MATH 27: NUMBER SYSTEMS FOR EDUCATORS



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Math 27: Number Systems for Educators

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This text was compiled on 06/12/2025

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Glossary

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Licensing

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CHAPTER OVERVIEW

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1.1: How to Read this Textbook

A Note From the Editor

This textbook is comprised of several different texts on teaching mathematics for elementary educators. The style of writing may be different as the chapters switch between authors, and the activities may differ. The purpose of using multiple texts for this book is to ensure that you have good content that covers all the objectives for the course. Your instructor has the ultimate choice on the activities for the course, so make sure you understand their expectations for you throughout this semester.

The course is intended to help you reflect on your own learning in mathematics, to give you a stronger foundation of basic mathematics for elementary school teaching, and to begin organizing your own lesson plans around teaching mathematics based on the common core state standards. Enjoy learning about one of the most beautiful subjects we get to teach!

Introduction

This introduction chapter gives you a summary of the textbook, a brief history of Mathematics Education, information on Common Core and resources you can use in your future classroom. Use this chapter for your discussions on Canvas and your Lesson Plans for this class. Take the time to read it on your own as it is good background knowledge for when you are a teacher.

Each chapter is broken into Lessons. Most lessons have Partner Activities, which we will complete and discuss in class. After each lesson is the Practice Problems, which you will complete as homework and bring questions to our next meeting. Each chapter begins with The Essential Question(s). We start this way since many textbooks for Elementary schools also start this way. Think of the Essential Question as the Big Idea of the Chapter, the overarching goal of the chapter.

This book is made for you, the future teacher. Write in this book. That is how it is designed. Take notes. Complete your homework here. Keep everything in this book so you have something to take to your future classroom.

Each chapter ends with a Methods of Teaching Extension.

Appendices

A. The Common Core State Standards
   1. Covers Grades K through 6
   2. You will use this for a few assignments throughout the semester
   3. Do not memorize this enormous document, but rather know how to use it to find information.

B. The Mathematical Practices for Teachers
   1. Shows you how to teach the mathematics
   2. Think of this as the standards for Teachers, guidelines to help you teach well

C. Answers to Partner Activities, use if you are absent

D. Answers to Practice Problems (homework answers, not step-by-step solutions)

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1.2: The Importance of Mathematics for Elementary School Students

At the beginning of each chapter, we will discuss WHY the mathematics is taught in elementary school and WHY it is important for teachers to understand it, besides we need to understand it since we are going to teach it.

What is Common Core?

A common *Common Core* misconception is that Common Core is a way of teaching. No. Common Core is just a list of standards, which are required to be taught during the duration of the school year. HOW you teach it is up to your principal, vice principal, department chair, your team, etc.

The state of California has adopted the common core standards for teaching in all California schools. There are several resources online that detail the standards:

- The California Department of Education has published the common core math standards in an electronic booklet https://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf
- The Common Core State Standards Initiative (part of the Federal Department of Education) has an indexed website for easy searching. Be aware that California's adopted standards may differ from the national standards. http://www.corestandards.org/Math/

Why teach math?

Mathematics is the study of patterns! Patterns are everywhere, and children are especially adept at identifying patterns as their brains are developing and learning. Color, shape, and number patterns are studied in the early elementary years. As students progress into arithmetic operations, they make connections with pattern and number, counting and adding, and these connections set them up for success in future mathematics learning. Children are natural mathematicians. Their rapidly growing brains are constantly connecting past learning with new, and the acquisition of knowledge and experience is a very joyous period from birth to elementary school.

Teachers' beliefs have a profound influence on their students' achievements [1]. It is very important that we as teachers understand our beliefs about learning and ensure that these beliefs allow every student - independent of race, ethnicity, socio-economic status, mental or physical ability - to thrive in the classroom.

Throughout this book/course, consider what your beliefs about student learning are. Think about why you believe what you believe, and don't be afraid to change or adjust them as you learn more about learning and about teaching!

The National Council of Teachers of Mathematics (NCTM) states:

> A strong foundation in mathematics, for each and every student from pre-K–12, is vital to our nation's economic stability, national security, workforce productivity, and full participation in our democratic society. Mathematical literacy is fundamental for adult numeracy, financial literacy, and everyday life. [2]

Matt Larson, Associate Superintendent of Instruction at Lincoln Public Schools in Nebraska and past-president of NCTM, wrote an opinion article in 2018 [3] about why we should teach mathematics. In this article, he considers several different reasons from multiple sources, including:

- Mathematics for life - knowing math can be personally satisfying and empowering
- Mathematics as part of cultural heritage - specifically, "valuing and developing a better understanding of each other and each others cultures, including the multiple contributions various cultures have made to mathematics
- Mathematics for the workplace
- Mathematics for the Scientific and Technical Community

Mathematics teaching tends to emphasize the necessary mathematics. By shifting our teaching to encompass the other reasons, we are empowering our students to enjoy learning mathematics and valuing their identities within their respective cultures, including their identities as mathematicians.

Avoiding Math Anxiety

Math anxiety is a special form of anxiety linked to higher brain activity that causes a fear of doing a math task, which then leads to avoidance. Psychologist Mark H. Ashcraft defines math anxiety as "a feeling of tension, apprehension, or fear that interferes with math performance." [4] Unfortunately, due to societal misconceptions of mathematics and math learning, including gender stereotypes, negative attitudes, and assumed difficulty of math, math anxiety persists for far too many people. As students with math anxiety encounter more challenges, their anxiety increases and often leads to avoidance.

There is a lot of research underway to understand how math anxiety develops, and it is known that math anxiety develops early and tends to get worse, not better. One of the best ways to combat math anxiety is to understand how the brain works and to try to "reprogram" the brain to lessen the negative emotional response to mathematics through developing a growth mindset around learning and writing about your emotions. To help prevent our young students from developing math anxiety, it is important that we don't perpetuate the misconceptions of mathematics and we encourage mistakes as part of the learning process.

At LPC, the Math Department hosts the "Conquering Math Anxiety" Smart Shop. If you want to learn more about math anxiety, consider attending one of these workshops!

If you want to learn more about Growth Mindset, there are a lot of resources for teachers online. One of the best ones for math is YouCubed.org, a site established by Jo Boaler at Stanford University.

Sources:

1. Tatto, Maria Teresa. (2019). The Influence of Teacher Education on Teacher Beliefs. Oxford Research Encyclopedia of Education.
2. National Council of Teachers of Mathematics. Policies and Recommendations.
3. Larson, Matthew. (2018). Why Teach Mathematics? NCTM Blog.
4. Ashcraft, Mark H. (2002). Math Anxiety: Personal, Educational, and Cognitive Consequences. Current Directions in Psychological Science. American Psychological Society. 181-185.

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1.3: How Did We Get Here?

A Brief History of the Creation and Need for Common Core Mathematics

Somewhere around 2007 to 2008, a few mathematicians got together and decided that students are not prepared for the 21st century. They spoke with physicists, engineers, business people and the like to figure out what the current student needs in this world of having all the information of the entire world in their pocket. Remember when people used to say “You need to learn this because you won’t be walking around with a calculator in your pocket”? Those days are gone forever. Today, with the right technology, anyone can solve complex equations (i.e. photo math) graph anything (see graphing apps, as there are many) or even ask google to do it for him or her.

However, what about analytical thinking and problem solving? The original reason why mathematics became a school requirement was to teach students how to think. Technology has not figured out a way to problem solve. That is where Common Core steps in. Yes, we are still teaching the basics. Fractions are not going anywhere, sorry. Nevertheless, we cannot use technology to solve problems. A computer still is stupid. Seriously. It follows the instructions, which comes from a human being. A human being needs to correctly give the instructions to the computer. Common Core challenges students to solve problems conceptually and discover the meaning behind the calculator. For example, think about dividing fractions. A trick that many people have learned is to change the division to multiplication and flip the second term. Then multiply. But why does that work? (We will explore this in Chapter 3.) Another example which Common Core wanted to stress is Mental Math. Make people smarter by forcing them to learn mental math tricks. Think about it, how do you train your body to run a marathon? You run a little bit each day and gradually improve your mileage and then gradually improve your speed. Your body is improving each day that you practice. The brain is the same. Work on mental math activities a little bit each day, and you are training your brain to be stronger and a better thinking. Math trains you to be smarter in general, which can help you master all other subjects.

The Impact of Common Core in Elementary School Mathematics

When Common Core first arrived it was daunting, from a teacher’s and parent’s perspective. Photos were floating around social media about bad math techniques. Parents and students disagreed on how to solve a problem. It was joked about in Disney’s *Incredibles 3*, where the Dad is trying to help his son do his homework and he becomes very frustrated that “they” changed math. The way subtraction is taught has changed. The way multiplication is taught has changed. The authors wanted students to understand the meaning behind subtraction and multiplication instead of just rote memorization. The biggest change was pushing down the curriculum. What students are learning now in elementary is around two years earlier than students before Common Core. Kindergarten is fully academic, no longer just learning to be social and learning how to play nice.

How has Common Core changed assessments?

The biggest change is having students complete Performance Tasks during their exam. The first part of an exam can be basic questions. The next part could have error analysis (find the mistake). There could be a matching portion as well. There are many methods to access students without having multiple-choice answers or simply asking to solve a math problem without any context to why. Then there are the Performance Tasks, which always show up on State Standardized Testing. A typical Performance Task starts with a situation, a more involved word problem. Students then have to go through a process of steps, answering questions along the way to eventually reach the overall question. There could be solving and graphing involved in the same Performance Task.

How has Common Core changed grading?

Another large change brought by Common Core is how teachers grade assessments. Before, we cared about the final answer. Right or wrong? If wrong, a teacher would look for ways to give partial credit. Now, we look at everything and assign points accordingly. For example, say a student is asked to perform a long division problem worth five points. Getting the right answer is one point out of five. HOW they got to the answer is worth four points. Did they move the decimal correctly? Did they subtract correctly? Did they make a mistake twice and ended up being lucky and getting to the right answer? Did they check their work by multiplying back?

Comparing the Old Traditional Standards to the New Common Core Standards: Kindergarten through 3rd Grade

The major change to 4th through 6th grade are more pre-algebra standards.

Information in the table is pulled from corestandards.org.

Table 1.3.1: Kindergarten

New: Common Core Standards	Old: California State Standards
Count to 100 by ones and 10s.	Count to 30 by ones.
Identify whether the number of objects in one group is greater than, less than or equal to the number of objects in another group.	Compare two or more sets of objects and identify which set is equal to, more than or less than the other.
Solve word problems that require addition and subtraction for problems with sums up to and including 10. Use objects or drawings to represent the problem.	Use objects to determine the answers to addition and subtraction problems.
Break numbers between 11 and 19 into two parts: 10 ones and some further ones. For example, 17 contains 10 ones and seven additional ones.	Not taught until 1st grade.
Put two shapes together to form a different shape. For example, place two triangles together to make a rectangle.	Not taught. New standard.
Moved to first grade.	Tell time and recite the days of the week.

Table 1.3.2: Grade 1

New: Common Core Standards	Old: California State Standards
Solve word problems that call for addition of three numbers whose sum is less than or equal to 20 by using objects, drawings or equations.	Replaces the old standard that required students to "commit to memory" addition equations with a sum of 20 or less and subtraction equations with a difference of 20 or less. Notably, the new standard does not require memorization.
Apply properties of operations (commutative and associative) as strategies to add and subtract.	New to 1st grade. The associative property (a + (b + c) = (a +b) + c) was previously introduced in 2nd grade. The commutative property (a + b = b + a) was not mentioned in the state standards for kindergarten through 3rd grade.
Determine the unknown number in an addition or subtraction equation. For example, 8 + x = 11.	New standard. Previously, students were not expected to find for "x" until after 3rd grade.
Given a two-digit number, mentally find 10 more or 10 less than that number, without having to count. Explain your reasoning.	Similar to the old standard that asked students to identify one more than, one less than, 10 more than and 10 less than a given number.
Tell and write time in hours and half hours using analog and digital clocks.	Previously introduced in kindergarten, although 1st grade students were also required to tell time to the nearest half hour under the old standards.
Eliminates previous 1st grade standards requiring students to understand weight, volume and the monetary value of coins, among other specific skills found in the old 1st grade standards. Also eliminates requirement that students memorize sets of numbers.	The old standards called for 1st grade students to work with weight, volume, classifying objects by color and size, estimating sums, committing math facts to memory, writing number sentences and understanding the value of coins.

Table 1.3.3: Grade 2

New: Common Core Standards	Old: California State Standards
Use addition and subtraction to solve one- and two-step word problems where the sum or difference is less than 100.	New standard. Multi-step word problems were not mentioned in the old standards for kindergarten through 3rd grade.
Easily solve addition and subtraction problems with a sum or difference of less than 20 in your head (7 - 4 = 3; 2 + 9 = 11; 14 + 3 = 17; etc.). By end of grade 2, memorize all the ways to add two one-digit numbers.	Similar to the old standard that required students to find the sum or difference of two, two-digit numbers in their heads (14 + 16 = 30; 12 + 5 = 17; 32 - 7 = 25). However, the old standard required students to add and subtract larger numbers.
Use addition to find the total number of objects arranged in equal rows. For example, if there are three rows of four, students should be able to add 4+4+4 to find the total, rather than counting each object. Students must also write an equation to represent how they found the total number of objects.	Replaces the old standards that focused on multiplication and division, which required students to do "repeated" addition and subtraction (2 + 2 + 2 + 2 = 8), form equal groups from a set of objects (sort eight blocks into four groups of two) and know the multiplication tables for twos, fives and 10s.
Explain strategies that can be used to make addition and subtraction easier. For example, for the problem 14 - 5 = x, a student might know that 15 - 5 = 10. Because 14 is one less than 15, the student could figure out that answer to the problem presented would also therefore be one less, or 14 - 5 = 9. Students should be able to use these strategies and explain why they work.	New standard. The old standards did not call for student explanations of specific math processes, nor did they call for specific instruction in developing strategies to add and subtract faster. Instead, a standard at the end of each grade level called for students to be able to "justify their reasoning" in general.
Understand that each number falls a set distance from zero on a number line. Use a diagram of a number line to find sums and differences of less than 100. For example, for the problem 82 - 17 = x, start at 82 on the number line and count down 17 spaces to determine that 82 - 17 = 65.	New standard. Common Core's focus on teaching children about the number line comes from research showing that familiarity with number lines improves mathematical performance in young children.
In a first introduction to the concept of area, students should understand that a rectangle can be made up of a grid of smaller squares. Count the squares that fit within a rectangle to find the total number of squares.	New standard. Fractions were introduced in 2nd grade under the old standards, but not in this format. Area was not introduced.

Table 1.3.4: Grade 3

New: Common Core Standards	Old: California State Standards
Students are required to solve two-step word problems using addition, subtraction, multiplication or division as needed.	Solve problems using two or more operations (addition, subtraction, multiplication or division), but does not specify solving word problems.
Understand that a fraction can be represented on a number line between zero and 1.	New standard. Previously, students were required to add and subtract fractions, but no mention is made of understanding that a fraction is less than 1.
Tell and write time to the nearest minute.	New to 3rd grade. Previously, this was a 2nd grade standard.
Measure and estimate liquid volumes and masses of objects.	Similar to old standard calling for students to estimate and measure the length, liquid volume and mass of an object.
Recognize that shapes in different categories (i.e. squares and rectangles) can share attributes (i.e. both have four sides) and that those shared attributes can define a larger category (i.e. quadrilaterals).	Similar to old standard calling for students to identify the attributes of quadrilaterals (i.e. parallel sides for a parallelogram, equal side lengths for a square), but with emphasis on how to sort shapes instead of on specific rules about shapes.

New: Common Core Standards	Old: California State Standards
Multiplication tables are not mentioned in the new standards.	Previously, students were required to memorize the multiplication tables for number 1-10.

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1.4: Showing Students' Thinking

Ways to Help Students “Show Their Thoughts” on Paper

Here are six methods, which can help students organize their thoughts. It is recommended that the teacher says to the students: “Show your thoughts” instead of “Show your work”. This change encourages students to write more on the paper.

Method 1: Inquiry

Before you teach a new concept, have students fill out the Inquiry Box below. Have the students write or draw any observations. (I see…) What inferences do they have? (I think…) Describe your prior knowledge. (I know…)

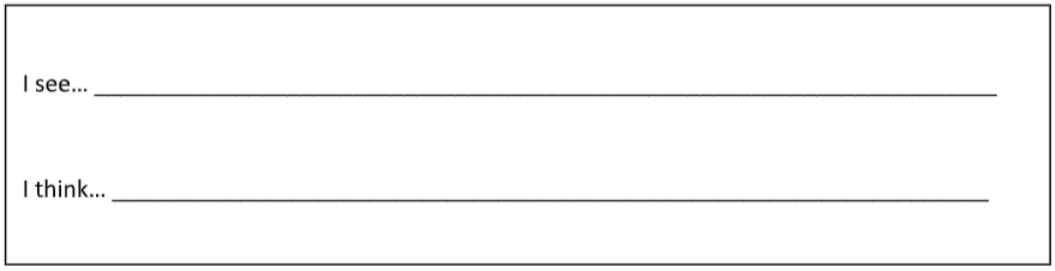


Figure 1.3.1: Inquiry Box

Method 2: Four-Square Chart

Use to review learned vocabulary.

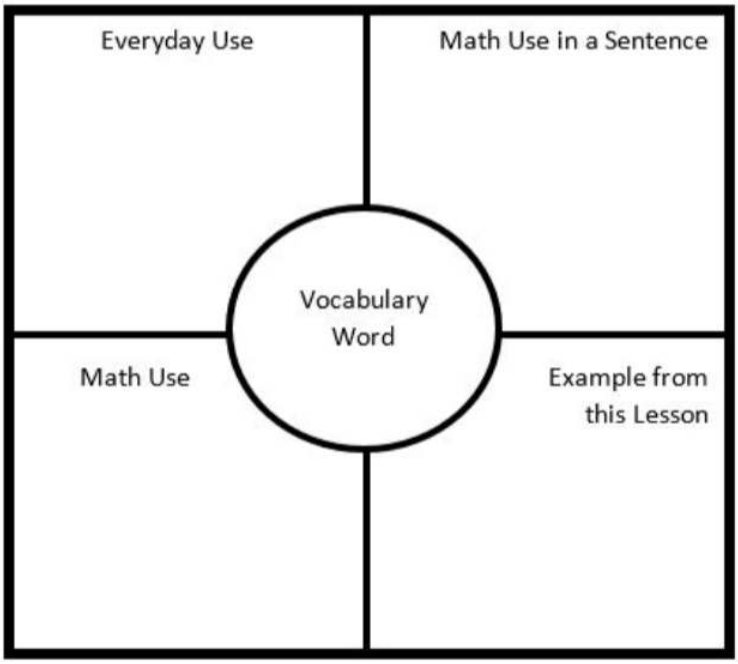


Figure 1.3.2: Four-Square Chart

Method 3: Note Taking

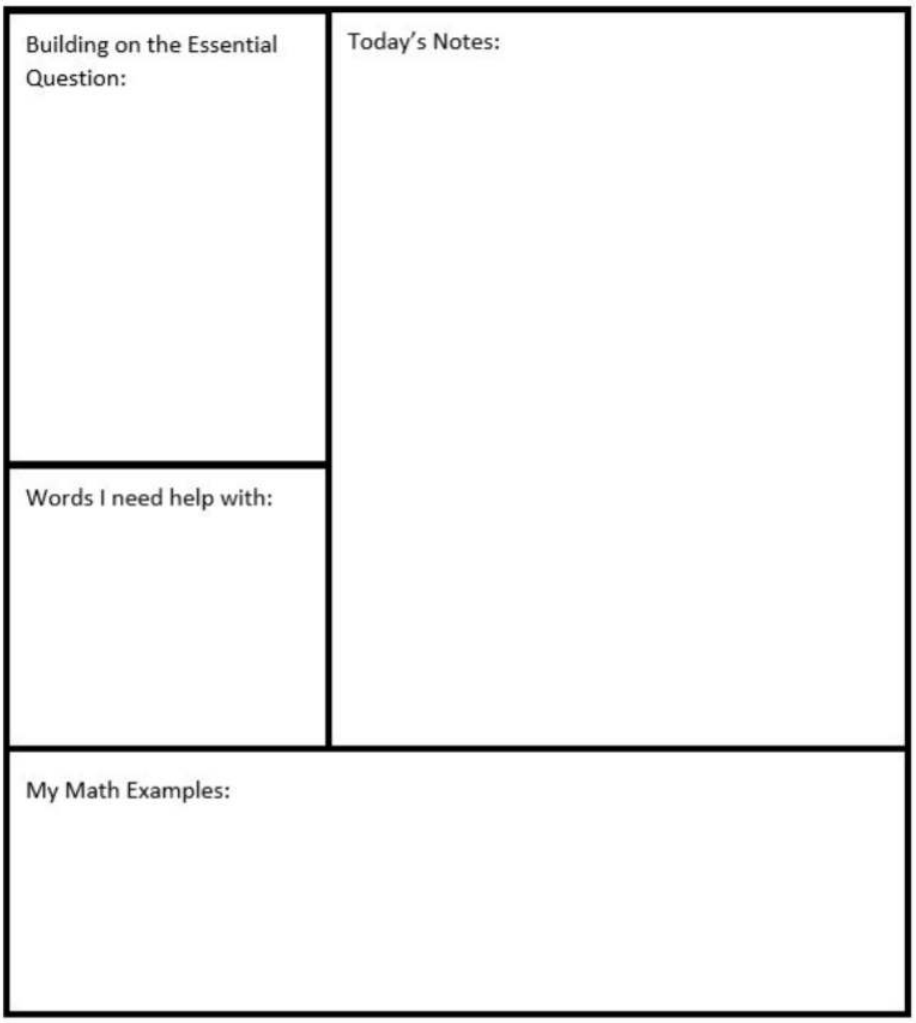


Figure 1.3.3: Sample Note taking

Method 4: Concept Web

Use a Concept Web to solve different types of problems which all have something in common. For example, “Fill in the Blank” can have students look at different patterns and fill in the missing piece of the pattern.

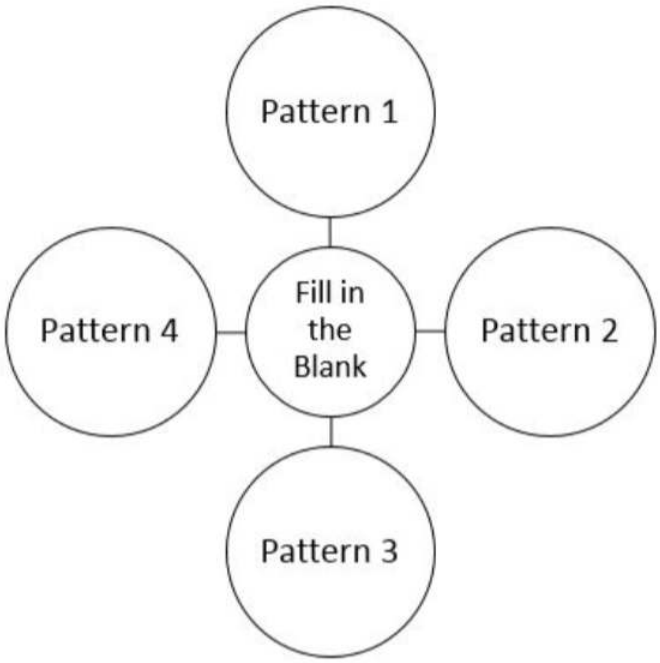


Figure 1.3.4: Concept Web

Method 5: Problem-Solving Strategy

Given a word problem, students can fill out this table to help them decode the word problem and figure out how to set up and solve the problem.

Understand	Solve
Plan	Check

Figure 1.3.5: Table to decode a word problem

Method 6: A Five-Step Formula

Use this method to solve complicated word problems.

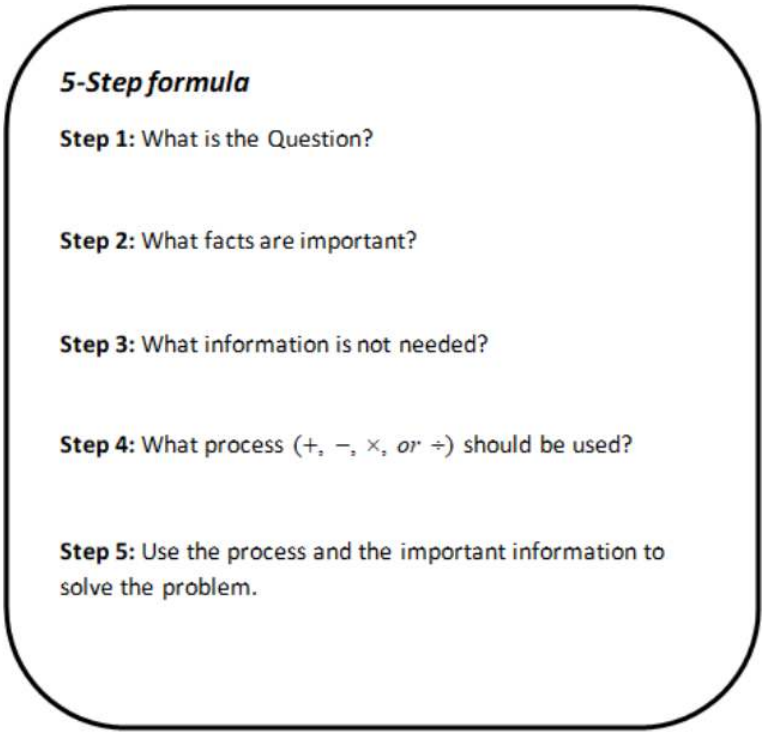


Figure 1.3.6: Five-Step formula

Classroom Instruction and Discussions

Method 1: Elbow Buddies

Kids get a kick out of this method. “Now talk with your elbow buddy,” is an instruction where they can only talk with the person who sits next to them, touching elbow to elbow. This works well when the students are seated in pairs or groups of four.

Method 2: Think-Pair-Share

This method is very popular, easy, and has been around for a long time – because it works! To start a deep discussion with your students they need to be prepared.

Pose the question to your students. Have them take some predetermined time to **THINK** about their answer. No talking. No writing. No researching. Just thinking. Have them share their thoughts with their Elbow Buddy. Only a **PAIR** of students should be whispering their thoughts back and forth. Then ask the class to raise their hands if they would like to **SHARE** what they thought of or what their Elbow Buddy thought of.

Method 3: Cooperative Learning

Cooperative Learning is very important in Common Core mathematics learning. Students need to share ideas. Having students work on group projects and group presentations will help them be successful in high school and beyond.

Method 4: Direct Instruction

Direct instruction is when the teacher is in the front of the classroom and the students are silently taking notes. There is very little interaction.

One of the founders/authors of the original Common Core idea said that Direct Instruction should be about 20% of total learning. 80% of the time, students should be working in some other fashion. Use direct instruction when the students need to learn a difficult topic.

Method 5: Guided Instruction

Guided Instruction is better than Direct Instruction. With Guided Instruction, the students are interacting with the teacher. The teacher will ask questions to lead the instruction and learning.

Method 6: Inquiry

With Inquiry, students come up with their own questions about what is presented in front of them. Most of the time, this method is used in Science, but it can also be used in mathematics.

Technology Resources for Your Classroom

And it is all FREE!! Or most times, the school will have a subscription, so it is free for you.

Khan Academy

Khan Academy is more than just mathematics. Many subjects are found on this free website. Not only will it be a semester long project for you, but I hope you use this website in your future classroom to help your future students keep up in mathematics.

See Canvas for Math 130's instructions for your semester long project

Google Classroom and Google Forms

In this technology society, it makes sense for students to take exams online. With Google Classroom, a Chromebook and Google Forms, students can take an exam online; where Google locks the Chromebook. Students are not able to visit any other websites while taking the assessment. If they try, Google will cut their assessment score in half and send the teacher an email.

You Tube

It has been said that one can learn anything on YouTube. And it is true. Use YouTube in your classroom to show videos which might help the students "by-in" to what you are trying to teach them.

edPuzzle

This is a wonderful resource. Teachers and students can use to make videos.

Dreambox

This is a fun game based website where the students have to solve grade level appropriate math problems to advance on to the next level and earn gold coins.

ABCya!

Another game based website where the students must solve the math problem to advance. Here the students' games can be assigned by Common Core standards.

Zipgrades

Zipgrades are the new version of scantrons. You use their paper (free download) and your phone to grade their scantron. Works great most of time, to be honest. But if your school does not have a scantron machine, this could be the next best option.

Desmos

Desmos.com is a mathematicians dream come true. It can graph virtually anything. This is a great teaching tool if your classroom has a smart board and you are teaching pre-algebra in 6th grade.

Geometers Sketchpad

Another great tool for teachers. Easily make geometry shapes. Ask your school to buy a license.

Kuta Software

Kuta makes "drill and kill" worksheets for grades 5 and 6 (it is mostly used for junior high and high school teachers.) Free, already made, worksheets can be found at kutasoftware.com. To make your own worksheets, have your school buy the license.

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CHAPTER OVERVIEW

2: Problem Solving

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- 2.2: Problem or Exercise?
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- 2.4: Beware of Patterns!
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2.1: Introduction to Problem Solving

The Common Core State Standards for Mathematics (http://www.corestandards.org/Math/Practice) identify eight "Mathematical Practices" — the kinds of expertise that all teachers should try to foster in their students, but they go far beyond any particular piece of mathematics content. They describe what mathematics is really about, and why it is so valuable for students to master. The very first Mathematical Practice is:

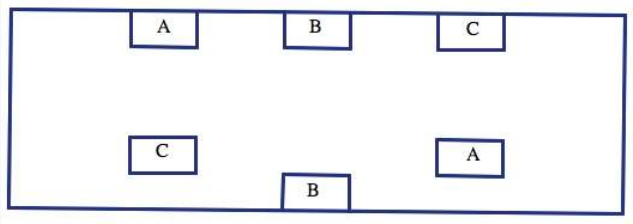
Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary.

This chapter will help you develop these very important mathematical skills, so that you will be better prepared to help your future students develop them. Let's start with solving a problem!

✔ (ABC)

Draw curves connecting A to A, B to B, and C to C. Your curves cannot cross or even touch each other,they cannot cross through any of the lettered boxes, and they cannot go outside the large box or even touch it's sides.



? Think / Pair / Share

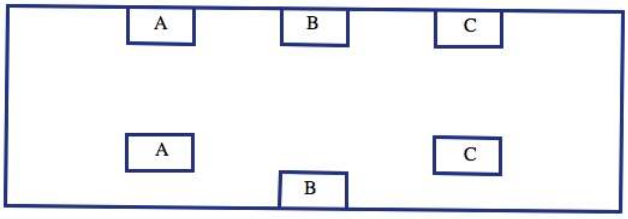
After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it).

- What did you try?
- What makes this problem difficult?
- Can you change the problem slightly so that it would be easier to solve?

Problem Solving Strategy 1 (Wishful Thinking).

Do you wish something in the problem was different? Would it then be easier to solve the problem?

For example, what if ABC problem had a picture like this:



Can you solve this case and use it to help you solve the original case? Think about moving the boxes around once the lines are already drawn.

Here is one possible solution.



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2.2: Problem or Exercise?

The main activity of mathematics is solving problems. However, what most people experience in most mathematics classrooms is practice exercises. An exercise is different from a problem.

In a **problem**, you probably don't know at first how to approach solving it. You don't know what mathematical ideas might be used in the solution. Part of solving a problem is understanding what is being asked, and knowing what a solution should look like. Problems often involve false starts, making mistakes, and lots of scratch paper!

In an **exercise**, you are often practicing a skill. You may have seen a teacher demonstrate a technique, or you may have read a worked example in the book. You then practice on very similar assignments, with the goal of mastering that skill.

> **Note**
>
> What is a problem for some people may be an exercise for other people who have more background knowledge! For a young student just learning addition, this might be a problem:
>
> $$\textit{Fill in the blank to make a true statement } \_\_\_ + 4 = 7.$$
>
> But for you, that is an exercise!

Both problems and exercises are important in mathematics learning. But we should never forget that the ultimate goal is to develop more and better skills (through exercises) so that we can solve harder and more interesting problems.

Learning math is a bit like learning to play a sport. You can practice a lot of skills:

- hitting hundreds of forehands in tennis so that you can place them in a particular spot in the court,
- breaking down strokes into the component pieces in swimming so that each part of the stroke is more efficient,
- keeping control of the ball while making quick turns in soccer,
- shooting free throws in basketball,
- catching high fly balls in baseball,

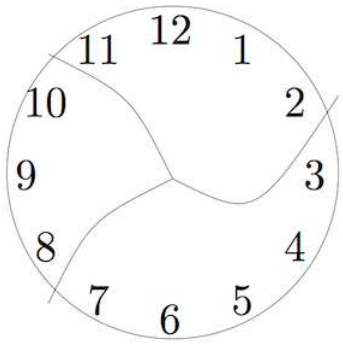
and so on.

But the point of the sport is to play the game. You practice the skills so that you are better at playing the game. In mathematics, solving problems is playing the game!

On Your Own

For each question below, decide if it is a *problem* or an *exercise*. (You do not need to solve the problems! Just decide which category it fits for you.) After you have labeled each one, compare your answers with a partner.

1. This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers.(Note: **Consecutive numbers** are whole numbers that appear one after the other, such as 1, 2, 3, 4 or 13, 14, 15.)



Can you break another clock into a different number of pieces so that the sums are consecutive numbers? Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2).

2. A soccer coach began the year with a $500 budget. By the end of December, the coach spent $450. How much money in the budget was not spent?
3. What is the product of 4,500 and 27?
4. Arrange the digits 1–6 into a “difference triangle” where each number in the row below is the difference of the two numbers above it.
5. Simplify the following expression:

$$\frac{2+2(5^3-4^2)^5-2^2}{2(5^3-4^2)}. \tag{2.2.1}$$

6. What is the sum of $\frac{5}{2}$ and $\frac{3}{13}$?
7. You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?



8. How many squares, of any possible size, are on a standard 8 × 8 chess board?
9. What number is 3 more than half of 20?
10. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1’s are separated by one digit, the 2’s are separated by two digits, the 3’s by three digits, and the 4’s by four digits.

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2.3: Problem Solving Strategies

Think back to the first problem in this chapter, the ABC Problem. What did you do to solve it? Even if you did not figure it out completely by yourself, you probably worked towards a solution and figured out some things that *did not* work.

Unlike exercises, there is never a simple recipe for solving a problem. You can get better and better at solving problems, both by building up your background knowledge and by simply practicing. As you solve more problems (and learn how other people solved them), you learn strategies and techniques that can be useful. But no single strategy works every time.

How to Solve It

George Pólya was a great champion in the field of *teaching*effective problem solving skills. He was born in Hungary in 1887, received his Ph.D. at the University of Budapest, and was a professor at Stanford University (among other universities). He wrote many mathematical papers along with three books, most famously, "How to Solve it." Pólya died at the age 98 in 1985. [1]



George Pólya, circa 1973

1. Image of Pólya by Thane Plambeck from Palo Alto, California (Flickr) [CC BY 2.0 (http://creativecommons.org/licenses/by/2.0)], via Wikimedia Commons ↵

In 1945, Pólya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems:

1. First, you have to understand the problem.
2. After understanding, then make a plan.
3. Carry out the plan.
4. Look back on your work. How could it be better?

This is all well and good, but how do you actually do these steps?!?! Steps 1. and 2. are particularly mysterious! How do you "make a plan?" That is where you need some tools in your toolbox, and some experience to draw upon.

Much has been written since 1945 to explain these steps in more detail, but the truth is that they are more art than science. This is where math becomes a creative endeavor (and where it becomes so much fun). We will articulate some useful problem solving strategies, but no such list will ever be complete. This is really just a start to help you on your way. The best way to become a skilled problem solver is to learn the background material well, and then to solve a lot of problems!

We have already seen one problem solving strategy, which we call "Wishful Thinking." Do not be afraid to change the problem! Ask yourself "what if" questions:

- What if the picture was different?
- What if the numbers were simpler?
- What if I just made up some numbers?

You need to be sure to go back to the original problem at the end, but wishful thinking can be a powerful strategy for getting started.

This brings us to the most important problem solving strategy of all:

Problem Solving Strategy 2 (Try Something!).

If you are really trying to solve a problem, the whole point is that you do not know what to do right out of the starting gate. You need to just try something! Put pencil to paper (or stylus to screen or chalk to board or whatever!) and try something. This is often an important step in understanding the problem; just mess around with it a bit to understand the situation and figure out what is going on.

And equally important: If what you tried first does not work, try something else! Play around with the problem until you have a feel for what is going on.

✔ (Payback)

Last week, Alex borrowed money from several of his friends. He finally got paid at work, so he brought cash to school to pay back his debts. First he saw Brianna, and he gave her 1/4 of the money he had brought to school. Then Alex saw Chris and gave him 1/3 of what he had left after paying Brianna. Finally, Alex saw David and gave him 1/2 of what he had remaining. Who got the most money from Alex?

? Think/Pair/Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What did you figure out about the problem? This problem lends itself to two particular strategies. Did you try either of these as you worked on the problem? If not, read about the strategy and then try it out before watching the solution.

Problem Solving Strategy 3 (Draw a Picture).

Some problems are obviously about a geometric situation, and it is clear you want to draw a picture and mark down all of the given information before you try to solve it. But even for a problem that is not geometric, like this one, thinking visually can help! Can you represent something in the situation by a picture?

Draw a square to represent all of Alex's money. Then shade 1/4 of the square — that's what he gave away to Brianna. How can the picture help you finish the problem?

After you have worked on the problem yourself using this strategy (or if you are completely stuck), you can watch someone else's solution.



Problem Solving Strategy 4 (Make Up Numbers).

Part of what makes this problem difficult is that it is about money, but there are no numbers given. That means the numbers must not be important. So just make them up!

You can work forwards: Assume Alex had some specific amount of money when he showed up at school, say $100. Then figure out how much he gives to each person. Or you can work backwards: suppose he has some specific amount left at the end, like $10.

Since he gave Chris half of what he had left, that means he had $20 before running into Chris. Now, work backwards and figure out how much each person got.

Watch the solution only after you tried this strategy for yourself.



If you use the "Make Up Numbers" strategy, it is really important to remember what the original problem was asking! You do not want to answer something like "Everyone got $10." That is not true in the original problem; that is an artifact of the numbers you made up. So after you work everything out, be sure to re-read the problem and answer what was asked!

✔ (Squares on a Chess Board)

How many squares, of any possible size, are on a 8 × 8 chess board? (The answer is not 64... It's a lot bigger!)

Remember Pólya's first step is to understand the problem. If you are not sure what is being asked, or why the answer is not just 64, be sure to ask someone!

? Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What did you figure out about the problem, even if you have not solved it completely?

It is clear that you want to draw a picture for this problem, but even with the picture it can be hard to know if you have found the correct answer. The numbers get big, and it can be hard to keep track of your work. Your goal at the end is to be absolutely positive that you found the right answer. You should never ask the teacher, "Is this right?" Instead, you should declare, "Here's my answer, and here is why I know it is correct!"

Problem Solving Strategy 5 (Try a Simpler Problem).

Pólya suggested this strategy: "If you can't solve a problem, then there is an easier problem you can solve: find it." He also said: "If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?" In this case, an 8 × 8 chess board is pretty big. Can you solve the problem for smaller boards? Like 1 × 1? 2 × 2? 3 × 3?

Of course the ultimate goal is to solve the original problem. But working with smaller boards might give you some insight and help you devise your plan (that is Pólya's step (2)).

Problem Solving Strategy 6 (Work Systematically).

If you are working on simpler problems, it is useful to keep track of what you have figured out and what changes as the problem gets more complicated.

For example, in this problem you might keep track of how many 1 × 1 squares are on each board, how many 2 × 2 squares on are each board, how many 3 × 3 squares are on each board, and so on. You could keep track of the information in a table:

size of board	# of 1 × 1 squares	# of 2 × 2 squares	# of 3 × 3 squares	# of 4 × 4 squares	…
1 by 1	1	0	0	0	
2 by 2	4	1	0	0	
3 by 3	9	4	1	0	
…					

Problem Solving Strategy 7 (Use Manipulatives to Help You Investigate).

Sometimes even drawing a picture may not be enough to help you investigate a problem. Having actual materials that you move around can sometimes help a lot!

For example, in this problem it can be difficult to keep track of which squares you have already counted. You might want to cut out 1 × 1 squares, 2 × 2 squares, 3 × 3 squares, and so on. You can actually move the smaller squares across the chess board in a systematic way, making sure that you count everything once and do not count anything twice.



Problem Solving Strategy 8 (Look for and Explain Patterns).

Sometimes the numbers in a problem are so big, there is no way you will actually count everything up by hand. For example, if the problem in this section were about a 100 × 100 chess board, you would not want to go through counting all the squares by hand! It would be much more appealing to find a pattern in the smaller boards and then extend that pattern to solve the problem for a 100 × 100 chess board just with a calculation.

? Think / Pair / Share

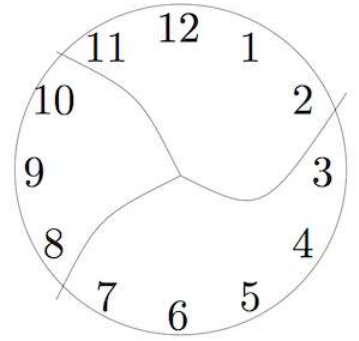
If you have not done so already, extend the table above all the way to an 8 × 8 chess board, filling in all the rows and columns. Use your table to find the total number of squares in an 8 × 8 chess board. Then:

- Describe all of the patterns you see in the table.
- Can you explain and justify any of the patterns you see? How can you be sure they will continue?
- What calculation would you do to find the total number of squares on a 100 × 100 chess board?

(We will come back to this question soon. So if you are not sure right now how to explain and justify the patterns you found, that is OK.)

✓ (Broken Clock)

This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. (**Consecutive numbers** are whole numbers that appear one after the other, such as 1, 2, 3, 4 or 13, 14, 15.)



Can you break another clock into a different number of pieces so that the sums are consecutive numbers? Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2.)

Remember that your first step is to understand the problem. Work out what is going on here. What are the sums of the numbers on each piece? Are they consecutive?

? Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What progress have you made?

Problem Solving Strategy 9 (Find the Math, Remove the Context).

Sometimes the problem has a lot of details in it that are unimportant, or at least unimportant for getting started. The goal is to find the underlying math problem, then come back to the original question and see if you can solve it using the math.

In this case, worrying about the clock and exactly how the pieces break is less important than worrying about finding consecutive numbers that sum to the correct total. Ask yourself:

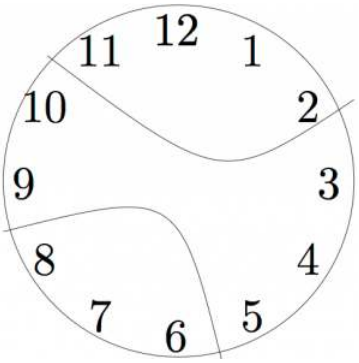
- What is the sum of all the numbers on the clock's face?
- Can I find two consecutive numbers that give the correct sum? Or four consecutive numbers? Or some other amount?
- How do I know when I am done? When should I stop looking?

Of course, solving the question about consecutive numbers is not the same as solving the original problem. You have to go back and see if the clock can actually break apart so that each piece gives you one of those consecutive numbers. Maybe you can solve the math problem, but it does not translate into solving the clock problem.

Problem Solving Strategy 10 (Check Your Assumptions).

When solving problems, it is easy to limit your thinking by adding extra assumptions that are not in the problem. Be sure you ask yourself: Am I constraining my thinking too much?

In the clock problem, because the first solution has the clock broken radially (all three pieces meet at the center, so it looks like slicing a pie), many people assume that is how the clock must break. But the problem does not require the clock to break radially. It might break into pieces like this:



Were you assuming the clock would break in a specific way? Try to solve the problem now, if you have not already.

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2.3.1: George Polya's Four Step Problem Solving Process

Step 1: Understand the Problem

- Do you understand all the words?
- Can you restate the problem in your own words?
- Do you know what is given?
- Do you know what the goal is?
- Is there enough information?
- Is there extraneous information?
- Is this problem similar to another problem you have solved?

Step 2: Devise a Plan: Below are some strategies one might use to solve a problem. Can one (or more) of the following strategies be used? (A strategy is defined as an artful means to an end.)

1. Guess and test.	11. Solve an equivalent problem.
2. Use a variable.	12. Work backwards
3. Draw a picture.	13. Use cases.
4. Look for a pattern.	14. Solve an equation.
5. Make a list.	15. Look for a formula.
6. Solve a simpler problem.	16. Do a simulation.
7. Draw a diagram.	17. Use a model
8. Use direct reasoning.	

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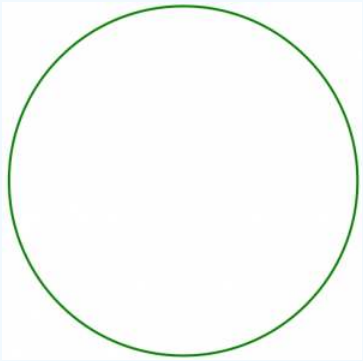
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2.4: Beware of Patterns!

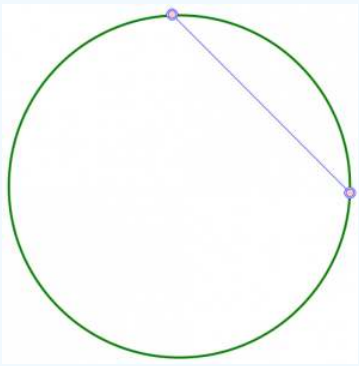
The “Look for Patterns” strategy can be particularly appealing, but you have to be careful! Do not forget the “and Explain” part of the strategy. Not all patterns are obvious, and not all of them will continue.

(Dots on a Circle)

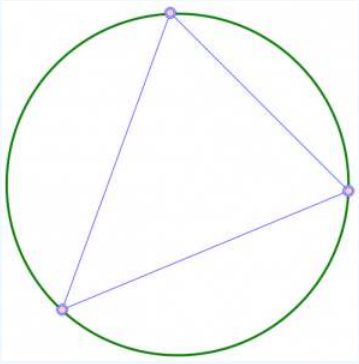
Start with a circle.



If I put two dots on the circle and connect them, the line divides the circle into two pieces.



If I put three dots on the circle and connect each pair of dots, the lines divides the circle into four pieces.



Suppose you put one hundred dots on a circle and connect each pair of dots, meaning every dot is connected to 99 other dots. How many pieces will you get? Lines may cross each other, but assume the points are chosen so that three or more lines never meet at a single point.

? Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What strategies did you try? What did you figure out? What questions do you still have?

The natural way to work on this problem is to use smaller numbers of dots and look for a pattern, right? If you have not already, try it. How many pieces when you have four dots? Five dots? How would you describe the pattern?

Now try six dots. You will want to draw a big circle and space out the six dots to make your counting easier. Then carefully count up how many pieces you get. It is probably a good idea to work with a partner so you can check each other's work. Make sure you count every piece once and do not count any piece twice. How can you be sure that you do that?

Were you surprised? For the first several steps, it seems to be the case that when you add a dot you double the number of pieces. But that would mean that for six dots, you should get 32 pieces, and you only get 30 or 31, depending on how the dots are arranged. No matter what you do, you cannot get 32 pieces. The pattern simply does not hold up.

Mathematicians love looking for patterns and finding them. We get excited by patterns. But we are also very skeptical of patterns! If we cannot explain why a pattern would occur, then we are not willing to just believe it.

For example, if my number pattern starts out: 2, 4, 8, ... I can find lots of ways to continue the pattern, each of which makes sense in some contexts. Here are some possibilities:

- 2, 4, 8, 2, 4, 8, 2, 4, 8, 2, 4, 8, ...

This is a a repeating pattern, cycling through the numbers 2, 4, 8 and then starting over with 2.

- 2, 4, 8, 32, 256, 8192, ...

To get the next number, multiply the previous two numbers together.

- 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
- 2, 4, 8, 14, 22, 32, 44, 58, 74 ...

? Think / Pair / Share

- For the last two patterns above, describe in words how the number sequence is being created.
- Find at least two other ways to continue the sequence 2, 4, 8, . . . that looks different from all the ones you have seen so far. Write your rule in words, and write the next five terms of the number sequence.

So how can you be sure your pattern fits the problem? You have to tie them together! Remember the "Squares on a Chess Board" problem? You might have noticed a pattern like this one:

If the chess board has 5 squares on a side, then there are

- 5 × 5 = 25 squares of size 1 × 1.
- 4 × 4 = 16 squares of size 2 × 2.
- 3 × 3 = 9 squares of side 3 × 3.
- 2 × 2 = 4 squares of size 4 × 4.
- 1 × 1 = 1 squares of size 5 × 5.

So there are a total of

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

squares on a 5 × 5 chess board. You can probably guess how to continue the pattern to any size board, but how can you be absolutely sure the pattern continues in this way? What if this is like "Dots on a Circle," and the obvious pattern breaks down after a few steps? You have to tie the pattern to the problem, so that it is clear why the pattern must continue in that way.

The first step in explaining a pattern is writing it down clearly. This brings us to another problem solving strategy.

Problem Solving Strategy 11 (Use a Variable!).

One of the most powerful tools we have is the use of a variable. If you find yourself doing calculations on things like "the number of squares," or "the number of dots," give those quantities a name! They become much easier to work with.

? Think/Pair/Share

For now, just work on describing the pattern with variables.

- Stick with a 5 × 5 chess board for now, and consider a small square of size $k \times k$. Describe the pattern: How many squares of size $k \times k$ fit on a chess board of size 5 × 5?
- What if the chess board is bigger? Based on the pattern above, how many squares of size $k \times k$ should fit on a chess board of size 10 × 10?
- What if you do not know how big the chess board is? Based on the pattern above, how many squares of size $k \times k$ should fit on a chess board of size $n \times n$?

Now comes the tough part: explaining the pattern. Let us focus on an 8 × 8 board. Since it measures 8 squares on each side, we can see that we get 8× 8 = 64 squares of size 1× 1. And since there is just a single board, we get just one square of size 8 × 8. But what about all the sizes in-between?

? Think/Pair/Share

Using the Chess Board video in the previous chapter as a model, work with a partner to carefully explain why the number of 3 × 3 squares will be $6 \cdot 6 = 36$, and why the number of 4 × 4 squares will be $5 \cdot 5 = 25$.

There are many different explanations other than what is found in the video. Try to find your own explanation.

Here is what a final justification might look like (*watch the Chess Board video as a concrete example of this solution*):

Solution (Chess Board Pattern).

Let n be the side of the chess board and let k be the side of the square. If the square is going to fit on the chess board at all, it must be true that $k \leq n$. Otherwise, the square is too big.

If I put the $k \times k$ square in the upper left corner of the chess board, it takes up k spaces across and there are $(n - k)$ spaces to the right of it. So I can slide the $k \times k$ square to the right $(n - k)$ times, until it hits the top right corner of the chess board. The square is in $(n - k + 1)$ different positions, counting the starting position.

If I move the $k \times k$ square back to the upper left corner, I can shift it down one row and repeat the whole process again. Since there are $(n - k)$ rows below the square, I can shift it down $(n - k)$ times until it hits the bottom row. This makes $(n - k + 1)$ total rows that the square moves across, counting the top row.

So, there are $(n - k + 1)$ rows with $(n - k + 1)$ squares in each row. That makes $(n - k + 1)^2$ total squares.

Thus, the solution is the sum of $(n - k + 1)^2$ for all $k \leq n$. In symbols:

$$\text{number of squares on an n} \times \text{n board} = \sum_{k=1}^{n} (n - k + 1)^2.$$

Once we are sure the pattern continues, we can use it to solve the problem. So go ahead!

- How many squares on a 10 × 10 chess board?
- What calculation would you do to solve that problem for a 100 × 100 chess board?

There is a number pattern that describes the number of pieces you get from the "Dots on a Circle" problem. If you want to solve the problem, go for it! Think about all of your problem solving strategies. But be sure that when you find a pattern, you can explain *why it is the right pattern for this problem*, and not just another pattern that seems to work but might not continue.

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2.5: Problem Bank

You have several problem solving strategies to work with. Here are the ones we have described so far (and you probably came up with even more of your own strategies as you worked on problems).

1. Wishful Thinking.
2. Try Something!
3. Draw a Picture.
4. Make Up Numbers.
5. Try a Simpler Problem.
6. Work Systematically.
7. Use Manipulatives to Help you Investigate.
8. Look for and Explain Patterns.
9. Find the Math, Remove the Context.
10. Check Your Assumptions.
11. Use a Variable.

Try your hand at some of these problems, keeping these strategies in mind. If you are stuck on a problem, come back to this list and ask yourself which of the strategies might help you make some progress.

Problem 6

You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?



Problem 7

You have five coins, no two of which weigh the same. In seven weighings on a balance scale, can you put the coins in order from lightest to heaviest? That is, can you determine which coin is the lightest, next lightest, . . . , heaviest.

Problem 8

You have ten bags of coins. Nine of the bags contain good coins weighing one ounce each. One bag contains counterfeit coins weighing 1.1 ounces each. You have a regular (digital) scale, not a balance scale. The scale is correct to one-tenth of an ounce. In one weighing, can you determine which bag contains the bad coins?

Problem 9

Suppose you have a balance scale. You have three different weights, and you are able to weigh every whole number from 1 gram to 13 grams using just those three weights. What are the three weights?

Problem 10

There are a bunch of coins on a table in front of you. Your friend tells you how many of the coins are heads-up. You are blindfolded and cannot see a thing, but you can move the coins around, and you can flip them over. However, you cannot tell just by feeling them if the coins are showing heads or tails. Your job: separate the coins into two piles so that the same number of heads are showing in each pile.

Problem 11

The digital root of a number is the number obtained by repeatedly adding the digits of the number. If the answer is not a one-digit number, add those digits. Continue until a one-digit sum is reached. This one digit is the digital root of the number.

For example, the digital root of 98 is 8, since 9 + 8 = 17 and 1 + 7 = 8.

Record the digital roots of the first 30 integers and find as many patterns as you can. Can you explain any of the patterns?

Problem 12

If this lattice were continued, what number would be directly to the right of 98? How can you be sure you're right?

	3		6		9		12		…
1	2	4	5	7	8	10	11	13	…

Problem 13

Arrange the digits 0 through 9 so that the first digit is divisible by 1, the first two digits are divisible by 2, the first three digits are divisible by 3, and continuing until you have the first 9 digits divisible by 9 and the whole 10-digit number divisible by 10.

Problem 14

There are 25 students and one teacher in class. After an exam, everyone high-fives everyone else to celebrate how well they did. How many high- fives were there?

Problem 15

In cleaning out your old desk, you find a whole bunch of 3¢ and 7¢ stamps. Can you make exactly 11¢ of postage? Can you make exactly 19¢ of postage? What is the largest amount of postage you cannot make?

Problem 16

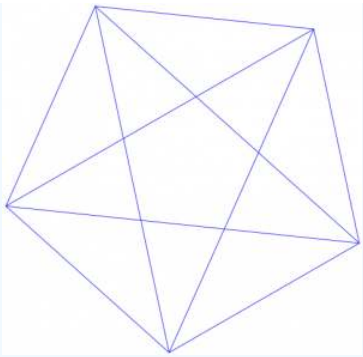
Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1's are separated by one digit, the 2's are separated by two digits, the 3's by three digits, and the 4's by four digits.

Problem 17

Kami has ten pockets and 44 dollar bills. She wants to have a different amount of money in each pocket. Can she do it?

Problem 18

How many triangles of all possible sizes and shapes are in this picture?



Problem 19

Arrange the digits 1–6 into a “difference triangle” where each number in the row below is the difference of the two numbers above it.

Example: Below is a difference triangle, but it does not work because it uses 1 twice and does not have a 6:

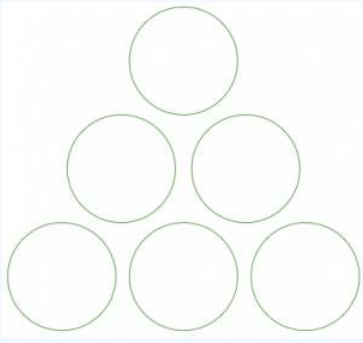
4		5		3
	1		2	
		1		

Problem 20

Certain pipes are sold in lengths of 6 inches, 8 inches, and 10 inches. How many different lengths can you form by attaching three sections of pipe together?

Problem 21

Place the digits 1, 2, 3, 4, 5, 6 in the circles so that the sum on each side of the triangle is 12. Each circle gets one digit, and each digit is used exactly once.



Problem 22

Find a way to cut a circular pizza into 11 pieces using just four straight cuts

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2.6: Careful Use of Language in Mathematics

This section might seem like a bit of a sidetrack from the idea of problem solving, but in fact it is not. Mathematics is a social endeavor. We do not just solve problems and then put them aside. Problem solving has (at least) three components:

1. Solving the problem. This involves a lot of scratch paper and careful thinking.
2. Convincing yourself that your solution is complete and correct. This involves a lot of self-check and asking yourself questions.
3. Convincing someone else that your solution is complete and correct. This usually involves writing the problem up carefully or explaining your work in a presentation.

If you are not able to do that last step, then you have not really solved the problem. We will talk more about how to write up a solution soon. Before we do that, we have to think about how mathematicians use language (which is, it turns out, a bit different from how language is used in the rest of life).

Mathematical Statements

Definition

A mathematical statement is a complete sentence that is either true or false, but not both at once.

So a "statement" in mathematics cannot be a question, a command, or a matter of opinion. It is a complete, grammatically correct sentence (with a subject, verb, and usually an object). It is important that the statement is either true or false, though you may not know which! (Part of the work of a mathematician is figuring out which sentences are true and which are false.)

Think / Pair / Share

For each English sentence below, decide if it is a mathematical statement or not. If it is, is the statement true or false (or are you unsure)? If it is not a mathematical statement, in what way does it fail?

1. Blue is the prettiest color.
2. 60 is an even number.
3. Is your dog friendly?
4. Honolulu is the capital of Hawaii.
5. This sentence is false.
6. All roses are red.
7. UH Manoa is the best college in the world.
8. 1/2 = 2/4.
9. Go to bed.
10. There are a total of 204 squares on an 8 × 8 chess board.

Now write three mathematical statements and three English sentences that fail to be mathematical statements.

Notice that "1/2 = 2/4" is a perfectly good mathematical statement. It does not look like an English sentence, but read it out loud. The subject is "1/2." The verb is "equals." And the object is "2/4." This is a very good test when you write mathematics: try to read it out loud. Even the equations should read naturally, like English sentences.

Statement (5) is different from the others. It is called a **paradox**: a statement that is self-contradictory. If it is true, then we conclude that it is false. (Why?) If it is false, then we conclude that it is true. (Why?) Paradoxes are no good as mathematical statements, because it cannot be true and it cannot be false.

And / or

Consider this sentence:

After work, I will go to the beach, or I will do my grocery shopping.

In everyday English, that probably means that if I go to the beach, I will not go shopping. I will do one or the other, but not both activities. This is called an "exclusive or."

We can usually tell from context whether a speaker means “either one or the other or both,” or whether he means “either one or the other but not both.” (Some people use the awkward phrase “and/or” to describe the first option.)

Remember that in mathematical communication, though, we have to be very precise. We cannot rely on context or assumptions about what is implied or understood.

Definition

In mathematics, the word “or” always means “one or the other or both.”

The word “and” always means “both are true.”

Think / Pair / Share

For each sentence below:

- Decide if the choice $x = 3$ makes the statement true or false.
- Choose a different value of that makes the statement true (or say why that is not possible).
- Choose a different value of that makes the statement false (or say why that is not possible).

1. x is odd or x is even.
2. x is odd and x is even.
3. x is prime or x is odd.
4. $x > 5$ or $x < 5$.
5. $x > 5$ and $x < 5$.
6. $x + 1 = 7$ or $x - 1 = 7$.
7. $x \cdot 1 = x$ or $x \cdot 0 = x$.
8. $x \cdot 1 = x$ and $x \cdot 0 = x$.
9. $x \cdot 1 = x$ or $x \cdot 0 = 0$.

Quantifiers

Problem 23 (All About the Benjamins)

You are handed an envelope filled with money, and you are told “Every bill in this envelope is a $100 bill.”

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Suppose you were given a different sentence: “There is a $100 bill in this envelope.”

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Think / Pair / Share

What is the difference between the two sentences? How does that difference affect your method to decide if the statement is true or false?

Some mathematical statements have this form:

- “Every time...”
- “For all numbers. . . ”
- “For every choice. . . ”

- "It's always true that. . . "

These are *universal* statements. Such statements claim that something is always true, no matter what.

- To prove a universal statement is false, you must find an example where it fails. This is called a **counterexample** to the statement.
- To prove a universal statement is true, you must either check every single case, or you must find a logical reason why it would be true. (Sometimes the first option is impossible, because there might be infinitely many cases to check. You would never finish!)

Some mathematical statements have this form:

- "Sometimes..."
- "There is some number. . . "
- "For some choice. . . "
- "At least once..."

These are *existential* statements. Such statements claim there is some example where the statement is true, but it may not always be true.

- To prove an existential statement is true, you may just find the example where it works.
- To prove an existential statement is false, you must either show it fails in every single case, or you must find a logical reason why it cannot be true. (Sometimes the first option is impossible!)

? Think / Pair / Share

For each statement below, do the following:

- Decide if it is a universal statement or an existential statement. (This can be tricky because in some statements the quantifier is "hidden" in the meaning of the words.)
- Decide if the statement is true or false, and do your best to justify your decision.

1. Every odd number is prime.
2. Every prime number is odd.
3. For all positive numbers x, $x^3 > x$.
4. There is some number x such that $x^3 = x$.
5. The points (1, 1), (2, 1), and (3, 0) all lie on the same line.
6. Addition (of real numbers) is commutative.
7. Division (of real numbers) is commutative.

Look back over your work. you will probably find that some of your arguments are sound and convincing while others are less so. In some cases you may "know" the answer but be unable to justify it. That is okay for now! Divide your answers into four categories:

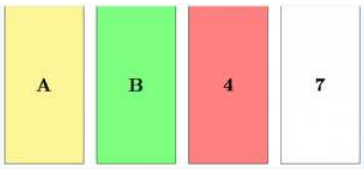
1. I am confident that the justification I gave is good.
2. I am not confident in the justification I gave.
3. I am confident that the justification I gave is not good, or I could not give a justification.
4. I could not decide if the statement was true or false.

Conditional Statements

✓ Problem 24 (Card Logic)

You have a deck of cards where each card has a letter on one side and a number on the other side. Your friend claims: "If a card has a vowel on one side, then it has an even number on the other side."

These cards are on a table.



Which cards must you flip over to be certain that your friend is telling the truth?

? Think / Pair / Share

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Try to come to agreement on an answer you both believe.

Here is another very similar problem, yet people seem to have an easier time solving this one:

✓ Problem 25 (IDs at a Party)

You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table, there are four young people:

- One person has a can of beer, another has a bottle of Coke, but their IDs happen to be face down so you cannot see their ages.
- You can, however, see the IDs of the other two people. One is under the drinking age, the other is above it. They both have fizzy clear drinks in glasses, and you are not sure if they are drinking soda water or gin and tonic.

Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

? Think / Pair / Share

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Compare these two problems. Which question is easier and why?

Definition

A **conditional statement** can be written in the form

If some statement then some statement.

Where the first statement is the **hypothesis** and the second statement is the **conclusion**.

? Think / Pair / Share

These are each conditional statements, though they are not all stated in “if/then” form. Identify the hypothesis of each statement. (You may want to rewrite the sentence as an equivalent “if/then” statement.)

1. If the tomatoes are red, then they are ready to eat.
   The tomatoes are red. / The tomatoes are ready to eat.
2. An integer n is even if it is a multiple of 2.
   n is even. / n is a multiple of 2.
3. If n is odd, then n is prime.
   n is odd. / n is prime.
4. The team wins when JJ plays.
   The team wins. / JJ plays.

Remember that a mathematical statement must have a definite truth value. It is either true or false, with no gray area (even though we may not be sure which is the case). How can you tell if a conditional statement is true or false? Surely, it depends on whether

the hypothesis and the conclusion are true or false. But how, exactly, can you decide?

The key is to think of a conditional statement like a promise, and ask yourself: under what condition(s) will I have broken my promise?

Example 2.6.1:

Here is a conditional statement:

> *If I win the lottery, then I'll give each of my students $1,000.*

There are four things that can happen:

- **True hypothesis, true conclusion:** I do win the lottery, and I do give everyone in class $1,000. I kept my promise, so the conditional statement is TRUE.
- **True hypothesis, false conclusion:** I do win the lottery, but I decide not to give everyone in class $1,000. I broke my promise, so the conditional statement is FALSE.
- **False hypothesis, true conclusion:** I do not win the lottery, but I am exceedingly generous, so I go ahead and give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.
- **False hypothesis, false conclusion:** I do not win the lottery, so I do not give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.

What can we conclude from this? **A conditional statement is false only when the hypothesis is true and the conclusion is false.** In every other instance, the promise (as it were) has not been broken. If a mathematical statement is not false, it must be true.

Example 2.6.2:

Here is another conditional statement:

> *If you live in Honolulu, then you live in Hawaii.*

Is this statement true or false? It seems like it should depend on who the pronoun "you" refers to, and whether that person lives in Honolulu or not. Let us think it through:

- Sookim lives in Honolulu, so the hypothesis is true. Since Honolulu is in Hawaii, she does live in Hawaii. The statement is true about Sookim, since both the hypothesis and conclusion are true.
- DeeDee lives in Los Angeles. The statement is true about DeeDee since the hypothesis is false.

So in fact it does not matter! The statement is true either way. The right way to understand such a statement is as a universal statement: "Everyone who lives in Honolulu lives in Hawaii."

This statement is true, and here is how you might justify it: "Pick a random person who lives in Honolulu. That person lives in Hawaii (since Honolulu is in Hawaii), so the statement is true for that person. I do not need to consider people who do not live in Honolulu. The statement is automatically true for those people, because the hypothesis is false!"

Example 2.6.3:

How do we show a (universal) conditional statement is false?

You need to give a specific instance where the hypothesis is true and the conclusion is false. For example:

> *If you are a good swimmer, then you are a good surfer.*

Do you know someone for whom the hypothesis is true (that person is a good swimmer) but the conclusion is false (the person is not a good surfer)? Then the statement is false!

Think / Pair / Share

For each conditional statement, decide if it is true or false. Justify your answer.

1. If $2 \times 2 = 4$ then $1 + 1 = 3$.
2. If $2 \times 2 = 5$ then $1 + 1 = 3$.
3. If $\pi > 3$ then all odd numbers are prime.
4. If $\pi < 3$ then all odd numbers are prime.
5. If a number has a 4 in the one's place, then the number is even.
6. If a number is even, then the number has a 4 in the one's place.
7. If the product of two numbers is 0, then one of the numbers is 0.
8. If the sum of two numbers is 0, then one of the numbers is 0.

Think / Pair / Share (Two truths and a lie)

On your own, come up with two conditional statements that are true and one that is false. Share your three statements with a partner, but do not say which are true and which is false. See if your partner can figure it out!

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2.7: Explaining Your Work

At its heart, mathematics is a social endeavor. Even if you work on problems all by yourself, you have not really solved the problem until you have explained your work to someone else, and they sign off on it. Professional mathematicians write journal articles, books, and grant proposals. Teachers explain mathematical ideas to their students both in writing and orally. Explaining your work is really an essential part of the problem-solving process, and probably should have been Pólya's step 5.

Writing in mathematics is different from writing poetry or an English paper. The goal of mathematical writing is not florid description, but clarity. If your reader does not understand, then you have not done a good job. Here are some tips for good mathematical writing.

Do Not Turn in Scratch Work: When you are solving problems and not exercises, you are going to have a lot of false starts. You are going to try a lot of things that do not work. You are going to make a lot of mistakes. You are going to use scratch paper. At some point (hopefully!) you will scribble down an idea that actually solves the problem. Hooray! That paper is not what you want to turn in or share with the world. Take that idea, and write it up carefully, neatly, and clearly. (The rest of these tips apply to that write-up.)

(Re)state the Problem: Do not assume your reader knows what problem you are solving. (Even if it is the teacher who assigned the problem!) If the problem has a very long description, you can summarize it. You do not have rewrite it word-for-word or give all of the details, but make sure the question is clear.

Clearly Give the Answer: It is not a bad idea to state the answer right up front, then show the work to justify your answer. That way, the reader knows what you are trying to justify as they read. It makes their job much easier, and making the reader's job easier should be one of your primary goals! In any case, the answer should be clearly stated somewhere in the write up, and it should be easy to find.

Be Correct: Of course, everyone makes mistakes as they are working on a problem. But we are talking about after you have solved the problem, when you are writing up your solution to share with someone else. The best writing in the world cannot save a wrong approach and a wrong answer. Check your work carefully. Ask someone else to read your solution with a critical eye.

Justify Your Answer: You cannot simply give an answer and expect your reader to "take your word for it." You have to explain how you know your answer is correct. This means "showing your work," explaining your reasoning, and justifying what you say. You need to answer the question, "How do you know your answer is right?"

Be Concise: There is no bonus prize for writing a lot in math class. Think clearly and write clearly. If you find yourself going on and on, stop, think about what you really want to say, and start over.

Use Variables and Equations: An equation can be much easier to read and understand than a long paragraph of text describing a calculation. Mathematical writing often has a lot fewer words (and a lot more equations) than other kinds of writing.

Define your Variables: If you use variables in the solution of your problem, always say what a variable stands for before you use it. If you use an equation, say where it comes from and why it applies to this situation. Do not make your reader guess!

Use Pictures: If pictures helped you solve the problem, include nice versions of those pictures in your final solution. Even if you did not draw a picture to solve the problem, it still might help your reader understand the solution. And that is your goal!

Use Correct Spelling and Grammar: Proofread your work. A good test is to read your work aloud (this includes reading the equations and calculations aloud). There should be complete, natural-sounding sentences. Be especially careful with pronouns. Avoid using "it" and "they" for mathematical objects; use the names of the objects (or variables) instead.

Format Clearly: Do not write one long paragraph. Separate your thoughts. Put complicated equations on a single displayed line rather than in the middle of a paragraph. Do not write too small. Do not make your reader struggle to read and understand your work.

Acknowledge Collaborators: If you worked with someone else on solving the problem, give them credit!

Here is a problem you've already seen:

Problem 16

Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1's are separated by one digit, the 2's are separated by two digits, the 3's by three digits, and the 4's by four digits.

Think / Pair / Share

Below you will find several solutions that were turned in by students. Using the criteria above, how would you score these solutions on a scale of 1 to 5? Give reasons for your answers.

Answer

(Solution 1). 41312432

This is the largest eight-digit b/c the #s 1, 2, 3, 4 & all separated by the given amount of spaces.

(Solution 2). 41312432

You have to have the 4 in the highest place and work down from there. However unable to follow the rules the 2 and the 1 in the 10k and 100k place must switch.

(Solution 3). 41312432

First, I had to start with the #4 because that is the largest digit I could start with to get the largest #. Then I had to place the next 4 five spaces away because I knew there had to be four digits separating the two 4's. Next, I place 1 in the second digit spot because 2 or 3 would interfere with the rule of how many digits could separate them, which allowed me to also place where the next 1 should be. I then placed the 3 because opening spaces showed me that I could fit three digits in between the two 3's. Lastly, I had to input the final 2's, which worked out because there were two digits separating them.

(Solution 4).

1×1

2xx2

3xxx3

4xxxx4
Answer: 41312432

(Solution 5).

4 3 2 4 3 2

4 2 2 4

4 1 3 1 4 3

*4 1 3 1 2 4 3 2

4 needs to be the first # to make it the biggest. Then check going down from next largest to smallest. Ex:

4 3 _____ ×

4 2 _____×

4 1 _____✓

(Solution 6). 41312432

I put 4 at the 10,000,000 place because the largest # should be placed at the highest value. Numbers 2 & 3 could not be placed in the 1,000,000 place because I wasn't able to separate the digits properly. So I ended up placing the #1 there. In the 100,000 place I put the #3 because it was the second highest number.

(Solution 7). 41312432

Since the problem asks you for the largest 8 digit #, I knew 4 had to be the first # since it's the greatest # of the set. To solve the rest of the problem, I used the guess and test method. I tried many different combinations. First using the #3 as the

second digit in the sequence, but came to no answer. Then the #2, but no combination I found correctly finished the sequence.I then finished with the #1 in the second digit in the sequence and was able to successfully fill out the entire #.

(Solution 8).

4 _ _ _ _ 4 _ _

4 has to be the first digit, for the number to be the largest possible. That means the other 4 has to be the 6th digit in the number, because 4's have to be separated by four digits.

4 _ 3 _ _ 4 3 _

3 must be the third digit, in order for the number to be largest possible. 3 cannot be the second digit because the other 3 would have to be the 6th digit in the number, but 4 is already there.

4 1 3 1 _ 4 3 _

1's must be separated by one digit, so the 1's can only be the 2nd and 4th digit in the number.

4 1 3 1 2 4 3 2

This leaves the 2s to be the 5th and 8th digits.

(Solution 9).

With the active rules, I tried putting the highest numbers as far left as possible. Through trying different combinations, I figured out that no two consecutive numbers can be touching in the first two digits. So I instead tried starting with the 4 then 1 then 3, since I'm going for the highest # possible.

My answer: 41312432

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2.8: The Last Step

A lot of people — from Polya to the writers of the Common Core State Standards and a lot of people in between — talk about problem solving in mathematics. One fact is rarely acknowledged, except by many professional mathematicians: Asking good questions is as valuable (and as difficult) as solving mathematical problems.

After solving a mathematical problem and explaining your solution to someone else, it is a very good mathematical habit to ask yourself: What other questions can I ask?

Example 2.8.1: Squares on a Chess Board

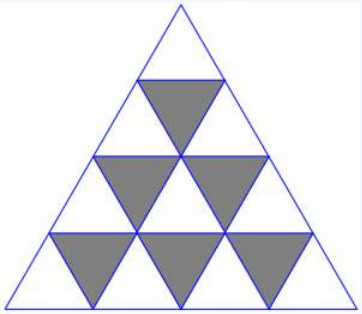
Recall Problem 3, "Squares on a Chess Board":

> *How many squares of any possible size are on a standard 8 × 8 chess board? (The answer is not 64! It's a lot bigger!)*

We have already talked about some obvious follow-up questions like "What about a 10 × 10 chess board? Or 100 × 100? Or $n \times n$?"

But there are a lot of interesting (and less obvious . . . and harder) questions you might ask:

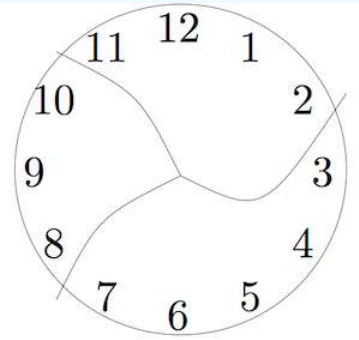
- How many rectangles of any size and shape can you find on a standard 8 × 8 chess board? (This is a lot harder, because the rectangles come in all different sizes, like 1 × 2 and 5 × 3. How could you possibly count them all?)
- How many triangles of any size and shape can you find in this picture?



Example 2.8.2: Broken Clock

Recall Problem 4, "Broken Clock":

> *This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. Can you break another clock into a different number of pieces so that the sums are consecutive numbers?*



The original problem only asks if you can find one other way. The obvious follow-up question: "Find every possibly way to break the clock into some number of pieces so that the sums of the numbers on each piece are consecutive numbers. Justify

that you have found every possibility.”

? Think / Pair / Share

Choose a problem from the Problem Bank (preferably a problem you have worked on, but that is not strictly necessary). What follow-up or similar questions could you ask?

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CHAPTER OVERVIEW

3: Numeration Systems and Bases

3.1: The Why
3.2: Units of Measurement
3.3: Historical Counting Systems
3.3.1: Introduction and Basic Number and Counting Systems
3.3.2: The Number and Counting System of the Inca Civilization
3.3.3: The Hindu-Arabic Number System
3.3.4: The Development and Use of Different Number Bases
3.3.5: The Mayan Numeral System
3.3.6: Roman Numerals
3.3.7: Exercises
3.4: Different Bases and Their Number Lines
3.5: Converting Between (our) Base 10 and Any Other Base (and vice versa)
3.6: Place Values with Different Bases
3.7: Operations in Different Bases

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3.1: The Why

The Essential Questions

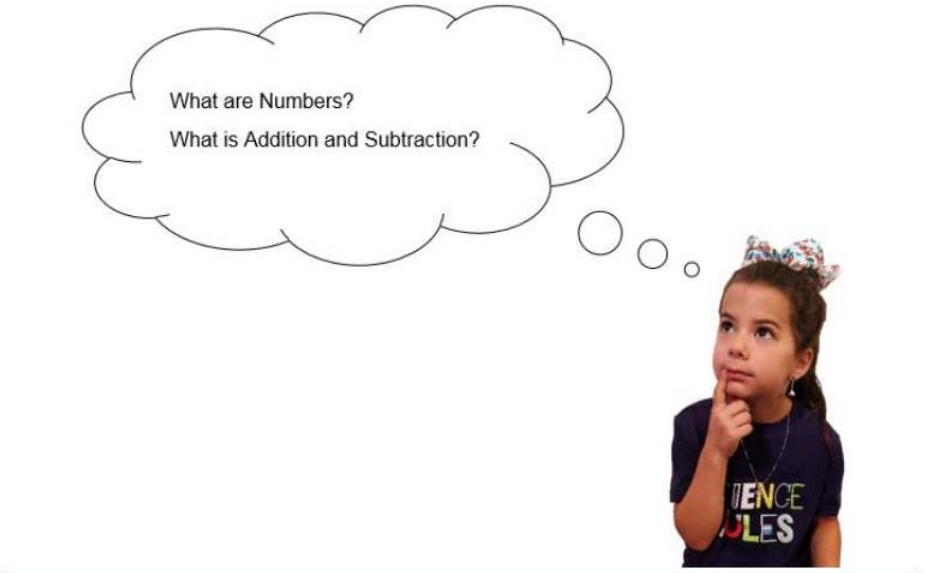


Figure 2.1.1

Why are Teachers Learning this Material?

Empathy.

Do you remember struggling with very basic math? Do you remember using your fingers to add two plus three? Chances are you do not, so this chapter will give you an opportunity to relearn basic math for the first time.

Why are Elementary School Students Learning this Mathematics?

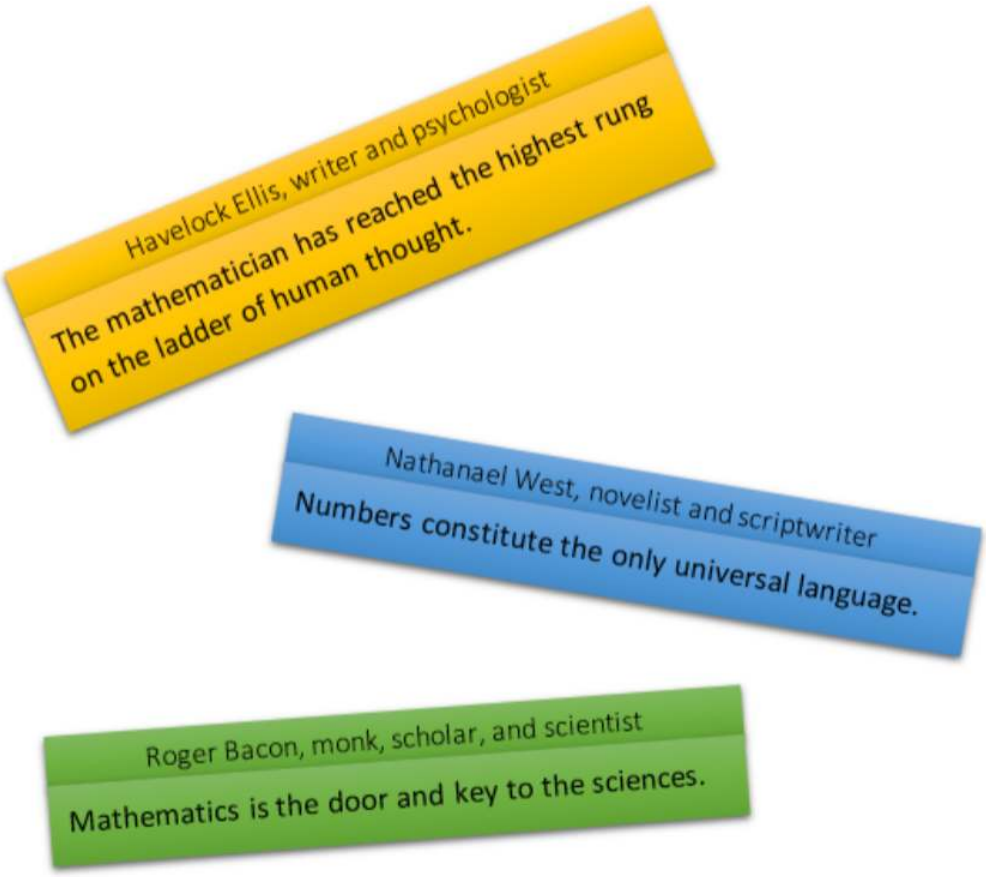


Figure 2.1.2: Several Quotes

Practice Problems

1. Why do you want to be a teacher?
2. If you had a choice (and most first year teachers do not), which grade level would you want to teach? Why?
3. Which mathematical topics were the most difficult for you in elementary school? Are they still difficult for you now?
4. Why do you think mathematics is very important subject to learn?

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3.2: Units of Measurement

How Common Core Changed Word Problems

Before the mathematicians wrote Common Core, they sat down with Physicists and asked what they would want from Math Teachers: The number one response was UNITS! Students who take physics are notorious for leaving off the units from their answers. Therefore, Common Core made learning units a standard, for grades one through five:

Common Core standards for Units

- **Grade 1**: Measure lengths indirectly and by iterating length units.
- **Grade 2**: Measure and estimate lengths in standard units.
- **Grade 3**: Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (L). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.
- **Grade 4**: Know relative sizes of measurement units within one system of units including $\mathrm{km}, \mathrm{m}, \mathrm{cm}; \mathrm{kg}, \mathrm{g}; \mathrm{lb}, \mathrm{oz.}$; $\mathrm{l}, \mathrm{ml}; \mathrm{hr}, \mathrm{min}, \mathrm{sec}$. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. For example, know that $1\mathrm{ft}.$ is 12 times as long as 1 in. Express the length of a 4 ft. snake as 48 in. Generate a conversion table for feet and inches listing the number pairs $(1, 12), (2, 24), (3, 36), \ldots$
- **Grade 5**: Convert among different-sized standard measurement units within a given measurement system (e.g., convert $5\mathrm{cm}$ to $0.05\mathrm{m}$), and use these conversions in solving multi-step, real world problems.

Mathematical Practices for Teachers 6: Attend to Precision (Did 1 Label My Answer?)

Example 3.2.1

Table 2.2.1: Physical quantity and Units

Type	Units
Height (how tall?)	Feet, meters, miles, yards, etc
Weight (how much?)	Pounds, kilograms, ounces, liters, quarts, etc
Speed (how fast?)	Miles per hour (mph) and kilometers per hour (kph) and feet per second (f/s or fps)
Temperature (how hot?)	Celsius C° and Fahrenheit F°

Example 3.2.2

Table 2.2.2: Quantity versus Quality

Quantifiable (Measureable units)	Qualitative (Descriptive Units)
Feet	Feelings
Gallons	Colors
Miles per Hour	Opinions
Weight	Taste
Money	Sound
Asking about something about a large group of people	Asking about something just about yourself

Practice Problems

Use Google to help you answer these questions:

1. How tall is the Eiffel Tower?
2. What is the average weight of a newborn baby?
3. What is the wealth of America?

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SECTION OVERVIEW

3.3: Historical Counting Systems

This is from a Math for Liberal Arts book, so the audience may be slightly different. Some of the content is redundant to other sections in this chapter, so be mindful of what your teacher wants to use.

Introduction to Historical Counting Systems

In the following section, we will try to focus on two main ideas. The first will be an examination of basic number and counting systems and the symbols that we use for numbers. We will look at our own modern (Western) number system as well those of a couple of selected civilizations to see the differences and diversity that is possible when humans start counting. The second idea we will look at will be base systems. By comparing our own base-ten (decimal) system with other bases, we will quickly become aware that the system that we are so used to, when slightly changed, will challenge our notions about numbers and what symbols for those numbers actually mean.

3.3.1: Introduction and Basic Number and Counting Systems

3.3.2: The Number and Counting System of the Inca Civilization

3.3.3: The Hindu-Arabic Number System

3.3.4: The Development and Use of Different Number Bases

3.3.5: The Mayan Numeral System

3.3.6: Roman Numerals

3.3.7: Exercises

Thumbnail: Roman Numerals. (Public Domain; Monaneko via Wikipedia)

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3.3.1: Introduction and Basic Number and Counting Systems

Introduction

As we begin our journey through the history of mathematics, one question to be asked is “Where do we start?” Depending on how you view mathematics or numbers, you could choose any of a number of launching points from which to begin. Howard Eves suggests the following list of possibilities.[i]

Where to start the study of the history of mathematics…

- At the first logical geometric “proofs” traditionally credited to Thales of Miletus (600 BCE).
- With the formulation of methods of measurement made by the Egyptians and Mesopotamians/Babylonians.
- Where prehistoric peoples made efforts to organize the concepts of size, shape, and number.
- In pre-human times in the very simple number sense and pattern recognition that can be displayed by certain animals, birds, etc.
- Even before that in the amazing relationships of numbers and shapes found in plants.
- With the spiral nebulae, the natural course of planets, and other universe phenomena.

We can choose no starting point at all and instead agree that mathematics has *always* existed and has simply been waiting in the wings for humans to discover. Each of these positions can be defended to some degree and which one you adopt (if any) largely depends on your philosophical ideas about mathematics and numbers.

Nevertheless, we need a starting point. Without passing judgment on the validity of any of these particular possibilities, we will choose as our starting point the emergence of the idea of number and the process of counting as our launching pad. This is done primarily as a practical matter given the nature of this course. In the following chapter, we will try to focus on two main ideas. The first will be an examination of basic number and counting systems and the symbols that we use for numbers. We will look at our own modern (Western) number system as well those of a couple of selected civilizations to see the differences and diversity that is possible when humans start counting. The second idea we will look at will be base systems. By comparing our own base-ten (decimal) system with other bases, we will quickly become aware that the system that we are so used to, when slightly changed, will challenge our notions about numbers and what symbols for those numbers actually mean.

Recognition of More vs. Less

The idea of numbers and the process of counting goes back far beyond when history began to be recorded. There is some archeological evidence that suggests that humans were counting as far back as 50,000 years ago.[ii] However, we do not really know how this process started or developed over time. The best we can do is to make a good guess as to how things progressed. It is probably not hard to believe that even the earliest humans had some sense of *more* and *less*. Even some small animals have been shown to have such a sense. For example, one naturalist tells of how he would secretly remove one egg each day from a plover’s nest. The mother was diligent in laying an extra egg every day to make up for the missing egg. Some research has shown that hens can be trained to distinguish between even and odd numbers of pieces of food.[iii] With these sorts of findings in mind, it is not hard to conceive that early humans had (at least) a similar sense of more and less. However, our conjectures about how and when these ideas emerged among humans are simply that; educated guesses based on our own assumptions of what might or could have been.

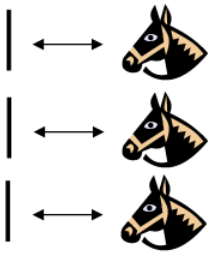
The Need for Simple Counting

As societies and humankind evolved, simply having a sense of more or less, even or odd, etc., would prove to be insufficient to meet the needs of everyday living. As tribes and groups formed, it became important to be able to know how many members were in the group, and perhaps how many were in the enemy’s camp. Certainly it was important for them to know if the flock of sheep or other possessed animals were increasing or decreasing in size. “Just how many of them do we have, anyway?” is a question that we do not have a hard time imagining them asking themselves (or each other).

In order to count items such as animals, it is often conjectured that one of the earliest methods of doing so would be with “tally sticks.” These are objects used to track the numbers of items to be counted. With this method, each “stick” (or pebble, or whatever counting device being used) represents one animal or object. This method uses the idea of ***one to one correspondence***. In a one to one correspondence, items that are being counted are uniquely linked with some counting tool.

In the picture to the right, you see each stick corresponding to one horse. By examining the collection of sticks in hand one knows how many animals should be present. You can imagine the usefulness of such a system, at least for smaller numbers of items to

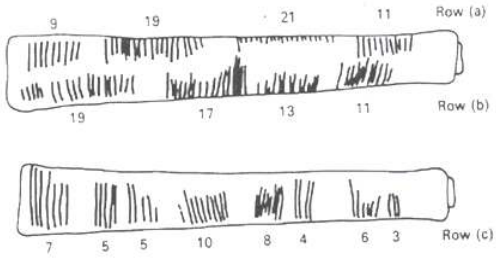
keep track of. If a herder wanted to "count off" his animals to make sure they were all present, he could mentally (or methodically) assign each stick to one animal and continue to do so until he was satisfied that all were accounted for.



Of course, in our modern system, we have replaced the sticks with more abstract objects. In particular, the top stick is replaced with our symbol "1," the second stick gets replaced by a "2" and the third stick is represented by the symbol "3," but we are getting ahead of ourselves here. These modern symbols took many centuries to emerge.

Another possible way of employing the "tally stick" counting method is by making marks or cutting notches into pieces of wood, or even tying knots in string (as we shall see later). In 1937, Karl Absolom discovered a wolf bone that goes back possibly 30,000 years. It is believed to be a counting device.[iv] Another example of this kind of tool is the Ishango Bone, discovered in 1960 at Ishango, and shown below.[v] It is reported to be between six and nine thousand years old and shows what appear to be markings used to do counting of some sort.

The markings on rows (a) and (b) each add up to 60. Row (b) contains the prime numbers between 10 and 20. Row (c) seems to illustrate for the method of doubling and multiplication used by the Egyptians. It is believed that this may also represent a lunar phase counter.



Spoken Words

As methods for counting developed, and as language progressed as well, it is natural to expect that spoken words for numbers would appear. Unfortunately, the developments of these words, especially those corresponding to the numbers from one through ten, are not easy to trace. Past ten, however, we do see some patterns:

Eleven comes from "ein lifon," meaning "one left over."

Twelve comes from "twe lif," meaning "two left over."

Thirteen comes from "Three and ten" as do fourteen through nineteen.

Twenty appears to come from "twe-tig" which means "two tens."

Hundred probably comes from a term meaning "ten times."

Written Numbers

When we speak of "written" numbers, we have to be careful because this could mean a variety of things. It is important to keep in mind that modern paper is only a little more than 100 years old, so "writing" in times past often took on forms that might look quite unfamiliar to us today.

As we saw earlier, some might consider wooden sticks with notches carved in them as writing as these are means of recording information on a medium that can be "read" by others. Of course, the symbols used (simple notches) certainly did not leave a lot of flexibility for communicating a wide variety of ideas or information.

Other mediums on which "writing" may have taken place include carvings in stone or clay tablets, rag paper made by hand (12^{th} century in Europe, but earlier in China), papyrus (invented by the Egyptians and used up until the Greeks), and parchments from animal skins. And these are just a few of the many possibilities.

These are just a few examples of early methods of counting and simple symbols for representing numbers. Extensive books, articles and research have been done on this topic and could provide enough information to fill this entire course if we allowed it to. The range and diversity of creative thought that has been used in the past to describe numbers and to count objects and people is staggering. Unfortunately, we don't have time to examine them all, but it is fun and interesting to look at one system in more detail to see just how ingenious people have been.

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[i] Eves, Howard; An Introduction to the History of Mathematics, p. 9.

[ii] Eves, p. 9.

[iii] McLeish, John; The Story of Numbers - How Mathematics Has Shaped Civilization, p. 7.

[iv] Bunt, Lucas; Jones, Phillip; Bedient, Jack; The Historical Roots of Elementary Mathematics, p. 2.

[v] http://www.math.buffalo.edu/mad/Ancient-Africa/mad_zaire-uganda.html

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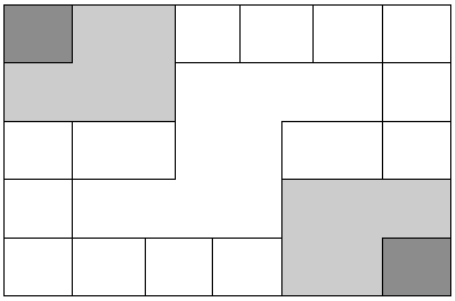
3.3.2: The Number and Counting System of the Inca Civilization

Background

There is generally a lack of books and research material concerning the historical foundations of the Americas. Most of the "important" information available concentrates on the eastern hemisphere, with Europe as the central focus. The reasons for this may be twofold: first, it is thought that there was a lack of specialized mathematics in the American regions; second, many of the secrets of ancient mathematics in the Americas have been closely guarded.[i] The Peruvian system does not seem to be an exception here. Two researchers, Leland Locke and Erland Nordenskiold, have carried out research that has attempted to discover what mathematical knowledge was known by the Incas and how they used the Peruvian quipu, a counting system using cords and knots, in their mathematics. These researchers have come to certain beliefs about the quipu that we will summarize here.

Counting Boards

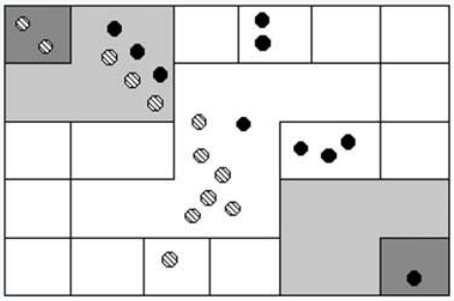
It should be noted that the Incas did not have a complicated system of computation. Where other peoples in the regions, such as the Mayans, were doing computations related to their rituals and calendars, the Incas seem to have been more concerned with the simpler task of record-keeping. To do this, they used what are called the "quipu" to record quantities of items. (We will describe them in more detail in a moment.) However, they first often needed to do computations whose results would be recorded on quipu. To do these computations, they would sometimes use a counting board constructed with a slab of stone. In the slab were cut rectangular and square compartments so that an octagonal (eight-sided) region was left in the middle. Two opposite corner rectangles were raised. Another two sections were mounted on the original surface of the slab so that there were actually three levels available. In the figure shown, the darkest shaded corner regions represent the highest, third level. The lighter shaded regions surrounding the corners are the second highest levels, while the clear white rectangles are the compartments cut into the stone slab.



Pebbles were used to keep accounts and their positions within the various levels and compartments gave totals. For example, a pebble in a smaller (white) compartment represented one unit. Note that there are 12 such squares around the outer edge of the figure. If a pebble was put into one of the two (white) larger, rectangular compartments, its value was doubled. When a pebble was put in the octagonal region in the middle of the slab, its value was tripled. If a pebble was placed on the second (shaded) level, its value was multiplied by six. And finally, if a pebble was found on one of the two highest corner levels, its value was multiplied by twelve. Different objects could be counted at the same time by representing different objects by different colored pebbles.

Example 1

Suppose you have the following counting board with two different kind of pebbles places as illustrated. Let the solid black pebble represent a dog and the striped pebble represent a cat. How many dogs are being represented?



Solution

- There are two black pebbles in the outer square regions…these represent 2 dogs.
- There are three black pebbles in the larger (white) rectangular compartments. These represent 6 dogs.
- There is one black pebble in the middle region…this represents 3 dogs.
- There are three black pebbles on the second level…these represent 18 dogs.

- Finally, there is one black pebble on the highest corner level…this represents 12 dogs.

We then have a total of $2+6+3+18+12=41$ dogs.

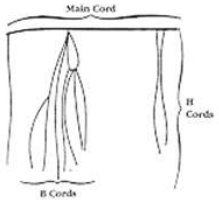
Try it Now 1

How many cats are represented on this board?

Answer

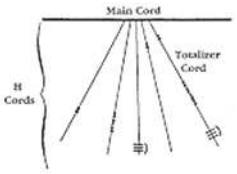
$1+6\times3+3\times6+2\times12=61$ cats.

The Quipu

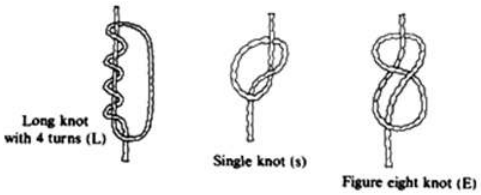


This kind of board was good for doing quick computations, but it did not provide a good way to keep a permanent recording of quantities or computations. For this purpose, they used the quipu. The quipu is a collection of cords with knots in them. These cords and knots are carefully arranged so that the position and type of cord or knot gives specific information on how to decipher the cord.

