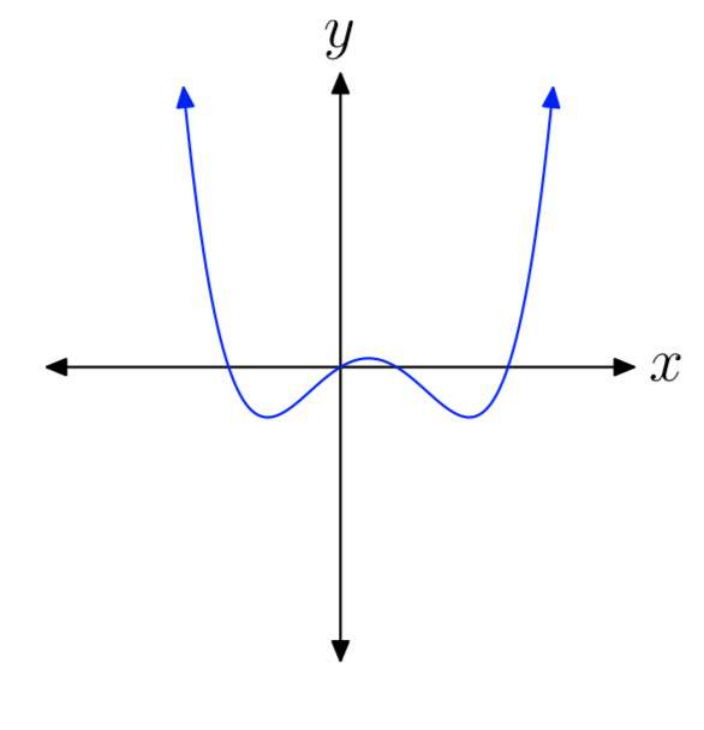
Exercise 2.1.19



Answer

odd, positive

Exercise 2.1.20



For each polynomial in **Exercises 21-30**, perform each of the following tasks.

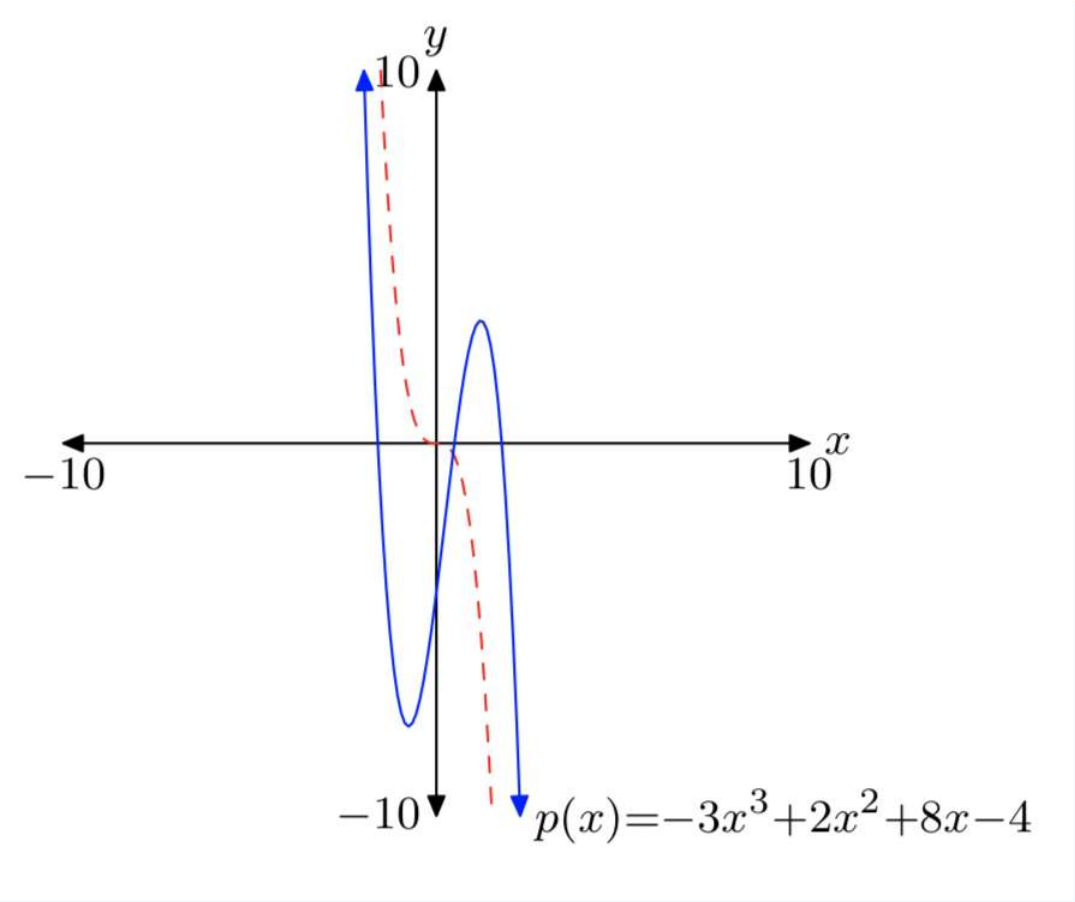
1. Predict the end-behavior of the polynomial by drawing a very rough sketch of the polynomial. Do this without the assistance of a calculator. The only concern here is that your graph show the correct end-behavior.
2. Draw the graph on your calculator, adjust the viewing window so that all "turning points" of the polynomial are visible in the viewing window, and copy the result onto your homework paper. As usual, label and scale each axis with xmin, xmax, ymin, and ymax. Does the actual end-behavior agree with your predicted end-behavior?

Exercise 2.1.21

$p(x) = -3x^3 + 2x^2 + 8x - 4$

Answer

Note that the leading term $-3x^3$ (dashed) has the same end-behavior as the polynomial p.



? Exercise 2.1.22

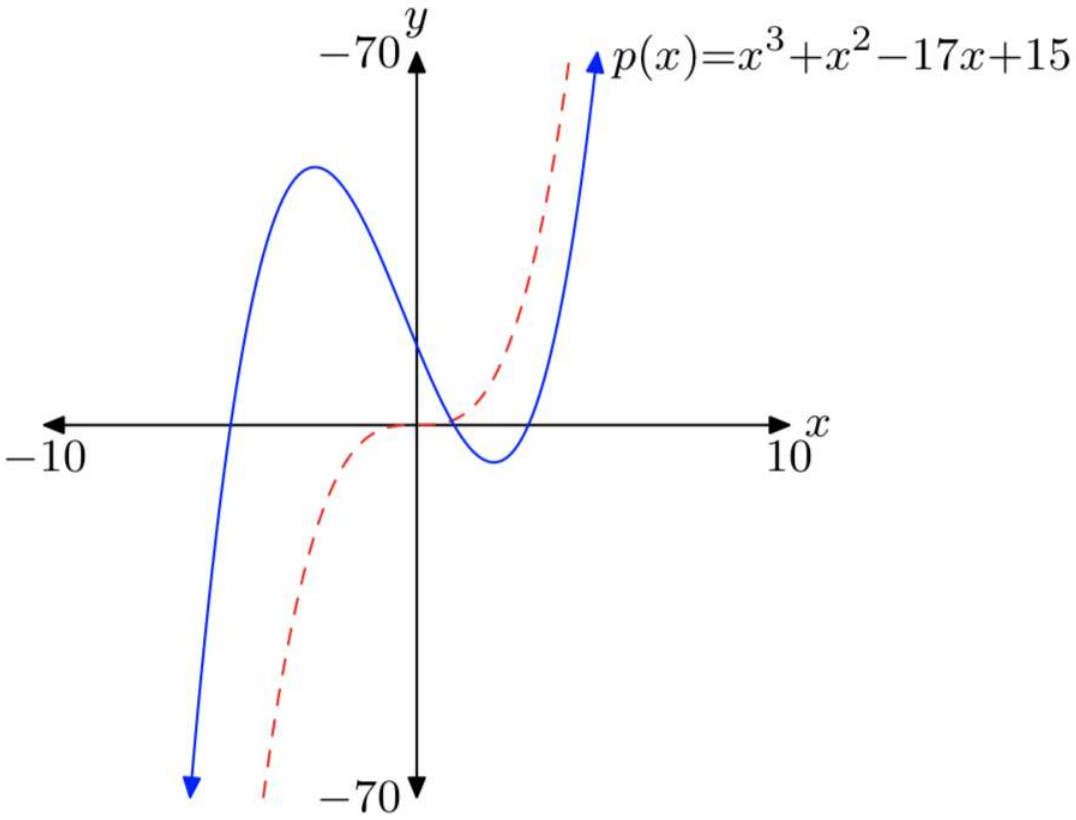
$p(x) = 2x^3 - 3x^2 + 4x - 8$

? Exercise 2.1.23

$p(x) = x^3 + x^2 - 17x + 15$

Answer

Note that the leading term x^3 (dashed) has the same end-behavior as the polynomial p.



Exercise 2.1.24

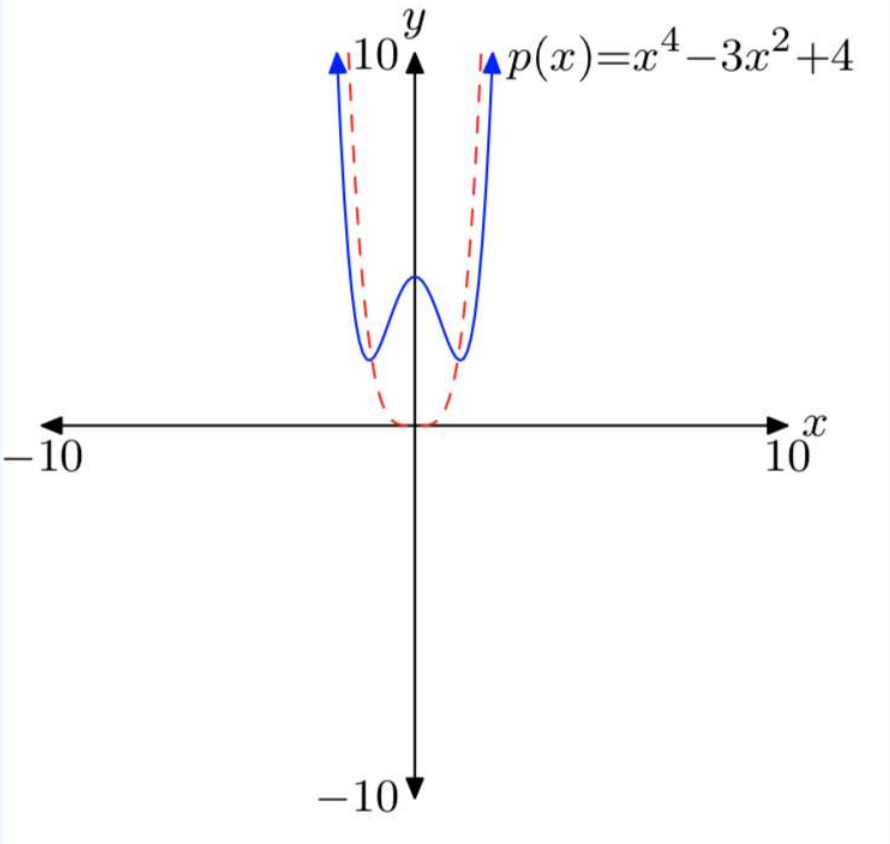
$p(x) = -x^4 + 2x^2 + 29x - 30$

Exercise 2.1.25

$p(x) = x^4 - 3x^2 + 4$

Answer

Note that the leading term x^4 (dashed) has the same end-behavior as the polynomial p.



Exercise 2.1.26

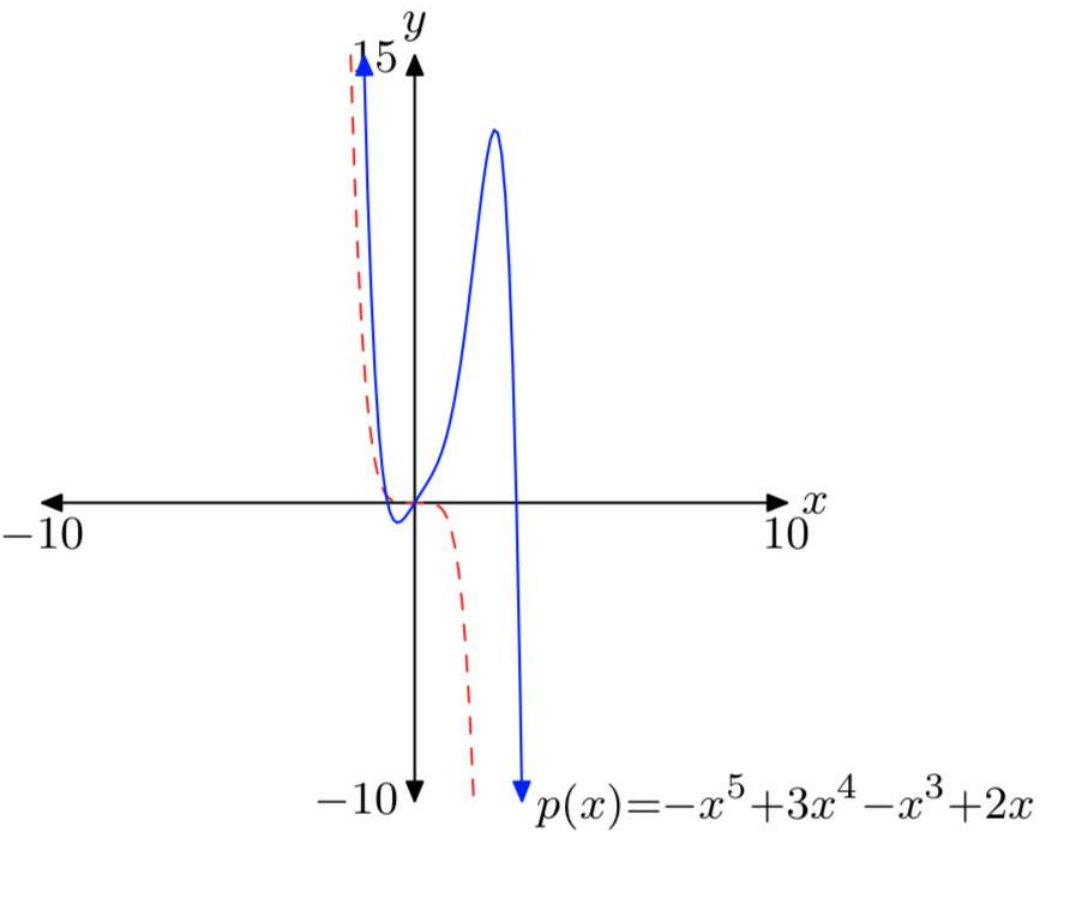
$p(x) = -x^4 + 8x^2 - 12$

Exercise 2.1.27

$p(x) = -x^5 + 3x^4 - x^3 + 2x$

Answer

Note that the leading term $-x^5$ (dashed) has the same end-behavior as the polynomial p.



Exercise 2.1.28

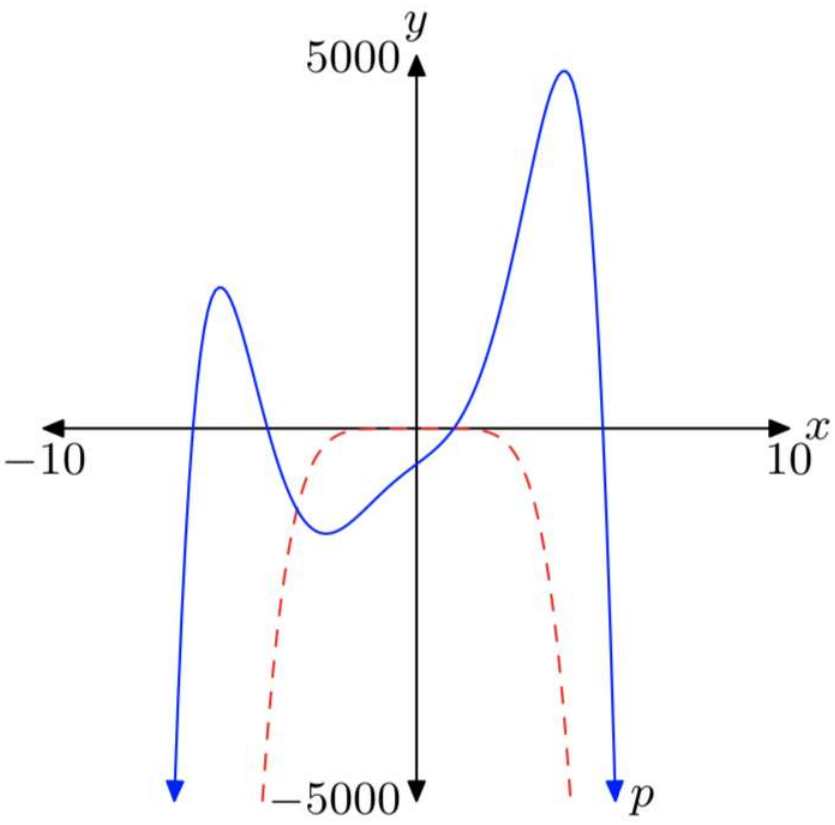
$p(x) = 2x^4 - 3x^3 + x - 10$

Exercise 2.1.29

$p(x) = -x^6 - 4x^5 + 27x^4 + 78x^3 + 4x^2 + 376x - 480$

Answer

Note that the leading term $-x^6$ (dashed) has the same end-behavior as the polynomial p.



? Exercise 2.1.30

$p(x) = x^5 - 27x^3 + 30x^2 - 124x + 120$

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2.2: Zeros of Polynomials

In the previous section we studied the end-behavior of polynomials. We know that a polynomial's end-behavior is identical to the end-behavior of its leading term. Our focus was concentrated on the far right- and left-ends of the graph and not upon what happens in-between. In this section, our focus shifts to the interior. There are two important areas of concentration: the local **maxima and minima** of the polynomial, and the location of the **x-intercepts** or zeros of the polynomial. In this section we concentrate on finding the zeros of the polynomial.

Zeros

Let's begin with a formal definition of the zeros of a polynomial.

Definition: Zero of the Polynomial

Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial with real coefficients. We say that a is a **zero of the polynomial** if and only if $p(a) = 0$.

The definition also holds if the coefficients are complex, but that's a topic for a more advanced course.

For example, -5 is a zero of the polynomial $p(x) = x^2 + 3x - 10$ because

$$\begin{aligned} p(-5) &= (-5)^2 + 3(-5) - 10 \\ &= 25 - 15 - 10 \\ &= 0 \end{aligned}$$

Similarly, -1 is a zero of the polynomial $p(x) = x^3 + 3x^2 - x - 3$ because

$$\begin{aligned} p(-1) &= (-1)^3 + 3(-1)^2 - (-1) - 3 \\ &= -1 + 3 + 1 - 3 \\ &= 0 \end{aligned}$$

Let's look at a more extensive example.

Example 2.2.1

Find the zeros of the polynomial defined by

$$p(x) = (x + 3)(x - 2)(x - 5).$$

Solution

At first glance, the function does not appear to have the form of a polynomial. However, two applications of the distributive property provide the product of the last two factors.

$$\begin{aligned} p(x) &= (x + 3)(x(x - 5) - 2(x - 5)) \\ &= (x + 3)\left(x^2 - 5x - 2x + 10\right) \\ &= (x + 3)\left(x^2 - 7x + 10\right) \end{aligned}$$

A third and fourth application of the distributive property reveals the nature of our function.

$$\begin{aligned} p(x) &= x\left(x^2 - 7x + 10\right) + 3\left(x^2 - 7x + 10\right) \\ &= x^3 - 7x^2 + 10x + 3x^2 - 21x + 30 \\ &= x^3 - 4x^2 - 11x + 30 \end{aligned}$$

Hence, p is clearly a polynomial. However, the original factored form provides quicker access to the zeros of this polynomial. Using Definition 1, we need to find values of x that make p(x) = 0. That is, we need to solve the equation

$$p(x) = 0$$

Of course, p(x) = (x + 3)(x − 2)(x − 5), so, equivalently, we need to solve the equation

$$(x+3)(x-2)(x-5)=0$$

By the zero product property, either

$$x+3=0 \quad \text{or} \quad x-2=0 \quad \text{or} \quad x-5=0$$

These are linear (first degree) equations, each of which can be solved independently. Thus, either

$$x=-3 \quad \text{or} \quad x=2 \quad \text{or} \quad x=5$$

Hence, the zeros of the polynomial p are −3, 2, and 5.

Let's use equation (4) to check that −3 is a zero of the polynomial p. Substitute −3 for x in $p(x)=x^3-4x^2-11x+30$.

$$\begin{aligned} p(-3) &= (-3)^3-4(-3)^2-11(-3)+30 \\ &= -27-36+33+30 \\ &= 0 \end{aligned}$$

This calculation verifies that −3 is a zero of the polynomial p. However, it is much easier to check that −3 is a zero of the polynomial using equation (3). Substitute −3 for x in p(x) = (x + 3)(x − 2)(x − 5).

$$\begin{aligned} p(-3) &= (-3+3)(-3-2)(-3-5) \\ &= (0)(-5)(-8) \\ &= 0 \end{aligned}$$

We'll leave it to our readers to check that 2 and 5 are also zeros of the polynomial p.

It's very important to note that once you know the linear (first degree) factors of a polynomial, the zeros follow with ease. In the last example, p(x) = (x+3)(x−2)(x−5), so the linear factors are x + 3, x − 2, and x − 5. Consequently, the zeros are −3, 2, and 5.

Before continuing, we take a moment to review an important multiplication pattern.

The Difference of Two Squares

A special multiplication pattern that appears frequently in this text is called the difference of two squares. Use the distributive property to expand (a + b)(a − b).

$$\begin{aligned} (a+b)(a-b) &= a(a-b)+b(a-b) \\ &= a^2-ab+ba-b^2 \end{aligned}$$

Since $ab=ba$, we have the following result.

Property 5: The Difference of Two Squares Pattern

$$(a+b)(a-b)=a^2-b^2$$

Thus, if you have two binomials with identical first and second terms, but the terms of one are separated by a plus sign, while the terms of the second are separated by a minus sign, then you multiply by squaring the first and second terms and separating these squares with a minus sign. Hence the name, the "difference of two squares."

For example,

$$(2x+3)(2x-3)=(2x)^2-(3)^2=4x^2-9$$

Note how we simply squared the matching first and second terms and then separated our squares with a minus sign. In similar fashion,

$$\begin{aligned} (x+5)(x-5) &= x^2-25 \\ (5x+4)(5x-4) &= 25x^2-16 \\ (3x-7)(3x+7) &= 9x^2-49 \end{aligned}$$

In each case, note how we squared the matching first and second terms, then separated the squares with a minus sign.

Once you've mastered multiplication using the "Difference of Squares" pattern, it is easy to factor using the same pattern. You simply reverse the procedure. For example

$$4x^2 - 9 = (2x + 3)(2x - 3)$$

We start by taking the square root of the two squares. Thus, the square root of $4x^2$ is 2x and the square root of 9 is 3. We then form two binomials with the results 2x and 3 as matching first and second terms, separating one pair with a plus sign, the other pair with a minus sign.

In similar fashion,

$$9x^2 - 49 = (3x + 7)(3x - 7)$$

Again, note how we take the square root of each term, form two binomials with the results, then separate one pair with a plus, the other with a minus.

We'll find the "Difference of Squares" pattern handy in what follows.

Finding Zeros by Factoring

We will now explore how we can find the zeros of a polynomial by factoring, followed by the application of the zero product property. It is important to understand that the polynomials of this section have been carefully selected so that you will be able to factor them using the various techniques that follow.

Let's explore factoring by grouping.

Example 2.2.2

Find the zeros of the polynomial

$$p(x) = x^3 + 2x^2 - 25x - 50$$

Solution

In Example 2.2.1 we learned that it is easy to spot the zeros of a polynomial if the polynomial is expressed as a product of linear (first degree) factors. In this example, the polynomial is not factored, so it would appear that the first thing we'll have to do is factor our polynomial.

Whenever you are presented with a four term expression, one thing you can try is factoring by grouping. So, with this thought in mind, let's factor an x out of the first two terms, then a −25 out of the second two terms.

$$\begin{aligned} p(x) &= x^3 + 2x^2 - 25x - 50 \\ &= x^2(x + 2) - 25(x + 2) \end{aligned}$$

Note that this last result is the difference of two terms. The polynomial is not yet fully factored as it is not yet a product of two or more factors. However, note that each of the two terms has a common factor of x + 2. Let's factor out this common factor.

$$p(x) = \left(x^2 - 25\right)(x + 2)$$

We've still not completely factored our polynomial. The first factor is the difference of two squares and can be factored further.

$$p(x) = (x + 5)(x - 5)(x + 2)$$

The polynomial p is now fully factored. To find the zeros of the polynomial p, we need to solve the equation

$$p(x) = 0$$

However, p(x) = (x + 5)(x − 5)(x + 2), so equivalently, we need to solve the equation

$$(x + 5)(x - 5)(x + 2) = 0$$

We can use the zero product property. Either

$$x+5=0 \quad \text{or} \quad x-5=0 \quad \text{or} \quad x+2=0$$

Again, each of these linear (first degree) equations can be solved independently. Either

$$x=-5 \quad \text{or} \quad x=5 \quad \text{or} \quad x=-2$$

Thus, the zeros of the polynomial p are −5, 5, and −2. We'll leave it to our readers to check these results.

Again, it is very important to realize that once the linear (first degree) factors are determined, the zeros of the polynomial follow. In this example, the linear factors are x + 5, x − 5, and x + 2. It immediately follows that the zeros of the polynomial are −5, 5, and −2.

In the next example, we will see that sometimes the first step is to factor out the greatest common factor.

Example 2.2.3

Find the zeros of the polynomial

$$p(x)=x^4+2x^3-16x^2-32x$$

Solution

To find the zeros of the polynomial, we need to solve the equation

$$p(x)=0$$

Equivalently, because $p(x)=x^4+2x^3-16x^2-32x$, we need to solve the equation

$$x^4+2x^3-16x^2-32x=0$$

Note that each term on the left-hand side has a common factor of x. Thus, our first step is to factor out this common factor of x.

$$x\left[x^3+2x^2-16x-32\right]=0$$

The four-term expression inside the brackets looks familiar. Let's try factoring by grouping. Factor an x^2 out of the first two terms, then a −16 from the third and fourth terms.

$$x\left[x^2(x+2)-16(x+2)\right]=0$$

We now have a common factor of x + 2, so we factor it out.

$$x\left[\left(x^2-16\right)(x+2)\right]=0$$

The brackets are no longer needed (multiplication is associative) so we leave them off, then use the difference of squares pattern to factor x^2-16.

$$x(x+4)(x-4)(x+2)=0$$

The zero product property tells us that either

$$x=0 \quad \text{or} \quad \text{or} \quad x+4=0 \quad \text{or} \quad x-4=0 \quad \text{or} \quad \text{or} \quad x+2=0$$

Each of these linear (first degree) factors can be solved independently. Either

$$x=0 \quad \text{or} \quad x=-4 \quad \text{or} \quad x=4 \quad \text{or} \quad x=-2$$

Thus, the zeros of the polynomial p are 0, −4, 4, and −2. We'll leave it to our readers to check these results.

Again, it is very important to note that once you've determined the linear (first degree) factors of a polynomial, then you know the zeros. In this case, the linear factors are x, x + 4, x − 4, and x + 2. Therefore, the zeros are 0, −4, 4, and −2, respectively. This discussion leads to a result called the Factor Theorem.

Factor Theorem

Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial with real coefficients. If x − a is a factor of the polynomial p(x), then a is a zero of the polynomial. That is, if x − a is a factor of the polynomial p(x), then p(a) = 0.

The upshot of all of these remarks is the fact that, if you know the linear factors of the polynomial, then you know the zeros. The converse is also true, but we will not need it in this course.

Let's examine the connection between the zeros of the polynomial and the x-intercepts of the graph of the polynomial.

The x-intercepts and the Zeros of a Polynomial

For the discussion that follows, let's assume that the independent variable is x and the dependent variable is y. Corresponding to these assignments, we will also assume that we've labeled the horizontal axis with x and the vertical axis with y, as shown in Figure 2.2.1.

The key fact for the remainder of this section is that a function is zero at the points where its graph crosses the x-axis. The phrases "function values" and "y-values" are equivalent (provided your dependent variable is y), so when you are asked where your function value is equal to zero, you are actually being asked "where is your y-value equal to zero?" Of course, y = 0 where the graph of the function crosses the horizontal axis (again, providing you are using the letter y for your dependent variable—labeling the vertical axis with y).

A polynomial is a function, so, like any function, a polynomial is zero where its graph crosses the horizontal axis. As you can see in Figure 2.2.1, the graph of the polynomial crosses the horizontal axis at x = −6, x = 1, and x = 5. Note that at each of these intercepts, the y-value (function value) equals zero. The zeros of the polynomial are −6, 1, and 5. Therefore the x-intercepts of the graph of the polynomial are located at (−6, 0), (1, 0), and (5, 0).

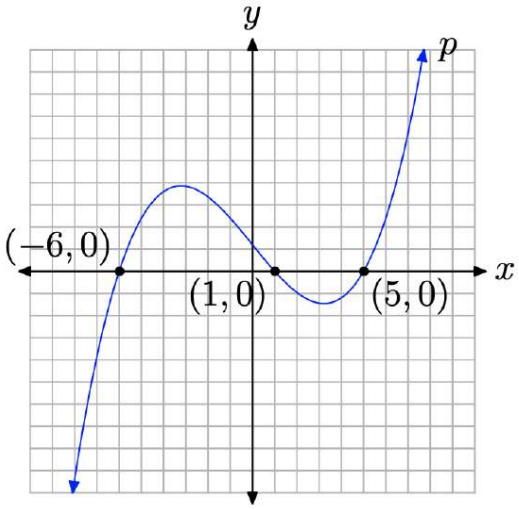


Figure 2.2.1. A function is zero where its graph crosses the horizontal axis.

Let's use these ideas to plot the graphs of several polynomials.

Example 2.2.4

Sketch the graph of the polynomial in Example 2.2.2.

Solution

In Example 2.2.2, the polynomial $p(x) = x^3 + 2x^2 - 25x - 50$ factored into linear factors

$$p(x) = (x + 5)(x - 5)(x + 2)$$

Consequently, the zeros of the polynomial were −5, 5, and −2. Thus, the x-intercepts of the graph of the polynomial are located at (−5, 0), (5, 0), and (−2, 0).

The polynomial $p(x) = x^3 + 2x^2 - 25x - 50$ has leading term x^3. Consequently, as we swing our eyes from left to right, the graph of the polynomial p must rise from negative infinity, wiggle through its x-intercepts, then continue to rise to positive infinity. We have no choice but to sketch a graph similar to that in Figure 2.2.2.

Note that there are two "turning points" of the polynomial in Figure 2.2.2. You might ask how we knew where to put these "turning points" of the polynomial. The answer is "we didn't know where to put them." We know they have to be there, but we don't know their precise location. That's why we haven't scaled the vertical axis, because without the aid of a calculator, it's hard to determine the precise location of the turning points shown in Figure 2.2.2.

However, note that knowledge of the end-behavior and the zeros of the polynomial allows us to construct a reasonable facsimile of the actual graph. If we want more accuracy than a rough approximation provides, such as the accuracy displayed in Figure 2.2.2, we'll have to use our graphing calculator, as demonstrated in Figure 2.2.3.

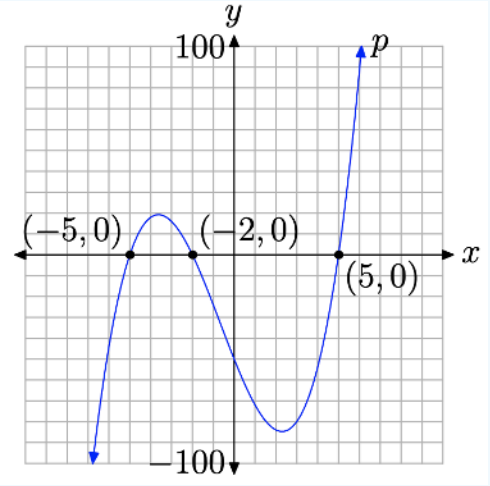


Figure 2.2.2. The graph rises from negative infinity, wiggles through its x-intercepts, then rises to positive infinity.

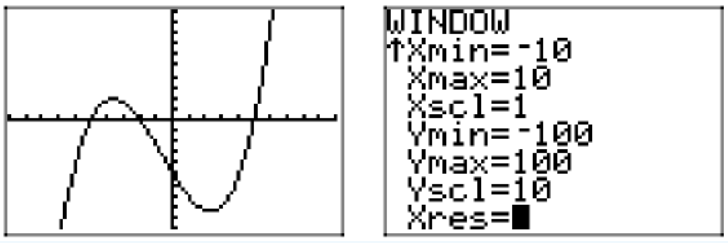


Figure 2.2.3. The graph of $p(x) = x^3 + 2x^2 - 25x - 50$ and the window settings used.

We'll have more to say about the "turning points" (relative extrema) in the next section. For now, let's continue to focus on the end-behavior and the zeros.

Let's look at another example.

Example 2.2.5

Sketch the graph of the polynomial in Example 2.2.3.

Solution

In Example 2.2.3, the polynomial $p(x) = x^4 + 2x^3 - 16x^2 - 32x$ factored into a product of linear factors

$$p(x) = x(x+4)(x-4)(x+2)$$

Consequently, the zeros of the polynomial are 0, −4, 4, and −2. Thus, the x-intercepts of the graph of the polynomial are located at (0, 0), (−4, 0), (4, 0) and (−2, 0).

The polynomial $p(x) = x^4 + 2x^3 - 16x^2 - 32x$ has leading term x^4. Consequently, as we swing our eyes from left to right, the graph of the polynomial p must fall from positive infinity, wiggle through its x-intercepts, then rise back to positive infinity. We have no choice but to sketch a graph similar to that in Figure 2.2.4.

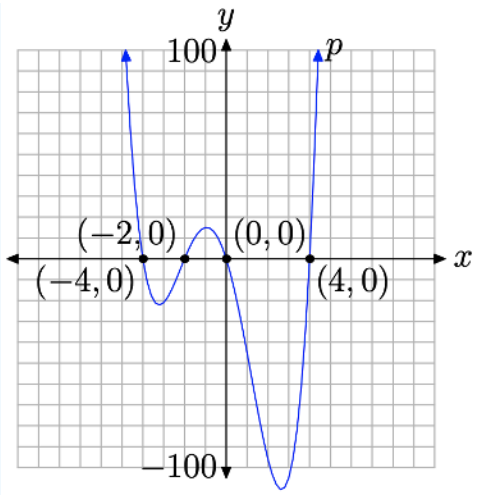


Figure 2.2.4. The graph falls from positive infinity, wiggles through its x-intercepts, then rises back to positive infinity.

Again, we can draw a sketch of the graph without the use of the calculator, using only the end-behavior and zeros of the polynomial. However, if we want the accuracy depicted in Figure 2.2.4, particularly finding correct locations of the "turning points," we'll have to resort to the use of a graphing calculator. This is shown in Figure 2.2.5.

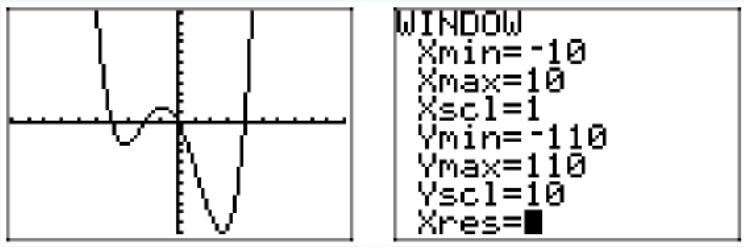


Figure 2.2.5. The graph of $p(x) = x^4 - 20x^2 + 64$ and the window settings used.

Let's look at a final example that requires factoring out a greatest common factor followed by the ac-test.

Example 2.2.6

Find the zeros of the polynomial

$$p(x) = 4x^3 - 2x^2 - 30x$$

Solution

First, notice that each term of this trinomial is divisible by 2x. This is the greatest common divisor, or equivalently, the greatest common factor. You should always look to factor out the greatest common factor in your first step.

$$\begin{aligned} p(x) &= 4x^3 - 2x^2 - 30x \\ &= 2x\left[2x^2 - x - 15\right] \end{aligned}$$

Next, compare the trinomial $2x^2 - x - 15$ with $ax^2 + bx + c$ and note that ac = −30. The integer pair {5, −6} has product −30 and sum −1. Rewrite the middle term of $2x^2 - x - 15$ in terms of this pair and factor by grouping.

$$\begin{aligned} p(x) &= 2x\left[2x^2 + 5x - 6x - 15\right] \\ &= 2x[x(2x+5) - 3(2x+5)] \\ &= 2x(x-3)(2x+5) \end{aligned}$$

To find the zeros, we need to solve the polynomial equation p(x) = 0, or equivalently,

$$2x(x-3)(2x+5) = 0$$

Using the zero product property, either

$$2x = 0, \quad \text{or} \quad x - 3 = 0, \quad \text{or} \quad 2x + 5 = 0$$

Each of these linear factors can be solved independently. Thus, either

$$x = 0, \quad \text{or} \quad x = 3, \quad \text{or} \quad x = -\frac{5}{2}$$

Thus, the zeros of the polynomial are 0, 3, and −5/2.

Alternatively, one can factor out a 2 from the third factor in equation (12).

$$\begin{aligned} p(x) &= 2x(x-3)(2)\left(x + \frac{5}{2}\right) \\ &= 4x(x-3)\left(x + \frac{5}{2}\right) \end{aligned}$$

In this form,

- x is a factor, so x = 0 is a zero,
- x − 3 is a factor, so x = 3 is a zero, and
- x + 5/2 is a factor, so x = −5/2 is a zero.

The leading term of $p(x) = 4x^3 - 2x^2 - 30x$ is $4x^2$, so as our eyes swing from left to right, the graph of the polynomial must rise from negative infinity, wiggle through its zeros, then rise to positive infinity. The graph must therefore be similar to that shown in Figure 2.2.6.

Again, the intercepts and end-behavior provide ample clues to the shape of the graph, but, if we want the accuracy portrayed in Figure 6, then we must rely on the graphing calculator. The graph and window settings used are shown in Figure 2.2.7.

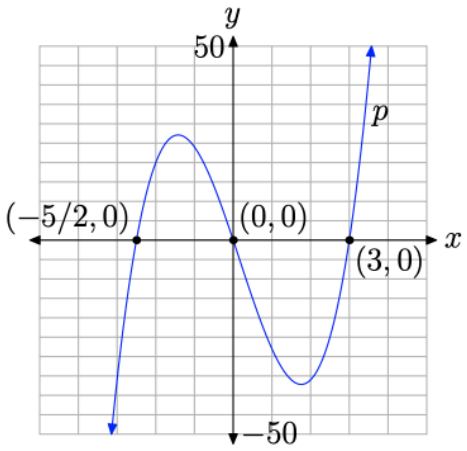


Figure 2.2.6. The graph rises from negative infinity, wiggles through its zeros, then rises to positive infinity

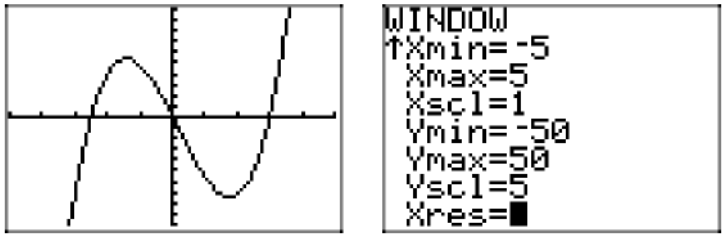


Figure 2.2.7. The graph of $p(x) = 4x^3 - 2x^2 - 30x$ and the window settings used.

Exercise

In **Exercises 1-6**, use direct substitution to show that the given value is a zero of the given polynomial.

Exercise 2.2.1

$p(x) = x^3 - 3x^2 - 13x + 15$, x = −3

Answer

p(−3) = (−3)3−3(−3)2−13(−3)+15 = 0

Exercise 2.2.2

$p(x) = x^3 - 2x^2 - 13x - 10$, x = −2

Exercise 2.2.3

$p(x) = x^4 - x^3 - 12x^2$, x = 4

Answer

$p(4) = 4^4 - 4^3 - 12(4)^2 = 0$

Exercise 2.2.4

$p(x) = x^4 - 2x^3 - 3x^2$, x = −1

Exercise 2.2.5

$p(x) = x^4 + x^2 - 20$, x = −2

Answer

$p(-2) = (-2)^4 + (-2)^2 - 20 = 0$

Exercise 2.2.6

$p(x) = x^4 + x^3 - 19x^2 + 11x + 30$, x = −1

In **Exercises 7-28**, identify all of the zeros of the given polynomial without the aid of a calculator. Use an algebraic technique and show all work (factor when necessary) needed to obtain the zeros.

Exercise 2.2.7

p(x) = (x−2)(x+4)(x−5)

Answer

−4, 2, and 5

Exercise 2.2.8

p(x) = (x−1)(x−3)(x+8)

Exercise 2.2.9

p(x) = −2(x−3)(x+4)(x−2)

Answer

−4, 2, and 3

Exercise 2.2.10

p(x) = −3(x+1)(x−1)(x−8)

Exercise 2.2.11

p(x) = x(x−3)(2x+1)

Answer

$-\frac{1}{2}$, 0, and 3

Exercise 2.2.12

p(x) = −3x(x+5)(3x−2)

Exercise 2.2.13

p(x) = −2(x+3)(3x−5)(2x+1)

Answer

−3, $-\frac{1}{2}$, and $\frac{5}{3}$

Exercise 2.2.14

p(x) = 3(x−2)(2x+5)(3x−4)

Exercise 2.2.15

$p(x) = 3x^3 + 5x^2 - 12x - 20$

Answer

−2, $-\frac{5}{3}$, and 2

Exercise 2.2.16

$p(x) = 3x^3 + x^2 - 12x - 4$

Exercise 2.2.17

$p(x) = 2x^3 + 5x^2 - 2x - 5$

Answer

$-\frac{5}{2}$, −1, and 1

Exercise 2.2.18

$p(x) = 2x^3 - 5x^2 - 18x + 45$

Exercise 2.2.19

$p(x) = x^4 + 4x^3 - 9x^2 - 36x$

Answer

0, −3, 3, and −4

Exercise 2.2.20

$p(x) = -x^4 + 4x^3 + x^2 - 4x$

Exercise 2.2.21

$p(x) = -2x^4 - 10x^3 + 8x^2 + 40x$

Answer

0, −2, 2, and −5

Exercise 2.2.22

$p(x) = 3x^4 + 6x^3 - 75x^2 - 150x$

Exercise 2.2.23

$p(x) = 2x^3 - 7x^2 - 15x$

Answer

$-\frac{3}{2}$, 0, and 5

Exercise 2.2.24

$p(x) = 2x^3 - x^2 - 10x$

Exercise 2.2.25

$p(x) = -6x^3 + 4x^2 + 16x$

Answer

$-\frac{4}{3}$, 0, and 2

Exercise 2.2.26

$p(x) = 9x^3 + 3x^2 - 30x$

Exercise 2.2.27

$p(x) = -2x^7 - 10x^6 + 8x^5 + 40x^4$

Answer

0, −2, 2, and −5

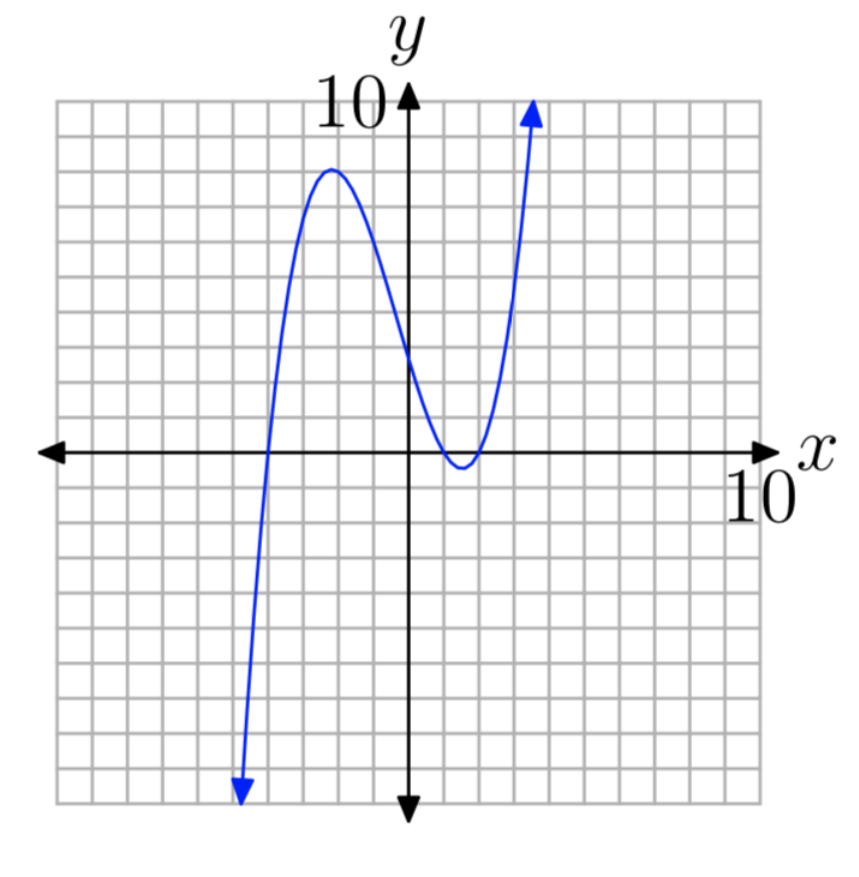
Exercise 2.2.28

$p(x) = 6x^5 - 21x^4 - 45x^3$

In **Exercises 29-34**, the graph of a polynomial is given. Perform each of the following tasks

1. Copy the image onto your homework paper. Label and scale your axes, then label each x-intercept with its coordinates.
2. Identify the zeros of the polynomial.

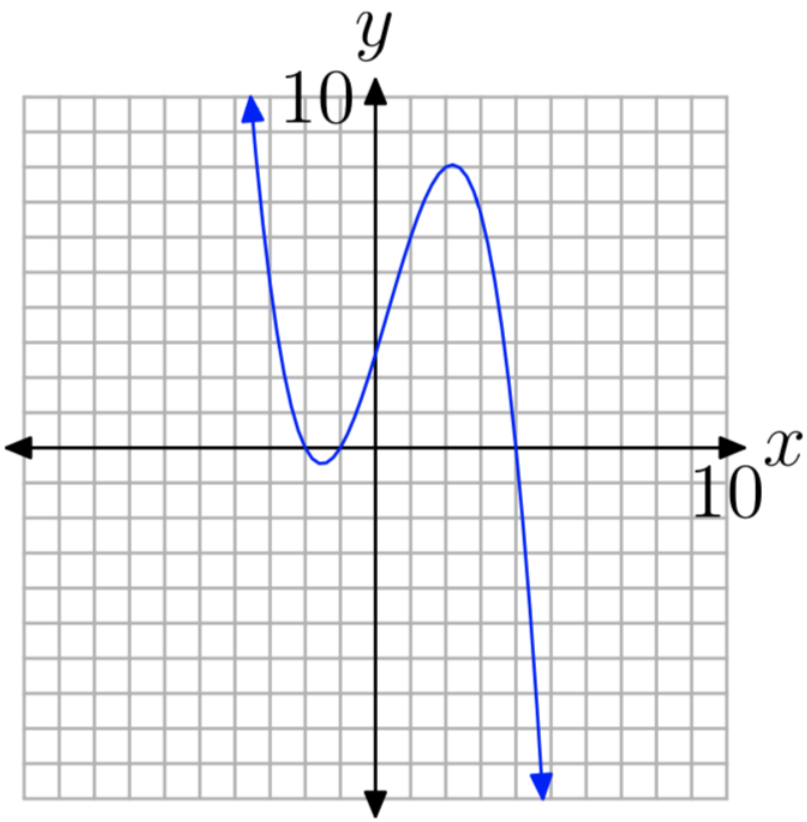
Exercise 2.2.29



Answer

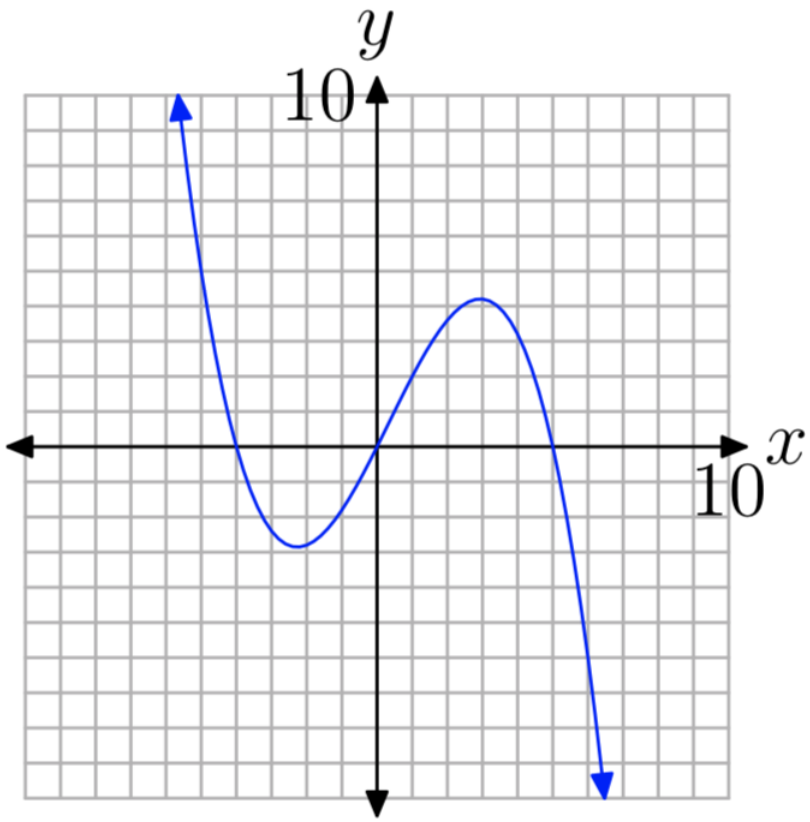
Zeros: −4, 1, and 2

Exercise 2.2.30



Exercise 2.2.31

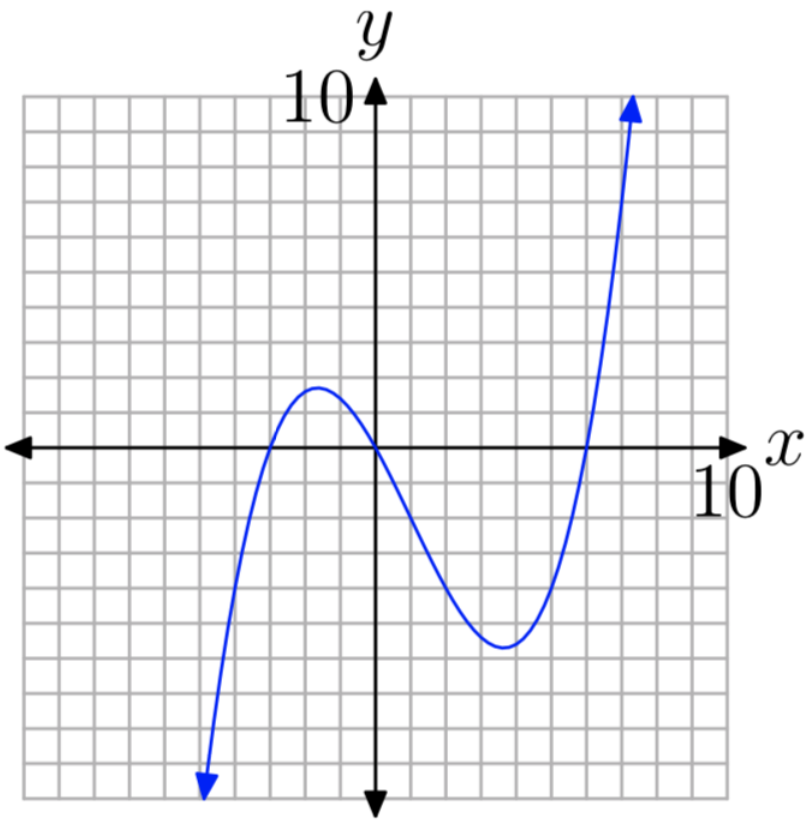
https://math.libretexts.org/@go/page/58416



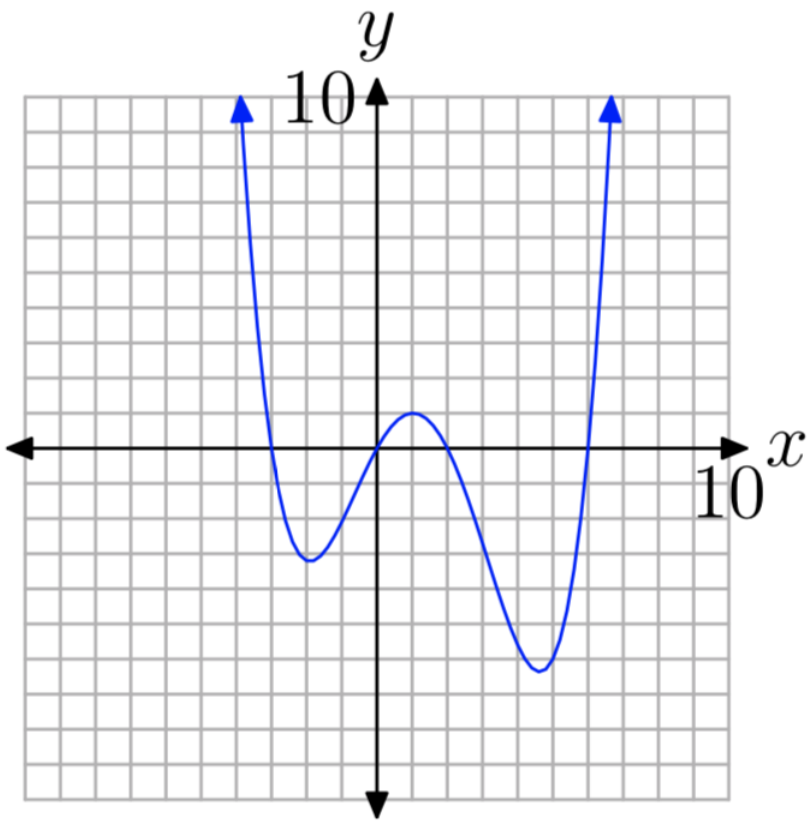
Answer

Zeros: −4, 0, and 5

? Exercise 2.2.32



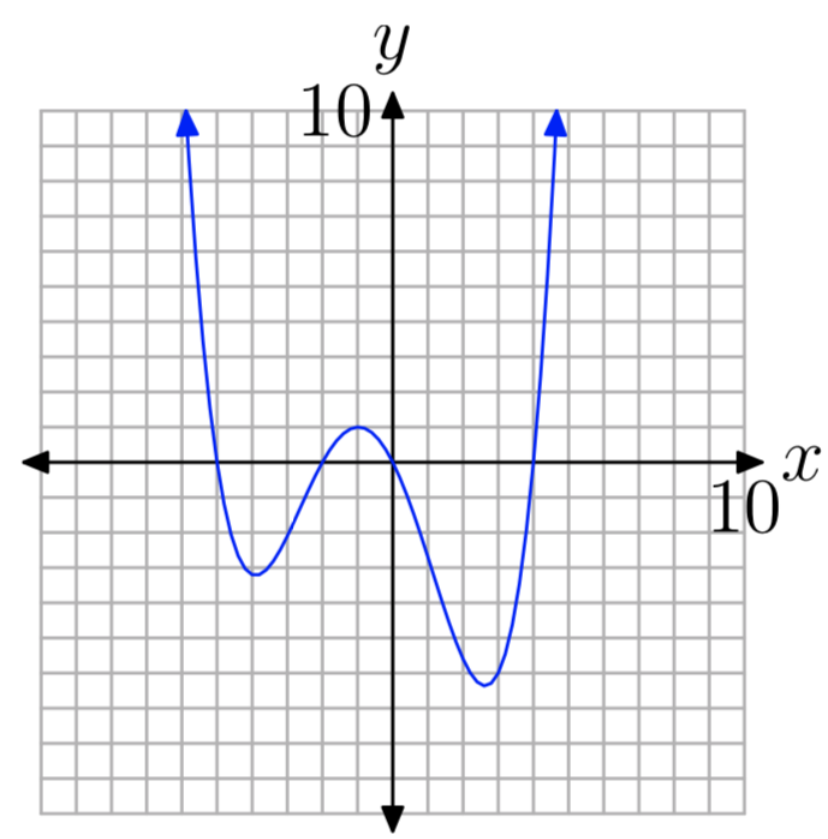
? Exercise 2.2.33



Answer

Zeros: 0, 6, −3, 2

Exercise 2.2.34



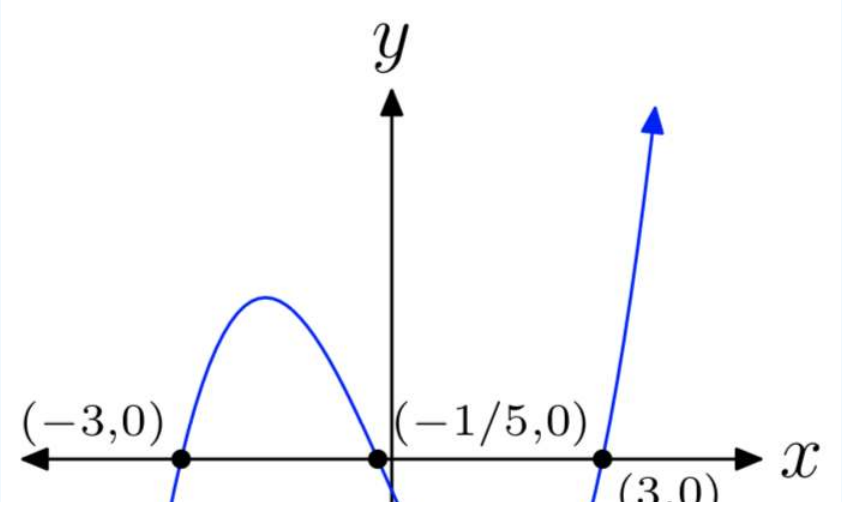
For each of the polynomials in **Exercises 35-46**, perform each of the following tasks.

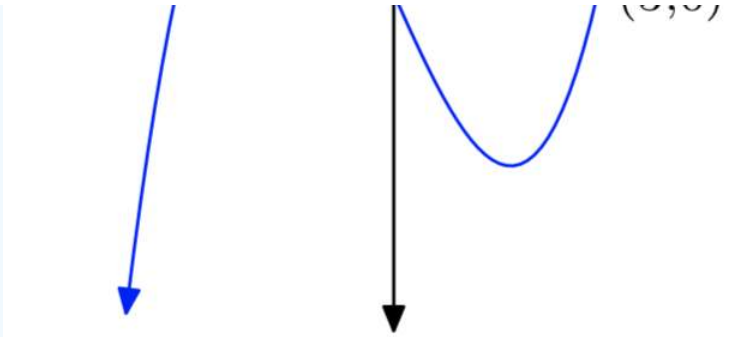
1. Factor the polynomial to obtain the zeros. Show your work.
2. Set up a coordinate system on graph paper. Label and scale the horizontal axis. Use the zeros and end-behavior to help sketch the graph of the polynomial without the use of a calculator.
3. Verify your result with a graphing calculator.

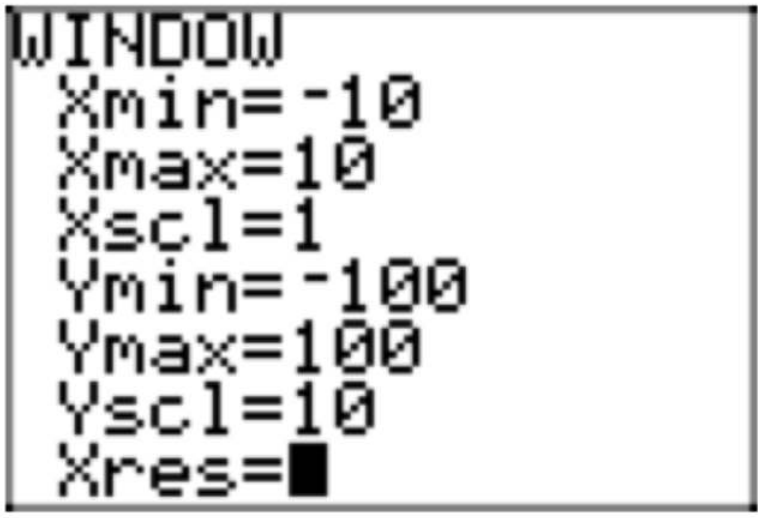
? Exercise 2.2.35

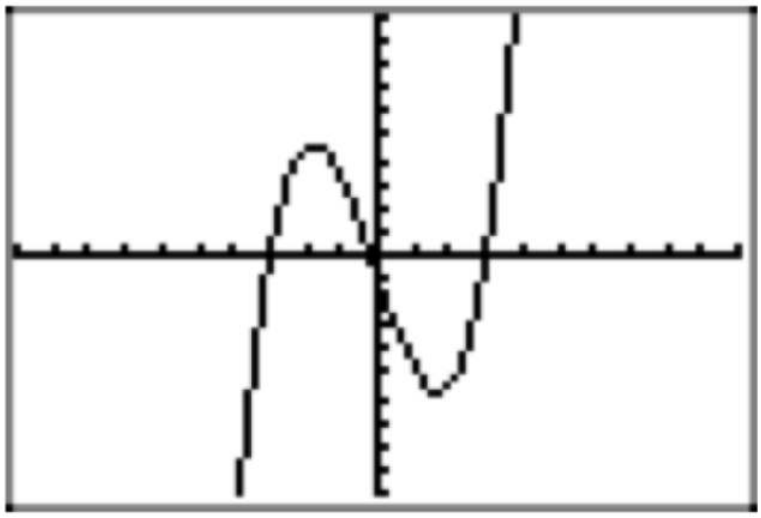
$p(x) = 5x^3 + x^2 - 45x - 9$

Answer






WINDOW
Xmin=-10
Xmax=10
Xscl=1
Ymin=-100
Ymax=100
Yscl=10
Xres=



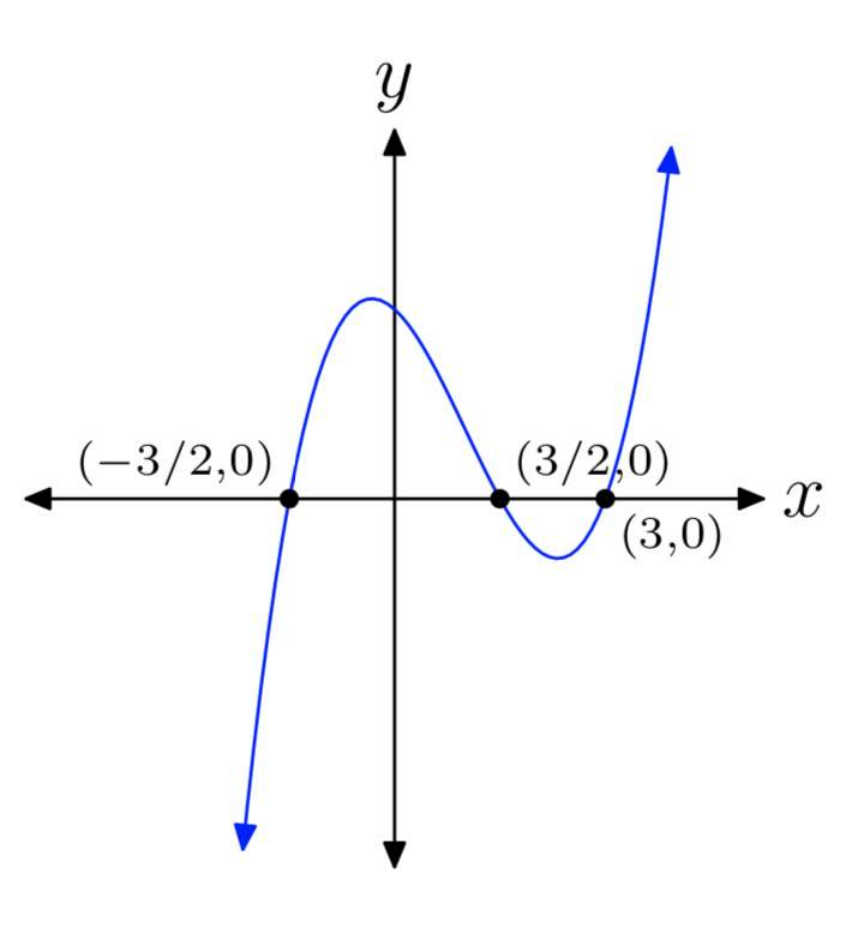
Exercise 2.2.36

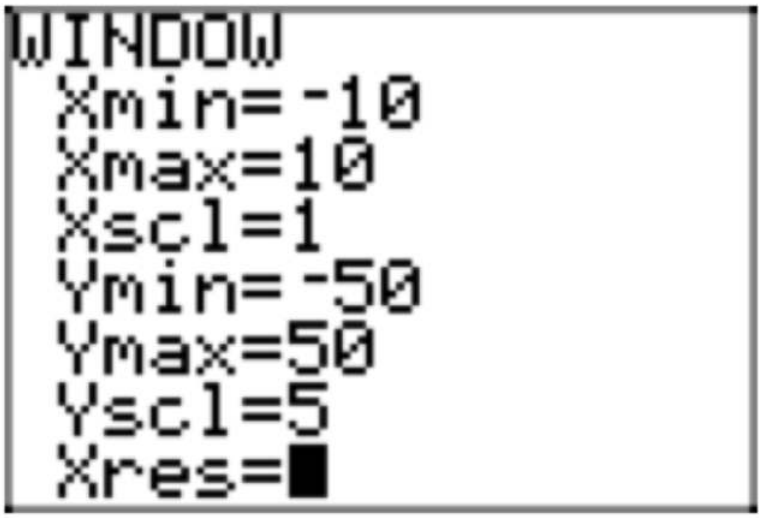
$p(x) = 4x^3 + 3x^2 - 64x - 48$

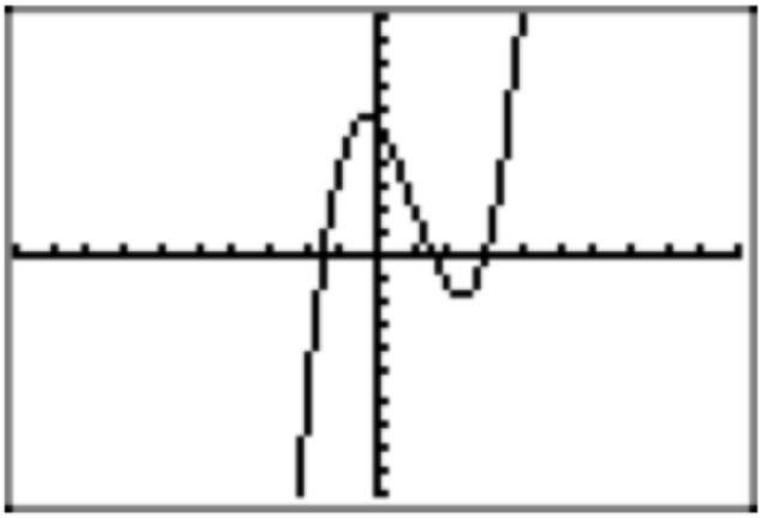
Exercise 2.2.37

$p(x) = 4x^3 - 12x^2 - 9x + 27$

Answer







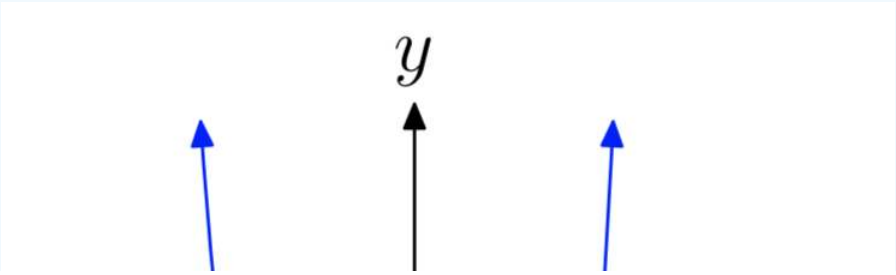
Exercise 2.2.38

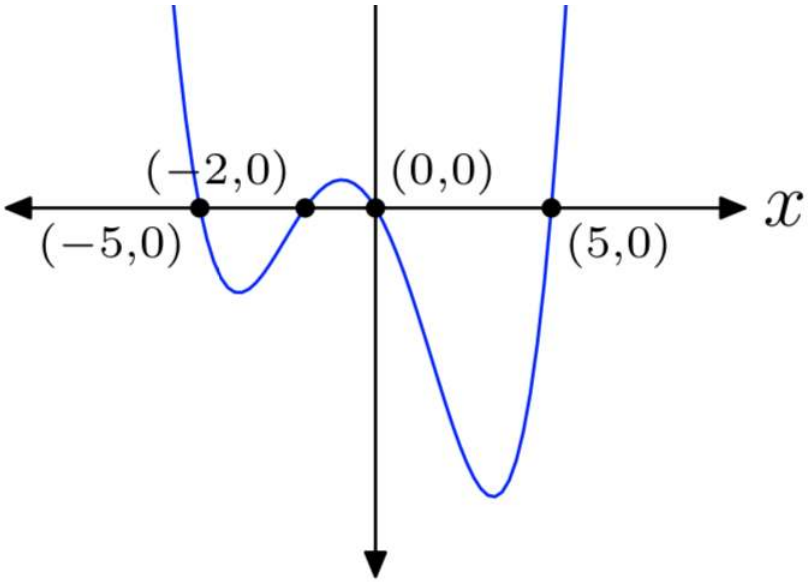
$p(x) = x^3 + x^2 - 16x - 16$

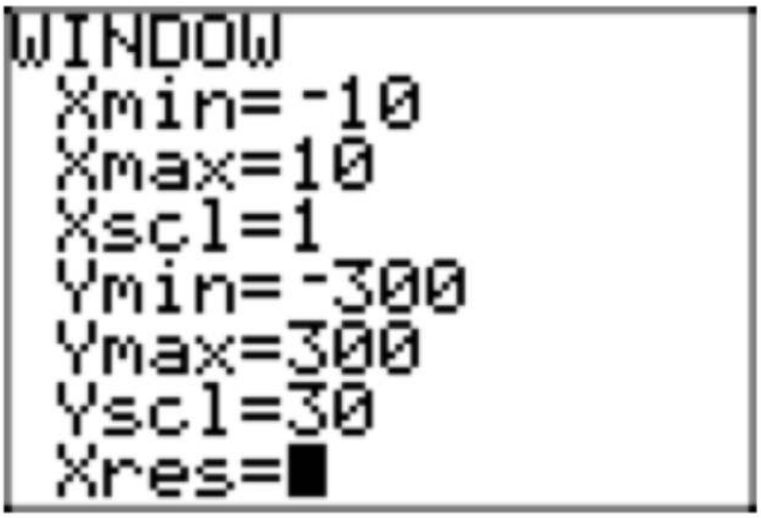
Exercise 2.2.39

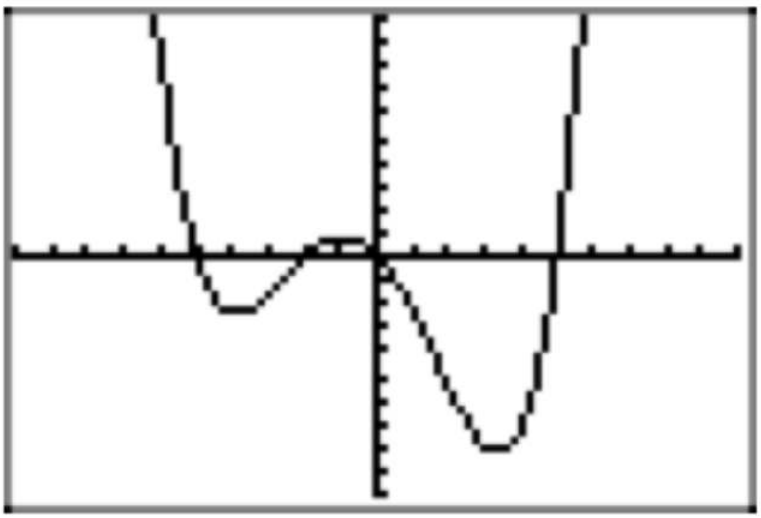
$p(x) = x^4 + 2x^3 - 25x^2 - 50x$

Answer




(−2,0)
(0,0)
x
(−5,0)
(5,0)


WINDOW
Xmin=-10
Xmax=10
Xscl=1
Ymin=-300
Ymax=300
Yscl=30
Xres=



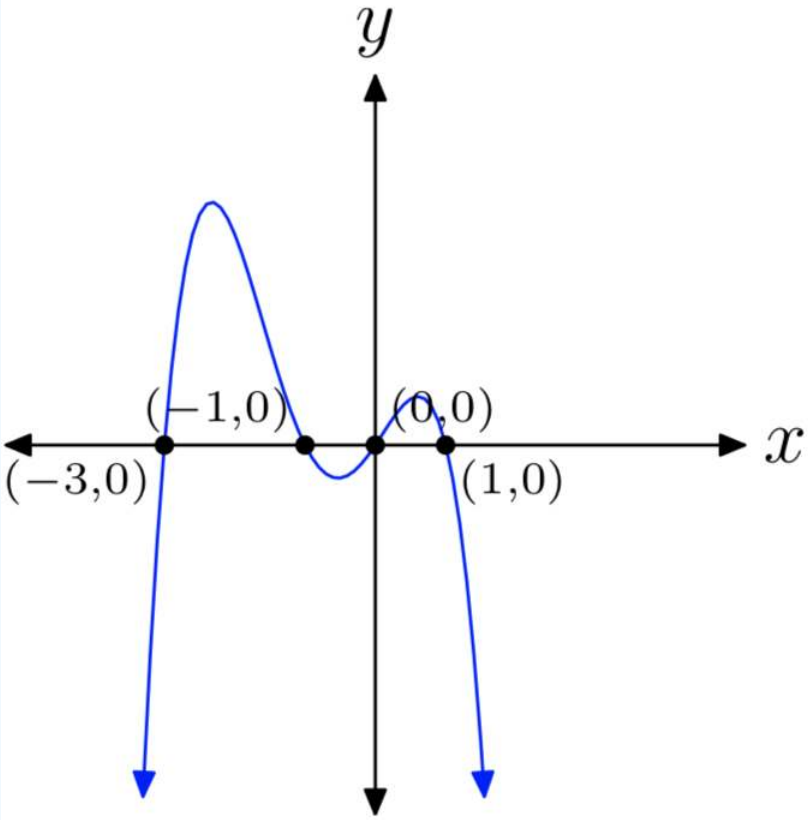
Exercise 2.2.40

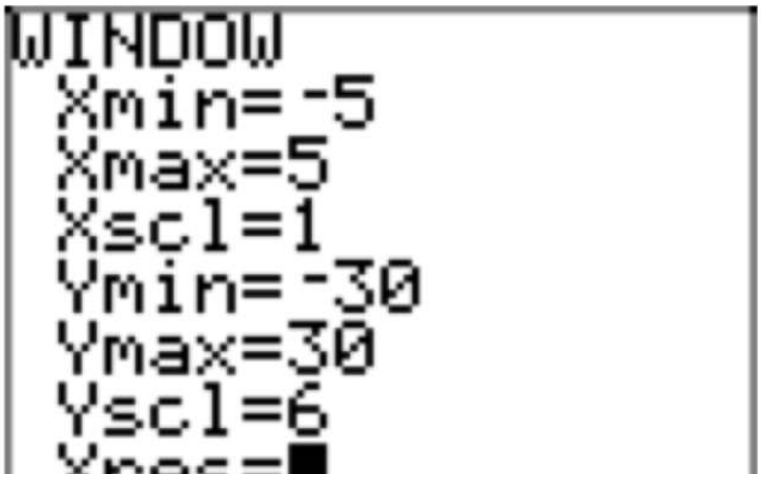
$p(x) = -x^4 - 5x^3 + 4x^2 + 20x$

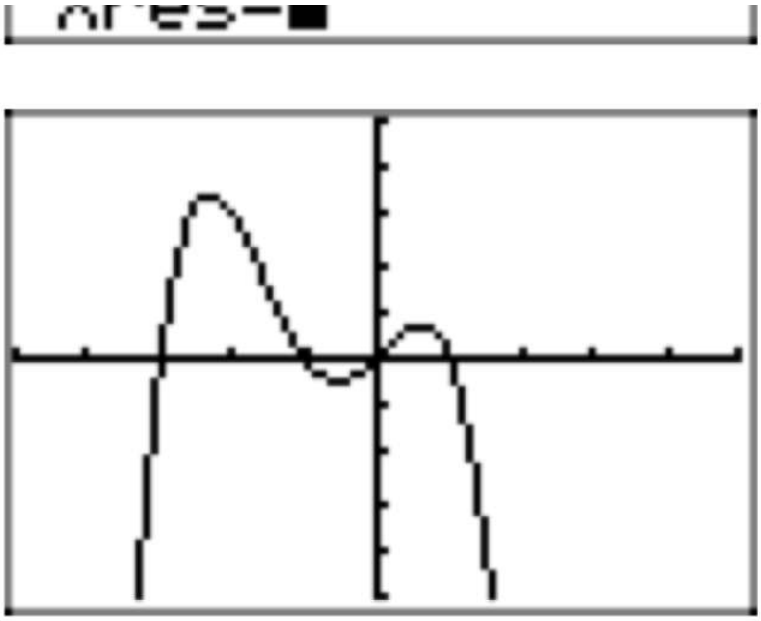
Exercise 2.2.41

$p(x) = -3x^4 - 9x^3 + 3x^2 + 9x$

Answer







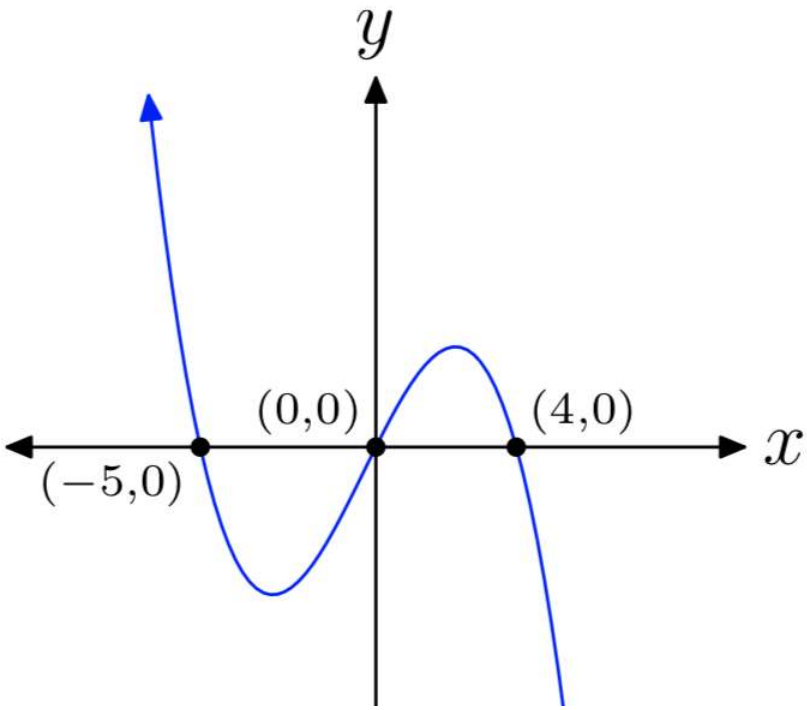
Exercise 2.2.42

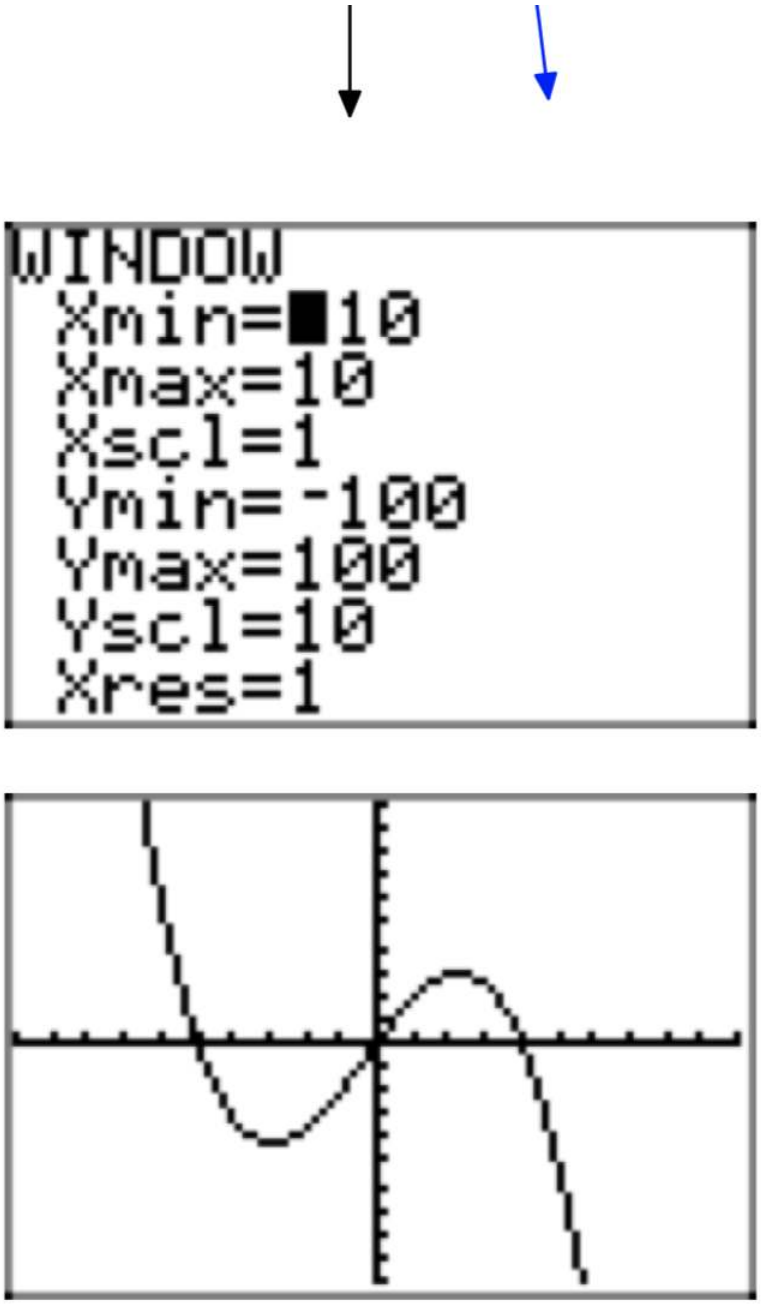
$p(x) = 4x^4 - 29x^2 + 25$

Exercise 2.2.43

$p(x) = -x^3 - x^2 + 20x$

Answer





Exercise 2.2.44

$p(x) = 2x^3 - 7x^2 - 30x$

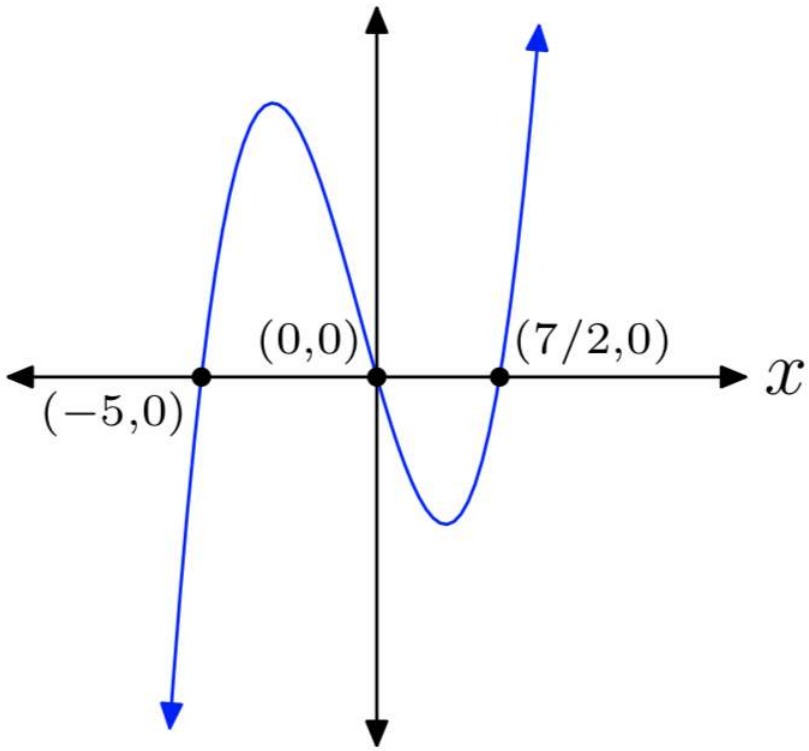
Exercise 2.2.45

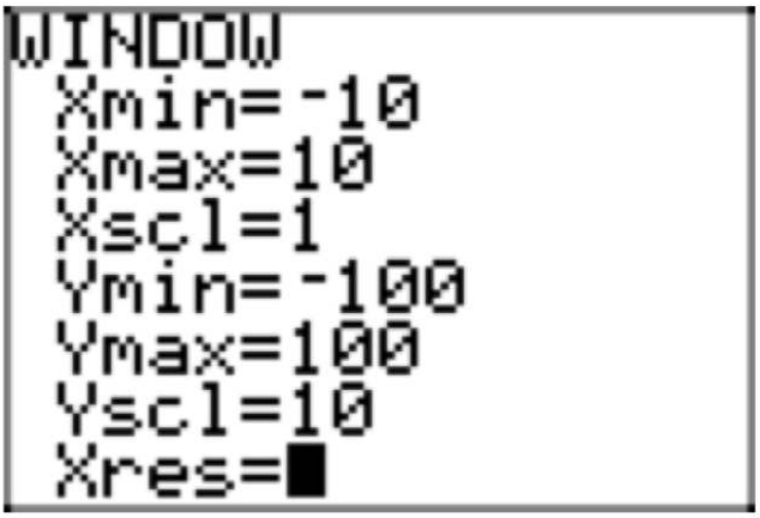
$p(x) = 2x^3 + 3x^2 - 35x$

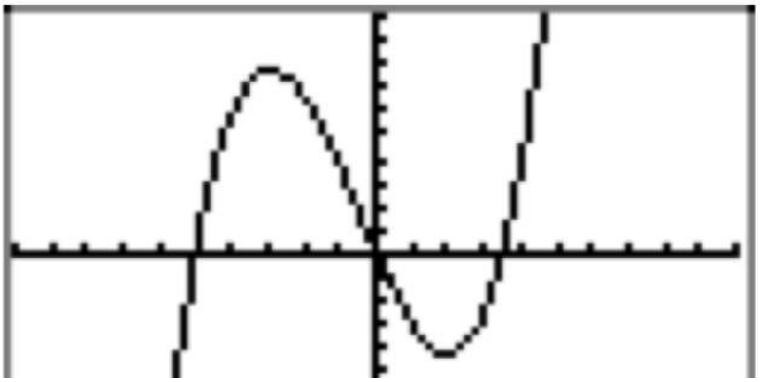
Answer

y

https://math.libretexts.org/@go/page/58416


(0,0)
(7/2,0)
(−5,0)
x


WINDOW
Xmin=-10
Xmax=10
Xscl=1
Ymin=-100
Ymax=100
Yscl=10
Xres=





? Exercise 2.2.46

$p(x) = -2x^3 - 11x^2 + 21x$

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SECTION OVERVIEW

2.3: Factoring

2.3.1: Prelude to Factoring

2.3.2: Greatest Common Factor and Factor by Grouping

2.3.2E: Exercises

2.3.3: Factor Trinomials

2.3.3E: Exercises

2.3.4: Factor Special Products

2.3.4E: Exercises

2.3.5: General Strategy for Factoring Polynomials

2.3.5E: Exercises

2.3.6: Polynomial Equations

2.3.6E: Exercises

2.3.7: Review Exercises

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2.3.1: Prelude to Factoring



Figure 2.3.1.1: Scientists use factoring to calculate growth rates of infectious diseases such as viruses. (credit: “FotoshopTofs” / Pixabay)

An epidemic of a disease has broken out. Where did it start? How is it spreading? What can be done to control it? Answers to these and other questions can be found by scientists known as epidemiologists. They collect data and analyze it to study disease and consider possible control measures. Because diseases can spread at alarming rates, these scientists must use their knowledge of mathematics involving factoring. In this chapter, you will learn how to factor and apply factoring to real-life situations.

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2.3.2: Greatest Common Factor and Factor by Grouping

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

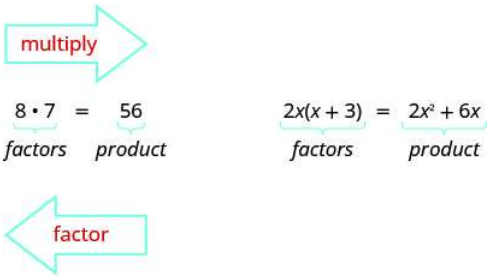
Be Prepared

Before you get started, take this readiness quiz.

1. Factor 56 into primes.
   If you missed this problem, review [link].
2. Find the least common multiple (LCM) of 18 and 24.
   If you missed this problem, review [link].
3. Multiply: $-3a(7a+8b)$.
   If you missed this problem, review [link].

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a **product**. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

GREATEST COMMON FACTOR

The **greatest common factor** (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We summarize the steps we use to find the greatest common factor.

FIND THE GREATEST COMMON FACTOR (GCF) OF TWO EXPRESSIONS.

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all expressions share.
4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three expressions.

Example 2.3.2.1

Find the greatest common factor of $21x^3,\ 9x^2,\ 15x$.

Answer

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$ $\text{GCF} = 3 \cdot x$
Multiply the factors.	$\text{GCF} = 3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

Example 2.3.2.2

Find the greatest common factor: $25m^4,\ 35m^3,\ 20m^2$.

Answer

$5m^2$

Example 2.3.2.3

Find the greatest common factor: $14x^3,\ 70x^2,\ 105x$.

Answer

$7x$

Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \cdot 6$ or $3 \cdot 4$. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with its factors, $3x(x + 5)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

DISTRIBUTIVE PROPERTY

If *a*, *b*, and *c* are real numbers, then

$$a(b+c) = ab + ac \quad \text{and} \quad ab + ac = a(b+c)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a **polynomial**? You just find the GCF of all the terms and write the polynomial as a product!

Example 2.3.2.4: How to Use the Distributive Property to factor a polynomial

Factor: $8m^3 - 12m^2n + 20mn^2$.

Answer

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $8m^3, 12m^2n, 20mn^2$	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $8m^3, 12m^2n, 20mn^2$ as products of their GCF, $4m$. $8m^3 = 4m \cdot 2m^2$ $12m^2n = 4m \cdot 3mn$ $20mn^2 = 4m \cdot 5n^2$	$8m^3 - 12m^2n + 20mn^2$ $4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4m(2m^2 - 3mn + 5n^2)$
Step 4. Check by multiplying the factors.		$4m(2m^2 - 3mn + 5n^2)$ $4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$ $8m^3 - 12m^2n + 20mn^2$ ✓

Example 2.3.2.5

Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

Answer

$3y^2(3x + 2x^2 + 7y)$

Example 2.3.2.6

Factor: $3p^3 - 6p^2q + 9pq^3$.

Answer

$3p(p^2 - 2pq + 3q^3)$

FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL.

1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the "reverse" Distributive Property to factor the expression.
4. Check by multiplying the factors.

FACTOR AS A NOUN AND A VERB

We use "factor" as both a noun and a verb:

Noun:	7 is a factor of 14
Verb:	factor 3 from $3a + 3$

Example 2.3.2.7

Factor: $5x^3 - 25x^2$.

Answer

Find the GCF of $5x^3$ and $25x^2$.		$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $GCF = 5 \cdot x \cdot x$
		$GCF = 5x^2$
		$5x^3 - 25x^2$
Rewrite each term.		$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.		$5x^2(x - 5)$
Check: $5x^2(x-5)$ $5x^2 \cdot x - 5x^2 \cdot 5$ $5x^3 - 25x^2$ ✓		

Example 2.3.2.8

Factor: $2x^3 + 12x^2$.

Answer

$2x^2(x+6)$

Example 2.3.2.9

Factor: $6y^3 - 15y^2$.

Answer

$3y^2(2y-5)$

Example 2.3.2.10

Factor: $8x^3y - 10x^2y^2 + 12xy^3$.

Answer

The GCF of $8x^3y$, $-10x^2y^2$, and $12xy^3$ is $2xy$.	$8x^3y = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y$ $10x^2y^2 = 2 \cdot 5 \cdot x \cdot x \cdot y \cdot y$ $12xy^3 = 2 \cdot 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$ $GCF = 2 \cdot x \cdot y$
	$GCF = 2xy$
	$8x^3y - 10x^2y^2 + 12xy^3$
Rewrite each term using the GCF, $2xy$.	$2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$
Factor the GCF.	$2xy(4x^2 - 5xy + 6y^2)$

Check: $$2xy(4x^2-5xy+6y^2)$$ $$2xy\cdot 4x^2-2xy\cdot 5xy+2xy\cdot 6y^2$$ $$8x^3y-10x^2y^2+12xy^3\checkmark$$	

Example 2.3.2.11

Factor: $15x^3y-3x^2y^2+6xy^3$.

Answer

$3xy(5x^2-xy+2y^2)$

Example 2.3.2.12

Factor: $8a^3b+2a^2b^2-6ab^3$.

Answer

$2ab(4a^2+ab-3b^2)$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

Example 2.3.2.13

Factor: $-4a^3+36a^2-8a$.

Answer

The leading coefficient is negative, so the GCF will be negative.

	$-4a^3+36a^2-8a$
Rewrite each term using the GCF, $-4a$.	$-4a\cdot a^2-(-4a)\cdot 9a+(-4a)\cdot 2$
Factor the GCF.	$-4a(a^2-9a+2)$
Check: $$-4a(a^2-9a+2)$$ $$-4a\cdot a^2-(-4a)\cdot 9a+(-4a)\cdot 2$$ $$-4a^3+36a^2-8a\checkmark$$	

Example 2.3.2.14

Factor: $-4b^3+16b^2-8b$.

Answer

$-4b(b^2-4b+2)$

Example 2.3.2.15

Factor: $-7a^3+21a^2-14a$.

Answer

$-7a(a^2-3a+2)$

So far our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.

Example 2.3.2.16

Factor: $3y(y+7)-4(y+7)$.

Answer

The GCF is the binomial $y+7$.

		$3y(y+7)-4(y+7)$
Factor the GCF, $(y+7)$.		$(y+7)(3y-4)$
Check on your own by multiplying.		

Example 2.3.2.17

Factor: $4m(m+3)-7(m+3)$.

Answer

$(m+3)(4m-7)$

Example 2.3.2.18

Factor: $8n(n-4)+5(n-4)$.

Answer

$(n-4)(8n+5)$

Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the **GCF** in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored. Just like some numbers are **prime**, some polynomials are prime.

Example 2.3.2.19: How to Factor a Polynomial by Grouping

Factor by grouping: $xy+3y+2x+6$.

Answer

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms? No, so let's separate the first two terms from the second two.	$xy + 3y + 2x + 6$ $xy + 3y + 2x + 6$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$y(x + 3) + 2x + 6$ $y(x + 3) + 2(x + 3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x + 3)$. Factor out the common factor.	$y(x + 3) + 2(x + 3)$ $(x + 3)(y + 2)$
Step 4. Check.	Multiply $(x + 3)(y + 2)$. Is the product the original expression?	$(x + 3)(y + 2)$ $xy + 2x + 3y + 6$ $xy + 3y + 2x + 6$ ✓

? Example 2.3.2.20

Factor by grouping: $xy+8y+3x+24$.

Answer

$(x+8)(y+3)$

? Example 2.3.2.21

Factor by grouping: $ab+7b+8a+56$.

Answer

$(a+7)(b+8)$

FACTOR BY GROUPING.

1. Group terms with common factors.
2. Factor out the common factor in each group.
3. Factor the common factor from the expression.
4. Check by multiplying the factors.

? Example 2.3.2.22

Factor by grouping: ⓐ $x^2+3x-2x-6$ ⓑ $6x^2-3x-4x+2$.

Answer

ⓐ

There is no GCF in all four terms.	$x^2+3x-2x-6$
Separate into two parts.	$x^2+3x \quad -2x-6$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x+3)-2(x+3)$
Factor out the common factor.	$(x+3)(x-2)$
Check on your own by multiplying.	

ⓑ

There is no GCF in all four terms.	$6x^2-3x-4x+2$
Separate into two parts.	$6x^2-3x \quad -4x+2$
Factor the GCF from both parts.	$3x(2x-1)-2(2x-1)$
Factor out the common factor.	$(2x-1)(3x-2)$
Check on your own by multiplying.	

? Example 2.3.2.23

Factor by grouping: ⓐ $x^2+2x-5x-10$ ⓑ $20x^2-16x-15x+12$.

Answer

ⓐ $(x-5)(x+2)$
ⓑ $(5x-4)(4x-3)$

? Example 2.3.2.24

Factor by grouping: ⓐ $y^2+4y-7y-28$ ⓑ $42m^2-18m-35m+15$.

Answer

ⓐ $(y+4)(y-7)$
ⓑ $(7m-3)(6m-5)$

Key Concepts

- **How to find the greatest common factor (GCF) of two expressions.**
  1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
  2. List all factors—matching common factors in a column. In each column, circle the common factors.
  3. Bring down the common factors that all expressions share.
  4. Multiply the factors.
- **Distributive Property:** If a, b and c are real numbers, then

$$a(b+c)=ab+ac \quad \text{and} \quad ab+ac=a(b+c)$$

  The form on the left is used to multiply. The form on the right is used to factor.
- **How to factor the greatest common factor from a polynomial.**
  1. Find the GCF of all the terms of the polynomial.
  2. Rewrite each term as a product using the GCF.
  3. Use the "reverse" Distributive Property to factor the expression.
  4. Check by multiplying the factors.
- **Factor as a Noun and a Verb:** We use "factor" as both a noun and a verb.

Noun:	7 is a factor of 14
Verb:	factor 3 from $3a+3$

- **How to factor by grouping.**
  1. Group terms with common factors.
  2. Factor out the common factor in each group.
  3. Factor the common factor from the expression.
  4. Check by multiplying the factors.

Glossary

factoring

Splitting a product into factors is called factoring.

greatest common factor

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

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2.3.2E: Exercises

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. $10p^3q, 12pq^2$

Answer

$2pq$

2. $8a^2b^3, 10ab^2$

3. $12m^2n^3, 30m^5n^3$

Answer

$6m^2n^3$

4. $28x^2y^4, 42x^4y^4$

5. $10a^3, 12a^2, 14a$

Answer

$2a$

6. $20y^3, 28y^2, 40y$

7. $35x^3y^2, 10x^4y, 5x^5y^3$

Answer

$5x^3y$

8. $27p^2q^3, 45p^3q^4, 9p^4q^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

9. $6m+9$

Answer

$3(2m+3)$

10. $14p+35$

11. $9n-63$

Answer

$9(n-7)$

12. $45b - 18$

13. $3x^2 + 6x - 9$

Answer

$3(x^2 + 2x - 3)$

14. $4y^2 + 8y - 4$

15. $8p^2 + 4p + 2$

Answer

$2(4p^2 + 2p + 1)$

16. $10q^2 + 14q + 20$

17. $8y^3 + 16y^2$

Answer

$8y^2(y + 2)$

18. $12x^3 - 10x$

19. $5x^3 - 15x^2 + 20x$

Answer

$5x(x^2 - 3x + 4)$

20. $8m^2 - 40m + 16$

21. $24x^3 - 12x^2 + 15x$

Answer

$3x(8x^2 - 4x + 5)$

22. $24y^3 - 18y^2 - 30y$

23. $12xy^2 + 18x^2y^2 - 30y^3$

Answer

$6y^2(2x + 3x^2 - 5y)$

24. $21pq^2 + 35p^2q^2 - 28q^3$

25. $20x^3y - 4x^2y^2 + 12xy^3$

Answer

$4xy(5x^2 - xy + 3y^2)$

26. $24a^3b+6a^2b^2-18ab^3$

27. $-2x-4$

Answer

$-2(x+4)$

28. $-3b+12$

29. $-2x^3+18x^2-8x$

Answer

$-2x(x^2-9x+4)$

30. $-5y^3+35y^2-15y$

31. $-4p^3q-12p^2q^2+16pq^2$

Answer

$-4pq(p^2+3pq-4q)$

32. $-6a^3b-12a^2b^2+18ab^2$

33. $5x(x+1)+3(x+1)$

Answer

$(x+1)(5x+3)$

34. $2x(x-1)+9(x-1)$

35. $3b(b-2)-13(b-2)$

Answer

$(b-2)(3b-13)$

36. $6m(m-5)-7(m-5)$

Factor by Grouping

In the following exercises, factor by grouping.

37. $ab+5a+3b+15$

Answer

$(b+5)(a+3)$

38. $cd+6c+4d+24$

39. $8y^2+y+40y+5$

Answer

$(y+5)(8y+1)$

40. $6y^2+7y+24y+28$

41. $uv-9u+2v-18$

Answer

$(u+2)(v-9)$

42. $pq-10p+8q-80$

43. $u^2-u+6u-6$

Answer

$(u-1)(u+6)$

44. $x^2-x+4x-4$

45. $9p^2-15p+12p-20$

Answer

$(3p-5)(3p+4)$

46. $16q^2+20q-28q-35$

47. $mn-6m-4n+24$

Answer

$(n-6)(m-4)$

48. $r^2-3r-r+3$

49. $2x^2-14x-5x+35$

Answer

$(x-7)(2x-5)$

50. $4x^2-36x-3x+27$

Mixed Practice

In the following exercises, factor.

51. $-18xy^2-27x^2y$

Answer

$-9xy(3x+2y)$

52. $-4x^3y^5-x^2y^3+12xy^4$

53. $3x^3 - 7x^2 + 6x - 14$

Answer

$(x^2 + 2)(3x - 7)$

54. $x^3 + x^2 - x - 1$

55. $x^2 + xy + 5x + 5y$

Answer

$(x + y)(x + 5)$

56. $5x^3 - 3x^2 + 5x - 3$

Writing Exercises

57. What does it mean to say a polynomial is in factored form?

Answer

Answers will vary.

58. How do you check result after factoring a polynomial?

59. The greatest common factor of 36 and 60 is 12. Explain what this means.

Answer

Answers will vary.

60. What is the GCF of y^4, y^5, and y^{10}? Write a general rule that tells you how to find the GCF of y^a, y^b, and y^c.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you

the help you need.

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2.3.3: Factor Trinomials

Learning Objectives

By the end of this section, you will be able to:

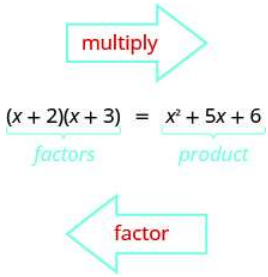
- Factor trinomials of the form x^2+bx+c
- Factor trinomials of the form ax^2+bx+c using trial and error
- Factor trinomials of the form ax^2+bx+c using the 'ac' method
- Factor using substitution

Before you get started, take this readiness quiz.

1. Find all the factors of 72.
   If you missed this problem, review [link].
2. Find the product: $(3y+4)(2y+5)$.
   If you missed this problem, review [link].
3. Simplify: $-9(6);\ -9(-6)$.
   If you missed this problem, review [link].

Factor Trinomials of the Form x^2+bx+c

You have already learned how to multiply binomials using **FOIL**. Now you'll need to "undo" this multiplication. To factor the trinomial means to start with the product, and end with the factors.



To figure out how we would factor a **trinomial** of the form x^2+bx+c, such as x^2+5x+6 and factor it to $(x+2)(x+3)$, let's start with two general binomials of the form $(x+m)$ and $(x+n)$.

	$(x+m)(x+n)$
Foil to find the product.	$x^2+mx+nx+mn$
Factor the GCF from the middle terms.	$x^2+(m+n)x+mn$
Our trinomial is of the form x^2+bx+c.	$\overbrace{x^2+(m+n)x+mn}^{x^2+bx+c}$

This tells us that to factor a trinomial of the form x^2+bx+c, we need two factors $(x+m)$ and $(x+n)$ where the two numbers m and n multiply to c and add to b.

Example 2.3.3.1: How to Factor a Trinomial of the form x^2+bx+c

Factor: $x^2+11x+24$.

Answer

Step 1. Write the factors as two binomials with first terms x.	Write two sets of parentheses and put x as the first term.	$x^2 + 11x + 24$ $(x \quad)(x \quad)$
Step 2. Find two numbers m and n that multiply to c, $m \cdot n = c$ add to b, $m + n = b$	Find two numbers that multiply to 24 and add to 11. **Factors of 24** / **Sum of factors** 1, 24 — $1 + 24 = 25$ 2, 12 — $2 + 12 = 14$ 3, 8 — $3 + 8 = 11$* 4, 6 — $4 + 6 = 10$	
Step 3. Use m and n as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$(x + 3)(x + 8)$
Step 4. Check by multiplying the factors.		$(x + 3)(x + 8)$ $x^2 + 8x + 3x + 24$ $x^2 + 11x + 24$ ✓

? Example 2.3.3.2

Factor: $q^2 + 10q + 24$.

Answer

$(q+4)(q+6)$

? Example 2.3.3.3

Factor: $t^2 + 14t + 24$.

Answer

$(t+2)(t+12)$

Let's summarize the steps we used to find the factors.

1. Write the factors as two binomials with first terms x. $x^2 + bx + c$ $(x \quad)(x \quad)$
2. Find two numbers m and n that
   - multiply to c, $m \cdot n = c$
   - add to b, $m + n = b$
3. Use m and n as the last terms of the factors. $(x+m)(x+n)$
4. Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

How do you get a *positive product* and a *negative sum*? We use two negative numbers.

? Example 2.3.3.4

Factor: $y^2 - 11y + 28$.

Answer

Again, with the positive last term, 28, and the negative middle term, $-11y$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

	$y^2-11y+28$
Write the factors as two binomials with first terms y.	$(y\quad)(y\quad)$
Find two numbers that: multiply to 28 and add to -11.	

Factors of 28	Sum of factors
$-1,\ -28$	$-1+(-28)=-29$
$-2,\ -14$	$-2+(-14)=-16$
$-4,\ -7$	$-4+(-7)=-11^*$

Use $-4,\ -7$ as the last terms of the binomials.	$(y-4)(y-7)$
Check:	
	$(y-4)(y-7)$
	$y^2-7y-4y+28$
	$y^2-11y+28\checkmark$

? Example 2.3.3.5

Factor: $u^2-9u+18$.

Answer

$(u-3)(u-6)$

? Example 2.3.3.6

Factor: $y^2-16y+63$.

Answer

$(y-7)(y-9)$

Now, what if the last term in the trinomial is negative? Think about **FOIL**. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

? Example 2.3.3.7

Factor: $2x+x^2-48$.

Answer

	$2x+x^2-48$
First we put the terms in decreasing degree order.	$x^2+2x-48$
Factors will be two binomials with first terms x.	$(x\quad)(x\quad)$

Factors of −48−48	Sum of factors

Factors of -48	Sum of factors
$-1,\ 48$	$-1+48=47$
$-2,\ 24$	$-2+24=22$
$-3,\ 16$	$-3+16=13$
$-4,\ 12$	$-4+12=8$
$-6,\ 8$	$-6+8=2^*$

Use $-6,\ 8$ as the last terms of the binomials. $(x-6)(x+8)$

Check:

$$(x-6)(x+8)$$
$$x^2-6q+8q-48$$
$$x^2+2x-48\checkmark$$

Example 2.3.3.8

Factor: $9m+m^2+18$.

Answer

$(m+3)(m+6)$

Example 2.3.3.9

Factor: $-7n+12+n^2$.

Answer

$(n-3)(n-4)$

Sometimes you'll need to factor trinomials of the form $x^2+bxy+cy^2$ with two variables, such as $x^2+12xy+36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x\cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain y. To get the coefficients b and c, you use the same process summarized in How To Factor trinomials.

Example 2.3.3.10

Factor: $r^2-8rs-9s^2$.

Answer

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$r^2-8rs-9s^2$$
$$(r\quad s)(r\quad s)$$

Note that the first terms are r, last terms contain s.

Find the numbers that multiply to -9 and add to -8.

Factors of -9	Sum of factors
$1,\ -9$	$-1+9=8$
$-1,\ 9$	$1+(-9)=-8^*$
$3,\ -3$	$3+(-3)=0$

Use $1,\ -9$ as coefficients of the last terms. $(r+s)(r-9s)$

Check:

$$(r-9s)(r+s)$$
$$r^2+rs-9rs-9s^2$$
$$r^2-8rs-9s^2 \checkmark$$

? Example 2.3.3.11

Factor: $a^2-11ab+10b^2$.

Answer

$(a-b)(a-10b)$

? Example 2.3.3.12

Factor: $m^2-13mn+12n^2$.

Answer

$(m-n)(m-12n)$

Some trinomials are prime. The only way to be certain a trinomial is **prime** is to list all the possibilities and show that none of them work.

? Example 2.3.3.13

Factor: $u^2-9uv-12v^2$.

Answer

We need u in the first term of each binomial and v in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$u^2-9uv-12v^2$$

Note that the first terms are u, last terms contain v. $(u\quad v)(u\quad v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	Sum of factors
$1, -12$	$1+(-12)=-11$
$-1, 12$	$-1+12=11$
$2, -6$	$2+(-6)=-4$
$-2, 6$	$-2+6=4$
$3, -4$	$3+(-4)=-1$
$-3, 4$	$-3+4=1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

? Example 2.3.3.14

Factor: $x^2-7xy-10y^2$.

Answer

prime

Example 2.3.3.15

Factor: $p^2+15pq+20q^2$.

Answer

prime

Let's summarize the method we just developed to factor trinomials of the form x^2+bx+c .

STRATEGY FOR FACTORING TRINOMIALS OF THE FORM x^2+bx+c

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

	x^2+bx+c	
	$(x+m)(x+n)$	
When c is positive, m and n have the same sign.		
b positive		b negative
m, n positive		m, n negative
x^2+5x+6		x^2-6x+8
$(x+2)(x+3)$		$(x-4)(x-2)$
same signs		same signs
When c is negative, m and n have the opposite sign.		
x^2+x-12		$x^2-2x-15$
$(x+4)(x-3)$		$(x-5)(x+3)$
opposite signs		opposite signs

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

Factor Trinomials of the form $ax^2 + bx + c$ using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form ax^2+bx+c .

Remember to always check for a **GCF** first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we've used so far. Let's do an example to see how this works.

ExAMPLE 2.3.3.16

Factor completely: $4x^3+16x^2-20x$.

Answer

Is there a greatest common factor? Yes, $GCF = 4x$. Factor it.	$4x^3 + 16x^2 - 20x$ $4x(x^2 + 4x - 5)$
Binomial, trinomial, or more than three terms? It is a trinomial. So "undo FOIL."	$4x(x \quad)(x \quad)$
Use a table like the one shown to find two numbers that multiply to -5 and add to 4.	$4x(x-1)(x+5)$

Factors of -5	Sum of factors
$-1, 5$ $1, -5$	$-1+5=4^*$ $1+(-5)=-4$

Check:

$$4x(x-1)(x+5)$$
$$4x(x^2+5x-x-5)$$
$$4x(x^2+4x-5)$$
$$4x^3+16x2-20x\checkmark$$

? ExAMPLE 2.3.3.17

Factor completely: $5x^3 + 15x^2 - 20x$.

Answer

$5x(x-1)(x+4)$

? ExAMPLE 2.3.3.18

Factor completely: $6y^3 + 18y^2 - 60y$.

Answer

$6y(y-2)(y+5)$

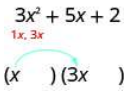
What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2 + 5x + 2$.

From our earlier work, we expect this will factor into two binomials.

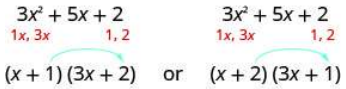
$$3x^2 + 5x + 2$$
$$(\quad)(\quad)$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x$, $3x$. We can place them in the binomials.

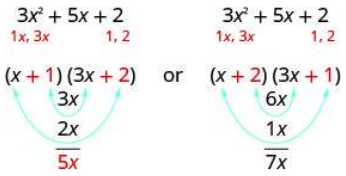


Check: Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1 and 2. But we now have two cases to consider as it will make a difference if we write 1, 2 or 2, 1.



Which factors are correct? To decide that, we multiply the inner and outer terms.



Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let's use FOIL to check.

$$(x+1)(3x+2)$$

$$3x^2+2x+3x+2$$

$$3x^2+5x+2\checkmark$$

Our result of the factoring is:

$$3x^2+5x+2$$

$$(x+1)(3x+2)$$

? ExAMPLE 2.3.3.19: How to Factor a Trinomial Using Trial and Error

Factor completely using trial and error: $3y^2+22y+7$.

Answer

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$
Step 2. Factor any GCF.	There is no GCF.	
Step 3. Find all the factor pairs of the first term.	The only of $3y^2$ are $1y$, $3y$. Since there is only one pair, we can put them in the parentheses.	$3y^2 + 22y + 7$ 1y, 3y $3y^2 + 22y + 7$ 1y, 3y $(y\ )(3y\ )$
Step 4. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2 + 22y + 7$ 1y, 3y 1, 7 $(y\ )(3y\ )$

Step 5. Test all the possible combinations of the factors until the correct product is found.	$3y^2 + 22y + 7$ 1y, 3y 1, 7 $(y + 1)(3y + 7)$ 3y, 7y 10y No! We need 22y $3y^2 + 22y + 7$ 1y, 3y 1, 7 $(y + 7)(3y + 1)$ 21y, +y 22y	$3y^2 + 22y + 7$ **Possible factors** / **Product** $(y + 1)(3y + 7)$ / $3y^2 + 10y + 7$ $(y + 7)(3y + 1)$ / $3y^2 + 22y + 7$
Step 6. Check by multiplying.		$(y + 7)(3y + 1)$ $3y^2 + 22y + 7$ ✓

? ExAMPLE 2.3.3.20

Factor completely using trial and error: $2a^2+5a+3$.