A quipu is made up of a main cord which has other cords (branches) tied to it. See pictures to the right. [ii]



Locke called the branches H cords. They are attached to the main cord. B cords, in turn, were attached to the H cords. Most of these cords would have knots on them. Rarely are knots found on the main cord, however, and tend to be mainly on the H and B cords. A quipu might also have a "totalizer" cord that summarizes all of the information on the cord group in one place.



Locke points out that there are three types of knots, each representing a different value, depending on the kind of knot used and its position on the cord. The Incas, like us, had a decimal (base-ten) system, so each kind of knot had a specific decimal value. The Single knot, pictured in the middle of the diagram[iii] was used to denote tens, hundreds, thousands, and ten thousands. They would be on the upper levels of the H cords. The figure-eight knot on the end was used to denote the integer "one." Every other integer from 2 to 9 was represented with a long knot, shown on the left of the figure. (Sometimes long knots were used to represents tens and hundreds.) Note that the long knot has several turns in it…the number of turns indicates which integer is being represented. The units (ones) were placed closest to the bottom of the cord, then tens right above them, then the hundreds, and so on.

In order to make reading these pictures easier, we will adopt a convention that is consistent. For the long knot with turns in it (representing the numbers 2 through 9), we will use the following notation: ≡⟩

The four horizontal bars represent four turns and the curved arc on the right links the four turns together. This would represent the number 4.

We will represent the single knot with a large dot (·) and we will represent the figure eight knot with a sideways eight (∞).

Example 2

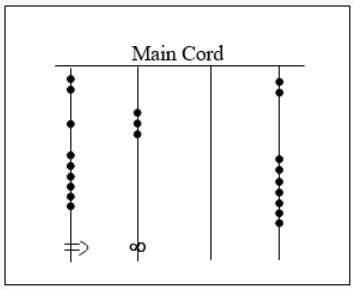
What number is represented on the cord shown?

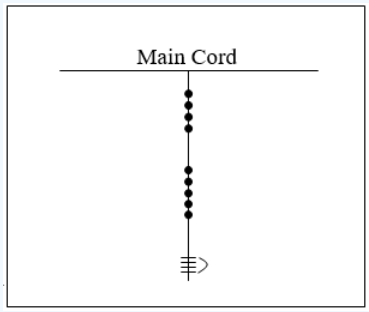
Solution

On the cord, we see a long knot with four turns in it…this represents four in the ones place. Then 5 single knots appear in the tens position immediately above that, which represents 5 tens, or 50. Finally, 4 single knots are tied in the hundreds, representing four 4 hundreds, or 400. Thus, the total shown on this cord is 454.

Try it Now 2

What numbers are represented on each of the four cords hanging from the main cord?





Answer

From left to right:

$\text{Cord } 1 = 2,162$

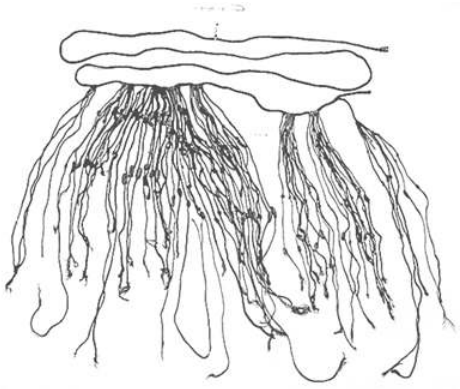
$\text{Cord } 2 = 301$

$\text{Cord } 3 = 0$

$\text{Cord } 4 = 2,070$

The colors of the cords had meaning and could distinguish one object from another. One color could represent llamas, while a different color might represent sheep, for example. When all the colors available were exhausted, they would have to be re-used. Because of this, the ability to read the quipu became a complicated task and specially trained individuals did this job. They were called Quipucamayoc, which means keeper of the quipus. They would build, guard, and decipher quipus.

As you can see from this photograph of an actual quipu, they could get quite complex.



There were various purposes for the quipu. Some believe that they were used to keep an account of their traditions and history, using knots to record history rather than some other formal system of writing. One writer has even suggested that the quipu replaced writing as it formed a role in the Incan postal system.[iv] Another proposed use of the quipu is as a translation tool. After the conquest of the Incas by the Spaniards and subsequent "conversion" to Catholicism, an Inca supposedly could use the quipu to confess their sins to a priest. Yet another proposed use of the quipu was to record numbers related to magic and astronomy, although this is not a widely accepted interpretation.

The mysteries of the quipu have not been fully explored yet. Recently, Ascher and Ascher have published a book, *The Code of the Quipu: A Study in Media, Mathematics, and Culture, which is* "an extensive elaboration of the logical-numerical system of the quipu."[v] For more information on the quipu, you may want to check out the following Internet link:

www.anthropology.wisc.edu/salomon/Chaysimire/khipus.php

We are so used to seeing the symbols 1, 2, 3, 4, etc. that it may be somewhat surprising to see such a creative and innovative way to compute and record numbers. Unfortunately, as we proceed through our mathematical education in grade and high school, we receive very little information about the wide range of number systems that have existed and which still exist all over the world. That's not to say our own system is not important or efficient. The fact that it has survived for hundreds of years and shows no sign of going away any time soon suggests that we may have finally found a system that works well and may not need further improvement, but only time will tell that whether or not that conjecture is valid or not. We now turn to a brief historical look at how our current system developed over history.

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[i] Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher,* Issue 60 (Oct., 1967), p. 623-28.

[ii] Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher,* Issue 60 (Oct., 1967), p. 623-28.

[iii] wiscinfo.doit.wisc.edu/chaysimire/titulo2/khipus/what.htm

[iv] Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher,* Issue 60 (Oct., 1967), p. 623-28.

[v] http://www.cs.uidaho.edu/~casey931/seminar/quipu.html

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3.3.3: The Hindu-Arabic Number System

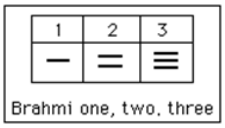
The Evolution of a System

Our own number system, composed of the ten symbols {0,1,2,3,4,5,6,7,8,9} is called the *Hindu-Arabic system*. This is a base-ten (decimal) system since place values increase by powers of ten. Furthermore, this system is positional, which means that the position of a symbol has bearing on the value of that symbol within the number. For example, the position of the symbol 3 in the number 435,681 gives it a value much greater than the value of the symbol 8 in that same number. We'll explore base systems more thoroughly later. The development of these ten symbols and their use in a positional system comes to us primarily from India.[i]

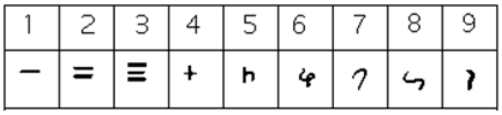


It was not until the 15th century that the symbols that we are familiar with today first took form in Europe. However, the history of these numbers and their development goes back hundreds of years. One important source of information on this topic is the writer al-Biruni, whose picture is shown here.[ii] Al-Biruni, who was born in modern day Uzbekistan, had visited India on several occasions and made comments on the Indian number system. When we look at the origins of the numbers that al-Biruni encountered, we have to go back to the third century B.C.E. to explore their origins. It is then that the Brahmi numerals were being used.

The Brahmi numerals were more complicated than those used in our own modern system. They had separate symbols for the numbers 1 through 9, as well as distinct symbols for 10, 100, 1000,…, also for 20, 30, 40,…, and others for 200, 300, 400, …, 900. The Brahmi symbols for 1, 2, and 3 are shown below.[iii]



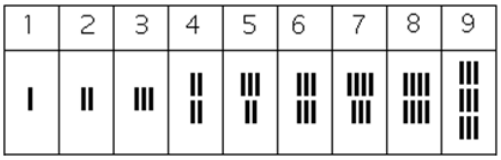
These numerals were used all the way up to the 4th century C.E., with variations through time and geographic location. For example, in the first century C.E., one particular set of Brahmi numerals took on the following form[iv]:



From the 4th century on, you can actually trace several different paths that the Brahmi numerals took to get to different points and incarnations. One of those paths led to our current numeral system, and went through what are called the Gupta numerals. The Gupta numerals were prominent during a time ruled by the Gupta dynasty and were spread throughout that empire as they conquered lands during the 4th through 6th centuries. They have the following form[v]:



How the numbers got to their Gupta form is open to considerable debate. Many possible hypotheses have been offered, most of which boil down to two basic types[vi]. The first type of hypothesis states that the numerals came from the initial letters of the names of the numbers. This is not uncommon…the Greek numerals developed in this manner. The second type of hypothesis states that they were derived from some earlier number system. However, there are other hypotheses that are offered, one of which is by the researcher Ifrah. His theory is that there were originally nine numerals, each represented by a corresponding number of vertical lines. One possibility is this:[vii]



Because these symbols would have taken a lot of time to write, they eventually evolved into cursive symbols that could be written more quickly. If we compare these to the Gupta numerals above, we can try to see how that evolutionary process might have taken

place, but our imagination would be just about all we would have to depend upon since we do not know exactly how the process unfolded.

The Gupta numerals eventually evolved into another form of numerals called the Nagari numerals, and these continued to evolve until the 11th century, at which time they looked like this:[viii]



Note that by this time, the symbol for 0 has appeared! The Mayans in the Americas had a symbol for zero long before this, however, as we shall see later in the chapter.

These numerals were adopted by the Arabs, most likely in the eighth century during Islamic incursions into the northern part of India.[ix] It is believed that the Arabs were instrumental in spreading them to other parts of the world, including Spain (see below).

Other examples of variations up to the eleventh century include:

Devangari, eighth century[x]:



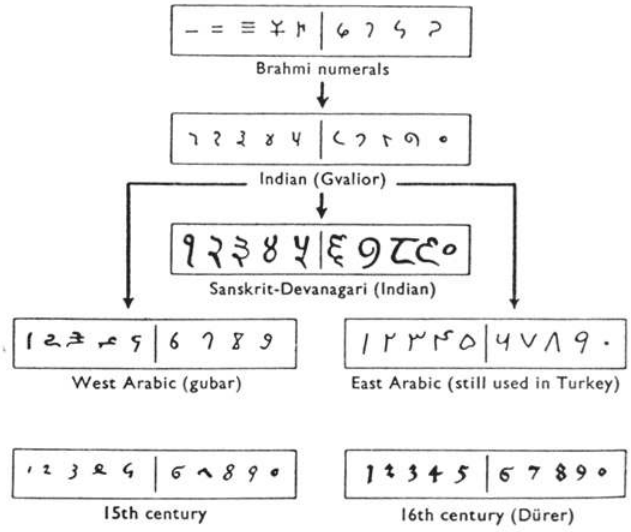
West Arab Gobar, tenth century[xi]:



Spain, 976 C.E.[xii]:



Finally, one more graphic[xiii] shows various forms of these numerals as they developed and eventually converged to the 15th century in Europe.



The Positional System

More important than the form of the number symbols is the development of the place value system. Although it is in slight dispute, the earliest known document in which the Indian system displays a positional system dates back to 346 C.E. However, some evidence suggests that they may have actually developed a positional system as far back as the first century C.E.

The Indians were not the first to use a positional system. The Babylonians (as we will see in Chapter 3) used a positional system with 60 as their base. However, there is not much evidence that the Babylonian system had much impact on later numeral systems,

except with the Greeks. Also, the Chinese had a base-10 system, probably derived from the use of a counting board[xiv]. Some believe that the positional system used in India was derived from the Chinese system.

Wherever it may have originated, it appears that around 600 C.E., the Indians abandoned the use of symbols for numbers higher than nine and began to use our familiar system where the position of the symbol determines its overall value.[xv] Numerous documents from the seventh century demonstrate the use of this positional system.

Interestingly, the earliest dated inscriptions using the system with a symbol for zero come from Cambodia. In 683, the 605th year of the Saka era is written with three digits and a dot in the middle. The 608th year uses three digits with a modern 0 in the middle.[xvi] The dot as a symbol for zero also appears in a Chinese work (*Chiu-chih li*). The author of this document gives a strikingly clear description of how the Indian system works:

Using the [Indian] numerals, multiplication and division are carried out. Each numeral is written in one stroke. When a number is counted to ten, it is advanced into the higher place. In each vacant place a dot is always put. Thus the numeral is always denoted in each place. Accordingly there can be no error in determining the place. With the numerals, calculations is easy…" ***[xvii]***

Transmission to Europe

It is not completely known how the system got transmitted to Europe. Traders and travelers of the Mediterranean coast may have carried it there. It is found in a tenth-century Spanish manuscript and may have been introduced to Spain by the Arabs, who invaded the region in 711 C.E. and were there until 1492.

In many societies, a division formed between those who used numbers and calculation for practical, every day business and those who used them for ritualistic purposes or for state business.[xviii] The former might often use older systems while the latter were inclined to use the newer, more elite written numbers. Competition between the two groups arose and continued for quite some time.



In a 14th century manuscript of Boethius' *The Consolations of Philosophy*, there appears a well-known drawing of two mathematicians. One is a merchant and is using an abacus (the "abacist"). The other is a Pythagorean philosopher (the "algorist") using his "sacred" numbers. They are in a competition that is being judged by the goddess of number. By 1500 C.E., however, the newer symbols and system had won out and has persevered until today. The Seattle Times recently reported that the Hindu-Arabic numeral system has been included in the book *The Greatest Inventions of the Past 2000 Years*.[xix]

One question to answer is *why* the Indians would develop such a positional notation. Unfortunately, an answer to that question is not currently known. Some suggest that the system has its origins with the Chinese counting boards. These boards were portable and it is thought that Chinese travelers who passed through India took their boards with them and ignited an idea in Indian mathematics.[xx] Others, such as G. G. Joseph propose that it is the Indian fascination with very large numbers that drove them to develop a system whereby these kinds of big numbers could easily be written down. In this theory, the system developed entirely within the Indian mathematical framework without considerable influence from other civilizations.

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[i] www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

[ii] www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Al-Biruni.html

[iii] www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

[iv] www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

[v] Ibid

[vi] Ibid

[vii] Ibid

[viii] Ibid

[ix] Katz, page 230

[x] Burton, David M., *History of Mathematics, An Introduction*, p. 254-255

[xi] Ibid

[xii] Ibid

[xiii] Katz, page 231.

[xiv] Ibid, page 230

[xv] Ibid, page 231.

[xvi] Ibid, page 232.

[xvii] Ibid, page 232.

[xviii] McLeish, p. 18

[xix] seattletimes.nwsource.com/news/health-science/html98/invs_20000201.html, Seattle Times, Feb. 1, 2000

[xx] Ibid, page 232.

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3.3.4: The Development and Use of Different Number Bases

Introduction and Basics

During the previous discussions, we have been referring to positional base systems. In this section of the chapter, we will explore exactly what a base system is and what it means if a system is "positional." We will do so by first looking at our own familiar, base-ten system and then deepen our exploration by looking at other possible base systems. In the next part of this section, we will journey back to Mayan civilization and look at their unique base system, which is based on the number 20 rather than the number 10.

A base system is a structure within which we count. The easiest way to describe a base system is to think about our own base-ten system. The base-ten system, which we call the "decimal" system, requires a total of ten different symbols/digits to write any number. They are, of course, 0, 1, 2, ….. 9.

The decimal system is also an example of a *positional* base system, which simply means that the position of a digit gives its place value. Not all civilizations had a positional system even though they did have a base with which they worked.

In our base-ten system, a number like 5,783,216 has meaning to us because we are familiar with the system and its places. As we know, there are six ones, since there is a 6 in the ones place. Likewise, there are seven "hundred thousands," since the 7 resides in that place. Each digit has a value that is explicitly determined by its position within the number. We make a distinction between digit, which is just a symbol such as 5, and a number, which is made up of one or more digits. We can take this number and assign each of its digits a value. One way to do this is with a table, which follows:

$5,000,000$	$=5\times1,000,000$	$=5\times10^6$	Five million
$+700,000$	$=7\times100,000$	$=7\times10^5$	Seven hundred thousand
$+80,000$	$=8\times10,000$	$=8\times10^4$	Eighty thousand
$+3,000$	$=3\times1000$	$=3\times10^3$	Three thousand
$+200$	$=2\times100$	$=2\times10^2$	Two hundred
$+10$	$=1\times10$	$=1\times10^1$	Ten
$+6$	$=6\times1$	$=6\times10^0$	Six
$5,783,216$	Five million, seven hundred eighty-three thousand, two hundred sixteen		

From the third column in the table we can see that each place is simply a multiple of ten. Of course, this makes sense given that our base is ten. The digits that are multiplying each place simply tell us how many of that place we have. We are restricted to having at most 9 in any one place before we have to "carry" over to the next place. We cannot, for example, have 11 in the hundreds place. Instead, we would carry 1 to the thousands place and retain 1 in the hundreds place. This comes as no surprise to us since we readily see that 11 hundreds is the same as one thousand, one hundred. Carrying is a pretty typical occurrence in a base system.

However, base-ten is not the only option we have. Practically any positive integer greater than or equal to 2 can be used as a base for a number system. Such systems can work just like the decimal system except the number of symbols will be different and each position will depend on the base itself.

Other Bases

For example, let's suppose we adopt a base-five system. The only modern digits we would need for this system are 0,1,2,3 and 4. What are the place values in such a system? To answer that, we start with the ones place, as most base systems do. However, if we were to count in this system, we could only get to four (4) before we had to jump up to the next place. Our base is 5, after all! What is that next place that we would jump to? It would not be tens, since we are no longer in base-ten. We're in a different numerical world. As the base-ten system progresses from 10^0 to10^1, so the base-five system moves from 5^0 to $5^1 = 5$. Thus, we move from the ones to the fives.

After the fives, we would move to the 5^2 place, or the twenty fives. Note that in base-ten, we would have gone from the tens to the hundreds, which is, of course, 10^2.

Let's take an example and build a table. Consider the number 30412 in base five. We will write this as 30412_5 , where the subscript 5 is not part of the number but indicates the base we're using. First off, note that this is NOT the number "thirty thousand, four hundred twelve." We must be careful not to impose the base-ten system on this number. Here's what our table might look like. We will use it to convert this number to our more familiar base-ten system.

	Base 5	This column coverts to base-ten	In Base-Ten
	3×5^4	$= 3 \times 625$	$= 1875$
+	0×5^3	$= 0 \times 125$	$= 0$
+	4×5^2	$= 4 \times 25$	$= 100$
+	1×5^1	$= 1 \times 5$	$= 5$
+	2×5^0	$= 2 \times 1$	$= 2$
		Total	1982

As you can see, the number 30412_5 is equivalent to 1,982 in base-ten. We will say $30412_5 = 1982_{10}$. All of this may seem strange to you, but that's only because you are so used to the only system that you've ever seen.

Example 3

Convert 6234_7 to a base 10 number.

Solution

We first note that we are given a base-7 number that we are to convert. Thus, our places will start at the ones (7^0), and then move up to the $7's, 49's$ (7^2) , etc. Here's the breakdown:

	Base 7	Convert	Base 10
	$= 6 \times 7^3$	$= 6 \times 343$	$= 2058$
+	$= 2 \times 7^2$	$= 2 \times 49$	$= 98$
+	$= 3 \times 7$	$= 3 \times 7$	$= 21$
+	$= 4 \times 1$	$= 4 \times 1$	$= 4$
		Total	2181

Thus $6234_7 = 2181_{10}$

Try it Now 3

Convert 41065_7 to a base 10 number.

Answer

$41065_7 = 9994_{10}$

Converting from Base 10 to Other Bases

Converting from an unfamiliar base to the familiar decimal system is not that difficult once you get the hang of it. It's only a matter of identifying each place and then multiplying each digit by the appropriate power. However, going the other direction can be a little trickier. Suppose you have a base-ten number and you want to convert to base-five. Let's start with some simple examples before we get to a more complicated one.

Example 4

Convert twelve to a base-five number.

Solution

We can probably easily see that we can rewrite this number as follows:

$$12 = (2 \times 5) + (2 \times 1)$$

Hence, we have two fives and 2 ones. Hence, in base-five we would write twelve as 22_5. Thus, $12_{10} = 22_5$

Example 5

Convert sixty-nine to a base-five number.

Solution

We can see now that we have more than 25, so we rewrite sixty-nine as follows:

$$69 = (2 \times 25) + (3 \times 5) + (4 \times 1)$$

Here, we have two twenty-fives, 3 fives, and 4 ones. Hence, in base five we have 234 . Thus, $69_{10} = 234_5$

Example 6

Convert the base-seven number 3261_7 to base 10

Solution

The powers of 7 are:

$$7^0 = 1$$
$$7^1 = 7$$
$$7^2 = 49$$
$$7^3 = 343$$

Etc...

$$3261_7 = (3 \times 343) + (2 \times 49) + (6 \times 7) + (1 \times 1) = 1170_{10}$$

Thus $3261_7 = 1170_{10}$

Try it Now 4

Convert 143 to base 5

Answer

$143_{10} = 1033_5$

Try it Now 5

Convert the base-three number 21021_3 to base 10.

Answer

$21021_3 = 196_{10}$

In general, when converting from base-ten to some other base, it is often helpful to determine the highest power of the base that will divide into the given number at least once. In the last example, $5^2 = 25$ is the largest power of five that is present in 69, so that was our starting point. If we had moved to $5^3 = 125$, then 125 would not divide into 69 at least once.

Converting from Base 10 to Base b

1. Find the highest power of the base b that will divide into the given number at least once and then divide.
2. Write down the whole number part, then use the remainder from division in the next step.
3. Repeat step two, dividing by the next highest power of the base b, writing down the whole number part (including 0), and using the remainder in the next step.
4. Continue until the remainder is smaller than the base. This last remainder will be in the “ones” place.
5. Collect all your whole number parts to get your number in base b notation.

Example 7

Convert the base-ten number 348 to base-five.

Solution

The powers of five are:

$5^0 = 1$
$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$

Etc..

since 348 is smaller than 625, but bigger than 125, we see that $5^3 = 125$ is the highest power of five present in 348. So we divide 125 into 348 to see how many of them there are:

$348 \div 125 = 2$ with remainder 98

We write down the whole part, 2, and continue with the remainder. There are 98 left over, so we see how many 25's (the next smallest power of five) there are in the remainder:

$98 \div 25 = 3$ with remainder 23

We write down the whole part, 2, and continue with the remainder. There are 23 left over, so we look at the next place, the 5's:

$23 \div 5 = 4$ with remainder 3

This leaves us with 3, which is less than our base, so this number will be in the "ones" place. We are ready to assemble our base-five number:

$348 = (2 \times 5^3) + (3 \times 5^2) + (4 \times 5^1) + (3 \times 1)$

Hence, our base-five number is 2343. We'll say that $348_{10} = 2343_5$

Example 8

Convert the base-ten number 4509 to base-seven.

Solution

The powers of 7 are:

$7^0 = 1$
$7^1 = 7$
$7^2 = 49$
$7^3 = 343$
$7^4 = 2401$
$7^5 = 16807$

Etc…

The highest power of 7 that will divide into 4509 is $7^4 = 2401$

With division, we see that it will go in 1 time with a remainder of 2108. So we have 1 in the 7^4 place.

The next power down is $7^3 = 343$, which goes into 2108 six times with a new remainder of 50. So we have 6 in the 7^3 place.

The next power down is $7^2 = 49$, which goes into 50 once with a new remainder of 1. So there is a 1 in the 7^2 place.

The next power down is 7^1 but there was only a remainder of 1, so that means there is a 0 in the 7 's place and we still have 1 as a remainder.

That, of course, means that we have 1 in the ones place.

$4509 \div 7^4 =$ 1 R 2108
$2108 \div 7^3 =$ 6 R 50
$50 \div 7^2 =$ 1 R 1
$1 \div 7^1 =$ 0 R 1
$1 \div 7^0 =$ 1

$4509_{10} = 16101_7$

Putting all of this together means that $4509_{10} = 16101_7$

? Try it Now 6

Convert 657_{10} to a base 4 number.

Answer

$657_{10} = 22101_4$

? Try it Now 7

Convert 8377_{10} to a base 8 number.

Answer

$8377_{10} = 20271_8$

Another Method For Converting From Base 10 to Other Bases

As you read the solution to this last example and attempted the "Try it Now" problems, you may have had to repeatedly stop and think about what was going on. The fact that you are probably struggling to follow the explanation and reproduce the process yourself is mostly due to the fact that the non-decimal systems are so unfamiliar to you. In fact, the only system that you are probably comfortable with is the decimal system.

As budding mathematicians, you should always be asking questions like "How could I simplify this process?" In general, that is one of the main things that mathematicians do…they look for ways to take complicated situations and make them easier or more familiar. In this section we will attempt to do that.

To do so, we will start by looking at our own decimal system. What we do may seem obvious and maybe even intuitive but that's the point. We want to find a process that we readily recognize works and makes sense to us in a familiar system and then use it to extend our results to a different, unfamiliar system.

Let's start with the decimal number, 4863_10. We will convert this number to base 10. Yeah, I know it's already in base 10, but if you carefully follow what we're doing, you'll see it makes things work out very nicely with other bases later on. We first note that the highest power of 10 that will divide into 4863 at least once is $10^3 = 1000$. *In general, this is the first step in our new process; we find the highest power that a given base that will divide at least once into our given number.*

We now divide 1000 into 4863:

$$4863 \div 1000 = 4.863$$

This says that there are four thousands in 4863 (obviously). However, it also says that there are 0.863 thousands in 4863. This fractional part is our remainder and will be converted to lower powers of our base (10). If we take that decimal and multiply by 10 (since that's the base we're in) we get the following:

$$0.863 \times 10 = 8.63$$

Why multiply by 10 at this point? We need to recognize here that 0.863 thousands is the same as 8.63 hundreds. Think about that until it sinks in.

$$(0.863)(1000) = 863$$

$$(8.63)(100) = 863$$

These two statements are equivalent. So, what we are really doing here by multiplying by 10 is rephrasing or converting from one place (thousands) to the next place down (hundreds).

$$0.863 \times 10 \Rightarrow 8.63$$

$$(\text{Parts of Thousands}) \times 10 \Rightarrow \text{Hundreds}$$

What we have now is 8 hundreds and a remainder of 0.63 hundreds, which is the same as 6.3 tens. We can do this again with the 0.63 that remains after this first step.

$$0.63 \times 10 \Rightarrow 6.3$$

$$\text{Hundreds} \times 10 \Rightarrow \text{Tens}$$

So we have six tens and 0.3 tens, which is the same as 3 ones, our last place value.

Now here's the punch line. Let's put all of the together in one place:

$$4863 \div 1000 = \text{④}.863$$
$$0.863 \times 10 = \text{⑧}.63$$
$$0.63 \times 10 = \text{⑥}.3$$
$$0.3 \times 10 = \text{③}.0$$

Note that in each step, the remainder is carried down to the next step and multiplied by 10, the base. Also, at each step, the whole number part, which is circled, gives the digit that belongs in that particular place. What is amazing is that this works for any base! So, to convert from a base 10 number to some other base, b, we have the following steps we can follow:

Converting from Base 10 to Base b: Another method

1. Find the highest power of the base b that will divide into the given number at least once and then divide.
2. Keep the whole number part, and multiply the fractional part by the base b.
3. Repeat step two, keeping the whole number part (including 0), carrying the fractional part to the next step until only a whole number result is obtained.
4. Collect all your whole number parts to get your number in base b notation.

We will illustrate this procedure with some examples.

Example 9

Convert the base 10 number, 348_{10}, to base 5.

Solution

This is actually a conversion that we have done in a previous example. The powers of five are:

$5^0 = 1$
$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$

Etc…

The highest power of five that will go into 348 at least once is 5^3

We divide by 125 and then proceed.

$$348 \div 5^3 = \text{②}.784$$
$$0.784 \times 5 = \text{③}.92$$
$$0.92 \times 5 = \text{④}0.6$$
$$0.6 \times 5 = \text{③}.0$$

By keeping all the whole number parts, from top bottom, gives 2343 as our base 5 number. Thus, $2343_5 = 348_{10}$

We can compare our result with what we saw earlier, or simply check with our calculator, and find that these two numbers really are equivalent to each other.

Example 10

Convert the base 10 number, 3007_{10}, to base 5.

Solution

The highest power of 5 that divides at least once into 3007 is $5^4 = 625$. Thus, we have:

$3007 \div 625 = (4).8112$
$0.8112 \times 5 = (4).056$
$0.056 \times 5 = (0).28$
$0.28 \times 5 = (1)0.4$
$0.4 \times 5 = (2)0.0$

This gives us that $3007_{10} = 44012_5$. Notice that in the third line that multiplying by 5 gave us 0 for our whole number part. We don't discard that! The zero tells us that a zero in that place. That is, there are no 5^2 's in this number

This last example shows the importance of using a calculator in certain situations and taking care to avoid clearing the calculator's memory or display until you get to the very end of the process.

Example 11

Convert the base 10 number, 63201_{10}, to base 7.

Solution

The powers of 7 are:

$7^0 = 1$
$7^1 = 7$
$7^2 = 49$
$7^3 = 343$
$7^4 = 2401$
$7^5 = 16807$

etc…

The highest power of 7 that will divide at least once into 63201 is 7^5. When we do the initial division on a calculator, we get the following:

$63201 \div 7^5 = 3.760397453$

The decimal part actually fills up the calculators display and we don't know if it terminates at some point or perhaps even repeats down the road. So if we clear our calculator at this point, we will introduce error that is likely to keep this process from ever ending. To avoid this problem, we leave the result in the calculator and simply subtract 3 from this to get the fractional part all by itself. DO NOT ROUND OFF! Subtraction and then multiplication by seven gives:

$63201 \div 7^5 = (3).760397453$
$0.760397453 \times 7 = (5).322782174$
$0.322782174 \times 7 = (2).259475219$
$0.259475219 \times 7 = (1).816326531$
$0.816326531 \times 7 = (5).714285714$
$0.714285714 \times 7 = (5).000000000$

Yes, believe it or not, that last product is exactly 5, *as long as you don't clear anything out on your calculator*. This gives us our final result: $63201_{10} = 352155_7$.

If we round, even to two decimal places in each step, clearing our calculator out at each step along the way, we will get a series of numbers that do not terminate, but begin repeating themselves endlessly. (Try it!) We end up with something that doesn't make any sense, at least not in this context. So be careful to use your calculator cautiously on these conversion problems.

Also, remember that if your first division is by 7^5, then you expect to have 6 digits in the final answer, corresponding to the places for $7^5, 7^4$ and so on down to 7^0. If you find yourself with more than 6 digits due to rounding errors, you know something went wrong

Try it Now 8

Convert the base 10 number, 9352_{10}, to base 5.

Answer

$9352_{10} = 244402_5$

Try it Now 9

Convert the base 10 number, 1500_{10}, to base 3.

Be careful not to clear your calculator on this one. Also, if you're not careful in each step, you may not get all of the digits you're looking for, so move slowly and with caution.

Answer

$1500_{10} = 2001120_3$

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3.3.5: The Mayan Numeral System

Background

As you might imagine, the development of a base system is an important step in making the counting process more efficient. Our own base-ten system probably arose from the fact that we have 10 fingers (including thumbs) on two hands. This is a natural development. However, other civilizations have had a variety of bases other than ten. For example, the Natives of Queensland used a base-two system, counting as follows: “one, two, two and one, two two’s, much.” Some Modern South American Tribes have a base-five system counting in this way: “one, two, three, four, hand, hand and one, hand and two,” and so on. The Babylonians used a base-sixty (sexigesimal) system. In this chapter, we wrap up with a specific example of a civilization that actually used a base system other than 10.

The Mayan civilization is generally dated from 1500 B.C.E to 1700 C.E. The Yucatan Peninsula (see map[i]) in Mexico was the scene for the development of one of the most advanced civilizations of the ancient world. The Mayans had a sophisticated ritual system that was overseen by a priestly class. This class of priests developed a philosophy with time as divine and eternal.[ii] The calendar, and calculations related to it, were thus very important to the ritual life of the priestly class, and hence the Mayan people. In fact, much of what we know about this culture comes from their calendar records and astronomy data. Another important source of information on the Mayans is the writings of Father Diego de Landa, who went to Mexico as a missionary in 1549.



1. Chichen Itza
2. Uxmal
3. Tulum
4. Palenque
5. Bonampak, Yaxchilan
6. Tikal
7. Altun Ha
8. Copán



There were two numeral systems developed by the Mayans - one for the common people and one for the priests. Not only did these two systems use different symbols, they also used different base systems. For the priests, the number system was governed by ritual. The days of the year were thought to be gods, so the formal symbols for the days were decorated heads,[iii] like the sample to the left[iv] Since the basic calendar was based on 360 days, the priestly numeral system used a mixed base system employing multiples of 20 and 360. This makes for a confusing system, the details of which we will skip.

The Mayan Number System

Instead, we will focus on the numeration system of the “common” people, which used a more consistent base system. As we stated earlier, the Mayans used a base-20 system, called the “vigesimal” system. Like our system, it is positional, meaning that the position of a numeric symbol indicates its place value. In the following table you can see the place value in its vertical format.[v]

Powers	Base-Ten Value	Place Name
20^7	$12,800,000,000$	Hablat
20^6	$64,000,000$	Alau
20^5	$3,200,000$	Kinchil
20^4	$160,000$	Cabal
20^3	$8,000$	Pic
20^2	400	Bak
20^1	20	Kal
20^0	1	Hun

In order to write numbers down, there were only three symbols needed in this system. A horizontal bar represented the quantity 5, a dot represented the quantity 1, and a special symbol (thought to be a shell) represented zero. The Mayan system may have been the first to make use of zero as a placeholder/number. The first 20 numbers are shown in the table to the right.[vi]

Unlike our system, where the ones place starts on the right and then moves to the left, the Mayan systems places the ones on the bottom of a vertical orientation and moves up as the place value increases.

When numbers are written in vertical form, there should never be more than four dots in a single place. When writing Mayan numbers, every group of five dots becomes one bar. Also, there should never be more than three bars in a single place…four bars would be converted to one dot in the next place up. It's the same as 10 getting converted to a 1 in the next place up when we carry during addition.

Number	Vertical Form	Number	Vertical Form
0		10	
1		11	
2		12	
3		13	
4		14	
5		15	
6		16	
7		17	
8		18	
9		19	

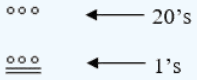
Example 12

What is the value of this number, which is shown in vertical form?



Solution

Starting from the bottom, we have the ones place. There are two bars and three dots in this place. Since each bar is worth 5, we have 13 ones when we count the three dots in the ones place. Looking to the place value above it (the twenties places), we see there are three dots so we have three twenties.



Hence we can write this number in base-ten as:

$$\begin{aligned}(3\times 20^1)+(13\times 20^0) &= (3\times 20)+(13\times 1)\\ &= 60+13\\ &= 73\end{aligned}$$

Example 13

What is the value of the following Mayan number?



Solution

This number has 11 in the ones place, zero in the 20's place, and 18 in the 20^2=400's place. Hence, the value of this number in base-ten is:

$$18\times 400+0\times 20+11\times 1=7211$$

Try it Now 10

Convert the Mayan number below to base 10.



Answer

1562

Example 14

Convert the base 10 number 3575_{10} to Mayan numerals.

Solution

This problem is done in two stages. First we need to convert to a base 20 number. We will do so using the method provided in the last section of the text. The second step is to convert that number to Mayan symbols.

The highest power of 20 that will divide into 3575 is $20^2 = 400$, so we start by dividing that and then proceed from there:

$$3575 \div 400 = 8.9375$$
$$0.9375 \times 20 = 18.75$$
$$(0.75 \times 20 = 15.0$$

This means that $3575_{10} = 8, 18, 15_{20}$

The second step is to convert this to Mayan notation. This number indicates that we have 15 in the ones position. That's three bars at the bottom of the number. We also have 18 in the 20's place, so that's three bars and three dots in the second position. Finally, we have 8 in the 400's place, so that's one bar and three dots on the top. We get the following



Note that in the previous example a new notation was used when we wrote $8, 18, 15_{20}$. The commas between the three numbers 8, 18, and 15 are now separating place values for us so that we can keep them separate from each other. This use of the comma is slightly different than how they're used in the decimal system. When we write a number in base 10, such as 7,567,323, the commas are used primarily as an aide to read the number easily but they do not separate single place values from each other. We will need this notation whenever the base we use is larger than 10.

Writing numbers with bases bigger than 10

When the base of a number is larger than 10, separate each "digit" with a comma to make the separation of digits clear.

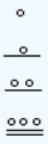
For example, in base 20, to write the number corresponding to $17 \times 20^2 + 6 \times 20^1 + \ 13 \times 20^0$, we'd write $17, 6, 13_{20}$.

Try it Now 11

Convert the base 10 number 10553_{10} to Mayan numerals.

Answer

$10553_{10} = 1, 6, 7, 13_{20}$



Try it Now 12

Convert the base 10 number 5617 to Mayan numerals.

Answer

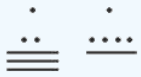
$5617_{10} = 14, 0, 17_{20}$. Note that there is a zero in the 20's place, so you'll need to use the appropriate zero symbol in between the ones and 400's places.

Adding Mayan Numbers

When adding Mayan numbers together, we'll adopt a scheme that the Mayans probably did not use but which will make life a little easier for us.

Example 15

Add, in Mayan, the numbers 37 and 29: [vii]



Solution

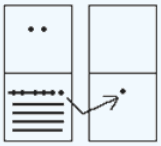
First draw a box around each of the vertical places. This will help keep the place values from being mixed up.



Next, put all of the symbols from both numbers into a single set of places (boxes), and to the right of this new number draw a set of empty boxes where you will place the final sum:

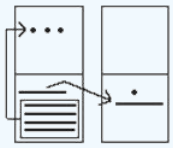


You are now ready to start carrying. Begin with the place that has the lowest value, just as you do with Arabic numbers. Start at the bottom place, where each dot is worth 1. There are six dots, but a maximum of four are allowed in any one place; once you get to five dots, you must convert to a bar. Since five dots make one bar, we draw a bar through five of the dots, leaving us with one dot which is under the four-dot limit. Put this dot into the bottom place of the empty set of boxes you just drew:

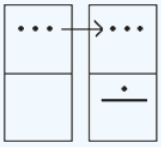


Now look at the bars in the bottom place. There are five, and the maximum number the place can hold is three. *Four bars are equal to one dot in the next highest place*.

Whenever we have four bars in a single place we will automatically convert that to a *dot* in the next place up. We draw a circle around four of the bars and an arrow up to the dots' section of the higher place. At the end of that arrow, draw a new dot. That dot represents 20 just the same as the other dots in that place. Not counting the circled bars in the bottom place, there is one bar left. One bar is under the three-bar limit; put it under the dot in the set of empty places to the right.



Now there are only three dots in the next highest place, so draw them in the corresponding empty box.



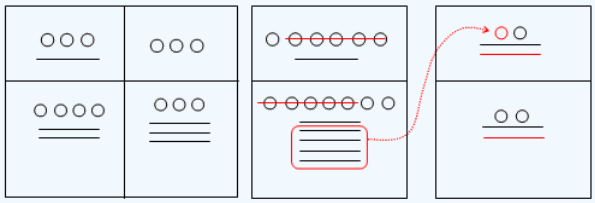
We can see here that we have 3 twenties (60), and 6 ones, for a total of 66. We check and note that $37+29=66$, so we have done this addition correctly. Is it easier to just do it in base-ten? Probably, but that's only because it's more familiar to you. Your task here is to try to learn a new base system and how addition can be done in slightly different ways than what you have seen in the past. Note, however, that the concept of carrying is still used, just as it is in our own addition algorithm.

? Try it Now 13

Try adding 174 and 78 in Mayan by first converting to Mayan numbers and then working entirely within that system. Do not add in base-ten (decimal) until the very end when you *check* your work.

Answer

A sample solution is shown.



Conclusion

In this chapter, we have briefly sketched the development of numbers and our counting system, with the emphasis on the "brief" part. There are numerous sources of information and research that fill many volumes of books on this topic. Unfortunately, we cannot begin to come close to covering all of the information that is out there.

We have only scratched the surface of the wealth of research and information that exists on the development of numbers and counting throughout human history. What is important to note is that the system that we use every day is a product of thousands of years of progress and development. It represents contributions by many civilizations and cultures. It does not come down to us from the sky, a gift from the gods. It is not the creation of a textbook publisher. It is indeed as human as we are, as is the rest of mathematics. Behind every symbol, formula and rule there is a human face to be found, or at least sought.

Furthermore, we hope that you now have a basic appreciation for just how interesting and diverse number systems can get. Also, we're pretty sure that you have also begun to recognize that we take our own number system for granted so much that when we try to adapt to other systems or bases, we find ourselves truly having to concentrate and think about what is going on.

[i] www.gorp.com/gorp/location/latamer/map_maya.htm

[ii] Bidwell, James; Mayan Arithmetic in *Mathematics Teacher*, Issue 74 (Nov., 1967), p. 762-68.

[iii] www.ukans.edu/~lctls/Mayan/numbers.html

[iv] www.ukans.edu/~lctls/Mayan/numbers.html

[v] Bidwell

[vi] www.vpds.wsu.edu/fair_95/gym/UM001.html

[vii] forum.swarthmore.edu/k12/mayan.math/mayan2.html

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3.3.6: Roman Numerals

Why study Roman Numerals?

Even though Roman Numerals are rare in today's society, they are still used and expected to be understood. They are taught in grades three through five, depending on the district. They can be seen in clocks, the Super Bowl, Film Credits for the copyright date like MCMLXII, preface of textbooks and others like Star Wars Episode VI and WWII.

2.4.1: Basic Table of Roman Numerals

Value	Symbol
1	I
5	V
10	X
50	L
100	C
500	D
1000	M

2.4.2: The Complete Table of Roman Numerals

	1	2	3	4	5	6	7	8	9
ONES	I	II	III	IV	V	VI	VII	VIII	IX
TENS	X	XX	XXX	XL	L	LX	LXX	LXXX	XC
HUNDREDS	C	CC	CCC	CD	D	DC	DCC	DCCC	CM
THOUSANDS	M	MM	MMM	$\overline{\text{IV}}$	$\overline{\text{V}}$	$\overline{\text{VI}}$	$\overline{\text{VII}}$	$\overline{\text{VIII}}$	$\overline{\text{IX}}$
TEN THOUSANDS	$\bar{X}$	$\overline{XX}$	$\overline{XXX}$	$\overline{XL}$	$\bar{L}$	$\overline{LX}$	$\overline{LXX}$	$\overline{LXXX}$	$\overline{XC}$
HUNDRED THOUSANDS	$\bar{C}$	$\overline{CC}$	$\overline{CCC}$	$\overline{CD}$	$\bar{D}$	$\overline{DC}$	$\overline{DCC}$	$\overline{DCCC}$	$\overline{\boldsymbol{CM}}$

Example 3.3.6.1

For all numbers except 4 and 9, we **ADD** the Roman Numerals together, in order from left to right, greatest value to lowest value.

$$\begin{aligned} 11 &= 10+1 = XI \\ 8 &= 5+1+1+1 = VIII \\ 123 &= 100+10+10+1+1+1 = CXXIII \\ 3816 &= 1000+1000+1000+500+100+100+100+10+5+1 = MMMDCCCXVI \\ 7002 &= 5000+1000+1000+1+1 = \overline{VIII}I \end{aligned}$$

Example 3.3.6.2

For any number that includes a 4 or a 9, we subtract. When we are looking at a Roman Number expression and we see a Roman character **OUT OF ORDER**, which is the clue to **SUBTRACT**!

$$9 = 10 - 1 = IX$$
$$4 = 5 - 1 = IV$$
$$1400 = 1000 + (500 - 100) = MCD$$
$$29452 = 10000 + 10000 + 10000 - 1000 + (500 - 100) + 50 + 1 + 1 = \overline{IXXX}CDLII$$

Partner Activity 1

Convert Between Roman Numerals and Our Current Decimal System:

1. MMXLV = _____
2. MDCCLXXXIX = _____
3. 1993 = _____
4. 5495 = _____

Practice Problems

Convert Between Roman Numerals and Our Current Decimal System:

1. 84759
2. MMXX
3. MCMLXXXII
4. 1764
5. 50000
6. $\overline{XX}LXXXIX$

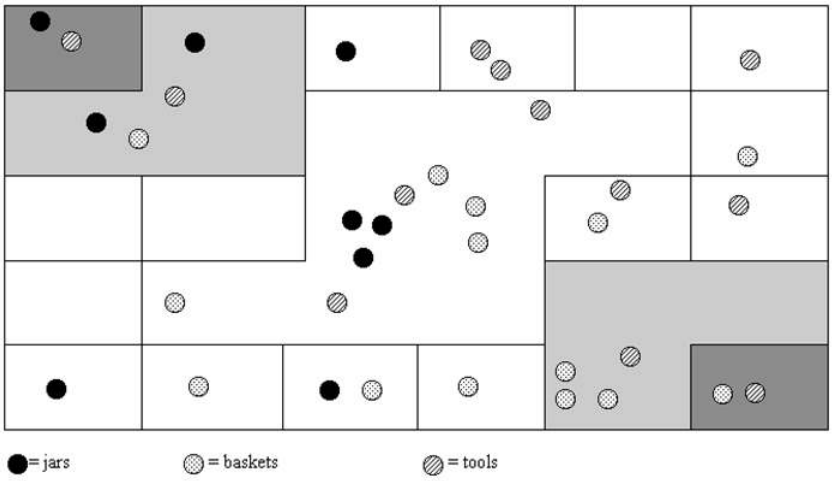
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3.3.7: Exercises

Skills

Counting Board And Quipu

1) In the following Peruvian counting board, determine how many of each item is represented. Please show all of your calculations along with some kind of explanation of how you got your answer. Note the key at the bottom of the drawing.



2) Draw a quipu with a main cord that has branches (H cords) that show each of the following numbers on them. (You should produce one drawing for this problem with the cord for part **a** on the left and moving to the right for parts **b** through **d**.)

a. 232	b. 5065
c. 23451	d. 3002

Basic Base Conversions

3) 423 in base 5 to base 10	4) 3044 in base 5 to base 10
5) 387 in base 10 to base 5	6) 2546 in base 10 to base 5
7) 110101 in base 2 to base 10	8) 11010001 in base 2 to base 10
9) 100 in base 10 to base 2	10) 2933 in base 10 to base 2
11) Convert 653 in base 7 to base 10	12) Convert 653 in base 10 to base 7
13) 3412 in base 5 to base 2	14) 10011011 in base 2 to base 5

(Hint: convert first to base 10 then to the final desired base)

The Caidoz System

Suppose you were to discover an ancient base-12 positional system made up twelve symbols. Let's call this base system the Caidoz system. Here are the symbols for each of the numbers 0 through 12:

0 = ♈	6 = ♎
1 = ♉	7 = ♏
2 = ♊	8 = ♐
3 = ♋	9 = ♑
4 = ♌	10 = ♒
5 = ♍	11 = ♓

Convert each of the following numbers in Caidoz to base 10

15) ♏♉♒ 16) ♐♋♈♓

17) ♎♌♊ 18) ♍♉♒♈

Convert the following base 10 numbers to Caidoz, using the symbols shown above.

19) 175	20) 3030
21) 10000	22) 5507

Mayan Conversions

Convert the following numbers to Mayan notation. Show your calculations used to get your answers.

23) 135	24) 234
25) 360	26) 1215
27) 10500	28) 1100000

Convert the following Mayan numbers to decimal (base-10) numbers. Show all calculations.

29. 

30. 

31. 

32. 

James Bidwell has suggested that Mayan addition was done by "simply combining bars and dots and carrying to the next higher place." He goes on to say, "After the combining of dots and bars, the second step is to exchange every five dots for one bar in the same position." After converting the following base 10 numbers into vertical Maya notation (in base 20, of course), perform the indicated addition:

33) 32 + 11	34) 82 + 15
35) 35 + 148	36) 2412 + 5000
37) 450 + 844	38) 10000 + 20000
39) 4500 + 3500	40) 130000 + 30000

41) Use the fact that the Mayans had a base-20 number system to complete the following multiplication table. The table entries should be in Mayan notation. Remember: Their zero looked like this…. *Xerox and then cut out the table below, fill it in, and paste it onto your homework assignment if you do not want to duplicate the table with a ruler.*

(To think about but not write up: Bidwell claims that only these entries are needed for "Mayan multiplication." What does he mean?)

Binary and Hexadecimal Conversions

Modern computers operate in a world of "on" and "off" electronic switches, so use a **binary** counting system – base 2, consisting of only two digits: 0 and 1.

Convert the following binary numbers to decimal (base-10) numbers.

42) 1001	43) 1101
44) 110010	45) 101110

Convert the following base-10 numbers to binary

46) 7	47) 12
48) 36	49) 27

Four binary digits together can represent any base-10 number from 0 to 15. To create a more human-readable representation of binary-coded numbers, hexadecimal numbers, base 16, are commonly used. Instead of using the $8,13,12_{16}$ notation used earlier, the letter A is used to represent the digit 10, B for 11, up to F for 15, so $8,13,12_{16}$ would be written as 8DC.

Convert the following hexadecimal numbers to decimal (base-10) numbers.

50) C3	51) 4D
52) 3A6	53) BC2

Convert the following base-10 numbers to hexadecimal

54) 152	55) 176
56) 2034	57) 8263

Exploration

58) What are the advantages and disadvantages of bases other than ten.

59) Supposed you are charged with creating a base-15 number system. What symbols would you use for your system and why? Explain with at least two specific examples how you would convert between your base-15 system and the decimal system.

60) Describe an interesting aspect of Mayan civilization that we did not discuss in class. Your findings must come from some source such as an encyclopedia article, or internet site and you must provide reference(s) of the materials you used (either the publishing information or Internet address).

61) For a Papuan tribe in southeast New Guinea, it was necessary to translate the bible passage John 5:5 “And a certain man was there, which had an infirmity 30 and 8 years” into “A man lay ill one man, both hands, five and three years.” Based on your own understanding of bases systems (and some common sense), furnish an explanation of the translation. Please use complete sentences to do so. (Hint: To do this problem, I am asking you to think about how base systems work, where they come from, and how they are used. You won’t necessarily find an “answer” in readings or such…you’ll have to think it through and come up with a reasonable response. Just make sure that you clearly explain why the passage was translated the way that it was.)

62) The Mayan calendar was largely discussed leading up to December 2012. Research how the Mayan calendar works, and how the counts are related to the number based they use.

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3.4: Different Bases and Their Number Lines

The way we count, the way we visualize numbers, all starts from a number line. As early as Kindergarten, students learn how to read and use a number line. Our current number system is called the Decimal Number System (or Hindu-Arabic). Notice that Decimal begins with DEC like DECahedron (a 10 sided polygon) and we have 10 characters in our numbering system: zero through nine. So, even though we live in a Base 10 system (because we have characters for zero through nine), the number ten is represented by two characters (digits), one and zero.

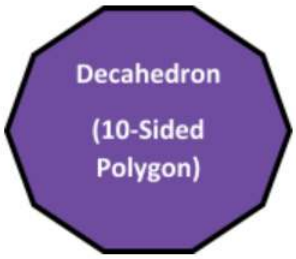


Figure 2.5.1: Decahedron

Below is the number line for **Base 10**. Notice how it is broken up into rows. We will be using the number lines as rows for the sake of our lesson. However, when you teach the number line to the elementary school students, the number line will be one continuous row (to set up for negative numbers in second grade) OR set up like below only going from 1 to 10 or 0 through 10.

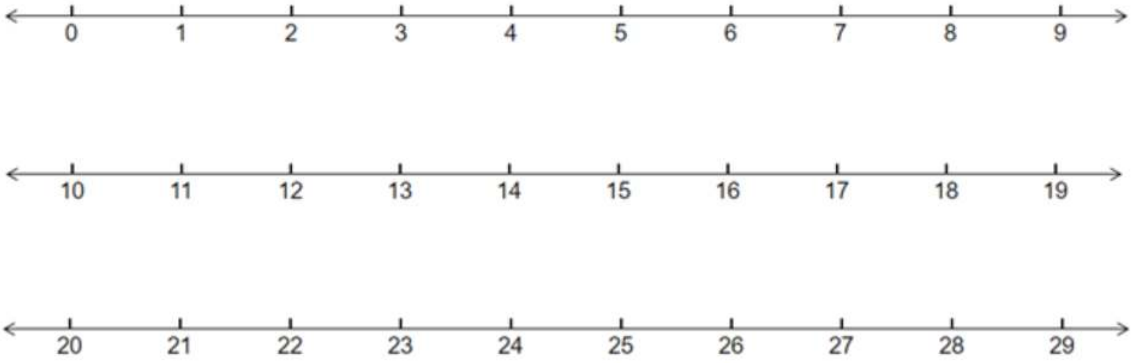


Figure 2.5.2: Number line for Base 10

There are an infinite amount of different bases and an infinite amount of corresponding number lines. Below are three different examples, written like we have Base 10 on the previous page.

Base 2 Number Line (reads from left to right, then top to bottom). The fifth number in the number line is 101_{two}.

Table 2.5.1: Base 2 Number Line

0	1
10	11
100	101
110	111
1000	1001
1010	1011
1100	1101
1110	1111
10000	10001

Base 8 Number Line (reads from left to right, then top to bottom). The 10^{th} number is 12_{eight}.

Table 2.5.2: Base 8 Number Line

0	1	2	3	4	5	6	7
10	11	12	13	14	15	16	17
20	21	22	23	24	25	26	27

Base 12 Number Line (reads from left to right, then top to bottom). The 15^{th} number is 13_{twelve}.

Table 2.5.3: Base 12 Number Line

0	1	2	3	4	5	6	7	8	9	A	B
10	11	12	13	14	15	16	17	18	19	1A	1B
20	21	22	23	24	25	26	27	28	29	2A	2B

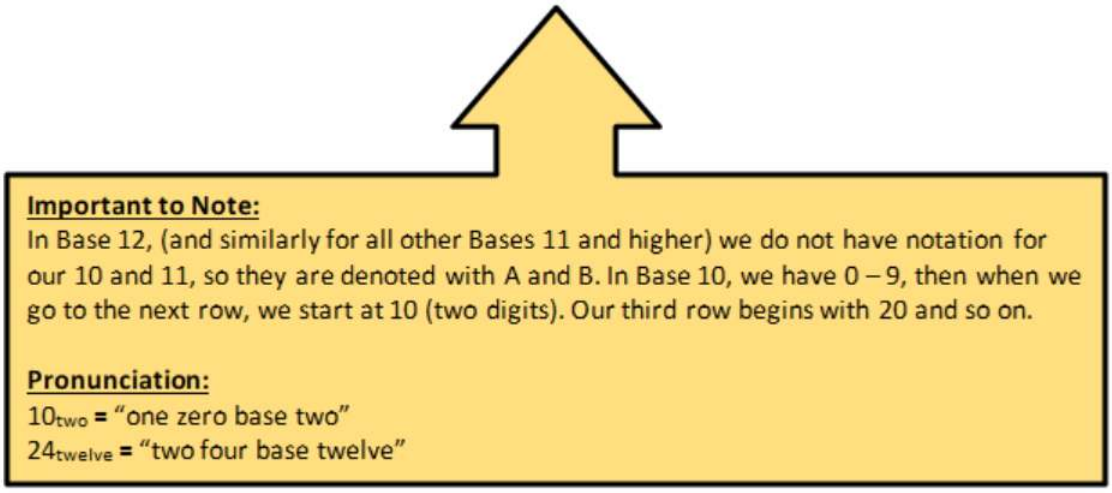


Figure 2.5.3

Where are different bases used?

Base 2: Computers use Base 2, just zeros and ones for all of their programming. Your phone operates in only zeros (OFF) and ones (ON).

Base 5: This was one of the very first systems of counting, since we have five fingers.

Base 8: Think of Base 8 as the mathematics for the Cartoon Universe. Many Cartoons have only eight fingers. We have 10 fingers and live in a Base 10 universe. Cartoon have eight fingers and live in a Base 8 universe. The Cartoon character does not have a character for eight or nine items, as we do in our universe.

Base 12: Base 12 is rare to find within history. However, we currently have 12 hours on the clock and 12 months in the year. There were a few tribes in Africa and India which used the duodecimal (base 12) system.

Base 20: The Mayans used a Base 20 system, and invented the concept of zero.

Base 60: An extreme example is base 60, which was used by the Babylonians (about 4000 years ago) which is now current day Iraq. They had characters for 1 – 59 items. (The concept of "zero" was not discovered yet.) However, this is where our current concept of 60 minutes and 60 seconds come from.

Practice Problems

1. What is wrong about 413_{four}?
2. How do you pronounce 542_{six}?
3. Write four rows worth of the number line of Base 3.

4. If you were not familiar with the Base 10 system, which base system do you think would work best for our society and culture?

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3.5: Converting Between (our) Base 10 and Any Other Base (and vice versa)

To convert any number in (our base) Base 10 to any other base, we must use basic division with remainders. Do not divide using decimals; the algorithm will not work.



Figure 2.6.1

Example 3.5.1

Convert from (our) Base 10 to (weird) Base _____

Change 236_{ten} to _____ $_{\text{five}}$

Solution

Keep **dividing by 5**, until your quotient is zero.

$$\begin{aligned} 236 \div 5 &= 47\ r\ \mathbf{1} \\ 47 \div 5 &= 9\ r\ \mathbf{2} \\ 9 \div 5 &= 1\ r\ \mathbf{4} \\ 1 \div 5 &= 0\ r\ \mathbf{1} \end{aligned}$$

Now write your remainders backwards!

Answer: 1421_{five}

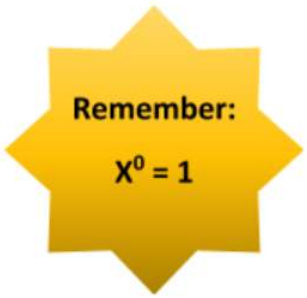


Figure 2.6.2

Example 3.5.2

Convert from (weird) Base ____ to (our) Base 10.

Solution

First, notice how to break down 602_{ten}:

$$602_{\text{ten}} : 602 = 6\left(10^2\right) + 0\left(10^1\right) + 2\left(10^0\right)$$

Now, use the same approach to change 602_{eight} into Base 10

$$6\left(8^2\right) + 0\left(8^1\right) + 2\left(8^0\right) = 386_{\text{ten}}$$

Example 3.5.3

Convert 5361_{seven} into Base 10.

Solution

$$5\left(7^3\right)+3\left(7^2\right)+6\left(7^1\right)+1\left(7^0\right)=1905_{\text{ten}}$$

Partner Activity 1

1. Convert the base 10 numbers into base 4
   a. 30_{ten} = _____ four
   b. 2103_{ten} = _____ four
   c. 16_{ten} = _____ four
2. Convert the base 5 numbers into base 10
   a. 30_{five} = _____ ten
   b. 2103_{five} = _____ten
   c. 16_{five} = _____ ten

Think carefully about 2c!

****For extra practice, click here.*

Practice Problems

1. Write the following Base 10 numbers into the new Base.
   a. 5567 into Base 9
   b. 12 into Base 4
   c. 100 into Base 3
   d. 73 into Base 2
2. Write the following numbers into Base 10.
   a. 64_{seven}
   b. 157_{eight}
   c. 1001001_{two}
   d. 84671_{eleven}

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3.6: Place Values with Different Bases

Table 2.7.1: Place Values with different bases

Base 10		Base 2		Base 8		Base 12	
10^0	ones	2^0	ones	8^0	ones	12^0	ones
10^1	tens	2^1	twos	8^1	eights	12^1	twelves
10^2	hundreds	2^2	fours	8^2	sixty-fours	12^2	one-hundred forty-fours
10^3	thousands	2^3	eights	8^3	five-hundred twelves	12^3	one-thousand seven-hundred twenty-eights

Notice that all the words are plural!

Note: $8^3 = 512$ and $12^3 = 1728$

Practice Problems

In our Base 10 system, the place value for 10,000 is called the ten thousands. Write the equivalent name for the same place value for:

1. Base 3
2. Base 5
3. Base 9
4. Base 14

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3.7: Operations in Different Bases

First, let us set up the number lines. We plan to add in base six and then (separate exercise) base three. We are not adding mixed bases in this book, but yes, it is a thing. Google it.

Second, add it vertically. Rewrite the problem. Carry your numbers that are greater than or equal to 10, just like you did in elementary school. But count using the number (actually count it with your finger, swallow your pride and do it).

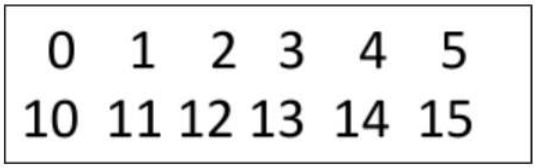


Figure 2.8.1: Base Six Number Line

Example 3.7.1

$$5415_{\text{SIX}} + 3042_{\text{SIX}}$$

$$\begin{array}{r} 5415_{\text{six}} \\ +3042_{\text{six}} \\ \hline 12501_{\text{six}} \end{array}$$

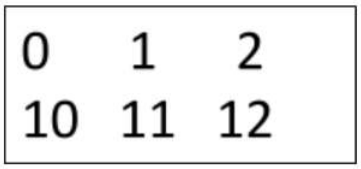


Figure 2.8.2: Base Three Number Line

Example 3.7.2

$$1202_{\text{THREE}} + 1022_{\text{THREE}}$$

$$\begin{array}{r} 1202_{\text{three}} \\ +1022_{\text{three}} \\ \hline 10001_{\text{three}} \end{array}$$

Now, we will subtract in different bases. For this example, we will use base seven and eight respectively. Below are the corresponding number lines. Again, use your fingers to count on the number line (everyone else is doing it). Borrow, when appropriate, like you did in elementary school.

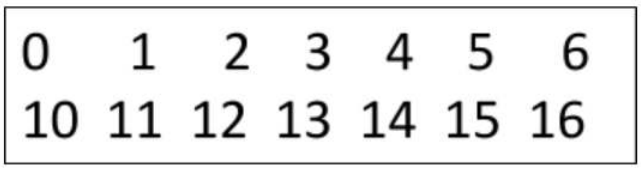


Figure 2.8.3: Base Seven Number Line

Example 3.7.3

$$5631234_{\text{SEVEN}} - 2564310_{\text{SEVEN}}$$

$$\begin{array}{r} 5631234_{\text{seven}} \\ -2564310_{\text{seven}} \\ \hline 3033624_{\text{seven}} \end{array}$$

0	1	2	3	4	5	6	7
10	11	12	13	14	15	16	17

Figure 2.8.4: Base Eight Number Line

Example 3.7.4

$$3741_{\text{EIGHT}} - 1465_{\text{EIGHT}}$$

$$\begin{array}{r} 3741_{\text{eight}} \\ -1465_{\text{eight}} \\ \hline 2254_{\text{eight}} \end{array}$$

Partner Activity 1

Add or subtract the following numbers:

1. $65_{\text{nine}} + 41_{\text{nine}} =$
2. $123_{\text{four}} + 301_{\text{four}} =$
3. $540_{\text{six}} - 23_{\text{six}} =$
4. $220_{\text{three}} - 122_{\text{three}} =$
5. $1001001_{\text{two}} - 1011_{\text{two}} =$

Practice Problems

Add or subtract the following expressions:

1. $435_{\text{six}} - 31_{\text{six}}$
2. $10010_{\text{two}} - 111_{\text{two}}$
3. $2100_{\text{three}} + 21_{\text{three}}$
4. $57643_{\text{eight}} + 24677_{\text{eight}}$

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CHAPTER OVERVIEW

4: Arithmetic and Mental Mathematics

4.1: The Why
4.2: Addition and Subtraction
4.3: Multiplication
4.4: Division
4.5: Estimation and Rounding
4.6: Personal Referents
4.7: Calculating Percentages
4.8: Extension - Methods of Teaching Mathematics

Thumbnail: Animated example of multi-digit long division. (CC BY-SA 3.0; Xanthoxyl via Wikipedia)

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4.1: The Why

The Essential Question

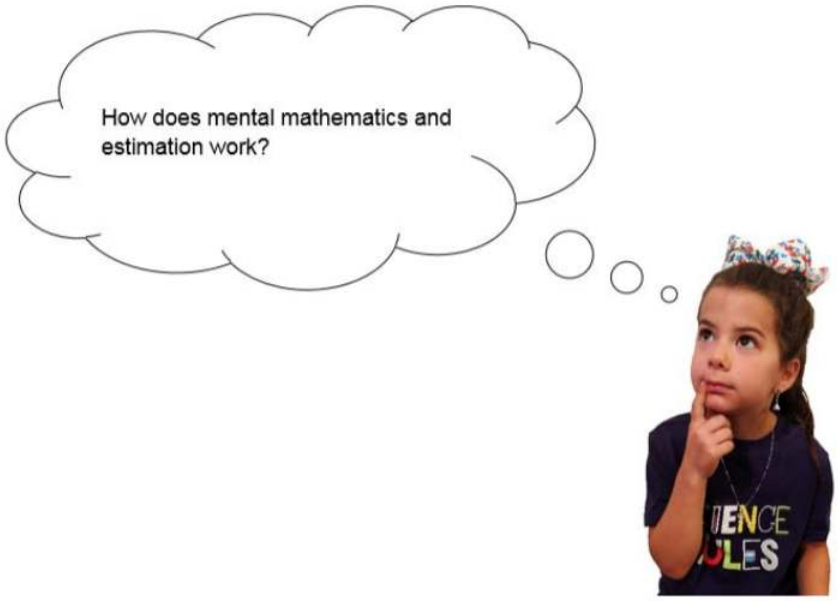


Figure 3.1.1

Why are Teachers Learning this Material?

Common Core has changed many of the procedures learned throughout this chapter. Most people teach the way they were taught. Most of us did not experience Common Core mathematics in Elementary school, so we need to relearn basic mathematics and retrain our brain on how to teach accurately to the Common Core State Standards. Common Core Mathematics might seem weird or confusing to you, but imagine learning it for the first time, without the prior knowledge. It will make more sense.

Why are Elementary School Students Learning this Mathematics?

One of the major components of Common Core for Elementary School Mathematics is calculating basic mathematics faster and mentally.

Practice Problems

1. List at least three times in your personal daily life where you use mental math (outside of school).
2. Why do you think it is important for elementary school students to be proficient in mental math and estimation?

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4.2: Addition and Subtraction

The Three Methods of Subtraction

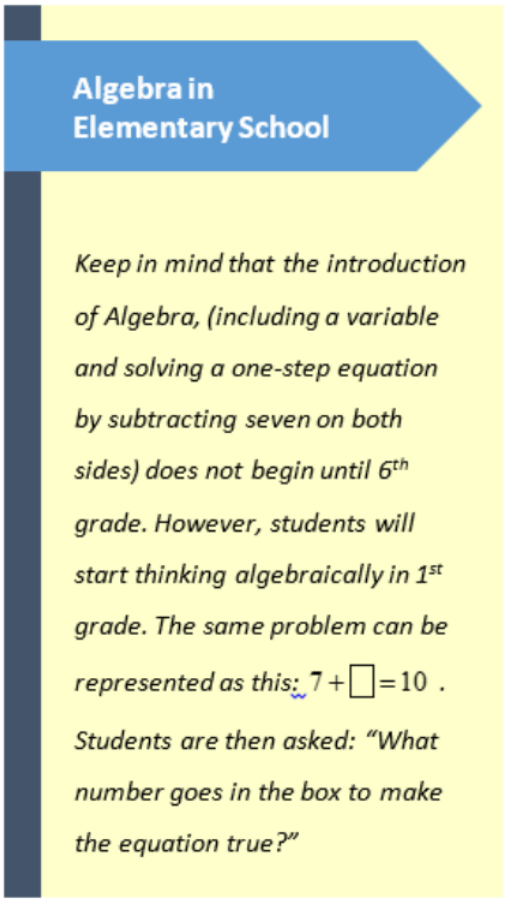


Figure 3.2.1

Example 4.2.1

The Three methods:

1. Missing Addends
   i. Represent with an algebraic equation
   ii. Example: "Mary has seven bananas, but she needs ten. How many more bananas does she need to purchase?"
   iii. $7+x=10 \rightarrow x=3$
2. Take-Away
   i. Straight forward subtraction
   ii. Example: "Jennifer has ten oranges. She sold three of them. How many oranges does she have left?"
   iii. 10-3=7
3. Comparison
   i. Comparing Separate Quantities
   ii. Example: "Cory has ten oranges and seven bananas. How many more oranges does Cory have than bananas?"
   iii. 10-7=3

Partner Activity 1

By yourself, solve 354-89, any way you like. Then compare and contrast your method to your partner's method. Be prepared to share your method with the class.

Partner Activity 2

Consider the work of nine 2nd graders, all solving 354-89, just like you did a minute ago. Grade each student, as if you were their teacher, using a scale from 1 – 5, 5 being the best. Having the correct answer is only one point out of five. The other four points come from the students' procedure and thoughts. Remember, even if you do not understand HOW they arrived at their correct answer, does not make their procedure incorrect.

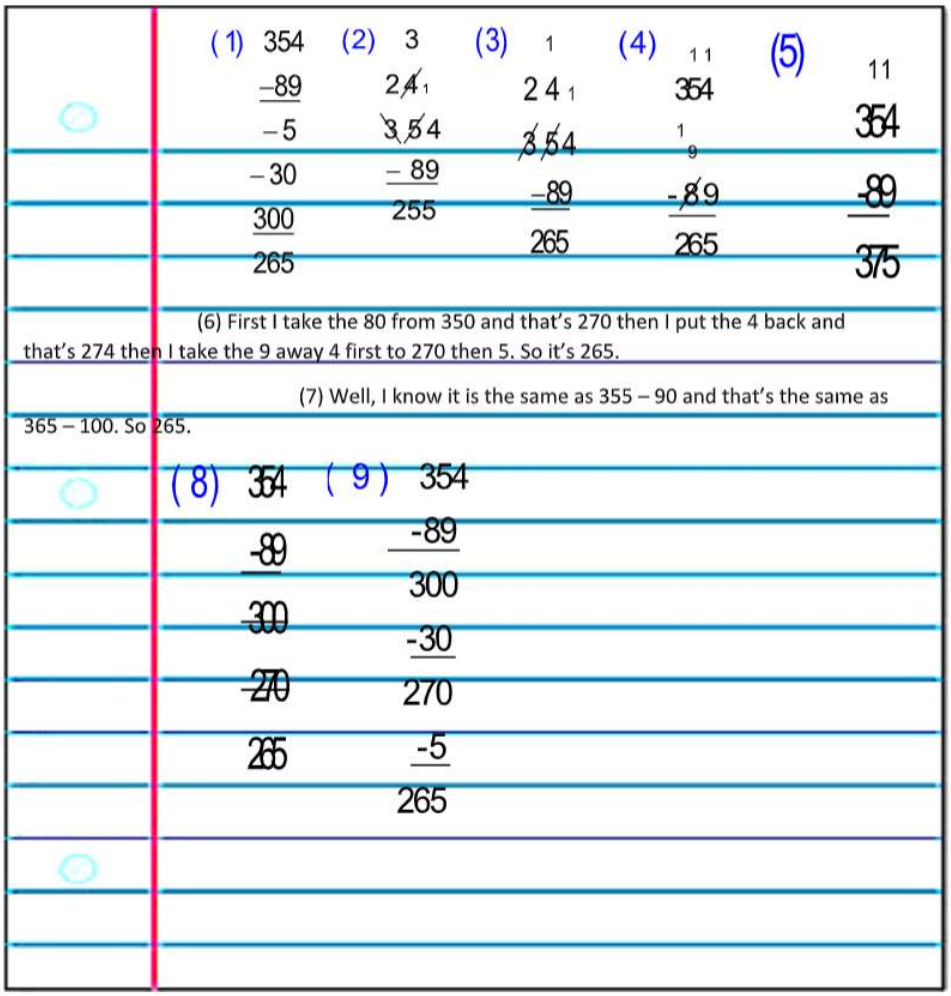


Figure 3.2.2

Example 4.2.2

Here is a real-life example of needing to subtract, but actually using addition:

Lance buys some supplies totaling $7.32. He hands the cashier a ten-dollar bill. His change is $2.68.

Solution

Instead of subtracting 10 – 7.32, the cashier will count UP:

"$7.32 + $1 + $1 + 25¢ + 25¢ + 10¢ + 5¢ + 1¢ + 1¢ + 1¢ = $10.00"

Example 4.2.3

Subtract 342 – 186 = 156 using a number line and count UP.

Solution

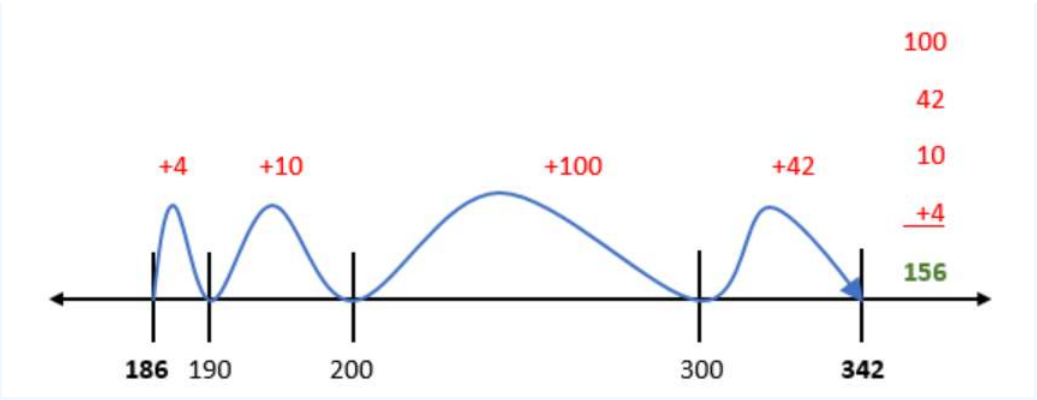


Figure 3.2.3

Why are the above examples in the Mental Math section of this textbook? Because doing these problems on paper enough times will train your brain to subtract with mental math and without borrowing.

Partner Activity 3

Subtract the following problems using the methods from either Example 2 or Example 3 above.

1. 753 – 345 = _____
2. 421 – 175 = _____

Practice Problems

Explain how to solve the following problems using mental math:

1. 56 + 81
2. 1000 – 284
3. 94 + 801
4. 762 – 451

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4.3: Multiplication

Definition: Multiplication

What is the definition of Multiplication? ***Repeated Addition***

Common Core has slightly changed the WHY behind Repeated Addition

Repeated addition

Example 4.3.1

1. Watermelons cost $3 per pound. How much does a watermelon cost which weighs five pounds?
   a. We could answer this question two different ways
      i. $5+5+5=5\times 3=\$15$ (3 groups of 5)
      ii. $3+3+3+3+3=3\times 5=\$15$ (5 groups of 3)
   b. Which method is correct? *The first one, only.*
   c. Why? *Two reasons.*
      i. Think farther ahead in math: $5^3=5\times 5\times 5$, which matches how we write. Do keep in mind that multiplication is communitive, and.
      ii. The watermelon is the subject of the problem. Its weight is the first issue. We pick up a watermelon and then look at the price. 5 pounds multiplies to $3 because we have 5 pounds for each dollar.
2. Are we able to use repeated addition for decimals like $0.3\times 6=1.8$
   a. Six boxes weigh 0.3 pounds each. What is the total weight?
      i. Are you able to write a repeated 6, 0.3 times? No. So instead we write:
      $6\times 0.3=0.3+0.3+0.3+0.3+0.3+0.3=1.8$
   b. What is 30% of 6 pounds?
      i. $6\times 0.3=0.3+0.3+0.3+0.3+0.3+0.3=1.8$ (6 groups of 0.3)

Sketching Multiplication

Common Core is very big on visualizing mathematics. Students are expected to make drawings to show how multiplication works.

Example 4.3.2

Sketch 3×2

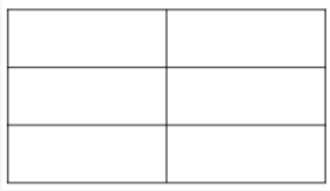


Figure 3.3.1

Sketch 2×3



Figure 3.3.2

Multiplying with Decimals

Example 4.3.3

Step 1: Multiply 7 to each digit in the top number from right to left.

Step 2: Multiply 4 to each digit in the top number from right to left.

Step 3: Multiply 2 to each digit in the top number from right to left.

Step 4: Add

Step 5: Move the decimal three spots to the left. Since there are three digits to the right of the decimals, we move the final decimal three spots to the left. (bold numbers)

$$\begin{array}{r} 123.8 \\ \times 2.47 \\ \hline 28666 \\ 49520 \\ +247600 \\ \hline 305.\mathbf{786} \end{array}$$

Practice Problems

1. Expand out 7×4
2. Expand out 2×6
3. Multiply 4.61×7.94 (do not round)
4. Multiply 516.4×0.347(do not round)

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4.4: Division

Model Division with Repeat-Subtraction

Example 4.4.1

Leslie has 15 cookies to share with her friends. Each friend wants three cookies. How many friends can Leslie share her cookies with?



Figure 3.4.1

Solution

$15-3=12-3=9-3=6-3=3-3=0$

There are five groups of cookies, so Leslie is able to share with five friends.

This procedure is called Measurement Division. Measurement is any movement across the number line. On the number line, we are starting at 15 and moving down to zero by three's.

Model Division with Sharing

Example 4.4.2

Charlie has 15 balloons and three children. How many balloons does each child receive?

Solution

Each child receives five balloons.

This procedure is called Sharing Equally OR Partitive Division. Think of it like using the distributive property. Charlie is equally distributing a balloon to each child. Once each child has one balloon, Charlie then equally distributes another round of balloons to each child. Charlie stops when he runs out of balloons.

Partner Activity 1 - Dividing by Zero

What is $5 \div 0$ also written as $\frac{5}{0}$? Why?

What is $0 \div 5$ also written as $\frac{0}{5}$? Why?

What is $0 \div 0$ also written as $\frac{0}{0}$? Why?

**How would you explain dividing by zero to a 4th grader??

Practice Problems

1. Sophie needs to pass out papers to her coworkers. She has 28 papers and each coworker needs four papers. How many coworkers does she have? What type of division is this?
2. Later that day, Sophie speaks to ten other coworkers who need different papers. She runs off 30 copies. How many papers did each coworker receive? What type of division is this?

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4.5: Estimation and Rounding

What is the difference between estimation and rounding? First, let us look at the definitions from Google:

Definition: Estimation

A rough calculation of the value, number, quantity, or extent of something.

Definition: Rounding

Alter (a number) to one less exact but more convenient for calculations.

Now let us look at some examples and determine which one is which:

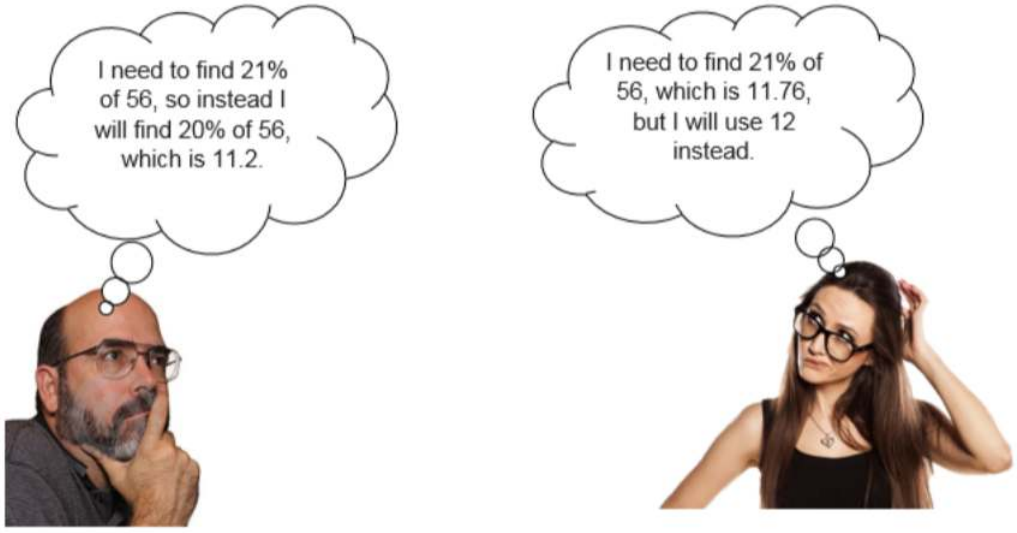


Figure 3.5.1

Irwin's example is **Estimation**. He is estimating an answer by using easier numbers.

Mary's example is **Rounding**. She is rounding up 0.76 to 1, to make the numbers easier for her.

One of the main components of Common Core was to make sure that students could perform mental math exercises with ease. Therefore, there is a strong emphasis on percentages, estimation, rounding and multiplication.

Mental Math Percentages

(use these to help you with Example 4.5.1*)*

10%: Move the decimal once to the left

5%: Find 10% and divide by 2

20%: Find 10% and multiply by 2

1%: Move the decimal twice to the left

Example 4.5.1

Restaurant Split

For dinner, on Friday night, Chris and Maddy dine at Seven Seas Seafood. They decide to leave their phones in the car, so that there are no distractions and actually have some real conversation! The dinner went very well and the bill comes. Maddy decided to be nice and pay the entire bill; after all, she did order the crab buffet. Now, her phone is in the car and she needs to calculate tip and total. Chris says, "I think you should tip 17%. The service was good, but not great. And do it quick, I want to go home."



Figure 3.5.2

Solution

Luckily, Maddy remembered how to use her amazing estimation skills and tipped $20.

$$17\% = 10\% + 5\% + 1\% + 1\%$$
$$10\% = \$11.658 \approx \$12$$
$$5\% \approx \$12 \div 2 = \$6$$
$$1\% = \$1.1658 \approx \$1$$
$$\$12 + \$6 + \$1 + \$1 = \$20$$

Partner Activity 1

Calculate tip for the following restaurant bills using the method presented in Example 4.5.1. Make sure you round to the nearest penny.

1. $85.12
2. $21.47
3. $96.01

Mental Math Multiplication

Example 4.5.2

Growing up, most of us learned the multiplication tables from zero to ten. Maybe, even some of you reached 11 and 12. But what about the higher numbers? Here is Common Core's answer.

We need to multiply **18 × 4**.

Solution

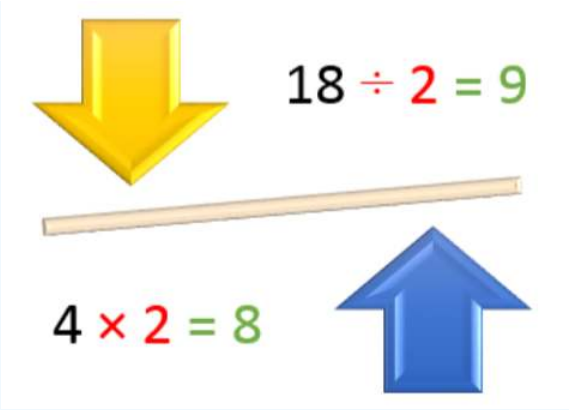


Figure 3.5.3

We can change the expression as long as we keep it balanced. Dividing 18 by 2 AND multiplying 4 by 2 will keep the expression balanced.

Therefore, 18 × 4 is the same as 9 X 8 = 72.

We do not have to use just two. Any number will work! As long as you keep it balanced!

Partner Activity 2

Here are some advanced multiplication expressions. By yourself, write an easier way to evaluate the expression, so you can use mental math. Then solve. Afterwards, compare your methods to your partner's method.

1. 24×3
2. 5×36
3. 16×4

Mental Math Subtraction Trick

Example 4.5.3

One of the hardest things for students to learn is subtracting when the first number (called the minuend) has lots of zeros, like 10,000. There can be confusion with borrowing: which zeros end up being tens and which zeros end up being nines? Common Core has an answer for that, too!

Solution

Let's try subtracting 1000 - 4983 the traditional way, with borrowing:

$$\begin{array}{cccccc} & & 9 & 9 & 9 \\ 0 & \not{1}0 & \not{1}0 & \not{1}0 & 10 \\ \not{1} & \not{0} & \not{0} & \not{0} & \not{0} \\ - & 4 & 9 & 8 & 3 \\ \hline & 5 & 0 & 1 & 7 \end{array}$$

Some students might see a problem like this and be intimidated by the amount of borrowing. An easier way to subtract: First subtract 1 from both numbers.

Going the Distance

1000 → 9999 and 4983 → 4982

$$\begin{array}{r} 9999 \\ -4982 \\ \hline 5017 \end{array}$$

No. More. Borrowing. EVER.

Note

Remember, subtraction is just measuring distance. We are finding the DISTANCE between 10000 and 4983, which is the same DISTANCE as 9999 and 4982

Partner Activity 3

Solve these subtraction problems WITHOUT borrowing! Think about what you should **add or subtract to both numbers.**

1. 1000 – 679
2. 513 – 284
3. 16452 – 999

Mental Math Addition and Subtraction in Schools Today

Common Core changed the way students add and subtract. Some schools still using carrying (addition) and borrowing (subtraction), like traditional mathematics. However, some schools encourage to teach the Common Core methods explained above, which will lead to mental math abilities, after enough practice.

Practice Problems

1. Estimation
   a. What is 11% of 150?
   b. What is 19% of 400?
2. Rounding
   a. Round 16.456 to the nearest hundredth.
   b. Round $49.347 to the nearest penny.
   c. Round 82.42 to the nearest whole number.
3. Tip
   a. The restaurant bill is $43.18. What is a 15% tip?
   b. The restaurant bill is $56.79. What is a 20% tip?
   c. The restaurant bill is $96.42. What is a 17% tip?
4. Multiply big numbers
   a. Use mental math to multiply 4×18.
   b. Use mental math to multiply 24×3.
   c. What did you notice about A and B?
5. Subtract from 10000
   a. Use the trick in Example $4.5.3$ to subtract 10000 – 9134.
   b. Use the trick in Example $4.5.3$ to subtract 100000 – 34678.

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4.6: Personal Referents

Think, for yourself, personally, how big areas or distances can be for you. It can be anything, as long as it makes sense to you! For example, the distance from Noel's house to the end of the block is ¼ of a mile. So when anybody says, "Let's take a walk, it is just half a mile", Noel knows it is the same distance as from his house to the end of the block and back. Another example, is where Janice knows that 300 people can fit comfortably in the gym. So, if someone says, I think I am going to invite about 300 people to my wedding, she can picture the gym filled up to capacity and have an idea about how large the wedding will be.

Partner Activity 1

Think about personal referents for yourself and then share with your partner:

1. 200 miles
2. 10 feet
3. 50 pounds

Practice Problems

Write a personal referent for 1 meter, 1 inch, 1 foot, and 1 yard.

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4.7: Calculating Percentages

(Mental Math verses pencil to paper)

30 is 20% of what number?

Mental math version: We know that a whole amount is 100%, so five 20%s are 100%. Therefore, we can multiply 30 by 5 and get 150, our answer.

But sometimes, it is not that easy. We need a formula:

$$\frac{is}{of} = \frac{\%}{100}$$

30 is 23% of what number? Use the proportion below, cross multiply, and solve for x. (Algebra)

$$\frac{30}{x} = \frac{23}{100}$$

Partner Activity 1

1. Find 16% of 54.
2. What percent of 97 is 43?
3. 21% of what number is 842.

Practice Problems

1. Find 94% of 329.
2. What percent of 14 is 13? (round to the nearest hundredth)
3. 64% of what number is 73. (round to the nearest hundredth)

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4.8: Extension - Methods of Teaching Mathematics

Part 1

What do you need to be a good (OR GREAT) mathematics teacher!

1. Know your subject matter.
2. Have patience with your students.
3. Be organized.
4. Have good classroom management skills

Part 2

In class, we will be watching videos of good and bad teaching practices. See Canvas for worksheet.

Part 3

Your homework assignment is to observe a teacher (either in person or on youtube.com) and write about their Classroom Management approach (good or bad). Turn this in via Canvas.

Part 4

Keep working on Khan Academy!

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CHAPTER OVERVIEW

5: Real Numbers

The teacher announced that the class would be learning more about rational numbers.

An English major raised his hand and said that numbers weren't rational After all; a number certainly couldn't think let alone be rational!

'The teacher said she didn't mean that kind of rational.

Another student said she'd heard numbers could be rational; as well as irrational; but she didn't think it had anything to do with thinking.

The teacher agreed. The root in the word 'rational' is ratio". A rational number is simply a number that can be written as the ratio of an integer and a nonzero integer.

Someone asked if that definition would be on the test, and if so, would it be on an English test or math test?

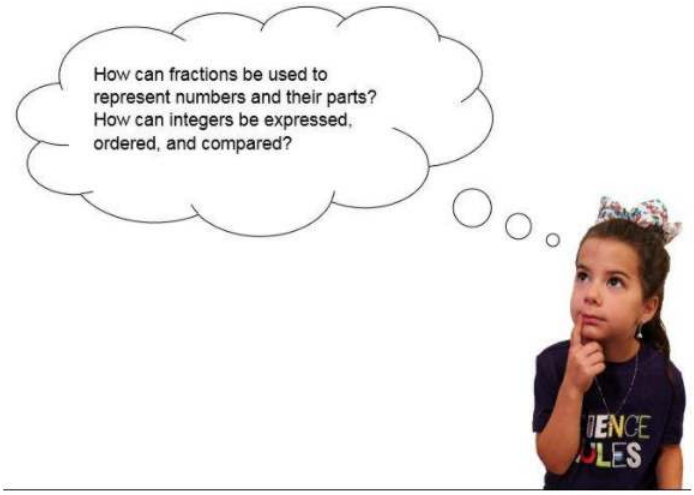
The teacher finally said that they were going to learn more about fractions today.

5.1: The Why
5.2: What are Fractions?
5.3: Add, Subtract, Multiply and Divide Fractions
5.4: Understanding Fractions With The C-Strips
5.5: Rational Numbers
5.6: Facts About Comparing Fractions
5.7: Decimals
5.8: Definition of Real Numbers and the Number Line
5.9: Models and Operations with Integers
5.10: Number comparisons using < ,>, and =
5.11: Homework

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5.1: The Why

The Essential Questions



BEFORE YOU BEGIN CHAPTER 4,
ANSWER THE ESSENTIAL QUESTIONS HERE:

Figure 4.1.1

Why are Elementary School Teachers Learning this Mathematics?

In a Wall Street Journal article from 2013, it states that understanding fractions in third, fourth and fifth grade is a predictor of long-term math achievement. Fractions are the foundation for advanced mathematics. Common Core mandates that fractions be mastered by the end of fifth grade.

Why are Elementary School Students Learning this Mathematics?

Fractions are a necessary component for students. Fractions are used outside of the classroom and career in areas like baking, sewing, construction and some sports. Studies have shown that students who do not master fractions end up having anxiety in their secondary and post-secondary mathematics classes; and end up not preforming as well as other students who did master fractions in elementary school.

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5.2: What are Fractions?

Fractions

$\frac{\text{Part}}{\text{Whole}} \rightarrow \frac{\text{Numerator}}{\text{Denominator}}$ ~~Denominator are just~~ Numerator Division!

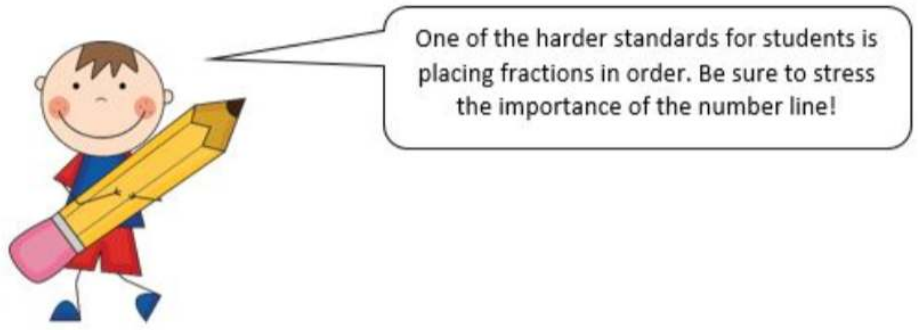


Figure 4.2.1

Example 5.2.1

Explain the meaning of $\frac{1}{4}$

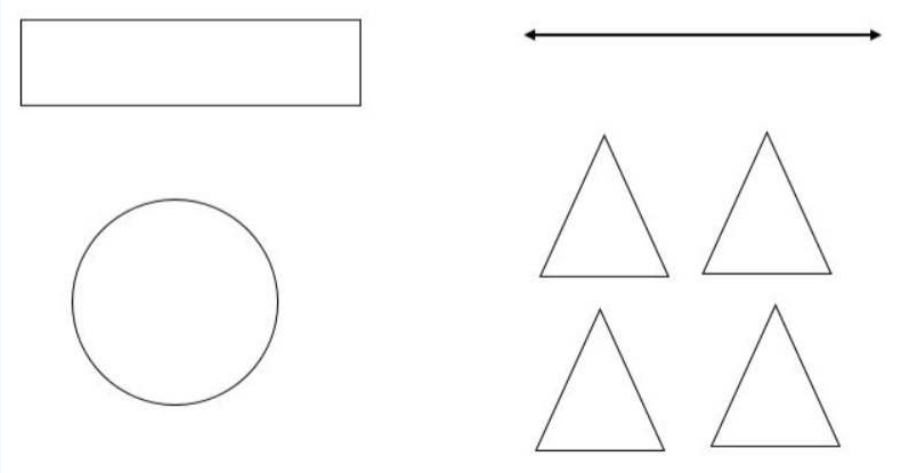


Figure 4.2.2

Example 5.2.2

Show $3\frac{5}{8}$



Figure 4.2.3

Example 5.2.3

Which is bigger? $\frac{1}{3}$ OR $\frac{1}{4}$

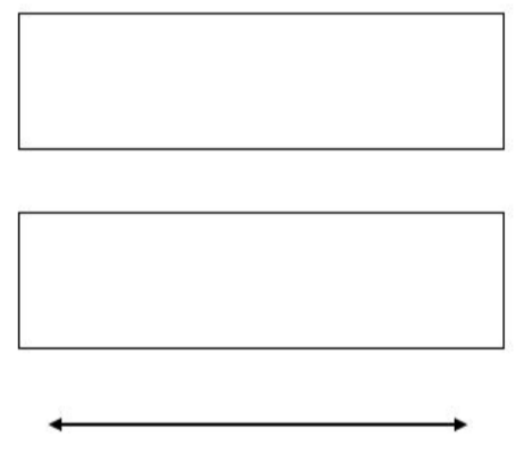


Figure 4.2.4

Example 5.2.4

Which is bigger? $\frac{4}{9}$ OR $\frac{5}{8}$

Solution

Use the LCM to make the common denominator. The LCM (9, 8) = 72.

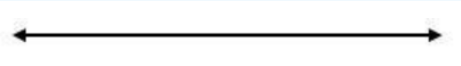


Figure 4.2.4

$$\frac{4}{9} = \frac{4 \times 8}{9 \times 8} = \frac{32}{72}$$
$$\frac{5}{8} = \frac{5 \times 9}{8 \times 9} = \frac{45}{72}$$

Since $\frac{45}{72} > \frac{32}{72}$, then $\frac{5}{8} > \frac{4}{9}$

Example 5.2.5

Equivalent Fractions. Show that $\frac{1}{2} = \frac{4}{8}$.

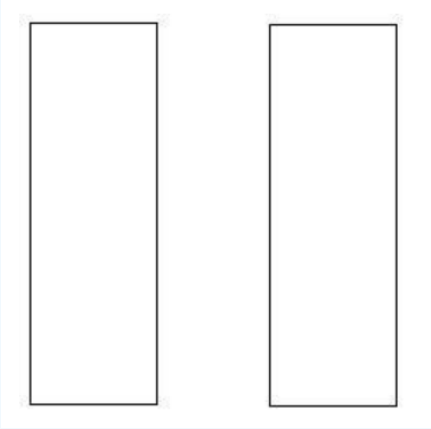


Figure 4.2.5

Example 5.2.6

Create Equivalent Fractions for $\frac{2}{3}$.

Solution

$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$

OR

$\frac{2}{3} = \frac{2 \times 11}{3 \times 11} = \frac{22}{33}$

Example 5.2.7

A 5th grade class has 11 girls and 13 boys. What fraction of the class has boys?

Solution

1. Find the total (whole): 11 + 13 = 24 students
2. Write your fraction: $\frac{\text{boys}}{\text{class}} = \frac{13}{24}$

Partner Activity 1

Describe ways of telling when a fraction is close to…

1. zero
2. One
3. one-half
4. one-third

Partner Activity 2

Organize the fractions by which it is closest to zero, one-half or one.

$\frac{3}{8} \quad \frac{5}{4} \quad \frac{2}{9} \quad \frac{4}{7} \quad \frac{1}{3}$

Practice Problems

1. Explain the meaning of $3\frac{1}{2}$ two different ways.
2. Put the following fractions in order from least to greatest: $\frac{4}{7}, \frac{2}{5}, \frac{1}{9}, \frac{12}{13}, \frac{3}{8}$

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5.3: Add, Subtract, Multiply and Divide Fractions

Partner Activity 1

Which expression would you rather add?

$$\frac{51}{684}+\frac{43}{684}+\frac{738}{684} \text{ OR } \frac{1}{8}+\frac{4}{5}+\frac{1}{9}$$

Explain to a 3rd grader why:

_____

_____

_____

_____

_____

Now, we will explore why fractions behave the way they do for adding, subtracting, multiplying and dividing:

Example 5.3.1: Add Fractions with a Drawing, Number Line, then with Common Denominators

Add $\frac{1}{2}+\frac{1}{3}$. Why 6 boxes?

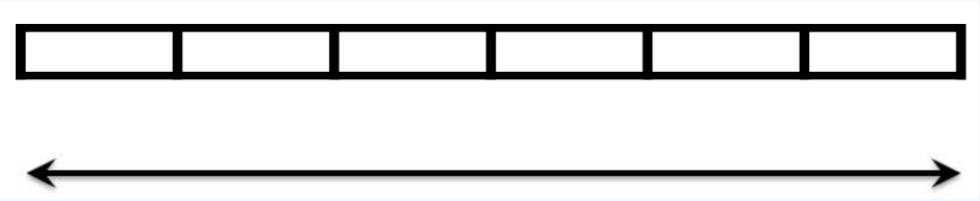


Figure 4.3.1

Example 5.3.2: Subtract Fractions with a Drawing, Number Line, then with Common Denominators

Subtract $\frac{1}{2}-\frac{1}{3}$. Why 6 boxes?

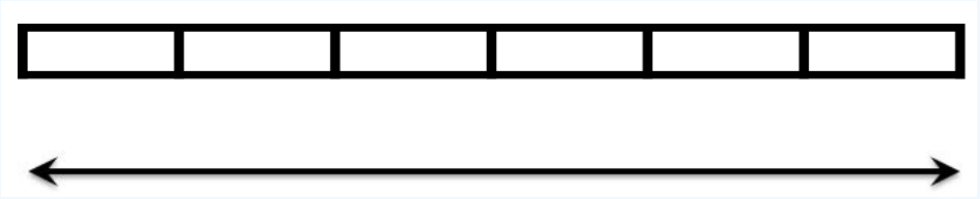


Figure 4.3.2

Example 5.3.3: Multiply Fractions with a Drawing, Number Line, then with Common Denominators

Add $\frac{1}{2}\times\frac{1}{3}$. Why 6 boxes?

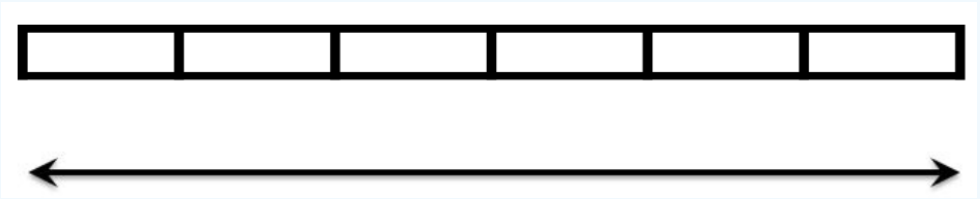


Figure 4.3.3

Example 5.3.4: Divide Fractions with a Drawing, Number Line, then with Common Denominators

Divide $\frac{1}{2} \div \frac{1}{3}$. "Think portions, when it comes to division! Why 6 boxes?

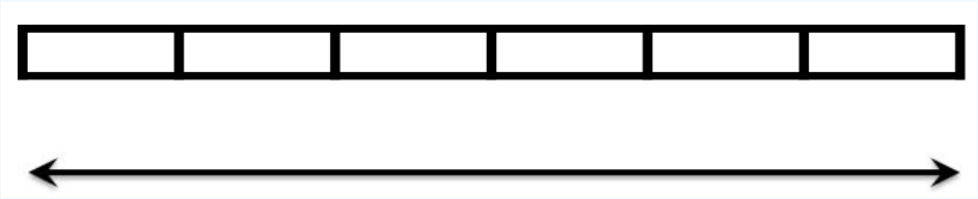


Figure 4.3.3

Example 5.3.5

Why does "multiply and flip the second fraction" work when dividing fractions?

Solution

We know that:

$$\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1$$

$$\begin{aligned} \frac{3}{4} \div \frac{2}{7} &= \left(\frac{3}{4} \times \frac{7}{2}\right) \div \left(\frac{2}{7} \times \frac{7}{2}\right) \\ &= \left(\frac{3}{4} \times \frac{7}{2}\right) \div 1 \\ &= \frac{3}{4} \times \frac{7}{2} \\ &= \frac{21}{8} \end{aligned}$$

Partner Activity 2

Which Operation is Correct?

A stretch of highway is $3\frac{1}{2}$ miles long. Each day, $\frac{2}{3}$ of a mile is repaved. How many days are needed to repave the entire section? How would you explain to a 5th grader which operation is correct?

Do we add?

$3\frac{1}{2} + \frac{2}{3} = \frac{7}{2} + \frac{2}{3} = \frac{21}{6} + \frac{4}{6} = \frac{25}{6} = 4\frac{1}{6}$ days

Do we subtract?

$3\frac{1}{2} - \frac{2}{3} = \frac{7}{2} - \frac{2}{3} = \frac{21}{6} - \frac{4}{6} = \frac{17}{6} = 2\frac{5}{6}$ days

Do we multiply?

$3\frac{1}{2} \times \frac{2}{3} = \frac{7}{2} \times \frac{2}{3} = \frac{14}{6} = 2\frac{2}{6} = 2\frac{1}{3}$ days

Do we divide?

$3\frac{1}{2} \div \frac{2}{3} = \frac{7}{2} \times \frac{3}{2} = \frac{21}{4} = 5\frac{1}{4}$ days

Practice Problems

Add, subtract, multiply or divide the expressions. Use any method.

1. $\frac{3}{4}+\frac{8}{7}$
2. $\frac{8}{7}-\frac{3}{4}$
3. $\frac{3}{4}\times\frac{8}{7}$
4. $\frac{3}{4}\div\frac{8}{7}$
5. $5\frac{2}{5}\div 3\frac{1}{6}$
6. $5\frac{2}{5}-3\frac{1}{6}$
7. $5\frac{2}{5}\times 3\frac{1}{6}$
8. $5\frac{2}{5}\div 3\frac{1}{6}$

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5.4: Understanding Fractions With The C-Strips

When dealing with fractions, the denominator tells you how many equal parts it takes to make 1 unit. The numerator tells you how many of those equal parts are taken.

Example 1

If H (hot pink) represents 1 unit, then which C-strip represents $\frac{3}{4}$?

Solution

The denominator is 4; so it takes 4 equal parts to make one unit, and each of those 4 equal parts = $\frac{1}{4}$. Find the C-strip such that a train of 4 of them is as long as one unit (H). Since a train of 4 light green C-strips (L) is the length of H (1 unit), then each light green C-strip makes up one part of a whole, and is therefore worth $\frac{1}{4}$. You need to find which C-strip represents $\frac{3}{4}$, so make a train of 3 light green C-strips and find the C-strip having this length. This would be the Blue (B) C-strip. Therefore, the answer is B.

For exercises 1 - 13, explain how to find the solution. Do each step using your C-strips.

a. State how many C-strips (each an equal part of the whole) make up one unit.

b. State which C-strip makes up one part of the whole.

c. State the fraction that the C-strip in part b represents.

d. State how many of the C-strips in part b you need to make into a train.

e. State which C-strip is the length of the train you made in part c, this is the answer!!!

Exercise Example

If H represents 1 unit, then which C-strip represents $\frac{3}{4}$?

a. **4**	b. **L**	c. $\frac{1}{4}$	d. **3**	e. **B**

Exercise 1

If H represents 1 unit, then which C-strip represents $\frac{1}{6}$?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 2

If H represents 1 unit, then which C-strip represents $\frac{5}{12}$?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 3

If O represents 1 unit, then which C-strip represents $\frac{2}{5}$?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 4

If B represents 1 unit, then which C-strip represents $\frac{2}{3}$?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 5

If D represents 1 unit, then which C-strip represents $\frac{5}{3}$?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 6

If N represents 1 unit, then which C-strip represents $\frac{3}{4}$?

a. ___	b. ___	c. ___	d. ___	e. ___

Example 2

If N (brown) represents $\frac{4}{5}$, then which C-strip represents 1 unit?

Solution

The denominator is 5, so it takes 5 equal parts to make up 1 whole unit, where each equal part is $\frac{1}{5}$. Since N is only $\frac{4}{5}$, then a train of only 4 of the 5 equal parts will be the length of N. A train of 4 red C-strips is the same length as N. So a red C-strip is one of the 5 equal parts that make up a whole. Since it takes 5 equal parts (5 reds) to make one unit, form a train of 5 reds and see which C-strip has this length. It is the orange C-strip (O). Therefore, the answer is O.

For exercises 7 - 11, explain how to find the solution. Do each step using your C-strips.

a. State how many C-strips will make up the named C-strip stated in the problem. Look at the numerator.

b. Which C-strip makes up one equal part?

c. State the fraction that the C-strip in part b represents. Look at the denominator.

d. State how many of the C-strips in part b will make up one unit.

e. Form the unit by making a train from the equal parts (C-strip in part b) and state which C-strip has the same length as that train.

Exercise Example

If N represents $\frac{4}{5}$, then which C-strip is 1 unit?

a. 4	b. R	c. 1/5	d. 5	e. O

Exercise 7

If D represents $\frac{1}{2}$, then which C-strip is 1 unit?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 8

If L represents $\frac{1}{3}$, then which C-strip is 1 unit?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 9

If P represents $\frac{2}{3}$, then which C-strip is 1 unit?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 10

If N represents $\frac{4}{5}$, then which C-strip is 1 unit?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 11

If R represents $\frac{2}{5}$, then which C-strip is 1 unit?

a. ___	b. ___	c. ___	d. ___	e. ___

The type of problems on this page are a little more challenging. They take more steps. From the first piece of information, figure out which C-strip is the whole unit – just like you did in problems 7 - 11. Then, start over using that unit C-rod, and figure out the second part of the question – just like you did in problems 1 - 16.

Example 3

If N represents $\frac{2}{3}$, then which C-strip represents $\frac{3}{4}$?

Solution

Begin these the same way the previous problems were done by first figuring out what the unit C-strip is. After doing the same steps you did for exercises 14-18, you will conclude that **H is the unit C-strip**. Now, on to part 2: Start over with H as the unit C-strip, and find $\frac{3}{4}$ the same way you did it for the first 13 exercises. The key is to start over by looking only at the stated unit C- strip (H in this case), and not getting that confused with the first part of the problem. In other words, now that you have determined that the unit is H, determine what is 3/4 (in relation to the unit - H). You will find that **the answer is B.**

For exercises 12-14, discuss how to find the solution. Do each step using your C-strips.

a. State which C-strip is one unit.

b. State which C-strip is the answer.

Example 3

If N represents $\frac{2}{3}$, then which C-strip represents $\frac{3}{4}$?

a. H	b. B

Exercise 12

If P represents $\frac{2}{3}$, then which C-strip represents $\frac{3}{2}$?

a. _____ b. _____

Exercise 13

If O represents $\frac{5}{6}$, then which C-strip represents $\frac{3}{4}$?

a. _____ b. _____

Exercise 14

If D represents $\frac{2}{3}$, then which C-strip represents $\frac{1}{3}$?

a. _____ b. _____

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5.5: Rational Numbers

Materials: Fraction Circles, Fraction Arrays, Multiple Strips, C-strips

Use your marked fraction circles to do these first few activities.

Exercise 1

Take one wedge from each fraction circle (1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/9, 1/10, and 1/12). One fraction is bigger than another if it covers more space than the other fraction. Compare them and put them in order from smallest to largest, and write them in order below, using the less than symbol (<).

Exercise 2

From what you discovered in exercise 1, circle the larger fraction in each case.

a. 1/90 or 1/95	b. 1/32 or 1/33

If you take 5 of the fraction pieces that say 1/8, then together you have 5/8. Use this fact to do the next exercise.

Exercise 3

Use the fraction circles to compare 5/6, 5/8, 5/9, 5/10, and 5/12. Put them in order from smallest to largest using the less than symbol (<).

Exercise 4

From what you discovered in exercise 3, circle the larger fraction in each case.

a. 15/37 or 15/40	b. 89/100 or 89/200

Exercise 5

Use the fraction array to order these fractions: 4/5, 4/6, 4/7, 4/8, 4/9, 4/10, 4/11 and 4/12. Put them in order from smallest to largest using the less than symbol (<).

Exercise 6

Explain the pattern you learned from doing exercises 1-5.

Exercise 7

Use fraction circles to compare these fractions: 1/2, 2/3, 3/4, 4/5, 5/6, 7/8, 8/9, 9/10, and 11/12. Put them in order from smallest to largest using the less than symbol (<).

Exercise 8

Use the fraction array to order these fractions: 1/2, 2/3, 3/4, 4/5, 5/6, 7/8, 8/9, 9/10, 11/12. Put them in order from smallest to largest using the less than symbol (<).

Exercise 9

From what you discovered in exercises 7 and 8, circle the larger fraction in each case.

a. 89/90 or 94/95	b. 56/57 or 31/33

Exercise 10

Order these fractions: 45/46, 71/72, 34/35, 99/100, 25/26, 13/14, 51/52. Put them in order from smallest to largest using the less than symbol (<).

Exercise 11

Explain the pattern you learned from doing exercises 7 - 10.

Exercise 12

Draw a model of your own that would help to convince someone why 4/5 was bigger than 2/3.

A rational number is simply the formal name for a fraction. Below is a formal definition of a rational number.

A **rational number** is a number that **can** be written as the quotient (ratio) of an integer, m, and a nonzero integer, n, which is written like this: $\frac{m}{n}$.

When written in this form, m is called the numerator and n is called the denominator. If GCF(m, n) = 1 (i.e., m and n have no factors in common), then the rational number is said to be in simplest (or reduced) form.

The following is one way to represent a rational number, $\frac{m}{n}$.

Divide a (defined) unit into an equal number of n parts or subsets. Which means n parts make up a whole. Each one of these parts (or subsets) represents $\frac{1}{n}$. Then, m of those parts represents $\frac{m}{n}$.

Examples

For each rational number, $\frac{m}{n}$,

a. Define a unit in terms of dots. State the value of each dot.
   i. Divide the unit into an equal number (the denominator) of parts.
   ii. Show a representation of $\frac{1}{n}$.
   iii. Show a representation of $\frac{m}{n}$.

b. Define the unit differently, and do the three parts (i, ii and iii) again.

Example 1: 3/8

a. Let one unit be defined by 8 dots, as shown below. Each dot = 1/8

. . . .
. . . .

i. The unit is divided into 8 equal parts, as shown below.



ii. Below is a representation of 1/8, which is one of the equal parts shown in part b.

.

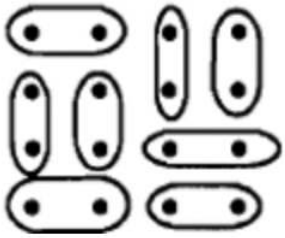
iii.Below is a representation of 3/8.

. . .

b. Let one unit be defined by 16 dots, as shown below. Each dot = 1/16

. . . .
. . . .
. . . .
. . . .

i. The unit is divided into 8 equal parts, as shown below.



ii. Below is a representation of 1/8, which is one of the equal parts shown in part b.

. .

iii.Below is a representation of 3/8.

. . . . . .

Notice there are 3 dots for the answer in part a, where each dot represents 1/8. So, the 3 dots represents the number 3/8. For part b, the answer has 6 dots. Since each dot represents 1/16, this also represents 6/16. Therefore, 3/8 and 6/16 must represent the same number. **Two fractions that represent the same number are called equivalent fractions.**

It is crucial to define your unit before beginning any exercise!!

Example 2: 2/5

a. Let one unit be defined by 5 dots, as shown below. Each dot = 1/5

. . . . .

i. The unit is divided into 5 equal parts, as shown below.



ii. Below is a representation of 1/5, which is one of the equal parts shown in part b.

.

iii.Below is a representation of 2/5.

. .

b. Let one unit be defined by 10 dots, as shown below. Each dot = 1/10

. . . . .
. . . . .

i. The unit is divided into 5 equal parts, as shown below.



ii. Below is a representation of 1/5,which is one of the equal parts shown in part b.

. .

iii.Below is a representation of 2/5

. . . .

Notice there are 2 dots for the answer in part a, where each dot represents 1/5. So, the 2 dots represents the number 2/5. For part b, the answer has 4 dots. Since each dot represents 1/10, this also represents 4/10. Therefore, 2/5 and 4/10 must represent the same number. **Two fractions that represent the same number are called equivalent fractions.**

IT IS CRUCIAL TO DEFINE YOUR UNIT BEFORE BEGINNING ANY EXERCISE!!

? Exercise 13

Use the method in the last two examples to show two different representations for 3/4.

? Exercise 14

Use your fraction arrays to determine all fractions on the fraction array that are equivalent to 1/2. Do this by finding 1/2 on the array, and seeing what other numbers are the same length.

? Exercise 15

Use your fraction arrays to determine all fractions on the fraction array that are equivalent to 2/3. Do this by finding 2/3 on the array, and seeing what other numbers are the same length.

You'll now be using your multiple strips to identify equivalent fractions. To use your strips, you need to cut out the strips by rows. To find fractions equivalent to 3/7, align the 3 strip above the 7 strip as shown below:

3	6	9	12	15	18	21	24	27	30	33	36
7	14	21	28	35	42	49	56	63	70	77	84

Now, you can see eleven other equivalent fractions for 3/7: 6/14, 9/21, 12/28, 15/35, 18/42, 21/49, 24/56, 27/63, 30/70, 33/77, and 36/84.

? Exercise 16

Use your multiple strips to write 6 fractions equivalent to 2/9

? Exercise 17

Use your multiple strips to write 6 fractions equivalent to 4/5

? Exercise 18

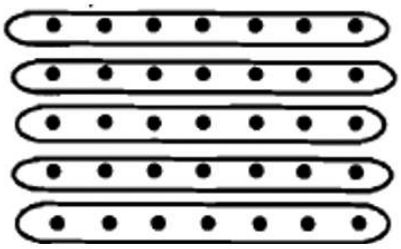
What is the rule for finding a fraction equivalent to a given fraction? Give an example how to find some fractions that are equivalent to 5/6.

We are now going to work again with models, using dots, to compare two fractions, add two fractions, subtract two fractions, or multiply or divide two fractions. For each problem, it is CRUCIAL that you begin each problem by explicitly stating the following:

1. Be specific about what you are using for the unit. It will be easiest if you use an array of dots, where the denominator of one fraction is the number of rows in the array, and the denominator of the other fraction is the number of columns in the array.

2. State the value of each dot, each column and each row.

Okay, let's go on to an example: **Use models to compare 2/5 and 3/7.**

Step 1: Let 1 unit = 5 rows of dots by 7 columns of dots, for a total of 35 dots, as shown below. Since there are 35 dots to a unit, each dot = 1/35 of a unit.	Step 2: Since there are 5 rows, the unit can be broken up into 5 equal parts by circling the rows, as shown below:  Therefore, each row is 1/5 of a unit. Notice there are 7 dots in 1/5 of a unit: 1/5 = ...	Step 3: Similarly, since there are 7 equal columns, each column is 1/7 of a unit: 1/7 = ...
Step 4: Now that we have properly defined the unit, we are ready to show what 2/5 looks like. Since 1/5 is 1 row of dots, then 2/5 must be 2 rows of dots. Therefore, 2/5 is shown below: . . . . Notice that 2/5 contains 14 dots, which is equivalent to 14/35.	Step 5: Similarly, we can show what 3/7 looks like. Since 1/7 is 1 column of dots, then 3/7 must be 3 columns of dots. Therefore, 3/7 is shown below: . . . . Notice that 3/7 contains 15 dots, which is equivalent to 15/35.	

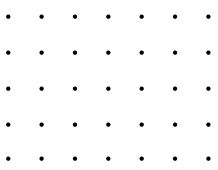
Since 2/5 contains less dots than 3/7, 2/5 must be less than 3/7. Answer: **2/5 < 3/7**.

? Exercise 19

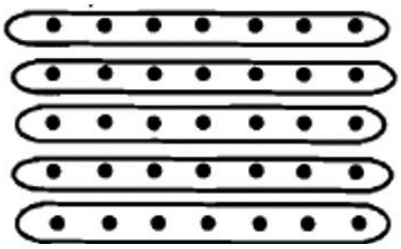
Compare 3/4 and 4/5 using models. Show **all** of the steps, and explain the procedure as shown in the previous example.

Okay, let's go on to an example of addition: **Use models to add 2/5 and 3/7.**

Step 1: Let 1 unit = 5 rows of dots by 7 columns of dots, for a total of 35 dots, as shown below. Since there are 35 dots to a unit, each dot = 1/35 of a unit.



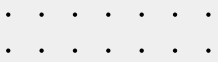
Step 2: Since there are 5 rows, the unit can be broken up into 5 equal parts by circling the rows, as shown below:



Therefore, each row is 1/5 of a unit. Notice there are 7 dots in 1/5 of a unit:
1/5 =

Step 3: Similarly, since there are 7 equal columns, each column is 1/7 of a unit:
1/7 =

Step 4: Now that we have properly defined the unit, we are ready to show what 2/5 looks like. Since 1/5 is 1 row of dots, then 2/5 must be 2 rows of dots. Therefore, 2/5 is shown below:



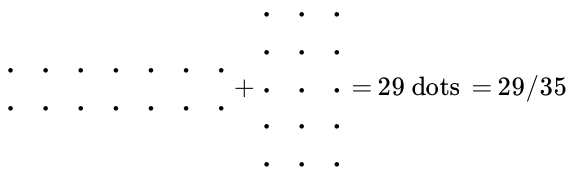
Notice that 2/5 contains 14 dots, which is equivalent to 14/35.

Step 5: Now, we add the two models together, as shown below. **The answer is 29/35.**



Notice that 3/7 contains 15 dots, which is equivalent to 15/35.

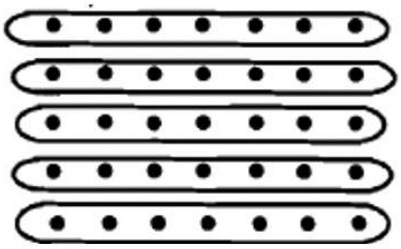
Step 5: Now, we add the two models together, as shown below. **The answer is 29/35.**



? Exercise 20

Add 1/4 and 2/5 using models. Show all of the steps, and explain the procedure as shown in the previous example.

Okay, let's go on to an example of subtraction: **Use models to do the following subtraction: 3/7 – 2/5.**

Step 1: Let 1 unit = 5 rows of dots by 7 columns of dots, for a total of 35 dots, as shown below. Since there are 35 dots to a unit, each dot = 1/35 of a unit. •	Step 2: Since there are 5 rows, the unit can be broken up into 5 equal parts by circling the rows, as shown below:  Therefore, each row is 1/5 of a unit. Notice there are 7 dots in 1/5 of a unit: 1/5 = • • • • • • •	Step 3: Similarly, since there are 7 equal columns, each column is 1/7 of a unit: 1/7 = • • • • •
Step 4: Now that we have properly defined the unit, we are ready to show what 2/5 looks like. Since 1/5 is 1 row of dots, then 2/5 must be 2 rows of dots. Therefore, 2/5 is shown below: • • • • • • • • • • • • • • Notice that 2/5 contains 14 dots, which is equivalent to 14/35.	Step 5: Similarly, we can show what 3/7 looks like. Since 1/7 is 1 column of dots, then 3/7 must be 3 columns of dots. Therefore, 3/7 is shown below: • • • • • • • • • • • • • • • Notice that 3/7 contains 15 dots, which is equivalent to 15/35.	

Now, we subtract, as shown below. **The answer is 1/35.**

$$\begin{matrix}\bullet\ \bullet\ \bullet\\ \bullet\ \bullet\ \bullet\\ \bullet\ \bullet\ \bullet\\ \bullet\ \bullet\ \bullet\\ \bullet\ \bullet\ \bullet\end{matrix} - \begin{matrix}\bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\end{matrix} = 1 \text{ dot } = 1/35$$

? Exercise 21

Do the following subtraction using models: 7/8 – 2/3. Show all of the steps, and explain the procedure as shown in the previous example.

Multiplying fractions is a little bit trickier. The repeated addition model doesn't make sense when you are multiplying two fractions. When you see two fractions multiplied together, like 2/5 · 3/7, think of this as 2/5 OF 3/7. In other words, you have to take 3/7 of a unit, and then you have to take 2/5 of that. Note that you have to represent the second fraction BEFORE you can do the multiplication.

Okay, let's go on to an example of multiplication: **Use models to do the following multiplication: 2/5 · 3/7.**

Step 1: Let 1 unit = 5 rows of dots by 7 columns of dots, for a total of 35 dots, as shown below. Since there are 35 dots to a unit, each dot = 1/35 of a unit.

Step 2: Since there are 5 rows, the unit can be broken up into 5 equal parts. Therefore, each row is 1/5 of a unit. Notice there are 7 dots in 1/5 of a unit:
1/5 =

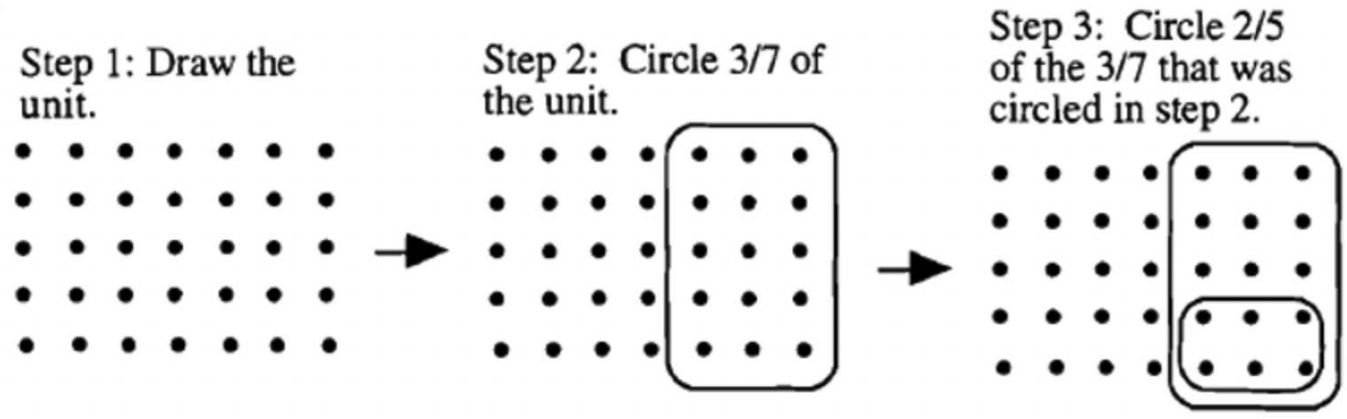
Step 3: Similarly, since there are 7 equal columns, each column is 1/7 of a unit:
1/7 =

Step 4: Now that we have properly defined the unit, we first have to show what 3/7 (the second number in the multiplication) looks like. Since 1/7 is 1 column of dots, then 3/7 must be 3 columns of dots. Therefore, 3/7 is shown below:

Step 5: This is where there is a difference in how you proceed. You now need to find 2/5 of the 3/7 that is shown in step 4. Since there are 5 rows of dots in step 4, each row (of 3 dots) represents 1/5. So, you only want 2 rows of the dots from step 4, as shown below.

This is the answer. There are 6 dots, so this represents 6/35.

Multiplication can be shown all on one unit by first showing the unit, second circling the part that represents the second fraction in the multiplication, and then circling the part that represents the first fraction in the multiplication. Below is the sequence of steps for this problem.



Okay, let's see what it looks like if we switch the order of the fractions: **Use models to do the following multiplication: $3/7 \cdot 2/5$.**

Step 1: Let 1 unit = 5 rows of dots by 7 columns of dots, for a total of 35 dots, as shown below. There are 35 dots to a unit, so each dot = 1/35 of a unit.

Step 2: Since there are 5 rows, the unit can be broken up into 5 equal parts. Therefore, each row is 1/5 of a unit. Notice there are 7 dots in 1/5 of a unit:
1/5 =

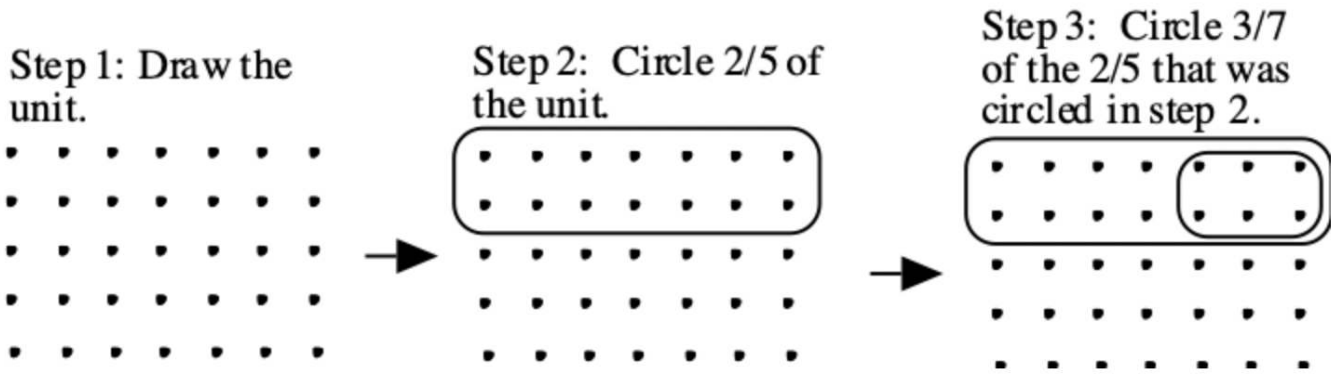
Step 3: Similarly, since there are 7 equal columns, each column is 1/7 of a unit:
1/7 =

Step 4: Now that we have properly defined the unit, we first have to show what 2/5 (the second number in the multiplication) looks like. Since 1/5 is 1 column of dots, then 2/5 must be 2 columns of dots. Therefore, 2/5 is shown below:

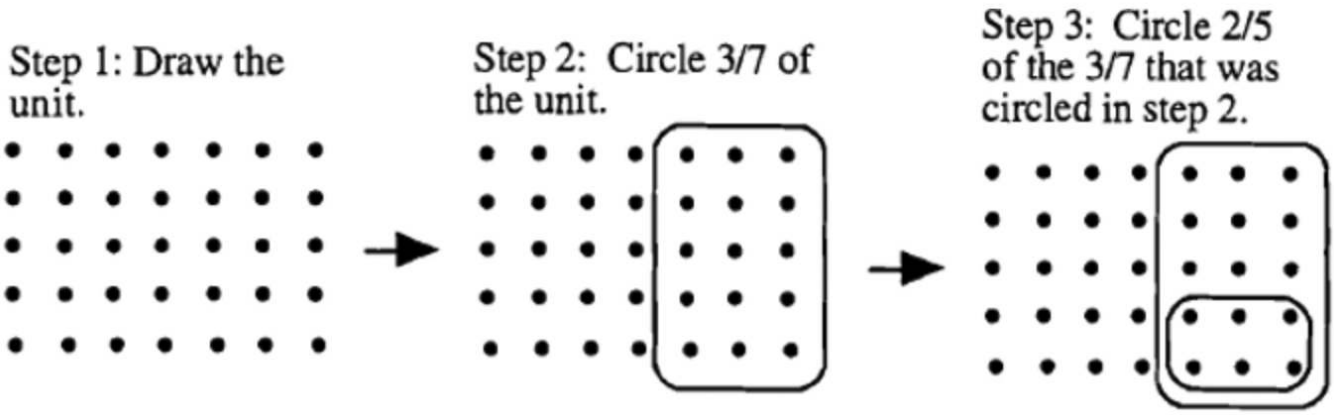
Step 5: You now need to find 3/7 of the 2/5 that is shown in step 4. Since there are 7 columns of dots in step 4, each column (of 2 dots) represents 1/7. So, you only want 3 rows of the dots from step 4, as shown below.

This is the answer. There are 6 dots, so this represents 6/35.

Again, this multiplication can be shown all on one unit by first showing the unit, second circling the part that represents the second fraction in the multiplication, and then circling the part that represents the first fraction in the multiplication. Below is the sequence of steps for this problem.



Although the answer is the same for 3/7 · 2/5, and 2/5 · 3/7, the sequence of steps is not. Note the difference between step 3 above (3/7 · 2/5) and step 3 below (2/5 · 3/7).



Exercise 22

Do the following multiplications using models. Show all of the steps, and explain the procedure as shown in the previous examples.

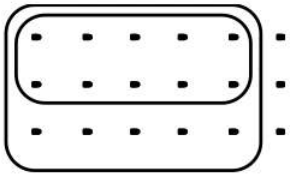
a. 4/5 · 2/3

b. 3/4 · 5/6

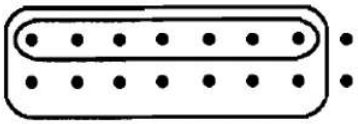
Exercise 23

By looking at the final drawing someone made to model a multiplication of two fractions, determine which multiplication was performed, and then state the answer. Circle which multiplication represents the correct choice.

a. 5/6 · 2/3 OR 2/3 · 5/6

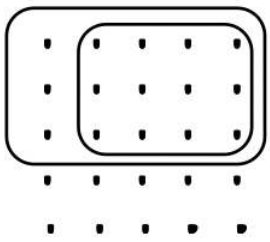


b. 1/2 · 7/8 OR 7/8 · 1/2

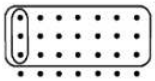


Exercise 24

If all of the dots shown for each problem represent 1 unit, determine the multiplication problem that someone did to get the answer, and state the answer.



a.



b.

Another model you can use to do multiplication is the C-strips, although it is somewhat limiting.

Example 1

Multiply $3/4 \cdot 1/2$.

Solution

This means 3/4 of 1/2 of a unit. So, we'll have to define an appropriate unit, first take 1/2 of the unit, and then take 3/4 of the answer obtained after taking 1/2. The choice for 1 unit using C-strips is to multiply the denominators together to get the length of the unit rod you need. In this case, 4 · 2 = 8, so choose the Brown C-strip as the unit.

Let 1 unit = N. First, take 1/2 of Brown (N), which is purple (P). Then, take 3/4 of purple, which is light green (L). The answer, L, represents 3/8, since the unit C-strip was N. Therefore, 3/4 · 1/2 = 3/8.

Example 2

Multiply $1/2 \cdot 3/4$.

Solution

This means 1/2 of 3/4 of a unit. So, we'll have to define an appropriate unit, first take 3/4 of the unit, and then take 1/2 of the answer obtained after taking 3/4. The choice for 1 unit using C-strips is to multiply the denominators together to get the length of the unit rod you need. In this case, 4 · 2 = 8, so choose the Brown C-strip as the unit.

Let 1 unit = N. First, take 3/4 of Brown (N), which is dark green (D). Then, take 1/42 of dark green (D), which is light green (L). The answer, L, represents 3/8, since the unit C-strip was N. Therefore, 1/2 · 3/4 = 3/8.

Example 3

Multiply $2/3 \cdot 4/5$.

Solution

This means 2/3 of 4/5 of a unit. So, we'll have to define an appropriate unit, first take 4/5 of the unit, and then take 2/3 of the answer obtained after taking 4/5. The choice for 1 unit using C-strips is to multiply the denominators together to get the length of the unit rod you need. In this case, 3 · 5 = 15. There isn't one C-strip that long, so use and orange + yellow as the unit.

Let 1 unit = O + Y. First, take 4/5 of O + Y, which is hot pink (H). Then, take 2/3 of hot pink (H), which is brown (N). The answer, N, represents 8/15, since the unit C-strip was O + Y. Therefore, 2/3 · 4/5 = 8/15.

Exercise 25

Use the C-strips to multiply 1/3 · 3/4. Explain the steps.

The process of doing multiplication using C-strips can be shown in a chart. I've shown the steps for the above three examples on the next page.

Exercise 26

Fill in the chart showing how to do the following multiplications using C-strips. The multiplication is in the first column. State an appropriate choice for the unit (name a C-strip, or sum of two C-strips) in the second column. Write the C-strip obtained after the first part of the multiplication (which is the second fraction as a part of the unit) in the third column. Then, do the final multiplication, and write the C-strip obtained in the fourth column. In the fifth column, write a fraction using C-strips putting the final unit obtained in the fourth column as the numerator, and the unit in the denominator. Then, in the last column, write the answer as a fraction. Do not simplify.

Example	$3/4 \cdot 1/2$	N	P	L	$\frac{L}{N}$	$\frac{3}{8}$
Example	$1/2 \cdot 3/4$	N	D	L	$\frac{L}{N}$	$\frac{3}{8}$
Example	$2/3 \cdot 4/5$	O+Y	H	N	$\frac{N}{O+R}$	$\frac{8}{15}$

a.	$1/3 \cdot 3/4$					
b.	$1/2 \cdot 1/4$					
c.	$3/2 \cdot 1/4$					
d.	$2/3 \cdot 1/2$					

On to Division with Rational Numbers...

Remember that the answer to the division $\mathbf{a \div b}$ can be obtained by answering this question: "**How many sets of b are contained in a**?"

Let's use this idea to find the answer to the following division problem:

Example 1

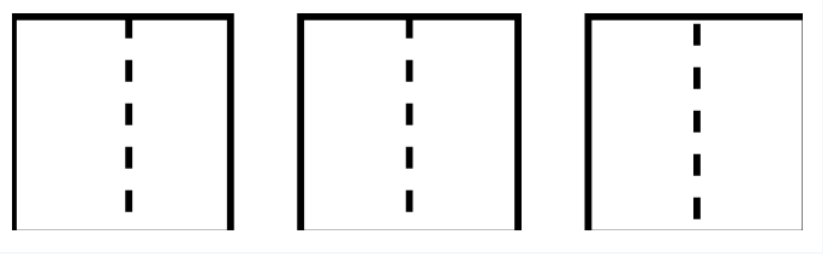
Use a model to compute: $\mathbf{3 \div 1/2}$

Solution

The answer to $\mathbf{3 \div 1/2}$ is the answer to this question: How many 1/2's are contained in 3? Below are two possible methods you can use to find the solution.

Method 1: Let 1 unit = 1 square. Then count how many 1/2's are contained in 3 squares.

Draw three squares to represent 3. Then, divide each square into 1/2's. Count how many 1/2's are in the 3 squares. From the model, it is clear there are 6 1/2's in 3 squares. So the answer to 3 ÷ 1/2 is **6**.



Method 2: Let the unit be defined similarly to how we did it for the previous problems.Multiply the denominators together to determine how many dots are in a unit. In this case, 3 is the same as 3/1. Let the unit = 1 row of dots by 2 columns of dots for a total of 2 dots.

Unit = • • Then represent the number 3 based on this unit, and the number 1/2 based on this unit.

3 = 3 sets of 2 dots = • • • • • • and 1/2 = 1/2 of the 2 dots = •

In the set representing 3, circle as many sets as possible that represent 1/2: ⊡ ⊡ ⊡ ⊡ ⊡ ⊡

Count how many 1/2's are in 3. There are **6** 1/2's in 3, so the answer is **6**.

Method 3: Let 1 unit = the dark green C-strip. Represent 3 and 1/2 as a C-strip based on this unit. Since 3 = 3 dark green C-strips, then 3 = the orange + brown C-strip (or 3 dark greens, or hot pink + dark green). Since 1/2 of the dark green C-strip is light green, then 1/2 = the light green C-strip.

We have to answer the question: How many 1/2's are in 3? Since 1/2 = L, and 3 = O + N, count how many light green C-strips make up the length of the O + N C-strip. There are **6**. Once again, the answer is **6**.

Unit:	D

3:	H	D

1/2	L

Here is how to get the answer. Show how many light greens are contained hot pink + dark green.

L	L	L	L	L	L

Notice that you begin by defining a unit C-strip, and then use that unit to define the two numbers in the division problem. But you do NOT refer back to the unit to compute the division problem. You simply count how many of the divisor (second C-strip) is in the dividend (first C-strip).

Example 2

Use a model to compute: $\mathbf{1/2 \div 1/4}$.

Solution

The answer to $\mathbf{1/2 \div 1/4}$ is the answer to this question: How many 1/4's are contained in 1/2?

Let 1 unit = 2 rows of dots by 4 columns of dots:



Then represent the number 1/2 based on this unit, and the number 1/4 based on this unit.

1/2 = and 1/4 =

In the set representing 1/2, circle as many sets as possible that represent 1/4: 

Count how many 1/4's are in 1/2. There are **2** 1/4's in 1/2, so the answer is **2**.

This can also be done using C-strips. Let the unit = N. Then 1/2 = P and 1/4 = R. Since there are **2** reds in the purple C-strip, 1/2 ÷ 1/4 = **2**.

Exercise 27

Use a model to do the following division: 1/5 ÷ 1/10. Use boxes, dots, or C-strips. First define the unit. Then explain and show all of the steps.

Exercise 28

Perform the following division using each of the methods (boxes, dots, C-strips):

1/3 ÷ 1/9. First define the unit. Then explain and show all of the steps.

Exercise 29

Perform the following division using each of the methods (boxes, dots, C-strips):

2/3 ÷ 1/6. First define the unit. Then explain and show all of the steps.

Exercise 30

Perform the following division using each of the methods (boxes, dots, C-strips):

3 ÷ 1/4. First define the unit. Then explain and show all of the steps.

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5.6: Facts About Comparing Fractions

Assume a and c are whole numbers, and b and d are counting numbers.

Two fractions, a/b and c/d, are *equivalent* if and only if ad = bc. In other words,

$$\frac{a}{b} - \frac{c}{d} \Leftrightarrow ad - bc$$

Note: $\Leftrightarrow$ means "if and only if", so if ad = bc, then the two fractions a/b and c/d are equivalent, and vice versa. Using this method to determine if two fractions are equivalent is called **comparing cross products**.

Examples: Determine if the following statements are true or false by comparing cross products. State whether the pair of fractions are equivalent or not.

Example 1

6/8 = 9/12

Solution

Since $6 \cdot 12 = 8 \cdot 9$, the statement is true. Therefore, 3/4 and 9/12 are equivalent fractions.

Example 2

15/24 = 10/18

Solution

Since $15 \cdot 18 \neq 24 \cdot 10$, the statement is false. Therefore, 15/24 and 10/18 are not equivalent fractions.

Exercise 31

Determine if the following statements are true or false by comparing cross products. State whether the pair of fractions are equivalent or not. Show your work and reasoning.

a. 14/25 = 28/50

b. 25/35 = 10/14

c. 12/16 = 25/40

The Fundamental Law of Fractions:

For any rational number, a/b and any integer c, a/b = ac/bc. The fractions a/b and ac/bc are called equivalent fractions.

Exercise 32

Write five equivalent fractions for each fraction given:

a. 2/3: _____

b. 5/7: _____

c. 3/8: _____

Two fractions, a/b and c/d, are *equivalent* if and only if the numerators are equal after writing each fraction with a common denominator.

Determine if the following statements are true or false. State whether the pair of fractions are equivalent or not.

Example 1

6/8 = 9/12

Solution

Write each fraction with a common denominator of 24 by applying the fundamental law of fractions: 6/8 = 18/24, and 9/12 = 18/24. Since the numerators are equal when each fraction is written with a common denominator, 6/8 and 9/12 are equivalent.

Example 2

15/24 = 10/18

Solution

Write each fraction with a common denominator of 72 by applying the fundamental law of fractions: 15/24 = 45/72, and 10/18 = 40/72. Since the numerators are not equal when each fraction is written with a common denominator, 15/24 and 10/18 are not equivalent.

Exercise 33

Determine if the following statements are true or false by writing each fraction with a common denominator. State whether the pair of fractions are equivalent or not. Show your work and reasoning.

a. 14/25 = 28/50

b. 25/35 = 10/14

c. 12/16 = 25/40

A fraction, a/b, is in *simplest* (reduced) *form* if GCF(a,b) = 1.

One way to simplify fractions is to prime factor the numerator and denominator, and then "cancel out" any common factors. Another way is to

(1) find the GCF of the numerator and denominator,

(2) rewrite both the numerator and denominator as the product of the GCF(a,b) and another factor, and then

(3) "cancel out" any common factors. When you cancel out everything from either the numerator and/or denominator, there is always a factor of 1 that is still there.

Example

Write 315/350 in simplest form by using each of the two methods just described.

Method 1: Prime factorization:

$$\frac{315}{350} = \frac{3 \times 3 \times 5 \times 7}{2 \times 5 \times 5 \times 7} = \frac{9}{10}$$

Method 2: GCF(315, 350) = 35:

$$\frac{315}{350} = \frac{9 \times 35}{10 \times 35} = \frac{9}{10}$$

Exercise 34

Write each fraction in simplest form **using each of the two methods**:

(1) prime factorization and

(2) finding GCF as shown in the previous example.

Show the actual factorization for each method, and then reduce to lowest terms.

a. Method 1: $\frac{378}{675}$

Method 2: $\frac{378}{675}$

b. Method 1: $\frac{247}{323}$

Method 2: $\frac{247}{323}$

Two fractions, a/b and c/d, are *equivalent* if the two fractions are equal when they are both written in simplest form.

Examples: Determine if the following statements are true or false. State whether the pair of fractions are equivalent or not.

Example 1

6/8 = 9/12

Solution

In simplest form, 6/8 = 3/4, and 9/12 = 3/4. Therefore, 6/8 and 9/12 are equivalent.

Example 2

15/24 = 10/18

Solution

In simplest form, 15/24 = 5/8, and 10/18 = 5/9. Therefore, 15/24 and 10/18 are not equivalent.

Exercise 35

35. Determine if the following statements are true or false by writing each fraction in simplest form. State whether the pair of fractions are equivalent or not. Show your work and reasoning.

a. 14/25 = 28/50

b. 25/35 = 10/14

c. 12/16 = 25/40

Below is a way to compare two fractions that are unequal by using cross products.

$$\frac{a}{b} < \frac{c}{d} \Leftrightarrow ad < bc \qquad \text{and} \qquad \frac{a}{b} > \frac{c}{d} \Leftrightarrow ad > bc$$

As you write the fractions, a/b < c/d, a is the first number written and d is the last number written. a and d are called the extremes (outside numbers when written as a/b < c/d), and their product is written on the left of the inequality sign when you take the cross products. The other two numbers, b and c, are called the means (when written as a/b < c/d, these are the inside numbers), and their product is written on the right of the inequality sign when you take the cross products. There are only three cases possible when comparing two fractions, a/b and c/d. Either the first fraction (a/b) is equal to the second (c/d), in which case ad = bc; the first (a/b) is less than the second (c/d), in which case ad < bc; or the first (a/b) is more than the second (c/d), in which case ad > bc.

Example 1

Compare 2/5 and 3/7 using cross products.

Solution

Multiply the extremes (2 and 7) and put on the left. Multiply the means (5 and 3) and put on the right. Compare the products. Since $2 \cdot 7 < 5 \cdot 3$, then 2/5 < 3/7.

Example 2

Compare 6/7 and 5/6 using cross products.

Solution

Multiply the outside numbers (6 and 6) and put on the left. Multiply the inside numbers (7 and 5) and put on the right. Compare the products.
Since 6·6 > 7·5, then 6/7 > 5/6.

Exercise 36

Use cross products to compare each of the following fractions. Use < or >.

a. 4/5 and 5/8	b. 12/35 and 11/18	c. 13/15 and 14/17

At the beginning of this exercise set, you compared several fractions using the fraction circles and fraction array. One problem was to compare 1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/9 , 1/10, and 1/12, and put them in order from smallest to largest using the less than symbol.

The answer was: 1/12 < 1/10 < 1/9 < 1/8 < 1/6 < 1/5 < 1/4 < 1/3 < 1/2

This can easily be checked using cross products. You can check one at a time: 1/12 < 1/10 since 10 < 12, and 1/10 < 1/9 because 9 < 10, etc.

Exercise 37

Check the validity of the answer you got for exercise 9a. Write the answer. Then check using cross products.

Exercise 38

Check the validity of the answer you got for exercise 9b. Write the answer. Then check using cross products.

Adding and Subtracting Fractions

In order to add or subtract fractions, the fractions must have a common denominator. Below is the rule for adding or subtracting fractions that have a common denominator.

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \qquad \text{and} \qquad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

In order to add fractions that do not have a common denominator, you must first rewrite each fraction as an equivalent fraction so that both fractions have a common denominator, OR you may use the following rule for addition of fractions.

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

Note: If you are going to add fractions by rewriting each fraction with a common denominator, it is usually preferable to find the least common denominator, which is the least common multiple of the two denominators. Whether or not you do this, you should always write the answer in simplest form after adding or subtracting.

Multiplying two fractions

The rule for multiplying two fractions is to multiply the numerators together to obtain the new numerator, and multiply the denominators together to obtain the new denominator.

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

It is easier to first "cancel out" any common factors before multiplying. This can be done by first prime factoring the numerators and denominators, and writing all the factors of the numerator in the numerator of the fraction, and all the factors of the denominator in the denominator of the fraction. Then, cancel any common factors before multiplying. If you do this, the fraction will already be in simplest form.

Dividing two fractions

To understand the rule for dividing two fractions, see if you can follow the reasoning below.

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{1} = \frac{a}{b} \cdot \frac{d}{c}$$

The rule for dividing two fractions is this: Dividing by a fraction is the same as multiplying by the reciprocal of that fraction. Then, use the rule for multiplying as just explained.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

? Exercise 39

Compute each of the following using the order of operations. Write each answer in simplest terms. Show all work.

a.

$$\frac{3}{4} \times \frac{4}{5} - \frac{5}{6} \div \frac{7}{3}$$

b.

$$\frac{5}{6} \div \frac{4}{5} + \frac{5}{6}\left(\frac{1}{3} - \frac{2}{5}\right)$$

Unlike integers, there are infinitely many rational numbers between any two rational numbers. It doesn't make sense to talk about two consecutive rational numbers. For instance, between 4/9 and 5/9, there are infinitely many rational numbers. One easy way to list a few is to write equivalent fractions for 7/9 and 8/9. A simple common denominator would be 90, so look at 70/90 and 80/90. It's easy to see these rational numbers between 70/90 and 80/90: 71/90, 72/90, 73/90, 74/90, 75/90, 76/90, 77/90, 78/90, 79/90. Of course, had I chosen equivalent fractions with a larger denominator (900 or 900,000, etc.), you would easily be able to list many more rational numbers between 7/9 and 8/9.

If you wanted to list just one rational number between 7/9 and 8/9, you could choose one of the ones from the above list, or you could simply find the midpoint (or average) of the two. To find the average of any two numbers, add and divide by 2. Since dividing by 2 is the same as multiplying by one-half, add and multiply by one-half. Note the average of 7/9 and 8/9 is 1/2(7/9 + 8/9) = 1/2(15/9) = 15/18 or 5/6.

? Exercise 40

Find 5 rational numbers, written with a common denominator, between 2/5 and 3/5.

? Exercise 41

Find the average of 5/11 and 6/11.

? Exercise 42

Find the average of 3/7 and 4/7

If you are asked to find several rational numbers between two rational numbers that do not have a common denominator, first you should rewrite each fraction as equivalent fractions having a common denominator. For instance, to find 4 rational numbers

between 4/7 and 5/6, first rewrite 4/7 and 5/6 with a common denominator of 42: 24/42 and 35/42. In this case, it is easy to find 4 rational numbers between 4/7 and 5/6: 25/42, 26/42, 27/42, etc.

Here's another one: Find 5 rational numbers between 3/4 and 4/5. First, rewrite 3/4 and 4/5 with a common denominator of 20: 15/20 and 16/20. Then, find equivalent fractions with a larger denominator than 20: 150/200 and 160/200 are easy choices, although you could have picked 90/120 and 96/120. In any case, there are infinitely many choices. I probably would choose 151/200, 152/200, 153/200, 154/200 and 155/200.

? Exercise 43

Find 3 rational numbers, written with a common denominator, between 1/3 and 2/5.

? Exercise 44

Find 3 rational numbers, written with a common denominator, between 1/2 and 1/3. (Be careful: Which number is smaller and which is larger?)

? Exercise 45

Find 5 rational numbers, written with a common denominator, between 5/6 and 4/5.

(Be careful: Which number is smaller and which is larger?)

? Exercise 46

Find the average of 5/8 and 6/7.

? Exercise 47

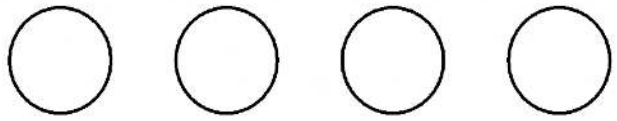
Find the average of 5/7 and 5/8.

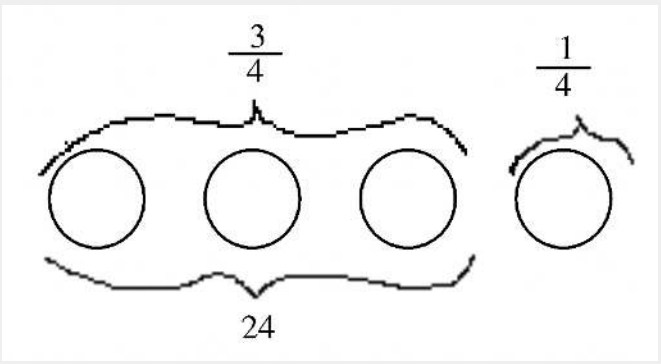
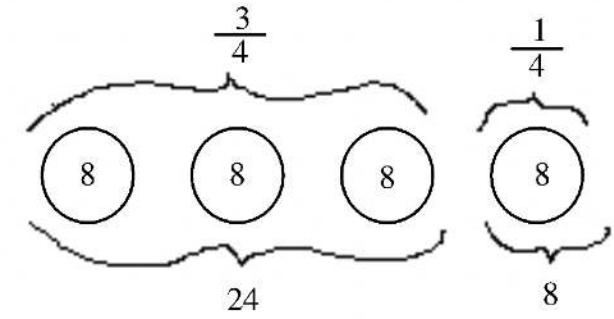
Now, we'll work on some word problems where we can use the meaning of fractions to easily solve the problems. We'll use the fraction circles as manipulatives.

Problem: One day, 24 of my students showed up for class. This only represented 3/4 of my students. How many students were enrolled in the class? How many were absent?

Solution: We know that 24 is 3/4 of the total students in the class. That means 4/4 makes up all the students. Get out the fraction circle where 4 equal parts make up a whole. You know three of those parts represent 24 students. If 3 of those parts represent 24 students, then each of the equal parts represents 8 students. Since 4 equal parts represents the whole class, then there must be $8 \cdot 4$, or 32 students enrolled. One of the equal parts represents the absent students, so 8 students are absent.

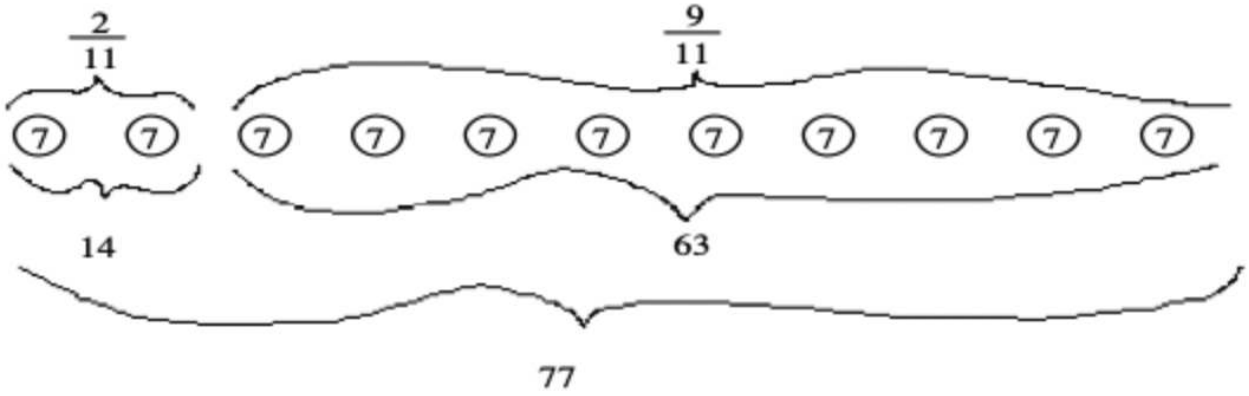
You could also show this pictorially – you see circles in place of the fraction circles.

Step 1: 4 equal parts make up a whole.	

Step 2: 3 equal parts, or 3/4, represents 24 students. The part left over represents 1/4 of the students.	
Step 3: If those three circles represent 24, then 8 must be in each circle. Now it is clear there are 32 students in the class, and 8 are absent.	

The picture in Step 3 is what the final picture looks like.

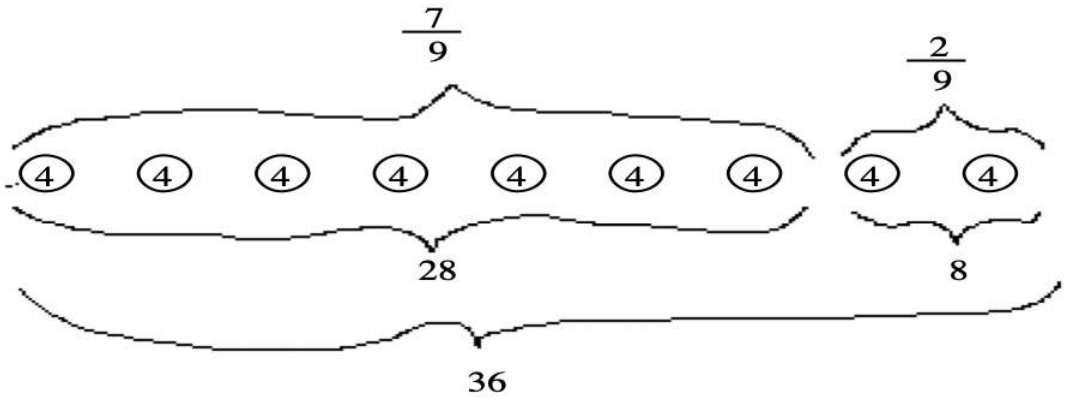
Here is another example. 14 teachers were absent one day. This represents 2/11 of the teachers working at the school. How many teachers work at the school? Since 11 equal parts make up a whole, and 2 of those parts represent 14, then each of the 11 equal parts represents 7 teachers. So, 11 of the equal parts represents 77 teachers.



Here is one more example:

One day, 7/9 of the people at a local business came to work. 36 people worked there. How many people came to work?

First, since 9 equal parts make up a whole, draw 9 circles. You know 7 of those represent 7/9 and 2 of those represent 2/9. You also know, the total equals 36, which means 4 goes into each equal part. Therefore, 28 came to work, and 8 did not. Here is the picture:



For each of the following word problems, draw model similar to the previous two examples to solve the problem. You may want to use fraction circles as well.

? Exercise 48

66 students in my class passed the first test. That represents 11/12 of my students. How many students did not pass?

? Exercise 49

36 students in first grade brought a lunchbox to school one day. This was 3/7 of the first graders. How many did not buy lunch?

? Exercise 50

I have 120 students this semester. 5/8 of my students are female. How many are female and how many are male?

? Exercise 51

Make up a word problem to solve using fractions. Make sure you ask a question. Then solve the problem using models. Explain how the model works.

Write the word problem here:

Solve the problem using models here:

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5.7: Decimals

The meaning of decimals is best understood once one has a real understanding of place value and fractions. The decimal notation we commonly use is an extension of place value in base ten. The decimal point indicates that succeeding digits represent tenths, hundredths, thousandths, etc. So a person must understand what these fractions mean in order to understand and make the connection to decimals.

The key to understanding the relationship between decimals and fractions (or mixed numbers) begins with READING the decimal correctly. Most people read 5.3 as "five point three" which doesn't help one understand its meaning. It should be read "five and three tenths." In doing so, the connection between the decimal 5.3 and the mixed number $5\frac{3}{10}$ is clear. Similarly, 18.035 should be read "Eighteen and thirty-five thousandths" and corresponds to the mixed number $18\frac{35}{1000}$. Here are a few more:

a. 0.309 and $\frac{309}{1000}$ are both correctly read "three hundred nine thousandths"
b. 10.04 and $10\frac{4}{100}$ are both correctly read "ten and four hundredths"

Note that the decimal point is read as "and". The decimal point (and the word "and") separates the whole part from the fractional part of a mixed number. This is the only correct use of the word "and" when reading numbers. 760 is read "seven hundred sixty." A common mistake is to read 760 as "seven hundred and sixty." If there is no decimal point, don't say the word "and".

People just learning about decimals (like elementary school children) should NOT read the number 5.4 as "five point four". This is a shortcut way of reading the number that is only appropriate to use once one really understands the connection between decimals and fractions. Remember that a number written in decimal form is really just a different way to write a mixed number where the denominator of the fractional part is a power of ten! The name of the fractional part (tenths, hundredths, thousandths, etc.) is the place value of the last digit of the number after the decimal point, which also happens to be the denominator of the number written in fractional form.

Look again at 0.309 and $\frac{309}{1000}$. In the decimal, there are three digits (or place values) after the decimal point. In the fraction, there are three zeroes after the 1 (which is the number 1000 in the denominator). The same applies to 18.035 and $18\frac{35}{1000}$. For 5.3 and $5\frac{3}{10}$, the decimal has one digit after the decimal point, and the fraction has one zero after the 1. It's best if students are allowed to discover this fact for themselves. It's always more meaningful to discover relationships (that often become rules) on your own. Students who are regularly asked to read decimals, fractions and mixed numbers the correct way are more likely to make this particular discovery by themselves.

Mixed numbers like 5.3 and 18.035 can also be written immediately as improper fractions. The denominator of the fraction will still be the same as if it were written as a mixed number. The numerator is the number without a decimal point at all. For instance, 5.3 can be written as $5\frac{3}{10}$ or $\frac{53}{10}$; and 18.035 can be written as $18\frac{35}{1000}$ or $\frac{18035}{1000}$.

Note that when you first write a decimal as a fraction, it isn't necessarily in simplest (or reduced) form.

? Exercise 1

On the line, write in words how to read each of the following decimals. Do not use the word "point". Then, underneath, write each decimal as a fraction. If the number is greater than or equal to 1, first write the fraction as a mixed number and then write it as an improper fraction. *Simplify any fraction that is not in simplest form.* **Show all steps** (including original fractions before simplifying).

a. 0.4 _____

0.4 =

b. 0.26 _____

0.26 =

c. 3.08 _____

3.08 =

d. 9.85 _____

9.85 =

e. 17.305 _____

17.305 =

Exercise 2

Write each proper fraction as a decimal. Write each improper fraction first as a mixed number (don't reduce), and then also as a decimal. Do not use your calculator.

a. $\frac{14}{100}$ = _____

b. $\frac{8}{1000}$ = _____

c. $\frac{435}{100}$ = _____ = _____

d. $\frac{5638}{10}$ = _____ = _____

e. $\frac{305}{100}$ = _____ = _____

The rule you may remember for multiplying fractions is to multiply the numbers together as if there were no decimal point, and then move the decimal point in from the right the total number of places it is in for both numbers. For instance, (8)(0.4) is done by multiplying 8 times 4 and then moving the decimal point in one place to get 3.2. Similarly, (0.06)(0.7) is done by multiplying 6 times 7 and moving the decimal point in three places (two for 0.06 and one more for 0.7 for a total of three) to get 0.042. Many people make this harder than it really is and don't realize they can easily multiply 0.3 and 0.4 in their head. It's as simple as $3 \cdot 4$ and moving the decimal in two places to get 0.12. Now, how about 1.1 times 1.2? It's simply 11 times 12 with the decimal moved in two places: 1.32.

Let's observe why this rule for multiplying decimals works by rewriting the numbers as fractions first. The key here is to write numbers greater than or equal to one as improper fractions. Then multiply (without canceling or reducing) the fractions. Lastly, rewrite the fraction as a decimal.

Exercise Example 1

(8)(0.4) = $\frac{8}{1} \cdot \frac{4}{10} = \frac{32}{10}$ = 3.2 (notice the decimal point is one place in for .4)

Exercise Example 2

(1.2)(1.01) = $\frac{12}{10} \cdot \frac{101}{100} = \frac{1212}{1000}$ = 1.212

(notice the decimal point is three places in, one for 1.2 plus two more for 1.01)

Exercise 3

Multiply the following decimals mentally, and write the answer on the blank. Then do it again by showing the same steps as shown in the previous two examples where each decimal is **first** written as a fraction, then multiply the numerators and denominators, and then convert that answer (don't simplify) to a decimal. Do not use your calculator.

a. (0.4)(.07) = _____

(0.4)(.07) =

b. (1.6)(0.2) = _____

(1.6)(0.2) =

c. (0.25)(0.3) = _____

(0.25)(0.3) =

d. (2.2)(0.3) = _____

(2.2)(0.3) =

Recognizing Equivalent Decimals and Comparing Decimals

Zeroes, which are behind a decimal's last non-zero digit, can be added or removed without changing the value of the decimal. If you look at some equivalent fraction, you'll see why this should be true. For example,

$$\frac{14}{10} = \frac{140}{1000} = \frac{1400}{10000} = \frac{14000}{100000}.$$

These are all equivalent to $\frac{14}{100}$ because the numerator and denominator was multiplied by some power of 10 (10, 100 or 1000) to get one of the other equivalent fractions. If we replace each of the four fractions above with their decimal equivalents, we get 0.14 = 0.140 = 0.1400 = 0.14000. Let's call any zeroes at the end of a decimal's last non-zero digit "**trailing zeroes**". Then, we could conclude that any number starting with .14 that has trailing zeroes is also equivalent, like 0.1400000.

Two decimals are equal only if one can be made to look identical to the other by adding or removing trailing zeroes. You can also determine if they are equivalent by removing any excess trailing zeroes from each to see if they are identical.

? Exercise 4

For the first decimal given, circle any of the next four decimals that are equal to it.

a.	1.900;	1.0900	1.9	1.90000	0.190
b.	4.034;	4.0340	4.0334	4.0034	4.3040

To compare two or more decimals that are not equal, but have the same number of digits after the decimal point, you can write each as a fraction with the same denominator and then compare the numerator. For instance 0.14 is less than 0.21 since fourteen hundredths is less than twenty-one hundredths. Basically, in this case, it's just like comparing whole numbers. You'll be able to determine which is larger by comparing each number as if there was no decimal point. But, keep in mind this only makes sense if the numbers you are looking at have the same number of digits after the decimal point.

? Exercise 5

Compare each of the following decimals using <, = or >.

a.	3.5	0.9
b.	35.06	35.0600
c.	0.089	0.098

To compare two or more decimals that are not equal that do not all have the same number of digits after the decimal point, first write each decimal with the same number of digits after the decimal point (by adding trailing zeroes to one or more if necessary). By doing that, you are comparing tenths with tenths, or hundredths with hundredth, etc., as you did in exercise 5.

? Exercise 6

Compare each of the following decimals using <, = or >.

a.	3.51	3.488
b.	35.061	35.35
c.	0.08933	0.0894

If a fraction is written with a power of ten in the denominator, it's basic to write the same number in decimal form. You did that in Exercise 2. Any fraction that is written with a power of 10 in the denominator can be written as a terminal decimal. This means that

it's possible to write the number with trailing zeroes. But what if the fraction doesn't have a power of 10 (like 10, 100, 1000, etc.) in the denominator? Sometimes those can be tricky!

If a fraction WITHOUT a power of ten in the denominator CAN be written as an equivalent fraction WITH a power of ten in the denominator, then it can be written as a terminating decimal.

For instance, $\frac{1}{2}$ can be written as $\frac{5}{10}$ (by multiplying the numerator and denominator by 5)

Therefore, $\frac{1}{2} = \frac{5}{10} = 0.5$ (remember to read this as five-tenths)

Well, that one wasn't too hard, but what about $\frac{7}{80}$? The question is whether you can multiply the denominator, 80, by something to get 10, 100, 1000, 10000, etc. There is no whole number you can multiply 80 by to get 10, 100 or 1000. But, if you multiply 80 times 125, it equals 100000. So, by multiplying both 7 and 80 by 125, we get the equivalent fraction $\frac{875}{10000}$ for $\frac{7}{80}$ which can now be written as the decimal 0.0875.

How about writing $\frac{5}{6}$ as a terminating decimal? Well, there is nothing you can multiply 6 by to get 10 or 100 or 1000 or 10000. Is there maybe some number we could multiply 6 by to get some higher power of ten? Well, that's a good question! Actually, there isn't, but how could you be sure? You certainly can't try every power of ten since there are an infinite number of them to try.

It would be nice if there was an easy way to determine if any given fraction could be written as a terminating decimal. The key is to consider the factors of the denominator of a fraction that can be written as a terminating decimal. If a fraction can be written as a terminating decimal, then there is some equivalent fraction where the denominator must be a power of ten: 10 or 100 or 1000, etc.

? Exercise 7

Write the prime factorization of each of the following:

a. 10 = _____

b. 100 = _____

c. 1000 = _____

d. 10000 = _____

e. 100000 = _____

? Exercise 8

What are the only prime factors of powers of 10? _____

? Exercise 9

If a power of ten has three factors of 5, how many factors of 2 does it have? _____

? Exercise 10

If a power of ten has two factors of 2, how many factors of 5 does it have? _____

Powers of ten only have 2s and 5s as its prime factors and nothing else.

Let's go back to our three numbers , $\frac{1}{2}$, $\frac{7}{80}$, and $\frac{5}{6}$, that we were trying to write as terminating decimals and analyze the situation.

$\frac{1}{2}$ is simplified, and in its prime factored form, there is exactly one 2 in the denominator. To write as an equivalent fraction with a denominator that is a power of ten, it must have only 2s and 5s as prime factors in the denominator, and the same number of each! Therefore, multiplying by one more 5 in the numerator and denominator did the trick!

Prime factor the numerator and denominator of this reduced fraction: $\frac{7}{80} = \frac{7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}$

We must determine if it is possible to multiply the denominator by something so that the resulting denominator will be made up of only 2s and 5s and the same number of each. Well, there are four factors of 2 and one factor of 5. Since we need the same number

of each factor, making an equivalent fraction by multiplying the numerator and denominator by three more factors of 5 will do the trick.

$$\frac{7}{80} = \frac{7}{2\cdot2\cdot2\cdot2\cdot5} \cdot \frac{5\cdot5\cdot5}{5\cdot5\cdot5} = \frac{875}{10000} = 0.0875$$

In both examples, note we either multiplied by extra factors of 2 or 5, but not both!

What about $\frac{5}{6}$? Well, it's reduced, and the prime factorization of the denominator is $2 \cdot 3$. No matter what the denominator is multiplied by, we'll be stuck with a factor of 3 in the denominator. Since the only prime factors of powers of 10 are 2 and 5, there can't be a prime factor of 3 in the denominator if we want to end up with only a power of 10 in the denominator. Therefore, since it is impossible to write $\frac{5}{6}$ as an equivalent fraction with a power of ten in the denominator, it cannot be written as a terminating decimal.

How about $\frac{63}{72}$? If we prime factor the denominator, we get $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$.

? Exercise 11

Well, what do you think? Can $\frac{63}{72}$ be written as a terminating decimal? _____

Explain why or why not.

One way to check that you are obtaining the correct results is to use a calculator. For $\frac{1}{2}$, we got 0.5, which you can check by doing the division $1 \div 2$ on your calculator.

? Exercise 12

Use your calculator to find the decimal equivalent for each of the following:

a. $\frac{7}{80}$ = _____

b. $\frac{63}{72}$ = _____

? Exercise 13

Hmmm, did you get 0.0875 for part a? _____

If you did part b on your calculator, did you get a terminating decimal of 0.875?

Is that what you expected? Why or why not?

The reason $\frac{63}{72}$ can be written as a terminating decimal is because in its ***simplified*** form, it only has 2's and/or 5's as its prime factors. Here is how to finish this problem by simplifying first, and then multiplying by any needed factors of 2 or 5 to get the same number of each:

$$\frac{63}{72} = \frac{3\cdot3\cdot7}{2\cdot2\cdot2\cdot3\cdot3} = \frac{7}{2\cdot2\cdot2} \cdot \frac{5\cdot5\cdot5}{5\cdot5\cdot5} = \frac{875}{10000} = 0.875$$

? Exercise 14

For each fraction, determine if it can be written as an equivalent fraction with a power of ten in the denominator. If a fraction cannot be written as a terminal decimal, ***explain*** why not. Otherwise, show **ALL** of the steps (as shown in the previous examples) to write it as a terminal decimal. The steps are listed below.

a. Simplify if possible
b. Prime factor the denominator
c. Multiply the numerator and denominator by an appropriate number of factors of 2 or 5 so that the denominator will be a power of 10
d. Simplify the numerator and denominator
e. Write as a terminating decimal

Then, **check your answer with a calculator** by taking the original fraction and dividing the numerator by the denominator. You should get the same decimal obtained by doing the five steps outlined above.

a. $\frac{3}{4}$

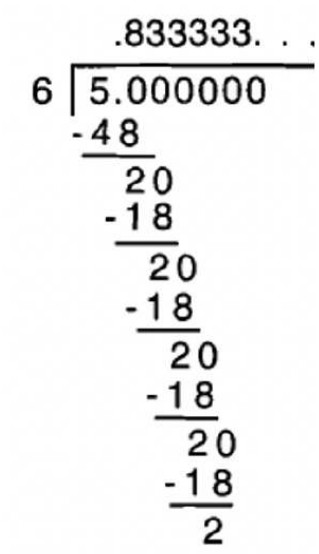
b. $\frac{9}{20}$

c. $\frac{9}{15}$

d. $\frac{18}{25}$

e. $\frac{5}{14}$

Okay, now we have to deal with those fractions that cannot be written as terminating decimals. Any simplified fraction that has at least one prime factor that isn't a 2 or 5 is in this category. Let's look at $\frac{5}{6}$ again. One way to write this as a decimal is to divide 5 by 6.



As you could see, I keep dividing 6 into 20, write 3, multiply to get 18, subtract from 20, get 2, bring down the 0, and start all over again. This can go on forever and ever. So, the 3's at the end will trail on forever. This is NOT a terminating decimal since the number cannot be written with trailing zeroes. In this case, there are trailing 3s. The **three dots at the end (called ellipses)** of the number are to show that the 3s repeat forever. The answer can be written as 0.8333333... or $0.8\bar{3}$. *Remember to put the ellipses (three dots) at the end of the number!* The bar over the 3 indicates that the 3 repeats forever and ever. Here are other ways to represent the same thing: $0.8\overline{33}$ or $0.83333\bar{3}$. In the first case, it says 33 repeats forever. In the second case, it says after the first three 3's, the 3 repeats forever. In both cases, when you write it out in long form, it looks like 0.8333333... so it's the same number. Usually, we'll write 0.8333333... or $0.8\bar{3}$.

? Exercise 15

Write 0.8333333... or $0.8\bar{3}$ in three more ways, different than $0.8\overline{33}$ *or* $0.83333\bar{3}$

_____ , _____, and _____

One thing to notice about what happened when we divided 5 by 6 is that I kept getting 2 after each subtraction in the division, this is like the remainder. Also remember that when you divide, each remainder must be less than or equal than what you are dividing by.

Exercise 16

Write the possible remainders for each number.

a. 6: _____

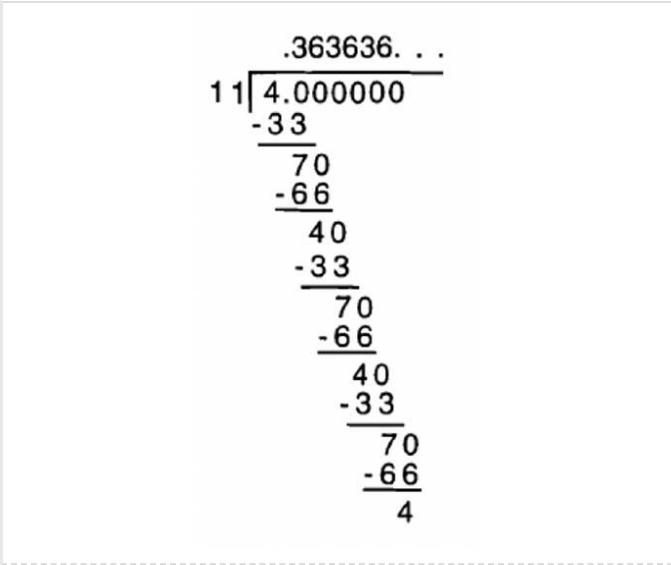
b. 7: _____

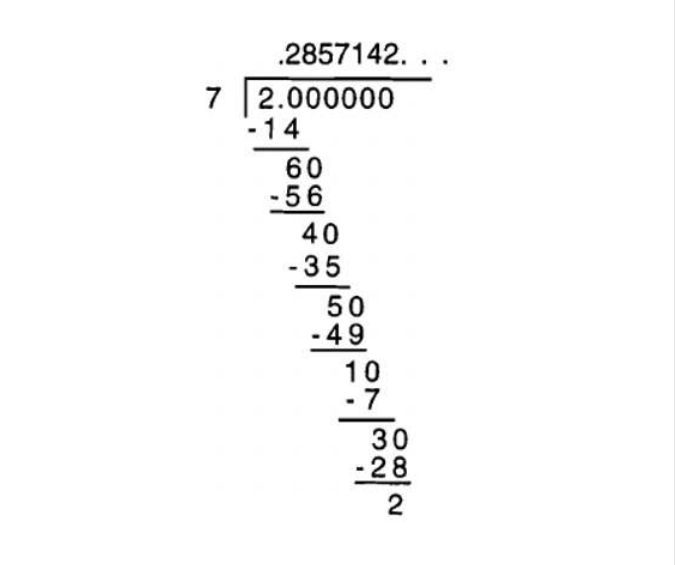
c. 9: _____

d. 11: _____

e. 3: _____

Let's now look at what happens when you do long division compute $\frac{4}{11}$ and $\frac{2}{7}$.





In both of these cases, eventually, you get a remainder that you got previously, so the computation repeats itself. In the case of dividing 4 by 11, two remainders come up before there is some repetition 4 and 7. From 16d, you should have realized that the only possible remainders for 11 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Now if you got a remainder of 0 when dividing, you'd have a terminating decimal. So, when dividing by 11, the most remainders you might get in a row before one repeats is 10 in a row. But in the case of $\frac{4}{11}$, the remainders repeat after the 4 and 7. This means you keep dividing into 40 or 70 (since you bring the zero down) and that's why in the quotient the digits start repeating. ***Remember to put the ellipses (three dots) at the end of the number if there is a pattern of repeating digits.***

Now in the case of dividing 2 by 7, the remainders you get as you go along are 6, then 4, then 5, then 1, then 3, and then 2. When you bring down the 0, you are dividing 7 into 20 again, and hence the remainders repeat. Notice that the sequence of remainders (6, 4, 5, 1, 3, and 2) is different than the sequence of digits that repeat in the quotient. The digits that repeat in the quotient are 285714, so $\frac{2}{7}$ = 0.285714285714... or $0.\overline{285714}$

Since there are only six possible remainders other than zero when dividing by 7, only a sequence of 6 digits could possibly repeat. In this case, all six possible remainders of 7 appeared in the long division.

There are many ways to express the infinite or repeating decimal 0.285714285714... or $0.\overline{285714}$ First of all, to establish what is repeating, you would want to see the sequence of digits repeating at least two times through. Therefore, 0.285714285714...is the shortest possible way of showing it when the ellipses (three dots) are used. If you simply wrote 0.285714..., it wouldn't be clear whether or not the 4 repeated or the 14 repeated, etc.

Here are a few other ways to write 0.285714285714... besides $0.\overline{285714}$

0.285714285714285714285714...

0.285714285714285714...

(in this case, you see it as the 857142 repeating from this point on)

0.2857142857142857...

(in this case, you see it as the 142857 repeating from this point on)

$0.2\overline{857142}8$

(in this case, you see it as the 571428 repeating from this point on)

? Exercise 17

Write 0.285714285714... two more ways using ellipses (the three dots) and two more ways using a bar over a repeating sequence of digits.

? Exercise 18

Determine which of the following is equivalent to 0.383432432432...

One way to do this is write the number out in long form by carefully continuing the pattern and then check the digits one place value at a time.

a. 0.3834324324...
b. 0.383432432...
c. 0.3834323432...
d. $0.3834\overline{3243}$
e. $0.383\overline{4324}$

Keep in mind that if you are finding a decimal equivalent for a fraction with 17 in the denominator that there may be up to sixteen digits in a row before you see any repetition. Now, obviously, there are only 10 digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) in base ten. Note that the sequence of digits repeating in the quotient is totally different than the possible remainders you might get when you divide by 17 since a remainder may be more than a single-digit number.

? Exercise 19

a. How many possible remainders could there be if you divide a number by 33?

b. Do long division to write $\frac{31}{33}$ as a repeating decimal using ellipses and then using the bar over the repeating digits on this line: _____

Show work here:

c. In the long division, how many digits repeat? _____

d. In the long division, what were the remainders you would keep getting if you continued the division forever and ever? _____

? Exercise 20

Use your calculator to write each as a repeating decimal. Write the answer two ways, first with a bar over the repeating digits, and then in long form, **using ellipses**. Note that your calculator may or may not round the last digit shown. It can't show digits repeating forever, so you have to be savvy enough to know whether the decimal showing on the display is terminal or if it is an approximation.

a. $\frac{5}{9}$ = _____ = _____
b. $\frac{5}{7}$ = _____ = _____
c. $\frac{1}{6}$ = _____ = _____
d. $\frac{2}{3}$ = _____ = _____

e. $\frac{7}{11}$ = _____ = _____

f. $\frac{5}{12}$ = _____ = _____

g. $\frac{8}{15}$ = _____ = _____

h. $\frac{16}{45}$ = _____ = _____

i. $\frac{5}{66}$ = _____ = _____

Exercise 21

If you are changing a simplified fraction to a decimal where the denominator is x, how many digits at most can repeat in the quotient? _____

Well, you've learned how to write terminating decimals as fractions, and how to write fractions as decimals. You should be able to determine whether a fraction can be written as a terminating or repeating decimal before doing the actual division.

Rational Numbers are defined to be numbers which can be expressed as the ratio of two integers. Fractions as we usually refer to them (without decimal points or square roots, etc. in the numerator or denominator) are rational numbers. Since all fractions can be written as terminating or nonterminating (infinite, repeating) decimals, then all terminating and repeating decimals are also rational numbers. In Exercise 1, you wrote some terminating decimals as fractions. Since all nonterminating decimals came from a fraction, now we want to be able to go the other way around and write a repeating decimal as a fraction. We've got a nifty trick for doing just that!

Let's say we wanted to write $0.\overline{72}$ or 0.727272727272... as a fraction.

First, notice this is a very different number from 0.72, since 0.72 is seventy-two hundredths, or $\frac{72}{100}$. This simplifies to $\frac{18}{25}$. So,0 .72 and $\frac{18}{25}$ are the same number. You can check to make sure $\frac{18}{25}$ is the correct fraction by using a calculator and dividing 18 by 25. The calculator should read 0.72, which is where we started.

Okay, so how do we write $0.\overline{72}$ as a fraction? Since this is a repeating decimal that has infinitely many digits, there is no single power of ten we can put in the denominator. The trick is to use algebra to write a number in a way that eliminates the infinitely repeating part of the decimal. First, let's write out $0.\overline{72}$ the long way:

$0.\overline{72}$ = .727272727272... and call it the number x. So x = 0.727272727272...

Remember that if you multiply a decimal by 10, the decimal point moves one place to the right, and when you multiply a decimal by 100, the decimal point moves two places to the right, etc.

Exercise 22

If x = 0.727272727272..., then write out what 10x, 100x and 1000x equals. Write it out **without** using the bar over the repeating digits. Use the ellipses (three dots).

10x = _____

100x = _____

1000x = _____

Exercise 23

Okay, let's work with x = 0.727272727272... and 100x = 72.727272727272...

a. In algebra, what is 100x - x? _____ (Hint: Subtract coefficients.)

b. Compute 100x - x another way: Line up the decimal points and subtract 0.727272727272... from 72.727272727272... Notice that if the decimals are lined up, the "tails" at the end of both repeating decimals are exactly the same so it should be easy to subtract. Show the work below.

c. Write an equation so that the answer to part a equals the answer to part b. Then use algebra to solve for x. Show work.

d. Simplify the fraction you got for x in part c. Show work.

e. Use a calculator to rewrite the fraction you got in part c as a decimal:

Is it equal to $0.\overline{72}$? _____

If so, you must have written $0.\overline{72}$ as the correct fraction!

Here's the trick for changing repeating decimals to simplified fractions:

Call the number you are trying to write as a decimal a variable, like n or x.

If there is one repeating digit, compute 10x; if there are two repeating digits, compute 100x; if there are three repeating digits, compute 1000x, etc. This aligns the repeating decimals up with each other so that the tail of x and the other number (10x, 100x, 1000x, etc.) is the same.

Then subtract x from the other number (10x, 100x, 1000x, etc.) The tails of both numbers will come off and you should have an algebraic problem to solve at this point. Make sure you write your answer as a reduced fraction with only integers in the numerator and denominator.

Here are two examples. *If you write the number in decimal form without the bar over the number, remember to put the ellipses (three dots) at the end!*

Write $0.\bar{2}$ as a simplified fraction.

Let x = $0.\bar{2}$ = .222222... Since only one digit is repeating, multiply x by 10.

Then 10x = 2.222222...

$$\begin{aligned} 10x &= 2.22222\ldots \\ -x &= .2222\ldots \\ \hline 9x &= 2 \end{aligned}$$

$$x = \frac{2}{9}$$

Use a calculator to check that $\frac{2}{9} = 0.\bar{2}$

Write $0.\overline{54}$ as a simplified fraction.

Let x = $0.\overline{54}$ = .545454... Since two digits are repeating, multiply x by 100.

Then 100x = 54.545454...

$$\begin{aligned} 100x &= 54.545454\ldots \\ -x &= .545454\ldots \\ \hline 99x &= 54 \end{aligned}$$

$$x = \frac{54}{99} = \frac{6}{11}$$

Use a calculator to check that $\frac{6}{11} = 0.\overline{54}$

? Exercise 24

Rewrite each of the following decimals as simplified fractions. For repeating decimals, use the technique shown in the previous examples. Then, check your answer using a calculator by dividing the numerator by the denominator to see if the result matches the original problem.

a. 0.4

b. $0.\bar{4}$ Begin by letting x = $0.\bar{4}$ or x = 0.44444...

c. 0.06 =

d. $0.\overline{06}$ Begin by letting x = $0.\overline{06}$ or x = 0.060606...

e. 0.9

f. $0.\bar{9}$ Begin by letting x = $0.\bar{9}$ or x = 0.9999... (this answer might surprise you)

g. 0.45

h. $0.\overline{45}$ Begin by letting x = $0.\overline{45}$ or x = 0.454545...

i. 0.084

j. $0.\overline{084}$ Begin by letting x = $0.\overline{084}$ or 0.084084084...

Sometimes, the arithmetic gets a little more challenging. Consider writing 0.14444... as a fraction. The 4 repeats starting two places after the decimal point. We plunge forward as before, but there's a little glitch at the end because one side of the equation will have a decimal point in it. If we were to divide by 9, the fraction will have a decimal point in the numerator so it isn't a reduced rational number; both the numerator and denominator have to be integers. Here are basic steps up until that point.

$10n = 1.444444\ldots$and $n = 0.144444$

Subtracting, we get $9n = 1.3$

One way to remedy this situation is to multiply both sides of the equation by 10 (or 100 or 1000 as needed) to eliminate the decimal. This is like clearing fractions by multiplying both sides of an equation by the least common denominator.

So multiply both sides of $9n = 1.3$ by 10 to get $90n = 13$. Then divide by 90 to get $\frac{13}{90}$.

Another way to remedy this situation is to write the right side of the equation as a fraction. Remember that 1.3 is $\frac{13}{10}$. So the equation is $9n = \frac{13}{10}$.

You can cross multiply to get $90n = 13$ and then divide by 90 to get $n = \frac{13}{90}$

In any case, remember that a reduced fraction is the ratio of two integers that are relatively prime.

? Exercise 25

Rewrite each repeating decimal, use the technique shown in the previous example, as a simplified fraction. Then, using your calculator, divide the numerator by the denominator and see if the result matches the original problem.

a. 0.02828...

b. 0.2888...

c. 0.00666...

d. 0.1011011...

e. 0.3999...

So, what did you think about the answer to 24f and 25e? Both of those are little mind-boggling. It's kind of hard to accept, but 0.999999... is really the same number as 1. It's not less than 1 it's exactly 1! For 24f, if you let $10n = 9.999\ldots$ and $n = 0.999\ldots$, then subtracting n from $10n$ yields $9n = 9$, so $n = 1$. That's hard to swallow, but it's the truth! Another way to see this is to realize that 1/3 + 2/3 = 1. But 1/3 = 0.333333.... and 2/3 = 0.666666... Then, 1/3 + 2/3 = 0.33333... + 0.66666... = 0.99999.... We know 1/4 + 2/3 = 1, so 0.99999... must also equal 1, as well. Oh, this stuff is just too cool!

For 0.39999..., the result of repeating 9s after a decimal makes 0.39999...= 0.4 = $\frac{4}{10} = \frac{2}{5}$.

So, basically, any number with a bunch of trailing 9s ends up being a terminating decimal.

All of the numbers we've been dealing with so far fractions, terminating decimals, and repeating decimals make up the rational numbers. Every rational number can be written as the ratio of two relatively prime integers, and can also be written as a terminating or repeating decimal. Conversely, every terminating and repeating decimal is a rational number.

Oh, but there's more...much more!

The rationals make up a very small part of the real numbers. To complete the real number system, we have to talk about the irrational numbers. Every real number is either rational or irrational. Those numbers which cannot be written as the ratio of two relatively prime integers is irrational. Those decimal numbers which neither terminate nor repeat are irrational. So, what do they look like?

One of the most commonly known irrational numbers is π. It is the number that is the ratio of the circumference of a circle to its diameter. Sounds like it's a ratio of two integers, but it's not! π can only be approximated. The most common approximation is 3.14 or 22/7. Neither of these is equal to π because both of these are rational numbers, and π is not!

Other irrational numbers are square roots of numbers that are not perfect squares, or cube roots of numbers that are not perfect cubes, etc.

For instance, these numbers are irrational: $\sqrt{5}, \sqrt{3}, \sqrt{12}, \sqrt{20}, \sqrt[3]{35}, \sqrt[4]{72}$

Exercise 26

Write five irrational numbers not already listed above _____

Exercise 27

Is $\sqrt{9}$ irrational? _____ Why or why not? _____

Another way to express an irrational number in decimal form is to make up a decimal that perhaps has some pattern to it, but never terminates or repeats. Two examples of this type are 2.12112111211112... and 5.010203040506070809010011012...

Exercise 28

Write four irrational numbers in decimal form that shows a clear pattern.

Exercise 29

Write both a rational number and an irrational number in decimal form that is between 0.53 and 0.54.

Exercise 30

Write both a rational number and an irrational number in decimal form that is between 0.53333... and 0.54444...

Rational _____

Irrational _____

Exercise 31

Classify each of the following numbers as rational or irrational.

a. 0.428222... _____

b. 0.283848... _____

c. $\frac{5}{13}$ _____

d. $\sqrt{80}$ _____

e. $\sqrt{100}$ _____

f. π _____

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5.8: Definition of Real Numbers and the Number Line

Definition: Real Numbers

Real numbers are the numbers that are normally used in real world math problems.

Here are common groups of numbers that are real numbers:

Whole Numbers:	$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$	Positive counting numbers plus zero
Integers:	$\ldots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots$	Positive and negative whole numbers
Rational Numbers:	$13,\ \frac{2}{7},\ \frac{-1}{3},\ -2,\ 1.32,\ -12.64$	Numbers that may be written as a b , where a and b are integers. Decimals are rational numbers.
Irrational Numbers:	$e,\ \sqrt{8},\ -\sqrt{11},\ \pi,\ 0.1234$	Numbers that can't be expressed as a b . Irrational numbers are numbers with non-repeating and never-ending decimals!

Note: Real numbers may be positive or negative and include 0 as shown above.

Definition: The Number Line

A line that extends horizontally with coordinates that correspond to real numbers. The number line helps measure the distance between the origin (0) to a real number. Here's an example of a number line:

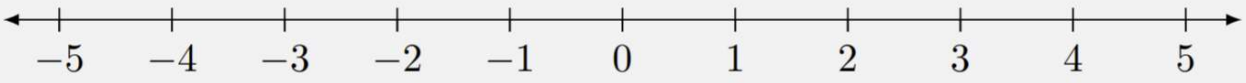


Figure 1.1.1

5.8.1: Reading the number line:

The origin corresponds to the number 0 in the number line.

To the left of the origin are the negative numbers.

To the right of the origin are the positive numbers.

Exercise 1.1.1

Graph the following numbers on the number line below: $-5,\ e,\ 3.5,\ -2.25,\ 7.01,\ -5.2,\ \sqrt{20},\ \pi$

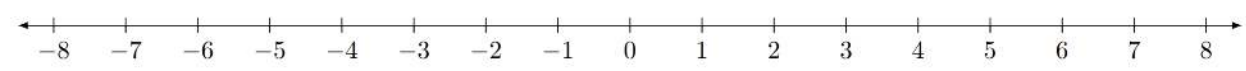


Figure 1.1.2

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5.9: Models and Operations with Integers

Groups of Numbers

The largest group is the Real Number System. The Real Numbers hold true for all numbers learned in elementary school. The only numbers, which do not belong in the Real Number group, are imaginary numbers, which are not learned until high school.

Within the Real Numbers are the Rational and Irrational Groups. Rational numbers are any number which can be written as a fraction, whereas any irrational number is non-terminating (never stops) and non-repeating. If a decimal has a pattern, it can be made into a fraction.

Integers, the next smallest group, include negative numbers, zero and positive numbers, without any fractions or decimals.

Whole numbers, like Integers, cannot have fractions or decimals. In addition, Whole Numbers start with zero include positive numbers. No negatives.

The smallest group of numbers are the Natural Numbers. They are like Whole Numbers, except without zero.