Answer

$(a+1)(2a+3)$

? ExAMPLE 2.3.3.21

Factor completely using trial and error: $4b^2+5b+1$.

Answer

$(b+1)(4b+1)$

FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING TRIAL AND ERROR.

1. Write the trinomial in descending order of degrees as needed.
2. Factor any GCF.
3. Find all the factor pairs of the first term.
4. Find all the factor pairs of the third term.
5. Test all the possible combinations of the factors until the correct product is found.
6. Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

? ExAMPLE 2.3.3.22

Factor completely using trial and error: $6b^2-13b+5$.

Answer

The trinomial is already in descending order.	$6b^2 - 13b + 5$
Find the factors of the first term.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $2b \cdot 3b$

Find the factors of the last term. Consider the signs. Since the last term, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $2b \cdot 3b$ $-1, -5$

Consider all the combinations of factors.

$6b^2 - 13b + 5$

Possible factors	Product
$(b-1)(6b-5)$	$6b^2 - 11b + 5$
$(b-5)(6b-1)$	$6b^2 - 31b + 5$
$(2b-1)(3b-5)$	$6b^2 - 13b + 5^*$
$(2b-5)(3b-1)$	$6b^2 - 17b + 5$

The correct factors are those whose product is the original trinomial. $(2b-1)(3b-5)$

Check by multiplying:

$$(2b-1)(3b-5)$$
$$6b^2 - 10b - 3b + 5$$
$$6b^2 - 13b + 5 \checkmark$$

? ExAMPLE 2.3.3.23

Factor completely using trial and error: $8x^2 - 14x + 3$.

Answer

$(2x-3)(4x-1)$

? ExAMPLE 2.3.3.24

Factor completely using trial and error: $10y^2 - 37y + 7$.

Answer

$(2y-7)(5y-1)$

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

? ExAMPLE 2.3.3.25

Factor completely using trial and error: $18x^2 - 37xy + 15y^2$.

Answer

The trinomial is already in descending order.	$18x^2 - 37xy + 15y^2$
Find the factors of the first term.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $2x \cdot 9x$ $3x \cdot 6x$

Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $\quad -1, -15$ $2x \cdot 9x$ $\quad -3, -5$ $3x \cdot 6x$

Consider all the combinations of factors.

$18x^2 - 37xy + 15y^2$	
Possible factors	**Product**
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^2$*
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. $(2x - 3y)(9x - 5y)$

Check by multiplying:

$$(2x - 3y)(9x - 5y)$$
$$18x^2 - 10xy - 27xy + 15y^2$$
$$18x^2 - 37xy + 15y^2 \checkmark$$

? ExAMPLE 2.3.3.26

Factor completely using trial and error $18x^2 - 3xy - 10y^2$.

Answer

$(3x + 2y)(6x - 5y)$

? ExAMPLE 2.3.3.27

Factor completely using trial and error: $30x^2 - 53xy - 21y^2$.

Answer

$(3x + y)(10x - 21y)$

Don't forget to look for a GCF first and remember if the leading coefficient is negative, so is the GCF.

? ExAMPLE 2.3.3.28

Factor completely using trial and error: $-10y^4 - 55y^3 - 60y^2$.

Answer

	$-10y^4 - 55y^3 - 60y^2$
Notice the greatest common factor, so factor it first.	$-5y^2(2y^2 + 11y + 12)$
Factor the trinomial.	$-5y^2\left(\begin{matrix} 2y^2 + 11y + 12 \\ y \cdot 2y \qquad\quad 1 \cdot 12 \\ \qquad\quad\;\; 2 \cdot 6 \\ \qquad\quad\;\; 3 \cdot 4 \end{matrix}\right)$

Consider all the combinations.

$2y^2 + 11y + 12$	
Possible factors	**Product**
$(y + 1)(2y + 12)$	Not an option
$(y + 12)(2y + 1)$	$2y^2 + 25y + 12$
$(y + 2)(2y + 6)$	Not an option
$(y + 6)(2y + 2)$	Not an option
$(y + 3)(2y + 4)$	Not an option
$(y + 4)(2y + 3)$	$2y^2 + 11y + 12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor -5^y2.

$$-5y^2(y+4)(2y+3)$$

Check by multiplying:

$$-5y^2(y+4)(2y+3)$$
$$-5y^2(2y^2+8y+3y+12)$$
$$-10y^4 - 55y^3 - 60y^2 \checkmark$$

? ExAMPLE 2.3.3.29

Factor completely using trial and error: $15n^3 - 85n^2 + 100n$.

Answer

$5n(n-4)(3n-5)$

? ExAMPLE 2.3.3.30

Factor completely using trial and error: $56q^3 + 320q^2 - 96q$.

Answer

$8q(q+6)(7q-2)$

Factor Trinomials of the Form $ax^2 + bx + c$ using the "ac" Method

Another way to factor trinomials of the form $ax^2 + bx + c$ is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

Example 2.3.3.31: How to Factor Trinomials using the “ac” Method

Factor using the “ac” method: $6x^2+7x+2$.

Answer

Step 1. Factor any GCF.	Is there a greatest common factor? No!	$6x^2+7x+2$
Step 2. Find the product ac.	$a \cdot c$ $6 \cdot 2$ 12	ax^2+bx+c $6x^2+7x+2$
Step 3. Find two numbers m and n that: Multiply to ac. $m \cdot n = a \cdot c$ Add to b. $m+n=b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3+4=7$	
Step 4. Split the middle term using m and n. ax^2+bx+c bx $ax^2+mx+nx+c$	Rewrite $7x$ as $3x+4x$. It would also give the same result if we used $4x+3x$. Notice that $6x^2+3x+4x+2$ is equal to $6x^2+7x+2$. We just split the middle term to get a more useful form.	$6x^2+7x+2$ $6x^2+3x+4x+2$
Step 5. Factor by grouping.		$3x(2x+1)+2(2x+1)$ $(2x+1)(3x+2)$
Step 6. Check by multiplying the factors.		$(2x+1)(3x+2)$ $6x^2+4x+3x+2$ $6x^2+7x+2$ ✓

Example 2.3.3.32

Factor using the “ac” method: $6x^2+13x+2$.

Answer

$(x+2)(6x+1)$

Example 2.3.3.33

Factor using the “ac” method: $4y^2+8y+3$.

Answer

$(2y+1)(2y+3)$

The “ac” method is summarized here.

FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING THE "ac" METHOD.

1. Factor any GCF.
2. Find the product ac.
3. Find two numbers m and n that:
   Multiply to ac $m \cdot n = a \cdot c$
   Add to b $m + n = b$
   $ax^2 + bx + c$
4. Split the middle term using m and n. $ax^2 + mx + nx + c$
5. Factor by grouping.
6. Check by multiplying the factors.

Don't forget to look for a common factor!

Example 2.3.3.34

Factor using the "'ac" method: $10y^2 - 55y + 70$.

Answer

Is there a greatest common factor?		
Yes. The GCF is 5.		$10y^2 - 55y + 70$
Factor it.		$5(2y^2 - 11y + 14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.		$ax^2 + bx + c$ $5(2y^2 - 11y + 14)$
Find the product ac.	$ac = 28$	
Find two numbers that multiply to ac	$(-4)(-7) = 28$	
and add to b.	$-4(-7) = -11$	
Split the middle term.		$5(2y^2 - 11y + 14)$
		$5(2y^2 - 7y - 4y + 14)$
Factor the trinomial by grouping.		
		$5(y - 2)(2y - 7)$
Check by multiplying all three factors. $5(y-2)(2y-7)$ $5(2y^2 - 7y - 4y + 14)$ $5(2y^2 - 11y + 14)$ $10y^2 - 55y + 70$✓		

Example 2.3.3.35

Factor using the "ac" method: $16x^2 - 32x + 12$.

Answer

$4(2x - 3)(2x - 1)$

Example 2.3.3.36

Factor using the "ac" method: $18w^2 - 39w + 18$.

Answer

$3(3w - 2)(2w - 3)$

Factor Using Substitution

Sometimes a trinomial does not appear to be in the $ax^2 + bx + c$ form. However, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c$ form. This is called **factoring by substitution**. It is standard to use u for the substitution.

In the $ax^2 + bx + c$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

ExAMPLE 2.3.3.37

Factor by substitution: $x^4 - 4x^2 - 5$.

Answer

The variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $(x^2)^2 = x^4$). If we let $u = x^2$, we can put our trinomial in the $ax^2 + bx + c$ form we need to factor it.

	$x^4 - 4x^2 - 5$
Rewrite the trinomial to prepare for the substitution.	$(x^2)^2 - 4(x^2) - 5$
Let $u = x^2$ and substitute.	$(u)^2 - 4(u) - 5$
Factor the trinomial.	$(u + 1)(u - 5)$
Replace u with x^2.	$(x^2 + 1)(x^2 - 5)$
Check: $(x^2 + 1)(x^2 - 5)$ $x^4 - 5x^2 + x^2 - 5$ $x^4 - 4x^2 - 5$✓	

ExAMPLE 2.3.3.38

Factor by substitution: $h^4 + 4h^2 - 12$.

Answer

$(h^2 - 2)(h^2 + 6)$

ExAMPLE 2.3.3.39

Factor by substitution: $y^4 - y^2 - 20$.

Answer

$(y^2 + 4)(y^2 - 5)$

Sometimes the expression to be substituted is not a monomial.

ExAMPLE 2.3.3.40

Factor by substitution: $(x-2)^2+7(x-2)+12$

Answer

The binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

	$(x-2)^2+7(x-2)+12$
Rewrite the trinomial to prepare for the substitution.	$(x-2)^2+7(x-2)+12$
Let $u=x-2$ and substitute.	$u^2+7u+12$
Factor the trinomial.	$(u+3)(u+4)$
Replace u with $x-2$.	$((x-2)+3)((x-2)+4)$
Simplify inside the parentheses.	$(x+1)(x+2)$

This could also be factored by first multiplying out the $(x-2)^2$ and the $7(x-2)$ and then combining like terms and then factoring. Most students prefer the substitution method.

ExAMPLE 2.3.3.41

Factor by substitution: $(x-5)^2+6(x-5)+8$.

Answer

$(x-3)(x-1)$

ExAMPLE 2.3.3.42

Factor by substitution: $(y-4)^2+8(y-4)+15$.

Answer

$(y-1)(y+1)$

Watch this video for additional instruction and practice with factoring.



Key Concepts

- **How to factor trinomials of the form x^2+bx+c.**
  1. Write the factors as two binomials with first terms x. $l)x^2+bx+c$ $(x\quad)(x\quad)$
  2. Find two numbers m and n that
     multiply to $c,\ m\cdot n=c$
     add to $b,\ m+n=b$
  3. Use m and n as the last terms of the factors. $(x+m)(x+n)$
  4. Check by multiplying the factors.
- **Strategy for Factoring Trinomials of the Form** x^2+bx+c : When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

  For trinomials of the form: $x^2+bx+c=(x+m)(x+n)$

  When c is positive, m and n must have the same sign (and this will be the sign of b).

  Examples: $x^2+5x+6=(x+2)(x+3)$, $x^2-6x+8=(x-4)(x-2)$

  When c is negative, m and n have opposite signs. The larger of m and n will have the sign of b.

  Examples: $x^2+x-12=(x+4)(x-3)$, $x^2-2x-15=(x-5)(x+3)$

  Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.
- **How to factor trinomials of the form ax^2+bx+c using trial and error.**
  1. Write the trinomial in descending order of degrees as needed.
  2. Factor any GCF.
  3. Find all the factor pairs of the first term.
  4. Find all the factor pairs of the third term.
  5. Test all the possible combinations of the factors until the correct product is found.
  6. Check by multiplying.
- **How to factor trinomials of the form ax^2+bx+c using the "ac" method.**

1. Factor any GCF.
2. Find the product ac.
3. Find two numbers m and n that:
   Multiply to ac. $m \cdot n = a \cdot c$
   Add to b. $m + n = b$
   $ax^2 + bx + c$
4. Split the middle term using m and n. $ax^2 + mx + nx + c$
5. Factor by grouping.
6. Check by multiplying the factors.

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2.3.3E: Exercises

Practice Makes Perfect

Factor Trinomials of the Form x^2+bx+c

In the following exercises, factor each trinomial of the form x^2+bx+c .

1. $p^2+11p+30$

Answer

$(p+5)(p+6)$

2. $w^2+10w+21$

3. $n^2+19n+48$

Answer

$(n+3)(n+16)$

4. $b^2+14b+48$

5. $a^2+25a+100$

Answer

$(a+5)(a+20)$

6. $u^2+101u+100$

7. $x^2-8x+12$

Answer

$(x-2)(x-6)$

8. $q^2-13q+36$

9. $y^2-18y+45$

Answer

$(y-3)(y-15)$

10. $m^2-13m+30$

11. x^2-8x+7

Answer

$(x-1)(x-7)$

12. y^2-5y+6

13. $5p-6+p^2$

Answer

$(p-1)(p+6)$

14. $6n-7+n^2$

15. $8-6x+x^2$

Answer

$(x-4)(x-2)$

16. $7x+x^2+6$

17. $x^2-12-11x$

Answer

$(x-12)(x+1)$

18. $-11-10x+x^2$

In the following exercises, factor each trinomial of the form $x^2+bxy+cy^2$.

19. $x^2-2xy-80y^2$

Answer

$(x+8y)(x-10y)$

20. $p^2-8pq-65q^2$

21. $m^2-64mn-65n^2$

Answer

$(m+n)(m-65n)$

22. $p^2-2pq-35q^2$

23. $a^2+5ab-24b^2$

Answer

$(a+8b)(a-3b)$

24. $r^2+3rs-28s^2$

25. $x^2-3xy-14y^2$

Answer

Prime

26. $u^2-8uv-24v^2$

27. $m^2 - 5mn + 30n^2$

Answer

Prime

28. $c^2 - 7cd + 18d^2$

Factor Trinomials of the Form $ax^2 + bx + c$ Using Trial and Error

In the following exercises, factor completely using trial and error.

29. $p^3 - 8p^2 - 20p$

Answer

$p(p-10)(p+2)$

30. $q^3 - 5q^2 - 24q$

31. $3m^3 - 21m^2 + 30m$

Answer

$3m(m-5)(m-2)$

32. $11n^3 - 55n^2 + 44n$

33. $5x^4 + 10x^3 - 75x^2$

Answer

$5x^2(x-3)(x+5)$

34. $6y^4 + 12y^3 - 48y^2$

35. $2t^2 + 7t + 5$

Answer

$(2t+5)(t+1)$

36. $5y^2 + 16y + 11$

37. $11x^2 + 34x + 3$

Answer

$(11x+1)(x+3)$

38. $7b^2 + 50b + 7$

39. $4w^2 - 5w + 1$

Answer

$(4w-1)(w-1)$

40. $5x^2 - 17x + 6$

41. $4q^2 - 7q - 2$

Answer

$(4q+1)(q-2)$

42. $10y^2 - 53y - 111$

43. $6p^2 - 19pq + 10q^2$

Answer

$(2p-5q)(3p-2q)$

44. $21m^2 - 29mn + 10n^2$

45. $4a^2 + 17ab - 15b^2$

Answer

$(4a-3b)(a+5b)$

46. $6u^2 + 5uv - 14v^2$

47. $-16x^2 - 32x - 16$

Answer

$-16(x+1)(x+1)$

48. $-81a^2 + 153a + 18$

49. $-30q^3 - 140q^2 - 80q$

Answer

$-10q(3q+2)(q+4)$

50. $-5y^3 - 30y^2 + 35y$

Factor Trinomials of the Form $ax^2 + bx + c$ using the 'ac' Method

In the following exercises, factor using the 'ac' method.

51. $5n^2 + 21n + 4$

Answer

$(5n+1)(n+4)$

52. $8w^2 + 25w + 3$

53. $4k^2 - 16k + 15$

Answer

$(2k-3)(2k-5)$

54. $5s^2-9s+4$

55. $6y^2+y-15$

Answer

$(3y+5)(2y-3)$

56. $6p^2+p-22$

57. $2n^2-27n-45$

Answer

$(2n+3)(n-15)$

58. $12z^2-41z-11$

59. $60y^2+290y-50$

Answer

$10(6y-1)(y+5)$

60. $6u^2-46u-16$

61. $48z^3-102z^2-45z$

Answer

$3z(8z+3)(2z-5)$

62. $90n^3+42n^2-216n$

63. $16s^2+40s+24$

Answer

$8(2s+3)(s+1)$

64. $24p^2+160p+96$

65. $48y^2+12y-36$

Answer

$12(4y-3)(y+1)$

66. $30x^2+105x-60$

Factor Using Substitution

In the following exercises, factor using substitution.

67. $x^4 - x^2 - 12$

Answer

$(x^2 + 3)(x^2 - 4)$

68. $x^4 + 2x^2 - 8$

69. $x^4 - 3x^2 - 28$

Answer

$(x^2 - 7)(x^2 + 4)$

70. $x^4 - 13x^2 - 30$

71. $(x - 3)^2 - 5(x - 3) - 36$

Answer

$(x - 12)(x + 1)$

72. $(x - 2)^2 - 3(x - 2) - 54$

73. $(3y - 2)^2 - (3y - 2) - 2$

Answer

$(3y - 4)(3y - 1)$

74. $(5y - 1)^2 - 3(5y - 1) - 18$

Mixed Practice

In the following exercises, factor each expression using any method.

75. $u^2 - 12u + 36$

Answer

$(u - 6)(u - 6)$

76. $x^2 - 14x - 32$

77. $r^2 - 20rs + 64s^2$

Answer

$(r - 4s)(r - 16s)$

78. $q^2 - 29qr - 96r^2$

79. $12y^2 - 29y + 14$

Answer

$(4y - 7)(3y - 2)$

80. $12x^2+36y-24z$

81. $6n^2+5n-4$

Answer

$(2n-1)(3n+4)$

82. $3q^2+6q+2$

83. $13z^2+39z-26$

Answer

$13(z^2+3z-2)$

84. $5r^2+25r+30$

85. $3p^2+21p$

Answer

$3p(p+7)$

86. $7x^2-21x$

87. $6r^2+30r+36$

Answer

$6(r+2)(r+3)$

88. $18m^2+15m+3$

89. $24n^2+20n+4$

Answer

$4(2n+1)(3n+1)$

90. $4a^2+5a+2$

91. x^4-4x^2-12

Answer

$(x^2+2)(x^2-6)$

92. x^4-7x^2-8

93. $(x+3)^2-9(x+3)-36$

Answer

$(x-9)(x+6)$

94. $(x+2)^2-25(x+2)-54$

Writing Exercises

95. Many trinomials of the form x^2+bx+c factor into the product of two binomials $(x+m)(x+n)$. Explain how you find the values of m and n.

Answer

Answers will vary.

96. Tommy factored x^2-x-20 as $(x+5)(x-4)$. Sara factored it as $(x+4)(x-5)$. Ernesto factored it as $(x-5)(x-4)$. Who is correct? Explain why the other two are wrong.

97. List, in order, all the steps you take when using the "ac" method to factor a trinomial of the form ax^2+bx+c.

Answer

Answers will vary.

98. How is the "ac" method similar to the "undo FOIL" method? How is it different?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor trinomials of the form x^2+bx+c			
factor trinomials of the form ax^2+bx+c using trial and error.			
factor trinomials of the form ax^2+bx+c with using the "ac" method.			
factor using substitution.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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2.3.4: Factor Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

Before you get started, take this readiness quiz.

1. Simplify: $(3x^2)^3$.
2. Multiply: $(m+4)^2$.
3. Multiply: $(x-3)(x+3)$.

We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$(3x+4)^2$$

$$(3x)^2+2(3x\cdot 4)+4^2$$

$$9x^2+24x+16$$

The trinomial $9x^2+24x+16$ is called a *perfect square trinomial*. It is the square of the binomial $3x+4$.

In this chapter, you will start with a perfect square trinomial and factor it into its **prime** factors. You could factor this **trinomial** using the methods described in the last section, since it is of the form ax^2+bx+c. But if you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work. Here is the pattern—the reverse of the binomial squares pattern.

PERFECT SQUARE TRINOMIALS PATTERN

If a and b are real numbers

$$a^2+2ab+b^2=(a+b)^2$$

$$a^2-2ab+b^2=(a-b)^2$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it the product, $2ab$? If everything checks, you can easily write the factors.

Example 2.3.4.1: How to Factor Perfect Square Trinomials

Factor: $9x^2+12x+4$.

Answer

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$? • Is the first term a perfect square? Write it as a square, a^2. • Is the last term a perfect square? Write it as a square, b^2. • Check the middle term. Is it $2ab$?	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$. Is 4 a perfect square? Yes—write it as $(2)^2$. Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$9x^2 + 12x + 4$ $(3x)^2$ $(3x)^2$ $(2)^2$ $2(3x)(2)$ $12x$
Step 2. Write the square of the binomial.		$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a + b)^2$ $(3x + 2)^2$
Step 3. Check.		$(3x + 2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓

? Example 2.3.4.2

Factor: $4x^2 + 12x + 9$.

Answer

$(2x + 3)^2$

? Example 2.3.4.3

Factor: $9y^2 + 24y + 16$.

Answer

$(3y + 4)^2$

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

FACTOR PERFECT SQUARE TRINOMIALS

Step 1.	Does the trinomial fit the pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
	Are the first and last terms perfect squares?		
	Write them as squares.	$(a)^2$ $(b)^2$	$(a)^2$ $(b)^2$
	Check the middle term. Is it $2ab$?	$2 \cdot a \cdot b$	$2 \cdot a \cdot b$
Step 2.	Write the square of the binomial.	$(a + b)^2$	$(a - b)^2$
Step 3.	Check by multiplying.		

We'll work one now where the middle term is negative.

Example 2.3.4.4

Factor: $81y^2 - 72y + 16$.

Answer

The first and last terms are squares. See if the middle term fits the pattern of a **perfect square** trinomial. The middle term is negative, so the binomial square would be $(a-b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2 \qquad (4)^2$
Check the middle term.	$(9y)^2 \qquad (4)^2$ $2(9y)(4)$ $72y$
Does it match $(a-b)^2$? Yes.	$a^2 \quad -2 \quad a \quad b + b^2$ $(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$
Write as the square of a binomial.	$(9y-4)^2$
Check by multiplying:	$(9y-4)^2$ $(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$ $81y^2 - 72y + 16$✓

Example 2.3.4.5

Factor: $64y^2 - 80y + 25$.

Answer

$(8y-5)^2$

Example 2.3.4.6

Factor: $16z^2 - 72z + 81$.

Answer

$(4z-9)^2$

The next example will be a perfect square trinomial with two variables.

Example 2.3.4.7

Factor: $36x^2 + 84xy + 49y^2$.

Answer

	$36x^2 + 84xy + 49y^2$
Test each term to verify the pattern.	$a^2 \quad +2 \quad a \quad b + \quad b^2$ $(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$
Factor.	$(6x+7y)^2$

Check by multiplying. $(6x+7y)^2$ $(6x)^2+2\cdot 6x\cdot 7y+(7y)^2$ $36x^2+84xy+49y^2$ ✓	

Example 2.3.4.8

Factor: $49x^2+84xy+36y^2$.

Answer

$(7x+6y)^2$

Example 2.3.4.9

Factor: $64m^2+112mn+49n^2$.

Answer

$(8m+7n)^2$

Remember the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a **GCF** in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

Example 2.3.4.10

Factor: $100x^2y-80xy+16y$.

Answer

	$100x^2y-80xy+16y$
Is there a GCF? Yes, $4y$, so factor it out.	$4y\left(25x^2-20x+4\right)$
Is this a perfect square trinomial?	
Verify the pattern.	$4y[(5x)^2-2\cdot 5x\cdot 2+2^2]$
Factor.	$4y(5x-2)^2$
Remember: Keep the factor $4y$ in the final product.	
Check: $4y(5x-2)^2$ $4y[(5x)2-2\cdot 5x\cdot 2+22]$ $4y(25x2-20x+4)$ 100x2y−80xy+16y\checkmark\]	

Example 2.3.4.11

Factor: $8x^2y - 24xy + 18y$.

Answer

$2y(2x-3)^2$

Example 2.3.4.12

Factor: $27p^2q + 90pq + 75q$.

Answer

$3q(3p+5)^2$

Factor Differences of Squares

The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here's an example:

$$\begin{array}{c} (a-b)(a+b) \\ (3x-4)(3x+4) \\ (a)^2-(b)^2 \\ (3x)^2-(4)^2 \\ 9x^2-16 \end{array}$$

A difference of squares factors to a product of conjugates.

DIFFERENCE OF SQUARES PATTERN

If a and b are real numbers,

$$a^2-b^2=(a-b)(a+b)$$

difference: a^2 (squares) $-$ b^2 (squares); $(a-b)(a+b)$ (conjugates)

Remember, "difference" refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

Example 2.3.4.13: How to Factor a Trinomial Using the Difference of Squares

Factor: $64y^2-1$.

Answer

Step 1. Does the binomial fit the pattern?		$64y^2-1$
• Is this a difference?	Yes	$64y^2-1$
• Are the first and last terms perfect squares?	Yes	
Step 2. Write them as squares.	Write them as x^2 and 2^2.	a^2-b^2 $(8y)^2-1^2$

Step 3. Write the product of conjugates.		$(a-b)(a+b)$ $(8y-1)(8y+1)$
Step 4. Check.		$(8y-1)(8y+1)$ $64y^2-1$ ✓

? Example 2.3.4.14

Factor: $121m^2-1$.

Answer

$(11m-1)(11m+1)$

? Example 2.3.4.15

Factor: $81y^2-1$.

Answer

$(9y-1)(9y+1)$

FACTOR DIFFERENCES OF SQUARES.

Step 1.	Does the binomial fit the pattern?	a^2-b^2
	Is this a difference?	____−____
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2-(b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression a^2+b^2 is prime!

The next example shows variables in both terms.

? Example 2.3.4.16

Factor: $144x^2-49y^2$.

Answer

	$144x^2-49y^2$
Is this a difference of squares? Yes.	$(12x)^2-(7y)^2$
Factor as the product of conjugates.	$(12x-7y)(12x+7y)$
Check by multiplying.	$(12x-7y)(12x+7y)$

Check by multiplying.

$(12x-7y)(12x+7y)$

$144x^2-49y^2$ ✓

Example 2.3.4.17

Factor: $196m^2 - 25n^2$.

Answer

$(14m - 5n)(14m + 5n)$

Example 2.3.4.18

Factor: $121p^2 - 9q^2$.

Answer

$(11p - 3q)(11p + 3q)$

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may "disguise" the difference of squares and you won't recognize the perfect squares until you factor the GCF.

Also, to completely factor the binomial in the next example, we'll factor a difference of squares twice!

Example 2.3.4.19

Factor: $48x^4y^2 - 243y^2$.

Answer

	$48x^4y^2 - 243y^2$
Is there a GCF? Yes, $3y^2$—factor it out!	$3y^2(16x^4 - 81)$
Is the binomial a difference of squares? Yes.	$3y^2\left((4x^2)^2 - (9)^2\right)$
Factor as a product of conjugates.	$3y^2(4x^2 - 9)(4x^2 + 9)$
Notice the first binomial is also a difference of squares!	$3y^2((2x)^2 - (3)^2)(4x^2 + 9)$
Factor it as the product of conjugates.	$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

The last factor, the sum of squares, cannot be factored.

Check by multiplying:

$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

$3y^2(4x^2 - 9)(4x^2 + 9)$

$3y^2(16x^4 - 81)$

$48x^4y^2 - 243y^2$ ✓

Example 2.3.4.20

Factor: $2x^4y^2 - 32y^2$.

Answer

$2y^2(x - 2)(x + 2)(x^2 + 4)$

Example 2.3.4.21

Factor: $7a^4c^2 - 7b^4c^2$.

Answer

$7c^2(a-b)(a+b)(a^2+b^2)$

The next example has a polynomial with 4 terms. So far, when this occurred we grouped the terms in twos and factored from there. Here we will notice that the first three terms form a perfect square trinomial.

Example 2.3.4.22

Factor: $x^2 - 6x + 9 - y^2$.

Answer

Notice that the first three terms form a perfect square trinomial.

	$x^2 - 6x + 9 - y^2$
Factor by grouping the first three terms.	$\underbrace{x^2 - 6x + 9}\ - y^2$
Use the perfect square trinomial pattern.	$(x-3)^2 - y^2$
Is this a difference of squares? Yes.	
Yes—write them as squares.	$\overset{a^2}{(x-3)^2} - \overset{b^2}{y^2}$
Factor as the product of conjugates.	$((x-3)-y)((x-3)+y)$
	$(x-3-y)(x-3+y)$

You may want to rewrite the solution as $(x-y-3)(x+y-3)$.

Example 2.3.4.23

Factor: $x^2 - 10x + 25 - y^2$.

Answer

$(x-5-y)(x-5+y)$

Example 2.3.4.24

Factor: $x^2 + 6x + 9 - 4y^2$.

Answer

$(x+3-2y)(x+3+2y)$

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

We'll check the first pattern and leave the second to you.

	$(a+b)\left(a^2-ab+b^2\right)$
Distribute.	$a\left(a^2-ab+b^2\right)+b\left(a^2-ab+b^2\right)$
Multiply.	$a^3-a^2b+ab^2+a^2b-ab^2+b^3$
Combine like terms.	a^3+b^3

SUM AND DIFFERENCE OF CUBES PATTERN

$$a^3+b^3=(a+b)(a^2-ab+b^2$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
same sign — opposite signs

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
same sign — opposite signs

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in Table.

n	1	2	3	4	5	6	7	8	9	10
n^3	1	8	27	64	125	216	343	512	729	1000

Example 2.3.4.25: How to Factor the Sum or Difference of Cubes

Factor: x^3+64.

Answer

Step 1. Does the binomial fit the sum or difference of cubes pattern? • Is it a sum or difference? • Are the first and last terms perfect cubes?	This is a sum. Yes.	$x^3 + 64$ $x^3 + 64$
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3.	$a^3 + b^3$ $x^3 + 4^3$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$(a + b)(a^2 - ab + b^2)$ $(x + 4)(x^2 - 4x + 4^2)$

Step 4. Simplify inside the parentheses.	It is already simplified.	$(x+4)(x^2-4x+16)$
Step 5. Check by multiplying the factors.		$\begin{array}{r} x^2-4x+16 \\ \times\quad x+4 \\ \hline 4x^2-16x+64 \checkmark \\ x^3-4x^2+16x \\ \hline x^3 \qquad\quad +64 \end{array}$

Example 2.3.4.26

Factor: x^3+27.

Answer

$(x+3)(x^2-3x+9)$

Example 2.3.4.27

Factor: y^3+8.

Answer

$(y+2)(y^2-2y+4)$

FACTOR THE SUM OR DIFFERENCE OF CUBES.

1. Does the binomial fit the sum or difference of cubes pattern?
   Is it a sum or difference?
   Are the first and last terms perfect cubes?
2. Write them as cubes.
3. Use either the sum or difference of cubes pattern.
4. Simplify inside the parentheses.
5. Check by multiplying the factors.

Example 2.3.4.28

Factor: $27u^3-125v^3$.

Answer

	$27u^3-125v^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(3u)^3}-\overset{b^3}{(5v)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{3u}-\overset{b}{5v}\right)\left(\overset{a^2}{(3u)^2}+\overset{ab}{3u\cdot 5v}+\overset{b^2}{(5v)^2}\right)$
Simplify.	$\left(\overset{a}{3u}-\overset{b}{5v}\right)\left(\overset{a^2}{9u^2}+\overset{ab}{15uv}+\overset{b^2}{25v^2}\right)$
Check by multiplying.	We'll leave the check to you.

Example 2.3.4.29

Factor: $8x^3 - 27y^3$.

Answer

$(2x - 3y)(4x^2 + 6xy + 9y^2)$

Example 2.3.4.30

Factor: $1000m^3 - 125n^3$.

Answer

$(10m - 5n)(100m^2 + 50mn + 25n^2)$

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

Example 2.3.4.31

Factor: $6x^3y + 48y^4$.

Answer

	$6x^3y + 48y^4$
Factor the common factor.	$6y\left(x^3 + 8y^3\right)$
This binomial is a sum The first and last terms are perfect cubes.	
Write the terms as cubes.	$6y\left(x^3 + (2y)^3\right)$
Use the sum of cubes pattern.	$6y\left(x + 2y\right)\left(x^2 - x \cdot 2y + (2y)^2\right)$
Simplify.	$6y(x + 2y)(x^2 - 2xy + 4y^2)$

Check:

To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by 6y.6y. We'll leave the multiplication for you.

Example 2.3.4.32

Factor: $500p^3 + 4q^3$.

Answer

$4(5p + q)(25p^2 - 5pq + q^2)$

Example 2.3.4.33

Factor: $432c^3 + 686d^3$.

Answer

$2(6c + 7d)(36c^2 - 42cd + 49d^2)$

The first term in the next example is a binomial cubed.

Example 2.3.4.34

Factor: $(x+5)^3-64x^3$.

Answer

	$(x+5)^3-64x^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(x+5)^3}-\overset{b^3}{(4x)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{(x+5)}-\overset{b}{4x}\right)\left(\overset{a^2}{(x+5)^2}+\overset{a}{(x+5)}\cdot\overset{b}{4x}+\overset{b^2}{(4x)^2}\right)$
Simplify.	$(x+5-4x)\left(x^2+10x+25+4x^2+20x+16x^2\right)$
	$(-3x+5)\left(21x^2+30x+25\right)$
Check by multiplying.	We'll leave the check to you.

Example 2.3.4.35

Factor: $(y+1)^3-27y^3$.

Answer

$(-2y+1)(13y^2+5y+1)$

Example 2.3.4.36

Factor: $(n+3)^3-125n^3$.

Answer

$(-4n+3)(31n^2+21n+9)$

Access this online resource for additional instruction and practice with factoring special products.

- Factoring Binomials-Cubes #2

Key Concepts

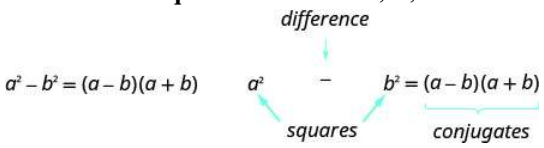
- **Perfect Square Trinomials Pattern:** If *a* and *b* are real numbers,

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

- **How to factor perfect square trinomials.**

Step 1.	Does the trinomial fit the pattern?	$a^2+2ab+b^2$	$a^2-2ab+b^2$
	Are the first and last terms perfect squares?		
	Write them as squares.	$(a)^2 \quad (b)^2$	$(a)^2 \quad (b)^2$
	Check the middle term. Is it $2ab$?	$\searrow 2\cdot a\cdot b \swarrow$	$\searrow 2\cdot a\cdot b \swarrow$
Step 2.	Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3.	Check by multiplying.		

- **Difference of Squares Pattern:** If a,ba,b are real numbers,



- **How to factor differences of squares.**

Step 1.	Does the binomial fit the pattern?	$a^2 - b^2$
	Is this a difference?	____−____
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2 - (b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

- **Sum and Difference of Cubes Pattern**

  $a^3 + b3 = (a+b)(a^2 - ab + b^2)$

  $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

- **How to factor the sum or difference of cubes.**

  1. Does the binomial fit the sum or difference of cubes pattern?
     Is it a sum or difference?
     Are the first and last terms perfect cubes?
  2. Write them as cubes.
  3. Use either the sum or difference of cubes pattern.
  4. Simplify inside the parentheses
  5. Check by multiplying the factors.

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2.3.4E: Exercises

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

1. $16y^2+24y+9$

Answer

$(4y+3)^2$

2. $25v^2+20v+4$

3. $36s^2+84s+49$

Answer

$(6s+7)^2$

4. $49s^2+154s+121$

5. $100x^2-20x+1$

Answer

$(10x-1)^2$

6. $64z^2-16z+1$

7. $25n^2-120n+144$

Answer

$(5n-12)^2$

8. $4p^2-52p+169$

9. $49x^2+28xy+4y^2$

Answer

$(7x+2y)^2$

10. $25r^2+60rs+36s^2$

11. $100y^2-52y+1$

Answer

$(50y-1)(2y-1)$

12. $64m^2-34m+1$

13. $10jk^2+80jk+160j$

Answer

$10j(k+4)^2$

14. $64x^2y-96xy+36y$

15. $75u^4-30u^3v+3u^2v^2$

Answer

$3u^2(5u-v)^2$

16. $90p^4+300p^4q+250p^2q^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

17. $25v^2-1$

Answer

$(5v-1)(5v+1)$

18. $169q^2-1$

19. $4-49x^2$

Answer

$(7x-2)(7x+2)$

20. $121-25s^2$

21. $6p^2q^2-54p^2$

Answer

$6p^2(q-3)(q+3)$

22. $98r^3-72r$

23. $24p^2+54$

Answer

$6(4p^2+9)$

24. $20b^2+140$

25. $121x^2-144y^2$

Answer

$(11x-12y)(11x+12y)$

26. $49x^2-81y^2$

27. $169c^2 - 36d^2$

Answer

$(13c - 6d)(13c + 6d)$

28. $36p^2 - 49q^2$

29. $16z^4 - 1$

Answer

$(2z - 1)(2z + 1)(4z^2 + 1)$

30. $m^4 - n^4$

31. $162a^4b^2 - 32b^2$

Answer

$2b^2(3a - 2)(3a + 2)(9a^2 + 4)$

32. $48m^4n^2 - 243n^2$

33. $x^2 - 16x + 64 - y^2$

Answer

$(x - 8 - y)(x - 8 + y)$

34. $p^2 + 14p + 49 - q^2$

35. $a^2 + 6a + 9 - 9b^2$

Answer

$(a + 3 - 3b)(a + 3 + 3b)$

36. $m^2 - 6m + 9 - 16n^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

37. $x^3 + 125$

Answer

$(x + 5)(x^2 - 5x + 25)$

38. $n^6 + 512$

39. $z^6 - 27$

Answer

$(z^2 - 3)(z^4 + 3z^2 + 9)$

40. $v^3 - 216$

41. $8 - 343t^3$

Answer

$(2-7t)(4+14t+49t^2)$

42. $125 - 27w^3$

43. $8y^3 - 125z^3$

Answer

$(2y-5z)(4y^2+10yz+25z^2)$

44. $27x^3 - 64y^3$

45. $216a^3 + 125b^3$

Answer

$(6a+5b)(36a^2-30ab+25b^2)$

46. $27y^3 + 8z^3$

47. $7k^3 + 56$

Answer

$7(k+2)(k^2-2k+4)$

48. $6x^3 - 48y^3$

49. $2x^2 - 16x^2y^3$

Answer

$2x^2(1-2y)(1+2y+4y^2)$

50. $-2x^3y^2 - 16y^5$

51. $(x+3)^3 + 8x^3$

Answer

$9(x+1)(x^2+3)$

52. $(x+4)^3 - 27x^3$

53. $(y-5)^3 - 64y^3$

Answer

$-(3y+5)(21y^2-30y+25)$

54. $(y-5)^3+125y^3$

Mixed Practice

In the following exercises, factor completely.

55. $64a^2-25$

Answer

$(8a-5)(8a+5)$

56. $121x^2-144$

57. $27q^2-3$

Answer

$3(3q-1)(3q+1)$

58. $4p^2-100$

59. $16x^2-72x+81$

Answer

$(4x-9)^2$

60. $36y^2+12y+1$

61. $8p^2+2$

Answer

$2(4p^2+1)$

62. $81x^2+169$

63. $125-8y^3$

Answer

$(5-2y)(25+10y+4y^2)$

64. $27u^3+1000$

65. $45n^2+60n+20$

Answer

$5(3n+2)^2$

66. $48q^3-24q^2+3q$

67. $x^2-10x+25-y^2$

Answer

$(x+y-5)(x-y-5)$

68. $x^2+12x+36-y^2$

69. $(x+1)^3+8x^3$

Answer

$(3x+1)(3x^2+1)$

70. $(y-3)^3-64y^3$

Writing Exercises

71. Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

Answer

Answers will vary.

72. How do you recognize the binomial squares pattern?

73. Explain why $n^2+25 \neq (n+5)^2$. Use algebra, words, or pictures.

Answer

Answers will vary.

74. Maribel factored $y^2-30y+81$ as $(y-9)^2$. Was she right or wrong? How do you know?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor perfect square trinomials.			
factor differences of squares.			
factor sums and differences of cubes.			

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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2.3.5: General Strategy for Factoring Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. The following chart summarizes all the factoring methods we have covered, and outlines a strategy you should use when factoring polynomials.

GENERAL STRATEGY FOR FACTORING POLYNOMIALS

GCF

Binomial	Trinomial	More than 3 terms
• **Difference of Squares** $a^2 - b^2 = (a - b)(a + b)$	• $x^2 + bx + c$ $(x \quad)(x \quad)$	• grouping
• **Sum of Squares** Sums of squares do not factor.	• $ax^2 + bx + c$ ◦ **'a' and 'c' squares** $(a + b)^2 = a^2 + 2ab + b^2$ $(a - b)^2 = a^2 - 2ab + b^2$	
• **Sum of Cubes** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	◦ **'ac' method**	
• **Difference of Cubes** $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$		

USE A GENERAL STRATEGY FOR FACTORING POLYNOMIALS.

1. Is there a greatest common factor?
   Factor it out.
2. Is the polynomial a binomial, trinomial, or are there more than three terms?
   If it is a binomial:
   - Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.
   - Is it a difference?
     Of squares? Factor as the product of conjugates.
     Of cubes? Use the difference of cubes pattern.

   If it is a trinomial:
   - Is it of the form $x^2 + bx + c$? Undo FOIL.
   - Is it of the form $ax^2 + bx + c$?
     If a and c are squares, check if it fits the trinomial square pattern.
     Use the trial and error or "ac" method.

   If it has more than three terms:
   - Use the grouping method.
3. Check.
   Is it factored completely?
   Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are **prime**!

? Example 2.3.5.1

Factor completely: $7x^3 - 21x^2 - 70x$.

Solution

$7x^3 - 21x^2 - 70x$	
Is there a GCF? Yes, $7x$.	
Factor out the GCF.	$7x(x^2 - 3x - 10)$
In the parentheses, is it a binomial, trinomial, or are there more terms?	
Trinomial with leading coefficient 1.	
"Undo" FOIL.	$7x(x \quad)(x \quad)$
	$7x(x+2)(x-5)$
Is the expression factored completely? Yes.	
Neither binomial can be factored.	
Check your answer.	
Multiply.	

$$7x(x+2)(x-5)$$
$$7x(x^2 - 5x + 2x - 10)$$
$$7x(x^2 - 3x - 10)$$
$$7x^3 - 21x^2 - 70x \checkmark$$

? Try It 2.3.5.1

Factor completely: $8y^3 + 16y^2 - 24y$.

Answer

$8y(y-1)(y+3)$

? Try It 2.3.5.2

Factor completely: $5y^3 - 15y^2 - 270y$.

Answer

$5y(y-9)(y+6)$

Be careful when you are asked to factor a binomial as there are several options!

? Example 2.3.5.2

Factor completely: $24y^2 - 150$

Solution

	$24y^2-150$
Is there a GCF? Yes, 6.	
Factor out the GCF.	$6(4y^2-25)$
In the parentheses, is it a binomial, trinomial or are there more than three terms? Binomial.	
Is it a sum? No.	
Is it a difference? Of squares or cubes? Yes, squares.	$6((2y)^2-(5)^2)$
Write as a product of conjugates.	$6(2y-5)(2y+5)$

Is the expression factored completely?
Neither binomial can be factored.

Check:

Multiply.

$6(2y-5)(2y+5)$

$6(4y^2-25)$
$24y^2-150$✓

? Try It 2.3.5.3

Factor completely: $16x^3-36x$.

Answer

$4x(2x-3)(2x+3)$

? Try It 2.3.5.4

Factor completely: $27y^2-48$.

Answer

$3(3y-4)(3y+4)$

The next example can be factored using several methods. Recognizing the trinomial squares pattern will make your work easier.

? Example 2.3.5.3

Factor completely: $4a^2-12ab+9b^2$.

Solution

$4a^2-12ab+9b^2$

Is there a GCF? No.

Is it a binomial, trinomial, or are there more terms?

Trinomial with $a \neq 1$. But the first term is a perfect square.

Is the last term a perfect square? Yes. $(2a)^2-12ab+(3b)^2$

Does it fit the pattern, $a^2-2ab+b^2$? Yes. $(2a)^2-12ab+(3b)^2$

$\searrow \quad -2(2a)(3b) \quad \swarrow$

Write it as a square. $(2a-3b)^2$

Is the expression factored completely? Yes.

The binomial cannot be factored.

Check your answer.

Multiply.

$$(2a-3b)^2$$
$$(2a)^2-2\cdot 2a\cdot 3b+(3b)^2$$
$$4a^2-12ab+9b^2 \checkmark$$

? Try It 2.3.5.5

Factor completely: $4x^2+20xy+25y^2$.

Answer

$(2x+5y)^2$

? Try It 2.3.5.6

Factor completely: $9x^2-24xy+16y^2$.

Answer

$(3x-4y)^2$

Remember, sums of squares do not factor, but sums of cubes do!

? Example 2.3.5.4

Factor completely $12x^3y^2+75xy^2$.

Solution

	$12x^3y^2+75xy^2$
Is there a GCF? Yes, $3xy^2$.	
Factor out the GCF.	$3xy^2(4x^2+25)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms? Binomial.	
Is it a sum? Of squares? Yes.	Sums of squares are prime.
Is the expression factored completely? Yes.	

Check:

Multiply.

$3xy^2(4x^2+25)$

$12x^3y^2+75xy^2$ ✓

? Try It 2.3.5.7

Factor completely: $50x^3y+72xy$.

Answer

$2xy(25x^2+36)$

? Try It 2.3.5.8

Factor completely: $27xy^3+48xy$.

Answer

$3xy(9y^2+16)$

When using the sum or difference of cubes pattern, being careful with the signs.

? Example 2.3.5.5

Factor completely: $24x^3+81y^3$.

Solution

Is there a GCF? Yes, 3.	$24x^3+81y^3$
Factor it out.	$3(8x^3+27y^3)$
In the parentheses, is it a binomial, trinomial, of are there more than three terms? Binomial.	
Is it a sum or difference? Sum.	
Of squares or cubes? Sum of cubes.	$3\left(\overset{a^3}{(2x)^3}+\overset{b^3}{(3y)^3}\right)$
Write it using the sum of cubes pattern.	$3\left(\overset{a}{2x}+\overset{b}{3y}\right)\left(\overset{a^2}{(2x)^2}-\overset{ab}{2x\cdot 3y}+\overset{b^2}{(3y)^3}\right)$
Is the expression factored completely? Yes.	$3(2x+3y)(4x^2-6xy+9y^2)$

Check by multiplying.

Try It 2.3.5.9

Factor completely: $250m^3+432n^3$.

Answer

$2(5m+6n)(25m^2-30mn+36n^2)$

Try It 2.3.5.10

Factor completely: $2p^3+54q^3$.

Answer

$2(p+3q)(p^2-3pq+9q^2)$

Example 2.3.5.6

Factor completely: $3x^5y-48xy$.

Solution

	$3x^5y-48xy$
Is there a GCF? Factor out $3xy$	$3xy(x^4-16)$
Is the binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$3xy\left((x^2)^2-(4)2\right)$
Factor it as a product of conjugates	$3xy(x^2-4)(x^2+4)$
The first binomial is again a difference of squares.	$3xy\left((x)^2-(2)^2\right)(x^2+4)$
Factor it as a product of conjugates.	$3xy(x-2)(x+2)(x^2+4)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	
$3xy(x-2)(x+2)(x^2+4)$	
$3xy(x^2-4)(x^2+4)$	
$3xy(x^4-16)$	
$3x^5y-48xy$✓	

Try It 2.3.5.11

Factor completely: $4a^5b-64ab$.

Answer

$4ab(a^2+4)(a-2)(a+2)$

Try It 2.3.5.12

Factor completely: $7xy^5-7xy$.

Answer

$7xy(y^2+1)(y-1)(y+1)$

Example 2.3.5.7

Factor completely: $4x^2+8bx-4ax-8ab$.

Solution

	$4x^2+8bx-4ax-8ab$
Is there a GCF? Factor out the GCF, 4.	$4(x^2+2bx-ax-2ab)$
There are four terms. Use grouping.	$4[x(x+2b)-a(x+2b)]4(x+2b)(x-a)$

Is the expression factored completely? Yes.

Check your answer.

Multiply.

$$4(x+2b)(x-a)$$
$$4(x^2-ax+2bx-2ab)$$
$$4x^2+8bx-4ax-8ab\checkmark$$

Try It 2.3.5.13

Factor completely: $6x^2-12xc+6bx-12bc$.

Answer

$6(x+b)(x-2c)$

Try It 2.3.5.14

Factor completely: $16x^2+24xy-4x-6y$.

Answer

$2(4x-1)(2x+3y)$

Taking out the complete GCF in the first step will always make your work easier.

Example 2.3.5.8

Factor completely: $40x^2y+44xy-24y$.

Solution

	$40x^2y+44xy-24y$
Is there a GCF? Factor out the GCF, $4y$.	$4y(10x^2+11x-6)$
Factor the trinomial with $a\neq 1$.	$4y(10x^2+11x-6)$
	$4y(5x-2)(2x+3)$

Is the expression factored completely? Yes.

Check your answer.

Multiply.

$$4y(5x-2)(2x+3)$$
$$4y(10x^2+11x-6)$$
$$40x^2y+44xy-24y\checkmark$$

Try It 2.3.5.15

Factor completely: $4p^2q-16pq+12q$.

Answer

$4q(p-3)(p-1)$

Try It 2.3.5.16

Factor completely: $6pq^2-9pq-6p$.

Answer

$3p(2q+1)(q-2)$

When we have factored a polynomial with four terms, most often we separated it into two groups of two terms. Remember that we can also separate it into a trinomial and then one term.

Example 2.3.5.9

Factor completely: $9x^2-12xy+4y^2-49$.

Solution

	$9x^2-12xy+4y^2-49$
Is there a GCF? No.	
With more than 3 terms, use grouping. Last 2 terms have no GCF. Try grouping first 3 terms.	$9x^2-12xy+4y^2-49$
Factor the trinomial with $a\neq 1$. But the first term is a perfect square.	
Is the last term of the trinomial a perfect square? Yes.	$(3x)^2-12xy+(2y)^2-49$
Does the trinomial fit the pattern, $a^2-2ab+b^2$? Yes.	$(3x)^2-12xy+(2y)^2-49$
	$\searrow \quad -2(3x)(2y) \quad \swarrow$
Write the trinomial as a square.	$(3x-2y)^2-49$
Is this binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$(3x-2y)^2-72$
Write it as a product of conjugates.	$((3x-2y)-7)((3x-2y)+7)$
	$(3x-2y-7)(3x-2y+7)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	
$(3x-2y-7)(3x-2y+7)$	
$9x^2-6xy-21x-6xy+4y^2+14y+21x-14y-49$	
$9x^2-12xy+4y^2-49\checkmark$	

Try It 2.3.5.17

Factor completely: $4x^2-12xy+9y^2-25$.

Answer

$(2x-3y-5)(2x-3y+5)$

? Try It 2.3.5.18

Factor completely: $16x^2 - 24xy + 9y^2 - 64$.

Answer

$(4x - 3y - 8)(4x - 3y + 8)$

Key Concepts

General Strategy for Factoring Polynomials

GCF

Binomial	Trinomial	More than 3 terms
• **Difference of Squares** $a^2 - b^2 = (a - b)(a + b)$	• $x^2 + bx + c$ $(x \quad)(x \quad)$	• grouping
• **Sum of Squares** Sums of squares do not factor.	• $ax^2 + bx + c$ ◦ **'a' and 'c' squares** $(a + b)^2 = a^2 + 2ab + b^2$ $(a - b)^2 = a^2 - 2ab + b^2$ ◦ **'ac' method**	
• **Sum of Cubes** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$		
• **Difference of Cubes** $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$		

- **How to use a general strategy for factoring polynomials.**
  1. Is there a greatest common factor?
     Factor it out.
  2. Is the polynomial a binomial, trinomial, or are there more than three terms?
     If it is a binomial:
     Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.
     Is it a difference?
     Of squares? Factor as the product of conjugates.
     Of cubes? Use the difference of cubes pattern.
     If it is a trinomial:
     Is it of the form $x^2 + bx + c$? Undo FOIL.
     Is it of the form $ax^2 + bx + c$?
     If *a* and *c* are squares, check if it fits the trinomial square pattern.
     Use the trial and error or "ac" method.
     If it has more than three terms:
     Use the grouping method.
  3. Check.
     Is it factored completely?
     Do the factors multiply back to the original polynomial?

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2.3.5E: Exercises

Practice Makes Perfect

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

1. $2n^2+13n-7$

Answer

$(2n-1)(n+7)$

2. $8x^2-9x-3$

3. a^5+9a^3

Answer

$a^3(a^2+9)$

4. $75m^3+12m$

5. $121r^2-s^2$

Answer

$(11r-s)(11r+s)$

6. $49b^2-36a^2$

7. $8m^2-32$

Answer

$8(m-2)(m+2)$

8. $36q^2-100$

9. $25w^2-60w+36$

Answer

$(5w-6)^2$

10. $49b^2-112b+64$

11. $m^2+14mn+49n^2$

Answer

$(m+7n)^2$

12. $64x^2+16xy+y^2$

13. $7b^2+7b-42$

Answer

$7(b+3)(b-2)$

14. $30n^2+30n+72$

15. $3x^4y-81xy$

Answer

$3xy(x-3)(x^2+3x+9)$

16. $4x^5y-32x^2y$

17. k^4-16

Answer

$(k-2)(k+2)(k^2+4)$

18. m^4-81

19. $5x5y^2-80xy^2$

Answer

$5xy^2(x^2+4)(x+2)(x-2)$

20. $48x^5y^2-243xy^2$

21. $15pq-15p+12q-12$

Answer

$3(5p+4)(q-1)$

22. $12ab-6a+10b-5$

23. $4x^2+40x+84$

Answer

$4(x+3)(x+7)$

24. $5q^2-15q-90$

25. $4u^5v+4u^2v^3$

Answer

$u^2(u+1)(u^2-u+1)$

26. $5m^4n+320mn^4$

27. $4c^2+20cd+81d^2$

Answer

prime

28. $25x^2+35xy+49y^2$

29. $10m^4-6250$

Answer

$10(m-5)(m+5)(m^2+25)$

30. $3v^4-768$

31. $36x^2y+15xy-6y$

Answer

$3y(3x+2)(4x-1)$

32. $60x^2y-75xy+30y$

33. $8x^3-27y^3$

Answer

$(2x-3y)(4x^2+6xy+9y^2)$

34. $64x^3+125y^3$

35. y^6-1

Answer

$(y+1)(y-1)(y^2-y+1)$

36. y^6+1

37. $9x^2-6xy+y^2-49$

Answer

$(3x-y+7)(3x-y-7)$

38. $16x^2-24xy+9y^2-64$

39. $(3x+1)^2-6(3x-1)+9$

Answer

$(3x-2)2$

40. $(4x-5)^2-7(4x-5)+12$

Writing Exercises

41. Explain what it mean to factor a polynomial completely.

Answer

Answers will vary.

42. The difference of squares $y^4 - 625$ can be factored as $(y^2 - 25)(y^2 + 25)$. But it is not completely factored. What more must be done to completely factor.

43. Of all the factoring methods covered in this chapter (GCF, grouping, undo FOIL, 'ac' method, special products) which is the easiest for you? Which is the hardest? Explain your answers.

Answer

Answers will vary.

44. Create three factoring problems that would be good test questions to measure your knowledge of factoring. Show the solutions.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize and use the appropriate method to factor a polynomial completely.			

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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2.3.6: Polynomial Equations

Learning Objectives

By the end of this section, you will be able to:

- Use the Zero Product Property
- Solve quadratic equations by factoring
- Solve equations with polynomial functions
- Solve applications modeled by polynomial equations

Before you get started, take this readiness quiz.

1. Solve: $5y-3=0$.
   If you missed this problem, review [link].
2. Factor completely: n^3-9n^2-22n.
   If you missed this problem, review [link].
3. If $f(x)=8x-16$, find $f(3)$ and solve $f(x)=0$.
   If you missed this problem, review [link].

We have spent considerable time learning how to factor polynomials. We will now look at polynomial equations and solve them using factoring, if possible.

A **polynomial equation** is an equation that contains a polynomial expression. The **degree of the polynomial equation** is the degree of the polynomial.

POLYNOMIAL EQUATION

A **polynomial equation** is an equation that contains a polynomial expression.

The **degree of the polynomial equation** is the degree of the polynomial.

We have already solved polynomial equations of **degree one**. Polynomial equations of degree one are linear equations are of the form $ax+b=c$.

We are now going to solve polynomial equations of **degree two**. A polynomial equation of degree two is called a **quadratic equation**. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0 \qquad 3y^2+4y=10 \qquad 64u^2-81=0 \qquad n(n+1)=42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get n^2+n.

The general form of a quadratic equation is $ax^2+bx+c=0$, with $a\neq 0$. (If $a=0$, then $0\cdot x^2=0$ and we are left with no quadratic term.)

QUADRATIC EQUATION

An equation of the form $ax^2+bx+c=0$ is called a quadratic equation.

$$a, b, \text{ and } c \text{ are real numbers and } a\neq 0$$

To solve quadratic equations we need methods different from the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Use the Zero Product Property

We will first solve some quadratic equations by using the **Zero Product Property**. The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

ZERO PRODUCT PROPERTY

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

We will now use the Zero Product Property, to solve a **quadratic equation**.

Example 2.3.6.1: How to Solve a Quadratic Equation Using the Zero Product Property

Solve: $(5n-2)(6n-1)=0$.

Answer

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(5n-2)(6n-1)=0$ $5n-2=0$ or $6n-1=0$
Step 2. Solve the linear equations.	Solve each equation.	$n=\frac{2}{5}$ $n=\frac{1}{6}$
Step 3. Check.	Substitute each solution separately into the original equation.	$n=\frac{2}{5}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$ $(2-2)\left(\frac{12}{5}-1\right)\stackrel{?}{=}0$ $0\cdot\frac{7}{5}\stackrel{?}{=}0$ $0=0$ ✓ $n=\frac{1}{6}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$ $\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$ $\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$ $0=0$ ✓

Example 2.3.6.2

Solve: $(3m-2)(2m+1)=0$.

Answer

$m=\frac{2}{3}$, $m=-\frac{1}{2}$

Example 2.3.6.3

Solve: $(4p+3)(4p-3)=0$.

Answer

$p=-\frac{3}{4}$, $p=\frac{3}{4}$

USE THE ZERO PRODUCT PROPERTY.

1. Set each factor equal to zero.
2. Solve the linear equations.
3. Check.

Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So we be sure to start with the quadratic equation in **standard form**, $ax^2+bx+c=0$. Then we factor the expression on the left.

Solve: $2y^2=13y+45$.

Answer

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	Write the equation in standard form.	$2y^2=13y+45$ $2y^2-13y-45=0$
Step 2. Factor the quadratic expression.	Factor $2y^2-13y+45$ $(2y+5)(y-9)$	$(2y+5)(y-9)=0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero. We have two linear equations.	$2y+5=0$ $y-9=0$
Step 4. Solve the linear equations.		$y=-\frac{5}{2}$ $y=9$
Step 5. Check. Substitute each solution separately into the original equation.	Substitute each solution separately into the original equation.	$y=-\frac{5}{2}$ $2y^2=13y+45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right)+45$ $2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right)+\frac{90}{2}$ $\frac{25}{2}=\frac{25}{2}$ ✓ $y=9$ $2y^2=13y+45$ $2(9)^2 \stackrel{?}{=} 13(9)+45$ $2(81) \stackrel{?}{=} 117+45$ $162=162$ ✓

Example 2.3.6.5

Solve: $3c^2=10c-8$.

Answer

$c=2,\ c=\frac{4}{3}$

Example 2.3.6.6

Solve: $2d^2 - 5d = 3$.

Answer

$d = 3,\ d = -12$

SOLVE A QUADRATIC EQUATION BY FACTORING.

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Check. Substitute each solution separately into the original equation.

Before we factor, we must make sure the **quadratic equation** is in **standard form**.

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Example 2.3.6.7

Solve: $169q^2 = 49$.

Answer

	$169x^2 = 49$
Write the quadratic equation in standard form.	$169x^2 - 49 = 0$
Factor. It is a difference of squares.	$(13x - 7)(13x + 7) = 0$
Use the Zero Product Property to set each factor to 0.	
	$13x - 7 = 0 \quad 13x + 7 = 0$
Solve each equation.	$13x = 7 \quad 13x = -7$
	$x = \frac{7}{13} \quad x = -\frac{7}{13}$

Check:

We leave the check up to you.

Example 2.3.6.8

Solve: $25p^2 = 49$.

Answer

$p = \frac{7}{5}, p = -\frac{7}{5}$

Example 2.3.6.9

Solve: $36x^2 = 121$.

Answer

$x = \frac{11}{6}, x = -\frac{11}{6}$

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the **Zero Product Property**, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Example 2.3.6.10

Solve: $(3x-8)(x-1)=3x$.

Answer

	$(3x-8)(x-1)=3x$
Multiply the binomials.	$3x^2-11x+8=3x$
Write the quadratic equation in standard form.	$3x^2-14x+8=0$
Factor the trinomial.	$(3x-2)(x-4)=0$
Use the Zero Product Property to set each factor to 0.	$3x-2=0 \quad x-4=0$
Solve each equation.	$3x=2 \quad x=4$
	$x=\frac{2}{3}$
Check your answers.	The check is left to you.

Example 2.3.6.11

Solve: $(2m+1)(m+3)=12m$.

Answer

$m=1,\ m=\frac{3}{2}$

Example 2.3.6.12

Solve: $(k+1)(k-1)=8$.

Answer

$k=3,\ k=-3$

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Example 2.3.6.13

Solve: $3x^2=12x+63$.

Answer

	$3x^2=12x+63$
Write the quadratic equation in standard form.	$3x^2-12x-63=0$
Factor the greatest common factor first.	$3(x^2-4x-21)=0$
Factor the trinomial.	$3(x-7)(x+3)=0$
Use the Zero Product Property to set each factor to 0.	$3\neq 0 \quad x-7=0 \quad x+3=0$
Solve each equation.	$3\neq 0 \quad x=7 \quad x=-3$
Check your answers.	The check is left to you.

Example 2.3.6.14

Solve: $18a^2-30=-33a$.

Answer

$a=-\frac{5}{2}, a=\frac{2}{3}$

Example 2.3.6.15

Solve: $123b = -6 - 60b^2$

Answer

$b = -2,\ b = -\frac{1}{20}$

The **Zero Product Property** also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree greater than two by using the Zero Product Property, just like we solved quadratic equations.

Example 2.3.6.16

Solve: $9m^3 + 100m = 60m^2$

Answer

	$9m^3 + 100m = 60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3 - 60m^2 + 100m = 0$
Factor the greatest common factor first.	$m(9m^2 - 60m + 100) = 0$
Factor the trinomial.	$m(3m - 10)^2 = 0$
Use the Zero Product Property to set each factor to 0.	
Solve each equation.	$m = 0 \quad 3m - 10 = 0$ $m = 0 \quad m = \frac{10}{3}$
Check your answers.	The check is left to you.

Example 2.3.6.17

Solve: $8x^3 = 24x^2 - 18x$.

Answer

$x = 0,\ x = \frac{3}{2}$

Example 2.3.6.18

Solve: $16y^2 = 32y^3 + 2y$.

Answer

$y = 0,\ y = 14$

Solve Equations with Polynomial Functions

As our study of polynomial functions continues, it will often be important to know when the function will have a certain value or what points lie on the graph of the function. Our work with the **Zero Product Property** will be help us find these answers.

Example 2.3.6.19

For the function $f(x) = x^2 + 2x - 2$,

ⓐ find x when $f(x) = 6$

ⓑ find two points that lie on the graph of the function.

Answer

ⓐ

	$f(x)=x^2+2x-2$
Substitute 6 for $f(x)$.	$6=x^2+2x-2$
Put the quadratic in standard form.	$x^2+2x-8=0$
Factor the trinomial.	$(x+4)(x-2)=0$
Use the zero product property.	$x+4=0 \quad \text{or} \quad x-2=0$
Solve.	$x=-4 \quad \text{or} \quad x=2$
Check:	

$f(x)=x^2+2x-2$	$f(x)=x^2+2x-2$
$f(-4)=(-4)^2+2(-4)-2$	$f(2)=2^2+2\cdot 2-2$
$f(-4)=16-8-2$	$f(2)=4+4-2$
$f(-4)=6$✓	$f(2)=6$✓

ⓑ Since $f(-4)=6$ and $f(2)=6$, the points $(-4,6)$ and $(2,6)$ lie on the graph of the function.

? Example 2.3.6.20

For the function $f(x)=x^2-2x-8$,

ⓐ find x when $f(x)=7$
ⓑ Find two points that lie on the graph of the function.

Answer

ⓐ $x=-3$ or $x=5$
ⓑ $(-3,7)$ $(5,7)$

? Example 2.3.6.21

For the function $f(x)=x^2-8x+3$,

ⓐ find x when $f(x)=-4$
ⓑ Find two points that lie on the graph of the function.

Answer

ⓐ $x=1$ or $x=7$
ⓑ $(1,-4)$ $(7,-4)$

The **Zero Product Property** also helps us determine where the function is zero. A value of x where the function is 0, is called a **zero of the function**.

ZERO OF A FUNCTION

For any function f, if $f(x)=0$, then x is a **zero of the function**.

When $f(x)=0$, the point $(x,0)$ is a point on the graph. This point is an x-**intercept** of the graph. It is often important to know where the graph of a function crosses the axes. We will see some examples later.

Example 2.3.6.22

For the function $f(x)=3x^2+10x-8$, find

ⓐ the zeros of the function,
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function

Answer

ⓐ To find the zeros of the function, we need to find when the function value is 0.

	$f(x)=3x^2+10x-8$
Substitute 0 for $f(x)$.	$0=3x^2+10x-8$
Factor the trinomial.	$(x+4)(3x-2)=0$
Use the zero product property.	$x+4=0 \quad \text{or} \quad 3x-2=0$
Solve.	$x=-4 \quad \text{or} \quad x=\frac{2}{3}$

ⓑ An x-intercept occurs when $y=0$. Since $f(-4)=0$ and $f(\frac{2}{3})=0$, the points $(-4,0)$ and $(\frac{2}{3},0)$ lie on the graph. These points are x-intercepts of the function.

ⓒ A y-intercept occurs when $x=0$. To find the y-intercepts we need to find $f(0)$.

	$f(x)=3x^2+10x-8$
Find $f(0)$ by substituting 0 for x.	$f(0)=3\cdot 0^2+10\cdot 0-8$
Simplify.	$f(0)=-8$

Since $f(0)=-8$, the point $(0,-8)$ lies on the graph. This point is the y-intercept of the function.

Example 2.3.6.23

For the function $f(x)=2x^2-7x+5$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

Answer

ⓐ $x=1$ or $x=\frac{5}{2}$
ⓑ $(1,0),\ (\frac{5}{2},0)$ ⓒ $(0,5)$

Example 2.3.6.24

For the function $f(x)=6x^2+13x-15$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

Answer

ⓐ $x=-3$ or $x=\frac{5}{6}$
ⓑ $(-3,0),\ (\frac{5}{6},0)$ ⓒ $(0,-15)$

Solve Applications Modeled by Polynomial Equations

The problem-solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to polynomial equations. We will copy the problem-solving strategy here so we can use it for reference.

USE A PROBLEM SOLVING STRATEGY TO SOLVE WORD PROBLEMS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using appropriate algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a polynomial equation.

Example 2.3.6.25

The product of two consecutive odd integers is 323. Find the integers.

Answer

Step 1. Read the problem.

Step 2. Identify what we are looking for. — We are looking for two consecutive integers.

Step 3. Name what we are looking for. — Let $n=$ the first integer.
$n+2=$ next consecutive odd integer

Step 4. Translate into an equation. Restate the problem in a sentence. — The product of the two consecutive odd integers is 323.
$n(n+2)=323$

Step 5. Solve the equation. $n^2+2n=323$

Bring all the terms to one side. $n^2+2n-323=0$

Factor the trinomial. $(n-17)(n+19)=0$

Use the Zero Product Property. $n-17=0$ $\quad$ $n+19=0$

Solve the equations. $n=17$ $\quad$ $n=-19$

There are two values for n that are solutions to this problem. So there are two sets of consecutive odd integers that will work.

If the first integer is $n=17$ then the next odd integer is

$$\begin{aligned} n+2 \\ 17+2 \\ 19 \\ 17,19 \end{aligned}$$

If the first integer is $n=-19$ then the next odd integer is

$$\begin{aligned} n+2 \\ -19+2 \\ -17 \\ -17,-19 \end{aligned}$$

Step 6. Check the answer.

The results are consecutive odd integers
$17,\ 19$ and $-19,\ -17$.

$17\cdot 19=323$✓ $\quad$ $-19(-17)=323$✓

Both pairs of consecutive integers are solutions.

Step 7. Answer the question — The consecutive integers are $17, 19$ and $-19, -17$.

? Example 2.3.6.26

The product of two consecutive odd integers is 255. Find the integers.

Answer

$-15, -17$ and $15, 17$

? Example 2.3.6.27

The product of two consecutive odd integers is 483 Find the integers.

Answer

$-23, -21$ and $21, 23$

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give positive results.

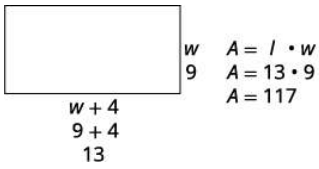
In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

? Example 2.3.6.28

A rectangular bedroom has an area 117 square feet. The length of the bedroom is four feet more than the width. Find the length and width of the bedroom.

Answer

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what you are looking for.	Let $w =$ the width of the bedroom.
The length is four feet more than the width.	$w + 4 =$ the length of the garden
Step 4. Translate into an equation.	
Restate the important information in a sentence.	The area of the bedroom is 117 square feet.
Use the formula for the area of a rectangle.	$A = l \cdot w$
Substitute in the variables.	$117 = (w + 4)w$
Step 5. Solve the equation Distribute first.	$117 = w^2 + 4w$
Get zero on one side.	$117 = w^2 + 4w$
Factor the trinomial.	$0 = w^2 + 4w - 117$
Use the Zero Product Property.	$0 = (w^2 + 13)(w - 9)$
Solve each equation.	$0 = w + 13 \quad 0 = w - 9$
Since w is the width of the bedroom, it does not make sense for it to be negative. We eliminate that value for w.	$\cancel{w = -13} \quad w = 9$
	$w = 9$ The width is 9 feet.
Find the value of the length.	$w + 4$ $9 + 4$ 13 The length is 13 feet.

Step 6. Check the answer. Does the answer make sense?  $A = l \cdot w$ $A = 13 \cdot 9$ $A = 117$ Yes, this makes sense.	
Step 7. Answer the question.	The width of the bedroom is 9 feet and the length is 13 feet.

? Example 2.3.6.29

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

Answer

The width is 5 feet and length is 6 feet.

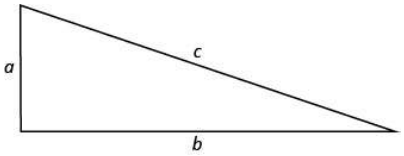
? Example 2.3.6.30

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

Answer

The length of the patio is 12 feet and the width 15 feet.

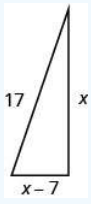
In the next example, we will use the Pythagorean Theorem $(a^2 + b^2 = c^2)$. This formula gives the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

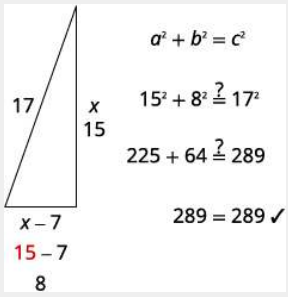
? Example 2.3.6.31

A boat's sail is in the shape of a right triangle as shown. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the sail.



Answer

Step 1. Read the problem

Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the sail.
Step 3. Name what you are looking for. One side is 7 less than the other.	Let $x =$ length of a side of the sail. $x - 7 =$ length of other side
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Substitute in the variables.	$x^2 + (x-7)^2 = 17^2$
Step 5. Solve the equation Simplify.	$x^2 + x^2 - 14x + 49 = 289$
	$2x^2 - 14x + 49 = 289$
It is a quadratic equation, so get zero on one side.	$2x^2 - 14x - 240 = 0$
Factor the greatest common factor.	$2(x^2 - 7x - 120) = 0$
Factor the trinomial.	$2(x-15)(x+8) = 0$
Use the Zero Product Property.	$2 \neq 0 \quad x - 15 = 0 \quad x + 8 = 0$
Solve.	$2 \neq 0 \quad x = 15 \quad x = -8$
Since x is a side of the triangle, $x = -8$ does not make sense.	$2 \neq 0 \quad x = 15 \quad \cancel{x = -8}$
Find the length of the other side.	
If the length of one side is then the length of the other side is	$x = 15$ $x - 7$ $15 - 7$ 8 is the length of the other side.
Step 6. Check the answer in the problem Do these numbers make sense? 	
Step 7. Answer the question	The sides of the sail are 8, 15 and 17 feet.

? Example 2.3.6.32

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle. The length of one side of the deck is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the deck.

Answer

5 feet and 12 feet

Example 2.3.6.33

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of the other leg. Find the lengths of the hypotenuse and the other leg.

Answer

24 feet and 25 feet

The next example uses the function that gives the height of an object as a function of time when it is thrown from 80 feet above the ground.

Example 2.3.6.34

Dennis is going to throw his rubber band ball upward from the top of a campus building. When he throws the rubber band ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, h, of the ball above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tell us when the ball hits the ground
ⓑ when the ball will be 80 feet above the ground
ⓒ the height of the ball at $t = 2$ seconds.

Answer

ⓐ The zeros of this function are found by solving $h(t) = 0$. This will tell us when the ball will hit the ground.

	$h(t) = 0$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 0$
Factor the GCF, -16.	$-16(t^2 - 4t - 5) = 0$
Factor the trinomial.	$-16(t-5)(t+1) = 0$
Use the Zero Product Property.	$t-5=0 \quad t+1=0$
Solve.	$t = 5 \quad t = -1$

The result $t = 5$ tells us the ball will hit the ground 5 seconds after it is thrown. Since time cannot be negative, the result $t = -1$ is discarded.

ⓑ The ball will be 80 feet above the ground when $h(t) = 80$.

	$h(t) = 80$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 80$
Subtract 80 from both sides.	$-16t^2 + 64t = 0$
Factor the GCF, $-16t$.	$-16t(t-4) = 0$
Use the Zero Product Property.	$-16t = 0 \quad t-4=0$
Solve.	$t = 0 \quad t = 4$
	The ball will be at 80 feet the moment Dennis tosses the ball and then 4 seconds later, when the ball is falling.

ⓒ To find the height ball at $t = 2$ seconds we find $h(2)$.

	$h(t) = -16t^2 + 64t + 80$
To find $h(2)$ substitute 2 for t.	$h(2) = -16(2)^2 + 64 \cdot 2 + 80$
Simplify.	$h(2) = 144$
	After 2 seconds, the ball will be at 144 feet.

? Example 2.3.6.35

Genevieve is going to throw a rock from the top a trail overlooking the ocean. When she throws the rock upward from 160 feet above the ocean, the function $h(t) = -16t^2 + 48t + 160$ models the height, h, of the rock above the ocean as a function of time, t. Find:

ⓐ the zeros of this function which tell us when the rock will hit the ocean
ⓑ when the rock will be 160 feet above the ocean.
ⓒ the height of the rock at $t = 1.5$ seconds.

Answer

ⓐ 5 ⓑ 0;3 ⓒ 196

? Example 2.3.6.36

Calib is going to throw his lucky penny from his balcony on a cruise ship. When he throws the penny upward from 128 feet above the ground, the function $h(t) = -16t^2 + 32t + 128$ models the height, h, of the penny above the ocean as a function of time, t. Find:

ⓐ the zeros of this function which is when the penny will hit the ocean
ⓑ when the penny will be 128 feet above the ocean.
ⓒ the height the penny will be at $t = 1$ seconds which is when the penny will be at its highest point.

Answer

ⓐ 4 ⓑ 0;2 ⓒ 144

Access this online resource for additional instruction and practice with quadratic equations.

- Beginning Algebra & Solving Quadratics with the Zero Property

Key Concepts

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

$$a, b, c \text{ are real numbers and } a \neq 0$$

- **Zero Product Property:** If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.
- **How to use the Zero Product Property**
  1. Set each factor equal to zero.
  2. Solve the linear equations.
  3. Check.
- **How to solve a quadratic equation by factoring.**
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.
  4. Solve the linear equations.
  5. Check. Substitute each solution separately into the original equation.
- **Zero of a Function:** For any function f, if $f(x) = 0$, then x is a zero of the function.
- **How to use a problem solving strategy to solve word problems.**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.

4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using appropriate algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

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2.3.6E: Exercises

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.

1. $(3a-10)(2a-7)=0$

Answer

$a=\frac{10}{3},\ a=\frac{7}{2}$

2. $(5b+1)(6b+1)=0$

3. $6m(12m-5)=0$

Answer

$m=0,\ m=\frac{5}{12}$

4. $2x(6x-3)=0$

5. $(2x-1)^2=0$

Answer

$x=\frac{1}{2}$

6. $(3y+5)^2=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

7. $5a^2-26a=24$

Answer

$a=-\frac{4}{5},\ a=6$

8. $4b^2+7b=-3$

9. $4m^2=17m-15$

Answer

$m=\frac{5}{4},\ m=3$

10. $n^2=5-6n$

11. $7a^2+14a=7a$

Answer

$a=-1,\ a=0$

12. $12b^2 - 15b = -9b$

13. $49m^2 = 144$

Answer

$m = \frac{12}{7},\ m = -\frac{12}{7}$

14. $625 = x^2$

15. $16y^2 = 81$

Answer

$y = -\frac{9}{4},\ y = \frac{9}{4}$

16. $64p^2 = 225$

17. $121n^2 = 36$

Answer

$n = -\frac{6}{11},\ n = \frac{6}{11}$

18. $100y^2 = 9$

19. $(x+6)(x-3) = -8$

Answer

$x = 2,\ x = -5$

20. $(p-5)(p+3) = -7$

21. $(2x+1)(x-3) = -4x$

Answer

$x = \frac{3}{2},\ x = -1$

22. $(y-3)(y+2) = 4y$

23. $(3x-2)(x+4) = 12x$

Answer

$x = \frac{3}{2},\ x = -1$

24. $(2y-3)(3y-1) = 8y$

25. $20x^2 - 60x = -45$

Answer

$x = -\frac{2}{3}$

26. $3y^2 - 18y = -27$

27. $15x^2 - 10x = 40$

Answer

$x = 2,\ x = -\frac{4}{3}$

28. $14y^2 - 77y = -35$

29. $18x^2 - 9 = -21x$

Answer

$x = -\frac{3}{2},\ x = \frac{1}{3}$

30. $16y^2 + 12 = -32y$

31. $16p^3 = 24p^2 - 9p$

Answer

$p = 0,\ p = \frac{3}{4}$

32. $m^3 - 2m^2 = -m$

33. $2x^3 + 72x = 24x^2$

Answer

$x = 0,\ x = 6$

34. $3y^3 + 48y = 24y^2$

35. $36x^3 + 24x^2 = -4x$

Answer

$x = 0,\ x = \frac{1}{3}$

36. $2y^3 + 2y^2 = 12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

37. For the function, $f(x) = x^2 - 8x + 8$, ⓐ find when $f(x) = -4$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x = 2$ or $x = 6$ ⓑ $(2, -4)$ $(6, -4)$

38. For the function, $f(x) = x^2 + 11x + 20$, ⓐ find when $f(x) = -8$ ⓑ Use this information to find two points that lie on the graph of the function.

39. For the function, $f(x) = 8x^2 - 18x + 5$, ⓐ find when $f(x) = -4$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x = \frac{3}{2}$ or $x = \frac{3}{4}$
ⓑ $(\frac{3}{2}, -4)$ $(\frac{3}{4}, -4)$

40. For the function, $f(x) = 18x^2 + 15x - 10$, ⓐ find when $f(x) = 15$ ⓑ Use this information to find two points that lie on the graph of the function.

In the following exercises, for each function, find: ⓐ the zeros of the function ⓑ the x-intercepts of the graph of the function ⓒ the y-intercept of the graph of the function.

41. $f(x) = 9x^2 - 4$

Answer

ⓐ $x = \frac{2}{3}$ or $x = -\frac{2}{3}$
ⓑ $(\frac{2}{3}, 0)$, $(-\frac{2}{3}, 0)$
ⓒ $(0, -4)$

42. $f(x) = 25x^2 - 49$

43. $f(x) = 6x^2 - 7x - 5$

Answer

ⓐ $x = \frac{5}{3}$ or $x = -\frac{1}{2}$
ⓑ $(\frac{5}{3}, 0)$, $(-\frac{1}{2}, 0)$
ⓒ $(0, -5)$

44. $f(x) = 12x^2 - 11x + 2$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

45. The product of two consecutive odd integers is 143. Find the integers.

Answer

$-13,\ -11$ and $11,\ 13$

46. The product of two consecutive odd integers is 195. Find the integers.

47. The product of two consecutive even integers is 168. Find the integers.

Answer

$-14,\ -12$ and $12,\ 14$

48. The product of two consecutive even integers is 288. Find the integers.

49. The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

Answer

-4 and 7

50. A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

51. The area of a bulletin board is 55 square feet. The length is four feet less than three times the width. Find the length and the width of the a bulletin board.

Answer

$5,\ 11$

52. A rectangular carport has area 150 square feet. The height of the carport is five feet less than twice its length. Find the height and the length of the carport.

53. A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

Answer

$6,\ 8$

54. A stained glass window is shaped like a right triangle. The hypotenuse is 15 feet. One leg is three more than the other. Find the lengths of the legs.

55. A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

Answer

$8,\ 15,\ 17$

56. A goat enclosure is in the shape of a right triangle. One leg of the enclosure is built against the side of the barn. The other leg is 4 feet more than the leg against the barn. The hypotenuse is 8 feet more than the leg along the barn. Find the three sides of the goat enclosure.

57. Juli is going to launch a model rocket in her back yard. When she launches the rocket, the function $h(t) = -16t^2 + 32t$ models the height, h, of the rocket above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tells us when the rocket will hit the ground. ⓑ the time the rocket will be 16 feet above the ground.

Answer

ⓐ 0, 2 ⓑ 1

58. Gianna is going to throw a ball from the top floor of her middle school. When she throws the ball from 48 feet above the ground, the function $h(t) = -16t^2 + 32t + 48$ models the height, h, of the ball above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 48 feet above the ground. ⓒ the height the ball will be at $t = 1$ seconds which is when the ball will be at its highest point.

Writing Exercises

59. Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

Answer

Answers will vary.

60. Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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2.3.7: Review Exercises

Chapter Review Exercises

Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

$12a^2b^3,\ 15ab^2$

Answer

$3ab^2$

$12m^2n^3, 42m^5n^3$

$15y^3,\ 21y^2,\ 30y$

Answer

$3y$

$45x^3y^2,\ 15x^4y,\ 10x^5y^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

$35y+84$

Answer

$7(5y+12)$

$6y^2+12y-6$

$18x^3-15x$

Answer

$3x(6x^2-5)$

$15m^4+6m^2n$

$4x^3-12x^2+16x$

Answer

$4x(x^2-3x+4)$

$-3x+24$

$-3x^3+27x^2-12x$

Answer

$-3x(x^2-9x+4)$

$3x(x-1)+5(x-1)$

Factor by Grouping

In the following exercises, factor by grouping.

$ax-ay+bx-by$

Answer

$(a+b)(x-y)$

$x^2y-xy^2+2x-2y$

$x^2+7x-3x-21$

Answer

$(x-3)(x+7)$

$4x^2-16x+3x-12$

m^3+m^2+m+1

Answer

$(m^2+1)(m+1)$

$5x-5y-y+x$

Factor Trinomials

Factor Trinomials of the Form x^2+bx+c

In the following exercises, factor each trinomial of the form x^2+bx+c .

$a^2+14a+33$

Answer

$(a+3)(a+11)$

$k^2-16k+60$

$m^2+3m-54$

Answer

$(m+9)(m-6)$

$x^2-3x-10$

In the following examples, factor each trinomial of the form $x^2+bxy+cy^2$.

$x^2+12xy+35y^2$

Answer

$(x+5y)(x+7y)$

$r^2+3rs-28s^2$

$a^2+4ab-21b^2$

Answer

$(a+7b)(a-3b)$

$p^2-5pq-36q^2$

$m^2-5mn+30n^2$

Answer

Prime

Factor Trinomials of the Form ax2+bx+cax2+bx+c Using Trial and Error

In the following exercises, factor completely using trial and error.

x^3+5x^2-24x

$3y^3-21y^2+30y$

Answer

$3y(y-5)(y-2)$

$5x^4+10x^3-75x^2$

$5y^2+14y+9$

Answer

$(5y+9)(y+1)$

$8x^2+25x+3$

$10y^2-53y-11$

Answer

$(5y+1)(2y-11)$

$6p^2-19pq+10q^2$

$-81a^2+153a+18$

Answer

$-9(9a-1)(a+2)$

Factor Trinomials of the Form ax2+bx+cax2+bx+c using the 'ac' Method

In the following exercises, factor.

$2x^2+9x+4$

$18a^2-9a+1$

Answer

$(3a-1)(6a-1)$

$15p^2+2p-8$

$15x^2+6x-2$

Answer

$(3x-1)(5x+2)$

$8a^2+32a+24$

$3x^2+3x-36$

Answer

$3(x+4)(x-3)$

$48y^2+12y-36$

$18a^2-57a-21$

Answer

$3(2a-7)(3a+1)$

$3n^4-12n^3-96n^2$

Factor using substitution

In the following exercises, factor using substitution.

x^4-13x^2-30

Answer

$(x^2-15)(x^2+2)$

$(x-3)^2-5(x-3)-36$

Factor Special Products

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

$25x^2+30x+9$

Answer

$(5x+3)^2$

$36a^2 - 84ab + 49b^2$

$40x^2 + 360x + 810$

Answer

$10(2x + 9)^2$

$5k^3 - 70k^2 + 245k$

$75u^4 - 30u^3v + 3u^2v^2$

Answer

$3u^2(5u - v)^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

$81r^2 - 25$

$169m^2 - n^2$

Answer

$(13m + n)(13m - n)$

$25p^2 - 1$

$9 - 121y^2$

Answer

$(3 + 11y)(3 - 11y)$

$20x^2 - 125$

$169n^3 - n$

Answer

$n(13n + 1)(13n - 1)$

$6p^2q^2 - 54p^2$

$24p^2 + 54$

Answer

$6(4p^2 + 9)$

$49x^2 - 81y^2$

$16z^4 - 1$

Answer

$(2z-1)(2z+1)(4z^2+1)$

$48m^4n^2-243n^2$

$a^2+6a+9-9b^2$

Answer

$(a+3-3b)(a+3+3b)$

$x^2-16x+64-y^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

a^3-125

Answer

$(a-5)(a^2+5a+25)$

b^3-216

$2m^3+54$

Answer

$2(m+3)(m^2-3m+9)$

$81m^3+3$

General Strategy for Factoring Polynomials

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

$24x^3+44x^2$

Answer

$4x^2(6x+11)$

$24a^4-9a^3$

$16n^2-56mn+49m^2$

Answer

$(4n-7m)^2$

$6a^2-25a-9$

$5u^4-45u^2$

Answer

$5u^2(u+3)(u-3)$

n^4-81

$64j^2+225$

Answer

prime

$5x^2+5x-60$

b^3-64

Answer

$(b-4)(b^2+4b+16)$

m^3+125

$2b^2-2bc+5cb-5c^2$

Answer

$(2b+5c)(b-c)$

$48x^5y^2-243xy^2$

$5q^2-15q-90$

Answer

$5(q+3)(q-6)$

$4u^5v+4u^2v^3$

$10m^4-6250$

Answer

$10(m-5)(m+5)(m^2+25)$

$60x^2y-75xy+30y$

$16x^2-24xy+9y^2-64$

Answer

$(4x-3y+8)(4x-3y-8)$

Polynomial Equations

Use the Zero Product Property

In the following exercises, solve.

$(a-3)(a+7)=0$

$(5b+1)(6b+1)=0$

Answer

$b=-\frac{1}{5},\ b=-\frac{1}{6}$

$6m(12m-5)=0$

$(2x-1)^2=0$

Answer

$x=\frac{1}{2}$

$3m(2m-5)(m+6)=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

$x^2+9x+20=0$

Answer

$x=-4,\ x=-5$

$y^2-y-72=0$

$2p^2-11p=40$

Answer

$p=-\frac{5}{2}, p=8$

$q^3+3q^2+2q=0$

$144m^2-25=0$

Answer

$m=\frac{5}{12},\ m=-\frac{5}{12}$

$4n^2=36$

$(x+6)(x-3)=-8$

Answer

$x=2,\ x=-5$

$(3x-2)(x+4)=12$

$16p^3=24p^2+9p$

Answer

$p = 0,\ p = \frac{3}{4}$

$2y^3 + 2y^2 = 12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

For the function, $f(x) = x^2 + 11x + 20$, ⓐ find when $f(x) = -8$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x = -7$ or $\backslash x = -4$
ⓑ $(-7, -8)$ $(-4, -8)$

For the function, $f(x) = 9x^2 - 18x + 5$, ⓐ find when $f(x) = -3$ ⓑ Use this information to find two points that lie on the graph of the function.

In each function, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒ the *y*-intercept of the graph of the function.

$f(x) = 64x^2 - 49$

Answer

ⓐ $x = \frac{7}{8}$ or $x = -\frac{7}{8}$
ⓑ $(\frac{7}{8}, 0),\ (-\frac{7}{8}, 0)$ ⓒ $(0, -49)$

$f(x) = 6x^2 - 13x - 5$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

The product of two consecutive numbers is 399. Find the numbers.

Answer

The numbers are -21 and -19 or 19 and 21.

The area of a rectangular shaped patio 432 square feet. The length of the patio is 6 feet more than its width. Find the length and width.

A ladder leans against the wall of a building. The length of the ladder is 9 feet longer than the distance of the bottom of the ladder from the building. The distance of the top of the ladder reaches up the side of the building is 7 feet longer than the distance of the bottom of the ladder from the building. Find the lengths of all three sides of the triangle formed by the ladder leaning against the building.

Answer

The lengths are 8, 15, and 17 ft.

Shruti is going to throw a ball from the top of a cliff. When she throws the ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐthe zeros of this

function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 80 feet above the ground. ⓒ the height the ball will be at $t = 2$ seconds which is when the ball will be at its highest point.

Chapter Practice Test

In the following exercises, factor completely.

$80a^2 + 120a^3$

Answer

$40a^2(2 + 3a)$

$5m(m - 1) + 3(m - 1)$

$x^2 + 13x + 36$

Answer

$(x + 7)(x + 6)$

$p^2 + pq - 12q^2$

$xy - 8y + 7x - 56$

Answer

$(x - 8)(y + 7)$

$40r^2 + 810$

$9s^2 - 12s + 4$

Answer

$(3s - 2)^2$

$6x^2 - 11x - 10$

$3x^2 - 75y^2$

Answer

$3(x + 5y)(x - 5y)$

$6u^2 + 3u - 18$

$x^3 + 125$

Answer

$(x + 5)(x^2 - 5x + 25)$

$32x^5y^2 - 162xy^2$

$6x^4 - 19x^2 + 15$

Answer

$(3x^2-5)(2x^2-3)$

$3x^3-36x^2+108x$

In the following exercises, solve

$5a^2+26a=24$

Answer

$a=\frac{4}{5},\ a=-6$

The product of two consecutive integers is 156. Find the integers.

The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

Answer

The width is 12 inches and the length is 14 inches.

Jing is going to throw a ball from the balcony of her condo. When she throws the ball from 80 feet above the ground, the function $h(t)=-16t^2+64t+80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 128 feet above the ground. ⓒ the height the ball will be at $t=4$ seconds.

For the function, $f(x)=x^2-7x+5$, ⓐ find when $f(x)=-7$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x=3$ or $x=4$ ⓑ $(3,-7)$ $(4,-7)$

For the function $f(x)=25x^2-81$, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒthe *y*-intercept of the graph of the function.

Glossary

degree of the polynomial equation

The degree of the polynomial equation is the degree of the polynomial.

polynomial equation

A polynomial equation is an equation that contains a polynomial expression.

quadratic equation

Polynomial equations of degree two are called quadratic equations.

zero of the function

A value of xx where the function is 0, is called a zero of the function.

Zero Product Property

The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero.

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2.4: Add and Subtract Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Determine the degree of polynomials
- Add and subtract polynomials
- Evaluate a polynomial function for a given value
- Add and subtract polynomial functions

Note

Before you get started, take this readiness quiz.

1. Simplify: $3x^2+3x+1+8x^2+5x+5$.
   If you missed this problem, review [link].
2. Subtract: $(5n+8)-(2n-1)$.
   If you missed this problem, review [link].
3. Evaluate: $4xy^2$ when $x=-2x$ and $y=5$..
   If you missed this problem, review [link].

Determine the Degree of Polynomials

We have learned that a *term* is a constant or the product of a constant and one or more variables. A **monomial** is an algebraic expression with one term. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial in one variable. Some examples of monomial in one variable are. Monomials can also have more than one variable such as and $-4a^2b^3c^2$.

Definition: MONOMIAL

A *monomial* is an algebraic expression with one term. A monomial in one variable is a term of the form ax^m, where a is a constant and m is a whole number.

A monomial, or two or more monomials combined by addition or subtraction, is a **polynomial**. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a **trinomial** has exactly three terms. There are no special names for polynomials with more than three terms.

Definition: POLYNOMIALS

- *polynomial*—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.
- *monomial*—A polynomial with exactly one term is called a monomial.
- *binomial*—A polynomial with exactly two terms is called a binomial.
- *trinomial*—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	$y+1$	$4a^2-7ab+2b^2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	$-13a^3b^2c$
Binomial	$a+7ba+7b$	$4x^2-y^2$	y^2-16	$3p^3q-9p^2q$
Trinomial	$x^2-7x+12$	$9m^2+2mn-8n^2$	$6k^4-k^3+8k$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special

polynomials and just call all the rest *polynomials*.

The **degree of a polynomial** and the degree of its terms are determined by the exponents of the variable. A monomial that has no variable, just a constant, is a special case. The **degree of a constant** is 0.

Definition: DEGREE OF A POLYNOMIAL

- The *degree of a term* is the sum of the exponents of its variables.
- The *degree of a constant* is 0.
- The *degree of a polynomial* is the highest degree of all its terms.

Let's see how this works by looking at several polynomials. We'll take it step by step, starting with monomials, and then progressing to polynomials with more terms. Let's start by looking at a monomial. The monomial $8ab^2$ has two variables a and b. To find the degree we need to find the sum of the exponents. The variable a doesn't have an exponent written, but remember that means the exponent is 1. The exponent of b is 2. The sum of the exponents, 1+2,1+2, is 3 so the degree is 3.



Here are some additional examples.

Monomials	14	$8ab^2$	$-9x^3y^5$	$-13a$
Degree	0	3	8	1
Binomial	$h+7$	$7b^2-3b$	x^2y^2-25	$4n^3-8n^2$
Degree of each term	1 0	2 1	4 0	3 2
Degree of polynomial	1	2	4	3
Trinomial	$x^2-12x+27$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of polynomial	2	2	6	4
Polynomial	$y-1$	$3y^2-2y-5$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of polynomial	1	2	4	

Working with polynomials is easier when you list the terms in descending order of degrees. When a polynomial is written this way, it is said to be in **standard form of a polynomial**. Get in the habit of writing the term with the highest degree first.

Example 2.4.1

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. $7y2-5y+3$
b. $-2a^4b^2$
c. $3x5-4x3-6x2+x-8$
d. $2y-8xy^3$
e. 15

Answer

	Polynomial	Number of terms	Type	Degree of terms	Degree of polynomial

	Polynomial	Number of terms	Type	Degree of terms	Degree of polynomial
ⓐ	$7y^2-5y+3$	3	Trinomial	2, 1, 0	2
ⓑ	$-2a^4b^2-2a^4b^2$	1	Monomial	4, 2	6
ⓒ	$3x5-4x3-6x2+x-8$	5	Polynomial	5, 3, 2, 1, 0	5
ⓐ	$7y^2-5y+3$	3	Trinomial	2, 1, 0	2
ⓓ	$2y-8xy^3$	2	Binomial	1, 4	4
ⓑ	$-2a^4b^2-2a^4b^2$	1	Monomial	4, 2	6
ⓔ	15	1	Monomial	0	0
ⓒ	$3x5-4x3-6x2+x-8$	5	Polynomial	5, 3, 2, 1, 0	5
ⓓ	$2y-8xy^3$	2	Binomial	1, 4	4
ⓔ	15	1	Monomial	0	0

? Example 2.4.2

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. -5
b. $8y^3-7y^2-y-3$
c. $-3x^2y-5xy+9xy^3$
d. $81m^2-4n^2$
e. $-3x^6y^3z$

Answer a

monomial, 0

Answer b

polynomial, 3

Answer c

trinomial, 3

Answer d

binomial, 2

Answer b

monomial, 10

? Example 2.4.3

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. $64k^3-8$
b. $9m^3+4m^2-2$
c. 56
d. $8a^4-7a^3b-6a^2b^2-4ab^3+7b^4$
e. $-p^4q^3$

Answer

ⓐbinomial, 3 ⓑ trinomial, 3 ⓒ monomial, 0 ⓓ polynomial, 4 ⓔ monomial, 7

Add and Subtract Polynomials

We have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficients.

Example 2.4.4

Add or subtract:

a. $25y^2+15y^2$
b. $16pq^3-(-7pq^3)$.

Answer a

	$25y^2+15y^2$
Combine like terms.	$40y^2$

Answer b

	$16pq^3-(-7pq^3)$
Combine like terms.	$23pq^3$

Example 2.4.5

Add or subtract:

a. $12q^2+9q^2$
b. $8mn^3-(-5mn^3)$.

Answer

ⓐ $21q^2$ ⓑ $13mn^3$

Example 2.4.6

Add or subtract:

a. $-15c^2+8c^2$
b. $-15y^2z^3-(-5y^2z^3)$

Answer

ⓐ $-7c^2$ ⓑ $-10y^2z^3$

Remember that like terms must have the same variables with the same exponents.

Example 2.4.7

Simplify:

a. $a^2+7b^2-6a^2$
b. $u^2v+5u^2-3v^2$

Answer

ⓐ Combine like terms.

$$a^2+7b^2-6a^2 \;=\; -5a^2+7b^2$$

ⓑ There are no like terms to combine. In this case, the polynomial is unchanged.

$$u^2v+5u^2-3v^2$$

Example 2.4.8

Add:

a. $8y^2+3z^2-3y^2$
b. $m^2n^2-8m^2+4n^2$

Answer

ⓐ $5y^2+3z^2$
ⓑ $m^2n^2-8m^2+4n^2$

Example 2.4.9

Add:

a. $3m^2+n^2-7m^2$
b. $pq^2-6p-5q^2$

Answer

ⓐ $-4m^2+n^2$
ⓑ $pq^2-6p-5q^2$

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Example 2.4.10

Find the sum: $(7y^2-2y+9)+(4y^2-8y-7)$.

Answer

Identify like terms.	$(\underline{\underline{7y^2}}-\underline{2y}+9)+(\underline{\underline{4y^2}}-\underline{8y}-7)$
Rewrite without the parentheses, rearranging to get the like terms together.	$\underline{\underline{7y^2+4y^2}}-\underline{2y-8y}+9-7$
Combine like terms.	$11y^2-10y+2$

Example 2.4.11

Find the sum: $(7x^2-4x+5)+(x^2-7x+3)$

Answer

$8x^2-11x+8$

Example 2.4.12

Find the sum: $(14y^2+6y-4)+(3y^2+8y+5)$

Answer

$17y^2+14y+1$

Be careful with the signs as you distribute while subtracting the polynomials in the next example.

Example 2.4.13

Find the difference: $(9w^2-7w+5)\ -\ (2w^2-4)$

Answer

	$(9w^2-7w+5)\ -\ (2w^2-4)$
Distribute and identify like terms.	$\underline{\underline{9w^2}}-\underline{7w}+5-\underline{\underline{2w^2}}+4$
Rearrange the terms.	$\underline{\underline{9w^2-2w^2}}-\underline{7w}+5+4$
Combine like terms.	$7w^2-7w+9$

Example 2.4.14

Find the difference: $(8x^2+3x-19)\ -\ (7x^2-14)$

Answer

x^2+3x-5

Example 2.4.15

Find the difference: $(9b^2-5b-4)\ -\ (3b^2-5b-7)$

Answer

$6b^2+3$

Example 2.4.16

Subtract $(p^2+10pq-2q^2)$ from (p^2+q^2).

Answer

	$(p^2+q^2)\ -\ (p^2+10pq-2q^2)$
Distribute and identify like terms.	$\underline{\underline{p^2}}+\underline{q^2}-\underline{\underline{p^2}}-10pq+\underline{2q^2}$
Rearrange the terms, putting like terms together.	$\underline{\underline{p^2-p^2}}-10pq+\underline{q^2+2q^2}$
Combine like terms.	$-10pq+3q^2$

Example 2.4.17

Subtract $(a^2+5ab-6b^2)$ from (a^2+b^2)

Answer

$-5ab+7b^2$

Example 2.4.18

Subtract $(m^2-7mn-3n^2)$ from (m^2+n^2).

Answer

7mn+4n^2

Example 2.4.19

Find the sum: $(u^2-6uv+5v^2)+(3u^2+2uv)$

Answer

	$(u^2-6uv+5v^2)+(3u^2+2uv)$
Distribute and identify like terms.	$\underline{\underline{u^2}}-\underline{6uv}+5v^2+\underline{\underline{\underline{3u^2}}}+\underline{2uv}$
Rearrange the terms to put like terms together.	$\underline{\underline{u^2}}+\underline{\underline{\underline{3u^2}}}-\underline{6uv}+\underline{2uv}+5v^2$
Combine like terms.	$4u^2-4uv+5v^2$

Example 2.4.20

Find the sum: $(3x^2-4xy+5y^2)+(2x^2-xy)$

Answer

$5x^2-5xy+5y^2$

Example 2.4.21

Find the sum: $(2x^2-3xy-2y^2)+(5x^2-3xy)$

Answer

$7x^2-6xy-2y^2$

When we add and subtract more than two polynomials, the process is the same.

Example 2.4.22

Simplify: $(a^3-a^2b)-(ab^2+b^3)+(a^2b+ab^2)$

Answer

	$(a^3-a^2b)-(ab^2+b^3)+(a^2b+ab^2)$
Distribute	$a^3-a^2b-ab^2-b^3+a^2b+ab^2$
Rearrange the terms to put like terms together.	$a^3-a^2b+a^2b-ab^2+ab^2-b^3$
Combine like terms.	a^3-b^3

Example 2.4.23

Simplify: $(x^3-x^2y)-(xy^2+y^3)+(x^2y+xy^2)$

Answer

x^3+y^3

Example 2.4.24

Simplify: $(p^3 - p^2q) + (pq^2 + q^3) - (p^2q + pq^2)$

Answer

$p^3 - 3p^2q + q^3$

Evaluate a Polynomial Function for a Given Value

A **polynomial function** is a function defined by a polynomial. For example, $f(x) = x^2 + 5x + 6$ and $g(x) = 3x - 4$ are polynomial functions, because $x^2 + 5x + 6$ and $3x - 4$ are polynomials.

Definition: POLYNOMIAL FUNCTION

A *polynomial function* is a function whose range values are defined by a polynomial.

In Graphs and Functions, where we first introduced functions, we learned that evaluating a function means to find the value of $f(x)$ for a given value of x. To evaluate a polynomial function, we will substitute the given value for the variable and then simplify using the order of operations.

Example 2.4.25

For the function $f(x) = 5x^2 - 8x + 4$ find:

a. $f(4)$
b. $f(-2)$
c. $f(0)$.

Answer

ⓐ

	$f(x) = 5x^2 - 8x + 4$
To find $f(4)$, substitute 4 for x.	$f(4) = 5(4)^2 - 8(4) + 4$
Simplify the exponents.	$f(4) = 5 \cdot 16 - 8(4) + 4$
Multiply.	$f(4) = 80 - 32 + 4$
Simplify.	$f(4) = 52$

ⓑ

	$f(x) = 5x^2 - 8x + 4$
To find $f(-2)$, substitute -2 for x.	$f(-2) = 5(-2)^2 - 8(-2) + 4$
Simplify the exponents.	$f(-2) = 5 \cdot 4 - 8(-2) + 4$
Multiply.	$f(-2) = 20 + 16 + 4$
Simplify.	$f(-2) = 40$

ⓒ

	$f(x) = 5x^2 - 8x + 4$
To find $f(0)$, substitute 0 for x.	$f(0) = 5(0)^2 - 8(0) + 4$
Simplify the exponents.	$f(0) = 5 \cdot 0 - 8(0) + 4$

Multiply.	$f(0) = 0 + 0 + 4$

Example 2.4.26

For the function $f(x) = 3x^2 + 2x - 15$, find

a. $f(3)$
b. $f(-5)$
c. $f(0)$.

Answer

ⓐ 18 ⓑ 50 ⓒ -15

Example 2.4.27

For the function $g(x) = 5x^2 - x - 4$, find

a. $g(-2)$
b. $g(-1)$
c. $g(0)$.

Answer

ⓐ 20 ⓑ 2 ⓒ -4

The polynomial functions similar to the one in the next example are used in many fields to determine the height of an object at some time after it is projected into the air. The polynomial in the next function is used specifically for dropping something from 250 ft.

Example 2.4.28

The polynomial function $h(t) = -16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 2$ seconds.

Answer

$$h(t) = -16t^2 + 250$$

To find $h(2)$, substitute $t = 2$.	$h(2) = -16(2)^2 + 250$
Simplify.	$h(2) = -16 \cdot 4 + 250$
Simplify.	$h(2) = -64 + 250$
Simplify.	$h(2) = 186$

After 2 seconds the height of the ball is 186 feet.

Example 2.4.29

The polynomial function $h(t) = -16t^2 + 150$ gives the height of a stone t seconds after it is dropped from a 150-foot tall cliff. Find the height after $t = 0$ seconds (the initial height of the object).

Answer

The height is 150 feet.

Example 2.4.30

The polynomial function $h(t) = -16t^2 + 175$ gives the height of a ball t seconds after it is dropped from a 175-foot tall bridge. Find the height after $t = 3$ seconds.

Answer

The height is 31 feet.

Add and Subtract Polynomial Functions

Just as polynomials can be added and subtracted, polynomial functions can also be added and subtracted.

Definition: ADDITION AND SUBTRACTION OF POLYNOMIAL FUNCTIONS

For functions $f(x)$ and $g(x)$,

$$(f+g)(x) = f(x) + g(x) \tag{2.4.1}$$

$$(f-g)(x) = f(x) - g(x) \tag{2.4.2}$$

Example 2.4.31

For functions $f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$, find:

a. $(f+g)(x)$
b. $(f+g)(3)$
c. $(f-g)(x)$
d. $(f-g)(-2)$.

Answer

ⓐ

	$(f+g)(x) = f(x) + g(x)$
Substitute $f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$.	$(f+g)(x) = (3x^2 - 5x + 7) + (x^2 - 4x - 3)$
Rewrite without the parentheses.	$(f+g)(x) = 3x^2 - 5x + 7 + x^2 - 4x - 3$
Put like terms together.	$(f+g)(x) = 3x^2 + x^2 - 5x - 4x + 7 - 3$
Combine like terms.	$(f+g)(x) = 4x^2 - 9x + 4$

ⓑ In part (a) we found $(f+g)(x)$ and now are asked to find $(f+g)(3)$.

$$(f+g)(x) = 4x^2 - 9x + 4$$

To find $(f+g)(3)$, substitute $x = 3$. $(f+g)(3) = 4(3)^2 - 9 \cdot 3 + 4$

$$(f+g)(3) = 4 \cdot 9 - 9 \cdot 3 + 4$$

$$(f+g)(3) = 36 - 27 + 4$$

Notice that we could have found $(f+g)(3)$ by first finding the values of $f(3)$ and $g(3)$ separately and then adding the results.

Find $f(3)$.	$f(x) = 3x^2 - 5x + 7$

	$f(3) = 3(3)^2 - 5(3) + 7$
	$f(3) = 19$
Find $g(3)$.	$g(x) = x^2 - 4x - 3$
	$g(3) = 3^2 - 4(3) - 3$
	$g(3) = -6$
Find $(f+g)(3)$.	$(f+g)(x) = f(x) + g(x)$
	$(f+g)(3) = f(3) + g(3)$
Substitute $f(3) = 19$ and $g(3) = -6$.	$(f+g)(3) = 19 + (-6)$
	$(f+g)(3) = 13$

ⓒ

	$(f-g)(x) = f(x) - g(x)$
Substitute $f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$.	$(f-g)(x) = (3x^2 - 5x + 7) - (x^2 - 4x - 3)$
Rewrite without the parentheses.	$(f-g)(x) = 3x^2 - 5x + 7 - x^2 + 4x + 3$
Put like terms together.	$(f-g)(x) = 3x^2 - x^2 - 5x + 4x + 7 + 3$
Combine like terms.	$(f-g)(x) = 2x^2 - x + 10$

ⓓ

	$(f-g)(x) = 2x^2 - x + 10$
To find $(f-g)(-2)$, substitute $x = -2$.	$(f-g)(-2) = 2(-2)^2 - (-2) + 10$
	$(f-g)(-2) = 2 \cdot 4 - (-2) + 10$
	$(f-g)(-2) = 20$

? Example 2.4.32

For functions $f(x) = 2x^2 - 4x + 3$ and $g(x) = x^2 - 2x - 6$, find: ⓐ $(f+g)(x)$ ⓑ $(f+g)(3)$ ⓒ $(f-g)(x)$ ⓓ $(f-g)(-2)$.

Answer

ⓐ $(f+g)(x) = 3x^2 - 6x - 3$

ⓑ $(f+g)(3) = 6$

ⓒ $(f-g)(x) = x^2 - 2x + 9$

ⓓ $(f-g)(-2) = 17$

? Example 2.4.33

For functions $f(x) = 5x^2 - 4x - 1$ and $g(x) = x^2 + 3x + 8$, find ⓐ $(f+g)(x)$ ⓑ $(f+g)(3)$ ⓒ $(f-g)(x)$ ⓓ $(f-g)(-2)$.

Answer

ⓐ $(f+g)(x) = 6x^2 - x + 7$

ⓑ $(f+g)(3)=58$

ⓒ $(f-g)(x)=4x^2-7x-9$

ⓓ $(f-g)(-2)=21$

Access this online resource for additional instruction and practice with adding and subtracting polynomials.

- Adding and Subtracting Polynomials

Key Concepts

- **Monomial**
  - A **monomial** is an algebraic expression with one term.
  - A monomial in one variable is a term of the form axm,axm, where *a* is a constant and *m* is a whole number.
- **Polynomials**
  - **Polynomial**—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.
  - **monomial** —A polynomial with exactly one term is called a monomial.
  - **binomial** — A polynomial with exactly two terms is called a binomial.
  - **trinomial** —A polynomial with exactly three terms is called a trinomial.
- **Degree of a Polynomial**
  - The **degree of a term** is the sum of the exponents of its variables.
  - The **degree of a constant** is 0.
  - The **degree of a polynomial** is the highest degree of all its terms.

Glossary

binomial

A binomial is a polynomial with exactly two terms.

degree of a constant

The degree of any constant is 0.

degree of a polynomial

The degree of a polynomial is the highest degree of all its terms.

degree of a term

The degree of a term is the sum of the exponents of its variables.

monomial

A monomial is an algebraic expression with one term. A monomial in one variable is a term of the form axm,axm, where *a* is a constant and *m* is a whole number.

polynomial

A monomial or two or more monomials combined by addition or subtraction is a polynomial.

standard form of a polynomial

A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial

A trinomial is a polynomial with exactly three terms.

polynomial function

A polynomial function is a function whose range values are defined by a polynomial.

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2.4E: Exercises

Practice Makes Perfect

Determine the Type of Polynomials

In the following exercises, determine if the polynomial is a **monomial**, **binomial**, **trinomial**, or **other polynomial**. Also give the **degree** of each polynomial.

1. ⓐ $47x^5-17x^2y^3+y^2$
ⓑ $5c^3+11c^2-c-8$
ⓒ $59ab+13b$
ⓓ 4
ⓔ $4pq+17$

Answer

ⓐ trinomial, degree 5
ⓑ other polynomial, degree 3
ⓒ binomial, degree 2
ⓓ monomial, degree 0
ⓔ binomial, degree 2

2. ⓐ x^2-y^2
ⓑ $-13c^4$
ⓒ $a^2+2ab-7b^2$
ⓓ $4x^2y^2-3xy+8$
ⓔ 19

3. ⓐ $8y-5x$
ⓑ $y^2-5yz-6z^2$
ⓒ $y^3-8y^2+2y-16$
ⓓ $81ab^4-24a^2b^2+3b$
ⓔ -18

Answer

ⓐ binomial, degree 1
ⓑ trinomial, degree 2
ⓒ other polynomial, degree 3
ⓓ trinomial, degree 5
ⓔ monomial, degree 0

4. ⓐ $11y^2$
ⓑ -73
ⓒ $6x^2-3xy+4x-2y+y^2$
ⓓ $4y^2+17z^2$
ⓔ $5c^3+11c^2-c-8$

5. ⓐ $5a^2+12ab-7b^2$
ⓑ $18xy^2z$
ⓒ $5x+2$
ⓓ $y^3-8y^2+2y-16$
ⓔ -24

Answer

ⓐ trinomial, degree 2
ⓑ monomial, degree 4
ⓒ binomial, degree 1
ⓓ other polynomial, degree 3
ⓔ monomial, degree 0

6. ⓐ $9y^3 - 10y^2 + 2y - 6$
ⓑ $-12p^3q$
ⓒ $a^2 + 9ab + 18b^2$
ⓓ $20x^2y^2 - 10a^2b^2 + 30$
ⓔ 17

7. ⓐ $14s - 29t$
ⓑ $z^2 - 5z - 6$
ⓒ $y^3 - 8y^2z + 2yz^2 - 16z^3$
ⓓ $23ab^2 - 14$
ⓔ -3

Answer

ⓐ binomial, degree 1
ⓑ trinomial, degree 2
ⓒ other polynomial, degree 3
ⓓ binomial, degree 3
ⓔ monomial, degree 0

8. ⓐ $15xy$
ⓑ 15
ⓒ $6x^2 - 3xy + 4x - 2y + y^2$
ⓓ $10p - 9q$
ⓔ $m^4 + 4m^3 + 6m^2 + 4m + 1$

Add and Subtract Polynomials

In the following exercises, add or subtract the monomials.

9. ⓐ $7x^2 + 5x^2$
ⓑ $4a - 9a$

Answer

ⓐ $12x^2$ ⓑ $-5a$

10. ⓐ $4y^3 + 6y^3$
ⓑ $-y - 5y$

11. ⓐ $-12w + 18w$
ⓑ $7x^2y - (-12x^2y)$

Answer

ⓐ $6w$
ⓑ $19x^2y$

12. ⓐ $-3m+9m$
ⓑ $15yz^2-(-8yz^2)$

13. $7x^2+5x^2+4a-9a$

Answer

$12x^2-5a$

14. $4y^3+6y^3-y-5y$

15. $-12w+18w+7x^2y-(-12x^2y)$

Answer

$6w+19x^2y$

16. $-3m+9m+15yz^2-(-8yz^2)$

17. ⓐ $-5b-17b$
ⓑ $3xy-(-8xy)+5xy$

Answer

ⓐ $-22b$
ⓑ $16xy$

18. ⓐ $-10x-35x$
ⓑ $17mn^2-(-9mn^2)+3mn^2$

19. ⓐ $12a+5b-22a$
ⓑ $pq^2-4p-3q^2$

Answer

ⓐ $-10a+5b$
ⓑ $pq^2-4p-3q^2$

20. ⓐ $14x-3y-13x$
ⓑ $a^2b-4a-5ab^2$

21. ⓐ $2a^2+b^2-6a^2$
ⓑ $x^2y-3x+7xy^2$

Answer

ⓐ $-4a^2+b^2$
ⓑ $x^2y-3x+7xy^2$

22. ⓐ $5u^2+4v^2-6u^2$
ⓑ $12a+8b$

23. ⓐ $xy^2-5x-5y^2$
ⓑ $19y+5z$

Answer

ⓐ $xy^2-5x-5y^2$
ⓑ $19y+5z$

24. $12a+5b-22a+pq^2-4p-3q^2$

25. $14x-3y-13x+a^2b-4a-5ab^2$

Answer

$x-3y+a^2b-4a-5ab^2$

26. $2a^2+b^2-6a^2+x^2y-3x+7xy^2$

27. $5u^2+4v^2-6u^2+12a+8b$

Answer

$-u^2+4v^2+12a+8b$

28. $xy^2-5x-5y^2+19y+5z$

29. Add: $4a, -3b, -8a$

Answer

$-4a-3b$

30. Add: $4x, 3y, -3x$

31. Subtract $5x^6$ from $-12x^6$

Answer

$-7x^6$

32. Subtract $2p^4$ from $-7p^4$

In the following exercises, add the polynomials.

33. $(5y^2+12y+4)+(6y^2-8y+7)$

Answer

$11y^2+4y+11$

34. $(4y^2+10y+3)+(8y^2-6y+5)$

35. $(x^2+6x+8)+(-4x^2+11x-9)$

Answer

$-3x^2+17x-1$

36. $(y^2+9y+4)+(-2y^2-5y-1)$

37. $(8x^2-5x+2)+(3x^2+3)$

Answer

$11x^2-5x+5$

38. $(7x^2-9x+2)+(6x^2-4)$

39. $(5a^2+8)+(a^2-4a-9)$

Answer

$6a^2-4a-1$

40. $(p^2-6p-18)+(2p^2+11)$

In the following exercises, subtract the polynomials.

41. $(4m^2-6m-3)-(2m^2+m-7)$

Answer

$2m^2-7m+4$

42. $(3b^2-4b+1)-(5b^2-b-2)$

43. $(a^2+8a+5)-(a^2-3a+2)$

Answer

$11a+3$

44. $(b^2-7b+5)-(b^2-2b+9)$

45. $(12s^2-15s)-(s-9)$

Answer

$12s^2-14s+9$

46. $(10r^2-20r)-(r-8)$

In the following exercises, subtract the polynomials.

47. Subtract $(9x^2+2)$ from $(12x^2-x+6)$

Answer

$3x^2-x+4$

48. Subtract $(5y^2-y+12)$ from $(10y^2-8y-20)$

49. Subtract $(7w^2-4w+2)$ from $(8w^2-w+6)$

Answer

w^2+3w+4

50. Subtract $(5x^2 - x + 12)$ from $(9x^2 - 6x - 20)$

In the following exercises, find the difference of the polynomials.

51. Find the difference of $(w^2 + w - 42)$ and $(w^2 - 10w + 24)$

Answer

$11w - 64$

52. Find the difference of $(z^2 - 3z - 18)$ and $(z^2 + 5z - 20)$

In the following exercises, add the polynomials.

53. $(7x^2 - 2xy + 6y^2) + (3x^2 - 5xy)$

Answer

$10x^2 - 7xy + 6y^2$

54. $(-5x^2 - 4xy - 3y^2) + (2x^2 - 7xy)$

55. $(7m^2 + mn - 8n^2) + (3m^2 + 2mn)$

Answer

$10m^2 + 3mn - 8n^2$

56. $(2r^2 - 3rs - 2s^2) + (5r^2 - 3rs)$

In the following exercises, add or subtract the polynomials.

57. $(a^2 - b^2) - (a^2 + 3ab - 4b^2)$

Answer

$-3ab + 3b^2$

58. $(m^2 + 2n^2) - (m^2 - 8mn - n^2)$

59. $(p^3 - 3p^2q) + (2pq^2 + 4q^3) - (3p^2q + pq^2)$

Answer

$p^3 - 6p^2q + pq^2 + 4q^3$

60. $(a^3 - 2a^2b) + (ab^2 + b^3) - (3a^2b + 4ab^2)$

61. $(x^3 - x^2y) - (4xy^2 - y^3) + (3x^2y - xy^2)$

Answer

$x^3 + 2x^2y - 5xy^2 + y^3$

62. $(x^3 - 2x^2y) - (xy^2 - 3y^3) - (x^2y - 4xy^2)$

Evaluate a Polynomial Function for a Given Value

In the following exercises, find the function values for each polynomial function.

63. For the function $f(x) = 8x^2 - 3x + 2$, find:
ⓐ $f(5)$ ⓑ $f(-2)$ ⓒ $f(0)$

Answer

ⓐ 187 ⓑ 40 ⓒ 2

64. For the function $f(x) = 5x^2 - x - 7$, find:
ⓐ $f(-4)$ ⓑ $f(1)$ ⓒ $f(0)$

65. For the function $g(x) = 4 - 36x$, find:
ⓐ $g(3)$ ⓑ $g(0)$ ⓒ $g(-1)$

Answer

ⓐ -104 ⓑ 4 ⓒ 40

66. For the function $g(x) = 16 - 36x^2$, find:
ⓐ $g(-1)$ ⓑ $g(0)$ ⓒ $g(2)$

In the following exercises, find the height for each polynomial function.

67. A painter drops a brush from a platform 75 feet high. The polynomial function $h(t) = -16t^2 + 75$ gives the height of the brush t seconds after it was dropped. Find the height after $t = 2$ seconds.

Answer

The height is 11 feet.

68. A girl drops a ball off the cliff into the ocean. The polynomial $h(t) = -16t^2 + 200$ gives the height of a ball t seconds after it is dropped. Find the height after $t = 3$ seconds.

69. A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of p dollars each is given by the polynomial function $R(p) = -4p^2 + 420p$. Find the revenue received when $p = 60$ dollars.

Answer

The revenue is $10,800.

70. A manufacturer of the latest basketball shoes has found that the revenue received from selling the shoes at a cost of p dollars each is given by the polynomial $R(p) = -4p^2 + 420p$. Find the revenue received when $p = 90$ dollars.

71. The polynomial $C(x) = 6x^2 + 90x$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and height 6 feet. Find the cost of producing a box with $x = 4$ feet.

Answer

The cost is $456.

72. The polynomial $C(x) = 6x^2 + 90x$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and height 4 feet. Find the cost of producing a box with $x = 6$ feet.

Add and Subtract Polynomial Functions

In each example, find ⓐ $(f+g)(x)$ ⓑ $(f+g)(2)$ ⓒ $(f-g)(x)$ ⓓ $(f-g)(3)$.

73. $f(x)=2x^2-4x+1$ and $g(x)=5x^2+8x+3$

Answer

ⓐ $(f+g)(x)=7x^2+4x+4$
ⓑ $(f+g)(2)=40$
ⓒ $(f-g)(x)=-3x^2-12x-2$
ⓓ $(f-g)(-3)=7$

74. $f(x)=4x^2-7x+3$ and $g(x)=4x^2+2x-1$

75. $f(x)=3x^3-x^2-2x+3$ and $g(x)=3x^3-7x$

Answer

ⓐ $(f+g)(x)=6x^3-x^2-9x+3$
ⓑ $(f+g)(2)=29$
ⓒ $(f-g)(x)=-x^2+5x+3$
ⓓ $(f-g)(-3)=-21$

76. $f(x)=5x^3-x^2+3x+4$ and $g(x)=8x^3-1$

Writing Exercises

77. Using your own words, explain the difference between a monomial, a binomial, and a trinomial.

Answer

Answers will vary.

78. Using your own words, explain the difference between a polynomial with five terms and a polynomial with a degree of 5.

79. Ariana thinks the sum $6y^2+5y^4$ is $11y^6$. What is wrong with her reasoning?

Answer

Answers will vary.

80. Is every trinomial a second-degree polynomial? If not, give an example.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify polynomials, monomials, binomials, and trinomials.			
determine the degree of polynomials.			
add and subtract monomials.			
add and subtract polynomials.			
evaluate a polynomial for a given value.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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2.5: Multiply Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Multiply monomials
- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a polynomial by a polynomial
- Multiply special products
- Multiply polynomial functions

Before you get started, take this readiness quiz.

1. Distribute: $2(x+3)$.
   If you missed this problem, review [link].
2. Simplify: a. 9^2 b. $(-9)^2$ c. -9^2.
   If you missed this problem, review [link].
3. Evaluate: $2x^2-5x+3$ for $x=-2$.
   If you missed this problem, review [link].

Multiply Monomials

We are ready to perform operations on polynomials. Since monomials are algebraic expressions, we can use the properties of exponents to multiply monomials.

Example 2.5.1

Multiply:

a. $(3x^2)(-4x^3)$
b. $\left(\frac{5}{6}x^3y\right)(12xy^2)$.

Answer a

	$(3x^2)(-4x^3)$
Use the Commutative Property to rearrange the terms.	$3\cdot(-4)\cdot x^2\cdot x^3$
	$-12x^5$

Answer b

	$\left(\frac{5}{6}x^3y\right)(12xy^2)$
Use the Commutative Property to rearrange the terms.	$\frac{5}{6}\cdot 12\cdot x^3\cdot x\cdot y\cdot y^2$
Multiply.	$10x^4y^3$

Example 2.5.2

Multiply:

a. $(5y^7)(-7y^4)$
b. $(25a^4b^3)(15ab^3)$

Answer a

$-35y^{11}$

Answer b

$375a^5b^6$

Example 2.5.3

Multiply:

a. $(-6b^4)(-9b^5)$
b. $(23r^5s)(12r^6s^7)$.

Answer a

$54b^9$

Answer b

$276r^{11}s^8$

Multiply a Polynomial by a Monomial

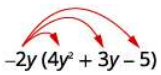
Multiplying a polynomial by a monomial is really just applying the Distributive Property.

Example 2.5.4

Multiply:

a. $-2y(4y^2+3y-5)$
b. $3x^3y(x^2-8xy+y^2)$.

Answer a

	
Distribute.	$-2y \cdot 4y^2 + (-2y) \cdot 3y - (-2y) \cdot 5$
Multiply.	$-8y^3 - 6y^2 + 10y$

Answer b

	$3x^3y(x^2-8xy+y^2)$
Distribute.	$3x^3y \cdot x^2 + (3x^3y) \cdot (-8xy) + (3x^3y) \cdot y^2$
Multiply.	$3x^5y - 24x^4y^2 + 3x^3y^3$

Example 2.5.5

Multiply:

a. $-3y(5y^2+8y^7)$
b. $4x^2y^2(3x^2-5xy+3y^2)$

Answer a

$-15y^3-24y^8$

Answer b

$12x^4y^2-20x^3y^3+12x^2y^4$

Example 2.5.6

Multiply:

a. $4x^2(2x^2-3x+5)$
b. $-6a^3b(3a^2-2ab+6b^2)$

Answer a

$8x^4-12x^3+20x^2$

Answer b

$-18a^5b+12a^4b^2-36a^3b^3$

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

Example 2.5.7

Multiply:

a. $(y+5)(y+8)$
b. $(4y+3)(2y-5)$.

Answer

ⓐ

	$(y+5)(y+8)$
Distribute $(y+8)$.	$y(y+8)+5(y+8)$
Distribute again.	$y^2+8y+5y+40$
Combine like terms.	$y^2+13y+40$

ⓑ

	$(4y+3)(2y-5)$
Distribute.	$4y(2y-5)+3(2y-5)$
Distribute again.	$8y^2-20y+6y-15$
Combine like terms.	$8y^2-14y-15$

Example 2.5.8

Multiply:

a. $(x+8)(x+9)$
b. $(3c+4)(5c-2)$.

Answer a

$x^2+17x+72$

Answer b

$15c^2+14c-8$

Example 2.5.9

Multiply:

a. $(5x+9)(4x+3)$
b. $(5y+2)(6y-3)$.

Answer a

$20x^2+51x+27$

Answer b

$30y^2-3y-6$

If you multiply binomials often enough you may notice a pattern. Notice that the first term in the result is the product of the *first* terms in each binomial. The second and third terms are the product of multiplying the two *outer* terms and then the two *inner* terms. And the last term results from multiplying the two *last* terms,

We abbreviate "First, Outer, Inner, Last" as FOIL. The letters stand for 'First, Outer, Inner, Last'. We use this as another method of multiplying binomials. The word FOIL is easy to remember and ensures we find all **four** products.

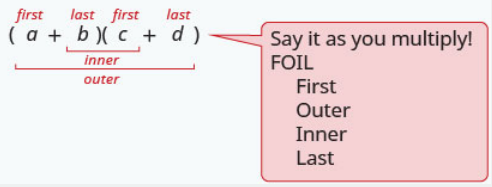
Let's multiply $(x+3)(x+7)$ using both methods.

Distributive Property	FOIL
$(x+3)(x+7)$	$(x+3)(x+7)$
$x(x+7)+3(x+7)$	
$x^2+7x+3x+21$ F O I L	$x^2+7x+3x+21$ F O I L
$x^2+10x+21$	$x^2+10x+21$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

DEFINITION: USE THE FOIL METHOD TO MULTIPLY TWO BINOMIALS.

Step 1. Multiply the *First* terms.
Step 2. Multiply the *Outer* terms.
Step 3. Multiply the *Inner* terms.
Step 4. Multiply the *Last* terms.
Step 5. Combine like terms, when possible.



When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

Example 2.5.10

Multiply:

a. $(y-7)(y+4)$
b. $(4x+3)(2x-5)$.

Answer

a.

$(y-7)(y+4)$

Step 1. Multiply the *First* terms. $y^2 + \_ + \_ + \_$ (F O I L)

Step 2. Multiply the *Outer* terms. $y^2 + 4y + \_ + \_$ (F O I L)

Step 3. Multiply the *Inner* terms. $y^2 + 4y - 7y + \_$ (F O I L)

Step 4. Multiply the *Last* terms. $y^2 + 4y - 7y - 28$ (F O I L)

Step 5. Combine like terms. $y^2 - 3y - 28$

b.

$(4x-3)(2x-5)$

Step 1. Multiply the *First* terms, $4x \cdot 2x$. $8x^2 + \_ + \_ + \_$ (F O I L)

Step 2. Multiply the *Outer* terms, $4x \cdot (-5)$. $8x^2 - 20x + \_ + \_$ (F O I L)

Step 3. Multiply the *Inner* terms, $3 \cdot 2x$. $8x^2 - 2x + 6x + \_$ (F O I L)

Step 4. Multiply the *Last* terms, $3 \cdot (-5)$. $8x^2 - 20x + 6x - 15$ (F O I L)

Step 5. Combine like terms. $8x^2 - 14x - 15$

Exercise 2.5.11

Multiply:

a. $(x-7)(x+5)$
b. $(3x+7)(5x-2)$.

Answer

a. $x^2 - 2x - 35$
b. $15x^2 + 29x - 14$

Exercise 2.5.12

Multiply:

a. $(b-3)(b+6)$
b. $(4y+5)(4y-10)$.

Answer

a. $b^2 + 3b - 18$
b. $16y^2 - 20y - 50$

The final products in the last example were trinomials because we could combine the two middle terms. This is not always the case.

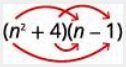
Example 2.5.13

Multiply:

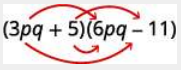
a. $(n^2+4)(n-1)$
b. $(3pq+5)(6pq-11)$.

Answer

a.

	$(n^2+4)(n-1)$
	
Step 1. Multiply the *First* terms.	$n^3+\_\_+\_\_+\_\_$ (F O I L)
Step 2. Multiply the *Outer* terms.	$n^3-n^2+\_\_+\_\_$ (F O I L)
Step 3. Multiply the *Inner* terms.	$n^3-n^2+4n+\_\_$ (F O I L)
Step 4. Multiply the *Last* terms.	n^3-n^2+4n-4 (F O I L)
Step 5. Combine like terms—there are none.	n^3-n^2+4n-4

b.

	$(3pq+5)(6pq-11)$
	
Step 1. Multiply the *First* terms.	$18p^2q^2+\_\_+\_\_+\_\_$ (F O I L)
Step 2. Multiply the *Outer* terms.	$18p^2q^2-33pq+\_\_+\_\_$ (F O I L)
Step 3. Multiply the *Inner* terms.	$18p^2q^2-33pq+30pq+\_\_$ (F O I L)
Step 4. Multiply the *Last* terms.	$18p^2q^2-33pq+30pq-55$ (F O I L)
Step 5. Combine like terms.	$18p^2q^2-3pq-55$

Example 2.5.14

Multiply:

a. $(x^2+6)(x-8)$
b. $(2ab+5)(4ab-4)$.

Answer

a. $x^3-8x^2+6x-48$
b. $8a^2b^2+12ab-20$

? Example 2.5.15

Multiply:

a. $(y^2+7)(y-9)$
b. $(2xy+3)(4xy-5)$.

Answer

a. $y^3-9y^2+7y-63$
b. $8x^2y^2+2xy-15$

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

23 ×46		
138	partial product	Start by multiplying 23 by 6 to get 138.
92	partial product	Next, multiply 23 by 4, lining up the partial product in the correct columns.
1058	product	Last you add the partial products.

Now we'll apply this same method to multiply two binomials.

? Example 2.5.16

Multiply using the Vertical Method: $(3y-1)(2y-6)$.

Answer

It does not matter which binomial goes on the top.

$$\begin{array}{r} 3y-1 \\ \times\ 2y-6 \\ \hline \end{array}$$

Multiply $3y-1$ by -6.	$-18y+6$	partial product
Multiply $3y-1$ by $2y$.	$6y^2-2y$	partial product
Add like terms.	$6y^2-20y+6$	

Notice the partial products are the same as the terms in the FOIL method.

$$\begin{array}{r} 3y-1 \\ \times\ 2y-6 \\ \hline -18y+6 \\ 6y^2-2y \\ \hline 6y^2-20x+6 \end{array}$$

$$\begin{array}{l} (3y-1)(2y-6) \\ 6y^2-2y-18y+6 \\ \hline 6y^2-20y+6 \end{array}$$

? Example 2.5.17

Multiply using the Vertical Method: $(5m-7)(3m-6)$.

Answer

$15m^2-51m+42$

Example 2.5.18

Multiply using the Vertical Method: $(6b-5)(7b-3)$.

Answer

$42b^2-53b+15$

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

DEFINITION: MULTIPLYING TWO BINOMIALS

To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Multiply a Polynomial by a Polynomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a polynomial by a polynomial. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method.

Example 2.5.19

Multiply $(b+3)(2b^2-5b+8)$ using ⓐ the Distributive Property and ⓑ the Vertical Method.

Answer

a.

	$(b+3)(2b^2-5b+8)$
Distribute.	$b(2b^2-5b+8)+3(2b^2-5b+8)$
Multiply.	$2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$2b^3+b^2-7b+24$

b. It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b^2-5b+8)$ by 3. Multiply $(2b^2-5b+8)$ by b.	$\begin{array}{r} 2b^2-5b+8 \\ \times \quad b+3 \\ \hline 6b^2-15b+24 \end{array}$
Add like terms.	$\begin{array}{r} 2b^3-5b^2+8b \\ \hline 2b^3+b^2-7b+24 \end{array}$

Example 2.5.20

Multiply $(y-3)(y^2-5y+2)$ using ⓐ the Distributive Property and ⓑ the Vertical Method.

Answer

a. $y^3-8y^2+17y-6$
b. $y^3-8y^2+17y-6$

Example 2.5.21

Multiply $(x+4)(2x^2-3x+5)$ using a) the Distributive Property and b) The Vertical Method.

Answer

a. and b. $2x^3+5x^2-7x+20$

We have now seen two methods you can use to multiply a polynomial by a polynomial. After you practice each method, you'll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

DEFINITION: MULTIPLYING A POLYNOMIAL BY A POLYNOMIAL

To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

Multiply Special Products

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and multiplying them, there is less work to do if you learn to use a pattern. Let's start by looking at three examples and look for a pattern.

Look at these results. Do you see any patterns?

$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

What about the number of terms? In each example we squared a binomial and the result was a trinomial.

$$(a+b)^2 = \_\_\_ + \_\_\_ + \_\_\_$$

Now look at the *first term* in each result. Where did it come from?

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

$$(a+b)^2 = a^2 + \_\_\_ + \_\_\_$$

To get the first term of the product, square the first term.

Where did the *last term* come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

$$(a+b)^2 = \_\_\_ + \_\_\_ + b^2$$

To get the last term of the product, square the last term.

Finally, look at the *middle term*. Notice it came from adding the "outer" and the "inner" terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

$$(a+b)^2 = \_\_\_ + 2ab + \_\_\_$$

$$(a-b)^2 = \_\_\_ - 2ab + \_\_\_$$

To get the middle term of the product, multiply the terms and double their product.

Putting it all together:

definition: BINOMIAL SQUARES PATTERN

If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

To square a binomial, square the first term, square the last term, double their product.

Example 2.5.22

Multiply: a. $(x+5)^2$ b. $(2x-3y)^2$.

Answer

a.

	$(x+5)^2$
Square the first term.	$x^2 + \_\_\_ + \_\_\_$
Square the last term.	$x^2 + \_\_\_ + 5^2$
Double their product.	$x^2 + 2 \cdot x \cdot 5 + 5^2$
Simplify.	$x^2 + 10x + 25$

b.

	$(2x-3y)^2$
Use the pattern.	$(2x)^2 - 2 \cdot 2x \cdot 3y + (3y)^2$
Simplify.	$4x^2 - 12xy + 9y^2$

Example 2.5.23

Multiply: a.$(x+9)^2$ b. $(2c-d)^2$.

Answer

a. $x^2+18x+81$
b. $4c^2-4cd+d^2$

Example 2.5.24

Multiply: a. $(y+11)^2$ b. $(4x-5y)^2$.

Answer

a. $y^2+22y+121$
b. $16x^2-40xy+25y^2$

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a **conjugate pair** and is of the form $(a-b)$, $(a+b)$.

definition: conjugate pair

A **conjugate pair** is two binomials of the form

$$(a-b),(a+b).$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier. Let's look for the pattern by using FOIL to multiply some conjugate pairs.

$(x+9)(x-9)$ $\quad$ $(y-8)(y+8)$ $\quad$ $(2x-5)(2x+5)$

$x^2-9x+9x-81$ $\quad$ $y^2+8y-8y-64$ $\quad$ $4x^2+10x-10x-25$

x^2-81 $\quad$ y^2-64 $\quad$ $4x^2-25$

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Each *first term* is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b)=a^2-\_\_\_$$

To get the first term, square the first term.

The *last term* came from multiplying the last terms, the square of the last term.

$$(a+b)(a-b)=a^2-b^2$$

To get the last term, square the last term.

Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form a^2-b^2. This is called a **difference of squares**.

This leads to the pattern:

definition: PRODUCT OF CONJUGATES PATTERN

If a and b are real numbers,

$(a-b)(a+b)=a^2-b^2$ $\quad$ $(a-b)(a+b)=a^2-b^2$ (conjugates; squares; difference)

The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, write it as a difference of squares.

Example 2.5.25

Multiply using the product of conjugates pattern: a. $(2x+5)(2x-5)$ b. $(5m-9n)(5m+9n)$.

Answer

a.

Are the binomials conjugates?	$(2x+5)(2x-5)$
It is the product of conjugates.	$(a+b)(a-b)$ $(2x+5)(2x-5)$
Square the first term, 2x.2x.	a^2-b^2 $(2x)^2-\_\_$
Square the last term, 5.5.	a^2-b^2 $(2x)^2-5^2$
Simplify. The product is a difference of squares.	a^2-b^2 $4x^2-25$

b.

	$(5m-9n)(5m+9n)$
This fits the pattern.	$(a-b)(a+b)$ $(5m-9n)(5m+9n)$
Use the pattern.	a^2-b^2 $(5m)^2-(9n)^2$
Simplify.	$25m^2-81n^2$

Example 2.5.26

Multiply: a. $(6x+5)(6x-5)$ b. $(4p-7q)(4p+7q)$.

Answer

a. $36x^2-25$
b. $16p^2-49q^2$

Example 2.5.27

Multiply: a. $(2x+7)(2x-7)$ b.$(3x-y)(3x+y)$.

Answer

a. $4x^2-49$ b. $9x^2-y^2$

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

COMPARING THE SPECIAL PRODUCT PATTERNS

Binomial Squares	Product of Conjugates
$(a+b)^2=a^2+2ab+b^2$	$(a-b)(a+b)=a^2-b^2$
$(a-b)^2=a^2-2ab+b^2$	
• Squaring a binomial	• Multiplying conjugates

Binomial Squares	Product of Conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

? Example 2.5.28

Choose the appropriate pattern and use it to find the product:

a. $(2x-3)(2x+3)$ b. $(8x-5)^2$ c. $(6m+7)^2$ d. $(5x-6)(6x+5)$.

Answer

a. $(2x-3)(2x+3)$

These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

	$(\underset{a}{2x}-\underset{b}{3})(\underset{a}{2x}+\underset{b}{3})$
Use the pattern.	$\underset{a^2}{(2x)^2}-\underset{b^2}{3^2}$
Simplify.	$4x^2-9$

b. $(8x-5)^2$

We are asked to square a binomial. It fits the binomial squares pattern.

	$(\underset{a}{8x}-\underset{b}{5})^2$
Use the pattern.	$\underset{a^2}{(8x)^2}-\underset{2ab}{2\cdot 8x\cdot 5}+\underset{b^2}{5^2}$
Simplify.	$64x^2-80x+25$

c. $(6m+7)^2$

Again, we will square a binomial so we use the binomial squares pattern.

	$(\underset{a}{6m}+\underset{b}{7})^2$
Use the pattern.	$\underset{a^2}{(6m)^2}+\underset{2ab}{2\cdot 6m\cdot 7}+\underset{b^2}{7^2}$
Simplify.	$36m^2+84m+49$

d. $(5x-6)(6x+5)$

This product does not fit the patterns, so we will use FOIL.

	$(5x-6)(6x+5)$
Use FOIL.	$30x^2+25x-36x-30$
Simplify.	$30x^2-11x-30$

Example 2.5.29

Choose the appropriate pattern and use it to find the product:

a. $(9b-2)(2b+9)$ b. $(9p-4)^2$ c. $(7y+1)^2$ d. $(4r-3)(4r+3)$.

Answer

a. FOIL; $18b^2+77b-18$
b. Binomial Squares; $81p^2-72p+16$
c. Binomial Squares; $49y^2+14y+1$
d. Product of Conjugates; $16r^2-9$

Example 2.5.30

Choose the appropriate pattern and use it to find the product:

a. $(6x+7)^2$ b. $(3x-4)(3x+4)$ c. $(2x-5)(5x-2)$ d. $(6n-1)^2$.

Answer

a. Binomial Squares; $36x^2+84x+49$ b. Product of Conjugates; $9x^2-16$ c. FOIL; $10x^2-29x+10$ d. Binomial Squares; $36n^2-12n+1$

Multiply Polynomial Functions

Just as polynomials can be multiplied, polynomial functions can also be multiplied.

MULTIPLICATION OF POLYNOMIAL FUNCTIONS

For functions $f(x)$ and $g(x)$,

$$(f\cdot g)(x)=f(x)\cdot g(x) \tag{2.5.1}$$

Example 2.5.31

For functions $f(x)=x+2$ and $g(x)=x^2-3x-4$, find:

a. $(f\cdot g)(x)$
b. $(f\cdot g)(2)$.

Answer

a.

	$(f\cdot g)(x)=f(x)\cdot g(x)$
Substitute for $f(x)$ and $g(x)$	$(f\cdot g)(x)=(x+2)(x^2-3x-4)$
Multiply the polynomials.	$(f\cdot g)(x)=x(x^2-3x-4)+2(x^2-3x-4)$
Distribute.	$(f\cdot g)(x)=x3-3x^2-4x+2x^2-6x-8$
Combine like terms.	$(f\cdot g)(x)=x3-x^2-10x-8$

b. In part a. we found $(f\cdot g)(x)$ and now are asked to find $(f\cdot g)(2)$.

	$(f\cdot g)(x)=x^3-x^2-10x-8$
To find $(f\cdot g)(2)$, substitute $x=2$.	$(f\cdot g)(2)=2^3-2^2-10\cdot 2-8$
	$(f\cdot g)(2)=8-4-20-8$
	$(f\cdot g)(2)=-24$

Example 2.5.32

For functions $f(x) = x - 5$ and $g(x) = x^2 - 2x + 3$, find

a. $(f \cdot g)(x)$
b. $(f \cdot g)(2)$.

Answer a

$(f \cdot g)(x) = x^3 - 7x^2 + 13x - 15$

Answer b

$(f \cdot g)(2) = -9$

Example 2.5.33

For functions $f(x) = x - 7$ and $g(x) = x^2 + 8x + 4$, find

a. $(f \cdot g)(x)$
b. $(f \cdot g)(2)$.

Answer a

$(f \cdot g)(x) = x^3 + x^2 - 52x - 28$

Answer a

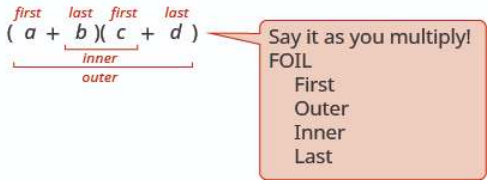
$(f \cdot g)(2) = -120$

Access this online resource for additional instruction and practice with multiplying polynomials.

- Introduction to special products of binomials

Key Concepts

- **How to use the FOIL method to multiply two binomials.**
  Step 1. Multiply the *First* terms.
  Step 2. Multiply the *Outer* terms.
  Step 3. Multiply the *Inner* terms.
  Step 4. Multiply the *Last* terms.
  Step 5. Combine like terms, when possible.



- **Multiplying Two Binomials:** To multiply binomials, use the:
  - Distributive Property
  - FOIL Method
- **Multiplying a Polynomial by a Polynomial:** To multiply a trinomial by a binomial, use the:
  - Distributive Property
  - Vertical Method
- **Binomial Squares Pattern**

$(a + b)^2 = a^2 + 2ab + b^2$

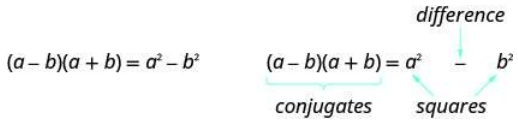
$\underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$

$(a - b)^2 = a^2 - 2ab + b^2$

$\underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$

If a and b are real numbers,

- **Product of Conjugates Pattern**
  If a, b are real numbers



The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, write it as a difference of squares.

- **Comparing the Special Product Patterns**

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)^2 = a^2 - 2ab + b^2$
$(a-b)(a+b) = a^2 - b^2$	
• Squaring a binomial	• Multiplying conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

- **Multiplication of Polynomial Functions:**
  - For functions $f(x)$ and $g(x)$,

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Glossary

conjugate pair

A conjugate pair is two binomials of the form $(a-b)$ and $(a+b)$. The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

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2.5E: Exercises

Practice Makes Perfect

Multiply Monomials

In the following exercises, multiply the monomials.

1. ⓐ $(6y^7)(-3y^4)$ ⓑ $(\frac{4}{7}rs^2)(\frac{1}{4}rs^3)$

2. ⓐ $(-10x^5)(-3x^3)$ ⓑ $(58x^3y)(24x^5y)$

Answer

ⓐ $30x^8$ ⓑ $15x^8y^2$

3. ⓐ $(-8u^6)(-9u)$ ⓑ $(\frac{2}{3}x^2y)(\frac{3}{4}xy^2)$

4. ⓐ $(-6c^4)(-12c)$ ⓑ $(\frac{3}{5}m^3n^2)(\frac{5}{9}m^2n^3)$

Answer

ⓐ $72c^5$ ⓑ $\frac{1}{3}m^5n^5$

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

5. ⓐ $-8x(x^2+2x-15)$ ⓑ $5pq^3(p^2-2pq+6q^2)$

6. ⓐ $-5t(t^2+3t-18)$ ⓑ $9r^3s(r^2-3rs+5s^2)$

Answer

ⓐ $-5t^3-15t^2+90t$
ⓑ $9sr^5-27s^2r^4+45s^3r^3$

7. ⓐ $-8y(y^2+2y-15)$ ⓑ $-4y^2z^2(3y^2+12yz-z^2)$

8. ⓐ $-5m(m^2+3m-18)$ ⓑ $-3x^2y^2(7x^2+10xy-y^2)$

Answer

ⓐ $-5m^3-15m^2+90m$
ⓑ $-21x^4y^2-30x^3y^3+3x^2y^4$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using ⓐ the Distributive Property; ⓑ the FOIL method; ⓒ the Vertical Method.

9. $(w+5)(w+7)$

10. $(y+9)(y+3)$

Answer

$y^2+12y+27$

11. $(4p+11)(5p-4)$

12. $(7q+4)(3q-8)$

Answer

$21q^2-44q-32$

In the following exercises, multiply the binomials. Use any method.

13. $(x+8)(x+3)$

14. $(y-6)(y-2)$

Answer

$y^2-8y+12$

15. $(2t-9)(10t+1)$

16. $(6p+5)(p+1)$

Answer

$6p^2+11p+5$

17. $(q-5)(q+8)$

18. $(m+11)(m-4)$

Answer

$m^2+7m-44$

19. $(7m+1)(m-3)$

20. $(3r-8)(11r+1)$

Answer

$33r^2-85r-8$

21. $(x^2+3)(x+2)$

22. $(y^2-4)(y+3)$

Answer

$y^3+3y^2-4y-12$

23. $(5ab-1)(2ab+3)$

24. $(2xy+3)(3xy+2)$

Answer

$6x^2y^2+13xy+6$

25. $(x^2+8)(x^2-5)$

26. $(y^2-7)(y^2-4)$

Answer

y^4-11y^2+28

27. $(6pq-3)(4pq-5)$

28. $(3rs-7)(3rs-4)$

Answer

$9r^2s^2-33rs+28$

Multiply a Polynomial by a Polynomial

In the following exercises, multiply using ⓐ the Distributive Property; ⓑ the Vertical Method.

29. $(x+5)(x^2+4x+3)$

30. $(u+4)(u^2+3u+2)$

Answer

$u^3+7u^2+14u+8$

31. $(y+8)(4y^2+y-7)$

32. $(a+10)(3a^2+a-5)$

Answer

$3a^3+31a^2+5a-50$

33. $(y^2-3y+8)(4y^2+y-7)$

34. $(2a^2-5a+10)(3a^2+a-5)$

Answer

$6a^4-13a^3+15a^2+35a-50$

Multiply Special Products

In the following exercises, multiply. Use either method.

35. $(w-7)(w^2-9w+10)$

36. $(p-4)(p^2-6p+9)$

Answer

$p^3-10p^2+33p-36$

37. $(3q+1)(q^2-4q-5)$

38. $(6r+1)(r^2-7r-9)$

Answer

$6r^3-41r^2-61r-9$

In the following exercises, square each binomial using the Binomial Squares Pattern.

39. $(w+4)^2$

40. $(q+12)^2$

Answer

$q^2+24q+144$

41. $(3x-y)^2$

42. $(2y-3z)^2$

Answer

$4y^2-12yz+9z^2$

43. $(y+\frac{1}{4})^2$

44. $(x+\frac{2}{3})^2$

Answer

$x^2+\frac{4}{3}x+\frac{4}{9}$

45. $(\frac{1}{5}x-\frac{1}{7}y)^2$

46. $(\frac{1}{8}x-\frac{1}{9}y)^2$

Answer

$\frac{1}{64}x^2-\frac{1}{36}xy+\frac{1}{81}y^2$

47. $(3x^2+2)^2$

48. $(5u^2+9)^2$

Answer

$25u^4+90u^2+81$

49. $(4y3-2)2$

50. $(8p3-3)2$

Answer

$64p^6-48p^3+9$

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

51. $(5k+6)(5k-6)$

52. $(8j+4)(8j-4)$

Answer

$64j^2-16$

53. $(11k+4)(11k-4)$

54. $(9c+5)(9c-5)$

Answer

$81c^2-25$

55. $(9c-2d)(9c+2d)$

56. $(7w+10x)(7w-10x)$

Answer

$49w^2-100x^2$

57. $(m+\frac{2}{3}n)(m-\frac{2}{3}n)$

58. $(p+\frac{4}{5}q)(p-\frac{4}{5}q)$

Answer

$p^2-\frac{16}{25}q^2$

59. $(ab-4)(ab+4)$

60. $(xy-9)(xy+9)$

Answer

x^2y^2-81

61. $(12p^3-11q^2)(12p^3+11q^2)$

62. $(15m^2-8n^4)(15m^2+8n^4)$

Answer

$225m^4-64n^8$

In the following exercises, find each product.

63. $(p-3)(p+3)$

64. $(t-9)^2$

Answer

$t^2 - 18t + 81$

65. $(m+n)^2$

66. $(2x+y)(x-2y)$

Answer

$2x^2 - 3xy - 2y^2$

67. $(2r+12)^2$

68. $(3p+8)(3p-8)$

Answer

$9p^2 - 64$

69. $(7a+b)(a-7b)$

70. $(k-6)^2$

Answer

$k^2 - 12k + 36$

71. $(a^5 - 7b)^2$

72. $(x^2+8y)(8x-y^2)$

Answer

$8x^3 - x^2y^2 + 64xy - 8y^3$

73. $(r^6+s^6)(r^6-s^6)$

74. $(y^4+2z)^2$

Answer

$y^8 + 4y^4z + 4z^2$

75. $(x^5+y^5)(x^5-y^5)$

76. $(m^3-8n)^2$

Answer

$m^6 - 16m^3n + 64n^2$

77. $(9p+8q)^2$

78. $(r^2-s^3)(r^3+s^2)$

Answer

$r^5 + r^2 s^2 - r^3 s^3 - s^5$

Mixed Practice

79. $(10y - 6) + (4y - 7)$

80. $(15p - 4) + (3p - 5)$

Answer

$18p - 9$

81. $(x^2 - 4x - 34) - (x^2 + 7x - 6)$

82. $(j^2 - 8j - 27) - (j^2 + 2j - 12)$

Answer

$-10j - 15$

83. $(\frac{1}{5} f^8)(20f^3)$

84. $(\frac{1}{4} d^5)(36d^2)$

Answer

$9d^7$

85. $(4a^3 b)(9a^2 b^6)$

86. $(6m^4 n^3)(7mn^5)$

Answer

$72m^5 n^8$

87. $-5m(m^2 + 3m - 18)$

88. $5q^3(q^2 - 2q + 6)$

Answer

$5q^5 - 10q^4 + 30q^3$

89. $(s - 7)(s + 9)$

90. $(y^2 - 2y)(y + 1)$

Answer

$y^3 - y^2 - 2y$

91. $(5x - y)(x - 4)$

92. $(6k-1)(k^2+2k-4)$

Answer

$6k^3-11k^2-26k+4$

93. $(3x-11y)(3x-11y)$

94. $(11-b)(11+b)$

Answer

$121-b^2$

95. $(rs-\frac{2}{7})(rs+\frac{2}{7})$

96. $(2x^2-3y^4)(2x^2+3y^4)$

Answer

$4x^4-9y^8$

97. $(m-15)^2$

98. $(3d+1)^2$

Answer

$9d^2+6d+1$

99. $(4a+10)^2$

100. $(3z+15)^2$

Answer

$9z^2-\frac{6}{5}z+\frac{1}{25}$

Multiply Polynomial Functions

101. For functions $f(x)=x+2$ and $g(x)=3x^2-2x+4$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-1)$

102. For functions $f(x)=x-1$ and $g(x)=4x^2+3x-5$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-2)$

Answer

ⓐ $(f\cdot g)(x)=4x^3-x^2-8x+5$
ⓑ $(f\cdot g)(-2)=-15$

103. For functions $f(x)=2x-7$ and $g(x)=2x+7$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-3)$

104. For functions $f(x)=7x-8$ and $g(x)=7x+8$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-2)$

Answer

ⓐ $(f\cdot g)(x)=49x^2-64$
ⓑ $(f\cdot g)(-2)=187$

105. For functions $f(x)=x^2-5x+2$ and $g(x)=x^2-3x-1$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-1)$

106. For functions $f(x)=x^2+4x-3$ and $g(x)=x^2+2x+4$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(1)$

Answer

ⓐ $(f\cdot g)(x)=x^4+6x^3+9x^2+10x-12$ ⓑ $(f\cdot g)(1)=14$

Writing Exercises

107. Which method do you prefer to use when multiplying two binomials: the Distributive Property or the FOIL method? Why? Which method do you prefer to use when multiplying a polynomial by a polynomial: the Distributive Property or the Vertical Method? Why?

108. Multiply the following:

$(x+2)(x-2)$

$(y+7)(y-7)$

$(w+5)(w-5)$

Explain the pattern that you see in your answers.

Answer

Answers will vary.

109. Multiply the following:

$(p+3)(p+3)$

$(q+6)(q+6)$

$(r+1)(r+1)$

Explain the pattern that you see in your answers.

110. Why does $(a+b)^2$ result in a trinomial, but $(a-b)(a+b)$ result in a binomial?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
multiply monomials.			
multiply a polynomial by a monomial.			
multiply a binomial by a binomial.			
multiply a polynomial by a polynomial.			
multiply special products.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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2.6: Extrema and Models

In the last section, we used end-behavior and zeros to sketch the graph of a given polynomial. We also mentioned that it takes a semester of calculus to learn an analytic technique used to calculate the "turning points" of the polynomial. That said, we'll still pursue the coordinates of the "turning points" in this section, but we will use the graphing calculator to assist us in this quest; and then we will use this technique with some applications.

Extrema

Before we begin, we'd first like to differentiate between local extrema and absolute extrema. This is best accomplished by means of an example. Consider, if you will, the graphs of three polynomial functions in Figure 1.

In the first figure, Figure $2.6.1a$, the point A is the "absolute" lowest point on the graph. Therefore, the y-value of point A is an absolute minimum value of the function.

In the second figure, Figure $2.6.1b$, there is no "absolute" highest point on the graph (the graph goes up to positive infinity), nor is there an "absolute" lowest point on the graph (the graph goes down to negative infinity). Therefore, this function has neither an absolute minimum nor an absolute maximum.

However, point B in Figure $2.6.1b$ is the highest point in its immediate neighborhood. If you wander too far to the right, there are points on the graph higher than point B, but locally point B is the highest point. Therefore, the y-value of point B is called a local maximum value of the function.

Similarly, point C in Figure $2.6.1b$ is the lowest point in its immediate neighborhood. If you wander too far to the left, there are points on the graph lower than point C, but, in its neighborhood, point C is the lowest point. Therefore, the y-value of point C is called a local minimum of the function.

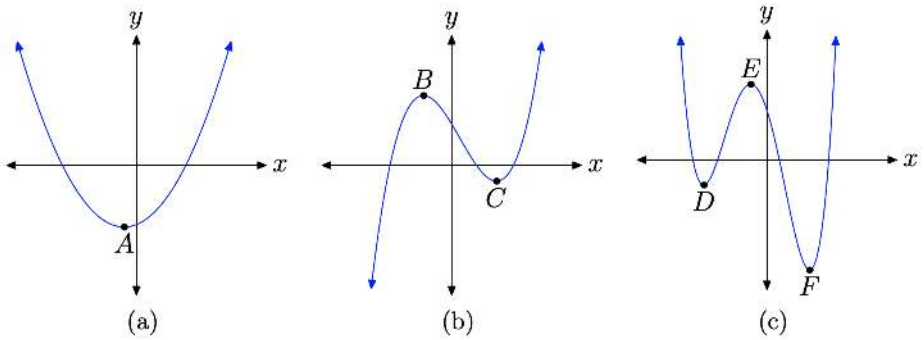


Figure 2.6.1: Differentiating between local and absolute extrema.

Finally, take a look at the graph in Figure $2.6.1c$. Point F is the "absolute" lowest point on the graph, so the y-value of point F is an absolute minimum of the function. On the other hand, there is no highest point on the graph in Figure $2.6.1$(c), as each end of the graph escapes to positive infinity. Hence, the function has no absolute maximum.

Locally, point D in Figure $2.6.1$(c) is the lowest point, so the y-value of point D is a local minimum of the function. Similarly, in its immediate neighborhood, point E is the highest point, so the y-value of point E is a local maximum.

We now present the formal definitions.

Definition: Absolute Maximum

Suppose that c is in the domain of a function f and $f(c) \geq f(x)$ for all x in the domain of f. Then we say that $f(c)$ is an **absolute maximum** of the function f. Similarly, if $f(c) \leq f(x)$ for all x in the domain of f, then $f(c)$ is an absolute minimum of the function f.

The definition of local extrema is less restrictive.

Definition: Local Maximum

Let c be in the domain of f. If $f(c) \geq f(x)$ for all x in a neighborhood containing c, then we say that f(c) is a **local maximum** of the function f. On the other hand, if $f(c) \leq f(x)$ for all x in a neighborhood containing c, then we say that f(c) is a local minimum of the function f.

When mathematicians say "a neighborhood containing c," they usually mean a small open interval (a, b) that contains c.

Let's explore the use of the graphing calculator in finding extrema.

Example 2.6.1

Consider the polynomial function defined by the equation

$$p(x) = 2(x-6)(x+2)(x+4).$$

Use the graphing calculator to help find and classify all extrema of this function.

Solution

First, how much of the graph can you draw without the use of a calculator? The linear factors of $p(x)$ are x − 6, x + 2, and x + 4, so the zeros are 6, −2, and −4, respectively. So we now know where the graph of p(x) crosses the x-axis.

To determine the end-behavior of $p(x)$, we need to determine the leading term. It is not necessary to completely expand the polynomial with the distributive property.3 A little thought quickly reveals that if we were to do just that, the leading term in this case would be $2x^3$. Consequently, as we sweep our eyes from left to right, the end-behavior of the polynomial should match that of its leading term $2x^3$, rising from negative infinity, wiggling through its x-intercepts, then rising to positive infinity. The only choice is a graph similar to that in Figure 2.6.2.

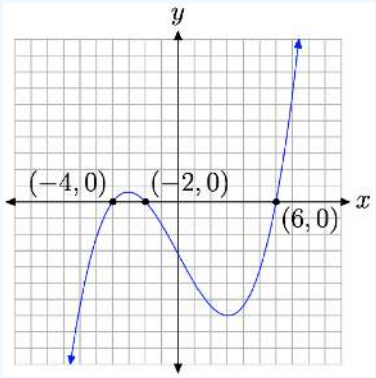


Figure 2.6.2: The graph of $p(x) = 2(x-6)(x+2)(x+4)$ rises from negative infinity, wiggles through its x-intercepts, then continues to rise to positive infinity.

Note that the graph achieves a local maximum somewhere near x = −3 and a local minimum at approximately x = 3. We can find better approximations of the local extrema by using the maximum and minimum utilities in the CALC menu of the graphing calculator.

- First, plot the graph of the polynomial p(x) = 2(x − 6)(x + 2)(x + 4), as shown in Figure 2.6.3(a), using the window parameters shown in Figure 2.6.3(b).
- Open the CALCULATE menu by pressing 2nd CALC. This reveals a menu of choices as shown in Figure 2.6.3(c). To start the utility to help find the local maximum near x = −3 (see Figure 2.6.2), press 4:maximum on the menu.
- The utility responds by asking for a "Left Bound." Use the arrow keys to move the cursor slightly left of the local maximum near x = −3, as shown in Figure 2.6.3(d), then press the ENTER key.

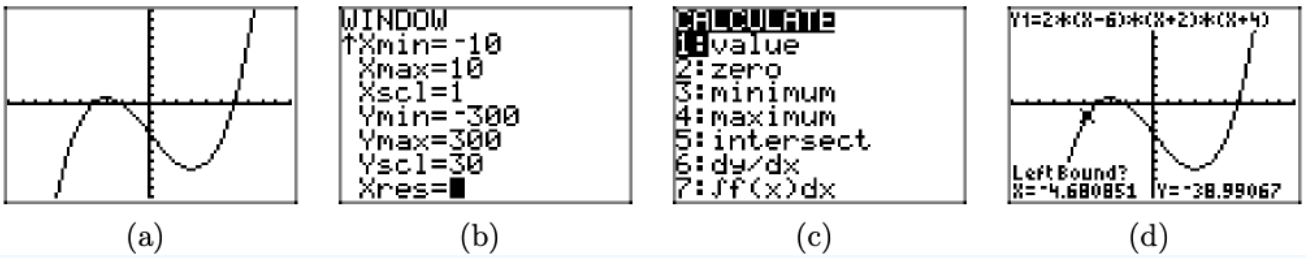


Figure 2.6.3. Using the graphing calculator to determine the local maximum.

- The utility responds by asking for a “Right Bound.” Use the arrow keys to move the cursor slightly to the right of the local maximum near x = −3, as shown in Figure 2.6.4(e), then press the ENTER key.
- The utility responds by asking for a “Guess.” Move the cursor so that it lies between the “Left Bound” and “Right Bound” made earlier, as shown in Figure 2.6.4(f), then press the ENTER key. Anywhere between the left bound and right bound (note marks at top of screen in Figure 2.6.4(f)) will do.

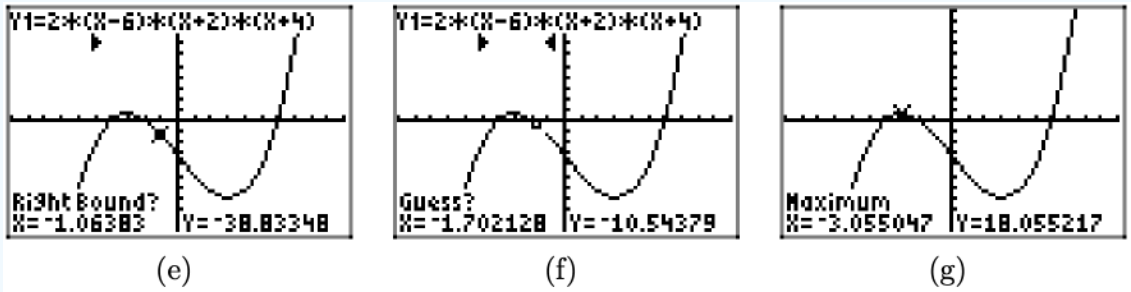


Figure 2.6.4. Using the graphing calculator to determine the local maximum (continued).

- The calculator responds by placing the cursor at the point where the local maximum occurs and reports its coordinates at the bottom of the screen, as shown in Figure 2.6.4(g).

The coordinates of the point where the local maximum occurs are approximately

$$(-3.055047, 18.055217)$$

We say that the function achieves a local maximum value of 18.055217 and that maximum occurs at $x \approx -3.055047$.

In a similar manner, one can use the minimum utility in the CALC menu to find the local minimum that occurs near x = 3, as shown in Figure 2.6.5.

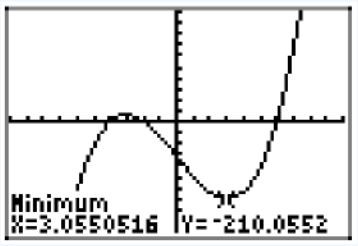


Figure 2.6.5. Use the minimum utility to find the local minimum.

The coordinates of the point where the local minimum occurs are approximately

$$(3.0550516, -210.0552)$$

We say that the function achieves a local minimum value of −210.0552 and that minimum occurs at $x \approx 3.0550516$.

Applications

In this section we will look at some applications that are modeled by polynomials.

Example 2.6.2

A square piece of cardboard measures 24 inches per side. John cuts four smaller squares from each corner of the cardboard, tossing the material aside. He then bends up the sides of the remaining cardboard to form an open box with no top. Find the dimensions of the squares cut from each corner of the original piece of cardboard so that John maximizes the resulting volume of the box.

Solution

Let x represent the length of the side of the square cut from each corner of the larger square (see Figure 2.6.6(a)). Because each side of the original square measures 24 inches, and we're cutting two lengths of x inches off each end, the resulting length and width of the box is 24 − 2x inches (see Figure 2.6.6(a) and/or (b)). When we toss away the square corners, then fold up the sides, we get a box with the dimensions shown in Figure 2.6.6(b).

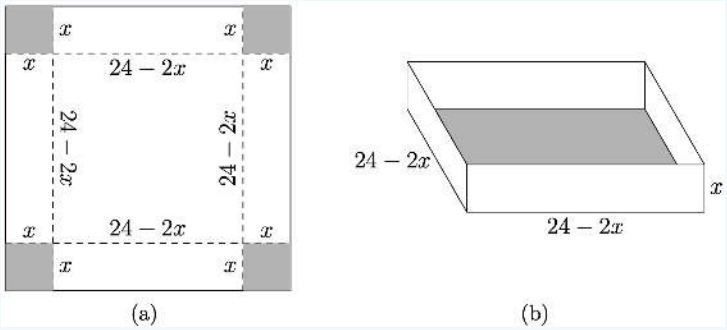


Figure 2.6.6: Material and resulting box.

Because the volume of a box is computed by taking the product of the length and width of the base, multiplied by the height of the box, the volume of the box is given by the formula

$$V = x(24 - 2x)(24 - 2x)$$

We can simplify equation (6) somewhat. Take a factor of 2 from each factor of 24−2x, as in

$$V = x(2)(12 - x)(2)(12 - x)$$

then combine factors to write

$$V = 4x(12 - x)^2$$

We see that x and $12 - x$ are linear factors of V. Hence, the zeros of V are 0 and 12, respectively. Because 12−x is used as a factor twice, 2 is a "double root," so the graph should be tangent to the x-axis at $x = 2$.

If we were to expand Equation (7) completely, we would get a polynomial with leading term $4x^3$. Hence, the end-behavior of our volume polynomial should match the end-behavior of its leading term, rising from negative infinity, wiggling through it zeros, then rising to positive infinity. However, because we have a "double root" at x = 2, we expect the graph to "kiss" the horizontal axis at this zero rather than pass through this zero.

Thus, the only possible shape the volume polynomial can assume is that shown in Figure 2.6.7.

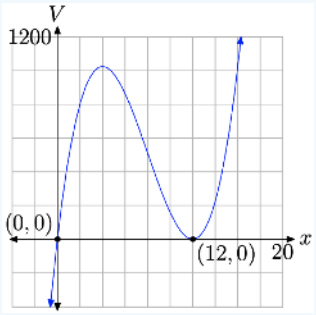


Figure 2.6.7: The volume of the box as a function of the edge length of the removed squares.

The domain of the polynomial defined by equation (7) is the set of all real numbers, or, in interval notation, $(-\infty, \infty)$. In Figure 2.6.7, if you project all the points on the graph onto the x-axis, the entire x-axis would be shaded, further indicating that the domain of the volume function is all real numbers.

However, this mathematical domain $(-\infty, \infty)$ ignores the fact that x represents the length of the square cut from each corner of the original square of cardboard (see Figure 2.6.6(a)). You cannot cut a square having a side of negative length. Upon further inspection, the largest square that could be cut from each corner would have an edge measuring 12 inches. Remember, you have to cut four squares, one from each corner, and the edge of the original square piece of cardboard measures just 24 inches. Thus, the problem constrains x to the interval [0, 12]. This domain is called the empirical domain, or, if you will, the practical domain.

Definition: Empirical Domain

The **empirical domain** of a function is a subset of the mathematical domain, restricted so as to satisfy the constraints of the model.

Thus, only a portion of the graph in Figure 2.6.7 makes sense for this application— the part that is drawn over the empirical domain [0, 12], as shown in Figure 2.6.8.

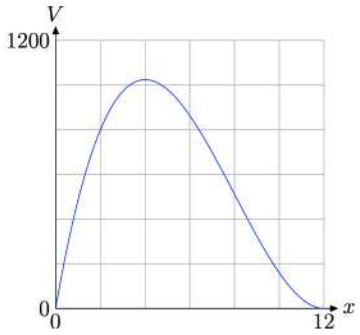


Figure 2.6.8: Sketching the volume function over the empirical domain [0, 12].

Remember, the original goal was to find the value of x that would maximize the volume of the box. A quick glance at the graph in Figure 2.6.8 shows that there is an absolute maximum (at least on the empirical domain [0, 12]) near x = 4. To obtain a better approximation, use the maximum utility in the CALC menu on your calculator, as we did to obtain the approximation shown in Figure 2.6.9(b).

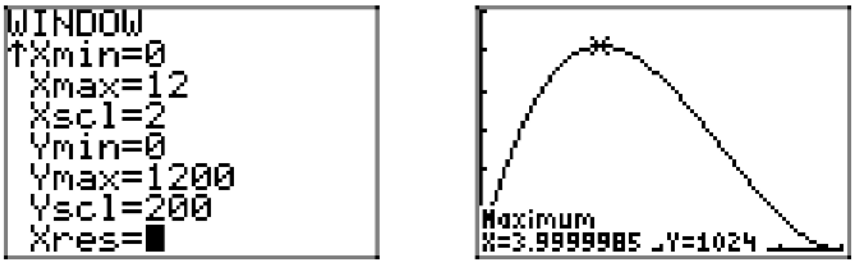


Figure 2.6.9. Finding the length of the edge that maximizes the volume of the box.

Indeed, it would appear that a maximum volume of 1024 cubic inches ($V = 1024\text{in}^3$) is attained at $x \approx 3.9999985$. It's probably safe to say that the maximum volume occurs if squares having sides of length 4 inches are cut from the corners of the original piece of cardboard. The 3.9999985 probably contains a bit of error due to roundoff error on the calculator. Indeed, it is highly likely that some readers will get exactly x = 4 when they use the maximum utility, depending on the bounds and initial guess used, so don't be worried if your calculator approximation differs slightly from ours.

Let's look at another example.

Example 2.6.3

Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the graph of the parabola $y = 4 - x^2$.

Solution

The graph of $y = 4 - x^2$ is a parabola that opens downward and is shifted upward 4 units. The right side of this equation factors

$$y = (2 + x)(2 - x)$$

so the zeros of this function are −2 and 2. Because the rectangle has its base on the x-axis and its other vertices are on the parabola lying above the x-axis, we need only sketch the parabola on the domain [−2, 2] (see Figure 2.6.10).

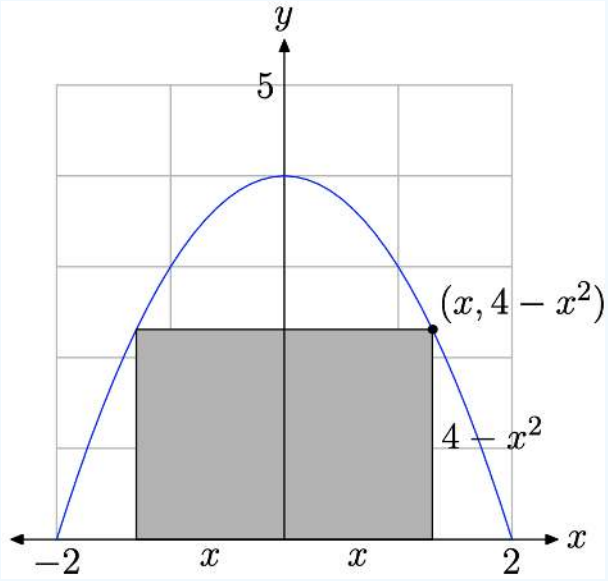


Figure 2.6.10: A rectangle inscribed under a parabola.

Because of symmetry, we can restrict x to the empirical domain [0, 2]. In Figure 2.6.10, note that we've selected a value of x from [0, 2], then plotted the point having this x-value on the parabola. Of course, the y-value of this point is $y = 4 - x^2$. Thus, the height of the rectangle is $4 - x^2$ and the base (or width) of the rectangle is twice x, or 2x. The area of the rectangle is given by

$$A = \text{ width } \cdot \text{ height}$$

Hence, the area A as a function of x is given by the polynomial

$$A = 2x\left(4 - x^2\right)$$

Note that equation (10) is a third degree polynomial having leading term $-2x^3$. Thus, the graph of the polynomial, as we sweep our eyes from left to right, must fall from positive infinity, wiggle through its x-intercepts, then continue falling to negative infinity.

We can factor equation (10) to obtain

$$A = 2x(2 + x)(2 - x)$$

Therefore, the zeros of the polynomial are 0, −2, and 2, respectively. Thus, the polynomial must have shape similar to that shown in Figure 2.6.11. Note that the graph has x-intercepts at (−2, 0), (0, 0), and (2, 0).

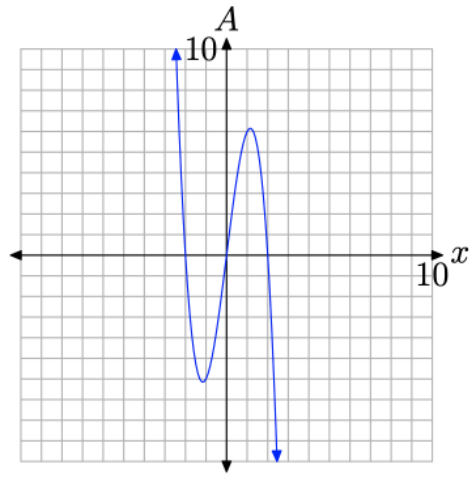


Figure 2.6.11. The graph of the polynomial A = 2x(2 + x)(2 − x).

Because of the practical nature of this problem, we need to restrict x to the empirical domain [0, 2], as discussed above (see Figure 2.6.10. The graph of A = 2x(2 + x)(2 − x), restricted to the domain [0, 2], is shown in Figure 2.6.12.

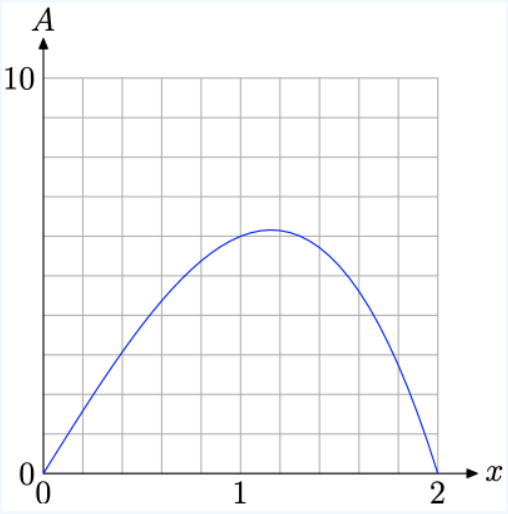


Figure 2.6.12. The graph of the polynomial A = 2x(2 + x)(2 − x) restricted to the empirical domain [0, 2].

It appears (see Figure 2.6.12) that A achieves an absolute maximum (at least on the empirical domain [0, 2]) near $x \approx 1.2$. To obtain a better approximation, use the maximum utility in the CALC menu of the graphing calculator, as we did to obtain the approximation shown in Figure 2.6.13(b).

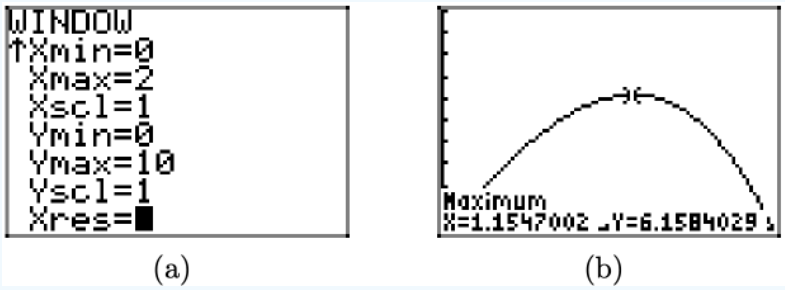


Figure 2.6.13. Use the maximum utility to obtain an approximation of the maximum value of A.

The result in Figure 2.6.13(b) shows that we achieve a rectangle of maximum area $A \approx 6.1584029$ if we choose $x \approx 1.1547002$ Remember, your answers may differ slightly according to the left and right bounds you select, your guess, and also due to the inherent roundoff error in all calculators.

Exercise

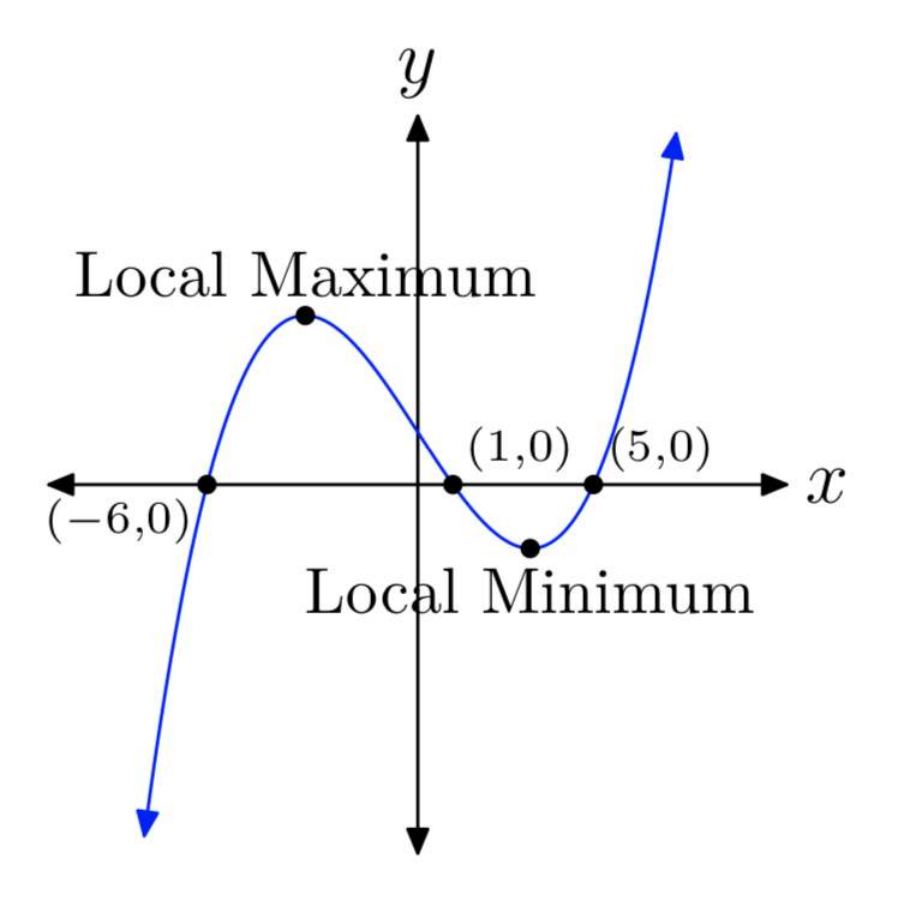
In **Exercises 1-8**, perform each of the following tasks for the given polynomial.

1. Without the aid of a calculator, use an algebraic technique to identify the zeros of the given polynomial. Factor if necessary.
2. On graph paper, set up a coordinate system. Label each axis, but scale only the x-axis. Use the zeros and the end-behavior to draw a “rough graph” of the given polynomial with- out the aid of a calculator.
3. Classify each local extrema as a relative minimum or relative maximum.Note: It is not necessary to find the coordinates of the relative extrema. Indeed, this would be difficult with- out a calculator. All that is required is that you label each extrema as a relative maximum or minimum

Exercise 2.6.1

p(x) = (x+6)(x−1)(x−5)

Answer



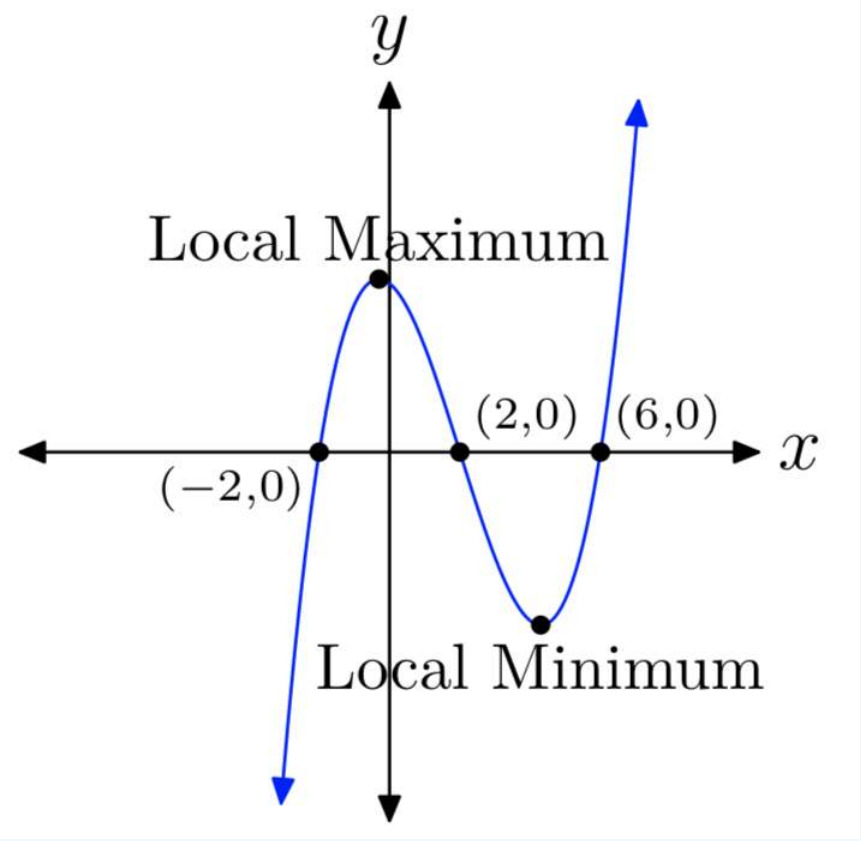
Exercise 2.6.2

p(x) = (x+2)(x−4)(x−7)

Exercise 2.6.3

$p(x) = x^3 - 6x^2 - 4x + 24$

Answer



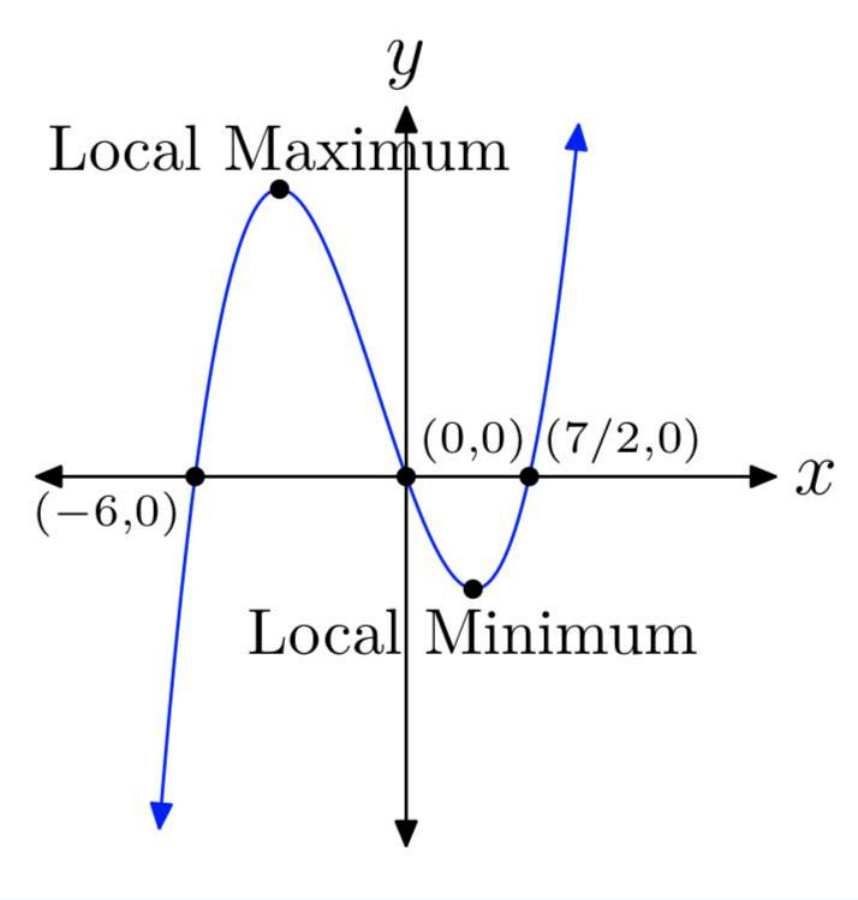
Exercise 2.6.4

$p(x) = x^3 + x^2 - 36x - 36$

Exercise 2.6.5

$p(x) = 2x^3 + 5x^2 - 42x$

Answer



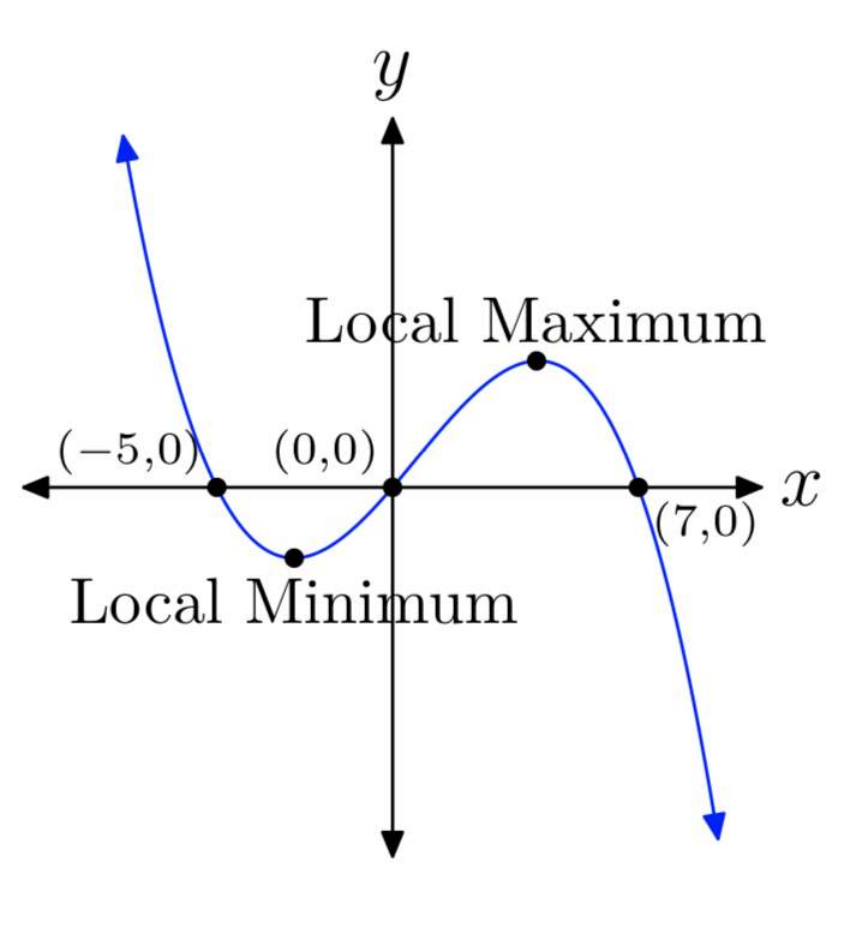
Exercise 2.6.6

$p(x) = 2x^3 - 3x^2 - 44x$

Exercise 2.6.7

$p(x) = -2x^3 + 4x^2 + 70x$

Answer



? Exercise 2.6.8

$p(x) = -6x^3 - 21x^2 + 90x$

In **Exercises 9-16**, perform each of the following tasks for the given polynomial.

1. Use a graphing calculator to draw the graph of the polynomial. Adjust the viewing window so that the extrema or "turning points" of the polynomial are visible in the viewing window. Copy the resulting image onto your home- work paper. Label and scale each axis with xmin, xmax, ymin, and ymax.
2. Use the maximum and/or minimum utility in your calculator's CALC menu to find the coordinates of the extrema. Label each extremum on your homework copy with its coordinates and state whether the extremum is a relative or absolute maximum or mini- mum.

? Exercise 2.6.9

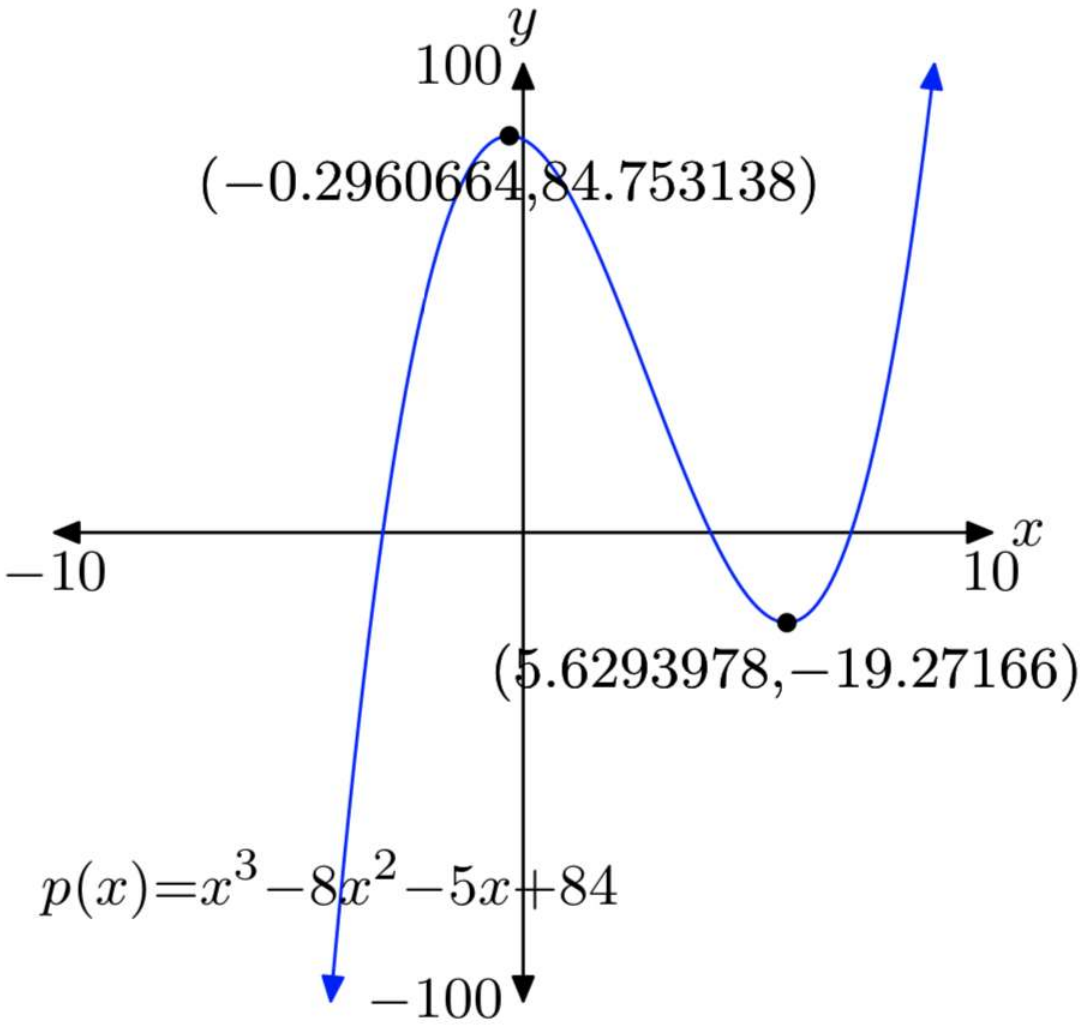
$p(x) = x^3 - 8x^2 - 5x + 84$

Answer

Relative max: (−0.2960664, 84.753138)

Relative min: (5.6293978, −19.27166)

Answers may differ slightly due to round-off error.



Exercise 2.6.10

$p(x) = x^3 + 3x^2 - 33x - 35$

Exercise 2.6.11

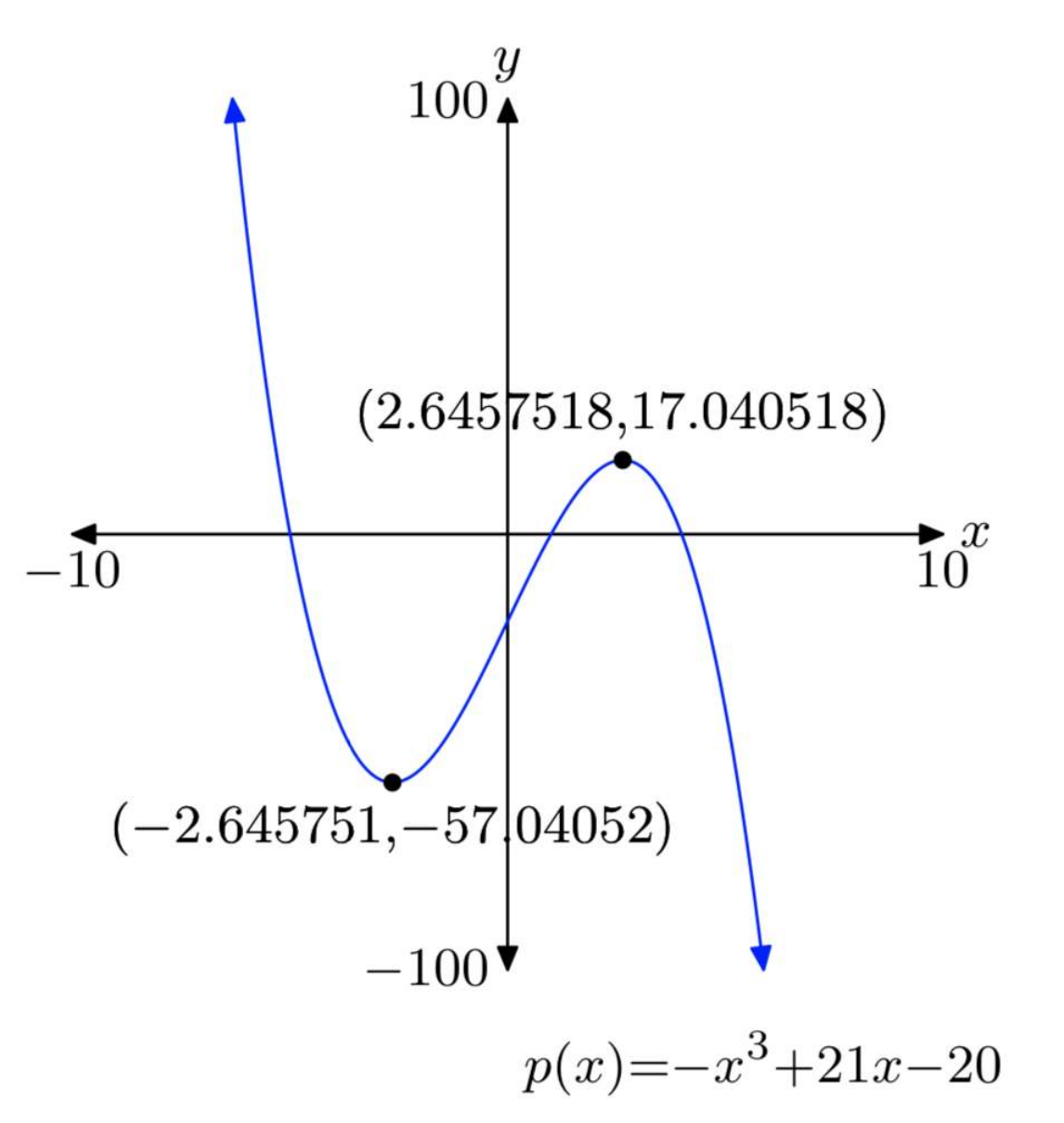
$p(x) = -x^3 + 21x - 20$

Answer

Relative min: $(-2.645751,\ -57.04052)$

Relative max: (2.6457518, 17.040518)

Answers may differ slightly due to round-off error.



Exercise 2.6.12

$p(x) = -x^3 + 5x^2 + 12x - 36$

Exercise 2.6.13

$p(x) = x^4 - 50x^2 + 49$

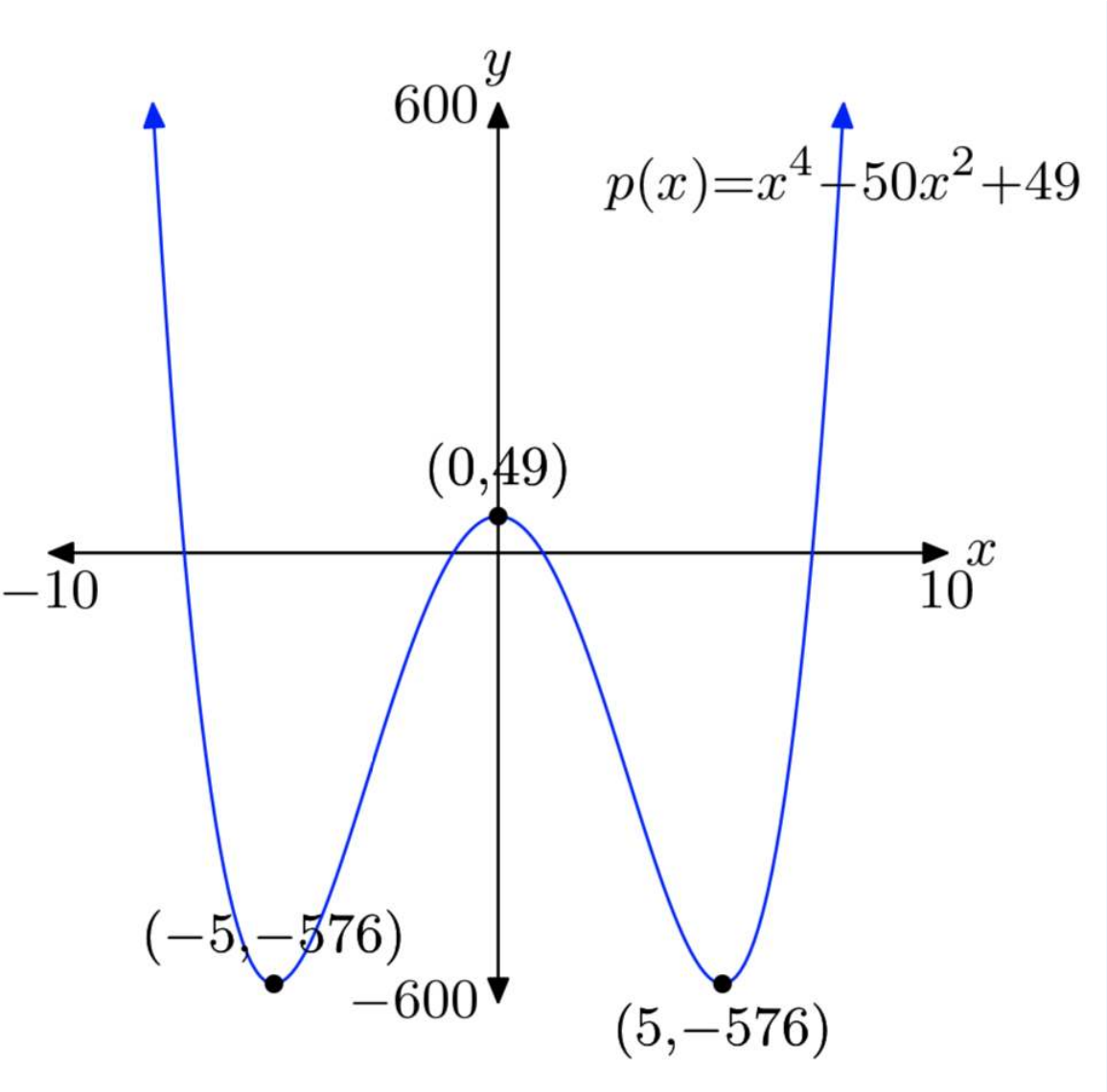
Answer

Absolute min: (−5, −576)

Relative max: (0, 49)

Absolute min: (5, −576)

Answers may differ slightly due to round-off error.



? Exercise 2.6.14

$p(x) = x^4 - 29x^2 + 100$

? Exercise 2.6.15

$p(x) = x^4 - 2x^3 - 39x^2 + 72x + 108$

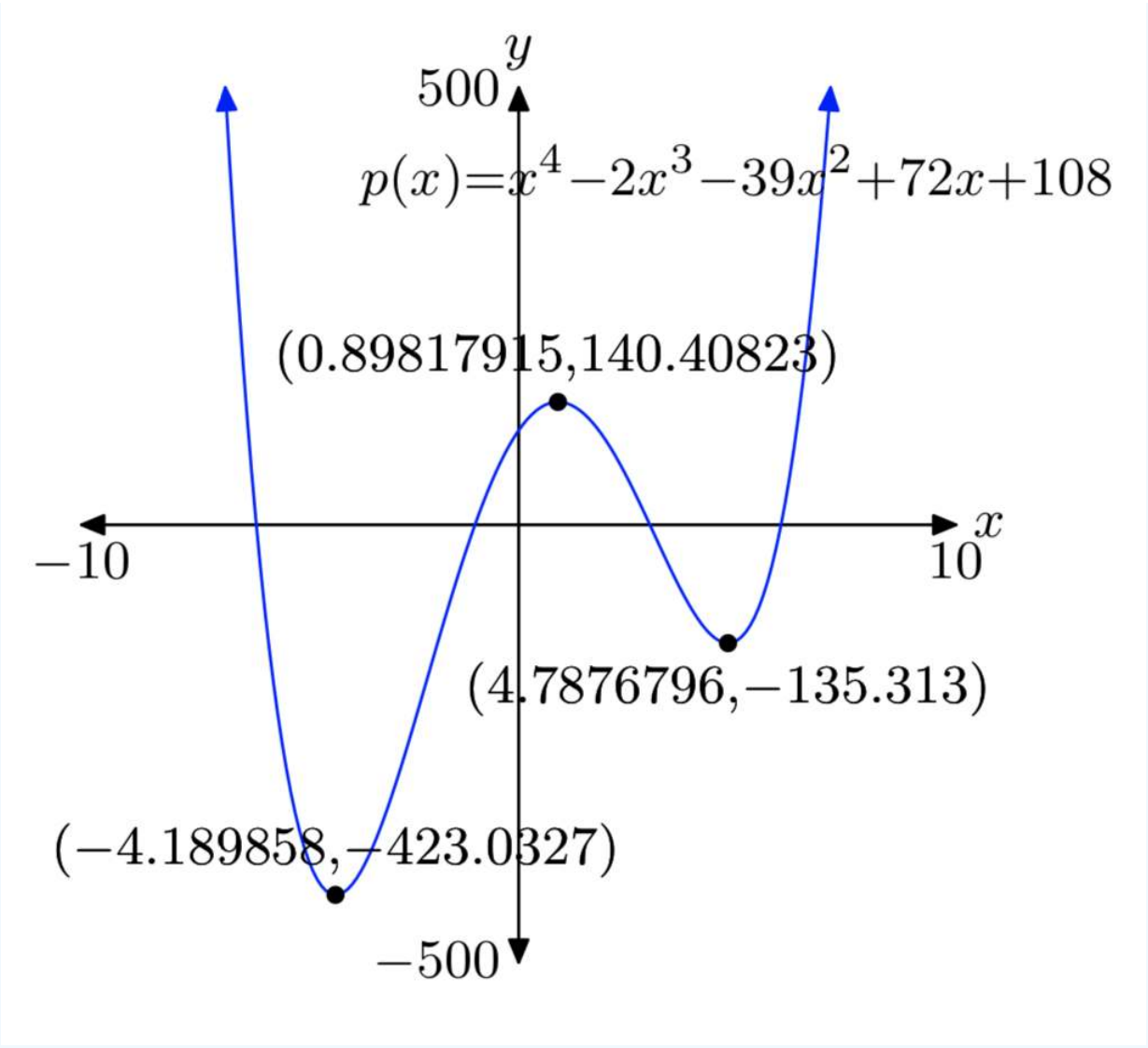
Answer

Absolute min: (−4.189858, −423.0327)

Relative max: (0.89817915, 140.40823)

Relative min: (4.7876796, −135.313)

Answers may differ slightly due to round-off error.



? Exercise 2.6.16

$p(x) = x^4 - 3x^3 - 31x^2 + 63x + 90$

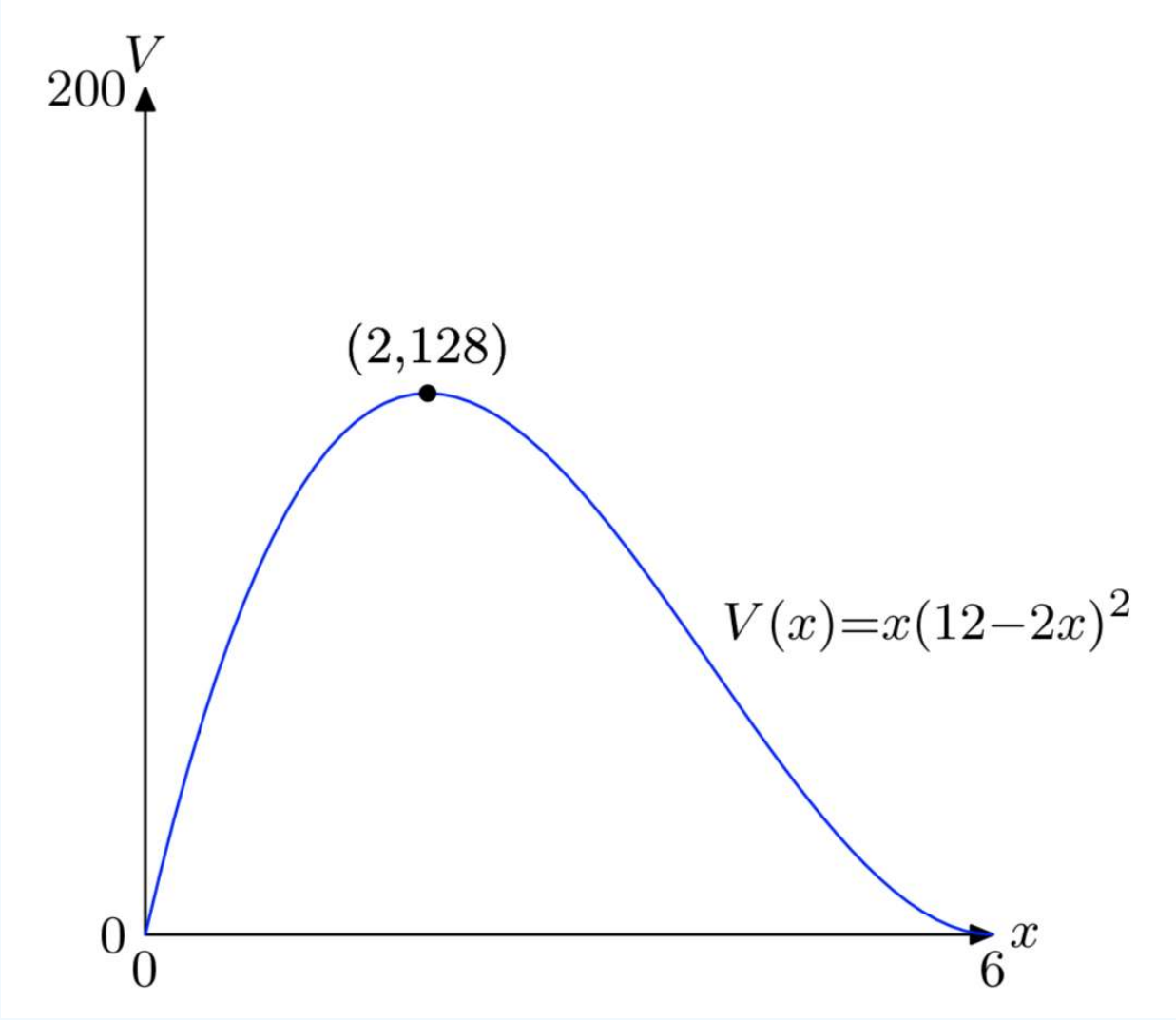
? Exercise 2.6.17

A square piece of cardboard measures 12 inches per side. Cherie cuts four smaller squares from each corner of the cardboard square, tossing the material aside. She then bends up the sides of the remaining cardboard to form an open box with no top. Find the dimensions of the squares cut from each corner of the original piece of cardboard so that Cherie maximizes the volume of the resulting box. Perform each of the following steps in your analysis.

1. Set up an equation that determines the volume of the box as a function of x, the length of the edge of each square cut from the four corners of the cardboard. Include any pictures used to determine this volume function.
2. State the empirical domain of the function created in part (**a**). Use your calculator to sketch the graph of the function over this empirical domain. Adjust the viewing window so that all extrema are visible in the viewing window.
3. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Use the maximum utility to find the coordinates of the absolute maximum on the function's empirical domain.
4. What are the measures of the four squares cut from each corner of the original cardboard? What is the maximum volume of the box?

Answer

1. $V = x(12 - 2x)^2$
2. [0, 6]
3. Absolute max: (2, 128)



4. Cut square 2 inches on a side to produce a box having value 128 in^3.

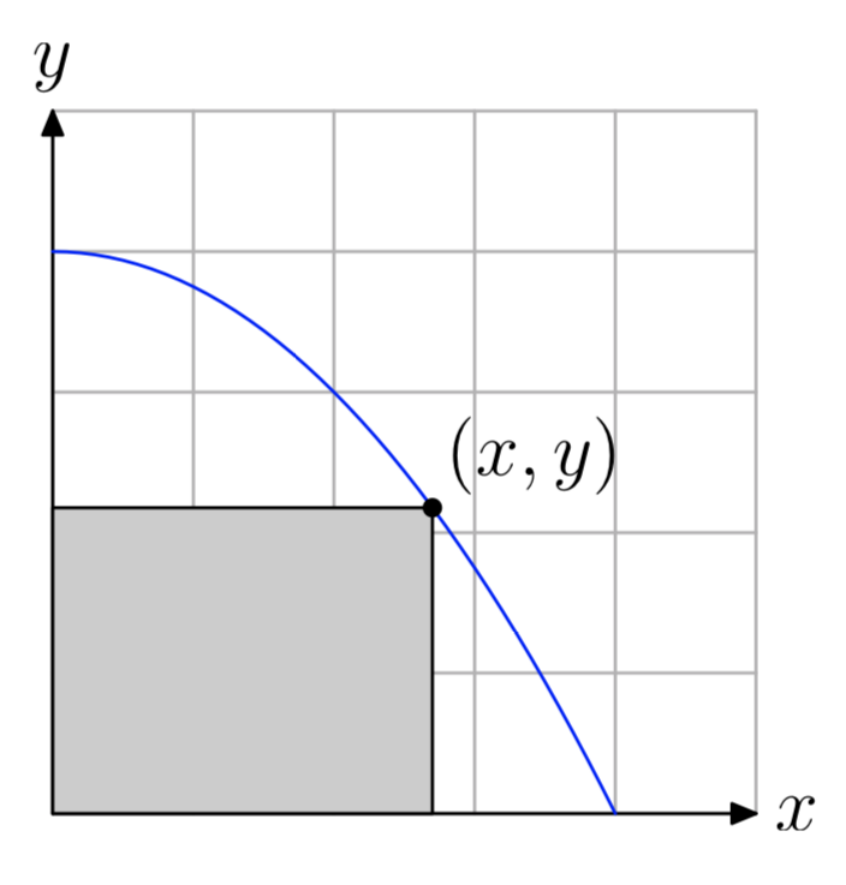
? Exercise 2.6.18

A rectangular piece of cardboard measures 8 inches by 12 inches. Schuyler cuts four smaller squares from each corner of the cardboard square, tossing the material aside. He then bends up the sides of the remaining cardboard to form an open box with no top. Find the dimensions of the squares cut from each corner of the original piece of cardboard so that Schuyler maximizes the volume of the resulting box. Perform each of the following steps in your analysis.

1. Set up an equation that determines the volume of the box as a function of x, the length of the edge of each square cut from the four corners of the cardboard. Include any pictures used to determine this volume function.
2. State the empirical domain of the function created in part (**a**). Use your calculator to sketch the graph of the function over this empirical domain. Adjust the viewing window so that all extrema are visible in the viewing window.
3. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Use the maximum utility to find the coordinates of the absolute maximum on the function's empirical domain.
4. What are the measures of the four squares cut from each corner of the original cardboard? What is the maximum volume of the box?

Exercise 2.6.19

Restrict the graph of the parabola $y = 4 - \frac{x^2}{4}$ to the first quadrant, then inscribe a rectangle inside the parabola, as shown in the figure that follows.

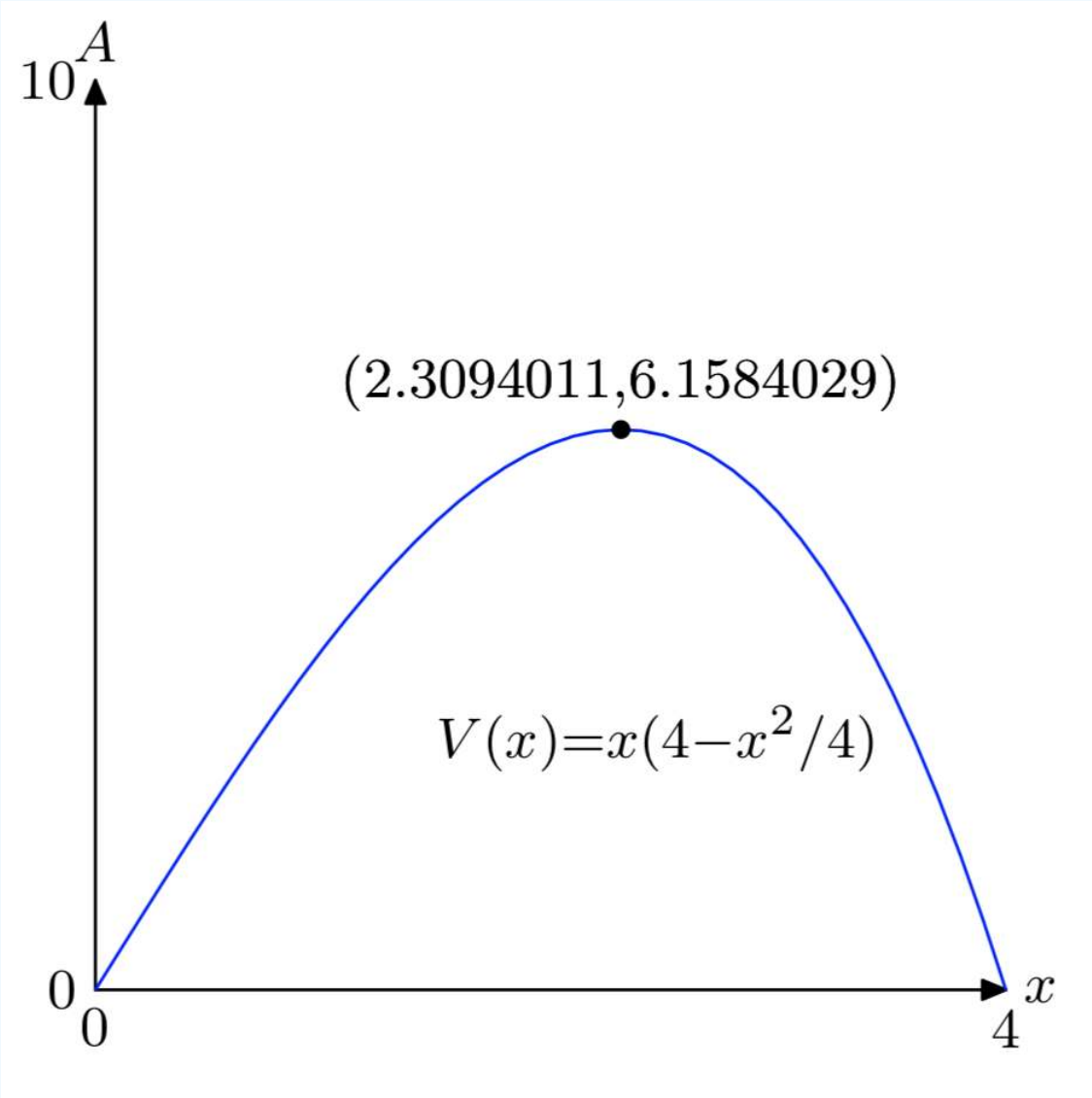


1. Express the area of the inscribed rectangle as a function of x.
2. State the empirical domain of the function defined in part (**1**). Use your calculator to graph the area function over its empirical domain. Adjust the window parameters so that all extrema are visible in the viewing window.
3. Copy the image in your viewing window to your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Use the maximum utility to find the coordinates of the absolute maximum on the function's empirical domain. Label your graph with this result.
4. What are the dimensions of the rectangle of maximum area?

Answer

1. $A = x(4 - \frac{x^2}{4})$
2. [0, 4]

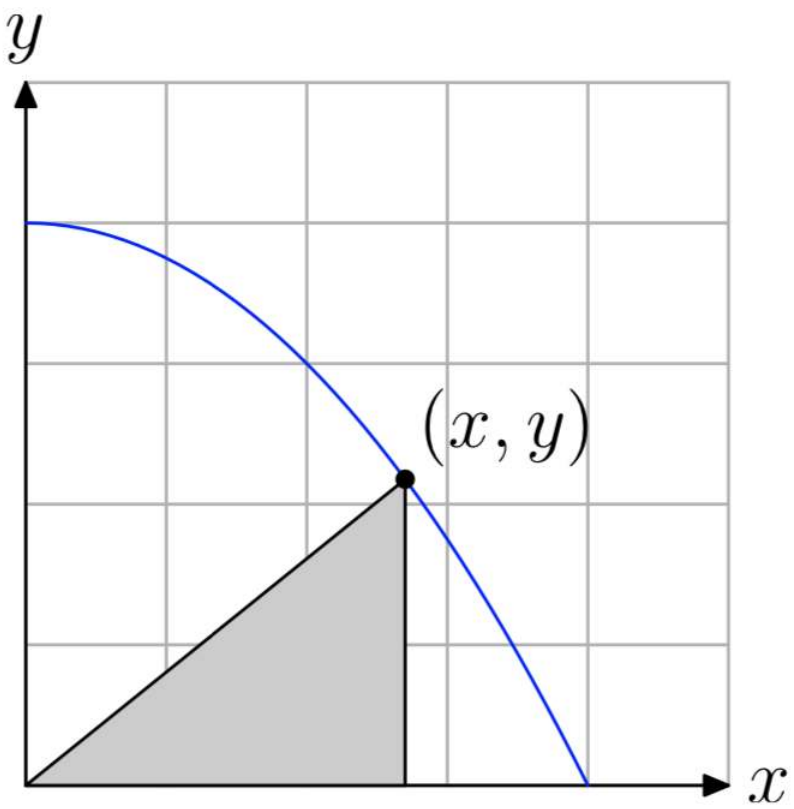
3. Absolute max: (2.3094011, 6.1584029)



4. x = 2.3094011, y = 2.6666666

Exercise 2.6.20

Restrict the graph of the parabola $y = 4 - \frac{x^2}{4}$ to the first quadrant, then inscribe a triangle inside the parabola, as shown in the figure that follows.



1. Express the area of the inscribed tri- angle as a function of x.
2. State the empirical domain of the function defined in part (**a**). Use your calculator to graph the area function over its empirical domain. Adjust the window parameters so that all extrema are visible in the viewing window.
3. Copy the image in your viewing window to your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Use the maximum utility to find the coordinates of the absolute maximum on the function's empirical domain. Label your graph with this result.
4. What are the length of the base and height of the triangle of maximum area?

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CHAPTER OVERVIEW

3: Chapter 3 - Rationals

3.1: Rational Expressions and Functions
3.2: Multiply and Divide Rational Expressions
3.2E: Exercises
3.3: Divide Polynomials
3.3E: Exercises
3.4: Add and Subtract Rational Expressions
3.4E: Exercises
3.5: Simplify Complex Rational Expressions
3.5E: Exercises
3.6: Solve Rational Equations
3.6E: Exercises
3.7: Rational Inequalities

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3.1: Rational Expressions and Functions

Twelve goals last season. Fifteen home runs. Nine touchdowns. Whatever the statistics, sports analysts know it. Their jobs depend on it. Compiling and analyzing sports data not only help fans appreciate their teams but also help owners and coaches decide which players to recruit, how to best use them in games, how much they should be paid, and which players to trade. Understanding this kind of data requires a knowledge of specific types of expressions and functions. In this chapter, you will work with rational expressions and perform operations on them. And you will use rational expressions and inequalities to solve real-world problems.



Figure 7.0.1: *American football is the most watched spectator sport in the United States. People around the country are constantly tracking statistics for football and other sports. (credit: "keijj44" / Pixabay)*

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3.2: Multiply and Divide Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions
- Multiply and divide rational functions

Before you get started, take this readiness quiz.

1. Simplify: $\frac{90y}{15y^2}$.
   If you missed this problem, review [link].
2. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
   If you missed this problem, review [link].
3. Divide: $\frac{12}{10} \div \frac{8}{25}$.
   If you missed this problem, review [link].

We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

RATIONAL EXPRESSION

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{24}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{24}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

1. Set the denominator equal to zero.
2. Solve the equation.

EXAMPLE 3.2.1

Determine the value for which each rational expression is undefined:

a. $\frac{8a^2b}{3c}$ b. $\frac{4b-3}{2b+5}$ c. $\frac{x+4}{x^2+5x+6}$.

Solution

The expression will be undefined when the denominator is zero.

a.

$$\frac{8a^2b}{3c}$$

Set the denominator equal to zero and solve for the variable.

$$3c=0$$

$$c=0$$

$\frac{8a^2b}{3c}$ is undefined for $c=0$

b.

$$\frac{4b-3}{2b+5}$$

Set the denominator equal to zero and solve for the variable.

$$2b+5=0$$

$$2b=-5$$

$$b=-\frac{5}{2}$$

$\frac{4b-3}{2b+5}$ is undefined for $b=-\frac{5}{2}$

c.

$$\frac{x+4}{x^2+5x+6}$$

Set the denominator equal to zero and solve for the variable.

$$x^2+5x+6=0$$

$$(x+2)(x+3)=0$$

$$x+2=0 \text{ or } x+3=0$$

$$x=-2 \text{ or } x=-3$$

$\frac{x+4}{x^2+5x+6}$ is undefined for $x=-2$ or $x=-3$

Try It 3.2.2

Determine the value for which each rational expression is undefined.

a. $\frac{3y^2}{8x}$ b. $\frac{8n-5}{3n+1}$ c. $\frac{a+10}{a^2+4a+3}$

Answer

a. $x = 0$

b. $n = -\frac{1}{3}$

c. $a = -1, a = -3$

Try It 3.2.3

Determine the value for which each rational expression is undefined.

a. $\frac{4p}{5q}$ b. $\frac{y-1}{3y+2}$ c. $\frac{m-5}{m^2+m-6}$

Answer

a. $q = 0$

b. $y = -\frac{2}{3}$

c. $m = 2, m = -3$

Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

SIMPLIFIED RATIONAL EXPRESSION

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,

$\frac{x+2}{x+3}$ is simplified because there are no common factors of $x+2$ and $x+3$.

$\frac{2x}{3x}$ is not simplified because x is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

EQUIVALENT FRACTIONS PROPERTY

If a, b, and c are numbers where $b \neq 0, c \neq 0$,

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0, c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$ $\frac{2}{5}$

We removed the common factors 3 and 7. They are factors of the product.

$\frac{3x\cancel{(x-9)}}{5\cancel{(x-9)}}$ where $x \neq 9$ $\frac{3x}{5}$

We removed the common factor $(x-9)$. It is a factor of the product.

$\frac{x+5}{x}$ **NO COMMON FACTORS**

While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

How to Simplify a Rational Expression

EXAMPLE 3.2.4

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$

Solution

Step 1. Factor the numerator and denominator completely.	Factor x^2+5x+6 and $x^2+8x+12$	$\frac{x^2+5x+6}{x^2+8x+12}$ $\frac{(x+2)(x+3)}{(x+2)(x+6)}$
Step 2. Simplify by dividing out common factors.	Remove the common factor $x+2$ from the numerator and the denominator.	$\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$ $\frac{(x+3)}{(x+6)}$ $x \neq -2 \quad x \neq -6$

Try It 3.2.5

Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

Answer

$\frac{x+1}{x-1}, x \neq 2, x \neq 1$

Try It 3.2.6

Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

Answer

$\frac{x-5}{x-1}, x \neq -2, x \neq 1$

We now summarize the steps you should follow to simplify rational expressions.

SIMPLIFY A RATIONAL EXPRESSION.

1. Factor the numerator and denominator completely.
2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

We'll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

EXAMPLE 3.2.7

Simplify: $\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$.