Modeling Signed Numbers

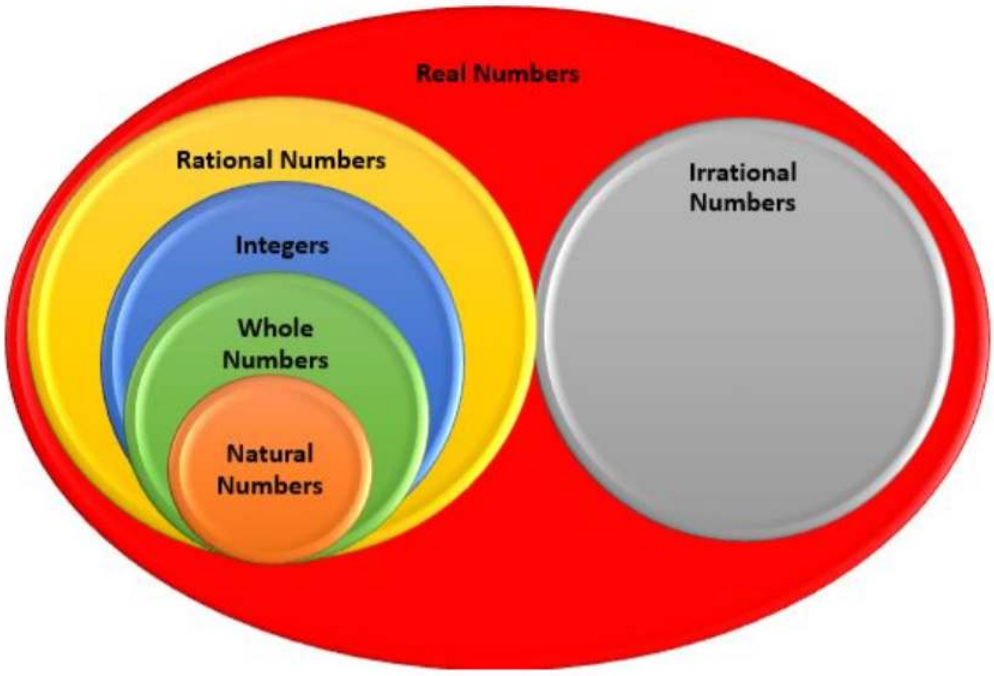


Figure 4.4.1

- Represents a Positive Number
- Represents a Negative Number

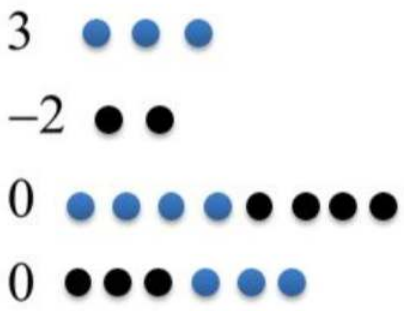


Figure 4.4.2: Modeling signed numbers

Operations with Signed Numbers

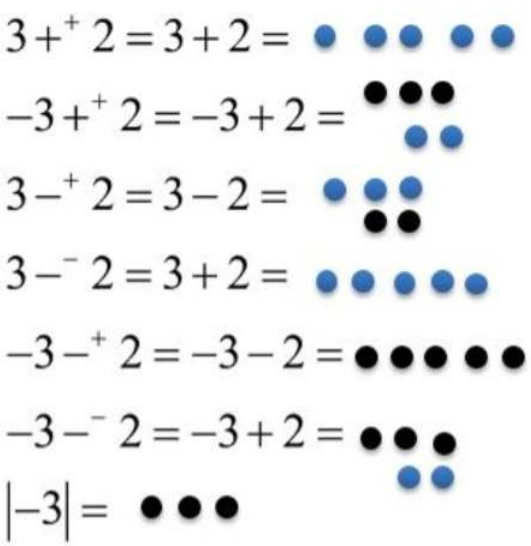


Figure 4.4.3: Operations with signed numbers

Operations on the Number Line

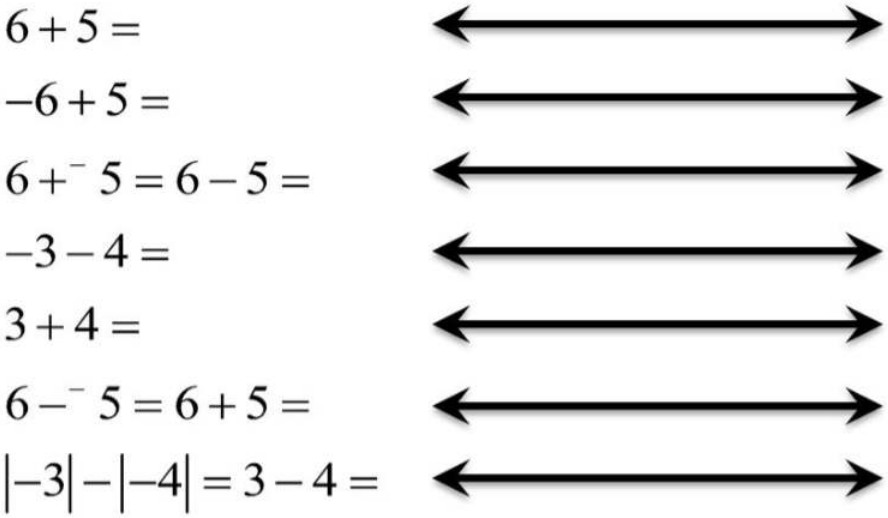


Figure 4.4.4: Operations of number line

Table of Numerical and Algebraic Properties

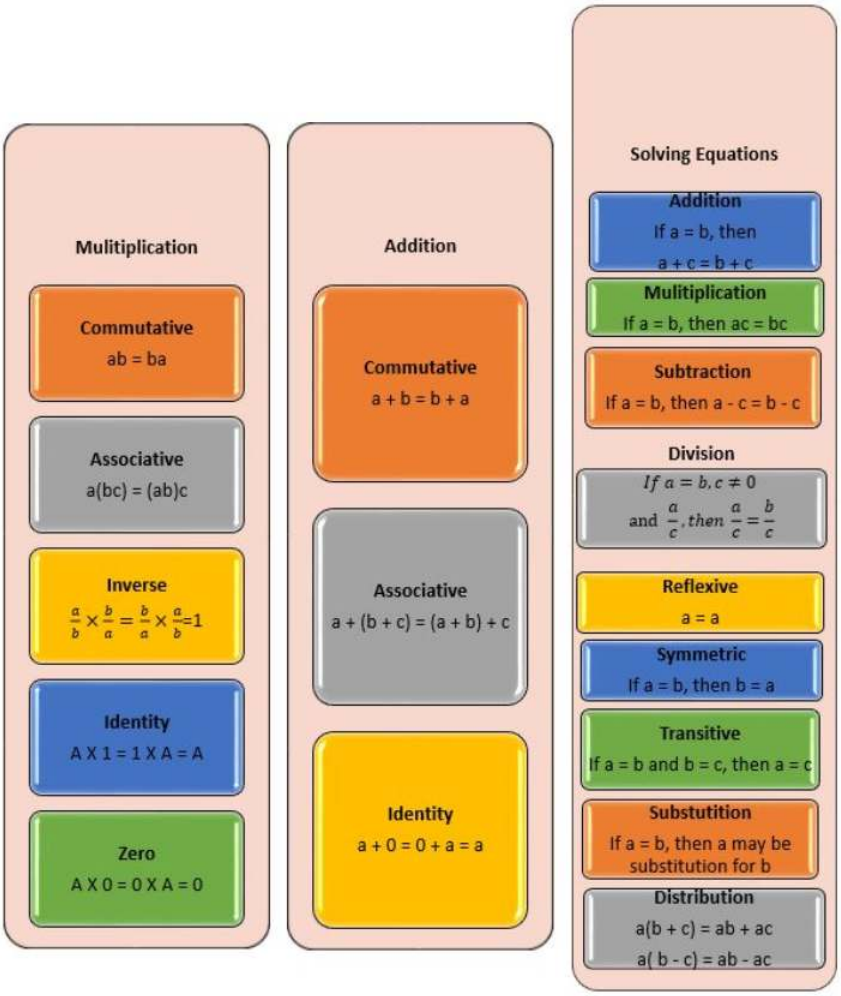


Figure 4.4.5: Table of Numerical and Algebraic properties

Practice Problems

Put the following numbers into their correct group(s).

1. 16
2. 3.14
3. π
4. $\sqrt{16}$
5. $\sqrt{23}$
6. -8
7. 0
8. 1

Add or subtract the following integers.

9. -16 + 45
10. 91 – (-56)
11. 64 – 341
12. 78 + (-89)
13. 232 + 44

State the property used for each equation.

14. 5(4 + 2) = 20 + 10

15. 4(3)(12) = 4(12)(3)
16. 42 = 42
17. If x = 3 and x = y, then y = 3

Extension: Methods of Teaching Mathematics

Part 1

Watch a full 45-minute lesson and write the corresponding lesson plan for it. See Canvas for more detailed instructions.

Part 2

Make sure you are working on Khan Academy throughout the semester.

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5.10: Number comparisons using < ,>, and =

The number line is to compare numbers. Here are the symbols used when comparing two or more numbers:

Symbols used	Read as
$<$	less than
$\leq$	less than or equal to
$>$	greater than
$\geq$	greater then or equal to
$=$	equals

Caution: $\leq$ and $\geq$ are used if the numbers that are being compared satisfy at least one of the two conditions.

Note: Moving further to the left of the origin, the numbers are decreasing in value. Moving further to the right of the origin, the numbers are increasing in value.

Examples 1.2.1

Statement	Reason
$-3 < -2$	Since -3 is further to the left of -2
$5 > 1$	Since 5 is further to the right of 1
$2 > -6$	Since 2 is further to the right of -6
$3 \geq 3$	3 is not greater than 3, but they are equal!

Hint: The symbol $>$ is pointing to the right direction. The numbers in the right direction are increasing so use this symbol when the first number is greater than the second number. Think similarly for less than.($<$ pointing left so less than)

Exercise 1.2.1

For the following exercises compare the following numbers and fill in the blank by using the appropriate symbol $<$, $>$, $\geq$, $\leq$ or $=$

1. $-1\_\_\_\_4$
2. $-4\_\_\_\_9$
3. $0\_\_\_\_-2$
4. $3\_\_\_\_3$
5. $-12\_\_\_\_12$
6. $4\_\_\_\_6$
7. $-1\_\_\_\_-1$
8. $5\_\_\_\_-2$
9. $35\_\_\_\_53$
10. $-29\_\_\_\_-37$

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5.11: Homework

HW #1

Do each of the following steps using your C-strips.

i. State how many C-strips (each an equal part of the whole) make up one unit.
ii. State which C-strip makes up one part of the whole.
iii. State the fraction that the C-strip in part b represents.
iv. State how many of the C-strips in part b you need to make into a train.
v. State which C-strip is the length of the train you made in part c

a. If S represents 1 unit, then which C-strip represents $\frac{7}{11}$?

b. If H represents 1 unit, then which C-strip represents $\frac{2}{3}$?

c. If P represents 1 unit, then which C-strip represents $\frac{3}{2}$?

d. If L represents 1 unit, then which C-strip represents 3 ?

e. If Y represents 1 unit, then which C-strip represents $\frac{6}{5}$?

f. If O represents 1 unit, then which C-strip represents $\frac{1}{2}$?

g. If B represents 1 unit, then which C-strip represents $\frac{4}{3}$?

HW #2

Do each step using your C-strips.

i. State how many C-strips will make up the named C-strip stated in the problem.
ii. Which C-strip makes up one equal part?
iii. State the fraction that the C-strip in part b represents.
iv. State how many of the C-strips in part b will make up one unit.
v. Form the unit by making a train from the equal parts (C-strip in part b) and state which C-strip has the same length as that train.

a. If O represents $\frac{5}{6}$, then which C-strip is 1 unit?

b. If W represents $\frac{1}{7}$, then which C-strip is 1 unit?

c. If D represents $\frac{3}{2}$, then which C-strip is 1 unit?

d. If N represents $\frac{4}{3}$, then which C-strip is 1 unit?

e. If D represents 3, then which C-strip is 1 unit?

f. If K represents $\frac{7}{9}$, then which C-strip is 1 unit?

HW #3

Do each step using your C-strips.

i. State which C-strip is one unit.
ii. State which C-strip is the answer.

a. If N represents $\frac{2}{3}$, then which C-strip represents $\frac{1}{4}$?

b. If D represents $\frac{3}{4}$, then which C-strip represents $\frac{3}{2}$?

c. If B represents $\frac{3}{2}$, this which C-strip represents $\frac{4}{3}$?

? HW #4

Use your fraction arrays to determine all fractions on the fraction array that are equivalent to 3/4. Do this by finding 3/4 on the array, and seeing what other numbers are the same length. Include a diagram.

? HW #5

Use your multiple strips to write 6 fractions equivalent to 5/6. Draw the strips.

? HW #6

Use your multiple strips to write 6 fractions equivalent to 3/8 Draw the strips.

? HW #7

Compare 3/8 and 1/3 using models. Show **all** of the steps, and explain the procedure as shown in this module.

? HW #8

Add 3/8 and 1/3 using models. Show all of the steps, and explain the procedure as shown in this module.

? HW #9

Do the following subtraction using models: 3/5 – 1/4. Show all of the steps, and explain the procedure as shown in this module.

? HW #10

Do the following multiplications using models. Show all of the steps, and explain the procedure as shown in this module.

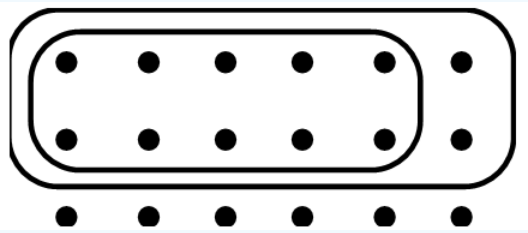
a. 3/8 · 2/5

b. 4/7 · 2/3

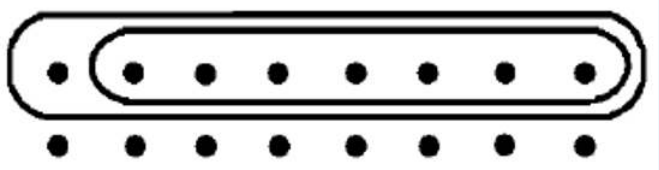
? HW #11

By looking at the final drawing someone made to model a multiplication of two fractions, determine which multiplication was performed, and then state the answer.

a. 5/6 · 2/3 OR 2/3 · 5/6

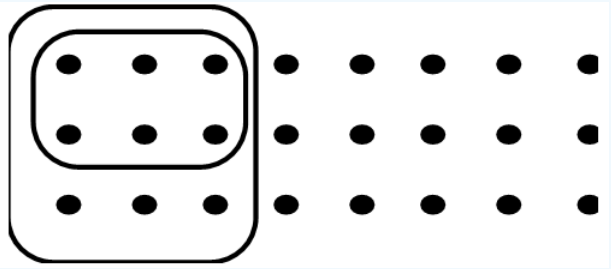


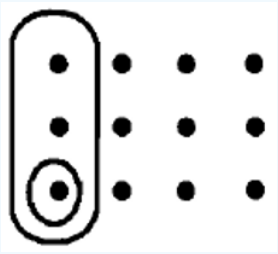
b. 1/2 · 7/8 OR 7/8 · 1/2



HW #12

If all of the dots shown for each problem represent 1 unit, determine the multiplication problem that someone did to get the answer, and state the answer.

a. 

b. 

HW #13

Fill in the chart showing how to do the following multiplications using C-strips. The multiplication is in the first column. State an appropriate choice for the unit (name a C-strip, or sum of two C-strips) in the second column. Write the C-strip obtained after the first part of the multiplication (which is the second fraction as a part of the unit) in the third column. Then, do the final multiplication, and write the C-strip obtained in the fourth column. In the fifth column, write a fraction using C-strips putting the final unit obtained in the fourth column as the numerator, and the unit in the denominator. Then, in the last column, write the answer as a fraction. Do not simplify.

a.	$\frac{1}{3} \cdot \frac{2}{3}$					
b.	$\frac{1}{2} \cdot \frac{5}{6}$					

HW #14

Perform the following division using the box and dot methods. First define the unit. Then explain and show all of the steps. Include diagrams.

a. 5 ÷ 1/3

b. 3/4 ÷ 1/3

HW #15

Determine if the following statements are true or false by comparing cross products.

a. 19/23 = 57/69

b. 24/37 = 68/91

HW #16

Write each fraction in simplest form **using each of the two methods:**

(1) prime factorization and

(2) finding GCF.

a. $\frac{216}{420}$

b. $\frac{195}{286}$

HW #17

Use cross products to compare each of the following fractions. Use < or >.

a. 18/23 and 5/8

b. 11/18 and 121/250

HW #18

Find 3 rational numbers, written with a common denominator, between 3/8 and 5/8.

HW #19

Find 3 rational numbers, written with a common denominator, between 1/2 and 4/7.

HW #20

a. 21 of John's students have cats at home. This represents 7/10 of John's students. How many students are in John's class? Solve the problem using models. Explain how the model works.

b. At an elementary school, 38 teachers drive alone to work. This represents 2/3 of the teachers. How many teachers work at the school? Solve the problem using models. Explain how the model works.

HW #21

Write in words how to read each of the following decimals.

a. 0.7

b. 0.67

c. 3.28

d. 19.835

HW #22

Multiply the following decimals mentally then do it again by showing the same steps as shown in this module..

a. (0.3)(0.8)

b. (1.2)(0.4)

c. (1.22)(2.3)

d. (3.2)(2.41)

HW #23

For each fraction, determine if it can be written as an equivalent fraction with a power of ten in the denominator. If a fraction cannot be written as a terminal decimal, **explain** why not. Otherwise, show **ALL** of the steps to write it as a terminal decimal.

a. $\frac{11}{16}$

b. $\frac{3}{125}$

c. $\frac{1}{12}$

d. $\frac{9}{40}$

e. $\frac{21}{56}$

HW #24

Rewrite each of the following decimals as simplified fractions. For repeating decimals, use the techniques shown in this module. Then, check your answer using a calculator by dividing the numerator by the denominator to see if the result matches the original problem.

a. $0.\overline{7}$

b. $0.\overline{72}$

c. $0.\overline{235}$

d. $0.2\overline{5}$

e. $0.3\overline{42}$

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CHAPTER OVERVIEW

6: Number Theory

6.1: The Why
6.2: Number Theory
6.3: Divisibility Rules
6.3.1: Digital Roots and Divisibility
6.4: Primes and GCF
6.5: The Greatest Common Factor
6.6: LCM and other Topics
6.7: The Least Common Multiple
6.8: Homework

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6.1: The Why

The Essential Question



Figure 5.1.1

Why are Teachers Learning this Material?

By the time many students reach college, applying the rules of Greatest Common Factor (GCF) and Least Common Multiple (LCM) are so ingrained in their minds, that they do not even realize they are using the rules anymore. They then began to forget the rules and just apply the steps needed to get the correct answer. For example, many times college students forget that getting like denominators for adding fractions does involve the LCM OR factoring out a number of a polynomial involves the GCF. This chapter is to remind them where their knowledge came from.

Why are Elementary School Students Learning this Mathematics?

The Greatest Common Factor (GCF) and Least Common Multiple (LCM) are important skills for elementary school students. The GCF allows students to reduce fractions. Mastering the GCF will help students later on with the distributive property (factoring) in Algebra 1 as well as word problems. Learning the LCM in elementary school allows students to add and subtract fractions with unlike denominators.

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6.2: Number Theory

Definition: Factors and Multiples

Let $mn = p$, then m and n are factors of p and p is a multiple of m and n

Hints about Factors and Multiples

Factors are always **smaller** than the given number, whereas **multiples** are always **bigger** than the given number.

Example 6.2.1

Find Factors and Multiples of 12

Solution

- Factors of 12: 1, 2, 3, 4, 6, & 12
- Multiples of 12: 12, 24, 36, 48, 60, …

Partner Activity 1

List all the factors and the first four multiples of 30.

Partner Activity 2 - Finding Primes

1. Below are the numbers from 0 to 99.
2. Cross out 0 and 1 (neither prime nor composite) and circle 2 (the first prime)
3. Cross out all multiples of 2.
4. Circle 3 (prime) and cross out all multiples of 3.
5. Circle 5 (prime) and cross out all multiples of 5.
6. Continue this exercise until each number is either crossed out or circled.
7. Write all your circled primes below.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59

60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79

80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99

This activity is called the **Sieve of Eratosthenes**, which is an ancient algorithm developed by Greek mathematician Eratosthenes for finding primes less than a given number n. In this activity we found all the primes less than 100.

Primes and Composites

Definition: Prime number

Any natural number, which has no factors other than 1 and itself.

Examples: 2, 3, 5, 7, 11, 13, 17, 19, 23, …

Definition: Composite Number

Any natural number, which has a factor other than 1 and itself.

Examples: 4, 6, 8, 9, 10, 12, 14, 15, …

Definition: Relatively Prime

Two or more numbers with no factors in common.

Examples: 7 and 8 **or** 15 and 4

Partner Activity 3

Categorize the following as Prime, Composite or Neither: 0, 1, 2, and any negative number

Prime Factorization (Factor Tree)

A composite number can be expressed as a unique product of prime numbers. A factor tree is demonstrated below, in which you think of two factors of the given number and write them below the number on branches. Repeat with the two new numbers until you reach a prime number. For a prime number, circle it (you can't break it down any farther). The circled numbers multiplied together create the prime factorization of the given number.

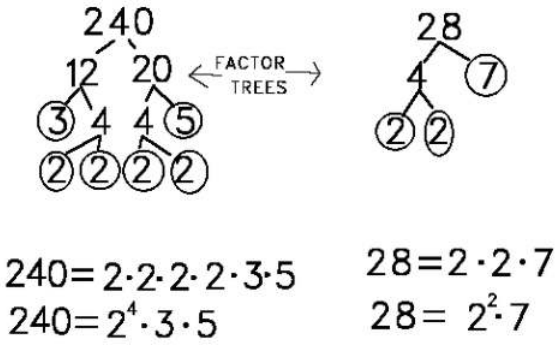


Figure 5.2.1: Sample Factor Tree

Prime factor trees are not unique, meaning two people can have two very different trees (think about for 28, instead of 4*7, you used 2*14). However, the END result is the same! The given number will result in the exact same prime factorization every single time. This is so important, it is given a special designation.

Fundamental Theorem of Arithmetic (Unique Factorization Theorem)

Every whole number (integer greater than 1) can be uniquely factored as a product of prime numbers.

Recall that order doesn't matter when we multiply (multiplication is commutative; 3*2 = 2*3), so this theorem says if we order the prime factors from smallest to largest, everyone will get the same answer of prime factors for a given number.

Partner Activity 4

Write the prime factorization:

1. 85
2. 350
3. 60

Practice Problems

List all the factors.

1. 56
2. 145
3. 32

List the first four multiplies.

4. 50
5. 23

6. 8

List the prime numbers.

7. Between 20 and 40
8. Between 60 and 80
9. Between 120 and 150

List the composite numbers.

10. Between 20 and 40
11. Between 60 and 80
12. Between 120 and 150

Write the prime factorization.

13. 540
14. 60
15. 125

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6.3: Divisibility Rules

Divisibility Rules

Table 5.3.1: Divisibility Rules

Divisible by ___?	The Trick!
2	Last digit is even
3	Add up the digits and if the sum is divisible by 3 then so is the original number
4	Divide last 2 digits by 4
5	Ends in 0 or 5
6	Rules for 2 and 3 work
8	Divide last 3 digits by 8
9	Add up the digits and if the sum is divisible by 9 then so is the original number
10	Ends in 0

Example 6.3.1

What divides evenly into 3495?

Solution

Table 5.3.2

Number	Check it!	Yes or No?
2	Is 3495 even?	no
3	Add the digits: 3 + 4 + 9 + 5 = 21 and 21 divides 3 evenly	yes
4	Can 95 divide 4 evenly?	no
5	Ends in 0 or 5?	yes
6	Divisible by both 2 AND 3?	no
8	Can 495 divide 8 evenly?	no
9	Add the digits: 3 + 4 + 9 + 5 = 21 and 21 does not divide 9 evenly	no
10	Ends in 0?	no



Figure 5.3.1

Practice Problems

Test if the numbers in the left column are divisible by the numbers in the top row. Put a “X” in the box where divisibility holds true. Show any work below the table.

Table 5.3.3

	2	3	4	5	6	8	9	10
67820								
512								
49								
3463								

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6.3.1: Digital Roots and Divisibility

It's helpful to understand what is meant by the **DIGITAL ROOT** of a number because they are used in divisibility tests, and are also used for checking arithmetic problem. A **DIGITAL ROOT** of a number is *one* of these digits: **0, 1, 2, 3, 4, 5, 6, 7 or 8**.

Definition: The DIGITAL ROOT of a number is the remainder obtained when a number is divided by 9.

? Exercise 1

Divide each of the following numbers by 9. Then, write the remainder.

a. 25	h. 8
b. 48	i. 54
c. 53	j. 74
d. 829	k. 481
e. 5402	l. 936
f. 3455	m. 8314
g. 47522	n. 647

Below is another easier way to find the **digital root** of a number.

Step 1: Add the individual digits of the number.

Step 2: If there is more than one digit after adding all the digits, repeat this process until you get a single digit. If the final sum is 9, write 0, because 9 and 0 are equivalent in digital roots (since the remainder is a number smaller than 9). That is why the digital root of a number is only one of these digits: 0, 1, 2, 3, 4, 5, 6, 7 or 8. Those are the only possible remainders that can be obtained when a number is divided by 9! It can't be 9. *The single digit you finally end up with is the **DIGITAL ROOT** of the number*.

Examples: Find the digital roots of the following numbers.

✓ Example 1: 34

34: Add the digits: 3 + 4 = 7

*The digital root of 34 is **7***

✓ Example 2: 321

321: Add the digits: 3 + 2 + 1 = 6

*The digital root of 321 is **6***

✓ Example 3: 58

58: Add the digits: 5 + 8 = 13 Add the digits again: 1 + 3 = 4

*The digital root of 58 is **4***

✓ Example 4: 97

97: Add the digits: 9 + 7 = 16 Add the digits again: 1 + 6 = 7

*The digital root of 97 is **7***

Example 5: 72

72: Add the digits: 7 + 2 = 9 In digital roots, 9 is the same as 0

*The digital root of 72 is **0***

Example 6: 346,721

346,721: Add the digits, we get 3 + 4 + 6 + 7 + 2 + 1= 23
Add the digits again: 2 + 3 = 5

*The digital root of 346,725 is **5***

Note that you **should not write 97 = 16 = 7.** 97 **IS NOT EQUAL** to 7!! The *digital root* of 97 is equal to 7. I use dashes, colons or arrows to record the digital root, so it looks something like this: 346,721 $\rightarrow$ 23 $\rightarrow$ 5. **Don't use equal signs!!!**

Another method for finding digital roots uses the fact that 9 and 0 are equivalent in digital roots. The process of finding digital roots is also called **Casting out Nines**. When you add the digits, you don't have to add the digit 9 or any combination of numbers that add up to 9 (like 2 and 7, or 5 and 4, or 2 and 3 and 4, etc.) – you can "cast out" all 9's. Cross them off; then add the remaining digits together. The following example illustrates how "casting out nines" simplifies the process of finding the digital root of a large number.

Example

Find the digital root of 5,624,398.

Without casting out nines: 5 + 6 + 2 + 4 + 3 + 9 + 8 = 37. Add again: 3 + 7 = 10. Add again: 1 + 0 = **1**.

Casting out Nines: Since 5 and 4 add up to 9, cross them off: $\not{5}, 6, \not{2}\not{4}, \not{3}\not{9}\not{8}$. Since 6 and 3 add up to 9, also cross them off: $\not{5}, \not{6}, \not{2}\not{4}, \not{3}\not{9}\not{8}$. Also cross off the 9: $\not{5}, \not{6}, \not{2}\not{4}, \not{3}\not{9}\not{8}$. The only digits to add are the 2 and 8, which is 10. The digital root of 10 is 1. So **1** is the digital root of 5,624,398 which is the same answer obtained without first casting out nines.

Below, the digital roots for examples 4, 5 and 6 on the previous page is computed again using casting out nines. Note that the digital root remains the same.

Example 4: 97

97: Cross off the 9. Only the 7 remains. The digital root is **7**.

Example 5: 72

72: 7 + 2 = 9, so cross them off. Therefore, the digital root is **0**.

Example 6: 346,721

346,721: Cross off the 3 and 6, and also the 7 and 2. The only digits to add are the 4 and 1. Therefore, the digital root is **5**.

Exercise 2

Find the digital roots of the following numbers, using either method. Remember that 9 is not a digital root. Write zero (0) if the sum is 9 (cast out the 9).

a. 25	h. 8
b. 48	i. 54
c. 53	j. 74

d. 829	k. 481
e. 5402	l. 936
f. 3455	m. 8314
g. 47522	n. 647

? Exercise 3

Did you obtain the same answers for exercises 1 and 2?

Note: This method of finding the digital root only has to do with the number 9. You can't find the remainder of a number when dividing by 7 by adding the digits.

Using Digital Roots is to Check Addition and Multiplication Problems.

To check an addition problem, add the digital roots of the addends. Then, check to see if the digital root of that sum is the same as the digital root of the actual sum of the addends. This works whether there are only two addends, or several addends.

Below are some examples of how to check addition. The actual addition is shown to the left. Arrows are used to show the digital roots of the addends and sum. To check, the digital roots of the addends are added together, and then the digital root of that sum is computed. Compare it to the digital root of the actual sum. If they are equal, put a check to indicate the answer is ***probably*** correct. *NOTE: There is a **slight** chance that the digital roots match, but the answer is still not correct due to some other mistake, like transposing digits. For instance, in example 1 below, it's possible someone might write down 1153 for the answer. The digital roots in the check would match. The possibility of this happening is slight, so we generally assume the problem was done correctly if the digital roots check.* On the other hand, if the digital roots do not check, you know it is wrong.

✓ Example 1

$$\begin{array}{r} 723 \rightarrow 3 \\ +412 \rightarrow 7 \\ \hline 1135 \rightarrow 1 \end{array}$$

Ck: 3 + 7 = 10 $\rightarrow$ 1 Correct!

Explanation: To check, add the digital root of the addends (3 + 7 = 10); then find the digital root of 10 (1). Verify that this equals the digital root of the sum, 1135 (1). Since it does, the addition problem was probably done correctly.

✓ Example 2

$$\begin{array}{r} 463 \rightarrow 4 \\ +529 \rightarrow 7 \\ \hline 1630 \rightarrow 1 \end{array}$$

Ck: 4 + 7 = 11 $\rightarrow$ 2 Wrong!

Explanation: To check, add the digital root of the addends (4 + 7 = 11); then find the digital root of 11 (2). This should equal the digital root of the sum, 1630 (1). Since it doesn't , there is a mistake and the addition problem was done incorrectly.

? Exercise 4

Someone did the following addition problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Note that the procedure is the same if there are more than two addends. Add the digital roots of all the addends. Show Work!

a. $\begin{array}{r} 4983 \\ +6829 \\ \hline 10802 \end{array}$	b. $\begin{array}{r} 5567 \\ +4987 \\ \hline 10554 \end{array}$	c. $\begin{array}{r} 3467 \\ +2541 \\ \hline 5908 \end{array}$
d. $\begin{array}{r} 8972 \\ 8876 \\ +5873 \\ \hline 23721 \end{array}$	e. $\begin{array}{r} 5873 \\ 3674 \\ +4763 \\ \hline 13260 \end{array}$	f. $\begin{array}{r} 4789 \\ 9835 \\ +5301 \\ \hline 19925 \end{array}$

For a multiplication problem, the check is similar, except to check, you **multiply** the digital roots of the numbers you are multiplying. To check addition, you ADD the digital roots of the addends and check against the digital root of the sum. To check multiplication, you MULTIPLY the digital root of the numbers being multiplied and check against the digital root of the product. Here are some examples that provide an answer someone might have written down after doing the multiplication problem on another piece of paper.

Example 1

$$\begin{array}{r} 85 \to 13 \to 4 \\ \times 63 \to 0 \\ \hline 5355 \to 18 \to 0 \end{array}$$

Ck: $4 \times 0 = 0$ Correct!

Explanation: To check, multiply the digital roots of the numbers you are multiplying ($4 \times 0 = 0$); then find the digital root of 0 (0). Verify that it equals the digital root of the product, 5355 (0). Since it does, the multiplication problem was probably done correctly.

Example 2

$$\begin{array}{r} 43 \to 7 \\ \times 52 \to 7 \\ \hline 2136 \to 3 \end{array}$$

Ck: $7 \times 7 = 49 \to 4$ Incorrect!

Explanation: To check, multiply the digital roots of the numbers you are multiplying ($7 \times 7 = 49$); then find the digital root of 49 (4). This should equal the digital root of the product, 2136 (3). Since it doesn't , there is a mistake. Therefore, the multiplication problem was done incorrectly.

Exercise 5

Someone did the following multiplication problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show Work!

a. $\begin{array}{r} 74 \\ \times 53 \\ \hline 3922 \end{array}$	b. $\begin{array}{r} 29 \\ \times 36 \\ \hline 944 \end{array}$	c. $\begin{array}{r} 68 \\ \times 28 \\ \hline 1904 \end{array}$
d. $\begin{array}{r} 65 \\ \times 79 \\ \hline 5125 \end{array}$	e. $\begin{array}{r} 81 \\ \times 55 \\ \hline 4455 \end{array}$	f. $\begin{array}{r} 48 \\ \times 52 \\ \hline 2596 \end{array}$
g. $\begin{array}{r} 569 \\ \times 61 \\ \hline 34709 \end{array}$	h. $\begin{array}{r} 653 \\ \times 524 \\ \hline 242172 \end{array}$	i. $\begin{array}{r} 2333 \\ \times 1203 \\ \hline 2806599 \end{array}$

Exercise 6

Add or multiply the following as indicated, then use digital roots to check the answer to each problem. Show Work!

a. $\begin{array}{r} 7362 \\ +5732 \\ \hline \end{array}$	b. $\begin{array}{r} 8308 \\ +956 \\ \hline \end{array}$
d. $\begin{array}{r} 6784 \\ +6835 \\ \hline \end{array}$	e. $\begin{array}{r} 9994 \\ +8721 \\ \hline \end{array}$
f. $\begin{array}{r} 57 \\ \times 8 \\ \hline \end{array}$	g. $\begin{array}{r} 34 \\ \times 7 \\ \hline \end{array}$
g. $\begin{array}{r} 87 \\ \times 52 \\ \hline \end{array}$	h. $\begin{array}{r} 825 \\ \times 13 \\ \hline \end{array}$

Using Digital Roots is to Check Subtraction Problems.

To check subtraction, you usually add the difference (the answer) to the subtrahend (the number that was subtracted) and see if you get the minuend (the number you subtracted from). In other words, to check that 215 – 134 = 81 was done correctly, simply add 81 and 134, which is 215. To check subtraction using digital roots, simply add the *digital root* of the difference to the *digital root* of the subtrahend and see if you get the *digital root* of the minuend. To check that 215 – 134 = 81 was done correctly, simply add the digital roots 81 and 134: 0 + 8 = 8. Verify that 8 is the digital root of the minuend, 215. It is! Here is another example. Let's say someone did the following subtraction problem: 5462 – 2873 = 2589. To check, add the digital roots of 2589 and 2873, or 6 + 2 to get 8. Since 8 is also the digital root of 5462, the answer is probably correct.

Here are some more examples of how to check subtraction using digital roots.

Example 1

$$\begin{array}{r} 7362 \to 0 \\ -5732 \to 8 \\ \hline 1630 \to 1 \end{array}$$

Ck: 1 + 8 = 9 → 0 √ Correct!

Explanation: To check, add the digital root of the difference and subtrahend (1 + 8 = 9). The digital root of 9 is 0. Verify that this equals the digital root of the minuend, 7632 (0). Since it does, the subtraction was probably done correctly.

Example 2

$$\begin{array}{r} 5073 \to 6 \\ -878 \to 5 \\ \hline 4205 \to 2 \end{array}$$

Ck: 2 + 5 = 7 WRONG!

Explanation: To check, add the digital root of the difference and subtrahend (2 + 5 = 7). This should equal the digital root of the minuend, 5073 (6). Since it doesn't, the subtraction was not done correctly.

Exercise 7

Someone did the following subtractions problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show Work!

a. $\begin{array}{r} 572 \\ -356 \\ \hline 216 \end{array}$	b. $\begin{array}{r} 296 \\ -189 \\ \hline 107 \end{array}$	c. $\begin{array}{r} 501 \\ -327 \\ \hline 284 \end{array}$

d. $\begin{array}{r} 3217 \\ -2391 \\ \hline 926 \end{array}$	e. $\begin{array}{r} 3546 \\ -1138 \\ \hline 2408 \end{array}$	f. $\begin{array}{r} 7502 \\ -4729 \\ \hline 2773 \end{array}$

Exercise 8

Subtract, add or multiply the following as indicated, then use digital roots to check the answer to each problem. Show Work!

a. $\begin{array}{r} 7362 \\ -5732 \end{array}$	b. $\begin{array}{r} 8308 \\ -956 \end{array}$
d. $\begin{array}{r} 6784 \\ -6335 \end{array}$	e. $\begin{array}{r} 9994 \\ -8721 \end{array}$
f. $\begin{array}{r} 557 \\ +348 \end{array}$	g. $\begin{array}{r} 834 \\ +767 \end{array}$
h. $\begin{array}{r} 48 \\ \times 6 \end{array}$	i. $\begin{array}{r} 13 \\ \times 29 \end{array}$

Using Digital Roots is to Check Division Problems.

To check division, you multiply the divisor (the number you are dividing by) by the quotient (the answer), and then add the remainder. It is correct if the answer obtained is equal to the dividend (the number you divided into). In other words, here is how to check this division, $431 \div 34 = 12$r. 23: Multiply 34 by 12 and add 23. Here are the steps: $34 \times 12 + 23 = 408 + 23 = 431$. Since 431 is the dividend, this problem was done correctly. To check division using digital roots, simply multiply the *digital root* of the divisor by the *digital root* of the quotient, and add the *digital root* of the remainder. If the digital root of that number equals the *digital root* of the dividend, it is probably correct.

Note: After the digital roots of the divisor and quotient are multiplied, you can first find the digital root of that product before adding the digital root of the remainder.

Here is how to check the division problem we did above by using digital roots. Multiply the digital root of the divisor, 34, by the digital root of the quotient, 12: $7 \times 3 = 21$. Then add the digital root of the remainder, 23: 21 + 5 = 26. Determine the digital root of that number, 26, which is 8. Check to see if 8 matches the digital root of the dividend, 431. It does. As stated in the note, you could determine the digital root of 21 before adding the digital root of the remainder. In this case, it would have looked like this: $7 \times 3 = 21 \rightarrow 3$. Then add the digital root of the remainder, 23: 3 + 5 = 8. Verify that this equals the digital root of the dividend, 431. It does. Therefore, the division was probably done correctly. More examples follow. The check without digital roots is shown to the right. The digital root check is shown under the division problem.

Example 1

Example 1:

$$\begin{array}{r} 0 \leftarrow \quad 45 \text{ r. } 15 \quad \rightarrow 6 \\ 0 \leftarrow 27 \,\big|\, 1230 \rightarrow 6 \end{array}$$

Digital Root Check:

0 x 0 = 0; 0 + 6 = 6 Correct!

Explanation: To check, multiply the digital root of the divisor, 27, by the digital root of the quotient, 45: 0 x 0 = 0. Add the digital root of the remainder, 15: 0 + 6 = 6. Since this equals the digital root of the dividend, 1230, the division was probably done correctly.

Check:

$$\begin{array}{r} 27 \\ \times\ 45 \\ \hline 135 \\ 108 \\ \hline 1215 \\ +15 \\ \hline 1230 \end{array}$$

Example 2

Example 2:

$$\begin{array}{r} 1 \leftarrow \quad 19 \text{ r. } 32 \quad \rightarrow 5 \\ 7 \leftarrow 34 \,\big|\, 678 \rightarrow 21 \rightarrow 3 \end{array}$$

Digital Root Check:

7 x 1 = 7; 7 + 5 = 12 --> 3 Correct!

Explanation: To check, multiply the digital root of the divisor, 34, by the digital root of the quotient, 19: 7 x 1 = 7. Add the digital root of the remainder, 32: 7 + 5 = 12 --> 3. Since this equals the digital root of the dividend, 678, the division was probably done correctly.

Check:

$$\begin{array}{r} 34 \\ \times\ 19 \\ \hline 646 \\ +32 \\ \hline 678 \end{array}$$

Below is one more example for you to study.

Example 3

Example 3:

$$\begin{array}{r} 6 \leftarrow \quad 213 \text{ r. } 271 \quad \rightarrow 1 \\ 8 \leftarrow 458 \,\big|\, 97831 \rightarrow 10 \rightarrow 1 \end{array}$$

Digital Root Check:

8 x 6 = 48 --> 12 --> 3; 3 + 1 = 4 WRONG!

Explanation: To check, multiply the digital root of the divisor, 458, by the digital root of the quotient, 213: 8 x 6 = 48 --> 12 --> 3. Add the digital root of the remainder, 271: 3 + 1 = 4. Since 4 does not equal the digital root of the dividend, 97831, the division was done incorrectly.

Check:

$$\begin{array}{r} 458 \\ \times\ 213 \\ \hline 97554 \\ +271 \\ \hline 97825 \end{array}$$

Incorrect!

Exercise 9

Someone did the following division problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show work. You don't need to do the actual check as shown to the right of the last three examples. Show work!

a.

$$73\overline{)5609}\quad 76 \text{ r. } 61$$

b.

$$123\overline{)6723}\quad 54 \text{ r. } 81$$

c.

$$26\overline{)5134}\quad 197 \text{ r. } 10$$

d.

$$78\overline{)7988}\quad 102 \text{ r. } 32$$

One more note of caution about using digital roots:

In problem 9d, a common mistake some people make is to forget to put the zero in the quotient, 102. Instead, the answer might have been written as 12 r. 32. Using digital roots, the answer would check. So, you should also check the reasonableness of the answer by approximating. For instance, 78×12 is about 80 times 10 or 800, which is not even close to equaling the dividend, 7988. Another time digital roots might fail is when someone transposes the digits of a number. If the answer to a problem was 465, and it was written as 456, using digital roots wouldn't detect the mistake.

Exercise 10

Do the following division problem, and check the answer using digital roots. Show work.

$23972 \div 156 =$ _____

Later in this module, we will explore prime numbers, composite numbers, the greatest common factor of two or more numbers, and the least common multiple of two or more numbers. Factoring is the method used to find the prime factorization of a composite number, and it can also be used to find the greatest common factor and least common multiple of a set of numbers. One of the problems in factoring large numbers is that sometimes it isn't clear if it is prime or composite. In other words, it isn't clear whether it has any factors other than 1 or itself. Most of us know that if the last digit of a numeral is even, then 2 will divide into it; or if it ends in 0, 10 will divide into it; or if it ends in 0 or 5, that 5 will divide into it. Sometimes, people even have trouble determining if relatively small numbers are prime. For instance, many people think 91 is prime, but in fact it is not. Knowing some divisibility tests makes the task easier, so we'll soon take some time to discuss divisibility tests for several numbers. First, we need to go over some notation concerning divisibility.

When you see 12/3, this means 12 "**divided by**" 3. The slash that slants to the right is another way to write the division sign, ÷. 12/3 (or $12 \div 3$) is a division problem, and the answer is 4.

Here is something altogether different. If I say "3 **divides** 12", I am making a *statement*. "3 divides 12" is not a division problem that needs to be done. It is a statement that happens to be true. The symbol used to represent the word "divides" is a vertical line. So, "3 **divides** 12" can be written "3|12". Again, this is a statement, a fact, not a division problem. The way to express "does not divide" is to put a slash through the symbol: $\not|$

So, how do I know "3 ***divides*** 12" is a true statement? The definition of "divides" follows:

Definition: **"a divides b" if there exists a whole number, n, such that an = b.** This is simply saying that **"a divides b" means a is a factor of b!** In shorthand notation, this is written **"a|b if there exists a whole number, n, such that an = b, or a|b means that a is a factor of b."**

Okay, so what exactly does "there exists a whole number, n, such that an = b" mean? Well, it means that the first number times some whole number equals the second number. Or you can think "a divides b" is true if $b \div a$ is a whole number.

Let's get back to why "3 **divides** 12" is a true statement. Hmmm...you can be formal and ask yourself: Is there a whole number n such that 3n = 12? Or, you can simply ask yourself: Is 3 a factor (or divisor) of 12? In either case, the answer is yes, so the statement is true.

Let's work a little more on the difference between a statement using "divides", and an actual division problem.

Examples: Determine if each of the following is a statement or if it is a division problem. If it is a statement, state if it is true or false and back up your answer. If it is a division problem, state the answer to the division problem.

✔ Example 1: 4|20

4|20

Solution

This is a true statement, because 4 is a factor of 20 (since $4 \cdot 5 = 20$)

✔ Example 2: 20/6

20/6

Solution

This is a division problem. The answer is 3 r. 2

✔ Example 3: 15|3

15|3

Solution

This is a false statement because 15 is not a factor of 3

✔ Example 4: 6 divides 20

6 divides 20

Solution

This is a false statement because 6 is not a factor of 20

✔ Example 5: 3 divides 21

3 divides 21

Solution

This is a true statement because 3 is a factor of 21 (since $3 \cdot 7 = 21$)

? Exercise 11

Determine if each of the following is a statement or if it is a division problem. If it is a statement, then decide if it is true or false and back up your answer. If it is a division problem, state the answer to the division problem. Study the examples on the previous page if you need help getting started.

a. 35/7
b. 35|7
c. 7|35:
d. 40/7
e. 56|8
f. 7|40:
g. 12 divides 60
h. 80 divided by 30
i. 70 divided by 5
j. 42 divides 3
k. 6 divides 42
l. 80 divided by 10
m. 100/2
n. 4|100
o. 4|90:
p. 25|5

Let's go back to the formal definition of "divides":

Definition: **a|b if there exists a whole number, n, such that an = b.**

We need to use this formal notation in order to do some proofs. Consider the following:

Example 1

Is the following statement true? If a|b and a|c, then a|(b+c).

a|b and a|c means that a is a factor of b and a is also a factor of c. We must determine if that necessarily implies that a is also a factor of the sum, b + c. The first line of strategy is to test it out on a few numbers, and see if you can find a counterexample. If you find a counterexample, the answer is no and you are done. If you can't find a counterexample, maybe it is true. If you think it is true, you must PROVE it by being general and formal.

I would start by picking any number for a, like a = 3, and then choose numbers for b and c for which 3 is a factor, like 15 and 18. Plug them in: "If 3|15 and 3|18, does 3|(15+18)?" Since 3|33, the answer is yes. It looks like this statement might be true.

Exercise 12

Choose different numbers for a, b and c in "If a|b and a|c, then a|(b+c)" to see if the statement seems to be true.

The statement, "If a|b and a|c, then a|(b+c)", happens to be true. You must write a proof to prove that it is always true. Here is one way to write a formal proof:

Example 1

Is the following statement true?: If a|b and a|c, then a|(b+c).

Solution

If a|b, then an = b for some whole number, n. If a|c, then am = c for some whole number, m. Using these substitutions for b and c, we get that a|(b + c) is true if a|(an + am) which is true if a is a factor of an + am. Factor: an + am = a(n + m). This clearly shows that a is indeed a factor of an + am. Therefore, if a|b and a|c, then a|(b+c).

? Exercise 13

Write a formal proof to show that the following is true: If x|y and x|z, then x|(y + z).

✓ Example 2

Is the following statement true? If a|(b+c), then a|b and a|c.

a|(b + c) means that a is a factor of the sum, b + c. The question asks if that necessarily implies that a is a factor of b and also a factor of c. The first line of strategy is to test it out on a few numbers, and see if you can find a counterexample, or if it looks like it is true. If you find a counterexample, the answer is no and you are done. If you think it is true, you must PROVE it by being general and formal.

Start by picking any number for a, like a = 3 and picking a number for the sum b + c, for which 3 is a factor, like 15 or 18, etc. Let b + c = 15. Note that there are many combinations of numbers that add up to 15: 1 + 14, 2 + 13, 3 + 12, 7 + 8, etc. You are being asked if 3 divides into the sum of any two whole numbers, will it necessarily divide into each individual addend as well? For instance, if you broke 15 up as the sum of 9 and 6, this would be the statement: "If 3|(9 + 6), then 3|9 and 3|6." Using these numbers, it is true and you haven't found a counterexample. Try splitting up 15 another way, perhaps as the sum of 10 and 5. Then, the statement becomes: "If 3|(10 + 5), then 3|10 and 3|5." The answer is no, and therefore these numbers may be used as a counterexample, which proves the statement is false. Remember: In order for a statement to be true, it must be for all values of a and b. On the other hand, only one counterexample showing it is not true is sufficient to prove a statement is false.

? Exercise 14

Consider the statement: "If a|(b+c), then a|b and a|c." Provide an example using numbers other than a = 3, b = 9 and c = 6 that makes it look like this statement might be true.

? Exercise 15

Consider the statement: "If a|(b+c), then a|b and a|c." Provide a counterexample using numbers other than a = 3, b = 10 and c = 5 to show this statement is false.

? Exercise 16

Consider the statement: "If a|c and b|c, then (a + b)|c." Provide a counterexample to show this statement is false.

? Exercise 17

Consider the statement: "If (a + b)|c, then a|c and b|c." Provide a counterexample to show this statement is false.

? Exercise 18

Prove that the following statement is true: "If a|b and a|c, then a|(bc)"

✓ Example

Prove that the following statement is true: "If a|b and a|(b + c), then a|c"

Solution

If a|b, then am = b for some whole number, m. If a|(b + c), then an = b + c for some whole number, n. Keep in mind that since b and c are positive, then (b + c) > b, which means n > m. We are trying to prove that a is a factor of c. Since am = b, we can substitute am for b into the equation an = b + c, which means an = am + c. Solving for c, this is equivalent to an – am = c. So if a is a factor of an – am, then a is a factor of c. Factor: an – am = a(n – m). This clearly shows a is a factor of an – am, which

means a is a factor of c. Therefore, the statement "If a|b and a|(b + c), then a|c" is true. (Note: n – m must be a whole number since n and m are whole numbers and n > m.)

? Exercise 19

Prove that the following statement is true: "If c|a and c|(a + b), then c|b"

Okay, finally on to the divisibility tests. We will use our new notation for divides (|).

All counting numbers can be considered factors of zero. In other words, for all counting numbers, m, m|0 is always true since there is always some number times m that equals zero, namely zero itself.

Assume n is a positive whole number. The following are divisibility tests to determine what numbers divide into n, or what numbers are factors of n.

Divisibility Test for 2: 2|n if the last digit of n is even (0, 2, 4, 6, or 8)

✓ Examples

Use the divisibility test for 2 to determine if the following is true or false

1. 2|97

This is false because the last digit (7) is not even

2. 2|356

This is true because the last digit (6) is even

Divisibility Test for 4: (think of 4 as 2^2)

4|n if 4|(the number represented by the last two digits of n)

✓ Examples

Use the divisibility test for 4 to determine if the following is true or false.

1. 4|527

This is false because 4 does not divide 27, (since 4 is not a factor of 27).

2. 4|25,356

This is true because 4|56.

3. 4|624

This is true because 4|24.

Divisibility Test for 8: (think of 8 as 2^3)

8|n if 8|(the number represented by the last three digits of n)

✓ Examples

Use the divisibility test for 8 to determine if the following is true or false.

1. 8|42,527

This is false because 8 does not divide 527

2. 8|25,336

This is true because 8|336

3. 8|7,624

This is true because 8 does divide 624, (since 8 is a factor of 624)

It should be clear to you that if a number is not divisible by 2, it is not divisible by 4 or 8; and if it is not divisible by 4, it is not divisible by 8. Conversely, if a number is divisible by 8, then it is divisible by both 2 and 4; and if it is divisible by 4, it is divisible by 2.

? Exercise 20

Use the divisibility tests for 2, 4 and 8 to determine if the following is true or false. Support your answer with a reason using the appropriate divisibility test.

a. 2|9,712 _____
b. 2|5,643 _____
c. 4|5,690 _____
d. 4|63,868 _____
e. 4|854,100 _____
f. 8|12,345,248 _____
g. 8| 54,094,422 _____

Divisibility Test for 5: 5|n if the last digit of n is 0 or 5

✔ Examples

Use the divisibility test for 5 to determine if the following is true or false.

1. 5|527

This is false because 7 is the last digit of 527

2. 5|25,335

This is true because 5 is the last digit of 25,335

3. 5|7,620

This is true because 0 is the last digit of 7,620

Divisibility Test for 10: 10|n if the last digit of n is 0

✔ Examples

Use the divisibility test for 10 to determine if the following is true or false.

1. 10|527

This is false because 7 is the last digit of 527

2. 10|25,335

This is false because 5 is the last digit of 25,335

3. 10|7,620

This is true because 0 is the last digit of 7,620

Exercise 21

Use the divisibility tests for 5 and 10 to determine if the following is true or false. Support your answer with a reason using the appropriate divisibility test.

a. 5|9,750 _____

b. 5|5,645 _____

c. 5|5,696 _____

d. 10|63,860 _____

e. 10|854,105 _____

Divisibility Test for 3: 3|n if 3|(the digital root of n)

Note: This is equivalent to saying 3|n if the digital root of n is 0, 3 or 6

Examples

Use the divisibility test for 3 to determine if the following is true or false.

1. 3|97

This is false because 3 does not divide 7, which is the digital root of 97

2. 3|356

This is false because 3 does not divide 5, which is the digital root of 356

3. 3|738

This is true because 3|0, where 0 is the digital root of 738

Exercise 22

Use the divisibility test for 3 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 3. Show work.

a. 3|9,750 _____

b. 3|5,645 _____

c. 3|5,696 _____

d. 3|63,860 _____

e. 3|854,115 _____

Divisibility Test for 9: 9|n if 9|(the digital root of n)

Note: **This is equivalent to saying that the digital root of n must equal zero.** Take a moment to think about this divisibility test for 9. In exercise 16, we discovered that the digital root of a number is the same as the remainder you obtain when you divide a number by 9. In order for a number to be divisible by 9, the remainder would be zero, which is exactly what the digital root would be.

Examples

Use the divisibility test for 9 to determine if the following is true or false.

1. 9|627

This is false because 9 does not divide 6, which is the digital root of 627.

2. 9|25,334

This is false since 9 does not divide 8, which is the digital root of 25,334.

3. 9|7,533

This is true because the digital root of 9 is 0.

Exercise 23

Use the divisibility test for 9 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 9. Show work.

a. 9|9,753 _____

b. 9|5,646 _____

c. 9|5,697 _____

d. 9|63,576 _____

e. 9|854,103 _____

Divisibility Test for 6: 6|n if 2|n AND 3|n. (Think of 6 as $2 \cdot 3$)

Important Note: A number is divisible by 6 only if it passes the divisibility test for **both** 2 and 3. If it doesn't pass one of the tests, then it is not divisible by 6. To show it is divisible by 6, you must show it passes both of the tests.

Examples

Use the divisibility test for 6 to determine if the following is true or false.

1. 6|627

This is false because 2 does not divide 7 (the last digit, which is not even).

2. 6|25,334

This is false because 3 does not divide 25,334, since 3 does not divide the digital root, which is 8.

3. 6|7,620

This is true because 2|7,620 (since the last digit, 0, is even) AND 3|7,620 (since 3|6, where 6 is the digital root of 7,620).

Exercise 24

Use the divisibility test for 6 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 6.

a. 6|9,753

b. 6|5,645

c. 6|5,696

d. 6|63,876

e. 6|854,103

Divisibility Test for 15: 15|n if 5|n AND 3|n. (Think of 15 as $5 \cdot 3$)

Important Note: A number is divisible by 15 only if it passes the divisibility test for both 5 and 3. If it doesn't pass one of the tests, then it is not divisible by 15. To show it is divisible by 15, you must show it passes both of the tests.

Examples

Use the divisibility test for 15 to determine if the following is true or false.

1. 15|623: This is false because 5 does not divide 623 (since the last digit is not 0 or 5).

2. 15|24,335: This is false because 3 does not divide 24,335, since 3 does not divide the digital root, which is 8.

3. 15|7,620: This is true because 5|7,620 (since the last digit is 0) AND 3|7,620 (since 3|6, where 6 is the digital root of 7,620).

Exercise 25

Use the divisibility test for 15 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 15.

a. 15|9,753

b. 15|6,645

c. 15|5,690

d. 15|63,872

e. 15|654,105

Divisibility Test for 7: This test isn't easy to describe. Below are the steps.

Step 1: Cross off the one's digit of the number to get a number with one less place value.

Step 2: Double the one's digit you crossed off and subtract from the new number obtained with the one's digit missing.

Step 3: If 7 divides the number you get after subtracting, it divides the original number. Otherwise, it doesn't. If you aren't sure, repeat the procedure on the new number by going back to step 1.

Examples

Use the divisibility test for 7 to determine if the following is true or false. The way you would show the steps using the numbers is shown to the right of the explanation.

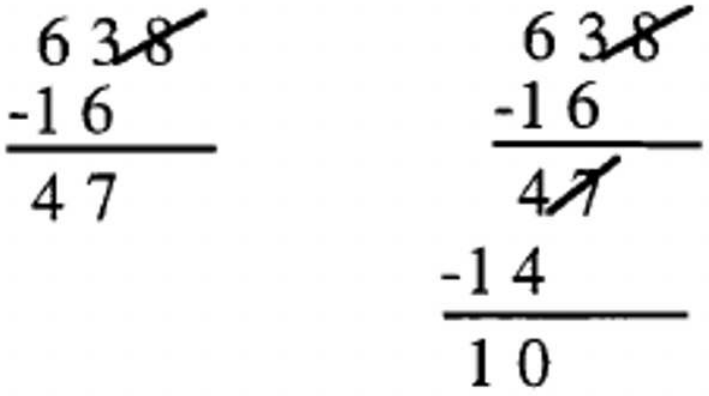
1. 7|91: Cross off the 1, double it (2), and subtract from what is left (9). The answer is 7 (9 – 2 = 7). Since 7|7, 7|91 is true.

$$\begin{array}{r} 9\not{1} \\ -2 \\ \hline 7 \end{array}$$

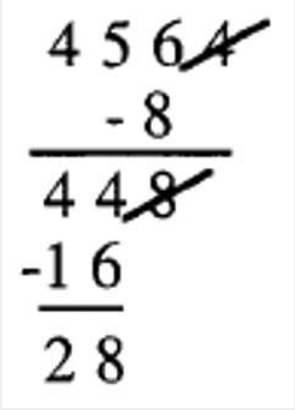
2. 7|96: Cross off the 6, double it (12), and subtract from what is left (9). When you subtract, ignore the sign (just do 12 – 9 = 3). Since 7 does not divide 3, 7|96 is false.

$$\begin{array}{r} 9\not{6} \\ -12 \\ \hline 3 \end{array}$$

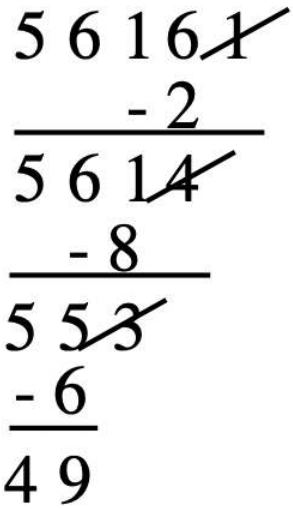
3. 7|638: Cross off the 8, double it (16) and subtract from what is left (63). The answer is 47 (63 – 16 = 47). Since 7 does not divide 47, then 7|638 is false. (If you weren't sure whether or not 7 divided 47, you can take it a step further; this is shown to the right.)



4. 7|4564: Cross off the 4, double it (8) and subtract from what is left (456). The answer is 448 (456 – 8 = 456). Cross off the 8, double it (16) and subtract from what is left (44). The answer is 28 (44 – 16 = 28). Since 7|28, then 7|4564 is true.



5. 7|56161: Cross off the 1, double it (2) and subtract from what is left (5616 – 2 = 5614). Cross off the 4, double it (8) and subtract from what is left (561 – 8 = 553). Cross off the 3, double it (6) and subtract from what is left (55 – 6 = 49). Since 7|49, then 7|56161 is true.



Exercise 26

Use the divisibility test for 7 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 7. Show work.

a. 7|833
b. 7|5,645
c. 7|4,795
d. 7|14,763

Divisibility Test for 11: 11|n if the difference between the sum of the digits in the places that are even powers of 10 and the sum of the digits in the places that are odd powers of 10 is divisible by 11.

This test is more confusing to describe than to do. What you do is add up every other digit. Then, add up the ones you skipped. Then, subtract these two numbers and see if 11 divides this number.

Examples

Use the divisibility test for 11 to determine if the following is true or false.

a. 11|4,365: Add 4 + 6 = 10 Add 3 + 5 = 8 Subtract 10 – 8 = 2 Since 11 doesn't divide 2, then 11 doesn't divide 4365. Therefore, 11|4365 is false.

b. 11|540,879,216: Add 5 + 0 + 7 + 2 + 6 = 20. Add 4 + 8 + 9 + 1 = 22. Subtract 22 – 20 = 2 Since 11 doesn't divide 2, 11|540,879,216 is false.

c. 11|542,879,216: Add 5 + 2 + 7 + 2 + 6 = 22 Add 4 + 8 + 9 + 1 = 22 Subtract 22 – 22 = 0 Since 11|0, then 11|542,879,216 is true.

d. 11|4,052,631: Add 4 + 5 + 6 + 1 = 16 Add 0 + 2 + 3 = 5 Subtract 16 – 5 = 11 Since 11|11, then 11|4,052,631 is true.

Exercise 27

Use the divisibility test for 11 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 11. Show work.

a. 11|9,053
b. 11|63,920,876
c. 11|568,696
d. 11|513,645
e. 11|803,003,808

Listed below are the divisibility tests that you should know. It's important to realize that each of the tests only work for the number specified. In other words, you can't use the divisibility test for 3 to determine is 7 divides a number. The divisibility test for 7 has nothing to do with digital roots!

Divisibility Test for 2: 2|n if the last digit of n is even (0, 2, 4, 6, or 8)

Divisibility Test for 3: 3|n if 3|(the digital root of n); OR 3|n if the digital root of n is 0, 3 or 6

Divisibility Test for 4: 4|n if 4|(the number represented by the last two digits of n)

Divisibility Test for 5: 5|n if the last digit of n is 0 or 5

Divisibility Test for 6: 6|n if 2|n AND 3|n. (Think of 6 as $2 \cdot 3$)

Divisibility Test for 7: The steps of this test are described below.

- Step 1: Cross off the one's digit of the number to get a number with one less place value.
- Step 2: Double the one's digit you crossed off and subtract from the new number without the one's digit.
- Step 3: If 7 divides the number you get after subtracting, it divides the original number. Otherwise, it doesn't. If you aren't sure, repeat the procedure on the new number by going back to step 1.

Divisibility Test for 8: 8|n if 8|(the number represented by the last three digits of n)

Divisibility Test for 9: 9|n if 9|(the digital root of n); OR 9|n if the digital root of n is zero

Divisibility Test for 10: 10|n if the last digit of n is 0

Divisibility Test for 11: 11|n if the difference between the sum of the digits in the places that are even powers of 10 and the sum of the digits in the places that are odd powers of 10 is divisible by 11

Divisibility Test for 15: 15|n if 5|n AND 3|n

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6.4: Primes and GCF

You will need: ***Centimeter Strips*** *(Material Cards 16A-16L)*

Prime Number Squares *(Material Cards 19A-19B)*

Composite Number Squares *(Material Cards 20A-20B)*

In these first few exercises, you'll be using C-strips to explore divisors, factors, prime numbers and composite numbers.

? Exercise 1

a. Take out the Hot Pink (H) C-strip. Use your C-strips (one color at a time) to see if you can form a train made up of C-strips of the same color equal in length to the hot pink C-strip in other words, see if you can do it with all whites (always possible for any train length), or all reds, or all light greens, etc. You should be able to make six different trains each train is made up of a single color. Draw a picture of each of these trains under the Hot Pink one shown. I've drawn two trains for you already one is simply the hot pink strip (a train consisting of just one C-strip), and a second is made up of three purple C-strips.

Train 1	H		
Train 2	P	P	P
Train 3			
Train 4			
Train 5			
Train 6			

b. Take each train and form it into a rectangle. Then, find a C-strip that fits across the width of the rectangle, which is the top if the C-strips of the train made into a rectangle are placed vertically. From this, you should be able to tell from which multiplication problem each train was formed. From this information, write an equation using C-strips and then translate to an equation with numbers. For instance, for train 2, first I would make a rectangle out of the three purple C-strips. Next, I would try to find a C-strip to fit across the top, which would be light green. Therefore, train 2 was formed from the multiplication $L \times P$: remember that since the train is formed with purple C-strips, P is the *second* letter in the multiplication. So, the equation in C-strips is $L \times P = H$, and the numerical equivalent is $3 \times 4 = 12$. Follow this same procedure for the other four trains you made in part a.

Train 1 illustrates the multiplication $W \times H = H$, or $1 \times 12 = 12$. (Note that if the hot pink strip is placed vertically, the white C-strip fits across the top.)

Train 2 illustrates the multiplication $L \times P = H$, or $3 \times 4 = 12$.

Train 3 illustrates the multiplication ____ × ____ = H, or ____ × ____ = 12.

Train 4 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Train 5 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Train 6 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

In exercise part 1b, the set of numbers placed in the blanks before the equal sign in the equations with numbers are called **factors** or **divisors** of 12. Note, that because of the commutative property of multiplication, each factor or divisor was listed twice.

c. List the numbers that are factors of 12 (length of the hot pink C-strip). Only list each number once.

Exercise 2

Use the same procedure used in exercise 1 to make all possible rectangles from a train having each of the following lengths. Use the C-strip(s) shown in the parenthesis to make a train having the given length. Then, after discovering the possible rectangles that can be made, list the factors (the actual numbers) of each number. Every number greater than 1 has at least two factors.

a. Factors of 2 (red C-strip): ____
b. Factors of 3 (light green C-strip): ____
c. Factors of 4 (purple C-strip): ____
d. Factors of 5 (yellow C-strip): ____
e. Factors of 6 (dark green C-strip): ____
f. Factors of 7 (black C-strip): ____
g. Factors of 8 (brown C-strip): ____
h. Factors of 9 (blue C-strip): ____
i. Factors of 10 (orange C-strip): ____
j. Factors of 11 (silver C-strip): ____
k. Factors of 13 (brown + yellow): ____
l. Factors of 14 (orange + purple): ____
m. Factors of 15 (black + brown): ____
n. Factors of 16 (blue + black): ____

Exercise 3

a. List the numbers from exercise 2 that have **exactly** 2 factors: _____

b. What do you notice about the 2 factors each of these number have? Is there a pattern or anything they have in common with each other?

c. List numbers from exercise 2 that have an odd number (3 or 5) of factors: _____

d. What do all of these numbers have in common?

Definition: A whole number that has exactly two different factors is called a **prime** number.

If a number is prime, its only factors are 1 and itself. In other words, if a number, p, is prime, its only factors are 1 and p. If you are using the C-strips and try to make a rectangle out of a train that has a length that is a prime number, the only possibility is when the width is 1 and the length is the length of the original train. That is because those are the only factors, and no other number divides into it.

NOTE: **1 is NOT prime** since it doesn't have two different factors *it only has one 1*.

Definition: Any whole number larger than 1 that is not prime is called a **composite** number.

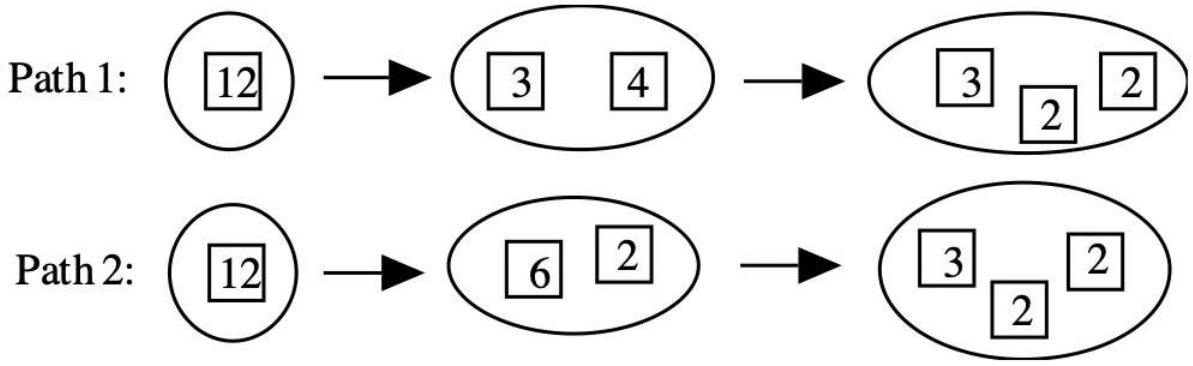
The whole numbers, 0 and 1, are neither prime nor composite. Any whole number larger than 1 is either prime or composite.

If a number is composite, it means that it has more than 2 factors, and can be written as a product of factors less than itself. For instance, 12 is not a prime number. It can be written as $2 \cdot 6$ or $3 \cdot 4$. This is called factoring, and $2 \cdot \cdot \cdot 6$ and $3 \cdot \cdot \cdot 4$ are only two ways of factoring 12.

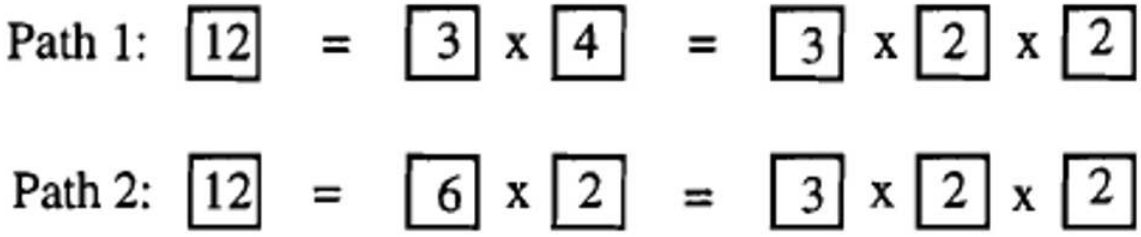
Get out your prime number squares and composite number squares. If you haven't already done so, color the prime number squares so that each prime number has a different color the 2's could be yellow, the 3's could be blue, the 5's could be green, the 7's could be purple, the 11's could be red, the 13's could be orange, the 17's could be pink, and so on. All of the composite number squares should remain white.

What we are going to do is factor numbers. Anytime we have a white number, we know it is composite and can be factored further. The ultimate goal will be to factor numbers into a product of primes, which means there will only be colored squares to represent each number.

Let's begin by factoring 12. Take out a white number square that says 12, and put it in front of you. Since it is white, it can be factored into a product of two smaller numbers. Someone might choose 3 and 4; another might choose 2 and 6. Replace 12 with the two squares you chose. For 12, it so happens that no matter which combination of two numbers you choose, one of the squares you choose will be colored, which means it is prime and can't be factored (or broken down) any further. But the other square will be white. Replace the white one with two other number squares by using smaller factors. In your pile, instead of a 12, you should have three squares, a 2, a 2 and a 3. There is no order it's just a group of three squares that if multiplied together represent 12. Below are two paths one might have taken to come up with the final solution. The arrows show one step after the other.



In the first path, the 12 was replaced with 3 and 4, and then the 4 was replaced with two 2's. In the second path, the 12 was replaced with a 6 and a 2, and then the 6 was replaced with a 2 and a 3. Using this model to prime factor, the numbers you end up should all be prime, and the understanding is that when the final (prime) factors are MULTIPLIED together, we obtain the original number we tried to factor. In other words, the prime factorization for 12 is $3 \cdot 2 \cdot 2$. Because of the commutative and associative properties of multiplication, the order of the factors is insignificant. Sometimes, for consistency, the factors are written in ascending or descending order, but this is not necessary unless you are instructed to write it in a particular order. If you are asked to write the prime factorization of 12, you might write $12 = 2 \cdot 2 \cdot 3$ (or $12 = 2^2 \cdot 3$). To check, multiply the prime factors to make sure that the product really is the number you set out to prime factor. For large numbers, you should use a calculator. Here is another way to show on paper the replacement process going on for factoring 12 using the number squares.



Example: Use the number squares to prime factor 210, and show the individual steps.

Note: Since there is not a number square for 210, write 210 on a blank square to begin.

Solution 1: $210 = 21 \cdots 10 = 3 \cdots 7 \cdots 2 \cdots 5$

Solution 2: $210 = 3 \cdots 70 = 2 \cdots 105 = 2 \cdots 3 \cdots 35 = 2 \cdots 3 \cdots 5 \cdots 7$

Solution 3: $210 = 30 \cdots 7 = 5 \cdots 6 \cdots 7 = 5 \cdots 2 \cdots 3 \cdots 7$

There are many other ways one might go about factoring 210, but in the end, there are 4 prime factors that when multiplied together equal 210. Because of the commutative and associative properties of multiplication, $3 \cdot \cdot \cdot 7 \cdot \cdot \cdot 2 \cdot \cdot \cdot 5 = 2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 5 \cdot \cdot \cdot 7 = 5 \cdot \cdot \cdot 2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 7$ Usually, the primes are written in ascending order (from the smallest factor to the largest factor. We write that the prime factorization of 210 is $2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 5 \cdot \cdot \cdot 7$.

This leads to a very important theorem. You can think of the prime numbers as building blocks for all whole numbers greater than 1. Every whole number greater than 1 is prime, or it can be expressed as the product of prime factors (called the prime factorization). The fact that any composite number can be written as a **unique** product of primes is so important that it is called the **Fundamental Theorem of Arithmetic**.

The Fundamental Theorem of Arithmetic:

Every composite number has exactly one unique prime factorization (except for the order in which you write the factors.)

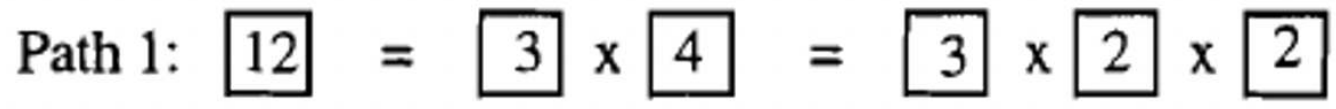
Note again that the order in which the factors are written doesn't matter. However, for consistency, they are usually written in ascending order (from smallest to largest). Also, exponents may be used if a factor is repeated in the prime factorization. For instance, the prime factorization of 12 is usually written as $2 \cdot \cdot \cdot 2 \cdot \cdot \cdot 3$ or $2^2 \cdot \cdot \cdot 3$

? Exercise 4

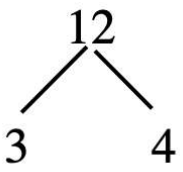
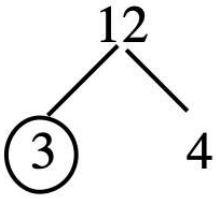
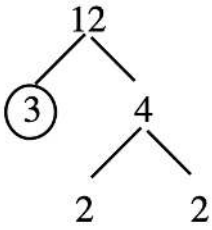
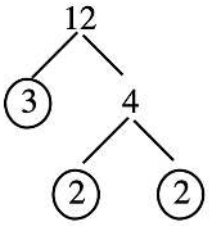
Use the number squares to prime factor each of the following composite numbers into a product of primes. Write the prime factorization so the factors are written in ascending order (from smallest to largest). None of these numbers is prime. **Show each of the individual steps one at a time, not just the final product.**

a. 45 = _____
b. 65 = _____
c. 200 = _____
d. 91 = _____
e. 76 = _____
f. 350 = _____
g. 189 = _____
h. 74 = _____
i. 512 = _____
j. 147 = _____

There are other methods commonly used to find the prime factorization. One uses a FACTOR TREE, which is similar to what you did with the prime and composite number squares. The difference is that it is done on paper, as opposed to using manipulatives. The number you are trying to factor is called the root, and is at the top. So, it's actually an upside down tree. If a number is not prime, you draw two branches down from that number and factor it as the product of any two factors. At the end of each branch is a smaller factor, which is called a leaf. If a leaf is prime, circle it, it is one of the factors in the prime factorization of the number. Otherwise, branch off again. When all of the leaves are circled prime numbers, you are done. The prime factorization of the root is the product of the leaves. Below is one way we factored 12, on the previous page, using squares.



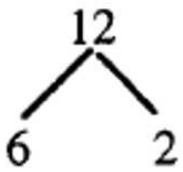
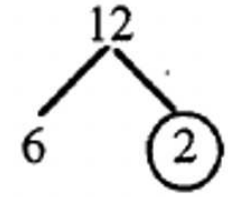
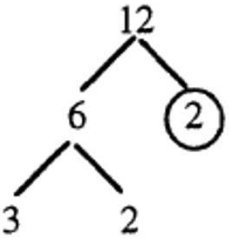
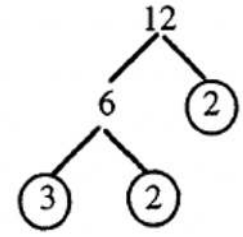
Here are the individual steps showing how one might use a factor tree to factor 12, similar to how it was factored using the squares, via Path 1. Note the similarity.

Step 1: Factor 12 as 3d4	Step 2: Circle 3 since it is prime	Step 3 Circle the 2s since they are prime	Step 4: Circle the 2s since they are prime	Step 5: The prime factorization of 12 is the product of the circled leaves 3 d2 d2
				

Below is the other way we factored 12 using squares.

Path 2: $\boxed{12} = \boxed{6} \times \boxed{2} = \boxed{3} \times \boxed{2} \times \boxed{2}$

Here are the individual steps showing how one might use a factor tree to factor 12, similar to how it was factored using the squares, via Path 2. Again, note the similarity.

Step 1: Facrtor 12 as 6×2	Step 2: Circle 2 since it is prime	Step 3: Factor 6 as 2×3	Step 4: Circle 3 and 2 since they are both prime.	Step 5: The prime factorization of 12 is the product of the circled leaves $3 \cdots 2 \cdots 2$
				

If you aren't sure whether a number is prime or composite and don't know how to start factoring, use the divisibility tests. See if it is possible to divide by 2, 3, 5, 7, 11, etc. Make **sure** there are no factors before you circle it and decide it's prime. You are going to repeat exercise 4 again, but this time, use a factor tree.

? Exercise 4

Use a factor tree to factor each of the following composite numbers into a product of primes. Write the prime factorization so the factors are written in ascending order (from smallest to largest). None of these numbers is prime. Show the individual steps. Show the factor tree under each problem.

a. 45 = _____	c. 200 = _____
b. 65 = _____	d. 91 = _____
e. 76 = _____	h. 74 = _____
f. 350 = _____	i. 512 = _____
g. 189 = _____	j. 147 = _____

The problem with trying to find the prime factorization is that sometimes it isn't obvious if a number you are trying to factor is prime or not. For instance, it is not immediately obvious whether or not 517 is prime or composite. This is where divisibility tests

come in handy. Actually, if you want to find the prime factorization of 517, you only need to check if 517 has any prime numbers as factors. You'll need to try one prime at a time. Before going on, let's list the first several prime numbers starting with 2. Note: 2 is the only even prime number. To make a list, start with 2, 3, 5, and check to see if the next odd number is prime or not. It is not prime if one of the prime numbers listed earlier is a factor.

? Exercise 5

List all of the prime numbers less than 100. You only need to use the divisibility tests for 2, 3, 5 and 7 at the most to check if any odd number less than 100 is prime. In other words, any composite number less than 100 has 2, 3, 5 or 7 as a factor. We'll discuss why after this exercise.

Consider the possible ways to factor 54 as a product of 2 factors:

$1 \cdots 54, 2 \cdots 27, 3 \cdots 18, 6 \cdots 9, 9 \cdots 6, 18 \cdots 3, 27 \cdots 2, 54 \cdots 1$

Note that if you start with the smallest factor (1) as the left factor, you start repeating half-way through the list. This "half-way" mark happens after you get to the square root of the number you are factoring. The square root of 54 is between 7 and 8. So if you list a factor bigger than 7, it would have shown up earlier in the list as a factor less than 7. So that means if I am trying to find the prime factorization of 54, I only need to check prime numbers up to and including at most 7.

So why does any composite number less than 100 have 2, 3, 5 or 7 as a factor? The next prime number after 7 is 11. Since 11^2, or 121, is greater than 100, if there was a prime number greater than or equal to 11 that was a factor of a number less than 100, then the other factor would have to be less than 11. Think about it. If both factors were bigger than 11, the product would be more than 121! This realization makes finding the prime factorization of a number much easier. This is how it works.

Since $3^2 = 9$, any composite number < 9 has 2 as a factor.

Since $5^2 = 25$, any composite number < 25 has 2 or 3 as a factor.

Since $7^2 = 49$, any composite number < 49 has 2, 3 or 5 as a factor.

Since $11^2 = 121$, any composite number < 121 has 2, 3, 5 or 7 as a factor.

Since $13^2 = 169$, any composite number < 169 has 2, 3, 5, 7 or 11 as a factor.

Since $17^2 = 289$, any composite number < 289 has 2, 3, 5, 7, 11 or 13 as a factor.

Since $19^2 = 361$, any composite number < 361 has 2, 3, 5, 7, 11, 13 or 17 as a factor.

Since $23^2 = 529$, any composite number < 529 has 2, 3, 5, 7, 11, 13, 17 or 19 as a factor.

Now, we'll verify that some of the above statements are true.

✓ Example

Since $3^2 = 9$, any composite number < 9 has 2 as a factor.

This can be verified by listing the composite numbers less than 9 and showing that 2 is a factor of each of those numbers: the only composite numbers less than 9 are 4, 6, and 8. Verify that 2 is a factor of each of these numbers: $4 = 2 \cdots 2, 6 = 2 \cdots 3, 8 = 2 \cdots 4$

✓ Example

Since $5^2 = 25$, any composite number < 25 has 2 or 3 as a factor.

The previous example verified that the composite numbers less than 9 have 2 or 3 as a factor (since we've shown they all have 2 as a factor). So we only need to list and verify that the composite numbers less than 25 have 2 or 3 as a factor. The only composite numbers between 9 and 25 are 10, 12, 14, 15, 16, 18, 20, 21, 22, and 24. Mentally verify that 2 or 3 is a factor of each of these numbers. Note: each number only needs to have 2 OR 3 as a factor, although it may have both. For instance, 2 is a factor of 10, 3 is a factor of 15, and both 2 and 3 are factors of 18. You only have to make sure at least one of those factors is a factor of each composite number.

? Exercise 6

Since $7^2 = 49$, any composite number < 49 has 2, 3 or 5 as a factor. List the composite numbers between 25 and 49, and mentally verify that 2, 3, or 5 is a factor of each of these numbers.

? Exercise 7

Since $11^2 = 121$, any composite number < 121 has 2, 3, 5 or 7 as a factor. List the composite numbers between 49 and 121, and mentally verify that 2, 3, 5 or 7 is a factor of each of these numbers.

It gets tedious trying to verify for larger numbers, but the pattern continues. In other words, if you listed all of the composite numbers less than 361, which is 19^2, then you could actually verify that every one of them has 2, 3, 5, 7, 11 or 13 as a factor.

So, to find if 517 is prime or composite, since 517 is less than 23^2, I only need to check if any of the following are factors: 2, 3, 5, 7, 11, 13, 17 or 19. Instead of checking every number up to 517, only these eight need to be checked. You can use divisibility tests for the first five primes, then use a calculator for the last three. If you find a factor early on, there is no need to keep going.

? Exercise 8

Find the prime factorization of 517. Do this by using the divisibility tests up to 11, if necessary. If you still don't find a factor of 517, use a calculator to see if 13, 17 or 19 is a factor. Write the prime factorization here:

Because I wrote the squares of the first several prime numbers on the previous page, I knew I didn't have to check any primes higher than 19. One way to figure out the highest prime number you **might** have to check is to take the square root of the number you are trying to prime factor.

? Exercise 9

On your calculator, find the square root of 517 rounded to the nearest tenth:

You should have gotten 22.7. This tells us 517 is not a perfect square. Next, we know that if 517 is composite, it must have a prime factor less than 22.7. So, we need to determine the highest whole number that is prime that is less than 22.7. Start with 22 not prime; then 21 not prime; then 20 not prime; then 19 prime! Therefore, we at most need to check the primes up to 17: 2, 3, 5, 7, 11, 13, 17 and 19. When you actually prime factored 517, did you notice 11 was a factor? Did you use the divisibility test for 11?

So, $517 = 11 \cdots 47$. Then, you look at the other factor, 47, and note it is prime, so you are done! By the way, 5 is the highest prime you'd have to check to see if 47 is prime!

What we have just described is a theorem called the Prime Factor Test. This is the formal way of stating that theorem.

Prime Factor Test: To test for prime factors of a number n, one need only search for prime factors p of n, where $p^2 \leq n$ (or $p \leq \sqrt{n}$)

? Exercise 10

For each number, determine the highest prime that might need to be checked to find the prime factorization of the number. Then, find the prime factorization. If it is prime, simply write the number itself, since that ***is*** the prime factorization.

a. 149
highest prime to check: _____ Prime factorization for 149:

b. 273
highest prime to check: _____ Prime factorization for 149:

c. 381
highest prime to check: _____ Prime factorization for 149:

d. 437 highest prime to check: _____ Prime factorization for 149:
e. 509 highest prime to check: _____ Prime factorization for 149:
f. 613 highest prime to check: _____ Prime factorization for 149:
g. 787 highest prime to check: _____ Prime factorization for 149:

Now, let's look at a range of numbers and figure out how to determine which ones are prime. For instance, let's determine which numbers between 350 and 370 are prime. First of all, only odd numbers in this range are prime. So, begin by listing the odd numbers as possibilities: 351, 353, 355, 357, 359, 361, 363, 365, 367 and 369. Next, note 355 and 365 can't be prime since it is divisible by 5. Now, you might use the divisibility test for 3 to cross off 351, 357, 363 and 369 note you can cross of multiples of 3 by crossing off every third odd number if you start at a multiple of 3. Now, our list is down to these possibilities: 353, 359, 361, 367 and 369. The highest prime you'd have to check is the prime number that is less than the square root of 369, which is 19. So, simply check the rest of the primes (7, 11, 13, 17 and 19 at most) on each of these numbers to determine which, if any, are prime. 353 is prime; 359 is prime; 361 is 19, 367 is prime. Therefore, 353, 359 and 367 are the numbers between 350 and 370 that are prime. Furthermore, you should be able to write the prime factorization for all the numbers between 350 and 370 that are composite.

? Exercise 11

Find the prime factorization for all the numbers between 280 and 295. If a number is prime, simply write the number itself.

a. 280 = _____	i. 288 = _____
b. 281 = _____	j. 289 = _____
c. 282 = _____	k. 290 = _____
d. 283 = _____	l. 291 = _____
e. 284 = _____	m. 292 = _____
f. 285 = _____	n. 293 = _____
g. 286 = _____	o. 294 = _____
h. 287 = _____	p. 295 = _____

Twin primes are two consecutive odd numbers that are prime. For instance, 5 and 7 are twin primes, 11 and 13 are twin primes, 17 and 19 are twin primes. There is no pattern to determine how often twin primes come up. One unsolved question in mathematics is if there are a finite number of or infinitely many sets of twin primes. Nobody knows.

? Exercise 12

From your work in exercise 11, list any sets of twin primes: _____

? Exercise 13

Is it possible to have three odd numbers in a row that are prime? Why or why not?

One use of prime factoring a set of numbers is so you can find the **greatest common factor** (GCF) and **least common multiple** (LCM) of a set of numbers. Many people have trouble distinguishing between the greatest common factor and the least common

multiple (LCM) because they don't think about what the words actually mean. The greatest common factor of a number is obviously a **factor**, but the adjective **common** describes that you want a factor that is common to all the numbers, and the adjective **greatest** describes that you want the very largest of the common factors of the numbers.

We are going to explore ways of finding the greatest common factor of two numbers, a and b. The notation to express this is GCF(a, b). It doesn't matter which number you list first in the parentheses – it's not an ordered pair. **The greatest common factor of a set of numbers is the largest number that is a factor of each number.**

We are going to explore different ways of finding the greatest common factor of two numbers.

First, we are going to explore one way to find the greatest common factor of 42 and 72.The notation to express this is GCF(42, 70). Remember: it doesn't matter which number you list first in the parentheses, *it's not an ordered pair*. GCF(42, 70) means the same thing as GCF(70, 42). **The greatest common factor of 42 and 70 is the largest number that is a factor of both 42 and 70.**

One way of doing this is to list every single factor of each number and then pick the biggest one that is a factor of each.

? Exercise 14

a. List all of the factors of 42 in ascending order: ____

b. List all of the factors of 70 in ascending order: ____

c. List all of the factors that are common to both 42 and 70: ____

d. List the greatest common factor of 42 and 70: ____

e. Fill in the blank: GCF(42, 70) = ____

Listing all of the factors of a given number is sometimes a difficult task. For instance, it's easy to miss a factor. One remedy is to prime factor the number first. To list all of the factors of 42, one might first prime factor 42 like this: $2 \cdot 3 \cdot 7$. For a number to be a factor of 42, it must be composed of the prime factors listed. Of course, 1 is always a factor. Next, you'd check 2, and then 3, which are both factors. 4 is not a factor because if it were, $2 \cdot 2$ would be in the prime factorization! It's clear to see 5 is not a factor. 6 is a factor, since $2 \cdot 3$ is in the prime factorization. Continuing on, 7 is a factor, but 8 is not because $2 \cdot 2 \cdot 2$ is not in the prime factorization of 42. Neither is 9 $(3 \cdot 3)$, $10 (2 \cdot 5)$, 11, $12 (2 \cdot 2 \cdot 3)$, or 13. But 14 is a factor of 42 since $2 \cdot 7$ is in the prime factorization. You can use this strategy as you check every number up to 42, but that is still a lot of numbers to check. Eventually, you'd get this list: 1, 2, 3, 6, 7, 14, 21, 42.

Here's a way to shorten the process a little more. Starting with the smallest factor 1, immediately list the other factor you'd have to multiply by that factor to get 42. So, we start with 1, 42. We check the next number, 2, and note it is a factor. To get the other factor that pairs up with 2, either divide 2 into 42, or simply look at the prime factorization of 42, with the 2 missing. There is $3 \cdot 7$ left, which is the other factor. So the list is now 1, 42, 2, 21. Continuing, we note 3 is a factor. To get the other factor, either divide 42 by 3, or do it the easy way, which is to see what factors are left in the prime factorization of 42 with the 3 missing. Since there is a 2 and a 7, then the number that pairs up with 3 is 14.

The list is now 1, 42, 2, 21, 3, 14. Next, we note 4 and 5 are not factors. 6 is a factor since 2 and 3 $(2 \cdot 3)$ is in the prime factorization. 7 is the number that pairs up with 6. So, the list is now: 1, 42, 2, 21, 3, 14, 6, 7. If you were to continue, the next number to check would be 7. Since $7^2 > 42$, you can stop. All the factors that are larger than 7 will already be in the list because there would have been a smaller factor that it paired up with already. Now, put the list of factors of 42 in ascending order: 1, 2, 3, 6, 7, 14, 21, 42.

The same procedure can be used to list all the factors of 70. First, write the prime factorization of 70: $2 \cdot 5 \cdot 7$. You would start off with 1 and 70: 1, 70. Next, it's clear 2 is a factor that pairs up with 35. The list is now: 1, 70, 2, 35. Next discard 3 and 4 as factors, and note 5 is a factor. The factors left are 2 and 7, which multiplied together is 14. So the list is 1, 70, 2, 35, 5, 14. Continuing on, note 6 is not a factor, and 7 is. 7 pairs up with 10. The list is now: 1, 70, 2, 35, 5, 14, 7, 10. Continuing on, note that 8 is not a factor. The next number to check would be 9. But $9^2 > 70$, so all the factors higher are already in the list. Writing the list in ascending order, we get: 1, 2, 5, 7, 10, 14, 35, 70.

Make a note that in both of these examples, 42 and 70 each had exactly 3 prime numbers in the prime factorization. Consider the prime factorization of 220: $2 \cdot \cdot \cdot 2 \cdot \cdot \cdot 5 \cdot \cdot \cdot 11$. Note that as you list factors, there may be one, two, or three other factors to multiply together to get the pair. 2 is a factor; its pair is $2 \cdot \cdot \cdot 5 \cdot \cdot \cdot 11$, or 110. 4 $(2 \cdot \cdot \cdot 2)$ is a factor; its pair is $5 \cdot \cdot \cdot 11$, or 55. 5 is a factor; its pair is $2 \cdot \cdot \cdot 2 \cdot \cdot \cdot 11$, or 44. 10 $(2 \cdot \cdot \cdot 5)$ is a factor; its pair is $2 \cdot 11$, or 22. 11 is a factor; its pair is $2 \cdot \cdot \cdot 2 \cdot \cdot \cdot 5$, or 20. As I check numbers larger than 11, I stop at 15 since 152 > 220. So, the list is: 1, 220, 2, 110, 4, 55, 5, 44, 10, 22, 11, 20. Writing these in ascending order, we get: 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, 220.

By the way, it's a lot more time-consuming for me to explain (and for you to read) how to list all the factors by prime factoring. But doing it is a lot easier! Now, do the next exercises. If you wish, use prime factorization to do parts a and b.

? Exercise 15

The goal of this problem is to find the greatest common factor of 92 and 115.

a. List all of the factors of 92 in ascending order:
b. List all of the factors of 115 in ascending order:
c. List all of the factors that are common to both 92 and 115:
d. List the **greatest** common factor of 92 and 115:
e. Fill in the blank: GCF(92, 115) =

? Exercise 16

The goal of this problem is to find the greatest common factor of 48, 54 and 63.

a. List all of the factors of 48 in ascending order:
b. List all of the factors of 54 in ascending order:
c. List all of the factors of 63 in ascending order:
d. List all of the factors that are common to 48, 54 and also 63:
e. List the **greatest** common factor of 48, 54 and 63:
f. Fill in the blank: GCF(48, 54 and 63) =

As you might have realized, listing factors can still be a very time-consuming way to find the greatest common factor, especially if the numbers are very large or have a lot of factors.

It's time to get out your colored prime number squares and white composite number squares again. We'll be using these to prime factor sets of numbers, which can then be used to find the greatest common factor of a set of numbers. This method is usually faster than the one we just used.

The goal of this next example is to find the greatest common factor of 42 and 72 using prime factorization with the prime and composite number squares.

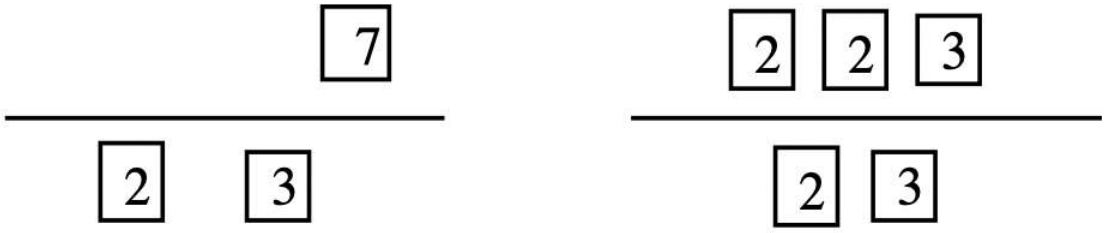
Step 1: Use the squares to prime factor 42 and 72.

$42 = 2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 7$ and $72 = 2 \cdot \cdot \cdot 2 \cdot \cdot \cdot 2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 3$

Step 2: Draw a line and put the prime factors of 42 above it. Next to that, draw a line and put the prime factors of 72 above it.



Step 3: Look at the prime factorization of the numbers. Move every factor they have in common under the line. They should have the same exact factors under the line, and they should have no common factors left above the line.



Step 4: The product of the factors under a line is the greatest common factor of the numbers. In this case, $2 \cdots 3$, or 6, is the greatest common factor of 42 and 72.

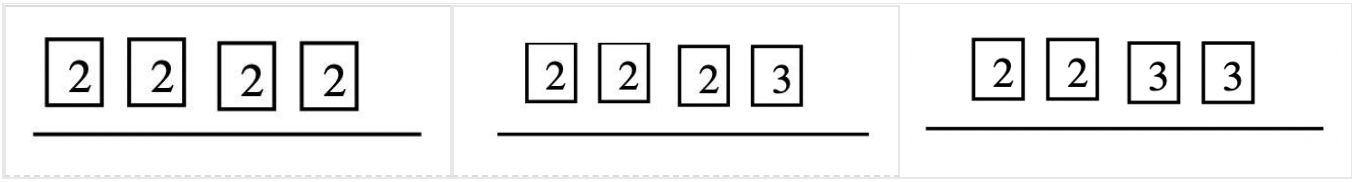
Step 5: Use the correct notation to write the answer: GCF(42, 70) = 6.

This next example used the same steps to find the greatest common factor of 16, 24 and 36.

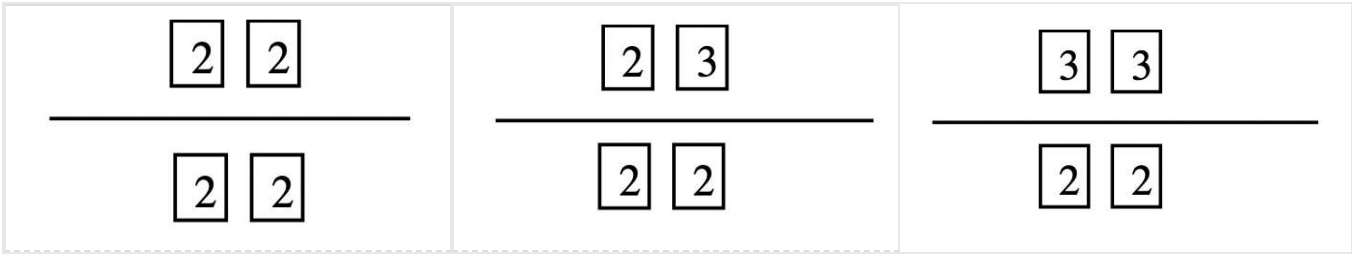
Step 1: Use the squares to prime factor 16, 24 and 36.

$16 = 2 \cdots 2 \cdots 2 \cdots 2, 24 = 2 \cdots 2 \cdots 2 \cdots 3 \text{and } 36 = 2 \cdots 2 \cdots 3 \cdots 3$

Step 2: Draw a line for each number and put the prime factors of each number above it.



Step 3: Look at the prime factorization of the numbers. Move every factor that all three numbers have in common. There should be no factors left above the line that all three numbers have in common.