Solution

$$\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$$

Factor the numerator and denominator, first factoring out the GCF.

$$\frac{3(a^2-4ab+4b^2)}{6(a^2-4b^2)}$$

$$\frac{3(a-2b)(a-2b)}{6(a+2b)(a-2b)}$$

Remove the common factors of $a-2b$ and 3.

$$\frac{\cancel{3}(a-2b)\ \cancel{(a-2b)}}{\cancel{3}\cdot 2(a+2b)\ \cancel{(a-2b)}}$$

$$\frac{a-2b}{2(a+2b)}$$

Try It 3.2.8

Simplify: $\frac{2x^2-12xy+18y^2}{3x^2-27y^2}$.

Answer

$$\frac{2(x-3y)}{3(x+3y)}$$

Try It 3.2.9

Simplify: $\frac{5x^2-30xy+25y^2}{2x^2-50y^2}$.

Answer

$$\frac{5(x-y)}{2(x+5y)}$$

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of a is $-a$ and $-a = -1 \cdot a$.

The numerical fraction, say $\frac{7}{-7}$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $\frac{a}{-a}$, whose numerator and denominator are opposites also simplifies to -1.

Let's look at the expression $b-a$.	$b-a$
Rewrite.	$-a+b$
Factor out -1.	$-1(a-b)$

This tells us that $b-a$ is the opposite of $a-b$.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\frac{a-b}{b-a}$ simplifies to -1.

OPPOSITES IN A RATIONAL EXPRESSION

The opposite of $a-b$ is $b-a$.

$$\frac{a-b}{b-a} = -1 \quad a \neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a \neq -b$, then $\frac{a+b}{b+a} = 1$.

EXAMPLE 3.2.10

Simplify: $\frac{x^2-4x-32}{64-x^2}$

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and the denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

Try It 3.2.11

Simplify: $\frac{x^2-4x-5}{25-x^2}$

Answer

$-\frac{x+1}{x+5}$

Try It 3.2.12

Simplify: $\dfrac{x^2+x-2}{1-x^2}$.

Answer

$-\dfrac{x+2}{x+1}$

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

MULTIPLICATION OF RATIONAL EXPRESSIONS

If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$, $x \neq 3$, and $x \neq 4$.

EXAMPLE 3.2.13: How to Multiply Rational Expressions

Simplify: $\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$.

Solution

Step 1. Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$ $\dfrac{2x}{(x-3)(x-4)} \cdot \dfrac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\dfrac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\dfrac{2 \cdot x \cdot (x-3)(x+3)}{2 \cdot 3 \cdot x \cdot x \cdot (x-3)(x-4)}$ $\dfrac{(x+3)}{3x(x-4)}$

Try It 3.2.14

Simplify: $\dfrac{5x}{x^2+5x+6} \cdot \dfrac{x^2-4}{10x}$.

Answer

$\dfrac{x-2}{2(x+3)}$

Try It 3.2.15

Simplify: $\frac{9x^2}{x^2+11x+30} \cdot \frac{x^2-36}{3x^2}$.

Answer

$\frac{3(x-6)}{x+5}$

MULTIPLY RATIONAL EXPRESSIONS.

1. Factor each numerator and denominator completely.
2. Multiply the numerators and denominators.
3. Simplify by dividing out common factors.

EXAMPLE 3.2.16

Multiply: $\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$.

Solution

$$\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$$

Factor the numerators and denominators and then multiply.

$$\frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$$

Simplify by dividing out common factors.

$$\frac{\cancel{(3a+1)}(a-3)\ \cancel{(a+5)}(a+5)}{(a-5)\ \cancel{(a+5)}\ \cancel{(3a+1)}(a-5)}$$

Simplify.

$$\frac{(a-3)(a+5)}{(a-5)(a-5)}$$

Rewrite $(a-5)(a-5)$ using an exponent.

$$\frac{(a-3)(a+5)}{(a-5)^2}$$

Try It 3.2.17

Simplify: $\frac{2x^2+5x-12}{x^2-16} \cdot \frac{x^2-8x+16}{2x^2-13x+15}$.

Answer

$\frac{x-4}{x-5}$

Try It 3.2.18

Simplify: $\frac{4b^2+7b-2}{1-b^2} \cdot \frac{b^2-2b+1}{4b^2+15b-4}$.

Answer

$$-\frac{(b+2)(b-1)}{(1+b)(b+4)}$$

Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

DIVISION OF RATIONAL EXPRESSIONS

If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

EXAMPLE 3.2.19: How to Divide Rational Expressions

Divide: $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Solution

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$ $\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$
Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \frac{2 \cdot 3}{(p-q)(p+q)}$
Step 3. Multiply the numerators and denominators.	Multiply the numerators and multiply the denominators.	$\frac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \cdot 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$ $\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Try It 3.2.20

Simplify: $\frac{x^3-8}{3x^2-6x+12} \div \frac{x^2-4}{6}$.

Answer

$$\frac{2(x^2+2x+4)}{(x+2)(x^2-2x+4)}$$

Try It 3.2.21

Simplify: $\frac{2z^2}{z^2-1} \div \frac{z^3-z^2+z}{z^3+1}$.

Answer

$\frac{2z}{z-1}$

DIVISION OF RATIONAL EXPRESSIONS

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

Recall from Use the Language of Algebra that a complex fraction is a fraction that contains a fraction in the numerator, the denominator or both. Also, remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

EXAMPLE 3.2.22

Divide: $\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$.

Solution

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign. $\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$

Rewrite as product of first times reciprocal of second. $\frac{6x^2-7x+2}{4x-8} \cdot \frac{x^2-5x+6}{2x^2-7x+3}$

Factor the numerators and the denominators, and then multiply. $\frac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$

Simplify by dividing out common factors. $\frac{\cancel{(2x-1)}(3x-2)\ \cancel{(x-2)}\ \cancel{(x-3)}}{4\ \cancel{(x-2)}\ \cancel{(2x-1)}\ \cancel{(x-3)}}$

Simplify. $\frac{3x-2}{4}$

Try It 3.2.23

Simplify: $\dfrac{\dfrac{3x^2+7x+2}{4x+24}}{\dfrac{3x^2-14x-5}{x^2+x-30}}$.

Answer

$\dfrac{x+2}{4}$

Try It 3.2.24

Simplify: $\dfrac{\dfrac{y^2-36}{2y^2+11y-6}}{\dfrac{2y^2-2y-60}{8y-4}}$.

Answer

$\dfrac{2}{y+5}$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then, we factor and multiply.

EXAMPLE 3.2.25

Perform the indicated operations: $\dfrac{3x-6}{4x-4}\cdot\dfrac{x^2+2x-3}{x^2-3x-10}\div\dfrac{2x+12}{8x+16}$.

Solution

	$\frac{3x-6}{4x-4}\cdot\frac{x^2+2x-3}{x^2-3x-10}\div\frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4}\cdot\frac{x^2+2x-3}{x^2-3x-10}\cdot\frac{8x+16}{2x+12}$
Factor the numerators and the denominators.	$\frac{3(x-2)}{4(x-1)}\cdot\frac{(x+3)(x-1)}{(x+2)(x-5)}\cdot\frac{8(x+2)}{2(x+6)}$
Multiply the fractions. Bringing the constants to the front will help when removing common factors.	
Simplify by dividing out common factors.	$\frac{3\cdot 8(x-2)(x+3)(x-1)(x+2)}{4\cdot 2(x-1)(x+2)(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

Try It 3.2.26

Perform the indicated operations: $\dfrac{4m+4}{3m-15}\cdot\dfrac{m^2-3m-10}{m^2-4m-32}\div\dfrac{12m-36}{6m-48}$.

Answer

$\dfrac{2(m+1)(m+2)}{3(m+4)(m-3)}$

Try It 3.2.27

Perform the indicated operations: $\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

Answer

$$\frac{(n+5)(n+9)}{2(n+6)(2n+3)}$$

Multiply and Divide Rational Functions

We started this section stating that a **rational expression** is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$. Similarly, we define a **rational function** as a function of the form $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

RATIONAL FUNCTION

A rational function is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

The domain of a rational function is all real numbers except for those values that would cause division by zero. We must eliminate any values that make $q(x) = 0$.

DETERMINE THE DOMAIN OF A RATIONAL FUNCTION.

1. Set the denominator equal to zero.
2. Solve the equation.
3. The domain is all real numbers excluding the values found in Step 2.

EXAMPLE 3.2.28

Find the domain of $R(x) = \frac{2x^2-14x}{4x^2-16x-48}$.

Solution

The domain will be all real numbers except those values that make the denominator zero. We will set the denominator equal to zero , solve that equation, and then exclude those values from the domain.

Set the denominator to zero.	$4x^2-16x-48=0$
Factor, first factor out the GCF.	$4(x^2-4x-12)=0$
	$4(x-6)(x+2)=0$
Use the Zero Product Property.	$4 \neq 0 \quad x-6=0 \quad x+2=0$
Solve.	$x=6 \quad x=-2$
	The domain of $R(x)$ is all real numbers where $x \neq 6$ and $x \neq -2$

Try It 3.2.29

Find the domain of $R(x) = \dfrac{2x^2 - 10x}{4x^2 - 16x - 20}$.

Answer

The domain of $R(x)$ is all real numbers where $x \neq 5$ and $x \neq -1$.

Try It 3.2.30

Find the domain of $R(x) = \dfrac{4x^2 - 16x}{8x^2 - 16x - 64}$.

Answer

The domain of $R(x)$ is all real numbers where $x \neq 4$ and $x \neq -2$.

To multiply rational functions, we multiply the resulting rational expressions on the right side of the equation using the same techniques we used to multiply rational expressions.

EXAMPLE 3.2.31

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{2x-6}{x^2-8x+15}$ and $g(x) = \dfrac{x^2-25}{2x+10}$.

Solution

$$R(x) = f(x) \cdot g(x)$$

$$R(x) = \frac{2x-6}{x^2-8x+15} \cdot \frac{x^2-25}{2x+10}$$

Factor each numerator and denominator. $$R(x) = \frac{2(x-3)}{(x-3)(x-5)} \cdot \frac{(x-5)(x+5)}{2(x+5)}$$

Multiply the numerators and denominators. $$R(x) = \frac{2(x-3)(x-5)(x+5)}{2(x-3)(x-5)(x+5)}$$

Remove common factors. $$R(x) = \frac{\cancel{2}\,\cancel{(x-3)}\,\cancel{(x-5)}\,\cancel{(x+5)}}{\cancel{2}\,\cancel{(x-3)}\,\cancel{(x-5)}\,\cancel{(x+5)}}$$

Simplify. $$R(x) = 1$$

Try It 3.2.32

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{3x-21}{x^2-9x+14}$ and $g(x) = \dfrac{2x^2-8}{3x+6}$.

Answer

$R(x) = 2$

Try It 3.2.33

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{x^2 - x}{3x^2 + 27x - 30}$ and $g(x) = \dfrac{x^2 - 100}{x^2 - 10x}$.

Answer

$$R(x) = \frac{1}{3}$$

To divide rational functions, we divide the resulting rational expressions on the right side of the equation using the same techniques we used to divide rational expressions.

EXAMPLE 3.2.34

Find $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x) = \dfrac{3x^2}{x^2 - 4x}$ and $g(x) = \dfrac{9x^2 - 45x}{x^2 - 7x + 10}$.

Solution

$$R(x) = \frac{f(x)}{g(x)}$$

Substitute in the functions $f(x),\ g(x)$.

$$R(x) = \frac{\dfrac{3x^2}{x^2 - 4x}}{\dfrac{9x^2 - 45x}{x^2 - 7x + 10}}$$

Rewrite the division as the product of $f(x)$ and the reciprocal of $g(x)$.

$$R(x) = \frac{3x^2}{x^2 - 4x} \cdot \frac{x^2 - 7x + 10}{9x^2 - 45x}$$

Factor the numerators and denominators and then multiply.

$$R(x) = \frac{3 \cdot x \cdot x \cdot (x-5)(x-2)}{x(x-4) \cdot 3 \cdot 3 \cdot x \cdot (x-5)}$$

Simplify by dividing out common factors.

$$R(x) = \frac{\cancel{3} \cdot \cancel{x} \cdot \cancel{x}\,\cancel{(x-5)}(x-2)}{\cancel{x}(x-4) \cdot \cancel{3} \cdot 3 \cdot \cancel{x}\,\cancel{(x-5)}}$$

$$R(x) = \frac{x-2}{3(x-4)}$$

Try It 3.2.35

Find $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x) = \dfrac{2x^2}{x^2 - 8x}$ and $g(x) = \dfrac{8x^2 + 24x}{x^2 + x - 6}$.

Answer

$$R(x) = \frac{x-2}{4(x-8)}$$

Try It 3.2.36

Find $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x) = \dfrac{15x^2}{3x^2 + 33x}$ and $g(x) = \dfrac{5x - 5}{x^2 + 9x - 22}$.

Answer

$$R(x) = \frac{x(x-2)}{x-1}$$

Key Concepts

- **Determine the values for which a rational expression is undefined.**
  1. Set the denominator equal to zero.
  2. Solve the equation.
- **Equivalent Fractions Property**
  If a, b, and c are numbers where $b \neq 0$, $c \neq 0$, then

  $$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$
- **How to simplify a rational expression.**
  1. Factor the numerator and denominator completely.
  2. Simplify by dividing out common factors.
- **Opposites in a Rational Expression**
  The opposite of $a - b$ is $b - a$.

  $$\frac{a-b}{b-a} = -1 \qquad a \neq b$$

  An expression and its opposite divide to -1.
- **Multiplication of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

  $$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$
- **How to multiply rational expressions.**
  1. Factor each numerator and denominator completely.
  2. Multiply the numerators and denominators.
  3. Simplify by dividing out common factors.
- **Division of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then

  $$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$
- **How to divide rational expressions.**
  1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
  2. Factor the numerators and denominators completely.
  3. Multiply the numerators and denominators together.
  4. Simplify by dividing out common factors.
- **How to determine the domain of a rational function.**
  1. Set the denominator equal to zero.
  2. Solve the equation.
  3. The domain is all real numbers excluding the values found in Step 2.

Glossary

rational expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

simplified rational expression

A simplified rational expression has no common factors, other than 1, in its numerator and denominator.

rational function

A rational function is a function of the form $R(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

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3.2E: Exercises

Practice Makes Perfect

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

1. a. $\frac{2x^2}{z}$ b. $\frac{4p-1}{6p-5}$ c. $\frac{n-3}{n^2+2n-8}$

Answer

a. $z=0$
b. $p=\frac{5}{6}$
c. $n=-4, n=2$

2. a. $\frac{10m}{11n}$ b. $\frac{6y+13}{4y-9}$ c. $\frac{b-8}{b^2-36}$

3. a. $\frac{4x^2y}{3y}$ b. $\frac{3x-2}{2x+1}$ c. $\frac{u-1}{u^2-3u-28}$

Answer

a. $y=0$
b. $x=-\frac{1}{2}$
c. $u=-4, u=7$

4. a. $\frac{5pq^2}{9q}$ b. $\frac{7a-4}{3a+5}$ c. $\frac{1}{x^2-4}$

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

5. $-\frac{44}{55}$

Answer

$-\frac{4}{5}$

6. $\frac{56}{63}$

7. $\frac{8m^3n}{12mn^2}$

Answer

$\frac{2m^2}{3n}$

8. $\frac{36v^3w^2}{27vw^3}$

9. $\dfrac{8n-96}{3n-36}$

Answer

$\dfrac{8}{3}$

10. $\dfrac{12p-240}{5p-100}$

11. $\dfrac{x^2+4x-5}{x^2-2x+1}$

Answer

$\dfrac{x+5}{x-1}$

12. $\dfrac{y^2+3y-4}{y^2-6y+5}$

13. $\dfrac{a^2-4}{a^2+6a-16}$

Answer

$\dfrac{a+2}{a+8}$

14. $\dfrac{y^2-2y-3}{y^2-9}$

15. $\dfrac{p^3+3p^2+4p+12}{p^2+p-6}$

Answer

$\dfrac{p^2+4}{p-2}$

16. $\dfrac{x^3-2x^2-25x+50}{x^2-25}$

17. $\dfrac{8b^2-32b}{2b^2-6b-80}$

Answer

$\dfrac{4b(b-4)}{(b+5)(b-8)}$

18. $\dfrac{-5c^2-10c}{-10c^2+30c+100}$

19. $\frac{3m^2+30mn+75n^2}{4m^2-100n^2}$

Answer

$\frac{3(m+5n)}{4(m-5n)}$

20. $\frac{5r^2+30rs-35s^2}{r^2-49s^2}$

21. $\frac{a-5}{5-a}$

Answer

-1

22. $\frac{5-d}{d-5}$

23. $\frac{20-5y}{y^2-16}$

Answer

$\frac{-5}{y+4}$

24. $\frac{4v-32}{64-v^2}$

25. $\frac{w^3+21}{6w^2-36}$

Answer

$\frac{w^2-6w+3}{6w-6}$

26. $\frac{v^3+125}{v^2-25}$

27. $\frac{z^2-9z+20}{16-z^2}$

Answer

$\frac{-z-5}{4+z}$

28. $\frac{a^2-5z-36}{81-a^2}$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

29. $\frac{12}{16} \cdot \frac{4}{10}$

Answer

$\frac{3}{10}$

30. $\frac{3}{25} \cdot \frac{16}{24}$

31. $\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2}$

Answer

$\frac{x^3}{8y}$

32. $\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3}$

33. $\frac{5p^2}{p^2-5p-36} \cdot \frac{p^2-16}{10p}$

Answer

$\frac{p(p-4)}{2(p-9)}$

34. $\frac{3q^2}{q^2+q-6} \cdot \frac{q^2-9}{9q}$

35. $\frac{2y^2-10y}{y^2+10y+25} \cdot \frac{y+5}{6y}$

Answer

$\frac{y-5}{3(y+5)}$

36. $\frac{z^2+3z}{z^2-3z-4} \cdot \frac{z-4}{z^2}$

37. $\frac{28-4b}{3b-3} \cdot \frac{b^2+8b-9}{b^2-49}$

Answer

$\frac{-4(b+9)}{3(b+7)}$

38. $\frac{72m-12m^2}{8m+32} \cdot \frac{m^2+10m+24}{m^2-36}$

39. $\frac{c^2-10c+25}{c^2-25} \cdot \frac{c^2+10c+25}{3c^2-14c-5}$

Answer

$\frac{c+5}{3c+1}$

40. $\frac{2d^2+d-3}{d^2-16} \cdot \frac{d^2-8d+16}{2d^2-9d-18}$

41. $\frac{2m^2-3m-2}{2m2+7m+3} \cdot \frac{3m^2-14m+15}{3m^2+17m-20}$

Answer

$\frac{(m-3)(m-2)}{(m+4)(m+3)}$

42. $\frac{2n^2-3n-14}{25-n^2} \cdot \frac{n^2-10n+25}{2n^2-13n+21}$

Divide Rational Expressions

In the following exercises, divide the rational expressions.

43. $\frac{v-5}{11-v} \div \frac{v^2-25}{v-11}$

Answer

$-\frac{1}{v+5}$

44. $\frac{10+w}{w-8} \div \frac{100-w^2}{8-w}$

45. $\frac{3s^2}{s^2-16} \div \frac{s^3-4s^2+16s}{s^3-64}$

Answer

$\frac{3s}{s+4}$

46. $\frac{r^2-9}{15} \div \frac{r^3-27}{5r^2+15r+45}$

47. $\frac{p^3+q^3}{3p^2+3pq+3q^2} \div \frac{p^2-q^2}{12}$

Answer

$\frac{4(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

48. $\dfrac{v^3-8w^3}{2v^2+4vw+8w^2} \div \dfrac{v^2-4w^2}{4}$

49. $\dfrac{x^2+3x-10}{4x} \div (2x^2+20x+50)$

Answer

$$\frac{x-2}{8x(x+5)}$$

50. $\dfrac{2y^2-10yz-48z^2}{2y-1} \div (4y^2-32yz)$

51. $\dfrac{\dfrac{2a^2-a-21}{5a+20}}{\dfrac{a^2+7a+12}{a^2+8a+16}}$

Answer

$$\frac{2a-7}{5}$$

52. $\dfrac{\dfrac{3b^2+2b-8}{12b+18}}{\dfrac{3b^2+2b-8}{2b^2-7b-15}}$

53. $\dfrac{\dfrac{12c^2-12}{2c^2-3c+1}}{\dfrac{4c+4}{6c^2-13c+5}}$

Answer

$$3(3c-5)$$

54. $\dfrac{\dfrac{4d^2+7d-2}{35d+10}}{\dfrac{d^2-4}{7d^2-12d-4}}$

For the following exercises, perform the indicated operations.

55. $\dfrac{10m^2+80m}{3m-9} \cdot \dfrac{m^2+4m-21}{m^2-9m+20} \div \dfrac{5m^2+10m}{2m-10}$

Answer

$$\frac{4(m+8)(m+7)}{3(m-4)(m+2)}$$

56. $\dfrac{4n^2+32n}{3n+2} \cdot \dfrac{3n^2-n-2}{n^2+n-30} \div \dfrac{108n^2-24n}{n+6}$

57. $\dfrac{12p^2+3p}{p+3} \div \dfrac{p^2+2p-63}{p^2-p-12} \cdot \dfrac{p-7}{9p^3-9p^2}$

Answer

$\dfrac{(4p+1)(p-4)}{3p(p+9)(p-1)}$

58. $\dfrac{6q+3}{9q^2-9q} \div \dfrac{q^2+14q+33}{q^2+4q-5} \cdot \dfrac{4q^2+12q}{12q+6}$

Multiply and Divide Rational Functions

In the following exercises, find the domain of each function.

59. $R(x) = \dfrac{x^3-2x^2-25x+50}{x^2-25}$

Answer

$x \neq 5$ and $x \neq -5$

60. $R(x) = \dfrac{x^3+3x^2-4x-12}{x^2-4}$

61. $R(x) = \dfrac{3x^2+15x}{6x^2+6x-36}$

Answer

$x \neq 2$ and $x \neq -3$

62. $R(x) = \dfrac{8x^2-32x}{2x^2-6x-80}$

For the following exercises, find $R(x) = f(x) \cdot g(x)$ where $f(x)$ and $g(x)$ are given.

63. $f(x) = \dfrac{6x^2-12x}{x^2+7x-18} \quad g(x) = \dfrac{x^2-81}{3x^2-27x}$

Answer

$R(x) = 2$

64. $f(x) = \dfrac{x^2-2x}{x^2+6x-16} \quad g(x) = \dfrac{x^2-64}{x^2-8x}$

65. $f(x) = \dfrac{4x}{x^2-3x-10} \quad g(x) = \dfrac{x^2-25}{8x^2}$

Answer

$R(x) = \dfrac{x+5}{2x(x+2)}$

66. $f(x) = \dfrac{2x^2+8x}{x^2-9x+20} \quad g(x) = \dfrac{x-5}{x^2}$

For the following exercises, find $R(x) = f(x)g(x)$ where $f(x)$ and $g(x)$ are given.

67. $f(x) = \dfrac{27x^2}{3x-21} \quad g(x) = \dfrac{3x^2+18x}{x^2+13x+42}$

Answer

$$R(x) = \frac{3x(x+7)}{x-7}$$

68. $f(x) = \dfrac{24x^2}{2x-8} \quad g(x) = \dfrac{4x^3+28x^2}{x^2+11x+28}$

69. $f(x) = \dfrac{16x^2}{4x+36} \quad g(x) = \dfrac{4x^2-24x}{x^2+4x-45}$

Answer

$$R(x) = \frac{x(x-5)}{x-6}$$

70. $f(x) = \dfrac{24x^2}{2x-4} \quad g(x) = \dfrac{12x^2+36x}{x^2-11x+18}$

Writing Exercises

71. Explain how you find the values of *x* for which the rational expression $\dfrac{x^2-x-20}{x^2-4}$ is undefined.

Answer

Answers will vary.

72. Explain all the steps you take to simplify the rational expression $\dfrac{p^2+4p-21}{9-p^2}$.

73. a. Multiply $\dfrac{7}{4} \cdot \dfrac{9}{10}$ and explain all your steps.
b. Multiply $\dfrac{n}{n-3} \cdot \dfrac{9}{n+3}$ and explain all your steps.
c. Evaluate your answer to part b. when $n = 7$. Did you get the same answer you got in part a.? Why or why not?

Answer

Answers will vary.

74. a. Divide $\dfrac{24}{5} \div 6$ and explain all your steps.
b. Divide $\dfrac{x^2-1}{x} \div (x+1)$ and explain all your steps.
c. Evaluate your answer to part b. when $x = 5$. Did you get the same answer you got in part a.? Why or why not?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
simplify rational expressions.			
multiply rational expressions.			
divide rational expressions.			
multiply and divide rational functions.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

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3.3: Divide Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Divide monomials
- Divide a polynomial by a monomial
- Divide polynomials using long division
- Divide polynomials using synthetic division
- Divide polynomial functions
- Use the remainder and factor theorems

Before you get started, take this readiness quiz.

1. Add: $\frac{3}{d}+\frac{x}{d}$.
   If you missed this problem, review [link].
2. Simplify: $\frac{30xy}{35xy}$.
   If you missed this problem, review [link].
3. Combine like terms: $8a^2+12a+1+3a^2-5a+4$.
   If you missed this problem, review [link].

Dividing Monomials

We are now familiar with all the properties of exponents and used them to multiply polynomials. Next, we'll use these properties to divide monomials and polynomials.

Example 3.3.1

Find the quotient: $54a^2b^3 \div (-6ab^5)$.

Solution

When we divide monomials with more than one variable, we write one fraction for each variable.

	$54a^2b^3 \div (-6ab^5)$
Rewrite as a fraction.	$\frac{54a^2b^3}{-6ab^5}$
Use fraction multiplication.	$\frac{54}{-6}\cdot\frac{a^2}{a}\cdot\frac{b^3}{b^5}$
Simplify and use the Quotient Property.	$-9\cdot a\cdot\frac{1}{b^2}$
Multiply.	$-\frac{9a}{b^2}$

Try It! 3.3.1

Find the quotient: $-72a^7b^3 \div (8a^{12}b^4)$.

Answer

$-\frac{9}{a^5b}$

Try It! 3.3.2

Find the quotient: $-63c^8d^3 \div (7c^{12}d^2)$.

Answer

$$\frac{-9d}{c^4}$$

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

Example 3.3.2

Find the quotient: $\frac{14x^7y^{12}}{21x^{11}y^6}$.

Solution

Be very careful to simplify $\frac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

	$\frac{14x^7y^{12}}{21x^{11}y^6}$
Simplify and use the Quotient Property.	$\frac{2y^6}{3x^4}$

Try It! 3.3.3

Find the quotient: $\frac{28x^5y^{14}}{49x^9y^{12}}$.

Answer

$$\frac{4y^2}{7x^4}$$

Try It! 3.3.4

Find the quotient: $\frac{30m^5n^{11}}{48m^{10}n^{14}}$.

Answer

$$\frac{5}{8m^5n^3}$$

Divide a Polynomial by a Monomial

Now that we know how to divide a monomial by a monomial, the next procedure is to divide a polynomial of two or more terms by a monomial. The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review fraction addition. The sum $\frac{y}{5}+\frac{2}{5}$ simplifies to $\frac{y+2}{5}$. Now we will do this in reverse to split a single fraction into separate fractions. For example, $\frac{y+2}{5}$ can be written $\frac{y}{5}+\frac{2}{5}$.

This is the "reverse" of fraction addition and it states that if *a*, *b*, and *c* are numbers where $c \neq 0$, then $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$. We will use this to divide polynomials by monomials.

definition: DIVISION OF A POLYNOMIAL BY A MONOMIAL

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Example 3.3.3

Find the quotient: $(18x^3y - 36xy^2) \div (-3xy)$.

Solution

	$(18x^3y - 36xy^2) \div (-3xy)$
Rewrite as a fraction.	$\frac{18x^3y - 36xy^2}{-3xy}$
Divide each term by the divisor. Be careful with the signs!	$\frac{18x^3y}{-3xy} - \frac{36xy^2}{-3xy}$
Simplify.	$-6x^2 + 12y$

Try It! 3.3.5

Find the quotient: $(32a^2b - 16ab^2) \div (-8ab)$.

Answer

$-4a + 2b$

Try It! 3.3.6

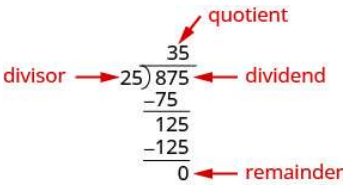
Find the quotient: $(-48a^8b^4 - 36a^6b^5) \div (-6a^3b^3)$.

Answer

$8a^5b + 6a^3b^2$

Divide Polynomials Using Long Division

Divide a polynomial by a binomial, we follow a procedure very similar to long division of numbers. So let's look carefully the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.



We check division by multiplying the quotient by the divisor. If we did the division correctly, the product should equal the dividend.

$$35 \cdot 25$$
$$875\checkmark$$

Now we will divide a trinomial by a binomial. As you read through the example, notice how similar the steps are to the numerical example above.

Example 3.3.4

Find the quotient: $(x^2+9x+20)\div(x+5)$.

Solution

	$(x^2+9x+20)\div(x+5)$
Write it as a long division problem. Be sure the dividend is in standard form.	$x+5\overline{)\,x^2+9x+20}$
Divide x^2 by x. It may help to ask yourself, "What do I need to multiply x by to get x^2?"	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \end{array}$
Put the answer, x, in the quotient over the x term. Multiply x times $x+5$. Line up the like terms under the dividend.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \\ \underline{x^2+5x} \end{array}$
Subtract x^2+5x from x^2+9x . You may find it easier to change the signs and then add. Then bring down the last term, 20.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+\ \ 9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \end{array}$
Divide $4x$ by x. It may help to ask yourself, "What do I need to multiply x by to get $4x$?" Put the answer, 4, in the quotient over the constant term.	$\begin{array}{r} x+\ 4 \\ x+5\overline{)\,x^2+\ \ 9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \end{array}$
Multiply 4 times $x+5$.	$\begin{array}{r} x+\ 4 \\ x+5\overline{)\,x^2+\ \ 9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \\ \underline{4x+20} \end{array}$
Subtract $4x+20$ from $4x+20$.	$\begin{array}{r} x+\ 4 \\ x+5\overline{)\,x^2+\ \ 9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \\ \underline{-4x+(-20)} \\ 0 \end{array}$
Check: Multiply the quotient by the divisor. $(x+4)(x+5)$ You should get the dividend. $x^2+9x+20\checkmark$	

Try It! 3.3.7

Find the quotient: $(y^2+10y+21)\div(y+3)$.

Answer

$y+7$

Try It! 3.3.8

Find the quotient: $(m^2+9m+20)\div(m+4)$.

Answer

$m+5$

When we divided 875 by 25, we had no remainder. But sometimes division of numbers does leave a remainder. The same is true when we divide polynomials. In the next example, we'll have a division that leaves a remainder. We write the remainder as a fraction with the divisor as the denominator.

Look back at the dividends in previous examples. The terms were written in descending order of degrees, and there were no missing degrees. The dividend in this example will be $x^4 - x^2 + 5x - 6$. It is missing an x^3 term. We will add in $0x^3$ as a placeholder.

Example 3.3.5

Find the quotient: $(x^4 - x^2 + 5x - 6) \div (x + 2)$.

Solution

Notice that there is no x^3 term in the dividend. We will add $0x^3$ as a placeholder.

	$(x^4 - x^2 + 5x - 6) \div (x + 2)$
Write it as a long division problem. Be sure the dividend is in standard form with placeholders for missing terms.	$x + 2 \overline{)x^4 + 0x^3 - x^2 + 5x - 6}$
Divide x^4 by x. Put the answer, x^3, in the quotient over the x^3 term. Multiply x^3 times $x + 2$. Line up the like terms. Subtract and then bring down the next term.	x^3 $x + 2 \overline{)x^4 + 0x^3 - x^2 + 5x - 6}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ It may be helpful to change the signs and add.
Divide $-2x^3$ by x. Put the answer, $-2x^2$, in the quotient over the x^2 term. Multiply $-2x^2$ times $x + 1$. Line up the like terms Subtract and bring down the next term.	$x^3 - 2x^2$ $x + 2 \overline{)x^4 + 0x^3 - x^2 + 5x - 6}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ $-(-2x^3 - 4x^2)$ $3x^2 + 5x$ It may be helpful to change the signs and add.
Divide $3x^2$ by x. Put the answer, $3x$, in the quotient over the x term. Multiply $3x$ times $x + 1$. Line up the like terms. Subtract and bring down the next term.	$x^3 - 2x^2 + 3x$ $x + 2 \overline{)x^4 + 0x^3 - x^2 + 5x - 6}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ $-(-2x^3 - 4x^2)$ $3x^2 + 5x$ $-(3x^2 + 6x)$ $-x - 6$ It may be helpful to change the signs and add.
Divide $-x$ by x. Put the answer, -1, in the quotient over the constant term. Multiply -1 times $x + 1$. Line up the like terms. Change the signs, add. Write the remainder as a fraction with the divisor as the denominator.	$x^3 - 2x^2 + 3x - 1 - \frac{4}{x+2}$ $x + 2 \overline{)x^4 + 0x^3 - x^2 + 5x - 6}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ $-(-2x^3 - 4x^2)$ $3x^2 + 5x$ $-(3x^2 + 6x)$ $-x - 6$ $-(-x - 2)$ -4 It may be helpful to change the signs and add.
To check, multiply $(x + 2)(x^3 - 2x^2 + 3x - 1 - 4x + 2)$. The result should be $x^4 - x^2 + 5x - 6$.	

Try It! 3.3.9

Find the quotient: $(x^4 - 7x^2 + 7x + 6) \div (x + 3)$.

Answer

$x^3-3x^2+2x+1+3x+3$

Try It! 3.3.10

Find the quotient: $(x^4-11x^2-7x-6)\div(x+3)$.

Answer

$x^3-3x^2-2x-1-3x+3$

In the next example, we will divide by $2a-3$. As we divide, we will have to consider the constants as well as the variables.

Example 3.3.6

Find the quotient: $(8a^3+27)\div(2a+3)$.

Solution

This time we will show the division all in one step. We need to add two placeholders in order to divide.

$(8a^3+27)\div(2a+3)$

$$
\begin{array}{rl}
 & 4a^2-6a+9 \\
2a+3\,\overline{)} & 8a^3+0a^2+0a+27 \\
 & \underline{-(8a^3+12a^2)} \quad \leftarrow 4a^2(2a+3) \\
 & -12a^2+0a \\
 & \underline{-(-12a^2-18a)} \quad \leftarrow 6a(2a+3) \\
 & 18a+27 \\
 & \underline{-(18a+27)} \quad \leftarrow 9(2a+3) \\
 & 0
\end{array}
$$

To check, multiply $(2a+3)(4a^2-6a+9)$.

The result should be $8a^3+27$.

Try It! 3.3.11

Find the quotient: $(x^3-64)\div(x-4)$.

Answer

$x^2+4x+16$

Try It! 3.3.12

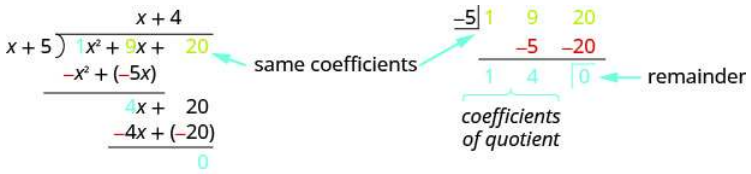
Find the quotient: $(125x^3-8)\div(5x-2)$.

Answer

$25x^2+10x+4$

Divide Polynomials using Synthetic Division

As we have mentioned before, mathematicians like to find patterns to make their work easier. Since long division can be tedious, let's look back at the long division we did in Example and look for some patterns. We will use this as a basis for what is called *synthetic division*. The same problem in the synthetic division format is shown next.



Synthetic division basically just removes unnecessary repeated variables and numbers. Here all the x and x^2 are removed. as well as the $-x^2$ and $-4x$ as they are opposite the term above.

- The first row of the synthetic division is the coefficients of the dividend. The -5 is the opposite of the 5 in the divisor.
- The second row of the synthetic division are the numbers shown in red in the division problem.
- The third row of the synthetic division are the numbers shown in blue in the division problem.

Notice the quotient and remainder are shown in the third row.

Synthetic division only works when the divisor is of the form $x - c$.

The following example will explain the process.

Example 3.3.7

Use synthetic division to find the quotient and remainder when $2x^3 + 3x^2 + x + 8$ is divided by $x + 2$.

Solution

Write the dividend with decreasing powers of x.	$2x^3 + 3x^2 + x + 8$
Write the coefficients of the terms as the first row of the synthetic division.	⌴ | 2 3 1 8
Write the divisor as $x - c$ and place c in the synthetic division in the divisor box.	–2 | 2 3 1 8
Bring down the first coefficient to the third row.	–2 | 2 3 1 8 2
Multiply that coefficient by the divisor and place the result in the second row under the second coefficient.	–2 | 2 3 1 8 –4 2
Add the second column, putting the result in the third row.	–2 | 2 3 1 8 –4 2 –1
Multiply that result by the divisor and place the result in the second row under the third coefficient.	–2 | 2 3 1 8 –4 2 2 –1
Add the third column, putting the result in the third row.	–2 | 2 3 1 8 –4 2 2 –1 3
Multiply that result by the divisor and place the result in the third row under the third coefficient.	–2 | 2 3 1 8 –4 2 –6 2 –1 3
Add the final column, putting the result in the third row.	–2 | 2 3 1 8 –4 2 –6 2 –1 3 | 2
The quotient is $2x^2 - 1x + 3$ and the remainder is 2.	

The division is complete. The numbers in the third row give us the result. The $2 \quad -1 \quad 3$ are the coefficients of the quotient. The quotient is $2x^2 - 1x + 3$. The 2 in the box in the third row is the remainder.

Check:

$$\begin{aligned} (\text{quotient})(\text{divisor}) + \text{remainder} &= \text{dividend} \\ (2x^2 - 1x + 3)(x + 2) + 2 &\stackrel{?}{=} 2x^3 + 3x^2 + x + 8 \\ 2x^3 - x^2 + 3x + 4x^2 - 2x + 6 + 2 &\stackrel{?}{=} 2x^3 + 3x^2 + x + 8 \\ 2x^3 + 3x^2 + x + 8 &= 2x^3 + 3x^2 + x + 8 \checkmark \end{aligned}$$

? Try It! 3.3.13

Use synthetic division to find the quotient and remainder when $3x^3 + 10x^2 + 6x - 2$ is divided by $x + 2$.

Answer

$3x^2 + 4x - 2;\ 2$

? Try It! 3.3.14

Use synthetic division to find the quotient and remainder when $4x^3 + 5x^2 - 5x + 3$ is divided by $x + 2$.

Answer

$4x^2 - 3x + 1; 1$

In the next example, we will do all the steps together.

✓ Example 3.3.8

Use synthetic division to find the quotient and remainder when $x^4 - 16x^2 + 3x + 12$ is divided by $x + 4$.

Solution

The polynomial $x^4 - 16x^2 + 3x + 12$ has its term in order with descending degree but we notice there is no x^3 term. We will add a 0 as a placeholder for the x^3 term. In $x - c$ form, the divisor is $x - (-4)$.

–4	1	0	–16	3	12
		–4	16	0	–12
	1	–4	0	3	0

We divided a 4th degree polynomial by a 1st degree polynomial so the quotient will be a 3rd degree polynomial.

Reading from the third row, the quotient has the coefficients $1 \quad -4 \quad 0 \quad 3$, which is $x^3 - 4x^2 + 3$. The remainder is 0.

? Try It! 3.3.15

Use synthetic division to find the quotient and remainder when $x^4 - 16x^2 + 5x + 20$ is divided by $x + 4$.

Answer

$x^3 - 4x^2 + 5;\ 0$

? Try It! 3.3.16

Use synthetic division to find the quotient and remainder when $x^4 - 9x^2 + 2x + 6$ is divided by $x + 3$.

Answer

$x^3 - 3x^2 + 2;\ 0$

Divide Polynomial Functions

Just as polynomials can be divided, polynomial functions can also be divided.

definition: DIVISION OF POLYNOMIAL FUNCTIONS

For functions $f(x)$ and $g(x)$, where $g(x) \neq 0$,

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example 3.3.9

For functions $f(x) = x^2 - 5x - 14$ and $g(x) = x + 2$, find:

a. $\left(\frac{f}{g}\right)(x)$

b. $\left(\frac{f}{g}\right)(-4)$.

Solution

ⓐ

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$\begin{array}{r} x - 7 \\ x + 2 \overline{)\, x^2 - 5x - 14} \\ \underline{-(x^2 + 2x)} \\ -7x - 14 \\ \underline{-(-7x - 14)} \\ 0 \end{array}$$

Substitute for $f(x)$ and $g(x)$. $\left(\frac{f}{g}\right)(x) = \frac{x^2 - 5x - 14}{x + 2}$

Divide the polynomials. $\left(\frac{f}{g}\right)(x) = x - 7$

ⓑ In part ⓐ we found $\left(\frac{f}{g}\right)(x)$ and now are asked to find $\left(\frac{f}{g}\right)(-4)$.

$\left(\frac{f}{g}\right)(x) = x - 7$

To find $\left(\frac{f}{g}\right)(-4)$, substitute $x = -4$. $\left(\frac{f}{g}\right)(-4) = -4 - 7$

$\left(\frac{f}{g}\right)(-4) = -11$

Try It! 3.3.17

For functions $f(x) = x^2 - 5x - 24$ and $g(x) = x + 3$, find:

a. $\left(\frac{f}{g}\right)(x)$

b. $\left(\frac{f}{g}\right)(-3)$.

Answer a

$\left(\frac{f}{g}\right)(x) = x - 8$

Answer b

$\left(\frac{f}{g}\right)(-3) = -11$

? Try It! 3.3.18

For functions $f(x) = x2 - 5x - 36$ and $g(x) = x + 4$, find:

a. $\left(\frac{f}{g}\right)(x)$

b. $\left(\frac{f}{g}\right)(-5)$.

Answer a

$\left(\frac{f}{g}\right)(x) = x - 9$

Answer b

$\left(\frac{f}{g}\right)(x) = x - 9$

Use the Remainder and Factor Theorem

Let's look at the division problems we have just worked that ended up with a remainder. They are summarized in the chart below. If we take the dividend from each division problem and use it to define a function, we get the functions shown in the chart. When the divisor is written as $x - c$, the value of the function at c, $f(c)$, is the same as the remainder from the division problem.

Dividend	Divisor $x - c$	Remainder	Function	$f(c)$
$x^4 - x^2 + 5x - 6$	$x - (-2)$	-4	$f(x) = x^4 - x^2 + 5x - 6$	-4
$3x^3 - 2x^2 - 10x + 8$	$x - 2$	4	$f(x) = 3x^3 - 2x^2 - 10x + 8$	4
$x^4 - 16x^2 + 3x + 15$	$x - (-4)$	3	$f(x) = x^4 - 16x^2 + 3x + 15$	3

To see this more generally, we realize we can check a division problem by multiplying the quotient times the divisor and add the remainder. In function notation we could say, to get the dividend $f(x)$, we multiply the quotient, $q(x)$ times the divisor, $x - c$, and add the remainder, r.

	$f(x) = q(x)(x - c) + r$
If we evaluate this at c, we get:	$f(c) = q(c)(c - c) + r$
	$f(c) = q(c)(0) + r$
	$f(c) = r$

This leads us to the Remainder Theorem.

Definition: REMAINDER THEOREM

If the polynomial function $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Example 3.3.10

Use the Remainder Theorem to find the remainder when $f(x) = x^3 + 3x + 19$ is divided by $x + 2$.

Solution

To use the Remainder Theorem, we must use the divisor in the $x - c$ form. We can write the divisor $x + 2$ as $x - (-2)$. So, our c is -2.

To find the remainder, we evaluate $f(c)$ which is $f(-2)$.

	$f(x) = x^3 + 3x + 19$
To evaluate $f(-2)$, substitute $x = -2$.	$f(-2) = (-2)^3 + 3(-2) + 19$
Simplify.	$f(-2) = -8 - 6 + 19$
	$f(-2) = 5$
	The remainder is 5 when $f(x) = x^3 + 3x + 19$ is divided by $x + 2$.
Check: Use synthetic division to check.	
$\begin{array}{r\|rrr} -2 & 1 & 0 & 3 & 19 \\ & & -2 & 4 & -14 \\ \hline & 1 & -2 & 7 & \boxed{5} \end{array}$	
The remainder is 5.	

Try It! 3.3.19

Use the Remainder Theorem to find the remainder when $f(x) = x^3 + 4x + 15$ is divided by $x + 2$.

Answer

-1

Try It! 3.3.20

Use the Remainder Theorem to find the remainder when $f(x) = x^3 - 7x + 12$ is divided by $x + 3$.

Answer

6

When we divided $8a^3 + 27$ by $2a + 3$ in Example the result was $4a^2 - 6a + 9$. To check our work, we multiply $4a2 - 6a + 9$ by $2a + 3$ to get $8a^3 + 27$.

$$(4a^2 - 6a + 9)(2a + 3) = 8a^3 + 27$$

Written this way, we can see that $4a^2 - 6a + 9$ and $2a + 3$ are factors of $8a^3 + 27$. When we did the division, the remainder was zero.

Whenever a divisor, $x - c$, divides a polynomial function, $f(x)$, and resulting in a remainder of zero, we say $x - c$ is a factor of $f(x)$.

The reverse is also true. If $x - c$ is a factor of $f(x)$ then $x - c$ will divide the polynomial function resulting in a remainder of zero.

We will state this in the Factor Theorem.

Definition: FACTOR THEOREM

For any polynomial function $f(x)$,

- if $x-c$ is a factor of $f(x)$, then $f(c)=0$
- if $f(c)=0$, then $x-c$ is a factor of $f(x)$

Example 3.3.11

Use the Remainder Theorem to determine if $x-4$ is a factor of $f(x)=x^3-64$.

Solution

The Factor Theorem tells us that $x-4$ is a factor of $f(x)=x^3-64$ if $f(4)=0$.

	$f(x)=x^3-64$
To evaluate $f(4)$ substitute $x=4$.	$f(4)=4^3-64$
Simplify.	$f(4)=64-64$
Subtract.	$f(4)=0$

Since $f(4)=0$, $x-4$ is a factor of $f(x)=x^3-64$.

Try It! 3.3.21

Use the Factor Theorem to determine if $x-5$ is a factor of $f(x)=x^3-125$.

Answer

yes

Try It! 3.3.22

Use the Factor Theorem to determine if $x-6$ is a factor of $f(x)=x^3-216$.

Answer

yes

Access these online resources for additional instruction and practice with dividing polynomials.

- Dividing a Polynomial by a Binomial
- Synthetic Division & Remainder Theorem

Key Concepts

- **Division of a Polynomial by a Monomial**
  - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.
- **Division of Polynomial Functions**
  - For functions $f(x)$ and $g(x)$, where $g(x)\neq 0$,
    $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$
- **Remainder Theorem**
  - If the polynomial function $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.
- **Factor Theorem:** For any polynomial function $f(x)$,
  - if $x-c$ is a factor of $f(x)$, then $f(c)=0$
  - if $f(c)=0$, then $x-c$ is a factor of $f(x)$

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3.3E: Exercises

Practice Makes Perfect

Divide Monomials

In the following exercises, divide the monomials.

1. $15r^4s^9 \div (15r^4s^9)$

2. $20m^8n^4 \div (30m^5n^9)$

Answer

$\frac{2m^3}{3n^5}$

3. $\frac{18a^4b^8}{-27a^9b^5}$

4. $\frac{45x^5y^9}{-60x^8y^6}$

Answer

$\frac{-3y^3}{4x^3}$

5. $\frac{(10m^5n^4)(5m^3n^6)}{25m^7n^5}$

6. $\frac{(-18p^4q^7)(-6p^3q^8)}{-36p^{12}q^{10}}$

Answer

$\frac{-3q^5}{p^5}$

7. $\frac{(6a^4b^3)(4ab^5)}{(12a^2b)(a^3b)}$

8. $\frac{(4u^2v^5)(15u^3v)}{(12u^3v)(u^4v)}$

Answer

$\frac{5v^4}{u^2}$

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial.

9. $(9n^4 + 6n^3) \div 3n$

10. $(8x^3 + 6x^2) \div 2x$

Answer

$4x^2+3x$

11. $(63m^4-42m^3)\div(-7m^2)$

12. $(48y^4-24y^3)\div(-8y^2)$

Answer

$-6y^2+3y$

13. $\dfrac{66x^3y^2-110x^2y^3-44x^4y^3}{11x^2y^2}$

14. $\dfrac{72r^5s^2+132r^4s^3-96r^3s^5}{12r^2s^2}$

Answer

$6r^3+11r^2s-8rs^3$

15. $10x^2+5x-4-5x$

16. $20y^2+12y-1-4y$

Answer

$-5y-3+\dfrac{1}{4y}$

Divide Polynomials using Long Division

In the following exercises, divide each polynomial by the binomial.

17. $(y^2+7y+12)\div(y+3)$

18. $(a^2-2a-35)\div(a+5)$

Answer

$a-7$

19. $(6m^2-19m-20)\div(m-4)$

20. $(4x^2-17x-15)\div(x-5)$

Answer

$4x+3$

21. $(q^2+2q+20)\div(q+6)$

22. $(p^2+11p+16)\div(p+8)$

Answer

$$p+3-\frac{8}{p+8}$$

23. $(3b^3+b^2+4)\div(b+1)$

24. $(2n^3-10n+28)\div(n+3)$

Answer

$$\frac{2n^2-6n+8+4}{n+3}$$

25. $(z^3+1)\div(z+1)$

26. $(m^3+1000)\div(m+10)$

Answer

$m^2-10m+100$

27. $(64x^3-27)\div(4x-3)$

28. $(125y^3-64)\div(5y-4)$

Answer

$25y^2+20x+16$

Divide Polynomials using Synthetic Division

In the following exercises, use synthetic Division to find the quotient and remainder.

29. $x^3-6x^2+5x+14$ is divided by $x+1$

30. $x^3-3x^2-4x+12$ is divided by $x+2$

Answer

$x^2-5x+6;\ 0$

31. $2x^3-11x^2+11x+12$ is divided by $x-3$

32. $2x^3-11x^2+16x-12$ is divided by $x-4$

Answer

$2x^2-3x+4;\ 4$

33. $x^4-5x^2+2+13x+3$ is divided by $x+3$

34. $x^4+x^2+6x-10$ is divided by $x+2$

Answer

$x^3-2x^2+5x-4;\ -2$

35. $2x^4-9x^3+5x^2-3x-6$ is divided by $x-4$

36. $3x^4-11x^3+2x^2+10x+6$ is divided by $x-3$

Answer

$3x^3-2x^2-4x-2;\ 0$

Divide Polynomial Functions

In the following exercises, divide.

37. For functions $f(x)=x^2-13x+36$ and $g(x)=x-4$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(-1)$

38. For functions $f(x)=x^2-15x+54$ and $g(x)=x-9$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(-5)$

Answer

ⓐ $\left(\frac{f}{g}\right)(x)=x-6$

ⓑ $\left(\frac{f}{g}\right)(-5)=-11$

39. For functions $f(x)=x^3+x^2-7x+2$ and $g(x)=x-2$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(2)$

40. For functions $f(x)=x^3+2x^2-19x+12$ and $g(x)=x-3$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(0)$

Answer

ⓐ $\left(\frac{f}{g}\right)(x)=x^2+5x-4$

ⓑ $\left(\frac{f}{g}\right)(0)=-4$

41. For functions $f(x)=x^2-5x+2$ and $g(x)=x^2-3x-1$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-1)$

42. For functions $f(x)=x^2+4x-3$ and $g(x)=x^2+2x+4$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(1)$

Answer

ⓐ $(f\cdot g)(x)=x^4+6x^3+9x^2+10x-12$; ⓑ $(f\cdot g)(1)=14$

Use the Remainder and Factor Theorem

In the following exercises, use the Remainder Theorem to find the remainder.

43. $f(x)=x^3-8x+7$ is divided by $x+3$

44. $f(x)=x^3-4x-9$ is divided by $x+2$

Answer

-9

45. $f(x) = 2x^3 - 6x - 24$ divided by $x - 3$

46. $f(x) = 7x^2 - 5x - 8$ divided by $x - 1$

Answer

-6

In the following exercises, use the Factor Theorem to determine if x−cx−c is a factor of the polynomial function.

47. Determine whether $x + 3$ a factor of $x^3 + 8x^2 + 21x + 18$

48. Determine whether $x + 4$ a factor of $x^3 + x^2 - 14x + 8$

Answer

no

49. Determine whether $x - 2$ a factor of $x^3 - 7x^2 + 7x - 6$

50. Determine whether $x - 3$ a factor of $x^3 - 7x^2 + 11x + 3$

Answer

yes

Writing Exercises

51. James divides $48y + 6$ by 6 this way: $\frac{48y + 6}{6} = 48y$. What is wrong with his reasoning?

52. Divide $\frac{10x^2 + x - 12}{2x}$ and explain with words how you get each term of the quotient.

Answer

Answer will vary

53. Explain when you can use synthetic division.

54. In your own words, write the steps for synthetic division for $x^2 + 5x + 6$ divided by $x - 2$.

Answer

Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section

I can...	Confidently	With some help	No-I don't get it!
divide monomials.			
divide a polynomial by a monomial.			
divide polynomials using long division.			
divide polynomials synthetic division.			
divide polynomial functions.			
use the Remainder and Factor Theorem.			

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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3.4: Add and Subtract Rational Expressions

Summary

By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{10}+\frac{8}{15}$.
   If you missed this problem, review [link].
2. Subtract: $\frac{3x}{4}-\frac{8}{9}$.
   If you missed this problem, review [link].
3. Subtract: $6(2x+1)-4(x-5)$.
   If you missed this problem, review [link].

Add and Subtract Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

RATIONAL EXPRESSION ADDITION AND SUBTRACTION

If p, q, and r are polynomials where $r\neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of x should be excluded in the next example?

Example 3.4.1

Add: $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$.

Solution

Since the denominator is $x+4$, we must exclude the value $x=-4$.

	$\frac{11x+28}{x+4}+\frac{x^2}{x+4},\ x\neq -4$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{11x+28+x^2}{x+4}$
Write the degrees in descending order.	$\frac{x^2+11x+28}{x+4}$
Factor the numerator.	$\frac{(x+4)(x+7)}{x+4}$
Simplify by removing common factors.	$\frac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$
Simplify.	$x+7$

The expression simplifies to $x+7$ but the original expression had a denominator of $x+4$ so $x\neq -4$.

Try It 3.4.2

Simplify: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

Answer

$x+2$

Try It 3.4.3

Simplify: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Answer

$x+3$

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator. Be careful of the signs when you subtract a binomial or trinomial.

Example 3.4.4

Subtract: $\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$.

Solution

	$\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the difference over the common denominator.	$\frac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\frac{(x-2)\ \cancel{(x-6)}}{(x+3)\ \cancel{(x-6)}}$
	$(x-2)(x+3)$

? Try It 3.4.5

Subtract: $\frac{4x^2-11x+8}{x^2-3x+2}-\frac{3x^2+x-3}{x^2-3x+2}$.

Answer

$\frac{x-11}{x-2}$

? Try It 3.4.6

Subtract: $\frac{6x^2-x+20}{x^2-81}-\frac{5x^2+11x-7}{x^2-81}$.

Answer

$\frac{x-3}{x+9}$

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works.

	$\frac{7}{d}+\frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d}+\frac{(-1)5}{(-1)(-d)}$
The denominators are the same.	$\frac{7}{d}+\frac{-5}{d}$

Simplify.	$\frac{2}{d}$

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

Example 3.4.7

Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Solution

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	$\frac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by removing common factors.	$\frac{(\cancel{m-1})(m-2)}{(\cancel{m-1})(m+1)}$
Simplify.	$\frac{m-2}{m+1}$

Try It 3.4.8

Subtract: $\frac{y^2-5y}{y^2-4}-\frac{6y-6}{4-y^2}$.

Answer

$\frac{y+3}{y+2}$

Try It 3.4.9

Subtract: $\frac{2n^2+8n-1}{n^2-1}-\frac{n^2-7n-1}{1-n^2}$.

Answer

$\frac{3n-2}{n-1}$

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example: $\frac{7}{12}+\frac{5}{18}$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number. We are now ready to add.

$$\frac{7}{12}+\frac{5}{18}$$
$$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$$
$$\frac{21}{36}+\frac{10}{36}$$

$$\begin{aligned} 12&=2\cdot 2\cdot 3\\ 18&=2\cdot \quad 3\cdot 3\\ \text{LCD}&=2\cdot 2\cdot 3\cdot 3\\ \text{LCD}&=36 \end{aligned}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

FIND THE LEAST COMMON DENOMINATOR OF RATIONAL EXPRESSIONS.

1. Factor each denominator completely.
2. List the factors of each denominator. Match factors vertically when possible.
3. Bring down the columns by including all factors, but do not include common factors twice.
4. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of xx should we exclude in this next example?

Example 3.4.10

a. Find the LCD for the expressions $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$ and b. rewrite them as equivalent rational expressions with the lowest common denominator.

Solution

a.

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.	
Factor each denominator completely, lining up common factors. Bring down the columns.	$x^2-2x-3=(x+1)(x-3)$ $x^2+4x+3=(x+1)(x+3)$ $\text{LCD}\quad =(x+1)(x-3)(x+3)$
Write the LCD as the product of the factors.	The LCD is $(x+1)(x-3)(x+3)$.

b.

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

Try It 3.4.11

a. Find the LCD for the expressions $\frac{2}{x^2-x-12}$, $\frac{1}{x^2-16}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x-4)(x+3)(x+4)$

b. $\frac{2x+8}{(x-4)(x+3)(x+4)}$, $\frac{x+3}{(x-4)(x+3)(x+4)}$

Try It 3.4.12

a. Find the LCD for the expressions $\frac{3x}{x^2-3x+10}$, $\frac{5}{x^2+3x+2}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x+2)(x-5)(x+1)$

b. $\frac{3x^2+3x}{(x+2)(x-5)(x+1)}$, $\frac{5x-25}{(x+2)(x-5)(x+1)}$

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

Example 3.4.13: How to Add Rational Expressions with Unlike Denominators

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes–go to step 2. • No–Rewrite each rational expression with the LCD. ◦ Find the LCD. ◦ Rewrite each rational expression as an equivalent rational expression with the LCD.	No. Find the LCD of $(x-3)$ and $(x-2)$. Change into equivalent rational expressions with the LCD, $(x-3)$ and $(x-2)$. Keep the denominators factored!	$x-3:(x-3)$ $x-2$: $(x-2)$ LCD : $(x-3)(x-2)$ $\frac{3}{x-3}+\frac{2}{x-2}$ $\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$ $\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	$\frac{5x-12}{(x-3)(x-2)}$

Try It 3.4.14

Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Answer

$$\frac{7x-4}{(x-2)(x+3)}$$

Try It 3.4.15

Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Answer

$$\frac{7m+25}{(m+3)(m+4)}$$

The steps used to add rational expressions are summarized here.

ADD OR SUBTRACT RATIONAL EXPRESSIONS.

1. Determine if the expressions have a common denominator.
   - **Yes** – go to step 2.
   - **No** – Rewrite each rational expression with the LCD.
     - Find the LCD.
     - Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as $\frac{3x-6}{(x-3)(x-2)}$ to be able to add it to $\frac{2x-6}{(x-2)(x-3)}$. Simplify *only* after you have combined the numerators.

Example 3.4.16

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $x^2-2x-3=(x+1)(x-3)$, $x^2+4x+3=(x+1)(x+3)$, $LCD=(x+1)(x-3)(x+3)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x+24}{(x+1)(x-3)(x+3)}$
	The numerator is prime, so there are no common factors.

Try It 3.4.17

Add: $\frac{1}{m^2-m-2}+\frac{5m}{m^2+3m+2}$.

Answer

$$\frac{5m^2-9m+2}{(m+1)(m-2)(m+2)}$$

Try It 3.4.18

Add: $\frac{2n}{n^2-3n-10}+\frac{6}{n^2+5n+6}$.

Answer

$$\frac{2n^2+12n-30}{(n+2)(n-5)(n+3)}$$

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

Example 3.4.19

Subtract: $\frac{8y}{y^2-16}-\frac{4}{y-4}$.

Solution

	$\frac{8y}{y^2-16}-\frac{4}{y-4}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $y^2-16=(y-4)(y+4)$, $y-4=y-4$, $LCD=(y-4)(y+4)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)}-\frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)}-\frac{4y+16}{(y-4)(y+4)}$

Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerator.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors	$\frac{4(\cancel{y-4})}{(\cancel{y-4})(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

Try It 3.4.20

Subtract: $\frac{2x}{x^2-4}-\frac{1}{x+2}$.

Answer

$\frac{1}{x-2}$

Try It 3.4.21

Subtract: $\frac{3}{z+3}-\frac{6z}{z^2-9}$.

Answer

$\frac{-3}{z-3}$

There are lots of negative signs in the next example. Be extra careful.

Example 3.4.22

Subtract: $\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$.

Solution

	$\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$
Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(-1)(2-n)}$
Write $(-1)(2-n)$ as $n-2$.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(n-2)}$
Simplify. Remember, $a-(-b)=a+b$.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)}{(n-2)}$
Do the rational expressions have a common denominator? No.	

Find the LCD. $n^2+n-6=(n-2)(n+3)$ $n-2=(n-2)$ $LCD=\quad(n-2)(n+3)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{n^2+6n+9}{(n-2)(n+3)}$
Add the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Simplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n(n+3)}{(n-2)(n+3)}$
Simplify.	$\frac{n}{(n-2)}$

Try It 3.4.23

Subtract: $\frac{3x-1}{x^2-5x-6}-\frac{2}{6-x}$.

Answer

$\frac{5x+1}{(x-6)(x+1)}$

Try It 3.4.24

Subtract: $\frac{-2y-2}{y^2+2y-8}-\frac{y-1}{2-y}$.

Answer

$\frac{y+3}{y+4}$

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

Example 3.4.25

Subtract: $\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$.

Solution

	$\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$
Factor the denominators.	$\frac{4}{(a+1)(a+5)}-\frac{3}{(a+2)(a+5)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $a^2+6a+5=(a+1)(a+5)$ $a^2+7a+10=(a+5)(a+2)$ $LCD=(a+1)(a+5)(a+2)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{4(a+2)}{(a+1)(a+5)(a+2)} - \frac{3(a+1)}{(a+2)(a+5)(a+1)}$
Simplify the numerators.	$\frac{4a+8}{(a+1)(a+5)(a+2)} - \frac{3a+3}{(a+2)(a+5)(a+1)}$
Subtract the rational expressions.	$\frac{4a+8-(3a+3)}{(a+1)(a+5)(a+2)}$
Simplify the numerator.	$\frac{4a+8-3a+3}{(a+1)(a+5)(a+2)}$
	$\frac{a+5}{(a+1)(a+5)(a+2)}$
Look for common factors.	$\frac{\cancel{(a+5)}}{(a+1)\cancel{(a+5)}(a+2)}$
Simplify.	$\frac{1}{(a+1)(a+2)}$

Try It 3.4.26

Subtract: $\frac{3}{b^2-4b-5} - \frac{2}{b^2-6b+5}$.

Answer

$$\frac{1}{(b+1)(b-1)}$$

Try It 3.4.27

Subtract: $\frac{4}{x^2-4} - \frac{3}{x^2-x-2}$.

Answer

$$\frac{1}{(x+2)(x+1)}$$

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example, we will start by factoring all three denominators to find their LCD.

Example 3.4.28

Simplify: $\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$.

Solution

	$\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $u-1=(u-1)$ $u=u$ $u^2-u=u(u-1)$ $LCD=u(u-1)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u \cdot u}{(u-1)u} + \frac{1 \cdot (u-1)}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u} + \frac{u-1}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2+u-1-2u+1}{u(u-1)}$
Simplify.	$\frac{2u^2-u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

Try It 3.4.29

Simplify: $\frac{v}{v+1} + \frac{3}{v-1} - \frac{6}{v^2-1}$.

Answer

$\frac{v+3}{v+1}$

Try It 3.4.30

Simplify: $\frac{3w}{w+2} + \frac{2}{w+7} - \frac{17w+4}{w^2+9w+14}$.

Answer

$\frac{3w}{w+7}$

Add and subtract rational functions

To add or subtract rational functions, we use the same techniques we used to add or subtract polynomial functions.

Example 3.4.31

Find $R(x) = f(x) - g(x)$ where $f(x) = \frac{x+5}{x-2}$ and $g(x) = \frac{5x+18}{x^2-4}$.

xolution

	$R(x) = f(x) - g(x)$
Substitute in the functions $f(x)$, $g(x)$.	$R(x) = \frac{x+5}{x-2} - \frac{5x+18}{x^2-4}$
Factor the denominators.	$R(x) = \frac{x+5}{x-2} - \frac{5x+18}{(x-2)(x+2)}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $x-2 = (x-2)$, $x^2-4 = (x-2)(x+2)$, $LCD = (x-2)(x+2)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$R(x)=\frac{(x+5)(x+2)}{(x-2)(x+2)}-\frac{5x+18}{(x-2)(x+2)}$
Write as one rational expression.	$R(x)=\frac{(x+5)(x+2)-(5x+18)}{(x-2)(x+2)}$
Simplify.	$R(x)=\frac{x^2+7x+10-5x-18}{(x-2)(x+2)}$
	$R(x)=\frac{x^2+2x-8}{(x-2)(x+2)}$
Factor the numerator, and remove common factors.	$R(x)=\frac{(x+4)(x-2)}{(x-2)(x+2)}$
Simplify.	$R(x)=\frac{(x+4)}{(x+2)}$

Try It 3.4.32

Find $R(x)=f(x)-g(x)$ where $f(x)=\frac{x+1}{x+3}$ and $g(x)=\frac{x+17}{x^2-x-12}$.

Answer

$\frac{x-7}{x-4}$

Try It 3.4.33

Find $R(x)=f(x)+g(x)$ where $f(x)=\frac{x-4}{x+3}$ and $g(x)=\frac{4x+6}{x^2-9}$.

Answer

$\frac{x^2-3x+18}{(x+3)(x-3)}$

Access this online resource for additional instruction and practice with adding and subtracting rational expressions.

- Add and Subtract Rational Expressions- Unlike Denominators

Key Concepts

- **Rational Expression Addition and Subtraction**
  If p, q, and r are polynomials where $r\neq 0$, then
  $$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$
- **How to find the least common denominator of rational expressions.**
  1. Factor each expression completely.
  2. List the factors of each expression. Match factors vertically when possible.
  3. Bring down the columns.
  4. Write the LCD as the product of the factors.
- **How to add or subtract rational expressions.**
  1. Determine if the expressions have a common denominator.
     - Yes – go to step 2.
     - No – Rewrite each rational expression with the LCD.
       - Find the LCD.

- Rewrite each rational expression as an equivalent rational expression with the LCD.

2. Add or subtract the rational expressions.
3. Simplify, if possible.

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3.4E: Exercises

Practice Makes Perfect

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, add.

1. $\frac{2}{15}+\frac{7}{15}$

Answer

$\frac{3}{5}$

2. $\frac{7}{24}+\frac{11}{24}$

3. $\frac{3c}{4c-5}+\frac{5}{4c-5}$

Answer

$\frac{3c+5}{4c-5}$

4. $\frac{7m}{2m+n}+\frac{4}{2m+n}$

5. $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

Answer

$r+8$

6. $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

7. $\frac{2w^2}{w^2-16}+\frac{8w}{w^2-16}$

Answer

$\frac{2w}{w-4}$

8. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

In the following exercises, subtract.

9. $\frac{9a^2}{3a-7}-\frac{49}{3a-7}$

Answer

$3a+7$

10. $\frac{25b^2}{5b-6} - \frac{36}{5b-6}$

11. $\frac{3m^2}{6m-30} - \frac{21m-30}{6m-30}$

Answer

$\frac{m-2}{2}$

12. $\frac{2n^2}{4n-32} - \frac{18n-16}{4n-32}$

13. $\frac{6p^2+3p+4}{p^2+4p-5} - \frac{5p^2+p+7}{p^2+4p-5}$

Answer

$\frac{p+3}{p+5}$

14. $\frac{5q^2+3q-9}{q^2+6q+8} - \frac{4q^2+9q+7}{q^2+6q+8}$

15. $\frac{5r^2+7r-33}{r^2-49} - \frac{4r^2+5r+30}{r^2-49}$

Answer

$\frac{r+9}{r+7}$

16. $\frac{7t^2-t-4}{t^2-25} - \frac{6t^2+12t-44}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add or subtract.

17. $\frac{10v}{2v-1} + \frac{2v+4}{1-2v}$

Answer

4

18. $\frac{20w}{5w-2} + \frac{5w+6}{2-5w}$

19. $\frac{10x^2+16x-7}{8x-3} + \frac{2x^2+3x-1}{3-8x}$

Answer

$x+2$

20. $\frac{6y^2+2y-11}{3y-7}+\frac{3y^2-3y+17}{7-3y}$

21. $\frac{z^2+6z}{z^2-25}-\frac{3z+20}{25-z^2}$

Answer

$\frac{z+4}{z-5}$

22. $\frac{a^2+3a}{a^2-9}-\frac{3a-27}{9-a^2}$

23. $\frac{2b^2+30b-13}{b^2-49}-\frac{2b^2-5b-8}{49-b^2}$

Answer

$\frac{4b-3}{b-7}$

24. $\frac{c^2+5c-10}{c^2-16}-\frac{c^2-8c-10}{16-c^2}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, a. find the LCD for the given rational expressions b. rewrite them as equivalent rational expressions with the lowest common denominator.

25. $\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$

Answer

a. $(x+2)(x-4)(x+3)$

b. $\frac{5x+15}{(x+2)(x-4)(x+3)}, \frac{2x^2+4x}{(x+2)(x-4)(x+3)}$

26. $\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$

27. $\frac{9}{z^2+2z-8}, \frac{4z}{z^2-4}$

Answer

a. $(z-2)(z+4)(z-4)$

b. $\frac{9z-36}{(z-2)(z+4)(z-4)}, \frac{4z^2-8z}{(z-2)(z+4)(z-4)}$

28. $\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$

29. $\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$

Answer

a. $(b+3)(b+3)(b-5)$

b. $\frac{4b-20}{(b+3)(b+3)(b-5)}, \frac{2b^2+6b}{(b+3)(b+3)(b-5)}$

30. $\frac{5}{c^2-4c+4}, \frac{3c}{c^2-7c+10}$

31. $\frac{2}{3d^2+14d-5}, \frac{5d}{3d^2-19d+6}$

Answer

a. $(d+5)(3d-1)(d-6)$

b. $\frac{2d-12}{(d+5)(3d-1)(d-6)}, \frac{5d^2+25d}{(d+5)(3d-1)(d-6)}$

32. $\frac{3}{5m^2-3m-2}, \frac{6m}{5m^2+17m+6}$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

33. $\frac{7}{10x^2y}+\frac{4}{15xy^2}$

Answer

$\frac{21y+8x}{30x^2y^2}$

34. $\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}$

35. $\frac{3}{r+4}+\frac{2}{r-5}$

Answer

$\frac{5r-7}{(r+4)(r-5)}$

36. $\frac{4}{s-7}+\frac{5}{s+3}$

37. $\frac{5}{3w-2}+\frac{2}{w+1}$

Answer

$$\frac{11w+1}{(3w-2)(w+1)}$$

38. $\frac{4}{2x+5}+\frac{2}{x-1}$

39. $\frac{2y}{y+3}+\frac{3}{y-1}$

Answer

$$\frac{2y^2+y+9}{(y+3)(y-1)}$$

40. $\frac{3z}{z-2}+\frac{1}{z+5}$

41. $\frac{5b}{a^2b-2a^2}+\frac{2b}{b^2-4}$

Answer

$$\frac{b(5b+10+2a^2)}{a^2(b-2)(b+2)}$$

42. $\frac{4}{cd+3c}+\frac{1}{d^2-9}$

43. $\frac{-3m}{3m-3}+\frac{5m}{m^2+3m-4}$

Answer

$$-\frac{m}{m+4}$$

44. $\frac{8}{4n+4}+\frac{6}{n^2-n-2}$

45. $\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$

Answer

$$\frac{3(r^2+6r+18)}{(r+1)(r+6)(r+3)}$$

46. $\frac{2s}{s^2+2s-8}+\frac{4}{s^2+3s-10}$

47. $\frac{t}{t-6}-\frac{t-2}{t+6}$

Answer

$$\frac{2(7t-6)}{(t-6)(t+6)}$$

48. $\frac{x-3}{x+6}-\frac{x}{x+3}$

49. $\frac{5a}{a+3}-\frac{a+2}{a+6}$

Answer

$\frac{4a^2+25a-6}{(a+3)(a+6)}$

50. $\frac{3b}{b-2}-\frac{b-6}{b-8}$

51. $\frac{6}{m+6}-\frac{12m}{m^2-36}$

Answer

$\frac{-6}{m-6}$

52. $\frac{4}{n+4}-\frac{8n}{n^2-16}$

53. $\frac{-9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

Answer

$\frac{p+2}{p+3}$

54. $\frac{-13q-8}{q^2+2q-24}-\frac{q+2}{4-q}$

55. $\frac{-2r-16}{r^2+6r-16}-\frac{5}{2-r}$

Answer

$\frac{3}{r-2}$

56. $\frac{2t-30}{t^2+6t-27}-\frac{2}{3-t}$

57. $\frac{2x+7}{10x-1}+3$

Answer

$\frac{4(8x+1)}{10x-1}$

58. $\frac{8y-4}{5y+2}-6$

59. $\dfrac{3}{x^2-3x-4}-\dfrac{2}{x^2-5x+4}$

Answer

$\dfrac{x-5}{(x-4)(x+1)(x-1)}$

60. $\dfrac{4}{x^2-6x+5}-\dfrac{3}{x^2-7x+10}$

61. $\dfrac{5}{x^2+8x-9}-\dfrac{4}{x^2+10x+9}$

Answer

$\dfrac{1}{(x-1)(x+1)}$

62. $\dfrac{3}{2x^2+5x+2}-\dfrac{1}{2x^2+3x+1}$

63. $\dfrac{5a}{a-2}+\dfrac{9}{a}-\dfrac{2a+18}{a^2-2a}$

Answer

$\dfrac{5a^2+7a-36}{a(a-2)}$

64. $\dfrac{2b}{b-5}+\dfrac{3}{2b}-\dfrac{2b-15}{2b^2-10b}$

65. $\dfrac{c}{c+2}+\dfrac{5}{c-2}-\dfrac{10c}{c^2-4}$

Answer

$\dfrac{c-5}{c+2}$

66. $\dfrac{6d}{d-5}+\dfrac{1}{d+4}+\dfrac{7d-5}{d^2-d-20}$

67. $\dfrac{3d}{d+2}+\dfrac{4}{d}-\dfrac{d+8}{d^2+2d}$

Answer

$\dfrac{3(d+1)}{d+2}$

68. $\dfrac{2q}{q+5}+\dfrac{3}{q-3}-\dfrac{13q+15}{q^2+2q-15}$

Add and Subtract Rational Functions

In the following exercises, find a. $R(x)=f(x)+g(x)$ b. $R(x)=f(x)-g(x)$.

69. $f(x)=\dfrac{-5x-5}{x^2+x-6}$ and $g(x)=\dfrac{x+1}{2-x}$

Answer

a. $R(x)=-\dfrac{(x+8)(x+1)}{(x-2)(x+3)}$

b. $R(x)=\dfrac{x+1}{x+3}$

70. $f(x)=\dfrac{-4x-24}{x^2+x-30}$ and $g(x)=\dfrac{x+7}{5-x}$

71. $f(x)=\dfrac{6x}{x^2-64}$ and $g(x)=\dfrac{3}{x-8}$

Answer

a. $R(x)=\dfrac{3(3x+8)}{(x-8)(x+8)}$

b. $R(x)=\dfrac{3}{x+8}$

72. $f(x)=\dfrac{5}{x+7}$ and $g(x)=\dfrac{10x}{x^2-49}$

Writing Exercises

73. Donald thinks that $\dfrac{3}{x}+\dfrac{4}{x}$ is $\dfrac{7}{2x}$. Is Donald correct? Explain.

Answer

Answers will vary.

74. Explain how you find the Least Common Denominator of x^2+5x+4 and x^2-16.

75. Felipe thinks $\dfrac{1}{x}+\dfrac{1}{y}$ is $\dfrac{2}{x+y}$.

a. Choose numerical values for *x* and *y* and evaluate $\dfrac{1}{x}+\dfrac{1}{y}$.

b. Evaluate $\dfrac{2}{x+y}$ for the same values of *x* and *y* you used in part a..

c. Explain why Felipe is wrong.

d. Find the correct expression for $1x+1y$.

Answer

a. Answers will vary.

b. Answers will vary.

c. Answers will vary.

d. $\dfrac{x+y}{x}$

76. Simplify the expression $\dfrac{4}{n^2+6n+9}-\dfrac{1}{n^2-9}$ and explain all your steps.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			
find the least common denominator of rational expressions.			
add and subtract rational expressions with unlike denominators.			
add or subtract rational functions.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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3.5: Simplify Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Before you get started, take this readiness quiz.

1. Simplify: $\dfrac{\frac{3}{5}}{\frac{9}{10}}$.
   If you missed this problem, review [link].
2. Simplify: $\dfrac{1-\frac{1}{3}}{4^2+4\cdot 5}$.
   If you missed this problem, review [link].
3. Solve: $\dfrac{1}{2x}+\dfrac{1}{4}=\dfrac{1}{8}$.
   If you missed this problem, review [link].

Complex Fraction

A **complex fraction** is a fraction in which the numerator and/or the denominator contains a fraction.

We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

Complex Rational Expression

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter.

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem:

$$\left(\frac{6x^2-7x+2}{4x-8}\right)\div\left(\frac{2x^2-8x+3}{x^2-5x+6}\right)$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form where one fraction is over one fraction. We then write it as if we were dividing two fractions.

Example 3.5.1

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$$

Solution

Rewrite the complex fraction as division.

$$\frac{6}{x-4}\div\frac{3}{x^2-16}$$

Rewrite as the product of first times the reciprocal of the second.Rewrite as the product of first times the reciprocal of the second.Rewrite as the product of first times the reciprocal of the second.

$$\frac{6}{x-4}\cdot\frac{x^2-16}{3}$$

Factor.

$$\frac{3\cdot 2}{x-4}\cdot\frac{(x-4)(x+4)}{3}$$

Multiply.

$$\frac{3\cdot 2(x-4)(x+4)}{3(x-4)}$$

Remove common factors.

$$\frac{\cancel{3}\cdot 2\ \cancel{(x-4)}(x+4)}{\cancel{3}\ \cancel{(x-4)}}$$

Simplify.

$$2(x+4)$$

Are there any value(s) of x that should not be allowed? The original complex rational expression had denominators of $x-4$ and x^2-16. This expression would be undefined if $x=4$ or $x=-4$.

Try It 3.5.1

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$$

Answer

$$\frac{2}{3(x-1)}$$

Try It 3.5.2

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$$

Answer

$$\frac{1}{2(x-3)}$$

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

Example 3.5.2

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$$

Solution

Simplify the numerator and denominator. Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.

$$\frac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$$

Simplify the numerator and denominator.

$$\frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$$

Rewrite the complex rational expression as a division problem.

$$\frac{3}{6}\div\frac{1}{6}$$

Multiply the first by the reciprocal of the second.

$$\frac{3}{6} \cdot \frac{6}{1}$$

Simplify.

$$3$$

? Try It 3.5.3

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$$

Answer

$$\frac{14}{11}$$

? Try It 3.5.4

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$$

Answer

$$\frac{10}{23}$$

We follow the same procedure when the complex rational expression contains variables.

✓ Example 3.5.3: How to Simplify a Complex Rational Expression using Division

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Solution

Step 1. Simplify the numerator.

We will simplify the sum in and denominator. the numerator and difference in the denominator.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Find a common denominator and add the fractions in the numerator.

$$\frac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$$

$$\frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$$

Find a common denominator and subtract the fractions in the denominator.

$$\frac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$$

We now have just one rational expression in the numerator and one in the denominator.

Step 2. Rewrite the complex rational expression as a division problem.

We write the numerator divided by the denominator.

$$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$$

Step 3. Divide the expressions.

Multiply the first by the reciprocal of the second.

$$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$$

Factor any expressions if possible.

$$\frac{xy(y+x)}{xy(x-y)(x+y)}$$

Remove common factors.

$$\frac{\cancel{xy}\ \cancel{(y+x)}}{\cancel{xy}(x-y)\ \cancel{(x+y)}}$$

Simplify.

$$\frac{1}{x-y}$$

? Try It 3.5.5

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$$

Answer

$$\frac{y+x}{y-x}$$

Try It 3.5.6

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$$

Answer

$$\frac{ab}{b-a}$$

We summarize the steps here.

How to simplify a complex rational expression by writing it as division.

1. Rewrite the complex rational expression as a division problem.
2. Divide the expressions.

Example 3.5.4

Simplify the complex rational expression by writing it as division:

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$$

Solution

Simplify the numerator and denominator. Find common denominators for the numerator and denominator.

$$\frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$$

Simplify the numerators.

$$\frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$$

Subtract the rational expressions in the numerator and add in the denominator.

$$\frac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$$

Simplify. (We now have one rational expression over one rational expression.)

$$\frac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$$

Rewrite as fraction division.

$$\frac{n^2+n}{n+5} \div \frac{2n}{(n+5)(n-5)}$$

Multiply the first times the reciprocal of the second.

$$\frac{n^2+n}{n+5} \cdot \frac{(n+5)(n-5)}{2n}$$

Factor any expressions if possible.

$$\frac{n(n+1)(n+5)(n-5)}{(n+5)2n}$$

Remove common factors.

$$\frac{\cancel{n}(n+1)\ \cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\ \cancel{n}}$$

Simplify.

$$\frac{(n+1)(n-5)}{2}$$

? Try It 3.5.7

Simplify the complex rational expression by writing it as division:

$$\frac{b-\frac{3b}{b+5}}{\frac{2}{b+5}+\frac{1}{b-5}}$$

Answer

$$\frac{b(b+2)(b-5)}{3b-5}$$

? Try It 3.5.8

Simplify the complex rational expression by writing it as division:

$$\frac{1-\frac{3}{c+4}}{\frac{1}{c+4}+\frac{c}{3}}$$

Answer

$$\frac{3}{c+3}$$

Simplify a Complex Rational Expression by Using the LCD

We "cleared" the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let's look at the complex rational expression we simplified one way in Example 7.4.2. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{LCD}{LCD}$ we are multiplying by 1, so the value stays the same.

Example 3.5.5

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$$

Solution

The LCD of all the fractions in the whole expression is 6.

Clear the fractions by multiplying the numerator and denominator by that LCD.

$$\frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$$

Distribute.

$$\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$$

Simplify.

$$\frac{2+1}{3-2}$$

$$\frac{3}{1}$$

$$3$$

Try It 3.5.9

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$$

Answer

$\frac{7}{3}$

Try It 3.5.10

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$$

Answer

$$\frac{10}{3}$$

We will use the same example as in Example 7.4.3. Decide which method works better for you.

Example 3.5.6: How to Simplify a Complex Rational Expressing using the LCD

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Solution

Step 1. Find the LCD of all fractions in the is complex rational expression.

The LCD of all the fractions xy.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Step 2. Multiply the numerator and denominator by the LCD.

Multiply both the numerator and denominator by xy.

$$\frac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$$

Step 3. Simplify the expression.

Distribute.

$$\frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$$

$$\frac{y+x}{x^2-y^2}$$

Simplify.

$$\frac{\cancel{(y+x)}}{(x-y)\ \cancel{(x+y)}}$$

Remove common factors.

$$\frac{1}{x-y}$$

Try It 3.5.11

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$$

Answer

$$\frac{b+a}{a^2+b^2}$$

Try It 3.5.12

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$$

Answer

$$\frac{y-x}{xy}$$

How to simplify a complex rational expression by using the LCD.

1. Multiply the numerator and denominator by the LCD.
2. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Example 3.5.7

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Solution

Find the LCD of all fractions in the complex rational expression. The LCD is:

$$x^2-36=(x+6)(x-6)$$

Multiply the numerator and denominator by the LCD.

$$\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$$

Simplify the expression.

Distribute in the denominator.

$$\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$$

Simplify.

$$\frac{\cancel{(x+6)}(x-6)\frac{2}{\cancel{x+6}}}{\cancel{(x+6)}\ \cancel{(x-6)}\left(\frac{4}{x-6}\right)-\ \cancel{(x+6)(x-6)}\left(\frac{4}{\cancel{(x+6)(x-6)}}\right)}$$

Simplify.

$$\frac{2(x-6)}{4(x+6)-4}$$

To simplify the denominator, distribute and combine like terms.

$$\frac{2(x-6)}{4x+20}$$

Factor the denominator.

$$\frac{2(x-6)}{4(x+5)}$$

Remove common factors.

$$\frac{\cancel{2}(x-6)}{\cancel{2}\cdot 2(x+5)}$$

Simplify.

$$\frac{x-6}{2(x+5)}$$

Notice that there are no more factors common to the numerator and denominator.

Try It 3.5.13

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}$$

Answer

$$\frac{3(x-2)}{5x+7}$$

Try It 3.5.14

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$$

Answer

$$\frac{x+21}{6x-43}$$

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

Example 3.5.8

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$$

Solution

Find the LCD of all fractions in the complex rational expression.

The LCD is $(m-3)(m-4)$.

Multiply the numerator and denominator by the LCD.

$$\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$$

Simplify.

$$\frac{\cancel{(m-3)(m-4)}\frac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\ \cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$$

Simplify.

$$\frac{4}{3(m-4)-2(m-3)}$$

Distribute.

$$\frac{4}{3m-12-2m+6}$$

Combine like terms.

$$\frac{4}{m-6}$$

Try It 3.5.15

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$$

Answer

$$\frac{3}{5x+22}$$

Try It 3.5.16

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$$

Answer

$$\frac{2\left(2y^2+13y+5\right)}{3y}$$

Example 3.5.9

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$$

Solution

Find the LCD of all fractions in the complex rational expression.

The LCD is $(y+1)(y-1)$.

Multiply the numerator and denominator by the LCD.

$$\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$$

Distribute in the denominator and simplify.

$$\frac{\cancel{(y+1)}(y-1)\frac{y}{\cancel{y+1}}}{(y+1)(y-1)(1)+(y+1)\ \cancel{(y-1)}\left(\frac{1}{\cancel{(y-1)}}\right)}$$

Simplify.

$$\frac{(y-1)y}{(y+1)(y-1)+(y+1)}$$

Simplify the denominator and leave the numerator factored.

$$\frac{y(y-1)}{y^2-1+y+1}$$

$$\frac{y(y-1)}{y^2+y}$$

Factor the denominator and remove factors common with the numerator.

$$\frac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$$

Simplify.

$$\frac{y-1}{y+1}$$

? Try It 3.5.17

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$$

Answer

$$\frac{x}{x+4}$$

? Try It 3.5.18

Simplify the complex rational expression by using the LCD:

$$\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$$

Answer

$$\frac{x(x+1)}{3(x-1)}$$

Access this online resource for additional instruction and practice with complex fractions.

- Complex Fractions

Key Concepts

- **How to simplify a complex rational expression by writing it as division.**
  1. Simplify the numerator and denominator.
  2. Rewrite the complex rational expression as a division problem.
  3. Divide the expressions.
- **How to simplify a complex rational expression by using the LCD.**
  1. Find the LCD of all fractions in the complex rational expression.
  2. Multiply the numerator and denominator by the LCD.
  3. Simplify the expression.

Glossary

complex rational expression

A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

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3.5E: Exercises

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify each complex rational expression by writing it as division.

1. $\dfrac{\dfrac{2a}{a+4}}{\dfrac{4a^2}{a^2-16}}$

Answer

$\dfrac{a-4}{2a}$

2. $\dfrac{\dfrac{3b}{b-5}}{\dfrac{b^2}{b^2-25}}$

3. $\dfrac{\dfrac{5}{c^2+5c-14}}{\dfrac{10}{c+7}}$

Answer

$\dfrac{1}{2(c-2)}$

4. $\dfrac{\dfrac{8}{d^2+9d+18}}{\dfrac{12}{d+6}}$

5. $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

Answer

$\dfrac{12}{13}$

6. $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$

7. $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

Answer

$$\frac{20}{57}$$

8. $\dfrac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

9. $\dfrac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

Answer

$$\frac{n^2+m}{m-n^2}$$

10. $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

11. $\dfrac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

Answer

$$\frac{rt}{t-r}$$

12. $\dfrac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

13. $\dfrac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

Answer

$$\frac{(x+1)(x-3)}{2}$$

14. $\dfrac{y-\frac{2y}{y-4}}{\frac{2}{y-4}+\frac{2}{y+4}}$

15. $\dfrac{2-\dfrac{2}{a+3}}{\dfrac{1}{a+3}+\dfrac{a}{2}}$

Answer

$\dfrac{4}{a+1}$

16. $\dfrac{4+\dfrac{4}{b-5}}{\dfrac{1}{b-5}+\dfrac{b}{4}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify each complex rational expression by using the LCD.

17. $\dfrac{\dfrac{1}{3}+\dfrac{1}{8}}{\dfrac{1}{4}+\dfrac{1}{12}}$

Answer

$\dfrac{11}{8}$

18. $\dfrac{\dfrac{1}{4}+\dfrac{1}{9}}{\dfrac{1}{6}+\dfrac{1}{12}}$

19. $\dfrac{\dfrac{5}{6}+\dfrac{2}{9}}{\dfrac{7}{18}-\dfrac{1}{3}}$

Answer

19

20. $\dfrac{\dfrac{1}{6}+\dfrac{4}{15}}{\dfrac{3}{5}-\dfrac{1}{2}}$

21. $\dfrac{\dfrac{c}{d}+\dfrac{1}{d}}{\dfrac{1}{d}-\dfrac{d}{c}}$

Answer

$\dfrac{c^2+c}{c-d^2}$

22. $\dfrac{\frac{1}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

23. $\dfrac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$

Answer

$\dfrac{pq}{q-p}$

24. $\dfrac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

25. $\dfrac{\frac{2}{x+5}}{\frac{3}{x-5}+\frac{1}{x^2-25}}$

Answer

$\dfrac{2x-10}{3x+16}$

26. $\dfrac{\frac{5}{y-4}}{\frac{3}{y+4}+\frac{2}{y^2-16}}$

27. $\dfrac{\frac{5}{z^2-64}+\frac{3}{z+8}}{\frac{1}{z+8}+\frac{2}{z-8}}$

Answer

$\dfrac{3z-19}{3z+8}$

28. $\dfrac{\frac{3}{s+6}+\frac{5}{s-6}}{\frac{1}{s^2-36}+\frac{4}{s+6}}$

29. $\dfrac{\frac{4}{a^2-2a-15}}{\frac{1}{a-5}+\frac{2}{a+3}}$

Answer

$$\frac{4}{3a-7}$$

30. $$\frac{\frac{5}{b^2-6b-27}}{\frac{3}{b-9}+\frac{1}{b+3}}$$

31. $$\frac{\frac{5}{c+2}-\frac{3}{c+7}}{\frac{5c}{c^2+9c+14}}$$

Answer

$$\frac{2c+29}{5c}$$

32. $$\frac{\frac{6}{d-4}-\frac{2}{d+7}}{\frac{2d}{d^2+3d-28}}$$

33. $$\frac{2+\frac{1}{p-3}}{\frac{5}{p-3}}$$

Answer

$$\frac{2p-5}{5}$$

34. $$\frac{\frac{n}{n-2}}{3+\frac{5}{n-2}}$$

35. $$\frac{\frac{m}{m+5}}{4+\frac{1}{m-5}}$$

Answer

$$\frac{m(m-5)}{(4m-19)(m+5)}$$

36. $$\frac{7+\frac{2}{q-2}}{\frac{1}{q+2}}$$

In the following exercises, simplify each complex rational expression using either method.

37. $\dfrac{\frac{3}{4}-\frac{2}{7}}{\frac{1}{2}+\frac{5}{14}}$

Answer

$\frac{13}{24}$

38. $\dfrac{\frac{v}{w}+\frac{1}{v}}{\frac{1}{v}-\frac{v}{w}}$

39. $\dfrac{\frac{2}{a+4}}{\frac{1}{a^2-16}}$

Answer

$2(a-4)$

40. $\dfrac{\frac{3}{b^2-3b-40}}{\frac{5}{b+5}-\frac{2}{b-8}}$

41. $\dfrac{\frac{3}{m}+\frac{3}{n}}{\frac{1}{m^2}-\frac{1}{n^2}}$

Answer

$\frac{3mn}{n-m}$

42. $\dfrac{\frac{2}{r-9}}{\frac{1}{r+9}+\frac{3}{r^2-81}}$

43. $\dfrac{x-\frac{3x}{x+2}}{\frac{3}{x+2}+\frac{3}{x-2}}$

Answer

$\frac{(x-1)(x-2)}{6}$

https://math.libretexts.org/@go/page/58444

44. $\dfrac{\dfrac{y}{y+3}}{2+\dfrac{1}{y-3}}$

Writing Exercises

45. In this section, you learned to simplify the complex fraction $\dfrac{\dfrac{3}{x+2}}{\dfrac{x}{x^2-4}}$ two ways: rewriting it as a division problem or multiplying the numerator and denominator by the LCD. Which method do you prefer? Why?

Answer

Answers will vary.

44. Efraim wants to start simplifying the complex fraction $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ by cancelling the variables from the numerator and denominator, $\dfrac{\frac{1}{\cancel{a}}+\frac{1}{\cancel{b}}}{\frac{1}{\cancel{a}}-\frac{1}{\cancel{b}}}$. Explain what is wrong with Efraim's plan.

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3.6: Solve Rational Equations

Learning Objectives

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Before you get started, take this readiness quiz.

1. Solve: $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$
   If you missed this problem, review Example 2.2.9.
2. Solve: $n^2-5n-36=0$
   If you missed this problem, review Example 6.6.4.
3. Solve the formula $5x+2y=10$ for y.
   If you missed this problem, review Example 2.4.10.

After defining the terms ‘expression’ and ‘equation’ earlier, we have used them throughout this book. We have simplified many kinds of expressions and solved many kinds of equations. We have simplified many rational expressions so far in this chapter. Now we will solve a **rational equation**.

Rational Equation

A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don’t end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

Example 3.6.1: How to Solve a Rational Equation

Solve:

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

Solution

Step 1. Note any value of the variable that would make any denominator zero.

If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}, x \neq 0$$

Step 2. Find the least common denominator of all denominators in the equation.

Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$

The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.

Multiply both sides of the equation by the LCD, $6x$.

$$6x \cdot \left(\frac{1}{x}+\frac{1}{3}\right)=6x \cdot \left(\frac{5}{6}\right)$$

Use the Distributive Property.

$$6x \cdot \frac{1}{x}+6x \cdot \frac{1}{3}=6x \cdot \left(\frac{5}{6}\right)$$

Simplify - and notice, no more fractions!

$$6+2x=5x$$

Step 4. Solve the resulting equation.

Simplify.

$$6=3x$$
$$2=x$$

Step 5. Check.

If any values found in Step 1 are algebraic solutions, discard them. Check any remaining solutions in the original equation.

We did not get 0 as an algebraic solution.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

We substitute $x=2$ into the original equation.

$$\frac{1}{2}+\frac{1}{3} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{3}{6}+\frac{2}{6} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{5}{6}=\frac{5}{6}\checkmark$$

The solution is $x=2$

Exercise 3.6.1

Solve:

$$\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$$

Answer

$$y=-\frac{15}{7}$$

Exercise 3.6.2

Solve:

$$\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$$

Answer

$$x=\frac{15}{3}$$

The steps of this method are shown.

how to Solve equations with rational expressions.

- Step 1. Note any value of the variable that would make any denominator zero.
- Step 2. Find the least common denominator of all denominators in the equation.
- Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
- Step 4. Solve the resulting equation.
- Step 5. Check:
  - If any values found in Step 1 are algebraic solutions, discard them.
  - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

Example 3.6.2: How to Solve a Rational Equation using the Zero Product Property

Solve:

$$1-\frac{5}{y}=-\frac{6}{y^2}$$

Solution

Note any value of the variable that would make any denominator zero.

$$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$$

Find the least common denominator of all denominators in the equation. The LCD is y^2.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Distribute.

$$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Multiply.

$$y^2-5y=-6$$

Solve the resulting equation. First write the quadratic equation in standard form.

$$y^2-5y+6=0$$

Factor.

$$(y-2)(y-3)=0$$

Use the Zero Product Property.

$$y-2=0 \text{ or } y-3=0$$

Solve.

$$y=2 \text{ or } y=3$$

Check. We did not get 0 as an algebraic solution.

Check $y=2$ and $y=3$in the original equation.

$$1-\frac{5}{y}=-\frac{6}{y^2} \qquad 1-\frac{5}{y}=-\frac{6}{y^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad \frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4} \qquad -\frac{2}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}=-\frac{3}{2}\surd \qquad -\frac{2}{3}=-\frac{2}{3}\surd$$

The solution is $y=2, y=3$

? Exercise 3.6.3

Solve:

$$1-\frac{2}{x}=\frac{15}{x^2}$$

Answer

$x=-3, x=5$

Exercise 3.6.4

Solve:

$$1-\frac{4}{y}=\frac{12}{y^2}$$

Answer

$y=-2, y=6$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Example 3.6.3

Solve:

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$

Solution

Note any value of the variable that would make any denominator zero.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{(x+2)(x-2)}, x\neq -2, x\neq 2$$

Find the least common denominator of all denominators in the equation. The LCD is $(x+2)(x-2)$.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(x+2)(x-2)\left(\frac{2}{x+2}+\frac{4}{x-2}\right)=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Distribute.

$$(x+2)(x-2)\frac{2}{x+2}+(x+2)(x-2)\frac{4}{x-2}=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Remove common factors.

$$\cancel{(x+2)}(x-2)\frac{2}{\cancel{x+2}}+(x+2)\ \cancel{(x-2)}\frac{4}{\cancel{x-2}}=\ \cancel{(x+2)(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$$

Simplify.

$$2(x-2)+4(x+2)=x-1$$

Distribute.

$$2x-4+4x+8=x-1$$

Solve.

$$\begin{aligned}6x+4&=x-1\\5x&=-5\\x&=-1\end{aligned}$$

Check: We did not get 2 or −2 as algebraic solutions.

Check $x=-1$ in the original equation.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$
$$\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$$
$$\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$
$$\frac{6}{3}-\frac{4}{3}\stackrel{?}{=}\frac{2}{3}$$
$$\frac{2}{3}=\frac{2}{3}\surd$$

The solution is $x=-1$.

Exercise 3.6.5

Solve:

$$\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$$

Answer

$x=\frac{2}{3}$

Exercise 3.6.6

Solve:

$$\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$$

Answer

$y=2$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

Example 3.6.4

Solve:

$$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$$

Solution

Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.

$$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1}, m\neq 4, m\neq 1$$

Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$$

Distribute.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$$

Remove common factors.

$$\cancel{(m-4)(m-1)}\left(\frac{m+11}{\cancel{(m-4)(m-1)}}\right)=\cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}}-(m-4)\ \cancel{(m-1)}\frac{3}{\cancel{m-1}}$$

Simplify.

$$m+11=5(m-1)-3(m-4)$$

Solve the resulting equation.

$$\begin{aligned} m+11 &= 5m-5-3m+12 \\ 4 &= m \end{aligned}$$

Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

? Exercise 3.6.7

Solve:

$$\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$$

Answer

There is no solution.

? Exercise 3.6.8

Solve:

$$\frac{y-6}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$$

Answer

There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

✓ Example 3.6.5

Solve:

$$\frac{y}{y+6}=\frac{72}{y^2-36}+4$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$$

Find the least common denominator. The LCD is $(y-6)(y+6)$

Clear the fractions.

$$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$$

Simplify.

$$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$$

Simplify.

$$y(y-6)=72+4\left(y^2-36\right)$$

Solve the resulting equation.

$$\begin{aligned} y^2-6y &= 72+4y^2-144 \\ 0 &= 3y^2+6y-72 \\ 0 &= 3\left(y^2+2y-24\right) \\ 0 &= 3(y+6)(y-4) \\ y &= -6, y=4 \end{aligned}$$

Check.

$y=-6$ is an extraneous solution. Check $y=4$ in the original equation.

$$\begin{aligned} \frac{y}{y+6} &= \frac{72}{y^2-36}+4 \\ \frac{4}{4+6} &\stackrel{?}{=} \frac{72}{4^2-36}+4 \\ \frac{4}{10} &\stackrel{?}{=} \frac{72}{-20}+4 \\ \frac{4}{10} &\stackrel{?}{=} -\frac{36}{10}+\frac{40}{10} \\ \frac{4}{10} &= \frac{4}{10}\surd \end{aligned}$$

The solution is $y=4$.

Exercise 3.6.9

Solve:

$$\frac{x}{x+4}=\frac{32}{x^2-16}+5$$

Answer

$x=3$

Exercise 3.6.10

Solve:

$$\frac{y}{y+8}=\frac{128}{y^2-64}+9$$

Answer

$y=7$

In some cases, all the algebraic solutions are extraneous.

Example 3.6.6

Solve:

$$\frac{x}{2x-2}-\frac{2}{3x+3}=\frac{5x^2-2x+9}{12x^2-12}$$

Solution

We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}$$

Note any value of the variable that would make any denominator zero.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}, x\neq 1, x\neq -1$$

Find the least common denominator. The LCD is $12(x-1)(x+1)$.

Clear the fractions.

$$12(x-1)(x+1)\left(\frac{x}{2(x-1)}-\frac{2}{3(x+1)}\right)=12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$$

Simplify.

$$6(x+1)\cdot x-4(x-1)\cdot 2=5x^2-2x+9$$

Simplify.

$$6x(x+1)-4\cdot 2(x-1)=5x^2-2x+9$$

Solve the resulting equation.

$$\begin{aligned} 6x^2+6x-8x+8 &= 5x^2-2x+9 \\ x^2-1 &= 0 \\ (x-1)(x+1) &= 0 \\ x &= 1 \text{ or } x=-1 \end{aligned}$$

Check.

$x=1$ and $x=-1$ are extraneous solutions.

The equation has no solution.

Exercise 3.6.11

Solve:

$$\frac{y}{5y-10}-\frac{5}{3y+6}=\frac{2y^2-19y+54}{15y^2-60}$$

Answer

There is no solution.

Exercise 3.6.12

Solve:

$$\frac{z}{2z+8}-\frac{3}{4z-8}=\frac{3z^2-16z-16}{8z^2+2z-64}$$

Answer

There is no solution.

Example 3.6.7

Solve:

$$\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}=\frac{2}{(x-3)(x+1)}, x\neq -1, x\neq \frac{1}{3}, x\neq 3$$

Find the least common denominator. The LCD is $(3x-1)(x+1)(x-3)$.

Clear the fractions.

$$(3x-1)(x+1)(x-3)\left(\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}\right)=(3x-1)(x+1)(x-3)\left(\frac{2}{(x-3)(x+1)}\right)$$

Simplify.

$$4(x+1)+3(x-3)=2(3x-1)$$

Distribute.

$$4x+4+3x-9=6x-2$$

Simplify.

$$7x-5=6x-2$$

$$x=3$$

The only algebraic solution was $x=3$, but we said that $x=3$ would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

Exercise 3.6.13

Solve:

$$\frac{15}{x^2+x-6}-\frac{3}{x-2}=\frac{2}{x+3}$$

Answer

There is no solution.

Exercise 3.6.14

Solve:

$$\frac{5}{x^2+2x-3}-\frac{3}{x^2+x-2}=\frac{1}{x^2+5x+6}$$

Answer

There is no solution.

Use Rational Functions

Working with functions that are defined by rational expressions often lead to rational equations. Again, we use the same techniques to solve them.

Example 3.6.8

For rational function, $f(x)=\dfrac{2x-6}{x^2-8x+15}$:

a. Find the domain of the function
b. Solve $f(x)=1$
c. Find the points on the graph at this function value.

Solution

a. The domain of a rational function is all real numbers except those that make the rational expression undefined. So to find them, we will set the denominator equal to zero and solve.

$$\begin{aligned} x^2-8x+15 &= 0 \\ (x-3)(x-5) &= 0 \quad \text{Factor the trinomial.} \\ x-3 &= 0 \quad \text{Use the Zero Product Property.} \\ x-5 &= 0 \quad \text{Use the Zero Product Property.} \\ x=3 \quad x &= 5 \quad \text{Solve.} \end{aligned}$$

The domain is all real numbers except $x\neq 3, x\neq 5$

b. $$f(x)=1$$

Substitute in the rational expression.

$$\frac{2x-6}{x^2-8x+15}=1$$

Factor the denominator.

$$\frac{2x-6}{(x-3)(x-5)}=1$$

Multiply both sides by the LCD, $(x-3)(x-5)$

$$(x-3)(x-5)\left(\frac{2x-6}{(x-3)(x-5)}\right)=(x-3)(x-5)(1)$$

Simplify.

$$2x-6=x^2-8x+15$$

Solve.

$$0=x^2-10x+21$$

Factor.

$$0 = (x-7)(x-3)$$

Use the Zero Product Property.

$$x-7=0 \quad x-3=0$$

Solve.

$$x=7 \quad x=3$$

c. The value of the function is 1 when $x=7, x=3$. So the points on the graph of this function when $f(x)=1$, will be $(7,1),(3,1)$. However, now we need to check for extraneous solutions. Since $x=3$ is not in the domain of the original function, we must remove $(3,1)$, since it would be considered extraneous.

? Exercise 3.6.15

For rational function, $f(x)=\dfrac{8-x}{x^2-7x+12}$

a. Find the domain of the function.
b. Solve $f(x)=3$.
c. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x \neq 3$ and $x \neq 4$
b. $x=2, x=\dfrac{14}{3}$
c. $(2,3), \left(\dfrac{14}{3},3\right)$

? Exercise 3.6.16

For rational function, $f(x)=\dfrac{x-1}{x^2-6x+5}$

a. Solve $f(x)=4$.
b. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x \neq 1$ and $x \neq 5$
b. $x=\dfrac{21}{4}$
c. $\left(\dfrac{21}{4},4\right)$

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

$$
\begin{aligned}
m &= \frac{y-y_1}{x-x_1} \\
m\,(x-x_1) &= \left(\frac{y-y_1}{x-x_1}\right)(x-x_1) \quad \text{Multiply both sides of the equation by } x-x_1. \\
m\,(x-x_1) &= y-y_1 \quad \text{Simplify.} \\
y-y_1 &= m\,(x-x_1) \quad \text{Rewrite the equation with the y terms on the left.}
\end{aligned}
$$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point $(2,3)$. We will add one more step to solve for y.

Example 3.6.9

Solve: $m = \dfrac{y-2}{x-3}$ for y.

Solution

$$m = \frac{y-2}{x-3}$$

Note any value of the variable that would make any denominator zero.

$$m = \frac{y-2}{x-3}, x \neq 3$$

Clear the fractions by multiplying both sides of the equation by the LCD, $x-3$.

$$(x-3)m = (x-3)\left(\frac{y-2}{x-3}\right)$$

Simplify.

$$xm-3m = y-2$$

Isolate the term with y.

$$xm-3m+2 = y$$

Exercise 3.6.17

Solve: $m = \dfrac{y-5}{x-4}$ for y.

Answer

$y = mx-4m+5$

Exercise 3.6.18

Solve: $m = \dfrac{y-1}{x+5}$ for y.

Answer

$y = mx+5m+1$

Remember to multiply both sides by the LCD in the next example.

Example 3.6.10

Solve: $\frac{1}{c}+\frac{1}{m}=1$ for c

Solution

$$\frac{1}{c}+\frac{1}{m}=1 \text{ for } c$$

Note any value of the variable that would make any denominator zero.

$$\frac{1}{c}+\frac{1}{m}=1, c\neq 0, m\neq 0$$

Clear the fractions by multiplying both sides of the equations by the LCD, cm.

$$cm\left(\frac{1}{c}+\frac{1}{m}\right)=cm(1)$$

Distribute.

$$cm\left(\frac{1}{c}\right)+cm\frac{1}{m}=cm(1)$$

Simplify.

$$m+c=cm$$

Collect the terms with c to the right.

$$m=cm-c$$

Factor the expression on the right.

$$m=c(m-1)$$

To isolate c, divide both sides by $m-1$.

$$\frac{m}{m-1}=\frac{c(m-1)}{m-1}$$

Simplify by removing common factors.

$$\frac{m}{m-1}=c$$

Notice that even though we excluded $c=0$ and $m=0$ from the original equation, we must also now state that $m\neq 1$.

Exercise 3.6.19

Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

Answer

$$a=\frac{b}{cb-1}$$

Exercise 3.6.20

Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y

Answer

$$y = \frac{3x}{x+6}$$

Media Access Additional Online Resources

Access this online resource for additional instruction and practice with equations with rational expressions.

- Equations with Rational Expressions

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3.6E: Exercises

Solve Rational Equations

In the following exercises, solve each rational equation.

1. $\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$

Answer

$a=10$

2. $\frac{6}{3}-\frac{2}{d}=\frac{4}{9}$

3. $\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$

Answer

$v=\frac{40}{21}$

4. $\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

5. $1-\frac{2}{m}=\frac{8}{m^2}$

Answer

$m=-2,\ m=4$

6. $1+\frac{4}{n}=\frac{21}{n^2}$

7. $1+\frac{9}{p}=\frac{-20}{p^2}$

Answer

$p=-5,\ p=-4$

8. $1-\frac{7}{q}=\frac{-6}{q^2}$

9. $\frac{5}{3v-2}=\frac{7}{4v}$

Answer

$v=14$

10. $\frac{8}{2w+1}=\frac{3}{w}$

11. $\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$

Answer

$x=-\frac{4}{5}$

12. $\frac{5}{y-9}+\frac{1}{y+9}=\frac{18}{y^2-81}$

13. $\frac{8}{z-10}-\frac{7}{z+10}=\frac{5}{z^2-100}$

Answer

$z=-145$

14. $\frac{9}{a+11}-\frac{6}{a-11}=\frac{6}{a^2-121}$

15. $\frac{-10}{q-2}-\frac{7}{q+4}=1$

Answer

$q=-18,\ q=-1$

16. $\frac{2}{s+7}-\frac{3}{s-3}=1$

17. $\frac{v-10}{v^2-5v+4}=\frac{3}{v-1}-\frac{6}{v-4}$

Answer

no solution

18. $\frac{w+8}{w^2-11w+28}=\frac{5}{w-7}+\frac{2}{w-4}$

19. $\frac{x-10}{x^2+8x+12}=\frac{3}{x+2}+\frac{4}{x+6}$

Answer

no solution

20. $\frac{y-5}{y^2-4y-5}=\frac{1}{y+1}+\frac{1}{y-5}$

21. $\frac{b+3}{3b}+\frac{b}{24}=\frac{1}{b}$

Answer

$b=-8$

22. $\frac{c+3}{12c}+\frac{c}{36}=\frac{1}{4c}$

23. $\frac{d}{d+3}=\frac{18}{d^2-9}+4$

Answer

$d=2$

24. $\frac{m}{m+5}=\frac{50}{m^2-25}+6$

25. $\frac{n}{n+2}-3=\frac{8}{n^2-4}$

Answer

$m=1$

26. $\frac{p}{p+7}-8=\frac{98}{p^2-49}$

27. $\frac{q}{3q-9}-\frac{3}{4q+12}=\frac{7q^2+6q+63}{24q^2-216}$

Answer

no solution

28. $\frac{r}{3r-15}-\frac{1}{4r+20}=\frac{3r^2+17r+40}{12r^2-300}$

29. $\frac{s}{2s+6}-\frac{2}{5s+5}=\frac{5s^2-3s-7}{10s^2+40s+30}$

Answer

$s=\frac{5}{4}$

30. $\frac{t}{6t-12}-\frac{5}{2t+10}=\frac{t^2-23t+70}{12t^2+36t-120}$

31. $\frac{2}{x^2+2x-8}-\frac{1}{x^2+9x+20}=\frac{4}{x^2+3x-10}$

Answer

$x=-\frac{4}{3}$

32. $\frac{5}{x^2+4x+3}+\frac{2}{x^2+x-6}=\frac{3}{x^2-x-2}$

33. $\frac{3}{x^2-5x-6}+\frac{3}{x^2-7x+6}=\frac{6}{x^2-1}$

Answer

no solution

34. $\dfrac{2}{x^2+2x-3}+\dfrac{3}{x^2+4x+3}=\dfrac{6}{x^2-1}$

Solve Rational Equations that Involve Functions

35. For rational function, $f(x)=\dfrac{x-2}{x^2+6x+8}$:

a. Find the domain of the function
b. Solve $f(x)=5$
c. Find the points on the graph at this function value

Answer

a. The domain is all real numbers except $x\neq -2$ and $x\neq -4$
b. $x=-3, x=-\dfrac{14}{5}$
c. $(-3,5), \left(-\dfrac{14}{5},5\right)$

36. For rational function, $f(x)=\dfrac{x+1}{x^2-2x-3}$:

a. Find the domain of the function
b. Solve $f(x)=1$
c. Find the points on the graph at this function value

37. For rational function, $f(x)=\dfrac{2-x}{x^2-7x+10}$:

a. Find the domain of the function
b. Solve $f(x)=2$
c. Find the points on the graph at this function value

Answer

a. The domain is all real numbers except $x\neq 2$ and $x\neq 5$
b. $x=\dfrac{9}{2}$
c. $\left(\dfrac{9}{2},2\right)$

38. For rational function, $f(x)=\dfrac{5-x}{x^2+5x+6}$:

a. Find the domain of the function
b. Solve $f(x)=3$
c. Find the points on the graph at this function value

Solve a Rational Equation for a Specific Variable

In the following exercises, solve:

39. $\dfrac{c}{r}=2\pi$ for r

Answer

$$r = \frac{C}{2\pi}$$

40. $\frac{I}{r} = P$ for r

41. $\frac{v+3}{w-1} = \frac{1}{2}$ for w

Answer

$$w = 2v+7$$

42. $\frac{x+5}{2-y} = \frac{4}{3}$ for y

43. $a = \frac{b+3}{c-2}$ for c

Answer

$$c = \frac{b+3+2a}{a}$$

44. $m = \frac{n}{2-n}$ for n

45. $\frac{1}{p} + \frac{2}{q} = 4$ for p

Answer

$$p = \frac{q}{4q-2}$$

46. $\frac{3}{s} + \frac{1}{t} = 2$ for s

47. $\frac{2}{v} + \frac{1}{5} = \frac{3}{w}$ for w

Answer

$$w = \frac{15v}{10+v}$$

48. $\frac{6}{x} + \frac{2}{3} = \frac{1}{y}$ for y

49. $\frac{m+3}{n-2} = \frac{4}{5}$ for n

Answer

$$n = \frac{5m+23}{4}$$

50. $r = \dfrac{s}{3-t}$ for t

51. $\dfrac{E}{c} = m^2$ for c

Answer

$$c = \frac{E}{m^2}$$

52. $\dfrac{R}{T} = W$ for T

53. $\dfrac{3}{x} - \dfrac{5}{y} = \dfrac{1}{4}$ for y

Answer

$$y = \frac{20x}{12-x}$$

54. $c = \dfrac{2}{a} + \dfrac{b}{5}$ for a

Writing Exercises

55. Your class mate is having trouble in this section. Write down the steps you would use to explain how to solve a rational equation.

Answer

Answers will vary.

56. Alek thinks the equation $\dfrac{y}{y+6} = \dfrac{72}{y^2-36} + 4$ has two solutions, $y = -6$ and $y = 4$. Explain why Alek is wrong.

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3.7: Rational Inequalities

In the past, we have discussed many types of inequalities, and one of the most critical parts of inequalities is we know the solutions are always intervals of numbers. Hence, we now discuss rational inequalities where the solutions are intervals, but we take one critical detail into account. Since rational inequalities and equations contain excluded values, we must take this into consideration when obtaining the solution. In fact, we use the excluded values and x-intercepts, if any, to determine the intervals in which make the inequality true. Again, with inequalities, we are trying to find intervals of numbers in which the statements is true.

Steps for solving rational inequalities

Step 1. Rewrite the inequality so that only zero is on the right side.

Step 2. Determine where the rational expression is *undefined* or *equals zero*.

Step 3. Graph the values found in **Step 2.** on a number line split into intervals.

Step 4. Take a test point within each interval and determine the sign of the result.

Step 5. Determine the solution, where the solution is the interval in which makes the inequality true.

Note

Be careful when determining the intervals. Recall, where the expression is undefined is not included in the interval, i.e., *always* use parenthesis for the values in which the expression is undefined.

Example 3.7.1

Solve $\dfrac{x-3}{x+1} > 0$.

Solution

Step 1. Rewrite the inequality so that only zero is on the right side.
Since $\dfrac{x-3}{x+1} > 0$ already has zero on the right side, this step is done.

Step 2. Determine where the rational expression is *undefined* or *equals zero*.
To obtain where the rational expression **equals zero**, we set the numerator equal to zero:

$$\begin{aligned} x-3 &= 0 \\ x &= 3 \end{aligned}$$

To obtain where the expression is **undefined**, we find its excluded value(s) by setting the denominator equal to zero:

$$\begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned}$$

Step 3. Graph the values found in **Step 2.** on a number line into split into intervals.
Label -1 and 3 on a blank real number line:



Figure 3.7.1

Step 4. Take a test point within each interval and determine the sign of the result.
We take test values on each side of -1 and 3. Let's choose fairly easy numbers such as $-2,\ 0,$ and 4. We plug-n-chug these numbers into $\dfrac{x-3}{x+1}$ and determine whether the value is positive or negative:

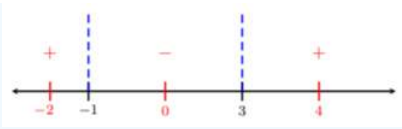


Figure 3.7.2

$$\text{letting } x=-2 \Longrightarrow \quad \frac{-2-3}{-2+1}=5>0$$
$$\text{letting } x=0 \Longrightarrow \quad \frac{0-3}{0+1}=-3<0$$
$$\text{letting } x=4 \Longrightarrow \quad \frac{4-3}{4+1}=0.2>0$$

Step 5. Determine the solution, where the solution is the interval in which makes the inequality true.
Since $\dfrac{x-3}{x+1}>0$ (from **Step 1.**), then we are looking for where the test values are positive. Looking at the number line above, we see these are the values to the left of -1 and to the right of 3. Thus, the solution is $(-\infty,-1)\cup(3,\infty)$.

Note

We always use a **parenthesis** for the value that is **excluded** since this value makes the rational expression undefined. Using a bracket or parenthesis for where the expression equals zero is determined by the original inequality sign, e.g., $<,>$ use parenthesis, and $\geq,\leq$ use brackets.

Example 3.7.2

Solve $\dfrac{2x+3}{x-2}\leq 1$.

Solution

Step 1. Rewrite the inequality so that only zero is on the right side.
We rewrite $\dfrac{2x+3}{x-2}\leq 1$ so that there is zero on the right side, and as one fraction:

$$\begin{aligned}
\frac{2x+3}{x-2} &\leq 1 && \text{Subtract 1 from each side}\\
\frac{2x+3}{x-2}-1 &\leq 0 && \text{Rewrite as one fraction where LCD: } (x-2)\\
\frac{2x+3}{x-2}-\frac{x-2}{x-2} &\leq 0 && \text{Subtract across numerators}\\
\frac{2x+3-x+2}{x-2} &\leq 0 && \text{Simplify}\\
\frac{x+5}{x-2} &\leq 0 && \text{We use this inequality to obtain the solution}
\end{aligned}$$

Step 2. Determine where the rational expression is *undefined* or *equals zero*.
To obtain where the rational expression **equals zero**, we set the numerator equal to zero:

$$\begin{aligned} x+5 &=0\\ x &=-5 \end{aligned}$$

To obtain the **excluded** values, we set the denominator equal to zero:

$$\begin{aligned} x-2 &=0\\ x &=2 \end{aligned}$$

Step 3. Graph the values found in **Step 2.** on a number line into split into intervals.
Label -5 and 2 on a blank number line:

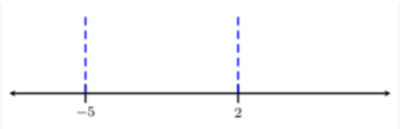


Figure 3.7.3

Step 4. Take a test point within each interval and determine the sign of the result.
We take test values on each side of -5 and 2. Let's choose fairly easy numbers such as $-6,\ 0,$ and 3. We plug these numbers into $\dfrac{x+5}{x-2}$ and determine whether the value is positive or negative:

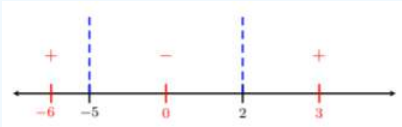


Figure 3.7.4

$$\text{letting } x=-6 \Longrightarrow \quad \frac{-6+5}{-6-2}=\frac{1}{8}>0$$
$$\text{letting } x=0 \Longrightarrow \quad \frac{0+5}{0-2}=-\frac{5}{2}<0$$
$$\text{letting } x=3 \Longrightarrow \quad \frac{3+5}{3-2}=8>0$$

Step 5. Determine the solution, where the solution is the interval in which makes the inequality true.
Since $\dfrac{x+5}{x-2}\le 0$ (from **Step 1.**), then we are looking for where the test values are negative or equal to zero. Looking at the number line above, we see these are the values between -5 and 2. Thus, the solution is $[-5,2)$.

We used a bracket on -5 since the original inequality was $\le$ and a parenthesis on 2 since 2 was an excluded value.

Rational Inequalities Homework

Solve the rational inequalities. Write the solution in interval notation.

? Exercise 3.7.1

$\dfrac{x-3}{x+1}<0$

? Exercise 3.7.2

$\dfrac{x-7}{x+6}>0$

? Exercise 3.7.3

$\dfrac{x-3}{x+1}<1$

? Exercise 3.7.4

$\dfrac{x+27}{x+4}<9$

? Exercise 3.7.5

$$x+\frac{45}{x}<14$$

Exercise 3.7.6

$$\frac{(x-9)(x+9)}{x}\geq 0$$

Exercise 3.7.7

$$\frac{(x+12)(x-4)}{x-1}\geq 0$$

Exercise 3.7.8

$$\frac{3x}{7-x}<x$$

Exercise 3.7.9

$$\frac{4x}{3-x}\geq 4x$$

Exercise 3.7.10

$$\frac{8}{x-3}>\frac{6}{x-1}$$

Find the values of x that satisfy the given condition for the function.

Exercise 3.7.11

Solve $R(x)\geq 0$ if $R(x)=\frac{x+6}{x-4}$.

Exercise 3.7.12

Solve $R(x)\leq 0$ if $R(x)=\frac{x-4}{x+8}$.

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CHAPTER OVERVIEW

4: Chapter 4 - Radicals

4.1: Use Radicals in Functions

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4.1: Use Radicals in Functions

Learning Objectives

By the end of this section, you will be able to:

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

Before you get started, take this readiness quiz.

1. Solve: $1-2x \geq 0$.
   If you missed this problem, review inequalities here.
2. For $f(x)=3x-4$, evaluate $f(2), f(-1), f(0)$
   If you missed this problem, review evaluate functions here.
3. Graph $f(x)=\sqrt{x}$. State the domain and range of the function in interval notation.
   If you missed this problem, review domain and range here.

Evaluate a Radical Function

In this section we will extend our previous work with functions to include radicals. If a function is defined by a radical expression, we call it a **radical function**.

- The square root function is $f(x)=\sqrt{x}$.
- The cube root function is $f(x)=\sqrt[3]{x}$.

Definition 4.1.1: radical function

A **radical function** is a function that is defined by a radical expression.

To evaluate a radical function, we find the value of $f(x)$ for a given value of x just as we did in our previous work with functions.

Example 4.1.1

For the function $f(x)=\sqrt{2x-1}$, find

a. $f(5)$
b. $f(-2)$

Solution:

a.

$$f(x)=\sqrt{2x-1}$$

To evaluate $f(5)$, substitute 5 for x.

$$f(5)=\sqrt{2\cdot 5-1}$$

Simplify.

$$f(5)=\sqrt{9}$$

Take the square root.

$$f(5)=3$$

b.

$$f(x)=\sqrt{2x-1}$$

To evaluate $f(-2)$, substitute -2 for x.

$$f(-2)=\sqrt{2(-2)-1}$$

Simplify.

$$f(-2)=\sqrt{-5}$$

Since the square root of a negative number is not a real number, the function undefined at $x=-2$. In in the future you will learn how to write negative numbers within radicals as imaginary numbers.

? Exercise 4.1.1

For the function $f(x)=\sqrt{3x-2}$, find

a. $f(6)$
b. $f(0)$

Answer

a. $f(6)=4$
b. undefined at $x=0$. In the future you will learn how to write negative numbers within radicals as imaginary numbers.

? Exercise 4.1.2

For the function $g(x)=\sqrt{5x+5}$, find

a. $g(4)$
b. $g(-3)$

Answer

a. $g(4)=5$
b. undefined at $f(-3)$. In in the future you will learn how to write negative numbers within radicals as imaginary numbers.

We follow the same procedure to evaluate cube roots.

✓ Example 4.1.2

For the function $g(x)=\sqrt[3]{x-6}$, find

a. $g(14)$
b. $g(-2)$

Solution:

a.

$$g(x)=\sqrt[3]{x-6}$$

To evaluate $g(14)$, substitute 14 for x.

$$g(14)=\sqrt[3]{14-6}$$

Simplify.

$$g(14)=\sqrt[3]{8}$$

Take the cube root.

$$g(14)=2$$

b.

$$g(x)=\sqrt[3]{x-6}$$

To evaluate $g(-2)$, substitute -2 for x.

$$g(-2) = \sqrt[3]{-2-6}$$

Simplify.

$$g(-2) = \sqrt[3]{-8}$$

Take the cube root.

$$g(-2) = -2$$

Exercise 4.1.3

For the function $g(x) = \sqrt[3]{3x-4}$, find

a. $g(4)$
b. $g(1)$

Answer

a. $g(4) = 2$
b. $g(1) = -1$

Exercise 4.1.4

For the function $h(x) = \sqrt[3]{5x-2}$, find

a. $h(2)$
b. $h(-5)$

Answer

a. $h(2) = 2$
b. $h(-5) = -3$

The next example has fourth roots.

Example 4.1.3

For the function $f(x) = \sqrt[4]{5x-4}$, find

a. $f(4)$
b. $f(-12)$

Solution:

a.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(4)$, substitute 4 for x.

$$f(4) = \sqrt[4]{5 \cdot 4 - 4}$$

Simplify.

$$f(4) = \sqrt[4]{16}$$

Take the fourth root.

$$f(4) = 2$$

b.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(-12)$, substitute -12 for x.

$$f(-12)=\sqrt[4]{5(-12)-4}$$

Simplify.

$$f(-12)=\sqrt[4]{-64}$$

Since the fourth root of a negative number is not a real number, the function is not defined at $x=-12$. In the future you will learn how to write negative numbers within radicals as imaginary numbers.

? Exercise 4.1.5

For the function $f(x)=\sqrt[4]{3x+4}$, find

a. $f(4)$
b. $f(-1)$

Answer

a. $f(4)=2$
b. $f(-1)=1$

? Exercise 4.1.6

For the function $g(x)=\sqrt[4]{5x+1}$, find

a. $g(16)$
b. $g(3)$

Answer

a. $g(16)=3$
b. $g(3)=2$

Find the Domain of a Radical Function

To find the **domain** and **range** of radical functions, we use our properties of radicals. For a radical with an even index, we said the radicand had to be greater than or equal to zero as even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Properties of $\sqrt[n]{a}$

When n is an **even** number and:

- $a\geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a<0$, then $\sqrt[n]{a}$ is not a real number.

When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.

So, to find the domain of a radical function with even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number.

Domain of a Radical Function

When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.

When the **index** of the radical is **odd**, the radicand can be any real number.

✓ Example 4.1.4

Find the domain of the function, $f(x)=\sqrt{3x-4}$. Write the domain in interval notation.

Solution:

Since the function, $f(x)=\sqrt{3x-4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0. We set the radicand to be greater than or equal to 0 and then solve to find the domain.

Solve.

$$\begin{aligned} 3x-4 &\geq 0 \\ 3x &\geq 4 \\ x &\geq \frac{4}{3} \end{aligned}$$

The domain of $f(x)=\sqrt{3x-4}$ is all values $x \geq \frac{4}{3}$ and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

Exercise 4.1.7

Find the domain of the function, $f(x)=\sqrt{6x-5}$. Write the domain in interval notation.

Answer

$\left[\frac{5}{6}, \infty\right)$

Exercise 4.1.8

Find the domain of the function, $f(x)=\sqrt{4-5x}$. Write the domain in interval notation.

Answer

$\left(-\infty, \frac{4}{5}\right]$

Example 4.1.5

Find the domain of the function, $g(x)=\sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution:

Solve the function, $g(x)=\sqrt{\frac{6}{x-1}}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

The radicand cannot be zero since the numerator is not zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive since the numerator is positive. We know a positive divided by a positive is positive.

We set $x-1>0$ and solve.

$$x-1>0$$

Solve.

$$x>1$$

Also, since the radicand is a fraction, we must realize that this denominator cannot be zero, since a zero in the denominator would cause the fraction to be undefined.

We solve $x-1=0$ to find the value that must be eliminated from the domain.

$$x-1=0$$

Solve.

$$x=1 \text{ so } x/neq1 \text{ in the domain.}$$

Putting this together we get the domain is $x>1$ and we write it as $(1, \infty)$.

Exercise 4.1.9

Find the domain of the function, $f(x)=\sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

Answer

$(-3,\infty)$

Exercise 4.1.10

Find the domain of the function, $h(x)=\sqrt{\frac{9}{x-5}}$. Write the domain in interval notation.

Answer

$(5,\infty)$

The next example involves a cube root and so will require different thinking.

Example 4.1.6

Find the domain of the function, $f(x)=\sqrt[3]{2x^2+3}$. Write the domain in interval notation.

Solution:

Since the function, $f(x)=\sqrt[3]{2x^2+3}$ has a radical with an index of 3, which is odd, we know the radicand can be any real number. This tells us the domain is any real number. In interval notation, we write $(-\infty,\infty)$.

The domain of $f(x)=\sqrt[3]{2x^2+3}$ is all real numbers and we write it in interval notation as $(-\infty,\infty)$.

Exercise 4.1.11

Find the domain of the function, $f(x)=\sqrt[3]{3x^2-1}$. Write the domain in interval notation.

Answer

$(-\infty,\infty)$

Exercise 4.1.12

Find the domain of the function, $g(x)=\sqrt[3]{5x-4}$. Write the domain in interval notation.

Answer

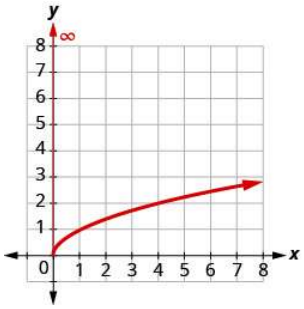
$(-\infty,\infty)$

Graph Radical Functions

Before we graph any radical function, we first find the **domain** of the function. For the function, $f(x)=\sqrt{x}$, the index is even, and so the radicand must be greater than or equal to 0.

This tells us the domain is $x\geq 0$ and we write this in interval notation as $[0,\infty)$.

Previously we used point plotting to graph the function, $f(x)=\sqrt{x}$. We chose x-values, substituted them in and then created a chart. Notice we chose points that are perfect squares in order to make taking the square root easier.



x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Figure 8.7.1

Once we see the graph, we can find the range of the function. The y-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

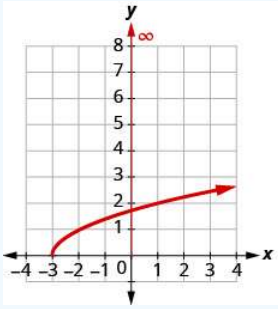
Example 4.1.7

For the function $f(x)=\sqrt{x+3}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 2, we know the radicand must be greater than or equal to zero. If $x+3 \geq 0$, then $x \geq -3$. This tells us the domain is all values $x \geq -3$ and written in interval notation as $[-3, \infty)$.
b. To graph the function, we choose points in the interval $[-3, \infty)$ that will also give us a radicand which will be easy to take the square root.



x	$f(x)=\sqrt{x+3}$	$(x, f(x))$
–3	0	(–3, 0)
–2	1	(–2, 1)
1	2	(1, 2)
6	3	(6, 3)

Figure 8.7.2

c. Looking at the graph, we see the y-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

Exercise 4.1.13

For the function $f(x)=\sqrt{x+2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[-2, \infty)$

b.

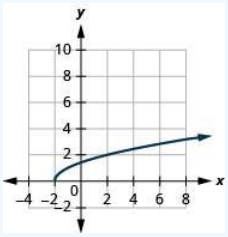


Figure 8.7.3

c. range: $[0, \infty)$

? Exercise 4.1.14

For the function $f(x) = \sqrt{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[2, \infty)$
b.

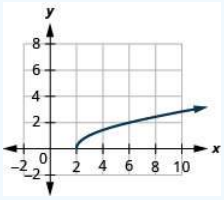


Figure 8.7.4

c. range: $[0, \infty)$

In our previous work graphing functions, we graphed $f(x) = x^3$ but we did not graph the function $f(x) = \sqrt[3]{x}$. We will do this now in the next example.

✓ Example 4.1.8

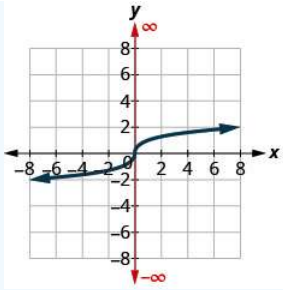
For the function, $f(x) = \sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 3, we know the radicand can be any real number. This tells us the domain is all real numbers and written in interval notation as $(-\infty, \infty)$

b. To graph the function, we choose points in the interval $(-\infty, \infty)$ that will also give us a radicand which will be easy to take the cube root.



x	$f(x)=\sqrt[3]{x}$	$(x, f(x))$
–8	–2	(–8, –2)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
8	2	(8, 2)

Figure 8.7.5

c. Looking at the graph, we see the y-values of the function are all real numbers. The range then is $(-\infty, \infty)$.

Exercise 4.1.15

For the function $f(x)=-\sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$
b.

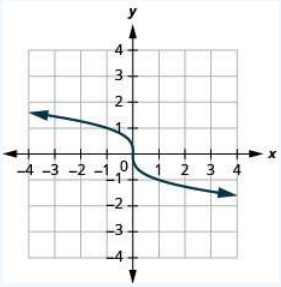


Figure 8.7.6

c. range: $(-\infty, \infty)$

Exercise 4.1.16

For the function $f(x)=\sqrt[3]{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$
b.

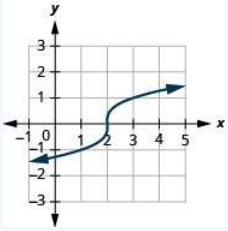


Figure 8.7.7

c. range: $(-\infty, \infty)$

Access these online resources for additional instruction and practice with radical functions.

- Domain of a Radical Function
- Domain of a Radical Function 2
- Finding Domain of a Radical Function

Key Concepts

- **Properties of $\sqrt[n]{a}$**
  - When n is an **even** number and:
    $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
    $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Domain of a Radical Function**
  - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
  - When the **index** of the radical is **odd**, the radicand can be any real number.

Glossary

radical function

A radical function is a function that is defined by a radical expression.

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4.1E: Exercises

Practice Makes Perfect

Exercise $4.1E.17$ Evaluate a Radical Function

In the following exercises, evaluate each function.

1. $f(x)=\sqrt{4x-4}$, find
   a. $f(5)$
   b. $f(0)$
2. $f(x)=\sqrt{6x-5}$, find
   a. $f(5)$
   b. $f(-1)$
3. $g(x)=\sqrt{6x+1}$, find
   a. $g(4)$
   b. $g(8)$
4. $g(x)=\sqrt{3x+1}$, find
   a. $g(8)$
   b. $g(5)$
5. $F(x)=\sqrt{3-2x}$, find
   a. $F(1)$
   b. $F(-11)$
6. $F(x)=\sqrt{8-4x}$, find
   a. $F(1)$
   b. $F(-2)$
7. $G(x)=\sqrt{5x-1}$, find
   a. $G(5)$
   b. $G(2)$
8. $G(x)=\sqrt{4x+1}$, find
   a. $G(11)$
   b. $G(2)$
9. $g(x)=\sqrt[3]{2x-4}$, find
   a. $g(6)$
   b. $g(-2)$
10. $g(x)=\sqrt[3]{7x-1}$, find
    a. $g(4)$
    b. $g(-1)$
11. $h(x)=\sqrt[3]{x^2-4}$, find
    a. $h(-2)$
    b. $h(6)$
12. $h(x)=\sqrt[3]{x^2+4}$, find
    a. $h(-2)$
    b. $h(6)$
13. For the function $f(x)=\sqrt[4]{2x^3}$, find
    a. $f(0)$
    b. $f(2)$
14. For the function $f(x)=\sqrt[4]{3x^3}$, find

a. $f(0)$
b. $f(3)$

15. For the function $g(x)=\sqrt[4]{4-4x}$, find
    a. $g(1)$
    b. $g(-3)$
16. For the function $g(x)=\sqrt[4]{8-4x}$, find
    a. $g(-6)$
    b. $g(2)$

Answer

1.
    a. $f(5)=4$
    b. no value at $x=0$
3.
    a. $g(4)=5$
    b. $g(8)=7$
5.
    a. $F(1)=1$
    b. $F(-11)=5$
7.
    a. $G(5)=2\sqrt{6}$
    b. $G(2)=3$
9.
    a. $g(6)=2$
    b. $g(-2)=-2$
11.
    a. $h(-2)=0$
    b. $h(6)=2\sqrt[3]{4}$
13.
    a. $f(0)=0$
    b. $f(2)=2$
15.
    a. $g(1)=0$
    b. $g(-3)=2$

? Exercise $4.1E.18$ Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

1. $f(x)=\sqrt{3x-1}$
2. $f(x)=\sqrt{4x-2}$
3. $g(x)=\sqrt{2-3x}$
4. $g(x)=\sqrt{8-x}$
5. $h(x)=\sqrt{\frac{5}{x-2}}$
6. $h(x)=\sqrt{\frac{6}{x+3}}$

7. $f(x)=\sqrt{\frac{x+3}{x-2}}$
8. $f(x)=\sqrt{\frac{x-1}{x+4}}$
9. $g(x)=\sqrt[3]{8x-1}$
10. $g(x)=\sqrt[3]{6x+5}$
11. $f(x)=\sqrt[3]{4x^2-16}$
12. $f(x)=\sqrt[3]{6x^2-25}$
13. $F(x)=\sqrt[4]{8x+3}$
14. $F(x)=\sqrt[4]{10-7x}$
15. $G(x)=\sqrt[5]{2x-1}$
16. $G(x)=\sqrt[5]{6x-3}$

Answer

1. $\left[\frac{1}{3},\infty\right)$
3. $\left(-\infty,\frac{2}{3}\right]$
5. $(2,\infty)$
7. $(-\infty,-3]\cup(2,\infty)$
9. $(-\infty,\infty)$
11. $(-\infty,\infty)$
13. $\left[-\frac{3}{8},\infty\right)$
15. $(-\infty,\infty)$

Exercise $4.1E.19$ graph radical functions

In the following exercises,

a. find the domain of the function
b. graph the function
c. use the graph to determine the range

1. $f(x)=\sqrt{x+1}$
2. $f(x)=\sqrt{x-1}$
3. $g(x)=\sqrt{x+4}$
4. $g(x)=\sqrt{x-4}$
5. $f(x)=\sqrt{x}+2$
6. $f(x)=\sqrt{x}-2$
7. $g(x)=2\sqrt{x}$
8. $g(x)=3\sqrt{x}$
9. $f(x)=\sqrt{3-x}$
10. $f(x)=\sqrt{4-x}$
11. $g(x)=-\sqrt{x}$
12. $g(x)=-\sqrt{x}+1$
13. $f(x)=\sqrt[3]{x+1}$
14. $f(x)=\sqrt[3]{x-1}$
15. $g(x)=\sqrt[3]{x+2}$
16. $g(x)=\sqrt[3]{x-2}$
17. $f(x)=\sqrt[3]{x}+3$
18. $f(x)=\sqrt[3]{x}-3$
19. $g(x)=\sqrt[3]{x}$
20. $g(x)=-\sqrt[3]{x}$

21. $f(x) = 2\sqrt[3]{x}$
22. $f(x) = -2\sqrt[3]{x}$

Answer

1.

a. domain: $[-1, \infty)$

b.

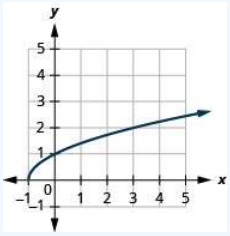


Figure 8.7.8

c. $[0, \infty)$

3.

a. domain: $[-4, \infty)$

b.

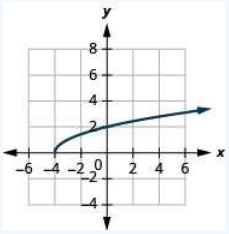


Figure 8.7.9

c. $[0, \infty)$

5.

a. domain: $[0, \infty)$

b.

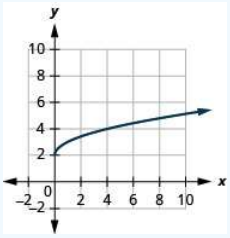


Figure 8.7.10

c. $[2, \infty)$

7.

a. domain: $[0, \infty)$

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b.

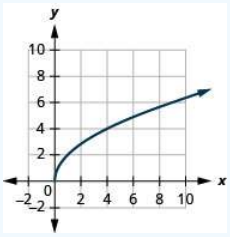


Figure 8.7.11

c. $[0, \infty)$

9.

a. domain: $(-\infty, 3]$

b.

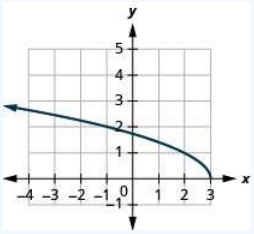


Figure 8.7.12

c. $[0, \infty)$

11.

a. domain: $[0, \infty)$

b.

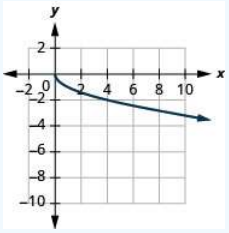


Figure 8.7.13

c. $(-\infty, 0]$

13.

a. domain: $(-\infty, \infty)$

b.

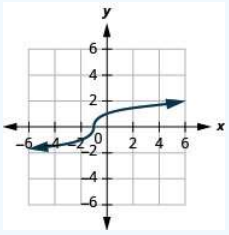


Figure 8.7.14

c. $(-\infty, \infty)$

15.

a. domain: $(-\infty, \infty)$

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b.

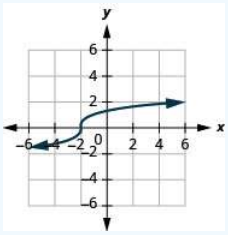


Figure 8.7.15

c. $(-\infty, \infty)$

17.

a. domain: $(-\infty, \infty)$

b.

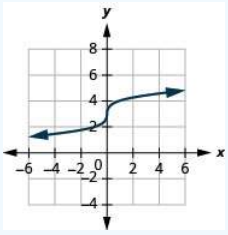


Figure 8.7.16

c. $(-\infty, \infty)$

19.

a. domain: $(-\infty, \infty)$

b.

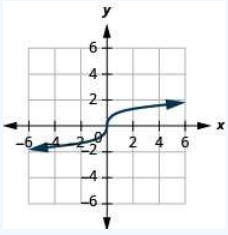


Figure 8.7.17

c. $(-\infty, \infty)$

21.

a. domain: $(-\infty, \infty)$

b.

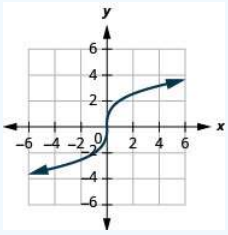


Figure 8.7.18

c. $(-\infty, \infty)$

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Exercise $4.1E.20$ writing exercises

1. Explain how to find the domain of a fourth root function.
2. Explain how to find the domain of a fifth root function.
3. Explain why $y = \sqrt[3]{x}$ is a function.
4. Explain why the process of finding the domain of a radical function with an even index is different from the process when the index is odd.

Answer

1. Answers may vary

3. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate a radical function.			
find the domain of a radical function.			
graph a radical function.			

Figure 8.7.19

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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4.2: Simplify Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

Before you get started, take this readiness quiz.

1. Simplify: $\frac{x^9}{x^4}$.
   If you missed this problem, review Example 5.13.
2. Simplify: $\frac{y^3}{y^{11}}$.
   If you missed this problem, review Example 5.13.
3. Simplify: $\left(n^2\right)^6$.
   If you missed this problem, review Example 5.17.

Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A **radical expression**, $\sqrt[n]{a}$, is considered simplified if it has no factors of m^n. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

Definition 4.2.1: Simplified Radical Expression

For real numbers a and m, and $n \geq 2$,

$\sqrt[n]{a}$ is considered simplified if a has no factors of m^n

For example, $\sqrt{5}$ is considered simplified because there are no perfect square factors in 5. But $\sqrt{12}$ is not simplified because 12 has a perfect square factor of 4.

Similarly, $\sqrt[3]{4}$ is simplified because there are no perfect cube factors in 4. But $\sqrt[3]{24}$ is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that

$$(ab)^n = a^n b^n. \tag{4.2.1}$$

The corresponding of **Product Property of Roots** says that

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}. \tag{4.2.2}$$

Definition 4.2.2: Product Property of nth Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, and $n \geq 2$ is an integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

We use the Product Property of Roots to remove all perfect square factors from a square root.

Example 4.2.1: Simplify square roots using the product property of roots

Simplify: $\sqrt{98}$.

Solution:

Step 1: Find the largest factor in the radicand that is a perfect power of the index.	We see that 49 is the largest factor of 98 that has a power of 2.	$\sqrt{98}$
Rewrite the radicand as a product of two factors, using that factor.	In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{49 \cdot 2}$
Step 2: Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{49} \cdot \sqrt{2}$
Step 3: Simplify the root of the perfect power.		$7\sqrt{2}$

Try It 4.2.1

Simplify: $\sqrt{48}$

Answer

$4\sqrt{3}$

Try It 4.2.2

Simplify: $\sqrt{45}$.

Answer

$3\sqrt{5}$

Notice in the previous example that the simplified form of $\sqrt{98}$ is $7\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index. The expression $7\sqrt{2}$ is very different from $\sqrt[7]{2}$.

Simplify a Radical Expression Using the Product Property

1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
2. Use the product rule to rewrite the radical as the product of two radicals.
3. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares, cubes, and fourth powers.

Example 4.2.2

Simplify:

a. $\sqrt{500}$
b. $\sqrt[3]{16}$

c. $\sqrt[4]{243}$

Solution:

a.

$$\sqrt{500}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{100 \cdot 5}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{100} \cdot \sqrt{5}$$

Simplify.

$$10\sqrt{5}$$

b.

$$\sqrt[3]{16}$$

Rewrite the radicand as a product using the greatest perfect cube factor. $2^3 = 8$

$$\sqrt[3]{8 \cdot 2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8} \cdot \sqrt[3]{2}$$

Simplify.

$$2\sqrt[3]{2}$$

c.

$$\sqrt[4]{243}$$

Rewrite the radicand as a product using the greatest perfect fourth power factor. $3^4 = 81$

$$\sqrt[4]{81 \cdot 3}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{81} \cdot \sqrt[4]{3}$$

Simplify.

$$3\sqrt[4]{3}$$

? Try It 4.2.3

Simplify: a. $\sqrt{288}$ b. $\sqrt[3]{81}$ c. $\sqrt[4]{64}$

Answer

a. $12\sqrt{2}$ b. $3\sqrt[3]{3}$ c. $2\sqrt[4]{4}$

? Try It 4.2.4

Simplify: a. $\sqrt{432}$ b. $\sqrt[3]{625}$ c. $\sqrt[4]{729}$

Answer

a. $12\sqrt{3}$ b. $5\sqrt[3]{5}$ c. $3\sqrt[4]{9}$

The next example is much like the previous examples, but with variables. Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

Example 4.2.3

Simplify:

a. $\sqrt{x^3}$
b. $\sqrt[3]{x^4}$
c. $\sqrt[4]{x^7}$

Solution:

a.

$$\sqrt{x^3}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{x^2 \cdot x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{x^2} \cdot \sqrt{x}$$

Simplify.

$$|x|\sqrt{x}$$

b.

$$\sqrt[3]{x^4}$$

Rewrite the radicand as a product using the largest perfect cube factor.

$$\sqrt[3]{x^3 \cdot x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$$

Simplify.

$$x\sqrt[3]{x}$$

c.

$$\sqrt[4]{x^7}$$

Rewrite the radicand as a product using the greatest perfect fourth power factor.

$$\sqrt[4]{x^4 \cdot x^3}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$$

Simplify.

$$|x|\sqrt[4]{x^3}$$

Try It 4.2.5

Simplify: a. $\sqrt{b^5}$ b. $\sqrt[4]{y^6}$ c. $\sqrt[3]{z^5}$

Answer

a. $b^2\sqrt{b}$ b. $|y|\sqrt[4]{y^2}$ c. $z\sqrt[3]{z^2}$

Try It 4.2.6

Simplify: a. $\sqrt{p^9}$ b. $\sqrt[5]{y^8}$ c. $\sqrt[6]{q^{13}}$

Answer

a. $p^4\sqrt{p}$ b. $p\sqrt[5]{p^3}$ c. $q^2\sqrt[6]{q}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

Example 4.2.4

Simplify:

a. $\sqrt{72n^7}$
b. $\sqrt[3]{24x^7}$
c. $\sqrt[4]{80y^{14}}$

Solution:

a.

$$\sqrt{72n^7}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{36n^6 \cdot 2n}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{36n^6} \cdot \sqrt{2n}$$

Simplify.

$$6\left|n^3\right|\sqrt{2n}$$

b.

$$\sqrt[3]{24x^7}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{8x^6 \cdot 3x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$$

Rewrite the first radicand as $\left(2x^2\right)^3$.

$$\sqrt[3]{\left(2x^2\right)^3} \cdot \sqrt[3]{3x}$$

Simplify.

$$2x^2\sqrt[3]{3x}$$

c.

$$\sqrt[4]{80y^{14}}$$

Rewrite the radicand as a product using perfect fourth power factors.

$$\sqrt[4]{16y^{12} \cdot 5y^2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{16y^{12}} \cdot \sqrt[4]{5y^2}$$

Rewrite the first radicand as $\left(2y^3\right)^4$.

$$\sqrt[4]{(2y^3)^4}\cdot\sqrt[4]{5y^2}$$

Simplify.

$$2\left|y^3\right|\sqrt[4]{5y^2}$$

Try It 4.2.7

Simplify: a. $\sqrt{32y^5}$ b. $\sqrt[3]{54p^{10}}$ c. $\sqrt[4]{64q^{10}}$

Answer

a. $4y^2\sqrt{2y}$ b. $3p^3\sqrt[3]{2p}$ c. $2q^2\sqrt[4]{4q^2}$

Try It 4.2.8

Simplify: a. $\sqrt{75a^9}$ b. $\sqrt[3]{128m^{11}}$ c. $\sqrt[4]{162n^7}$

Answer

a. $5a^4\sqrt{3a}$ b. $4m^3\sqrt[3]{2m^2}$ c. $3|n|\sqrt[4]{2n^3}$

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

Example 4.2.5

Simplify:

a. $\sqrt{63u^3v^5}$
b. $\sqrt[3]{40x^4y^5}$
c. $\sqrt[4]{48x^4y^7}$

Solution:

a.

$$\sqrt{63u^3v^5}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{9u^2v^4\cdot 7uv}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{9u^2v^4}\cdot\sqrt{7uv}$$

Rewrite the first radicand as $\left(3uv^2\right)^2$.

$$\sqrt{(3uv^2)^2}\cdot\sqrt{7uv}$$

Simplify.

$$3|u|v^2\sqrt{7uv}$$

b.

$$\sqrt[3]{40x^4y^5}$$

Rewrite the radicand as a product using the largest perfect cube factor.

$$\sqrt[3]{8x^3y^3\cdot 5xy^2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8x^3y^3} \cdot \sqrt[3]{5xy^2}$$

Rewrite the first radicand as $(2xy)^3$.

$$\sqrt[3]{(2xy)^3} \cdot \sqrt[3]{5xy^2}$$

Simplify.

$$2xy\sqrt[3]{5xy^2}$$

c.

$$\sqrt[4]{48x^4y^7}$$

Rewrite the radicand as a product using the largest perfect fourth power factor.

$$\sqrt[4]{16x^4y^4 \cdot 3y^3}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{16x^4y^4} \cdot \sqrt[4]{3y^3}$$

Rewrite the first radicand as $(2xy)^4$.

$$\sqrt[4]{(2xy)^4} \cdot \sqrt[4]{3y^3}$$

Simplify.

$$2|xy|\sqrt[4]{3y^3}$$

Try It 4.2.9

Simplify:

a. $\sqrt{98a^7b^5}$
b. $\sqrt[3]{56x^5y^4}$
c. $\sqrt[4]{32x^5y^8}$

Answer

a. $7|a^3|b^2\sqrt{2ab}$
b. $2xy\sqrt[3]{7x^2y}$
c. $2|x|y^2\sqrt[4]{2x}$

Try It 4.2.10

Simplify:

a. $\sqrt{180m^9n^{11}}$
b. $\sqrt[3]{72x^6y^5}$
c. $\sqrt[4]{80x^7y^4}$

Answer

a. $6m^4|n^5|\sqrt{5mn}$
b. $2x^2y\sqrt[3]{9y^2}$
c. $2|xy|\sqrt[4]{5x^3}$

Example 4.2.6

Simplify:

a. $\sqrt[3]{-27}$
b. $\sqrt[4]{-16}$

Solution:

a.

$$\sqrt[3]{-27}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{(-3)^3}$$

Take the cube root.

$$-3$$

b.

$$\sqrt[4]{-16}$$

There is no real number n where $n^4 = -16$.

Not a real number

Try It 4.2.11

Simplify:

a. $\sqrt[3]{-64}$
b. $\sqrt[4]{-81}$

Answer

a. -4
b. no real number

Try It 4.2.12

Simplify:

a. $\sqrt[3]{-625}$
b. $\sqrt[4]{-324}$

Answer

a. $-5\sqrt[3]{5}$
b. no real number

We have seen how to use the **order of operations** to simplify some expressions with radicals. In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer since one term contains a radical and the other does not. The next example also includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

Example 4.2.7

Simplify:

a. $3+\sqrt{32}$
b. $\dfrac{4-\sqrt{48}}{2}$

Solution:

a.

$$3+\sqrt{32}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$3+\sqrt{16\cdot 2}$$

Rewrite the radical as the product of two radicals.

$$3+\sqrt{16}\cdot\sqrt{2}$$

Simplify.

$$3+4\sqrt{2}$$

The terms cannot be added as one has a radical and the other does not. Trying to add an integer and a radical is like trying to add an integer and a variable. They are not like terms!

b.

$$\frac{4-\sqrt{48}}{2}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\frac{4-\sqrt{16\cdot 3}}{2}$$

Rewrite the radical as the product of two radicals.

$$\frac{4-\sqrt{16}\cdot\sqrt{3}}{2}$$

Simplify.

$$\frac{4-4\sqrt{3}}{2}$$

Factor the common factor from the numerator.

$$\frac{4(1-\sqrt{3})}{2}$$

Remove the common factor, 2, from the numerator and denominator.

$$\frac{\cancel{2}\cdot 2(1-\sqrt{3})}{\cancel{2}}$$

Simplify.

$$2(1-\sqrt{3})$$

? Try It 4.2.13

Simplify:

a. $5+\sqrt{75}$

b. $\dfrac{10-\sqrt{75}}{5}$

Answer

a. $5+5\sqrt{3}$

b. $2-\sqrt{3}$

? Try It 4.2.14

Simplify:

a. $2+\sqrt{98}$

b. $\dfrac{6-\sqrt{45}}{3}$

Answer

a. $2+7\sqrt{2}$
b. $2-\sqrt{5}$

Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

Example 4.2.8

Simplify:

a. $\sqrt{\dfrac{45}{80}}$
b. $\sqrt[3]{\dfrac{16}{54}}$
c. $\sqrt[4]{\dfrac{5}{80}}$

Solution:

a.

$$\sqrt{\frac{45}{80}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt{\frac{5\cdot 9}{5\cdot 16}}$$

Simplify the fraction by removing common factors.

$$\sqrt{\frac{9}{16}}$$

Simplify. Note $\left(\dfrac{3}{4}\right)^2=\dfrac{9}{16}$.

$$\frac{3}{4}$$

b.

$$\sqrt[3]{\frac{16}{54}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt[3]{\frac{2\cdot 8}{2\cdot 27}}$$

Simplify the fraction by removing common factors.

$$\sqrt[3]{\frac{8}{27}}$$

Simplify. Note $\left(\dfrac{2}{3}\right)^3=\dfrac{8}{27}$.

$$\frac{2}{3}$$

c.

$$\sqrt[4]{\frac{5}{80}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt[4]{\frac{5 \cdot 1}{5 \cdot 16}}$$

Simplify the fraction by removing common factors.

$$\sqrt[4]{\frac{1}{16}}$$

Simplify. Note $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

$$\frac{1}{2}$$

? Try It 4.2.15

Simplify:

a. $\sqrt{\frac{75}{48}}$

b. $\sqrt[3]{\frac{54}{250}}$

c. $\sqrt[4]{\frac{32}{162}}$

Answer

a. $\frac{5}{4}$

b. $\frac{3}{5}$

c. $\frac{2}{3}$

? Try It 4.2.16

Simplify:

a. $\sqrt{\frac{98}{162}}$

b. $\sqrt[3]{\frac{24}{375}}$

c. $\sqrt[4]{\frac{4}{324}}$

Answer

a. $\frac{7}{9}$

b. $\frac{2}{5}$

c. $\frac{1}{3}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the **Quotient Property** to simplify under the radical. We divide the like bases by subtracting their exponents,

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

Example 4.2.9

Simplify:

a. $\sqrt{\frac{m^6}{m^4}}$

b. $\sqrt[3]{\frac{a^8}{a^5}}$

c. $\sqrt[4]{\frac{a^{10}}{a^2}}$

Solution:

a.

$$\sqrt{\frac{m^6}{m^4}}$$

Simplify the fraction inside the radical first. Divide the like bases by subtracting the exponents.

$$\sqrt{m^2}$$

Simplify.

$$|m|$$

b.

$$\sqrt[3]{\frac{a^8}{a^5}}$$

Use the Quotient Property of exponents to simplify the fraction under the radical first.

$$\sqrt[3]{a^3}$$

Simplify.

$$a$$

c.

$$\sqrt[4]{\frac{a^{10}}{a^2}}$$

Use the Quotient Property of exponents to simplify the fraction under the radical first.

$$\sqrt[4]{a^8}$$

Rewrite the radicand using perfect fourth power factors.

$$\sqrt[4]{(a^2)^4}$$

Simplify.

$$a^2$$

Try It 4.2.17

Simplify:

a. $\sqrt{\frac{a^8}{a^6}}$
b. $\sqrt[4]{\frac{x^7}{x^3}}$
c. $\sqrt[4]{\frac{y^{17}}{y^5}}$

Answer

a. $|a|$
b. $|x|$
c. y^3

Try It 4.2.18

Simplify:

a. $\sqrt{\frac{x^{14}}{x^{10}}}$
b. $\sqrt[3]{\frac{m^{13}}{m^7}}$
c. $\sqrt[5]{\frac{n^{12}}{n^2}}$

Answer

a. x^2
b. m^2
c. n^2

Remember the **Quotient to a Power Property**? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

Definition 4.2.3

Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Example 4.2.10 how to simplify the quotient of radical expressions

Simplify: $\sqrt{\frac{27m^3}{196}}$

Solution:

Step 1: Simplify the fraction in the radicand, if possible.

$\frac{27m^3}{196}$ cannot be simplified.

$$\sqrt{\frac{27m^3}{196}}$$

Step 2: Use the Quotient Property to rewrite the radical as the quotient of two radicals.

We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.

$$\frac{\sqrt{27m^3}}{\sqrt{196}}$$

Step 3: Simplify the radicals in the numerator and the denominator.

$9m^2$ and 196 are perfect squares.

$$\frac{\sqrt{9m^2}\cdot\sqrt{3m}}{\sqrt{196}}$$

$$\frac{3m\sqrt{3m}}{14}$$

? Try It 4.2.19

Simplify: $\sqrt{\frac{24p^3}{49}}$.

Answer

$$\frac{2|p|\sqrt{6p}}{7}$$

? Try It 4.2.20

Simplify: $\sqrt{\frac{48x^5}{100}}$.

Answer

$$\frac{2x^2\sqrt{3x}}{5}$$

Simplify a Square Root Using the Quotient Property

1. Simplify the fraction in the radicand, if possible.
2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
3. Simplify the radicals in the numerator and the denominator.

✓ Example 4.2.11

Simplify:

a. $\sqrt{\frac{45x^5}{y^4}}$

b. $\sqrt[3]{\frac{24x^7}{y^3}}$

c. $\sqrt[4]{\frac{48x^{10}}{y^8}}$

Solution:

a.

$$\sqrt{\frac{45x^5}{y^4}}$$

We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.

$$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$$

Simplify.

$$\frac{3x^2\sqrt{5x}}{y^2}$$

b.

$$\sqrt[3]{\frac{24x^7}{y^3}}$$

The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.

$$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$$

Rewrite each radicand as a product using perfect cube factors.

$$\frac{\sqrt[3]{8x^6\cdot 3x}}{\sqrt[3]{y^3}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[3]{(2x^2)^3}\cdot\sqrt[3]{3x}}{\sqrt[3]{y^3}}$$

Simplify.

$$\frac{2x^2\sqrt[3]{3x}}{y}$$

c.

$$\sqrt[4]{\frac{48x^{10}}{y^8}}$$

The fraction in the radicand cannot be simplified.

$$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$$

Use the Quotient Property to write as two radicals. Rewrite each radicand as a product using perfect fourth power factors.

$$\frac{\sqrt[4]{16x^8\cdot 3x^2}}{\sqrt[4]{y^8}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[4]{(2x^2)^4}\cdot\sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$$

Simplify.

$$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$$

Try It 4.2.21

Simplify:

a. $\sqrt{\frac{80m^3}{n^6}}$

b. $\sqrt[3]{\frac{108c^{10}}{d^6}}$

c. $\sqrt[4]{\frac{80x^{10}}{y^4}}$

Answer

a. $\frac{4|m|\sqrt{5m}}{|n^3|}$

b. $\frac{3c^3\sqrt[3]{4c}}{d^2}$

c. $\frac{2x^2\sqrt[4]{5x^2}}{|y|}$

Try It 4.2.22

Simplify:

a. $\sqrt{\frac{54u^7}{v^8}}$

b. $\sqrt[3]{\frac{40r^3}{s^6}}$

c. $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$

Answer

a. $\frac{3u^3\sqrt{6u}}{v^4}$

b. $\frac{2r\sqrt[3]{5}}{s^2}$

c. $\frac{3\left|m^3\right|\sqrt[4]{2m^2}}{|n^3|}$

Be sure to simplify the fraction in the radicand first, if possible.

Example 4.2.12

Simplify:

a. $\sqrt{\frac{18p^5q^7}{32pq^2}}$

b. $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$

c. $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

Solution:

a.

$$\sqrt{\frac{18p^5q^7}{32pq^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt{\frac{9p^4q^5}{16}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$$

Simplify.

$$\frac{3p^2q^2\sqrt{q}}{4}$$

b.

$$\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt[3]{\frac{8x^3y^5}{27}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt[3]{8x^3y^5}}{\sqrt[3]{27}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt[3]{8x^3y^3}\cdot\sqrt[3]{y^2}}{\sqrt[3]{27}}$$

Simplify.

$$\frac{2xy\sqrt[3]{y^2}}{3}$$

c.

$$\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt[4]{\frac{a^5b^4}{16}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt[4]{a^5b^4}}{\sqrt[4]{16}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt[4]{a^4b^4}\cdot\sqrt[4]{a}}{\sqrt[4]{16}}$$

Simplify.

$$\frac{|ab|\sqrt[4]{a}}{2}$$

? Try It 4.2.23

Simplify:

a. $\sqrt{\frac{50x^5y^3}{72x^4y}}$

b. $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$

c. $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

Answer

a. $\frac{5|y|\sqrt{x}}{6}$

b. $\frac{2xy\sqrt[3]{y^2}}{3}$

c. $\frac{|ab|\sqrt[4]{a}}{2}$

? Try It 4.2.24

Simplify:

a. $\sqrt{\frac{48m^7n^2}{100m^5n^8}}$

b. $\sqrt[3]{\frac{54x^7y^5}{250x^2y^2}}$

c. $\sqrt[4]{\frac{32a^9b^7}{162a^3b^3}}$

Answer

a. $\frac{2|m|\sqrt{3}}{5\,|n^3|}$

b. $\frac{3xy\sqrt[3]{x^2}}{5}$

c. $\frac{2|ab|\sqrt[4]{a^2}}{3}$

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the **Quotient Property** again, to combine them into one radical. We will then look to see if we can simplify the expression.

Example 4.2.13

Simplify:

a. $\dfrac{\sqrt{48a^7}}{\sqrt{3a}}$

b. $\dfrac{\sqrt[3]{-108}}{\sqrt[3]{2}}$

c. $\dfrac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$

Solution:

a.

$$\frac{\sqrt{48a^7}}{\sqrt{3a}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt{\frac{48a^7}{3a}}$$

Simplify the fraction under the radical.

$$\sqrt{16a^6}$$

Simplify.

$$4\left|a^3\right|$$

b.

$$\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt[3]{\frac{-108}{2}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{-54}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{(-3)^3\cdot 2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{(-3)^3}\cdot\sqrt[3]{2}$$

Simplify.

$$-3\sqrt[3]{2}$$

c.

$$\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt[4]{\frac{96x^7}{3x^2}}$$

Simplify the fraction under the radical.

$$\sqrt[4]{32x^5}$$

Rewrite the radicand as a product using perfect fourth power factors.

$$\sqrt[4]{16x^4} \cdot \sqrt[4]{2x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$$

Simplify.

$$2|x|\sqrt[4]{2x}$$

? Try It 4.2.25

Simplify:

a. $\dfrac{\sqrt{98z^5}}{\sqrt{2z}}$

b. $\dfrac{\sqrt[3]{-500}}{\sqrt[3]{2}}$

c. $\dfrac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$

Answer

a. $7z^2$

b. $-5\sqrt[3]{2}$

c. $3|m|\sqrt[4]{2m^2}$

? Try It 4.2.26

Simplify:

a. $\dfrac{\sqrt{128m^9}}{\sqrt{2m}}$

b. $\dfrac{\sqrt[3]{-192}}{\sqrt[3]{3}}$

c. $\dfrac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$

Answer

a. $8m^4$

b. -4

c. $3|n|\sqrt[4]{2}$

Access these online resources for additional instruction and practice with simplifying radical expressions.

- Simplifying Square Root and Cube Root with Variables
- Express a Radical in Simplified Form-Square and Cube Roots with Variables and Exponents
- Simplifying Cube Roots

Key Concepts

- **Simplified Radical Expression**
  - For real numbers a, m and $n \geq 2$
    $\sqrt[n]{a}$ is considered simplified if a has no factors of m^n
- **Product Property of nth Roots**

- For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$
  $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

- **How to simplify a radical expression using the Product Property**
  1. Find the largest factor in the radicand that is a perfect power of the index.
     Rewrite the radicand as a product of two factors, using that factor.
  2. Use the product rule to rewrite the radical as the product of two radicals.
  3. Simplify the root of the perfect power.
- **Quotient Property of Radical Expressions**
  - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- **How to simplify a radical expression using the Quotient Property.**
  1. Simplify the fraction in the radicand, if possible.
  2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
  3. Simplify the radicals in the numerator and the denominator.

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4.2E: Exercises

Practice Makes Perfect

Exercise SET A: use the product property to simplify radical expressions

In the following exercises, use the Product Property to simplify radical expressions.

1. $\sqrt{27}$
2. $\sqrt{80}$
3. $\sqrt{125}$
4. $\sqrt{96}$
5. $\sqrt{147}$
6. $\sqrt{450}$
7. $\sqrt{800}$
8. $\sqrt{675}$
9. a. $\sqrt[4]{32}$
   b. $\sqrt[5]{64}$
10. a. $\sqrt[3]{625}$
    b. $\sqrt[6]{128}$
11. a. $\sqrt[5]{64}$
    b. $\sqrt[3]{256}$
12. a. $\sqrt[4]{3125}$
    b. $\sqrt[3]{81}$

Answer

1. $3\sqrt{3}$
3. $5\sqrt{5}$
5. $7\sqrt{3}$
7. $20\sqrt{2}$
9. a. $2\sqrt[4]{2}$
   b. $2\sqrt[5]{2}$
11. a. $2\sqrt[5]{2}$
    b. $4\sqrt[3]{4}$

Exercise SET B: use the product property to simplify radical expressions

In the following exercises, simplify using absolute value signs as needed.

1. a. $\sqrt{y^{11}}$
   b. $\sqrt[3]{r^5}$
   c. $\sqrt[4]{s^{10}}$
2. a. $\sqrt{m^{13}}$
   b. $\sqrt[5]{u^7}$
   c. $\sqrt[6]{v^{11}}$
3. a. $\sqrt{n^{21}}$
   b. $\sqrt[3]{q^8}$

c. $\sqrt[8]{n^{10}}$

4. a. $\sqrt{r^{25}}$
   b. $\sqrt[5]{p^8}$
   c. $\sqrt[4]{m^5}$
5. a. $\sqrt{125r^{13}}$
   b. $\sqrt[3]{108x^5}$
   c. $\sqrt[4]{48y^6}$
6. a. $\sqrt{80s^{15}}$
   b. $\sqrt[5]{96a^7}$
   c. $\sqrt[6]{128b^7}$
7. a. $\sqrt{242m^{23}}$
   b. $\sqrt[4]{405m10}$
   c. $\sqrt[5]{160n^8}$
8. a. $\sqrt{175n^{13}}$
   b. $\sqrt[5]{512p^5}$
   c. $\sqrt[4]{324q^7}$
9. a. $\sqrt{147m^7n^{11}}$
   b. $\sqrt[3]{48x^6y^7}$
   c. $\sqrt[4]{32x^5y^4}$
10. a. $\sqrt{96r^3s^3}$
    b. $\sqrt[3]{80x^7y^6}$
    c. $\sqrt[4]{80x^8y^9}$
11. a. $\sqrt{192q^3r^7}$
    b. $\sqrt[3]{54m^9n^{10}}$
    c. $\sqrt[4]{81a^9b^8}$
12. a. $\sqrt{150m^9n^3}$
    b. $\sqrt[3]{81p^7q^8}$
    c. $\sqrt[4]{162c^{11}d^{12}}$
13. a. $\sqrt[3]{-864}$
    b. $\sqrt[4]{-256}$
14. a. $\sqrt[5]{-486}$
    b. $\sqrt[6]{-64}$
15. a. $\sqrt[5]{-32}$
    b. $\sqrt[8]{-1}$
16. a. $\sqrt[3]{-8}$
    b. $\sqrt[4]{-16}$
17. a. $5+\sqrt{12}$
    b. $\dfrac{10-\sqrt{24}}{2}$
18. a. $8+\sqrt{96}$
    b. $\dfrac{8-\sqrt{80}}{4}$
19. a. $1+\sqrt{45}$
    b. $\dfrac{3+\sqrt{90}}{3}$
20. a. $3+\sqrt{125}$
    b. $\dfrac{15+\sqrt{75}}{5}$

Answer

1.

a. $|y^5|\sqrt{y}$
b. $r\sqrt[3]{r^2}$
c. $s^2\sqrt[4]{s^2}$

3.

a. $n^{10}\sqrt{n}$
b. $q^2\sqrt[3]{q^2}$
c. $|n|\sqrt[8]{n^2}$

5.

a. $5r^6\sqrt{5r}$
b. $3x\sqrt[3]{4x^2}$
c. $2|y|\sqrt[4]{3y^2}$

7.

a. $11\,|m^{11}|\sqrt{2m}$
b. $3m^2\sqrt[4]{5m^2}$
c. $2n\sqrt[5]{5n^3}$

9.

a. $7\,|m^3n^5|\sqrt{3mn}$
b. $2x^2y^2\sqrt[3]{6y}$
c. $2|xy|\sqrt[4]{2x}$

11.

a. $8\,|qr^3|\sqrt{3qr}$
b. $3m^3n^3\sqrt[3]{2n}$
c. $3a^2b^2\sqrt[4]{a}$

13.

a. $-6\sqrt[3]{4}$
b. not real

15.

a. -2
b. not real

17.

a. $5+2\sqrt{3}$
b. $5-\sqrt{6}$

19.

a. $1+3\sqrt{5}$
b. $1+\sqrt{10}$

Exercise Set C: use the quotient property to simplify radical expressions

In the following exercises, use the Quotient Property to simplify square roots.

1. a. $\sqrt{\frac{45}{80}}$

b. $\sqrt{\frac{8}{27}}$

c. $\sqrt{\frac{1}{81}}$

2. a. $\sqrt{\frac{72}{98}}$

b. $\sqrt{\frac{24}{81}}$

c. $\sqrt{\frac{6}{96}}$

3. a. $\sqrt{\frac{100}{36}}$

b. $\sqrt{\frac{81}{375}}$

c. $\sqrt{\frac{1}{256}}$

4. a. $\sqrt{\frac{121}{16}}$

b. $\sqrt{\frac{16}{250}}$

c. $\sqrt{\frac{32}{162}}$

5. a. $\sqrt{\frac{x^{10}}{x^6}}$

b. $\sqrt{\frac{p^{11}}{p^2}}$

c. $\sqrt{\frac{q^{17}}{q^{13}}}$

6. a. $\sqrt{\frac{p^{20}}{p^{10}}}$

b. $\sqrt{\frac{d^{12}}{d^7}}$

c. $\sqrt{\frac{m^{12}}{m^4}}$

7. a. $\sqrt{\frac{y^4}{y^8}}$

b. $\sqrt{\frac{u^{21}}{u^{11}}}$

c. $\sqrt{\frac{v^{30}}{v^{12}}}$

8. a. $\sqrt{\frac{q^8}{q^{14}}}$

b. $\sqrt{\frac{r^{14}}{r^5}}$

c. $\sqrt{\frac{c^{21}}{c^9}}$

9. $\sqrt{\frac{96x^7}{121}}$

10. $\sqrt{\frac{108y^4}{49}}$
11. $\sqrt{\frac{300m^5}{64}}$
12. $\sqrt{\frac{125n^7}{169}}$
13. $\sqrt{\frac{98r^5}{100}}$
14. $\sqrt{\frac{180s^{10}}{144}}$
15. $\sqrt{\frac{28q^6}{225}}$
16. $\sqrt{\frac{150r^3}{256}}$
17. a. $\sqrt{\frac{75r^9}{s^8}}$
    b. $\sqrt[3]{\frac{54a^8}{b^3}}$
    c. $\sqrt[4]{\frac{64c^5}{d^4}}$
18. a. $\sqrt{\frac{72x^5}{y^6}}$
    b. $\sqrt[5]{\frac{96r^{11}}{s^5}}$
    c. $\sqrt[6]{\frac{128u^7}{v^{12}}}$
19. a. $\sqrt{\frac{28p^7}{q^2}}$
    b. $\sqrt[3]{\frac{81s^8}{t^3}}$
    c. $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$
20. a. $\sqrt{\frac{45r^3}{s^{10}}}$
    b. $\sqrt[3]{\frac{625u^{10}}{v^3}}$
    c. $\sqrt[4]{\frac{729c^{21}}{d^8}}$
21. a. $\sqrt{\frac{32x^5y^3}{18x^3y}}$
    b. $\sqrt[3]{\frac{5x^6y^9}{40x^5y^3}}$
    c. $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$
22. a. $\sqrt{\frac{75r^6s^8}{48rs^4}}$
    b. $\sqrt[3]{\frac{24x^8y^4}{81x^2y}}$

c. $\sqrt[4]{\frac{32m^9n^2}{162mn^2}}$

23. a. $\sqrt{\frac{27p^2q}{108p^4q^3}}$

b. $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$

c. $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

24. a. $\sqrt{\frac{50r^5s^2}{128r^2s^6}}$

b. $\sqrt[3]{\frac{24m^9n^7}{375m^4n}}$

c. $\sqrt[4]{\frac{81m^2n^8}{256m^1n^2}}$

25. a. $\frac{\sqrt{45p^9}}{\sqrt{5q^2}}$

b. $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$

c. $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

26. a. $\frac{\sqrt{80q^5}}{\sqrt{5q}}$

b. $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

c. $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

27. a. $\frac{\sqrt{50m^7}}{\sqrt{2m}}$

b. $\sqrt[3]{\frac{1250}{2}}$

c. $\sqrt[4]{\frac{486y^9}{2y^3}}$

28. a. $\frac{\sqrt{72n^{11}}}{\sqrt{2n}}$

b. $\sqrt[3]{\frac{162}{6}}$

c. $\sqrt[4]{\frac{160r^{10}}{5r^3}}$

Answer

1.

a. $\frac{3}{4}$

b. $\frac{2}{3}$

c. $\frac{1}{3}$

3.

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a. $\frac{5}{3}$
b. $\frac{3}{5}$
c. $\frac{1}{4}$

5.

a. x^2
b. p^3
c. $|q|$

7.

a. $\frac{1}{y^2}$
b. u^2
c. $|v^3|$

9. $\frac{4\left|x^3\right|\sqrt{6x}}{11}$

11. $\frac{10m^2\sqrt{3m}}{8}$

13. $\frac{7r^2\sqrt{2r}}{10}$

15. $\frac{2\left|q^3\right|\sqrt{7}}{15}$

17.

a. $\frac{5r^4\sqrt{3r}}{s^4}$
b. $\frac{3a^2\sqrt[3]{2a^2}}{|b|}$
c. $\frac{2|c|\sqrt[4]{4c}}{|d|}$

19.

a. $\frac{2\left|p^3\right|\sqrt{7p}}{|q|}$
b. $\frac{3s^2\sqrt[3]{3s^2}}{t}$
c. $\frac{2\left|p^3\right|\sqrt[4]{4p^3}}{|q^3|}$

21.

a. $\frac{4|xy|}{3}$
b. $\frac{y^2\sqrt[3]{x}}{2}$
c. $\frac{|ab|\sqrt[4]{a}}{4}$

23.

a. $\frac{1}{2|pq|}$

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b. $\frac{2cd\sqrt[5]{2d^2}}{5}$

c. $\frac{|mn|\sqrt[6]{2}}{2}$

25.

a. $\frac{3p^4\sqrt{p}}{|q|}$

b. $2\sqrt[4]{2}$

c. $2x\sqrt[5]{2x}$

27.

a. $5\left|m^3\right|$

b. $5\sqrt[3]{5}$

c. $3|y|\sqrt[4]{3y^2}$

Exercise SET D: writing exercises

1. Explain why $\sqrt{x^4}=x^2$. Then explain why $\sqrt{x^{16}}=x^8$.
2. Explain why $7+\sqrt{9}$ is not equal to $\sqrt{7+9}$.
3. Explain how you know that $\sqrt[5]{x^{10}}=x^2$.
4. Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Answer

1. Answers may vary

3. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify radical expressions.			
use the Quotient Property to simplify radical expressions.			

Figure 8.2.1

b. After reviewing this checklist, what will you do to become confident for all objectives?

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4.3: Simplify Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15}+\frac{5}{12}$.
   If you missed this problem, review Example 1.28.
2. Simplify: $(4x^2y^5)^3$.
   If you missed this problem, review Example 5.18.
3. Simplify: 5^{-3}.
   If you missed this problem, review Example 5.14.

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n=a^{m\cdot n}$ when m and n are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number p such that $(8^p)^3=8$. We will use the Power Property of Exponents to find the value of p.

$$(8^p)^3=8$$

Multiple the exponents on the left.

$$8^{3p}=8$$

Write the exponent 1 on the right.

$$8^{3p}=8^1$$

Since the bases are the same, the exponents must be equal.

$$3p=1$$

Solve for p.

$$p=\frac{1}{3}$$

So $\left(8^{\frac{1}{3}}\right)^3=8$. But we know also $(\sqrt[3]{8})^3=8$. Then it must be that $8^{\frac{1}{3}}=\sqrt[3]{8}$.

This same logic can be used for any positive integer exponent n to show that $a^{\frac{1}{n}}=\sqrt[n]{a}$.

Definition 4.3.1: Rational Exponent $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n\geq 2$, then

$$a^{\frac{1}{n}}=\sqrt[n]{a}$$

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

Example 4.3.1

Write as a radical expression:

a. $x^{\frac{1}{2}}$
b. $y^{\frac{1}{3}}$
c. $z^{\frac{1}{4}}$

Solution:

We want to write each expression in the form $\sqrt[n]{a}$.

a.

$$x^{\frac{1}{2}}$$

The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.

$$\sqrt{x}$$

b.

$$y^{\frac{1}{3}}$$

The denominator of the exponent is 3, so the index is 3.

$$\sqrt[3]{y}$$

c.

$$z^{\frac{1}{4}}$$

The denominator of the exponent is \4, so the index is 4.

$$\sqrt[4]{z}$$

Exercise 4.3.1

Write as a radical expression:

a. $t^{\frac{1}{2}}$
b. $m^{\frac{1}{3}}$
c. $r^{\frac{1}{4}}$

Answer

a. $\sqrt{t}$
b. $\sqrt[3]{m}$
c. $\sqrt[4]{r}$

Exercise 4.3.2

Write as a radical expression:

a. $b^{\frac{1}{6}}$
b. $z^{\frac{1}{5}}$
c. $p^{\frac{1}{4}}$

Answer

a. $\sqrt[6]{b}$
b. $\sqrt[5]{z}$
c. $\sqrt[4]{p}$

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

Example 4.3.2

Write with a rational exponent:

a. $\sqrt{5y}$
b. $\sqrt[3]{4x}$
c. $3\sqrt[4]{5z}$

Solution:

We want to write each radical in the form $a^{\frac{1}{n}}$

a.

$$\sqrt{5y}$$

No index is shown, so it is 2.

The denominator of the exponent will be 2.

Put parentheses around the entire expression $5y$.

$$(5y)^{\frac{1}{2}}$$

b.

$$\sqrt[3]{4x}$$

The index is 3, so the denominator of the exponent is 3. Include parentheses $(4x)$.

$$(4x)^{\frac{1}{3}}$$

c.

$$3\sqrt[4]{5z}$$

The index is 4, so the denominator of the exponent is 4. Put parentheses only around the $5z$ since 3 is not under the radical sign.

$$3(5z)^{\frac{1}{4}}$$

Exercise 4.3.3

Write with a rational exponent:

a. $\sqrt{10m}$
b. $\sqrt[5]{3n}$
c. $3\sqrt[4]{6y}$

Answer

a. $(10m)^{\frac{1}{2}}$
b. $(3n)^{\frac{1}{5}}$
c. $3(6y)^{\frac{1}{4}}$

Exercise 4.3.4

Write with a rational exponent:

a. $\sqrt[7]{3k}$
b. $\sqrt[4]{5j}$
c. $8\sqrt[3]{2a}$

Answer

a. $(3k)^{\frac{1}{7}}$
b. $(5j)^{\frac{1}{4}}$
c. $8(2a)^{\frac{1}{3}}$

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

Example 4.3.3

Simplify:

a. $25^{\frac{1}{2}}$
b. $64^{\frac{1}{3}}$
c. $256^{\frac{1}{4}}$

Solution:

a.

$$25^{\frac{1}{2}}$$

Rewrite as a square root.

$$\sqrt{25}$$

Simplify.

$$5$$

b.

$$64^{\frac{1}{3}}$$

Rewrite as a cube root.

$$\sqrt[3]{64}$$

Recognize 64 is a perfect cube.

$$\sqrt[3]{4^3}$$

Simplify.

$$4$$

c.

$$256^{\frac{1}{4}}$$

Rewrite as a fourth root.

$$\sqrt[4]{256}$$

Recognize 256 is a perfect fourth power.

$$\sqrt[4]{4^4}$$

Simplify.

4

Exercise 4.3.5

Simplify:

a. $36^{\frac{1}{2}}$
b. $8^{\frac{1}{3}}$
c. $16^{\frac{1}{4}}$

Answer

a. 6
b. 2
c. 2

Exercise 4.3.6

Simplify:

a. $100^{\frac{1}{2}}$
b. $27^{\frac{1}{3}}$
c. $81^{\frac{1}{4}}$

Answer

a. 10
b. 3
c. 3

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

Example 4.3.4

Simplify:

a. $(-16)^{\frac{1}{4}}$
b. $-16^{\frac{1}{4}}$
c. $(16)^{-\frac{1}{4}}$

Solution:

a.

$$(-16)^{\frac{1}{4}}$$

Rewrite as a fourth root.

$$\sqrt[4]{-16}$$
$$\sqrt[4]{(-2)^4}$$

Simplify.

No real solution

b.

$$-16^{\frac{1}{4}}$$

The exponent only applies to the 16. Rewrite as a fourth root.

$$-\sqrt[4]{16}$$

Rewrite 16 as 2^4

$$-\sqrt[4]{2^4}$$

Simplify.

$$-2$$

c.

$$(16)^{-\frac{1}{4}}$$

Rewrite using the property $a^{-n}=\frac{1}{a^n}$.

$$\frac{1}{(16)^{\frac{1}{4}}}$$

Rewrite as a fourth root.

$$\frac{1}{\sqrt[4]{16}}$$

Rewrite 16 as 2^4.

$$\frac{1}{\sqrt[4]{2^4}}$$

Simplify.

$$\frac{1}{2}$$

Exercise 4.3.7

Simplify:

a. $(-64)^{-\frac{1}{2}}$
b. $-64^{\frac{1}{2}}$
c. $(64)^{-\frac{1}{2}}$

Answer

a. No real solution
b. -8
c. $\frac{1}{8}$

Exercise 4.3.8

Simplify:

a. $(-256)^{\frac{1}{4}}$
b. $-256^{\frac{1}{4}}$
c. $(256)^{-\frac{1}{4}}$

Answer

a. No real solution
b. -4
c. $\frac{1}{4}$

Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and $(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

This leads us to the following defintion.

Definition 4.3.2: Rational Exponent $a^{\frac{m}{n}}$

For any positive integers m and n,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

Example 4.3.5

Write with a rational exponent:

a. $\sqrt{y^3}$
b. $(\sqrt[3]{2x})^4$
c. $\sqrt{\left(\frac{3a}{4b}\right)^3}$

Solution:

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$

a.

Let

$$\sqrt[2]{y^3} \tag{4.3.1}$$

The numerator of the exponent is the exponent 3.
The denominator of the exponent is the index of the radical, 2.

$$y^{\frac{3}{2}} \tag{4.3.2}$$

b.

Let

$$(\sqrt[3]{2x})^4 \tag{4.3.3}$$

The numerator of the exponent is the exponent 4.
The denominator of the exponent is the index of the radical, 3.

$$(2x)^{\frac{4}{3}} \tag{4.3.4}$$

c.

Let

$$\sqrt{\left(\frac{3a}{4b}\right)^3} \tag{4.3.5}$$

The numerator of the exponent is the exponent 3.
The denominator of the exponent is the index of the radical, 2.

$$\left(\frac{3a}{4b}\right)^{\frac{3}{2}} \tag{4.3.6}$$

Exercise 4.3.9

Write with a rational exponent:

a. $\sqrt{x^5}$
b. $(\sqrt[4]{3y})^3$
c. $\sqrt{\left(\frac{2m}{3n}\right)^5}$

Answer

a. $x^{\frac{5}{2}}$
b. $(3y)^{\frac{3}{4}}$
c. $\left(\frac{2m}{3n}\right)^{\frac{5}{2}}$

Exercise 4.3.10

Write with a rational exponent:

a. $\sqrt[5]{a^2}$
b. $(\sqrt[3]{5ab})^5$
c. $\sqrt{\left(\frac{7xy}{z}\right)^3}$

Answer

a. $a^{\frac{2}{5}}$
b. $(5ab)^{\frac{5}{3}}$
c. $\left(\frac{7xy}{z}\right)^{\frac{3}{2}}$

Remember that $a^{-n} = \frac{1}{a^n}$. The negative sign in the exponent does not change the sign of the expression.

Example 4.3.6

Simplify:

a. $125^{\frac{2}{3}}$
b. $16^{-\frac{3}{2}}$
c. $32^{-\frac{2}{5}}$

Solution:

We will rewrite the expression as a radical first using the defintion, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

a.

$$125^{\frac{2}{3}}$$

The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.

$$(\sqrt[3]{125})^2$$

Simplify.

$$(5)^2$$

$$25$$

b. We will rewrite each expression first using $a^{-n} = \frac{1}{a^n}$ and then change to radical form.

$$16^{-\frac{3}{2}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{16^{\frac{3}{2}}}$$

Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.

$$\frac{1}{(\sqrt{16})^3}$$

Simplify.

$$\frac{1}{4^3}$$

$$\frac{1}{64}$$

c.

$$32^{-\frac{2}{5}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{32^{\frac{2}{5}}}$$

Change to radical form.

$$\frac{1}{(\sqrt[5]{32})^2}$$

Rewrite the radicand as a power.

$$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$$

Simplify.

$$\frac{1}{2^2}$$

$$\frac{1}{4}$$

? Exercise 4.3.11

Simplify:

a. $27^{\frac{2}{3}}$
b. $81^{-\frac{3}{2}}$
c. $16^{-\frac{3}{4}}$

Answer

a. 9
b. $\frac{1}{729}$
c. $\frac{1}{8}$

Exercise 4.3.12

Simplify:

a. $4^{\frac{3}{2}}$

b. $27^{-\frac{2}{3}}$

c. $625^{-\frac{3}{4}}$

Answer

a. 8

b. $\frac{1}{9}$

c. $\frac{1}{125}$

Example 4.3.7

Simplify:

a. $-25^{\frac{3}{2}}$

b. $-25^{-\frac{3}{2}}$

c. $(-25)^{\frac{3}{2}}$

Solution:

a.

$$-25^{\frac{3}{2}}$$

Rewrite in radical form.

$$-(\sqrt{25})^3$$

Simplify the radical.

$$-(5)^3$$

Simplify.

$$-125$$

b.

$$-25^{-\frac{3}{2}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$.

$$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$$

Rewrite in radical form.

$$-\left(\frac{1}{(\sqrt{25})^3}\right)$$

Simplify the radical.

$$-\left(\frac{1}{(5)^3}\right)$$

Simplify.

$$-\frac{1}{125}$$

c.

$$(-25)^{\frac{3}{2}}$$

Rewrite in radical form.

$$(\sqrt{-25})^3$$

There is no real number whose square root is -25.

Not a real number.

Exercise 4.3.13

Simplify:

a. $-16^{\frac{3}{2}}$
b. $-16^{-\frac{3}{2}}$
c. $(-16)^{-\frac{3}{2}}$

Answer

a. -64
b. $-\frac{1}{64}$
c. Not a real number

Exercise 4.3.14

Simplify:

a. $-81^{\frac{3}{2}}$
b. $-81^{-\frac{3}{2}}$
c. $(-81)^{-\frac{3}{2}}$

Answer

a. -729
b. $-\frac{1}{729}$
c. Not a real number

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponents here to have them for reference as we simplify expressions.

Properties of Exponents

If a and b are real numbers and m and n are rational numbers, then

Product Property

$$a^m \cdot a^n = a^{m+n}$$

Power Property

$$(a^m)^n = a^{m \cdot n}$$

Product to a Power

$$(ab)^m = a^m b^m$$

Quotient Property

$$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$$

Zero Exponent Definition

$$a^0 = 1, a \neq 0$$

Quotient to a Power Property

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

Negative Exponent Property

$$a^{-n} = \frac{1}{a^n}, a \neq 0$$

We will apply these properties in the next example.

Example 4.3.8

Simplify:

a. $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$

b. $\left(z^9\right)^{\frac{2}{3}}$

c. $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

Solution

a. The Product Property tells us that when we multiple the same base, we add the exponents.

$$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$$

The bases are the same, so we add the exponents.

$$x^{\frac{1}{2}+\frac{5}{6}}$$

Add the fractions.

$$x^{\frac{8}{6}}$$

Simplify the exponent.

$$x^{\frac{4}{3}}$$

b. The Power Property tells us that when we raise a power to a power, we multiple the exponents.

$$\left(z^9\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiple the exponents.

$$z^{9 \cdot \frac{2}{3}}$$

Simplify.

$$z^6$$

c. The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

$$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$$

To divide with the same base, we subtract the exponents.

$$\frac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$$

Simplify.

$$\frac{1}{x^{\frac{4}{3}}}$$

Exercise 4.3.15

Simplify:

a. $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$

b. $\left(x^{6}\right)^{\frac{4}{3}}$

c. $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$

Answer

a. $x^{\frac{3}{2}}$

b. x^{8}

c. $\frac{1}{x}$

Exercise 4.3.16

Simplify:

a. $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$

b. $\left(m^{9}\right)^{\frac{2}{9}}$

c. $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$

Answer

a. $y^{\frac{11}{8}}$

b. m^{2}

c. $\frac{1}{d}$

Sometimes we need to use more than one property. In the next example, we will use both the **Product to a Power Property** and then the **Power Property**.

Example 4.3.9

Simplify:

a. $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$

b. $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$

Solution:

a.

$$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

First we use the Product to a Power Property.

$$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

Rewrite 27 as a power of 3.

$$\left(3^{3}\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiple the exponents.

$$\left(3^2\right)\left(u^{\frac{1}{3}}\right)$$

Simplify.

$$9u^{\frac{1}{3}}$$

b.

$$\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

First we use the Product to a Power Property.

$$\left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

To raise a power to a power, we multiply the exponents.

$$mn^{\frac{3}{4}}$$

Exercise 4.3.17

Simplify:

a. $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$

b. $\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)^{\frac{2}{3}}$

Answer

a. $8x^{\frac{1}{5}}$

b. $x^{\frac{1}{2}}y^{\frac{1}{3}}$

Exercise 4.3.18

Simplify:

a. $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$

b. $\left(a^{\frac{3}{2}}b^{\frac{1}{2}}\right)^{\frac{4}{3}}$

Answer

a. $729n^{\frac{3}{5}}$

b. $a^2b^{\frac{2}{3}}$

We will use both the **Product Property** and the **Quotient Property** in the next example.

Example 4.3.10

Simplify:

a. $\frac{x^{\frac{3}{4}}\cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$

b. $\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$

Solution:

a.

$$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$$

Use the Product Property in the numerator, add the exponents.

$$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$$

Use the Quotient Property, subtract the exponents.

$$x^{\frac{8}{4}}$$

Simplify.

$$x^2$$

b.

$$\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$$

Use the Quotient Property, subtract the exponents.

$$\left(\frac{16x^{\frac{6}{3}}}{y^{\frac{6}{6}}}\right)^{\frac{1}{2}}$$

Simplify.

$$\left(\frac{16x^2}{y}\right)^{\frac{1}{2}}$$

Use the Product to a Power Property, multiply the exponents.

$$\frac{4x}{y^{\frac{1}{2}}}$$

Exercise 4.3.19

Simplify:

a. $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$

b. $\left(\frac{25m^{\frac{1}{6}}n^{\frac{11}{6}}}{m^{\frac{2}{3}}n^{-\frac{1}{6}}}\right)^{\frac{1}{2}}$

Answer

a. m^2

b. $\frac{5n}{m^{\frac{1}{4}}}$

Exercise 4.3.20

Simplify:

a. $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$

b. $\left(\frac{27x^{\frac{4}{5}}y^{\frac{1}{6}}}{x^{\frac{1}{5}}y^{-\frac{5}{6}}}\right)^{\frac{1}{3}}$

Answer

a. u^3

b. $3x^{\frac{1}{5}}y^{\frac{1}{3}}$

Access these online resources for additional instruction and practice with simplifying rational exponents.

- Review-Rational Exponents
- Using Laws of Exponents on Radicals: Properties of Rational Exponents

Key Concepts

- **Rational Exponent $a^{\frac{1}{n}}$**
  - If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- **Rational Exponent $a^{\frac{m}{n}}$**
  - For any positive integers m and n,
    $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- **Properties of Exponents**
  - If a, b are real numbers and m, n are rational numbers, then
    - **Product Property** $a^m \cdot a^n = a^{m+n}$
    - **Power Property** $(a^m)^n = a^{m \cdot n}$
    - **Product to a Power** $(ab)^m = a^m b^m$
    - **Quotient Property** $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
    - **Zero Exponent Definition** $a^0 = 1, a \neq 0$
    - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
    - **Negative Exponent Property** $a^{-n} = \frac{1}{a^n}, a \neq 0$

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4.3E: Exercises

Practice Makes Perfect

Exercise SET A: simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

1. a. $x^{\frac{1}{2}}$ b. $y^{\frac{1}{3}}$ c. $z^{\frac{1}{4}}$
2. a. $r^{\frac{1}{2}}$ b. $s^{\frac{1}{3}}$ c. $t^{\frac{1}{4}}$
3. a. $u^{\frac{1}{5}}$ b. $v^{\frac{1}{9}}$ c. $w^{\frac{1}{20}}$
4. a. $g^{\frac{1}{7}}$ b. $h^{\frac{1}{5}}$ c. $j^{\frac{1}{25}}$

Answer

1. a. $\sqrt{x}$ b. $\sqrt[3]{y}$ c. $\sqrt[4]{z}$
3. a. $\sqrt[5]{u}$ b. $\sqrt[9]{v}$ c. $\sqrt[20]{w}$

Exercise Set B: simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt[7]{x}$ b. $\sqrt[9]{y}$ c. $\sqrt[5]{f}$
2. a. $\sqrt[8]{4}$ b. $\sqrt[10]{s}$ c. $\sqrt[4]{t}$
3. a. $\sqrt[3]{7c}$ b. $\sqrt[7]{12d}$ c. $2\sqrt[4]{6b}$
4. a. $\sqrt[4]{5x}$ b. $\sqrt[8]{9y}$ c. $7\sqrt[5]{3z}$
5. a. $\sqrt{21p}$ b. $\sqrt[4]{8q}$ c. $4\sqrt[6]{36r}$
6. a. $\sqrt[3]{25a}$ b. $\sqrt{3b}$ c. $\sqrt[8]{40c}$

Answer

1. a. $x^{\frac{1}{7}}$ b. $y^{\frac{1}{9}}$ c. $f^{\frac{1}{5}}$
3. a. $(7c)^{\frac{1}{4}}$ b. $(12d)^{\frac{1}{7}}$ c. $2(6b)^{\frac{1}{4}}$
5. a. $(21p)^{\frac{1}{2}}$ b. $(8q)^{\frac{1}{4}}$ c. $4(36r)^{\frac{1}{6}}$

Exercise SET C: simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, simplify.

1. a. $81^{\frac{1}{2}}$ b. $125^{\frac{1}{3}}$ c. $64^{\frac{1}{2}}$
2. a. $625^{\frac{1}{4}}$ b. $243^{\frac{1}{5}}$ c. $32^{\frac{1}{5}}$
3. a. $16^{\frac{1}{4}}$ b. $16^{\frac{1}{2}}$ c. $625^{\frac{1}{4}}$
4. a. $64^{\frac{1}{3}}$ b. $32^{\frac{1}{5}}$ c. $81^{\frac{1}{4}}$
5. a. $(-216)^{\frac{1}{3}}$ b. $-216^{\frac{1}{3}}$ c. $(216)^{-\frac{1}{3}}$
6. a. $(-1000)^{\frac{1}{3}}$ b. $-1000^{\frac{1}{3}}$ c. $(1000)^{-\frac{1}{3}}$
7. a. $(-81)^{\frac{1}{4}}$ b. $-81^{\frac{1}{4}}$ c. $(81)^{-\frac{1}{4}}$
8. a. $(-49)^{\frac{1}{2}}$ b. $-49^{\frac{1}{2}}$ c. $(49)^{-\frac{1}{2}}$
9. a. $(-36)^{\frac{1}{2}}$ b. $-36^{\frac{1}{2}}$ c. $(36)^{-\frac{1}{2}}$
10. a. $(-16)^{\frac{1}{4}}$ b. $-16^{\frac{1}{4}}$ c. $16^{-\frac{1}{4}}$
11. a. $(-100)^{\frac{1}{2}}$ b. $-100^{\frac{1}{2}}$ c. $(100)^{-\frac{1}{2}}$

12. a. $(-32)^{\frac{1}{5}}$ b. $(243)^{-\frac{1}{5}}$ c. $-125^{\frac{1}{3}}$

Answer

1. a. 9 b. 5 c. 8
3. a. 2 b. 4 c. 5
5. a. -6 b. -6 c. $\frac{1}{6}$
7. a. not real b. -3 c. $\frac{1}{3}$
9. a. not real b. -6 c. $\frac{1}{6}$
11. a. not real b. -10 c. $\frac{1}{10}$

Exercise SET D: simplify expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt{m^5}$ b. $(\sqrt[3]{3y})^7$ c. $\sqrt[5]{\left(\frac{4x}{5y}\right)^3}$
2. a. $\sqrt[4]{r^7}$ b. $(\sqrt[5]{2pq})^3$ c. $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$
3. a. $\sqrt[5]{u^2}$ b. $(\sqrt[3]{6x})^5$ c. $\sqrt[4]{\left(\frac{18a}{5b}\right)^7}$
4. a. $\sqrt[3]{a}$ b. $(\sqrt[4]{21v})^3$ c. $\sqrt[4]{\left(\frac{2xy}{5z}\right)^2}$

Answer

1. a. $m^{\frac{5}{2}}$ b. $(3y)^{\frac{7}{3}}$ c. $\left(\frac{4x}{5y}\right)^{\frac{3}{5}}$
3. a. $u^{\frac{2}{5}}$ b. $(6x)^{\frac{5}{3}}$ c. $\left(\frac{18a}{5b}\right)^{\frac{7}{4}}$

Exercise SET E: simplify expressions with $a^{\frac{m}{n}}$

In the following exercises, simplify.

1. a. $64^{\frac{5}{2}}$ b. $81^{\frac{-3}{2}}$ c. $(-27)^{\frac{2}{3}}$
2. a. $25^{\frac{3}{2}}$ b. $9^{-\frac{3}{2}}$ c. $(-64)^{\frac{2}{3}}$
3. a. $32^{\frac{2}{5}}$ b. $27^{-\frac{2}{3}}$ c. $(-25)^{\frac{1}{2}}$
4. a. $100^{\frac{3}{2}}$ b. $49^{-\frac{5}{2}}$ c. $(-100)^{\frac{3}{2}}$
5. a. $-9^{\frac{3}{2}}$ b. $-9^{-\frac{3}{2}}$ c. $(-9)^{\frac{3}{2}}$
6. a. $-64^{\frac{3}{2}}$ b. $-64^{-\frac{3}{2}}$ c. $(-64)^{\frac{3}{2}}$

Answer

1. a. $32,768$b. $\frac{1}{729}$ c. 9
3. a. 4 b. $\frac{1}{9}$ c. not real
5. a. -27 b. $-\frac{1}{27}$ c. not real

Exercise SET F: use the laws of exponents to simplify expressions with rational exponents

In the following exercises, simplify. Assume all variables are positive.

1. a. $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$ b. $\left(p^{12}\right)^{\frac{3}{4}}$ c. $\dfrac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$
2. a. $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$ b. $\left(b^{15}\right)^{\frac{3}{5}}$ c. $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$
3. a. $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$ b. $\left(x^{12}\right)^{\frac{2}{3}}$ c. $\dfrac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$
4. a. $q^{\frac{2}{3}} \cdot q^{\frac{5}{6}}$ b. $\left(h^{6}\right)^{\frac{4}{3}}$ c. $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$
5. a. $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$ b. $\left(a^{\frac{1}{3}}b^{\frac{2}{3}}\right)^{\frac{3}{2}}$
6. a. $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$ b. $\left(m^{\frac{4}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{4}}$
7. a. $\left(16u^{\frac{1}{3}}\right)^{\frac{3}{4}}$ b. $\left(4p^{\frac{1}{3}}q^{\frac{1}{2}}\right)^{\frac{3}{2}}$
8. a. $\left(625n^{\frac{8}{3}}\right)^{\frac{3}{4}}$ b. $\left(9x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}$
9. a. $\dfrac{r^{\frac{5}{2}} \cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$ b. $\left(\dfrac{36s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$
10. a. $\dfrac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$ b. $\left(\dfrac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$
11. a. $\dfrac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$ b. $\left(\dfrac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$
12. a. $\dfrac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$ b. $\left(\dfrac{16m^{\frac{1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{4}}$

Answer

1. a. $c^{\frac{7}{8}}$ b. p^9 c. $\frac{1}{r}$
3. a. $y^{\frac{5}{4}}$ b. x^8 c. $\dfrac{1}{m}$
5. a. $81q^2$ b. $a^{\frac{1}{2}}b$
7. a. $8u^{\frac{1}{4}}$ b. $8p^{\frac{1}{2}}q^{\frac{3}{4}}$
9. a. $r^{\frac{7}{2}}$ b. $\dfrac{6s}{t}$
11. a. c^2 b. $\dfrac{2x}{3y}$

Exercise SET G: writing exercises

1. Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.
2. Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

Answer

1. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with $a^{\frac{1}{n}}$.			
simplify expressions with $a^{\frac{m}{n}}$.			
use the Laws of Exponents to simply expressions with rational exponents.			

Figure 8.3.4

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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4.4: Simplify Expressions with Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Before you get started, take this readiness quiz.

1. Simplify: a. $(-9)^2$ b. -9^2 c. $(-9)^3$
   If you missed this problem, review Example 2.21.
2. Round 3.846 to the nearest hundredth.
   If you missed this problem, review Example 1.34.
3. Simplify: a. $x^3 \cdot x^3$ b. $y^2 \cdot y^2 \cdot y^2$ c. $z^3 \cdot z^3 \cdot z^3 \cdot z^3$
   If you missed this problem, review Example 5.12.

Simplify Expressions with Roots

In Foundations, we briefly looked at square roots. Remember that when a real number n is multiplied by itself, we write n^2 and read it 'n^2 squared'. This number is called the **square** of n, and n is called the **square root**. For example,

13^2 is read "13 squared"

169 is called the square of 13, since $13^2 = 169$

13 is a square root of 169

Definition 4.4.1: Square and Square Root of a Number

Square

If $n^2 = m$, then m is the **square** of n.

Square Root

If $n^2 = m$, then n is a **square root** of m.

Notice $(-13)^2 = 169$ also, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of m. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Definition 4.4.2: Square Root Notation

$\sqrt{m}$ is read "the square root of m."

If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

radical sign $\longrightarrow \sqrt{m} \longleftarrow$ radicand

Figure 8.1.1

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{169} = 13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{169} = -13$.

Example 4.4.1

Simplify:

a. $\sqrt{144}$
b. $-\sqrt{289}$

Solution:

a.

$$\sqrt{144}$$

Since $12^2 = 144$.

$$12$$

b.

$$-\sqrt{289}$$

Since $17^2 = 289$ and the negative is in front of the radical sign.

$$-17$$

Exercise 4.4.1

Simplify:

a. $-\sqrt{64}$
b. $\sqrt{225}$

Answer

a. -8
b. 15

Exercise 4.4.2

Simplify:

a. $\sqrt{100}$
b. $-\sqrt{121}$

Answer

a. 10
b. -11

Can we simplify $-\sqrt{49}$? Is there a number whose square is -49?

$$(\_\_\_)^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

Example 4.4.2

Simplify:

a. $\sqrt{-196}$
b. $-\sqrt{64}$

Solution:

a.

$$\sqrt{-196}$$

There is no real number whose square is -196.

$\sqrt{-196}$ is not a real number.

b.

$$-\sqrt{64}$$

The negative is in front of the radical.

$$-8$$

Exercise 4.4.3

Simplify:

a. $\sqrt{-169}$
b. $-\sqrt{81}$

Answer

a. not a real number
b. -9

Exercise 4.4.4

Simplify:

a. $-\sqrt{49}$
b. $\sqrt{-121}$

Answer

a. -7
b. not a real number

So far we have only talked about squares and square roots. Let's now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth power
n^5	n to the fifth power

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from -5 to 5. See *Figure 8.1.2*

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Figure 8.1.2

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We'll copy the row with the powers of -2 to help you see this.

n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Even power
Positive result

Odd power
Negative result

Figure 8.1.3

We will now extend the square root definition to higher roots.

Definition 4.4.3: Nth Root of a Number

If $b^n = a$, then b is an nth root of a.

The principal n^{th} root of a is written $\sqrt[n]{a}$.

The n is called the **index** of the radical.

Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We can refer to *Figure 8.1.2* to help find higher roots.

$$4^3 = 64 \qquad \sqrt[3]{64} = 4$$
$$3^4 = 81 \qquad \sqrt[4]{81} = 3$$
$$(-2)^5 = -32 \qquad \sqrt[5]{-32} = -2$$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

Properties of $\sqrt[n]{a}$

When n is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an odd number, $\sqrt[n]{a}$ is a real number for all the values of a.

We will apply these properties in the next two examples.

Example 4.4.3

Simplify:

a. $\sqrt[3]{64}$
b. $\sqrt[4]{81}$
c. $\sqrt[5]{32}$

Solution:

a.

$$\sqrt[3]{64}$$

Since $4^3 = 64$.

$$4$$

b.

$$\sqrt[4]{81}$$

Since $(3)^4 = 81$.

$$3$$

c.

$$\sqrt[5]{32}$$

Since $(2)^5 = 32$.

$$2$$

Exercise 4.4.5

Simplify:

a. $\sqrt[3]{27}$
b. $\sqrt[4]{256}$
c. $\sqrt[5]{243}$

Answer

a. 3
b. 4
c. 3

Exercise 4.4.6

Simplify:

a. $\sqrt[3]{1000}$
b. $\sqrt[4]{16}$
c. $\sqrt[5]{243}$

Answer

a. 10
b. 2
c. 3

In this example be alert for the negative signs as well as even and odd powers.

Example 4.4.4

Simplify:

a. $\sqrt[3]{-125}$
b. $\sqrt[4]{16}$
c. $\sqrt[5]{-243}$

Solution:

a.

$$\sqrt[3]{-125}$$

Since $(-5)^3 = -125$.

$$-5$$

b.

$$\sqrt[4]{16}$$

Think, $(?)^4 = -16$. No real number raised to the fourth power is negative.

Not a real number.

c.

$$\sqrt[5]{-243}$$

Since $(-3)^5 = -243$.

$$-3$$

Exercise 4.4.7

Simplify:

a. $\sqrt[3]{-27}$
b. $\sqrt[4]{-256}$
c. $\sqrt[5]{-32}$

Answer

a. -3
b. not real
c. -2

Exercise 4.4.8

Simplify:

a. $\sqrt[3]{-216}$
b. $\sqrt[4]{-81}$
c. $\sqrt[5]{-1024}$

Answer

a. -6
b. not real
c. -4

Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $\sqrt{4} = 2$, what is the value of $\sqrt{21}$ or $\sqrt[3]{50}$? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

To get a numerical estimate of a square root, we look for **perfect square** numbers closest to the radicand. To find an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.

Number	Square Root
4	2
9	3
16	4
25	5

11 $\sqrt{11}$

$9 < 11 < 16$

$3 < \sqrt{11} < 4$

Number	Cube Root
8	2
27	3
64	4
125	5

91 $\sqrt[3]{91}$

$64 < 91 < 125$

$4 < \sqrt[3]{91} < 5$

Figure 8.1.4

Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

https://math.libretexts.org/@go/page/58454

Example 4.4.5

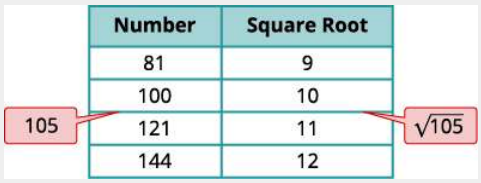
Estimate each root between two consecutive whole numbers:

a. $\sqrt{105}$
b. $\sqrt[3]{43}$

Solution:

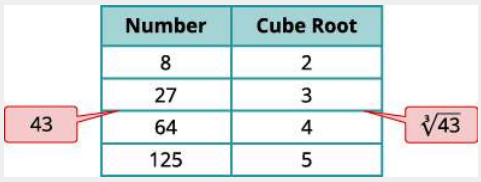
a. Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

Table 8.1.1

	$\sqrt{105}$
	
Locate 105 between two consecutive perfect squares.	$100 < 105 < 121$
$\sqrt{105}$ is between their square roots.	$10 < \sqrt{105} < 11$

b. Similarly we locate 43 between two perfect cube numbers.

Table 8.1.2

	$\sqrt[3]{43}$
	
Locate 43 between two consecutive perfect cubes.	$27 < 43 < 64$
$\sqrt[3]{43}$ is between their cube roots.	$3 < \sqrt[3]{43} < 4$

Exercise 4.4.9

Estimate each root between two consecutive whole numbers:

a. $\sqrt{38}$
b. $\sqrt[3]{93}$

Answer

a. $6 < \sqrt{38} < 7$
b. $4 < \sqrt[3]{93} < 5$

Exercise 4.4.10

Estimate each root between two consecutive whole numbers:

a. $\sqrt{84}$
b. $\sqrt[3]{152}$

Answer

a. $9 < \sqrt{84} < 10$
b. $5 < \sqrt[3]{152} < 6$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the $\sqrt{x}$ key on your calculator. To find a cube root, or any root with higher index, you will use the $\sqrt[y]{x}$ key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that

$$\sqrt{5} \approx 2.236067978 \text{rounded to two decimal places is } \sqrt{5} \approx 2.24$$

$$\sqrt[4]{93} \approx 3.105422799 \text{rounded to two decimal places is } \sqrt[4]{93} \approx 3.11$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$(2.236067978)^2 = 5.000000002 \qquad (3.105422799)^4 = 92.999999991$$
$$(2.24)^2 = 5.0176 \qquad (3.11)^4 = 93.54951841$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

Example 4.4.6

Round to two decimal places:

a. $\sqrt{17}$
b. $\sqrt[3]{49}$
c. $\sqrt[4]{51}$

Solution:

a.

$$\sqrt{17}$$

Use the calculator square root key.

$$4.123105626\ldots$$

Round to two decimal places.

$$4.12$$

$$\sqrt{17} \approx 4.12$$

b.

$$\sqrt[3]{49}$$

Use the calculator $\sqrt[y]{x}$ key.

$$3.659305710\ldots$$

Round to two decimal places.

$$3.66$$

$$\sqrt[3]{49} \approx 3.66$$

c.

$$\sqrt[4]{51}$$

Use the calculator $\sqrt[y]{x}$ key.

$$2.6723451177\ldots$$

Round to two decimal places.

$$2.67$$

$$\sqrt[4]{51} \approx 2.67$$

Exercise 4.4.11

Round to two decimal places:

1. $\sqrt{11}$
2. $\sqrt[3]{71}$
3. $\sqrt[4]{127}$

Answer

a. ≈ 3.32
b. ≈ 4.14
c. ≈ 3.36

Exercise 4.4.12

Round to two decimal places:

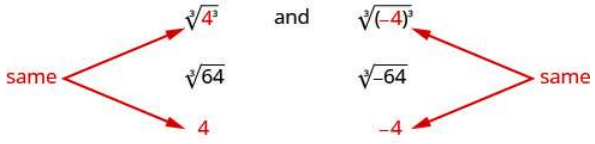
1. $\sqrt{13}$
2. $\sqrt[3]{84}$
3. $\sqrt[4]{98}$

Answer

a. ≈ 3.61
b. ≈ 4.38
c. ≈ 3.15

Simplify Variable Expressions with Roots

The odd root of a number can be either positive or negative. For example,

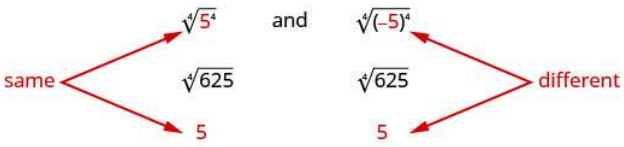


In either case, when n is odd, $\sqrt[n]{a^n} = a$.

Figure 8.1.13

But what about an even root? We want the principal root, so $\sqrt[4]{625} = 5$.

But notice,



Here we see that sometimes when n is even, $\sqrt[n]{a^n} \neq a$.

Figure 8.1.14

How can we make sure the fourth root of -5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when n is even $\sqrt[n]{a^n} = |a|$. This guarantees the principal root is positive.

Definition 4.4.4: Simplifying Odd and Even Roots

For any integer $n \geq 2$,

$$\text{when the index } n \text{ is odd } \sqrt[n]{a^n} = a$$

$$\text{when the index } n \text{ is even } \sqrt[n]{a^n} = |a|$$

We must use the **absolute value** signs when we take an even root of an expression with a variable in the radical.

Example 4.4.7

Simplify:

a. $\sqrt{x^2}$
b. $\sqrt[3]{n^3}$
c. $\sqrt[4]{p^4}$
d. $\sqrt[5]{y^5}$

Solution:

a. We use the absolute value to be sure to get the positive root.

$$\sqrt{x^2}$$

Since the index n is even, $\sqrt[n]{a^n} = |a|$.

b. This is an odd indexed root so there is no need for an absolute value sign.

$$\sqrt[3]{m^3}$$

Since the index is n is odd, $\sqrt[n]{a^n} = a$.

$$m$$

c.

$$\sqrt[4]{p^4}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$|p|$$

d.

$$\sqrt[5]{y^5}$$

Since the index n is odd, $\sqrt[n]{a^n} = a$.

$$y$$

Exercise 4.4.13

Simplify:

a. $\sqrt{b^2}$
b. $\sqrt[3]{w^3}$
c. $\sqrt[4]{m^4}$
d. $\sqrt[5]{q^5}$

Answer

a. $|b|$
b. w
c. $|m|$
d. q

Exercise 4.4.14

Simplify:

a. $\sqrt{y^2}$
b. $\sqrt[3]{p^3}$
c. $\sqrt[4]{z^4}$
d. $\sqrt[5]{q^5}$

Answer

a. $|y|$
b. p
c. $|z|$
d. q

What about square roots of higher powers of variables? The Power Property of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

Looking now at the square root.

$$\sqrt{a^{2m}}$$

Since $(a^m)^2 = a^{2m}$.

$$\sqrt{(a^m)^2}$$

Since n is even $\sqrt[n]{a^n} = |a|$.

$$|a^m|$$

So $\sqrt{a^{2m}} = |a^m|$.

We apply this concept in the next example.

Example 4.4.8

Simplify:

a. $\sqrt{x^6}$
b. $\sqrt{y^{16}}$

Solution:

a.

$$\sqrt{x^6}$$

Since $\left(x^3\right)^2 = x^6$.

$$\sqrt{(x^3)^2}$$

Since the index n is even $\sqrt{a^n} = |a|$.

$$|x^3|$$

b.

$$\sqrt{y^{16}}$$

Since $\left(y^8\right)^2 = y^{16}$.

$$\sqrt{(y^8)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$y^8$$

In this case the absolute value sign is not needed as y^8 is positive.

Exercise 4.4.15

Simplify:

a. $\sqrt{y^{18}}$
b. $\sqrt{z^{12}}$

Answer

a. $|y^9|$
b. z^6

Exercise 4.4.16

Simplify:

a. $\sqrt{m^4}$
b. $\sqrt{b^{10}}$

Answer

a. m^2
b. $|b^5|$

The next example uses the same idea for higher roots.

Example 4.4.9

Simplify:

a. $\sqrt[3]{y^{18}}$
b. $\sqrt[4]{z^8}$

Solution:

a.

$$\sqrt[3]{y^{18}}$$

Since $\left(y^6\right)^3 = y^{18}$.

$$\sqrt[3]{\left(y^6\right)^3}$$

Since n is odd, $\sqrt[n]{a^n} = a$.

$$y^6$$

b.

$$\sqrt[4]{z^8}$$

Since $\left(z^2\right)^4 = z^8$.

$$\sqrt[4]{\left(z^2\right)^4}$$

Since z^2 is positive, we do not need an absolute value sign.

$$z^2$$

Exercise 4.4.17

Simplify:

a. $\sqrt[4]{u^{12}}$
b. $\sqrt[3]{v^{15}}$

Answer

a. $|u^3|$
b. v^5

Exercise 4.4.18

Simplify:

a. $\sqrt[5]{c^{20}}$
b. $\sqrt[6]{d^{24}}$

Answer

a. c^4
b. d^4

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}} = |a^m|$ works in much the same way.

$$\sqrt{16r^{22}} = 4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^2 = 16r^{22}.$$

But notice $\sqrt{25u^8} = 5u^4$ and no absolute value sign is needed as u^4 is always positive.

Example 4.4.10

Simplify:

a. $\sqrt{16n^2}$
b. $-\sqrt{81c^2}$

Solution:

a.

$$\sqrt{16n^2}$$

Since $(4n)^2 = 16n^2$.

$$\sqrt{(4n)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$4|n|$$

b.

$$-\sqrt{81c^2}$$

Since $(9c)^2 = 81c^2$.

$$-\sqrt{(9c)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$-9|c|$$

Exercise 4.4.19

Simplify:

a. $\sqrt{64x^2}$
b. $-\sqrt{100p^2}$

Answer

a. $8|x|$
b. $-10|p|$

Exercise 4.4.20

Simplify:

a. $\sqrt{169y^2}$
b. $-\sqrt{121y^2}$

Answer

a. $13|y|$
b. $-11|y|$

This example just takes the idea farther as it has roots of higher index.

Example 4.4.11

Simplify:

a. $\sqrt[3]{64p^6}$
b. $\sqrt[4]{16q^{12}}$

Solution:

a.

$$\sqrt[3]{64p^6}$$

Rewrite $64p^6$ as $\left(4p^2\right)^3$.

$$\sqrt[3]{\left(4p^2\right)^3}$$

Take the cube root.

$$4p^2$$

b.

$$\sqrt[4]{16q^{12}}$$

Rewrite the radicand as a fourth power.

$$\sqrt[4]{\left(2q^3\right)^4}$$

Take the fourth root.

$$2|q^3|$$

Exercise 4.4.21

Simplify:

a. $\sqrt[3]{27x^{27}}$
b. $\sqrt[4]{81q^{28}}$

Answer

a. $3x^9$
b. $3|q^7|$

Exercise 4.4.22

Simplify:

a. $\sqrt[3]{125q^9}$
b. $\sqrt[5]{243q^{25}}$

Answer

a. $5p^3$
b. $3q^5$

The next examples have two variables.

Example 4.4.12

Simplify:

a. $\sqrt{36x^2y^2}$
b. $\sqrt{121a^6b^8}$
c. $\sqrt[3]{64p^{63}q^9}$

Solution:

a.

$$\sqrt{36x^2y^2}$$

Since $(6xy)^2 = 36x^2y^2$

$$\sqrt{(6xy)^2}$$

Take the square root.

$$6|xy|$$

b.

$$\sqrt{121a^6b^8}$$

Since $\left(11a^3b^4\right)^2 = 121a^6b^8$

$$\sqrt{\left(11a^3b^4\right)^2}$$

Take the square root.

$$11\left|a^3\right|b^4$$

c.

$$\sqrt[3]{64p^{63}q^9}$$

Since $\left(4p^{21}q^3\right)^3 = 64p^{63}q^9$

$$\sqrt[3]{(4p^{21}q^3)^3}$$

Take the cube root.

$$4p^{21}q^3$$

? Exercise 4.4.23

Simplify:

a. $\sqrt{100a^2b^2}$
b. $\sqrt{144p^{12}q^{20}}$
c. $\sqrt[3]{8x^{30}y^{12}}$

Answer

a. $10|ab|$
b. $12p^6q^{10}$
c. $2x^{10}y^4$

? Exercise 4.4.24

Simplify:

a. $\sqrt{225m^2n^2}$
b. $\sqrt{169x^{10}y^{14}}$
c. $\sqrt[3]{27w^{36}z^{15}}$

Answer

a. $15|mn|$
b. $13\,|x^5y^7|$
c. $3w^{12}z^5$

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- Simplifying Variables Exponents with Roots using Absolute Values

Key Concepts

- **Square Root Notation**
  - $\sqrt{m}$ is read 'the square root of m'
  - If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

    Figure 8.1.1
  - The square root of m, $\sqrt{m}$, is a positive number whose square is m.
- **n^{th} Root of a Number**
  - If $b^n = a$, then b is an nth root of a.
  - The principal n^{th} root of a is written $\sqrt[n]{a}$.
  - n is called the *index* of the radical.
- **Properties of $\sqrt[n]{a}$**
  - When n is an even number and
    - $a \geq 0$, then $\sqrt[n]{a}$ is a real number
    - $a < 0$, then $\sqrt[n]{a}$ is not a real number
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Simplifying Odd and Even Roots**

- For any integer $n \geq 2$,
  - when n is odd $\sqrt[n]{a^n} = a$
  - when n is even $\sqrt[n]{a^n} = |a|$
- We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Glossary

square of a number

If $n^2 = m$, then m is the square of n.

square root of a number

If $n^2 = m$, then n is a square root of m.

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4.4E: Exercises

Simplifying Expressions with Roots

In the following exercises, simplify.

1. a. $\sqrt{64}$ b. $-\sqrt{81}$

Answer

a. 8 b. -9

2. a. $\sqrt{169}$ b. $-\sqrt{100}$

3. a. $\sqrt{196}$ b. $-\sqrt{1}$

Answer

a. 14 b. -1

4. a. $\sqrt{144}$ b. $-\sqrt{121}$

5. a. $\sqrt{\frac{4}{9}}$ b. $-\sqrt{0.01}$

Answer

a. $\frac{2}{3}$ b. -0.1

6. a. $\sqrt{\frac{64}{121}}$ b. $-\sqrt{0.16}$

7. a. $\sqrt{-121}$ b. $-\sqrt{289}$

Answer

a. not a real number b. -17

8. a. $-\sqrt{400}$ b. $\sqrt{-36}$

9. a. $-\sqrt{225}$ b. $\sqrt{-9}$

Answer

a. -15 b. not a real number

10. a. $\sqrt{-49}$ b. $-\sqrt{256}$

11. a. $\sqrt[3]{216}$ b. $\sqrt[4]{256}$

Answer

a. 6 b. 4

12. a. $\sqrt[3]{27}$ b. $\sqrt[4]{16}$ c. $\sqrt[5]{243}$

13. a. $\sqrt[3]{512}$ b. $\sqrt[4]{81}$ c. $\sqrt[5]{1}$

Answer

a. 8 b. 3 b. 1

14. a. $\sqrt[3]{125}$ b. $\sqrt[4]{1296}$ c. $\sqrt[5]{1024}$

15. a. $\sqrt[3]{-8}$ b. $\sqrt[4]{-81}$ c. $\sqrt[5]{-32}$

Answer

a. -2 b. not a real number c. -2

16. a. $\sqrt[3]{-64}$ b. $\sqrt[4]{-16}$ c. $\sqrt[5]{-243}$

17. a. $\sqrt[3]{-125}$ b. $\sqrt[4]{-1296}$ c. $\sqrt[5]{-1024}$

Answer

a. -5 b. not a real number c. -4

18. a. $\sqrt[3]{-512}$ b. $\sqrt[4]{-81}$ c. $\sqrt[5]{-1}$

In the following exercises, estimate each root by giving the interval of two consecutive whole numbers in which the root lies.

19. a. $\sqrt{70}$ b. $\sqrt[3]{71}$

Answer

a. $8 < \sqrt{70} < 9$ b. $4 < \sqrt[3]{71} < 5$

20. a. $\sqrt{55}$ b. $\sqrt[3]{119}$

21. a. $\sqrt{200}$ b. $\sqrt[3]{137}$

Answer

a. $14 < \sqrt{200} < 15$ b. $5 < \sqrt[3]{137} < 6$

22. a. $\sqrt{172}$ b. $\sqrt[3]{200}$

In the following exercises, approximate each root and round to two decimal places.