Step 4: The product of the factors under a line is the greatest common factor of the numbers. In this case, $2 \cdots 2$, or 4, is the greatest common factor of 16, 24 and 36.

Step 5: Use the correct notation to write the answer: GCF(16, 24, 36) = 4.

? Exercise 17

Use the steps shown in the previous two examples to find the greatest common factor of each problem. Show a picture of how the problem looks at step 3 where the prime factorization of the greatest common factor is shown under the line of each number you first prime factored.

a. GCF(42, 70) = _____ (This answer should agree with Exercise 14e.)
Show work below:

b. GCF(92, 115) = _____ (This answer should agree with Exercise 15e.)
Show work below:

c. GCF(48, 54, 63) = _____ (This answer should agree with Exercise 16f.)
Show work below:

d. GCF(306, 340) = _____
Show work below:

e. GCF(125, 275, 400) = _____
Show work below:

f. GCF(126, 168, 210) = _____
Show work below:

Exercise 18

Let's say someone prime factored three large numbers, X, Y and Z like this:

$X = 2^5 \cdots 3^4 \cdots 7^2 \cdots 11^8 \cdots 13^3$	$Y = 2^4 \cdots 3^5 \cdots 5^3 \cdots 7^7 \cdots 13^2$	$Z = 2^6 \cdots 3 \cdots 5^5 \cdots 11^3 \cdots 13^4$

a. State the prime factorization (exponential notation is okay) of the greatest common factor of X, Y and Z. GCF(X, Y, Z) =

b. Explain how you did part a.

One way to do the previous exercise is to begin by listing the common prime factors (without exponents) of X, Y and Z. They are 2, 3 and 13. So, as a start, write $2 \cdots 3 \cdots 13$. Next, determine how many of each prime factor is common to X, Y and Z. Since X has 5 factors of 2, Y has 4 factors of 2, and Z has 6 factors of 2, they only have 4 factors of 2 in common. Similarly, they only have one factor of 3 in common, and 2 factors of 13 in common. Put these exponents on the factors and you've got the GCF of the three numbers: GCF(X, Y, Z) = $2^4 \cdots 3 \cdots 13^2$

Did you notice that if you list the factors they have in common without exponents, you put on the smallest exponent they have in common for each prime?

Exercise 19

Write the prime factorization of the greatest common factor of the set of numbers. For those that are factored with letters, assume each letter represents a different prime number.

a. If $X = 2^4 \cdots 3^2 \cdots 7^6 \cdots 11^3 \cdots 13^2$ and $Y = 2^5 \cdots 3^6 \cdots 5^4 \cdots 7^6 \cdots 13^3$
GCF(X, Y) = _____

b. If $X = 3^4 \cdots 5^2 \cdots 7^6$ and $Y = 2^5 \cdots 3^6 \cdots 5^6 \cdots 7^3$ and $Z = 2^4 \cdots 3^5 \cdots 5^4$
GCF(X, Y, Z) = _____

c. If $X = a^4 \cdots b^2 \cdots c^6 \cdots d^3 \cdots e^2$ and $Y = a^5 \cdots c^3 \cdots d^4 \cdots e^6 \cdots f^3$
GCF(X, Y) = _____

d. If $X = a^4 \cdots b \cdots c^4 \cdots d^3$ and $Y = a^5 \cdots b^3 \cdots d^4 \cdots e^6$ and $Z = a^2 \cdots c^3 \cdots d^7 \cdots f^3$
GCF(X, Y, Z) = _____

If two numbers have no factors in common, they are called **relatively prime**. In other words, **if GCF(a, b) = 1, then a and b are relatively prime**. The numbers, a and b, might both be prime, both be composite, or one might be prime and the other composite.

Exercise 20

Give an example of two composite numbers that are relatively prime:

Exercise 21

Write a prime number and composite number that are relatively prime:

Exercise 22

Assume GCF(28, x) = 1

a. List any prime numbers that are not factors of x: _____

b. Give three examples (numbers) of what x could equal: _____

Exercise 23

If x and y are different prime numbers, GCF(x, y) = _____

Exercise 24

If m is a whole number, find the following:

a. GCF(2m, 3m) = _____	b. GCF (4m, 10m) = _____
c. GCF(m, m) = _____	d. GCF(m, 1) = _____
e. GCF(m, 0) = _____	

Trying to find the greatest common factor of two large numbers by prime factorization is sometimes quite time-consuming. There are two other algorithms you can use that we'll use. One is called **The Old Chinese Method**, and the other is **The Euclidean Algorithm**. Both of these methods use a fact we can prove using what we know about divisibility. First, let's look at some examples.

Exercise 25

Compute each of the following:

a. List the common factors of 42 and 72: _____
b. List the common factors of 42 and 30 (which is 72 – 42): _____
c. List the common factors of 30 and 12 (which is 42 – 30): _____
d. List the common factors of 12 and 18 (which is 30 – 12): _____
e. List the common factors of 12 and 6 (which is 18 – 12): _____
f. List the common factors of 6 and 6 (which is 12 – 6): _____
g. List the common factors of 6 and 0 (which is 6 – 6): _____

From your work in exercise 25, compute the following:

a. GCF(42, 72) = ____	b. GCF(42, 30) = ____
c. GCF(30, 12) = ____	d. GCF(12, 18) = ____
e. GCF(12, 6) = ____	f. GCF(6, 6) = ____

g. GCF(6, 0) = ____	

The previous two exercises illustrate that the greatest common factor of two numbers is equal to the greatest common factor of the smaller number, and the difference of the original two numbers; i.e., **if x $\geq$ y, then GCF(x, y) = GCF(y, x – y)**.

Prove the following theorem: Let a $\geq$ b. If c|a and c|b, then c|(a – b).

Solution: If c|a, then cn=a for some whole number, n. If c|b, then cm=b for some whole number, m. Using these substitutions for a and b, we get that c|(a – b) is true if c|(cn – cm) which is true if c is a factor of cn – cm. Factor: cn – cm = c(n – m). This clearly shows that c is indeed a factor of cn – cm. Therefore, if c|a and c|b, then c|(a – b).

This theorem can be used to show that **if a $\geq$ b, then GCF(a, b) = GCF(b, a – b)**. The above theorem states that if c is a factor of two numbers, then it is also a factor of their difference. Hence, if c is a common factor of a and b, where a $\geq$ b, then c is also a common factor of b and a – b. Since every common factor of a and b is also a common factor of ba and a – b, the pairs (a, b) and (b, a – b) have the same common factors. So, GCF(a, b) and GCF(b, a – b) must also be the same number.

The **Old Chinese Method** employs the fact that **GCF(a, b) = GCF(b, a – b)**.

Note three more properties:

GCF(x, x) = x: GCF(x, x) states that the greatest common factor of the same two numbers is itself. That should be clear since x is the greatest factor of each number, so each has x as the greatest common factor.

GCF(x, 0) = x: GCF(x, 0) = x, is true since every number is a factor of zero. So, since x is the greatest factor of x, then x is the greatest common factor of x and 0.

GCF(x, 1) = 1: GCF(x, 1) = 1 is true since 1 is a factor of every number, including x, and 1 is the only factor of 1. Therefore, 1 must be the greatest common factor of 1 and x.

Old Chinese Method of finding the greatest common factor of two numbers:

Write the GCF of the two numbers in parentheses (remember the order of the numbers is irrelevant). Let that equal the GCF of the smaller of the two numbers, and the difference of the original two numbers. If the numbers in the parentheses are the same, that number is the GCF; if one the new numbers is 1, 1 is the GCF. Otherwise, repeat the process until the two numbers are the same, or 1 is one of the numbers. An example is shown below. On the right is an explanation of how I obtained the two new numbers in parentheses. The new numbers are underlined in the explanation. You do not need to write that out.

✔ Example

GCF(546, 390)	= GCF(390, 156)	(*390* is smaller than 546, *156* = 546 – 390)
	= GCF(156, 234)	(*156* is smaller than 390, *234* = 390 – 156)
	= GCF(156, 78)	(*156* is smaller than 234, *78* = 234 – 156)
	= GCF(78, 78)	(*78* is smaller than 156, *78* = 156 – 78)
	= ***78***	78 is the GCF(78, 78); The answer is a number!

Therefore, GCF(546, 390) = 78.

Check: First, make sure 78 is a factor of 546 and 390: $546 = 78 \cdots 7$ and $390 = 78 \cdots 5$. Second, check to make sure the other factors of each (7 and 5) are relatively prime. If so, then 78 is not only a factor of 546 and 390, but is in fact the greatest common factor of each since they have no other common factors (because 7 and 5 have no common factors).

Another example is shown below. On the right is an explanation of how I obtained the two new numbers (underlined) in parentheses. You do not need to write that out.

Example

GCF(1200, 504)	= GCF(504, 696)	(*504* is smaller than 1200, *696* = 1200 – 504)
	= GCF(504, 192)	(*504* is smaller than 696, *192* = 696 – 504)
	= GCF(192, 312)	(*192* is smaller than 504, *312* = 504 – 192)
	= GCF(192, 120)	(*192* is smaller than 312, *120* = 312 – 192)
	= GCF(120, 72)	(*120* is smaller than 192, *72* = 192 – 120)
	= GCF(72, 48)	(*72* is smaller than 120, *48* = 120 – 72)
	= GCF(48, 24)	(*48* is smaller than 72, *24* = 72 – 48)
	= GCF(24, 24)	(*24* is smaller than 48, *24* = 48 – 24)
	= ***24***	24 is the GCF(24, 24); The answer is a number!

Therefore, GCF(1200, 504) = 24.

Check: First, make sure 24 is a factor of 1200 and 504: 1200 = $24 \cdots 50$ and $502 = 24 \cdots 21$. Second, check to make sure the other factors of each (25 and 21) are relatively prime. If so, then 24 is not only a factor of 1200 and 504, but is in fact the greatest common factor of each since they have no other common factors (because 25 and 21 have no common factors.)

Here is one more example:

Example

GCF(667, 437)	= GCF(437, 230)	
	= GCF(230, 207)	
	= GCF(207, 23)	
	= GCF(23, 184)	
	= GCF(23, 161)	
	= GCF(23, 138)	
	= GCF(23, 115)	
	= GCF(23, 92)	
	= GCF(23, 69)	
	= GCF(23, 46)	
	= GCF(23, 23)	
	= ***23***	Remember: The answer is a number!

Therefore, GCF(667, 437) = 23.

Check: First, make sure 23 is a factor of 667 and 437: $667 = 23 \cdots 29$ and $437 = 23 \cdots 19$. Second, check to make sure the other factors of each (29 and 19) are relatively prime. If so, then 23 is not only a factor of 667 and 437, but is in fact the greatest common factor of each since they have no other common factors (because 29 and 19 have no common factors.)

A comment: You could have obtained the greatest common factor of the previous three examples by prime factoring. Often, this is a lengthy process for large numbers that look like they might be prime, as in the last example. Therefore, the Old Chinese provides an alternative way to obtain the greatest common factor. After doing a few using this method, we'll explore another alternate method, called the Euclidean Algorithm, which is related to, but usually takes less steps than, the Old Chinese Method.

? Exercise 27

Use the Old Chinese Method to compute the greatest common factor of the numbers given. Use correct notation, and show each step. State the answer. Then, show how you check your answer. Use the previous example as a model to do these problems.

a.

GCF(143, 91)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= _____	

b.

GCF(468, 378)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= _____	

c.

GCF(504, 180)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= _____	

Using the Chinese Method could be quite tedious. Take a look at the following example:

Example

GCF(1200, 504)	= GCF(504, 696)	(*504* is smaller than 1200, *696* = 1200 – 504)
	= GCF(504, 192)	(*504* is smaller than 696, *192* = 696 – 504)
	= GCF(192, 312)	(*192* is smaller than 504, *312* = 504 – 192)
	= GCF(192, 120)	(*192* is smaller than 312, *120* = 312 – 192)
	= GCF(120, 72)	(*120* is smaller than 192, *72* = 192 – 120)
	= GCF(72, 48)	(*72* is smaller than 120, *48* = 120 – 72)
	= GCF(48, 24)	(*48* is smaller than 72, *24* = 72 – 48)
	= GCF(24, 24)	(*24* is smaller than 48, *24* = 48 – 24)
	= ***24***	24 is the GCF(24, 24); The answer is a number!

In the beginning of this example, GCF(1200, 504), we had to subtract 504 twice until we got GCF(504, 192). Then, notice once we wrote GCF(504, 192), we had to subtract 192 twice until we got GCF(192, 120). Basically, we do repeated subtraction until we get a number smaller than the one we are subtracting. Repeated subtraction is actually division. Note that $1200 \div 504 = 2$ remainder 192. On the second step, we see GCF(504, 192), which has the **smaller** of the numbers in the original parentheses (504), and the **remainder** after dividing the larger number (1200) by the smaller number (504). Note that $504 \div 192 = 2$ remainder 120. Look at the fourth step: we see GCF(192, 120), which has the **smaller** of the numbers in GCF(504, 192), and the **remainder** after dividing the larger number (504) by the smaller number (192). The Euclidean Algorithm uses division instead of repeated subtraction to shorten the steps.

How to use the Euclidean Algorithm to find the greatest common factor of two numbers:

Write the GCF of the two numbers in parentheses (remember the order of the numbers is irrelevant). The smaller of the two numbers will be one of the numbers in the next parentheses. To get the other number, divide the larger number by the smaller number, and put the **remainder** in parentheses. If the the smaller number is a factor of the larger number, that means it will divide evenly, so there will be no remainder. That means the remainder is 0. Remember to put the remainder in the parentheses, not the quotient! If one of the new numbers in the parentheses is zero, the other number is the GCF; if one the new numbers is 1, 1 is the GCF. Otherwise, repeat the process until one of the two numbers is 0 or 1. An example is shown below. This is the first example we did using the Old Chinese Method. You might want to look back and compare. On the right is an explanation of how I obtained the two new numbers in parentheses.

Example

GCF(546, 390)	= GCF(390, 156)	(*390* is smaller than 546, $546 \div 390 = 1$ r *156*)
	= GCF(156, 78)	(*156* is smaller than 390, $390 \div 156 = 2$ r *78*)
	= GCF(78, 0)	(78 is smaller than 156, $156 \div 78 = 2$ r *0*)
	= ***78***	***Put the remainder (0), NOT 2, in the parentheses!***

		78 is the GCF(78, 78); The answer is a number!

Therefore, GCF(546, 390) = 78

Check: First, make sure 78 is a factor of 546 and 390: $546 = 78 \backslash = \cdots 7$ and $390 = 78 \cdots 5$. Second, check to make sure the other factors of each (7 and 5) are relatively prime. If so, then 78 is not only a factor of 546 and 390, but is in fact the greatest common factor of each since they have no other common factors (because 7 and 5 have no common factors

Below is another example – we did this earlier using the Old Chinese Method. On the right is an explanation of how I obtained the two new numbers in parentheses. You do not need to write that out.

Example

GCF(667, 437)	= GCF(437, 230)	(*437* is smaller than 667, $667 \div 437 = 1$ r *230*)
	= GCF(230, 207)	(*230* is smaller than 437, $437 \div 230 = 1$ r *207*)
	= GCF(207, 23)	(*207* is smaller than 230, $230 \div 207$ = 1 r *23*)
	= GCF(23, 0)	(*23* is smaller than 207, $207 \div 23$ = 9 r *0*)
	= ***23***	***Put the remainder (0), NOT 9, in the parentheses!***
		Remember: The answer is a number!

Therefore, GCF(667, 437) = 23

Check: First, make sure 23 is a factor of 667 and 437: 667 = $23 \cdots 29$ and $437 = 23 \cdots 19$. Second, check to make sure the other factors of each (29 and 19) are relatively prime. If so, then 23 is not only a factor of 667 and 437, but is in fact the greatest common factor of each since they have no other common factors (because 29 and 19 have no common factors.)

A comment: You could have obtained the greatest common factor of the previous three examples by prime factoring, the Old Chinese Method or the Euclidean Algorithm.

CAUTION: THE MOST COMMON MISTAKE PEOPLE MAKE WHEN USING THE EUCLIDEAN ALGORITHM IS ON THE LAST STEP, WHEN THE SMALLER NUMBER DIVIDES EVENLY INTO THE LARGER NUMBER, WHICH *MEANS THE REMAINDER IS ZERO*. IT DOESN'T MATTER WHAT THE QUOTIENT IS – IT'S THE *REMAINDER* THAT MATTERS – *0* GOES IN THE PARENTHESES!!! Also, zero is not a factor of any number except zero, so the GCF can't be zero. On the other hand, every number is a factor of zero. So, when zero is one of the numbers in parentheses, the other number is the GCF. **REMEMBER TO ALWAYS CHECK YOUR Answer!!!**

Before going on, I'm going to remind you of a quick and easy way to find out the quotient and remainder by using a simple calculator when doing these division problems, where you need to find the remainder. If you can already do it easily or your calculator figures it out for you, skip on down to Exercise 29. Let's say you wanted to divide $5263 \div 360$. When you do this on your calculator, it shows up something like 14.619444. This indicates that there are 14 360's in 5263, but the remainder isn't evident. At least, you know 14 is the quotient. To find the remainder on your calculator, key in 14×360 - 5263 and the number showing is the remainder if you ignore the negative sign! In this case, the remainder is 223. Remember that the remainder must be less than what you originally divided by – less than 360 in this case. Think about why this process works and try it on the next few problems.

Exercise 28

Use a calculator to find the quotient and remainder for these division problems.

a. $9876 \div 255 =$	b. $1509 \div 164 =$
c. $333 \div 46 =$	d. $4657 \div 579 =$

Exercise 29

Use the Euclidean Algorithm to compute the greatest common factor of the numbers given. Use correct notation, and show each step. State the answer. Then, show how you check your answer. Use the previous example as a model to do these problems. These are the same exercises you did using the Old Chinese Method in exercise 27. You might want to compare the two methods when you are done. Of course, the answer should be the same.

a.

GCF(143, 91)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= _____	

b.

GCF(468, 378)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= _____	

c.

GCF(504, 180)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= _____	

Exercise 30

Find GCF(418, 88) using the three methods discussed:

a) by prime factorization;
b) by the Old Chinese Method;
c) by the Euclidean Algorithm
d) Write the answer, and show how to check the answer.

a. Use prime factorization	c. Use the Euclidean Algorithm
b. Use the Old Chinese Method	d. Answer: GCF(418, 88) = _____ Check your answer

Exercise 31

Find GCF(527, 465) using the three methods discussed:

a) by prime factorization;
b) by the Old Chinese Method;
c) by the Euclidean Algorithm
d) Write the answer, and show how to check the answer.

a. Use prime factorization	c. Use the Euclidean Algorithm
b. Use the Old Chinese Method	d. Answer: GCF(527, 465) = _____ Check your answer

Exercise 32

Find GCF(353, 213) using the three methods discussed:

a) by prime factorization;
b) by the Old Chinese Method;
c) by the Euclidean Algorithm
d) Write the answer, and show how to check the answer.

a. Use prime factorization	c. Use the Euclidean Algorithm
b. Use the Old Chinese Method	d. Answer: GCF(353, 213) = _____ Check your answer

Now that we've covered a lot about prime numbers and factoring, we are going to revisit the divisibility tests one more time.

Exercise 33

Note that the divisibility test for 6 utilized two other divisibility tests, the one for 2 and the one for 3. Also, note that the divisibility test for 15 utilized two other divisibility tests, the one for 5 and 3. Why do you think that works? What is the procedure for figuring out what tests to use?

Fact: If each prime in the prime factorization of a composite number, c, is listed only once, the divisibility test for that composite number is this: c|n if each of the prime factors of c divide n.

The prime factorization for 6 is $2 \cdot 3$. Since both primes are only listed once, the divisibility test for 6 is the union of the divisibility tests for 2 and 3.

The prime factorization for 15 is $5 \cdot 3$. Since both primes are only listed once, the divisibility test for 15 is the union of the divisibility tests for 5 and 3.

Exercise 34

Write the divisibility test for 14:

Exercise 35

Use the divisibility test for 14 to see which of the following is true. Show work.

a. 14|742
b. 14|968
c. 14|483

Exercise 36

The prime factorization for 20 is $2 \cdots 2 \cdots 5$. The divisibility test for 2, 2 and 5 does not work since 20|70 is false even though 2|70 and 2|70 and 5|70. Why doesn't it work?

Exercise 37

This is the divisibility test for 20: 20|n if 4|n and 5|n. Why do you think this works?

Exercise 38

Try to write the divisibility tests for each of the following numbers:

a. 12|n if
b. 18|n if

Divisibility Test for a composite number: **Assume P_1, P_2, P_3, P_4, P_5,... are all different prime numbers. (You can think of P_1 as the first prime (2), P_2 as the second prime (3), P_3 as the third prime (5), P_4 as the fourth prime (7), P_5 as the fifth prime (11), etc. Assume the prime factorization of a number, c, is $(P_a)^x \cdots (P_b)^y \cdots \cdots \cdots (P_c)^z$ where $P_a \neq P_b \neq P_c$, etc. Then, c|n if $(P_a)^x$|n AND $(P_b)^y$|n AND $(P_c)^z$|n, etc.**

Example

The divisibility test for the following numbers is done by first factoring them.

1) Divisibility test for 26: Since 26 = $2 \cdots 13$: **26|n if 2|n and 13|n.**

2) Divisibility test for 12: Since 12 = $2^2 \cdots 3$: **12|n if 4|n and 3|n.**

3) Divisibility test for 24: Since 24 = $2^3 \cdots 3$: **24|n if 8|n and 3|n.**

4) Divisibility test for 45: Since 45 = $3^2 \cdots 5$: **45|n if 9|n and 5|n.**

5) Divisibility test for a number, c, whose prime factorization is $2^3 \cdots 3^2 \cdots 5^2 \cdots 11$: **c|n if 8|n AND 9|n and 25|n and 11|n**

7) Divisibility test for a number, b, whose prime factorization is $3^2 \cdots 5 \cdots 7^2 \cdots 11^2$: **b|n if 9|n AND 5|n and 49|n and 121|n**

Exercise 39

Write the divisibility test for the following numbers:

a. 35: ____
b. 28: ____

c. 75: ____
d. 56: ____
e. A number, c, whose prime factorization is $2^2 \cdots 3^3 \cdots 5^2 \cdots 11$: ____
f. A number, d, whose prime factorization is $2^4 \cdots 3 \cdots 5 \cdots 11$: ____

? Exercise 40

This exercise is for you to think about what you'd do if someone asked you to find the LEAST common factor of a set of numbers, instead of the greatest common factor.

a. Find the least common factor of 55 and 66: ____
b. Find the least common factor of 10 and 12: ____
c. Find the least common factor of 1 and 8: ____
d. Find the least common factor of 3, 6, and 9: ____
e. Find the least common factor of m and n: ____
f. Do you think it is a useful question to find the least common factor of a set of numbers? Why or why not? Explain your answer.
g. What is the least common factor of any set of numbers? ____

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6.5: The Greatest Common Factor

Definition: Greatest Common Factor (GCF)

The Greatest Common Factor (GCF) of two or more whole numbers is the largest number that is a factor of the two or more numbers.

To find the GCF, we **work backwards**, following the three steps below.

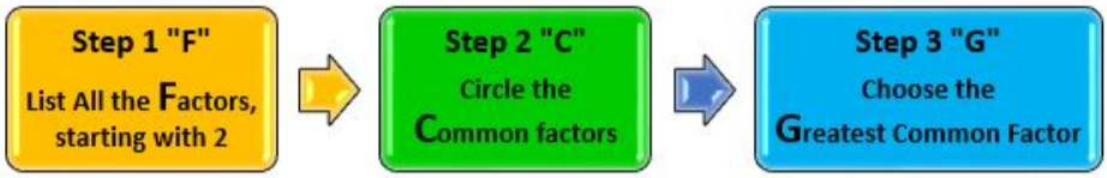


Figure 5.4.1: Steps to find GCF

Example 6.5.1

Find the GCF of 54 and 60

Solution

Step 1: List all the Factors, starting with 2

54 → 2, 3, 6, 9, 18, 27, 54

60 → 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60

Step 2: Circle the Common factors

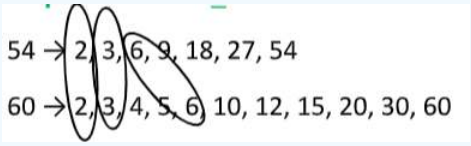


Figure 5.4.2

Step 3: Choose the Greatest circled pair

Answer: GCF (54, 60) = 6

Example 6.5.2

Write the Simplest Form of the Fraction using the GCF $\frac{140}{165} = ?.\ \mathrm{GCF}(140, 165) = 5$

Solution

Divide top and bottom by 5.

$$\frac{140}{165} = \frac{140 \div 5}{165 \div 5} = \frac{28}{33}$$

Partner Activity 1

A forest ranger needs to remove three tree trucks by cutting the trunks into equal lengths. If the lengths of the tree trunks are six feet, eight feet and 12 feet, what is the length of the longest log that can be cut?



Figure 5.4.3

Partner Activity 2

Rhys wants to organize his sports cards in packets for each type of sport. Each packet has the same number of cards. If he has 24 baseball cards, 60 hockey cards, and 48 football cards, find the greatest number of cards in each packet.



Figure 5.4.4

Practice Problems

Find the Greatest Common Factor (GCF).

1. 40 and 90
2. 75 and 25
3. 168 and 85
4. 90, 120, and 150
5. 135, 225, and 405

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6.6: LCM and other Topics

You will need: ***Prime Number Squares*** *(Material Cards 19A-19B)*

Composite Number Squares *(Material Cards 20A-20B)*

There are a ***finite*** *number of factors* for any given number. On the other hand, there are an ***infinite*** *number of nonzero multiples* of a number. For instance the list of multiples of 2 are all of the even numbers. It's impossible to list all of them, because this is an infinite set. A multiple of a number c is nc where n is a whole number. In other words, to find a multiple of c, multiply c by a whole number. Although 0 is a multiple of every number, we usually omit listing it as a multiple. The (nonzero) multiples of 6 are listed here: 6, 12, 18, 24, 30, 36, 42, ...

To list the multiples of a number, you begin with the number itself, then the number times 2, then the number times 3, etc. Since multiplication is repeated addition, you could repeatedly add the number to get the next multiple.

? Exercise 1

a. List the first 10 multiples of 8:
b. List the first 10 multiples of 12:
c. Of the list you produced in parts a and b, list the multiples that 8 and 12 have in common:
d. From part c, what is the *smallest multiple* that 8 and 12 have in common?
e. Is there a greatest common multiple that 8 and 12 have in common? Is so, what is it? Before you answer, remember that in a and b, you only listed the *first 10* multiples of 8 and 12.

Just like with the greatest common factor (GCF), many people don't really think about what **LEAST COMMON MULTIPLE** ***means***. Every number has an infinite number of multiples. Every set of numbers has an infinite number of multiples in common. The **LEAST COMMON MULTIPLE** is the ***smallest*** **multiple** they have in **common**.

In exercise 1, we write LCM(8, 12) = 24. The order of the numbers in the parentheses is irrelevant. So, LCM(8, 12) = LCM(12, 8). It is NOT an ordered pair. Also, you can find the least common multiple of a large set of numbers. Soon, you'll find how easy it is to find the least common multiple of several numbers; e.g., LCM(2, 3, 4, 5, 6, 7, 8, 9, 10). In this case, listing several multiples of each of these numbers until you find the smallest one all of them have in common is not the efficient way to find it, so you'll be exploring another way to find it using prime factorization.

? Exercise 2

a. List the first 15 multiples of 4:
b. List the first 15 multiples of 6:
c. List the first 15 multiples of 10:
d. Of the list you produced in parts a, b and c, list the multiples all three numbers (4, 6 and 10) have in common:
e. From part d, complete the following: LCM(4, 6, 10) =

The problem with trying to find the least common multiple this way is that you may have to write a lengthy list of multiples of each number until you finally find a multiple they all have in common. For instance, if you were asked to find the LCM(59,61), you wouldn't find a multiple they each have in common until you wrote 61 multiples of 59 and 59 multiples of 61. That is because these are prime numbers, so the least common multiple is their product: $59 \cdots 61$. This is the case with any numbers that are **relatively prime** as well. (Remember 4 and 15 are relatively prime, although neither is prime.)

If x and y have no factors in common, then GCF(x,y) = 1 and LCM(x,y) = xy.

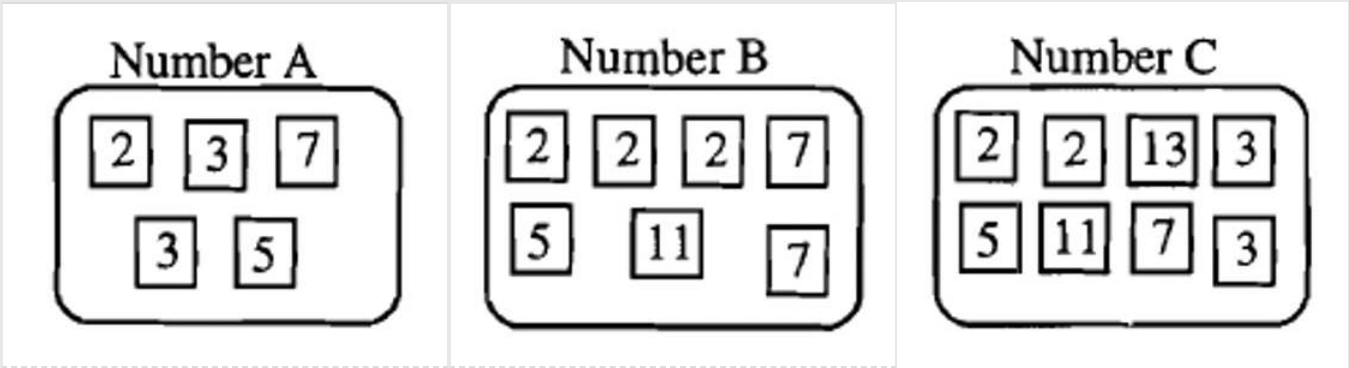
The following is also true:

If GCF(x,y) = 1, then x and y have no factors in common, and LCM(x,y) = xy.

If LCM(x,y) = xy, then x and y have no factors in common, and GCF(x,y) = 1.

A more efficient way to find the least common multiple of a set of numbers is to find the prime factorization of each number, and then BUILD the least common multiple. Get out your prime number squares to do the following exercises.

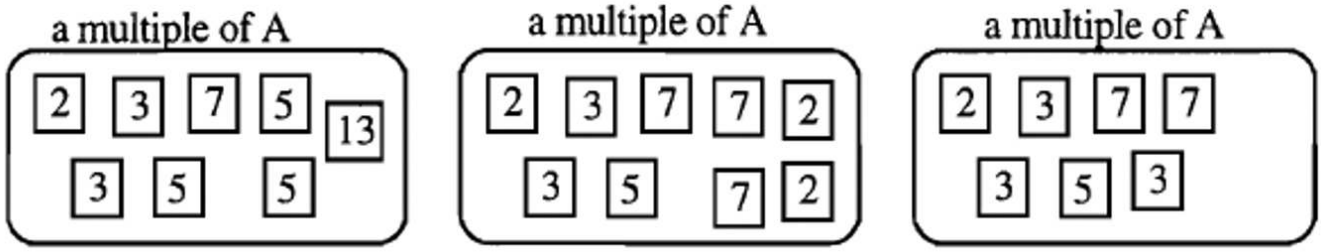
Below is the prime factorization of three numbers, A, B and C.



Below is how to write each of these numbers in prime factorization form:

$A = 2 \cdots 3^2 \cdots 5 \cdots 7$	$B = 2^3 \cdots 5 \cdots 7^2 \cdots 11$	$C = 2^2 \cdots 3^2 \cdots 5 \cdots 7 \cdots 11 \cdots 13$

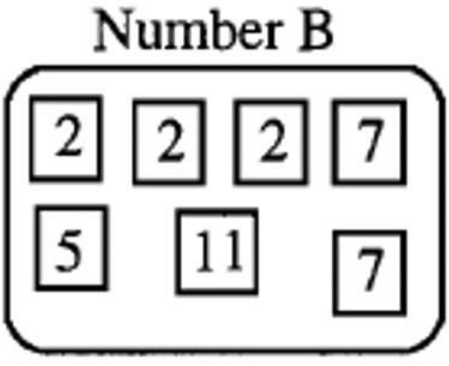
One way to make a multiple of A is to simply add extra factors to factors of A. *Note: A itself is a multiple of A, so you don't* ***have*** *to add any extra factors to get a multiple.* There are an infinite number of possibilities for multiples of A. Below are three multiples of A:



In prime factorization form, the first multiple of A above is written: $2 \cdots 3^2 \cdots 5^3 \cdots 7 \cdots 13$

? Exercise 3

Below is the prime factorization of a number, B



a. Form the number B with your prime number squares. Form a multiple of B by "throwing in" one or more prime number squares as factors to the original prime factorization of B shown above. Show a picture of the multiple of B that you formed.

Write the prime factorization of this multiple: _____

b. Form the number B with your prime number squares. Form a multiple of B, *different* from the one you formed in A, by "throwing in" one or more prime number squares as factors to the original prime factorization of B shown above. Show a picture of the multiple of B that you formed:

Write the prime factorization of this multiple of B: _____

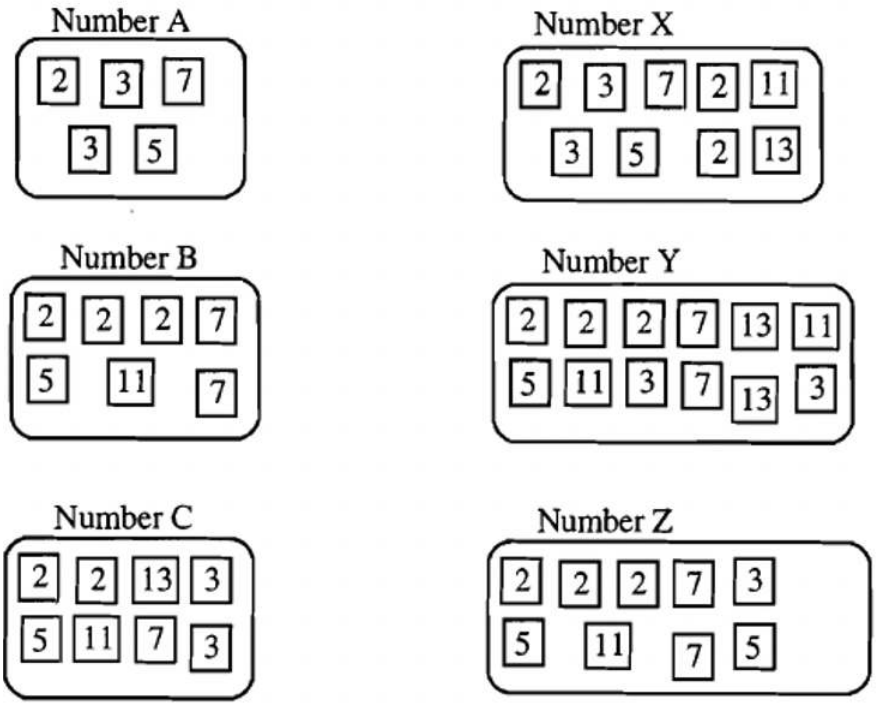
c. Form the number B with your prime number squares. Form another multiple of B by "throwing in" one or more prime number squares as factors to the original prime factorization of B shown above. Write the prime factorization of this multiple of B: _____

Number C is shown below toward the middle of the page. Use it to do exercise 4.

? Exercise 4

a. Form the number C with your prime number squares. Form a multiple of C by adding one or more prime number squares as factors. Write the prime factorization of the multiple of C you formed:
b. Form the number C with your prime number squares. Form a different multiple of C by adding one or more prime number squares as factors. Write the prime factorization of the multiple of C you formed:
c. Form the number C with your prime number squares. Form another multiple of C by adding one or more prime number squares as factors. Write the prime factorization of the multiple of C you formed:

NOTE: X is a multiple of M if M is a factor of X. So, you have a multiple of a number if the prime factors of the number itself are factors in the multiple. It's as if you can "see" the number in a multiple. We are going to determine if any of the numbers shown on the right are multiples of A, B or C.

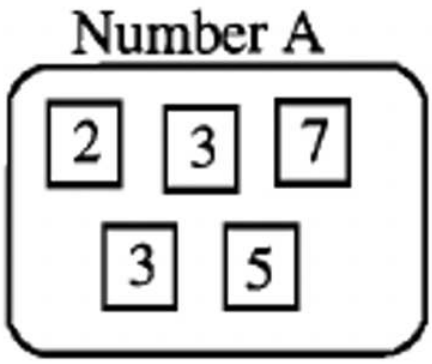


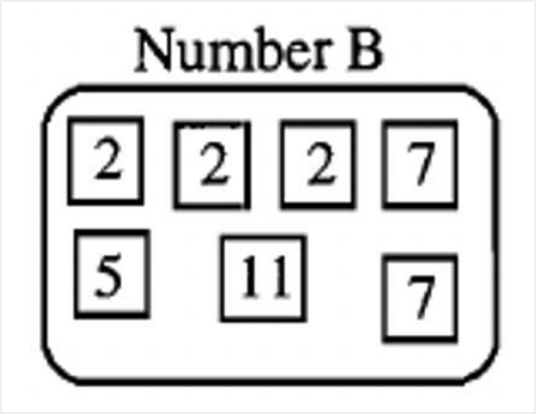
To decide if X, Y or Z is a multiple of A, see if each of X, Y or Z has the prime factors of A as a subset. In other words, A has one factor of 2, two factors of 3, one factor of 5 and one factor of 7. Any number containing those factors is a multiple of A. A multiple of A may have more factors of A, but can't be missing any factors of A.

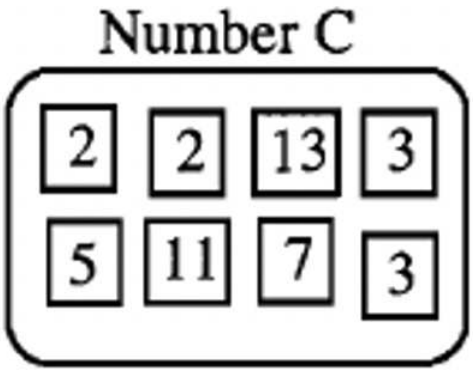
Exercise 5

a. List any number on the right (X, Y and/or Z) that is a multiple of A:
b. List any number on the right (X, Y and/or Z) that is a multiple of B:
c. List any number on the right (X, Y and/or Z) that is a multiple of C:

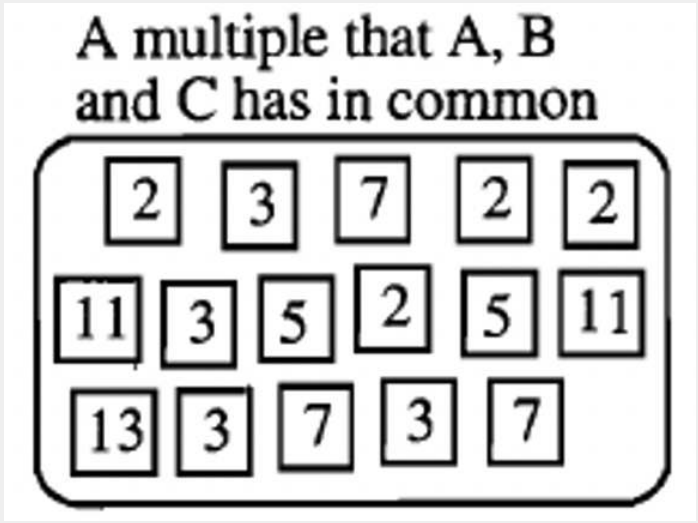
None of the numbers X, Y or Z was a multiple of all three of the numbers, A, B and C. Numbers A, B and C are shown below. Below that are only two examples of multiples that A, B and C have in common.

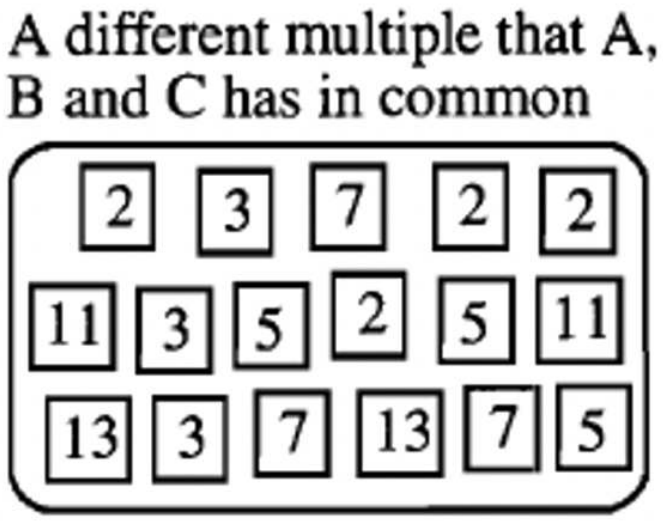






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? Exercise 6

Use prime number squares to form three other multiples that A, B and C have in common. Write the prime factorization of each of the multiples you formed.

a. _____

b. _____

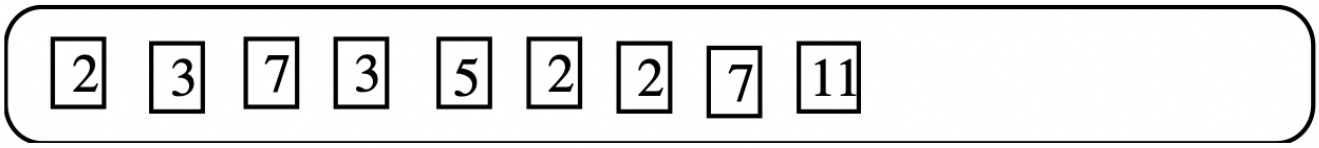
c. _____

Now, we are going to use the prime factorization to build the ***LEAST* COMMON MULTIPLE** of a set of numbers. The least common multiple is a multiple that the numbers have in common, but that has the least number of factors possible. The multiples formed above have 16 and 17 factors, respectively. To build a least common multiple, we only add in a factor if it is necessary.

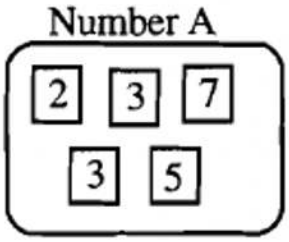
Let's say we want to build the least common multiple of A, B and C. In order to be a multiple of A, we need to have the factors of A. So, we start ***building*** the multiple by putting in the factors of A:

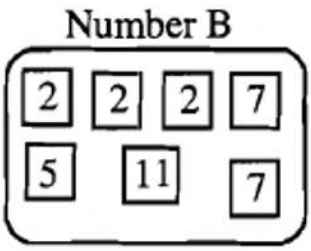


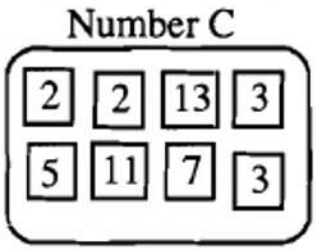
The factors of B must also be in the multiple. B has three factors of 2, but there is only one factor of 2 in the multiple so far. So, the common multiple will need two more factors of 2. B has two factors of 7, but there is only one factor of 7 in the multiple so far. So, the common multiple will need one more factor of 7. B has one factor of 5, which is already there. B also has one factor of 11. That is not there, so must be put in. Therefore, two factors of 2, one factor of 7 and a factor of 11 must be joined with the other factors in the common multiple. Once this is done, we have built the least common multiple of A and B:



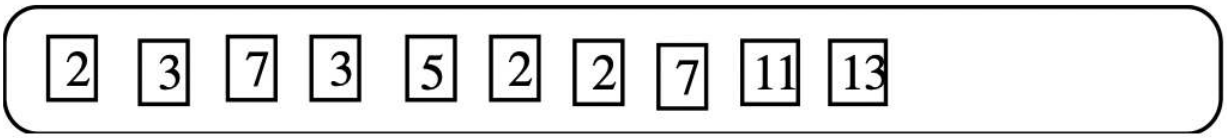
The prime factorizations of A, B and C are shown below for your convenience.







We have built the least common multiple of A and B. To find the least common multiple of A, B and C, we need to make sure the factors of C are also in the multiple. C has two factors of 2, which are already in the multiple, two factors of 3, which are already in the multiple, one factor each of 5, 7 and 11, each of which is already in the multiple, and a factor of 13, which is not in the multiple. So 13 is the only factor that needs to be joined with the factors already in the multiple we are building.



The above is a multiple of A, B and C. We only have we built a multiple, we actually built the **LEAST** COMMON MULTIPLE of A, B and C. Note it is a multiple of A, B *and* C; therefore it is a *common* multiple. It is the *least* common multiple because if any of the factors were removed, it wouldn't be a multiple of one of the numbers. For instance, if a factor of 2 was removed, it would not be a multiple of A. If a factor of 3 was removed, it would not be a multiple of A or C. If a factor of 7 was removed, it would not be a multiple of B, etc. If a factor of 11 was removed, it would not be a multiple of B or C. If a factor of 13 was removed, it would not be a multiple of C.

We write: LCM(A, B, C) = $2^3 \cdots 3^2 \cdots 5 \cdots 7^2 \cdots 11 \cdots 13$

? Exercise 7

Build the least common multiple of A and B using prime number squares. Then, write the prime factorization of the least common multiple of A and B. Let $A = 2^2 \cdots 3^5 \cdots 5 \cdots 13$ and $B = 2^2 \cdots 3^2 \cdots 7^3 \cdots 11^2 \cdots 13$

LCM(A, B) = ____

Exercise 8

Build the least common multiple of A, B and C using prime number squares. Then, write the prime factorization of the least common multiple of A and B.

Let A = $2^2 \cdots 11 \cdots 19$ and $B = 2 \cdots 3^2 \cdots 7 \cdots 11^2$ and $C = 2^2 \cdots 3^4 \cdots 7^3 \cdots 13^2 \cdots 19$

LCM(A, B, C) = _____

Exercise 9

Build the least common multiple of A, B and C using prime number squares. Then, write the prime factorization of the least common multiple of A and B.

Let $A = 2 \cdots 5^3 \cdots 11 \cdots 19$ and $B = 2^4 \cdots 3^6 \cdots 5^2 \cdots 23^2$ and $C = 5^4 \cdots 7^6 \cdots 11^2 \cdots 23$

LCM(A, B, C) = _____

Look back at the answer from exercise 9.

Let $A = 2 \cdots 5^3 \cdots 11 \cdots 19$ and $B = 2 \cdots 3^6 \cdots 5^2 \cdots 23^2$ and $C = 5^4 \cdots 7^6 \cdots 11^2 \cdots 23$

Answer: LCM(A, B, C) = $2 \cdots 3^6 \cdots 5^4 \cdots 7^6 \cdots 11^2 \cdots 19 \cdots 23^2$

If the numbers are prime factored using exponents, then the least common multiple contains each of the prime factors shown in any of the numbers. The exponent on each of the prime numbers is the highest exponent found on that prime number factor in the prime factorization of the numbers.

For instance, the prime numbers in the prime factorization of A, B and C are 2, 3, 5, 7, 11, 19 and 23. So begin by writing the product of these prime numbers: $2 \cdots 3 \cdots 5 \cdots 7 \cdots 11 \cdots 19 \cdots 23$

2 is a factor in A and B. The highest power of 2 found in either prime factorization of A and B is 1. 3 is a factor found in B, as 3^6. So, the highest power of 3 is 6. 5 is a factor of all three numbers. In A, the exponent on 5 is 3; in B, the exponent on 5 is 2; in C, the exponent on 5 is 4. So, the highest power of 5 is 4. Similarly, do the same for all of the other prime factors. Write the highest exponent on the factor.

Therefore, LCM(A, B, C) = $2 \cdots 3^6 \cdots 5^4 \cdots 7^6 \cdots 11^2 \cdots 19 \cdots 23^2$

Exercise 10

Find the greatest common factor and least common multiple for each set of numbers, written in prime factorization. Assume a, b, c, d, and e are different prime numbers.

a. Let $A = 2^2 \cdots 3^5 \cdots 5 \cdots 13$ and $B = 2^2 \cdots 3^2 \cdots 7^3 \cdots 11^2 \cdots 13$

GCF(A, B) = _____

LCM(A, B) = _____

b. Let $A = 2^2 \cdots 11 \cdots 19$ and $B = 2 \cdots 3^2 \cdots 7 \cdots 11^2$ and $C = 2^2 \cdots 3^4 \cdots 7^3 \cdots 13^2 \cdots 19$

GCF(A, B, C) = _____

LCM(A, B, C) = _____

c. Let $X = a^5 \cdots b^4 \cdots c^5 \cdots d$ and $Y = b^2 \cdots c^3 \cdots d^2 \cdots e^2$ and $Z = a^2 \cdots c^4 \cdots d^3 \cdots e^2$

GCF(X, Y, Z) = _____

LCM(X, Y, Z) = _____

d. Let $X = a^6 \cdots b^3 \cdots c \cdots d$ and $Y = b^4 \cdots c^4 \cdots d^3 \cdots e^7$ and $Z = a^2 \cdots c^4 \cdots d^2 \cdots e^3$

GCF(X, Y, Z) = _____

LCM(X, Y, Z) = _____

So far, the prime factorization of numbers has been given, and all you had to do was to build the prime factorization of the least common multiple. Now, your job will be to first prime factor numbers; then you can build the least common multiple from the prime factorizations. In the end, the least common multiple is a number. Multiply the factors in the least common multiple to find the one number that is the least common multiple.

Example

Find the least common multiple of 15, 18 and 20

Solution

Prime factor each of these numbers. $15 = 3 \cdot \cdot \cdot 5, 18 = 2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 3$, and $20 = 2 \cdot \cdot \cdot 2 \cdot \cdot \cdot 5$

Build the least common multiple by first "throwing in" the prime factors that make up the prime factorization of 15; then "throw in" any prime factors needed for 18; next, "throw in" any prime factors needed for 20. We get

LCM(15, 18, 20) = $3 \cdot \cdot \cdot 5 \cdot \cdot \cdot 2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 2 = 2^2 \cdot \cdot \cdot 3^2 \cdot \cdot \cdot 5$= **180**

Exercise 11

Find the least common multiples. Show the prime factorization of each number, and how you use it to build the least common multiple.

a. LCM (6, 8, 10) = _____

b. LCM (25, 35, 40) = _____

c. LCM (49, 91, 26) = _____

d. LCM(56, 24, 30) = _____

e. LCM(22, 34, 55) = _____

Note how easy it is to find the least common multiple of a larger set of numbers using prime factorization. We'll find: LCM(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14)

We don't need to expressly write down the prime factorization of the 12 numbers because it's easy enough to do that in our head. For instance, the prime factorization of 4 is $2 \cdot \cdot \cdot 2$, the prime factorization of 10 is $2 \cdot \cdot \cdot 5$, and the prime factorization of 13 is 13.

We build up the LCM of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14 by first making sure the prime factorization of 2 is there.

Step 1: Make sure the prime factorization 2 is there. **So far, LCM: 2**

Step 2: Make sure the prime factorization 3 is there. We need to "throw in" a 3 to the LCM. **So far, LCM: $2 \cdot \cdot \cdot 3$**

Step 3: Make sure the prime factorization 4 is there. We need to "throw in" a 2 to the LCM. **So far, LCM: $2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 2$**

Step 4: Make sure the prime factorization 5 is there. We need to "throw in" a 5 to the LCM. **So far, LCM: $2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 2 \cdot \cdot \cdot 5$**

Step 5: Make sure the prime factorization 6 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM:** $2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 2 \cdot \cdot \cdot 5$

Step 6: Make sure the prime factorization 7 is there. We need to "throw in" a 7 to the LCM. **So far, LCM: $2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 2 \cdot \cdot \cdot 5 \cdot \cdot \cdot 7$**

Step 7: Make sure the prime factorization 8 is there. We need to "throw in" a 2 to the LCM. **So far, LCM: $2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 2 \cdot \cdot \cdot 5 \cdot \cdot \cdot 7 \cdot \cdot \cdot 2$**

Step 8: Make sure the prime factorization 9 is there. We need to "throw in" a 3 to the LCM. **So far, LCM: $2 \cdot \cdot \cdot 3 \cdot \cdot \cdot 2 \cdot \cdot \cdot 5 \cdot \cdot \cdot 7 \cdot \cdot \cdot 2 \cdot \cdot \cdot 3$**

Step 9: Make sure the prime factorization 10 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM: $2 \cdot 3 \cdot 2 \cdot 5 \cdot 7 \cdot 2 \cdot 3$**

Step 10: Make sure the prime factorization 11 is there. We need to "throw in" an 11 to the LCM. **So far, LCM: $2 \cdot 3 \cdot 2 \cdot 5 \cdot 7 \cdot 2 \cdot 3 \cdot 11$**

Step 11: Make sure the prime factorization 12 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM: $2 \cdot 3 \cdot 2 \cdot 5 \cdot 7 \cdot 2 \cdot 3 \cdot 11$**

Step 12: Make sure the prime factorization 13 is there. We need to "throw in" a 13 to the LCM. **So far, LCM: $2 \cdot 3 \cdot 2 \cdot 5 \cdot 7 \cdot 2 \cdot 3 \cdot 11 \cdot 13$**

Step 13: Make sure the prime factorization 14 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM: $2 \cdot 3 \cdot 2 \cdot 5 \cdot 7 \cdot 2 \cdot 3 \cdot 11 \cdot 13$**

That's it! You have now built up the LEAST common multiple of all 12 numbers! You can check to make sure that the prime factorization of each of the 12 numbers is indeed in the prime factorization of the least common multiple you built. Furthermore, if any of the prime factors were removed, it wouldn't be a multiple of all 12 numbers. Therefore, the prime factorization of the LCM is as shown below:

LCM(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14) = $2 \cdot 3 \cdot 2 \cdot 5 \cdot 7 \cdot 2 \cdot 3 \cdot 11 \cdot 13$

If these factors are multiplied together, LCM(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14) = 360, 360.

The prime factorization of the LCM may have been written with the factors in ascending order using exponents: $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$

? Exercise 12

Use prime factorization to build the least common multiple. Show the prime factorization as the number is built. Then, multiply the factors to find the answer.

a. LCM(3, 4, 6, 7, 9, 10, 12, 14, 15, 18, 20) = _____ = _____

b. LCM(2, 3, 9, 11, 14, 15, 16, 17, 18, 22) = _____ = _____

c. LCM(5, 6, 7, 8, 12, 14, 15, 17, 18, 25) = _____ = _____

d. LCM(15, 18, 20, 25, 30, 35, 42, 45) = _____ = _____

Let's say you know the greatest common factor of 165 and some other number was 3, and the least common multiple of the same two numbers was 15,015. How would you figure out what the other number was?

First, it's a good idea to write down what you know. Let N be the other number.

Then, GCF(165, N) = 3 and LCM (165, N) = 15,015.

Since 3 is a factor, and in fact the greatest common factor, of both 165 and N, then each number can be written as 3 times something, and the other factor you obtain for 165 should be relatively prime to the other factor you obtain for N, *since 3 is the **greatest** common factor of 165 and N.*

So, let's rewrite, LCM (165,N) = 15,015 like this: LCM (3×55, $3 \times$ _____) = 15,015

In other words, I know N = $3 \times$ _____, but I have to figure out what goes in the blank to figure out what N is. If you want to introduce another variable, like M, instead of writing a blank, that works just as well. It's up to you.

To find the LCM of 3×55 and $3 \times$ _____ , where the 55 and the number on the blank have no common factors, you would multiply $3 \times 55 \times$ _____. But we know the product should be 15,015. So, the number that must go on the blank must be 91, since $3 \times 55 \times 91 = 15,015$. So, now we can figure out what N is: $N = 3 \times 91 = 273$.

Let's see if this makes sense by first rewriting 165 and 273 either in prime factored form or as the GCF(165,273) times something; then we'll figure out the GCF and LCM from the factored form, and see if it agrees with our original problem.

$165 = 3 \times 55$ and 273 = 3×91. First, make sure 3 is really a factor of each number!

Now, convince yourself that 3 really is the **greatest** common factor of 165 and 273 by checking to see that the other factor of one number is relatively prime (no factors in common) with the other factor of the other number. So, ask yourself if 55 and 91 have any factors in common (besides 1) if you aren't sure, then prime factor each of these numbers first ($55 = 5 \times 11$ and $91 = 7 \times 13$); note that they have no factors in common.

Since 3 is the greatest common factor, then the least common multiple is obtained by multiplying $3 \times 55 \times 91$, which is 15,015 which is what it should be according to the original information given. Therefore, 273 really was the number we were looking for.

N = *273*

There is another way to do the above problem using the following property:

For any *two* numbers, m and n, the following is always true:

$m \times n = GCF(m, n) \times LCM(m, n)$

This equation is *not true* for more than two numbers $r \times s \times t \neq GCF(r, s, t) \times LCM(r, s, t)$

Example

Verify the above property for numbers 15 and 18.

Since GCF(15, 18) = 3 and LCM(15, 18) = 90, we want to verify that the product of the two numbers, 15 and 18, equals the product of the GCF and LCM of the two numbers.

$15 \cdots 18 = 270$; GCF(15, 18) $\cdots$ LCM(15, 18) = $3 \cdots 90 = 270$

Therefore, $15 \cdots 18 =$ GCF(15, 18) $\cdots$ LCM(15, 18)

Example

Show that the above property does not work for three numbers.

Counterexample: Use the numbers 4, 6 and 8. GCF(4, 6, 8) = 2 and LCM(4, 6, 8) = 24

$4 \cdots 6 \cdots 8 = 192$; GCF(4, 6, 8) $\cdots$ LCM(4, 6, 8) = $2 \cdots 24 = 48$

Clearly, $4 \cdots 6 \cdots 8 \neq$ GCF(4, 6, 8) $\cdots$ LCM(4, 6, 8)

Example

Find N if GCF(165, N) = 3 and LCM (165, N) = 15,015

In this case, the two numbers are 165 and N, the GCF is 3 and the LCM is 15,015. Plug these values into the equation shown above in bold.

$165N = 3 \times 15,015$. Divide both sides by 165 to find N: N = $\frac{3 \times 15015}{165} = 273$

The property can also be used to find the LCM of two numbers if you know the GCF. For instance, if you were asked to find the GCF and LCM of 24 and 30. You can easily find the GCF, which is 6. To find the LCM, multiply the two numbers together, and divide by the GCF. (This should make sense to you intuitively if you think about it: You wouldn't list the GCF twice as you build the LCM. Also, the GCF will cancel into either of the two numbers since it is a factor of each.) So, the LCM(24, 30) = $(24 \cdot 30) \div 6 = 120$.

Exercise 13

If GCF(1176, 288) = 24, find the LCM(1176, 288)

Exercise 14

Find X if GCF(2940, X) = 105 and LCM(2940, X) = 79,380

Exercise 15

If the greatest common factor of 3,211 and another number is 247 and the least common multiple of the same two numbers is 48,165, then what is the other number?

Exercise 16

If you aren't given at least one of the numbers, there might be more that one possible solution. Figure out the possibilities for a and b if all you know is GCF(a, b) = 2 and LCM(a, b) = 20

The next few problems are a review of GCF and LCM. You now have several methods you can use to find the GCF and LCM.

17-20 Find the greatest common factor of each of the following pairs of numbers using prime factorization, the Old Chinese Method or the Euclidean Algorithm. Then, find the LCM of each pair. Show all work.

Exercise 17

a. GCF (693, 546) = _____

b. LCM (693, 546) = _____

Exercise 18

a. GCF (2117, 2555) = _____

b. LCM (2117, 2555) =

Exercise 19

a. GCF (1369, 10693) = _____

b. LCM (1369, 10693) =

Exercise 20

a. GCF (24300, 14406) = _____

b. LCM (24300, 14406) =

Exercise 21

Make up a problem where the GCF of 2 different 3-digit numbers is 32. For instance, GCF (x, y) = 32. Find an x and y that will work.

Exercise 22

Make up a problem where the GCF of two different 4-digit numbers is 32. In other words, find 2 numbers, a and b, such that GCF (a, b) = 32.

? Exercise 23

Make up a problem where the GCF of 2 different 3-digit numbers is 35. For instance, GCF (x, y) = 35. Find an x and y that will work.

? Exercise 24

Make up a problem where the GCF of two different 4-digit numbers is 28. In other words, find 2 numbers, a and b, such that GCF (a, b) = 28.

All even numbers are multiples of 2. Therefore, every even number can be written as the product of 2 and an integer. Symbolically, we can write every even number can be written in the form: 2k, where k is an integer. For example, 12 = 2(6), 20 = 2(10), 58 = 2(29), etc.

All odd numbers can be written as one more than an even number. Since an even number can be written as 2k, then symbolically, we write that every odd number can be written in the form: 2k + 1, where k is an integer. For example, 15 = 2(7) + 1, 41 = 2(20) + 1, etc.

Any integer that can be written in the form 2k is even and any integer that cannot be written in the form 2k is not even. Any integer that can be written in the form 2k + 1 is odd and any integer that cannot be written in the form 2k + 1 is not odd.

We'll be doing some proofs about even and odd numbers. The only way to express an even number, in general, is by 2k. The variable can be any letter. If you want to express more than one even number, you must use a new variable like 2m or 2n. The same is true for odd numbers.

For the examples and exercises that follow, assume all variables are integers.

EXAMPLES: State which of the following always represents an even number, which of the following always represents an odd number, and which are sometimes even and sometimes odd.

✓ Example 1: 6n+14

6n + 14

Solution

6n + 14 = 2(3n + 7), which is in the form of an even number. Therefore, 6n + 14 will always represent an even number (since n is an integer).

✓ Example 2: 4n+23

4n + 23

Solution

4n + 22 + 1 = 2(2n + 11) + 1, which is in the form of an odd number. Therefore, 6n + 14 will always represent an odd number.

✓ Example 3: 5n+2

5n + 2

Solution

It is impossible to write 5n + 2 in the form 2k or 2k + 1. Therefore, it can't be determined. It is sometimes even and sometimes odd.

Note: You can verify these answers by plugging in an even number for n and then an odd number for n, and see if the conclusion makes sense. In example 1, if n = 2, then 6n + 14 = 6(2) + 14 = 28 (even). If n = 3, then 6(3) + 14 = 32 (even). So, the result was even when n was replaced by either an even or an odd number. Do the same verification for example 2 and 3 in the space below:

? Exercise 25

State which of the following always represents an even number, which of the following always represents an odd number, and which are sometimes even and sometimes odd. Justify your answer.

a. 8n + 20

b. 10k + 9

c. 5x + 2

✔ Example 1

Formally prove that the sum of two odd numbers is even.

IMPORTANT Note: Don't define the two odd numbers to be the same odd number. You must be general, and assume they may be two different odd numbers! Use different variables to be the most general.

Solution

Let 2n+1 = one odd number, and let 2m+1 = another odd number.

The sum is: 2n+1 + 2m+1= 2n+2m+2 = 2(n+m+1), which is in the form of an even number. Therefore, the sum of 2 odd numbers is even.

✔ Example 2

Formally prove that the product of two odd numbers is odd.

Solution

Let 2n+1 = one odd number, and let 2m+1 = another odd number.

The sum is: (2n+1) (2m+1)= 4nm + 2m + 2n + 1 = 2(2nm + n + m) +1, which is in the form of an odd number. Therefore, the product of 2 odd numbers is odd.

? Exercise 26

Formally prove that the sum of two even numbers is even.

? Exercise 27

Formally prove that the sum of two odd numbers is even.

? Exercise 28

Formally prove that the sum of an even number and an odd number is odd.

? Exercise 29

Formally prove that the product of two even numbers is even.

? Exercise 30

Formally prove that the product of two odd numbers is odd.

Exercise 31

Formally prove that the product of an even number and an odd number is even.

We are going to explore a way to find the sum of several consecutive whole numbers.

Example 1

1 + 2 + 3 + 4 + 5 + 6

Example 2

The sum of the first 100 counting numbers would take a long time to write out and compute. We write it in the following way:
1 + 2 + 3 + . . . + 98 + 99 + 100

Example 3

55 + 56 + 57 + . . . + 128 + 129 + 130

The first example is fairly easy to compute. But as more and more numbers are added, the computation becomes cumbersome. The first two examples start with the number, 1. Our first goal will be to find a pattern, and then a formula for adding any set of consecutive counting numbers starting with 1.

Even though we know the answer to example 1, we'll use that to find a pattern for the sum for other sums. Let X = the sum we are looking for (1 + 2 + 3 + 4 + 5 + 6). Notice what happens if X is written down twice – first in ascending order, then in descending order – and then the two rows are added by adding lined-up columns:

$$\begin{aligned} X &= 1+2+3+4+5+6 \\ +X &= 6+5+4+3+2+1 \\ 2X &= 7+7+7+7+7+7 \end{aligned}$$

The left side equals 2X, which is twice the actual sum we want. How many columns of numbers are on the right hand side of the equals sign? _____ Note: the number of columns is the same as the amount of numbers in the actual sum we are trying to find. What does each column on the right side of the equals sign add up to? _____ Since the right hand side of the equal sign is repeated addition, you can obtain the answer by using multiplication. What multiplication problem is this? _____ Since the left side equals the right side (2X = 42), then X = 21. Even if you didn't represent the sum using a variable, you would divide the right side by 2 because the sum is twice as large since the numbers in the sum were added twice.

Let's use this same technique to add the first 100 consecutive counting numbers. Let N = the actual sum. You fill in the third row by adding the left side, and then adding all of the columns on the right side.

$$\begin{aligned} N &= 1 + 2 + 3+\ldots+98+99+100 \\ +N &= 100 + 99 + 98+\ldots+3 +2+ 1 \end{aligned}$$

How many columns (it's the same answer as the amount of numbers in the sum) are there on the right? _____ Since each column adds up to the same number, this is repeated addition. State the number that the right side adds up to: _____ That is twice as big as the actual sum, so what does the sum (1 + 2 + 3 + . . . + 98 + 99 + 100) equal? _____

Hopefully, you got the answer of 5,050!

Exercise 32

Use the previous technique to find the sum of the first 80 counting numbers. Show work

Let's try to find a formula for finding the first n counting numbers using the same technique. Before writing the sum, what is the counting number preceding n? _____ What is the counting number preceding that number? _____

Let X = the sum of the first n counting numbers. Below, we'll use the same techique to find the sum. Fill in the sum on the left side of the equals side and the sum of each column on the right.

What does each column on the right add up to? _____ Again, the right side is a repeated addition again. Multiply the number of columns (which is equal to the amount of numbers in the sum of the first n counting numbers) by the number each column adds up to.

What does the right side equal? _____ This is twice as large as the actual sum we want.

What does the actual sum of the first n counting numbers equal? _____

Hopefully, you got this answer: $\frac{n(n+1)}{2}$

Of course, the answer depends on n. Let's using this formula to compute the sum of the first six counting numbers, which was the first example we did. In this example, n = 6. So, plugging n = 6 into the formula, we get $(6 \cdot 7)$ / 2 = 21. Same answer!

Use the formula to find the sum of the first 80 counting numbers.

? Exercise 34

Use the formula to find this sum: 1 + 2 + 3 + . . . + 248 + 249 + 250.

What if you wanted to find this sum? 55 + 56 + 57 + . . . + 128 + 129 + 130

If you used the formula, then you would have the sum of all the numbers from 1 to 130 instead of only the ones from 55 on. One strategy is to use the formula and then subtract off the extra numbers you added. For instance, if you add all the numbers from 1 - 130, the extra numbers added that are not part of the sum are: 1 + 2 + 3 + . . . + 52 + 53 + 54. But this sum is easy to compute since we can use the formula to get this sum! So here is the strategy:

55 + 56 + 57 + . . . + 128 + 129 + 130

= (the sum of the first 130 counting numbers) – (the sum of the first 54 counting numbers)

= (1 + 2 + 3 + . . . + 128 + 129 + 130) – (1 + 2 + 3 + . . . + 52 + 53 + 54)

= $\frac{130(131)}{2} - \frac{54(55)}{2}$ = 8,515 - 1,485 = 7,030

✓ Example

Find the sum: 81 + 82 + 83 + . . . + 198 + 199 + 200

Solution

Subtract the sum of the first 80 counting numbers from the sum of the first 200 counting numbers: 200(201)/2 – 80(81)/2 = 20,100 – 3,240 = 16,860.

? Exercise 35

Find the following sums using the strategy just shown. Show work.

a. 51 + 52 + 53 + . . . + 98 + 99 + 100

b. 146 + 147 + 148 + . . . + 561 + 562 + 563

c. 500 + 501 + 502 + . . . + 798 + 799 + 800

There is a different strategy we can use to find the sum of consecutive whole numbers that do not begin with the number, 1. Let's look at another way to compute the sum of these whole numbers: 55 + 56 + 57 + . . . + 128 + 129 + 130. We'll begin by using the same type of strategy we used at the beginning of this topic. First, let X = the sum. Write the sum down twice – first in ascending

order, then in descending order – and then add the two rows by adding the individual lined-up columns. This is shown on the next page.

$$\begin{array}{rccccccc} X = & 55+ & 56+ & 57+ & \ldots+ & 128+ & 129+ & 130 \\ X = & \underline{130}+ & \underline{129}+ & \underline{128}+ & \ldots+ & \underline{57}+ & \underline{56}+\underline{55} & \\ 2X = & 185+ & 185+ & 185+ & \ldots+ & 185+ & 185+ & 185 \end{array}$$

At this point, we can see the similarity to how we derived the sum of the first n counting numbers, but there is one big difference here. It's not clear exactly how many addends of 185 are on the right-hand side of the equals sign. Most people will say that there are 130 – 55, or 75 of them, but actually that reasoning (subtracting the first number from the last number) is not correct. For instance, if you were adding the numbers from 1 to 130, most people will agree that there are 130 numbers being added, which is NOT the same answer you would get if you subtracted the first number from the last number since 130 – 1 only equals 129. Look at simpler example: 17 + 18 + 19 + 20. It's clear there are 4 numbers in the sum, but if you did the subtraction 20 – 17, you would get 3, which is the incorrect answer. One way to figure out how many numbers are in the sum is to decide how many numbers are missing from the sum if it did begin with 1. Look at the sum again: 17 + 18 + 19 + 20 has the first 16 numbers missing, so instead of 20 numbers in the sum (which is how many there would be if we started at 1), there are 20 – 16 numbers in the sum. Okay, back to figuring out how many numbers are actually in the sum above.

How many numbers are in the sum: 55 + 56 + 57 + . . . + 128 + 129 + 130? _____

So, the right side of the equation has 76 185's added together, which is 76 · · · 185, which equals 14,060. But, this is twice as big as the actual sum, so after dividing by 2, we get the actual sum of 7,030. This is the same answer we got when we did this problem on the previous page using a different strategy.

? Exercise 36

Find the following sums using the strategy just shown. Show work.

a. 51 + 52 + 53 + . . . + 98 + 99 + 100

b. 146 + 147 + 148 + . . . + 561 + 562 + 563

c. 500 + 501 + 502 + . . . + 798 + 799 + 800

d. Challenge: k + (k + 1) + (k + 2) + . . . + (n - 2) + (n - 1) + n

So far, you have learned how to easily find the sum of several consecutive whole numbers. Let's take it one step further. What if the sum you want to find are numbers that are not consecutive. Depending on the the sequence of numbers, the sum may or may not be easy to find. We are going to look at sums that have consecutive numbers added together in disguise. For instance, look at the following sum:

7 + 14 + 21 + . . . + 693 + 700

? Exercise 37

What do you notice about the numbers added together?

? Exercise 38

Hopefully, you noticed that all of the numbers were multiples of seven, or perhaps you noticed you add seven to each number to get the next number. Take the sum and rewrite the problem by factoring out a 7; just fill in the blanks below:

7 + 14 + 21 + . . .+ 693 + 700 = 7 (____ + ____ + ____ + . . . + ____ + ____)

? Exercise 39

Now, you should be able to figure out the sum of the numbers in parentheses. ***Show your work to figure out the sum***. Then answer a, b and c.

a. What is the sum of the numbers in parentheses? _____

b. So, the sum 7 + 14 + 21 + 28 + . . . + 700 becomes 7 _____

c. Therefore, 7 + 14 + 21 + 28 + . . . + 700 = _____

Exercise 40

Find the following sums. ***Show your work.***

a. 8 + 16 + 24 + . . . + 992 + 1000 = _____

b. 11 + 22 + 33 + . . . + 935+ 946 = _____

c. 20 + 40 + 60 + 80 + . . . + 2980 + 3000 = _____

Okay, just one more twist...this puts it all together. Consider this sum:

112 + 116 + 120 + . . . + 524 + 528

Exercise 41

This time, what do you notice about the numbers added together?

Exercise 42

Hopefully, you noticed all the numbers were multiples of four, or that you added four to each number to get the next number. Take the sum and rewrite the problem by factoring out a 4; just fill in the blanks below:

112 + 116 + 120 + . . . + 524 + 528 = 4 (____ + ____ + ____ + . . . + _____ + _____)

Exercise 43

Now, you should be able to figure out the sum of the numbers in parentheses. Note that the sum does not start with a 1. ***Show your work*** to figure out the sum. Then answer a, b and c.

a. What is the sum of the numbers in parentheses? _____

b. So, the sum 112 + 116 + 120 + . . . + 524 + 528 becomes 4 _____

c. Therefore, 112 + 116 + 120 + . . . + 524 + 528 = _____

Exercise 44

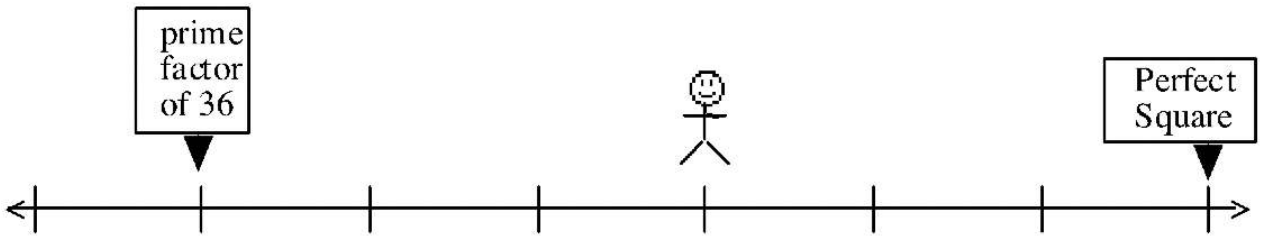
Find the following sums. ***Show your work.***

a. 85 + 90 + 95 + . . . + 735 + 740 = _____

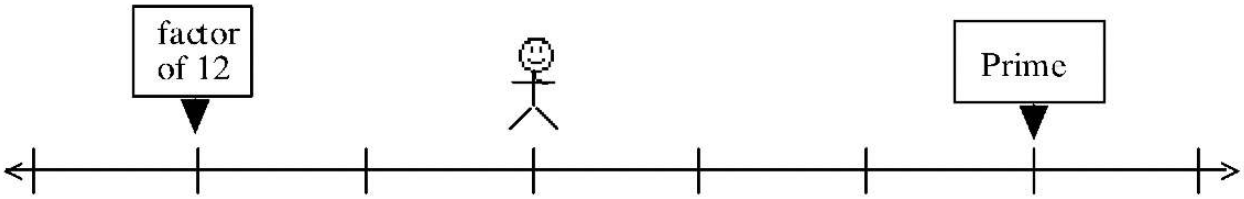
b. 430 + 473 + 516 + . . . + 2838+ 2881 = _____

Here are some problems for you to figure out what numbers are on the number line. For each problem, figure out where on the number line (what number) the man might be standing. Where there is more than one possibility, only list numbers between 1 and 1000.

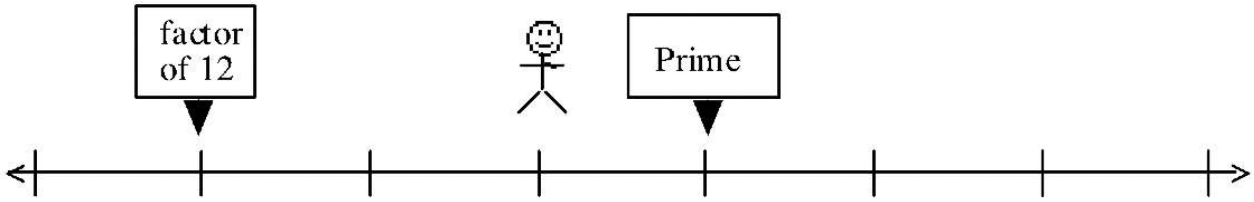
Exercise 45



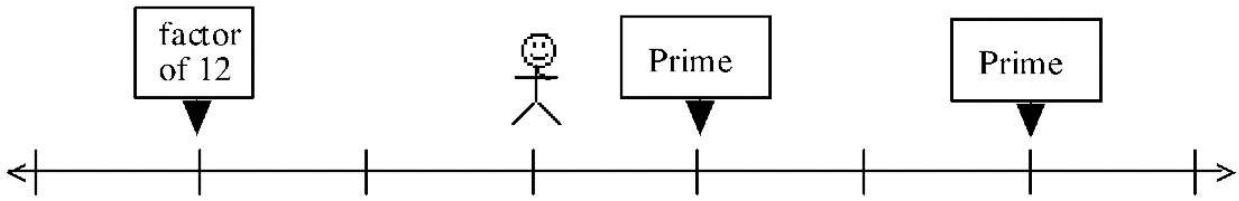
? Exercise 46



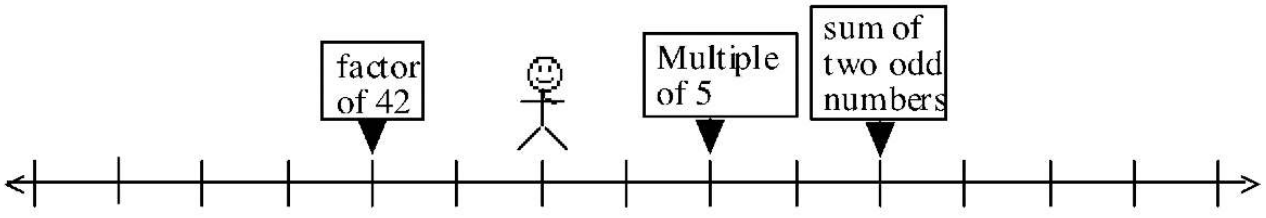
? Exercise 47



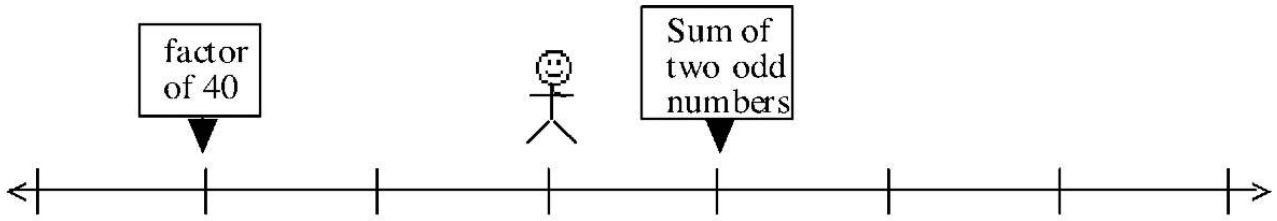
? Exercise 48



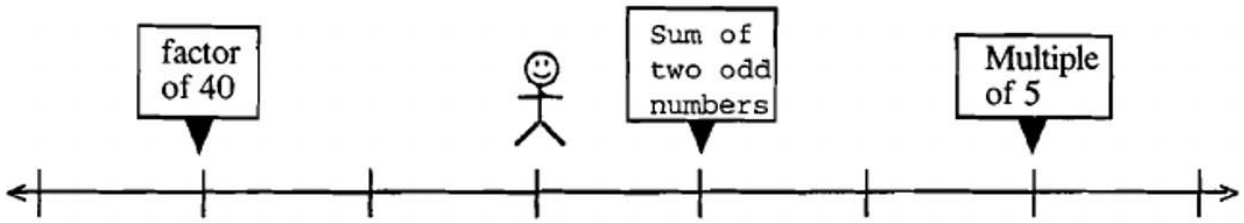
? Exercise 49



Exercise 50

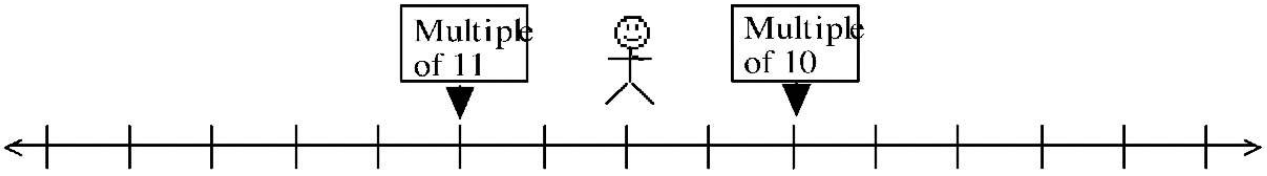


Exercise 51

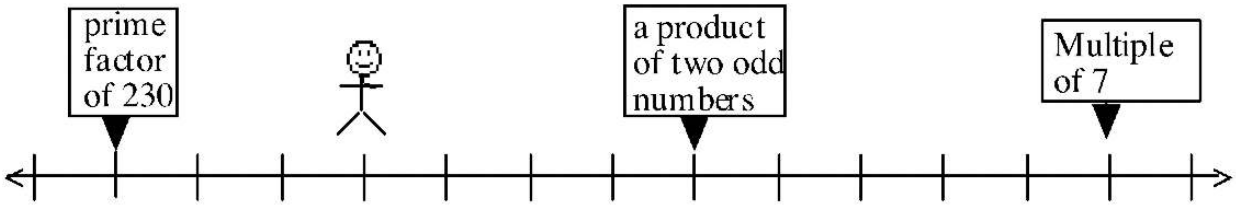


Exercise 52

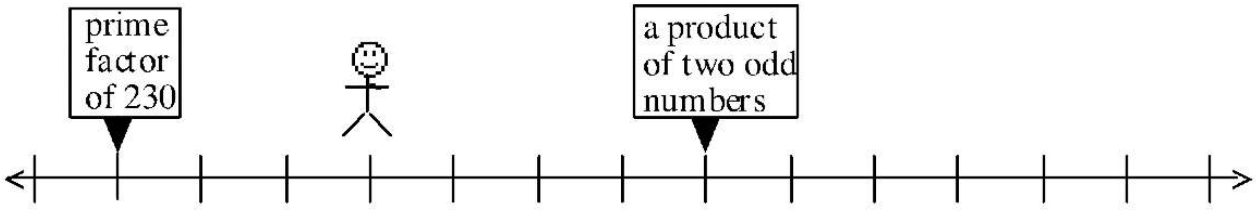
(There are actually nine possibilities between 1 and 1000 for this problem.)



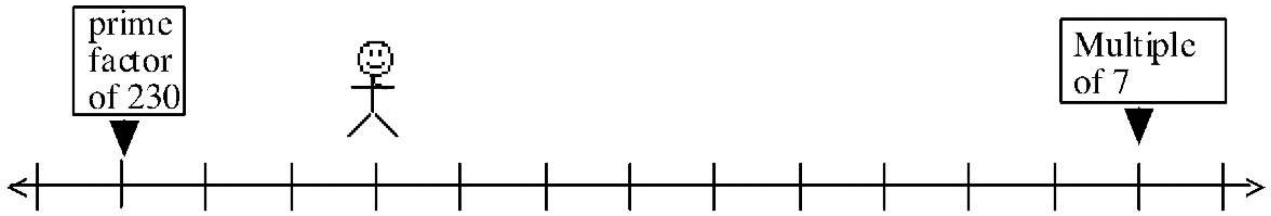
Exercise 53



Exercise 54



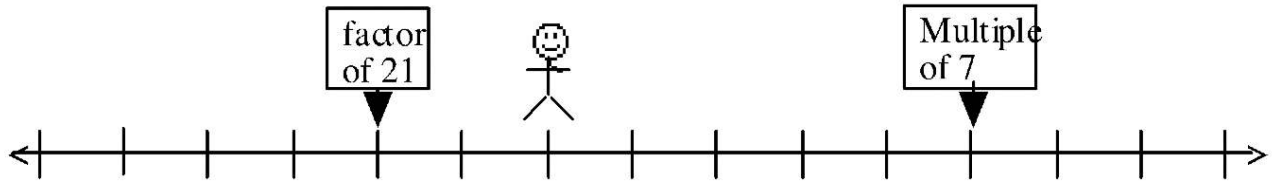
Exercise 55



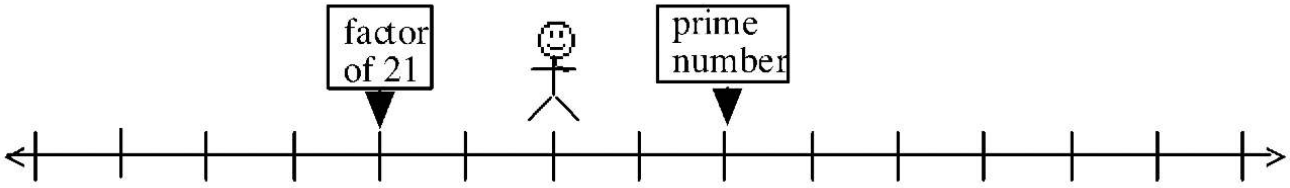
Exercise 56

Which of the clues in #53 was was not needed if the man is only standing on one single number? #54 and #55 should help you answer this. For instance, if you got more than one possibility for #54 or #55, then that problem didn't provide enough clues. For the one that gave exactly one possibility, that was enough clues, so the clue from #53 that was missing wasn't really needed.

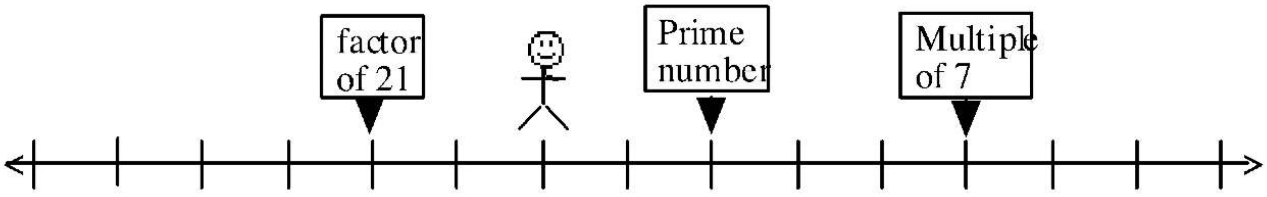
Exercise 57



Exercise 58



Exercise 59



Exercise 60

If the man in exercise #59 is standing on only one number in this problem, are all three clues given needed?

Yes or No:_____ Explain:

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6.7: The Least Common Multiple

Definition: Least Common Multiple (LCM)

The LCM is the smallest number that is a multiple of all numbers, excluding zero.

To find the LCM, we **work backwards**, following the three steps below.

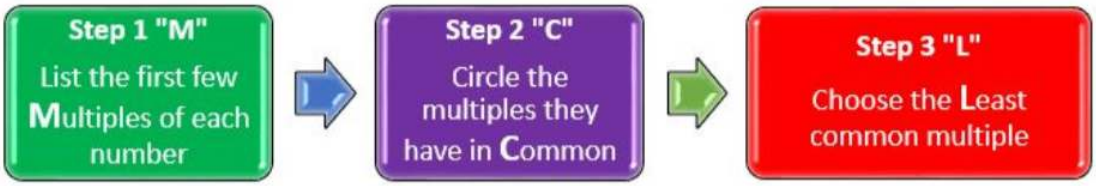


Figure 5.5.1: Steps to calculate LCM

Example 6.7.1

Find the LCM of 54 and 30

Solution

Step 1: List the Multiples

54 → 54, 108, 162, 216, 270, 324,…

30 → 30, 60, 90, 120, 150, 180, 210, 240, 270, 300,…

Step 2: Circle the Common multiple(s)

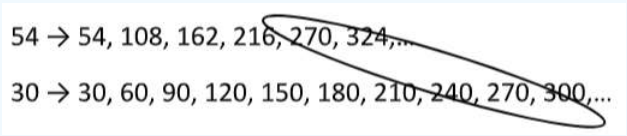


Figure 5.5.2

Step 3: Choose the Least common multiple = 270

Partner Activity 1

A recipe for peanut butter cookies will make 15 cookies. A recipe for chocolate cookies will make two dozen cookies. If you want to have the same number of each type of cookie, what is the least number of each that you will need to make using complete recipes?



Figure 5.5.3: Peanut butter cookies

Example 6.7.2

Add (and Subtract) Fractions Using the LCM and GCF

Solution



Figure 5.5.4

$$\frac{11}{12}+\frac{7}{20}=\frac{11\times5}{12\times5}+\frac{7\times3}{20\times3}=\frac{55}{60}+\frac{21}{60}=\frac{55+21}{60}=\frac{76}{60}=\frac{76\div4}{60\div4}=\frac{19}{15}$$

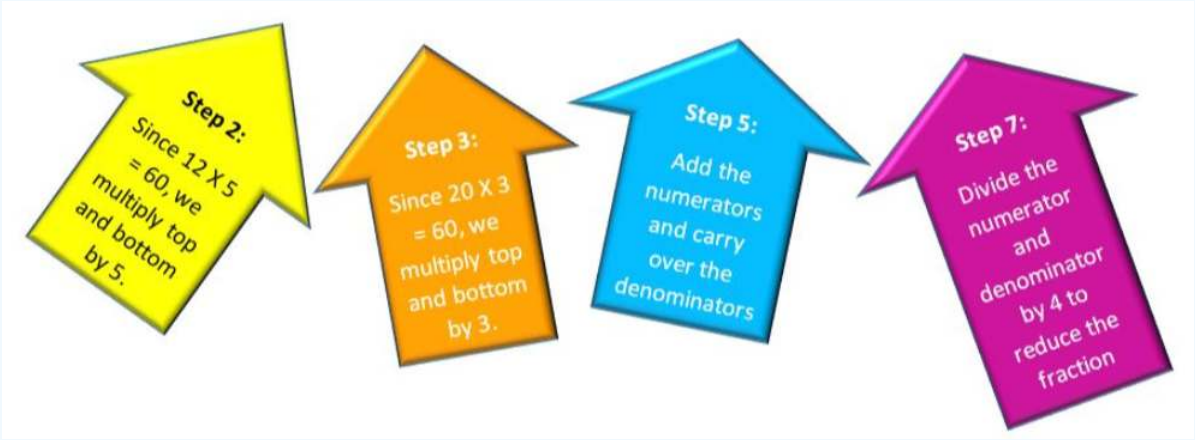


Figure 5.5.5

Partner Activity 2

Add or subtract the following fractions:

1. $\frac{5}{6}-\frac{3}{4}$
2. $\frac{2}{3}-\frac{1}{2}$
3. $6\frac{4}{5}+7\frac{1}{5}$

Practice Problems

Find the Least Common Multiple (LCM).

1. 40 and 90
2. 75 and 25
3. 168 and 85
4. 90, 120, and 150
5. 135, 225, and 405

Add or Subtract fractions.

6. $\frac{7}{12}+\frac{8}{5}$
7. $\frac{3}{16}-\frac{2}{9}$
8. $\frac{34}{26}+\frac{10}{13}$

Extension: Methods of Teaching Mathematics

Part 1

Using the standard and topic you choose from earlier this semester, write a full 45-minute lesson plan. See Canvas for more detailed instructions.

Part 2

Make sure you are working on Khan Academy throughout the semester.

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6.8: Homework

- Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)
- Start a new module on the front side of a new page and write the module number on the top center of the page.
- Answers without supporting work will receive no credit.
- Some solutions are given in the solutions manual.
- You may work with classmates but do your own work.

? HW #1

Find GCF(252, 350) using:

a. prime factorization;
b. Old Chinese Method
c. Euclidean Algorithm
d. Compute LCM(252,350)

? HW #2

Find GCF(140, 315) using:

a. prime factorization;
b. Old Chinese Method
c. Euclidean Algorithm
d. Compute LCM(252,350)

? HW #3

Use the Euclidean Algorithm to compute the greatest common factor of the numbers given. Use correct notation, and show each step. Then, show how you check your answer. Also, compute the LCM of the two numbers.

a. GCF(3525, 658) LCM(3525, 658)
b. GCF(1075, 1548) LCM(1075, 1548)

? HW #4

a. If 6|482__354 What digits could go on the blank? Explain.
b. If 6|482354__ What digits could go on the blank? Explain.

? HW #5

State whether each of the following statements is true or false. If it is false, provide a counterexample. If it is true, provide an example.

a. If (a + b)|c, then a|c and b|c

b. If a|b and a|c, then a|(bc)

c. If a|b and a|(b + c), then a|c

d. If a|bc, then a|b and a|c

e. If a|b and a|c, then a|(b + c)

HW #6

Write the prime factorization for the following numbers. If it is prime, write "prime" and explain how you know it is prime.

a. 371	b. 429	c. 197	d. 287

HW #7

Assume m and n are composite whole numbers in each of the following. Find the following. Then provide an example using numbers for m (and n where used). Remember not to use prime numbers in your example.

a. GCF(m,m) =

b. LCM(m,m) =

c. GCF(m,0) =

d. GCF(m,1) =

e. If GCF(m,n) = 1, then LCM(m,n) =

f. If GCF(m,n) = m, then LCM(m,n) =

g. If LCM(m,n) = mn, then GCF(m,n) =

HW #8

a. Formally prove that the sum of two odd numbers is even.
b. Formally prove that the product of two odd numbers is odd.

HW #9

Find the following sums using methods from this module: Show all work

a. 1 + 2 + 3 + . . . + 313 + 314 + 315 =

b. 111 + 112 + 113 + . . . + 287 + 288 + 289 =

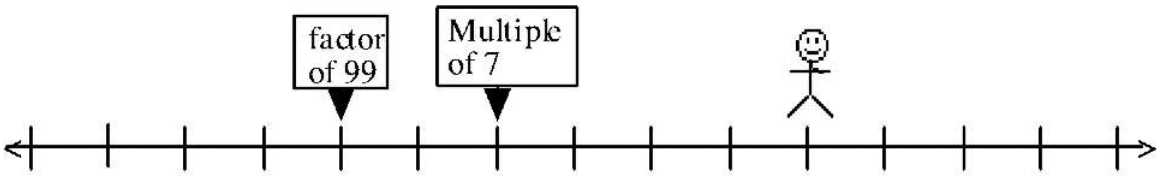
c. 15 + 30 + 45 + . . . + 900 + 915 + 930 =

d. 102 + 105 + 108 + . . . + 300 + 303 + 306 =

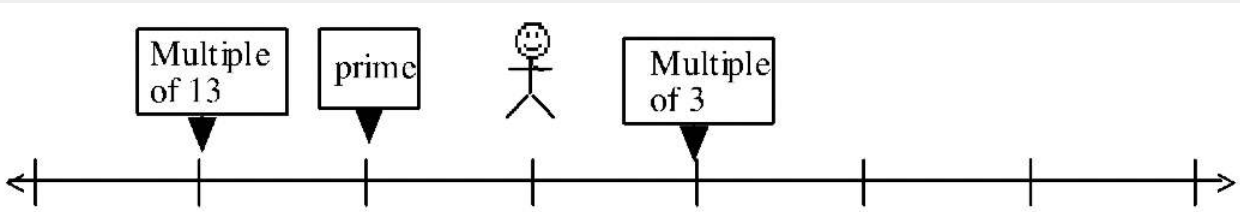
HW #10

On each number line, state all whole number possibilities less than 100 that the man could be standing on.

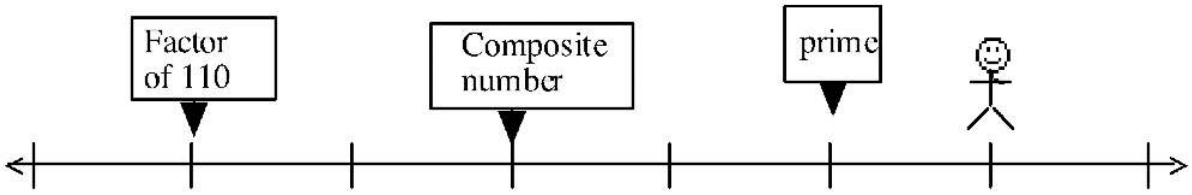
a.