23. a. $\sqrt{19}$ b. $\sqrt[3]{89}$ c. $\sqrt[4]{97}$

Answer

a. ≈ 4.36 b. ≈ 4.46 c. ≈ 3.14

24. a. $\sqrt{21}$ b. $\sqrt[3]{93}$ c. $\sqrt[4]{101}$

25. a. $\sqrt{53}$ b. $\sqrt[3]{147}$ c. $\sqrt[4]{452}$

Answer

a. ≈ 7.28 b. ≈ 5.28 c. ≈ 4.61

26. a. $\sqrt{47}$ b. $\sqrt[3]{163}$ c. $\sqrt[4]{527}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

27. a. $\sqrt[5]{u^5}$ b. $\sqrt[8]{v^8}$

Answer

a. u b. $|v|$

28. a. $\sqrt[3]{a^3}$ b. $\sqrt[9]{b^9}$

29. a. $\sqrt[4]{y^4}$ b. $\sqrt[7]{m^7}$

Answer

a. $|y|$ b. m

30. a. $\sqrt[8]{k^8}$ b. $\sqrt[6]{p^6}$

31. a. $\sqrt{x^6}$ b. $\sqrt{y^{16}}$

Answer

a. $|x^3|$ b. y^8

32. a. $\sqrt{a^{14}}$ b. $\sqrt{w^{24}}$

33. a. $\sqrt{x^{24}}$ b. $\sqrt{y^{22}}$

Answer

a. x^{12} b. $|y^{11}|$

34. a. $\sqrt{a^{12}}$ b. $\sqrt{b^{26}}$

35. a. $\sqrt[3]{x^9}$ b. $\sqrt[4]{y^{12}}$

Answer

a. x^3 b. $|y^3|$

36. a. $\sqrt[5]{a^{10}}$ b. $\sqrt[3]{b^{27}}$

37. a. $\sqrt[4]{m^8}$ b. $\sqrt[5]{n^{20}}$

Answer

a. m^2 b. n^4

38. a. $\sqrt[6]{r^{12}}$ b. $\sqrt[3]{s^{30}}$

39. a. $\sqrt{49x^2}$ b. $-\sqrt{81x^{18}}$

Answer

a. $7|x|$ b. $-9|x^9|$

40. a. $\sqrt{100y^2}$ b. $-\sqrt{100m^{32}}$

41. a. $\sqrt{121m^{20}}$ b. $-\sqrt{64a^2}$

Answer

a. $11m^{10}$ b. $-8|a|$

42. a. $\sqrt{81x^{36}}$ b. $-\sqrt{25x^2}$

43. a. $\sqrt[4]{16x^8}$ b. $\sqrt[6]{64y^{12}}$

Answer

a. $2x^2$ b. $2y^2$

44. a. $\sqrt[3]{-8c^9}$ b. $\sqrt[3]{125d^{15}}$

45. a. $\sqrt[3]{216a^6}$ b. $\sqrt[5]{32b^{20}}$

Answer

a. $6a^2$ b. $2b^4$

46. a. $\sqrt[7]{128r^{14}}$ b. $\sqrt[4]{81s^{24}}$

47. a. $\sqrt{144x^2y^2}$ b. $\sqrt{169w^8y^{10}}$ c. $\sqrt[3]{8a^{51}b^6}$

Answer

a. $12|xy|$ b. $13w^4\,|y^5|$ c. $2a^{17}b^2$

48. a. $\sqrt{196a^2b^2}$ b. $\sqrt{81p^{24}q^6}$ c. $\sqrt[3]{27p^{45}q^9}$

49. a. $\sqrt{121a^2b^2}$ b. $\sqrt{9c^8d^{12}}$ c. $\sqrt[3]{64x^{15}y^{66}}$

Answer

a. $11|ab|$ b. $3c^4d^6$ c. $4x^5y^{22}$

50. a. $\sqrt{225x^2y^2z^2}$ b. $\sqrt{36r^6s^{20}}$ c. $\sqrt[3]{125y^{18}z^{27}}$

Writing Exercises

51. Why is there no real number equal to $\sqrt{-64}$?

Answer

Since the square of any real number is positive, it's not possible for a real number to square to -64.

52. What is the difference between 9^2 and $\sqrt{9}$?

53. Explain what is meant by the nth root of a number.

Answer

If you raise this root to the nth power, it will give you back the original number (under the radical).

54. Explain the difference of finding the nth root of a number when the index is even compared to when the index is odd.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with roots.			
estimate and approximate roots.			
simplify variable expressions with roots.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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4.5: Add, Subtract, and Multiply Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

Before you get started, take this readiness quiz.

1. Add: $3x^2+9x-5-(x^2-2x+3)$.
   If you missed this problem, review Example 5.5.
2. Simplify: $(2+a)(4-a)$.
   If you missed this problem, review Example 5.28.
3. Simplify: $(9-5y)^2$.
   If you missed this problem, review Example 5.31.

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Definition 4.5.1: Like Radicals

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x+8x$ is $11x$.Similarly we add $3\sqrt{x}+8\sqrt{x}$ and the result is $11\sqrt{x}$.

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

Example 4.5.1

Simplify:

a. $2\sqrt{2}-7\sqrt{2}$
b. $5\sqrt[3]{y}+4\sqrt[3]{y}$
c. $7\sqrt[4]{x}-2\sqrt[4]{y}$

Solution:

a.

$$2\sqrt{2}-7\sqrt{2}$$

Since the radicals are like, we subtract the coefficients.

$$-5\sqrt{2}$$

b.

$$5\sqrt[3]{y}+4\sqrt[3]{y}$$

Since the radicals are like, we add the coefficients.

$$9\sqrt[3]{y}$$

c.

$$7\sqrt[4]{x}-2\sqrt[4]{y}$$

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

Exercise 4.5.1

Simplify:

a. $8\sqrt{2}-9\sqrt{2}$
b. $4\sqrt[3]{x}+7\sqrt[3]{x}$
c. $3\sqrt[4]{x}-5\sqrt[4]{y}$

Answer

a. $-\sqrt{2}$
b. $11\sqrt[3]{x}$
c. $3\sqrt[4]{x}-5\sqrt[4]{y}$

Exercise 4.5.2

Simplify:

a. $5\sqrt{3}-9\sqrt{3}$
b. $5\sqrt[3]{y}+3\sqrt[3]{y}$
c. $5\sqrt[4]{m}-2\sqrt[3]{m}$

Answer

a. $-4\sqrt{3}$
b. $8\sqrt[3]{y}$
c. $5\sqrt[4]{m}-2\sqrt[3]{m}$

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

Example 4.5.2

Simplify:

a. $2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$
b. $\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$

Solution:

a.

$$2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$$

Since the radicals are like, we combine them.

$$0\sqrt{5n}$$

Simplify.

$$0$$

b.

$$\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$$

Since the radicals are like, we combine them.

$$2\sqrt[4]{3xy}$$

Exercise 4.5.3

Simplify:

a. $\sqrt{7x}-7\sqrt{7x}+4\sqrt{7x}$
b. $4\sqrt[4]{5xy}+2\sqrt[4]{5xy}-7\sqrt[4]{5xy}$

Answer

a. $-2\sqrt{7x}$
b. $-\sqrt[4]{5xy}$

Exercise 4.5.4

Simplify:

a. $4\sqrt{3y}-7\sqrt{3y}+2\sqrt{3y}$
b. $6\sqrt[3]{7mn}+\sqrt[3]{7mn}-4\sqrt[3]{7mn}$

Answer

a. $-\sqrt{3y}$
b. $3\sqrt[3]{7mn}$

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

Example 4.5.3

Simplify:

a. $\sqrt{20}+3\sqrt{5}$
b. $\sqrt[3]{24}-\sqrt[3]{375}$
c. $\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$

Solution:

a.

$$\sqrt{20}+3\sqrt{5}$$

Simplify the radicals, when possible.

$$\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$$
$$2\sqrt{5}+3\sqrt{5}$$

Combine the like radicals.

$$5\sqrt{5}$$

b.

$$\sqrt[3]{24}-\sqrt[3]{375}$$

Simplify the radicals.

$$\sqrt[3]{8}\cdot\sqrt[3]{3}-\sqrt[3]{125}\cdot\sqrt[3]{3}$$
$$2\sqrt[3]{3}-5\sqrt[3]{3}$$

Combine the like radicals.

$$-3\sqrt[3]{3}$$

c.

$$\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$$

Simplify the radicals.

$$\frac{1}{2}\sqrt[4]{16}\cdot\sqrt[4]{3}-\frac{2}{3}\sqrt[4]{81}\cdot\sqrt[4]{3}$$
$$\frac{1}{2}\cdot 2\cdot\sqrt[4]{3}-\frac{2}{3}\cdot 3\cdot\sqrt[4]{3}$$
$$\sqrt[4]{3}-2\sqrt[4]{3}$$

Combine the like radicals.

$$-\sqrt[4]{3}$$

Exercise 4.5.5

Simplify:

a. $\sqrt{18}+6\sqrt{2}$
b. $6\sqrt[3]{16}-2\sqrt[3]{250}$
c. $\frac{2}{3}\sqrt[3]{81}-\frac{1}{2}\sqrt[3]{24}$

Answer

a. $9\sqrt{2}$
b. $2\sqrt[3]{2}$
c. $\sqrt[3]{3}$

Exercise 4.5.6

Simplify:

a. $\sqrt{27}+4\sqrt{3}$
b. $4\sqrt[3]{5}-7\sqrt[3]{40}$
c. $\frac{1}{2}\sqrt[3]{128}-\frac{5}{3}\sqrt[3]{54}$

Answer

a. $7\sqrt{3}$
b. $-10\sqrt[3]{5}$
c. $-3\sqrt[3]{2}$

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption thoughout the rest of this chapter.

Example 4.5.4

Simplify:

a. $9\sqrt{50m^2}-6\sqrt{48m^2}$
b. $\sqrt[3]{54n^5}-\sqrt[3]{16n^5}$

Solution:

a.

$$9\sqrt{50m^2}-6\sqrt{48m^2}$$

Simplify the radicals.

$$9\sqrt{25m^2}\cdot\sqrt{2}-6\sqrt{16m^2}\cdot\sqrt{3}$$

$$9 \cdot 5m \cdot \sqrt{2} - 6 \cdot 4m \cdot \sqrt{3}$$

$$45m\sqrt{2} - 24m\sqrt{3}$$

The radicals are not like and so cannot be combined.

b.

$$\sqrt[3]{54n^5} - \sqrt[3]{16n^5}$$

Simplify the radicals.

$$\sqrt[3]{27n^3} \cdot \sqrt[3]{2n^2} - \sqrt[3]{8n^3} \cdot \sqrt[3]{2n^2}$$

$$3n\sqrt[3]{2n^2} - 2n\sqrt[3]{2n^2}$$

Combine the like radicals.

$$n\sqrt[3]{2n^2}$$

? Exercise 4.5.7

Simplify:

a. $\sqrt{32m^7} - \sqrt{50m^7}$
b. $\sqrt[3]{135x^7} - \sqrt[3]{40x^7}$

Answer

a. $-m^3\sqrt{2m}$
b. $x^2\sqrt[3]{5x}$

? Exercise 4.5.8

Simplify:

a. $\sqrt{27p^3} - \sqrt{48p^3}$
b. $\sqrt[3]{256y^5} - \sqrt[3]{32n^5}$

Answer

a. $-p\sqrt{3p}$
b. $4y\sqrt[3]{4y^2} - 2n\sqrt[3]{4n^2}$

Multiply Radical Expressions

We have used the **Product Property of Roots** to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots 'in reverse' to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

Definition 4.5.2 Product Property of Roots

For any real numbers, $\sqrt[n]{a}$ and $\sqrt[b]{n}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

When we multiply two radicals they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is $12xy$. Keep this in mind as you do these examples.

Example 4.5.5

Simplify:

a. $(6\sqrt{2})(3\sqrt{10})$
b. $(-5\sqrt[3]{4})(-4\sqrt[3]{6})$

Solution:

a.

$$(6\sqrt{2})(3\sqrt{10})$$

Multiply using the Product Property.

$$18\sqrt{20}$$

Simplify the radical.

$$18\sqrt{4}\cdot\sqrt{5}$$

Simplify.

$$18\cdot 2\cdot\sqrt{5}$$
$$36\sqrt{5}$$

b.

$$(-5\sqrt[3]{4})(-4\sqrt[3]{6})$$

Multiply using the Product Property.

$$20\sqrt[3]{24}$$

Simplify the radical.

$$20\sqrt[3]{8}\cdot\sqrt[3]{3}$$

Simplify.

$$20\cdot 2\cdot\sqrt[3]{3}$$
$$40\sqrt[3]{3}$$

Exercise 4.5.9

Simplify:

a. $(3\sqrt{2})(2\sqrt{30})$
b. $(2\sqrt[3]{18})(-3\sqrt[3]{6})$

Answer

a. $12\sqrt{15}$
b. $-18\sqrt[3]{2}$

Exercise 4.5.10

Simplify:

a. $(3\sqrt{3})(3\sqrt{6})$
b. $(-4\sqrt[3]{9})(3\sqrt[3]{6})$

Answer

a. $27\sqrt{2}$
b. $-36\sqrt[3]{2}$

We follow the same procedures when there are variables in the radicands.

Example 4.5.6

Simplify:

a. $\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$
b. $\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$

Solution:

a.

$$\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$$

Multiply.

$$40\sqrt{18p^4}$$

Simplify the radical.

$$40\sqrt{9p^4}\cdot\sqrt{2}$$

Simplify.

$$40\cdot 3p^2\cdot\sqrt{3}$$

$$120p^2\sqrt{3}$$

b. When the radicands involve large numbers, it is often advantageous to factor them in order to find the perfect powers.

$$\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$$

Multiply.

$$6\sqrt[4]{4\cdot 5\cdot 4\cdot 7y^5}$$

Simplify the radical.

$$6\sqrt[4]{16y^4}\cdot\sqrt[4]{35y}$$

Simplify.

$$6\cdot 2y\sqrt[4]{35y}$$

Multiply.

$$12y\sqrt[4]{35y}$$

Exercise 4.5.11

Simplify:

a. $\left(6\sqrt{6x^2}\right)\left(8\sqrt{30x^4}\right)$
b. $\left(-4\sqrt[4]{12y^3}\right)\left(-\sqrt[4]{8y^3}\right)$

Answer

a. $36x^3\sqrt{5}$
b. $8y\sqrt[4]{3y^2}$

Exercise 4.5.12

Simplify:

a. $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$
b. $\left(-4\sqrt[4]{9a^3}\right)\left(3\sqrt[4]{27a^2}\right)$

Answer

a. $144y^2\sqrt{5y}$
b. $-36\sqrt[4]{3a}$

Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the **Distributive Property** to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

Example 4.5.7

Simplify:

a. $\sqrt{6}(\sqrt{2}+\sqrt{18})$
b. $\sqrt[3]{9}(5-\sqrt[3]{18})$

Solution:

a.

$$\sqrt{6}(\sqrt{2}+\sqrt{18})$$

Multiply.

$$\sqrt{12}+\sqrt{108}$$

Simplify.

$$\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$$

Simplify.

$$2\sqrt{3}+6\sqrt{3}$$

Combine like radicals.

$$8\sqrt{3}$$

b.

$$\sqrt[3]{9}(5-\sqrt[3]{18})$$

Distribute.

$$5\sqrt[3]{9}-\sqrt[3]{162}$$

Simplify.

$$5\sqrt[3]{9}-\sqrt[3]{27}\cdot\sqrt[3]{6}$$

Simplify.

$$5\sqrt[3]{9}-3\sqrt[3]{6}$$

Exercise 4.5.13

Simplify:

a. $\sqrt{6}(1+3\sqrt{6})$
b. $\sqrt[3]{4}(-2-\sqrt[3]{6})$

Answer

a. $18+\sqrt{6}$
b. $-2\sqrt[3]{4}-2\sqrt[3]{3}$

Exercise 4.5.14

Simplify:

a. $\sqrt{8}(2-5\sqrt{8})$
b. $\sqrt[3]{3}(-\sqrt[3]{9}-\sqrt[3]{6})$

Answer

a. $-40+4\sqrt{2}$
b. $-3-\sqrt[3]{18}$

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

Example 4.5.8

Simplify:

a. $(3-2\sqrt{7})(4-2\sqrt{7})$
b. $(\sqrt[3]{x}-2)(\sqrt[3]{x}+4)$

Solution:

a.

$$(3-2\sqrt{7})(4-2\sqrt{7})$$

Multiply.

$$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$$

Simplify.

$$12-6\sqrt{7}-8\sqrt{7}+28$$

Combine like terms.

$$40-14\sqrt{7}$$

b.

$$(\sqrt[3]{x}-2)(\sqrt[3]{x}+4)$$

Multiply.

$$\sqrt[3]{x^2}+4\sqrt[3]{x}-2\sqrt[3]{x}-8$$

Combine like terms.

$$\sqrt[3]{x^2}+2\sqrt[3]{x}-8$$

Exercise 4.5.15

Simplify:

a. $(6-3\sqrt{7})(3+4\sqrt{7})$
b. $(\sqrt[3]{x}-2)(\sqrt[3]{x}-3)$

Answer

a. $-66+15\sqrt{7}$
b. $\sqrt[3]{x^2}-5\sqrt[3]{x}+6$

Exercise 4.5.16

Simplify:

a. $(2-3\sqrt{11})(4-\sqrt{11})$
b. $(\sqrt[3]{x}+1)(\sqrt[3]{x}+3)$

Answer

a. $41-14\sqrt{11}$
b. $\sqrt[3]{x^2}+4\sqrt[3]{x}+3$

Example 4.5.9

Simplify: $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$

Solution:

$$(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$$

Multiply.

$$3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$$

Simplify.

$$6+12\sqrt{10}-\sqrt{10}-20$$

Combine like terms.

$$-14+11\sqrt{10}$$

Exercise 4.5.17

Simplify: $(5\sqrt{3}-\sqrt{7})(\sqrt{3}+2\sqrt{7})$

Answer

$1+9\sqrt{21}$

Exercise 4.5.18

Simplify: $(\sqrt{6}-3\sqrt{8})(2\sqrt{6}+\sqrt{8})$

Answer

$-12-20\sqrt{3}$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

Special Products

Binomial Squares

$$(a+b)^2=a^2+2ab+b^2$$
$$(a-b)^2=a^2-2ab+b^2$$

Product of Conjugates

$$(a+b)(a-b)=a^2-b^2$$

We will use the special product formulas in the next few examples. We will start with the **Product of Binomial Squares Pattern**.

Example 4.5.10

Simplify:

a. $(2+\sqrt{3})^2$
b. $(4-2\sqrt{5})^2$

Solution:

a.

Table 8.4.1

	$(a+b)^2$ $(2+\sqrt{3})^2$
Multiply using the Product of Binomial Squares Pattern.	$a^2+2ab+b^2$ $2^2+2\cdot 2\cdot\sqrt{3}+(\sqrt{3})^2$
Simplify.	$4+4\sqrt{3}+3$
Combine like terms.	$7+4\sqrt{3}$

b.

Table 8.4.2

	$(a-b)^2$ $(4-2\sqrt{5})^2$
Multiple, using the Product of Binomial Squares Pattern.	$a^2-2ab+b^2$ $4^2-2\cdot 4\cdot 2\sqrt{5}+(2\sqrt{5})^2$
Simplify.	$16-16\sqrt{5}+4\cdot 5$
	$16-16\sqrt{5}+20$
Combine like terms.	$36-16\sqrt{5}$

Exercise 4.5.19

Simplify:

a. $(10+\sqrt{2})^2$
b. $(1+3\sqrt{6})^2$

Answer

a. $102+20\sqrt{2}$
b. $55+6\sqrt{6}$

Exercise 4.5.20

Simplify:

a. $(6-\sqrt{5})^2$
b. $(9-2\sqrt{10})^2$

Answer

a. $41-12\sqrt{5}$
b. $121-36\sqrt{10}$

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

Example 4.5.11

Simplify: $(5-2\sqrt{3})(5+2\sqrt{3})$

Solution:

Table 8.4.3

	$(a-b)(a+b)$ $(5-2\sqrt{3})(5+2\sqrt{3})$
Multiply using the Product of Conjugates Pattern.	a^2-b^2 $5^2-(2\sqrt{3})^2$
Simplify.	$25-4\cdot 3$
	13

Exercise 4.5.21

Simplify: $(3-2\sqrt{5})(3+2\sqrt{5})$

Answer

-11

Exercise 4.5.22

Simplify: $(4+5\sqrt{7})(4-5\sqrt{7})$

Answer

-159

Access these online resources for additional instruction and practice with adding, subtracting, and multiplying radical expressions.

- Multiplying Adding Subtracting Radicals
- Multiplying Special Products: Square Binomials Containing Square Roots
- Multiplying Conjugates

Key Concepts

- **Product Property of Roots**
  - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$ $\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$ and $\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}$
- **Special Products**

Binomial Squares	Product of Conjugates
$(a+b)^2=a^2+2ab+b^2$	$(a+b)(a-b)=a^2-b^2$
$(a-b)^2=a^2-2ab+b^2$	

Glossary

like radicals

Like radicals are radical expressions with the same index and the same radicand.

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4.5E: Exercises

Practice Makes Perfect

Exercise A: add and subtract radical expressions

In the following exercises, simplify. Assume all variables are greater than or equal to zero so that absolute values are not needed.

1. a. $8\sqrt{2}-5\sqrt{2}$ b. $5\sqrt[3]{m}+2\sqrt[3]{m}$ c. $8\sqrt[4]{m}-2\sqrt[4]{n}$
2. a. $7\sqrt{2}-3\sqrt{2}$ b. $7\sqrt[3]{p}+2\sqrt[3]{p}$ c. $5\sqrt[3]{x}-3\sqrt[3]{x}$
3. a. $3\sqrt{5}+6\sqrt{5}$ b. $9\sqrt[3]{a}+3\sqrt[3]{a}$ c. $5\sqrt[4]{2z}+\sqrt[4]{2z}$
4. a. $4\sqrt{5}+8\sqrt{5}$ b. $\sqrt[3]{m}-4\sqrt[3]{m}$ c. $\sqrt{n}+3\sqrt{n}$
5. a. $3\sqrt{2a}-4\sqrt{2a}+5\sqrt{2a}$ b. $5\sqrt[4]{3ab}-3\sqrt[4]{3ab}-2\sqrt[4]{3ab}$
6. a. $\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$ b. $8\sqrt[4]{11cd}+5\sqrt[4]{11cd}-9\sqrt[4]{11cd}$
7. a. $8\sqrt{3c}+2\sqrt{3c}-9\sqrt{3c}$ b. $2\sqrt[3]{4pq}-5\sqrt[3]{4pq}+4\sqrt[3]{4pq}$
8. a. $3\sqrt{5d}+8\sqrt{5d}-11\sqrt{5d}$ b. $11\sqrt[3]{2rs}-9\sqrt[3]{2rs}+3\sqrt[3]{2rs}$
9. a. $\sqrt{27}-\sqrt{75}$ b. $\sqrt[3]{40}-\sqrt[3]{320}$ c. $\frac{1}{2}\sqrt[4]{32}+\frac{2}{3}\sqrt[4]{162}$
10. a. $\sqrt{72}-\sqrt{98}$ b. $\sqrt[3]{24}+\sqrt[3]{81}$ c. $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$
11. a. $\sqrt{48}+\sqrt{27}$ b. $\sqrt[3]{54}+\sqrt[3]{128}$ c. $6\sqrt[4]{5}-\frac{3}{2}\sqrt[4]{320}$
12. a. $\sqrt{45}+\sqrt{80}$ b. $\sqrt[3]{81}-\sqrt[3]{192}$ c. $\frac{5}{2}\sqrt[4]{80}+\frac{7}{3}\sqrt[4]{405}$
13. a. $\sqrt{72a^5}-\sqrt{50a^5}$ b. $9\sqrt[4]{80p^4}-6\sqrt[4]{405p^4}$
14. a. $\sqrt{48b^5}-\sqrt{75b^5}$ b. $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$
15. a. $\sqrt{80c^7}-\sqrt{20c^7}$ b. $2\sqrt[4]{162r^{10}}+4\sqrt[4]{32r^{10}}$
16. a. $\sqrt{96d^9}-\sqrt{24d^9}$ b. $5\sqrt[4]{243s^6}+2\sqrt[4]{3s^6}$
17. $3\sqrt{128y^2}+4y\sqrt{162}-8\sqrt{98y^2}$
18. $3\sqrt{75y^2}+8y\sqrt{48}-\sqrt{300y^2}$

Answer

1. a. $3\sqrt{2}$ b. $7\sqrt[3]{m}$ c. $6\sqrt[4]{m}$
3. a. $9\sqrt{5}$ b. $12\sqrt[3]{a}$ c. $6\sqrt[4]{2z}$
5. a. $4\sqrt{2a}$ b. 0
7. a. $\sqrt{3c}$ b. $\sqrt[3]{4pq}$
9. a. $-2\sqrt{3}$ b. $-2\sqrt[3]{5}$ c. $3\sqrt[4]{2}$

11. a. $7\sqrt{3}$ b. $7\sqrt[3]{2}$ c. $3\sqrt[4]{5}$

13. a. $a^2\sqrt{2a}$ b. 0

15. a. $2c^3\sqrt{5c}$ b. $14r^2\sqrt[4]{2r^2}$

17. $4y\sqrt{2}$

Exercise B: multiply radical expressions

In the following exercises, simplify.

1. a. $(-2\sqrt{3})(3\sqrt{18})$

   b. $(8\sqrt[3]{4})(-4\sqrt[3]{18})$

   c.

2. a. $(-4\sqrt{5})(5\sqrt{10})$

   b. $(-2\sqrt[3]{9})(7\sqrt[3]{9})$

   c.

3. a. $(5\sqrt{6})(-\sqrt{12})$

   b. $(-2\sqrt[4]{18})(-\sqrt[4]{9})$

   c.

4. a. $(-2\sqrt{7})(-2\sqrt{14})$

   b. $(-3\sqrt[4]{8})(-5\sqrt[4]{6})$

   c.

5. a. $\left(4\sqrt{12z^3}\right)\left(3\sqrt{9z}\right)$

   b. $\left(5\sqrt[3]{3x^3}\right)\left(3\sqrt[3]{18x^3}\right)$

c.

6. a. $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$

   b. $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

   c.

7. a. $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$

   b. $\left(2\sqrt[4]{8y^2}\right)\left(-2\sqrt[4]{12y^3}\right)$

   c.

8. a. $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$

   b. $\left(-\sqrt[4]{6b^3}\right)\left(3\sqrt[4]{8b^3}\right)$

   c.

Answer

1.

   a. $-18\sqrt{6}$

   b. $-64\sqrt[3]{9}$

   c.

3.

   a. $-30\sqrt{2}$

   b. $6\sqrt[4]{2}$

c.

5.

a. $72z^2\sqrt{3}$

b. $45x^2\sqrt[3]{2}$

c.

7.

a. $-42z^5\sqrt{2z}$

b. $-8y\sqrt[4]{6y}$

Exercise C: use polynomial multiplication to multiply radical expressions

In the following exercises, multiply.

1. a. $\sqrt{7}(5+2\sqrt{7})$

   b. $\sqrt[3]{6}(4+\sqrt[3]{18})$

   c.

2. a. $\sqrt{11}(8+4\sqrt{11})$

   b. $\sqrt[3]{3}(\sqrt[3]{9}+\sqrt[3]{18})$

   c.

3. a. $\sqrt{11}(-3+4\sqrt{11})$

   b. $\sqrt[4]{3}(\sqrt[4]{54}+\sqrt[4]{18})$

   c.

4. a. $\sqrt{2}(-5+9\sqrt{2})$

   b. $\sqrt[4]{2}(\sqrt[4]{12}+\sqrt[4]{24})$

c.

5. $(7+\sqrt{3})(9-\sqrt{3})$

6. $(8-\sqrt{2})(3+\sqrt{2})$

7. a. $(9-3\sqrt{2})(6+4\sqrt{2})$

   b. $(\sqrt[3]{x}-3)(\sqrt[3]{x}+1)$

   c.

8. a. $(3-2\sqrt{7})(5-4\sqrt{7})$

   b. $(\sqrt[3]{x}-5)(\sqrt[3]{x}-3)$

   c.

9. a. $(1+3\sqrt{10})(5-2\sqrt{10})$

   b. $(2\sqrt[3]{x}+6)(\sqrt[3]{x}+1)$

   c.

10. a. $(7-2\sqrt{5})(4+9\sqrt{5})$

    b. $(3\sqrt[3]{x}+2)(\sqrt[3]{x}-2)$

    c.

11. $(\sqrt{3}+\sqrt{10})(\sqrt{3}+2\sqrt{10})$

12. $(\sqrt{11}+\sqrt{5})(\sqrt{11}+6\sqrt{5})$

13. $(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$

14. $(4\sqrt{6}+7\sqrt{13})(8\sqrt{6}-3\sqrt{13})$

15. a. $(3+\sqrt{5})^2$

    b. $(2-5\sqrt{3})^2$

    c.

16. a. $(4+\sqrt{11})^2$

    b. $(3-2\sqrt{5})^2$

    c.

17. a. $(9-\sqrt{6})^2$

    b. $(10+3\sqrt{7})^2$

    c.

18. a. $(5-\sqrt{10})^2$

    b. $(8+3\sqrt{2})^2$

    c.

19. $(4+\sqrt{2})(4-\sqrt{2})$

20. $(7+\sqrt{10})(7-\sqrt{10})$

21. $(4+9\sqrt{3})(4-9\sqrt{3})$

22. $(1+8\sqrt{2})(1-8\sqrt{2})$

23. $(12-5\sqrt{5})(12+5\sqrt{5})$

24. $(9-4\sqrt{3})(9+4\sqrt{3})$

25. $(\sqrt[3]{3x}+2)(\sqrt[3]{3x}-2)$

26. $(\sqrt[3]{4x}+3)(\sqrt[3]{4x}-3)$

Answer

1.

a. $14+5\sqrt{7}$

b. $4\sqrt[3]{6}+3\sqrt[3]{4}$

c.

3.

a. $44-3\sqrt{11}$

b. $3\sqrt[4]{2}+\sqrt[4]{54}$

c.

5. $60+2\sqrt{3}$

7.

a. $30+18\sqrt{2}$

b. $\sqrt[3]{x^2}-2\sqrt[3]{x}-3$

c.

9.

a. $-54+13\sqrt{10}$

b. $2\sqrt[3]{x^2}+8\sqrt[3]{x}+6$

c.

11. $23+3\sqrt{30}$

13. $-439-2\sqrt{77}$

15.

a. $14+6\sqrt{5}$

b. $79-20\sqrt{3}$

17.

a. $87-18\sqrt{6}$

b. $163+60\sqrt{7}$

19. 14

21. -227

23. 19

25. $\sqrt[3]{9x^2}-4$

Exercise D: mixed practice

1. $\frac{2}{3}\sqrt{27}+\frac{3}{4}\sqrt{48}$
2. $\sqrt{175k^4}-\sqrt{63k^4}$
3. $\frac{5}{6}\sqrt{162}+\frac{3}{16}\sqrt{128}$
4. $\sqrt[3]{24}+\sqrt[3]{81}$
5. $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$
6. $8\sqrt[4]{13}-4\sqrt[4]{13}-3\sqrt[4]{13}$
7. $5\sqrt{12c^4}-3\sqrt{27c^6}$
8. $\sqrt{80a^5}-\sqrt{45a^5}$
9. $\frac{3}{5}\sqrt{75}-\frac{1}{4}\sqrt{48}$
10. $21\sqrt[3]{9}-2\sqrt[3]{9}$
11. $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$
12. $11\sqrt{11}-10\sqrt{11}$
13. $\sqrt{3}\cdot\sqrt{21}$
14. $(4\sqrt{6})(-\sqrt{18})$
15. $(7\sqrt[3]{4})(-3\sqrt[3]{18})$
16. $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$
17. $(\sqrt{29})^2$
18. $(-4\sqrt{17})(-3\sqrt{17})$
19. $(-4+\sqrt{17})(-3+\sqrt{17})$
20. $\left(3\sqrt[4]{8a^2}\right)\left(\sqrt[4]{12a^3}\right)$

21. $(6-3\sqrt{2})^2$

22. $\sqrt{3}(4-3\sqrt{3})$

23. $\sqrt[3]{3}(2\sqrt[3]{9}+\sqrt[3]{18})$

24. $(\sqrt{6}+\sqrt{3})(\sqrt{6}+6\sqrt{3})$

Answer

1. $5\sqrt{3}$
3. $9\sqrt{2}$
5. $-\sqrt[4]{5}$
7. $10c^2\sqrt{3}-9c^3\sqrt{3}$
9. $2\sqrt{3}$
11. $17q^2$
13. $3\sqrt{7}$
15. $-42\sqrt[3]{9}$
17. 29
19. $29-7\sqrt{17}$
21. $72-36\sqrt{2}$
23. $6+3\sqrt[3]{2}$

Exercise E: writing exercises

1. Explain when a radical expression is in simplest form.
2. Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.
3. a. Explain why $(-\sqrt{n})^2$ is always non-negative, for $n \geq 0$.
   b. Explain why $-(\sqrt{n})^2$ is always non-positive, for $n \geq 0$.
4. Use the binomial square pattern to simplify $(3+\sqrt{2})^2$. Explain all your steps.

Answer

1. Answers will vary
3. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract radical expressions.			
multiply radical expressions.			
use polynomial multiplication to multiply radical expressions.			

Figure 8.4.14

https://math.libretexts.org/@go/page/58457

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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4.6: Divide Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

Before you get started, take this readiness quiz.

1. Simplify: $\frac{30}{48}$.
   If you missed this problem, review Example 1.24.
2. Simplify: $x^2 \cdot x^4$.
   If you missed this problem, review Example 5.12.
3. Multiply: $(7+3x)(7-3x)$.
   If you missed this problem, review Example 5.32.

Divide Radical Expressions

We have used the **Quotient Property of Radical Expressions** to simplify roots of fractions. We will need to use this property 'in reverse' to simplify a fraction with radicals. We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

Definition 4.6.1: Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

Example 4.6.1

Simplify:

a. $\frac{\sqrt{72x^3}}{\sqrt{162x}}$

b. $\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$

Solution:

a.

$$\frac{\sqrt{72x^3}}{\sqrt{162x}}$$

Rewrite using the quotient property,

$$\sqrt{\frac{72x^3}{162x}}$$

Remove common factors.

$$\sqrt{\frac{\cancel{18}\cdot 4\cdot x^2\cdot \cancel{x}}{\cancel{18}\cdot 9\cdot \cancel{x}}}$$

Simplify.

$$\sqrt{\frac{4x^2}{9}}$$

Simplify the radical.

$$\frac{2x}{3}$$

b.

$$\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$$

Rewrite using the quotient property, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$.

$$\sqrt[3]{\frac{32x^2}{4x^5}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{\frac{8}{x^3}}$$

Simplify the radical.

$$\frac{2}{x}$$

Exercise 4.6.1

Simplify:

a. $\frac{\sqrt{50s^3}}{\sqrt{128s}}$

b. $\frac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}}$

Answer

a. $\frac{5s}{8}$

b. $\frac{2}{a}$

Exercise 4.6.2

Simplify:

a. $\frac{\sqrt{75q^5}}{\sqrt{108q}}$

b. $\frac{\sqrt[3]{72b^2}}{\sqrt[3]{9b^5}}$

Answer

a. $\frac{5q^2}{6}$

b. $\frac{2}{b}$

Example 4.6.2

Simplify:

a. $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$

b. $\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$

Solution:

a.

$$\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$$

Rewrite using the quotient property.

$$\sqrt{\frac{147ab^8}{3a^3b^4}}$$

Remove common factors in the fraction.

$$\sqrt{\frac{49b^4}{a^2}}$$

Simplify the radical.

$$\frac{7b^2}{a}$$

b.

$$\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$$

Rewrite using the quotient property.

$$\sqrt[3]{\frac{-250mn^{-2}}{2m^{-2}n^4}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{\frac{-125m^3}{n^6}}$$

Simplify the radical.

$$-\frac{5m}{n^2}$$

Exercise 4.6.3

Simplify:

a. $\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$

b. $\frac{\sqrt[3]{-128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}}$

Answer

a. $\frac{9x^2}{y^2}$
b. $\frac{-4x}{y}$

Exercise 4.6.4

Simplify:

a. $\frac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$
b. $\frac{\sqrt[3]{-81pq^{-1}}}{\sqrt[3]{3p^{-2}q^5}}$

Answer

a. $\frac{10n^3}{m}$
b. $\frac{-3p}{q^2}$

Example 4.6.3

Simplify: $\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$

Solution:

$$\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$$

Rewrite using the quotient property.

$$\sqrt{\frac{54x^5y^3}{3x^2y}}$$

Remove common factors in the fraction.

$$\sqrt{18x^3y^2}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{9x^2y^2\cdot 2x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{9x^2y^2}\cdot\sqrt{2x}$$

Simplify.

$$3xy\sqrt{2x}$$

Exercise 4.6.5

Simplify: $\frac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$

Answer

$4xy\sqrt{2x}$

Exercise 4.6.6

Simplify: $\frac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$

Answer

$4ab\sqrt{3b}$

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator. This process is still used today, and is useful in other areas of mathematics, too.

Definition 4.6.2: Rationalizing the Denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a **radical expression** is not considered simplified if the radicand contains a fraction.

Simplified Radical Expressions

A radical expression is considered simplified if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

Example 4.6.4

Simplify:

a. $\frac{4}{\sqrt{3}}$

b. $\sqrt{\frac{3}{20}}$

c. $\frac{3}{\sqrt{6x}}$

Solution:

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

Table 8.5.1

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

Table 8.5.2

	$\sqrt{\frac{3}{20}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{3}}{\sqrt{20}}$
Simplify the denominator.	$\frac{\sqrt{3}}{2\sqrt{5}}$
Multiply the numerator and denominator by $\sqrt{5}$.	$\frac{\sqrt{3} \cdot \sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}}$
Simplify.	$\frac{\sqrt{15}}{2 \cdot 5}$
Simplify.	$\frac{\sqrt{15}}{10}$

c.

Table 8.5.3

	$\frac{3}{\sqrt{6x}}$
Multiply the numerator and denominator by $\sqrt{6x}$.	$\frac{3 \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}}$
Simplify.	$\frac{3\sqrt{6x}}{6x}$
Simplify.	$\frac{\sqrt{6x}}{2x}$

? Exercise 4.6.7

Simplify:

a. $\frac{5}{\sqrt{3}}$

b. $\sqrt{\frac{3}{32}}$

c. $\frac{2}{\sqrt{2x}}$

Answer

a. $\frac{5\sqrt{3}}{3}$

b. $\frac{\sqrt{6}}{8}$

c. $\dfrac{\sqrt{2x}}{x}$

Exercise 4.6.8

Simplify:

a. $\dfrac{6}{\sqrt{5}}$

b. $\sqrt{\dfrac{7}{18}}$

c. $\dfrac{5}{\sqrt{5x}}$

Answer

a. $\dfrac{6\sqrt{5}}{5}$

b. $\dfrac{\sqrt{14}}{6}$

c. $\dfrac{\sqrt{5x}}{x}$

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

We will follow a similar process to rationalize higher roots. To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index. When we simplify the new radical, the denominator will no longer have a radical.

For example,

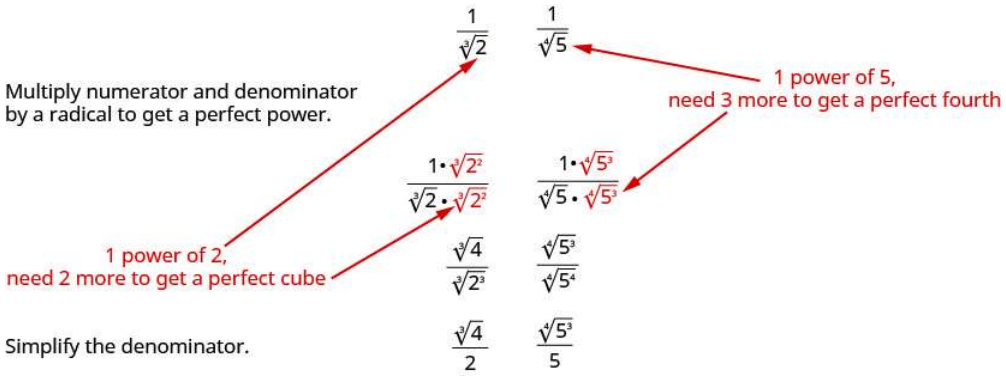


Figure 8.5.14

We will use this technique in the next examples.

Example 4.6.5

Simplify:

a. $\dfrac{1}{\sqrt[3]{6}}$

b. $\sqrt[3]{\dfrac{7}{24}}$

c. $\dfrac{3}{\sqrt[3]{4x}}$

Solution:

To rationalize a denominator with a cube root, we can multiply by a cube root that will give us a perfect cube in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

Table 8.5.4

	$\frac{1}{\sqrt[3]{6}}$
The radical in the denominator has one factor of 6. Multiply both the numerator and denominator by $\sqrt[3]{6^2}$, which gives us 2 more factors of 6.	$\frac{1 \cdot \sqrt[3]{6^2}}{\sqrt[3]{6} \cdot \sqrt[3]{6^2}}$
Multiply. Notice the radicand in the denominator has 3 powers of 6.	$\frac{\sqrt[3]{6^2}}{\sqrt[3]{6^3}}$
Simplify the cube root in the denominator.	$\frac{\sqrt[3]{36}}{6}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

Table 8.5.5

	$\sqrt[3]{\frac{7}{24}}$
The fraction is not a perfect cube, so rewrite using the Quotient Property.	$\frac{\sqrt[3]{7}}{\sqrt[3]{24}}$
Simplify the denominator.	$\frac{\sqrt[3]{7}}{2\sqrt[3]{3}}$
Multiply the numerator and denominator by $\sqrt[3]{3^2}$. This will give us 3 factors of 3.	$\frac{\sqrt[3]{7} \cdot \sqrt[3]{3^2}}{2\sqrt[3]{3} \cdot \sqrt[3]{3^2}}$
Simplify.	$\frac{\sqrt[3]{63}}{2\sqrt[3]{3^3}}$
Remember, $\sqrt[3]{3^3} = 3$.	$\frac{\sqrt[3]{63}}{2 \cdot 3}$
Simplify.	$\frac{\sqrt[3]{63}}{6}$

c.

Table 8.5.6

	$\frac{3}{\sqrt[3]{4x}}$
Rewrite the radicand to show the factors.	$\frac{3}{\sqrt[3]{2^2 \cdot x}}$
Multiple the numerator and denominator by $\sqrt[3]{2 \cdot x^2}$. This will get us 3 factors of 2 and 3 factors of x.	$\frac{3 \cdot \sqrt[3]{2 \cdot x^2}}{\sqrt[3]{2^2 x} \cdot \sqrt[3]{2 \cdot x^2}}$
Simplify.	$\frac{3\sqrt[3]{2x^2}}{\sqrt[3]{2^3 x^3}}$
Simplify the radical in the denominator.	$\frac{3\sqrt[3]{2x^2}}{2x}$

Exercise 4.6.9

Simplify:

a. $\frac{1}{\sqrt[3]{7}}$

b. $\sqrt[3]{\frac{5}{12}}$

c. $\frac{5}{\sqrt[3]{9y}}$

Answer

a. $\frac{\sqrt[3]{49}}{7}$

b. $\frac{\sqrt[3]{90}}{6}$

c. $\frac{5\sqrt[3]{3y^2}}{3y}$

Exercise 4.6.10

Simplify:

a. $\frac{1}{\sqrt[3]{2}}$

b. $\sqrt[3]{\frac{3}{20}}$

c. $\frac{2}{\sqrt[3]{25n}}$

Answer

a. $\frac{\sqrt[3]{4}}{2}$

b. $\frac{\sqrt[3]{150}}{10}$

c. $\frac{2\sqrt[3]{5n^2}}{5n}$

Example 4.6.6

Simplify:

a. $\frac{1}{\sqrt[4]{2}}$

b. $\sqrt[4]{\frac{5}{64}}$

c. $\frac{2}{\sqrt[4]{8x}}$

Solution:

To rationalize a denominator with a fourth root, we can multiply by a fourth root that will give us a perfect fourth power in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

Table 8.5.7

	$\frac{1}{\sqrt[4]{2}}$
The radical in the denominator has one factor of 2. Multiply both the numerator and denominator by $\sqrt[4]{2^3}$, which gives us 3 more factors of 2.	$\frac{1 \cdot \sqrt[4]{2^3}}{\sqrt[4]{2} \cdot \sqrt[4]{2^3}}$
Multiply. Notice the radicand in the denominator has 4 powers of 2.	$\frac{\sqrt[4]{8}}{\sqrt[4]{2^4}}$
Simplify the fourth root in the denominator.	$\frac{\sqrt[4]{8}}{2}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

Table 8.5.8

	$\sqrt[4]{\frac{5}{64}}$
The fraction is not a perfect fourth power, so rewrite using the Quotient Property.	$\frac{\sqrt[4]{5}}{\sqrt[4]{64}}$
Rewrite the radicand in the denominator to show the factors.	$\frac{\sqrt[4]{5}}{\sqrt[4]{2^6}}$
Simplify the denominator.	$\frac{\sqrt[4]{5}}{2\sqrt[4]{2^2}}$
Multiply the numerator and denominator by $\sqrt[4]{2^2}$. This will give us 4 factors of 2.	$\frac{\sqrt[4]{5} \cdot \sqrt[4]{2^2}}{2\sqrt[4]{2^2} \cdot \sqrt[4]{2^2}}$
Simplify.	$\frac{\sqrt[4]{5} \cdot \sqrt[4]{4}}{2\sqrt[4]{2^4}}$
Remember, $\sqrt[4]{2^4} = 2$.	$\frac{\sqrt[4]{20}}{2 \cdot 2}$
Simplify.	$\frac{\sqrt[4]{20}}{4}$

c.

Table 8.5.9

	$\frac{2}{\sqrt[4]{8x}}$
Rewrite the radicand to show the factors.	$\frac{2}{\sqrt[4]{2^3 \cdot x}}$
Multiply the numerator and denominator by $\sqrt[4]{2 \cdot x^3}$. This will get us 4 factors of 2 and 4 factors of x.	$\frac{2 \cdot \sqrt[4]{2 \cdot x^3}}{\sqrt[4]{2^3 x} \cdot \sqrt[4]{2 \cdot x^3}}$
Simplify.	$\frac{2\sqrt[4]{2x^3}}{\sqrt[4]{2^4 x^4}}$
Simplify the radical in the denominator.	$\frac{2\sqrt[4]{2x^3}}{2x}$
Simplify the fraction.	$\frac{\sqrt[4]{2x^3}}{x}$

Exercise 4.6.11

Simplify:

a. $\frac{1}{\sqrt[4]{3}}$

b. $\sqrt[4]{\frac{3}{64}}$

c. $\frac{3}{\sqrt[4]{125x}}$

Answer

a. $\frac{\sqrt[4]{27}}{3}$

b. $\frac{\sqrt[4]{12}}{4}$

c. $\frac{3\sqrt[4]{5x^3}}{5x}$

Exercise 4.6.12

Simplify:

a. $\frac{1}{\sqrt[4]{5}}$

b. $\sqrt[4]{\frac{7}{128}}$

c. $\frac{4}{\sqrt[4]{4x}}$

Answer

a. $\frac{\sqrt[4]{125}}{5}$

b. $\frac{\sqrt[4]{224}}{8}$

c. $\frac{\sqrt[4]{64x^3}}{x}$

Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the **Product of Conjugates Pattern** to **rationalize the denominator**.

$$(a-b)(a+b) \qquad (2-\sqrt{5})(2+\sqrt{5})$$
$$a^2-b^2 \qquad 2^2-(\sqrt{5})^2$$
$$4-5$$
$$-1$$

When we multiple a binomial that includes a square root by its conjugate, the product has no square roots.

Example 4.6.7

Simplify: $\frac{5}{2-\sqrt{3}}$

Solution:

Table 8.5.10

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$\frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{1}$
Simplify.	$5(2+\sqrt{3})$

? Exercise 4.6.13

Simplify: $\frac{3}{1-\sqrt{5}}$.

Answer

$-\frac{3(1+\sqrt{5})}{4}$

? Exercise 4.6.14

Simplify: $\frac{2}{4-\sqrt{6}}$.

Answer

$\frac{4+\sqrt{6}}{5}$

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

✔ Example 4.6.8

Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution:

Table 8.5.11

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}$
Multiply the conjugates in the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}$
Simplify the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$

Exercise 4.6.15

Simplify: $\dfrac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Answer

$$\frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{x-2}$$

Exercise 4.6.16

Simplify: $\dfrac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$

Answer

$$\frac{\sqrt{10}(\sqrt{y}+\sqrt{3})}{y-3}$$

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

Example 4.6.9

Simplify: $\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution:

Table 8.5.12

	$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$\frac{(\sqrt{x}+\sqrt{7})^2}{x-7}$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

Exercise 4.6.17

Simplify: $\dfrac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

Answer

$$\frac{(\sqrt{p}+\sqrt{2})^2}{p-2}$$

? Exercise 4.6.18

Simplify: $\dfrac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$

Answer

$$\frac{(\sqrt{q}-\sqrt{10})^2}{q-10}$$

Key Concepts

- **Quotient Property of Radical Expressions**
  - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then, $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$
- **Simplified Radical Expressions**
  - A radical expression is considered simplified if there are:
    - no factors in the radicand that have perfect powers of the index
    - no fractions in the radicand
    - no radicals in the denominator of a fraction

Glossary

rationalizing the denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

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4.6E: Exercises

Practice Makes Perfect

Exercise SET A: divide square roots

In the following exercises, simplify.

1. a. $\frac{\sqrt{128}}{\sqrt{72}}$ b. $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$
2. a. $\frac{\sqrt{48}}{\sqrt{75}}$ b. $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$
3. a. $\frac{\sqrt{200m^5}}{\sqrt{98m}}$ b. $\frac{\sqrt[3]{54y^2}}{\sqrt[3]{2y^5}}$
4. a. $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$ b. $\frac{\sqrt[3]{54y}}{\sqrt[3]{16y^4}}$
5. a. $\frac{\sqrt{75r^3}}{\sqrt{108r^7}}$ b. $\frac{\sqrt[3]{24x^7}}{\sqrt[3]{81x^4}}$
6. a. $\frac{\sqrt{196q}}{\sqrt{484q^5}}$ b. $\frac{\sqrt[3]{16m^4}}{\sqrt[3]{54m}}$
7. a. $\frac{\sqrt{108p^5q^2}}{\sqrt{3p^3q^6}}$ b. $\frac{\sqrt[3]{-16a^4b^{-2}}}{\sqrt[3]{2a^{-2}b}}$
8. a. $\frac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$ b. $\frac{\sqrt[3]{-375y^4z^2}}{\sqrt[3]{3y^{-2}z^4}}$
9. a. $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$ b. $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$
10. a. $\frac{\sqrt{810c^{-3}d^7}}{\sqrt{1000cd}}$ b. $\frac{\sqrt[3]{24a^7b^{-1}}}{\sqrt[3]{-81a^{-2}b^2}}$
11. $\frac{\sqrt{56x^5y^4}}{\sqrt{2xy^3}}$
12. $\frac{\sqrt{72a^3b^6}}{\sqrt{3ab^3}}$
13. $\frac{\sqrt[3]{48a^3b^6}}{\sqrt[3]{3a^{-1}b^3}}$
14. $\frac{\sqrt[3]{162x^{-3}y^6}}{\sqrt[3]{2x^3y^{-2}}}$

Answer

1. a. $\frac{4}{3}$ b. $\frac{4}{3}$
3. a. $\frac{10m^2}{7}$ b. $\frac{3}{y}$
5. a. $\frac{5}{6r^2}$ b. $\frac{2x}{3}$

7. a. $\frac{6p}{q^2}$ b. $-\frac{2a^2}{b}$

9. a. $\frac{8m^4}{3n^4}$ b. $-\frac{2x^2}{3y^2}$

11. $4x^4\sqrt{7y}$

13. $2ab\sqrt[3]{2a}$

Exercise SET B: Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

15. a. $\frac{10}{\sqrt{6}}$ b. $\sqrt{\frac{4}{27}}$ c. $\frac{10}{\sqrt{5x}}$

16. a. $\frac{8}{\sqrt{3}}$ b. $\sqrt{\frac{7}{40}}$ c. $\frac{8}{\sqrt{2y}}$

17. a. $\frac{6}{\sqrt{7}}$ b. $\sqrt{\frac{8}{45}}$ c. $\frac{12}{\sqrt{3p}}$

18. a. $\frac{4}{\sqrt{5}}$ b. $\sqrt{\frac{27}{80}}$ c. $\frac{18}{\sqrt{6q}}$

19. a. $\frac{1}{\sqrt[3]{5}}$ b. $\sqrt[3]{\frac{5}{24}}$ c. $\frac{4}{\sqrt[3]{36a}}$

20. a. $\frac{1}{\sqrt[3]{3}}$ b. $\sqrt[3]{\frac{5}{32}}$ c. $\frac{7}{\sqrt[3]{49b}}$

21. a. $\frac{1}{\sqrt[3]{11}}$ b. $\sqrt[3]{\frac{7}{54}}$ c. $\frac{3}{\sqrt[3]{3x^2}}$

22. a. $\frac{1}{\sqrt[3]{13}}$ b. $\sqrt[3]{\frac{3}{128}}$ c. $\frac{3}{\sqrt[3]{6y^2}}$

23. a. $\frac{1}{\sqrt[4]{7}}$ b. $\sqrt[4]{\frac{5}{32}}$ c. $\frac{4}{\sqrt[4]{4x^2}}$

24. a. $\frac{1}{\sqrt[4]{4}}$ b. $\sqrt[4]{\frac{9}{32}}$ c. $\frac{6}{\sqrt[4]{9x^3}}$

25. a. $\frac{1}{\sqrt[4]{9}}$ b. $\sqrt[4]{\frac{25}{128}}$ c. $\frac{6}{\sqrt[4]{27a}}$

26. a. $\frac{1}{\sqrt[4]{8}}$ b. $\sqrt[4]{\frac{27}{128}}$ c. $\frac{16}{\sqrt[4]{64b^2}}$

Answer

15. a. $\frac{5\sqrt{6}}{3}$ b. $\frac{2\sqrt{3}}{9}$ c. $\frac{2\sqrt{5x}}{x}$

17. a. $\frac{6\sqrt{7}}{7}$ b. $\frac{2\sqrt{10}}{15}$ c. $\frac{4\sqrt{3p}}{p}$

19. a. $\frac{\sqrt[3]{25}}{5}$ b. $\frac{\sqrt[3]{45}}{6}$ c. $\frac{2\sqrt[3]{6a^2}}{3a}$

21. a. $\frac{\sqrt[3]{121}}{11}$ b. $\frac{\sqrt[3]{28}}{6}$ c. $\frac{\sqrt[3]{9x}}{x}$

23. a. $\frac{\sqrt[4]{343}}{7}$ b. $\frac{\sqrt[4]{40}}{4}$ c. $\frac{2\sqrt[4]{4x^2}}{x}$

25. a. $\frac{\sqrt[4]{9}}{3}$ b. $\frac{\sqrt[4]{50}}{4}$ c. $\frac{2\sqrt[4]{3a^2}}{a}$

Exercise SET C: Rationalize a Two Term Denominator

In the following exercises, simplify.

27. $\frac{8}{1-\sqrt{5}}$

28. $\frac{7}{2-\sqrt{6}}$

29. $\frac{6}{3-\sqrt{7}}$

30. $\frac{5}{4-\sqrt{11}}$

31. $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

32. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

33. $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

34. $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

35. $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

36. $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

37. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

38. $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Answer

27. $-2(1+\sqrt{5})$

29. $3(3+\sqrt{7})$

31. $\frac{\sqrt{3}(\sqrt{m}+\sqrt{5})}{m-5}$

33. $\frac{\sqrt{2}(\sqrt{x}+\sqrt{6})}{x-6}$

35. $\frac{(\sqrt{r}+\sqrt{5})^2}{r-5}$

37. $\frac{(\sqrt{x}+2\sqrt{2})^2}{x-8}$

Exercise SET D: writing exercises

1. a. Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.

   b. Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.

   c. Why are the two methods of simplifying square roots different?

2. Explain what is meant by the word rationalize in the phrase, "rationalize a denominator."
3. Explain why multiplying $\sqrt{2x}-3$ by its conjugate results in an epxression with no radicals.
4. Explain why multiplying $\frac{7}{\sqrt[3]{x}}$ by $\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$ does not rationalize the denominator.

Answer

1. Answers will vary
3. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
divide radical expressions.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

Figure 8.5.63

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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4.7: Solve Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Before you get started, take this readiness quiz.

1. Simplify: $(y-3)^2$.
   If you missed this problem, review Example 5.31.
2. Solve: $2x-5=0$.
   If you missed this problem, review Example 2.2.
3. Solve $n^2-6n+8=0$.
   If you missed this problem, review Example 6.45.

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Definition: Radical Equation

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the nth power. This will eliminate the radical.

$$\text{For } a \geq 0, (\sqrt[n]{a})^n = a \text{ .}$$

Example 4.7.1 how to solve a radical equation

Solve: $\sqrt{5n-4}-9=0$.

Solution:

Table 8.6.1

Step 1: Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4}-9=0$ $\sqrt{5n-4}-9+9=0+9$ $\sqrt{5n-4}=9$
Step 2: Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$(\sqrt{5n-4})^2=(9)^2$
Step 3: Solve the new equation.	Remember, $(\sqrt{a})^2=a$.	$5n-4=81$ $5n=85$ $n=17$

Step 4: Check the answer in the original equation.		Check the answer. $\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9=0$ $0=0$ The solution is $n=17$.

Exercise 4.7.1

Solve: $\sqrt{3m+2}-5=0$.

Answer

$m=\frac{23}{3}$

Exercise 4.7.2

Solve: $\sqrt{10z+1}-2=0$.

Answer

$z=\frac{3}{10}$

Solve a Radical Equation With One Radical

1. Isolate the radical on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Solve the new equation.
4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

Example 4.7.2

Solve: $\sqrt{9k-2}+1=0$.

Solution:

Table 8.6.2

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 to both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

Exercise 4.7.3

Solve: $\sqrt{2r-3}+5=0$.

Answer

no solution

Exercise 4.7.4

Solve: $\sqrt{7s-3}+2=0$.

Answer

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Definition: Binomial Squares

$(a+b)^2=a^2+2ab+b^2$
$(a-b)^2=a^2-2ab+b^2$

Don't forget the middle term!

Example 4.7.3

Solve: $\sqrt{p-1}+1=p$.

Solution:

Table 8.6.3

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right, Then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the Zero Product Property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers.	
$p=1$ $\sqrt{p-1}+1=p$ $\sqrt{1-1}+1\stackrel{?}{=}1$ $\sqrt{0}+1\stackrel{?}{=}1$ $1=1$ ✓ $p=2$ $\sqrt{p-1}+1=p$ $\sqrt{2-1}+1\stackrel{?}{=}2$ $\sqrt{1}+1\stackrel{?}{=}2$ $2=2$ ✓	

The solutions are $p=1, p=2$.

Exercise 4.7.5

Solve: $\sqrt{x-2}+2=x$.

Answer

$x = 2, x = 3$

Exercise 4.7.6

Solve: $\sqrt{y-5}+5=y$.

Answer

$y = 5, y = 6$

When the index of the radical is 3, we cube both sides to remove the radical.

$$(\sqrt[3]{a})^3 = a$$

Example 4.7.4

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution:

Table 8.6.4

	$\sqrt[3]{5x+1}+8=4$
To isolate the radical, subtract 8 from both sides.	$\sqrt[3]{5x+1}=-4$
Cube both sides of the equation.	$(\sqrt[3]{5x+1})^3=(-4)^3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	
$x=-13$ $\quad \sqrt[3]{5x+1}+8=4$ $\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$ $\sqrt[3]{-64}+8\stackrel{?}{=}4$ $-4+8\stackrel{?}{=}4$ $4=4\checkmark$	
	The solution is $x=-13$.

Exercise 4.7.7

Solve: $\sqrt[3]{4x-3}+8=5$

Answer

$x = -6$

Exercise 4.7.8

Solve: $\sqrt[3]{6x-10}+1=-3$

Answer

$x = -9$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n = a^{m \cdot n}$, we have for example,

$$\left(x^{\frac{1}{2}}\right)^2 = x, \left(x^{\frac{1}{3}}\right)^3 = x$$

Remember, $x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{1}{3}} = \sqrt[3]{x}$.

Example 4.7.5

Solve: $(3x-2)^{\frac{1}{4}} + 3 = 5$.

Solution:

Table 8.6.5

	$(3x-2)^{\frac{1}{4}} + 3 = 5$
To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}} = 2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4 = (2)^4$
Simplify.	$3x-2 = 16$
Solve the equation.	$3x = 18$
	$x = 6$
Check the answer.	
$x=6 \quad (3x-2)^{\frac{1}{4}}+3=5$ $(3\cdot 6-2)^{\frac{1}{4}}+3 \stackrel{?}{=} 5$ $(16)^{\frac{1}{4}}+3 \stackrel{?}{=} 5$ $2+3 \stackrel{?}{=} 5$ $5=5$ ✓	
	The solution is $x = 6$.

Exercise 4.7.9

Solve: $(9x+9)^{\frac{1}{4}} - 2 = 1$

Answer

$x = 8$

Exercise 4.7.10

Solve: $(4x-8)^{\frac{1}{4}} + 5 = 7$

Answer

$x = 6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an **extraneous solution**!

Example 4.7.6

Solve: $\sqrt{r+4}-r+2=0$.

Solution:

Table 8.6.6

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$(\sqrt{r+4})^2=(r-2)^2$
Simplify and then solve the equation.	$r+4=r^2-4r+4$
If it is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the Zero Product Property.	$0=r \quad 0=r-5$
Solve the equation.	$r=0 \quad r=5$
Check your answer.	
$r=0$, $\sqrt{r+4}-r+2=0$ $\sqrt{0+4}-0+2\stackrel{?}{=}0$ $\sqrt{4}+2\stackrel{?}{=}0$ $4\neq 0$ $r=5$, $\sqrt{r+4}-r+2=0$ $\sqrt{5+4}-5+2\stackrel{?}{=}0$ $\sqrt{9}-3\stackrel{?}{=}0$ $0=0\checkmark$	The solution is $r=5$.
	$r=0$ is an extreneous solution.

Exercise 4.7.11

Solve: $\sqrt{m+9}-m+3=0$

Answer

$m=7$

Exercise 4.7.12

Solve: $\sqrt{n+1}-n+1=0$.

Answer

$n=3$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

Example 4.7.7

Solve: $3\sqrt{3x-5}-8=4$.

Solution:

Table 8.6.7

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$

Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$(\sqrt{3x-5})^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$
Check the answer.	
$x=7$ $3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$ $3\sqrt{21-5}-8\stackrel{?}{=}4$ $3\sqrt{16}-8\stackrel{?}{=}4$ $3(4)-8\stackrel{?}{=}4$ $4=4$ ✓	
	The solution is $x=7$.

? Exercise 4.7.13

Solve: $2\sqrt{4a+4}-16=16$.

Answer

$a=63$

? Exercise 4.7.14

Solve: $3\sqrt{2b+3}-25=50$

Answer

$b=311$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

✓ Example 4.7.8

Solve: $\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$.

Solution:

The radical terms are isolated.

$$\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$$

Since the index is 3, cube both sides of the equation.

$$(\sqrt[3]{4x-3})^3=(\sqrt[3]{3x+2})^3$$

Simplify, then solve the new equation.

$$\begin{aligned} 4x-3 &= 3x+2 \\ x-3 &= 2 \\ x &= 5 \end{aligned}$$

The solution is $x=5$.

Check the answer.

We leave it to you to show that 5 checks!

? Exercise 4.7.15

Solve: $\sqrt[3]{5x-4}=\sqrt[3]{2x+5}$.

Answer

$x=3$

? Exercise 4.7.16

Solve: $\sqrt[3]{7x+1}=\sqrt[3]{2x-5}$.

Answer

$x=-\frac{6}{5}$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

✓ Example 4.7.9 how to solve a radical equation

Solve: $\sqrt{m}+1=\sqrt{m+9}$.

Solution:

Table 8.6.8

Step 1: Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Step 2: Raise both sides of the equation to the power of the index.	We square both sides. Simplify--be very careful as you multiply!	$(\sqrt{m}+1)^2=(\sqrt{m+9})^2$
Step 3: Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$\begin{aligned} m+2\sqrt{m}+1 &= m+9 \\ 2\sqrt{m} &= 8 \\ \sqrt{m} &= 4 \\ (\sqrt{m})^2 &= (4)^2 \\ m &= 16 \end{aligned}$
Step 4: Check the answer in the original equation.		$\begin{aligned} \sqrt{m}+1 &= \sqrt{m+9} \\ \sqrt{16}+1 &\stackrel{?}{=} \sqrt{16+9} \\ 4+1 &\stackrel{?}{=} 5 \\ 5 &= 5 \end{aligned}$ The solution is $m=16$.

Exercise 4.7.17

Solve: $3-\sqrt{x}=\sqrt{x-3}$.

Answer

$x=4$

Exercise 4.7.18

Solve: $\sqrt{x}+2=\sqrt{x+16}$.

Answer

$x=9$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern in $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example 4.7.10

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution:

Table 8.6.9

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right is isolated. Square both sides.	$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$.	$\left(6\sqrt{q-2}\right)^2=\left(3q-6\right)^2$ $a^2 - 2ab + b^2$ $6^2\left(\sqrt{q-2}\right)^2=(3q)^2-2\cdot 3q\cdot 6+6^2$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$
Factor the right side.	$0=9(q^2-8q+12)$ $0=9(q-6)(q-2)$

Use the Zero Product Property.	$q-6=0 \quad q-2=0$ $q=6 \quad q=2$
The checks are left to you.	The solutions are $q=6$ and $q=2$.

Exercise 4.7.19

Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$

Answer

$x=5$

Exercise 4.7.20

Solve: $\sqrt{x}+2=\sqrt{3x+4}$

Answer

$x=0x=4$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a Problem Solving Strategy for Applications With Formulas

1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of **gravity** on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Definition: Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula

$$t=\frac{\sqrt{h}}{4}$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h=64$ into the formula.

Table 8.6.10

	$t=\frac{\sqrt{h}}{4}$
	$t=\frac{\sqrt{64}}{4}$

Take the square root of 64.	$t=\frac{8}{4}$
Simplify the fraction.	$t=2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example 4.7.11

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution:

Table 8.6.11

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking.	The time it takes for the sunglasses to reach the river.
Step 3: **Name** what we are looking.	Let (t=\) time.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$
Step 5: **Solve the equation**.	$t=\frac{20}{4}$
	$t=5$
Step 6: **Check** the answer in the problem and make sure it makes sense.	$5\stackrel{?}{=}\frac{\sqrt{400}}{4}$ $5\stackrel{?}{=}\frac{20}{4}$ $5=5\checkmark$
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7: **Answer** the equation.	It will take 5 seconds for the sunglasses to reach the river.

Exercise 4.7.21

A helicopter dropped a rescue package from a height of $1,296$ feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Answer

9 seconds

Exercise 4.7.22

A window washer dropped a squeegee from a platform 196 feet above the sidewalk. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Answer

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the **speed**, in miles per hour, a car was going before applying the brakes.

Definition: Skid Marks and Speed of a Car

If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

Example 4.7.12

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

Table 8.6.12

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking for.	The speed of a car.
Step 3: **Name** what we are looking for.	Let $s =$ the speed.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5: **Solve the equation**.	$s = \sqrt{4{,}560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \stackrel{?}{\approx} \sqrt{24(190)}$ $67.5 \stackrel{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277...$ ✓
	The speed of the car before the brakes were applied was 67.5 miles per hour.

Exercise 4.7.23

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

42.7mph

Exercise 4.7.24

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

54.1mph

Access these online resources for additional instruction and practice with solving radical equations.

- Solving an Equation Involving a Single Radical
- Solving Equations with Radicals and Rational Exponents
- Solving Radical Equations
- Solve Radical Equations
- Radical Equation Application

Key Concepts

- **Binomial Squares**
  $(a+b)^2 = a^2 + 2ab + b^2$
  $(a-b)^2 = a^2 - 2ab + b^2$
- **Solve a Radical Equation**
  1. Isolate one of the radical terms on one side of the equation.
  2. Raise both sides of the equation to the power of the index.
  3. Are there any more radicals?
     If yes, repeat Step 1 and Step 2 again.
     If no, solve the new equation.
  4. Check the answer in the original equation.
- **Problem Solving Strategy for Applications with Formulas**
  1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  2. Identify what we are looking for.
  3. Name what we are looking for by choosing a variable to represent it.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Falling Objects**
  - On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
  - If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Glossary

radical equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

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4.7E: Exercises

Practice Makes Perfect

Exercise SET A: Solve Radical Equations

In the following exercises, solve.

1. $\sqrt{5x-6}=8$
2. $\sqrt{4x-3}=7$
3. $\sqrt{5x+1}=-3$
4. $\sqrt{3y-4}=-2$
5. $\sqrt[3]{2x}=-2$
6. $\sqrt[3]{4x-1}=3$
7. $\sqrt{2m-3}-5=0$
8. $\sqrt{2n-1}-3=0$
9. $\sqrt{6v-2}-10=0$
10. $\sqrt{12u+1}-11=0$
11. $\sqrt{4m+2}+2=6$
12. $\sqrt{6n+1}+4=8$
13. $\sqrt{2u-3}+2=0$
14. $\sqrt{5v-2}+5=0$
15. $\sqrt{u-3}+3=u$
16. $\sqrt{v-10}+10=v$
17. $\sqrt{r-1}=r-1$
18. $\sqrt{s-8}=s-8$
19. $\sqrt[3]{6x+4}=4$
20. $\sqrt[3]{11x+4}=5$
21. $\sqrt[3]{4x+5}-2=-5$
22. $\sqrt[3]{9x-1}-1=-5$
23. $(6x+1)^{\frac{1}{2}}-3=4$
24. $(3x-2)^{\frac{1}{2}}+1=6$
25. $(8x+5)^{\frac{1}{3}}+2=-1$
26. $(12x-5)^{\frac{1}{3}}+8=3$
27. $(12x-3)^{\frac{1}{4}}-5=-2$
28. $(5x-4)^{\frac{1}{4}}+7=9$
29. $\sqrt{x+1}-x+1=0$
30. $\sqrt{y+4}-y+2=0$
31. $\sqrt{z+100}-z=-10$
32. $\sqrt{w+25}-w=-5$

https://math.libretexts.org/@go/page/58461

33. $3\sqrt{2x-3}-20=7$

34. $2\sqrt{5x+1}-8=0$

35. $2\sqrt{8r+1}-8=2$

36. $3\sqrt{7y+1}-10=8$

Answer

1. $m=14$
3. no solution
5. $x=-4$
7. $m=14$
9. $v=17$
11. $m=\frac{7}{2}$
13. no solution
15. $u=3, u=4$
17. $r=1, r=2$
19. $x=10$
21. $x=-8$
23. $x=8$
25. $x=-4$
27. $x=7$
29. $x=3$
31. $z=21$
33. $x=42$
35. $r=3$

Exercise SET B: Solve Radical Equations with Two Radicals

In the following exercises, solve.

37. $\sqrt{3u+7}=\sqrt{5u+1}$

38. $\sqrt{4v+1}=\sqrt{3v+3}$

39. $\sqrt{8+2r}=\sqrt{3r+10}$

40. $\sqrt{10+2c}=\sqrt{4c+16}$

41. $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$

42. $\sqrt[3]{8x-5}=\sqrt[3]{3x+5}$

43. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

44. $\sqrt[3]{x^2-x+18}=\sqrt[3]{2x^2-3x-6}$

45. $\sqrt{a}+2=\sqrt{a+4}$

46. $\sqrt{r}+6=\sqrt{r+8}$

47. $\sqrt{u}+1=\sqrt{u+4}$

48. $\sqrt{x}+1=\sqrt{x+2}$

49. $\sqrt{a+5}-\sqrt{a}=1$
50. $-2=\sqrt{d-20}-\sqrt{d}$
51. $\sqrt{2x+1}=1+\sqrt{x}$
52. $\sqrt{3x+1}=1+\sqrt{2x-1}$
53. $\sqrt{2x-1}-\sqrt{x-1}=1$
54. $\sqrt{x+1}-\sqrt{x-2}=1$
55. $\sqrt{x+7}-\sqrt{x-5}=2$
56. $\sqrt{x+5}-\sqrt{x-3}=2$

Answer

37. $u=3$
39. $r=-2$
41. $x=1$
43. $x=-8, x=2$
45. $a=0$
47. $u=\frac{9}{4}$
49. $a=4$
51. $x=0\ x=4$
53. $x=1\ x=5$
55. $x=9$

Exercise SET C: Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

57. **Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.
58. **Landscaping** Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.
59. **Gravity** A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.
60. **Gravity** A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.
61. **Accident investigation** The skid marks for a car involved in an accident measured 216 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.
62. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

57. 8.7 feet
59. 4.7 seconds
61. 72 feet

Exercise SET D: Writing Exercises

63. Explain why an equation of the form $\sqrt{x}+1=0$ has no solution.

64. a. Solve the equations $\sqrt{r+4}-r+2=0$.
    b. Explain why one of the "solutions" that was found was not actually a solution to the equation.

Answer

63. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve radical equations.			
solve radical equations with two radicals.			
use radicals in applications.			

Figure 8.6.42

b. After reviewing this checklist, what will you do to become confident for all objectives?

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4.8: Use the Complex Number System

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

Before you get started, take this readiness quiz.

1. Given the numbers $-4, -\sqrt{7}, 0.\overline{5}, \frac{7}{3}, 3, \sqrt{81}$, list the
   a. rational numbers
   b. irrational numbers
   c. real numbers

   If you missed this problem, review Example 1.42.
2. Multiply: $(x-3)(2x+5)$.
   If you missed this problem, review Example 5.28.
3. Rationalize the denominator: $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$
   If you missed this problem, review Example 5.32.

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify $\sqrt{-1}$, we are looking for a real number x so that $x^2 = -1$. Since all real numbers squared are positive numbers, there is no real number that equals -1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like $\sqrt{5}$. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as the number whose square is -1.

Definition 4.8.1

The **imaginary unit** i is the number whose square is -1.

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

We will use the imaginary unit to simplify the square roots of negative numbers.

Definition 4.8.2

Square Root of a Negative Number

If b is a positive real number, then

$$\sqrt{-b} = \sqrt{b}i$$

We will use this definition in the next example. Be careful that it is clear that the i is not under the radical. Sometimes you will see this written as $\sqrt{-b} = i\sqrt{b}$ to emphasize the i is not under the radical. But the $\sqrt{-b} = \sqrt{b}i$ is considered standard form.

Example 4.8.1

Write each expression in terms of i and simplify is possible:

a. $\sqrt{-25}$
b. $\sqrt{-7}$
c. $\sqrt{-12}$

Solution:

a.

$$\sqrt{-25}$$

Use the definition of the square root of negative numbers.

$$\sqrt{25}i$$

Simplify.

$$5i$$

b.

$$\sqrt{-7}$$

Use the definition of the square root of negative numbers.

$$\sqrt{7}i$$

Simplify.

Be careful that it is clear that i is not under the radical sign.

c.

$$\sqrt{-12}$$

Use the definition of the square root of negative numbers.

$$\sqrt{12}i$$

Simplify $\sqrt{12}$.

$$2\sqrt{3}i$$

Exercise 4.8.1

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-81}$
b. $\sqrt{-5}$
c. $\sqrt{-18}$

Answer

a. $9i$
b. $\sqrt{5}i$
c. $3\sqrt{2}i$

Exercise 4.8.2

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-36}$
b. $\sqrt{-3}$
c. $\sqrt{-27}$

Answer

a. $6i$
b. $\sqrt{3}i$
c. $3\sqrt{3}i$

Now that we are familiar with the imaginary number i, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form $a+bi$, where a, b are real numbers. We call a the real part and b the imaginary part.

Definition 4.8.3

A **complex number** is of the form $a+bi$, where a and b are real numbers.

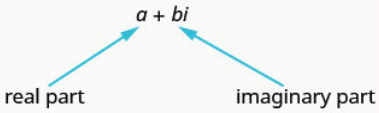


Figure 8.8.1

A complex number is in standard form when written as $a+bi$, where a and b are real numbers.

If $b=0$, then $a+bi$ becomes $a+0\cdot i=a$, and is a real number.

If $b\neq 0$, then $a+bi$ is an imaginary number.

If $a=0$, then $a+bi$ becomes $0+bi=bi$, and is called a pure imaginary number.

We summarize this here.

Table 8.8.1

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$R	$0+bi$ bi	Pure imaginary numbe4

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $\sqrt{-b}=\sqrt{b}i$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.

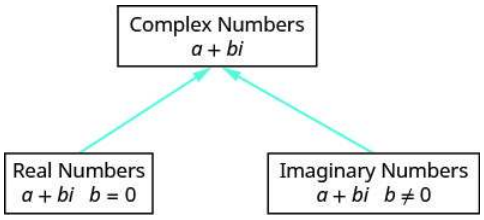


Figure 8.8.2

Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

Example 4.8.2

Add: $\sqrt{-12}+\sqrt{-27}$.

Solution:

$$\sqrt{-12}+\sqrt{-27}$$

Use the definition of the square root of negative numbers.

$$\sqrt{12}i+\sqrt{27}i$$

Simplify the square roots.

$$2\sqrt{3}i+3\sqrt{3}i$$

Add.

$$5\sqrt{3}i$$

Exercise 4.8.3

Add: $\sqrt{-8}+\sqrt{-32}$.

Answer

$6\sqrt{2}i$

Exercise 4.8.4

Add: $\sqrt{-27}+\sqrt{-48}$

Answer

$7\sqrt{3}i$

Remember to add both the real parts and the imaginary parts in this next example.

Example 4.8.3

Simplify:

a. $(4-3i)+(5+6i)$
b. $(2-5i)-(5-2i)$

Solution:

a.

$$(4-3i)+(5+6i)$$

Use the Associative Property to put the real parts and the imaginary parts together.

$$(4+5)+(-3i+6i)$$

Simplify.

$$9+3i$$

b.

$$(2-5i)-(5-2i)$$

Distribute.

$$2-5i-5+2i$$

Use the Associative Property to put the real parts and the imaginary parts together.

$$2-5-5i+2i$$

Simplify.

$$-3-3i$$

Exercise 4.8.5

Simplify:

a. $(2+7i)+(4-2i)$
b. $(8-4i)-(2-i)$

Answer

a. $6+5i$
b. $6-3i$

Exercise 4.8.6

Simplify:

a. $(3-2i)+(-5-4i)$
b. $(4+3i)-(2-6i)$

Answer

a. $-2-6i$
b. $2+9i$

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

Example 4.8.4

Multiply: $2i(7-5i)$

Solution:

$$2i(7-5i)$$

Distribute.

$$14i-10i^2$$

Simplify i^2.

$$14i-10(-1)$$

Multiply.

$$14i+10$$

Write in standard form.

$$10+14i$$

Exercise 4.8.7

Multiply: $4i(5-3i)$.

Answer

$12+20i$

Exercise 4.8.8

Multiply: $-3i(2+4i)$.

Answer

$12-6i$

In the next example, we multiply the binomials using the **Distributive Property** or **FOIL**.

Example 4.8.5

Multiply: $(3+2i)(4-3i)$.

Solution:

$$(3+2i)(4-3i)$$

Use FOIL.

$$12-9i+8i-6i^2$$

Simplify i^2 and combine like terms.

$$12-i-6(-1)$$

Multiply.

$$12-i+6$$

Combine the real parts.

$$18-i$$

Exercise 4.8.9

Multiple: $(5-3i)(-1-2i)$.

Answer

$-11-7i$

Exercise 4.8.10

Multiple: $(-4-3i)(2+i)$.

Answer

$-5-10i$

In the next example, we could use FOIL or the **Product of Binomial Squares Pattern**.

Example 4.8.6

Multiply: $(3+2i)^2$

Solution:

Table 8.8.2

	$(a+b)^2$ $(3+2i)^2$
Use the Product of Binomial Squares Pattern, $(a+b)^2=a^2+2ab+b^2$.	$a^2+2 \quad a \quad b+b^2$ $3^2+2 \cdot 3 \cdot 2i+(2i)^2$
Simplify.	$9+12i+4i^2$
Simplify i^2.	$9+12i+4(-1)$
Simplify.	$5+12i$

Exercise 4.8.11

Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

Answer

$-21+20i$

Exercise 4.8.12

Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

Answer

$9-40i$

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $\sqrt{-b}=\sqrt{b}i$.This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

Example 4.8.7

Multiply: $\sqrt{-36}\cdot\sqrt{-4}$.

Solution:

To multiply square roots of negative numbers, we first write them as complex numbers.

$$\sqrt{-36}\cdot\sqrt{-4}$$

Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.

$$\sqrt{36}i\cdot\sqrt{4}i$$

Simplify.

$$6i\cdot 2i$$

Multiply.

$$12i^2$$

Simplify i^2 and multiply.

$$-12$$

Exercise 4.8.13

Multiply: $\sqrt{-49} \cdot \sqrt{-4}$.

Answer

-14

Exercise 4.8.14

Multiply: $\sqrt{-36} \cdot \sqrt{-81}$.

Answer

-54

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

Example 4.8.8

Multiply: $(3-\sqrt{-12})(5+\sqrt{-27})$.

Solution:

To multiply square roots of negative numbers, we first write them as complex numbers.

$$(3-\sqrt{-12})(5+\sqrt{-27})$$

Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.

$$(3-2\sqrt{3}i)(5+3\sqrt{3}i)$$

Use FOIL.

$$15+9\sqrt{3}i-10\sqrt{3}i-6\cdot 3i^2$$

Combine like terms and simplify i^2.

$$15-\sqrt{3}i-6\cdot(-3)$$

Multiply and combine like terms.

$$33-\sqrt{3}i$$

Exercise 4.8.15

Multiply: $(4-\sqrt{-12})(3-\sqrt{-48})$.

Answer

$-12-22\sqrt{3}i$

Exercise 4.8.16

Multiply: $(-2+\sqrt{-8})(3-\sqrt{-18})$.

Answer

$6+12\sqrt{2}i$

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b),(a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Definition 4.8.4

A **complex conjugate pair** is of the form $a+bi, a-bi$.

We will multiply a complex conjugate pair in the next example.

Example 4.8.9

Multiply: $(3-2i)(3+2i)$.

Solution:

$$(3-2i)(3+2i)$$

Use FOIL

$$9+6i-6i-4i^2$$

Combine like terms and simplify i^2.

$$9-4(-1)$$

Multiply and combine like terms.

$$13$$

Exercise 4.8.17

Multiply: $(4-3i)\cdot(4+3i)$.

Answer

25

Exercise 4.8.18

Multiply: $(-2+5i)\cdot(-2-5i)$.

Answer

29

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$ .The result is called a **difference of squares**. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the **Product of Conjugates Pattern**.

$$(a-b)(a+b)$$

$$(3-2i)(3+2i)$$

$$a^2-b^2$$

$$(3)^2-(2i)^2$$

$$9-4i^2$$

$$9-4(-1)$$

$$13$$

Figure 8.8.8

Notice this is the same result we found in Example 8.8.9.

When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

$$(a-bi)(a+bi)$$
$$a^2-(bi)^2$$
$$a^2-b^2i^2$$
$$a^2-b^2(-1)$$
$$a^2+b^2$$

This leads us to the Product of Complex Conjugates Pattern: $(a-bi)(a+bi)=a^2+b^2$

Definition 4.8.5

Product of Complex Conjugates

If a and b are real numbers, then

$$(a-bi)(a+bi)=a^2+b^2$$

Example 4.8.10

Multiply using the Product of Complex Conjugates Pattern: $(8-2i)(8+2i)$.

Solution:

Table 8.8.3

	$(a-b)(a+b)$ $(8-2i)(8+2i)$
Use the Product of Complex Conjugates Pattern, $(a-bi)(a+bi)=a^2+b^2$.	a^2+b^2 8^2+2^2
Simplify the squares.	$64+4$
Add.	68

Exercise 4.8.19

Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

Answer

109

Exercise 4.8.20

Multiply using the Product of Complex Conjugates Pattern: $(-5+4i)(-5-4i)$.

Answer

41

Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

Example 4.8.11 how to divide complex numbers

Divide: $\frac{4+3i}{3-4i}$.

Solution:

Table 8.8.4

Step 1: Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$
Step 2: Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	$\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$
Step 3: Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator. Combine like terms. Simplify. Write the result in standard form.	$\frac{12+16i+9i+12i^2}{9+16}$ $\frac{12+25i-12}{25}$ $\frac{25i}{25}$ i

Exercise 4.8.21

Divide: $\frac{2+5i}{5-2i}$.

Answer

i

Exercise 4.8.22

Divide: $\frac{1+6i}{6-i}$.

Answer

i

We summarize the steps here.

How to Divide Complex Numbers

1. Write both the numerator and denominator in standard form.
2. Multiply both the numerator and denominator by the complex conjugate of the denominator.
3. Simplify and write the result in standard form.

Example 4.8.12

Divide, writing the answers in standard form: $\frac{-3}{5+2i}$.

Solution:

$$\frac{-3}{5+2i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{-3(5-2i)}{(5+2i)(5-2i)}$$

Multiply in the numerator and use the Product of Complex Conjugates Pattern in the denominator.

$$\frac{-15+6i}{5^2+2^2}$$

Simplify.

$$\frac{-15+6i}{29}$$

Write in standard form.

$$-\frac{15}{29}+\frac{6}{29}i$$

Exercise 4.8.23

Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

Answer

$\frac{4}{17}+\frac{16}{17}i$

Exercise 4.8.24

Divide, writing the answer in standard form: $\frac{-2}{-1+2i}$.

Answer

$\frac{2}{5}+\frac{4}{5}i$

Be careful as you find the conjugate of the denominator.

Example 4.8.13

Divide: $\frac{5+3i}{4i}$.

Solution:

$$\frac{5+3i}{4i}$$

Write the denominator in standard form.

$$\frac{5+3i}{0+4i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(5+3i)(0-4i)}{(0+4i)(0-4i)}$$

Simplify.

$$\frac{(5+3i)(-4i)}{(4i)(-4i)}$$

Multiply.

$$\frac{-20i-12i^2}{-16i^2}.$$

Simplify the i^2.

$$\frac{-20i+12}{16}$$

Rewrite in standard form.

$$\frac{12}{16}-\frac{20}{16}i$$

Simplify the fractions.

$$\frac{3}{4}-\frac{5}{4}i$$

Exercise 4.8.25

Divide: $\frac{3+3i}{2i}$.

Answer

$\frac{3}{2}-\frac{3}{2}i$

Exercise 4.8.26

Divide: $\frac{2+4i}{5i}$.

Answer

$\frac{4}{5}-\frac{2}{5}i$

Simplify Powers of i

The powers of i make an interesting pattern that will help us simplify higher powers of i. Let's evaluate the powers of i to see the pattern.

$$\begin{array}{cccc} i^1 & i^2 & i^3 & i^4 \\ i & -1 & i^2\cdot i & i^2\cdot i^2 \\ & & -1\cdot i & (-1)(-1) \\ & & -i & 1 \end{array}$$

$$\begin{array}{cccc} i^5 & i^6 & i^7 & i^8 \\ i^4\cdot i & i^4\cdot i^2 & i^4\cdot i^3 & i^4\cdot i^4 \\ 1\cdot i & 1\cdot i^2 & 1\cdot i^3 & 1\cdot 1 \\ i & i^2 & i^3 & 1 \\ & -1 & -i & \end{array}$$

We summarize this now.

$$\begin{array}{ll} i^1=i & i^5=i \\ i^2=-1 & i^6=-1 \\ i^3=-i & i^7=-i \\ i^4=1 & i^8=1 \end{array}$$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of i. Since $i^4=1$, we rewrite each power, i^n, as a product using i^4 to a power and another power of i.

We rewrite it in the form $i^n=\left(i^4\right)^q\cdot i^r$, where the exponent, q, is the quotient of n divided by 4 and the exponent, r, is the remainder from this division. For example, to simplify i^{57}, we divide 57 by 4 and we get 14 with a remainder of 1. In other words, $57=4\cdot 14+1$. So we write $i^{57}=\left(1^4\right)^{14}\cdot i^1$ and then simplify from there.

$$\begin{array}{r|l} 14 & i^{57} \\ 4\overline{)57} & \left(1^4\right)^{14}\cdot i^1 \\ \underline{4} & \\ 17 & 1\cdot i \\ \underline{16} & \\ 1 & i \end{array}$$

Figure 8.8.13

Example 4.8.14

Simplify: i^{86}.

Solution:

$$i^{86}$$

Divide 86 by 4 and rewrite i^{86} in the $i^n=\left(i^4\right)^q\cdot i^r$ form.

$$\left(1^4\right)^{21}\cdot i^2$$

$$\begin{array}{rl} 21 & \text{(divide the exponent by the number of repeats in the imaginary number i)} \\ 4\overline{)86} & \\ \underline{8} & \text{(four goes into eight twice with zero remainders)} \\ 06 & \\ \underline{4} & \text{(four goes into 6 one time with 2 remainders)} \\ 2 & \text{(remainder is 2 so calculate the imaginary number with an exponent of two)} \end{array} \tag{4.8.1}$$

Simplify.

$$(1)^{21} \cdot (-1)$$

Simplify.

$$-1$$

? Exercise 4.8.27

Simplify: i^{74}.

Answer

-1

? Exercise 4.8.28

Simplify: i^{92}.

Answer

1

Access these online resources for additional instruction and practice with the complex number system.

- Expressing Square Roots of Negative Numbers with i
- Subtract and Multiply Complex Numbers
- Dividing Complex Numbers
- Rewriting Powers of i

Key Concepts

- **Square Root of a Negative Number**
  - If b is a positive real number, then \(\sqrt{-b}=\sqrt{b} i\

Table 8.8.1

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$	$0+bi$ bi	Pure imaginary number

- A complex number is in **standard form** when written as $a + bi$, where a, b are real numbers.

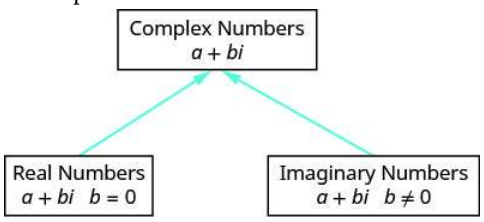


Figure 8.8.2

- **Product of Complex Conjugates**
  - If a, b are real numbers, then $(a-bi)(a+bi) = a^2 + b^2$
- **How to Divide Complex Numbers**
  1. Write both the numerator and denominator in standard form.
  2. Multiply the numerator and denominator by the complex conjugate of the denominator.
  3. Simplify and write the result in standard form.

Glossary

complex conjugate pair

A complex conjugate pair is of the form $a+bi, a-bi$.

complex number

A complex number is of the form $a+bi$, where a and b are real numbers. We call a the real part and b the imaginary part.

complex number system

The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit

The imaginary unit i is the number whose square is -1. $i^2 = -1$ or $i = \sqrt{-1}$.

standard form

A complex number is in standard form when written as $a+bi$, where a, b are real numbers.

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4.8E: Exercises

Practice Makes Perfect

Exercises 1 - 4: Evaluate a Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

1. a. $\sqrt{-16}$ b. $\sqrt{-11}$ c. $\sqrt{-8}$
2. a. $\sqrt{-121}$ b. $\sqrt{-1}$ c. $\sqrt{-20}$
3. a. $\sqrt{-100}$ b. $\sqrt{-13}$ c. $\sqrt{-45}$
4. a. $\sqrt{-49}$ b. $\sqrt{-15}$ c. $\sqrt{-75}$

Answer

1. a. $4i$ b. $i\sqrt{11}$ c. $2i\sqrt{2}$
3. a. $10i$ b. $i\sqrt{13}$ c. $3i\sqrt{5}$

ExerciseS 5 - 21: Add or Subtract Complex Numbers

In the following exercises, add or subtract, putting the answer in $a+bi$ form.

5. $\sqrt{-75}+\sqrt{-48}$
6. $\sqrt{-12}+\sqrt{-75}$
7. $\sqrt{-50}+\sqrt{-18}$
8. $\sqrt{-72}+\sqrt{-8}$
9. $(1+3i)+(7+4i)$
10. $(6+2i)+(3-4i)$
11. $(8-i)+(6+3i)$
12. $(7-4i)+(-2-6i)$
13. $(1-4i)-(3-6i)$
14. $(8-4i)-(3+7i)$
15. $(6+i)-(-2-4i)$
16. $(-2+5i)-(-5+6i)$
17. $(5-\sqrt{-36})+(2-\sqrt{-49})$
18. $(-3+\sqrt{-64})+(5-\sqrt{-16})$
19. $(-7-\sqrt{-50})-(-32-\sqrt{-18})$
20. $(-5+\sqrt{-27})-(-4-\sqrt{-48})$

Answer

5. $0+\left(9\sqrt{3}\right)i$
7. $0+\left(8\sqrt{2}\right)i$
9. $8+7i$
11. $14+2i$
13. $-2+2i$
15. $8+5i$

17. $7-13i$

19. $25-\left(2\sqrt{2}\right)i$

? ExerciseS 21 - 28: Multiply Complex Numbers

In the following exercises, multiply, putting the answer in $a+bi$ form.

21. $4i(5-3i)$

22. $2i(-3+4i)$

23. $-6i(-3-2i)$

24. $-i(6+5i)$

25. $(4+3i)(-5+6i)$

26. $(-2-5i)(-4+3i)$

27. $(-3+3i)(-2-7i)$

28. $(-6-2i)(-3-5i)$

Answer

21. $12+20i$

23. $-12+18i$

25. $-38+9i$

27. $27+15i$

? ExerciseS 29 - 32: Multiply Complex Numbers

In the following exercises, multiply using the Product of Binomial Squares Pattern, putting the answer in $a+bi$ form.

29. $(3+4i)^2$

30. $(-1+5i)^2$

31. $(-2-3i)^2$

32. $(-6-5i)^2$

Answer

29. $-7+24i$

31. $-5-12i$

? Exercises 33 - 46: Multiply Complex Numbers

In the following exercises, multiply, putting the answer in $a+bi$ form.

33. $\sqrt{-25}\cdot\sqrt{-36}$

34. $\sqrt{-4}\cdot\sqrt{-16}$

35. $\sqrt{-9}\cdot\sqrt{-100}$

36. $\sqrt{-64}\cdot\sqrt{-9}$

37. $(-2-\sqrt{-27})(4-\sqrt{-48})$

38. $(5-\sqrt{-12})(-3+\sqrt{-75})$

39. $(2+\sqrt{-8})(-4+\sqrt{-18})$

40. $(5+\sqrt{-18})(-2-\sqrt{-50})$

41. $(2-i)(2+i)$

42. $(4-5i)(4+5i)$

43. $(7-2i)(7+2i)$

44. $(-3-8i)(-3+8i)$

Answer

33. $30i=0+30i$

35. $-30=-30+0i$

37. $-44+\left(4\sqrt{3}\right)i$

39. $-20-\left(2\sqrt{2}\right)i$

41. $5=5+0i$

43. $53=53+0i$

? ExerciseS 45 - 49: Multiply Complex Numbers

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

45. $(7-i)(7+i)$

46. $(6-5i)(6+5i)$

47. $(9-2i)(9+2i)$

48. $(-3-4i)(-3+4i)$

Answer

45. 50

47. 85

? ExerciseS 49 - 60: Divide Complex Numbers

In the following exercises, divide, putting the answer in $a+bi$ form.

49. $\dfrac{3+4i}{4-3i}$

50. $\dfrac{5-2i}{2+5i}$

51. $\dfrac{2+i}{3-4i}$

52. $\dfrac{3-2i}{6+i}$

53. $\dfrac{3}{2-3i}$

54. $\dfrac{2}{4-5i}$

55. $\dfrac{-4}{3-2i}$

56. $\frac{-1}{3+2i}$

57. $\frac{1+4i}{3i}$

58. $\frac{4+3i}{7i}$

59. $\frac{-2-3i}{4i}$

60. $\frac{-3-5i}{2i}$

Answer

49. $i=0+i$

51. $\frac{2}{25}+\frac{11}{25}i$

53. $\frac{6}{13}+\frac{9}{13}i$

55. $-\frac{12}{13}-\frac{8}{13}i$

57. $\frac{4}{3}-\frac{1}{3}i$

59. $-\frac{3}{4}+\frac{1}{2}i$

? ExerciseS 61 - 68: Simplify Powers of i

In the following exercises, simplify.

61. i^{41}

62. i^{39}

63. i^{66}

64. i^{48}

65. i^{128}

66. i^{162}

67. i^{137}

68. i^{255}

Answer

61. $i^{41}=i^{40}\cdot i=\left(i^4\right)^{10}\cdot i=i$

63. $i^{66}=i^{64}\cdot i^2=\left(i^4\right)^{16}\cdot(-1)=-1$

65. $i^{128}=\left(i^4\right)^{32}=1$

67. $i^{137}=i^{136}\cdot i=\left(i^4\right)^{34}\cdot i=1\cdot i=i$

? ExerciseS 69 - 72: Writing Exercises

69. Explain the relationship between real numbers and complex numbers.

70. Aniket multiplied as follows and he got the wrong answer. What is wrong with his reasoning?

$\sqrt{-7}\cdot\sqrt{-7}$

$\sqrt{49}$

7

71. Why is $\sqrt{-64} = 8i$ but $\sqrt[3]{-64} = -4$.

72. Explain how dividing complex numbers is similar to rationalizing a denominator.

Answer

69. Answers may vary

71. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate the square root of a negative number.			
add or subtract complex numbers.			
multiply complex numbers.			
divide complex numbers.			
simplify powers of *i*.			

Figure 8.8.15

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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CHAPTER OVERVIEW

5: Chapter 5 - Quadratics

- 5.1: Solve Quadratic Equations Using the Square Root Property
  - 5.1E: Exercises
- 5.2: Solve Quadratic Equations by Completing the Square
  - 5.2E: Exercises
- 5.3: Solve Quadratic Equations Using the Quadratic Formula
  - 5.3E: Exercises
- 5.4: Solve Quadratic Equations in Quadratic Form
  - 5.4E: Exercises
- 5.5: Graph Quadratic Functions Using Properties
  - 5.5E: Graph Quadratic Functions Using Properties (Exercises)
- 5.6: Graph Quadratic Functions Using Transformations
  - 5.6E: Exercises
- 5.7: Solve Quadratic Inequalities
  - 5.7E: Exercises
- 5.8: Solve Applications of Quadratic Equations
  - 5.8.1: Exercises

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5.1: Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{128}$.
   If you missed this problem, review Example 8.13.
2. Simplify: $\sqrt{\frac{32}{5}}$.
   If you missed this problem, review Example 8.50.
3. Factor: $9x^2 - 12x + 4$.
   If you missed this problem, review Example 6.23.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the Form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

$$x^2 = 9$$

Put the equation in standard form.

$$x^2 - 9 = 0$$

Factor the difference of squares.

$$(x-3)(x+3) = 0$$

Use the Zero Produce Property.

$$x - 3 = 0 \quad x - 3 = 0$$

Solve each equation.

$$x = 3 \quad x = -3$$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4, x = -4$ and $x = 5, x = -5$

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

If $n^2 = m$, then n is a square root of m.

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Definition 5.1.1

Square Root Property

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

$$\begin{aligned} & & x^2 &= 9 \\ &\text{Use the Square Root Property.} & x &= \pm\sqrt{9} \\ & & x &= \pm 3 \end{aligned}$$

So $x = 3$ or $x = -3$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

$$x^2 = 7$$

Use the Square Root Property. $x = \sqrt{7}, \quad x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

Example 5.1.1 How to Solve a Quadratic Equation of the form $ax^2 - k$ Using the Square Root Property

Solve: $x^2 - 50 = 0$.

Solution:

Table 9.1.1

Step 1: Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$
Step 2: Use the Square Root Property.	Remember to write the $\pm$ symbol.	$x = \pm\sqrt{50}$
Step 3: Simplify the radical.	Rewrite to show two solutions.	$x = \pm\sqrt{25}\cdot\sqrt{2}$ $x = \pm 5\sqrt{2}$ $x = 5\sqrt{2},\ x = -5\sqrt{2}$
Step 4: Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^2 - 50 = 0$ $(5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25\cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$ $x^2 - 50 = 0$ $(-5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25\cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$

Exercise 5.1.1

Solve: $x^2 - 48 = 0$.

Answer

$x = 4\sqrt{3}, x = -4\sqrt{3}$

Exercise 5.1.2

Solve: $y^2 - 27 = 0$.

Answer

$y = 3\sqrt{3}, y = -3\sqrt{3}$

The steps to take to use the **Square Root Property** to solve a quadratic equation are listed here.

Solve a Quadratic Equation Using the Square Root Property

1. Isolate the quadratic term and make its coefficient one.
2. Use Square Root Property.
3. Simplify the radical.
4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

Example 5.1.2

Solve: $3z^2 = 108$.

Solution:

Table 9.1.2

	$3z^2 = 108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$
Simplify.	$z^2 = 36$
Use the Square Root Property.	$z = \pm\sqrt{36}$
Simplify the radical.	$z = \pm 6$
Rewrite to show two solutions.	$z = 6, \quad z = -6$
Check the solutions: $3z^2 = 108$ $\quad$ $3z^2 = 108$ $3(6)^2 \stackrel{?}{=} 108$ $\quad$ $3(-6)^2 \stackrel{?}{=} 108$ $3(36) \stackrel{?}{=} 108$ $\quad$ $3(36) \stackrel{?}{=} 108$ $108 = 108$ ✓ $\quad$ $108 = 108$ ✓ Figure 9.1.1	

Exercise 5.1.3

Solve: $2x^2 = 98$.

Answer

$x = 7, x = -7$

Exercise 5.1.4

Solve: $5m^2 = 80$.

Answer

$m = 4, m = -4$

The Square Root Property states 'If $x^2 = k$,' What will happen if $k < 0$? This will be the case in the next example.

Example 5.1.3

Solve: $x^2 + 72 = 0$.

Solution:

Table 9.1.3

	$x^2 + 72 = 0$
Isolate the quadratic term.	$x^2 = -72$
Use the Square Root Property.	$x = \pm\sqrt{-72}$
Simplify using complex numbers.	$x = \pm\sqrt{72}i$
Simplify the radical.	$x = \pm 6\sqrt{2}i$
Rewrite to show two solutions	$x = 6\sqrt{2}i, x = -6\sqrt{2}i$
Check the solutions: $x^2 + 72 = 0$ $(6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0$ $(6)^2 \cdot (\sqrt{2})^2 \cdot i^2 + 72 \stackrel{?}{=} 0$ $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ $0 = 0$ ✓	$x^2 + 72 = 0$ $(-6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0$ $(-6)^2 \cdot (\sqrt{2})^2 \cdot i^2 + 72 \stackrel{?}{=} 0$ $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ $0 = 0$ ✓

Exercise 5.1.5

Solve: $c^2 + 12 = 0$.

Answer

$c = 2\sqrt{3}i, \quad c = -2\sqrt{3}i$

Exercise 5.1.6

Solve: $q^2 + 24 = 0$.

Answer

$c = 2\sqrt{6}i, \quad c = -2\sqrt{6}i$

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

Example 5.1.4

Solve: $\frac{2}{3}u^2 + 5 = 17$.

Solution:

Table 9.1.4

	$\frac{2}{3}u^2+5=17$
Isolate the quadratic term.	$\frac{2}{3}u^2=12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2}\cdot\frac{2}{3}u^2=\frac{3}{2}\cdot 12$
Simplify.	$u^2=18$
Use the Square Root Property.	$u=\pm\sqrt{18}$
Simplify the radical.	$u=\pm\sqrt{9\cdot 2}$
Simplify.	$u=\pm 3\sqrt{2}$
Rewrite to show two solutions.	$u=3\sqrt{2},\quad u=-3\sqrt{2}$
Check: $\frac{2}{3}u^2+5=17$ $\frac{2}{3}(3\sqrt{2})^2+5\stackrel{?}{=}17$ $\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$ $12+5\stackrel{?}{=}17$ $17=17\checkmark$	$\frac{2}{3}u^2+5=17$ $\frac{2}{3}(-3\sqrt{2})^2+5\stackrel{?}{=}17$ $\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$ $12+5\stackrel{?}{=}17$ $17=17\checkmark$

? Exercise 5.1.7

Solve: $\frac{1}{2}x^2+4=24$.

Answer

$x=2\sqrt{10}, x=-2\sqrt{10}$

? Exercise 5.1.8

Solve: $\frac{3}{4}y^2-3=18$.

Answer

$y=2\sqrt{7}, y=-2\sqrt{7}$

The solutions to some equations may have fractions inside the radicals. When this happens, we must **rationalize the denominator**.

✓ Example 5.1.5

Solve: $2x^2-8=41$.

Solution:

Table 9.1.5

	$2x^2-8=41$
Isolate the quadratic term.	$2x^2=49$
Divide by 2 to make the coefficient 1.	$\frac{2x^2}{2}=\frac{49}{2}$
Simplify.	$x^2=\frac{49}{2}$

Use the Square Root Property.	$x=\pm\sqrt{\frac{49}{2}}$
Rewrite the radical as a fraction of square roots.	$x=\pm\frac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$x=\pm\frac{\sqrt{49}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$
Simplify.	$x=\pm\frac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$x=\frac{7\sqrt{2}}{2},\quad x=-\frac{7\sqrt{2}}{2}$
Check: We leave te check for you.	

Exercise 5.1.9

Solve: $5r^2-2=34$.

Answer

$r=\frac{6\sqrt{5}}{5},\quad r=-\frac{6\sqrt{5}}{5}$

Exercise 5.1.10

Solve: $3t^2+6=70$.

Answer

$t=\frac{8\sqrt{3}}{3},\quad t=-\frac{8\sqrt{3}}{3}$

Solve Quadratic Equation of the Form $a(x-h)^2=k$ Using the Square Root Property

We can use the **Square Root Property** to solve an equation of the form $a(x-h)^2=k$ as well. Notice that the quadratic term, x, in the original form $ax^2=k$ is replaced with $(x-h)$.

$ax^2=k$ $\quad$ $a(x-h)^2=k$

Figure 9.1.20

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x-h)^2$.

Example 5.1.6

Solve: $4(y-7)^2=48$.

Solution:

Table 9.1.6

	$4(y-7)^2=48$
Divide both sides by the coefficient 4.	$(y-7)^2=12$
Use the Square Root Property on the binomial.	$y-7=\pm\sqrt{12}$
Simplify the radical.	$y-7=\pm2\sqrt{3}$
Solve for y.	$y=7\pm2\sqrt{3}$

Rewrite to show two solutions.	$y = 7 + 2\sqrt{3}$ $y = 7 - 2\sqrt{3}$

Check:

$4(y-7)^2 = 48$ | $4(y-7)^2 = 48$

$4(7+2\sqrt{3}-7)^2 \stackrel{?}{=} 48$ | $4(7-2\sqrt{3}-7)^2 \stackrel{?}{=} 48$

$4(2\sqrt{3})^2 \stackrel{?}{=} 48$ | $4(-2\sqrt{3})^2 \stackrel{?}{=} 48$

$4(12) \stackrel{?}{=} 48$ | $4(12) \stackrel{?}{=} 48$

$48 = 48$ ✓ | $48 = 48$ ✓

Figure 9.1.21

? Exercise 5.1.11

Solve: $3(a-3)^2 = 54$.

Answer

$a = 3 + 3\sqrt{2}, \quad a = 3 - 3\sqrt{2}$

? Exercise 5.1.12

Solve: $2(b+2)^2 = 80$.

Answer

$b = -2 + 2\sqrt{10}, \quad b = -2 - 2\sqrt{10}$

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

✓ Example 5.1.7

Solve: $\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$.

Solution:

$$\left(x - \tfrac{1}{3}\right)^2 = \tfrac{5}{9}$$

Use the Square Root Property.

$$x - \tfrac{1}{3} = \pm\sqrt{\tfrac{5}{9}}$$

Rewrite the radical as a fraction of square roots.

$$x - \tfrac{1}{3} = \pm\tfrac{\sqrt{5}}{\sqrt{9}}$$

Simplify the radical.

$$x - \tfrac{1}{3} = \pm\tfrac{\sqrt{5}}{3}$$

Solve for x.

$$x = \tfrac{1}{3} \pm \tfrac{\sqrt{5}}{3}$$

Rewrite to show two solutions.

$$x = \tfrac{1}{3} + \tfrac{\sqrt{5}}{3}, x = \tfrac{1}{3} - \tfrac{\sqrt{5}}{3}$$

Check:

We leave the check for you.

Exercise 5.1.13

Solve: $\left(x-\frac{1}{2}\right)^2=\frac{5}{4}$.

Answer

$x=\frac{1}{2}+\frac{\sqrt{5}}{2}, x=\frac{1}{2}-\frac{\sqrt{5}}{2}$

Exercise 5.1.14

Solve: $\left(y+\frac{3}{4}\right)^2=\frac{7}{16}$.

Answer

$y=-\frac{3}{4}+\frac{\sqrt{7}}{4}, y=-\frac{3}{4}-\frac{\sqrt{7}}{4}$

We will start the solution to the next example by isolating the binomial term.

Example 5.1.8

Solve: $2(x-2)^2+3=57$.

Solution:

$$2(x-2)^2+3=57$$

Subtract 3 from both sides to isolate the binomial term.

$$2(x-2)^2=54$$

Divide both sides by 2.

$$(x-2)^2=27$$

Use the Square Root Property.

$$x-2=\pm\sqrt{27}$$

Simplify the radical.

$$x-2=\pm 3\sqrt{3}$$

Solve for x.

$$x=2\pm 3\sqrt{3}$$

Rewrite to show two solutions.

$$x=2+3\sqrt{3}, x=2-3\sqrt{3}$$

Check:

We leave the check for you.

Exercise 5.1.15

Solve: $5(a-5)^2+4=104$.

Answer

$a=5+2\sqrt{5}, a=5-2\sqrt{5}$

Exercise 5.1.16

Solve: $3(b+3)^2-8=88$.

Answer

$b=-3+4\sqrt{2}, \quad b=-3-4\sqrt{2}$

Sometimes the solutions are complex numbers.

Example 5.1.9

Solve: $(2x-3)^2=-12$.

Solution:

$$(2x-3)^2=-12$$

Use the Square Root Property.

$$2x-3=\pm\sqrt{-12}$$

Simplify the radical.

$$2x-3=\pm 2\sqrt{3}i$$

Add 3 to both sides.

$$2x=3\pm 2\sqrt{3}i$$

Divide both sides by 2.

$$x=\frac{3\pm 2\sqrt{3}i}{2}$$

Rewrite in standard form.

$$x=\frac{3}{2}\pm\frac{2\sqrt{3}i}{2}$$

Simplify.

$$x=\frac{3}{2}\pm\sqrt{3}i$$

Rewrite to show two solutions.

$$x=\frac{3}{2}+\sqrt{3}i, x=\frac{3}{2}-\sqrt{3}i$$

Check:

We leave the check for you.

Exercise 5.1.17

Solve: $(3r+4)^2=-8$.

Answer

$r=-\frac{4}{3}+\frac{2\sqrt{2}i}{3}, r=-\frac{4}{3}-\frac{2\sqrt{2}i}{3}$

Exercise 5.1.18

Solve: $(2t-8)^2=-10$.

Answer

$t=4+\frac{\sqrt{10}i}{2}, t=4-\frac{\sqrt{10}i}{2}$

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Example 5.1.10

Solve: $4n^2+4n+1=16$.

Solution:

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

Table 9.1.7

	$4n^2+4n+1=16$
Factor the perfect square trinomial.	$(2n+1)^2=16$
Use the Square Root Property.	$2n+1=\pm\sqrt{16}$
Simplify the radical.	$2n+1=\pm 4$
Solve for n.	$2n=-1\pm 4$
Divide each side by 2.	$\frac{2n}{2}=\frac{-1\pm 4}{2}$ $n=\frac{-1\pm 4}{2}$
Rewrite to show two solutions.	$n=\frac{-1+4}{2}, n=\frac{-1-4}{2}$
Simplify each equation.	$n=\frac{3}{2}, \quad n=-\frac{5}{2}$
Check: $4n^2+4n+1=16$ $4\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $9+6+1\stackrel{?}{=}16$ $16=16$ ✓ $4n^2+4n+1=16$ $4\left(-\frac{5}{2}\right)^2+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{25}{4}\right)+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $25-10+1\stackrel{?}{=}16$ $16=16$ ✓	

Figure 9.1.22

Exercise 5.1.19

Solve: $9m^2-12m+4=25$.

Answer

$m=\frac{7}{3}, \quad m=-1$

Exercise 5.1.20

Solve: $16n^2+40n+25=4$.

Answer

$n=-\frac{3}{4}, \quad n=-\frac{7}{4}$

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- Solving Quadratic Equations: The Square Root Property

- Using the Square Root Property to Solve Quadratic Equations

Key Concepts

- Square Root Property
  - If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$or $x = \pm\sqrt{k}$

  How to solve a quadratic equation using the square root property.

  1. Isolate the quadratic term and make its coefficient one.
  2. Use Square Root Property.
  3. Simplify the radical.
  4. Check the solutions.

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5.1E: Exercises

Practice Makes Perfect

? ExerciseS 1 - 22: Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

1. $a^2 = 49$
2. $b^2 = 144$
3. $r^2 - 24 = 0$
4. $t^2 - 75 = 0$
5. $u^2 - 300 = 0$
6. $v^2 - 80 = 0$
7. $4m^2 = 36$
8. $3n^2 = 48$
9. $\frac{4}{3}x^2 = 48$
10. $\frac{5}{3}y^2 = 60$
11. $x^2 + 25 = 0$
12. $y^2 + 64 = 0$
13. $x^2 + 63 = 0$
14. $y^2 + 45 = 0$
15. $\frac{4}{3}x^2 + 2 = 110$
16. $\frac{2}{3}y^2 - 8 = -2$
17. $\frac{2}{5}a^2 + 3 = 11$
18. $\frac{3}{2}b^2 - 7 = 41$
19. $7p^2 + 10 = 26$
20. $2q^2 + 5 = 30$
21. $5y^2 - 7 = 25$
22. $3x^2 - 8 = 46$

Answer

1. $a = \pm 7$
3. $r = \pm 2\sqrt{6}$
5. $u = \pm 10\sqrt{3}$
7. $m = \pm 3$
9. $x = \pm 6$
11. $x = \pm 5i$
13. $x = \pm 3\sqrt{7}i$
15. $x = \pm 9$
17. $a = \pm 2\sqrt{5}$

19. $p = \pm\frac{4\sqrt{7}}{7}$

21. $y = \pm\frac{4\sqrt{10}}{5}$

Exercises 23 - 46: Solve Quadratic Equations of the Form $a(x-h)^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

23. $(u-6)^2 = 64$

24. $(v+10)^2 = 121$

25. $(m-6)^2 = 20$

26. $(n+5)^2 = 32$

27. $\left(r-\frac{1}{2}\right)^2 = \frac{3}{4}$

28. $\left(x+\frac{1}{5}\right)^2 = \frac{7}{25}$

29. $\left(y+\frac{2}{3}\right)^2 = \frac{8}{81}$

30. $\left(t-\frac{5}{6}\right)^2 = \frac{11}{25}$

31. $(a-7)^2+5 = 55$

32. $(b-1)^2-9 = 39$

33. $4(x+3)^2-5 = 27$

34. $5(x+3)^2-7 = 68$

35. $(5c+1)^2 = -27$

36. $(8d-6)^2 = -24$

37. $(4x-3)^2+11 = -17$

38. $(2y+1)^2-5 = -23$

39. $m^2-4m+4 = 8$

40. $n^2+8n+16 = 27$

41. $x^2-6x+9 = 12$

42. $y^2+12y+36 = 32$

43. $25x^2-30x+9 = 36$

44. $9y^2+12y+4 = 9$

45. $36x^2-24x+4 = 81$

46. $64x^2+144x+81 = 25$

Answer

23. $u = 14, u = -2$

25. $m = 6 \pm 2\sqrt{5}$

27. $r = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$

29. $y = -\frac{2}{3} \pm \frac{2\sqrt{2}}{9}$

31. $a = 7 \pm 5\sqrt{2}$

33. $x = -3 \pm 2\sqrt{2}$

35. $c = -\frac{1}{5} \pm \frac{3\sqrt{3}}{5}i$

37. $x = \frac{3}{4} \pm \frac{\sqrt{7}}{2}i$

39. $m = 2 \pm 2\sqrt{2}$

41. $x = 3 + 2\sqrt{3}, x = 3 - 2\sqrt{3}$

43. $x = -\frac{3}{5}, x = \frac{9}{5}$

45. $x = -\frac{7}{6}, x = \frac{11}{6}$

ExerciseS 47 - 68: Mixed Practice

In the following exercises, solve using the Square Root Property.

47. $2r^2 = 32$

48. $4t^2 = 16$

49. $(a-4)^2 = 28$

50. $(b+7)^2 = 8$

51. $9w^2 - 24w + 16 = 1$

52. $4z^2 + 4z + 1 = 49$

53. $a^2 - 18 = 0$

54. $b^2 - 108 = 0$

55. $\left(p - \frac{1}{3}\right)^2 = \frac{7}{9}$

56. $\left(q - \frac{3}{5}\right)^2 = \frac{3}{4}$

57. $m^2 + 12 = 0$

58. $n^2 + 48 = 0$

59. $u^2 - 14u + 49 = 72$

60. $v^2 + 18v + 81 = 50$

61. $(m-4)^2 + 3 = 15$

62. $(n-7)^2 - 8 = 64$

63. $(x+5)^2 = 4$

64. $(y-4)^2 = 64$

65. $6c^2 + 4 = 29$

66. $2d^2 - 4 = 77$

67. $(x-6)^2 + 7 = 3$

68. $(y-4)^2 + 10 = 9$

Answer

47. $r = \pm 4$

49. $a = 4 \pm 2\sqrt{7}$

51. $w = 1, w = \frac{5}{3}$

53. $a = \pm 3\sqrt{2}$

55. $p = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$

57. $m = \pm 2\sqrt{2i}$

59. $u = 7 \pm 6\sqrt{2}$

61. $m = 4 \pm 2\sqrt{3}$

63. $x = -3, x = -7$

65. $c = \pm \frac{5\sqrt{6}}{6}$

67. $x = 6 \pm 2i$

? ExerciseS 69 - 70: Writing exercises

69. In your own words, explain the Square Root Property.

70. In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x+2)^2 = 16$.

Answer

69. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x-h)^2 = k$ using the square root property.			

Figure 9.1.23

Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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5.2: Solve Quadratic Equations by Completing the Square

Learning Objectives

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2+bx+c=0$ by completing the square
- Solve quadratic equations of the form $ax^2+bx+c=0$ by completing the square

Before you get started, take this readiness quiz.

1. Expand: $(x+9)^2$.
   If you missed this problem, review Example 5.32.
2. Factor $y^2-14y+49$.
   If you missed this problem, review Example 6.9.
3. Factor $5n^2+40n+80$.
   If you missed this problem, review Example 6.14.

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y-7)^2=12$ because the left side was a perfect square.

$$\begin{aligned}(y-7)^2&=12\\ y-7&=\pm\sqrt{12}\\ y-7&=\pm 2\sqrt{3}\\ y&=7\pm 2\sqrt{3}\end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x-k)^2$ in order to use the Square Root Property.

$$\begin{aligned}x^2-10x+25&=18\\ (x-5)^2&=18\end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

$$\begin{array}{cc}(x+9)^2 & (y-7)^2\\ (x+9)(x+9) & (y-7)(y-7)\\ x^2+9x+9x+81 & y^2-7y-7y+49\\ x^2+18x+81 & y^2-14y+49\end{array}$$

We restate the patterns here for reference.

Definition 5.2.1: Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2=a^2+2ab+b^2$$

$$\underbrace{(a+b)^2}_{\text{(binomial)}^2}=\underbrace{a^2}_{\text{(first term)}^2}+\underbrace{2ab}_{2\cdot\text{(product of terms)}}+\underbrace{b^2}_{\text{(second term)}^2}$$

$$(a-b)^2=a^2-2ab+b^2$$

$$\underbrace{(a-b)^2}_{\text{(binomial)}^2}=\underbrace{a^2}_{\text{(first term)}^2}-\underbrace{2ab}_{2\cdot\text{(product of terms)}}+\underbrace{b^2}_{\text{(second term)}^2}$$

We can use this pattern to “make” a perfect square.

We will start with the expression x^2+6x. Since there is a plus sign between the two terms, we will use the $(a+b)^2$ pattern, $a^2+2ab+b^2=(a+b)^2$.

$$\underset{a^2+2ab+b^2}{x^2+6x+\_}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of x^2+6x is a square, x^2. This tells us that $a=x$.

$$\underset{a^2+\quad 2ab\quad +b^2}{x^2+2\cdot x\cdot b+b^2}$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b=3$.

$$\underset{a^2+\quad 2ab\quad +b^2}{x^2+2\cdot 3\cdot x+\_}$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2=9$.

$$\underset{a^2+2ab+b^2}{x^2+6x+9}$$

We can now factor.

$$\underset{(a+b)^2}{(x+3)^2}$$

So we found that adding 9 to x^2+6x ‘completes the square’, and we write it as $(x+3)^2$.

Howto: Complete a Square of x^2+bx

1. Identify b, the coefficient of x.
2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
3. Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx.
4. Factor the perfect square trinomial, writing it as a binomial squared.

Example 5.2.1

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. x^2-26x
b. y^2-9y
c. $n^2+\frac{1}{2}n$

Solution:

a.

	$\underset{x^2-bx}{x^2-26x}$
The coefficient of x is -26.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-26)\right)^2$ $(13)^2$ 169	
Add 169 to the binomial to complete the square.	$x^2-26x+169$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x-13)^2$

b.

	$x^2 - bx$ $y^2 - 9y$
The coefficient of y is -9.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-9)\right)^2$ $\left(-\frac{9}{2}\right)^2$ $\frac{81}{4}$	
Add $\frac{81}{4}$ to the binomial to complete the square.	$y^2 - 9y + \frac{81}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(y - \frac{9}{2}\right)^2$

c.

	$x^2 + bx$ $n^2 + \frac{1}{2}n$
The coefficient of n is $\frac{1}{2}$.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$ $\left(\frac{1}{4}\right)^2$ $\frac{1}{16}$	
Add $\frac{1}{16}$ to the binomial to complete the square.	$n^2 + \frac{1}{2}n + \frac{1}{16}$
Rewrite as a binomial square.	$\left(n + \frac{1}{4}\right)^2$

Exercise 5.2.1

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. $a^2 - 20a$
b. $m^2 - 5m$
c. $p^2 + \frac{1}{4}p$

Answer

a. $(a-10)^2$
b. $\left(b - \frac{5}{2}\right)^2$
c. $\left(p + \frac{1}{8}\right)^2$

Exercise 5.2.2

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. $b^2 - 4b$
b. $n^2 + 13n$
c. $q^2 - \frac{2}{3}q$

Answer

a. $(b-2)^2$

b. $\left(n+\frac{13}{2}\right)^2$

c. $\left(q-\frac{1}{3}\right)^2$

Solve Quadratic Equations of the Form $x^2+bx+c=0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a **quadratic equation** by **completing the square** too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2+6x=40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

	$x^2+6x=40$
	$x^2+6x+\_\_=40+\_\_$
	$x^2+6x+9=40+9$
Add 9 to both sides to complete the square.	$(x+3)^2=49$

Now the equation is in the form to solve using the **Square Root Property**! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

Example 5.2.2 How to Solve a Quadratic Equation of the Form $x^2+bx+x=0$ by Completing the Square

Solve by completing the square: $x^2+8x=48$.

Solution:

Step 1: Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$x^2+bx \quad c$ $x^2+8x=48$
Step 2: Find $\left(\frac{1}{2}\cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2=16$ Add 16 to BOTH sides of the equation.	$x^2+8x+\underline{\quad}=48$ $\left(\frac{1}{2}\cdot 8\right)^2$ $x^2+8x+16=48+16$
Step 3: Factor the perfect square trinomial as a binomial square.	$x^2+8x+16=(x+4)^2$ Add the terms on the right.	$(x+4)^2=64$
Step 4: Use the Square Root Property.		$x+4=\pm\sqrt{64}$
Step 5: Simplify the radical and then solve the two resulting equations.		$x+4=\pm 8$ $x+4=8 \quad x+4=-8$ $x=4 \quad x=-12$
Step 6: Check the solutions.	Put each answer in the original equation to check. Substitute $x=4$ and $x=-12$.	$x^2+8x=48$ $(4)^2+8(4)\stackrel{?}{=}48$ $16+32\stackrel{?}{=}48$ $48=48$ $x^2+8x=48$ $(-12)^2+8(-12)\stackrel{?}{=}48$ $144-96\stackrel{?}{=}48$ $48=48$

Exercise 5.2.3

Solve by completing the square: $x^2+4x=5$.

Answer

$x=-5, x=-1$

Exercise 5.2.4

Solve by completing the square: $y^2-10y=-9$.

Answer

$y=1, y=9$

The steps to solve a quadratic equation by completing the square are listed here.

Solve a Quadratic Equation of the Form $x^2+bx+c=0$ by Completing the Square

1. Isolate the variable terms on one side and the constant terms on the other.
2. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
3. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
4. Use the Square Root Property.
5. Simplify the radical and then solve the two resulting equations.
6. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

Example 5.2.3

Solve by completing the square: $x^2+4x=-21$.

Solution:

	$x^2+4x=-21$ (x^2: b, x: c)
The variable terms are on the left side. Take half of 4 and square it.	$x^2+4x+\underset{\left(\frac{1}{2}\cdot 4\right)^2}{\_\_\_\_}=-21$
$\left(\frac{1}{2}(4)\right)^2=4$	
Add 4 to both sides.	$x^2+4x+4=-21+4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+2)^2=-17$
Use the Square Root Property.	$x+2=\pm\sqrt{-17}$
Simplifying using complex numbers.	$x+2=\pm\sqrt{17}\,i$
Subtract 2 from each side.	$x=-2\pm\sqrt{17}\,i$
Rewrite to show two solutions.	$x=-2+\sqrt{17}\,i, \quad x=-2-\sqrt{17}\,i$
We leave the check to you.	

Exercise 5.2.5

Solve by completing the square: $y^2 - 10y = -35$.

Answer

$y = 5 + \sqrt{15}i, y = 5 - \sqrt{15}i$

Exercise 5.2.6

Solve by completing the square: $z^2 + 8z = -19$.

Answer

$z = -4 + \sqrt{3}i, z = -4 - \sqrt{3}i$

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

Example 5.2.4

Solve by completing the square: $y^2 - 18y = -6$.

Solution:

	$\overset{x^2}{y^2} - \overset{bx}{18y} = \overset{c}{-6}$
The variable terms are on the left side. Take half of -18 and square it.	
$\left(\frac{1}{2}(-18)\right)^2 = 81$	$y^2 - 18y + \underset{\left(\frac{1}{2}\cdot(-18)\right)^2}{\_\_\_\_\_} = -6$
Add 81 to both sides.	$y^2 - 18y + 81 = -6 + 81$
Factor the perfect square trinomial, writing it as a binomial squared.	$(y-9)^2 = 75$
Use the Square Root Property.	$y - 9 = \pm\sqrt{75}$
Simplify the radical.	$y - 9 = \pm 5\sqrt{3}$
Solve for y.	$y = 9 \pm 5\sqrt{3}$
Check. $y^2 - 18y = -6$ $(9+5\sqrt{3})^2 - 18(9+5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓ $y^2 - 18y = -6$ $(9-5\sqrt{3})^2 - 18(9-5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓ Figure 9.2.35	$y^2 - 18y = -6$ $(9+5\sqrt{3})^2 - 18(9+5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓ $y^2 - 18y = -6$ $(9-5\sqrt{3})^2 - 18(9-5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓

Another way to check this would be to use a calculator. Evaluate $y^2 - 18y$ for both of the solutions. The answer should be -6.

Exercise 5.2.7

Solve by completing the square: $x^2 - 16x = -16$.

Answer

$x = 8 + 4\sqrt{3}, x = 8 - 4\sqrt{3}$

Exercise 5.2.8

Solve by completing the square: $y^2+8y=11$.

Answer

$y=-4+3\sqrt{3}, y=-4-3\sqrt{3}$

We will start the next example by isolating the variable terms on the left side of the equation.

Example 5.2.5

Solve by completing the square: $x^2+10x+4=15$.

Solution:

	$x^2+10x+4=15$
Isolate the variable terms on the left side. Subtract 4 to get the constant terms on the right side.	$x^2+10x=11$
Take half of 10 and square it.	
$\left(\frac{1}{2}(10)\right)^2=25$	$x^2-10x+\underset{\left(\frac{1}{2}\cdot(10)\right)^2}{\_\_\_\_\_}=11$
Add 25 to both sides.	$x^2+10x+25=11+25$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+5)^2=36$
Use the Square Root Property.	$x+5=\pm\sqrt{36}$
Simplify the radical.	$x+5=\pm 6$
Solve for x.	$x=-5\pm 6$
Rewrite to show two solutions.	$x=-5+6, \quad x=-5-6$
Solve the equations.	$x=1, \quad x=-11$
Check: $x^2+10x+4=15$ $(1)^2+10(1)+4\stackrel{?}{=}15$ $1+10+4\stackrel{?}{=}15$ $15=15$ ✓ $x^2+10x+4=15$ $(-11)^2+10(-11)+4\stackrel{?}{=}15$ $121+110+4\stackrel{?}{=}15$ $15=15$ ✓	

Figure 9.2.47

Exercise 5.2.9

Solve by completing the square: $a^2+4a+9=30$.

Answer

$a=-7, a=3$

Exercise 5.2.10

Solve by completing the square: $b^2+8b-4=16$.

Answer

$b=-10, b=2$

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

Example 5.2.6

Solve by completing the square: $n^2=3n+11$.

Solution:

	$n^2=3n+11$
Subtract $3n$ to get the variable terms on the left side.	$n^2-3n=11$
Take half of -3 and square it.	
$\left(\frac{1}{2}(-3)\right)^2=\frac{9}{4}$	$n^2-3n+\underset{\left(\frac{1}{2}\cdot(-3)\right)^2}{\_\_\_\_}=11$
Add $\frac{9}{4}$ to both sides.	$n^2-3n+\frac{9}{4}=11+\frac{9}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(n-\frac{3}{2}\right)^2=\frac{44}{4}+\frac{9}{4}$
Add the fractions on the right side.	$\left(n-\frac{3}{2}\right)^2=\frac{53}{4}$
Use the Square Root Property.	$n-\frac{3}{2}=\pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n-\frac{3}{2}=\pm\frac{\sqrt{53}}{2}$
Solve for n.	$n=\frac{3}{2}\pm\frac{\sqrt{53}}{2}$
Rewrite to show two solutions.	$n=\frac{3}{2}+\frac{\sqrt{53}}{2}, \quad n=\frac{3}{2}-\frac{\sqrt{53}}{2}$
Check: We leave the check for you!	

Exercise 5.2.11

Solve by completing the square: $p^2=5p+9$.

Answer

$p=\frac{5}{2}+\frac{\sqrt{61}}{2}, p=\frac{5}{2}-\frac{\sqrt{61}}{2}$

Exercise 5.2.12

Solve by completing the square: $q^2=7q-3$.

Answer

$q = \frac{7}{2} + \frac{\sqrt{37}}{2}, q = \frac{7}{2} - \frac{\sqrt{37}}{2}$

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the **Zero Product Property** since it says "If $a \cdot b = 0$, then $a = 0$ or $b = 0$." Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

Example 5.2.7

Solve by completing the square: $(x-3)(x+5) = 9$.

Solution:

	$(x-3)(x+5) = 9$
We multiple the binomials on the left.	$x^2 + 2x - 15 = 9$
Add 15 to isolate the constant terms on the right.	$x^2 + 2x \quad = 24$
Take half of 2 and square it.	
$\left(\frac{1}{2} \cdot (2)\right)^2 = 1$	$x^2 + 2x + \underset{\left(\frac{1}{2} \cdot (2)\right)^2}{\_\_\_\_} = 24$
Add 1 to both sides.	$x^2 + 2x + 1 = 24 + 1$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+1)^2 = 25$
Use the Square Root Property.	$x + 1 = \pm\sqrt{25}$
Solve for x.	$x = -1 \pm 5$
Rewrite to show two solutions.	$x = -1 + 5, x = -1 - 5$
Simplify.	$x = 4, \quad x = -6$
Check: We leave the check for you!	

Exercise 5.2.13

Solve by completing the square: $(c-2)(c+8) = 11$.

Answer

$c = -9, c = 3$

Exercise 5.2.14

Solve by completing the square: $(d-7)(d+3) = 56$.

Answer

$d = 11, d = -7$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

The process of **completing the square** works best when the coefficient of x^2 is 1, so the left side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

Example 5.2.8

Solve by completing the square: $3x^2 - 12x - 15 = 0$.

Solution:

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2 - 12x - 15 = 0$
Factor out the greatest common factor.	$3(x^2 - 4x - 5) = 0$
Divide both sides by 3 to isolate the trinomial with coefficient 1.	$\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$
Simplify.	$x^2 - 4x - 5 = 0$
Add 5 to get the constant terms on the right side.	$x^2 - 4x = 5$
Take half of 4 and square it.	
$\left(\frac{1}{2}(-4)\right)^2 = 4$	$x^2 - 4x + \underset{\left(\frac{1}{2}\cdot(4)\right)^2}{\_\_\_\_} = 5$
Add 4 to both sides.	$x^2 - 4x + 4 = 5 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x-2)^2 = 9$
Use the Square Root Property.	$x - 2 = \pm\sqrt{9}$
Solve for x.	$x - 2 = \pm 3$
Rewrite to show two solutions.	$x = 2 + 3,\ x = 2 - 3$
Simplify.	$x = 5,\quad x = -1$
Check: $x = 5$: $3x^2 - 12x - 15 = 0$; $3(5)^2 - 12(5) - 15 \stackrel{?}{=} 0$; $75 - 60 - 15 \stackrel{?}{=} 0$; $0 = 0$ ✓ — $x = -1$: $3x^2 - 12x - 15 = 0$; $3(-1)^2 - 12(-1) - 15 \stackrel{?}{=} 0$; $3 + 12 - 15 \stackrel{?}{=} 0$; $0 = 0$ ✓	

Figure 9.2.80

Exercise 5.2.15

Solve by completing the square: $2m^2 + 16m + 14 = 0$.

Answer

$m = -7, m = -1$

Exercise 5.2.16

Solve by completing the square: $4n^2 - 24n - 56 = 8$.

Answer

$n = -2, n = 8$

To complete the square, the coefficient of the x^2 must be 1. When the **leading coefficient** is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

Example 5.2.9

Solve by completing the square: $2x^2 - 3x = 20$.

Solution:

To complete the square we need the coefficient of x^2 to be one. We will divide both sides of the equation by the coefficient of x^2. Then we can continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it.	
$\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$	$x^2 - \frac{3}{2}x + \underset{\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2}{\_\_\_\_} = 10$
Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for x.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show two solutions.	$x = \frac{3}{4} + \frac{13}{4}, \quad x = \frac{3}{4} - \frac{13}{4}$
Simplify.	$x = 4, \qquad x = -\frac{5}{2}$
Check: We leave the check for you!	

Exercise 5.2.17

Solve by completing the square: $3r^2 - 2r = 21$.

Answer

$r = -\frac{7}{3}, r = 3$

Exercise 5.2.18

Solve by completing the square: $4t^2 + 2t = 20$.

Answer

$t = -\frac{5}{2}, t = 2$

Now that we have seen that the coefficient of x^2 must be 1 for us to complete the square, we update our procedure for solving a **quadratic equation** by completing the square to include equations of the form $ax^2 + bx + c = 0$.

Howto: Solve a Quadratic Equation of the Form $ax^2 + bx + c = 0$ by Completing the Square

1. Divide by aa to make the coefficient of x^2 term 1.
2. Isolate the variable terms on one side and the constant terms on the other.
3. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
5. Use the Square Root Property.
6. Simplify the radical and then solve the two resulting equations.
7. Check the solutions.

Example 5.2.10

Solve by completing the square: $3x^2 + 2x = 4$.

Solution:

Again, our first step will be to make the coefficient of x^2 one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2 + 2x = 4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2 + 2x}{3} = \frac{4}{3}$
Simplify.	$x^2 + \frac{2}{3}x = \frac{4}{3}$
Take half of $\frac{2}{3}$ and square it.	
$\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2 = \frac{1}{9}$	$x^2 + \frac{2}{3}x + \underset{\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2}{\_\_\_\_} = \frac{4}{3}$
Add $\frac{1}{9}$ to both sides.	$x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x + \frac{1}{3}\right)^2 = \frac{12}{9} + \frac{1}{9}$
Use the Square Root Property.	$x + \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x + \frac{1}{3} = \pm\frac{\sqrt{13}}{3}$
Solve for x.	$x = -\frac{1}{3} \pm \frac{\sqrt{13}}{3}$
Rewrite to show two solutions.	$x = -\frac{1}{3} + \frac{\sqrt{13}}{3}, \quad x = -\frac{1}{3} - \frac{\sqrt{13}}{3}$
Check: We leave the check for you!	

? Exercise 5.2.19

Solve by completing the square: $4x^2+3x=2$.

Answer

$x=-\frac{3}{8}+\frac{\sqrt{41}}{8}, x=-\frac{3}{8}-\frac{\sqrt{41}}{8}$

? Exercise 5.2.20

Solve by completing the square: $3y^2-10y=-5$.

Answer

$y=\frac{5}{3}+\frac{\sqrt{10}}{3}, y=\frac{5}{3}-\frac{\sqrt{10}}{3}$

Access these online resources for additional instruction and practice with completing the square.

- Completing Perfect Square Trinomials
- Completing the Square 1
- Completing the Square to Solve Quadratic Equations
- Completing the Square to Solve Quadratic Equations: More Examples
- Completing the Square 4

Key Concepts

- Binomial Squares Pattern
  If a and b are real numbers,

$$(a+b)^2=a^2+2ab+b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

$$(a-b)^2=a^2-2ab+b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

- How to Complete a Square
  1. Identify b, the coefficient of x.
  2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
  3. Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx
  4. Rewrite the trinomial as a binomial square
- How to solve a quadratic equation of the form $ax^2+bx+c=0$ by completing the square.
  1. Divide by a to make the coefficient of x^2 term 1.
  2. Isolate the variable terms on one side and the constant terms on the other.
  3. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
  4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.
  5. Use the Square Root Property.
  6. Simplify the radical and then solve the two resulting equations.
  7. Check the solutions.

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5.2E: Exercises

Practice Makes Perfect

? ExerciseS 1 - 4: Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

1. a. m^2-24m
   b. x^2-11x
   c. $p^2-\frac{1}{3}p$
2. a. n^2-16n
   b. y^2+15y
   c. $q^2+\frac{3}{4}q$
3. a. p^2-22p
   b. y^2+5y
   c. $m^2+\frac{2}{5}m$
4. a. q^2-6q
   b. x^2-7x
   c. $n^2-\frac{2}{3}n$

Answer

1. a. $(m-12)^2$ b. $\left(x-\frac{11}{2}\right)^2$ c. $\left(p-\frac{1}{6}\right)^2$

3. a. $(p-11)^2$ b. $\left(y+\frac{5}{2}\right)^2$ c. $\left(m+\frac{1}{5}\right)^2$

? ExerciseS 5 - 28: Solve Quadratic Equations of the Form $x^2+bx+c=0$ by Completing the Square

In the following exercises, solve by completing the square.

5. $u^2+2u=3$
6. $z^2+12z=-11$
7. $x^2-20x=21$
8. $y^2-2y=8$
9. $m^2+4m=-44$
10. $n^2-2n=-3$
11. $r^2+6r=-11$
12. $t^2-14t=-50$
13. $a^2-10a=-5$
14. $b^2+6b=41$
15. $x^2+5x=2$
16. $y^2-3y=2$
17. $u^2-14u+12=-1$
18. $z^2+2z-5=2$
19. $r^2-4r-3=9$
20. $t^2-10t-6=5$

21. $v^2 = 9v + 2$

22. $w^2 = 5w - 1$

23. $x^2 - 5 = 10x$

24. $y^2 - 14 = 6y$

25. $(x + 6)(x - 2) = 9$

26. $(y + 9)(y + 7) = 80$

27. $(x + 2)(x + 4) = 3$

28. $(x - 2)(x - 6) = 5$

Answer

5. $u = -3, u = 1$

7. $x = -1, x = 21$

9. $m = -2 \pm 2\sqrt{10}i$

11. $r = -3 \pm \sqrt{2}i$

13. $a = 5 \pm 2\sqrt{5}$

15. $x = -\frac{5}{2} \pm \frac{\sqrt{33}}{2}$

17. $u = 1, u = 13$

19. $r = -2, r = 6$

21. $v = \frac{9}{2} \pm \frac{\sqrt{89}}{2}$

23. $x = 5 \pm \sqrt{30}$

25. $x = -7, x = 3$

27. $x = -5, x = -1$

? ExerciseS 29 - 40: Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

29. $3m^2 + 30m - 27 = 6$

30. $2x^2 - 14x + 12 = 0$

31. $2n^2 + 4n = 26$

32. $5x^2 + 20x = 15$

33. $2c^2 + c = 6$

34. $3d^2 - 4d = 15$

35. $2x^2 + 7x - 15 = 0$

36. $3x^2 - 14x + 8 = 0$

37. $2p^2 + 7p = 14$

38. $3q^2 - 5q = 9$

39. $5x^2 - 3x = -10$

40. $7x^2 + 4x = -3$

Answer

29. $m = -11, m = 1$

31. $n = 1 \pm \sqrt{14}$

33. $c = -2, c = \frac{3}{2}$

35. $x = -5, x = \frac{3}{2}$

37. $p = -\frac{7}{4} \pm \frac{\sqrt{161}}{4}$

39. $x = \frac{3}{10} \pm \frac{\sqrt{191}}{10} i$

? ExerciseS 41 - 42: writing exercises

41. Solve the equation $x^2 + 10x = -25$

a. by using the Square Root Property
b. by Completing the Square
c. Which method do you prefer? Why?

42. Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

Answer

41. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.			
solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.			

Figure 9.2.103

b. After reviewing this checklist, what will you do to become confident for all objectives?

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5.3: Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Before you get started, take this readiness quiz.

1. Evaluate b^2-4ab when $a=3$ and $b=-2$.
2. Simplify $\sqrt{108}$.
3. Simplify $\sqrt{50}$.

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes'. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for x.

We start with the standard form of a quadratic equation and solve it for x by completing the square.

	$ax^2+bx+c=0, \quad a\neq 0$
Isolate the variable terms on one side.	$ax^2+bx \quad =-c$
Make the coefficient of x^2 equal to 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x \quad =-\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x \quad =-\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The final equation is called the "Quadratic Formula."

Definition 5.3.1: Quadratic Formula

The solutions to a **quadratic equation** of the form $ax^2+bx+c=0$, where $a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{5.3.1}$$

To use the **Quadratic Formula**, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the Quadratic Formula (Equation 5.3.1) is an equation. Make sure you use both sides of the equation.

Example 5.3.1 How to Solve a Quadratic Equation Using the Quadratic Formula

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

Solution:

Step 1: Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+9x-5=0$ $a=2, b=9, c=-5$
Step 2: Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $a=2, b=9, c=-5$	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$
Step 3: Simplify the fraction, and solve for x.		$x=\frac{-9\pm\sqrt{81-(-40)}}{4}$ $x=\frac{-9\pm\sqrt{121}}{4}$ $x=\frac{-9\pm 11}{4}$ $x=\frac{-9+11}{4}$ $\quad x=\frac{-9-11}{4}$ $x=\frac{2}{4}$ $\quad x=\frac{-20}{4}$ $x=\frac{1}{2}$ $\quad x=-5$

Step 4: Check the solutions.	Put each answer in the original equation to check. Substitute $x=\frac{1}{2}$ and $x=-5$.	$2x^2+9x-5=0$ $2\left(\frac{1}{2}\right)^2+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+0\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $\frac{1}{2}+\frac{9}{2}-5\stackrel{?}{=}0$ $\frac{10}{2}-5\stackrel{?}{=}0$ $5-5\stackrel{?}{=}0$ $0=0$ $2x^2+9x-5=0$ $2(-5)^2+9(-5)-5\stackrel{?}{=}0$ $2\cdot25-45-5\stackrel{?}{=}0$ $50-45-5\stackrel{?}{=}0$ $0=0$

Exercise 5.3.1

Solve by using the Quadratic Formula: $3y^2-5y+2=0$.

Answer

$y=1, y=\frac{2}{3}$

Exercise 5.3.2

Solve by using the Quadratic Formula: $4z^2+2z-6=0$.

Answer

$z=1, z=-\frac{3}{2}$

HowTo: Solve a Quadratic Equation Using the Quadratic Formula

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, and c.
2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.
3. Simplify.
4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with "$x=$".

Example 5.3.2

Solve by using the Quadratic Formula: $x^2-6x=-5$.

Solution:

	$x^2-6x=-5$

Write the equation in standard form by adding 5 to each side.	$x^2-6x+5=0$
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2-6x+5=0$
Identify the values of a, b, c.	$a=1, b=-6, c=5$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(-6)\pm\sqrt{(-6)^2-4\cdot 1\cdot(5)}}{2\cdot 1}$
Simplify.	$x=\frac{6\pm\sqrt{36-20}}{2}$ $x=\frac{6\pm\sqrt{16}}{2}$ $x=\frac{6\pm 4}{2}$
Rewrite to show two solutions.	$x=\frac{6+4}{2}, \quad x=\frac{6-4}{2}$
Simplify.	$x=\frac{10}{2}, \quad x=\frac{2}{2}$
	$x=5, \quad x=1$
Check: $x^2-6x+5=0$ $\quad$ $x^2-6x+5=0$ $5^2-6\cdot 5+5\stackrel{?}{=}0$ $\quad$ $1^2-6\cdot 1+5\stackrel{?}{=}0$ $25-30+5\stackrel{?}{=}0$ $\quad$ $1-6+5\stackrel{?}{=}0$ $0=0$ ✓ $\quad$ $0=0$ ✓	

? Exercise 5.3.3

Solve by using the Quadratic Formula: $a^2-2a=15$.

Answer

$a=-3, a=5$

? Exercise 5.3.4

Solve by using the Quadratic Formula: $b^2+24=-10b$.

Answer

$b=-6, b=-4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the **Quadratic Formula**. If we get a **radical** as a solution, the final answer must have the radical in its simplified form.

✓ Example 5.3.3

Solve by using the Quadratic Formula: $2x^2+10x+11=0$.

Solution:

	$2x^2+10x+11=0$
This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+10x+11=0$

Identify the values of a, b and c.	$a = 2,\ b = 10,\ c = 11$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \dfrac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$
Simplify.	$x = \dfrac{-10 \pm \sqrt{100 - 88}}{4}$
	$x = \dfrac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \dfrac{-10 \pm 2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x = \dfrac{2(-5 \pm \sqrt{3})}{4}$ $x = \dfrac{2(-5 \pm \sqrt{3})}{4}$
Remove the common factors.	$x = \dfrac{-5 \pm \sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \dfrac{-5 + \sqrt{3}}{2}, \quad x = \dfrac{-5 - \sqrt{3}}{2}$
Check: We leave the check for you!	

? Exercise 5.3.5

Solve by using the Quadratic Formula: $3m^2 + 12m + 7 = 0$.

Answer

$$m = \frac{-6 + \sqrt{15}}{3}, m = \frac{-6 - \sqrt{15}}{3}$$

? Exercise 5.3.6

Solve by using the Quadratic Formula: $5n^2 + 4n - 4 = 0$.

Answer

$$n = \frac{-2 + 2\sqrt{6}}{5}, n = \frac{-2 - 2\sqrt{6}}{5}$$

When we substitute a, b, and c into the Quadratic Formula and the **radicand** is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

✔ Example 5.3.4

Solve by using the Quadratic Formula: $3p^2 + 2p + 9 = 0$.

Solution:

Table 9.3.5

	$3p^2 + 2p + 9 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $3p^2 + 2p + 9 = 0$

Identify the values of a, b, c.	$a=3,\ b=2,\ c=9$
Write the Quadratic Formula.	$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$p=\frac{-(2)\pm\sqrt{(2)^2-4\cdot 3\cdot(9)}}{2\cdot 3}$
Simplify.	$p=\frac{-2\pm\sqrt{4-108}}{6}$
	$p=\frac{-2\pm\sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p=\frac{-2\pm\sqrt{104}\,i}{6}$
Simplify the radical.	$p=\frac{-2\pm 2\sqrt{26}\,i}{6}$
Factor the common factor in the numerator.	$p=\frac{2\left(-1\pm\sqrt{26}\,i\right)}{6}$
Remove the common factors.	$p=\frac{-1\pm\sqrt{26}\,i}{3}$
Rewrite in standard $a+bi$ form.	$p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$
Write as two solutions.	$p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3},\quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$

? Exercise 5.3.7

Solve by using the Quadratic Formula: $4a^2-2a+8=0$.

Answer

$$a=\frac{1}{4}+\frac{\sqrt{31}}{4}i,\quad a=\frac{1}{4}-\frac{\sqrt{31}}{4}i$$

? Exercise 5.3.8

Solve by using the Quadratic Formula: $5b^2+2b+4=0$.

Answer

$$b=-\frac{1}{5}+\frac{\sqrt{19}}{5}i,\quad b=-\frac{1}{5}-\frac{\sqrt{19}}{5}i$$

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2+bx+c=0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

✓ Example 5.3.5

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

Solution:

Our first step is to get the equation in standard form.

Table 9.3.6

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$

This equation is now in standard form.	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the values of a, b, c.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot 1\cdot(4)}}{2\cdot 1}$
Simplify.	$x=\frac{-6\pm\sqrt{36-16}}{2}$
	$x=\frac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\frac{-6\pm 2\sqrt{5}}{2}$
Factor the common factor in the numerator.	$x=\frac{2(-3\pm 2\sqrt{5})}{2}$
Remove the common factors.	$x=-3\pm 2\sqrt{5}$
Write as two solutions.	$x=-3+2\sqrt{5},\ x=-3-2\sqrt{5}$
Check: We leave the check for you!	

? Exercise 5.3.9

Solve by using the Quadratic Formula: $x(x+2)-5=0$.

Answer

$x=-1+\sqrt{6}, x=-1-\sqrt{6}$

? Exercise 5.3.10

Solve by using the Quadratic Formula: $3y(y-2)-3=0$.

Answer

$y=1+\sqrt{2}, y=1-\sqrt{2}$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

✓ Example 5.3.6

Solve by using the Quadratic Formula: $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$.

Solution:

Our first step is to clear the fractions.

Table 9.3.7

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$

Multiply.	$3u^2 + 4u = 2$
Subtract 2 to get the equation in standard form.	$ax^2 + bx + c = 0$ $3u^2 + 4u - 2 = 0$
Identify the values of a, b, and c.	$a = 3,\ b = 4,\ c = -2$
Write the Quadratic Formula.	$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$u = \frac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$
Simplify.	$u = \frac{-4 \pm \sqrt{16 + 24}}{6}$
	$u = \frac{-4 \pm \sqrt{40}}{6}$
Simplify the radical.	$u = \frac{-4 \pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u = \frac{2(-2 \pm \sqrt{10})}{6}$
Remove the common factors.	$u = \frac{-2 \pm \sqrt{10}}{3}$
Rewrite to show two solutions.	$u = \frac{-2 + \sqrt{10}}{3}, \quad u = \frac{-2 - \sqrt{10}}{3}$
Check: We leave the check for you!	

? Exercise 5.3.11

Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

Answer

$c = \frac{2+\sqrt{7}}{3}, \quad c = \frac{2-\sqrt{7}}{3}$

? Exercise 5.3.12

Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Answer

$d = \frac{9+\sqrt{33}}{4}, d = \frac{9-\sqrt{33}}{4}$

Think about the equation $(x-3)^2 = 0$. We know from the **Zero Product Property** that this equation has only one solution, $x = 3$.

We will see in the next example how using the **Quadratic Formula** to solve an equation whose standard form is a perfect square **trinomial** equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0, which leads to only one solution.

Example 5.3.7

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution:

Table 9.3.8

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$ax^2 + bx + c = 0$ $4x^2 - 20x + 25 = 0$
Identify the values of a, b, and c.	$a = 4,\ b = -20,\ c = 25$
Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$
	$x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x - 5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

Exercise 5.3.13

Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

Answer

$r = -5$

Exercise 5.3.14

Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

Answer

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Definition 5.3.2

Discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

Figure 9.3.85

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Table 9.3.9

Quadratic Equation (in standard form)	Discriminat $b^2 - 4ac$	Value of the Discriminant	Number and Type of Solutions
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5)$ 121	$+$	2 real
$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25$ 0	0	1 real
$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9$ -104	$-$	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions**. $x = \frac{-b \pm \sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution**. $x = \frac{-b \pm \sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions**. $x = \frac{-b \pm \sqrt{-}}{2a}$

Figure 9.3.86

Using the Discriminant $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Example 5.3.8

Determine the number of solutions to each quadratic equation.

a. $3x^2 + 7x - 9 = 0$
b. $5n^2 + n + 4 = 0$
c. $9y^2 - 6y + 1 = 0$

Solution:

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

a.

$$3x^2 + 7x - 9 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 3, \quad b = 7, \quad c = -9$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(7)^2-4\cdot 3\cdot(-9)$$

Simplify.

$$\begin{array}{c} 49+108 \\ 157 \end{array}$$

Since the discriminant is positive, there are 2 real solutions to the equation.

b.

$$5n^2+n+4=0$$

The equation is in standard form, identify a, b, and c.

$$a=5,\quad b=1,\quad c=4$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(1)^2-4\cdot 5\cdot 4$$

Simplify.

$$\begin{array}{c} 1-80 \\ -79 \end{array}$$

Since the discriminant is negative, there are 2 complex solutions to the equation.

c.

$$9y^2-6y+1=0$$

The equation is in standard form, identify a, b, and c.

$$a=9,\quad b=-6,\quad c=1$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(-6)^2-4\cdot 9\cdot 1$$

Simplify.

$$\begin{array}{c} 36-36 \\ 0 \end{array}$$

Since the discriminant is 0, there is 1 real solution to the equation.

? Exercise 5.3.15

Determine the number and type of solutions to each quadratic equation.

a. $8m^2-3m+6=0$
b. $5z^2+6z-2=0$
c. $9w^2+24w+16=0$

Answer

a. 2 complex solutions
b. 2 real solutions

c. 1 real solution

Exercise 5.3.16

Determine the number and type of solutions to each quadratic equation.

a. $b^2+7b-13=0$
b. $5a^2-6a+10=0$
c. $4r^2-20r+25=0$

Answer

a. 2 real solutions
b. 2 complex solutions
c. 1 real solution

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2=k$ or $a(x-h)^2=k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

Identify the Most Appropriate Method to Solve a Quadratic Equation

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2=k$ or $a(x-h)^2=k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

Example 5.3.9

Identify the most appropriate method to use to solve each quadratic equation.

a. $5z^2=17$
b. $4x^2-12x+9=0$
c. $8u^2+6u=11$

Solution:

a.

$$5z^2=17$$

Since the equation is in the $ax^2=k$, the most appropriate method is to use the Square Root Property.

b.

$$4x^2-12x+9=0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

c.

$$8u^2+6u=11$$

Put the equation in standard form.

$$8u^2+6u-11=0$$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Exercise 5.3.17

Identify the most appropriate method to use to solve each quadratic equation.

a. $x^2+6x+8=0$
b. $(n-3)^2=16$
c. $5p^2-6p=9$

Answer

a. Factoring
b. Square Root Property
c. Quadratic Formula

Exercise 5.3.18

Identify the most appropriate method to use to solve each quadratic equation.

a. $8a^2+3a-9=0$
b. $4b^2+4b+1=0$
c. $5c^2=125$

Answer

a. Quadratic Formula
b. Factoring or Square Root Property
c. Square Root Property

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula
- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions
- Discriminant in Quadratic Formula

Key Concepts

- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2+bx+c=0, a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, c.
  2. Write the Quadratic Formula. Then substitute in the values of a, b, c.
  3. Simplify.

4. Check the solutions.

- Using the Discriminant, b^2-4ac, to Determine the Number and Type of Solutions of a Quadratic Equation
  - For a quadratic equation of the form $ax^2+bx+c=0, a\neq 0$,
    - If $b^2-4ac>0$, the equation has 2 real solutions.
    - If $b^2-4ac=0$, the equation has 1 real solution.
    - If $b^2-4ac<0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.
  1. Try Factoring first. If the quadratic factors easily, this method is very quick.
  2. Try the **Square Root Property** next. If the equation fits the form $ax^2=k$ or $a(x-h)^2=k$, it can easily be solved by using the Square Root Property.
  3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

Glossary

discriminant

In the Quadratic Formula, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, the quantity b^2-4ac is called the discriminant.

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5.3E: Exercises

Practice Makes Perfect

? ExerciseS 1 - 32: Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. $4m^2+m-3=0$
2. $4n^2-9n+5=0$
3. $2p^2-7p+3=0$
4. $3q^2+8q-3=0$
5. $p^2+7p+12=0$
6. $q^2+3q-18=0$
7. $r^2-8r=33$
8. $t^2+13t=-40$
9. $3u^2+7u-2=0$
10. $2p^2+8p+5=0$
11. $2a^2-6a+3=0$
12. $5b^2+2b-4=0$
13. $x^2+8x-4=0$
14. $y^2+4y-4=0$
15. $3y^2+5y-2=0$
16. $6x^2+2x-20=0$
17. $2x^2+3x+3=0$
18. $2x^2-x+1=0$
19. $8x^2-6x+2=0$
20. $8x^2-4x+1=0$
21. $(v+1)(v-5)-4=0$
22. $(x+1)(x-3)=2$
23. $(y+4)(y-7)=18$
24. $(x+2)(x+6)=21$
25. $\frac{1}{4}m^2+\frac{1}{12}m=\frac{1}{3}$
26. $\frac{1}{3}n^2+n=-\frac{1}{2}$
27. $\frac{3}{4}b^2+\frac{1}{2}b=\frac{3}{8}$
28. $\frac{1}{9}c^2+\frac{2}{3}c=3$
29. $16c^2+24c+9=0$
30. $25d^2-60d+36=0$
31. $25q^2+30q+9=0$

32. $16y^2+8y+1=0$

Answer

1. $m=-1, m=\frac{3}{4}$
3. $p=\frac{1}{3}, p=2$
5. $p=-4, p=-3$
7. $r=-3, r=11$
9. $u=\frac{-7\pm\sqrt{73}}{6}$
11. $a=\frac{3\pm\sqrt{3}}{2}$
13. $x=-4\pm2\sqrt{5}$
15. $y=-\frac{2}{3}, y=-1$
17. $x=-\frac{3}{4}\pm\frac{\sqrt{15}}{4}i$
19. $x=\frac{3}{8}\pm\frac{\sqrt{7}}{8}i$
21. $v=2\pm2\sqrt{2}$
23. $y=-4, y=7$
25. $m=1, m=\frac{-4}{3}$
27. $b=\frac{-2\pm\sqrt{22}}{6}$
29. $c=-\frac{3}{4}$
31. $q=-\frac{3}{5}$

? ExerciseS 33 - 36 Use the Discriminant to Predict the Number of Real Solutions of a Quadratic Equation

In the following exercises, determine the number of real solutions for each quadratic equation.

33. a. $4x^2-5x+16=0$
    b. $36y^2+36y+9=0$
    c. $6m^2+3m-5=0$
34. a. $9v^2-15v+25=0$
    b. $100w^2+60w+9=0$
    c. $5c^2+7c-10=0$
35. a. $r^2+12r+36=0$
    b. $8t^2-11t+5=0$
    c. $3v^2-5v-1=0$
36. a. $25p^2+10p+1=0$
    b. $7q^2-3q-6=0$
    c. $7y^2+2y+8=0$

Answer

33. a. no real solutions b. 1 c. 2

35. a. 1 b. no real solutions c. 2

? ExerciseS 37 - 40: Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

37. a. $x^2-5x-24=0$
    b. $(y+5)^2=12$
    c. $14m^2+3m=11$
38. a. $(8v+3)^2=81$
    b. $w^2-9w-22=0$
    c. $4n^2-10=6$
39. a. $6a^2+14=20$
    b. $\left(x-\frac{1}{4}\right)^2=\frac{5}{16}$
    c. $y^2-2y=8$
40. a. $8b^2+15b=4$
    b. $\frac{5}{9}v^2-\frac{2}{3}v=1$
    c. $\left(w+\frac{4}{3}\right)^2=\frac{2}{9}$

Answer

37. a. Factor b. Square Root c. Quadratic Formula

39. a. Quadratic Formula b. Square Root c. Factor

? ExerciseS 41 - 42: Writing Exercises

41. Solve the equation $x^2+10x=120$
    a. by completing the square
    b. using the Quadratic Formula
    c. Which method do you prefer? Why?
42. Solve the equation $12y^2+23y=24$
    a. by completing the square
    b. using the Quadratic Formula
    c. Which method do you prefer? Why?

Answer

41. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

Figure 9.3.87

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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5.4: Solve Quadratic Equations in Quadratic Form

Learning Objectives

By the end of this section, you will be able to:

- Solve equations in quadratic form

Before you get started

... take this readiness quiz.

1. Factor by substitution: $y^4 - y^2 - 20$.
2. Factor by substitution: $(y-4)^2 + 8(y-4) + 15$.
3. Simplify
   a. $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$
   b. $\left(x^{\frac{1}{3}}\right)^2$
   c. $\left(x^{-1}\right)^2$

Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the $ax^2 + bx + c$ form. So we factored by substitution allowing us to make it fit the $ax^2 + bx + c$ form. We used the standard u for the substitution.

To factor the expression $x^4 - 4x^2 - 5$, we noticed the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$.) So we let $u = x^2$ and factored.

	$x^4 - 4x^2 - 5$
	$\left(x^2\right)^2 - 4\left(x^2\right) - 5$
Let $u = x^2$ and substitute.	$u^2 - 4u - 5$
Factor the trinomial.	$(u+1)(u-5)$
Replace u with x^2.	$\left(x^2+1\right)\left(x^2-5\right)$

Similarly, sometimes an equation is not in the $ax^2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

Notice that in the quadratic equation $ax^2 + bx + c = 0$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard u to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form $ax^2 + bx + c = 0$, we say the original equation was of **quadratic form**.

The next example shows the steps for solving an equation in quadratic form.

Example 5.4.1 How to Solve Equations in Quadratic Form

Solve: $6x^4 - 7x^2 + 2 = 0$

Solution:

Step 1: Identify a substitution that will put the equation in quadratic form.	Since $\left(x^2\right)^2 = x^4$, we let $u = x^2$.	$6x^4 - 7x^2 + 2 = 0$

Step 2: Rewrite the equation with the substitution to put it in quadratic form.	Rewrite to prepare for the substitution. Substitute $u = x^2$.	$6\left(x^2\right)^2 - 7x^2 + 2 = 0$ $6u^2 - 7u + 2 = 0$
Step 3: Solve the quadratic equation for u.	We can solve by factoring. Use the Zero Product Property.	$(2u-1)(3u-2) = 0$ $2u - 1 = 0, 3u - 2 = 0$ $2u = 1, 3u = 2$ $u = \frac{1}{2} u = \frac{2}{3}$
Step 4: Substitute the original variable back into the results, using the substitution.	Replace u with x^2.	$x^2 = \frac{1}{2} \quad x^2 = \frac{2}{3}$
Step 5: Solve for the original variable.	Solve for x, using the Square Root Property.	$x = \pm\sqrt{\frac{1}{2}} \quad x = \pm\sqrt{\frac{2}{3}}$ $x = \pm\frac{\sqrt{2}}{2} \quad x = \pm\frac{\sqrt{6}}{3}$ There are four solutions. $x = \frac{\sqrt{2}}{2} \quad x = \frac{\sqrt{6}}{3}$ $x = -\frac{\sqrt{2}}{2} \quad x = -\frac{\sqrt{6}}{3}$
Step 6: Check the solutions.	Check all four solutions. We will show one check here.	$x = \frac{\sqrt{2}}{2}$ $6x^4 - 7x^2 + 2 = 0$ $6\left(\frac{\sqrt{2}}{2}\right)^4 - 7\left(\frac{\sqrt{2}}{2}\right)^2 + 2 \stackrel{?}{=} 0$ $6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right)^2 + 2 \stackrel{?}{=} 0$ $\frac{3}{2} - \frac{7}{2} + \frac{4}{2} \stackrel{?}{=} 0$ $0 = 0$ We leave the other checks to you!

? Exercise 5.4.1

Solve: $x^4 - 6x^2 + 8 = 0$.

Answer

$x = \sqrt{2}, x = -\sqrt{2}, x = 2, x = -2$

? Exercise 5.4.2

Solve: $x^4 - 11x^2 + 28 = 0$.

Answer

$x = \sqrt{7}, x = -\sqrt{7}, x = 2, x = -2$

We summarize the steps to solve an equation in quadratic form.

Solve Equations in Quadratic Form

1. Identify a substitution that will put the equation in quadratic form.
2. Rewrite the equation with the substitution to put it in quadratic form.
3. Solve the quadratic equation for u.
4. Substitute the original variable back into the results, using the substitution.

5. Solve for the original variable.
6. Check the solutions.

In the next example, the binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

Example 5.4.2

Solve: $(x-2)^2+7(x-2)+12=0$.

Solution:

	$(x-2)^2+7(x-2)+12=0$
Prepare for the substitution.	$(x-2)^2+7(x-2)+12=0$
Let $u=x-2$ and substitute.	$u^2+7u+12=0$
Solve by factoring.	$(u+3)(u+4)=0$ $u+3=0, \quad u+4=0$ $u=-3, \quad u=-4$
Replace u with $x-2$.	$x-2=-3, \quad x-2=-4$
Solve for x.	$x=-1, \quad x=-2$
Check: $x=-1$ $(x-2)^2+7(x-2)+12=0$ $(-1-2)^2+7(-1-2)+12 \stackrel{?}{=} 0$ $(-3)^2+7(-3)+12 \stackrel{?}{=} 0$ $9-21+12 \stackrel{?}{=} 0$ $0=0$ ✓ $x=-2$ $(x-2)^2+7(x-2)+12=0$ $(-2-2)^2+7(-2-2)+12 \stackrel{?}{=} 0$ $(-4)^2+7(-4)+12 \stackrel{?}{=} 0$ $16-28+12 \stackrel{?}{=} 0$ $0=0$ ✓ Figure 9.4.14	

Exercise 5.4.3

Solve: $(x-5)^2+6(x-5)+8=0$.

Answer

$x=3, x=1$

Exercise 5.4.4

Solve: $(y-4)^2+8(y-4)+15=0$.

Answer

$y=-1, y=1$

In the next example, we notice that $(\sqrt{x})^2=x$. Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

Example 5.4.3

Solve: $x-3\sqrt{x}+2=0$.

Solution:

The $\sqrt{x}$ in the middle term, is squared in the first term $(\sqrt{x})^2=x$. If we let $u=\sqrt{x}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$x-3\sqrt{x}+2=0$
Rewrite the trinomial to prepare for the substitution.	$(\sqrt{x})^2-3\sqrt{x}+2=0$
Let $u=\sqrt{x}$ and substitute.	$u^2-3u+2=0$
Solve by factoring.	$(u-2)(u-1)=0$ $u-2=0, \quad u-1=0$ $u=2, \quad u=1$ \)
Replace u with $\sqrt{x}$.	$\sqrt{x}=2, \quad \sqrt{x}=1$
Solve for x, by squaring both sides.	$x=4, \quad x=1$
Check: $x=4$: $x-3\sqrt{x}+2=0$; $4-3\sqrt{4}+2\stackrel{?}{=}0$; $4-6+2\stackrel{?}{=}0$; $0=0$ ✓ $x=1$: $x-3\sqrt{x}+2=0$; $1-3\sqrt{1}+2\stackrel{?}{=}0$; $1-3+2\stackrel{?}{=}0$; $0=0$ ✓	

Exercise 5.4.5

Solve: $x-7\sqrt{x}+12=0$.

Answer

$x=9, x=16$

Exercise 5.4.6

Solve: $x-6\sqrt{x}+8=0$.

Answer

$x=4, x=16$

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

Example 5.4.4

Solve: $x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$.

Solution:

The $x^{\frac{1}{3}}$ in the middle term is squared in the first term $\left(x^{\frac{1}{3}}\right)^2=x^{\frac{2}{3}}$. If we let $u=x^{\frac{1}{3}}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$

Rewrite the trinomial to prepare for the substitution.	$\left(x^{\frac{1}{3}}\right)^2-2\left(x^{\frac{1}{3}}\right)-24=0$
Let $u=x^{\frac{1}{3}}$	$u^2-2u-24=0$
Solve by factoring.	$(u-6)(u+4)=0$ $u-6=0, \quad u+4=0$ $u=6, \quad u=-4$
Replace u with $x^{\frac{1}{3}}$.	$x^{\frac{1}{3}}=6, \quad x^{\frac{1}{3}}=-4$
Solve for x by cubing both sides.	$\left(x^{\frac{1}{3}}\right)^3=(6)^3, \quad \left(x^{\frac{1}{3}}\right)^3=(-4)^3$ $x=216, \quad x=-64$
Check: $x=216$ $x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$ $(216)^{\frac{2}{3}}-2(216)^{\frac{1}{3}}-24\stackrel{?}{=}0$ $36-12-24\stackrel{?}{=}0$ $0=0$ ✓ $x=-64$ $x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$ $(-64)^{\frac{2}{3}}-2(-64)^{\frac{1}{3}}-24\stackrel{?}{=}0$ $16+8-24\stackrel{?}{=}0$ $0=0$ ✓ Figure 9.4.33	

Exercise 5.4.7

Solve: $x^{\frac{2}{3}}-5x^{\frac{1}{3}}-14=0$.

Answer

$x=-8, x=343$

Exercise 5.4.8

Solve: $x^{\frac{1}{2}}+8x^{\frac{1}{4}}+15=0$.

Answer

$x=81, x=625$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

Example 5.4.5

Solve: $3x^{-2}-7x^{-1}+2=0$.

Solution:

The x^{-1} in the middle term is squared in the first term $\left(x^{-1}\right)^2=x^{-2}$. If we let $u=x^{-1}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$3x^{-2}-7x^{-1}+2=0$
Rewrite the trinomial to prepare for the substitution.	$3(x^{-1})^2-7(x^{-1})+2=0$
Let $u=x^{-1}$ and substitute.	$3u^2-7u+2=0$
Solve by factoring.	$(3u-1)(u-2)=0$
	$3u-1=0, \quad u-2=0$

	$u=\frac{1}{3}, \quad u=2$
Replace u with x^{-1}.	$x^{-1}=\frac{1}{3}, \quad x^{-1}=2$
Solve for x by taking the reciprocal since $x^{-1}=\frac{1}{x}$.	$x=3, \quad x=\frac{1}{2}$
Check: $x=3$ $3x^{-2}-7x^{-1}+2=0$ $3(3)^{-2}-7(3)^{-1}+2\stackrel{?}{=}0$ $3\left(\frac{1}{9}\right)-7\left(\frac{1}{3}\right)+2\stackrel{?}{=}0$ $\frac{1}{3}-\left(\frac{7}{3}\right)+\frac{6}{3}\stackrel{?}{=}0$ $0=0$ ✓ $x=\frac{1}{2}$ $3x^{-2}-7x^{-1}+2=0$ $3\left(\frac{1}{2}\right)^{-2}-7\left(\frac{1}{2}\right)^{-1}+2\stackrel{?}{=}0$ $3(4)-7(2)+2\stackrel{?}{=}0$ $12-14+2\stackrel{?}{=}0$ $0=0$ ✓ Figure 9.4.42	

? Exercise 5.4.9

Solve: $8x^{-2}-10x^{-1}+3=0$.

Answer

$x=\frac{4}{3}, x=2$

? Exercise 5.4.10

Solve: $6x^{-2}-23x^{-1}+20=0$.

Answer

$x=\frac{2}{5}, x=\frac{3}{4}$

Access this online video for additional instruction and practice with solving quadratic equations: https://www.youtube.com/watch?v=7X-CZMbpxuw

Key Concepts

- How to solve equations in quadratic form.
  1. Identify a substitution that will put the equation in quadratic form.
  2. Rewrite the equation with the substitution to put it in quadratic form.
  3. Solve the quadratic equation for u.
  4. Substitute the original variable back into the results, using the substitution.
  5. Solve for the original variable.
  6. Check the solutions.

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5.4E: Exercises

Practice Makes Perfect

Exercise $5.4E.11$ Solve equations in quadratic form

In the following exercises, solve.

1. $x^4-7x^2+12=0$
2. $x^4-9x^2+18=0$
3. $x^4-13x^2-30=0$
4. $x^4+5x^2-36=0$
5. $2x^4-5x^2+3=0$
6. $4x^4-5x^2+1=0$
7. $2x^4-7x^2+3=0$
8. $3x^4-14x^2+8=0$
9. $(x-3)^2-5(x-3)-36=0$
10. $(x+2)^2-3(x+2)-54=0$
11. $(3y+2)^2+(3y+2)-6=0$
12. $(5y-1)^2+3(5y-1)-28=0$
13. $\left(x^2+1\right)^2-5\left(x^2+1\right)+4=0$
14. $\left(x^2-4\right)^2-4\left(x^2-4\right)+3=0$
15. $2\left(x^2-5\right)^2-5\left(x^2-5\right)+2=0$
16. $2\left(x^2-5\right)^2-7\left(x^2-5\right)+6=0$
17. $x-\sqrt{x}-20=0$
18. $x-8\sqrt{x}+15=0$
19. $x+6\sqrt{x}-16=0$
20. $x+4\sqrt{x}-21=0$
21. $6x+\sqrt{x}-2=0$
22. $6x+\sqrt{x}-1=0$
23. $10x-17\sqrt{x}+3=0$
24. $12x+5\sqrt{x}-3=0$
25. $x^{\frac{2}{3}}+9x^{\frac{1}{3}}+8=0$
26. $x^{\frac{2}{3}}-3x^{\frac{1}{3}}=28$
27. $x^{\frac{2}{3}}+4x^{\frac{1}{3}}=12$
28. $x^{\frac{2}{3}}-11x^{\frac{1}{3}}+30=0$
29. $6x^{\frac{2}{3}}-x^{\frac{1}{3}}=12$
30. $3x^{\frac{2}{3}}-10x^{\frac{1}{3}}=8$
31. $8x^{\frac{2}{3}}-43x^{\frac{1}{3}}+15=0$
32. $20x^{\frac{2}{3}}-23x^{\frac{1}{3}}+6=0$
33. $x-8x^{\frac{1}{2}}+7=0$
34. $2x-7x^{\frac{1}{2}}=15$
35. $6x^{-2}+13x^{-1}+5=0$
36. $15x^{-2}-26x^{-1}+8=0$
37. $8x^{-2}-2x^{-1}-3=0$
38. $15x^{-2}-4x^{-1}-4=0$

Answer

1. $x=\pm\sqrt{3}, x=\pm 2$
3. $x=\pm\sqrt{15}, x=\pm\sqrt{2}i$

5. $x = \pm 1, x = \frac{\pm\sqrt{6}}{2}$

7. $x = \pm\sqrt{3}, x = \pm\frac{\sqrt{2}}{2}$

9. $x = -1, x = 12$

11. $x = -\frac{5}{3}, x = 0$

13. $x = 0, x = \pm\sqrt{3}$

15. $x = \pm\frac{11}{2}, x = \pm\frac{\sqrt{22}}{2}$

17. $x = 25$

19. $x = 4$

21. $x = \frac{1}{4}$

23. $x = \frac{1}{25}, x = \frac{9}{4}$

25. $x = -1, x = -512$

27. $x = 8, x = -216$

29. $x = \frac{27}{8}, x = -\frac{64}{27}$

31. $x = 27, x = 64,000$

33. $x = 1, x = 49$

35. $x = -2, x = -\frac{3}{5}$

37. $x = -2, x = \frac{4}{3}$

Exercise $5.4E.12$ writing exercises

1. Explain how to recognize an equation in quadratic form.
2. Explain the procedure for solving an equation in quadratic form.

Answer

1. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve equations in quadratic form.			

Figure 9.4.43

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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5.5: Graph Quadratic Functions Using Properties

Learning Objectives

By the end of this section, you will be able to:

- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions using properties
- Solve maximum and minimum applications

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
   If you missed this problem, review Example 3.54.
2. Solve: $2x^2 + 3x - 2 = 0$.
   If you missed this problem, review Example 6.45.
3. Evaluate $-\frac{b}{2a}$ when $a = 3$ and $b = -6$.
   If you missed this problem, review Example 1.21.

Recognize the Graph of a Quadratic Function

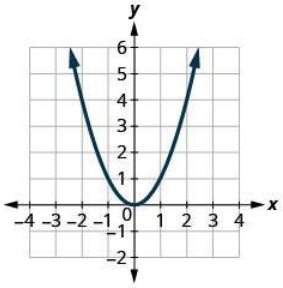
Previously we very briefly looked at the function $f(x) = x^2$, which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form $f(x) = ax^2 + bx + c$ if $a \neq 0$. We call this kind of function a quadratic function.

Definition 5.5.1

A **quadratic function**, where a, b, and c are real numbers and $a \neq 0$, is a function of the form

$$f(x) = ax^2 + bx + c$$

We graphed the quadratic function $f(x) = x^2$ by plotting points.



x	$f(x) = x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Figure 9.6.1

Every quadratic function has a graph that looks like this. We call this figure a **parabola**. Let's practice graphing a parabola by plotting a few points.

Example 5.5.1

Graph: $f(x) = x^2 - 1$.

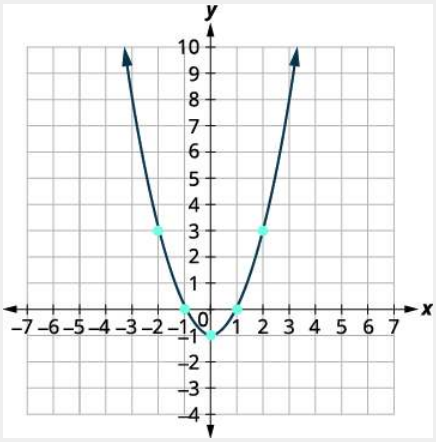
Solution:

We will graph the function by plotting points.

Choose integer values for x, substitute them into the equation and simplify to find $f(x)$. Record the values of the ordered pairs in the chart.

$f(x) = x^2 - 1$	
x	$f(x)$
0	–1
1	0
–1	0
2	3
–2	3

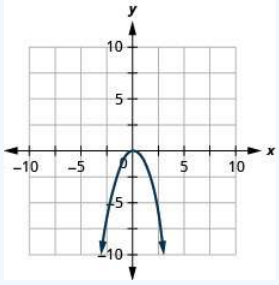
Plot the points, and then connect them with a smooth curve. The result will be the graph of the function $f(x) = x^2 - 1$.



Exercise 5.5.1

Graph $f(x) = -x^2$.

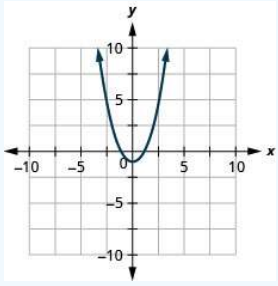
Answer



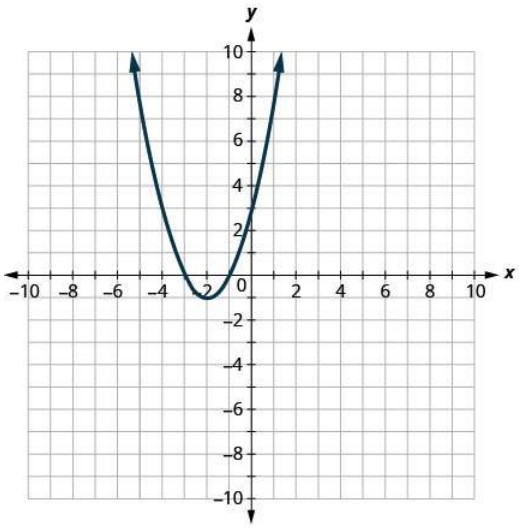
Exercise 5.5.2

Graph $f(x) = x^2 - 1$.

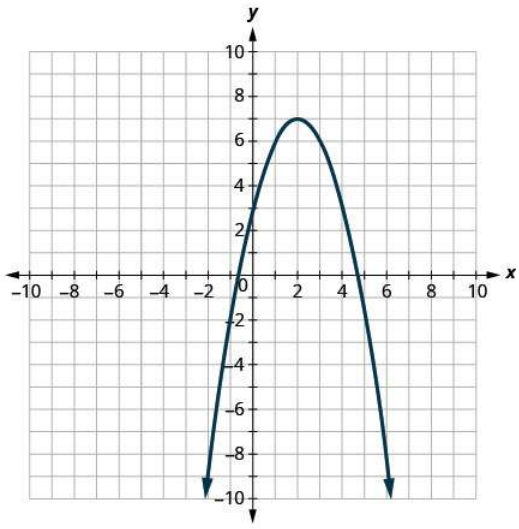
Answer



All graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$ are parabolas that open upward or downward. See *Figure 9.6.6*



$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$
$a > 0$
opens upward



$f(x) = ax^2 + bx + c$
$f(x) = -x^2 + 4x + 3$
$a < 0$
opens downward

Notice that the only difference in the two functions is the negative sign before the quadratic term (x^2 in the equation of the graph in *Figure 9.6.6*). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Definition 5.5.2

Parabola Orientation

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if

- $a > 0$, the parabola opens upward
- $a < 0$, the parabola opens downward

Example 5.5.2

Determine whether each parabola opens upward or downward:

a. $f(x) = -3x^2 + 2x - 4$
b. $f(x) = 6x^2 + 7x - 9$

Solution:

a. Find the value of a.

$f(x) = ax^2 + bx + c$
$f(x) = -3x^2 + 2x - 4$
$a = -3$

Since the a is negative, the parabola will open downward.

b. Find the value of a.

$f(x) = ax^2 + bx + c$
$f(x) = 6x^2 + 7x - 9$
$a = 6$

Since the a is positive, the parabola will open upward.

Exercise 5.5.3

Determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = 2x^2 + 5x - 2$
b. $f(x) = -3x^2 - 4x + 7$

Answer

a. up
b. down

Exercise 5.5.4

Determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = -2x^2 - 2x - 3$
b. $f(x) = 5x^2 - 2x - 1$

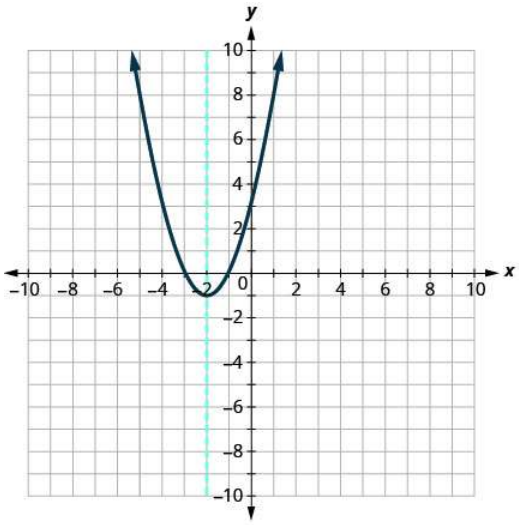
Answer

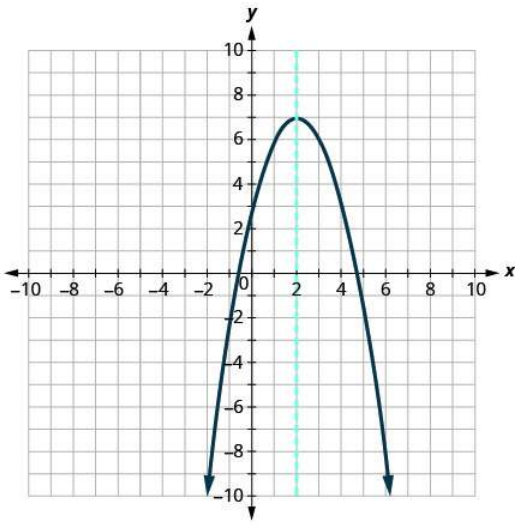
a. down
b. up

Find the Axis of Symmetry and Vertex of a Parabola

Look again at *Figure 9.6.10*. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry.





The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

$f(x) = ax^2 + bx + c$

$f(x) = x^2 + 4x + 3$

axis of symmetry

$x = -\frac{b}{2a}$

$x = -\frac{4}{2 \cdot 1}$

$x = -2$

$f(x) = ax^2 + bx + c$

$f(x) = -x^2 + 4x + 3$

axis of symmetry

$x = -\frac{b}{2a}$

$x = -\frac{4}{2(-1)}$

$x = 2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its x-coordinate is $-\frac{b}{2a}$. To find the y-coordinate of the vertex we substitute the value of the x-coordinate into the quadratic function.

$f(x) = x^2 + 4x + 3$

axis of symmetry is $x = -2$

vertex is $(-2, \_\_\_)$

$f(x) = x^2 + 4x + 3$

$f(x) = (-2)^2 + 4(-2) + 3$

$f(x) = -1$

vertex is $(-2, -1)$

$f(x) = -x^2 + 4x + 3$

axis of symmetry is $x = 2$

vertex is $(2, \_\_\_)$

$f(x) = -x^2 + 4x + 3$

$f(x) = -(2)^2 + 4(2) + 3$

$f(x) = 7$

vertex is $(2, 7)$

Axis of Symmetry and Vertex of a Parabola

The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$
- the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

Example 5.5.3

For the graph of $f(x) = 3x^2 - 6x + 2$ find:

a. the axis of symmetry
b. the vertex

Solution:

a.

	$f(x) = ax^2 + bx + c$ $f(x) = 3x^2 - 6x + 2$
The axis of symmetry is the vertical line $x = -\frac{b}{2a}$.	
Substitute the values a, b into the equation.	$x = -\frac{-6}{2 \cdot 3}$
Simplify.	$x = 1$
	The axis of symmetry is the line $x = 1$.

b.

	$f(x) = 3x^2 - 6x + 2$
The vertex is a point on the line of symmetry, so its x-coordinate will be $x = 1$. Find $f(1)$.	$f(1) = 3(1)^2 - 6(1) + 2$
Simplify.	$f(1) = 3 \cdot 1 - 6 + 2$

The result is the y-coordinate.	$f(1) = -1$
	The vertex is $(1, -1)$.

Exercise 5.5.5

For the graph of $f(x) = 2x^2 - 8x + 1$ find:

a. the axis of symmetry
b. the vertex

Answer

a. $x = 2$
b. $(2, -7)$

Exercise 5.5.6

For the graph of $f(x) = 2x^2 - 4x - 3$ find:

a. the axis of symmetry
b. the vertex

Answer

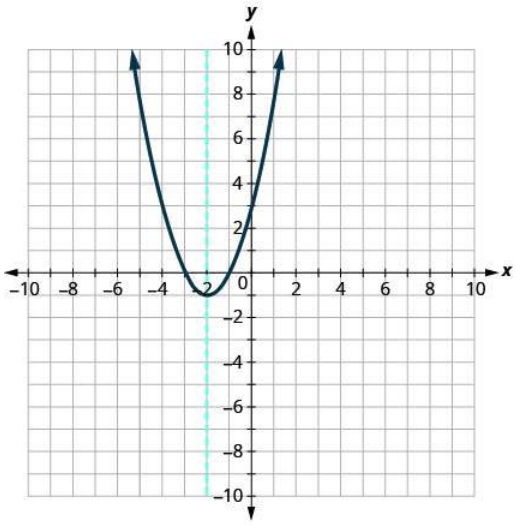
a. $x = 1$
b. $(1, -5)$

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the x- and y-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the y-intercept the value of x is zero. So to find the y-intercept, we substitute $x = 0$ into the function.

Let's find the y-intercepts of the two parabolas shown in *Figure 9.6.20*.

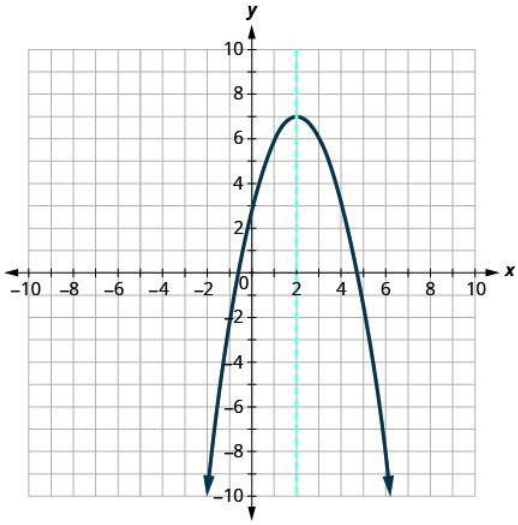


$f(x) = x^2 + 4x + 3$

$x = 0 \quad f(0) = 0^2 + 4 \cdot 0 + 3$

$f(0) = 3$

y-intercept $(0, 3)$



$f(x) = -x^2 + 4x + 3$

$x = 0 \quad f(0) = -0^2 + 4 \cdot 0 + 3$

$f(0) = 3$

y-intercept $(0, 3)$

An x-intercept results when the value of $f(x)$ is zero. To find an x-intercept, we let $f(x)=0$. In other words, we will need to solve the equation $0=ax^2+bx+c$ for x.

$$f(x)=ax^2+bx+c$$
$$0=ax^2+bx+c$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the x-intercepts of the two parabolas we looked at. First we will find the x-intercepts of the parabola whose function is $f(x)=x^2+4x+3$.

	$f(x)=x^2+4x+3$
Let $f(x)=0$.	$0=x^2+4x+3$
Factor.	$0=(x+1)(x+3)$
Use the Zero Product Property.	$x+1=0 \quad x+3=0$
Solve.	$x=-1 \quad x=-3$
	The x-intercepts are $(-1,0)$ and $(-3,0)$.

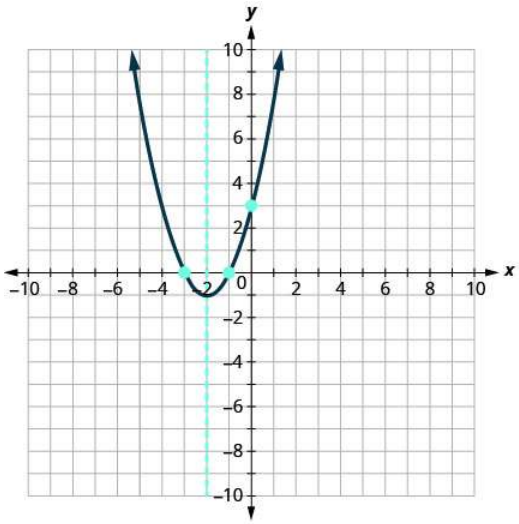
Now we will find the x-intercepts of the parabola whose function is $f(x)=-x^2+4x+3$.

	$f(x)=-x^2+4x+3$
Let $f(x)=0$.	$0=-x^2+4x+3$
This quadratic does not factor, so we use the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$a=-1, b=4, c=3$	$x=\frac{-4\pm\sqrt{4^2-4(-1)(3)}}{2(-1)}$
Simplify.	$x=\frac{-4\pm\sqrt{28}}{-2}$
	$x=\frac{-4\pm2\sqrt{7}}{-2}$
	$x=\frac{-2(2\pm\sqrt{7})}{-2}$
	$x=2\pm\sqrt{7}$
	The x-intercepts are $(2+\sqrt{7},0)$ and $(2-\sqrt{7},0)$.

We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph,

$$(2+\sqrt{7},0)\approx(4.6,0) \quad (2-\sqrt{7},0)\approx(-0.6,0)$$

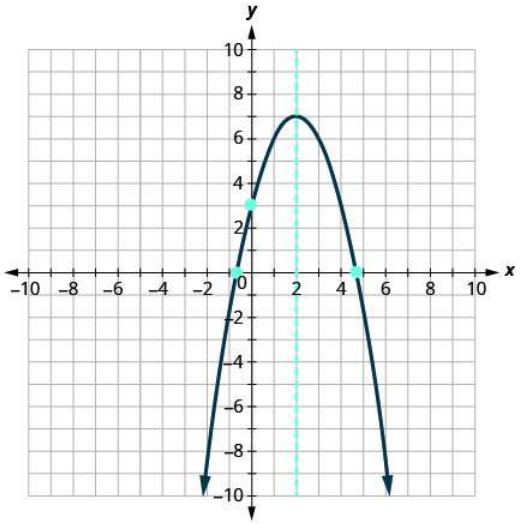
Do these results agree with our graphs? See *Figure 9.6.34*



$f(x) = x^2 + 4x + 3$

y-intercept $(0, 3)$

x-intercepts $(-1, 0)$ and $(-3, 0)$



$f(x) = -x^2 + 4x + 3$

y-intercept $(0, 3)$

x-intercepts $(2 + \sqrt{7}, 0) \approx (4.6, 0)$

$(2 - \sqrt{7}, 0) \approx (-0.6, 0)$

Find the Intercepts of a Parabola

To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

y-intercept

Let $x = 0$ and solve for $f(x)$.

x-intercepts

Let $f(x) = 0$ and solve for x

Example 5.5.4

Find the intercepts of the parabola whose function is $f(x) = x^2 - 2x - 8$.

Solution:

To find the y-intercept, let $x = 0$ and solve for $f(x)$.	$f(x) = x^2 - 2x - 8$
	$f(0) = 0^2 - 2 \cdot 0 - 8$
	$f(0) = -8$
	When $x = 0$, then $f(0) = -8$. The y-intercept is the point $(0, -8)$.
To find the x-intercept, let $f(x) = 0$ and solve for x.	$f(x) = x^2 - 2x - 8$
	$0 = x^2 - 2x - 8$
Solve by factoring.	$0 = (x - 4)(x + 2)$
	$0 = x - 4 \quad 0 = x + 2$
	$4 = x \quad -2 = x$
	When $f(x) = 0$, then $x = 4$ or $x = -2$. The x-intercepts are the points $(4, 0)$ and $(-2, 0)$.

Exercise 5.5.7

Find the intercepts of the parabola whose function is $f(x) = x^2 + 2x - 8$.

Answer

y -intercept: $(0, -8)$ x -intercepts $(-4, 0), (2, 0)$

Exercise 5.5.8

Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 12$.

Answer

y -intercept: $(0, -12)$ x -intercepts $(-2, 0), (6, 0)$

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for x and the results were the solutions to the equation.

We are now looking at quadratic functions of the form $f(x) = ax^2 + bx + c$. The graphs of these functions are parabolas. The x-intercepts of the parabolas occur where $f(x) = 0$.

For example:

Quadratic equation

$$\begin{aligned} x^2 - 2x - 15 &= 0 \quad \text{Let } f(x) = 0 \\ (x-5)(x+3) &= 0 \\ x - 5 = 0 \quad x + 3 &= 0 \\ x = 5 \quad x &= -3 \end{aligned}$$

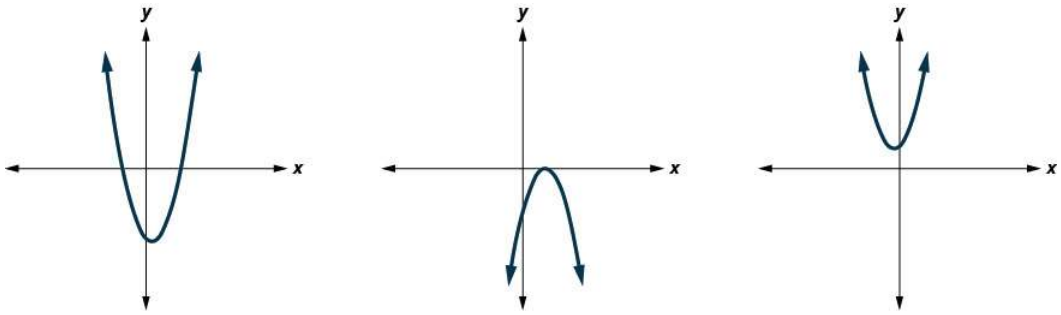
Quadratic function

$$\begin{aligned} f(x) &= x^2 - 2x - 15 \\ 0 &= x^2 - 2x - 15 \\ 0 &= (x-5)(x+3) \\ x - 5 = 0 \quad & x + 3 = 0 \\ x = 5 \quad & x = -3 \\ (5, 0) \text{ and } &(-3, 0) \\ x \text{ -intercepts} & \end{aligned}$$

The solutions of the quadratic function are the x values of the x-intercepts.

Earlier, we saw that quadratic equations have $2, 1$, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the x-intercepts of the graphs, the number of x-intercepts is the same as the number of solutions.

Previously, we used the **discriminant** to determine the number of solutions of a quadratic function of the form $ax^2 + bx + c = 0$. Now we can use the discriminant to tell us how many x-intercepts there are on the graph.



$b^2 - 4ac > 0$
Two solutions
Two x-intercepts

$b^2 - 4ac = 0$
One solution
One x-intercept

$b^2 - 4ac < 0$
No real solution
No x-intercept

Before you to find the values of the x-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Example 5.5.5

Find the intercepts of the parabola for the function $f(x) = 5x^2 + x + 4$.

Solution:

	$f(x) = 5x^2 + x + 4$
To find the y-intercept, let $x = 0$ and solve for $f(x)$.	$f(0) = 5 \cdot 0^2 + 0 + 4$
	$f(0) = 4$
	When $x = 0$, then $f(0) = 4$. The y-intercept is the point $(0, 4)$.
To find the x-intercept, let $f(x) = 0$ and solve for x.	$f(x) = 5x^2 + x + 4$
	$0 = 5x^2 + x + 4$
Find the value of the discriminant to predict the number of solutions which is also the number of x-intercepts.	
$b^2 - 4ac$ $1^2 - 4 \cdot 5 \cdot 4$ $1 - 80$ -79	
	Since the value of the discriminant is negative, there is no real solution to the equation. There are no x-intercepts.

Exercise 5.5.9

Find the intercepts of the parabola whose function is $f(x) = 3x^2 + 4x + 4$.

Answer

y-intercept: $(0, 4)$ no x-intercept

Exercise 5.5.10

Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 5$

Answer

y-intercept: $(0,-5)$ x-intercepts $(-1,0),(5,0)$

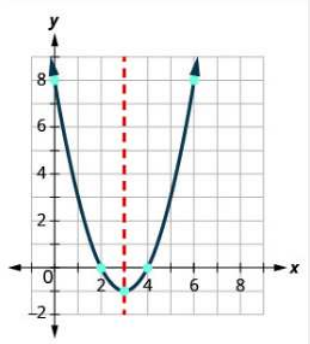
Graph Quadratic Functions Using Properties

Now we have all the pieces we need in order to graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

Example 5.5.6 How to Graph a Quadratic Function Using Properties

Graph $f(x)=x^2-6x+8$ by using its properties.

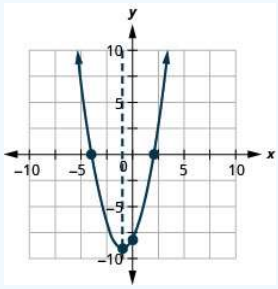
Solution:

Step 1: Determine whether the parabola opens upward or downward.	Look a in the equation $f(x)=x^2-6x+8$ Since a is positive, the parabola opens upward.	$f(x)=x^2-6x+8$ $a=1, b=-6, c=8$ **The parabola opens upward.**
Step 2: Find the axis of symmetry.	$f(x)=x^2-6x+8$ The axis of symmetry is the line $x=-\frac{b}{2a}$.	Axis of Symmetry $x=-\frac{b}{2a}$ $x=-\frac{(-6)}{2\cdot 1}$ $x=3$ **The axis of symmetry is the line $x=3$.**
Step 3: Find the vertex.	The vertex is on the axis of symmetry. Substitute $x=3$ into the function.	Vertex $f(x)=x^2-6x+8$ $f(3)=(3)^2-6(3)+8$ $f(3)=-1$ **The vertex is $(3,-1)$.**
Step 4: Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We find $f(0)$. We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x=3$. A point 3 units to the right of the axis of symmetry has $x=6$.	y-intercept $f(x)=x^2-6x+8$ $f(0)=(0)^2-6(0)+8$ $f(0)=8$ **The y-intercept is $(0,8)$.** Point symmetric to y-intercept: **The point is $(6,8)$.**
Step 5: Find the x-intercepts. Find additional points if needed.	We solve $f(x)=0$. We can solve this quadratic equation by factoring.	x-intercepts $f(x)=x^2-6x+8$ $0=x^2-6x+8$ $0=(x-2)(x-4)$ $x=2 or x=4$ **The x-intercepts are $(2,0)$ and $(4,0)$.**
Step 6: Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Exercise 5.5.11

Graph $f(x) = x^2 + 2x - 8$ by using its properties.

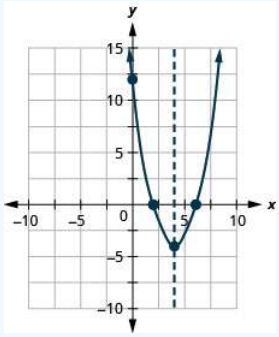
Answer



Exercise 5.5.12

Graph $f(x) = x^2 - 8x + 12$ by using its properties.

Answer



We list the steps to take in order to graph a quadratic function here.

To Graph a Quadratic Function Using Properties

1. Determine whether the parabola opens upward or downward.
2. Find the equation of the axis of symmetry.
3. Find the vertex.
4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
5. Find the x-intercepts. Find additional points if needed.
6. Graph the parabola.

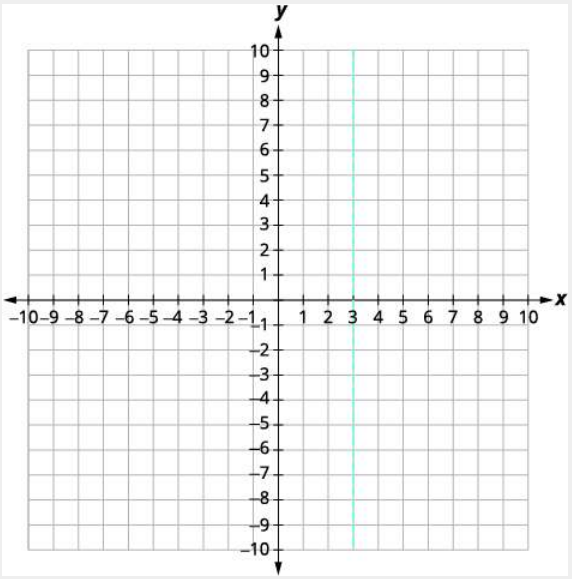
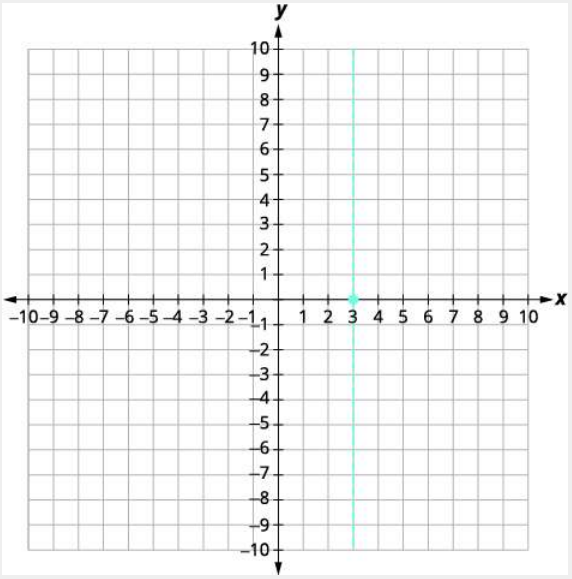
We were able to find the x-intercepts in the last example by factoring. We find the x-intercepts in the next example by factoring, too.

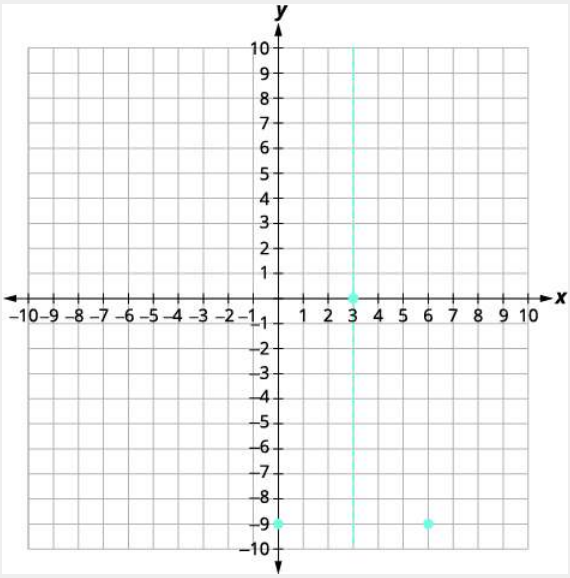
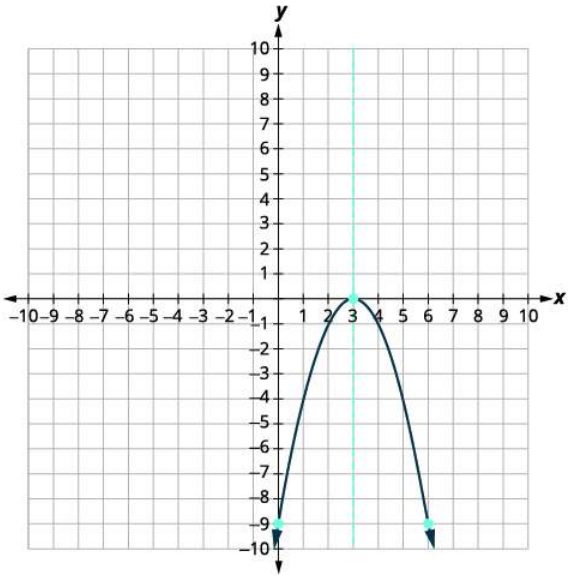
Example 5.5.7

Graph $f(x) = -x^2 + 6x - 9$ by using its properties.

Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
	

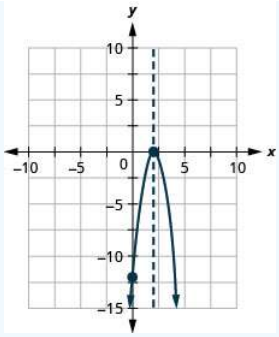
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{6}{2(-1)}$
	$x = 3$
	The axis of symmetry is $x = 3$. The vertex is on the line $x = 3$.
	
Find $f(3)$.	$f(x) = -x^2 + 6x - 9$
	$f(3) = -3^2 + 6 \cdot 3 - 9$
	$f(3) = -9 + 18 - 9$
	$f(3) = 0$
	The vertex is $(3, 0)$.
	
The y-intercept occurs when $x = 0$. Find $f(0)$.	$f(x) = -x^2 + 6x - 9$
Substitute $x = 0$.	$f(0) = -0^2 + 6 \cdot 0 - 9$

Simplify.	$f(0) = -9$
The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$.	
	Point symmetric to the y-intercept is $(6, -9)$
The x-intercept occurs when $f(x) = 0$.	$f(x) = -x^2 + 6x - 9$
Find $f(x) = 0$.	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for x.	$0 = 3$
Connect the points to graph the parabola.	

? Exercise 5.5.13

Graph $f(x) = 3x^2 + 12x - 12$ by using its properties.

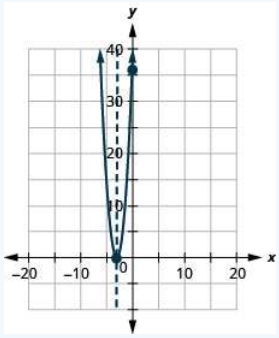
Answer



Exercise 5.5.14

Graph $f(x) = 4x^2 + 24x + 36$ by using its properties.

Answer



For the graph of $f(x) = -x^2 + 6x - 9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

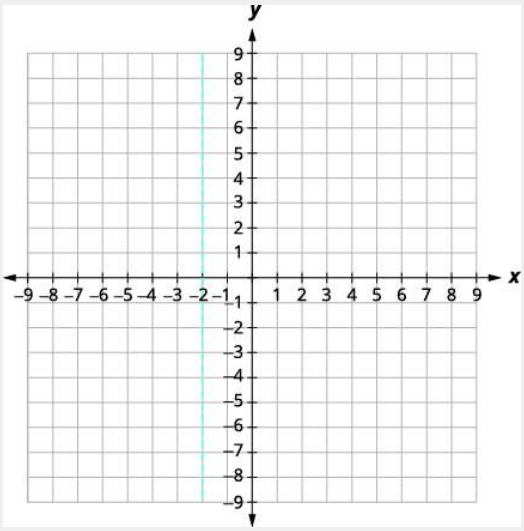
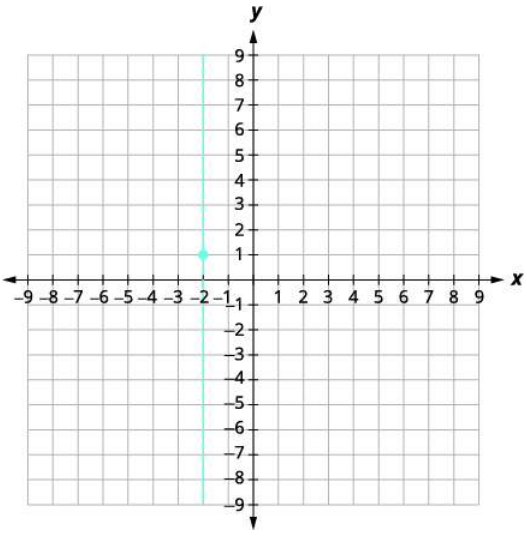
How many x-intercepts would you expect to see on the graph of $f(x) = x^2 + 4x + 5$?

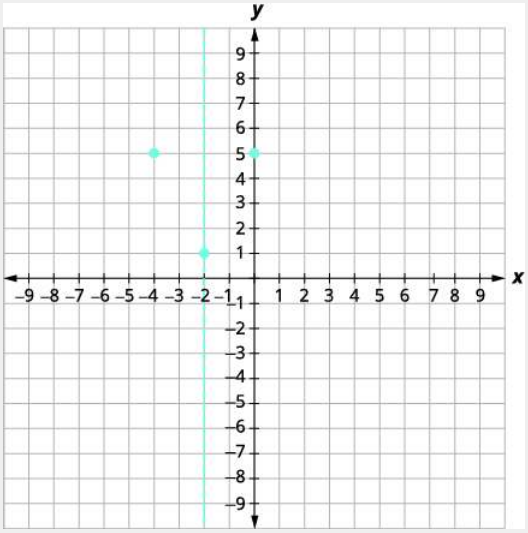
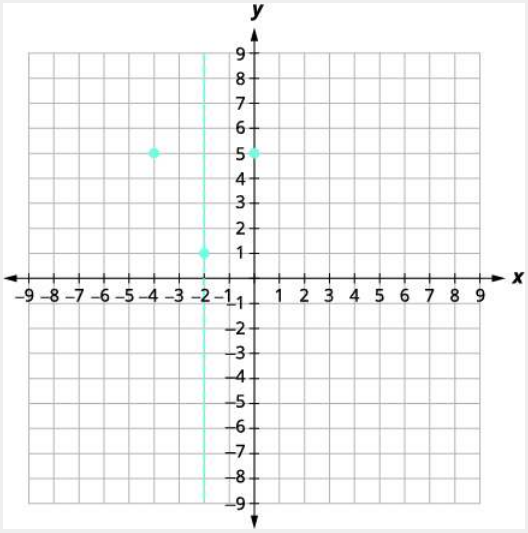
Example 5.5.8

Graph $f(x) = x^2 + 4x + 5$ by using its properties.

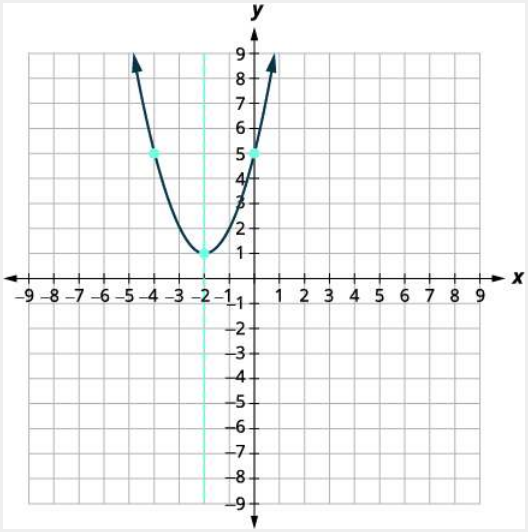
Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{4}{(2)1}$
	$x = -2$
	The equation of the axis of symmetry is \(x=-2).

	
The vertex is on the line $x = -2$.	
Find $f(x)$ when $x = -2$.	$f(x) = x^2 + 4x + 5$
	$f(-2) = (-2)^2 + 4(-2) + 5$
	$f(-2) = 4 - 8 + 5$
	$f(-2) = 1$
	The vertex is $(-2, 1)$.
	
The y-intercept occurs when $x = 0$.	$f(x) = x^2 + 4x - 5$
Find $f(0)$.	$f(0) = 5$
Simplify.	$f(0) = 5$
	The y-intercept is $(0, 5)$.

The point $(-4, 5)$ is two units to the left of the line of symmetry. The point to units to the right of the line of symmetry is \((0,5)\.	
	Point symmetric to the y-intercept is $(-4, 5)$.
The x-intercept occurs when $f(x) = 0$.	
Find $f(x) = 0$.	$0 = x^2 + 4x + 5$
Test the discriminant.	
	$b^2 - 4ac$
	$4^2 - 4 \cdot 1 \cdot 5$
	$16 - 20$
	-4
Since the value of the discriminant is negative, there is no real solution and so no x-intercept.	

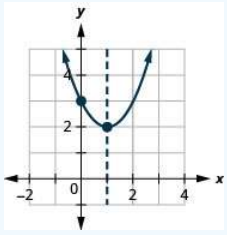
Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.



Exercise 5.5.15

Graph $f(x) = x^2 - 2x + 3$ by using its properties.

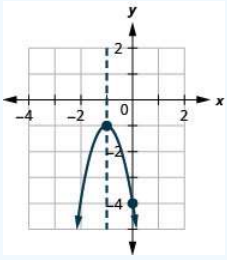
Answer



Exercise 5.5.16

Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.

Answer



Finding the y-intercept by finding $f(0)$ is easy, isn't it? Sometimes we need to use the **Quadratic Formula** to find the x-intercepts.

Example 5.5.9

Graph $f(x) = 2x^2 - 4x - 3$ by using its properties.

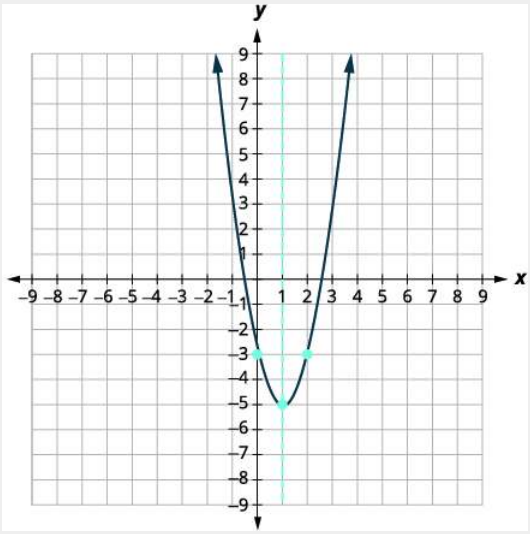
Solution:

$f(x) = ax^2 + bx + c$

$f(x) = 2x^2 - 4x - 3$

Since a is 2, the parabola opens upward.	
To find the equation of the axis of symmetry, use $x=-\frac{b}{2a}$.	$x=-\frac{b}{2a}$
	$x=-\frac{-4}{2\cdot 2}$
	$x=1$
	The equation of the axis of symmetry is $x=1$.
The vertex is on the line $x=1$.	$f(x)=2x^2-4x-3$
Find $f(1)$.	$f(x)=2(1)^2-4(1)-3$
	$f(1)=2-4-3$
	\(\f(1)=-5)
	The vertex is $(1,-5)$.
The y-intercept occurs when $x=0$.	$f(x)=2x^2-4x-3$
Find $f(0)$.	$f(0)=2(0)^2-4(0)-3$
Simplify.	$f(0)=-3$
	The y-intercept is $(0,-3)$.
The point $(0,-3)$ is one unit to the left of the line of symmetry.	Point symmetric to the y-intercept is $(2,-3)$
The point one unit to the right of the line of symmetry is $(2,3)$.	
The x-intercept occurs when $y=0$.	$f(x)=2x^2-4x-3$
Find $f(x)=0$.	$0=2x^2-4x-3$
Use the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Substitute in the values of a, b and c.	$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(3)}}{2(2)}$
Simplify.	$x=\frac{-4\pm\sqrt{16+24}}{4}$
Simplify inside the radical.	$x=\frac{4\pm\sqrt{40}}{4}$
Simplify the radical.	$x=\frac{4\pm2\sqrt{10}}{4}$
Factor the GCF.	$x=\frac{2(2\pm\sqrt{10})}{4}$
Remove common factors.	$x=\frac{2\pm\sqrt{10}}{2}$
Write as two equations.	$x=\frac{2+\sqrt{10}}{2}, \quad x=\frac{2-\sqrt{10}}{2}$
Approximate the values.	$x\approx 2.5, \quad x\approx -0.6$
	The approximate values of the x-intercepts are $(2.5,0)$ and $(-0.6,0)$.

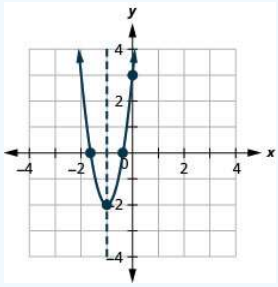
Graph the parabola using the points found.



Exercise 5.5.17

Graph $f(x) = 5x^2 + 10x + 3$ by using its properties.

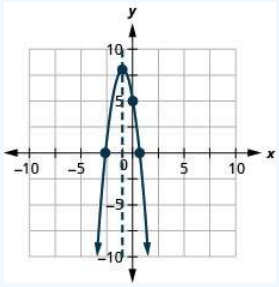
Answer



Exercise 5.5.18

Graph $f(x) = -3x^2 - 6x + 5$ by using its properties.

Answer



Solve Maximum and Minimum Applications

Knowing that the **vertex** of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The *y*-coordinate of the vertex is the **minimum** value of a parabola that opens upward. It is the **maximum** value of a parabola that opens downward. See *Figure 9.6.124*.

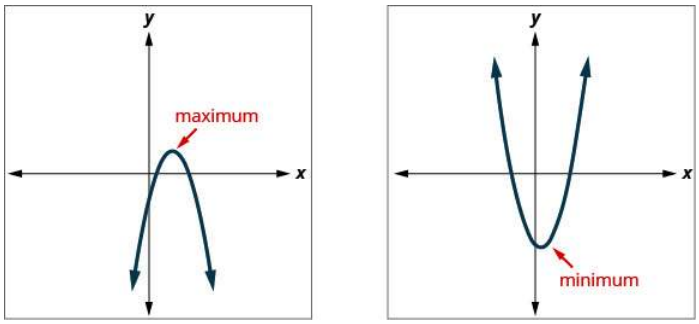


Figure 9.6.124

Minimum or Maximum Values of a Quadratic Function

The ***y*-coordinate of the vertex** of the graph of a quadratic function is the

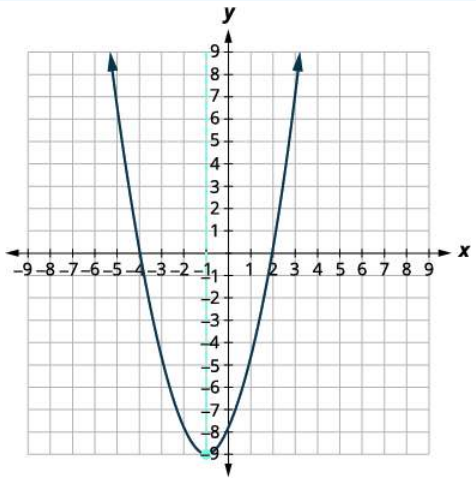
- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

Example 5.5.10

Find the minimum or maximum value of the quadratic function $f(x) = x^2 + 2x - 8$.

Solution:

	$f(x) = x^2 + 2x - 8$
Since a is positive, the parabola opens upward. The quadratic equation has a minimum.	
Find the equation of the axis of symmetry.	$x = -\frac{b}{2a}$
	$x = -\frac{2}{2\times 1}$
	$x = -1$
	The equation of the axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$f(x) = x^2 + 2x - 8$
Find $f(-1)$.	$f(-1) = (-1)^2 + 2(-1) - 8$
	$f(-1) = 1 - 2 - 8$
	$f(-1) = -9$
	The vertex is $(-1, -9)$.
Since the parabola has a minimum, the y-coordinate of the vertex is the minimum y-value of the quadratic equation. The minimum value of the quadratic is -9 and it occurs when $x = -1$.	



Show the graph to verify the result.

Exercise 5.5.19

Find the maximum or minimum value of the quadratic function $f(x) = x^2 - 8x + 12$.

Answer

The minimum value of the quadratic function is -4 and it occurs when $x = 4$.

Exercise 5.5.20

Find the maximum or minimum value of the quadratic function $f(x) = -4x^2 + 16x - 11$.

Answer

The maximum value of the quadratic function is 5 and it occurs when $x = 2$.

We have used the formula

$$h(t) = -16t^2 + v_0 t + h_0$$

to calculate the height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds .

This formula is a quadratic function, so its graph is a parabola. By solving for the coordinates of the vertex (t, h), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

Example 5.5.11

The quadratic equation $h(t) = -16t^2 + 176t + 4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

a. How many seconds will it take the volleyball to reach its maximum height?
b. Find the maximum height of the volleyball.

Solution:

$$h(t) = -16t^2 + 176t + 4$$

Since a is negative, the parabola opens downward. The quadratic function has a maximum.

a. Find the equation of the axis of symmetry.

$$t = -\frac{b}{2a}$$
$$t = -\frac{176}{2(-16)}$$
$$t = 5.5$$

The equation of the axis of symmetry is $t = 5.5$.

The vertex is on the line $t = 5.5$.

The maximum occurs when $t = 5.5$ seconds.

b. Find $h(5.5)$.

$$h(t) = -16t^2 + 176t + 4$$
$$h(t) = -16(5.5)^2 + 176(5.5) + 4$$

Use a calculator to simplify.

$$h(t) = 488$$

The vertex is $(5.5, 488)$.

Since the parabola has a maximum, the h-coordinate of the vertex is the maximum value of the quadratic function.

The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

? Exercise 5.5.21

Solve, rounding answers to the nearest tenth.

The quadratic function $h(t) = -16t^2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

Answer

It will take 4 seconds for the stone to reach its maximum height of 288 feet.

? Exercise 5.5.22

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of $h(t) = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height?

Answer

It will take 6.5 seconds for the rocket to reach its maximum height of 676 feet.

Key Concepts

- Parabola Orientation
  - For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if
    - $a > 0$, the parabola opens upward.
    - $a < 0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:
  - the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
  - the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$.
  - the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
  - To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

- **y-intercept**
  - Let $x = 0$ and solve for $f(x)$.
- **x-intercepts**
  - Let \(f(x)=0\) and solve for x.

- How to graph a quadratic function using properties.
  1. Determine whether the parabola opens upward or downward.
  2. Find the equation of the axis of symmetry.
  3. Find the vertex.
  4. Find the y-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  5. Find the x-intercepts. Find additional points if needed.
  6. Graph the parabola.
- Minimum or Maximum Values of a Quadratic Equation
  - The y-coordinate of the vertex of the graph of a quadratic equation is the
  - *minimum* value of the quadratic equation if the parabola opens *upward*.
  - *maximum* value of the quadratic equation if the parabola opens *downward*.

Glossary

quadratic function

A quadratic function, where a, b, and c are real numbers and $a \neq 0$, is a function of the form $f(x) = ax^2 + bx + c$.

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5.5E: Graph Quadratic Functions Using Properties (Exercises)

Practice Makes Perfect

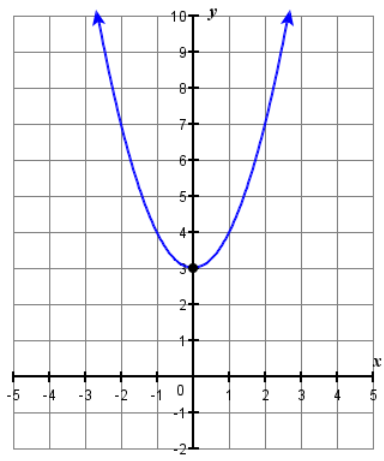
Exercises 1 - 4: Recognize the Graph of a Quadratic Function

In the following exercises, graph the functions by plotting points.

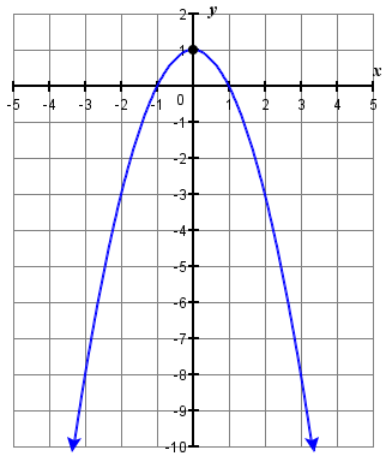
1. $f(x) = x^2 + 3$
2. $f(x) = x^2 - 3$
3. $y = -x^2 + 1$
4. $f(x) = -x^2 - 1$

Answer

1.



3.



Exercises 5 - 8: Recognize the Graph of a Quadratic Function

For each of the following exercises, determine if the parabola opens up or down.

5. a. $f(x) = -2x^2 - 6x - 7$ b. $f(x) = 6x^2 + 2x + 3$
6. a. $f(x) = 4x^2 + x - 4$ b. $f(x) = -9x^2 - 24x - 16$

7. a. $f(x) = -3x^2 + 5x - 1$ b. $f(x) = 2x^2 - 4x + 5$

8. a. $f(x) = x^2 + 3x - 4$ b. $f(x) = -4x^2 - 12x - 9$

Answer

5. a. down b. up

7. a. down b. up

Exercises 9 - 12: Find the Axis of Symmetry and Vertex of a Parabola

In the following functions, find

a. The equation of the axis of symmetry
b. The vertex of its graph

9. $f(x) = x^2 + 8x - 1$

10. $f(x) = x^2 + 10x + 25$

11. $f(x) = -x^2 + 2x + 5$

12. $f(x) = -2x^2 - 8x - 3$

Answer

9. a. Axis of symmetry: $x = -4$ b. Vertex: $(-4, -17)$

11. a. Axis of symmetry: $x = 1$ b. Vertex: $(1, 2)$

Exercises 13 - 24: Find the Intercepts of a Parabola

In the following exercises, find the intercepts of the parabola whose function is given.

13. $f(x) = x^2 + 7x + 6$

14. $f(x) = x^2 + 10x - 11$

15. $f(x) = x^2 + 8x + 12$

16. $f(x) = x^2 + 5x + 6$

17. $f(x) = -x^2 + 8x - 19$

18. $f(x) = -3x^2 + x - 1$

19. $f(x) = x^2 + 6x + 13$

20. $f(x) = x^2 + 8x + 12$

21. $f(x) = 4x^2 - 20x + 25$

22. $f(x) = -x^2 - 14x - 49$

23. $f(x) = -x^2 - 6x - 9$

24. $f(x) = 4x^2 + 4x + 1$

Answer

13. y-intercept: $(0, 6)$; x-intercept(s): $(-1, 0), (-6, 0)$

15. y-intercept: $(0, 12)$; x-intercept(s): $(-2, 0), (-6, 0)$

17. y-intercept: $(0, -19)$; x-intercept(s): none

19. y-intercept: $(0, 13)$; x-intercept(s): none

21. y-intercept: $(0, -16)$; x-intercept(s): $(\frac{5}{2}, 0)$

23. y-intercept: $(0, 9)$; x-intercept(s): $(-3, 0)$

Exercises 25 - 42: Graph Quadratic Functions Using Properties

In the following exercises, graph the function by using its properties.