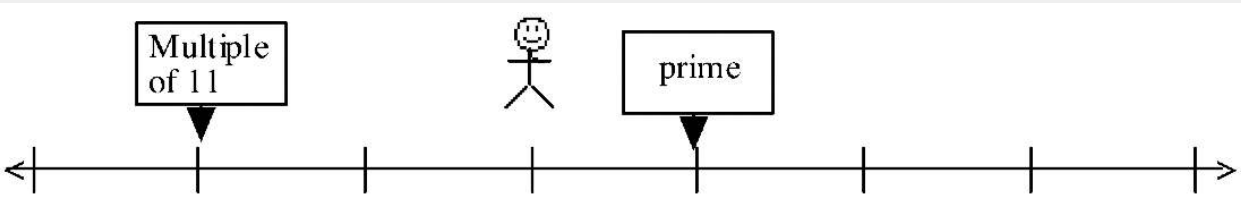
b.



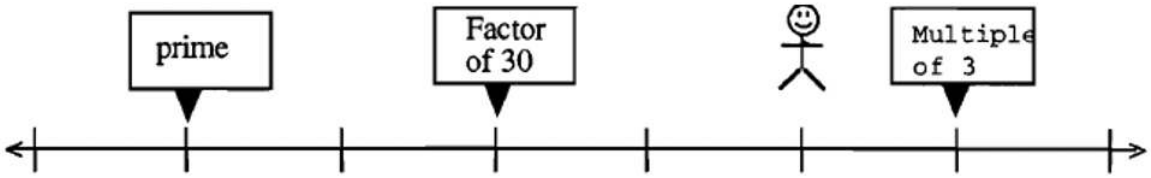
c.



d.



e.



HW #11

The factors of a number that are less than the number itself are called *proper factors*. For instance, the proper factors of 10 are 1, 2 and 5. A number is classified as deficient if the sum of its proper factors is less than the number itself. 10 is a deficient number since $1 + 2 + 5 < 10$. A number is classified as abundant if the sum of its proper factors is greater than the number itself. For instance, the proper factors of 18 are 1, 2, 3, 6, and 9. 18 is a deficient number since $1 + 2 + 3 + 6 + 9 > 18$. A number is classified as perfect if the sum of its proper factors equals the number itself. For each number, list its proper factors. Then find the sum of its proper factors. Then, classify each number as deficient, abundant or perfect.

a. Proper factors of 6: _____ ; Sum: _____ ; Classification:

b. Proper factors of 7: _____ ; Sum: _____ ; Classification:

c. Proper factors of 8: _____ ; Sum: _____ ; Classification:

d. Proper factors of 9: _____ ; Sum: _____ ; Classification:

e. Proper factors of 11: _____ ; Sum: _____ ; Classification:

f. Proper factors of 12: _____ ; Sum: _____ ; Classification:

? HW #12

Are prime numbers deficient, perfect, or abundant? _____ Explain why.

? HW #13

Answer true or false for each of the following. If it is true, provide an example. If it is false, provide a counterexample.

a. Every prime number is odd.

b. If a number is divisible by 6, then it is divisible by 2 and 3.

c. If a number is divisible by 2 and 6, then it is divisible by 12.

d. If a number is divisible by 3 and 4, then it is divisible by 12.

e. If $a \neq b$, then GCF(a, b) < LCM(a, b).

f. If 6 is a factor of mn, then 6 is a factor of m or a factor of n.

g. If 5 is a factor of mn, then 5 is a factor of m or a factor of n.

? HW #14

Can the sum of two odd prime numbers be a prime number? Explain why or why not.

? HW #15

Find the least common multiple of the following sets of numbers:

a. LCM(2, 4, 5, 7, 8, 12, 14, 15)

b. LCM(3, 4, 6, 8, 9, 10, 12, 18)

? HW #16

If GCF(30, x) = 6 and LCM(30, x) = 180, then what is x? (Hint: see page 65)

? HW #17

The theory of biorhythm states that your physical cycle is 23 days long, your emotional cycle is 28 days long, your intellectual cycle is 33 days long. If your cycles all occur on the same day, how many days until your cycles again occur on the same day? About how many years is this?

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CHAPTER OVERVIEW

7: Geometry

> Geometry is the art of good reasoning from bad drawings. – Henri Poincaré

7.1: The Why
7.2: Introduction
7.3: Tangrams
7.4: Triangles and Quadrilaterals
7.5: Polygons
7.6: Polygons
7.7: Linear Unit Conversions
7.8: Platonic Solids
7.9: Painted Cubes
7.10: Symmetry
7.11: Area, Surface Area and Volume
7.12: Area, Surface Area and Volume Formulas
7.13: Geometry in Art and Science
7.14: Problem Bank

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7.1: The Why

The Essential Questions



BEFORE YOU BEGIN CHAPTER 6,
ANSWER THE ESSENTIAL QUESTIONS HERE:

Figure 5.5.1

Why are Teachers Learning this Material?

Many people study Geometry twice in their life. The first is in elementary school and the second is one year in high school. Therefore, by the time a person is ready to class on teaching mathematics it has already been many years since any formal type of Geometry has been practiced in a classroom setting.

Why are Elementary School Students Learning this Mathematics?

Geometry is a unique branch of mathematics, which gives students a real-life (outside of the classroom) approach to mathematics. The students learn spatial reasoning and problem solving skills. From kindergarten to 6th grade, students learn about shapes and solids.

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7.2: Introduction

The word “geometry” comes from the ancient Greek words “geo” meaning Earth and “metron” meaning measurement. It is probably the oldest field of mathematics, because of its usefulness in calculating lengths, areas, and volumes of everyday objects.

The study of geometry has evolved a great deal during the last 3,000 years or so. Like all of mathematics, what’s really important in geometry is reasoning, making sense of problems, and justifying your solutions.

The mathematician Henri Poincaré said that

> *Geometry is the art of good reasoning from bad drawings.*

This insight should guide your study in this chapter. You should never trust a drawing. You might find that one line segment appears to be longer than another, or an angle looks like it might be 90 degrees. But “appears to be” and “looks like” are simply not good enough. You have to reason through the situation and figure out what you know for sure and why you know it.

? Think / Pair / Share

Reflect on your learning of geometry in the past. What is geometry really about? Also think about these questions:

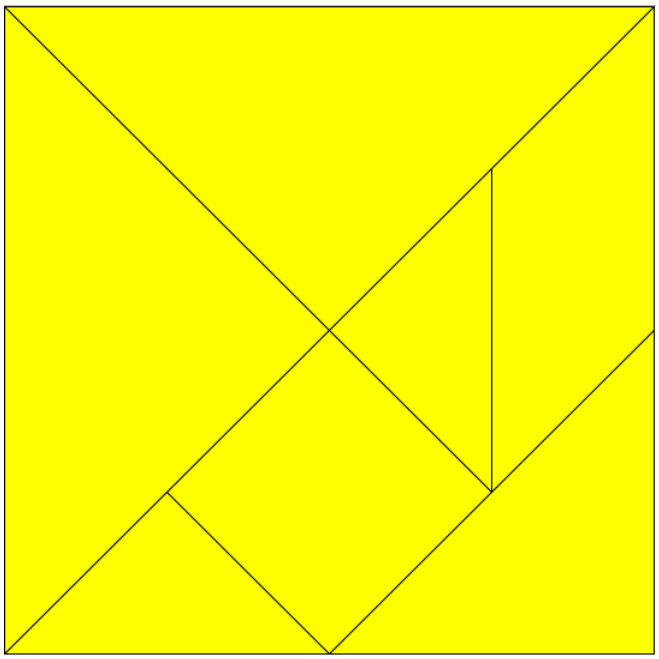
- What is a point?
- What is a line? A segment? A ray?
- What is a plane?
- What is a circle?
- What is an angle?
- Which of these basic objects can be measured? How are they measured? What kinds of tools are useful?

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7.3: Tangrams

Tangrams are a seven-piece geometric puzzle that dates back at least to the Song Dynasty in China (about 1100 AD). Below[1] you will find the seven puzzle pieces. Make a careful copy (a photocopy or printout is best), cut out the puzzle pieces, and then use them to solve the problems in this section.

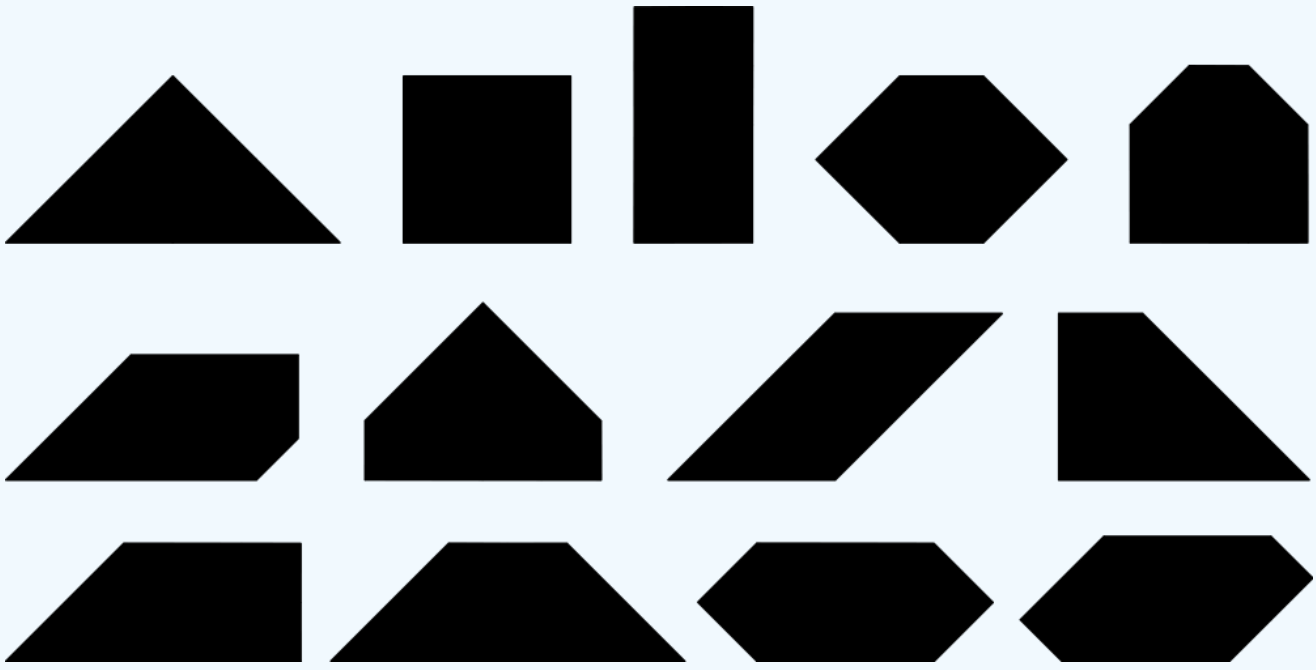


Whenever you solve a tangram puzzle, your job is to **use all seven pieces to form the shape**. They should fit together like puzzle pieces, sitting flat on the table; **no overlapping** of the pieces is allowed.

You can trace around your solutions to remember what you have done and to have a record of your work.

Problem 1

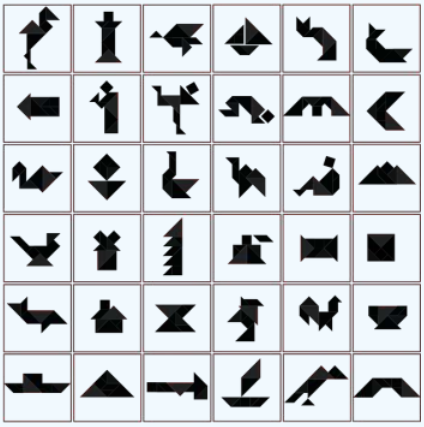
Use your tangram pieces to build the following designs[2]. How many can you make?



(These are all separate challenges. Each one requires all seven pieces. Once you solve one, trace your solution. Then try to solve another one.)

Problem 2

Use your tangram pieces to build the following designs[3]. How many can you make?



(These are all separate challenges. Each one requires all seven pieces. Once you solve one, trace your solution. Then try to solve another one.)

Think / Pair / Share

- Which tangram problems were easier and which were harder: making “real life” objects like cats and people, or purely mathematical objects like the rectangle?
- What do you think made one kind of problem easier or harder?

1. Image of tangram puzzle from Wikimedia Commons, public domain. ↵
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7.4: Triangles and Quadrilaterals

? Think / Pair / Share

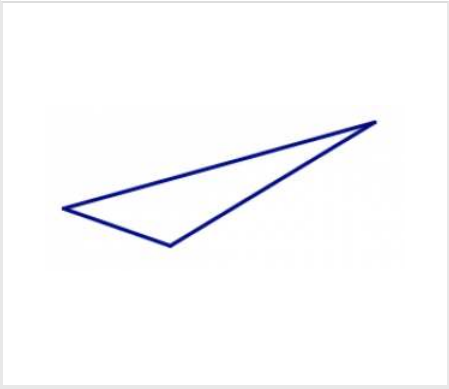
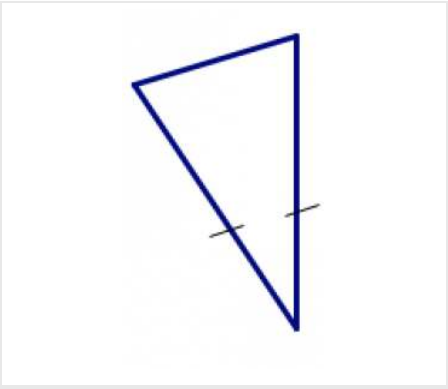
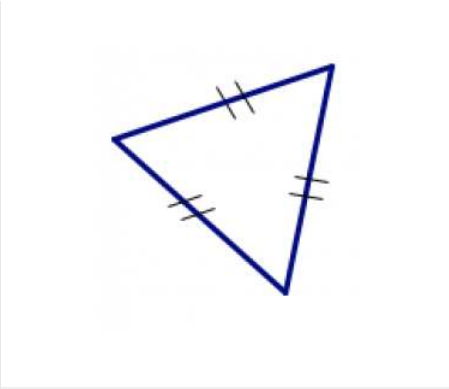
Follow these directions on your own:

- Draw any triangle on your paper.
- Draw a second triangle that is different in some way from your first one. Write down a sentence or two to say how it is different.
- Draw a third triangle that is different from both of your other two. Describe how it is different.
- Draw two more triangles, different from all the ones that came before.

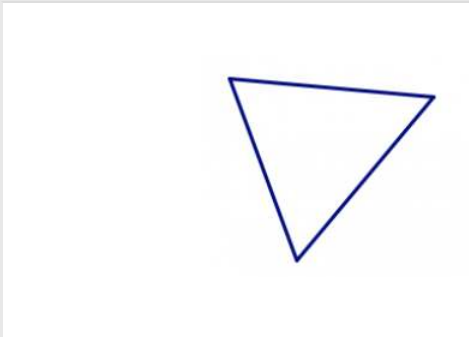
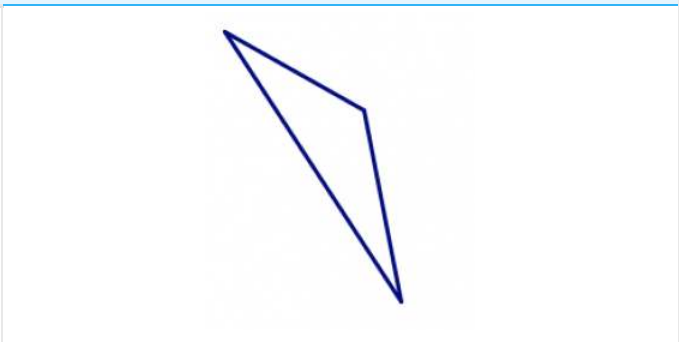
Compare your triangles and descriptions with a partner. To make "different" triangles, you have to change some feature of the triangle. Make a list of the features that you or your partner changed.

Triangles are classified according to different properties. The point of learning geometry is not to learn a lot of vocabulary, but it's useful to use the correct terms for objects, so that we can communicate clearly. Here's a quick dictionary of some types of triangles.

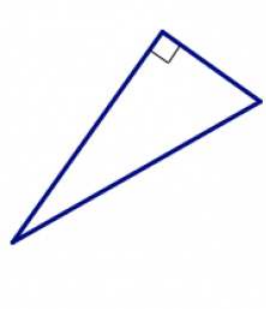
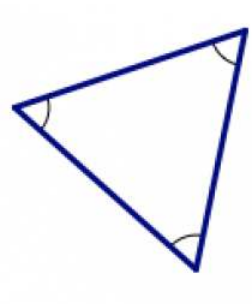
Classification by sides

scalene	isosceles	equilateral
		
all sides have different lengths	*two sides have the same length*	*all three sides have the same length*

Classification by angles

acute	obtuse
	
all interior angles measure less than 90°	*one interior angle measures more than 90°*

right	equiangular

right	equiangular
	
one interior angle measures exactly 90°	*all interior angles have the same measure*

Remember that "geometry is the art of good reasoning from bad drawings." That means you can't always trust your eyes. If you look at a picture of a triangle and one side looks like it's longer than another, that may just mean the drawing was done a bit sloppily.

Notation: Tick marks

Mathematicians either write down measurements or use tick marks to indicate when sides and angles are supposed to be equal.

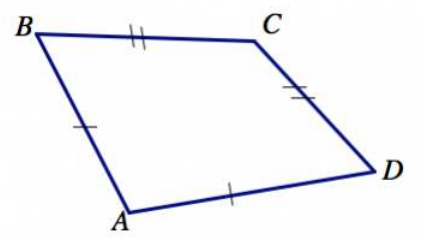
If two sides have the same measurement or the same number of tick marks, you must believe they are equal and work out the problem accordingly, *even if it doesn't look that way to your eyes*.

You can see examples of these in some of the pictures above. Another example is the little square used to indicate a right angle in the picture of the right triangle.

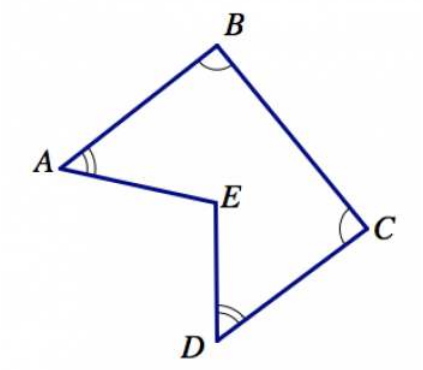
On Your Own

Work on the following exercises on your own or with a partner.

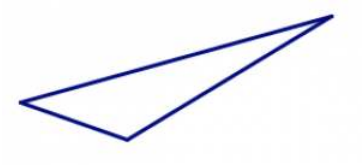
1. In the picture below, which sides are understood to have the same length (even if it doesn't look that way in the drawing)?



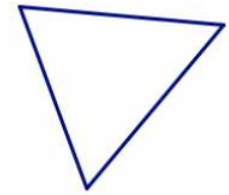
2. In the picture below, which angles are understood to have the same measure (even if if doesn't look that way in the drawing)?



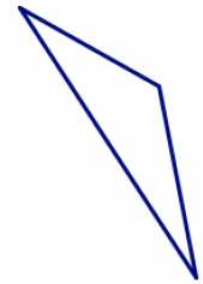
3. Here is a scalene triangle. Sketch two more scalene triangles, each of which is different from the one shown here in some way.



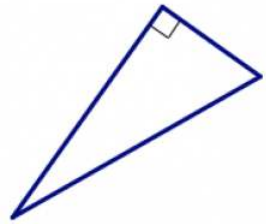
4. Here is an acute triangle. Sketch two more acute triangles, each of which is different from the one shown here in some way.



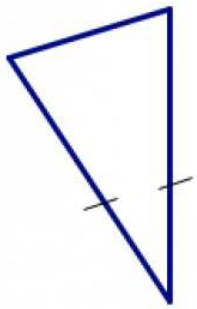
5. Here is an obtuse triangle. Sketch two more obtuse triangles, each of which is different from the one shown here in some way.



6. Here is a right triangle. Sketch two more right triangles, each of which is different from the one shown here in some way. Be sure to indicate which angle is 90°.



7. Here is an isosceles triangle. Sketch two more isosceles triangles, each of which is different from the one shown here in some way. Use tick marks to indicate which sides are equal.

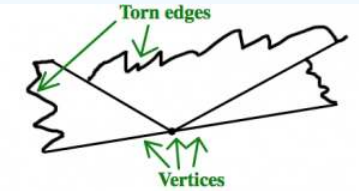


Angle Sum

? Think / Pair / Share

By now, you have drawn several different triangles on your paper. Choose one of your triangles, and follow these directions:

- Using scissors, cut the triangle out.
- Tear (do not cut) off the corners, and place the three vertices together. Your should have something that looks a bit like this picture:



What do you notice? What does this suggest about the angles in a triangle?

You may remember learning that the sum of the angles in any triangle is 180°. In your class, you now have lots of examples of triangles where the sum of the angles seems to be 180°. But remember, our drawings are not exact. How can we be sure that our eyes are not deceiving us? How can we be sure that the sum of the angles in a triangle isn't 181° or 178°, but is really 180° on the nose in every case?

? Think / Pair / Share

What would convince you beyond all doubt that the sum of the angles in any triangle is 180°? Would testing lots of cases be enough? How many is enough? Could you ever test every possible triangle?

History: Euclid's axioms

Often high school geometry teachers prove the sum of the angles in a triangle is 180°, usually using some facts about parallel lines. But (maybe surprisingly?) it's just as good to take this as an *axiom*, as a given fact about how geometry works, and go from there. Perhaps this is less satisfying than proving it from some other statement, and if you're curious you can certainly find a proof or your instructor can share one with you.

Note

In about 300BC, Euclid[1] was the first mathematician (as far as we know) who tried to write down careful axioms and then build from those axioms rigorous proofs of mathematical truths.



Euclid

Euclid had five axioms for geometry, the first four of which seemed pretty obvious to mathematicians. People felt they were reasonable assumptions from which to build up geometric truths:

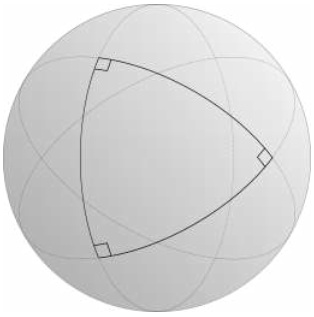
1. Given two points, you can connect them with a straight line segment.
2. Given a line segment, you can extend it as far as you like in either direction, making a line.
3. Given a line segment, you can draw a circle having that segment as a radius.
4. All right angles are congruent.

The fifth postulate bothered people a bit more. It was originally stated in more flowery language, but it was equivalent to this statement:

5. The sum of the angles in a triangle is 180°.

It's easy to see why this fifth axiom caused such a ruckus in mathematics. It seemed much less obvious than the other four, and mathematicians felt like they were somehow cheating if they just assumed it rather than proving it had to be true. Many mathematicians spent many, many years trying to prove this fifth axiom from the other axioms, but they couldn't do it. And with good reason: There are other kinds of geometries where the first four axioms are true, but the fifth one is not!

For example, if you do geometry on a sphere — like a basketball or more importantly on the surface of the Earth — rather than on a flat plane, the first four axioms are true. But triangles are a little strange on the surface of the earth. Every triangle you can draw on the surface of the earth has an angle sum strictly *greater than* 180°. In fact, you can draw a triangle on the Earth that has three right angles[2], making an angle sum of 270°.



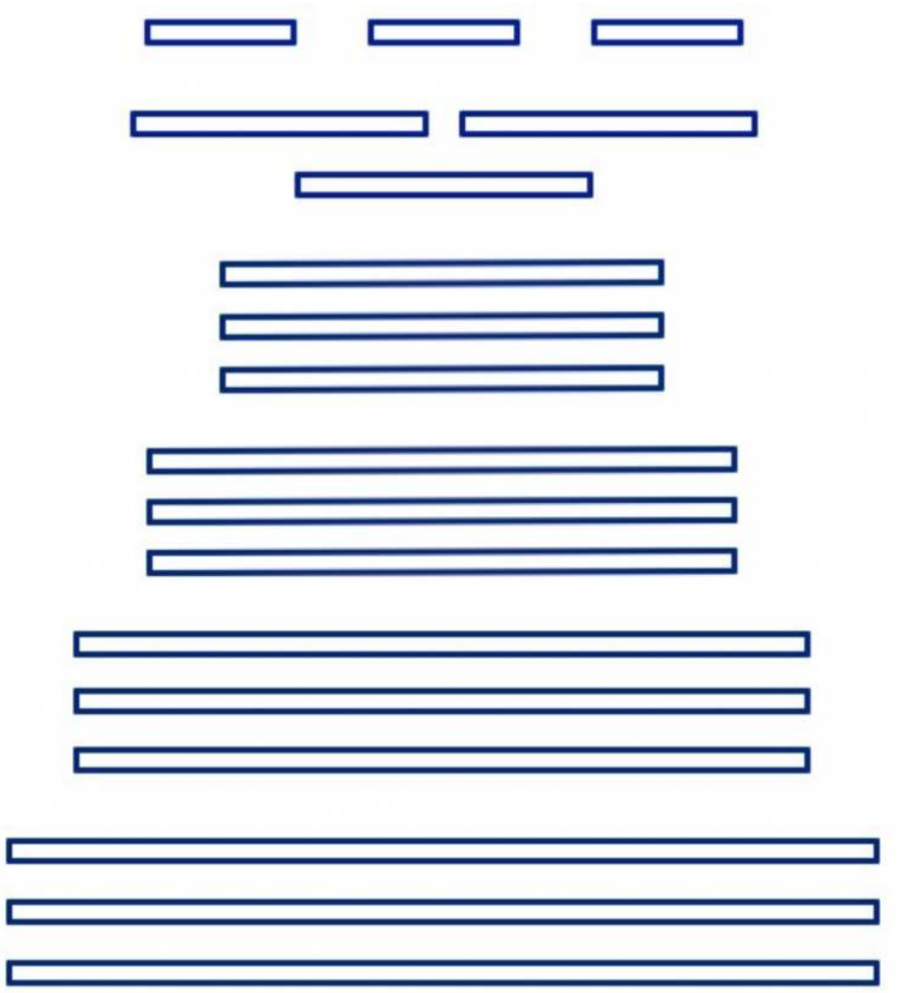
Triangle with three right angles on a sphere.

On a sphere like the Earth, the angle sum is not constant among all triangles. Bigger triangles have bigger angle sums, and smaller triangles have smaller angle sums, but even tiny triangles have angle sums that are greater than 180°.

The geometry you study in school is called *Euclidean geometry*; it is the geometry of a flat plane, of a flat world. It's a pretty good approximation for the little piece of the Earth that we see at any given time, but it's not the only geometry out there!

Triangle Inequality

Make a copy of these strips of paper and cut them out. They have lengths from 1 unit to 6 units. You may want to color the strips, write numbers on them, or do something that makes it easy to keep track of the different lengths.



Problem 3

Repeat the following process several times (at least 10) and keep track of the results (a table has been started for you).

- Pick three strips of paper. (The lengths do not have to be all different from each other; that's why you have multiple copies of each length.)
- Try to make a triangle with those three strips, and decide if you think it is possible or not. (Don't overlap the strips, cut them, or fold them. The length of the strips should be the length of the sides of the triangle.)

Length 1	Length 2	Length 3	Triangle?
4	3	2	yes
4	2	1	no
4	2	2	??

Your goal is to come up with a **rule that describes when three lengths will make a triangle** and when they will not. Write down the rule in your own words.

Think / Pair / Share

Compare your rule with other students. Then use your rule to answer the following questions. Keep in mind the goal is not to try to build the triangle, but to **predict the outcome** based on your rule.

- Suppose you were asked to make a triangle with sides 40 units, 40 units, and 100 units long. Do you think you could do it? Explain your answer.
- Suppose you were asked to make a triangle with sides 2.5 units, 2.6 units, and 5 units long. Do you think you could do it? Explain your answer.

You probably came up with some version of this statement:

Theorem: Triangle Inequality

The sum of the lengths of two sides in a triangle is greater than the length of the third side.

Of course, we know that in geometry we should not believe our eyes. You need to look for an *explanation*. Why does your statement make sense?

Remember that "geometry is the art of good reasoning from bad drawings." Our materials weren't very precise, so how can we be sure this rule we've come up with is is correct?

Well in this case, the rule is really just the same as the saying "the shortest distance between two points is a straight line." In fact, this is exactly what we mean by the words *straight line* in geometry.

SSS Congruence

We say that two triangles (or any two geometric objects) are *congruent* if they are exactly the same shape and the same size. That means that if you could pick one of them up and move it to put down on the other, they would exactly overlap.

Problem 4

Repeat the following process several times and keep track of the results.

- Pick three strips of paper that will definitely form a triangle.
- Try to make two *different* (non-congruent) triangles with the same three strips of paper. Record if you were able to do so.

Problem 5

Repeat the following process several times and keep track of the results.

- Pick four strips of paper and form a quadrilateral with them. (If your four strips do not form a quadrilateral, pick another four strips.)
- Try to make two *different* (non-congruent) quadrilaterals with the same four strips of paper. Record if you were able to do so.

Think / Pair / Share

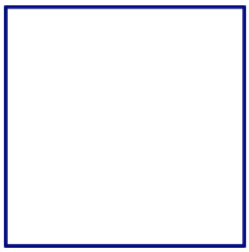
What do you notice from Problems 4 and 5? Can you make a general statement to describe what's going on? Can you explain why your statement makes sense?

You probably came up with some version of this statement:

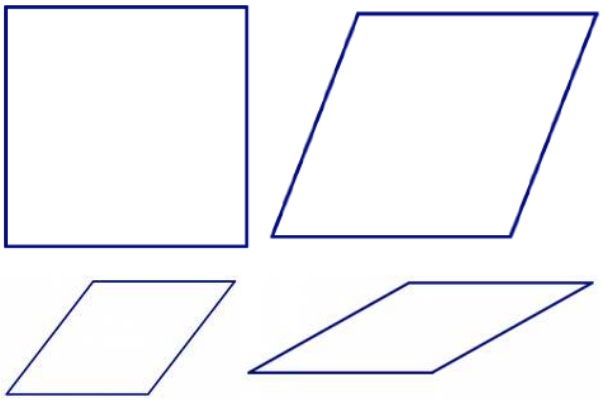
Theorem: SSS (side-side-side) Congruence

If two triangles have the same side lengths, then the triangles are congruent.

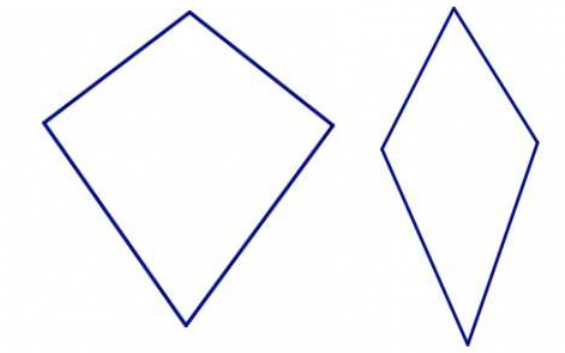
This most certainly is not true for quadrilaterals. For example, if you choose four strips that are all the same length, you can make a square:



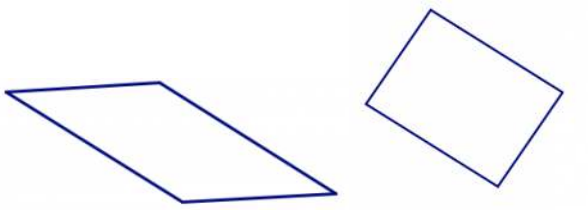
But you can also squish that square into a non-square rhombus. (Try it!)



If you don't choose four lengths that are all the same, in addition to "squishing" the shape, you can rearrange the sides to make different (non-congruent) shapes. (Try it!)



These two quadrilaterals have the same four side lengths in the same order.



These two quadrilaterals have the same four side lengths as the two above, but the sides are in a different order.

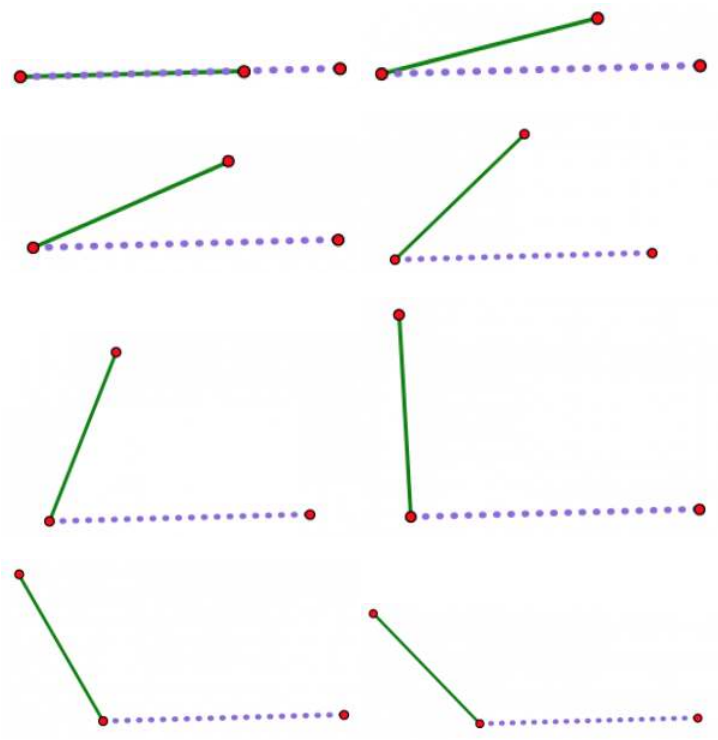
But this can't happen with triangles. Why not? Well, certainly you can't rearrange the three sides. That would be just the same as rotating the triangle or flipping it over, but not making a new shape.

Why can't the triangles "squish" the way a quadrilateral (and other shapes) can? Here's one way to understand it. Imagine you pick two of your three lengths and lay them on top of each other, hinged at one corner.



This shows a longer purple dashed segment and a shorter green segment. The two segments are hinged at the red dot on the left.

Now imagine opening up the hinge a little at a time.



As the hinge opens up, the two non-hinged endpoints get farther and farther apart. Whatever your third length is (assuming you are actually able to make a triangle with your three lengths), there is **exactly one position** of the hinge where it will just exactly fit to close off the triangle. No other position will work.

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1. Portrait of Euclid from Wikimedia Commons, licensed under the Creative Commons Attribution 4.0 International license. ↵
2. Image by Coyau / Wikimedia Commons, via Wikimedia Commons, licensed under Creative Commons Attribution-Share Alike 3.0 Unported. ↵

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7.5: Polygons

It can seem like the study of geometry in elementary school is nothing more than learning a bunch of definitions and then classifying objects. In this part, you'll explore some problem solving and reasoning activities that are based in geometry. But definitions are still important! So let's start with this one.

Definition

A **polygon** is:

1. a plane figure
2. that is bounded by a finite number of straight line segments
3. in which each segment meets exactly two others, one at each of its endpoints.

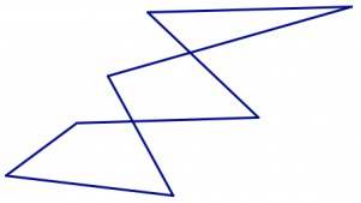
Think / Pair / Share

Just as the first step in problem solving is to understand the problem, the first step in reading a mathematical definition is to understand the definition.

- Use the definition above to draw several examples of figures that are definitely polygons. (You should be able to say why your example fits the definition.)
- Draw several non-examples as well: shapes that are definitely not polygons. (You should be able to say which part of the definition fails for your non-examples.)

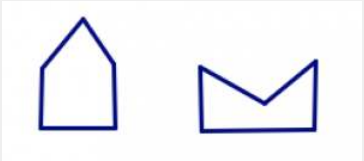
A few comments about polygons:

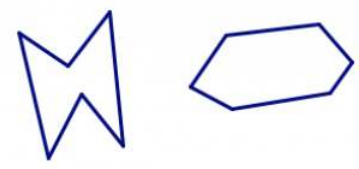
- The line segments that make up a polygon are called its **edges** and the points where they meet are called its **vertices** (singular: **vertex**).
- Because of properties (2) and (3) in the definition, the boundaries of polygons are not self-intersecting.



Not a polygon.

- Polygons are named based on the number of sides they have.

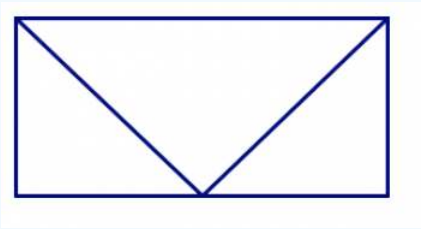
name	# of sides	examples
triangle	3	
quadrilateral	4	
pentagon	5	

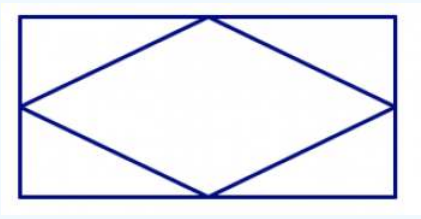
hexagon	6	
heptagon	7	
octagon	8	
nonagon	9	
decagon	10	

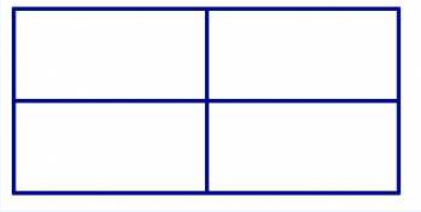
- In general, we call a polygon with *n* sides an *n-gon*.

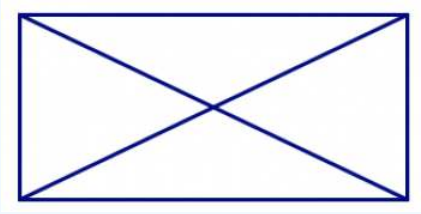
Problem 6

In the pictures below, there are polygons hidden in the design. In each design, find all of the triangles, quadrilaterals, pentagons, and hexagons. How can you be sure you've found them all and haven't counted any twice?









Angle Sum

You know that the sum of the interior angles in any triangle is 180°. Can you say anything about the angles in other polygons?

You probably know that rectangles have four 90° angles. So *if if all quadrilaterals have the same interior angle sum*, it must be 360° (since 4 × 90° = 360°).

But notice: We don't necessarily have any reason to believe this constant sum would be true. Remember that SSS congruence is true for triangles, but not for any other polygons. Triangles are special, and we shouldn't assume that true statements about triangles will hold true for other shapes.

? Think / Pair / Share

Any quadrilateral can be split into two triangles, where the vertices of the triangles all coincide with the vertices of the quadrilateral:



Use the pictures above to carefully explain why all quadrilaterals do, indeed, have an angle sum of 360°.

On Your Own

Work on the following exercises on your own or with a partner.

1. Draw several different pentagons on your paper. Show that each of them can be split into exactly three triangles in such a way that the vertices of the triangles all coincide with the vertices of the pentagon.
2. Use the fact that every pentagon can be split into three triangles in this way to find the sum of the angles in any pentagon.
3. Draw several different hexagons on your paper. Show that each of them can be split into exactly four triangles so that the vertices of the triangles all coincide with the vertices of the hexagon.
4. Use the fact that every hexagon can be split into four triangles in this way to find the sum of the angles in any hexagon.

✔ Problem 7

Use your work on the exercises above to complete this general statement:

Angle Sum in Polygons

The sum of the interior angles in an n-gon (a polygon with n sides) is

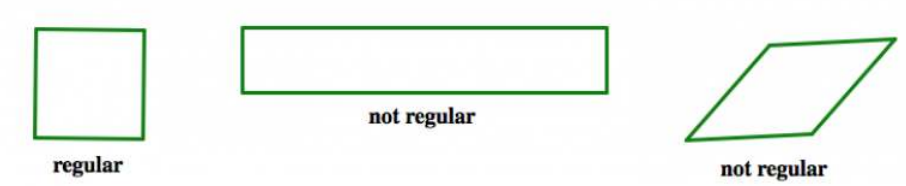
_____.

Explain how you know your statement is true.

Definition

A **regular polygon** has all sides the same length and all angles the same measure.

For example, squares are regular quadrilaterals — all four sides are the same length, and all four angles measure 90°. But a non-square rectangle is *not regular*. Even though all of the angles are 90°, the sides are not all the same length. Similarly, a non-square rhombus is *not regular*. Even though the sides of a rhombus are all the same length, the angles can be different.



✔ Problem 8

Since a square is a regular quadrilateral, you know that every angle in a regular quadrilateral measure 90°. What about angles in other regular polygons?

1. What is the measure of each angle in a regular triangle? Explain how you know you are right.

2. What is the measure of each angle in a regular pentagon? Explain how you know you are right.
3. What is the measure of each angle in a regular hexagon? Explain how you know you are right.
4. What is the measure of each angle in a regular *n-gon*? Explain how you know you are right.

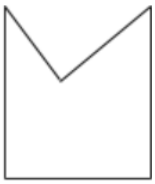
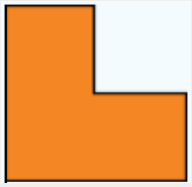
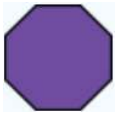
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7.6: Polygons

Below is a table of polygons. There are an infinite amount of polygons, but the following are the shapes taught in elementary school.

Table 6.2.1: Polygons

Number of Sides	Name	Irregular Polygon	Regular Polygon
3 sides	Triangle		
4 sides	Quadrilateral		
5 sides	Pentagon		
6 sides	Hexagon		
8 sides	Octagon		

Definition: Regular Polygon

A shape whose sides have the same length and whose angles have the same measure.

Definition: Irregular Polygon

A shape whose sides differ in length or have angles of different measure.

Hierarchy of Polygons

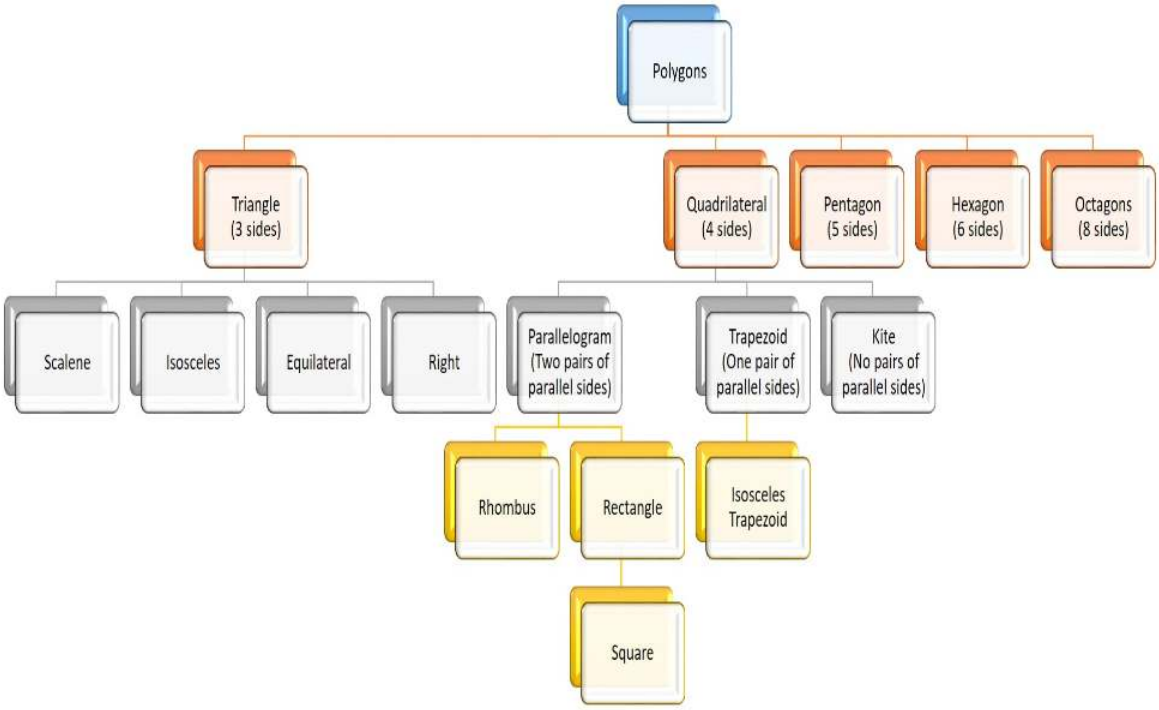


Figure 6.2.1: Hierarchy of Polygons

Polygon Definitions

Definition: Kite

A quadrilateral with two consecutive sides having equal lengths and the other two sides also have equal lengths.

Definition: Trapezoid

A quadrilateral with at least one pair of opposite sides parallel.

Definition: Isosceles Trapezoid

A trapezoid with both angles next to one of the parallel sides having the same size.

Definition: Parallelogram

A trapezoid with pairs of opposite sides parallel.

Definition: Rectangle

A parallelogram with a right angle.

Definition: Rhombus

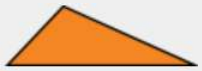
A quadrilateral with all sides being the same.

Definition: Square

A rectangle that has four equal sides.

Types of Triangles

Table 6.2.2: Triangles

Name	Definition	Triangle
SIDES		
Equilateral	All three sides are equal	
Isosceles	Only two sides are equal	
Scalene	All three sides are different in length	
ANGLES		
Acute	Each angle is less than 900	
Right	One angle is 900	
Obtuse	One angle is more than 900	

Partner Activity 1

Draw the following triangles

a. Isosceles right triangle
b. Scalene obtuse triangle
c. Equilateral right triangle

Partner Activity 2

1. Is a rectangle a square? Is a square a rectangle?
2. Multiple Choice: Which one is NOT a name for the figure below?
   a. Polygon
   b. Quadrilateral
   c. Parallelogram
   d. Trapezoid



Figure 6.2.2

3. What is the difference between a regular and irregular polygon?

Facts about Angles

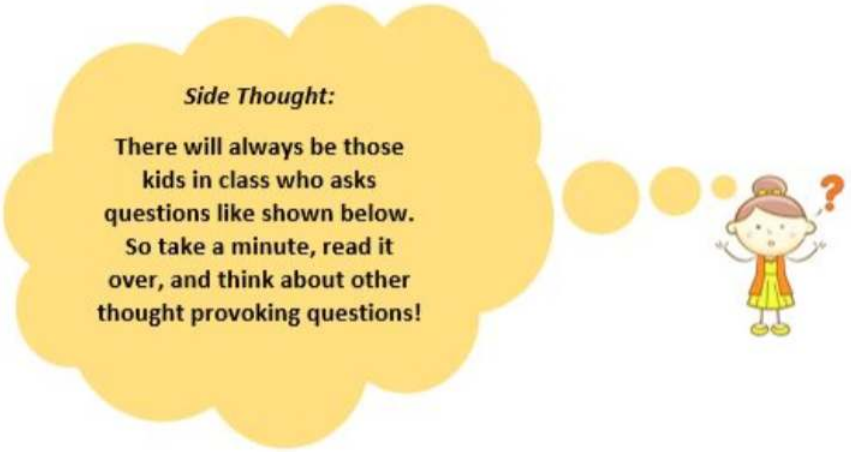


Figure 6.2.3

- Angles in a triangle add up to 1800
- An angle forming a straight line is also 1800
- Any quadrilateral (4-sided figure) is 3600
- Angles which round a point add up to 3600
- The two base angles of an isosceles triangle are equal

Why does a triaangle add up to 180°

A full circle is 360°. Half of a circle, called a semicircle, would then be 180°. The diameter (a line which passes through the center of the circle) of the semicircle is then also 180°. Therefore, all straight lines are 180°. See the figure below. Knowing that all straight lines are 180°, we look at the figure below of the line and triangle.

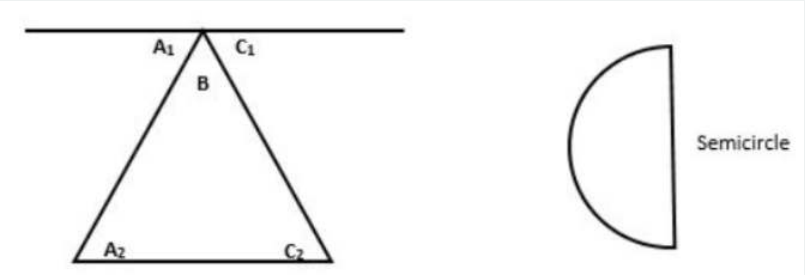


Figure 6.2.4

Since a line is 180°, we know that angles A_1, B, and C_1 must add up to 180°. A theorem (proven statement) in Geometry states that alternate (opposite sides) interior angles are congruent (equal). Angles A_1 and A_2 are alternate interior, cut by the transversal (line) connecting angle A_2 to the straight line. Angles C_1 and C_2 follow a similar approach.
Since the measures of angles $A_1 = A_2$, $C_1 = C_2$, and $A_1 + B + C_1 = 180$, then by substitution, $A_2 + B + C_2 = 180$. Therefore, triangle A_2BC_2 adds up to 180°.

Partner Activity 3

The sum of the interior angles of any polygon is represented by: $180(n-2)$.

1. Find the sum of the interior angles of a triangle, using the formula.
2. Find the sum of the interior angles of a pentagon, using the formula.
3. Find the sum of the interior angles of a 15-sided polygon, using the formula.
4. What is the sum of the EXTERIOR angles of a pentagon?

Complementary and Supplementary Angles

Definition: Complementary Angles

Complementary angles are any two angles with a sum of 900. See angles C and D below.



Figure 6.2.5: Complementary angles

Definition: Supplementary Angles

Supplementary angles are any two angles with a sum of 1800. See angles A and B below.

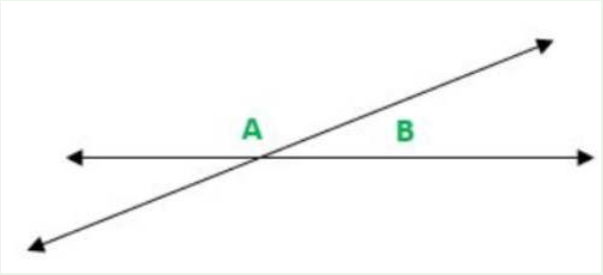


Figure 6.2.6: Supplementary angles

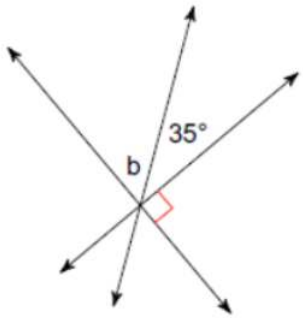
Partner Activity 4

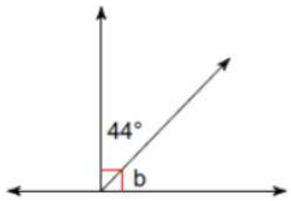
1. You have two supplementary angles. One angle is 300. What is the measure of the other angle?
2. One angle is complementary to another angle. The first one is 490. What is the measure of the second angle?

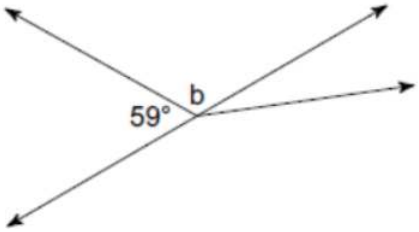
Practice Problems

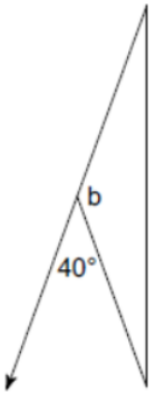
(Problems 1 – 4) Find the measure of angle b.

1.

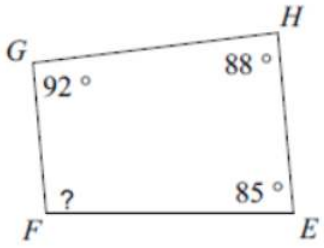


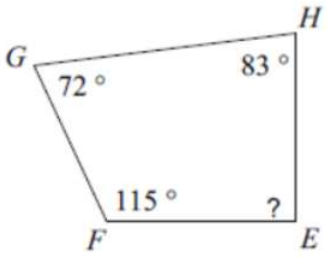
2. 

3. 

4. 

(Problems 5 – 6) Find the measure of each angle indicated.

5. 

6. 

(Problems 7 – 10) Classify each angle as acute, obtuse, right or straight.

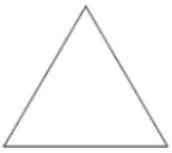
7. 121°
8. 180°

9. 

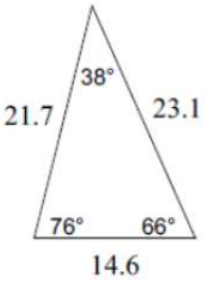
10. 

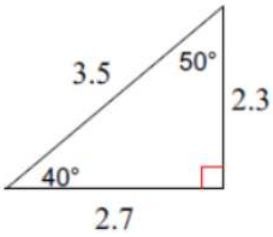
(Problems 11 – 12) Classify each triangle by its angles.

11. 

12. 

(Problems 13 – 14) Classify each triangle by its angles and sides.

13. 

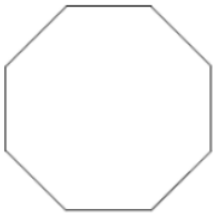
14. 

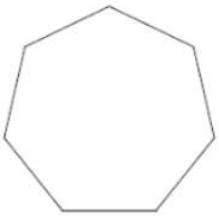
(Problems 15 – 16) Sketch an example of the type of triangle described.

15. Acute Isosceles

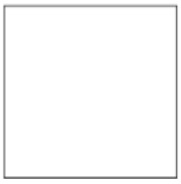
16. Right Obtuse

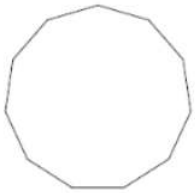
(Problems 17 – 18) Write the name of each polygon.

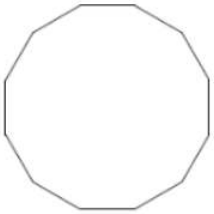
17. 

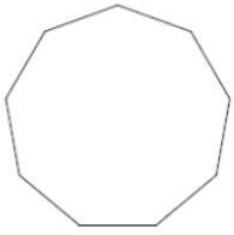
18. 

(Problems 19 – 22) Find the interior angle sum for each polygon. Round your answer to the nearest tenth, if necessary.

19. 

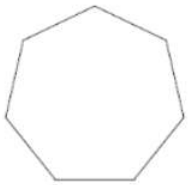
20. 

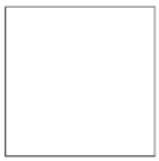
21. 

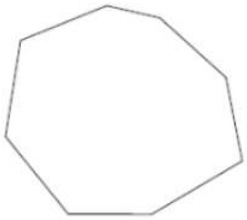
22. 

(Problems 23 – 26) State if the polygon is regular or irregular.

23. 

24. 

25. 

26. 

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7.7: Linear Unit Conversions

Example 7.7.1

Convert 2 miles into _____ feet.

Solution

Since 1 mile = 5280 feet. Therefore 2 miles = 2(5280 feet) = 10560 feet

Example 7.7.2

Convert 15 yards into _____ miles.

Solution

Since 1 mile = 5280 feet and 3 feet = 1 yard. Follow the math below:

$$\frac{15 \text{ yards}}{1} \times \frac{3 \text{ feet}}{1 \text{ yard}} \times \frac{1 \text{ mile}}{5280 \text{ feet}} = \frac{15 \text{ yards}}{1} \times \frac{3 \text{ feet}}{1 \text{ yard}} \times \frac{1 \text{ mile}}{5280 \text{ feet}}$$
$$= \frac{45}{5280} \text{ miles}$$

Partner Activity 1

1. Convert 20 inches into _____ yards.
2. Convert 16 miles into _____ feet.

Practice Problems

1. Convert 42 feet into ____ miles.
2. Convert 81 inches into _____ yards.
3. Convert 34 miles into _____ yards.
4. Convert 91 yards into _____ inches.

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7.8: Platonic Solids

Of course, we live in a three-dimensional world (at least!), so only studying flat geometry doesn't make a lot of sense. Why not think about some three-dimensional objects as well?

Definition

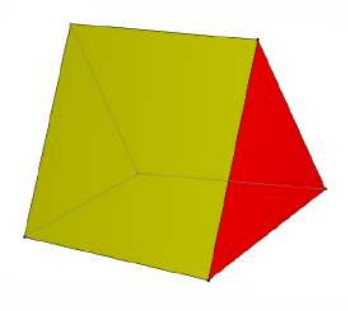
A **polyhedron** is a solid (3-dimensional) figure bounded by polygons. A polyhedron has **faces** that are flat polygons, straight **edges** where the faces meet in pairs, and **vertices** where three or more edges meet.

The plural of polyhedron is **polyhedra.**

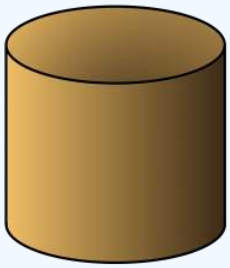
Think / Pair / Share

Look at the pictures of solids below, and decide which are polyhedra and which are not. You should be able to say why each figure does or does not fit the definition.

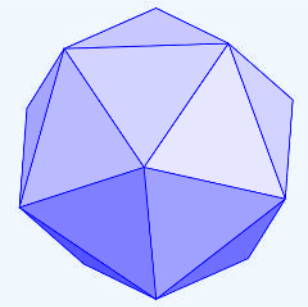
a. [1]



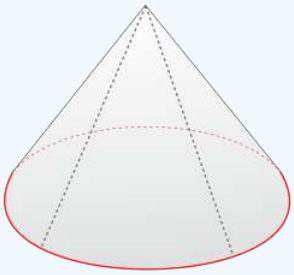
b. [2]



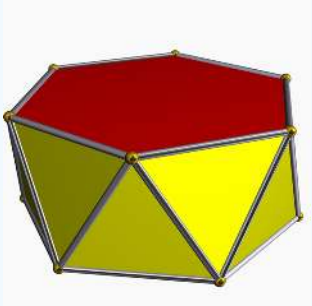
c. [3]



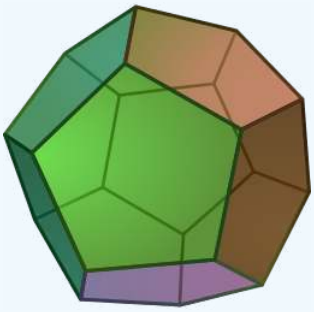
d. [4]



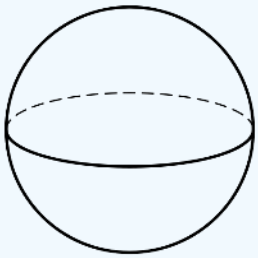
e. [5]



f. [6]



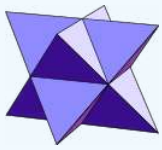
g. [7]



h. [8]



i. [9]



Remember that a *regular polygon* has all sides the same length and all angles the same measure. There is a similar (if slightly more complicated) notion of *regular* for solid figures.

Definition

A r**egular polyhedron** has faces that are all *identical (congruent) regular polygons*. All vertices are also identical (the same number of faces meet at each vertex).

Regular polyhedra are also called **Platonic solids** (named for Plato).

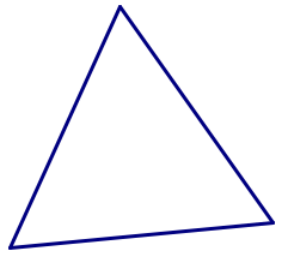
If you fix the number of sides and their length, there is one and only one regular polygon with that number of sides. That is, every regular quadrilateral is a square, but there can be different sized squares. Every regular octagon looks like a stop sign, but it may be scaled up or down. Your job in this section is to figure out what we can say about regular polyhedra.

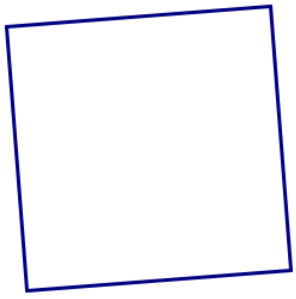
On Your Own

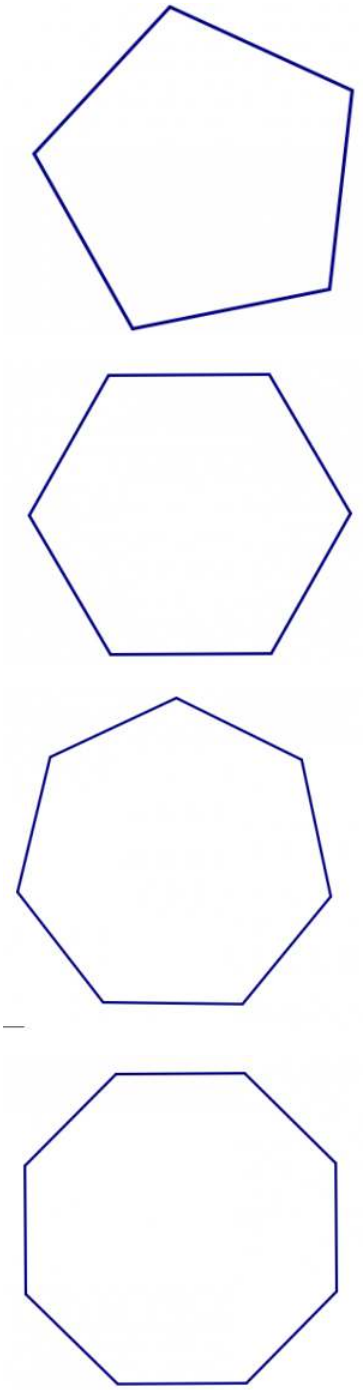
Work on the following exercises on your own or with a partner. You will need to make lots of copies of the regular polygons below. Copy and cut out at least:

- 40 copies of the equilateral triangle,
- 15 copies of the square,
- 20 copies of the regular pentagon, and
- 10 copies each of the hexagon, heptagon, and octagon.

You will also need some tape.







1. In any polyhedron, at least three polygons meet at each vertex. Start with the equilateral triangles: Put **three** of them together meeting at a vertex and tape them together. Then close them up so they form a solid shape. Can you complete this shape into a platonic solid? Be sure to check that at every vertex you have **exactly three** triangles meeting.
2. Now repeat this process, but start with **four** equilateral triangles around a single vertex. Then close them up so they form a solid shape. Can you complete this into a platonic solid? Be sure to check that at every vertex you have **exactly four** triangles meeting.
3. Repeat this process with **five** equilateral triangles, then six, then seven, and so on. Keep going until you are convinced you understand what's happening with Platonic solids that have triangular faces.
4. When you are done with triangular faces, move on to square faces. Work systematically: Try to build a Platonic solid with three squares at each vertex, then four, then five, etc. Keep going until you can make a definitive statement about Platonic solids with

square faces.

5. Repeat this process with the other regular polygons you cut out: pentagons, hexagons, heptagons, and octagons.

You must have noticed that the situation for Platonic solids is quite different from the situation for regular polygons. There are *infinitely many* regular polygons (even if you don't account for size). There is a regular polygon with n sides for every value of n bigger than 2. But for solids, we have the following (perhaps surprising) result.

Theorem

There are exactly five Platonic solids.

The key fact is that for a three-dimensional solid to close up and form a polyhedron, there must be less than 360° around each vertex. Otherwise, it either lies flat (if there is exactly 360°) or folds over on itself (if there is more than 360°).

Problem 9

Based on your work in the exercises, you should be able to write a convincing justification of the Theorem above. Here's a sketch, and you should fill in the explanations.

1. If a Platonic solid has faces that are equilateral triangles, then fewer than 6 faces must meet at each vertex. Why?
2. If a Platonic solid has square faces, then three faces can meet at each vertex, but not more than that. Why?
3. If a Platonic solid has faces that are regular pentagons, then three faces can meet at each vertex, but not more than that. Why?
4. Regular hexagons cannot be used as the faces for a Platonic solid. Why?
5. Similarly, regular n-gons for n bigger than 6 cannot be used as the faces for a Platonic solid. Why?

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6. Image DTR CC-BY-SA-3.0], via Wikimedia Commons ↵
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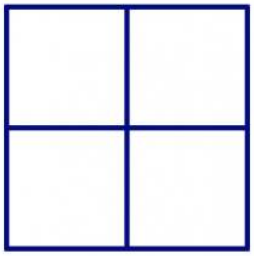
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7.9: Painted Cubes

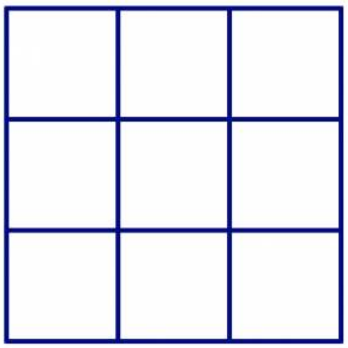
You can build up squares from smaller squares:



1 × 1 square

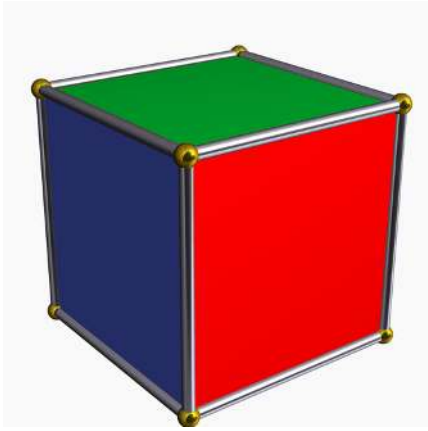


2 × 2 square

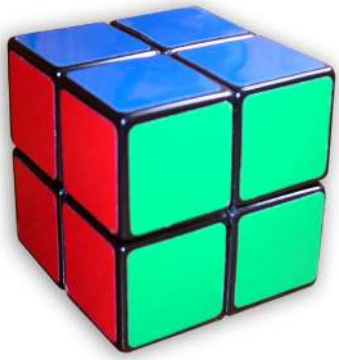


3 × 3 square

In a similar way, you can build up cubes from smaller cubes:



1 × 1 × 1 cube[1]



2 × 2 × 2 cube[2]



3 × 3 × 3 cube[3]

Think / Pair / Share

We call a 1 × 1 × 1 cube a **unit cube**.

- How many unit cubes are in a 2 × 2 × 2 cube?
- How many unit cubes are in a 3 × 3 × 3 cube?
- How many unit cubes are in a $n \times n \times n$ cube?

Explain your answers.

Problem 10

Imagine you build a 3 × 3 × 3 cube from 27 small white unit cubes. Then you take your cube and dip it into a bucket of bright blue paint. After the cube dries, you take it apart, separating the small unit cubes.

1. After you take the cube apart, some of the unit cubes are still all white (no blue paint). How many? How do you know you are right?
2. After you take the cube apart, some of the unit cubes have blue paint on just one face. How many? How do you know you are right?

3. After you take the cube apart, some of the unit cubes have blue paint on two faces. How many? How do you know you are right?
4. After you take the cube apart, some of the unit cubes have blue paint on three faces. How many? How do you know you are right?
5. After you take the cube apart, do any of the unit cubes have blue paint on more than three faces? How many? How do you know you are right?

Problem 11

Generalize your work on Problem 10. What if you started with a 2 × 2 × 2 cube? Answer the same questions. What about a 4 × 4 × 4 cube? How about an $n \times n \times n$ cube? Be sure to justify what you say.

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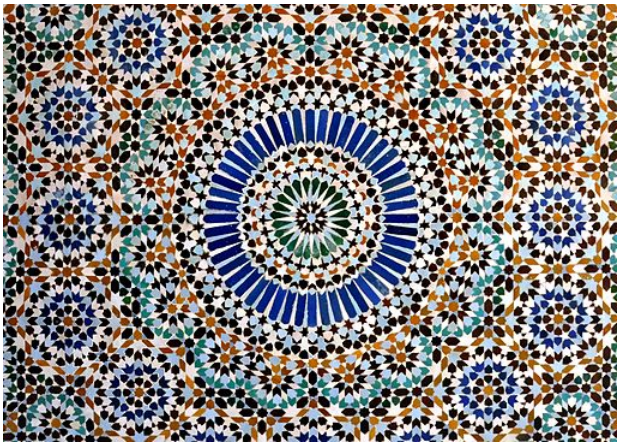
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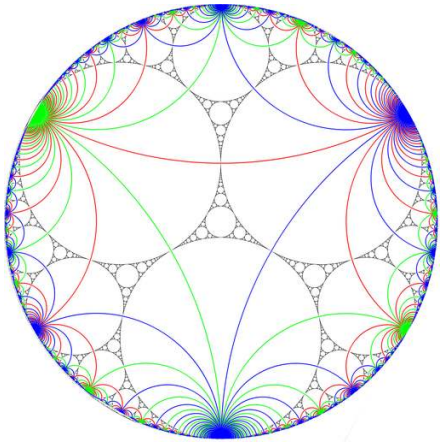
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7.10: Symmetry

Mathematicians use symmetry in all kinds of situations. There can be symmetry in calculations, for example. But the most recognizable kinds of symmetry are those in geometric designs.

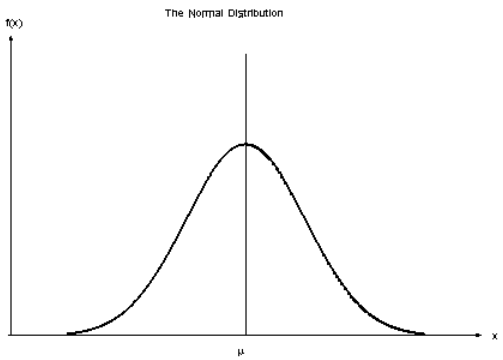
Geometric and real-world objects can have different kinds of symmetries[1].





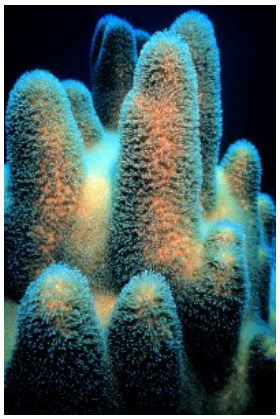


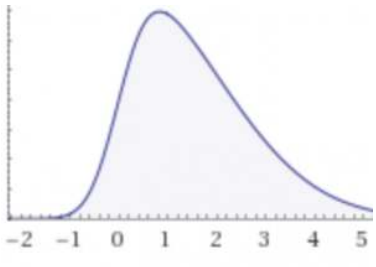


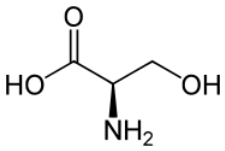


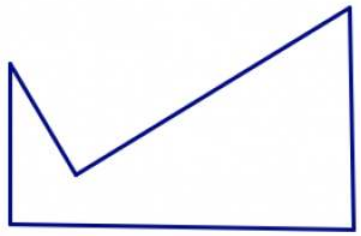


Or they might have no symmetry[2] at all.





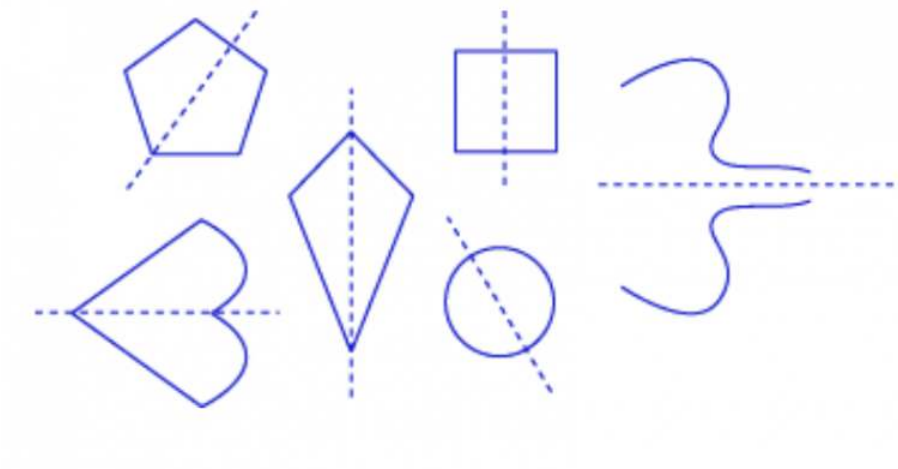


? Think / Pair / Share

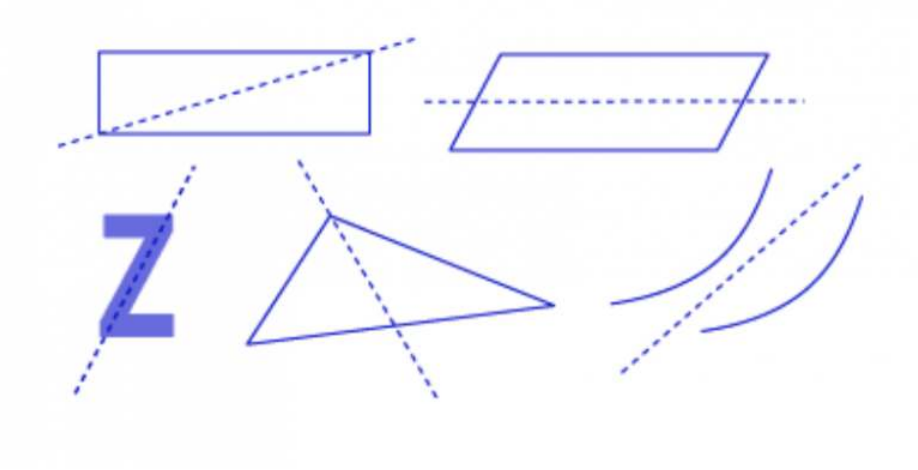
- What do you already know about the idea of *symmetry*? What does it mean to say a design is *symmetric*?
- Do you know about different types of symmetry? What types?
- Can you give examples of real-world objects that are symmetric? What about objects that are not symmetric?

Line Symmetry

If you can flip a figure over a line — this is called *reflecting* the figure — and then it appears unchanged, then the figure has **reflection symmetry** or **line symmetry**. A **line of symmetry** divides an object into two mirror-image halves. The dashed lines below are lines of symmetry:



Compare with the dashed lines below. Though they do cut the figures in half, they don't create mirror-image halves. These are **not** lines of symmetry:



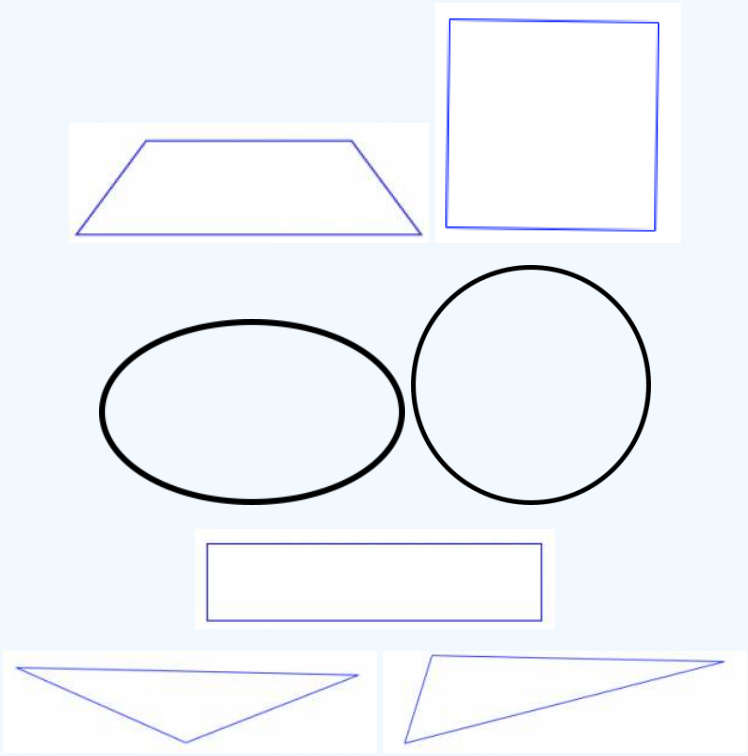
Think / Pair / Share

Look at the first set of pictures at the start of this chapter. Do any of them have lines of symmetry? How can you tell?

Problem 12

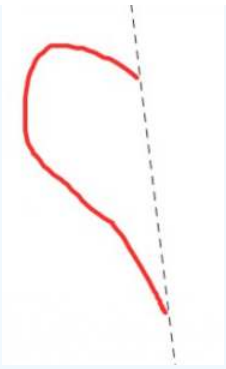
For each of the figures[3] below:

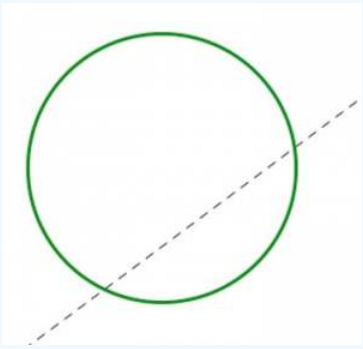
1. Decide if it has any lines of symmetry. If not, how do you know?
2. If it does have one or more lines of symmetry, find / describe all of them. Explain how you did it.



Problem 13

Each picture below shows **half** of a design with line symmetry. The line of symmetry (dashed) is shown. Can you complete the design? Explain how you did it.

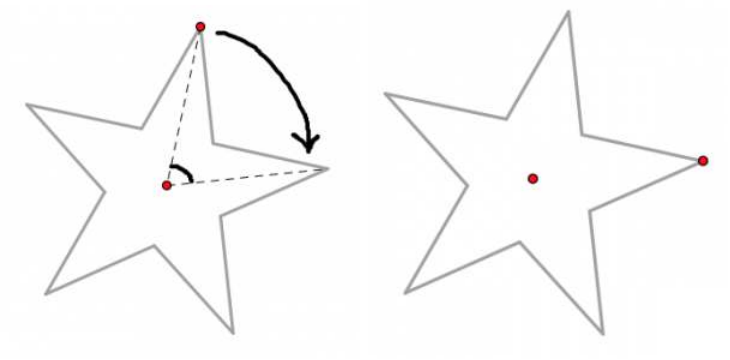




Rotational Symmetry

If you can turn a figure around a center point less than a full circle — this is called a *rotation* — and the figure appears unchanged, then the figure has **rotational symmetry**. The point around which you rotate is called the center of rotation, and the smallest angle you need to turn is called the angle of rotation.

This star has rotational symmetry of 72°, and the center of rotation is the center of the star. One point is marked to help you visualize the rotation.



? Think / Pair / Share

- How can you be certain that the angle of rotation for the star is exactly 72°?
- Look at the first set of pictures at the start of this chapter. Do any of them have rotational symmetry? How can you tell?

✔ Problem 14

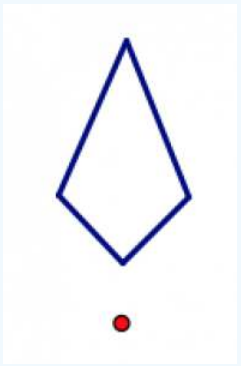
Each of the figures below has rotational symmetry. Find the center of rotation and the angle of rotation. Explain your thinking.

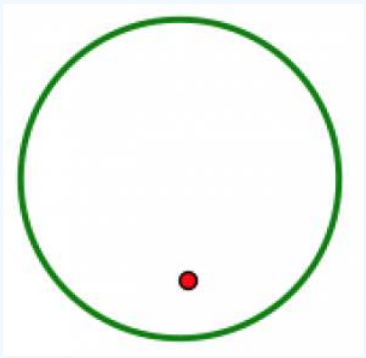




Problem 15

Each picture below shows part of a design with a marked center of rotation and an angle of rotation given. Can you complete the design so that it has the correct rotational symmetry? Explain how you did it.

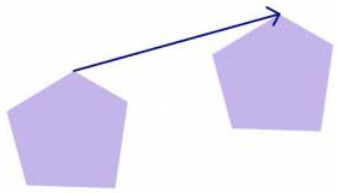




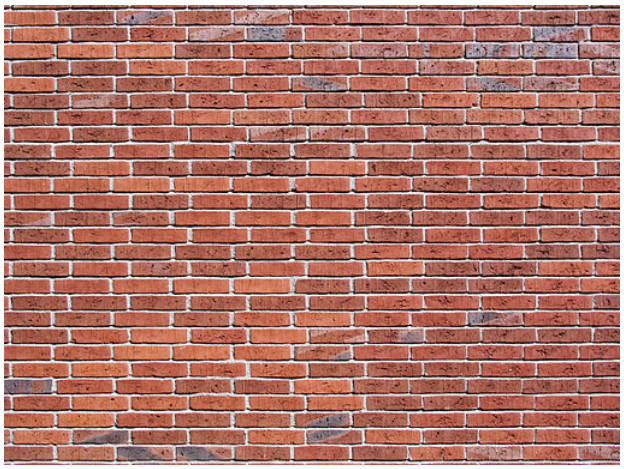
90° 45°

Translational Symmetry

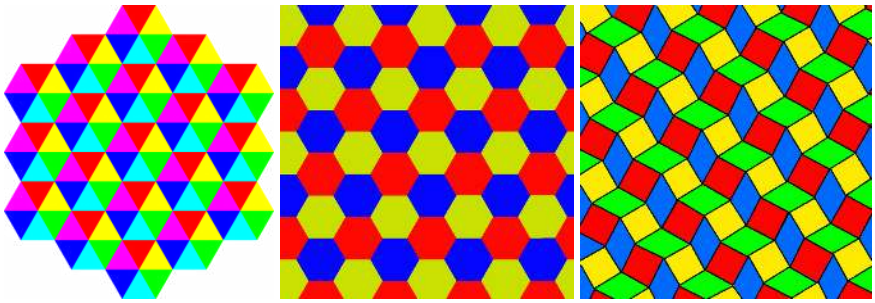
A **translation** (also called a slide) involves moving a figure in a specific direction for a specific distance. A **vector** (a line segment with an arrow on one end) can be used to describe a translation, because the vector communicates both a distance (the length of the segment) and a direction (the direction the arrow points).



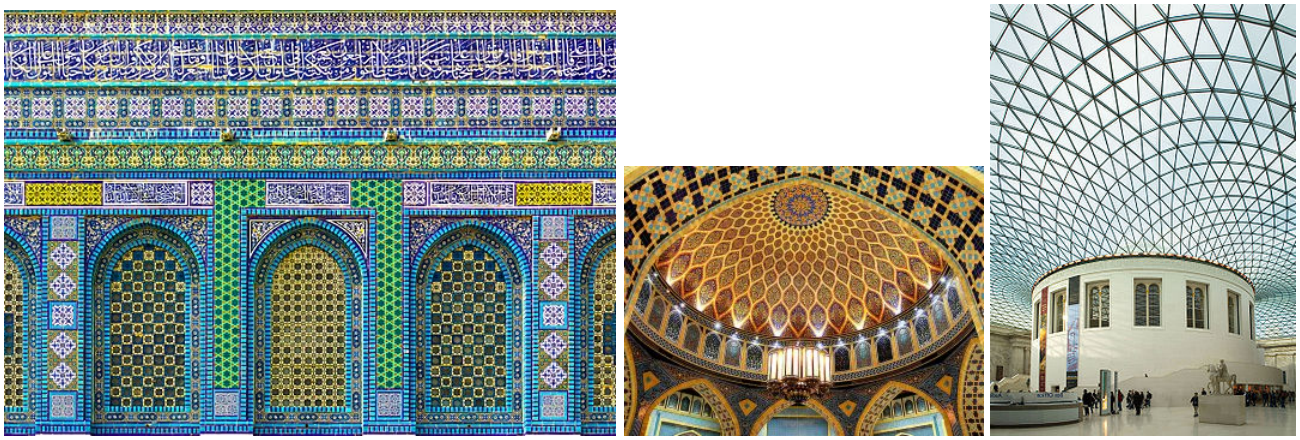
A design has **translational symmetry** if you can perform a translation on it and the figure appears unchanged. A brick wall[4] has translational symmetry in lots of directions!



The brick wall is one example of a *tessellation*[5], which you'll learn more about in the next chapter.



You can see translation symmetry in lots of places. It's in architecture and design[6].



It's in art, most famously that by M.C. Escher. (You might want to visit http://www.mcescher.com/gallery/symmetry/ and browse the "Symmetry" gallery.)

And it appears in traditional Hawaiian and other Polynesian tattoo[7]designs.





? Think / Pair / Share

- On each of the pictures with translational symmetry above, sketch a vector to indicate the direction and distance of the translational symmetry.
- Create your own design with translational symmetry. Explain how you did it.

---

1. Mosaic image by MarcCooperUK (Flickr: Paris central mosque) [CC BY 2.0], via Wikimedia Commons. Apollonian Circle Packing by Tomruen (Own work) [CC BY-SA 3.0], via Wikimedia Commons. Butterfly by Bernard DUPONT from FRANCE (Swallowtail Butterfly (Papilio oribazus)) [CC BY-SA 2.0], via Wikimedia Commons. Starfish by Paul Shaffner [CC BY 2.0], via Wikimedia Commons. Normal distribution from Wikimedia Commons [Public domain]. Water drop from pixababy.com [CC0 Creative Commons]. ↵
2. Pillar coral, wave, and molecule from Wikimedia commons [Public domain]. Head of a woman by Pablo Picasso, image from Gandalf's Gallery on flickr [CC-BY-NC-SA 2.0] ↵
3. Circle and ellipse by Paris 16 (Own work) [CC BY-SA 4.0], via Wikimedia Commons ↵
4. Image by I, Xauxa [CC-BY-SA-3.0], via Wikimedia Commons ↵
5. Triangular tessellation from pixababy [CC0]. Hexagonal and rhombic tessellations from Wikimedia Commons [Public domain]. ↵
6. Tile at Jerusalem temple by Andrew Shiva / Wikipedia, via Wikimedia Commons [CC BY-SA 4.0]. Mosque by Hisham Binsuwaif via flickr [CC BY-SA 2.0]. British Museum great court by Andrew Dunn, http://www.andrewdunnphoto.com/ (Own work) [CC BY-SA 2.0], via Wikimedia Commons ↵
7. Royal Hawaiian officer via Wikimedia Commons [Public domain]. Shoulder and arm tattoos by Micael Faccio on flicker [CC BY-2.0]. ↵

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7.11: Area, Surface Area and Volume

Definition: Area

The extent or measurement of a surface or piece of land. (2 dimensional)



Figure 6.3.1: 2 dimensional land

Definition: Surface Area

The area of such an outer part or uppermost layer. (3 dimensional)

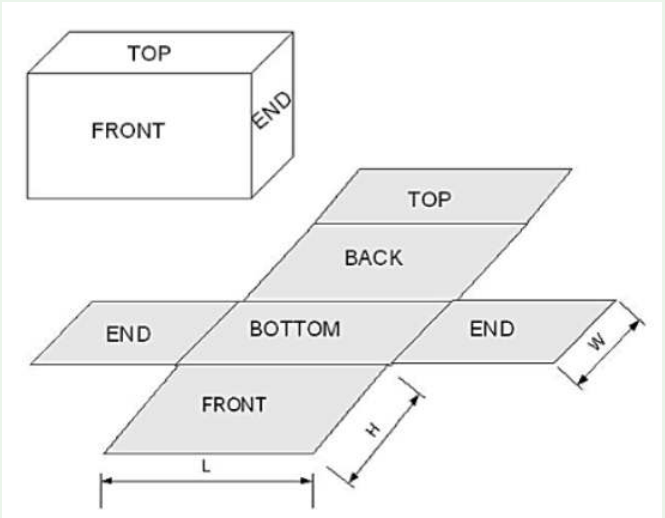


Figure 6.3.2: Surface area (3 dimensional)

Definition: Volume

The amount of space that a substance or object occupies, or that is enclosed within a container, especially when great. (3 dimensional)



Figure 6.3.3: Amount of space occupied by an object (3 dimensional)

Partner Activity 1

1. Why is area "squared"? i.e. $15\ \text{cm}^2$
2. Why is volume "cubed"? i.e. $40\ \text{liters}^3$

Partner Activity 2

Think *inside* the box and approximate the Shaded Area: (area of a square is base times height)

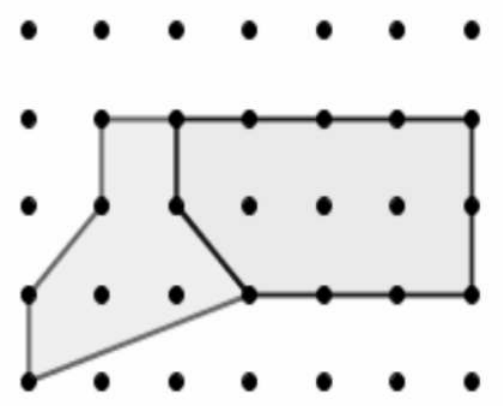


Figure 6.3.4

Partner Activity 3

Think *around* the box (surface area) and approximate the Shaded Area: (How many sides are not seen in the picture, which must be included in the final answer?)

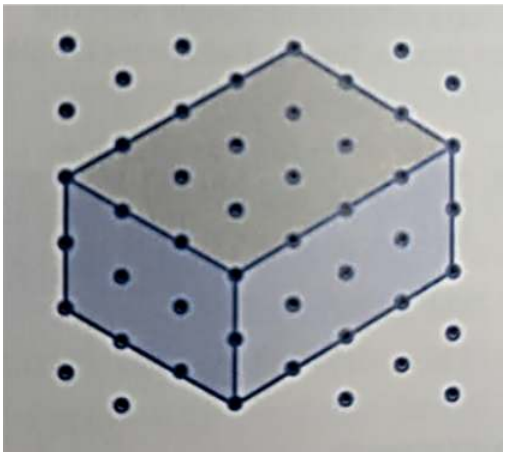


Figure 6.3.5

Partner Activity 4

Think *inside* the box (volume) and approximate the Shaded Area: Volume is base time's height times width.

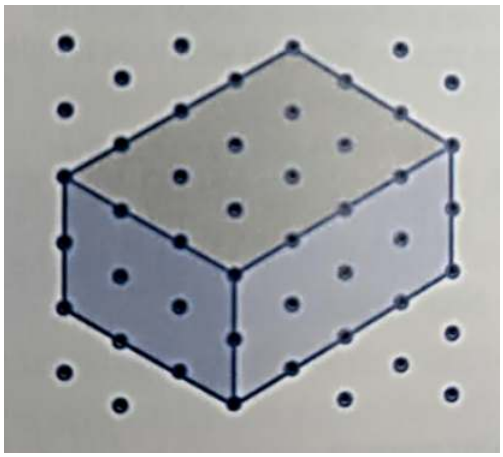
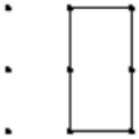


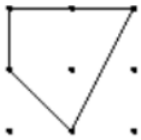
Figure 6.3.6

Practice Problems

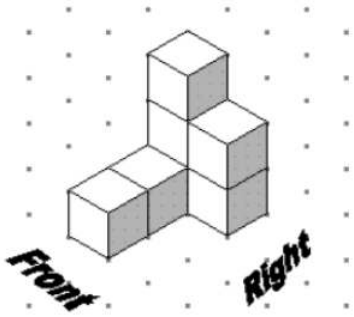
1. Explain the difference between area, surface area, and volume.
2. Estimate the area of the following shapes:

a. 

b. 

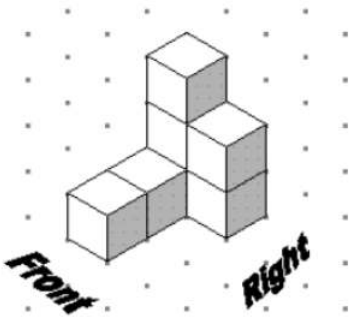
c. 

3. Find the Surface Area of the following shapes:

a. 

b. 

4. Find the Volume of the following shapes:

a. 

b. 

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7.12: Area, Surface Area and Volume Formulas

Area formulas

Let b = base

Let h = height

Let s = side

Let r = radius

Table 6.5.1: Area formulas

Shape Name	Shape	Area Formula
Rectangle		$A = bh$
Square		$A = bh$ $A = s^2$
Parallelogram		$A = bh$
Triangle		$A = \frac{1}{2}bh$
Circle		$A = \pi r^2$
Trapezoid		$A = \frac{1}{2}h\left(b_1 + b_2\right)$

Surface Area Formulas

Variables:

SA = Surface Area

B = area of the base of the figure

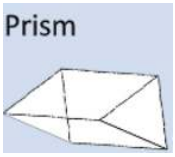
P = perimeter of the base of the figure

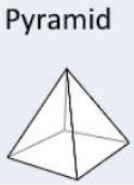
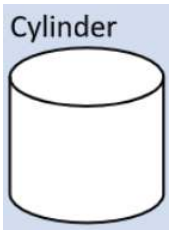
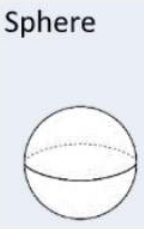
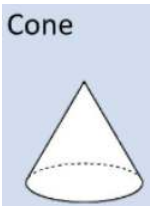
h = height

s = slant height

r = radius

Table 6.5.2: Surface Area formulas

Geometric Figure	Surface Area Formula	Surface Area Meaning
	$SA = 2B + Ph$	Find the area of each face. Add up all areas.

Geometric Figure	Surface Area Formula	Surface Area Meaning
	$SA = B + \frac{1}{2}sP$	Find the area of each face. Add up all areas.
	$SA = 2B + 2\pi rh$	Find the area of the base, times 2, then add the areas to the areas of the rectangle, which is the circumference times the height.
	$SA = 4\pi r^2$	Find the area of the great circle and multiply it by 4.
	$SA = B + \pi rS$	Find the area of the base and add the product of the radius times the slant height times PI.

Volume Formulas

Variables:

SA = Surface Area

B = area of the base of the figure

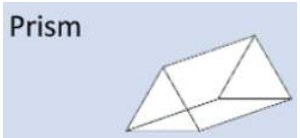
P = perimeter of the base of the figure

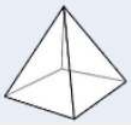
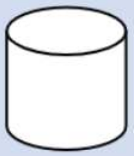
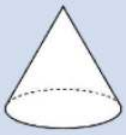
h = height

s = slant height

r = radius

Table 6.5.3: Volume formulas

Geometric Figure	Volume Formula	Volume Meaning
	$V = Bh$	Find the area of the base and multiply it by the height

Geometric Figure	Volume Formula	Volume Meaning
Pyramid 	$V = \frac{1}{3}Bh$	Find the area of the base and multiply it by 1/3 of the height.
Cylinder 	$V = Bh$	Find the area of the base and multiply it by the height.
Sphere 	$V = \frac{4}{3}\pi r^3$	Find the area of the great circle and multiply it by the radius and then multiply it by 4/3.
Cone 	$V = \frac{1}{3}Bh$	Find the area of the base and multiply it by 1/3 of the height.

Example 7.12.1

Find the area of a circle with diameter of 14 feet.

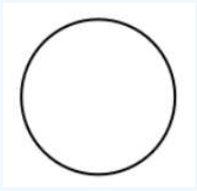


Figure 6.5.1

Solution

$$\begin{aligned} A &= \pi r^2 \\ &= \pi (7)^2 \\ &= 49\pi \text{feet}^2 \\ &= 153.86 \text{feet}^2 \end{aligned}$$

Example 7.12.2

Find the area of a trapezoid with a height of 12 inches, and bases of 24 and 10 inches.

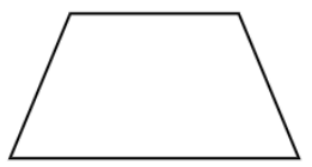


Figure 6.5.2

Solution

$$
\begin{aligned}
A &= \frac{1}{2}h\left(b_1+b_2\right) \\
&= \frac{1}{2}(12)(24+10) \\
&= 6(34) \\
&= 204 \text{ inches}^2
\end{aligned}
$$

Example 7.12.3

Find the surface area of a cone with a slant height of 8 cm and a radius of 3 cm.

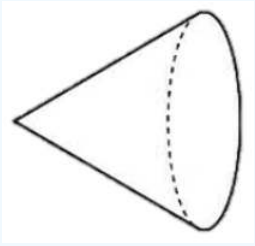


Figure 6.5.3

Solution

$$
\begin{aligned}
SA &= B+\pi rS \\
&= \left(\pi r^2\right)+\pi rs \\
&= \left(\pi\left(3^2\right)\right)+\pi(3)(8) \\
&= 9\pi+24\pi \\
&= 33\pi\text{cm}^2 \\
&= 103.62\text{cm}^2
\end{aligned}
$$

Example 7.12.4

Find the surface area of a rectangular pyramid with a slant height of 10 yards, a base width (b) of 8 yards and a base length (h) of 12 yards.

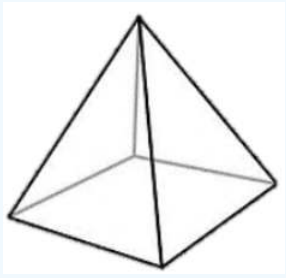


Figure 6.5.4

Solution

$$
\begin{aligned}
SA &= B+\frac{1}{2}sP \\
&= (bh)+\frac{1}{2}s(2b+2h) \\
&= (8)(12)+\frac{1}{2}(10)(2(8)+2(12)) \\
&= 96+\frac{1}{2}(10)(16+24) \\
&= 96+5(40) \\
&= 296 \text{ yards}^2
\end{aligned}
$$

Example 7.12.5

Find the volume of a sphere with a diameter of 6 meters.