25. $f(x) = x^2 + 6x + 5$
26. $f(x) = x^2 + 4x - 12$
27. $f(x) = x^2 + 4x + 3$
28. $f(x) = x^2 - 6x + 8$
29. $f(x) = 9x^2 + 12x + 4$
30. $f(x) = -x^2 + 8x - 16$
31. $f(x) = -x^2 + 2x - 7$
32. $f(x) = 5x^2 + 2$
33. $f(x) = 2x^2 - 4x + 1$
34. $f(x) = 3x^2 - 6x - 1$
35. $f(x) = 2x^2 - 4x + 2$
36. $f(x) = -4x^2 - 6x - 2$
37. $f(x) = -x^2 - 4x + 2$
38. $f(x) = x^2 + 6x + 8$
39. $f(x) = 5x^2 - 10x + 8$
40. $f(x) = -16x^2 + 24x - 9$
41. $f(x) = 3x^2 + 18x + 20$
42. $f(x) = -2x^2 + 8x - 10$

Answer

25.

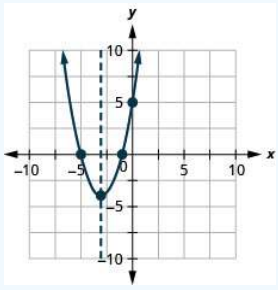


Figure 9.6.136

27.

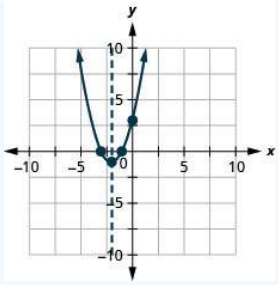


Figure 9.6.137

29.

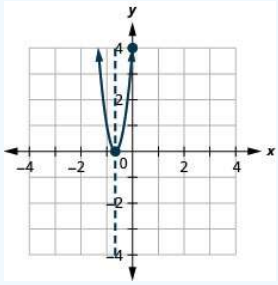


Figure 9.6.138

31.

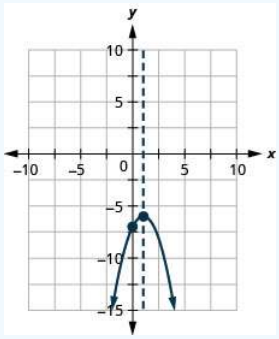


Figure 9.6.139

33.

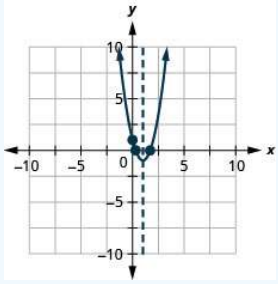


Figure 9.6.140

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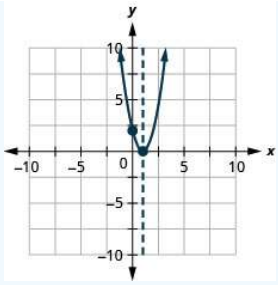


Figure 9.6.141

37.

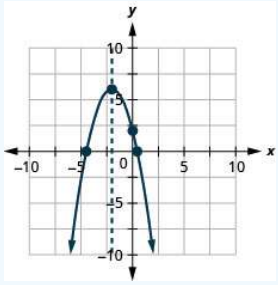


Figure 9.6.142

39.

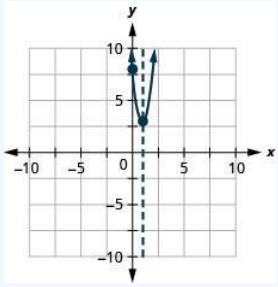


Figure 9.6.143

41.

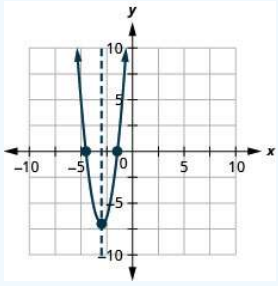


Figure 9.6.144

? Exercises 43 - 48: Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value of each function.

43. $f(x) = 2x^2 + x - 1$

44. $y = -4x^2 + 12x - 5$

45. $y = x^2 - 6x + 15$

46. $y = -x^2 + 4x - 5$

https://math.libretexts.org/@go/page/58474

47. $y = -9x^2 + 16$

48. $y = 4x^2 - 49$

Answer

43. The minimum value is $-\frac{9}{8}$ when $x = -\frac{1}{4}$.

45. The maximum value is 6 when $x = 3$.

47. The maximum value is 16 when $x = 0$.

Exercises 49 - 60: Solve Maximum and Minimum Applications

In the following exercises, solve. Round answers to the nearest tenth.

49. An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic function $h(t) = -16t^2 + 168t + 45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.

50. A stone is thrown vertically upward from a platform that is 20 feet height at a rate of 160 ft/sec. Use the quadratic function $h(t) = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

51. A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic function $h(t) = -16t^2 + 109t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

52. A ball is thrown vertically upward from the ground with an initial velocity of 122 ft/sec. Use the quadratic function $h(t) = -16t^2 + 122t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

53. A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

54. A retailer who sells backpacks estimates that by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic function $R(x) = -x^2 + 100x$ is used to find the R, received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

55. A retailer who sells fashion boots estimates that by selling them for x dollars each, he will be able to sell $70 - x$ boots a week. Use the quadratic function $R(x) = -x^2 + 70x$ to find the revenue received when the average selling price of a pair of fashion boots is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue per day.

56. A cell phone company estimates that by charging x dollars each for a certain cell phone, they can sell $8 - x$ cell phones per day. Use the quadratic function $R(x) = -x^2 + 8x$ to find the revenue received per day when the selling price of a cell phone is x. Find the selling price that will give them the maximum revenue per day, and then find the amount of the maximum revenue.

57. A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A(x) = x(240 - 2x)$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

58. A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic function $A(x) = x(100 - 2x)$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

59. A land owner is planning to build a fenced in rectangular patio behind his garage, using his garage as one of the "walls." He wants to maximize the area using 80 feet of fencing. The quadratic function $A(x) = x(80 - 2x)$ gives the area of the patio, where x is the width of one side. Find the maximum area of the patio.

60. A family of three young children just moved into a house with a yard that is not fenced in. The previous owner gave them 300 feet of fencing to use to enclose part of their backyard. Use the quadratic function $A(x) = x(300 - 2x)$ to determine the maximum area of the fenced in yard.

Answer

49. In 5.3 sec the arrow will reach maximum height of 486 ft.

51. In 3.4 seconds the ball will reach its maximum height of 185.6 feet.

53. 20 computers will give the maximum of $\$400$ in receipts.

55. He will be able to sell 35 pairs of boots at the maximum revenue of $\$1,225$

57. The length of the side along the river of the corral is 120 feet and the maximum area is $7,200$ square feet.

59. The maximum area of the patio is 800 feet.

? Exercises 61 - 64: Writing Exercises

61. How do the graphs of the functions $f(x) = x^2$ and $f(x) = x^2 - 1$ differ? We graphed them at the start of this section. What is the difference between their graphs? How are their graphs the same?

62. Explain the process of finding the vertex of a parabola.

63. Explain how to find the intercepts of a parabola.

64. How can you use the discriminant when you are graphing a quadratic function?

Answer

1. Answers will vary.

3. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

Figure 9.6.145

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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5.6: Graph Quadratic Functions Using Transformations

Learning Objectives

By the end of this section, you will be able to:

- Graph quadratic functions of the form $f(x) = x^2 + k$
- Graph quadratic functions of the form $f(x) = (x - h)^2$
- Graph quadratic functions of the form $f(x) = ax^2$
- Graph quadratic functions using transformations
- Find a quadratic function from its graph

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
   If you missed this problem, review Example 3.54.
2. Factor completely: $y^2 - 14y + 49$.
   If you missed this problem, review Example 6.24.
3. Factor completely: $2x^2 - 16x + 32$.
   If you missed this problem, review Example 6.26.

Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the last section, we learned how to graph quadratic functions using their properties. Another method involves starting with the basic graph of $f(x) = x^2$ and 'moving' it according to information given in the function equation. We call this graphing quadratic functions using transformations.

In the first example, we will graph the quadratic function $f(x) = x^2$ by plotting points. Then we will see what effect adding a constant, k, to the equation will have on the graph of the new function $f(x) = x^2 + k$.

Example 5.6.1

Graph $f(x) = x^2$, $g(x) = x^2 + 2$, and $h(x) = x^2 - 2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

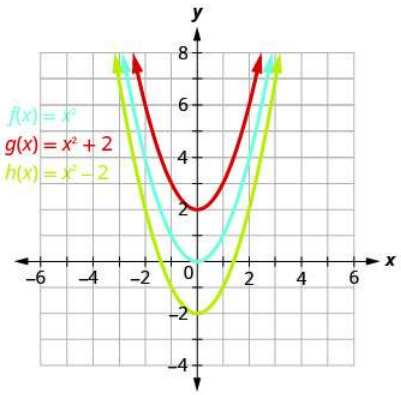
Solution:

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = x^2 + 2$	$(x, g(x))$	$h(x) = x^2 - 2$	$(x, h(x))$
–3	9	(–3, 9)	9 + 2	(–3, 11)	9 – 2	(–3, 7)
–2	4	(–2, 4)	4 + 2	(–2, 6)	4 – 2	(–2, 2)
–1	1	(–1, 1)	1 + 2	(–1, 3)	1 – 2	(–1, 1)
0	0	(0, 0)	0 + 2	(0, 2)	0 – 2	(0, –2)
1	1	(1, 1)	1 + 2	(1, 3)	1 – 2	(1, –1)
2	4	(2, 4)	4 + 2	(2, 6)	4 – 2	(2, 2)
3	9	(3, 9)	9 + 2	(3, 11)	9 – 2	(3, 7)

Figure 9.7.1

The $g(x)$ values are two more than the $f(x)$ values. Also, the $h(x)$ values are two less than the $f(x)$ values. Now we will graph all three functions on the same rectangular coordinate system.



The graph of $g(x) = x^2 + 2$ is the same as the graph of $f(x) = x^2$ but shifted up 2 units.

The graph of $h(x) = x^2 - 2$ is the same as the graph of $f(x) = x^2$ but shifted down 2 units.

Figure 9.7.2

The graph of $g(x) = x^2 + 2$ is the same as the graph of $f(x) = x^2$ but shifted up 2 units.

The graph of $h(x) = x^2 - 2$ is the same as the graph of $f(x) = x^2$ but shifted down 2 units.

Exercise 5.6.1

a. Graph $f(x) = x^2, g(x) = x^2 + 1,$ and $h(x) = x^2 - 1$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the function has on the basic parabola.

Answer

a.

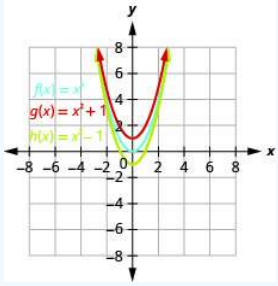


Figure 9.7.3

b. The graph of $g(x) = x^2 + 1$ is the same as the graph of $f(x) = x^2$ but shifted up 1 unit. The graph of $h(x) = x^2 - 1$ is the same as the graph of $f(x) = x^2$ but shifted down 1 unit.

Exercise 5.6.2

a. Graph $f(x) = x^2, g(x) = x^2 + 6,$ and $h(x) = x^2 - 6$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the function has on the basic parabola.

Answer

a.

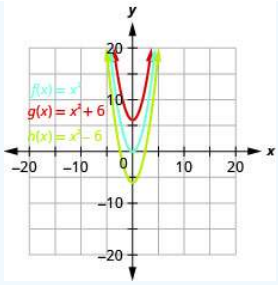


Figure 9.7.4

b. The graph of $h(x) = x^2 + 6$ is the same as the graph of $f(x) = x^2$ but shifted up 6 units. The graph of $h(x) = x^2 - 6$ is the same as the graph of $f(x) = x^2$ but shifted down 6 units.

The last example shows us that to graph a quadratic function of the form $f(x) = x^2 + k$, we take the basic parabola graph of $f(x) = x^2$ and vertically shift it up $(k > 0)$ or shift it down $(k < 0)$.

This transformation is called a vertical shift.

Graph a Quadratic Function of the Form $f(x) = x^2 + k$ Using a Vertical Shift

The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.

- If $k > 0$, shift the parabola vertically up k units.
- If $k < 0$, shift the parabola vertically down $|k|$ units.

Now that we have seen the effect of the constant, k, it is easy to graph functions of the form $f(x) = x^2 + k$. We just start with the basic parabola of $f(x) = x^2$ and then shift it up or down.

It may be helpful to practice sketching $f(x) = x^2$ quickly. We know the values and can sketch the graph from there.

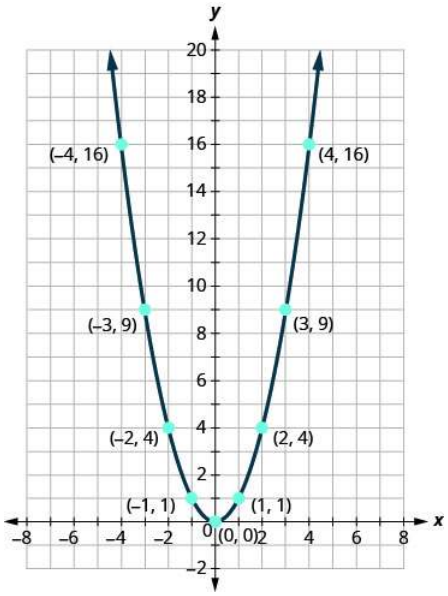


Figure 9.7.5

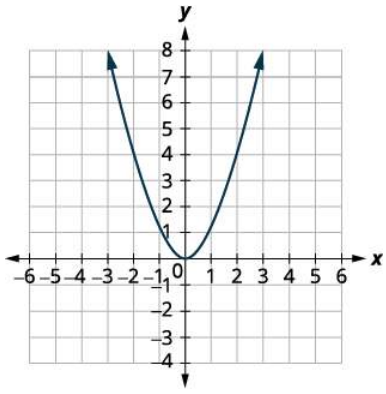
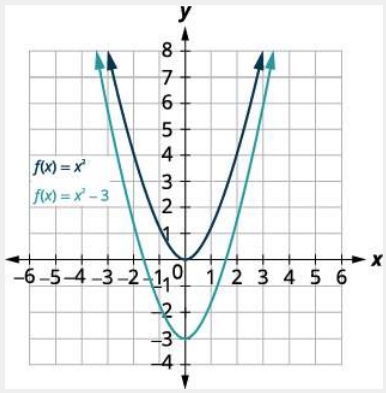
Once we know this parabola, it will be easy to apply the transformations. The next example will require a vertical shift.

Example 5.6.2

Graph $f(x) = x^2 - 3$ using a vertical shift.

Solution:

Table 9.7.1

We first draw the graph of $f(x) = x^2$ on the grid.	
Determine k.	$f(x) = x^2 + k$ $f(x) = x^2 - 3$
	$k = -3$
Shift the graph $f(x) = x^2$ down 3.	

Exercise 5.6.3

Graph $f(x) = x^2 - 5$ using a vertical shift.

Answer

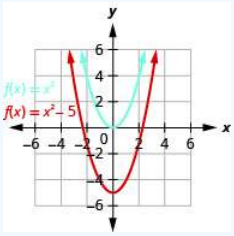


Figure 9.7.10

Exercise 5.6.4

Graph $f(x) = x^2 + 7$ using a vertical shift.

Answer

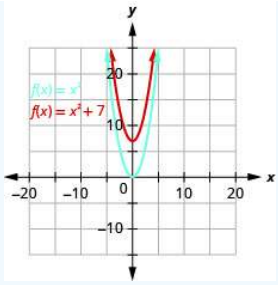


Figure 9.7.11

Graph Quadratic Functions of the Form $f(x) = (x - h)^2$

In the first example, we graphed the quadratic function $f(x) = x^2$ by plotting points and then saw the effect of adding a constant k to the function had on the resulting graph of the new function $f(x) = x^2 + k$.

We will now explore the effect of subtracting a constant, h, from x has on the resulting graph of the new function $f(x) = (x - h)^2$.

Example 5.6.3

Graph $f(x) = x^2, g(x) = (x - 1)^2,$ and $h(x) = (x + 1)^2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

Solution:

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = (x - 1)^2$	$(x, g(x))$	$h(x) = (x + 1)^2$	$(x, h(x))$
–3	9	(–3, 9)	16	(–3, 16)	4	(–3, 4)
–2	4	(–2, 4)	9	(–2, 9)	1	(–2, 1)
–1	1	(–1, 1)	4	(–1, 4)	0	(–1, 0)
0	0	(0, 0)	1	(0, 1)	1	(0, 1)
1	1	(1, 1)	0	(1, 0)	4	(1, 4)
2	4	(2, 4)	1	(2, 1)	9	(2, 9)
3	9	(3, 9)	4	(3, 4)	16	(3, 16)

Figure 9.7.12

The $g(x)$ values and the $h(x)$ values share the common numbers $0, 1, 4, 9,$ and 16, but are shifted.

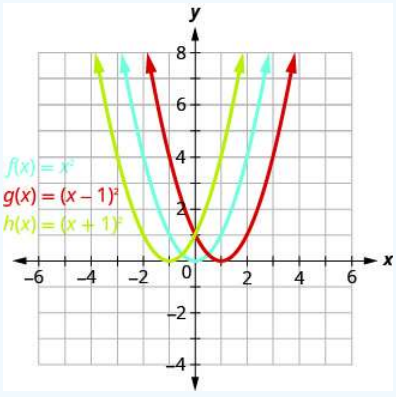


Figure 9.7.13

The graph of $g(x) = (x - 1)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 1 unit.

The graph of $h(x) = (x + 1)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 1 unit.

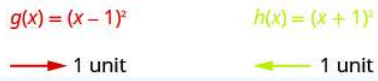


Figure 9.7.14

Exercise 5.6.5

a. Graph $f(x) = x^2, g(x) = (x + 2)^2,$ and $h(x) = (x - 2)^2$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the function has on the basic parabola.

Answer

a.

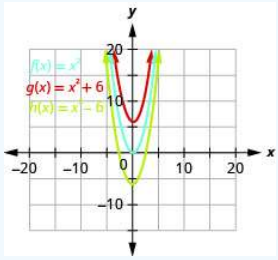


Figure 9.7.15

b. The graph of $g(x) = (x + 2)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 2 units. The graph of $h(x) = (x - 2)^2$ is the same as the graph of $f(x) = x^2$ but shift right 2 units.

Exercise 5.6.6

a. Graph $f(x) = x^2, g(x) = x^2 + 5,$ and $h(x) = x^2 - 5$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the function has on the basic parabola.

Answer

a.

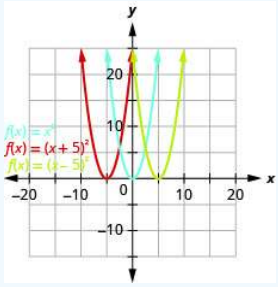


Figure 9.7.16

b. The graph of $g(x) = (x + 5)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 5 units. The graph of $h(x) = (x - 5)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 5 units.

The last example shows us that to graph a quadratic function of the form $f(x) = (x - h)^2$, we take the basic parabola graph of $f(x) = x^2$ and shift it left $(h > 0)$ or shift it right $(h < 0)$.

This transformation is called a horizontal shift.

Graph a Quadratic Function of the Form $f(x) = (x - h)^2$ Using a Horizontal Shift

The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.

- If $h > 0$, shift the parabola horizontally left h units.
- If $h < 0$, shift the parabola horizontally right $|h|$ units.

Now that we have seen the effect of the constant, h, it is easy to graph functions of the form $f(x) = (x - h)^2$. We just start with the basic parabola of $f(x) = x^2$ and then shift it left or right.

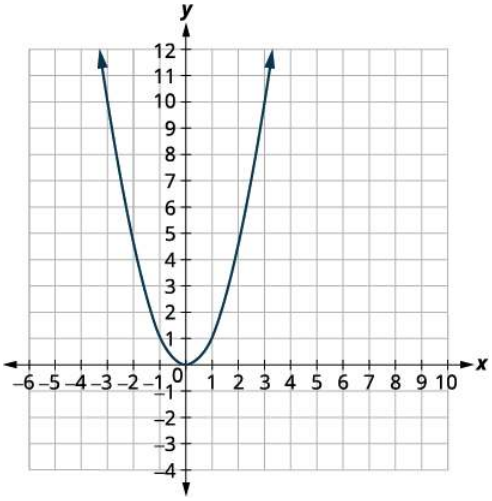
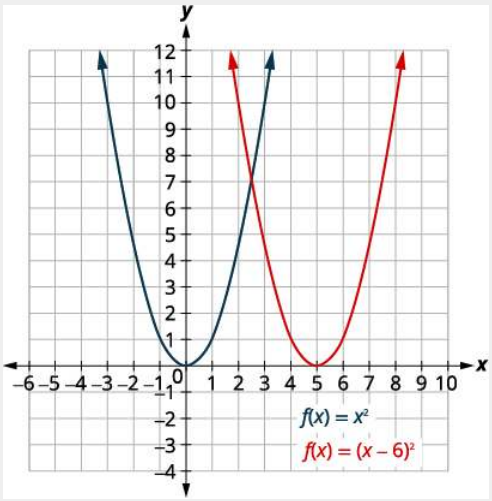
The next example will require a horizontal shift.

Example 5.6.4

Graph $f(x) = (x - 6)^2$ using a horizontal shift.

Solution:

Table 9.7.2

We first draw the graph of $f(x) = x^2$ on the grid.	
Determine h.	$f(x) = (x - h)^2$ $f(x) = (x - 6)^2$
	$h = 6$
Shift the graph $f(x) = x^2$ to the right 6 units.	

? Exercise 5.6.7

Graph $f(x) = (x-4)^2$ using a horizontal shift.

Answer

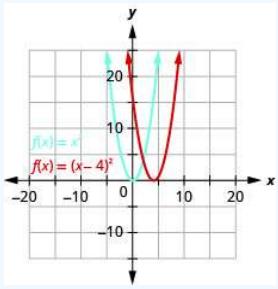


Figure 9.7.21

? Exercise 5.6.8

Graph $f(x) = (x+6)^2$ using a horizontal shift.

Answer

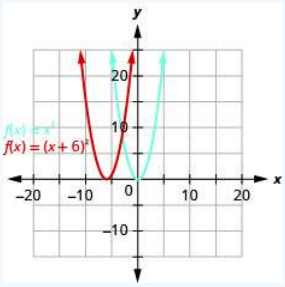


Figure 9.7.22

Now that we know the effect of the constants h and k, we will graph a quadratic function of the form $f(x) = (x-h)^2 + k$ by first drawing the basic parabola and then making a horizontal shift followed by a vertical shift. We could do the vertical shift followed by the horizontal shift, but most students prefer the horizontal shift followed by the vertical.

✔ Example 5.6.5

Graph $f(x) = (x+1)^2 - 2$ using transformations.

Solution:

This function will involve two transformations and we need a plan.

Let's first identify the constants h, k.

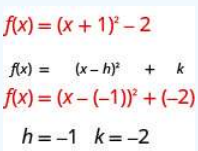


Figure 9.7.23

The h constant gives us a horizontal shift and the k gives us a vertical shift.

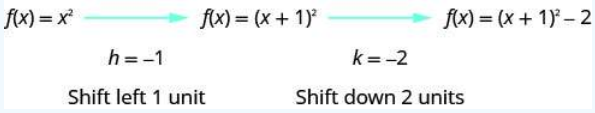


Figure 9.7.24

We first draw the graph of $f(x) = x^2$ on the grid.

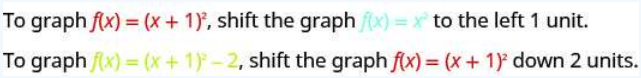


Figure 9.7.25

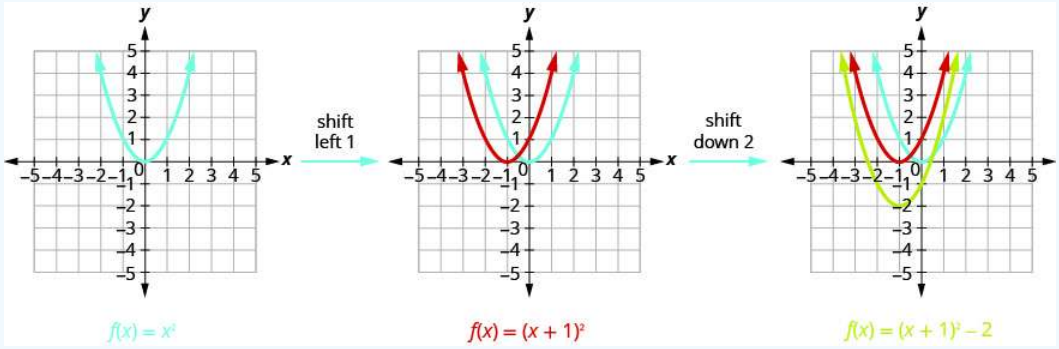


Figure 9.7.26

Exercise 5.6.9

Graph $f(x) = (x+2)^2 - 3$ using transformations.

Answer

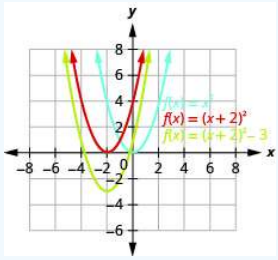


Figure 9.7.27

Exercise 5.6.10

Graph $f(x) = (x-3)^2 + 1$ using transformations.

Answer

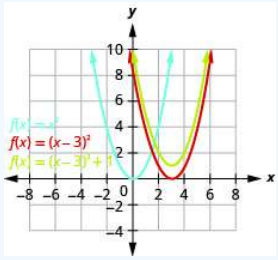


Figure 9.7.28

Graph Quadratic Functions of the Form $f(x) = ax^2$

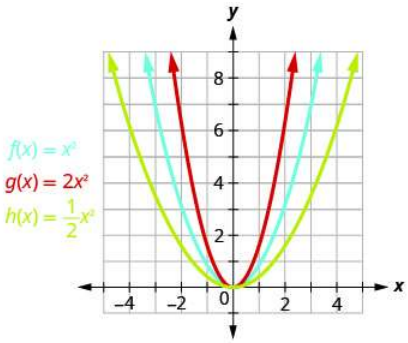
So far we graphed the quadratic function $f(x) = x^2$ and then saw the effect of including a constant h or k in the equation had on the resulting graph of the new function. We will now explore the effect of the coefficient a on the resulting graph of the new function $f(x) = ax^2$.

Let's look at the quadratic functions $f(x) = x^2$, $g(x) = 2x^2$ and $h(x) = \frac{1}{2}x^2$.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = 2x^2$	$(x, g(x))$	$h(x) = \frac{1}{2}x^2$	$(x, h(x))$
–2	4	(–2, 4)	2 • 4	(–2, 8)	$\frac{1}{2}$ • 4	(–2, 2)
–1	1	(–1, 1)	2 • 1	(–1, 2)	$\frac{1}{2}$ • 1	$\left(-1, \frac{1}{2}\right)$
0	0	(0, 0)	2 • 0	(0, 0)	$\frac{1}{2}$ • 0	(0, 0)
1	1	(1, 1)	2 • 1	(1, 2)	$\frac{1}{2}$ • 1	$\left(1, \frac{1}{2}\right)$
2	4	(2, 4)	2 • 4	(2, 8)	$\frac{1}{2}$ • 4	(2, 2)

Figure 9.7.29

If we graph these functions, we can see the effect of the constant a, assuming $a > 0$.



The graph of the function $g(x) = 2x^2$ is "skinnier" than the graph of $f(x) = x^2$.

The graph of the function $h(x) = \frac{1}{2}x^2$ is "wider" than the graph of $f(x) = x^2$.

Figure 9.7.30

To graph a function with constant a it is easiest to choose a few points on $f(x) = x^2$ and multiply the y-values by a.

Graph of a Quadratic Function of the Form $f(x) = ax^2$

The coefficient a in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.

- If $0 < |a| < 1$, the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$.
- If $|a| > 1$, the graph of $f(x) = ax^2$ will be "skinnier" than the graph of $f(x) = x^2$.

Example 5.6.6

Graph $f(x) = 3x^2$.

Solution:

We will graph the functions $f(x) = x^2$ and $g(x) = 3x^2$ on the same grid. We will choose a few points on $f(x) = x^2$ and then multiply the y-values by 3 to get the points for $g(x) = 3x^2$.

	$f(x) = x^2$	$g(x) = 3x^2$	
x	$(x, f(x))$	$(x, g(x))$	
–2	(–2, 4)	(–2, 12)	$3 \cdot 4 = 12$
–1	(–1, 1)	(–1, 3)	$3 \cdot 1 = 3$
0	(0, 0)	(0, 0)	$3 \cdot 0 = 0$
1	(1, 1)	(1, 3)	$3 \cdot 1 = 3$
2	(2, 4)	(2, 12)	$3 \cdot 4 = 12$

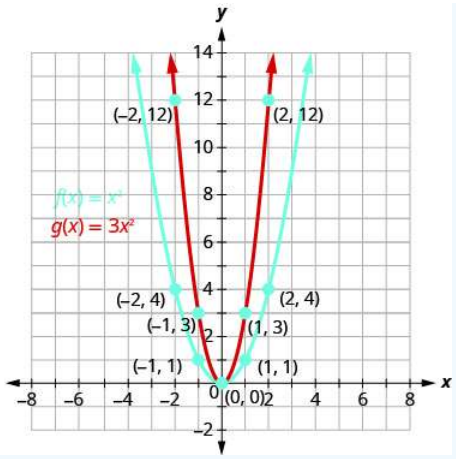


Figure 9.7.31

Exercise 5.6.11

Graph $f(x) = -3x^2$.

Answer

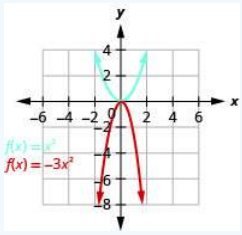


Figure 9.7.32

Exercise 5.6.12

Graph $f(x) = 2x^2$.

Answer

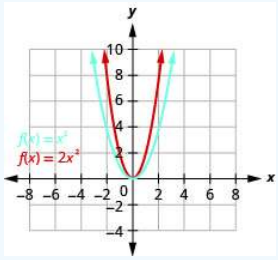


Figure 9.7.33

Graph Quadratic Functions Using Transformations

We have learned how the constants a, h, and k in the functions, $f(x) = x^2 + k$, $f(x) = (x - h)^2$, and $f(x) = ax^2$ affect their graphs. We can now put this together and graph quadratic functions $f(x) = ax^2 + bx + c$ by first putting them into the form $f(x) = a(x - h)^2 + k$ by completing the square. This form is sometimes known as the vertex form or standard form.

We must be careful to both add and subtract the number to the SAME side of the function to complete the square. We cannot add the number to both sides as we did when we completed the square with quadratic equations.

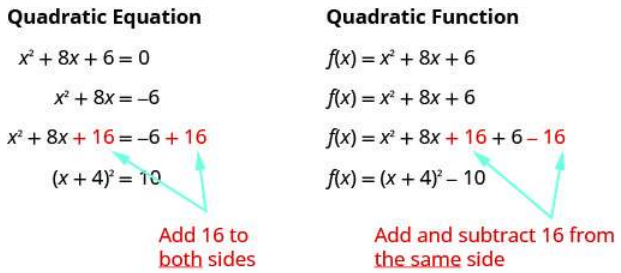


Figure 9.7.34

When we complete the square in a function with a coefficient of x^2 that is not one, we have to factor that coefficient from just the x-terms. We do not factor it from the constant term. It is often helpful to move the constant term a bit to the right to make it easier to focus only on the x-terms.

Once we get the constant we want to complete the square, we must remember to multiply it by that coefficient before we then subtract it.

Example 5.6.7

Rewrite $f(x) = -3x^2 - 6x - 1$ in the $f(x) = a(x-h)^2 + k$ form by completing the square.

Solution:

Table 9.7.3

	$f(x) = -3x^2 - 6x - 1$
Separate the x terms from the constant.	$f(x) = -3x^2 - 6x \quad - 1$
Factor the coefficient of x^2, -3.	$f(x) = -3(x^2 + 2x) \quad - 1$
Prepare to complete the square.	$f(x) = -3(x^2 + 2x \quad) \; - 1$
Take half of 2 and then square it to complete the square $(\frac{1}{2} \cdot 2)^2 = 1$	
The constant 1 completes the square in the parentheses, but the parentheses is multiplied by -3. So we are really adding -3. We must then add 3 to not change the value of the function.	$f(x) = -3(x^2 + 2x + 1) - 1 + 3$ $-3 \cdot 1 = -3$ so add 3
Rewrite the trinomial as a square and subtract the constants.	$f(x) = -3(x + 1) + 2$
The function is now in the $f(x) = a(x-h)^2 + k$ form.	$f(x) = \quad a(x-h)^2 \quad + k$ $f(x) = -3(x + 1)^2 + 2$

Exercise 5.6.13

Rewrite $f(x) = -4x^2 - 8x + 1$ in the $f(x) = a(x-h)^2 + k$ form by completing the square.

Answer

$f(x) = -4(x+1)^2 + 5$

Exercise 5.6.14

Rewrite $f(x) = 2x^2 - 8x + 3$ in the $f(x) = a(x-h)^2 + k$ form by completing the square.

Answer

$f(x) = 2(x-2)^2 - 5$

Once we put the function into the $f(x)=(x-h)^2+k$ form, we can then use the transformations as we did in the last few problems. The next example will show us how to do this.

Example 5.6.8

Graph $f(x)=x^2+6x+5$ by using transformations.

Solution:

Step 1: Rewrite the function in $f(x)=a(x-h)^2+k$ vertex form by completing the square.

Table 9.7.4

	$f(x)=x^2+6x+5$
Separate the x terms from the constant.	$f(x)=x^2+6x+5$
Take half of 6 and then square it to complete the square. $(\frac{1}{2}\cdot 6)^2=9$	
We both add 9 and subtract 9 to not change the value of the function.	$f(x)=x^2+6x+9+5-9$
Rewrite the trinomial as a square and subtract the constants.	$f(x)=(x+3)^2-4$
The function is now in the $f(x)=(x-h)^2+k$ form.	$f(x)=(x-h)^2+k$ $f(x)=(x+3)^2-4$

Step 2: Graph the function using transformations.

Looking at the h, k values, we see the graph will take the graph of $f(x)=x^2$ and shift it to the left 3 units and down 4 units.

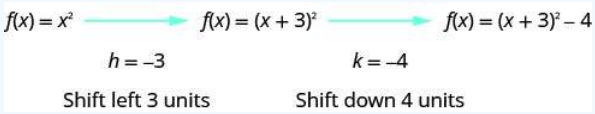


Figure 9.7.47

We first draw the graph of $f(x)=x^2$ on the grid.

To graph $f(x)=(x+3)^2$, shift the graph $f(x)=x^2$ to the left 3 units.

To graph $f(x)=(x+3)^2-4$, shift the graph $f(x)=(x+3)^2$ down 4 units.

Figure 9.7.48

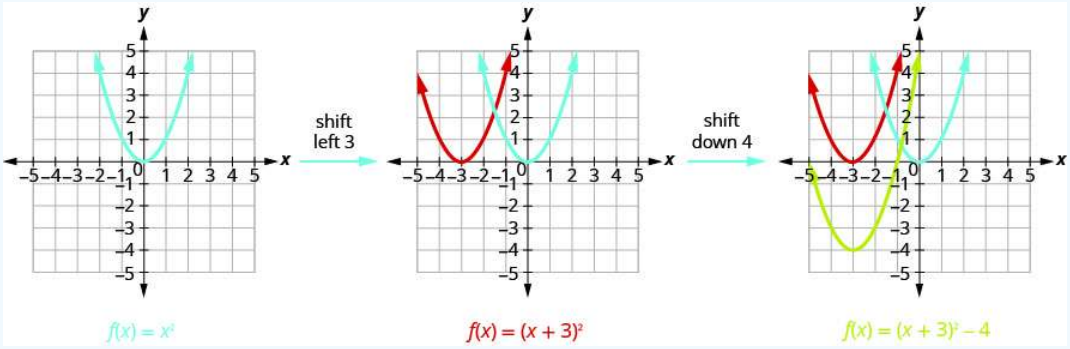


Figure 9.7.49

Exercise 5.6.15

Graph $f(x)=x^2+2x-3$ by using transformations.

Answer

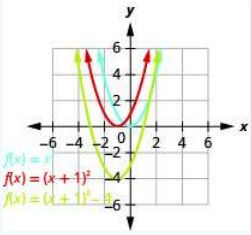


Figure 9.7.50

? Exercise 5.6.16

Graph $f(x) = x^2 - 8x + 12$ by using transformations.

Answer

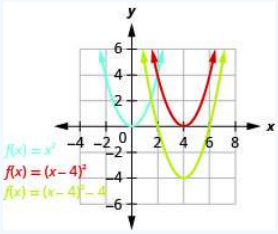


Figure 9.7.51

We list the steps to take a graph a quadratic function using transformations here.

Graph a Quadratic Function Using Transformations

1. Rewrite the function in $f(x) = a(x-h)^2 + k$ form by completing the square.
2. Graph the function using transformations.

✓ Example 5.6.9

Graph $f(x) = -2x^2 - 4x + 2$ by using transformations.

Solution:

Step 1: Rewrite the function in $f(x) = a(x-h)^2 + k$ vertex form by completing the square.

Table 9.7.5

	$f(x) = -2x^2 - 4x + 2$
Separate the x terms from the constant.	$f(x) = -2x^2 - 4x + 2$
We need the coefficient of x^2 to be one. We factor -2 from the x-terms.	$f(x) = -2(x^2 + 2x) + 2$
Take half of 2 and then square it to complete the square. $(\frac{1}{2} \cdot 2)^2 = 1$	
We add 1 to complete the square in the parentheses, but the parentheses is multiplied by -2. So we are really adding -2. To not change the value of the function we add 2.	$f(x) = -2(x^2 + 2x + 1) + 2 + 2$
Rewrite the trinomial as a square ad subtract the constants.	$f(x) = -2(x+1)^2 + 4$
The function is now in the $f(x) = a(x-h)^2 + k$ form.	$f(x) = a(x-h)^2 + k$ $f(x) = -2(x+1)^2 + 4$

Step 2: Graph the function using transformations.

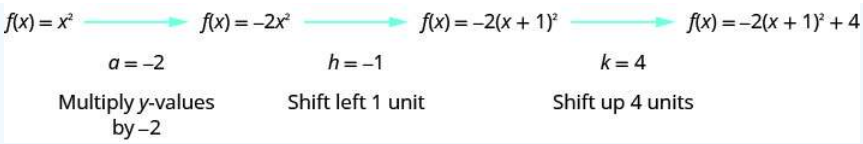


Figure 9.7.58

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = -2x^2$, multiply the y-values in parabola of $f(x) = x^2$ by -2.

To graph $f(x) = -2(x + 1)^2$, shift the graph $f(x) = -2x^2$ to the left 1 unit.

To graph $f(x) = -2(x + 1)^2 + 4$, shift the graph $f(x) = (x + 1)^2$ up 4 units.

Figure 9.7.59

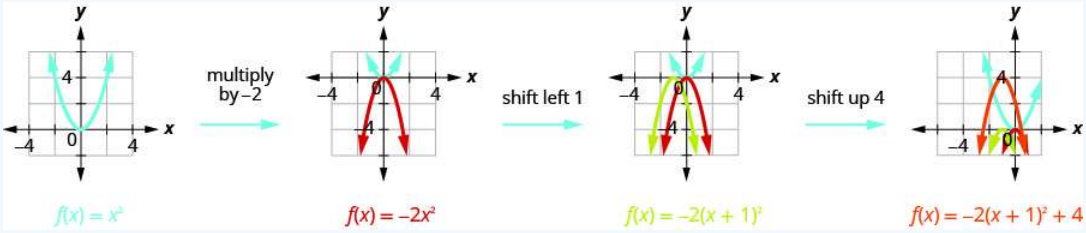


Figure 9.7.60

Exercise 5.6.17

Graph $f(x) = -3x^2 + 12x - 4$ by using transformations.

Answer

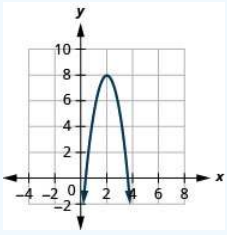


Figure 9.7.61

Exercise 5.6.18

Graph $f(x) = -2x^2 + 12x - 9$ by using transformations.

Answer

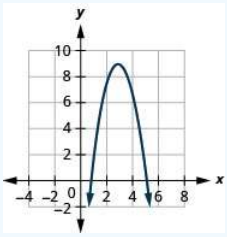


Figure 9.7.62

Now that we have completed the square to put a quadratic function into $f(x) = a(x - h)^2 + k$ form, we can also use this technique to graph the function using its properties as in the previous section.

If we look back at the last few examples, we see that the vertex is related to the constants h and k.

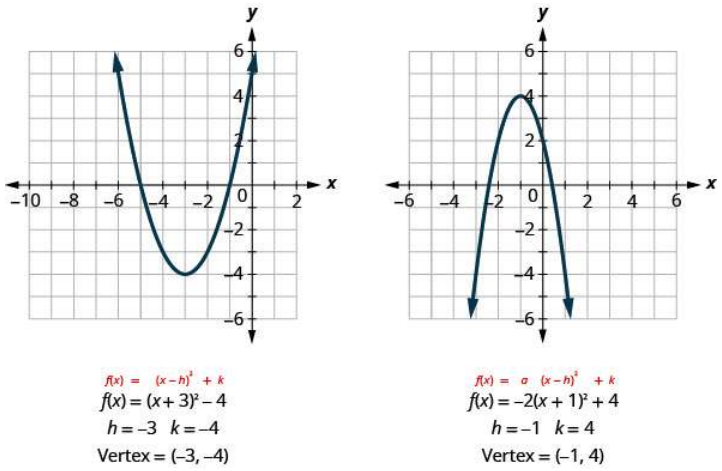


Figure 9.7.63

In each case, the vertex is (h, k). Also the **axis of symmetry** is the line $x = h$.

We rewrite our steps for graphing a quadratic function using properties for when the function is in $f(x) = a(x-h)^2 + k$ form.

Graph a Quadratic Function in the Form $f(x) = a(x-h)^2 + k$ Using Properties

1. Rewrite the function $f(x) = a(x-h)^2 + k$ form.
2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.
3. Find the axis of symmetry, $x = h$.
4. Find the vertex, $(h, k$.
5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
6. Find the x-intercepts.
7. Graph the parabola.

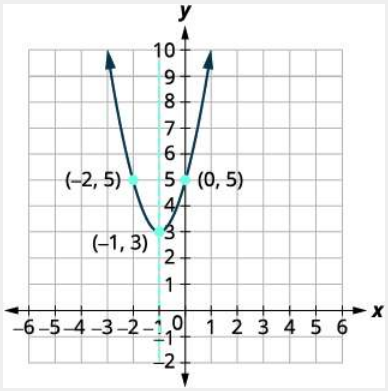
Example 5.6.10

a. Rewrite $f(x) = 2x^2 + 4x + 5$ in $f(x) = a(x-h)^2 + k$ form
b. Graph the function using properties

Solution:

Table 9.7.6

Rewrite the function in $f(x) = a(x-h)^2 + k$ form by completing the square.	$f(x) = 2x^2 + 4x + 5$
	$f(x) = 2\left(x^2 + 2x\right) + 5$
	$f(x) = 2\left(x^2 + 2x + 1\right) + 5 - 2$
	$f(x) = 2(x+1)^2 + 3$
Identify the constants a, h, k.	$a = 2 h = -1 k = 3$
Since $a = 2$, the parabola opens upward.	
The axis of symmetry is $x = h$.	The axis of symmetry is $x = -1$.
The vertex is (h, k).	The vertex is $(-1, 3)$.
Find the y-intercept by finding $f(0)$.	$f(0) = 2 \cdot 0^2 + 4 \cdot 0 + 5$
	$f(0) = 5$
	y-intercept $(0, 5)$

Find the point symmetric to $(0, 5)$ across the axis of symmetry.	$(-2, 5)$
Find the x-intercepts.	The discriminant is negative, so there are no x-intercepts. Graph the parabola.
	

Exercise 5.6.19

a. Rewrite $f(x) = 3x^2 - 6x + 5$ in $f(x) = a(x-h)^2 + k$ form
b. Graph the function using properties

Answer

a. $f(x) = 3(x-1)^2 + 2$
b.

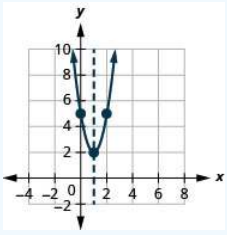


Figure 9.7.66

Exercise 5.6.20

a. Rewrite $f(x) = -2x^2 + 8x - 7$ in $f(x) = a(x-h)^2 + k$ form
b. Graph the function using properties

Answer

a. $f(x) = -2(x-2)^2 + 1$
b.

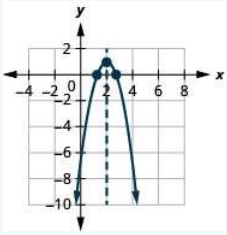


Figure 9.7.67

Find a Quadratic Function from its Graph

So far we have started with a function and then found its graph.

Now we are going to reverse the process. Starting with the graph, we will find the function.

Example 5.6.11

Determine the quadratic function whose graph is shown.

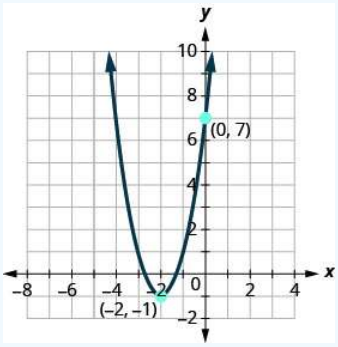


Figure 9.7.68

Solution:

Since it is quadratic, we start with the $f(x)=a(x-h)^2+k$ form.

The vertex, (h,k), is $(-2,-1)$ so $h=-2$ and $k=-1$.

$$f(x)=a(x-(-2))^2-1$$

To find a, we use the y-intercept, $(0,7)$.

So $f(0)=7$.

$$7=a(0+2)^2-1$$

Solve for a.

$$\begin{aligned} 7&=4a-1\\ 8&=4a\\ 2&=a \end{aligned}$$

Write the function.

$$f(x)=a(x-h)^2+k$$

Substitute in $h=-2, k=-1$ and $a=2$.

$$f(x)=2(x+2)^2-1$$

Exercise 5.6.21

Write the quadratic function in $f(x)=a(x-h)^2+k$ form whose graph is shown.

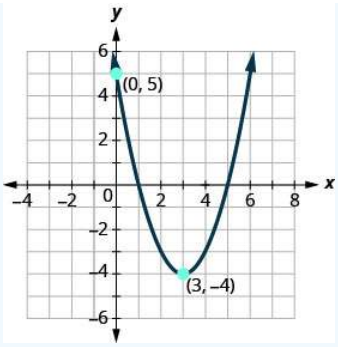


Figure 9.7.69

Answer

$f(x) = (x-3)^2 - 4$

? Exercise 5.6.22

Determine the quadratic function whose graph is shown.

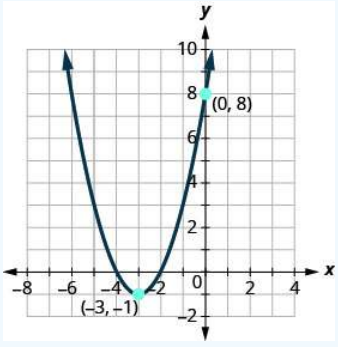


Figure 9.7.70

Answer

$f(x) = (x+3)^2 - 1$

Access these online resources for additional instruction and practice with graphing quadratic functions using transformations.

- Function Shift Rules Applied to Quadratic Functions
- Changing a Quadratic from Standard Form to Vertex Form
- Using Transformations to Graph Quadratic Functions
- Finding Quadratic Equation in Vertex Form from Graph

Key Concepts

- Graph a Quadratic Function of the form $f(x) = x^2 + k$ Using a Vertical Shift
  - The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.
    - If $k > 0$, shift the parabola vertically up k units.
    - If $k < 0$, shift the parabola vertically down $|k|$ units.
- Graph a Quadratic Function of the form $f(x) = (x-h)^2$ Using a Horizontal Shift
  - The graph of $f(x) = (x-h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.
    - If $h > 0$, shift the parabola horizontally left h units.
    - If $h < 0$, shift the parabola horizontally right $|h|$ units.
- Graph of a Quadratic Function of the form $f(x) = ax^2$

- The coefficient a in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.
  If $0 < |a| < 1$, then the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$.
  If $|a| > 1$, then the graph of $f(x) = ax^2$ will be "skinnier" than the graph of $f(x) = x^2$.

- How to graph a quadratic function using transformations
  1. Rewrite the function in $f(x) = a(x-h)^2 + k$ form by completing the square.
  2. Graph the function using transformations.

- Graph a quadratic function in the vertex form $f(x) = a(x-h)^2 + k$ using properties
  1. Rewrite the function in $f(x) = a(x-h)^2 + k$ form.
  2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.
  3. Find the axis of symmetry, $x = h$.
  4. Find the vertex, (h, k).
  5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
  6. Find the x-intercepts, if possible.
  7. Graph the parabola.

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5.6E: Exercises

Practice Makes Perfect

Exercise 5.6E.23 Graph Quadratic Functions of the Form $f(x) = x^2 = k$

In the following exercises,

a. Graph the quadratic functions on the same rectangular coordinate system
b. Describe what effect adding a constant, k, to the function has on the basic parabola.

1. $f(x) = x^2, g(x) = x^2 + 4, \text{ and } h(x) = x^2 - 4$
2. $f(x) = x^2, g(x) = x^2 + 7, \text{ and } h(x) = x^2 - 7$

Answer

1.

a.

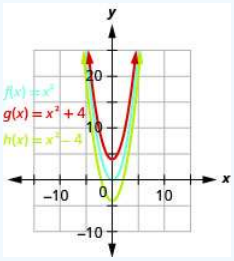


Figure 9.7.71

b. The graph of $g(x) = x^2 + 4$ is the same as the graph of $f(x) = x^2$ but shifted up 4 units. The graph of $h(x) = x^2 - 4$ is the same as the graph of $f(x) = x^2$ but shift down 4 units.

Exercise 5.6E.24 Graph Quadratic Functions of the Form $f(x) = x^2 = k$

In the following exercises, graph each function using a vertical shift.

1. $f(x) = x^2 + 3$
2. $f(x) = x^2 - 7$
3. $g(x) = x^2 + 2$
4. $g(x) = x^2 + 5$
5. $h(x) = x^2 - 4$
6. $h(x) = x^2 - 5$

Answer

1.

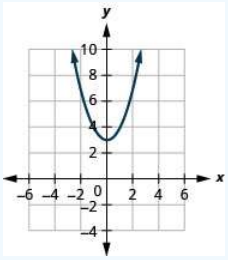


Figure 9.7.72

3.

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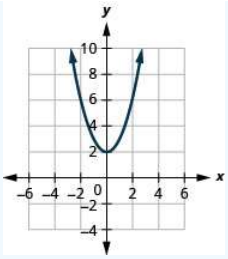


Figure 9.7.73

5.

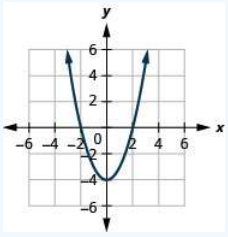


Figure 9.7.74

Exercise $5.6E.25$ Graph Quadratic Functions of the Form $f(x) = (x - h)^2$

In the following exercises,

a. Graph the quadratic functions on the same rectangular coordinate system
b. Describe what effect adding a constant, h, inside the parentheses has

1. $f(x) = x^2, g(x) = (x-3)^2,$ and $h(x) = (x+3)^2$
2. $f(x) = x^2, g(x) = (x+4)^2,$ and $h(x) = (x-4)^2$

Answer

1.

a.

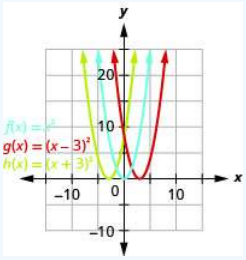


Figure 9.7.75

b. The graph of $g(x) = (x-3)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 3 units. The graph of $h(x) = (x+3)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 3 units.

Exercise $5.6E.26$ Graph Quadratic Functions of the Form $f(x) = (x - h)^2$

In the following exercises, graph each function using a horizontal shift.

1. $f(x) = (x-2)^2$
2. $f(x) = (x-1)^2$
3. $f(x) = (x+5)^2$
4. $f(x) = (x+3)^2$
5. $f(x) = (x-5)^2$

6. $f(x) = (x+2)^2$

Answer

1.

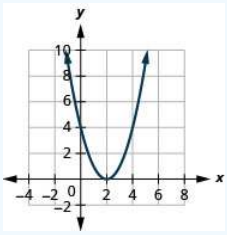


Figure 9.7.76

3.

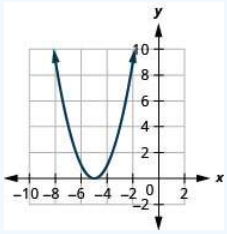


Figure 9.7.77

5.

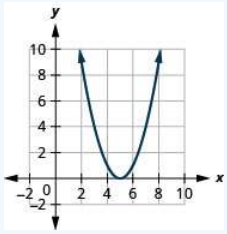


Figure 9.7.78

Exercise $5.6E.27$ Graph Quadratic Functions of the Form $f(x) = (x-h)^2$

In the following exercises, graph each function using transformations.

1. $f(x) = (x+2)^2 + 1$
2. $f(x) = (x+4)^2 + 2$
3. $f(x) = (x-1)^2 + 5$
4. $f(x) = (x-3)^2 + 4$
5. $f(x) = (x+3)^2 - 1$
6. $f(x) = (x+5)^2 - 2$
7. $f(x) = (x-4)^2 - 3$
8. $f(x) = (x-6)^2 - 2$

Answer

1.

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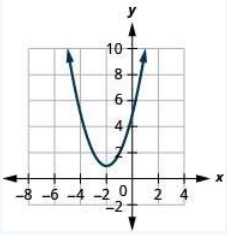


Figure 9.7.79

3.

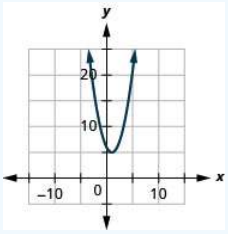


Figure 9.7.80

5.

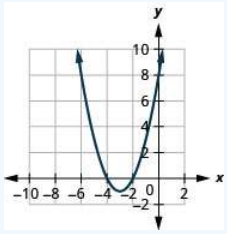


Figure 9.7.81

7.

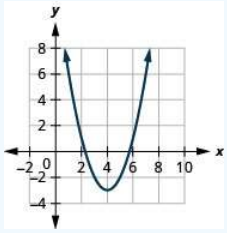


Figure 9.7.82

? Exercise $5.6E.28$ Graph Quadratic Functions of the Form $f(x) = ax^2$

In the following exercises, graph each function.

1. $f(x) = -2x^2$
2. $f(x) = 4x^2$
3. $f(x) = -4x^2$
4. $f(x) = -x^2$
5. $f(x) = \frac{1}{2}x^2$
6. $f(x) = \frac{1}{3}x^2$
7. $f(x) = \frac{1}{4}x^2$
8. $f(x) = -\frac{1}{2}x^2$

Answer

1.

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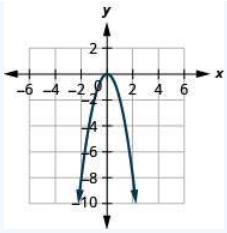


Figure 9.7.83

3.

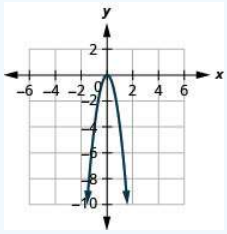


Figure 9.7.84

5.

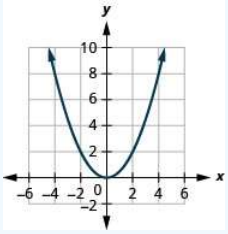


Figure 9.7.85

7.

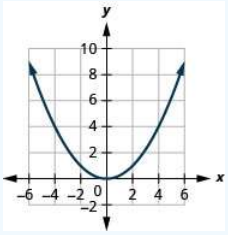


Figure 9.7.86

? Exercise $5.6E.29$ Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x)=a(x-h)^2+k$ form by completing the square.

1. $f(x)=-3x^2-12x-5$
2. $f(x)=2x^2-12x+7$
3. $f(x)=3x^2+6x-1$
4. $f(x)=-4x^2-16x-9$

Answer

1. $f(x)=-3(x+2)^2+7$
3. $f(x)=3(x+1)^2-4$

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Exercise $5.6E.30$ Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x)=a(x-h)^2+k$ form
b. Graph it by using transformations

1. $f(x)=x^2+6x+5$
2. $(x)=x^2+4x-12$
3. $f(x)=x^2+4x-12$
4. $f(x)=x^2-6x+8$
5. $f(x)=x^2-6x+15$
6. $f(x)=x^2+8x+10$
7. $f(x)=-x^2+8x-16$
8. $f(x)=-x^2+2x-7$
9. $f(x)=-x^2-4x+2$
10. $f(x)=-x^2+4x-5$
11. $f(x)=5x^2-10x+8$
12. $f(x)=3x^2+18x+20$
13. $f(x)=2x^2-4x+1$
14. $f(x)=3x^2-6x-1$
15. $f(x)=-2x^2+8x-10$
16. $f(x)=-3x^2+6x+1$

Answer

1.

a. f(x)=(x+3)^{2}-4
b.

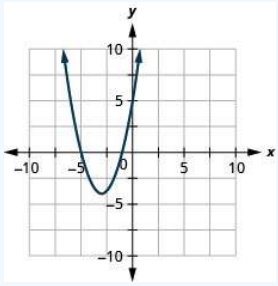


Figure 9.7.87

3.

a. $f(x)=(x+2)^2-1$
b.

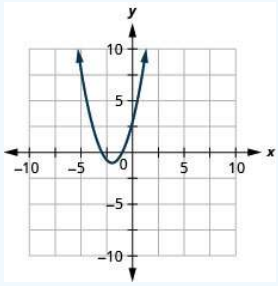


Figure 9.7.88

5.

a. $f(x) = (x-3)^2 + 6$

b.

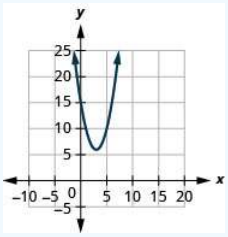


Figure 9.7.89

7.

a. $f(x) = -(x-4)^2 + 0$

b.

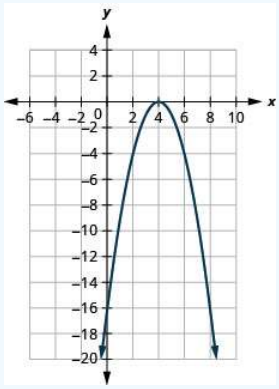


Figure 9.7.90

9.

a. $f(x) = -(x+2)^2 + 6$

b.

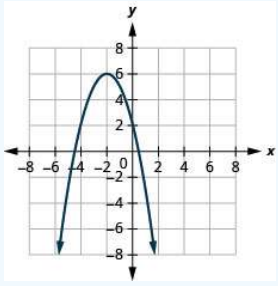


Figure 9.7.91

11.

a. $f(x) = 5(x-1)^2 + 3$

b.

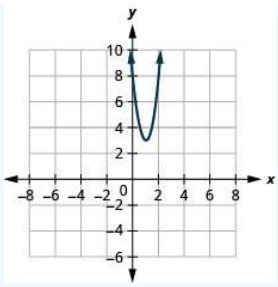


Figure 9.7.92

13.

a. $f(x) = 2(x-1)^2 - 1$

b.

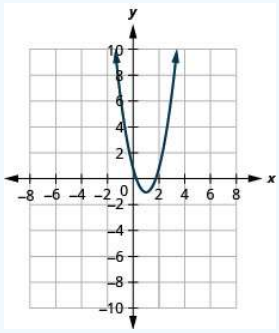


Figure 9.7.93

15.

a. $f(x) = -2(x-2)^2 - 2$

b.

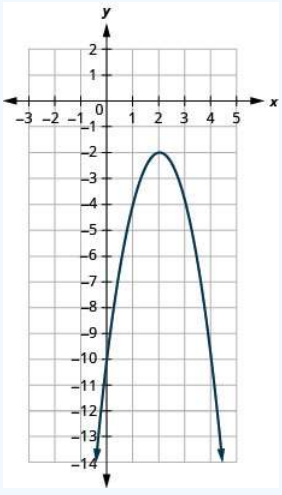


Figure 9.7.94

Exercise $5.6E.31$ Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x) = a(x-h)^2 + k$ form

b. Graph it using properties

1. $f(x) = 2x^2 + 4x + 6$
2. $f(x) = 3x^2 - 12x + 7$

3. $f(x) = -x^2 + 2x - 4$
4. $f(x) = -2x^2 - 4x - 5$

Answer

1.

a. $f(x) = 2(x+1)^2 + 4$

b.

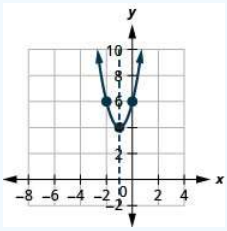


Figure 9.7.95

3.

a. $f(x) = -(x-1)^2 - 3$

b.

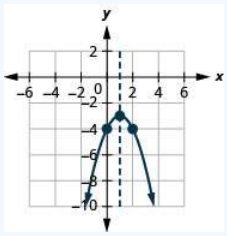


Figure 9.7.96

Exercise $5.6E.32$ Matching

In the following exercises, match the graphs to one of the following functions:

a. $f(x) = x^2 + 4$
b. $f(x) = x^2 - 4$
c. $f(x) = (x+4)^2$
d. $f(x) = (x-4)^2$
e. $f(x) = (x+4)^2 - 4$
f. $f(x) = (x+4)^2 + 4$
g. $f(x) = (x-4)^2 - 4$
h. $f(x) = (x-4)^2 + 4$

1.

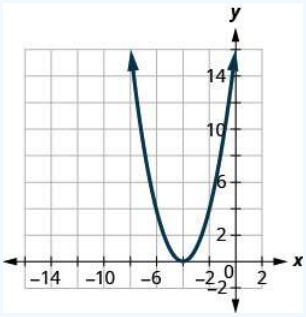


Figure 9.7.97

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2.

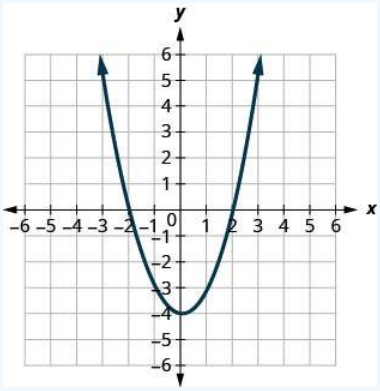


Figure 9.7.98

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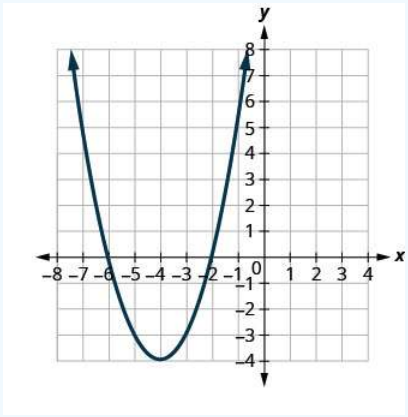


Figure 9.7.99

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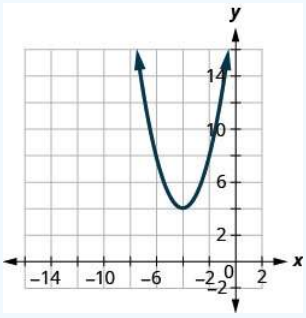


Figure 9.7.100

5.

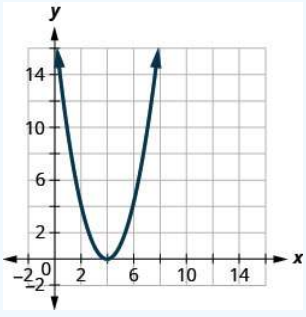


Figure 9.7.101

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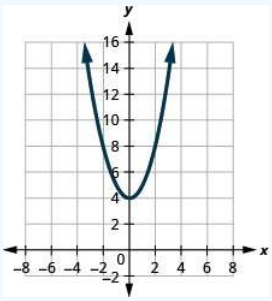


Figure 9.7.102

7.

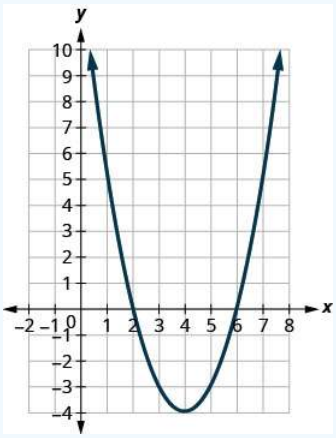


Figure 9.7.103

8.

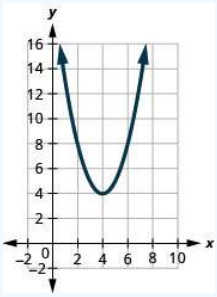


Figure 9.7.104

Answer

1. c
3. e
5. d
7. g

? Exercise $5.6E.33$ Find a Quadratic Function from its Graph

In the following exercises, write the quadratic function in $f(x) = a(x-h)^2 + k$ form whose graph is shown.

1.

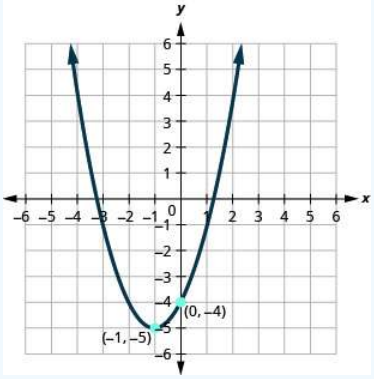


Figure 9.7.105

2.

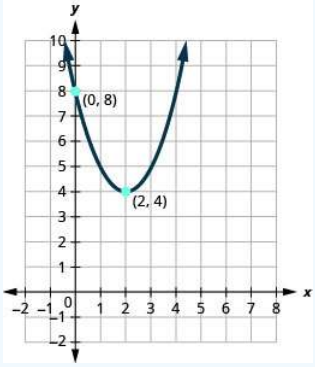


Figure 9.7.106

3.

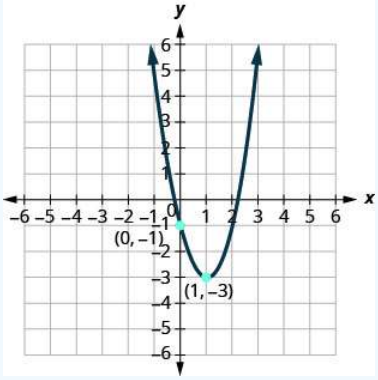


Figure 9.7.107

4.

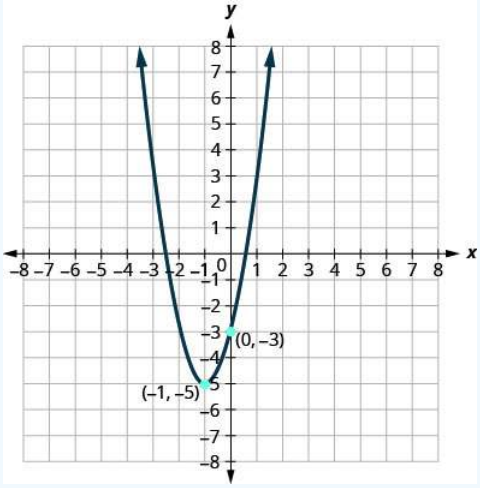


Figure 9.7.108

Answer

1. $f(x) = (x+1)^2 - 5$
3. $f(x) = 2(x-1)^2 - 3$

? Exercise $5.6E.34$ Writing Exercise

1. Graph the quadratic function $f(x) = x^2 + 4x + 5$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?
2. Graph the quadratic function $f(x) = 2x^2 - 4x - 3$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

Answer

1. Answers may vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph Quadratic Functions of the form $f(x) = x^2 + k$.			
graph Quadratic Functions of the form $f(x) = (x - h)^2$.			
graph Quadratic Functions of the form $f(x) = ax^2$.			
graph Quadratic Functions Using Transformations.			
find a Quadratic Function from its Graph.			

Figure 9.7.109

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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5.7: Solve Quadratic Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic inequalities graphically
- Solve quadratic inequalities algebraically

Before you get started, take this readiness quiz.

1. Solve: $2x-3=0$.
   If you missed this problem, review Example 2.2.
2. Solve: $2y^2+y=15$.
   If you missed this problem, review Example 6.45.
3. Solve $\frac{1}{x^2+2x-8}>0$
   If you missed this problem, review Example 7.56.

We have learned how to solve linear inequalities and rational inequalities previously. Some of the techniques we used to solve them were the same and some were different. We will now learn to solve inequalities that have a quadratic expression. We will use some of the techniques from solving linear and rational inequalities as well as quadratic equations. We will solve quadratic inequalities two ways —both graphically and algebraically.

Solve Quadratic Inequalities Graphically

A **quadratic equation** is in standard form when written as $ax^2+bx+c=0$. If we replace the equal sign with an inequality sign, we have a **quadratic inequality** in standard form.

Definition 5.7.1: Quadratic Inequality

A **quadratic inequality** is an inequality that contains a quadratic expression. The standard form of a quadratic inequality is written:

$$ax^2+bx+c<0 \quad ax^2+bx+c\le 0$$
$$ax^2+bx+c>0 \quad ax^2+bx+c\ge 0$$

The graph of a quadratic function $f(x)=ax^2+bx+c=0$ is a parabola. When we ask when is $ax^2+bx+c<0$, we are asking when is $f(x)<0$. We want to know when the parabola is below the x-axis.

When we ask when is $ax^2+bx+c>0$, we are asking when is $f(x)>0$. We want to know when the parabola is above the y-axis.

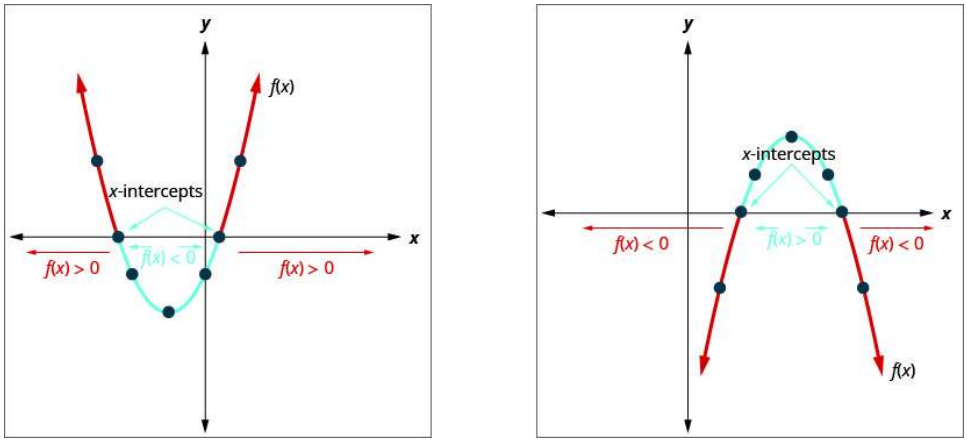


Figure 9.8.1

Example 5.7.1: How to Solve a Quadratic Inequality Graphically

Solve $x^2-6x+8<0$ graphically. Write the solution in interval notation.

Solution:

Step 1: Write the quadratic inequality in standard form.

The inequality is in standard form.

$$x^2-6x+8<0$$

Step 2: Graph the function $f(x)=ax^2+bx+c$ using properties or transformations.

We will graph using the properties.

$$f(x)=x^2-6x+8$$

Look at a in the equation.

$$a=1, b=-6, c=8$$

$f(x)=x^2-6x+8$

Since a is positive, the parabola opens upward.

The parabola opens upward.



Figure 9.8.2

$f(x)=x^2-6x+8$

The axis of symmetry is the line $x=-\frac{b}{2a}$.

Axis of Symmetry

$$x=-\frac{b}{2a}$$
$$x=-\frac{(-6)}{2\cdot 1}$$
$$x=3$$

The axis of symmetry is the line $x=3$.

The vertex is on the axis of symmetry. Substitute $x=3$ into the function.

Vertex

$$f(x)=x^2-6x+8$$
$$f(3)=(3)^2-6(3)+8$$
$$f(3)=-1$$

The vertex is $(3,-1)$.

We find $f(0)$

y-intercept

$$f(x)=x^2-6x+8$$
$$f(0)=(0)^2-6(0)+8$$
$$f(0)=8$$

The y-intercept is $(0,8)$.

We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x=3$. A point 3 units to the right of the axis of symmetry has $x=6$.

Point symmetric to y-intercept

The point is $(6,8)$.

We solve $f(x) = 0$.

x-intercepts

We can solve this quadratic equation by factoring.

$$\begin{aligned} f(x) &= x^2 - 6x + 8 \\ 0 &= x^2 - 6x + 8 \\ 0 &= (x-2)(x-4) \\ x &= 2 \text{ or } x = 4 \end{aligned}$$

The x-intercepts are $(2,0)$ and $(4,0)$.

We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.

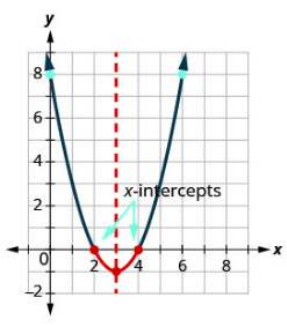


Figure 9.8.3

Step 3: Determine the solution from the graph.

$x^2 - 6x + 8 < 0$

The inequality asks for the values of x which make the function less than 0. Which values of x make the parabola below the x-axis.

We do not include the values 2, 4 as the inequality is less than only.

The solution, in interval notation, is $(2, 4)$.

Exercise 5.7.1

a. Solve $x^2 + 2x - 8 < 0$ graphically
b. Write the solution in interval notation

Answer

a.

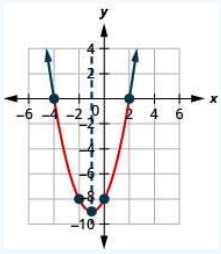


Figure 9.8.4

b. $(-4, 2)$

Exercise 5.7.2

a. Solve $x^2 - 8x + 12 \geq 0$ graphically
b. Write the solution in interval notation

Answer

a.

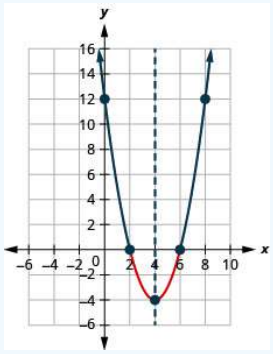


Figure 9.8.5

b. $(-\infty, 2] \cup [6, \infty)$

We list the steps to take to solve a quadratic inequality graphically.

Solve a Quadratic Inequality Graphically

1. Write the quadratic inequality in standard form.
2. Graph the function $f(x) = ax^2 + bx + c$.
3. Determine the solution from the graph.

In the last example, the parabola opened upward and in the next example, it opens downward. In both cases, we are looking for the part of the parabola that is below the x-axis but note how the position of the parabola affects the solution.

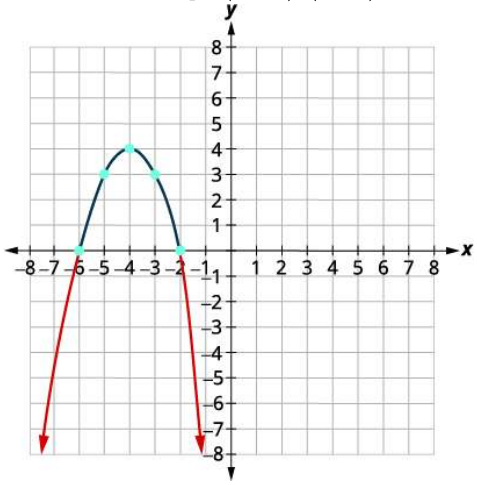
Example 5.7.2

Solve $-x^2 - 8x - 12 \leq 0$ graphically. Write the solution in interval notation.

Solution:

Table 9.8.1

The quadratic inequality in standard form.	$-x^2 - 8x - 12 \leq 0$
Graph the function $f(x) = -x^2 - 8x - 12$	The parabola opens downward.  Figure 9.8.6
Find the line of symmetry.	$x = -\frac{b}{2a}$ $x = -\frac{-8}{2(-1)}$ $x = -4$
Find the vertex.	$f(x) = -x^2 - 8x - 12$ $f(-4) = -(-4)^2 - 8(-4) - 12$ $f(-4) = -16 + 32 - 12$ $f(-4) = 4$ Vertex $(-4, 4)$
Find the x-intercepts. Let $f(x) = 0$.	$f(x) = -x^2 - 8x - 12$ $0 = -x^2 - 8x - 12$
Factor: Use the Zero Product Property.	$0 = -1(x + 6)(x + 2)$ $x = -6 \quad x = -2$

Graph the parabola.	x-intercepts $(-6, 0), (-2.0)$  Figure 9.8.7
Determine the solution from the graph. We include the x-intercepts as the inequality is "less than or equal to."	$(-\infty, -6] \cup [-2, \infty)$

? Exercise 5.7.3

a. Solve $-x^2 - 6x - 5 > 0$ graphically
b. Write the solution in interval notation

Answer

a.

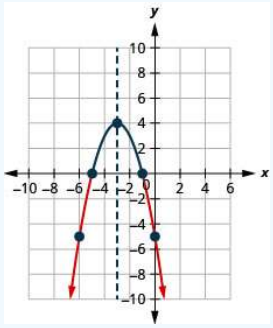


Figure 9.8.8

b. $(-5, -1)$

? Exercise 5.7.4

a. Solve $-x^2 + 10x - 16 \leq 0$ graphically
b. Write the solution in interval notation

Answer

a.

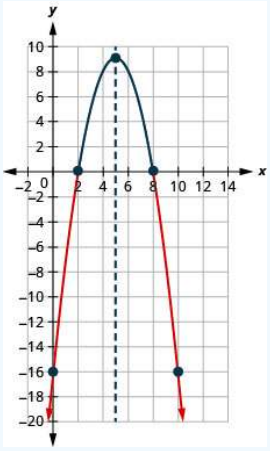


Figure 9.8.9

b. $(-\infty, 2] \cup [8, \infty)$

Solve Quadratic Inequalities Algebraically

The algebraic method we will use is very similar to the method we used to solve rational inequalities. We will find the critical points for the inequality, which will be the solutions to the related quadratic equation. Remember a polynomial expression can change signs only where the expression is zero.

We will use the **critical points** to divide the number line into intervals and then determine whether the quadratic expression will be positive or negative in the interval. We then determine the solution for the inequality.

Example 5.7.3: How to Solve Quadratic Inequalities Algebraically

Solve $x^2 - x - 12 \geq 0$ algebraically. Write the solution in interval notation.

Solution:

Table 9.8.2

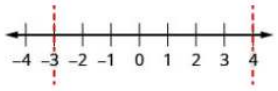
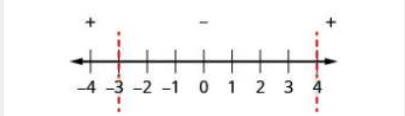
Step 1: Write the quadratic inequality in standard form.	The inequality is in standard form.	$x^2 - x - 12 \geq 0$
Step 2: Determine the critical points--the solutions to the related quadratic equation.	Change the inequality sign to an equal sign and then solve the equation.	$x^2 - x - 12 = 0$ $(x+3)(x-4) = 0$ $x+3=0 \quad x-4=0$ $x=-3 \quad x=4$
Step 3: Use the critical points to divide the number line into intervals.	Use -3 and 4 to divide the number line into intervals.	
Step 4: Above the number line show the sign of each quadratic expression using test points from each interval substituted from the original inequality.	Test: $x=-5$ $x=0$ $x=5$	$x^2 - x - 12$ $\quad$ $x^2 - x - 12$ $\quad$ $x^2 - x -$ $(-5)^2 - (-5) - 12$ $\quad$ $0^2 - 0 - 12$ $\quad$ $5^2 - 5 -$ 18 $\quad$ -12 $\quad$ 8 

Figure 9.8.11

Step 5: Determine the intervals where the inequality is correct. Write the solution in interval notation.	$x^2 - x - 12 \geq 0$ The inequality is positive in the first and last intervals and equals 0 at the points $-4, 3$.	The solution, in interval notation, is $(-\infty, -3] \cup [4, \infty)$.

? Exercise 5.7.5

Solve $x^2 + 2x - 8 \geq 0$ algebraically. Write the solution in interval notation.

Answer

$(-\infty, -4] \cup [2, \infty)$

? Exercise 5.7.6

Solve $x^2 - 2x - 15 \leq 0$ algebraically. Write the solution in interval notation.

Answer

$[-3, 5]$

In this example, since the expression $x^2 - x - 12$ factors nicely, we can also find the sign in each interval much like we did when we solved rational inequalities. We find the sign of each of the factors, and then the sign of the product. Our number line would like this:

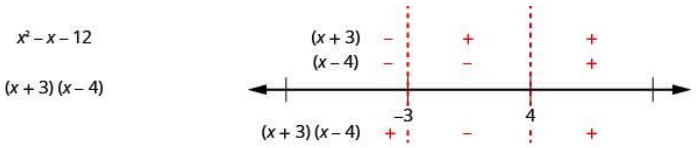


Figure 9.8.12

The result is the same as we found using the other method.

We summarize the steps here.

Solve a Quadratic Inequality Algebraically

1. Write the quadratic inequality in standard form.
2. Determine the critical points—the solutions to the related quadratic equation.
3. Use the critical points to divide the number line into intervals.
4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.
5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

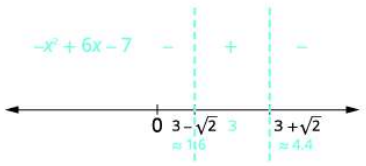
✓ Example 5.7.4

Solve $x^2 + 6x - 7 \geq 0$ algebraically. Write the solution in interval notation.

Solution:

Table 9.8.3

Write the quadratic inequality in standard form.	$-x^2 + 6x - 7 \geq 0$
Multiply both sides of the inequality by -1. Remember to reverse the inequality sign.	$x^2 - 6x + 7 \leq 0$
Determine the critical points by solving the related quadratic equation.	$x^2 - 6x + 7 = 0$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Then substitute in the values of a, b, c.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (7)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{8}}{2}$
Simplify the radical.	$x = \frac{6 \pm 2\sqrt{2}}{2}$
Remove the common factor, 2.	$x = \frac{2(3 \pm \sqrt{2})}{2}$ $x = 3 \pm \sqrt{2}$ $x = 3 + \sqrt{2} \quad x = 3 - \sqrt{2}$ $x \approx 1.6 \quad x \approx 4.4$
Use the critical points to divide the number line into intervals. Test numbers from each interval in the original inequality.	
Determine the intervals where the inequality is correct. Write the solution in interval notation.	$-x^2 + 6x - 7 \geq 0$ in the middle interval $[3 - \sqrt{2}, 3 + \sqrt{2}]$

? Exercise 5.7.7

Solve $-x^2 + 2x + 1 \geq 0$ algebraically. Write the solution in interval notation.

Answer

$[-1 - \sqrt{2}, -1 + \sqrt{2}]$

? Exercise 5.7.8

Solve $-x^2 + 8x - 14 < 0$ algebraically. Write the solution in interval notation.

Answer

$(-\infty, 4 - \sqrt{2}) \cup (4 + \sqrt{2}, \infty)$

The solutions of the quadratic inequalities in each of the previous examples, were either an interval or the union of two intervals. This resulted from the fact that, in each case we found two solutions to the corresponding quadratic equation $ax^2 + bx + c = 0$. These two solutions then gave us either the two x-intercepts for the graph or the two critical points to divide the number line into intervals.

This correlates to our previous discussion of the number and type of solutions to a quadratic equation using the discriminant.

For a quadratic equation of the form $ax^2 + bc + c = 0, a \neq 0$.

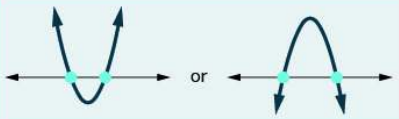
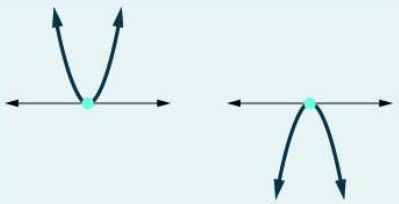
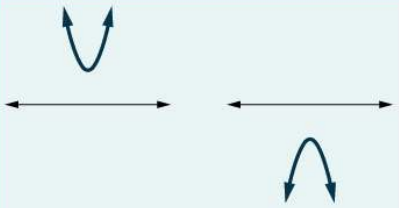
Discriminant	Number/Type of solution	Typical Graph
$b^2 - 4ac > 0$	2 real solutions 2 x-intercepts on graph	
$b^2 - 4ac = 0$	1 real solution 1 x-intercept on graph	
$b^2 - 4ac < 0$	2 complex solutions No x-intercept	

Figure 9.8.14

The last row of the table shows us when the parabolas never intersect the x-axis. Using the Quadratic Formula to solve the quadratic equation, the radicand is a negative. We get two complex solutions.

In the next example, the quadratic inequality solutions will result from the solution of the quadratic equation being complex.

Example 5.7.5

Solve, writing any solution in interval notation:

a. $x^2 - 3x + 4 > 0$
b. $x^2 - 3x + 4 \leq 0$

Solution:

a.

Table 9.8.4

Write the quadratic inequality in standard form.	$-x^2 - 3x + 4 > 0$
Determine the critical points by solving the related quadratic equation.	$x^2 - 3x + 4 = 0$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (4)}}{2 \cdot 1}$
Simplify.	$x = \frac{3 \pm \sqrt{-7}}{2}$
Simplify the radicand.	$x = \frac{3 \pm \sqrt{7}i}{2}$
The complex solutions tell us the parabola does not intercept the x-axis. Also, the parabola opens upward. This tells us that the parabola is completely above the x-axis.	Complex solutions 

Figure 9.8.15

We are to find the solution to $x^2-3x+4>0$. Since for all values of x the graph is above the x-axis, all values of x make the inequality true. In interval notation we write $(-\infty,\infty)$.

b. Write the quadratic inequality in standard form.

$$x^2-3x+4\leq 0$$

Determine the critical points by solving the related quadratic equation.

$$x^2-3x+4=0$$

Since the corresponding quadratic equation is the same as in part (a), the parabola will be the same. The parabola opens upward and is completely above the x-axis—no part of it is below the x-axis.

We are to find the solution to $x^2-3x+4\leq 0$. Since for all values of x the graph is never below the x-axis, no values of x make the inequality true. There is no solution to the inequality.

Exercise 5.7.9

Solve and write any solution in interval notation:

a. $-x^2+2x-4\leq 0$
b. $-x^2+2x-4\geq 0$

Answer

a. $(-\infty,\infty)$
b. no solution

Exercise 5.7.10

Solve and write any solution in interval notation:

a. $x^2+3x+3<0$
b. $x^2+3x+3>0$

Answer

a. no solution
b. $(-\infty,\infty)$

Key Concepts

- Solve a Quadratic Inequality Graphically
  1. Write the quadratic inequality in standard form.
  2. Graph the function $f(x)=ax^2+bx+c$ using properties or transformations.
  3. Determine the solution from the graph.
- How to Solve a Quadratic Inequality Algebraically
  1. Write the quadratic inequality in standard form.
  2. Determine the critical points -- the solutions to the related quadratic equation.
  3. Use the critical points to divide the number line into intervals.
  4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.
  5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

Glossary

quadratic inequality

A quadratic inequality is an inequality that contains a quadratic expression.

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5.7E: Exercises

Practice Makes Perfect

Exercise $5.7E.11$ Solve Quadratic Inequalities Graphically

In the following exercises,

a. Solve graphically
b. Write the solution in interval notation

1. $x^2+6x+5>0$
2. $x^2+4x-12<0$
3. $x^2+4x+3\leq 0$
4. $x^2-6x+8\geq 0$
5. $-x^2-3x+18\leq 0$
6. $-x^2+2x+24<0$
7. $-x^2+x+12\geq 0$
8. $-x^2+2x+15>0$

Answer

1.

a.

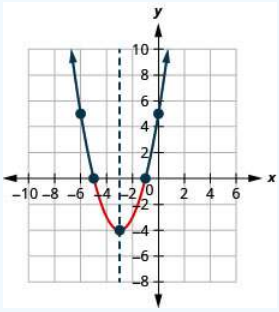


Figure 9.8.16

b. $(-\infty,-5)\cup(-1,\infty)$

3.

a.

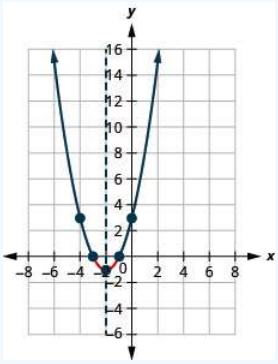


Figure 9.8.17

b. $[-3,-1]$

5.

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a.

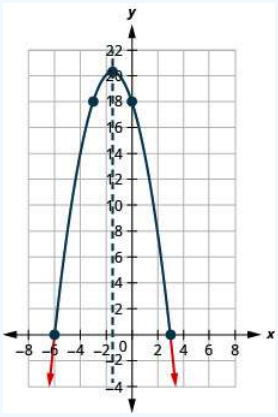


Figure 9.8.18

b. $(-\infty, -6] \cup [3, \infty)$

7.

a.

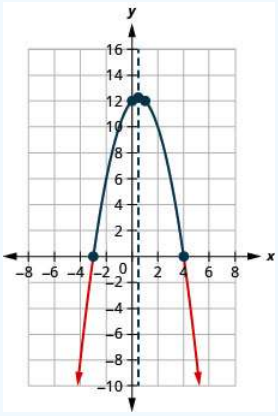


Figure 9.8.19

b. $[-3, 4]$

Exercise $5.7E.12$ Solve Quadratic Inequalities Graphically

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

1. $x^2 + 3x - 4 \geq 0$
2. $x^2 + x - 6 \leq 0$
3. $x^2 - 7x + 10 < 0$
4. $x^2 - 4x + 3 > 0$
5. $x^2 + 8x > -15$
6. $x^2 + 8x < -12$
7. $x^2 - 4x + 2 \leq 0$
8. $-x^2 + 8x - 11 < 0$
9. $x^2 - 10x > -19$
10. $x^2 + 6x < -3$
11. $-6x^2 + 19x - 10 \geq 0$
12. $-3x^2 - 4x + 4 \leq 0$
13. $-2x^2 + 7x + 4 \geq 0$
14. $2x^2 + 5x - 12 > 0$
15. $x^2 + 3x + 5 > 0$

16. $x^2-3x+6\leq 0$
17. $-x^2+x-7>0$
18. $-x^2-4x-5<0$
19. $-2x^2+8x-10<0$
20. $-x^2+2x-7\geq 0$

Answer

1. $(-\infty,-4]\cup[1,\infty)$
3. $(2,5)$
5. $(-\infty,-5)\cup(-3,\infty)$
7. $[2-\sqrt{2},2+\sqrt{2}]$
9. $(-\infty,5-\sqrt{6})\cup(5+\sqrt{6},\infty)$
11. $\left(-\infty,-\frac{5}{2}\right]\cup\left[-\frac{2}{3},\infty\right)$
13. $\left[-\frac{1}{2},4\right]$
15. $(-\infty,\infty)$
17. no solution
19. $(-\infty,\infty)$

Exercise $5.7E.13$ Writing Exercises

1. Explain critical points and how they are used to solve quadratic inequalities algebraically.
2. Solve $x^2+2x\geq 8$ both graphically and algebraically. Which method do you prefer, and why?
3. Describe the steps needed to solve a quadratic inequality graphically.
4. Describe the steps needed to solve a quadratic inequality algebraically.

Answer

1. Answers may vary.
3. Answers may vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic inequalities graphically.			
solve quadratic inequalities algebraically.			

Figure 9.8.20

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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5.8: Solve Applications of Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve applications modeled by quadratic equations

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is -100. Find the numbers.
   If you missed this problem, review Example 2.18.
2. Solve: $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.
   If you missed this problem, review Example 7.35.
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review Example 2.34.

Solve Applications Modeled by Quadratic Equations

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications.

Let's first summarize the methods we now have to solve quadratic equations.

Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As you solve each equation, choose the method that is most convenient for you to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.

Use a Problem-Solving Strategy

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We have solved number applications that involved consecutive even and odd integers, by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

	Consecutive even integers
	$64, 66, 68$
n	1st even integer
$n+2$	2nd consecutive even integer
$n+4$	3rd consecutive even integer

	Consecutive odd integers
	$77, 79, 81$
n	1st odd integer
$n+2$	2nd consecutive odd integer
$n+4$	3rd consecutive odd integer

Some applications of odd or even consecutive integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

Example 5.8.1

The product of two consecutive odd integers is 195. Find the integers.

Solution:

Step 1: **Read** the problem

Step 2: **Identify** what we are looking for.

We are looking for two consecutive odd integers.

Step 3: **Name** what we are looking for.

Let $n =$ the first odd integer.

$n+2 =$ the next odd integer.

Step 4: **Translate** into an equation. State the problem in one sentence.

"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.

Translate into an equation.

$$n(n+2) = 195$$

Step 5: **Solve** the equation. Distribute.

$$n^2 + 2n = 195$$

Write the equation in standard form.

$$n^2 + 2n - 195 = 0$$

Factor.

$$(n+15)(n-13) = 0$$

Use the Zero Product Property.

$$n+15 = 0 \quad n-13 = 0$$

Solve each equation.

$$n = -15, \quad n = 13$$

There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.

First odd integer $n = 13$	First odd integer $n = -15$
next odd integer $n+2$	next odd integer $n+2$
$13+2$	$-15+2$
15	-13

Step 6: **Check** the answer.

Do these pairs work? Are they consecutive odd integers?

$13, 15$ yes
$-13, -15$ yes

Is their product 195?

$13 \cdot 15 = 195$ yes

$-13(-15) = 195$ yes

Step 7: **Answer** the question.

Two consecutive odd integers whose product is 195 are $13, 15$ and $-13, -15$.

Exercise 5.8.1

The product of two consecutive odd integers is 99. Find the integers.

Answer

The two consecutive odd integers whose product is 99 are $9, 11$, and $-9, -11$.

Exercise 5.8.2

The product of two consecutive even integers is 168. Find the integers.

Answer

The two consecutive even integers whose product is 128 are $12, 14$ and $-12, -14$.

We will use the formula for the area of a triangle to solve the next example.

Definition 5.8.1

Area of a Triangle

For a triangle with base, b, and height, h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

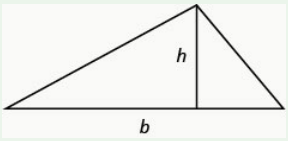


Figure 9.5.1

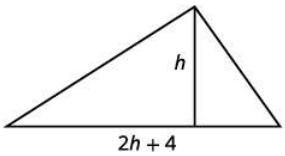
Recall that when we solve geometric applications, it is helpful to draw the figure.

Example 5.8.2

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of 120 square feet and the architect wants the base to be 4 feet more than twice the height. Find the base and height of the window.

Solution:

Table 9.5.1

Step 1: **Read** the problem. Draw a picture.	
Step 2: **Identify** what we are looking for.	We are looking for the base and height.
Step 3: **Name** what we are looking for.	Let $h =$ the height of the triangle. $2h + 4 =$ the base of the triangle.

Step 4: **Translate** into an equation. We know the area. Write the formula for the area of a triangle.	$A=\frac{1}{2}bh$
Step 5: **Solve** the equation. Substitute in the values.	$120=\frac{1}{2}(2h+4)h$
Distribute.	$120=h^2+2h$
This is a quadratic equation, rewrite it in standard form.	$h^2+2h-120=0$
Factor.	$(h-10)(h+12)=0$
Use the Zero Product Property.	$h-10=0 \quad h+12=0$
Simplify.	$h=10, \quad h=\cancel{-12}$
Since h is the height of a window, a value of $h=-12$ does not make sense.	
The height of the triangle $h=10$.	
The base of the triangle $2h+4$. $2\cdot 10+4$ 24	
Step 6: **Check** the answer. Does a triangle with height 10 and base 24 have area 120? Yes.	
Step 7: **Answer** the question.	The height of the triangular window is 10 feet and the base is 24 feet.

? Exercise 5.8.3

Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.

Answer

The height of the triangle is 12 inches and the base is 76 inches.

? Exercise 5.8.4

If a triangle that has an area of 110 square feet has a base that is two feet less than twice the height, what is the length of its base and height?

Answer

The height of the triangle is 11 feet and the base is 20 feet.

In the two preceding examples, the number in the radical in the **Quadratic Formula** was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

Definition 5.8.2

Area of a Rectangle

For a rectangle with length, L, and width, W, the area, A, is given by the formula $A=LW$.

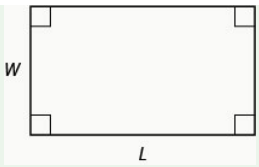


Figure 9.5.3

Example 5.8.3

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than 3 times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution:

Table 9.5.2

Step 1: **Read** the problem. Draw a picture.	
Step 2: **Identify** what we are looking for.	We are looking for the length and width.
Step 3: **Name** what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle
Step 4: **Translate** into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Step 5: **Solve** the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$
This is a quadratic equation; rewrite it in standard form. Solve the equation using the Quadratic Formula.	$ax^2 + bx + c = 0$ $3w^2 - w - 150 = 0$
Identify the a, b, c values.	$a = 3,\ b = -1,\ c = -150$
Write the Quadratic Formula.	$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$
Simplify.	$w = \frac{1 \pm \sqrt{1 + 1800}}{6}$ $w = \frac{1 \pm \sqrt{1801}}{6}$ Figure 9.5.13
Rewrite to show two solutions.	$w = \frac{1 + \sqrt{1801}}{6}, \quad w = \frac{1 - \sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w \approx 7.2, \quad \cancel{w \approx -6.9}$ Width $w \approx 7.2$ Length $\approx 3w - 1$ $\approx 3(7.2) - 1$ ≈ 20.6
Step 6: **Check** the answer. Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to 150.	
Step 7: **Answer** the question.	The width of the rectangle is approximately 7.2 feet and the length is approximately 20.6 feet.

Exercise 5.8.5

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

Answer

The length of the garden is approximately 18 feet and the width 11 feet.

Exercise 5.8.6

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length.What are the length and width of the tablecloth to the nearest tenth of a foot?

Answer

The length of the tablecloth is approximately 11.8feet and the width 6.8 feet.

The **Pythagorean Theorem** gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Definition 5.8.3

Pythagorean Theorem

In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2+b^2=c^2$.

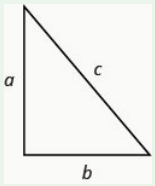


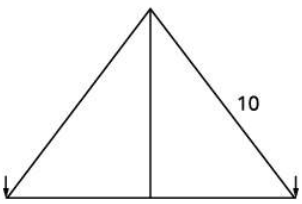
Figure 9.5.16

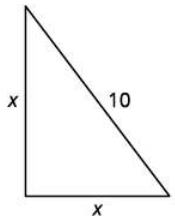
Example 5.8.4

Rene is setting up a holiday light display. He wants to make a 'tree' in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution:

Table 9.5.3

Step 1: **Read** the problem. Draw a picture.	
Step 2: **Identify** what we are looking for.	We are looking for the height of the pole.

Step 3: **Name** what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x =$ the height of the pole. $x =$ the distance from pole to stake Each side is a right triangle. We draw a picture of one of them.  Figure 9.5.18
Step 4: **Translate** into an equation. We can use the Pythagorean Theorem to solve for x. Write the Pythagorean Theorem.	$a^2+b^2=c^2$
Step 5: **Solve** the equation. Substitute.	$x^2+x^2=10^2$
Simplify.	$2x^2=100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2}=\frac{100}{2}$
Simplify.	$x^2=50$
Use the Square Root Property.	$x=\pm\sqrt{50}$
Simplify the radical.	$x=\pm 5\sqrt{2}$
Rewrite to show two solutions.	$x=5\sqrt{2}, \quad x=-5\sqrt{2}$
	If we approximate this number to the nearest tenth with a calculator, we find $x \approx 7.1$.
Step 6: **Check** the answer. Check on your own in the Pythagorean Theorem.	
Step 7: **Answer** the question.	The pole should be about 7.1 feet tall.

? Exercise 5.8.7

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

Answer

The length of the flag pole's shadow is approximately 6.3 feet and the height of the flag pole is 18.9 feet.

? Exercise 5.8.8

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

Answer

The distance between the opposite corners is approximately 7.2 feet.

The height of a projectile shot upward from the ground is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Definition 5.8.4

The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula

$$h = -16t^2 + v_0 t$$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

Example 5.8.5

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution:

Table 9.5.4

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking for.	We are looking for the number of seconds, which is time.
Step 3: **Name** what we are looking for.	Let t = the number of seconds.
Step 4: **Translate** into an equation. Use the formula.	$h = -16t^2 + v_0 t$
Step 5: **Solve** the equation. We know the velocity v_0 is 130 feet per second. The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in standard form. Solve the equation using the Quadratic Formula.	$ax^2 + bx + c = 0$ $16t^2 - 130t + 260 = 0$
Identify the values of a, b, c.	$a = 16,\ b = -130,\ c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$ Figure 9.5.26
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32}, \quad t = \frac{130 - \sqrt{260}}{32}$
Approximate the answer with a calculator.	$t \approx 4.6$ seconds, $t \approx 3.6$ seconds
Step 6: **Check** the answer. The check is left to you.	
Step 7: **Answer** the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

Exercise 5.8.9

An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h = -16t^2 + v_0 t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth.

Answer

The arrow will reach 180 feet on its way up after 3 seconds and again on its way down after approximately 3.8 seconds.

Exercise 5.8.10

A man throws a ball into the air with a velocity of 96 ft/s. Use the formula $h = -16t^2 + v_0 t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth.

Answer

The ball will reach 48 feet on its way up after approximately $.6$ second and again on its way down after approximately 5.4 seconds.

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate	•	Time	=	Distance

Figure 9.5.29

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

Some uniform motion problems are also modeled by quadratic equations.

Example 5.8.6

Professor Smith just returned from a conference that was $2,000$ miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

Solution:

This is a uniform motion situation. A diagram will help us visualize the situation.

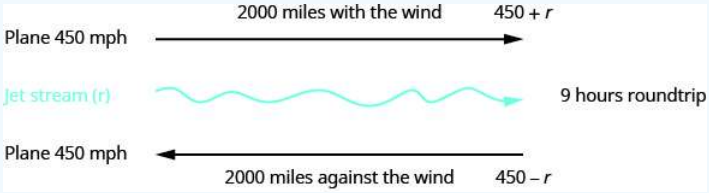


Figure 9.5.30

We fill in the chart to organize the information.

We are looking for the speed of the jet stream. Let $r =$ the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is $450 + r$.

When the plane flies against the wind, the wind decreases its speed and the rate is $450 - r$.

Table 9.5.5

Write in the rates. Write in the distances. Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th>Type</th><th>Rate</th><th>·</th><th>Time</th><th>=</th><th>Distance</th></tr><tr><td>Headwind</td><td>$450 - r$</td><td></td><td>$\frac{2000}{450-r}$</td><td></td><td>2000</td></tr><tr><td>Tailwind</td><td>$450 + r$</td><td></td><td>$\frac{2000}{450+r}$</td><td></td><td>2000</td></tr><tr><td></td><td></td><td></td><td>9</td><td></td><td></td></tr></table>
We know the times add to 9 and so we write our equation.	$\frac{2000}{450-r} + \frac{2000}{450+r} = 9$
We multiply both sides by the LCD.	$(450-r)(450+r)\left(\frac{2000}{450-r} + \frac{2000}{450+r}\right) = 9(450-r)(450+r)$
Simplify.	$2000(450+r) + 2000(450-r) = 9(450-r)(450+r)$
Factor the 2, 000.	$2000(450+r+450-r) = 9\left(450^2 - r^2\right)$
Solve.	$2000(900) = 9\left(450^2 - r^2\right)$
Divide by 9.	$2000(100) = 450^2 - r^2$
Simplify.	$200000 = 202500 - r^2$ $-2500 = -r^2$ \ $50 = r$ The speed of the jet stream is 50 mph.
Check: Is 50 mph a reasonable speed for the jet stream? Yes. If the plane is traveling 450 mph and the wind is 50 mph, Tailwind $450 + 50 = 500\text{mph} \quad \frac{2000}{500} = 4$ hours Headwind $450 - 50 = 400\text{mph} \quad \frac{2000}{400} = 5$ hours The times add to 9 hours, so it checks.	

The speed of the jet stream was 50 mph.

Exercise 5.8.11

MaryAnne just returned from a visit with her grandchildren back east. The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

Answer

The speed of the jet stream was 100 mph.

Exercise 5.8.12

Gerry just returned from a cross country trip. The trip was 3000 miles from his home and his total time in the airplane for the round trip was 11 hours. If the plane was flying at a rate of 550 miles per hour, what was the speed of the jet stream?

Answer

The speed of the jet stream was 50 mph.

Work applications can also be modeled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations.We'll use a similar scenario now.

Example 5.8.7

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 12 hours more than Press #2 to do the job and when both presses are running they can print the job in 8 hours. How long does it take for each press to print the job alone?

Solution:

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

Table 9.5.6

Let $x =$ the number of hours for Press #2 to complete the job. Enter the hours per job for Press #1, Press #2, and when they work together.	<table><tr><th></th><th>Number of hours needed to complete the job.</th><th>Part of job completed/hour</th></tr><tr><td>Press #1</td><td>$x+12$</td><td>$\frac{1}{x+12}$</td></tr><tr><td>Press #2</td><td>x</td><td>$\frac{1}{x}$</td></tr><tr><td>Together</td><td>8</td><td>$\frac{1}{8}$</td></tr></table>
The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together. Translate to an equation.	Work completed by Press #1 + Press #2 = Together $\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$
Solve.	$\frac{1}{x+12}+\frac{1}{x}=\frac{1}{8}$
Multiply by the LCD, $8x(x+12)$.	$8x(x+12)\left(\frac{1}{x+12}+\frac{1}{x}\right)=\left(\frac{1}{8}\right)8x(x+12)$
Simplify.	$8x+8(x+12)=x(x+12)$ $8x+8x+96=x^2+12x$ Figure 9.5.37 $0=x^2-4x-96$ Figure 9.5.38
Solve.	$0=(x-12)(x+8)$ $x-12=0, x+8=0$ Figure 9.5.40 $x=12, x=-8$ hours Figure 9.5.41
Since the idea of negative hours does not make sense, we use the values $x=12$.	$12+12$ 24 hours 12 12 hours Figure 9.5.43
Write our sentence answer.	Press #1 would take 24 hours and Press #2 would take 12 hours to do the job alone.

Exercise 5.8.13

The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours more than Press #2 to do the job and when both presses are running they can print the job in 4 hours. How long does it take for each press to print the job alone?

Answer

Press #1 would take 12 hours, and Press #2 would take 6 hours to do the job alone.

Exercise 5.8.14

Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes 3 hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in 2 hours. How long does it take for each hose to fill the hot tub?

Answer

The red hose take 6 hours and the green hose take 3 hours alone.

Access these online resources for additional instruction and practice with solving applications modeled by quadratic equations.

- Word Problems Involving Quadratic Equations
- Quadratic Equation Word Problems
- Applying the Quadratic Formula

Key Concepts

- Methods to Solve Quadratic Equations
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to use a Problem-Solving Strategy.
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- Area of a Triangle
  - For a triangle with base, b, and height, h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

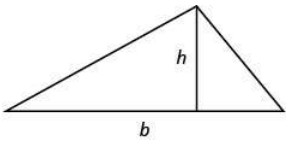


Figure 9.5.1

- Area of a Rectangle
  - For a rectangle with length,L, and width, W, the area, A, is given by the formula $A = LW$.

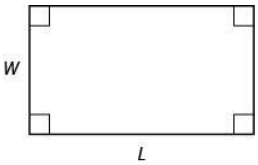


Figure 9.5.3

- Pythagorean Theorem
  - In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.

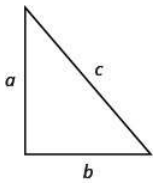


Figure 9.5.16

- Projectile motion
  - The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula $h = -16t^2 + v_0 t$.

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5.8.1: Exercises

Practice Makes Perfect

Exercise 5.8.1.15 Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method.

1. The product of two consecutive odd numbers is 255. Find the numbers.
2. The product of two consecutive even numbers is 360. Find the numbers.
3. The product of two consecutive even numbers is 624. Find the numbers.
4. The product of two consecutive odd numbers is 1,023. Find the numbers.
5. The product of two consecutive odd numbers is 483. Find the numbers.
6. The product of two consecutive even numbers is 528. Find the numbers.

Answer

1. Two consecutive odd numbers whose product is 255 are 15 and 17, and −15 and −17.

3. The first and second consecutive odd numbers are 24 and 26, and −26 and −24.

5. Two consecutive odd numbers whose product is 483 are 21 and 23, and −21 and −23.

Exercise 5.8.1.16 Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method. Round your answers to the nearest tenth, if needed.

1. A triangle with area 45 square inches has a height that is two less than four times the base Find the base and height of the triangle.
2. The base of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the base and height of the triangle.
3. The area of a triangular flower bed in the park has an area of 120 square feet. The base is 4 feet longer that twice the height. What are the base and height of the triangle?
4. A triangular banner for the basketball championship hangs in the gym. It has an area of 75 square feet. What is the length of the base and height , if the base is two-thirds of the height?
5. The length of a rectangular driveway is five feet more than three times the width. The area is 50 square feet. Find the length and width of the driveway.
6. A rectangular lawn has area 140 square yards. Its length is six yards less than twice its width. What are the length and width of the lawn?
7. A rectangular table for the dining room has a surface area of 24 square feet. The length is two more feet than twice the width of the table. Find the length and width of the table.
8. The new computer has a surface area of 168 square inches. If the the width is 5.5 inches less that the length, what are the dimensions of the computer?
9. The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.
10. The hypotenuse of a right triangle is 10 cm long. One of the triangle's legs is three times as the length of the other leg . Round to the nearest tenth. Find the lengths of the three sides of the triangle.
11. A rectangular garden will be divided into two plots by fencing it on the diagonal. The diagonal distance from one corner of the garden to the opposite corner is five yards longer than the width of the garden. The length of the garden is three times the width. Find the length of the diagonal of the garden.

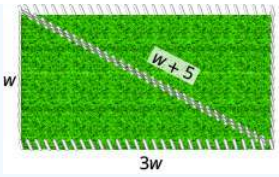


Figure 9.5.44

12. Nautical flags are used to represent letters of the alphabet. The flag for the letter, O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The hypotenuse of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.

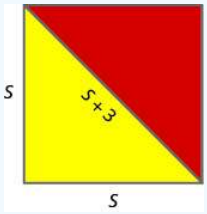


Figure 9.5.45

13. Gerry plans to place a 25-foot ladder against the side of his house to clean his gutters. The bottom of the ladder will be 5 feet from the house.How for up the side of the house will the ladder reach?

14. John has a 10-foot piece of rope that he wants to use to support his 8-foot tree. How far from the base of the tree should he secure the rope?

15. An arrow is shot vertically upward at a rate of $v_0 = 220$ feet per second. Use the projectile formula $h = -16t^2 + v_0 t$, to determine when the height of the arrow will be 400 feet.

16. A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when the height of the firework rocket will be 1200 feet.

17. A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h = -16t^2 + v_0 t + 8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h = 0$?)

18. A stone is dropped from a 196-foot platform. Use the formula $h = -16t^2 + v_0 t + 196$ to determine how many seconds it will take for the stone to hit the ground. (Since the stone is dropped, $v_0 = 0$.)

19. The businessman took a small airplane for a quick flight up the coast for a lunch meeting and then returned home. The plane flew a total of 4 hours and each way the trip was 200 miles. What was the speed of the wind that affected the plane which was flying at a speed of 120 mph?

20. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 300 miles. If the plane was flying at 125 mph, what was the speed of the wind that affected the plane?

21. Roy kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

22. Rick paddled up the river, spent the night camping, and then paddled back. He spent 10 hours paddling and the campground was 24 miles away. If Rick kayaked at a speed of 5 mph, what was the speed of the current?

23. Two painters can paint a room in 2 hours if they work together. The less experienced painter takes 3 hours more than the more experienced painter to finish the job. How long does it take for each painter to paint the room individually?

24. Two gardeners can do the weekly yard maintenance in 8 minutes if they work together. The older gardener takes 12 minutes more than the younger gardener to finish the job by himself. How long does it take for each gardener to do the weekly yard maintenance individually?

25. It takes two hours for two machines to manufacture $10,000$ parts. If Machine #1 can do the job alone in one hour less than Machine #2 can do the job, how long does it take for each machine to manufacture $10,000$ parts alone?

26. Sully is having a party and wants to fill his swimming pool. If he only uses his hose it takes 2 hours more than if he only uses his neighbor's hose. If he uses both hoses together, the pool fills in 4 hours. How long does it take for each hose to fill the pool?

Answer

1. The width of the triangle is 5 inches and the height is 18 inches.
3. The base is 24 feet and the height of the triangle is 10 feet.
5. The length of the driveway is 15.0 feet and the width is 3.3 feet.
7. The length of table is 8 feet and the width is 3 feet.
9. The length of the legs of the right triangle are 3.2 and 9.6 cm.
11. The length of the diagonal fencing is 7.3 yards.
13. The ladder will reach 24.5 feet on the side of the house.
15. The arrow will reach 400 feet on its way up in 2.2 seconds and on the way down in 11.6 seconds.
17. The bullet will take 70 seconds to hit the ground.
19. The speed of the wind was 49 mph.
21. The speed of the current was 4.3 mph.
23. The less experienced painter takes 6 hours and the experienced painter takes 3 hours to do the job alone.
25. Machine #1 takes 3.6 hours and Machine #2 takes 4.6 hours to do the job alone.

? Exercise 5.8.1.17 writing exercises

1. Make up a problem involving the product of two consecutive odd integers.
   a. Start by choosing two consecutive odd integers. What are your integers?
   b. What is the product of your integers?
   c. Solve the equation $n(n+2)=p$, where p is the product you found in part (b).
   d. Did you get the numbers you started with?
2. Make up a problem involving the product of two consecutive even integers.
   a. Start by choosing two consecutive even integers. What are your integers?
   b. What is the product of your integers?
   c. Solve the equation $n(n+2)=p$, where p is the product you found in part (b).
   d. Did you get the numbers you started with?

Answer

1. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

Figure 9.5.46

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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Glossary

Sample Word 1 | Sample Definition 1

Glossary

Sample Word 1 | Sample Definition 1

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