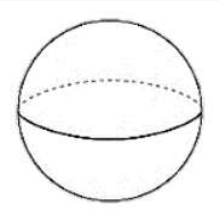


Figure 6.5.5

Solution

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(3)^3 \\ &= \frac{4}{3}(27\pi) \\ &= 36\pi \text{ meters}^3 \\ &= 113.04 \text{ meters}^3 \end{aligned}$$

Partner Activity 1

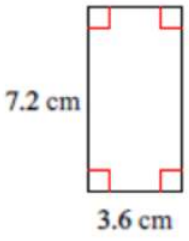
1. Find the area of a triangle with a base of 40 inches and a height of 60 inches.
2. Find the area of a square with a side of 15 feet.
3. Find the surface area of Earth, which has a diameter of 7917.5 miles. Use 3.14 for PI.
4. Find the volume of a can a soup, which has a radius of 2 inches and a height of 3 inches. Use 3.14 for PI.

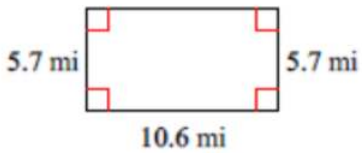
Practice Problems

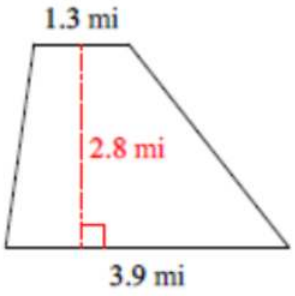
(Problems 1 – 4) Find the area of each circle with the given parameters. Use 3.14 for PI. Round your answer to the nearest tenth.

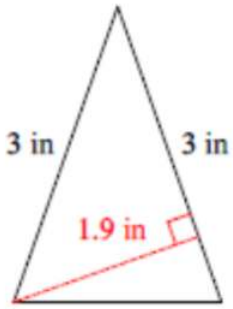
1. Radius = 9 cm
2. Diameter = 6 miles
3. Radius = 8.6 cm
4. Diameter = 14 meters

(Problems 5 – 8) Find the area of each polygon. Round answers to the nearest tenth.

5. 

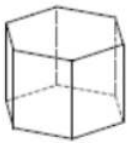
6. 

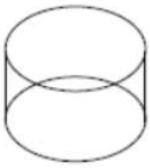
7. 

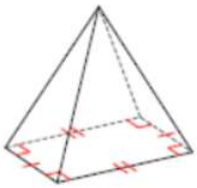
8. 

(Problems 9 – 12) Name each figure.

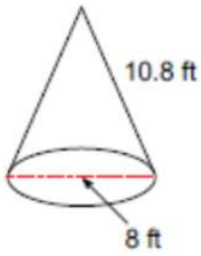
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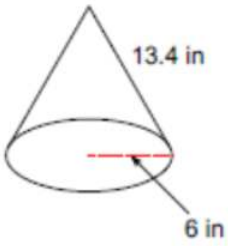
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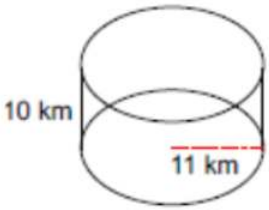
11. 

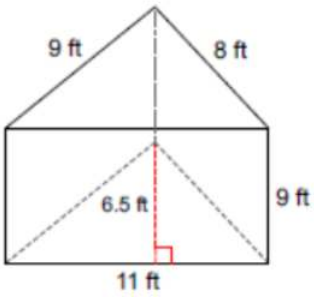
12. 

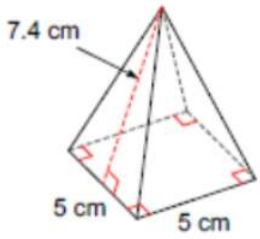
(Problems 13 – 17) Find the surface area of each figure. Leave your answers in terms of PI, if the answer contains PI. Round all other answers to the nearest hundredth.

13. 

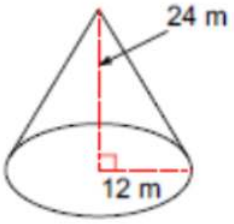
14. 

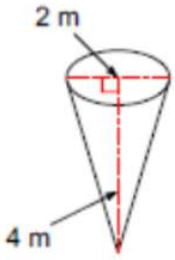
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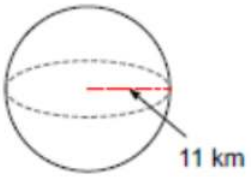
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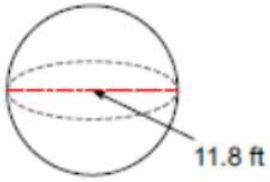
17. 

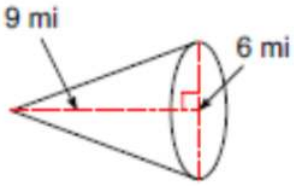
(Problems 18 – 25) Find the volume of each figure. Leave your answers in terms of PI, for answers that contain PI. Round all other answers to the nearest hundredth.

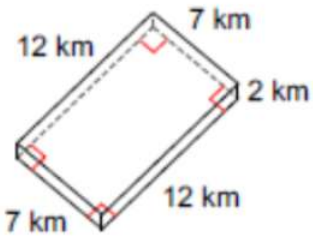
18. 

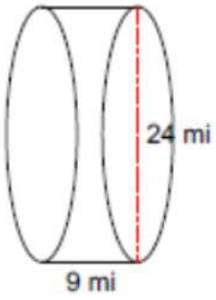
19. 

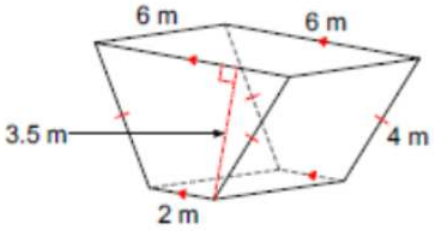
20. 

21. 

22. 

23. 

24. 

25. 

Extension: Methods of Teaching Mathematics

Part 1

Assessments:

1. What is the Difference between Formative and Summative Assessments? Which One is More Important?
2. Formative Assessment Examples and When to Use Them
3. Summative Assessment Examples and When to Use Them

Part 2

Write a Formative and Summative Assessment for Your Lesson Plan

Part 3

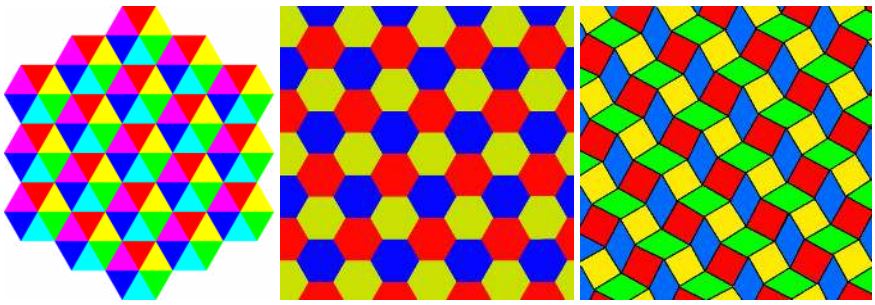
Make sure you are working on Khan Academy throughout the semester.

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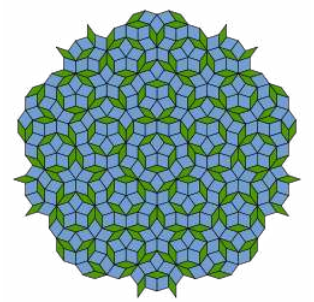
7.13: Geometry in Art and Science

Tessellations

A *tessellation*[1] is a design using one ore more geometric shapes with no overlaps and no gaps. The idea is that the design could be continued infinitely far to cover the whole plane (though of course we can only draw a small portion of it).



Many tessellations have translational symmetry, but it's not strictly necessary. The *Penrose tiling* shown below[2] does not have any translational symmetry.

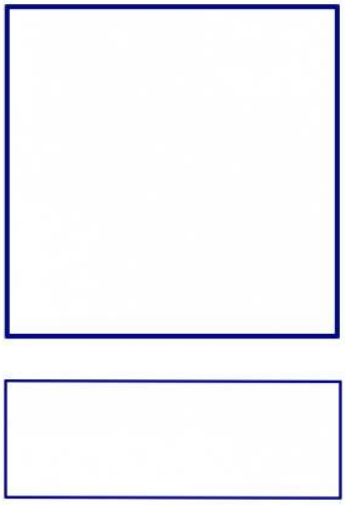


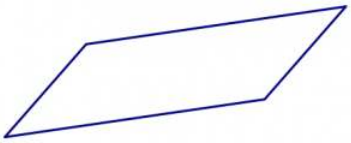
It's actually much harder to come up with these "aperiodic" tessellations than to come up with ones that have translational symmetry. So we'll focus on how to make symmetric tessellations.

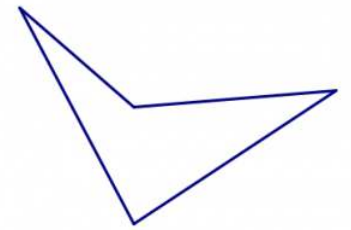
The first two tessellations above were made with a single geometric shape (called a *tile)* designed so that they can fit together without gaps or overlaps. The third design uses two basic tiles. Tessellations are often called *tilings*, and that's what you should think about: If I had tiles made in this shape, could I use them to tile my kitchen floor? Or would it be impossible?

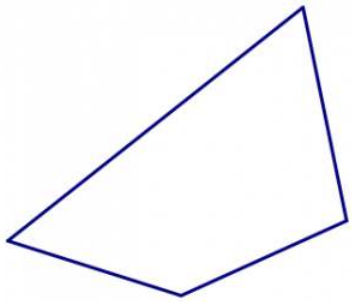
On Your Own

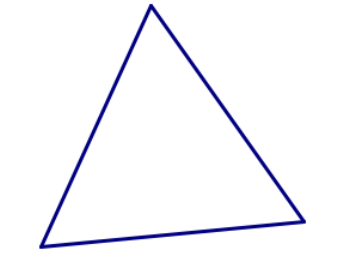
Work on these exercises on your own or with a partner. You will need lots of copies (maybe 10–15 each) of each shape below. In each problem, focus on just a single tile for making your tessellation.

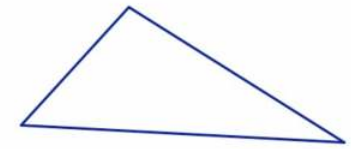


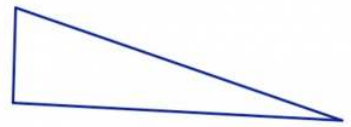


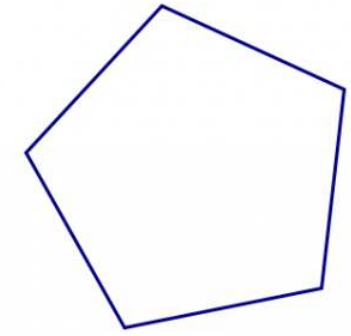


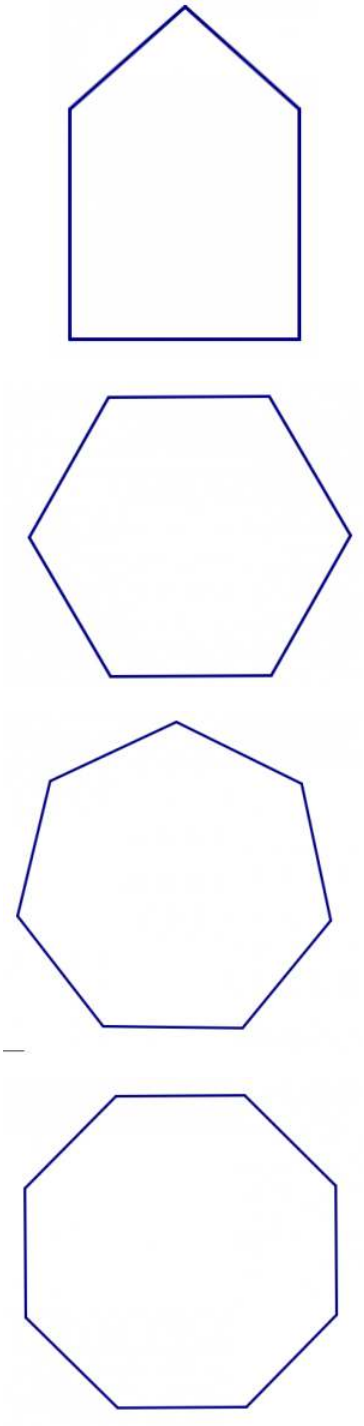












1. Start with the square tile. Can you fit the squares together in a pattern that could be continued forever, with no gaps and no overlaps? Can you do it in more than one way?
2. Now try one of the triangular tiles. Can you use many copies of a single triangle to tessellate the plane?
3. Repeat this process with each of the other tiles. Keep track of your findings.

? Think / Pair / Share

- Which of the tiles given above tessellate, and which do not?
- Do you have any conjectures based on this experience, about which shapes will tessellate and which will not?

Escher Drawings

The artist M.C. Escher created many works of art inspired by mathematics, including some very beautiful tessellations. Below you will see some images[3] inspired by his work. You can view the real thing at http://www.mcescher.com/ in the “Symmetry” gallery.

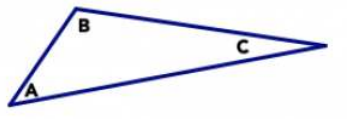


You can make your own Escher-like drawings using some facts that you learned while studying tessellations.

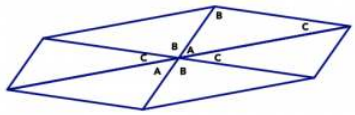
Theorem: Tessellations

Any triangle will tessellate. So will any quadrilateral.

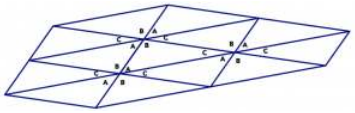
The explanation for this comes down to what you know about the sums of angles. The sum of the angles in a triangle is 180°.



So if you make six copies of a single triangle and put them together at a point so that each angle appears twice, there will be a total of 360° around the point, meaning the triangles fit together perfectly with no gaps and no overlaps.



You can then repeat this at every vertex, using more and more copies of the same triangles.



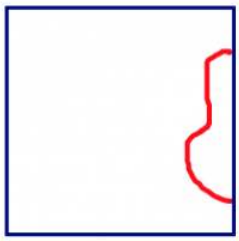
? Think / Pair / Share

- Use the fact that the sum of the angles in any quadrilateral is 360° to explain why every quadrilateral will tessellate.
- Use angles to explain why regular hexagons will tessellate.
- Explain why regular pentagons will not tessellate.

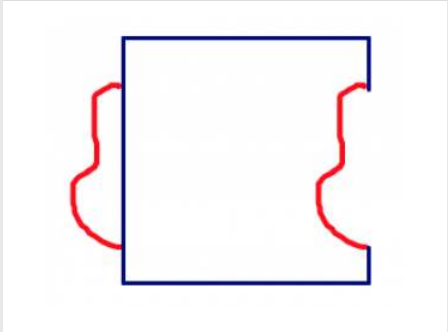
On Your Own

Work on the following exercises on your own or with a partner. Here’s how you can create your own Escher-like drawings.

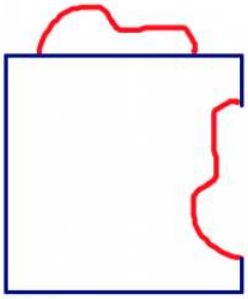
1. Select your basic tile. The first time you do this, it’s easiest to start with a simple shape that you know will tessellate, like an equilateral triangle, a square, or a regular hexagon.
2. Draw a “squiggle” on one side of your basic tile.



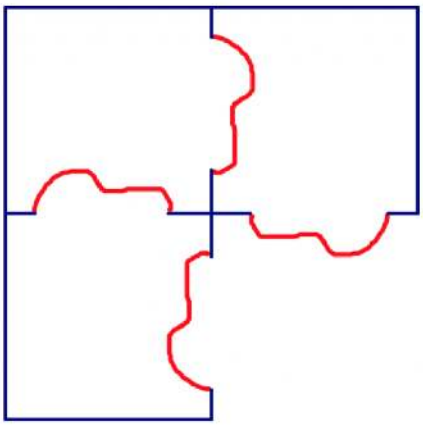
3. Cut out the squiggle, and move it to another side of your shape. You can either translate it straight across or rotate it.



or



4. It's important that the cut-out lines up along the new edge in the same place that it appeared on its original edge.
5. Tape the squiggle into its new location. This is your basic tile. On a large piece of paper, trace around your tile. Then move it the same way you moved the squiggle (translate or rotate) so that the squiggle fits in exactly where you cut it out.



6. The shape will still tessellate, so go ahead and fill up your paper.
7. Now get creative. Color in your basic shape to look like something — an animal? a flower? a colorful blob? Add color and design throughout the tessellation to transform it into your own Escher-like drawing.
8. If you want to try a more complicated version, cut two different squiggles out of two different sides, and move them both.

Building Towers

For this activity, you will need some construction materials:

- You'll need lots of toothpicks.
- You'll also need something to connect the toothpicks together. The best material for this is mini marshmallows; you can stick the ends of the toothpicks into the marshmallows to connect them. You can also use pieces of clay, bits of gummy candies, or other similar (sticky) material.

Problem 16

Try this as a warm-up activity. Grab exactly six toothpicks. Your job is to make four triangles using all six toothpicks. You cannot break any of the toothpicks or add any other materials besides the marshmallow connectors.

Problem 17

Now comes the main challenge. You have ten minutes to build the *tallest free-standing structure* that you can make. "Free-standing" means that it will stand up on its own. You can't hold it up or lean it against something. When the ten minutes are up, back away from your tower and measure its height.

Think / Pair / Share

Look at your own tower and at other students' towers. Talk about these questions:

- What design choices led to taller free-stranding structures? Why do you think that is?
- If you had another ten minutes to try this activity again, what would you do differently and why?

---

1. Triangular tessellation from pixababy [CC0]. Hexagonal and rhombic tessellations from Wikimedia Commons [Public domain]. ↵
2. Image via Wikimedia Commons [Public donmain]. ↵
3. Images from flickr [CC BY-NC-SA 2.0]. Birds by Sharon Drummond. Lizard tiles by Ben Lawson. ↵

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7.14: Problem Bank

Problem 18

In the Tangrams chapter, you first saw all 7 tangram pieces arranged into a square.

1. If the large square you made with all seven pieces is one whole, assign a (fractional) value to each of the seven tangram pieces. Justify your answers.
2. The tangram puzzle contains a small square. If the small square (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
3. The tangram set contains two large triangles. If a large triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
4. The tangram set contains one medium triangle. If the medium triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
5. The tangram set contains two small triangles. If a small triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.

Problem 19

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. A right triangle that is scalene.
2. A right triangle that is isosceles.
3. A right triangle that is equilateral.

Problem 20

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An acute triangle that is scalene.
2. An acute triangle that is isosceles.
3. An acute triangle that is equilateral.

Problem 21

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An obtuse triangle that is scalene.
2. An obtuse triangle that is isosceles.
3. An obtuse triangle that is equilateral.

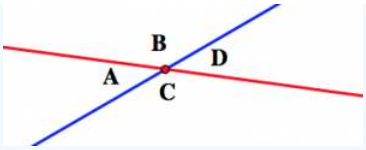
Problem 22

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An equiangular triangle that is scalene.
2. An equiangular triangle that is isosceles.
3. An equiangular triangle that is equilateral.

Problem 23

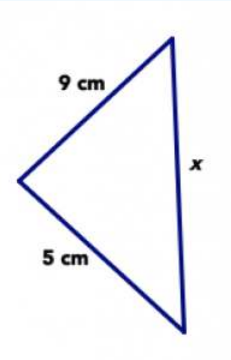
Look at the picture below, which shows two lines intersecting. Angles A and D are called “vertical angles,” and so are angles B and C.



Use this drawing to explain why vertical angles must have the same measure. (Hint: what is the sum of the measures of angle A angle B? How do you know?)

Problem 24

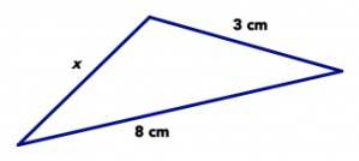
Answer the following questions about the triangle below. Be sure to focus on what you *know for sure* and not what the picture *looks like*.



1. Could it be true that $x = 4$ cm? Explain your answer.
2. Could it be true that $x = 20$ cm? Explain your answer.
3. Give three possible values of x, based on the information in the picture.

Problem 25

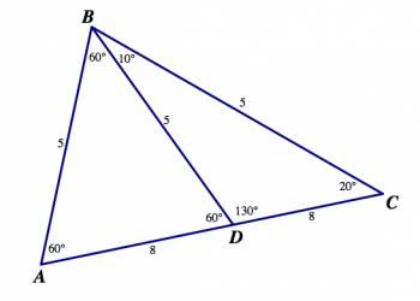
Answer the following questions about the triangle below. Be sure to focus on what you know for sure and not what the picture looks like.



1. If $x = 3$ cm, the triangle is isosceles. Is this possible? Explain your answer.
2. If $x = 8$ cm, the triangle is isosceles. Is this possible? Explain your answer.
3. Give three *impossible* values of x, based on the information in the picture.

Problem 26

Prof. Faber drew this picture on the board, saying it showed three triangles: $\triangle ABC$, $\triangle ABD$, and $\triangle CBD$. Side lengths and angle measurements are shown for each of the triangles.



There are **lots of mistakes** in this picture. Use what you know about side lengths and angles in triangles to find all the mistakes you can. For each mistake, say what is wrong with the picture, and why it's a mistake. Explain your thinking as clearly as you can.

Problem 27

Because of SSS congruence, triangles are exceptionally sturdy. This means they are used frequently in architecture and design to provide supports for buildings, bridges, and other man-made objects. Take your camera with you, and find several places in your neighborhood or near your campus that use triangular supports. Snap a picture, and describe what the structure is and where you see the triangles.

Problem 28

It is possible to create designs that have multiple symmetries. See if you can find images (or create your own!) that have both:

1. reflection symmetry and rotational symmetry,
2. reflection symmetry and translational symmetry, and
3. rotational symmetry and translational symmetry.

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CHAPTER OVERVIEW

8: Additional Activities

This page is under construction.

8.1: 1. Power of Patterns- Domino Tiling
8.2: 2. Regular Tiling
8.3: 3. Festive Folding
8.4: 4. Platonic Solids
8.5: 5. Egyptian Pizza
8.6: Semi Regular Tilings

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8.1: 1. Power of Patterns- Domino Tiling

This page is a draft and is under active development.



Power of Patterns: Domino Tiling

Grade 5-6

OVERVIEW & PURPOSE

Students gain an understanding of visualizing problems and explore the mathematical world of tiling.

OBJECTIVES

1. Use a systematic approach to discover the pattern of the different tiling (on 2 x n rectangle)
2. Use knowledge of Fibonacci numbers to help create a proof

MATERIALS NEEDED

1. Dominoes (6 dominoes per group)
2. Paper

ACTIVITY

1. Introduce the concept of tiling to students: focusing on tiling 2 x n rectangles with dominoes. (Provide an explanation of “2 x n” and examples of how dominoes can be moved)
2. Rules:
    1. There can be no gaps or overlapping on the rectangle
    2. Rotating tiles to create different variations counts as another way of tiling. An example below: these are considered two different tiling



3. Ask students to create a T-Table to keep track of their tiling. Labeled: ‘n’ to indicate how many dominoes they are using and ‘number of tilings’ to indicate the number of possible tiling.
4. Students will work through the table. Encourage students to come up with a formula for their answer. With prior knowledge: students will discover the sequence is the Fibonacci sequence.

SOLVING

- Fibonacci sequence: You add the two previous numbers and continue in the same pattern. (E.g. if your first two numbers are 1 and 2, you add 1+2 = 3, which makes 3 your third number.
- Go over the various tiling patterns as a class

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8.2: 2. Regular Tiling

This page is a draft and is under active development.



Goal: To appreciate polygons and support the idea that there are three regular polygons that be tessellated.

Terminology:

- A polygon is a closed 2-dimensional figure with straight sides
  - An n-gon is a polygon with exactly n sides
  - A regular n-gon is a polygon with exactly n sides, where all sides are of equal length and all interior angles of the polygon are equal. The sum of the interior angles of a regular n-gon is 180°(n - 2). It follows that each interior angle must measure 180°(n - 2)/n. So:
    - A regular 3-gon is an equilateral triangle. Each interior angle is 60°
    - A regular 4-gon is a square. Each interior angle is 90°
    - A regular 5-gon is a regular pentagon. Each interior angle is 108°
    - A regular 6-gon is a regular hexagon. Each interior angle is 120°
    - A regular 7-gon is a regular heptagon. Each interior angle is 900/7°, or approximately 128.6°
    - A regular 8-gon is a regular octagon. Each interior angle is 135°

Activity:

Suppose I want to tape the same regular n-gons together to make 2-dimensional shapes. What are my options? I don't want to bend or fold the n-gons. Let's just concentrate on the corners of these objects.

Fact:

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8.3: 3. Festive Folding

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Holiday Decorations (as an excuse to problem solve in geometry!)

References:

- The Mathlab.com, Making the Dice of the Gods, http://www.themathlab.com/wonders/godsdice/godsdice.htm (as of Oct. 6, 2012)
    - Aunt Annie's Crafts, Platonic solids, http://www.auntannie.com/Geometric/PlatonicSolids/ (as if Oct. 6, 2012).

Grade level: This activity can be tailored to various grade levels

Outcomes: Students will construct holiday decorations (really the Platonic solids). A large variety of geometric topics can be built into the lesson and/or discussed afterword. Any platonic solid can be made but I'll stick to the cube and icosahedrons. The cube because it is geometrically the simplest and makes nice gift boxes and an icosahedron because I think it is visually the most striking.

Motivational strategies:

- The teacher gets to choose the appropriate amount of mathematics to put into the activity and where to put the mathematics in. You can measure angles and lengths, discuss properties of circles and the regular polygons, apply the Pythagorean theorem to build a cube of a particular size, use the law of cosines or sine law to build an icosahedron of a particular size, discuss surface area and of course talk about the Platonic solids and other polyhedra.

Materials/Equipment:

- Cardstock, Bristol board, used greeting cards or some other stuff but not too thick paper. You'll need lots of it for a class.
    - White glue.

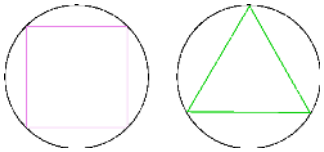
Process: You might want to look at the websites in the references to begin to see potential final versions of the decorations and make one or two on your own to get a feeling for time involved and difficulty. They are really just the Platonic solids with or without flaps. Unlike the website descriptions, however, we'll use a circle cutter to speed the activity along. On the next page, I will outline the directions for making the decorations. After that, I will discuss the potential geometry you can discuss before, after or during the construction of the decorations.

CONSTRUCTION DIRECTIONS:

Step 1: Fix a radius on your circle cutter and mass produce a lot of circles. For example, if students were making cubes they'd need 6 per cube, if they make the icosahedrons they'll

need 20 circles (in my opinion icosahedrons with flaps are the nicest to make!) Keeping the radius fairly small (eg 4 inches on my non-metric circle cutter) would let you get away with using up less paper and makes a nice sized decoration. Produce some extra circles for errors and a geometry discussion with students.

Step 2: Take one of the circles and inscribe the template square (for cubes) or equilateral triangle (for icosahedrons). (Fold to get the center of the circle and for the square, the folds locate the vertices for the square. For the triangle, you can pull out a protractor and measure off 3 120 degree angles to get the vertices.) Cut out the template shape.

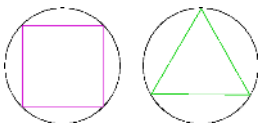


Step 3: Trace this shape and trace it out on as many circles as you need. Work fast – slight imperfections are ok. Now take a rule and a nail (or something else a little sharp) and quickly trace over the lines to break the surface of the cardstock. Fold the sides up to get little 'cups' with a square or triangular base. I have gone through steps one to three and made over 120 circles folded into cups in one hour.

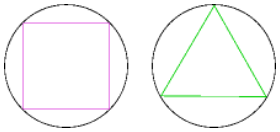
Step 4: Bring nets of the cube and icosahedrons (or pictures from the websites mentioned) to help guide the students in putting the shape together. Students will be gluing the flaps together to form the cube or icosahedrons. You have the options of gluing the flaps out or flaps in. The icosahedrons look really neat with the flaps out. Given them the white glue and lead them through putting it together. Again, make one yourself in advance so you can get a feel for how much guidance your students will need and if they need an older helper to finish it off.

Some ideas for the geometry:

1. Before you begin, show students a finished decoration and how it is actually constructed from squares/ equilateral triangles inscribed in circles. Show them a circle cutter and discuss how it works (a compass with a blade is much cooler than a compass with a pencil in it!)
2. Give them a blank circle and have them problem solve how to find its center and inscribe a square in it. (Folding will do it). If they know about angles and protractors, ask them how to inscribe an equilateral triangle into it and ask why it works. You have the option (if appropriate for your grade level) to go into the general idea of inscribing regular polygons into circles. The regular n sided polygon can be thought of as made up of n identical triangles each triangle having two sides of length the radius of the circle and angle (360/n) degrees. (See picture below) From here you can get to the formula for the interior angles of a regular polygon.



3. For lower grades, you can simply ask them to check the angles are equal (and what they are) and check that the side lengths are the same to confirm we have an equilateral triangle/square.
4. Ask: I want to make a cube with side length X what radius should I set the circle cutter to get the cube? You need the Pythagorean Theorem to answer that one! Warning: If your circle cutter is not in metric it is a little ugly. (8.3 inches is harder to explain than 8.3 cm) (See below)
5. Ask: I want to make an icosahedron with side length X, what radius should I set the circle cutter to get the equilateral triangles. You need the Law of Cosines or Law of Sines to get that one! (see below)



6. You can talk about area and surface area. How much paper was wasted? What is the total surface area of the resulting cube/icosahedrons. (One could always find the area of an equilateral triangle by measuring).
7. And of course, you can talk about the Platonic solids as well! In fact you can also construct other polyhedra (the Archimedean solids for example) using this technique. I'd need the Pythagorean theorem to construct a cuboctahedron.



Contributors

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8.4: 4. Platonic Solids

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Goal: To appreciate polygons and support the idea that there are exactly 5 platonic solids.

Terminology:

- A polygon is a closed 2-dimensional figure with straight sides
  - An n-gon is a polygon with exactly n sides
  - A regular n-gon is a polygon with exactly n sides, where all sides are of equal length and all interior angles of the polygon are equal. The sum of the interior angles of a regular n-gon is 180°(n - 2). It follows that each interior angle must measure 180°(n - 2)/n. So:
    - A regular 3-gon is an equilateral triangle. Each interior angle is 60°
    - A regular 4-gon is a square. Each interior angle is 90°
    - A regular 5-gon is a regular pentagon. Each interior angle is 108°
    - A regular 6-gon is a regular hexagon. Each interior angle is 120°
    - A regular 7-gon is a regular heptagon. Each interior angle is 900/7°, or approximately 128.6°
    - A regular 8-gon is a regular octagon. Each interior angle is 135°

Activity:

Suppose I want to tape regular n-gons together to make 3-dimensional shapes. I can make a cube, for example, by taping squares together. What are my options? I don't want to bend or fold the n-gons. Let's just concentrate on the corners of these objects.

Fact: To make a corner I'll need at least 3 regular n-gons.

Try making corners out of 3 n-gons. Which ones will work? Justify your conclusions.

Now try using four n-gons to make corners. Which ones will work? Justify your conclusions.

What about using five n-gons? Justify your conclusions.

Can we make corners out of six or more n-gons? Justify your conclusions.

A platonic solid is a 3-dimensional object made by taping together regular n-gons in such a way that each corner is the same, and has the same number of n-gons around it. Using the data you've gathered, please complete the following statement:

I have found that there are _____ ways to tape regular n-gons together to make the corners of a platonic solid. Therefore, there are at most _____ platonic solids.



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8.5: 5. Egyptian Pizza

This page is a draft and is under active development.

Pizza Fair & Square
Grade 5 / Math & Art Integration
~ 1.5 Math Blocks

Big Ideas/Rationale/Essential Understandings/Inquiry

Number Sense

Number sense is an intuition about numbers. It develops when students connect numbers to their own real-life experiences and when students use benchmarks and referents. Number sense can be developed by providing rich mathematical tasks that allow students to make connections to their experiences and their previous learning. (Program of Studies: Mathematics)

Outcomes

Program of Studies: Mathematics Kindergarten to Grade 12 ***General Outcome***: Develop number sense. ***Specific Outcomes***: 8. Demonstrate an understanding of fractions less than or equal to one by using concrete, pictorial and symbolic representation to:

- Name and record fractions for the parts of a whole or a set
- Compare and order fractions
- Model and explain that different wholes, two identical fractions may not represent the same quantity
- Provide examples of where fractions are used
- [C, CN, PS, R, V]

Assessment

What to look for:

- Procedural Knowledge – Students can identify, extend, and create Egyptian fractions
- Problem Solving Skills – Students can use different strategies to create and solve Egyptian fraction problems

Student Name	**Conceptual Understanding:** *Demonstrates and explains:*	**Procedural Knowledge:** *Identifies, describes,*	**Problem Solving Skills:** *Creates and solves problems*	**Communication:** *Records and explains reasoning and*

	- Egyptian fraction rule - Using manipulatives to show understanding	*extends the concept of Egyptian fractions to solve puzzle*	*using appropriate strategies - equally dividing parts, following a '1/n strategy'*	*procedures clearly and completely, including appropriate terminology*

** For the assessment above use approved levels, symbols, or numeric ratings.*

Teacher involvement:

- Watch an interactive math story about fractions for a recap:
- https://www.youtube.com/watch?v=Rr9kN2fQYPw
- The teacher will walk around and visit pairs/groups to check up on progress and discuss their strategies (implementing differentiation).
- The teacher will use the rubric as a formative assessment.

Differentiation for Learners

- For all students, especially ELL (English Language Learners) students, working with partners and group discussions will provide an opportunity for students to learn from each other, and a deeper understanding of appropriate strategies to use.
- For all students, especially ELL students and visual learners, clear step-by-step oral instructions along with a visual demonstration of the instructions will support their understanding (examples of visual artifacts shown below). Using manipulatives and other resources are excellent ways of representing patterns visually.
- Manipulatives: Magnetic Fraction Circles, Plastic Fraction Circles, etc.

Resources/Materials

- *Interactive math story:* **https://www.youtube.com/watch?v=Rr9kN2fQYPw**
- Manipulatives: Fraction Circles (pictures above)
- Math journal
- Pencil/eraser

Introduction/Hook

Students will be asked to gather at the carpet area to watch an interactive math story about fractions: **https://www.youtube.com/watch?v=Rr9kN2fQYPw**

Time	Teacher Activities	Learner Activities

	Use the control signal (clapping hand rhythm) to get the students attention. Gather the students to the carpet area.	Students respond to the control signal. Students are sitting, quiet, and actively listening to instructions.
10 min.	Watch the video: • Ask questions during the reading: - What do you notice? ◦ Have you seen this in real life? - Can you guess the fraction? • Ask questions after the reading: - What have you learned? ◦ What fractions have you noticed? ◦ Did you see any Egyptian fractions? How did you know it was an Egyptian fraction?	Students are sitting, quiet, and actively listening and participating.

Development

Students will be given the instructions. Students will solve a math puzzle by dividing the number of pizzas between friends equally using the concept of Egyptian fractions.

Math Puzzle:

Time	Teacher Activities	Learner Activities

50 min	Questions: Suppose you ordered 5 pizzas to share among 8 friends for a party. *Pause for a minute and decide with your students how they would solve this problem before carrying on....* What if there were only 4 pizzas, not 5 to be split amongst 8 friends? Explain to students how if that was the case then they would all get at least half a loaf, so you would use 4 of the pizzas to give all 8 of them half a pizza each. But in this question, we have one whole pizza left. Now it is easy to divide one pizza into 8, so they get an extra eighth of a pizza each and all the pizzas are divided equally between the 5 friends. On the picture here (Let's assume we have rectangularly shaped pizzas) they each receive one red part ($^1/_2$ a pizza) and one green part ($^1/_8$ of a pizza): and $^5/_8 = {}^1/_2 + {}^1/_8$	Students work in groups to discuss the math puzzle and brainstorm strategies they would go about to solving the problem. Students participate in *think, pair, share,* to discuss their thinking to the class. Students use manipulates to try to solve this puzzle. Students first work independently, and then in pairs to solve the puzzle. Students that quickly solve the puzzle are given more challenging puzzles to complete using Egyptian fractions: • Suppose you had 3 pizzas to share between 4 people, how would you do this? • How about 2 pizzas shared amongst 5 people? • Or 4 pizzas shared among 5 people?

Mathematical Method/Background

Mathematical Method: ***Fibonacci's Greedy Algorithm***

1. Using Fibonacci's Greedy algorithm to find Egyptian fractions with a sum of unit fractions is as follows:

- Choose the largest unit fraction we can, write it down and subtract it
- Repeat this on the remainder until we find the remainder is itself a unit fraction not equal to one already written down.
- At this point, we could stop or else continue splitting the unit fraction into smaller fractions.
- To use this method to find a set of unit fractions that sum to 1:
- So we would start with 1/2 as the largest unit fraction less than 1:
- 1 = 1/2 + (1/2 remaining)

- so we repeat the process on the remainder: the largest fraction less than 1/2 is 1/3:
- 1 = 1/2 + 1/3 + (1/6 remaining).
- We could stop now or else continue with 1/7 as the largest unit fraction less than 1/6 ...
- 1 = 1/2 + 1/3 + 1/7 + ...

Mathematical Background:

1. The Egyptians of 3000 BC had an interesting way to represent fractions. Although they had a notation for $^1/_2$ and $^1/_3$ and $^1/_4$ and so on (these are called **reciprocals** or **unit fractions** since they are $^1/_n$ for some number n), their notation did not allow them to write $^2/_5$ or $^3/_4$ or $^4/_7$ as we would today.
2. It turns out that Egyptian fractions are not only a very practical solution to *some* everyday problems today but are interesting in their own right. They had practical uses in the ancient Egyptian method of multiplying and dividing, and every fraction $^t/_b$ can always be written as an Egyptian fraction
3. Remember that

o $^t/_b$**<1** and o if t=1 the problem is solved since $^t/_b$ is already a unit fraction, so o we are interested in those fractions where **t>1**.

The method is to find the **biggest unit fraction we can and take it from** $^t/_b$ and hence its other name - the **greedy algorithm**.

With what is left, we repeat the process. We will show that this series of unit fractions always decreases, never repeats a fraction and eventually will stop. Such processes are now called *algorithms* and this is an example of a *greedy algorithm* since we (greedily) take the largest unit fraction we can and then repeat on the remainder.

Closure

Total Time: 5 minutes

Time	Teacher Activities	Learner Activities
5 min.	• Invite students to share their solution and identify the strategies they used to find the solution. Did they use manipulatives? What technique helped them solve the puzzle? What problems were challenging, and why?	Students are participating in group discussion.



Source:

Special Thanks to Noura Ismail.

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8.6: Semi Regular Tilings

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CHAPTER OVERVIEW

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9.1: Appendix A- Common Core State Standards, Mathematics K-6

The California Common Core State Standards, Mathematics, K Through 6

Decoding the Standards: CCSS.MATH.CONTENT.K.CC.B.4.C

Common Core State Standards. Mathematics. Content. Kindergarten. Counting and Cardinality. B = second group in Counting and Cardinality. 4 = 4th standard listed in Counting and Cardinality. C = third part of standard B.4

Kindergarten

Counting & Cardinality

Know number names and the count sequence.

CCSS.MATH.CONTENT.K.CC.A.1
Count to 100 by ones and by tens.

CCSS.MATH.CONTENT.K.CC.A.2
Count forward beginning from a given number within the known sequence (instead of having to begin at 1).

CCSS.MATH.CONTENT.K.CC.A.3
Write numbers from 0 to 20. Represent a number of objects with a written numeral 0-20 (with 0 representing a count of no objects).

Count to tell the number of objects.

CCSS.MATH.CONTENT.K.CC.B.4
Understand the relationship between numbers and quantities; connect counting to cardinality.

- CCSS.MATH.CONTENT.K.CC.B.4.A
  When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.
- CCSS.MATH.CONTENT.K.CC.B.4.B
  Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.
- CCSS.MATH.CONTENT.K.CC.B.4.C
  Understand that each successive number name refers to a quantity that is one larger.

CCSS.MATH.CONTENT.K.CC.B.5
Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1-20, count out that many objects.

Compare numbers.

CCSS.MATH.CONTENT.K.CC.C.6
Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.1

CCSS.MATH.CONTENT.K.CC.C.7
Compare two numbers between 1 and 10 presented as written numerals.

Operations & Algebraic Thinking

Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

CCSS.MATH.CONTENT.K.OA.A.1
Represent addition and subtraction with objects, fingers, mental images, drawings1, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

CCSS.MATH.CONTENT.K.OA.A.2
Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.

CCSS.MATH.CONTENT.K.OA.A.3
Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).

CCSS.MATH.CONTENT.K.OA.A.4
For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.

CCSS.MATH.CONTENT.K.OA.A.5
Fluently add and subtract within 5.

Number & Operations in Base Ten

Work with numbers 11-19 to gain foundations for place value.

CCSS.MATH.CONTENT.K.NBT.A.1
Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (such as 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Measurement & Data

Describe and compare measurable attributes.

CCSS.MATH.CONTENT.K.MD.A.1
Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.

CCSS.MATH.CONTENT.K.MD.A.2
Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

Classify objects and count the number of objects in each category.

CCSS.MATH.CONTENT.K.MD.B.3
Classify objects into given categories; count the numbers of objects in each category and sort the categories by count.

Geometry

Identify and describe shapes.

CCSS.MATH.CONTENT.K.G.A.1
Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.

CCSS.MATH.CONTENT.K.G.A.2
Correctly name shapes regardless of their orientations or overall size.

CCSS.MATH.CONTENT.K.G.A.3
Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Analyze, compare, create, and compose shapes.

CCSS.MATH.CONTENT.K.G.B.4
Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).

CCSS.MATH.CONTENT.K.G.B.5
Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.

CCSS.MATH.CONTENT.K.G.B.6
Compose simple shapes to form larger shapes. *For example, "Can you join these two triangles with full sides touching to make a rectangle*?"

Grade 1

Operations & Algebraic Thinking

Represent and solve problems involving addition and subtraction.

CCSS.MATH.CONTENT.1.OA.A.1
Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

CCSS.MATH.CONTENT.1.OA.A.2
Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Understand and apply properties of operations and the relationship between addition and subtraction.

CCSS.MATH.CONTENT.1.OA.B.3
Apply properties of operations as strategies to add and subtract.2 *Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)*

CCSS.MATH.CONTENT.1.OA.B.4
Understand subtraction as an unknown-addend problem. *For example, subtract 10 - 8 by finding the number that makes 10 when added to 8.*

Add and subtract within 20.

CCSS.MATH.CONTENT.1.OA.C.5
Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).

CCSS.MATH.CONTENT.1.OA.C.6
Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 - 4 = 13 - 3 - 1 = 10 - 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 - 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).

Work with addition and subtraction equations.

CCSS.MATH.CONTENT.1.OA.D.7
Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 - 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.

CCSS.MATH.CONTENT.1.OA.D.8
Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = _ - 3, 6 + 6 = _.*

Number & Operations in Base Ten

Extend the counting sequence.

CCSS.MATH.CONTENT.1.NBT.A.1
Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.

Understand place value.

CCSS.MATH.CONTENT.1.NBT.B.2
Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

CCSS.MATH.CONTENT.1.NBT.B.2.A
10 can be thought of as a bundle of ten ones — called a "ten."

CCSS.MATH.CONTENT.1.NBT.B.2.B
The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

CCSS.MATH.CONTENT.1.NBT.B.2.C
The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

CCSS.MATH.CONTENT.1.NBT.B.3
Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.

Use place value understanding and properties of operations to add and subtract.

CCSS.MATH.CONTENT.1.NBT.C.4
Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

CCSS.MATH.CONTENT.1.NBT.C.5
Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.

CCSS.MATH.CONTENT.1.NBT.C.6
Subtract multiples of 10 in the range 10-90 from multiples of 10 in the range 10-90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Measurement & Data

Measure lengths indirectly and by iterating length units.

CCSS.MATH.CONTENT.1.MD.A.1
Order three objects by length; compare the lengths of two objects indirectly by using a third object.

CCSS.MATH.CONTENT.1.MD.A.2
Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps.

Tell and write time.

CCSS.MATH.CONTENT.1.MD.B.3
Tell and write time in hours and half-hours using analog and digital clocks.

Represent and interpret data.

CCSS.MATH.CONTENT.1.MD.C.4
Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

Geometry

Reason with shapes and their attributes.

CCSS.MATH.CONTENT.1.G.A.1
Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.

CCSS.MATH.CONTENT.1.G.A.2
Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.

CCSS.MATH.CONTENT.1.G.A.3
Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*,

and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

Grade 2

Operations & Algebraic Thinking

Represent and solve problems involving addition and subtraction.

CCSS.MATH.CONTENT.2.OA.A.1
Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Add and subtract within 20.

CCSS.MATH.CONTENT.2.OA.B.2
Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

Work with equal groups of objects to gain foundations for multiplication.

CCSS.MATH.CONTENT.2.OA.C.3
Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.

CCSS.MATH.CONTENT.2.OA.C.4
Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Number & Operations in Base Ten

Understand place value.

CCSS.MATH.CONTENT.2.NBT.A.1
Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

> CCSS.MATH.CONTENT.2.NBT.A.1.A
> 100 can be thought of as a bundle of ten tens — called a "hundred."
>
> CCSS.MATH.CONTENT.2.NBT.A.1.B
> The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).

CCSS.MATH.CONTENT.2.NBT.A.2
Count within 1000; skip-count by 5s, 10s, and 100s.

CCSS.MATH.CONTENT.2.NBT.A.3
Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

CCSS.MATH.CONTENT.2.NBT.A.4
Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.

Use place value understanding and properties of operations to add and subtract.

CCSS.MATH.CONTENT.2.NBT.B.5
Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

CCSS.MATH.CONTENT.2.NBT.B.6
Add up to four two-digit numbers using strategies based on place value and properties of operations.

CCSS.MATH.CONTENT.2.NBT.B.7
Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

CCSS.MATH.CONTENT.2.NBT.B.8
Mentally add 10 or 100 to a given number 100-900, and mentally subtract 10 or 100 from a given number 100-900.

CCSS.MATH.CONTENT.2.NBT.B.9
Explain why addition and subtraction strategies work, using place value and the properties of operations.

Measurement & Data

Measure and estimate lengths in standard units.

CCSS.MATH.CONTENT.2.MD.A.1
Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

CCSS.MATH.CONTENT.2.MD.A.2
Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

CCSS.MATH.CONTENT.2.MD.A.3
Estimate lengths using units of inches, feet, centimeters, and meters.

CCSS.MATH.CONTENT.2.MD.A.4
Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

Relate addition and subtraction to length.

CCSS.MATH.CONTENT.2.MD.B.5
Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

CCSS.MATH.CONTENT.2.MD.B.6
Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram.

Work with time and money.

CCSS.MATH.CONTENT.2.MD.C.7
Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m.

CCSS.MATH.CONTENT.2.MD.C.8
Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using $ and ¢ symbols appropriately. Example: If you have 2 dimes and 3 pennies, how many cents do you have?

Represent and interpret data.

CCSS.MATH.CONTENT.2.MD.D.9
Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.

CCSS.MATH.CONTENT.2.MD.D.10
Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems1 using information presented in a bar graph.

Geometry

Reason with shapes and their attributes.

CCSS.MATH.CONTENT.2.G.A.1

Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces. Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

CCSS.MATH.CONTENT.2.G.A.2

Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

CCSS.MATH.CONTENT.2.G.A.3

Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Grade 3

Operations & Algebraic Thinking

Represent and solve problems involving multiplication and division.

CCSS.MATH.CONTENT.3.OA.A.1

Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as 5 × 7.*

CCSS.MATH.CONTENT.3.OA.A.2

Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.*

CCSS.MATH.CONTENT.3.OA.A.3

Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.1

CCSS.MATH.CONTENT.3.OA.A.4

Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = _ ÷ 3, 6 × 6 = ?*

Understand properties of multiplication and the relationship between multiplication and division.

CCSS.MATH.CONTENT.3.OA.B.5

Apply properties of operations as strategies to multiply and divide.2*Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (Associative property of multiplication.) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.)*

CCSS.MATH.CONTENT.3.OA.B.6

Understand division as an unknown-factor problem. *For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.*

Multiply and divide within 100.

CCSS.MATH.CONTENT.3.OA.C.7

Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

Solve problems involving the four operations, and identify and explain patterns in arithmetic.

CCSS.MATH.CONTENT.3.OA.D.8

Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.3

CCSS.MATH.CONTENT.3.OA.D.9

Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of

operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends*.

Number & Operations in Base Ten

Use place value understanding and properties of operations to perform multi-digit arithmetic.

CCSS.MATH.CONTENT.3.NBT.A.1
Use place value understanding to round whole numbers to the nearest 10 or 100.

CCSS.MATH.CONTENT.3.NBT.A.2
Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

CCSS.MATH.CONTENT.3.NBT.A.3
Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

Number & Operations—Fractions

Develop understanding of fractions as numbers.

CCSS.MATH.CONTENT.3.NF.A.1
Understand a fraction 1/*b* as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction *a*/*b* as the quantity formed by *a* parts of size 1/*b*.

CCSS.MATH.CONTENT.3.NF.A.2
Understand a fraction as a number on the number line; represent fractions on a number line diagram.

CCSS.MATH.CONTENT.3.NF.A.2.A
Represent a fraction 1/*b* on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts. Recognize that each part has size 1/*b* and that the endpoint of the part based at 0 locates the number 1/*b* on the number line.

CCSS.MATH.CONTENT.3.NF.A.2.B
Represent a fraction *a*/*b* on a number line diagram by marking off a lengths 1/*b* from 0. Recognize that the resulting interval has size *a*/*b* and that its endpoint locates the number *a*/*b* on the number line.

CCSS.MATH.CONTENT.3.NF.A.3
Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

CCSS.MATH.CONTENT.3.NF.A.3.A
Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

CCSS.MATH.CONTENT.3.NF.A.3.B
Recognize and generate simple equivalent fractions, e.g., 1/2 = 2/4, 4/6 = 2/3. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

CCSS.MATH.CONTENT.3.NF.A.3.C
Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form 3 = 3/1; recognize that 6/1 = 6; locate 4/4 and 1 at the same point of a number line diagram.*

CCSS.MATH.CONTENT.3.NF.A.3.D
Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Measurement & Data

Solve problems involving measurement and estimation.

CCSS.MATH.CONTENT.3.MD.A.1
Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

CCSS.MATH.CONTENT.3.MD.A.2

Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.2

Represent and interpret data.

CCSS.MATH.CONTENT.3.MD.B.3

Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets*.

CCSS.MATH.CONTENT.3.MD.B.4

Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units— whole numbers, halves, or quarters.

Geometric measurement: understand concepts of area and relate area to multiplication and to addition.

CCSS.MATH.CONTENT.3.MD.C.5

Recognize area as an attribute of plane figures and understand concepts of area measurement.

CCSS.MATH.CONTENT.3.MD.C.5.A

A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.

CCSS.MATH.CONTENT.3.MD.C.5.B

A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

CCSS.MATH.CONTENT.3.MD.C.6

Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

CCSS.MATH.CONTENT.3.MD.C.7

Relate area to the operations of multiplication and addition.

CCSS.MATH.CONTENT.3.MD.C.7.A

Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

CCSS.MATH.CONTENT.3.MD.C.7.B

Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

CCSS.MATH.CONTENT.3.MD.C.7.C

Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.

CCSS.MATH.CONTENT.3.MD.C.7.D

Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

Geometric measurement: recognize perimeter.

CCSS.MATH.CONTENT.3.MD.D.8

Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Geometry

Reason with shapes and their attributes.

CCSS.MATH.CONTENT.3.G.A.1

Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four

sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

CCSS.MATH.CONTENT.3.G.A.2

Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape*.

Grade 4

Operations & Algebraic Thinking

Use the four operations with whole numbers to solve problems.

CCSS.MATH.CONTENT.4.OA.A.1

Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

CCSS.MATH.CONTENT.4.OA.A.2

Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.1

CCSS.MATH.CONTENT.4.OA.A.3

Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Gain familiarity with factors and multiples.

CCSS.MATH.CONTENT.4.OA.B.4

Find all factor pairs for a whole number in the range 1-100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1-100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1-100 is prime or composite.

Generate and analyze patterns.

CCSS.MATH.CONTENT.4.OA.C.5

Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way*.

Number & Operations in Base Ten

Generalize place value understanding for multi-digit whole numbers.

CCSS.MATH.CONTENT.4.NBT.A.1

Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. *For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division*.

CCSS.MATH.CONTENT.4.NBT.A.2

Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.

CCSS.MATH.CONTENT.4.NBT.A.3

Use place value understanding to round multi-digit whole numbers to any place.

Use place value understanding and properties of operations to perform multi-digit arithmetic.

CCSS.MATH.CONTENT.4.NBT.B.4

Fluently add and subtract multi-digit whole numbers using the standard algorithm.

CCSS.MATH.CONTENT.4.NBT.B.5

Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies

based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

CCSS.MATH.CONTENT.4.NBT.B.6

Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Number & Operations—Fractions

Extend understanding of fraction equivalence and ordering.

CCSS.MATH.CONTENT.4.NF.A.1

Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

CCSS.MATH.CONTENT.4.NF.A.2

Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Build fractions from unit fractions.

CCSS.MATH.CONTENT.4.NF.B.3

Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

CCSS.MATH.CONTENT.4.NF.B.3.A

Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

CCSS.MATH.CONTENT.4.NF.B.3.B

Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. *Examples: 3/8 = 1/8 + 1/8 + 1/8 ; 3/8 = 1/8 + 2/8 ; 2 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8.*

CCSS.MATH.CONTENT.4.NF.B.3.C

Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.

CCSS.MATH.CONTENT.4.NF.B.3.D

Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

CCSS.MATH.CONTENT.4.NF.B.4

Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

CCSS.MATH.CONTENT.4.NF.B.4.A

Understand a fraction a/b as a multiple of $1/b$. *For example, use a visual fraction model to represent 5/4 as the product 5 × (1/4), recording the conclusion by the equation 5/4 = 5 × (1/4).*

CCSS.MATH.CONTENT.4.NF.B.4.B

Understand a multiple of a/b as a multiple of 1/b, and use this understanding to multiply a fraction by a whole number. *For example, use a visual fraction model to express 3 × (2/5) as 6 × (1/5), recognizing this product as 6/5. (In general, n × (a/b) = (n × a)/b.)*

CCSS.MATH.CONTENT.4.NF.B.4.C

Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. *For example, if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?*

Understand decimal notation for fractions, and compare decimal fractions.

CCSS.MATH.CONTENT.4.NF.C.5

Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.2 *For example, express 3/10 as 30/100, and add 3/10 + 4/100 = 34/100.*

CCSS.MATH.CONTENT.4.NF.C.6

Use decimal notation for fractions with denominators 10 or 100. *For example, rewrite 0.62 as 62/100; describe a length as 0.62 meters; locate 0.62 on a number line diagram.*

CCSS.MATH.CONTENT.4.NF.C.7

Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.

Measurement & Data

Solve problems involving measurement and conversion of measurements.

CCSS.MATH.CONTENT.4.MD.A.1

Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. *For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...*

CCSS.MATH.CONTENT.4.MD.A.2

Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

CCSS.MATH.CONTENT.4.MD.A.3

Apply the area and perimeter formulas for rectangles in real world and mathematical problems. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*

Represent and interpret data.

CCSS.MATH.CONTENT.4.MD.B.4

Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*

Geometric measurement: understand concepts of angle and measure angles.

CCSS.MATH.CONTENT.4.MD.C.5

Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

> CCSS.MATH.CONTENT.4.MD.C.5.A
>
> An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.
>
> CCSS.MATH.CONTENT.4.MD.C.5.B
>
> An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.

CCSS.MATH.CONTENT.4.MD.C.6

Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

CCSS.MATH.CONTENT.4.MD.C.7

Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Geometry

Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

CCSS.MATH.CONTENT.4.G.A.1
Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

CCSS.MATH.CONTENT.4.G.A.2
Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

CCSS.MATH.CONTENT.4.G.A.3
Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

Grade 5

Operations & Algebraic Thinking

Write and interpret numerical expressions.

CCSS.MATH.CONTENT.5.OA.A.1
Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

CCSS.MATH.CONTENT.5.OA.A.2
Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. *For example, express the calculation "add 8 and 7, then multiply by 2" as 2 × (8 + 7). Recognize that 3 × (18932 + 921) is three times as large as 18932 + 921, without having to calculate the indicated sum or product.*

Analyze patterns and relationships.

CCSS.MATH.CONTENT.5.OA.B.3
Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. *For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.*

Number & Operations in Base Ten

Understand the place value system.

CCSS.MATH.CONTENT.5.NBT.A.1
Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.

CCSS.MATH.CONTENT.5.NBT.A.2
Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

CCSS.MATH.CONTENT.5.NBT.A.3
Read, write, and compare decimals to thousandths.

CCSS.MATH.CONTENT.5.NBT.A.3.A
Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., 347.392 = 3 × 100 + 4 × 10 + 7 × 1 + 3 × (1/10) + 9 × (1/100) + 2 × (1/1000).

CCSS.MATH.CONTENT.5.NBT.A.3.B
Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.

CCSS.MATH.CONTENT.5.NBT.A.4

Use place value understanding to round decimals to any place.

Perform operations with multi-digit whole numbers and with decimals to hundredths.

CCSS.MATH.CONTENT.5.NBT.B.5

Fluently multiply multi-digit whole numbers using the standard algorithm.

CCSS.MATH.CONTENT.5.NBT.B.6

Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

CCSS.MATH.CONTENT.5.NBT.B.7

Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Number & Operations—Fractions

Use equivalent fractions as a strategy to add and subtract fractions.

CCSS.MATH.CONTENT.5.NF.A.1

Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)*

CCSS.MATH.CONTENT.5.NF.A.2

Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. *For example, recognize an incorrect result 2/5 + 1/2 = 3/7, by observing that 3/7 < 1/2.*

Apply and extend previous understandings of multiplication and division.

CCSS.MATH.CONTENT.5.NF.B.3

Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. *For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?*

CCSS.MATH.CONTENT.5.NF.B.4

Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

CCSS.MATH.CONTENT.5.NF.B.4.A

Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$. *For example, use a visual fraction model to show (2/3) × 4 = 8/3, and create a story context for this equation. Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = (ac)/(bd).*

CCSS.MATH.CONTENT.5.NF.B.4.B

Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

CCSS.MATH.CONTENT.5.NF.B.5

Interpret multiplication as scaling (resizing), by:

CCSS.MATH.CONTENT.5.NF.B.5.A

Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

CCSS.MATH.CONTENT.5.NF.B.5.B

Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

CCSS.MATH.CONTENT.5.NF.B.6

Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

CCSS.MATH.CONTENT.5.NF.B.7

Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.1

CCSS.MATH.CONTENT.5.NF.B.7.A

Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. *For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3.*

CCSS.MATH.CONTENT.5.NF.B.7.B

Interpret division of a whole number by a unit fraction, and compute such quotients. *For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4.*

CCSS.MATH.CONTENT.5.NF.B.7.C

Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins?*

Measurement & Data

Convert like measurement units within a given measurement system.

CCSS.MATH.CONTENT.5.MD.A.1

Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Represent and interpret data.

CCSS.MATH.CONTENT.5.MD.B.2

Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*

Geometric measurement: understand concepts of volume.

CCSS.MATH.CONTENT.5.MD.C.3

Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

CCSS.MATH.CONTENT.5.MD.C.3.A

A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.

CCSS.MATH.CONTENT.5.MD.C.3.B

A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.

CCSS.MATH.CONTENT.5.MD.C.4

Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

CCSS.MATH.CONTENT.5.MD.C.5

Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.

CCSS.MATH.CONTENT.5.MD.C.5.A

Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

CCSS.MATH.CONTENT.5.MD.C.5.B

Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.

CCSS.MATH.CONTENT.5.MD.C.5.C

Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Geometry

Graph points on the coordinate plane to solve real-world and mathematical problems.

CCSS.MATH.CONTENT.5.G.A.1

Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., *x*-axis and *x*-coordinate, *y*-axis and *y*-coordinate).

CCSS.MATH.CONTENT.5.G.A.2

Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

Classify two-dimensional figures into categories based on their properties.

CCSS.MATH.CONTENT.5.G.B.3

Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.

CCSS.MATH.CONTENT.5.G.B.4

Classify two-dimensional figures in a hierarchy based on properties.

Grade 6

Ratios & Proportional Relationships

Understand ratio concepts and use ratio reasoning to solve problems.

CCSS.MATH.CONTENT.6.RP.A.1

Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*

CCSS.MATH.CONTENT.6.RP.A.2

Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."*1

CCSS.MATH.CONTENT.6.RP.A.3

Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

CCSS.MATH.CONTENT.6.RP.A.3.A

Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and

plot the pairs of values on the coordinate plane. Use tables to compare ratios.

CCSS.MATH.CONTENT.6.RP.A.3.B

Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?*

CCSS.MATH.CONTENT.6.RP.A.3.C

Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

CCSS.MATH.CONTENT.6.RP.A.3.D

Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

The Number System

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

CCSS.MATH.CONTENT.6.NS.A.1

Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?*.

Compute fluently with multi-digit numbers and find common factors and multiples.

CCSS.MATH.CONTENT.6.NS.B.2

Fluently divide multi-digit numbers using the standard algorithm.

CCSS.MATH.CONTENT.6.NS.B.3

Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

CCSS.MATH.CONTENT.6.NS.B.4

Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor. *For example, express 36 + 8 as 4 (9 + 2).*.

Apply and extend previous understandings of numbers to the system of rational numbers.

CCSS.MATH.CONTENT.6.NS.C.5

Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

CCSS.MATH.CONTENT.6.NS.C.6

Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

CCSS.MATH.CONTENT.6.NS.C.6.A

Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., -(-3) = 3, and that 0 is its own opposite.

CCSS.MATH.CONTENT.6.NS.C.6.B

Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

CCSS.MATH.CONTENT.6.NS.C.6.C

Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

CCSS.MATH.CONTENT.6.NS.C.7

Understand ordering and absolute value of rational numbers.

CCSS.MATH.CONTENT.6.NS.C.7.A

Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. *For example, interpret -3 > -7 as a statement that -3 is located to the right of -7 on a number line oriented from left to right.*

CCSS.MATH.CONTENT.6.NS.C.7.B

Write, interpret, and explain statements of order for rational numbers in real-world contexts. *For example, write* $-3^\circ C > -7^\circ C$ *to express the fact that* $-3^\circ C$ *is warmer than* $-7^\circ C$.

CCSS.MATH.CONTENT.6.NS.C.7.C

Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. *For example, for an account balance of -30 dollars, write |-30| = 30 to describe the size of the debt in dollars.*

CCSS.MATH.CONTENT.6.NS.C.7.D

Distinguish comparisons of absolute value from statements about order. *For example, recognize that an account balance less than -30 dollars represents a debt greater than 30 dollars.*

CCSS.MATH.CONTENT.6.NS.C.8

Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

Expressions & Equations

Apply and extend previous understandings of arithmetic to algebraic expressions.

CCSS.MATH.CONTENT.6.EE.A.1

Write and evaluate numerical expressions involving whole-number exponents.

CCSS.MATH.CONTENT.6.EE.A.2

Write, read, and evaluate expressions in which letters stand for numbers.

CCSS.MATH.CONTENT.6.EE.A.2.A

Write expressions that record operations with numbers and with letters standing for numbers. *For example, express the calculation "Subtract y from 5" as 5 - y.*

CCSS.MATH.CONTENT.6.EE.A.2.B

Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. *For example, describe the expression 2 (8 + 7) as a product of two factors; view (8 + 7) as both a single entity and a sum of two terms.*

CCSS.MATH.CONTENT.6.EE.A.2.C

Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). *For example, use the formulas* $V = s^3$ *and* $A = 6s^2$ *to find the volume and surface area of a cube with sides of length s = 1/2.*

CCSS.MATH.CONTENT.6.EE.A.3

Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3 (2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6 (4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

CCSS.MATH.CONTENT.6.EE.A.4

Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.*

Reason about and solve one-variable equations and inequalities.

CCSS.MATH.CONTENT.6.EE.B.5

Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

CCSS.MATH.CONTENT.6.EE.B.6

Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

CCSS.MATH.CONTENT.6.EE.B.7

Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

CCSS.MATH.CONTENT.6.EE.B.8

Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x <$ c have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

Represent and analyze quantitative relationships between dependent and independent variables.

CCSS.MATH.CONTENT.6.EE.C.9

Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation d = 65t to represent the relationship between distance and time.

Geometry

Solve real-world and mathematical problems involving area, surface area, and volume.

CCSS.MATH.CONTENT.6.G.A.1

Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

CCSS.MATH.CONTENT.6.G.A.2

Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = l\ w\ h$ and $V = b\ h$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

CCSS.MATH.CONTENT.6.G.A.3

Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

CCSS.MATH.CONTENT.6.G.A.4

Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

Statistics & Probability

Develop understanding of statistical variability.

CCSS.MATH.CONTENT.6.SP.A.1

Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. *For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.*

CCSS.MATH.CONTENT.6.SP.A.2

Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center,

spread, and overall shape.

CCSS.MATH.CONTENT.6.SP.A.3

Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

Summarize and describe distributions.

CCSS.MATH.CONTENT.6.SP.B.4

Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

CCSS.MATH.CONTENT.6.SP.B.5

Summarize numerical data sets in relation to their context, such as by:

CCSS.MATH.CONTENT.6.SP.B.5.A

Reporting the number of observations.

CCSS.MATH.CONTENT.6.SP.B.5.B

Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.

CCSS.MATH.CONTENT.6.SP.B.5.C

Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.

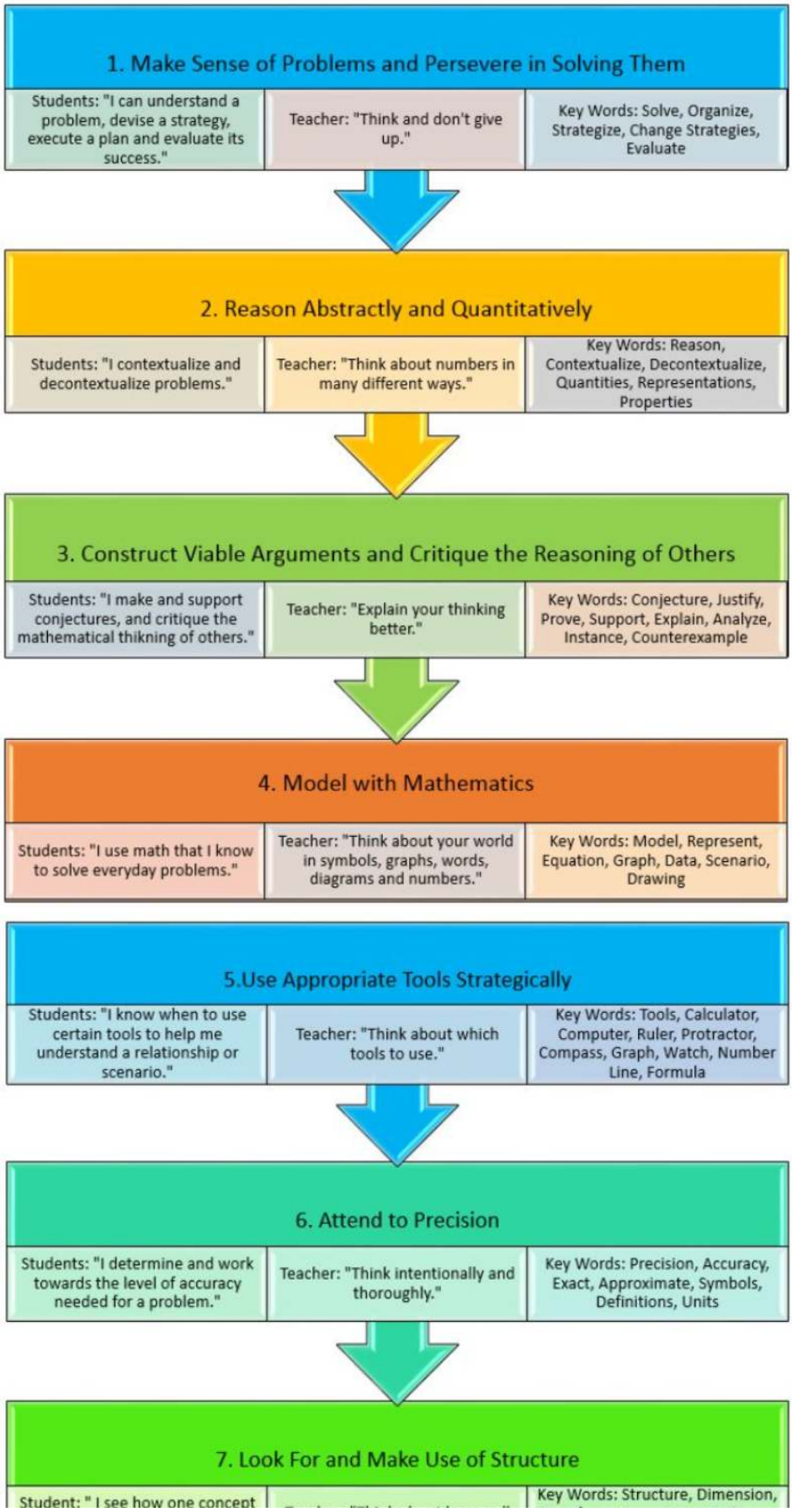
CCSS.MATH.CONTENT.6.SP.B.5.D

Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.

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9.2: Appendix B- Mathematical Practices for Teachers

The Mathematical Practices for Teachers


1. Make Sense of Problems and Persevere in Solving Them
Students: "I can understand a problem, devise a strategy, execute a plan and evaluate its success."
Teacher: "Think and don't give up."
Key Words: Solve, Organize, Strategize, Change Strategies, Evaluate
2. Reason Abstractly and Quantitatively
Students: "I contextualize and decontextualize problems."
Teacher: "Think about numbers in many different ways."
Key Words: Reason, Contextualize, Decontextualize, Quantities, Representations, Properties
3. Construct Viable Arguments and Critique the Reasoning of Others
Students: "I make and support conjectures, and critique the mathematical thikning of others."
Teacher: "Explain your thinking better."
Key Words: Conjecture, Justify, Prove, Support, Explain, Analyze, Instance, Counterexample
4. Model with Mathematics
Students: "I use math that I know to solve everyday problems."
Teacher: "Think about your world in symbols, graphs, words, diagrams and numbers."
Key Words: Model, Represent, Equation, Graph, Data, Scenario, Drawing
5.Use Appropriate Tools Strategically
Students: "I know when to use certain tools to help me understand a relationship or scenario."
Teacher: "Think about which tools to use."
Key Words: Tools, Calculator, Computer, Ruler, Protractor, Compass, Graph, Watch, Number Line, Formula
6. Attend to Precision
Students: "I determine and work towards the level of accuracy needed for a problem."
Teacher: "Think intentionally and thoroughly."
Key Words: Precision, Accuracy, Exact, Approximate, Symbols, Definitions, Units
7. Look For and Make Use of Structure
Student: " I see how one concept
Key Words: Structure, Dimension,

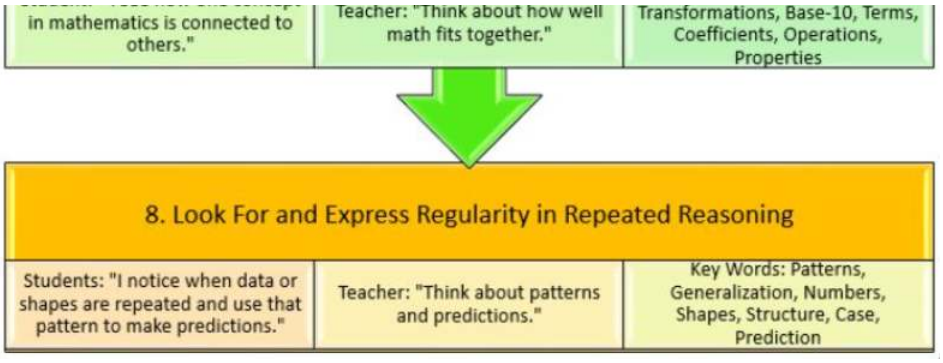


Figure 7.2.1

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9.3: Appendix C- Solutions for Partner Activities

Solutions for Partner Activities

Chapter 2

Section 2.3

Partner Activity 1

Step 1: Add up both lengths to find the total length

$$2\frac{3}{4}+1\frac{1}{2}=\frac{11}{4}+\frac{3}{2}=\frac{11}{4}+\frac{6}{4}=\frac{17}{4}=4\frac{1}{4}$$

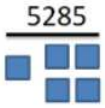
Step 2: Divide the part by the whole

$$\frac{\text{Sam}}{\text{Total}}=\frac{2\frac{3}{4}}{4\frac{1}{4}}=2\frac{3}{4}\div 4\frac{1}{4}=\frac{11}{4}\div\frac{17}{4}=\frac{11}{4}\times\frac{4}{17}=\frac{11}{17}\approx 0.65=65\%\approx\frac{2}{3}$$

Partner Activity 2

$$\frac{1\text{ hour}}{84\text{ miles }+92\text{ miles}}=\frac{1\text{ hour}}{176\text{ miles}}$$
$$\rightarrow\frac{132\text{ miles needed in one hour}}{176\text{ miles total in one hour}}=\frac{3}{4}\text{ hour }=45\text{ minutes}$$

Partner Activity 3



$$\rightarrow 5285\div 5=1057=\text{ first number}$$
$$1057\times 4=4228=\text{ second number}$$
$$\text{Check it! } 1057+4228=5285$$

Section 2.4

Partner Activity 1

1. MMXLV = 1000+1000+(50-10)+5 = **2045**
2. MDCCLXXXIX = 1000+500+200+50+30+(10-1) = **1789**
3. 1993 = 1000+(1000-100)+(100-10)+1+1+1 = **MCMXCIII**
4. 5495 = 5000+(500-100)+(100-10)+5 = **VCDXCV**

Section 2.6

Partner Activity 1

1.
   a. 132_{four}
   b. 200313_{four}
   c. 100_{four}
2.
   a. 15
   b. 278
   c. Undefined

Section 2.8

Partner Activity 1

1. 116_{nine}
2. 1030_{four}
3. 513_{six}
4. 21_{three}
5. 111110_{two}

Chapter 3

Section 3.2

Partner Activity 1

Answer: 265

Partner Activity 2

Answers vary

Partner Activity 3

1. 408
2. 246

Section 3.4

Partner Activity 1

$$5 \div 0 = \text{ undefined}$$
$$0 \div 5 = 0$$
$$0 \div 0 = \text{ indeterminate}$$

We know that $18 \div 6 = 3$ because $3 \times 6 = 18$, therefore. Using a similar approach, we know that $5 \div 0 \neq 0$, since $0 \times 0 \neq 5$

Section 3.5

Partner Activity 1

1. \$14.47
2. \$3.65
3. \$16.32

Partner Activity 2

1. $12 \times 6 = 72$
2. $10 \times 18 = 180$
3. $8 \times 8 = 64$

Partner Activity 3

1. 321
2. 229
3. 15453

Section 3.6

Partner Activity 1

Answers vary. Samples are below:

1. Here to Baker, CA
2. The length of my bedroom
3. The weight of a 2nd grader

Section 3.7

Partner Activity 1

1. 309.26
2. 92.86%
3. 114.06

Chapter 4

Section 4.2

Partner Activity 1

1. The denominator is VERY MUCH less than the numerator, i.e. $\frac{1}{10000}$
2. Both the denominator and the numerator are large numbers and they are very close in distance, i.e. $\frac{999}{1000}$
3. The numerator is roughly half of the denominator, i.e. $\frac{15}{31}$
4. The numerator is roughly a third of the denominator, i.e. $\frac{21}{60}$

Partner Activity 2

$$\frac{3}{8} \to \frac{1}{2}, \frac{5}{4} \to 1, \frac{2}{9} \to 0, \frac{4}{7} \to \frac{1}{2}, \frac{1}{3} \to \frac{1}{2}$$

Section 4.3

Partner Activity 1

$$\frac{51}{684} + \frac{43}{684} + \frac{738}{684} = \frac{832}{684}$$

OR

$$\frac{1}{8} + \frac{4}{5} + \frac{1}{9} = \frac{1}{360} + \frac{\quad}{360} + \frac{\quad}{360} = \frac{45}{360} + \frac{288}{360} + \frac{40}{360} = \frac{373}{360}$$

Partner Activity 2

Division works. The rest do not.

Chapter 5

Section 5.2

Partner Activity 1

Factors: 1, 2, 3, 5, 6, 10, 15, 30

First four multiples: 30, 60, 90, 120

Partner Activity 2

Primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97

Partner Activity 3

0, 1, negative = neither. 2 is the first prime number.

Partner Activity 4

1. $85 = 5 \cdot 17$
2. $350 = 2 \cdot 5^2 \cdot 7$
3. $60 = 2^2 \cdot 3 \cdot 5$

Section 5.4

Partner Activity 1

2 feet; The GCF of (6, 8, 12) = 2

Partner Activity 2

Each packet contains 12 cards; The GCF of (24, 60, 48) = 12

Section 5.5

Partner Activity 1

120 cookies; The LCM of (15, 24) = 120

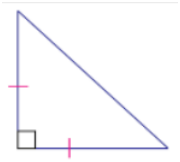
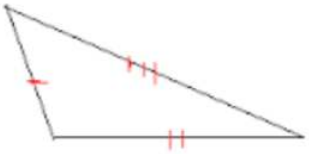
Partner Activity 2

1. $\frac{1}{12}$
2. $\frac{1}{6}$
3. 14

Chapter 6

Section 6.2

Partner Activity 1

1. 
2. 
3. Does not exist

Partner Activity 2

1. A square is a rectangle
2. Parallelogram
3. A regular has all same sides and all same angles

Partner Activity 3

1. 180
2. 540
3. 2340
4. 360

Partner Activity 4

1. 150
2. 41

Section 6.3

Partner Activity 1

1. Area is base times height, two dimensions
2. Volume is base times height times width, three dimensions

Partner Activity 2

About 12 units^2

Partner Activity 3

52 units^2

Partner Activity 4

24 units^3

Section 6.4

Partner Activity 1

1. $\frac{5}{9}$ yards
2. 84480 feet

Section 6.5

Partner Activity 1

1. 1200 inches^2
2. 225 feet^2
3. $196838560.5 \text{ miles}^2$
4. 37.68 inches^3

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9.4: Appendix D- Solutions to Practice Problems

Chapter 2

Section 2.2

1. 1063 feet
2. 5 pounds, 8 ounces
3. $100 trillion

Section 2.3

1. There are 5 trikes and 2 bikes
2. 21 cents
3. 56 seconds
4. 7 pieces; yes
5. $\frac{1}{5}$ are chickens
6. $198.65

Section 2.4

1. $\overline{VIIIV}DCCLIX$
2. 2020
3. 1982
4. MDCCLXIV
5. $\bar{L}$
6. 20089

Section 2.5

1. The character 4 does not exist in base four
2. “Five-four-two base six”
3. 0, 1, 2 10, 11, 12, 20, 21, 22 100, 101, 102
4. Answers vary

Section 2.6

1.
   a. 7565_{nine}
   b. 30_{four}
   c. 10201_{three}
   d. 1001001_{two}
2.
   a. 46_{ten}
   b. 111_{ten}
   c. 73_{ten}
   d. 123256_{ten}

Section 2.7

1. Eighty-ones
2. Six hundred twenty-fives
3. Six thousand sixty-ones
4. Thirty eight thousand four hundred sixteens

Section 2.8

1. 404_{six}
2. 1011_{two}

3. 2121_{three}
4. 104542_{eight}

Chapter 3

Section 3.2

1. 50 + 80 + 6 + 1 = 137
2. 1000 – 300 + 16 = 716
3. 95 + 800 = 895
4. 761 – 450 = 760 – 450 + 1 =311

Section 3.3

1. 7 + 7 + 7 + 7
2. 2 + 2 + 2 + 2 + 2 + 2
3. 36.6034
4. 179.1908

Section 3.4

1. 7 coworkers; repeat subtraction
2. 3 pieces of paper; Sharing

Section 3.5

1. The answers below are precise. Your answers must be close.
    a. 16.5
    b. 76
2. These answers must be precise.
    a. 16.46
    b. $49.35
    c. 82
3. These answers are precise. Your answer must be close.
    a. $6.48
    b. $11.36
    c. $16.39
4. Your answers must be exact.
    a. 72
    b. 72
    c. Same answers
5. Your answers must be exact.
    a. 866
    b. 65322

Section 3.6

Answers vary. Here are some sample answers:

1 meter = the height of a short woman

1 inch = distance from nail to join on my thumb

1 foot = length from my elbow to my wrist

1 yard = The height of preschooler

Section 3.7

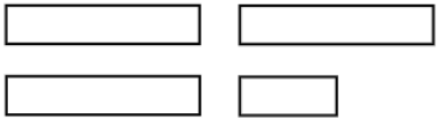
1. 309.26
2. 92.86%
3. 114.06

Chapter 4

Section 4.2

1. Two possible ways to show $3\frac{1}{2}$

   a. Rectangles:

   b. Number Line:

2. $\frac{1}{9}, \frac{3}{8}, \frac{2}{5}, \frac{4}{7}, \frac{12}{13}$

Section 4.3

1. $\frac{53}{28}$
2. $\frac{11}{28}$
3. $\frac{6}{7}$
4. $\frac{3}{224}$
5. $\frac{257}{30}$ or $8\frac{17}{30}$
6. $\frac{67}{30}$ or $2\frac{7}{30}$
7. $\frac{171}{10}$ or $17\frac{1}{10}$
8. $\frac{162}{95}$ or $1\frac{67}{95}$

Section 4.4

1. Natural, Whole, Integers, Rational and Real
2. Rational and Real
3. Irrational
4. Natural, Whole, Integers, Rational, and Real
5. Irrational
6. Integers, Rational, Real
7. Whole, Integers, Rational and Real
8. Natural, Whole, Integers, Rational and Real
9. 29
10. 147
11. -277
12. -11
13. 276
14. Distributive Property
15. Communitive Property of Multiplication
16. Reflexive Property

17. Transitive Property

Chapter 5

Section 5.2

1. 1, 2, 4, 7, 8, 14, 28, 56
2. 1, 5, 29, 145
3. 1,2,4,8,16,32
4. 50, 100, 150, 200
5. 23, 46, 69, 92
6. 8, 16, 24, 32
7. 23,29,31,37
8. 61,67,71,73,79
9. 127, 131, 137, 139, 149
10. 20, 21, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40
11. 60, 62, 63, 64, 65, 66, 68, 69, 70, 72, 74, 75, 76, 77, 78, 80
12. 120, 121, 122, 123, 124, 125, 126, 128, 129, 130, 132, 133, 134, 135, 136, 138, 140, 141, 142, 143, 144, 145, 146, 147, 148, 150
13. $2^2 \cdot 3^3 \cdot 5$
14. $2^2 \cdot 3 \cdot 5$
15. 5^3

Section 5.3

Table 7.4.1

	2	3	4	5	6	8	9	10
67820	X		X	X				X
512	X		X			X		
49								
3463								

Section 5.4

1. 10
2. 25
3. 1
4. 30
5. 45

Section 5.5

1. 360
2. 75
3. 14280
4. 1800
5. 2025
6. $\frac{131}{60}$
7. $-\frac{7}{144}$
8. $\frac{27}{13}$

Chapter 6

Section 6.2

1. 55
2. 46
3. 121
4. 140
5. 95
6. 90
7. Obtuse
8. Straight
9. Acute
10. Right
11. Acute
12. Equiangular
13. Acute scalene
14. Right scalene
15. 
16. Does not exist
17. Octagon
18. Heptagon
19. 360
20. 1617
21. 1800
22. 1260
23. Irregular
24. Regular
25. Regular
26. Irregular

Section 6.3

1. Area is amount inside a two dimensional shape, surface area is the area around a three dimensional shape, volume is the amount of space inside a three dimensional shape
2.
   a. 2 units^2
   b. About 2 units^2
   c. About 2.5 units^2
3.
   a. 28 units^2
   b. 24 units^2
4.
   a. 7 units^3
   b. 6 units^3

Section 6.4

1. 0.00795455 miles or $\frac{7}{880}$ miles

2. 2.25 yards or $\frac{9}{4}$ yards
3. 59840 yards
4. 3276 inches

Section 6.5

1. $254.3\ \text{cm}^2$
2. $28.3\ \text{miles}^2$
3. $232.2\ \text{cm}^2$
4. $153.9\ \text{meters}^2$
5. $25.9\ \text{cm}^2$
6. $60.4\ \text{miles}^2$
7. $7.3\ \text{miles}^2$
8. $2.9\ \text{inches}^2 2$
9. Triangular pyramid
10. Hexagonal prism
11. Cylinder
12. Rectangular pyramid
13. $59.2\pi\ \text{ft}^2$
14. $116.4\pi\ \text{in}^2$
15. $462\pi\ \text{km}^2$
16. $323.5\ \text{ft}^2$
17. $99\ \text{cm}^2$
18. $1152\pi\ \text{m}^3$
19. $1.33\pi\ \text{m}^3$
20. $1774.67\pi\ \text{km}^3$
21. $273.84\pi\ \text{ft}^3$
22. $27\pi\ \text{mi}^3$
23. 168km^3
24. $1296\pi\ \text{mi}^3$
25. $84\ \text{m}^3$

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SECTION OVERVIEW

9.5: Appendix E- Material Cards (Harland)

https://www.dropbox.com/home/LIbrete...s_reduced.docx

9.5.1: Coins

9.5.2: A-Blocks

9.5.3: Value Label Cards

9.5.4: Models for Base Two

9.5.5: Unit Blocks

9.5.6: Base Two Blocks

9.5.7: Base Three Blocks

9.5.8: Base Four Blocks

9.5.9: Base Five Blocks

9.5.10: Base Six Blocks

9.5.11: Base Seven Blocks

9.5.12: Base Eight Blocks

9.5.13: Base Nine Blocks

9.5.14: Base Ten Blocks

9.5.15: Base Eleven Blocks

9.5.16: Base Twelve Blocks

9.5.17: Supplementary Longs

9.5.18: Centimeter Strips

9.5.19: Counters

9.5.20: Number Squares

9.5.21: Fraction Circles

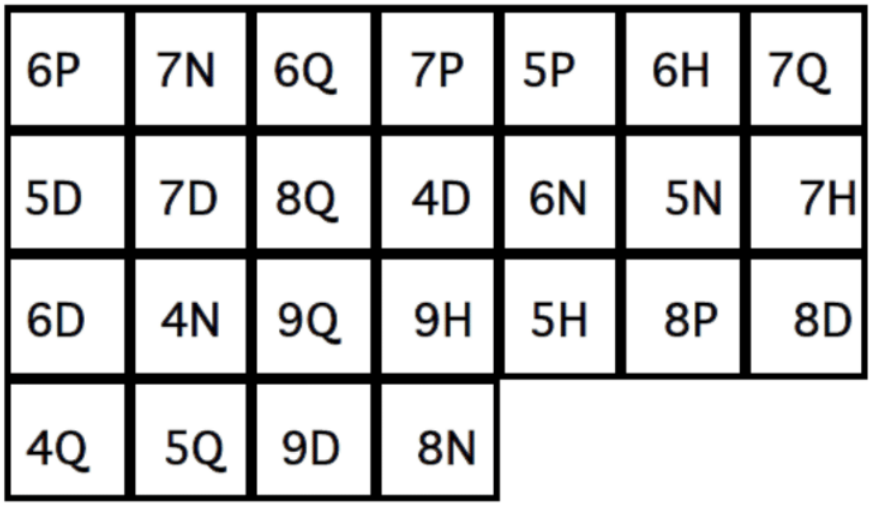
9.5.22: Strips and Arrays

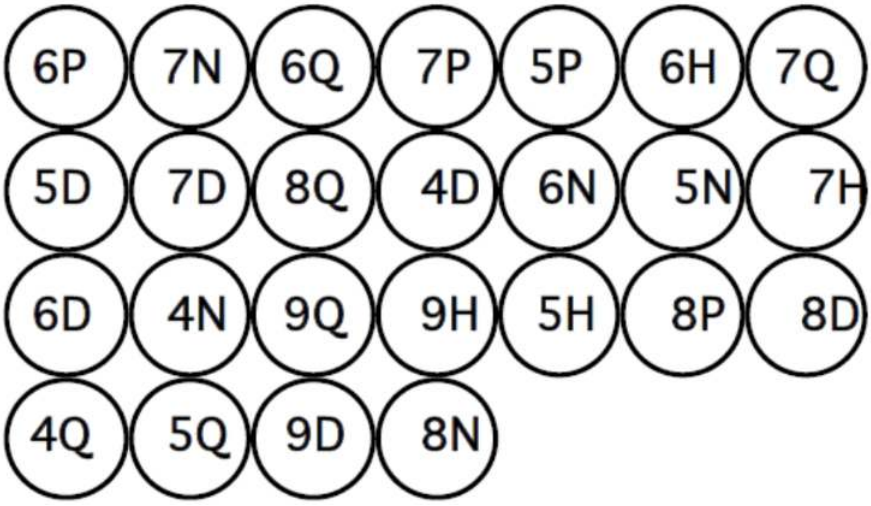
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9.5.1: Coins

Below are two sets of coins. You can either make square coins, which are easier and faster to cut out or you can make round coins. **You only need one set.** If you lose any pieces, you can always use the second set. In any case, they are all the same size even though they represent pennies, nickels, dimes, quarters and half dollars. These represent coins from 1964-1969. The number represents the last digit of the year from 1964 through 1969 (**4, 5, 6, 7, 8** or **9**) and the letter represents the first letter of the denomination (**P,N,D,Q** or **H**). For instance, **6P** would stand for a 1966 penny.

You might want to keep these in an envelope and paperclip it to your workbook so they are readily available for use.





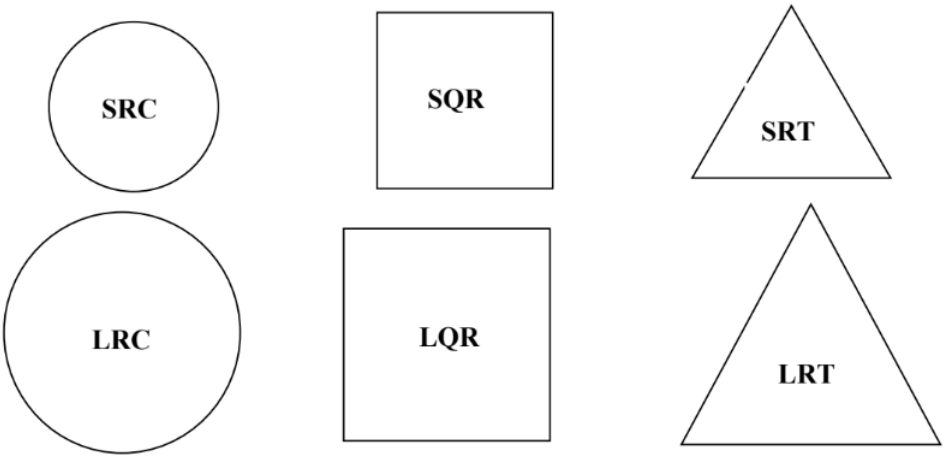
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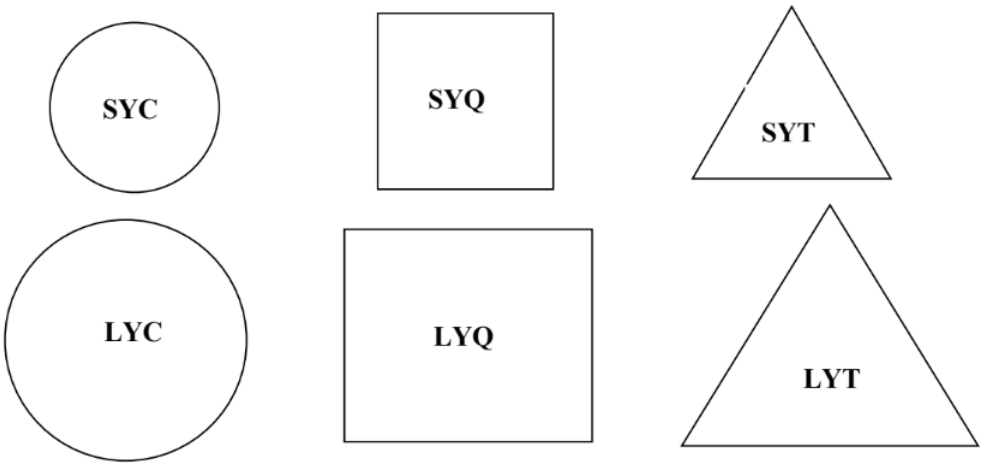
9.5.2: A-Blocks

The A-Blocks you'll be making will consist of 24 objects. There are two sizes (small and large), four colors (red, yellow, green and blue) and three shapes (circle, square and triangle).Each of the 24 elements is labeled with a three letter code. The first letter refers to its size (**S** or **L**), the second letter refers to its color (**R, Y, G** or **B**) and the last letter refers to its shape (**C, Q** or **T**). We are using **Q** to represent the square. For instance, **SGT** is the code for the small green triangle

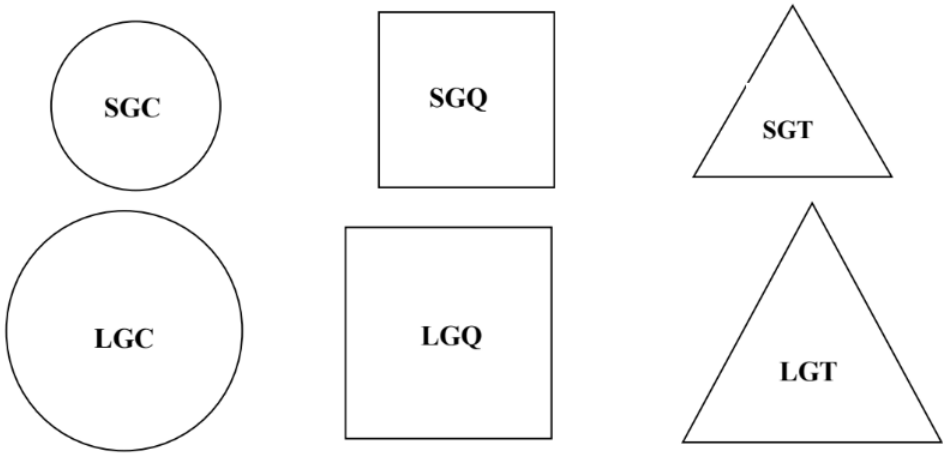
2A Red A-Blocks



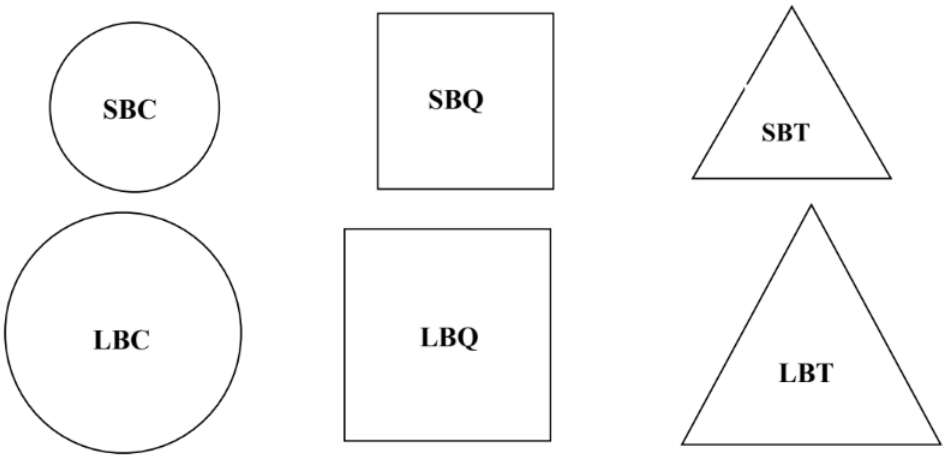
2B Yellow A-Blocks



2C Green A-Blocks



2D Blue A-Blocks



2E White

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9.5.3: Value Label Cards

Below are two sets of value label cards to be used with the set of A-blocks in various exercises. Each card has a value on it. The first set has the full value written out and the second set has the abbreviations. Use whichever set you prefer to work with. There is a negative value card for each positive value card. By negative, I am referring to the complement of a set. For instance, there is a subset of **BLUE (B)** A-blocks. There are six elements in this subset The negative value card would be **NOT BLUE** ($\mathbf{B}^{c}$) and this represents the set of all A-blocks that are not blue. There are 18 elements in that subset. **LARGE** and **SMALL** are complements of each other so there aren't separate cards for their negative values. There are some extra cards left blank so that you can fill in some creative labels like $\mathbf{B} \cap \mathbf{T}$, $\mathbf{B} \cup \mathbf{T}$ or some other combinations needed for certain exercises. These cards need to be cut out and saved along with your A-blocks. Preferably, this is on cardstock paper or laminated. If not and these labels are too flimsy, you can make some labels on index cards so they are sturdier.

RED	NOT RED
BLUE	NOT BLUE
GREEN	NOT GREEN
YELLOW	NOT YELLOW
CIRCLE	NOT CIRCLE
SQUARE	NOT SQUARE
TRIANGLE	NOT TRIANGLE
LARGE	SMALL

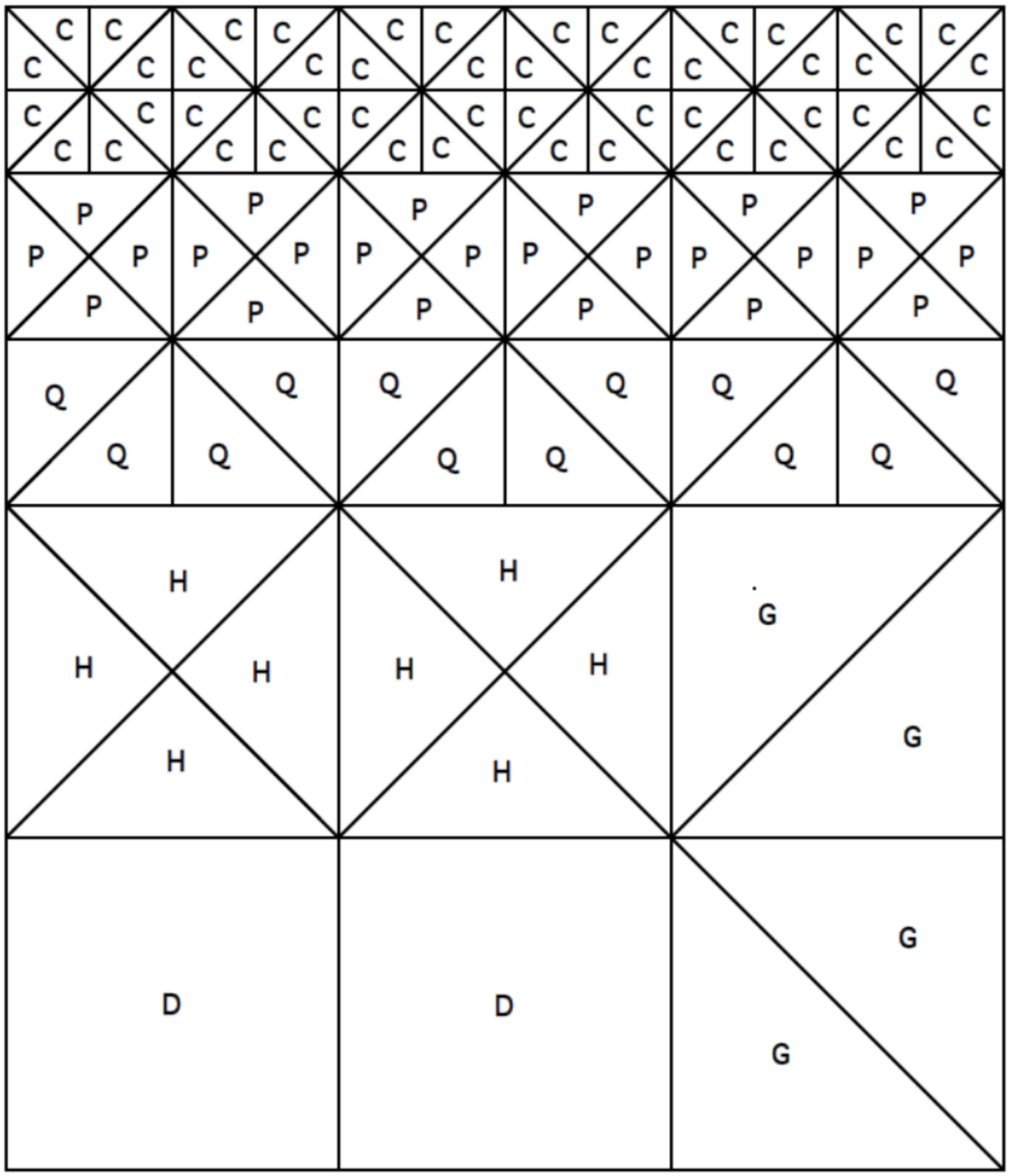
R	R^{c}
B	B^{c}
G	G^{c}
Y	Y^{c}
C	C^{c}
Q	Q^{c}
T	T^{c}
L	S

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9.5.4: Models for Base Two

Cut along all of the line segments to make modelsof cups (**C**), pints (**P**) and quarts (**Q**), half-gallons (**H**), gallons (**G**) and double-gallons (**D**).

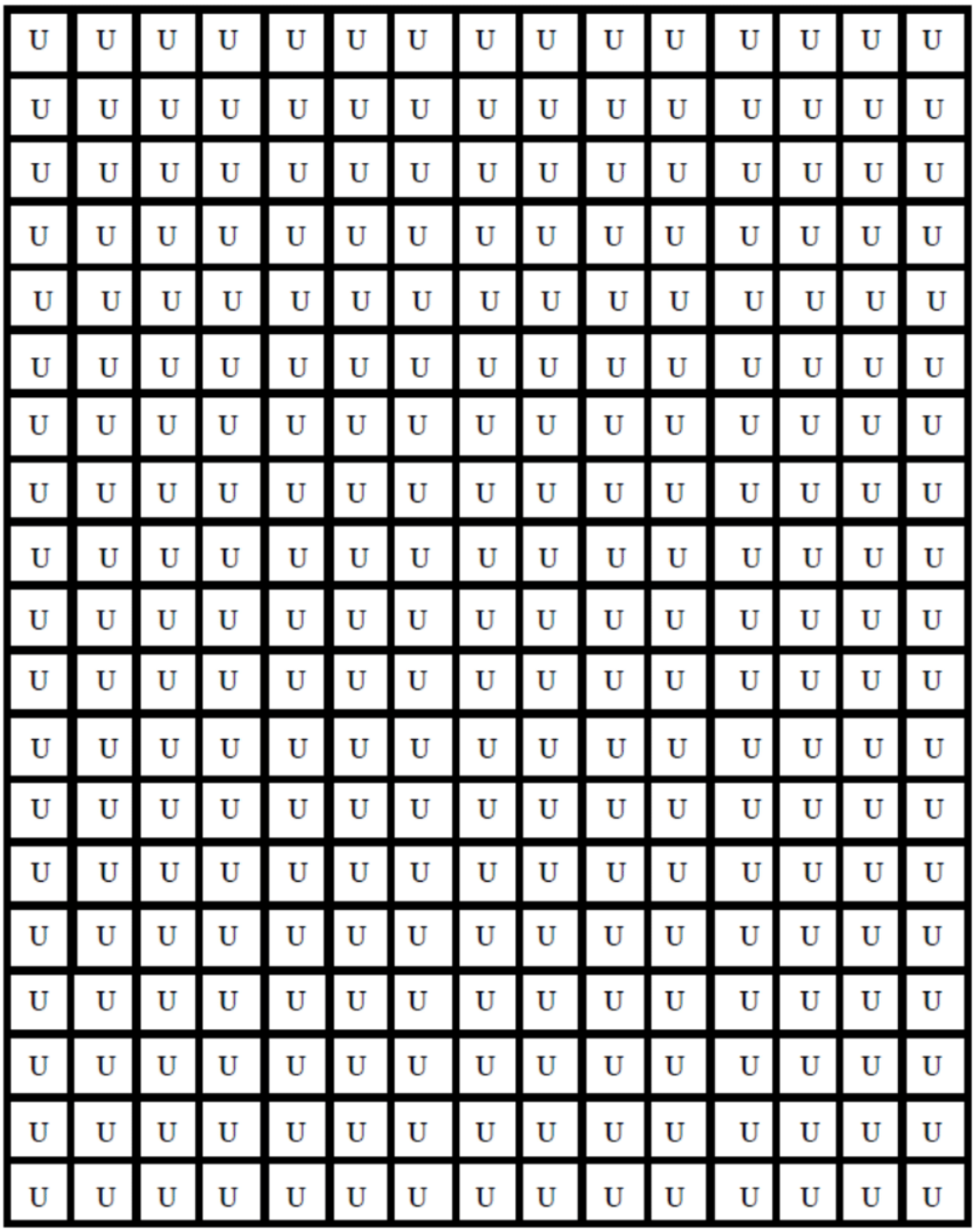


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9.5.5: Unit Blocks

A set of Unit Blocks are on this material card. Cut along all solid line segments. Each piece is labeled "U". These can be used as the units with any set of Base Blocks. **There are more unit blocks here than you need to cut out. Cut out at least 60 units.** ***If you prefer, you can make extra copies of this sheet, and cut out more unit blocks so that each of the Base Blocks that follows has its own units in its baggie.***



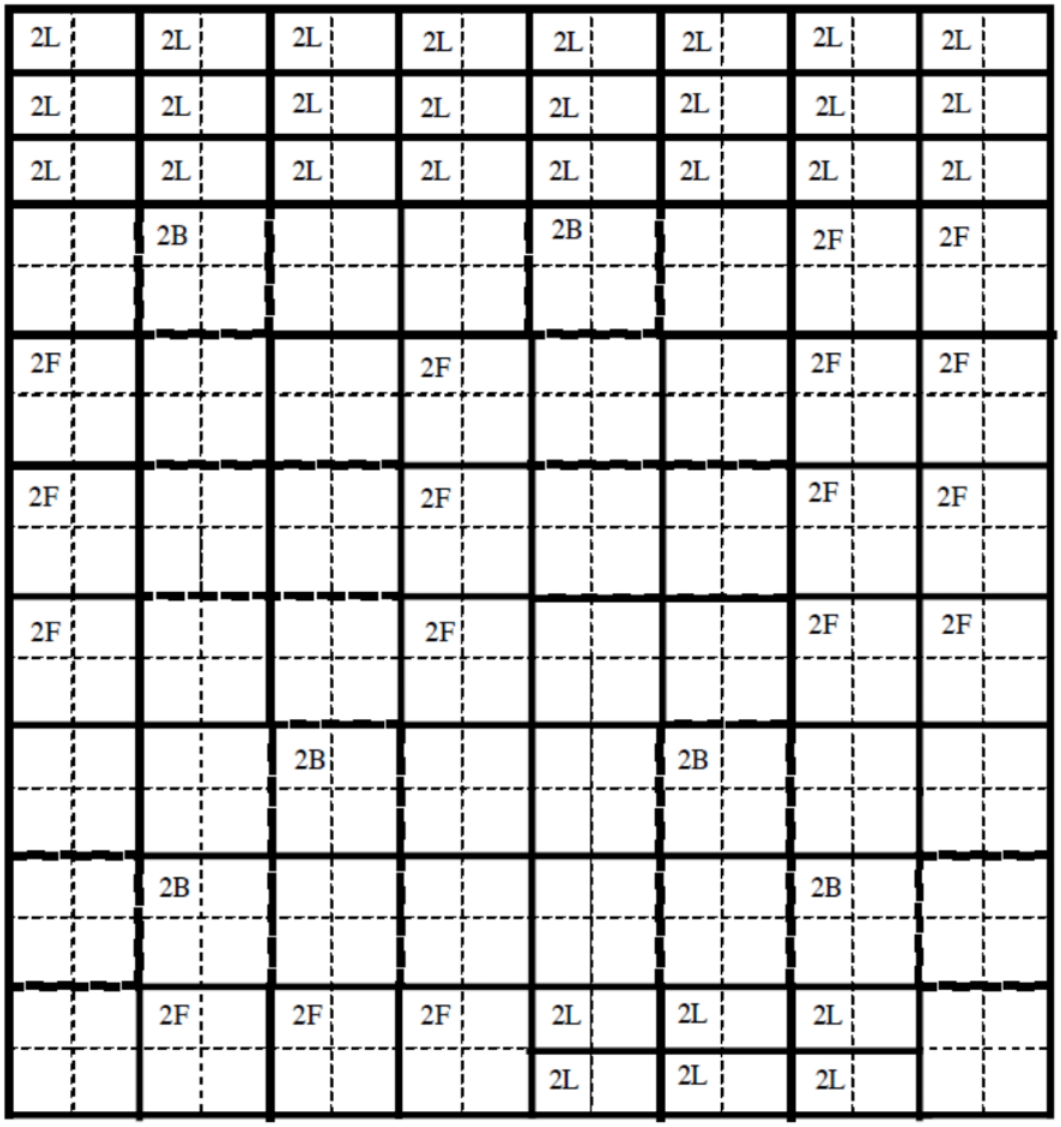
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9.5.6: Base Two Blocks

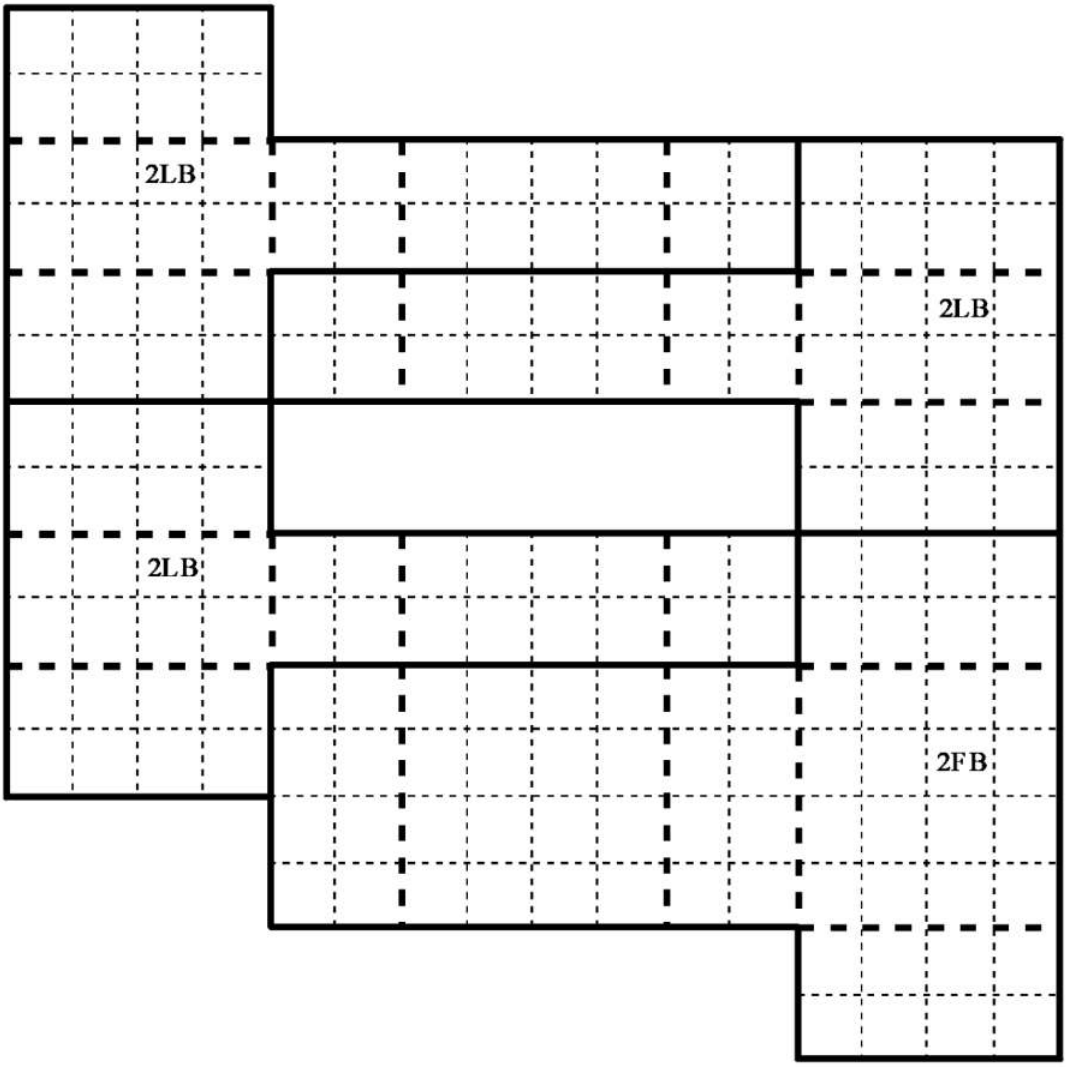
Longs (2L), Flats (2F), Blocks (2B)

A set of Base Two blocks are on this and the next material card (5B). Cut along all solid line segments. Each piece is labeled. A Base Two Long is 1 cm by 1 cm and is labeled "2L". A Base Two Flat is 2 cm by 2 cm and is labeled "2F". A Base Two Block is 2 cm by 2 cm by 2cm and is labeled "2B". Fold these 3-dimensional blocks along the dark dotted lines; then tape to make a cube. Directions for other blocks are on the next page. Use the unit blocks from Material Card 4 with this set of Base Blocks.



Long Blocks (2LB) and Flat Blocks (2FB)

This page contains the remainder of the set of Base Two blocks. Cut along all solid line segments. Each piece is labeled. A Base Two Long Block is 4 cm by 2 cm by 2 cm and is labeled "2LB". The Base Two Flat Block is 4 cm by 4 cm by 2 cm and is labeled "2FB". Fold these 3-dimensional blocks along the dark dotted lines; then tape to make rectangular solids.



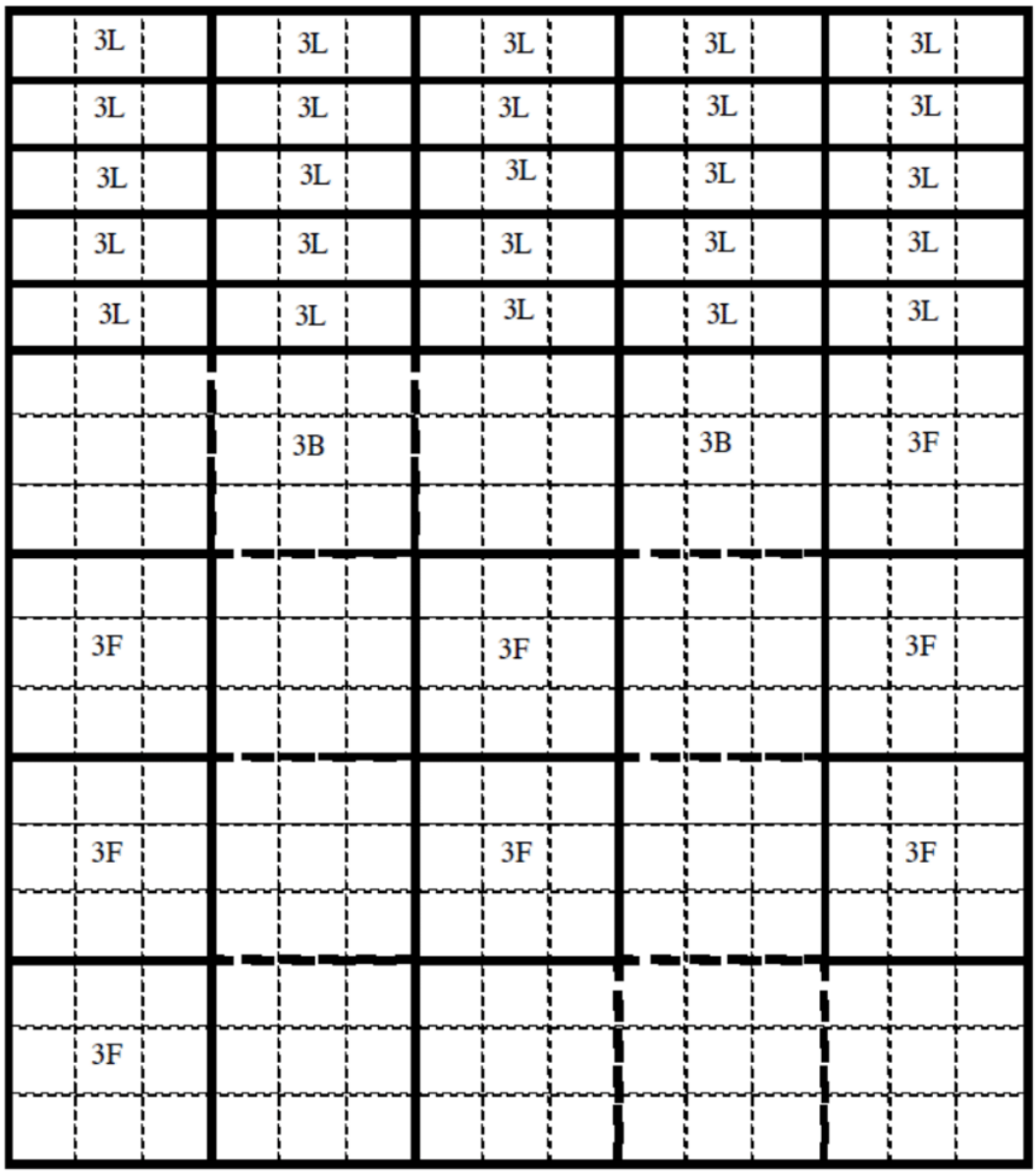
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9.5.7: Base Three Blocks

Longs (3L), Flats (3F) and Blocks (3B)

This page contains a set of Base Three blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks.. Cut along all solid line segments. Each piece is labeled. A Base Three Long is 1 cm by 3 cm and is labeled "3L". A Base Three Flat is 3 cm by 3 cm and is labeled "2F". A Base Three Block is 3 cm by 3cm by 3 cm and is labeled "3B". Fold these 3-dimensional blocks along the dark dotted lines; then tape to make a cube.



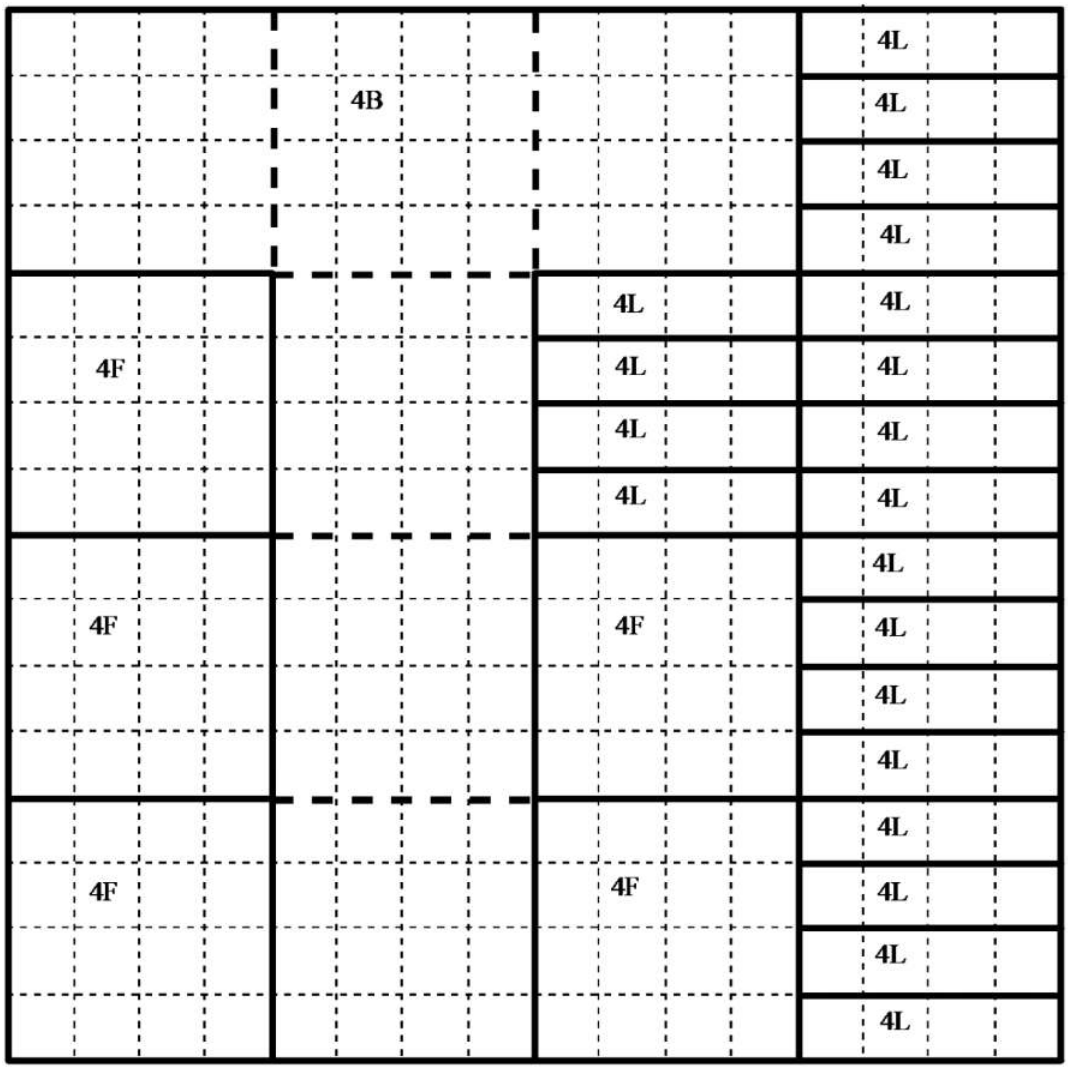
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9.5.8: Base Four Blocks

Longs (4L), Flats (4F) and Blocks (4B)

This page contains a set of Base Four blocks.Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Four Long is 1 cm by 4 cm and is labeled "4L". A Base Four Flat is 4 cm by 4 cm and is labeled"4F". The Base Four Block is 4 cm by 4 cm by 4 cm and is labeled "4B". Fold this 3-dimensional block along the dark dotted lines; then tape to make a cube.



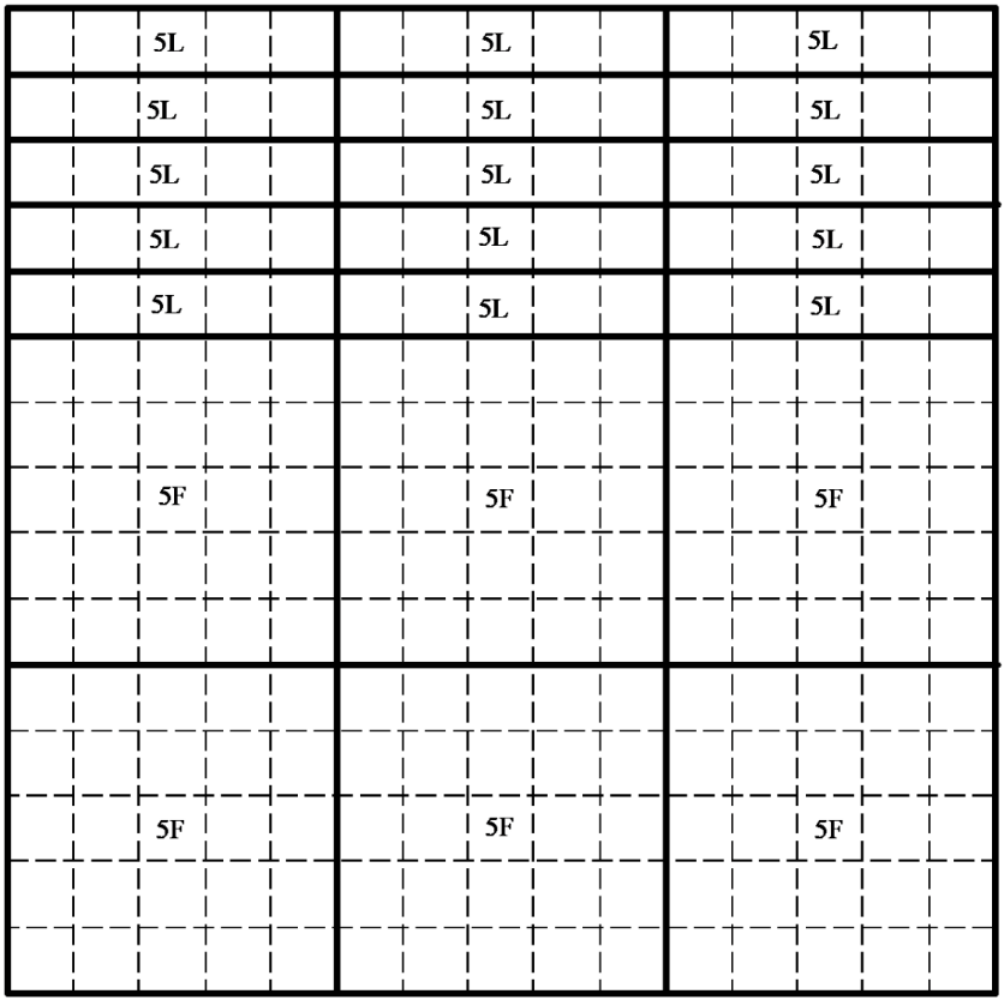
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9.5.9: Base Five Blocks

Longs (5L) and Flats (5F)

This page contains a set of Base Five blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Five Long is 1 cm by 5 cm and is labeled "5L". A Base Five Flat is 5 cm by 5 cm and is labeled "5F".



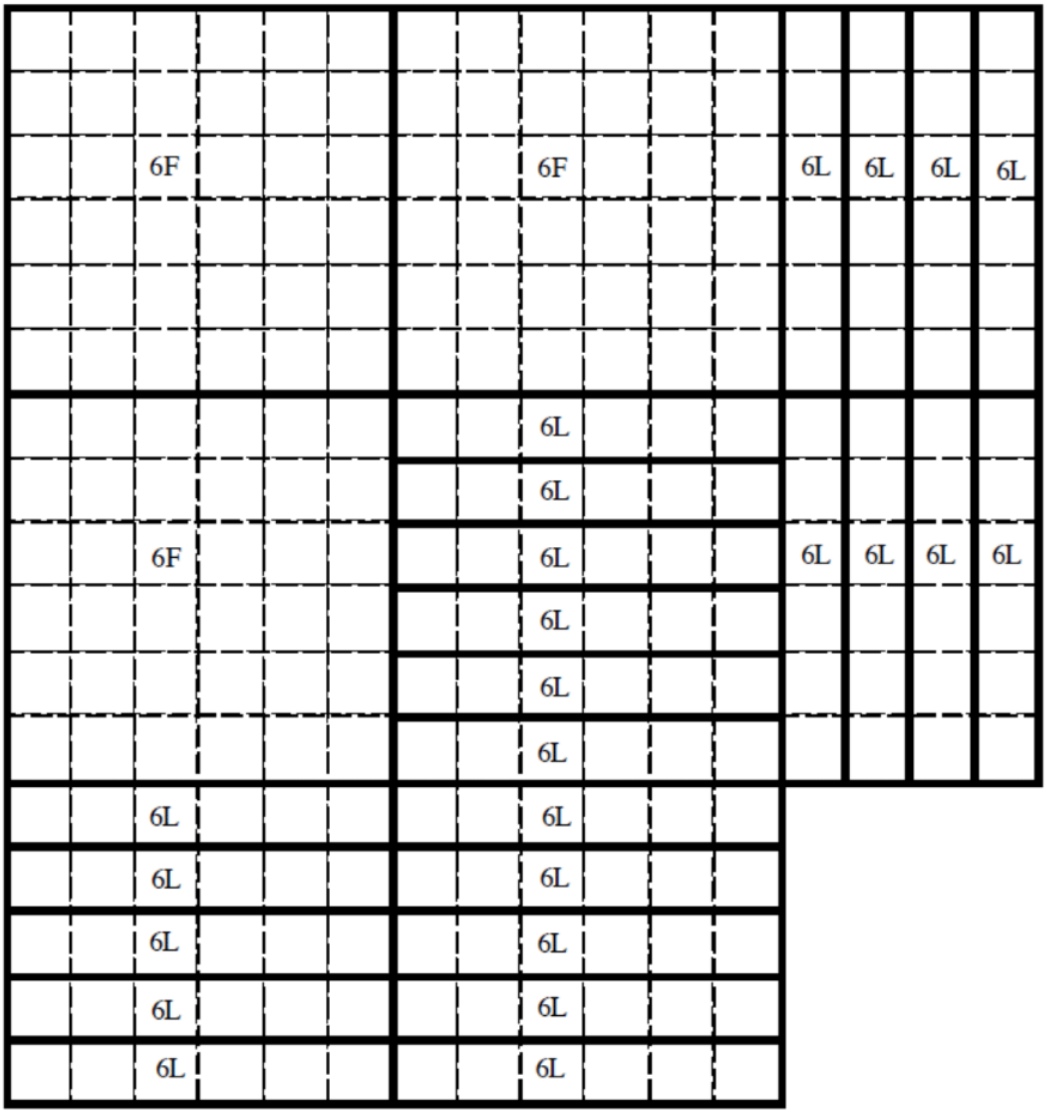
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9.5.10: Base Six Blocks

Longs (6L) and Flats (6F)

This page contains a set of Base Six blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Six Long is 1 cm by 6 cm and is labeled "6L". A Base Six Flat is 6 cm by 6 cm and is labeled "6F".



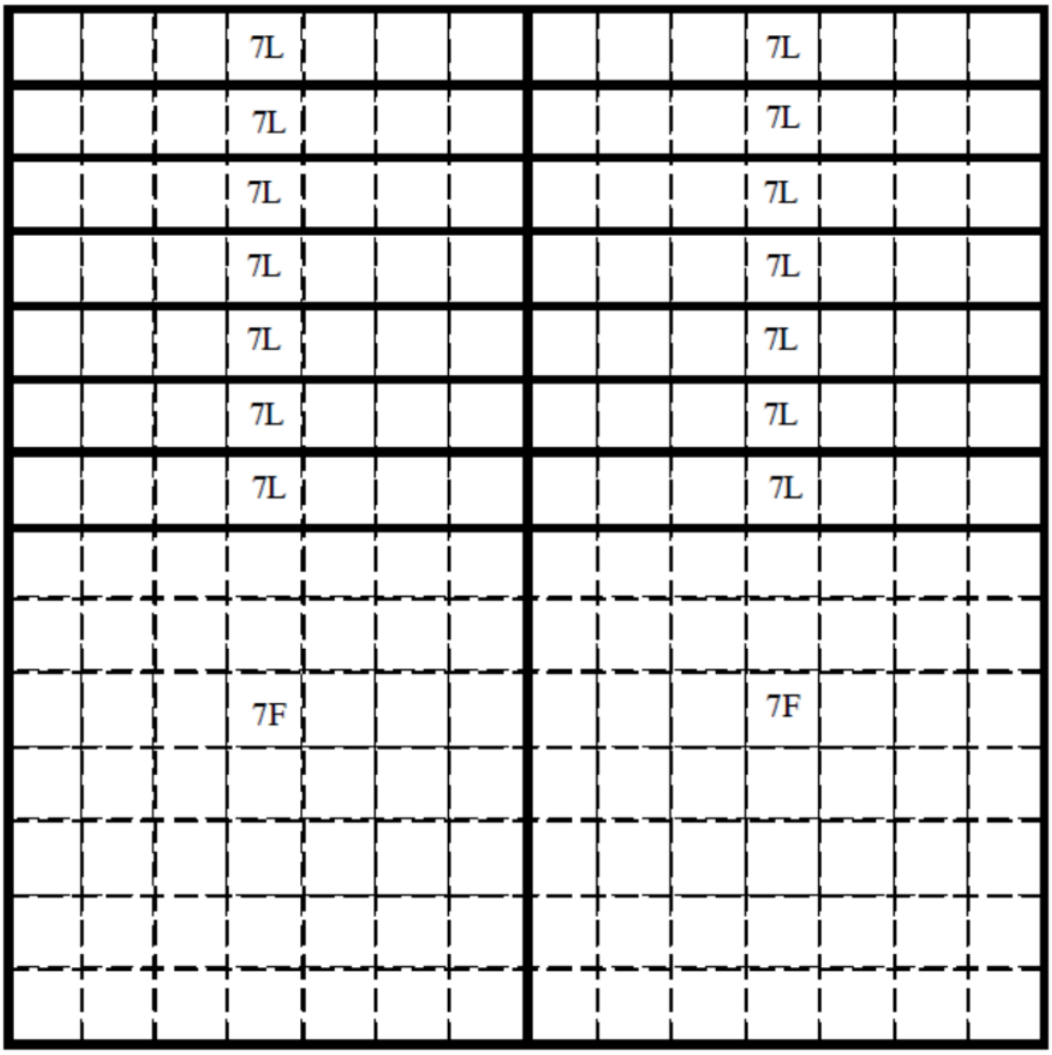
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9.5.11: Base Seven Blocks

Longs (7L) and Flats (7F)

This page contains a set of Base Seven blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Seven Long is 1 cm by 7 cm and is labeled "7L". A Base Seven Flat is 7 cm by 7 cm and is labeled "7F".

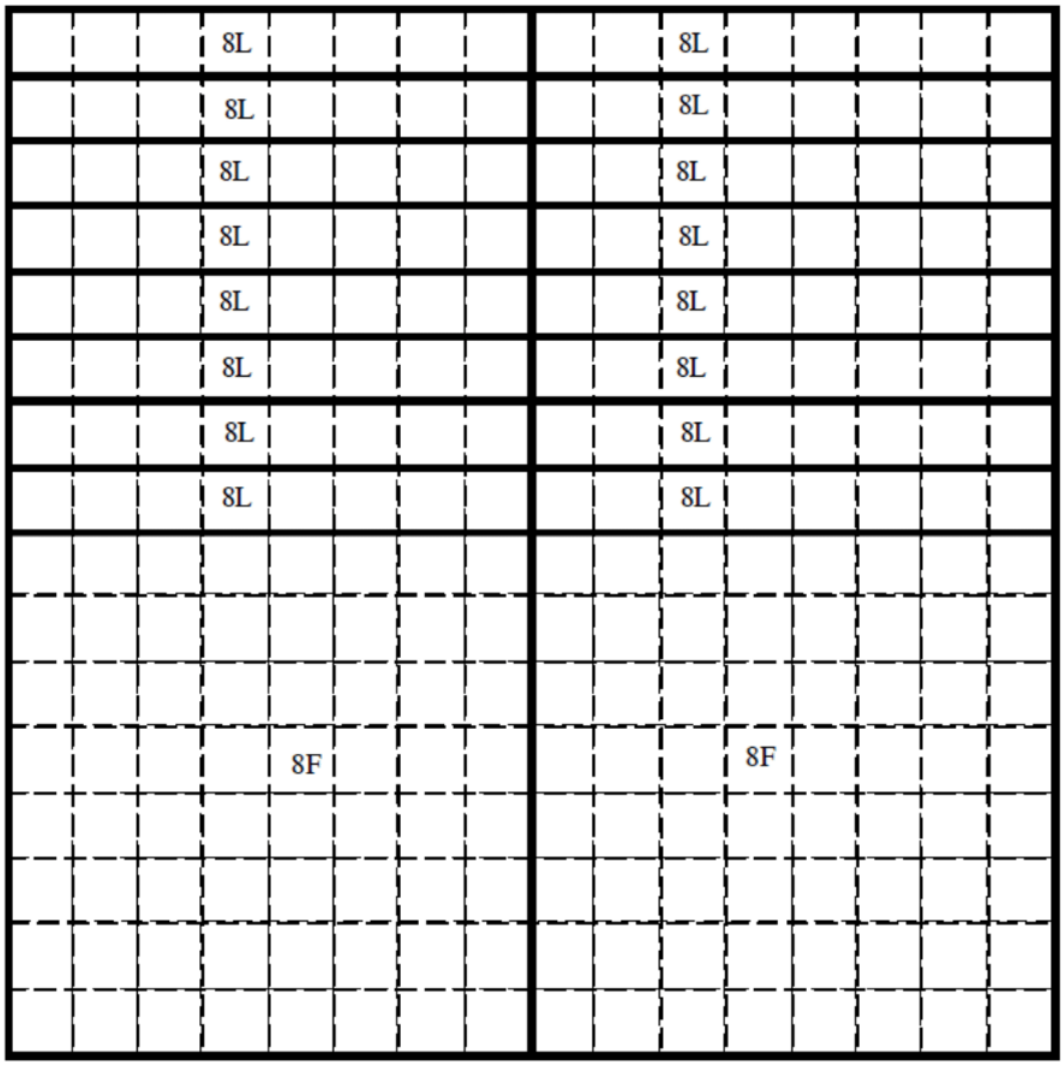


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9.5.12: Base Eight Blocks

This page contains a set of Base Eight blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Eight Long is 1 cm by 8 cm and is labeled "8L". A Base Eight Flat is 8 cm by 8 cm and is labeled "8F".

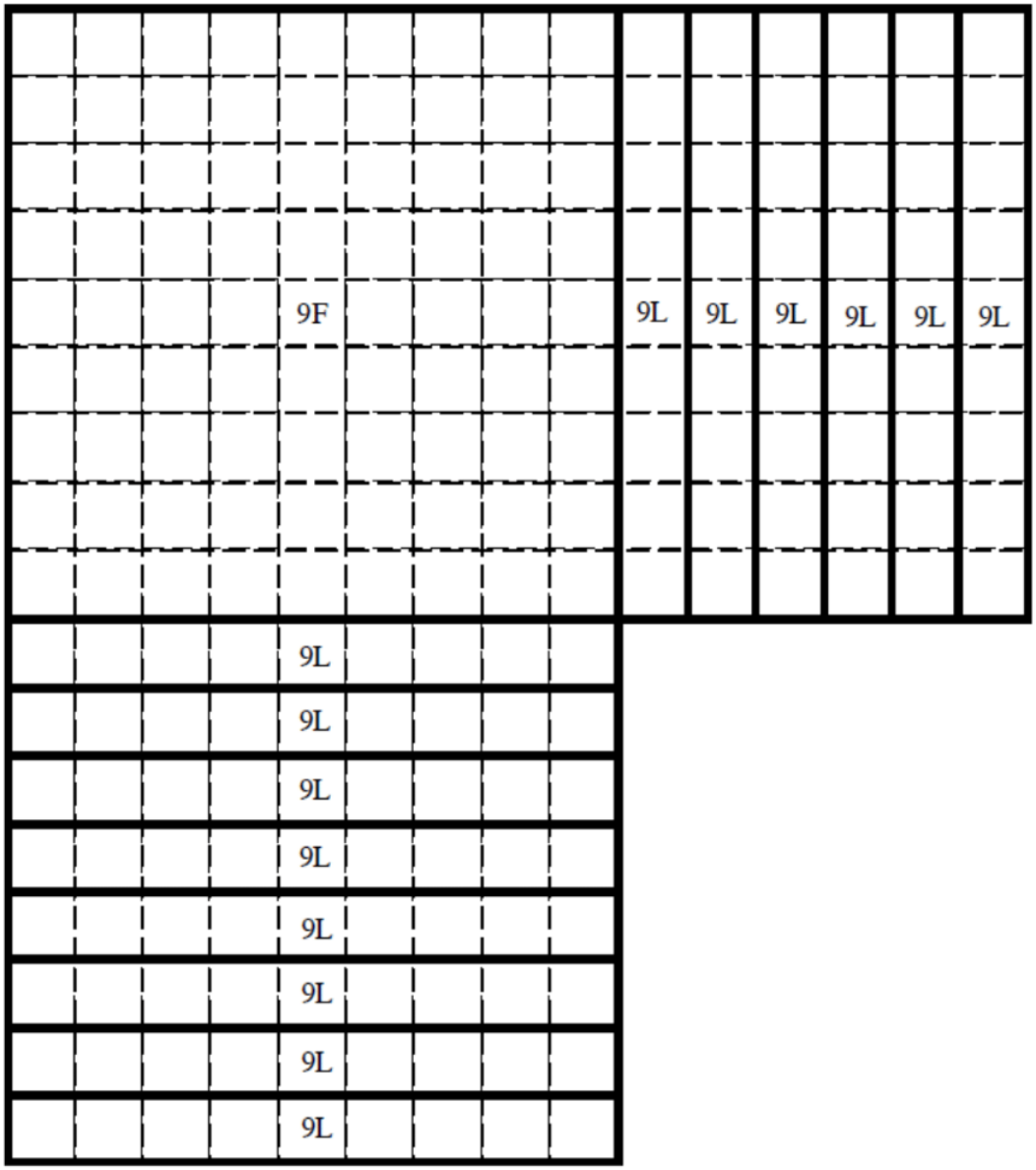


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9.5.13: Base Nine Blocks

This page contains a set of Base Nine blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Nine Long is 1 cm by 9 cm and is labeled "9L". The Base Nine Flat is 9 cm by 9 cm and is labeled "9F". Extra longs for Base Nine are on a separate Supplementary Material Card for Cards 12-15, found after Card 15.



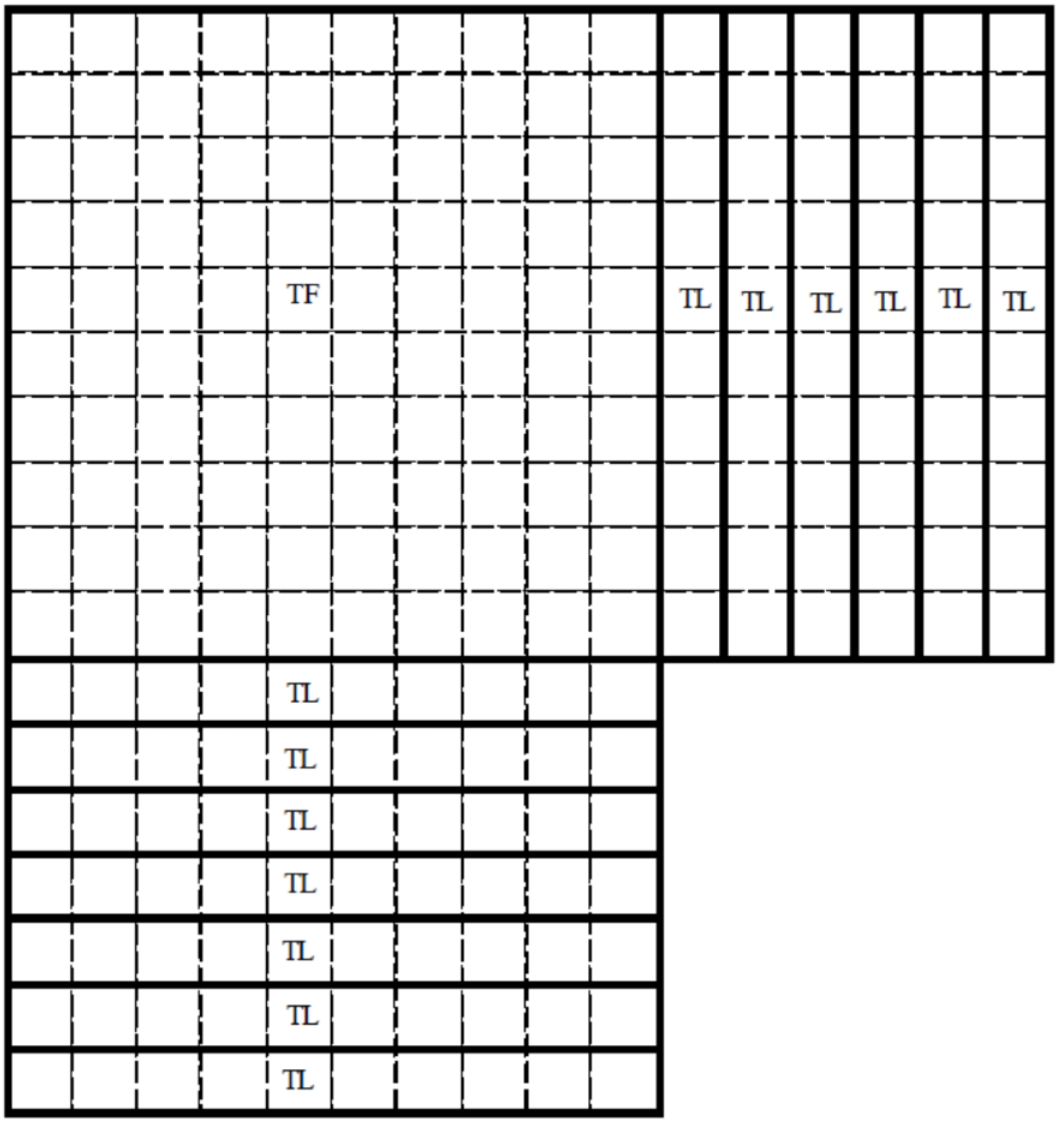
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9.5.14: Base Ten Blocks

Longs (TL) and Flats (TF)

This page contains a set of Base Ten blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Ten Long is 1 cm by 10 cm and is labeled "TL". The Base Ten Flat is 10 cm by 10 cm and is labeled "TF". Extra longs for Base Ten are on a separate Supplementary Material Card for Cards 12-15, found after Card 15.



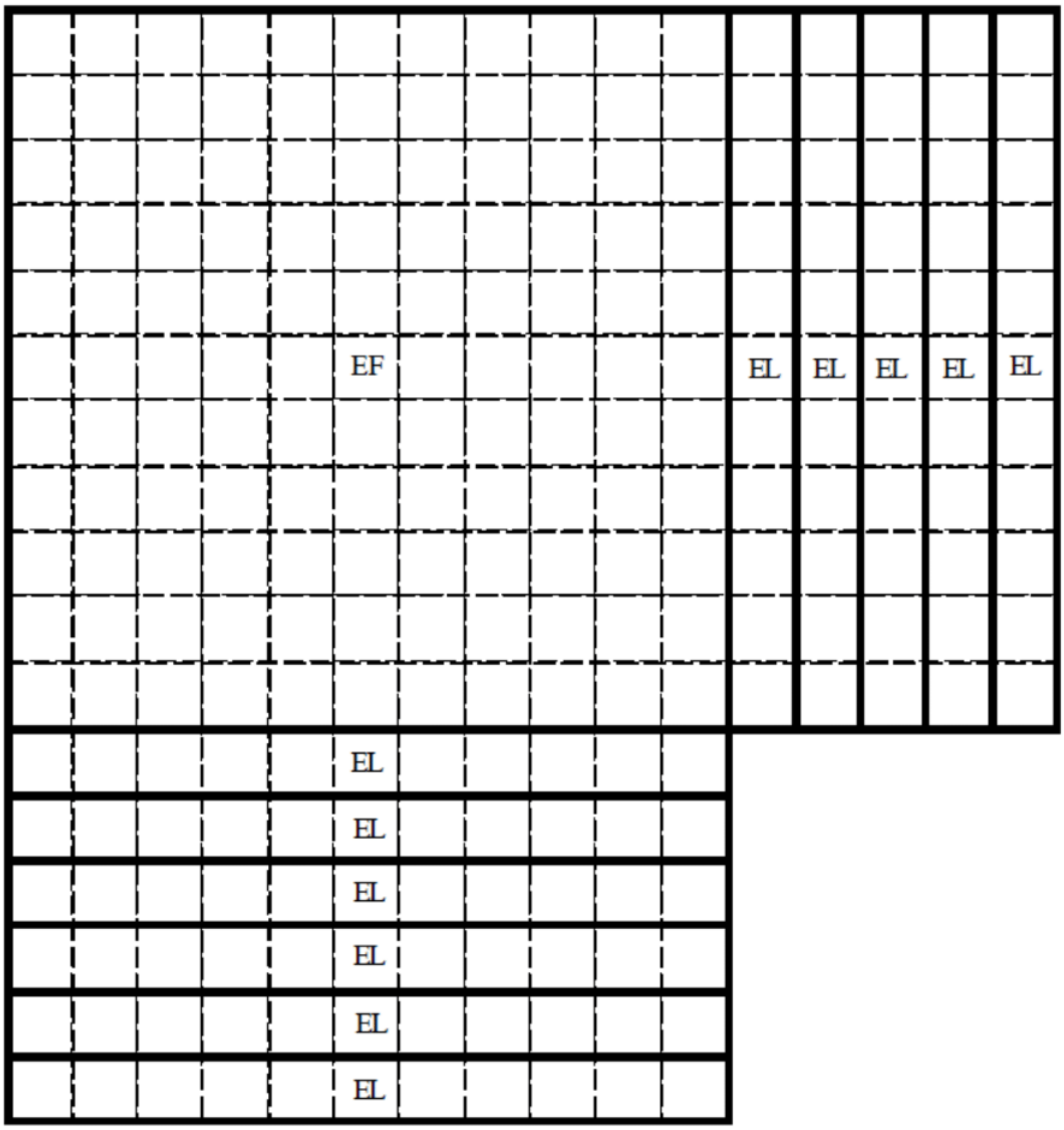
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9.5.15: Base Eleven Blocks

Longs (EL) and Flats (EF)

This page contains a set of Base Eleven blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments.Each piece is labeled. A Base Eleven Long is 1 cm by 11 cm and is labeled "EL". The Base Eleven Flat is 11 cm by 11 cm and is labeled "EF". Extra longs for Base Eleven are on a separate Supplementary Material Card for Cards 12-15, found after Card 15.



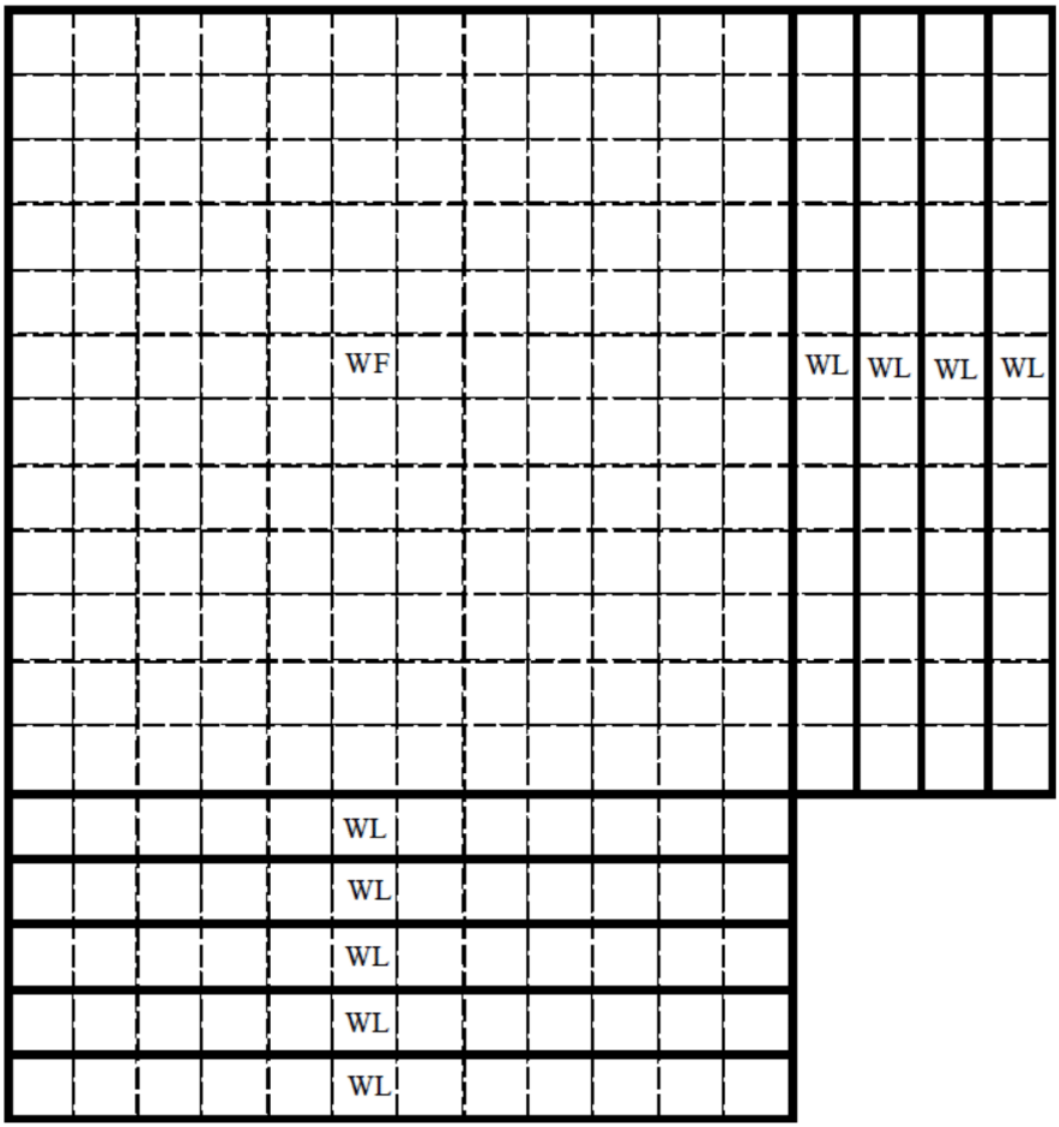
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9.5.16: Base Twelve Blocks

Longs (WL) and Flats (WF)

This page contains a set of Base Twelve blocks. Use the unit blocks from Material Card 9 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled.A Base Twelve Long is 1 cm by 12 cm and is labeled "WL". The Base Twelve Flat is 12 cm by 12 cm and is labeled "WF". Extra longs for Base Twelve are on a separate Supplementary Material Card for Cards 12-15, found on the next sheet.

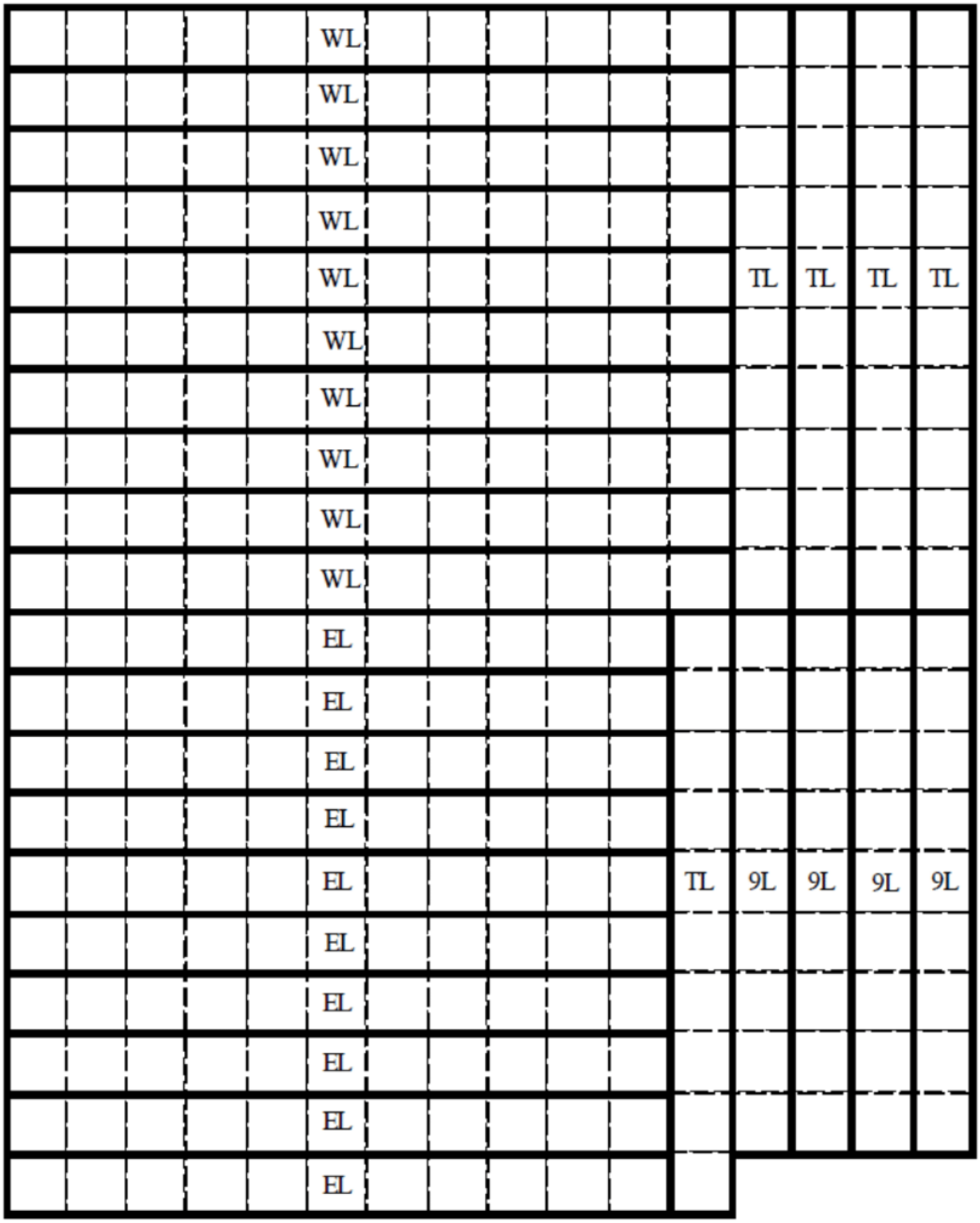


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9.5.17: Supplementary Longs

For Base Nine (9L), Base Ten (TL), Base Eleven (EL) and Base Twelve (WL)



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9.5.18: Centimeter Strips

White Centimeter Strips

The dimensions of each strip are one centimeter by one centimeter. The abbreviation is **W**.

W	W	W	W	W	W	W	W	W	W	W	W	W	W	W
W	W	W	W	W	W	W	W	W	W	W	W	W	W	W
W	W	W	W	W	W	W	W	W	W	W	W	W	W	W
W	W	W	W	W	W	W	W	W	W	W	W	W	W	W

Red Centimeter Strips

The dimensions of each strip are two centimeters by one centimeter. The abbreviation is **R**.

R	R	R	R	R	R	R	R
R	R	R	R	R	R	R	R
R	R	R	R	R	R	R	R
R	R	R	R	R	R	R	R

Light Green Centimeter Strips

The dimensions of each strip are three centimeters by one centimeter. The abbreviation is **L**.

L	L	L	L	L
L	L	L	L	L
L	L	L	L	L
L	L	L	L	L
L	L	L	L	L

Purple Centimeter Strips

The dimensions of each strip are four centimeters by one centimeter. The abbreviation is **P**.

P	P	P	P
P	P	P	P
P	P	P	P
P	P	P	P
P	P	P	P

Yellow Centimeter Strips

The dimensions of each strip are five centimeters by one centimeter. The abbreviation is **Y**.

Y	Y	Y
Y	Y	Y
Y	Y	Y
Y	Y	Y
Y	Y	Y

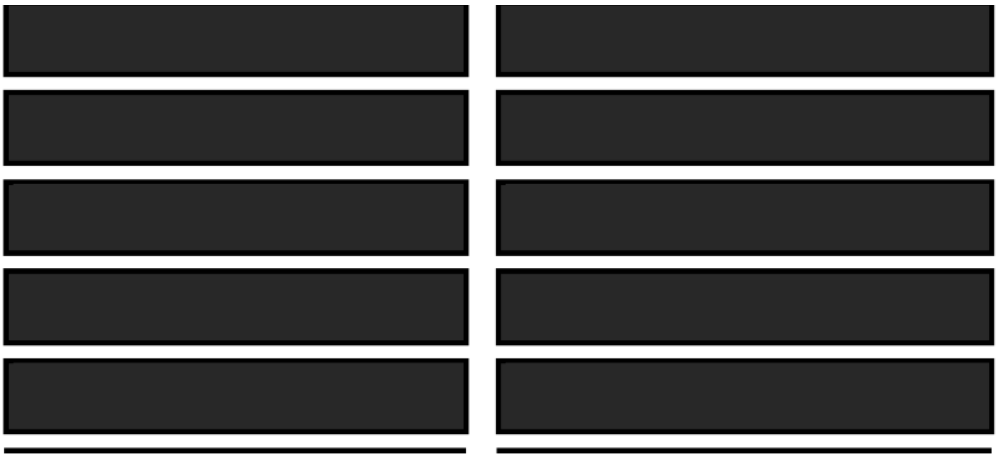
Dark Green Centimeter Strips

The dimensions of each strip are six centimeters by one centimeter. The abbreviation is **D**.

D	D	D
D	D	D
D	D	D
D	D	D
D	D	D

Black Centimeter Strips

The dimensions of each strip are seven centimeters by one centimeter. The abbreviation is **K**



Tan or Light Brown Centimeter Strips

The dimensions of each strip are eight centimeters by one centimeter. The abbreviation is **N**.

N	**N**
N	**N**
N	**N**
N	**N**
N	**N**

Blue Centimeter Strips

The dimensions of each strip are nine centimeters by one centimeter. The abbreviation is **B**.

B	**B**
B	**B**
B	**B**
B	**B**
B	**B**

Orange Centimeter Strips

The dimensions of each strip are ten centimeters by one centimeter. The abbreviation is **O**.

O
O
O
O
O

Silver or Gray Centimeter Strips

The dimensions of each strip are eleven centimeters by one centimeter. The abbreviation is **S.**

S
S
S
S
S

Hot Pink Centimeter Strips

The dimensions of each strip are twelve centimeters by one centimeter. The abbreviation is **H**.

H
H
H
H
H

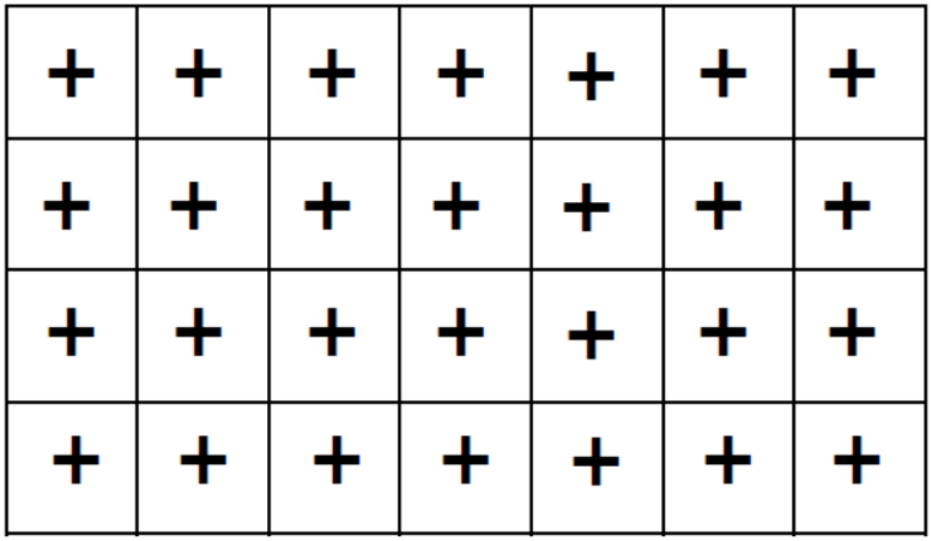
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9.5.19: Counters

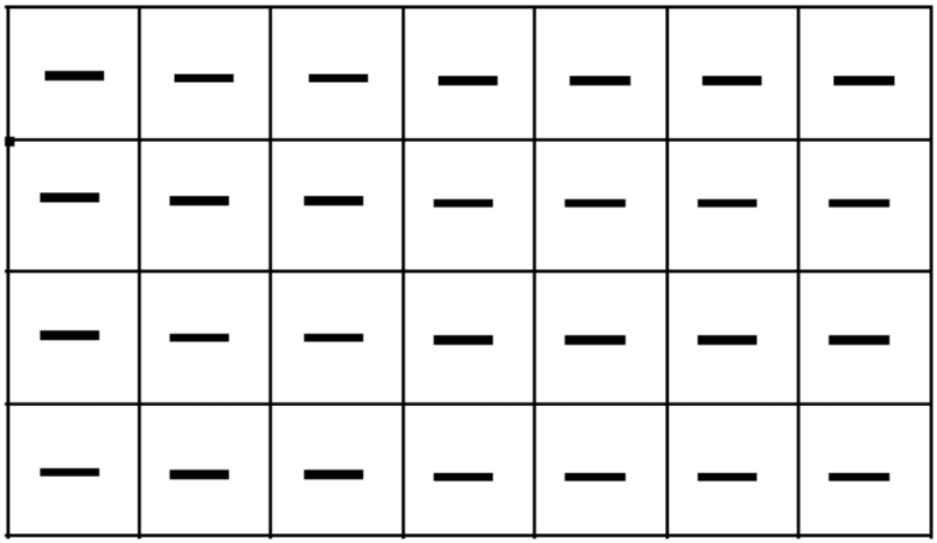
Postive Green Counters

Each counter represents a positive one (+1). THese are used to perform the four basic operations (addition, subtraction, multiplication and division) on the integers (positive and negative whole numbers and zero).



Negative Red Counters

Each counter represents a negative one (-1). These are used to perform the four basic operations (addition, subtraction, multiplication and division) on tthe integers (positive and negative whole numbers and zero).



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9.5.20: Number Squares

Prime Number Squares (2 - 23)

2	3	5	7	11	13	17	19	23
2	3	5	7	11	13	17	19	23
2	3	5	7	11	13	17	19	23
2	3	5	7	11	13	17	19	23
2	3	5	7	11	13	17	19	23

Prime Number Squares (29 - 59)

29	31	37	41	43	47	53	59
29	31	37	41	43	47	53	59
29	31	37	41	43	47	53	59
29	31	37	41	43	47	53	59
29	31	37	41	43	47	53	59

Composite Squares (4 - 40)

4	4	4	6	6	6	8	8
8	9	9	9	10	10	10	12
12	12	14	14	14	15	15	15
16	16	16	18	18	18	20	20
20	21	21	21	22	22	22	24
24	24	25	25	25	26	26	27
27	27	28	28	28	30	30	30
32	32	32	33	33	33	34	34
34	35	35	35	36	36	36	38
38	38	39	39	39	40	40	40

Composite Squares (42 - 75)

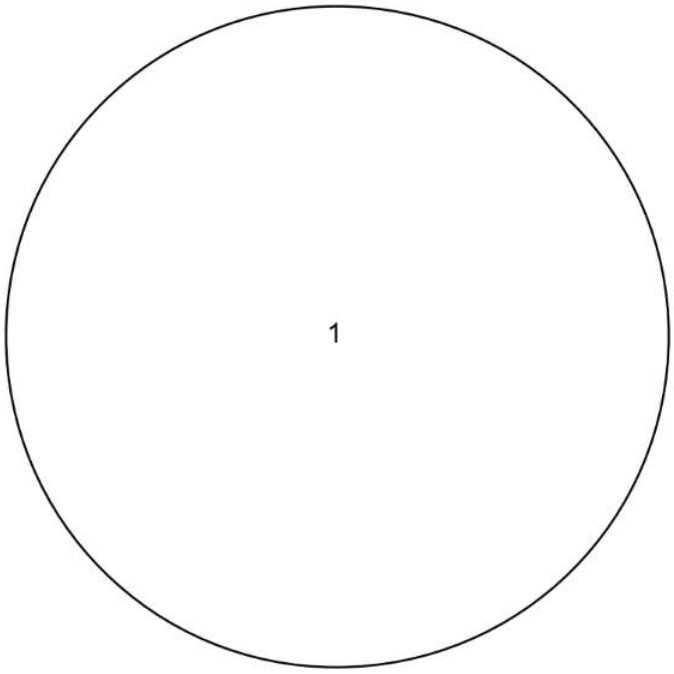
42	42	42	44	44	44	45	45
45	46	46	46	48	48	48	49
49	49	50	50	50	51	51	51
52	52	52	54	54	54	55	55
55	56	56	56	57	57	57	58
58	58	60	60	60	62	62	62
63	63	63	64	64	64	65	65
65	66	66	66	68	68	68	69
69	69	70	70	70	72	72	72
74	74	74	75	75	75	26	

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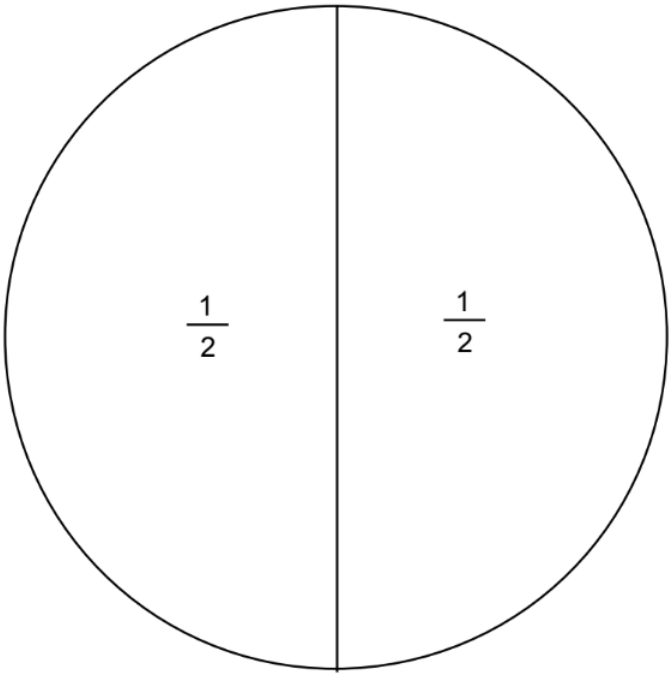
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9.5.21: Fraction Circles

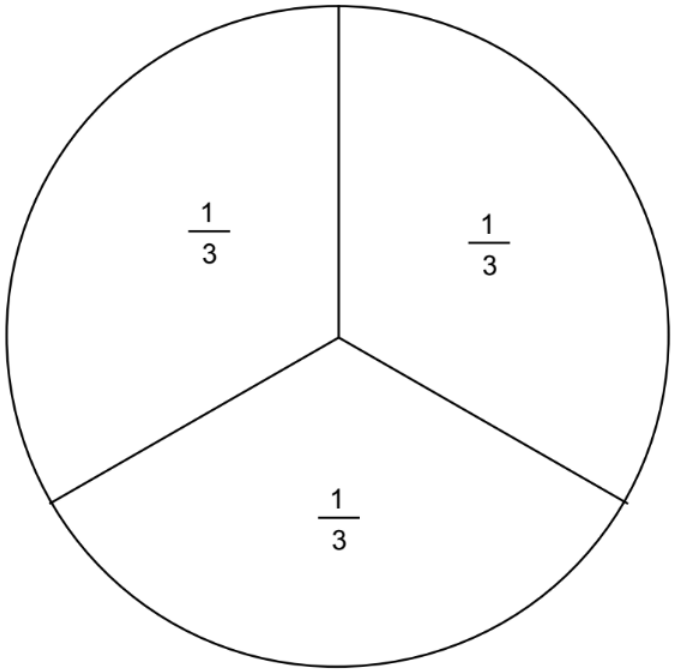
White Fraction Circle



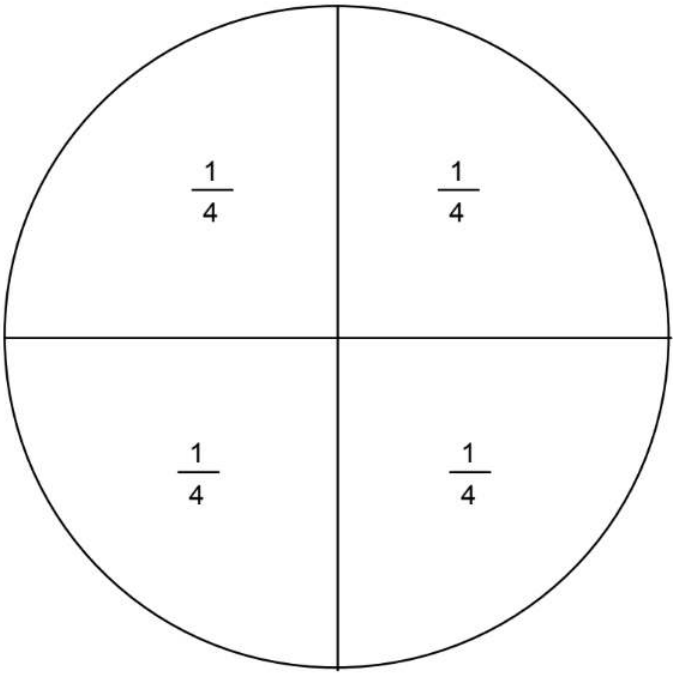
Red Fraction Circle (halves)



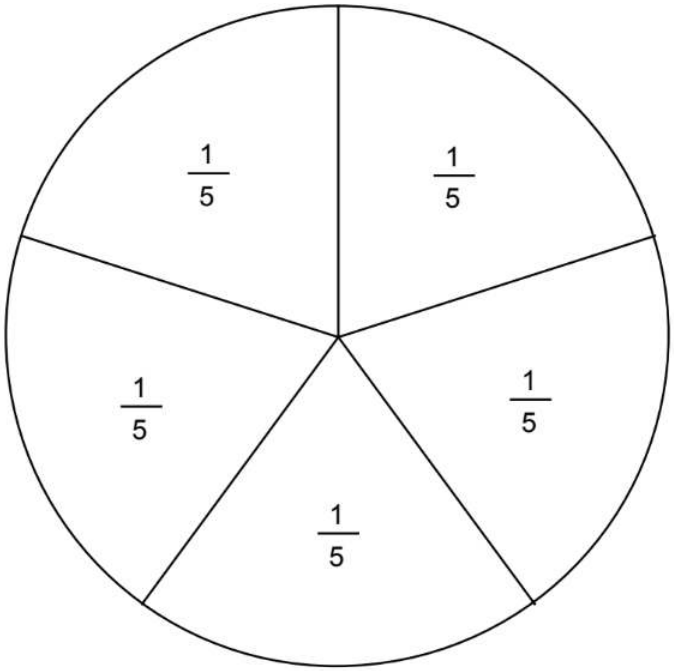
Light Green Circle (thirds)



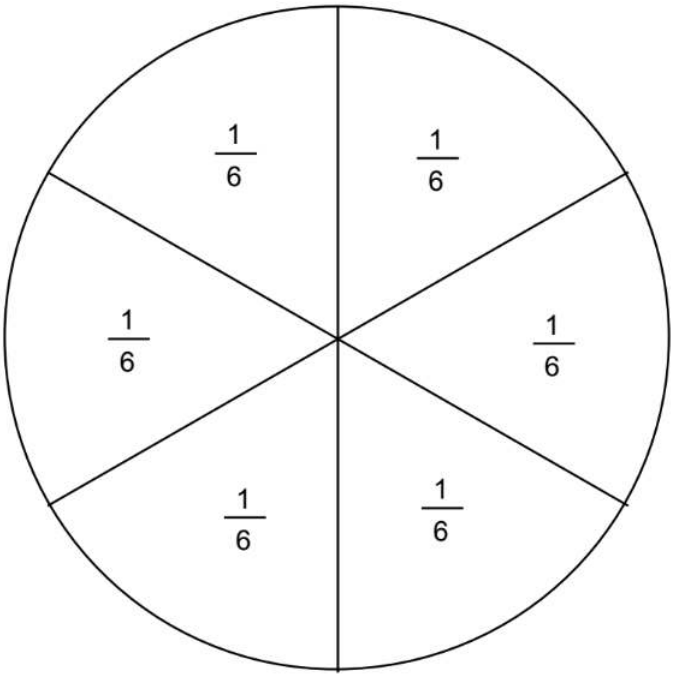
Purple Circle (fourths)



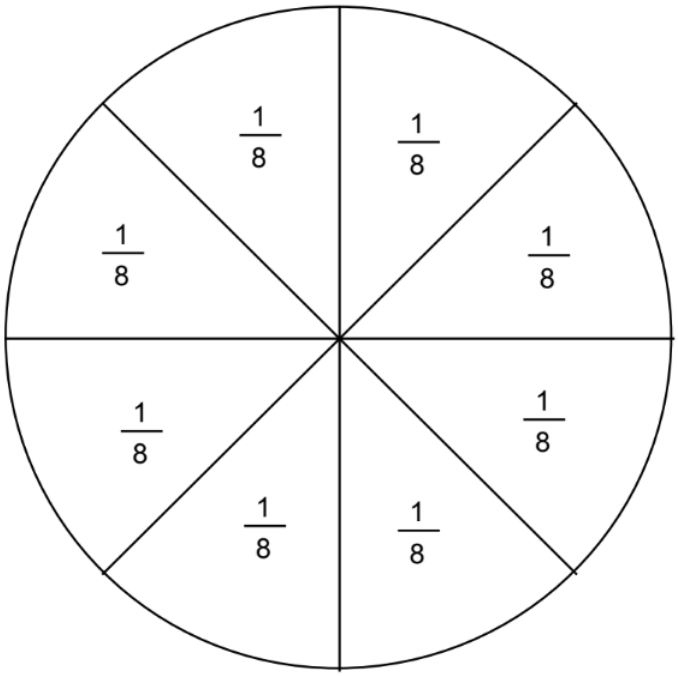
Yellow (fifths)



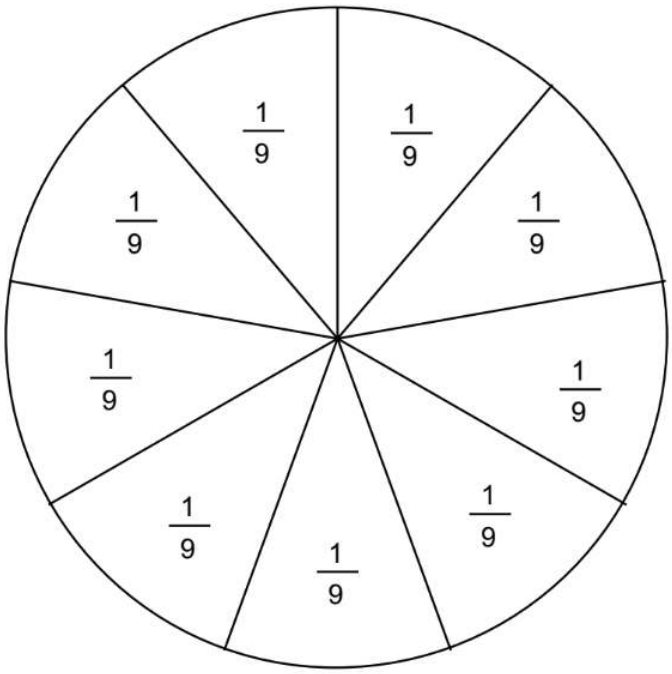
Dark Green (sixths)



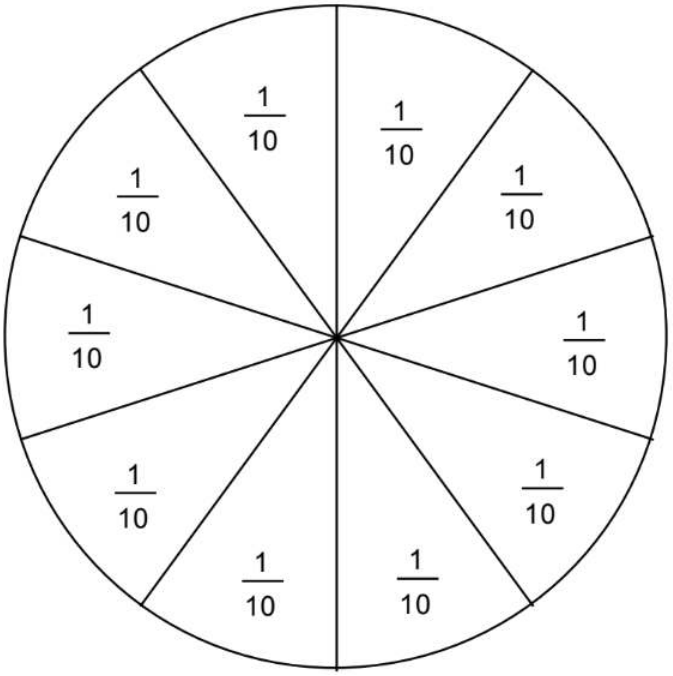
Brown (eights)



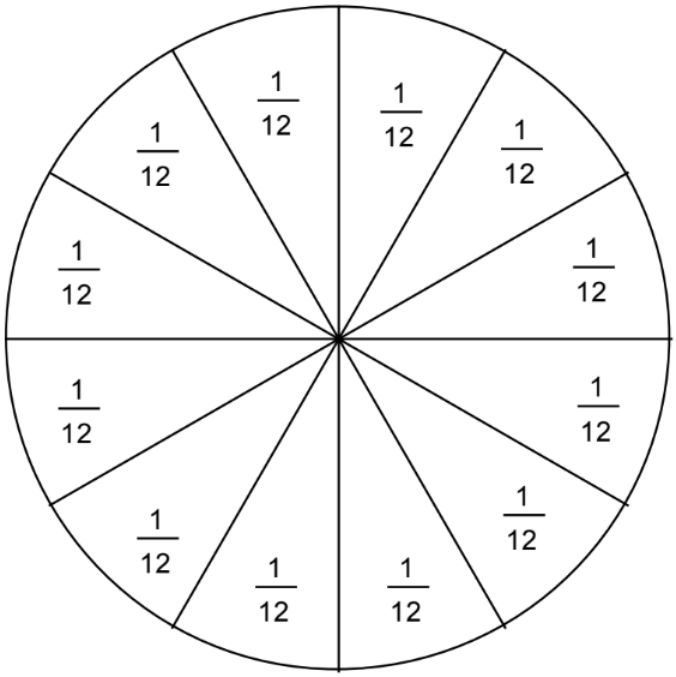
Blue (ninths)



Orange (tenths)



Hot Pink (twelfths)

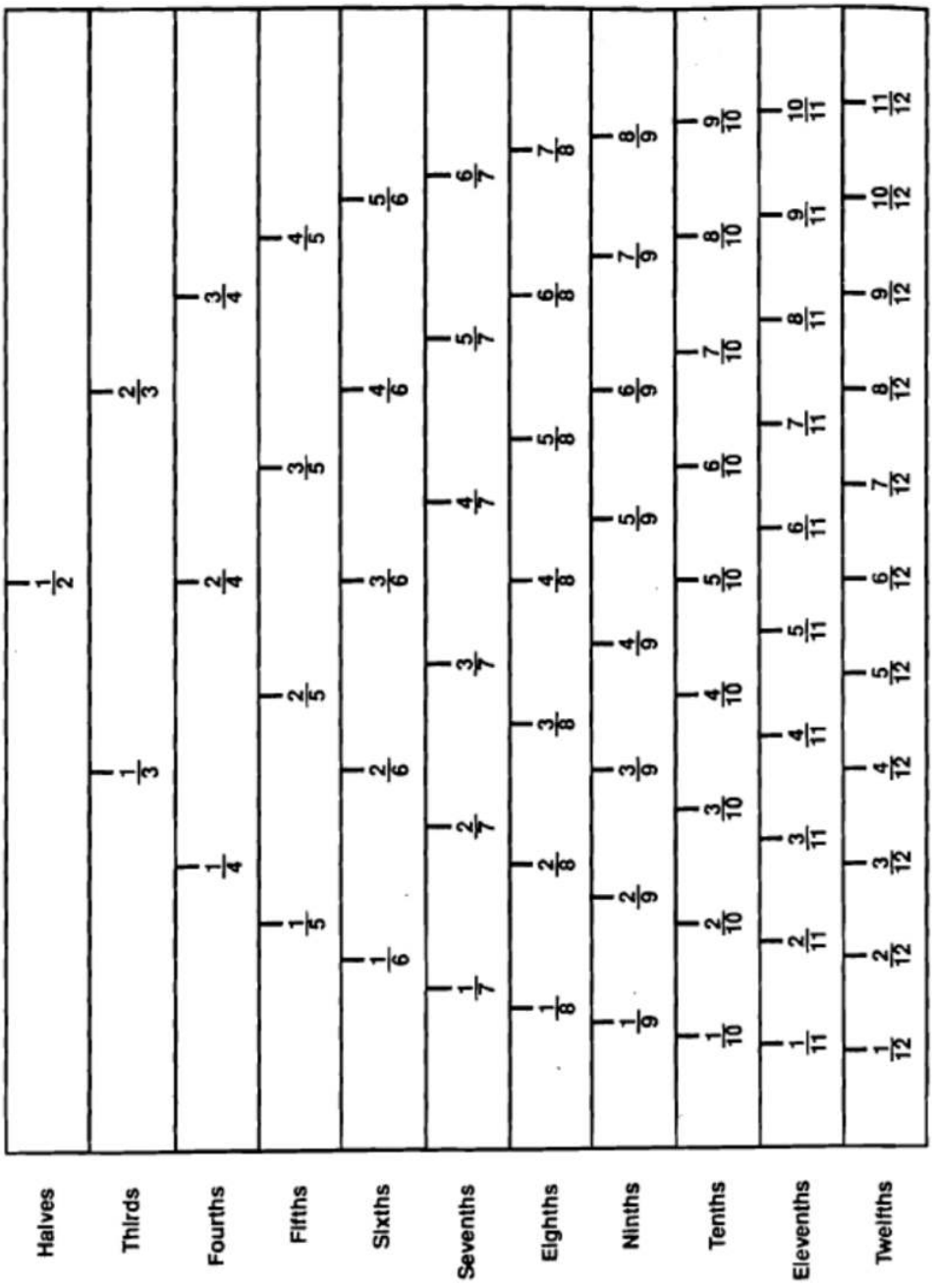


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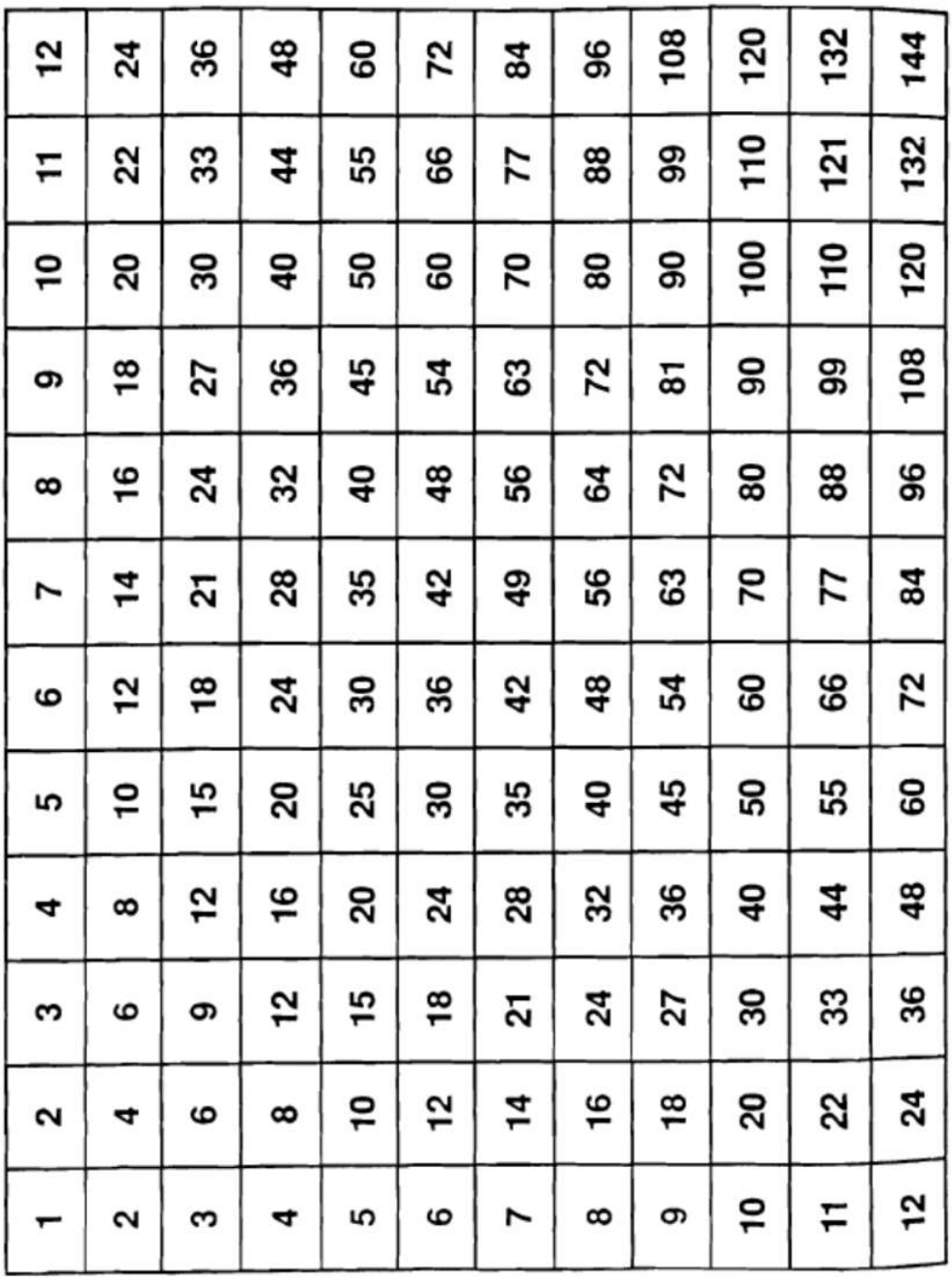
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Fraction Arrays

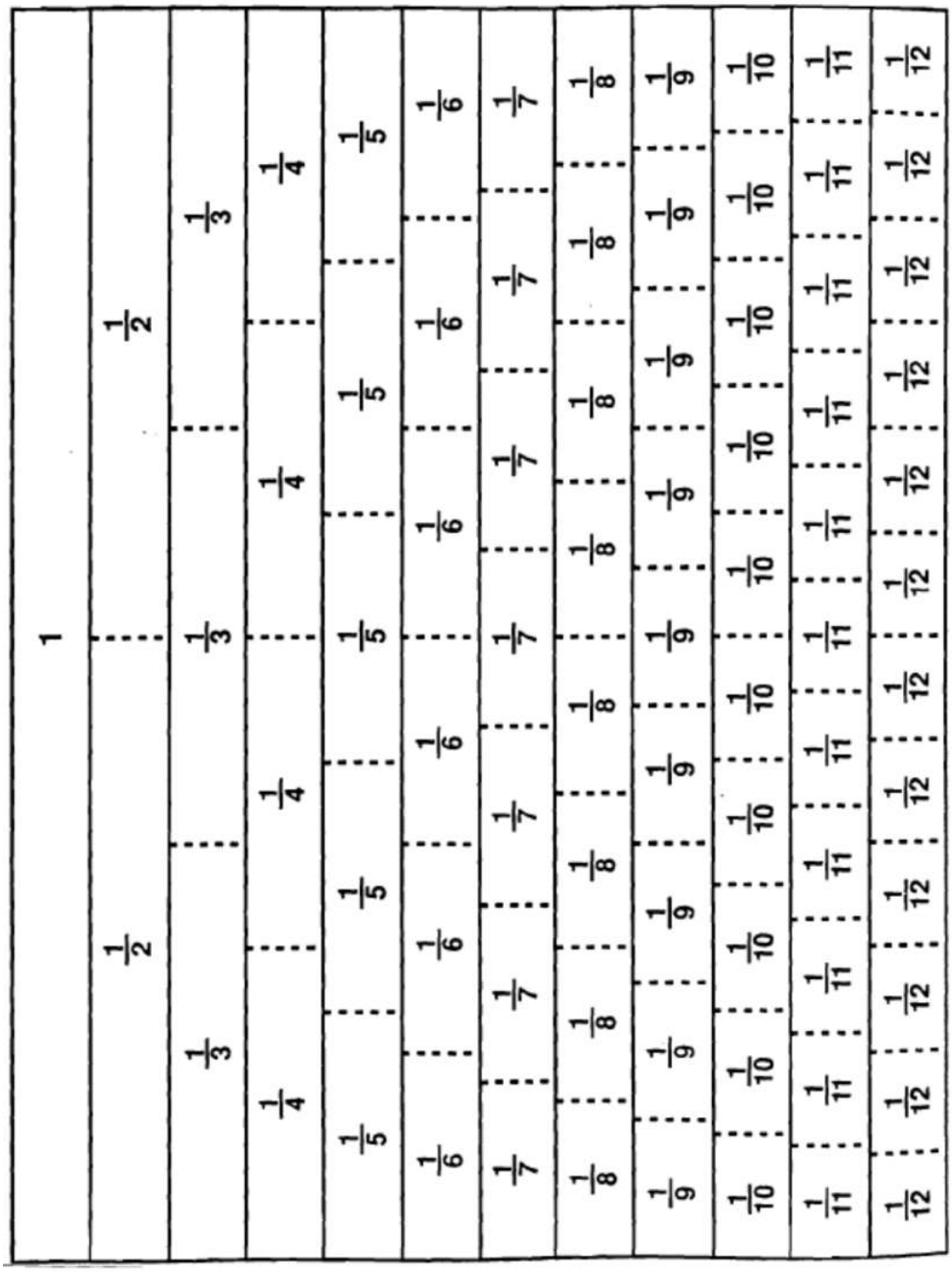


Multiple Strips



1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Fraction Strips



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